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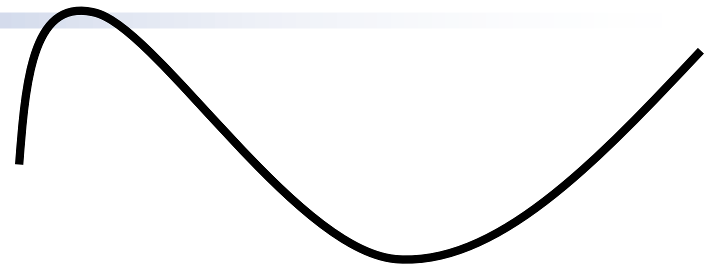
Spline Representations

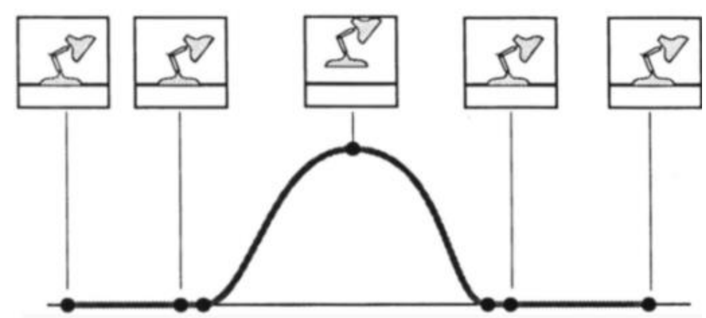
Contents

Spline Representations

- A spline is a smooth curve defined mathematically using a set of constraints
- Splines have many uses:
  - 2D illustration
  - Fonts
  - 3D Modelling
  - Animation



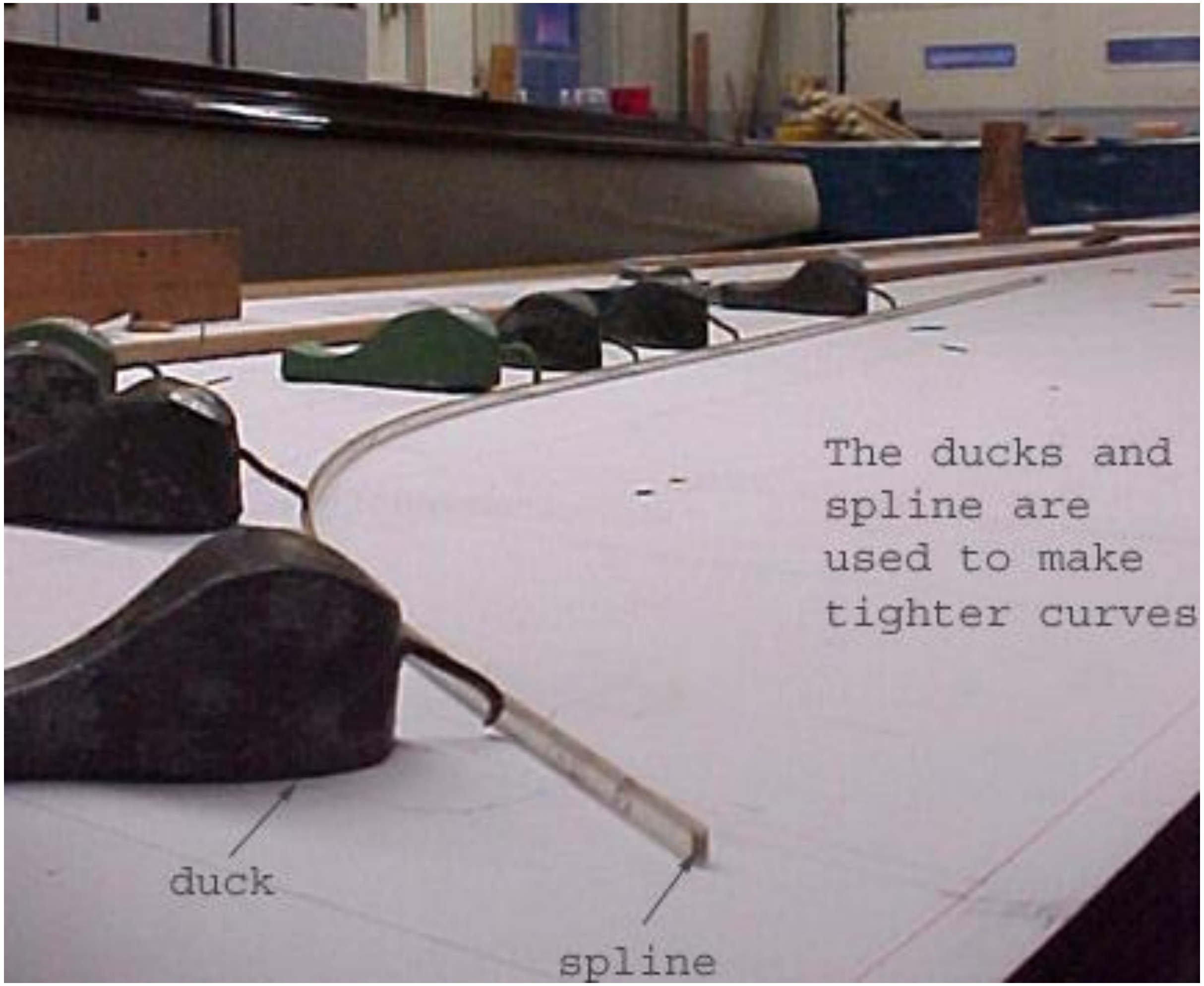
"Manifold Splines", X. Gu, Y. He & H. Qin, Solid and Physics Modeling 2005.



ACM © 1987 "Principles of traditional animation applied to 3D computer animation"

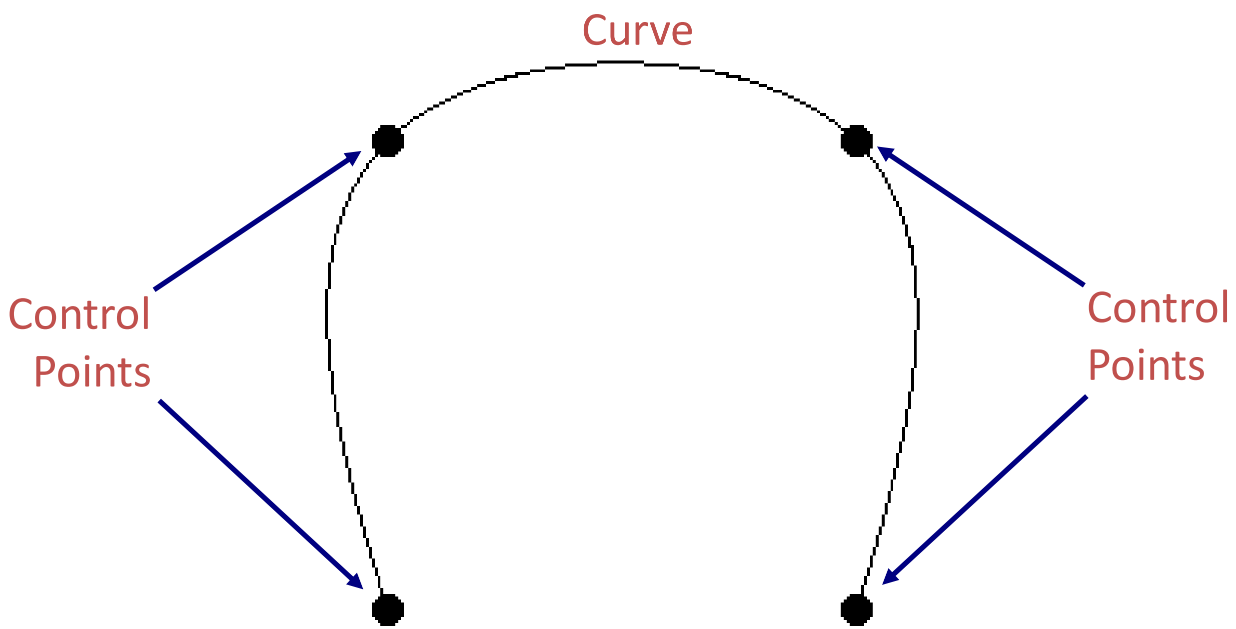
Physical Splines

- Physical splines are used in car/boat design



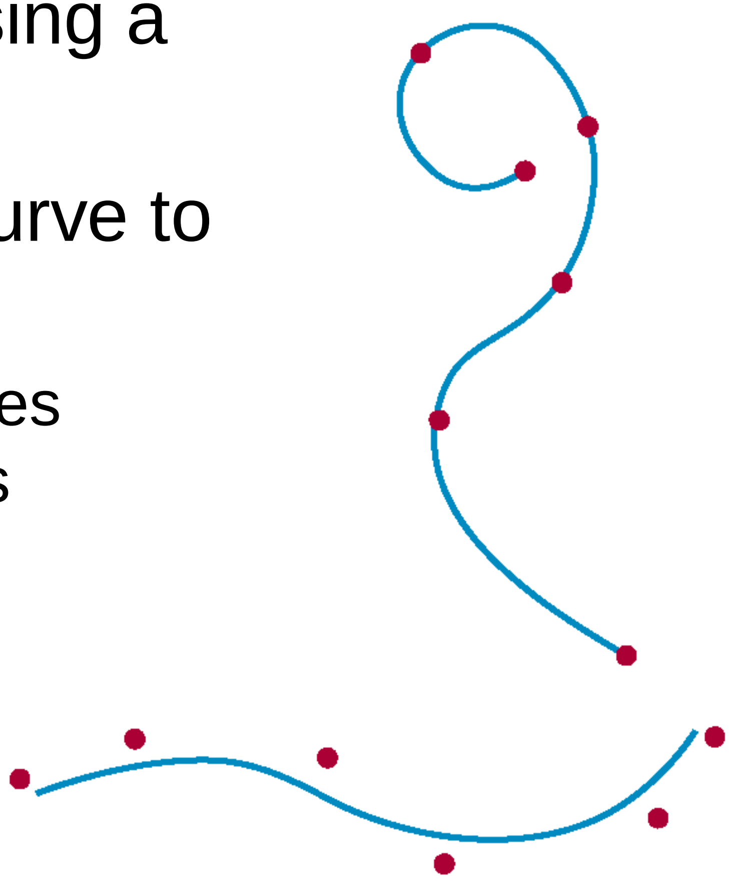
Big Idea

- User specifies control points
- Defines a smooth curve



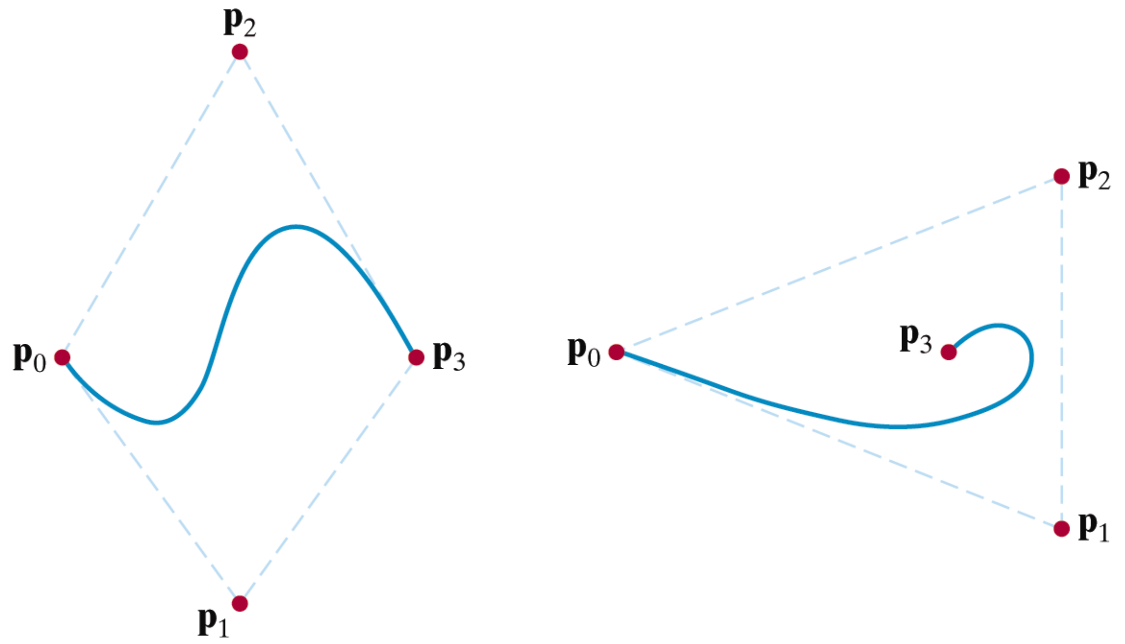
Interpolation vs Approximation

- A spline curve is specified using a set of **control points**
- There are two ways to fit a curve to these points:
  - **Interpolation -** the curve passes through all of the control points
  - **Approximation -** the curve does not pass through all of the control points



Convex Hulls

- The boundary formed by the set of control points for a spline is known as a **convex hull**
- Think of an elastic band stretched around the control points



Parametric Continuity Conditions

- We can impose various Continuity Conditions at the connection points.
- section of a spline curve is described with a set of parametric coordinate functions of the form

$$x = x(u), \quad y = y(u), \qquad z = z(u), \qquad u_1 \le u \le u_2$$

- We set parametric continuity by matching the parametric derivatives of adjoining curve sections at their common boundary.

Parametric Continuity Conditions Contd.

- **Zero-order parametric continuity**, represented as C° continuity, means simply that the curves meet. That is, the values of x, y and z evaluated at u_2 for the first curve section are equal, respectively, to the values of x, y, and z evaluated at u_1 for the next curve section.
- **First-order parametric continuity**, referred to C^1 continuity, means that the first parametric derivatives (tangent lines) of the coordinate functions for two successive curve sections are equal in their joining point.
- **Second-order parametric continuity**, or C^2 continuity, means that both the first and second parametric derivatives of the two curve sections are the same at the intersection.

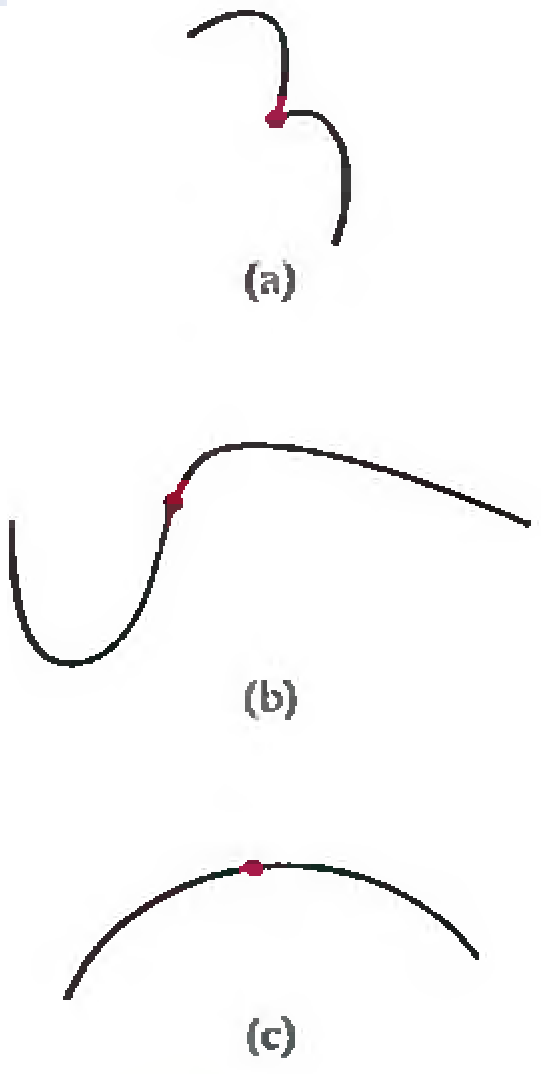
Parametric Continuity Conditions Contd.

- Piecewise construction of a curve by joining two curve segments using different orders of continuity:

a. zero-order continuity only,

b. first-order continuity,

c. second-order continuity.



Bézier Spline Curves

- A spline approximation method developed by the French engineer Pierre Bézier for use in the design of Renault car bodies
- A Bézier curve can be fitted to any number of control points – although usually 4 are used

Bézier Spline Curves (cont…)

- Consider the case of n+1 control points denoted as $p_k=(x_k, y_k, z_k)$ where k varies from 0 to n
- The coordinate positions are blended to produce the position vector $P(u)$ which describes the path of the Bézier polynomial function between p_0 and p_n

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \leq u \leq 1$$

Bézier Spline Curves (cont…)

- The Bézier blending functions $BEZ_{k,n}(u)$ are the *Bernstein polynomials*

$$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$$

- where parameters $C(n,k)$ are the binomial coefficients

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Bézier Spline Curves (cont…)

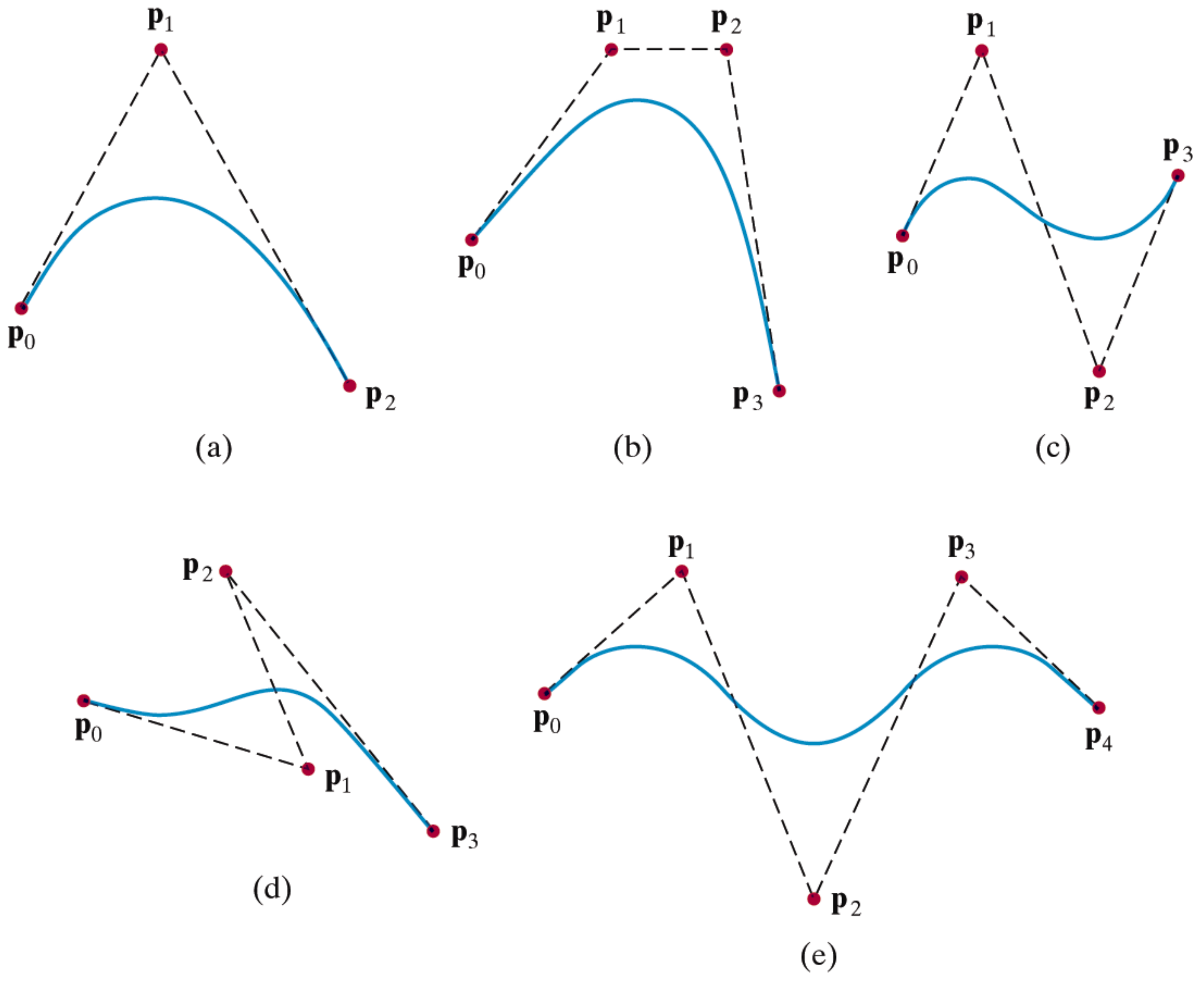
- So, the individual curve coordinates can be given as follows

$$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$

Bézier Spline Curves (cont…)



Important Properties of Bézier Curves

The first and last control points are the first and last point on the curve

- $P(0) = p_0$
- $P(1) = p_n$

The curve lies within the convex hull as the Bézier blending functions are all positive and sum to 1

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

Important Properties of Bézier Curves

- Values for the parametric first derivatives of a Bezier curve at the endpoints can be calculated from control-point coordinates as

$$P'(0) = -n.p_0 + n.p_1$$
$$P'(1) = -n.p_{n-1} + n.p_n$$

- From these expressions, we see that the slope at the beginning of the curve is along the line joining the first two control points,
- and the slope at the end of the curve is along the line joining the last two endpoints.

Cubic Bézier Curve

- Many graphics packages restrict Bézier curves to have only 4 control points (i.e. n = 3)
- The blending functions when n = 3 are simplified as follows:

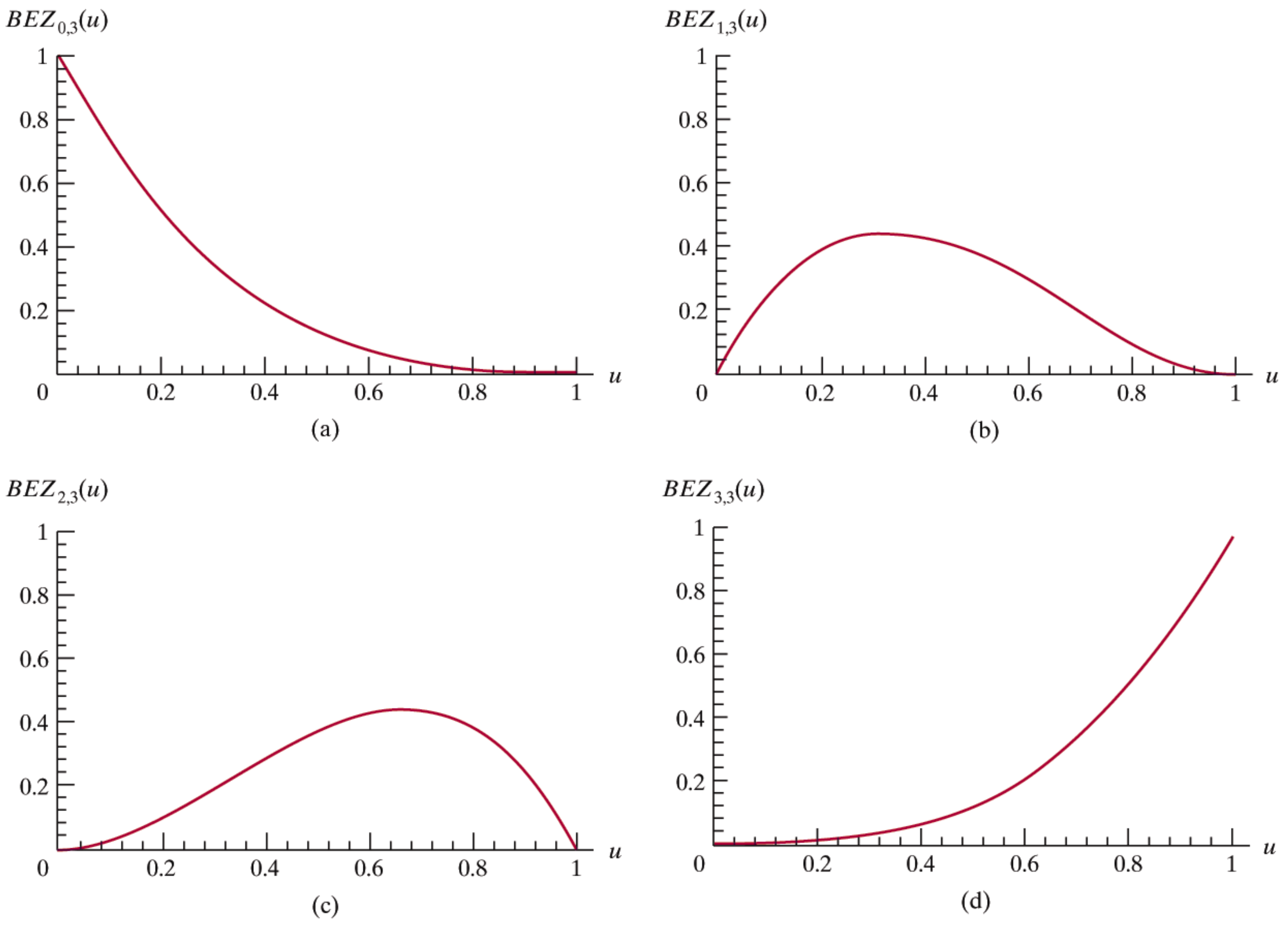
$$BEZ_{0,3} = (1-u)^3$$

$$BEZ_{1,3} = 3u(1-u)^2$$

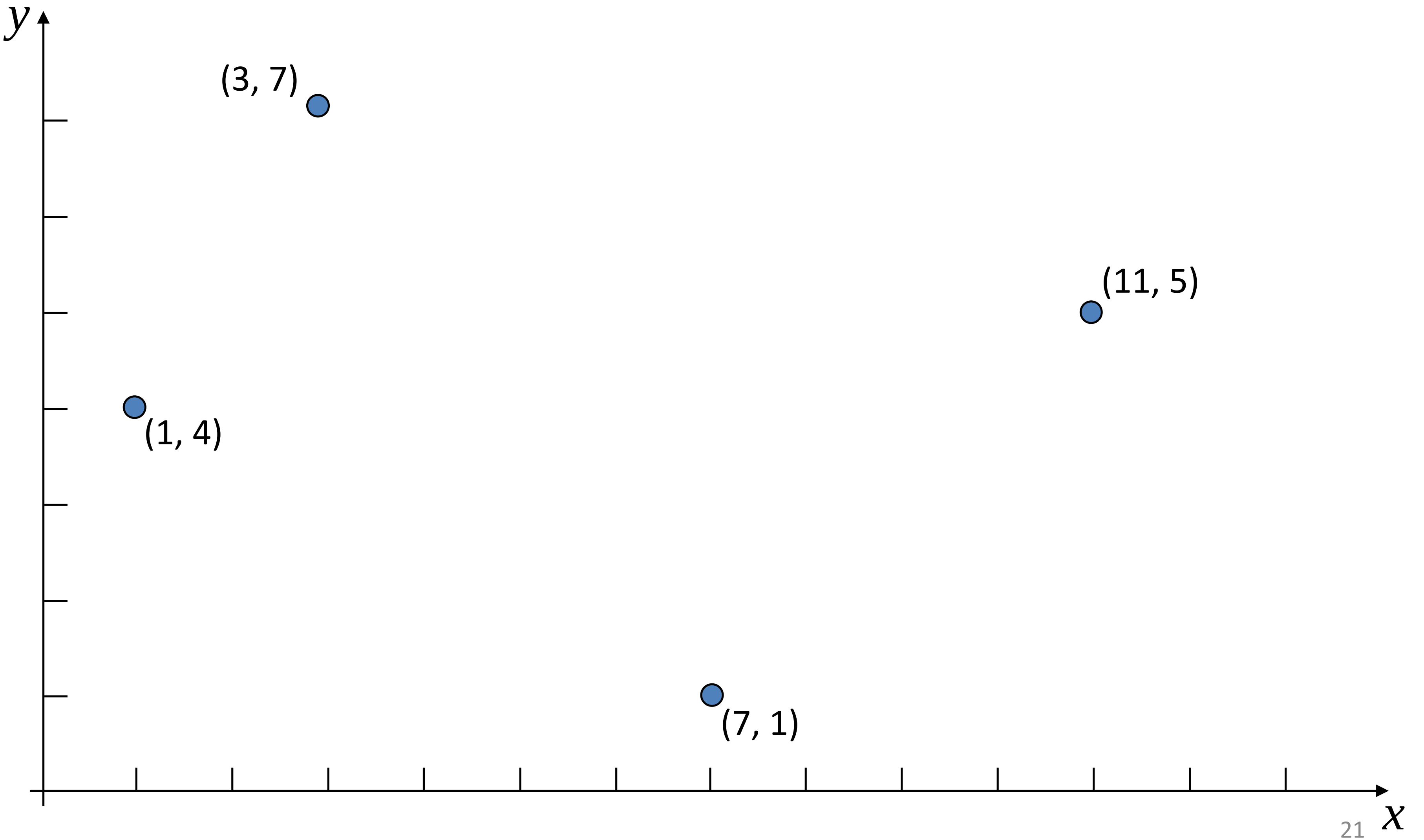
$$BEZ_{2,3} = 3u^2(1-u)$$

$$BEZ_{3,3} = u^3$$

Cubic Bézier Blending Functions



Bézier Spline Curve Exercise



Summary

- We had a look at spline curves and in particular Bézier curves
- The whole point is that the spline functions give us an approximation to a smooth curve