(3) Mathematical Expectation

(Book: Chapter 4, pg 111-137)
Mean of a Random Variable:

Definition:
Let \( X \) be a random variable with probability distribution \( f(x) \). The mean or expected value of \( X \) is:

\[
\mu = E(X) = \sum_{X} X f(X) \quad \text{if } X \text{ is discrete}
\]

\[
\mu = E(X) = \int_{-\infty}^{\infty} X f(X) dX \quad \text{if } X \text{ is continuous} \quad (1)
\]
Properties of the Expectation:

1. \( E (a) = a \), where \( a \) is a constant
2. \( E (aX) = aE(X) \)
3. \( E (aX + b) = aE(X) + b \)
Ex (1):

Find the expected number of chemists on a committee of 3 selected at random from 4 chemists and 3 biologists.

Find: $E(5)$, $E(3x)$, $E(2x-1)$
Solution:

Let $X$ represent the number of chemists on the committee. The probability distribution of $X$ is given by:

$$f(x) = \binom{4}{x} \binom{3}{3-x} \binom{7}{3} , \quad X = 0, 1, 2, 3$$
\[
\begin{align*}
 f(0) &= \frac{\binom{4}{0} \binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}, \\
 f(1) &= \frac{\binom{4}{1} \binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}, \\
 f(2) &= \frac{\binom{4}{2} \binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}, \\
 f(3) &= \frac{\binom{4}{3} \binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}.
\end{align*}
\]
<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>1/35</td>
<td>12/35</td>
<td>18/35</td>
<td>4/35</td>
<td>1</td>
</tr>
<tr>
<td>x f(x)</td>
<td>0</td>
<td>12/35</td>
<td>36/35</td>
<td>12/35</td>
<td>60/35=1.71</td>
</tr>
</tbody>
</table>

\[
E(X) = \mu_X = \sum x f(x) = \frac{60}{35} = 1.71
\]

\[
E(5) = 5
\]

\[
E(3x) = 3E(x) = 3\left(\frac{60}{35}\right) = 5.143
\]

\[
E(2x - 1) = 2E(x) - 1 = 2\left(\frac{60}{35}\right) - 1 = 2.429
\]

See Ex 4.1 pg 113
Let $X$ be a random variable that denotes the life in hours of a certain electronic device. The probability density function is given by:

$$f(x) = \begin{cases} \frac{20000}{X^3} , & X > 100 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected life of this type of device
Solution:

\[
\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{100}^{\infty} X \left(\frac{20000}{X^3}\right) \, dX
\]

\[
= \int_{100}^{\infty} \frac{20000}{X^2} \, dX = 20000 \int_{100}^{\infty} X^{-2} \, dX
\]

\[
= 20000 \left(\frac{X^{-1}}{-1}\right)_{100}^{\infty} = 20000[(100)^{-1} - (\infty)^{-1}]
\]

\[
= \frac{20000}{100} - \frac{20000}{\infty} = 200 - 0 = 200
\]
EX 4.4 pg 115:

Suppose that the number of cars $X$ that pass through a car wash between 4 P.M. and 5 P.M. on any sunny Friday has the following probability distribution:

<table>
<thead>
<tr>
<th>$X$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(X)$</td>
<td>1/12</td>
<td>1/12</td>
<td>1/4</td>
<td>1/4</td>
<td>1/6</td>
<td>1/6</td>
</tr>
</tbody>
</table>

Let $g(x) = 2x - 1$ represent the amount of money in dollars, paid to the attendant by the manager. Find the attendant's expected earning for this particular time period.
Solution:

<table>
<thead>
<tr>
<th>X</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(X)</td>
<td>1/12</td>
<td>1/12</td>
<td>1/4</td>
<td>1/4</td>
<td>1/6</td>
<td>1/6</td>
<td>1</td>
</tr>
<tr>
<td>X f(x)</td>
<td>4/12</td>
<td>5/12</td>
<td>6/4</td>
<td>7/4</td>
<td>8/6</td>
<td>9/6</td>
<td>164/24</td>
</tr>
</tbody>
</table>

\[ E(g(x)) = E(2x - 1) = 2E(X) - 1 = \]
\[ = 2 \left( \frac{164}{24} \right) - 1 = 12.67 \]
Ex (4.5 pg 115):

Let $X$ be a random variable with density function:

$$f(x) = \begin{cases} \frac{X^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected value of $g(x) = 4x + 3$
Solution:

\[ E(X) = \int_{-1}^{2} x\left(\frac{x^2}{3}\right) \, dx = \int_{-1}^{2} \left(\frac{x^3}{3}\right) \, dx = \left. \left(\frac{x^4}{12}\right) \right|_{-1}^{2} \]

\[ = \frac{1}{12} [2^4 - (-1)^4] = \frac{1}{12} (16 - 1) = \frac{15}{12} \]

\[ E(g(x)) = E(4x + 3) = 4E(X) + 3 = 4\left(\frac{15}{12}\right) + 3 = 8 \]
Variance:

Definition:

Let \( X \) be a random variable with probability distribution \( f(x) \) and mean \( \mu \). The variance of \( X \) is denoted by \( V(x) \) or \( \sigma^2_x \):

\[
V(x) = \sigma^2 = E(x - \mu)^2 = \sum_{\forall x} (x - \mu)^2 f(x) = E(X^2) - (E(X))^2 \quad \text{if } x \text{ is discrete (2)}
\]

\[
V(x) = \sigma^2 = E(x - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = E(X^2) - (E(X))^2 \quad \text{if } x \text{ is continuous (3)}
\]

where:

\[
E(x^2) = \begin{cases} 
\sum_{\forall x} x^2 f(x) & \text{if } x \text{ is discrete} \\
\int_{-\infty}^{\infty} x^2 f(x)dx & \text{if } x \text{ is continuous}
\end{cases}
\]
Properties of the variance:

1. \( V(a) = 0 \) where \( a \) is a constant

2. \( V(aX) = a^2 V(X) \)

3. \( V(aX + b) = a^2 V(X) + 0 \)

The Standard Deviation:

The positive square root of the variance, \( \sigma \) is called the standard deviation of \( X \) which is given by:

\[
\sigma_x = \sqrt{V(x)} = \sqrt{E(x - \mu_x)^2}
\]
Ex (4.8 pg 120):
The probability distribution for company A is given by:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
</tr>
</tbody>
</table>

and for company B is given by:

<table>
<thead>
<tr>
<th>Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(y)</td>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Show that the variance of the probability distribution for company B is greater than that of company A.
Solution:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>x f(x)</td>
<td>0.3</td>
<td>0.8</td>
<td>0.9</td>
<td>2</td>
</tr>
<tr>
<td>f(x)x^2</td>
<td>0.3</td>
<td>1.6</td>
<td>2.7</td>
<td>4.6</td>
</tr>
</tbody>
</table>

σ^2 = E(x^2) − (E(x))^2 = 4.6 − 4 = 0.6, σ = .77
\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Y & 0 & 1 & 2 & 3 & 4 & \sum \\
\hline
f(y) & 0.2 & 0.1 & 0.3 & 0.3 & 0.1 & 1 \\
\hline
Y f(y) & 0 & 0.1 & 0.6 & 0.9 & 0.4 & 2 \\
\hline
y^2 f(y) & 0 & 0.1 & 1.2 & 2.7 & 1.6 & 5.6 \\
\hline
\end{tabular}
\end{table}

$$\sigma^2 = E(y^2) - (E(y))^2 = 5.6 - 4 = 1.6, \sigma = 1.26$$

Note that $\sigma_y^2$ is greater than $\sigma_x^2$. 
Ex (4.10 pg 121):
The weekly demand for a drinking-water product, in thousands of liters from a local chain of efficiency stores having the probability density:

\[
f(x) = \begin{cases} 
2(x - 1), & 1 < X < 2 \\
0, & \text{otherwise} 
\end{cases}
\]

Find the mean and variance of \( x \).
Solution:

\[
\mu = \int_{1}^{2} 2x (x - 1) \, dx = 2 \int_{1}^{2} (x^2 - x) \, dx = 2 \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_1^2 = 2\left[ \frac{8}{3} - 2 - \frac{1}{3} - \frac{1}{2} \right] = 2\left( \frac{8 - 6}{3} - \frac{2 - 3}{6} \right) = 2\left( \frac{2}{3} + \frac{1}{6} \right) = \frac{5}{3}
\]

\[
E(X^2) = \int_{1}^{2} 2x^2 (x - 1) \, dx = 2 \int_{1}^{2} (x^3 - x^2) \, dx = 2 \left[ \frac{x^4}{4} - \frac{x^3}{3} \right]_1^2 = 2\left[ (4 - \frac{2}{8}) - (\frac{1}{4} - \frac{1}{3}) \right] = \frac{17}{6}
\]

\[
\sigma^2 = E(x^2) - (E(x))^2 = \frac{17}{6} - \left( \frac{5}{3} \right)^2 = 1/18
\]
Ex 4.18 pg 129:

Let $X$ be a random variable having the density function:

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the variance of the random variable $g(x) = 4x + 3$. 
Solution:

\[ V(g(x)) = V(4x+3) = 16 \]

\[ V(x) = 16[E(x^2) - (E(x))^2] \]

\[ E(x) = \int_{-1}^{2} x \frac{x^2}{3} \, dx = \frac{x^4}{12} \bigg|_{-1}^{2} = \frac{16}{12} - \frac{1}{12} = \frac{15}{12} = \frac{5}{4} \]

\[ E(x^2) = -1 \int_{-1}^{2} x^2 \left( \frac{x^2}{3} \right) \, dx = \frac{x^5}{15} \bigg|_{-1}^{2} = \frac{32}{15} - (-1) = \frac{11}{5} \]

\[ V(x) = E(x^2) - (E(x))^2 = \frac{11}{5} - \left( \frac{5}{4} \right)^2 = \frac{11}{5} - \frac{25}{16} = \frac{176 - 125}{80} = 0.6375 \]

\[ V(g(x)) = V(4x+3) = 16V(x) + 0 = 16(0.6375) = 10.2 \]
4.3 Means and Variance of Linear Combinations of Random Variables (pg 128):
The expected value of the sum or difference of two or more functions of a random variable $X$ is the sum or difference of the expected values of the functions. That is

$$E(g(x) \pm h(x)) = E(g(x)) \pm E(h(x)) \quad (6)$$
Ex4.19 pg 129 :

Let $X$ be a random variable with probability distribution as follows:

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1/3</td>
<td>1/2</td>
<td>0</td>
<td>1/6</td>
</tr>
</tbody>
</table>

Find the expected value of $y = (x - 1)^2$. 
Solution:

\[ E(y) = E(x-1)^2 = E(x^2 - 2x + 1) = E(x^2) - 2E(x) + 1 \]

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \sum )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1/3</td>
<td>1/2</td>
<td>0</td>
<td>1/6</td>
<td>1</td>
</tr>
<tr>
<td>( X f(x) )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>3/6</td>
<td>1</td>
</tr>
<tr>
<td>( X^2 f(x) )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>9/6</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ E(y) = 2 - 2(1) + 1 = 1 \]

See ex 4.17 pg 128.
Ex 4.20 pg 130:
Find the expected value for \( g(x) = x^2 + x - 2 \),
where \( X \) has the density function:

\[
f(x) = \begin{cases} 
2(x-1) & , \quad 1 < x < 2 \\ 
0 & , \quad \text{otherwise}
\end{cases}
\]
Solution:

\[ E(x) = \int_{1}^{2} x^2 (x-1) \, dx = 2 \int_{1}^{2} (x^2 - x) \, dx = 2(\frac{x^3}{3} - \frac{x^2}{2}) \bigg|_{1}^{2} = \]

\[ = 2\left(\frac{2}{3} + \frac{1}{6}\right) = \frac{5}{3} \]

\[ E(x^2) = \int_{1}^{2} 2x^2 (x-1) \, dx = 2 \int_{1}^{2} (x^3 - x^2) \, dx = 2(\frac{x^4}{4} - \frac{x^3}{3}) \bigg|_{1}^{2} = \]

\[ = 2\left[(4 - \frac{8}{3}) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = 2\left(\frac{12 - 8}{3} - \frac{3 - 4}{12}\right) = 2\left(\frac{4}{3} + \frac{1}{12}\right) = \frac{17}{6} \]

\[ E(x^2 + x - 2) = E(x^2) + E(x) - 2 = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2} \]
4.4 Chebyshev's Theorem (pg 135):
The probability that any random variable $X$ will assume a value within $K$ standard deviations of the mean $\mu_x$ is at least $\left(1 - \frac{1}{K^2}\right)$.
That is:

$$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2} \quad (7)$$
Ex (4.27 pg 137):

A random variable $X$ has a mean $\mu = 8$, a variance $\sigma^2 = 9$ and an unknown probability distribution. Find:

(a) $P(-4 < X < 20)$

(b) $P(|X - 8| \geq 6)$
Solution:

(a) \( P(-4 < X < 20) = P[8 - (k)(3) < X < 8 + (k)(3)] \rightarrow \)
\[ 8 - 3k = -4 \rightarrow 8 + 4 = 3k \rightarrow 12 = 3k \rightarrow k = 4 \]

\[ P(-4 < X < 20) \geq 1 - \frac{1}{16} \rightarrow P(-4 < X < 20) \geq \frac{15}{16} \]

(b) \( P(|X - 8| \geq 6) = 1 - P(|x - 8| < 6) = 1 - P(-6 < (X - 8) < 6) \)
\[ = 1 - P(-6 + 8 < X < 6 + 8) = 1 - P(2 < X < 14) \]

\[ 1 - P(2 < X < 14) \geq 1 - \frac{1}{k^2} \rightarrow 1 - 1 + \frac{1}{k^2} \geq P(2 < X < 14) \]
\[ \rightarrow \frac{1}{k^2} \geq P(2 < X < 14) \rightarrow P(2 < X < 14) \leq \frac{1}{k^2} \]

\[ 2 = 8 - 3K \rightarrow 3K = 8 - 2 = 6 \rightarrow K = 2 \quad \text{or} \]
\[ 14 = 8 + 3k \rightarrow 14 - 8 = 3K \rightarrow 6 = 3K \rightarrow K = 2 \]

\[ P(2 < x < 14) \leq \frac{1}{4} \]