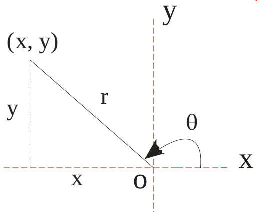
Chapter 16 (Summary)

Definitions of trigonometric functions:

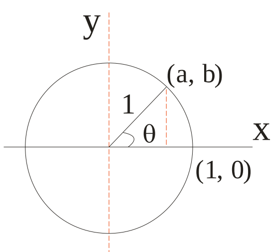


$\sin\theta = \frac{y}{r}$, $\csc\theta = \frac{r}{y}$, $\cos\theta = \frac{x}{r}$, $\sec\theta = \frac{r}{x}$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$

$\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$ *or* $\sin\theta = \frac{1}{\csc\theta}$, $\cos\theta = \frac{1}{\sec\theta}$, $\tan\theta = \frac{1}{\cot\theta}$

Derive the identity: $\sin^2\theta + \cos^2\theta = 1$

Consider the unit circle (radius $r = 1$) as shown.



$$\frac{a}{1} = \cos\theta \Rightarrow a = \cos\theta \qquad (1)$$

$$\frac{b}{1} = \sin\theta \Rightarrow b = \sin\theta \qquad (2)$$

Squaring and adding equation (1) and (2)

$a^2 + b^2 = \cos^2\theta + \sin^2\theta$

$Since \quad a^2 + b^2 = 1$

$$\therefore \sin^2\theta + \cos^2\theta = 1 \qquad (3)$$

$$(or \quad \cos^2\theta = 1 - \sin^2\theta)$$

Dividing both sides of equation (3) by $\cos^2\theta$

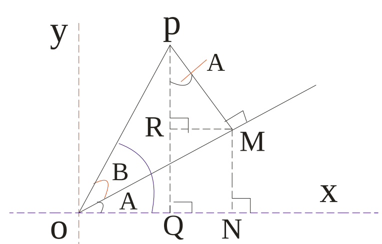
$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \sec^2\theta \qquad \Rightarrow 1 + \tan^2\theta = \sec^2\theta \quad (4)$$

Dividing both sides of equation (3) by $\sin^2\theta$

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = co\sec^2\theta \qquad \Rightarrow 1 + \cot^2\theta = co\sec^2\theta \quad (5)$$

Sum and Difference Formulas

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ *and* $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$



Let A and B be two acute angles as shown. PQ and MN are perpendicular to the x-axis. PM is perpendicular to OM and MR is perpendicular to PQ. Note that angle MPR is equal to angle A ($\angle MPR = \angle A$).

$$\sin(A+B)=\frac{PQ}{OP}=\frac{PR+RQ}{OP}\Rightarrow\sin(A+B)=\frac{PR}{OP}+\frac{RQ}{OP}\Rightarrow\sin(A+B)=\frac{PR}{OP}+\frac{MN}{OP}$$

These two fractions (on the right hand side (RHS)) do not define functions of either A or B. However, if we multiply numerator and denominator of the first fraction by PM and second fraction by OM we get on RHS (right hand side)

$$\frac{PR}{OP}\times\frac{PM}{PM}+\frac{MN}{OP}\times\frac{OM}{OM}=\frac{PR}{PM}\times\frac{PM}{OP}+\frac{MN}{OM}\times\frac{OM}{OP}=\cos A\sin B+\sin A\cos B$$

$$\therefore\sin(A+B)=\cos A\sin B+\sin A\cos B\quad or\quad \sin(A+B)=\sin A\cos B+\cos A\sin B \qquad \text{(A1)}$$

For $\cos(A+B)$

$$\cos(A+B)=\frac{OQ}{OP}=\frac{ON-QN}{OP}\Rightarrow\cos(A+B)=\frac{ON}{OP}-\frac{QN}{OP}\Rightarrow\cos(A+B)=\frac{ON}{OP}-\frac{RM}{OP}$$

These two fractions (on the right hand side (RHS)) do not define functions of either A or B. However, if we multiply numerator and denominator of the first fraction by OM and second fraction by PM we get on RHS (right hand side)

$$\cos(A+B)=\frac{ON}{OP}\times\frac{OM}{OM}-\frac{RM}{OP}\times\frac{PM}{PM}\Rightarrow\cos(A+B)=\frac{ON}{OM}\times\frac{OM}{OP}-\frac{RM}{PM}\times\frac{PM}{OP}$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \qquad \text{(A2)}$$

Since $\sin(-B)=-\sin B$ *and* $\cos(-B)=\cos B$

From equation (A1), $\sin(A+B)=\sin A\cos B+\cos A\sin B$ put $B=-B$

$$\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B)\quad\Rightarrow\sin(A-B)=\sin A\cos B-\cos A\sin B \qquad \text{(A3)}$$

From equation (A2), $\cos(A+B)=\cos A\cos B-\sin A\sin B$ put $B=-B$

$$\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B)\quad\Rightarrow\cos(A-B)=\cos A\cos B+\sin A\sin B \qquad \text{(A4)}$$

Show that $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}\Rightarrow\tan(A+B)=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

For RHS, dividing numerator and denominator by $\cos A\cos B$, $\tan(A+B)=\dfrac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}$

$$\Rightarrow\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \qquad \text{(A5)}$$

Similarly, $\Rightarrow\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$ (A6)

Euler's Formula

It shows a relationship between trigonometric function and complex exponential function.

$$e^{i\theta}=\cos\theta+i\sin\theta \qquad \text{(a1)}$$

Putting $\theta=-\theta$, $e^{-i\theta}=\cos\theta-i\sin\theta$ (a2)

Adding equation (a1) and (a2) we get $e^{i\theta}+e^{-i\theta}=2\cos\theta\quad\Rightarrow\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$ (a3)

Subtracting equation (a2) from (a1) we get $e^{i\theta}-e^{-i\theta}=2i\sin\theta\quad\Rightarrow\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ (a4)

Putting $\theta=-\theta$ in equation (a3): $\cos(-\theta)=\dfrac{e^{-i\theta}+e^{i\theta}}{2}=\cos\theta\quad\therefore\cos(-\theta)=\cos\theta$

Putting $\theta = -\theta$ in equation (a4): $\sin(-\theta) = \dfrac{e^{-i\theta} - e^{i\theta}}{2i} = -\left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right) = -\sin\theta \quad \therefore \sin(-\theta) = -\sin\theta$

Double-Angle Formula

We know that, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Let $A = B$; $\sin(A+A) = \sin A \cos A + \cos A \sin A \quad \Rightarrow \sin(2A) = 2\sin A \cos A$

Similarly,

$\cos(A+B) = \cos A \cos B - \sin A \sin B \Rightarrow \cos(A+A) = \cos A \cos A - \sin A \sin A \quad \Rightarrow \cos(2A) = \cos^2 A - \sin^2 A$

As $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$, using in above equation

$\cos 2A = \cos^2 A - \sin^2 A \Rightarrow \cos 2A = 1 - \sin^2 A - \sin^2 A \quad \Rightarrow \cos 2A = 1 - 2\sin^2 A$

$\cos 2A = \cos^2 A - \sin^2 A \Rightarrow \cos 2A = \cos^2 A - (1 - \cos^2 A) \Rightarrow \cos 2A = 2\cos^2 A - 1$

Half-Angle Formula

- The half-angle formulas can be obtained from the double-angle formulas by properly arranging terms.

$\cos 2x = 1 - 2\sin^2 x \quad \Rightarrow 2\sin^2 x = 1 - \cos 2x \quad \Rightarrow \sin^2 x = \dfrac{1-\cos 2x}{2} \Rightarrow \sin x = \pm\sqrt{\dfrac{1-\cos 2x}{2}}$

Let $x = \dfrac{A}{2}$, $\sin\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1-\cos A}{2}}$ Similarly, $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1+\cos A}{2}}$

Exercises / Section 16.1 (page 493)

Problem 7, 9, 13: change the expression to an equivalent involving sines and cosines and simplify if possible.
Problem 17, 21, 27, 29, 31: use the fundamental identities to simplify the problems. Convert to an expression involving sines and cosines if nexessary.

(Problems solved in class # 7, 13, 17, 21, 29) **Home work. (9, 27, 31)**

P # 7. $\cot x + \dfrac{1}{\sin x}$

P# 9. $\tan\theta\cos\theta\cot\theta$ P # 13. $1 - \sec^2\theta$ P # 17. $\dfrac{1}{1-\cos^2\theta}$ P # 21. $\dfrac{\tan\theta\csc\theta}{\sec\theta}$

P # 27. $\csc^2\alpha - \cot^2\alpha$ P # 29. $\dfrac{1+\tan^2 x}{\cos x}$ Problem # 31. $\cot\theta\cos^2\theta + \cot\theta\sin^2\theta$

Exercises / Section 16.2 (page 497-499)

(Problems solved in class # 5, 15, 35, 47) **Home work. (9, 27, 39) Prove the given identities**

Problem # 5. $\dfrac{\cos^2\beta}{\sin\beta} + \sin\beta = scs\beta$ Problem # 9. $\dfrac{1+\tan^2\omega}{1+\cot^2\omega} = \tan^2\omega$ Problem # 15. $\dfrac{\sin\beta + \tan\beta}{1+\cos\beta} = \tan\beta$

Problem # 27. $\dfrac{\tan\theta}{\csc\theta - \cot\theta} - \dfrac{\sin\theta}{\csc\theta + \cot\theta} = \sec\theta + \cos\theta$ Problem # 35. $\cos^4 x - \sin^4 x = 2\cos^2 x - 1$

Problem # 39. $\dfrac{\tan x - \sin x}{\sin^3 x} = \dfrac{\sec x}{1+\cos x}$ Problem # 47.In some problems on the motion of a pendulum, the expression $\dfrac{1}{\sqrt{1-\cos x}}$ arises. Show that this expression is equivalent to $\dfrac{\sqrt{1+\cos x}}{\sin x}$.

Exercises / Section 16.3 (page 503-505)

(Problems solved in class # 19, 37, 55) **Home work. (15, 35, 47)**
Problem 15, 19: write each expression as a single term.
Problem 35, 37: write each expression as a function of x or 2x.

P# 15. $\cos 3x \cos x + \sin 3x \sin x$ Problem # 19. $\sin(x+y)\cos y - \cos(x+y)\sin y$ P # 35. $\sin\left(x+\frac{\pi}{4}\right)$

Problem # 37. $\tan\left(x+\frac{\pi}{4}\right)$ Problem # 47. Prove the identity: $\tan(x-y)-\tan(y-x)=\frac{2(\tan x-\tan y)}{1+\tan x \tan y}$

Problem # 55. If a force $F_0 \cos\omega t$ is applied to a weight oscillating on a spring, then the energy supplied to the system can be written in the form $E = A\omega F_0 \cos(\omega t-\gamma)\cos\omega t$. Show that $E = A\omega F_0\left(\cos^2\omega t\cos\gamma+\cos\omega t\sin\omega t\sin\gamma\right)$

Exercises / Section 16.4 (page 509-510)

(Problems solved in class # 11, 33, 39) **Home work. (9, 29, 35)**

P # 9. Find $\cos 2\theta$, given that $\sin\theta = \frac{2}{5}$, θ in quadrant II. **P** # 11. Find $\cos 2\theta$, given that $\cos\theta = -\frac{3}{7}$, θ in quadrant III.

Prove the given identities: Problem # 29. $\frac{\cos 2\theta+\cos\theta+1}{\sin 2\theta+\sin\theta}=\cot\theta$ Problem # 33. $\frac{1+\cos 2\omega}{\sin 2\omega}=\cot\omega$

Problem # 35. $\frac{\csc^2\theta-2}{\csc^2\theta}=\cos 2\theta$ Problem # 39. An axle is placed through the center of a circular disk at an angle α. The magnitude T of the torque on the bearings holding the axel has the form $T = k\omega^2\sin\alpha\cos\alpha$, where ω is the angular velocity. Show that $T=\frac{1}{2}k\omega^2\sin 2\alpha$

Exercises / Section 16.5 (page 514 - 515)

(Problems solved in class # 9, 21, 27) **Home work. (7, 19, 25)**

Problem # 7. Find $\sin\left(\frac{\theta}{2}\right)$, given that $\cos\theta=-\frac{24}{25}$, θ in quadrant III. Problem # 9. Find $\sin\left(\frac{\theta}{2}\right)$, given that $\cos\theta=\frac{5}{13}$, θ in quadrant IV. (Problem 19, 21 eliminate the exponent) P # 19. $\cos^2 2x$ P # 21. $2\sin^2 3x$

Prove the given identities: Problem # 25. $\frac{\sin 2\theta}{2\sin\theta}=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$ Problem # 27. $\csc^2\theta=\frac{2}{1-\cos 2\theta}$

Solved Examples

Example # 1 Prove the identity: $\frac{1}{(\sin x)(\sec x)(\cot x)}=1$

Solution: L. H. S: $\frac{1}{(\sin x)(\sec x)(\cot x)}=\frac{1}{\sin x\times\frac{1}{\cos x}\times\frac{\cos x}{\sin x}}=1=R.H.S \quad \therefore L.H.S=R.H.S$

Example # 2 Prove the identity: $\frac{\cos x \operatorname{cosec} x}{\tan x}=\cot^2 x$

Solution: L. H. S: $\frac{\cos x\times\operatorname{cosec} x}{\tan x}=\frac{\cos x\times\frac{1}{\sin x}}{\frac{\sin x}{\cos x}}=\frac{\cos x}{\sin x}\times\frac{\cos x}{\sin x}=\frac{\cos^2 x}{\sin^2 x}=\cot^2 x=R.H.S \quad \therefore L.H.S=R.H.S$

Example # 3 Prove the identity: $(\sec x-\tan x)(\sec x+\tan x)=1$

Solution: L. H. S: $(\sec x-\tan x)(\sec x+\tan x)=\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)=\left(\frac{1-\sin x}{\cos x}\right)\left(\frac{1+\sin x}{\cos x}\right)$

$=\left(\frac{1-\sin^2 x}{\cos^2 x}\right)=\frac{\cos^2 x}{\cos^2 x}=1=R.H.S \quad \therefore L.H.S=R.H.S$