BMT 222 I

Chap. #5 and chap #7

Addition and subtraction) Basic Technical Mathematics with Calculus Peter K. F. Kuhfitting

Chapter 5 Factoring and Fractions (Summary August 2018

Exercises / Section 5.8 (page 183-185)

- Combine the given fractions and simplify.

Problem # 1. $\frac{1}{2}-\frac{1}{18}+\frac{5}{9}$, Problem # 11: $\frac{2}{xy}-\frac{1}{x}-\frac{y^2+2x-2y}{xy(x-y)}$

Problem # 23: $\frac{a+b}{b}-\frac{a^2}{b(a+2b)}+\frac{a}{(a+2b)}$ Problem # 35: $\frac{1}{2x^2+3xy+y^2}-\frac{1}{x^2+4xy+3y^2}+\frac{1}{2x^2+7xy+3y^2}$

Problem # 9: $\frac{x+3y}{x-y}-\frac{x-3y}{x+y}$ Problem # 15: $\frac{2}{x-3}+\frac{1}{x+2}-\frac{2x-1}{(x-3)(x+2)}$

Problem # 29 ; $\frac{3}{(x-y)(x+2y)}-\frac{1}{(x+y)(x+2y)}+\frac{1}{(y-x)(x+y)}$

(Problems solved in class # 1, 11, 23, 35) HW: Problem # 9, Problem # 15, Problem # 29,

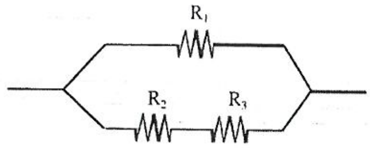
Problem # P1: $\frac{1}{x-y}\left(\frac{x}{y}-\frac{y}{x}\right)$ (Answer: $\frac{x+y}{xy}$) Problem # P2: $\frac{2}{w(w+1)}+\frac{3}{w^2}$ (Answer: $\frac{5w+3}{w^2(w+1)}$)

Exercises / Section 5.9 (page 188-189)

- Simplify the complex fractions.

Problem # 7: $\frac{1-\frac{16}{x^2}}{1+\frac{4}{x}}$, Problem # 19: $\frac{\frac{1}{E-1}+\frac{1}{E-2}}{1+\frac{1}{E-2}}$ Problem # 23: $\frac{\frac{x}{x-2}-\frac{2}{(x-1)(x-2)}}{\frac{(x-4)}{(x-1)}}$

Problem # 31 The total resistance of the circuit shown in the figure is given by $R=\frac{1}{\frac{1}{R_1}+\frac{1}{R_2+R_3}}$, simplify the expression for R.



Simplify the complex fractions: Problem # 5: $\frac{3-\frac{1}{x}}{9-\frac{1}{x^2}}$ Problem # 15: $\frac{w-\frac{w}{w-5}}{w-\frac{6}{w-5}}$ Problem # 21: $\frac{\frac{k}{k+1}-\frac{6}{(k+1)^2}}{1-\frac{9}{(k+1)^2}}$

(Problems solved in class # 7, 19, 23, 31) HW: Problem # 5, Problem # 15, Problem # 21,

Problem # P3:- $\frac{\frac{2}{x-2}+\frac{1}{x}}{\frac{3x}{x-5}-\frac{2}{x-5}}$ (Answer: $\frac{x-5}{x(x-2)}$) Problem # P4: $2-\frac{m}{1-\frac{1-m}{-m}}$ (Answer: $2-m^2$)

Problem # P5: $\frac{5-\frac{1}{x+2}}{1+\frac{3}{1+\frac{3}{x}}}$ (Answer: $\frac{(5x+9)(x+3)}{(x+2)(4x+3)}$)

Exercise 5.8 (Page 183-185)

Combine the given fractions and simplify:

Problem # 1: $\frac{1}{2} - \frac{1}{18} + \frac{5}{9}$

$$= \frac{9-1+10}{18} = \frac{18}{18} = \boxed{1}$$

Problem # 11: $\frac{2}{xy} - \frac{1}{x} - \frac{y^2+2x-2y}{xy(x-y)}$

$$= \frac{2(x-y) - y(x-y) - (y^2+2x-2y)}{xy(x-y)}$$

$$= \frac{\cancel{2x}-\cancel{2y}-xy+\cancel{y^2}-\cancel{y^2}-\cancel{2x}+\cancel{2y}}{xy(x-y)}$$

$$= \frac{-\cancel{xy}}{\cancel{xy}(x-y)} = -\frac{1}{x-y}$$

$$= \boxed{\frac{1}{y-x}}$$

Problem # 23

$$\frac{a+b}{b} - \frac{a^2}{b(a+2b)} + \frac{a}{(a+2b)}$$

$$= \frac{(a+b)(a+2b) - a^2 + ab}{b(a+2b)}$$

$$= \frac{\cancel{a^2}+2ab+ab+2b^2-\cancel{a^2}+ab}{b(a+2b)}$$

$$= \frac{4ab+2b^2}{b(a+2b)} = \frac{\cancel{b}(4a+2b)}{\cancel{b}(a+2b)}$$

$$= \frac{4a+2b}{a+2b}$$

Problem # 35

$$\frac{1}{2x^2+3xy+y^2} - \frac{1}{x^2+4xy+3y^2} + \frac{1}{2x^2+7xy+3y^2}$$

factorizing

$$= \frac{1}{2x^2+2xy+xy+y^2} - \frac{1}{x^2+3xy+xy+3y^2} + \frac{1}{2x^2+6xy+xy+3y^2}$$

$$= \frac{1}{2x(x+y)+y(x+y)} - \frac{1}{x(x+3y)+y(x+3y)} + \frac{1}{2x(x+3y)+y(x+3y)}$$

$$= \frac{1}{(x+y)(2x+y)} - \frac{1}{(x+3y)(x+y)} + \frac{1}{(x+3y)(2x+y)}$$

$$= \frac{x+3y-(2x+y)+x+y}{(x+y)(2x+y)(x+3y)}$$

$$= \frac{\cancel{x}+3y-\cancel{2x}-\cancel{y}+\cancel{x}+\cancel{y}}{(x+y)(2x+y)(x+3y)}$$

$$= \frac{3y}{(x+y)(2x+y)(x+3y)}$$

Problem # P1

$$\frac{1}{x-y}\left(\frac{x}{y} - \frac{y}{x}\right)$$

$$= \frac{1}{x-y}\left(\frac{x^2-y^2}{xy}\right)$$

$$= \frac{1}{\cancel{x-y}} \times \frac{(x+y)\cancel{(x-y)}}{xy}$$

$$= \frac{x+y}{xy}$$

Simplify the complex fraction

Problem # 7

$$\frac{1-\frac{16}{x^2}}{1+\frac{4}{x}}$$

$$= \frac{\frac{x^2-16}{x^2}}{\frac{x+4}{x}}$$

$$= \frac{x^2-16}{x^2} \times \frac{x}{x+4}$$

$$= \frac{\cancel{(x+4)}(x-4)}{x} \times \frac{1}{\cancel{x+4}}$$

$$= \frac{x-4}{x}$$

Problem #19

$$\frac{\frac{1}{E-1}+\frac{1}{E-2}}{1+\frac{1}{E-2}}$$

$$= \frac{\frac{E-2+E-1}{(E-1)(E-2)}}{\frac{E-2+1}{E-2}}$$

$$= \frac{2E-3}{(E-1)\cancel{(E-2)}} \times \frac{\cancel{(E-2)}}{E-1}$$

$$= \frac{2E-3}{(E-1)^2}$$

$$= \frac{2E-3}{E^2-2E+1}$$

Problem # 23

$$\frac{\frac{x}{x-2}-\frac{2}{(x-1)(x-2)}}{\frac{x-4}{(x-1)}}$$

$$= \frac{\frac{x(x-1)-2}{(x-1)(x-2)}}{\frac{(x-4)}{(x-1)}}$$

$$= \frac{x^2-x-2}{(x-1)(x-2)} \times \frac{(x-1)}{x-4}$$

$$= \frac{(x^2-x-2)}{(x-2)(x-4)}$$

$$= \frac{x^2-2x+x-2}{(x-2)(x-4)}$$

$$= \frac{x(x-2)+1(x-2)}{(x-2)(x-4)}$$

$$= \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-4)}$$

$$= \frac{x+1}{x-4}$$

Problem #31

$$R = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2+R_3}}$$

$$= \frac{1}{\frac{R_2+R_3+R_1}{R_1(R_2+R_3)}}$$

$$R = \frac{R_1(R_2+R_3)}{R_1+R_2+R_3}$$

Problem # P5

$$\frac{5-\frac{1}{x+2}}{1+\frac{3}{1+\frac{3}{x}}}$$

$$= \frac{\frac{5(x+2)-1}{x+2}}{1+\frac{3}{\frac{x+3}{x}}} = \frac{\frac{5x+10-1}{x+2}}{1+\frac{3x}{x+3}}$$

$$= \frac{\frac{5x+9}{x+2}}{\frac{x+3+3x}{x+3}} = \frac{\frac{5x+9}{x+2}}{\frac{4x+3}{x+3}} = \frac{(5x+9)}{(x+2)} \times \frac{(x+3)}{(4x+3)}$$

$$\therefore \frac{(5x+9)(x+3)}{(x+2)(4x+3)}$$

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Basic Technical Mathematics with Calculus
Peter K. F. Kuhfitting

The acceleration of the system is

$$a = \frac{(w_1 - w_2)g}{w_1 + w_2}$$

Write an expression for $gT(1/a)$ and simplify.

36. In the study of the dispersion of X rays, the expression

$$A = \frac{Ne^2}{\pi m(f_0^2 - f^2)}$$

arises. Multiply A by $m(f_0 + f)$ and simplify.

37. A perfectly flexible cable, suspended from two points at the same height, hangs under its own weight. The tension T_0 at its lowest point is

$$T_0 = \frac{w(s^2 - 4H^2)}{8H}$$

where s is the length of the cable, H the sag, and w the weight per unit length. Find an expression for $T_0[H/(s - 2H)]$.

5.8 Addition and Subtraction of Fractions

In considering the addition and subtraction of algebraic fractions, recall from arithmetic that the fractions $\frac{1}{3}$ and $\frac{1}{6}$ can be added if $\frac{1}{3}$ is changed to $\frac{2}{6}$, so that

$$\frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

The number 6 is called the **lowest common denominator (LCD)**.

Since algebraic fractions are added by the same rules, let us first state the definition of the lowest common denominator.

Lowest common denominator: The lowest common denominator (LCD) of two or more fractions is an expression that is divisible by every denominator and does not have any more factors than needed to satisfy this condition.

The LCD can be found by the procedure described next.

To construct the lowest common denominator for a set of algebraic fractions, factor each of the denominators. Then the LCD is the product of the factors of the denominators, each with an exponent equal to the largest of the exponents of any of these factors

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To add (or subtract) two or more fractions, we find the LCD for the fractions and change each fraction to an equivalent fraction having the LCD for its denominator. Next, we add (or subtract) the numerators of the fractions, placing the result over the LCD. Finally, we simplify the result fraction.

Example #1 Example #1

Combine

$$\frac{5}{6a^2bc} - \frac{4}{15ab^2c} - \frac{3}{20abc^2}$$

Solution. In terms of prime factors, the different denominators are

$$6a^2bc = 2 \cdot 3 \cdot a^2 \cdot b \cdot c$$
$$15ab^2c = 3 \cdot 5 \cdot a \cdot b^2 \cdot c$$
$$20abc^2 = 2^2 \cdot 5 \cdot a \cdot b \cdot c^2$$

To construct the LCD, observe that the factors are 2, 3, 5, a, b, and c. The largest exponent on the factor 2 is 2, the largest exponent on the factor 3 is 1, and so forth. So the LCD is given by

$$\text{LCD} = 2^2 \times 3 \times 5 \times a^2b^2c^2 = 60a^2b^2c^2$$

Now we write the fractions so that they all have the same denominators. For example, since $60a^2b^2c^2 = (6a^2bc)(10bc)$, we get for the first fraction

$$\frac{5}{6a^2bc} = \frac{5}{6a^2bc} \cdot \frac{10bc}{10bc} = \frac{50bc}{60a^2b^2c^2}$$

The other fractions are adjusted similarly:

$$\frac{5(10bc)}{6a^2bc(10bc)} - \frac{4(4ac)}{15ab^2c(4ac)} - \frac{3(3ab)}{20abc^2(3ab)}$$
$$= \frac{50bc}{60a^2b^2c^2} - \frac{16ac}{60a^2b^2c^2} - \frac{9ab}{60a^2b^2c^2}$$
$$= \frac{50bc - 16ac - 9ab}{60a^2b^2c^2}$$

The procedure for adding or subtracting fractions containing polynomials is similar.

Example #2 Example #2

Combine

$$\frac{x}{x - y} - \frac{x^2}{x^2 - y^2}$$

5. $\frac{3x+3}{4x} - \frac{1}{2x} - \frac{1}{x}$

6. $\frac{x-1}{5y} + \frac{x}{10y} - \frac{7x}{30y}$

7. $\frac{3a}{4b} - \frac{4a}{9b} + \frac{a-3}{36b}$

8. $\frac{x}{x+y} - \frac{x-y}{x}$

9. $\frac{x+3y}{x-y} - \frac{x-3y}{x+y}$

10. $\frac{x}{2x+y} - \frac{x}{x+y}$

11. $\frac{2}{xy} - \frac{1}{x} - \frac{y^2+2x-2y}{xy(x-y)}$

12. $\frac{x}{x-3y} - \frac{y}{2x+y}$

13. $\frac{x+y}{x+2y} + \frac{x}{x-y}$

14. $\frac{3}{x-2} - \frac{4}{x-3}$

15. $\frac{2}{x-3} + \frac{1}{x+2} - \frac{2x-1}{(x-3)(x+2)}$

16. $\frac{x}{(x+y)(x+2y)} + \frac{1}{x+y}$

17. $\frac{1}{x+3y} + \frac{4y}{(x-3y)(x-y)}$

18. $\frac{x}{2x-y} - \frac{y}{y-3x}$

19. $\frac{2x}{3x-y} - \frac{2y}{3y-2x}$

20. $\frac{1}{x} + \frac{2}{y} - \frac{1}{x+y}$

21. $\frac{1}{x} - \frac{1}{y} - \frac{y}{x(x+y)}$

22. $\frac{y}{x+y} - \frac{2y^2}{x(x+y)} + \frac{x+2y}{x}$

23. $\frac{a+b}{b} - \frac{a^2}{b(a+2b)} + \frac{a}{a+2b}$

24. $\frac{3ab}{a^2-b^2} + \frac{a}{b-a} + \frac{a-b}{a+b}$

25. $\frac{A}{A+B} - \frac{B^2}{A(A-B)} + \frac{2B}{A}$

26. $\frac{3s-t}{s(s-t)} - \frac{1}{s+t} + \frac{1}{s}$

27. $\frac{n}{n-m} - \frac{m}{n+m} - \frac{2m^2}{n^2-m^2}$

28. $\frac{4}{(R+3r)(R+r)} - \frac{1}{(R+3r)(R+2r)} - \frac{1}{(R+2r)(R+r)}$

29. $\frac{3}{(x-y)(x+2y)} - \frac{1}{(x+y)(x+2y)} + \frac{1}{(y-x)(x+y)}$

30. $\frac{2}{x+1} - \frac{2x}{x^2-1} - \frac{1}{x-1}$

31. $\frac{x}{x-2} - \frac{2}{x+2} - \frac{x^2}{x^2-4}$

32. $\frac{T_0}{T_0-2} - \frac{2}{T_0+2} - \frac{2T_0^2}{4-T_0^2}$

33. $\frac{c}{c+2d} - \frac{c}{2c+d}$

34. $\frac{2}{x^2-y^2} - \frac{3}{x^2+xy-2y^2} + \frac{1}{x^2+3xy+2y^2}$

35. $\frac{1}{2x^2+3xy+y^2} - \frac{1}{x^2+4xy+3y^2} + \frac{1}{2x^2+7xy+3y^2}$

*If x is the distance from one *

36. A light rod of length 1 is clamped at both ends and carries a load W at the center. If x is the end of the rod, then the deflection is

$$y = \frac{W}{k}\left(\frac{x^3}{12} - \frac{x^2}{16}\right), \qquad 0 \le x \le \frac{1}{2}$$

Where k is a constant. Write y as a single fraction

where k is a constant. Write y as a single fraction

Common errors (1) Failing to change all the signs when subtracting the numerator. For example, in step (5.18) of Example 3, both signs in the numerator of the middle fraction are changed to become $-6x - 6$.

(2) Forgetting that

$$\frac{1}{x} + \frac{1}{y} \neq \frac{1}{x+y}$$

(3) Forgetting that

$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$$

In case (2) the correct procedure is

$$\frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{xy} = \frac{x+y}{xy}$$

In case (3) the fraction

$$\frac{1}{x+y}$$

is already in simplest form and cannot be split up.

Example 4 Two perfectly elastic balls collide with a common velocity v. If their respective masses are m and M, then the velocity of m after the collision is

$$v\left(\frac{M}{M+m} - \frac{m}{M+m}\right) - \frac{2vm}{M+m}$$

Simplify this expression.

Solution.

$$v\left(\frac{M}{M+m} - \frac{m}{M+m}\right) - \frac{2vm}{M+m}$$
$$= \frac{vM}{M+m} - \frac{vm}{M+m} - \frac{2vm}{M+m}$$
$$= \frac{vM - vm - 2vm}{M+m}$$
$$= \frac{vM - 3vm}{M+m}$$

$\frac{t\ (5)}{t}$

Exercises / Section 5.8

In Exercises 1–35, combine the given fractions and simplify.

1. $\frac{1}{2} - \frac{1}{18} + \frac{5}{9}$

2. $\frac{5}{36} - \frac{3}{108} + \frac{1}{9}$

3. $\frac{2x+1}{9} + \frac{x}{2} - \frac{x}{6}$

4. $\frac{1}{x} - \frac{3}{2x} - \frac{1}{3x}$

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$$\frac{\frac{1}{\cancel{x+y}}\cdot\frac{(\cancel{x-y})(\cancel{x+y})}{1}+\frac{1}{\cancel{x-y}}\cdot\frac{(\cancel{x-y})(x+y)}{1}}{\frac{x}{(\cancel{x-y})(\cancel{x+y})}\cdot\frac{(\cancel{x-y})(\cancel{x+y})}{1}}$$

$$= \frac{(x-y)+(x+y)}{x} = \frac{2x}{x} = 2$$

The remaining examples further illustrate the alternate technique discussed in the previous example.

Example #3

Simplify the fraction

$$\frac{\frac{R}{R+3}+\frac{R}{R^2-9}}{\frac{1}{R-3}+1}$$

Solution. Since $R^2 - 9 = (R-3)(R+3)$, it follows that

$$\text{LCD} = (R-3)(R+3)$$

As before, multiplying the numerator and denominator of the complex fraction by the LCD will reduce the fraction directly.

$$\frac{\frac{R}{\cancel{R+3}}\cdot\frac{(R-3)(\cancel{R+3})}{1}+\frac{R}{\cancel{R^2-9}}\cdot\frac{\cancel{(R-3)(R+3)}}{1}}{\frac{1}{\cancel{R-3}}\cdot\frac{(\cancel{R-3})(R+3)}{1}+1\cdot(R-3)(R+3)}$$

$$= \frac{R(R-3)+R}{(R+3)+(R-3)(R+3)}$$

$$= \frac{R^2-3R+R}{R+3+R^2-9}$$

$$= \frac{R^2-2R}{R^2+R-6}$$

$$= \frac{R(\cancel{R-2})}{(R+3)(\cancel{R-2})}$$

$$= \frac{R}{R+3}$$

Example #4

Just as electrical components can be connected in parallel, blood vessels that branch out and come together again are said to be connected in parallel.

Each such vessel offers a certain resistance to the flow of blood. If r_1, r_2 and r_3 are the respective resistance forces of the blood vessels in parallel, then the combined resistance is given by

$$\frac{1}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$$

Simplify this expression.

Solution.

$$\frac{1}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}} = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}\cdot\frac{r_1r_2r_3}{r_1r_2r_3}$$

$$= \frac{r_1r_2r_3}{r_2r_3+r_1r_3+r_1r_2}$$

Exercises / Section 5.9

In Exercises 1–26, simplify the complex fractions.

1. $\dfrac{1+\frac{1}{3}}{2+\frac{2}{3}}$
2. $\dfrac{\frac{1}{7}+\frac{2}{7}}{1+\frac{2}{7}}$
3. $\dfrac{1+\frac{1}{3}}{2-\frac{1}{6}}$
4. $\dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{5}{6}}$
5. $\dfrac{3-\frac{1}{x}}{9-\frac{1}{x^2}}$
6. $\dfrac{\frac{1}{x}-2}{4-\frac{1}{x^2}}$
7. $\dfrac{1-\frac{16}{x^2}}{1+\frac{4}{x}}$
8. $\dfrac{1-\frac{9}{x^2}}{1-\frac{3}{x}}$
9. $\dfrac{\frac{1}{C_1}+\frac{1}{C_2}}{\frac{1}{C_1C_2}}$
10. $\dfrac{v_1-\frac{v_2^2}{v_1}}{1-\frac{v_2}{v_1}}$
11. $\dfrac{1+\frac{3}{x}-\frac{10}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}}$
12. $\dfrac{2-\frac{11}{y}-\frac{6}{y^2}}{2-\frac{5}{y}-\frac{3}{y^2}}$
13. $\dfrac{3-\frac{17}{h}+\frac{10}{h^2}}{3+\frac{10}{h}-\frac{8}{h^2}}$
14. $\dfrac{s-\frac{3s}{s+6}}{s+\frac{9}{s+6}}$
15. $\dfrac{w-\frac{w}{w-5}}{w-\frac{6}{w-5}}$
16. $\dfrac{z+\frac{4z}{z-3}}{z-\frac{4}{z-3}}$
17. $\dfrac{\frac{1}{\beta+1}+\frac{1}{\beta^2-1}}{\frac{\beta}{\beta-1}}$
18. $\dfrac{a-\frac{3a+2}{a+2}}{a-\frac{2a+1}{a+2}}$

Section 5.4 (page 166)

1. $(x + 2y)^2$ **3.** $(3x - 4y)^2$ **5.** $(x - 1)(x - 3)$ **7.** $(x - 4)(x + 3)$ **9.** $2(a - 2b)^2$
11. $2(x + 6)(x + 1)$ **13.** $(x - 6y)(x + y)$ **15.** $(D + 7)(D - 2)$ **17.** $(2x - y)(x - y)$
19. $(5x - y)(x - 2y)$ **21.** $(4x + y)(x + 3y)$ **23.** $(2x - 3y)(3x + 4y)$ **25.** $(5w_1 - 2w_2)(w_1 - 4w_2)$
27. $8(L - 3C)(L + 2C)$ **29.** $2(3f - 4g)^2$ **31.** $x^2(x - 2)^2$ **33.** not factorable
35. $(a + b - 3)(a + b + 2)$ **37.** $(n + m - 2)(n + m - 1)$ **39.** $(2a + 2b - 1)(a + b - 4)$
41. $(f_1 + 2f_2)^2(f_1 + 2f_2 - 1)$ **43.** $(1 - x + y)(1 + x - y + x^2 - 2xy + y^2)$ **45.** $(7a - 2b)(4a + b)$
47. $(5x - y)(8x + 3y)$ **49.** $(3\alpha - 2\beta)(4\alpha - 5\beta)$ **51.** $t = 3$ sec **53.** $t = 1.33$ sec

Section 5.5 (page 169)

1. $(x - y)(a + b)$ **3.** $(x + 3y)(2x + 1)$ **5.** $(a - b)(4c + 1)$ **7.** $(x - y)(5b - 1)$
9. $2(x + y)(a - c)$ **11.** $3(R - r)(a - 2b)$ **13.** $(x + y)(x - y - z)$ **15.** $(x + y)(a - x + y)$
17. $(x - y)(x + y - 2z)$ **19.** $(x - y)(x + 4y - 1)$ **21.** $(2x - y - z)(2x - y + z)$
23. $(x + 2y - z)(x + 2y + z)$ **25.** $(3a - 2b - c)(3a + 2b + c)$ **27.** $(x - y)(3 - x - 4y)$
29. $(x + 2y)(a - x + 3y)$ **31.** $a(A - 1)(Aa - 4)$

Section 5.6 (page 174)

1. $\dfrac{x}{2}$ **3.** $\dfrac{a}{2x}$ **5.** x **7.** $\dfrac{2}{x}$ **9.** $x + 4$ **11.** $x^2 - xy - y^2$ **13.** $\dfrac{x - y}{3}$
15. $R^2 + 2R + 4$ **17.** $x - 4$ **19.** $i + 5$ **21.** $\dfrac{3x - 4}{x - 3}$ **23.** $\dfrac{7x + 2}{2x + 1}$ **25.** $\dfrac{m_1 + m_2}{m_1 - 2m_2}$
27. -1 **29.** $\dfrac{2L + C}{3L - 4C}$ **31.** $\dfrac{x - 4}{x - 6}$ **33.** $\dfrac{a - 4b}{2a - b}$ **35.** $\dfrac{1}{2x + 1}$ **37.** $\dfrac{1}{2t + 1}$ **39.** $\dfrac{1}{3t + 1}$
41. $\dfrac{1}{3E - 2}$ **43.** $P - Q$ **45.** $\dfrac{M^2 + m^2}{(M + m)^2}$ **47.** $2t - 6$ (amperes)

Section 5.7 (page 177)

1. $\dfrac{3xy}{2zw^2}$ **3.** $\dfrac{8x^2y}{3a^2b}$ **5.** $\dfrac{6}{5}bcxy$ **7.** $\dfrac{dx}{ac^2}$ **9.** $2x(x - y)$ **11.** $\dfrac{x + y}{x - y}$ **13.** $-\dfrac{a + 2b}{3}$
15. $-(1 + 3a)(x^2 - 3xy + 9y^2)$ **17.** $\dfrac{4(2a - b)}{(3x - 1)(a + b)}$ **19.** $\dfrac{(3v_0 - 4)(v_1 + 6)}{(2v_0 - 3)(2v_1 - 1)}$
21. $\dfrac{(3T + J)(5K - 6)}{(4T + J)(K - 7)}$ **23.** $\dfrac{x - 3y}{(x + y)(x^2 - 2xy + 4y^2)}$ **25.** $\dfrac{x - y}{x - 2y}$ **27.** $\dfrac{1}{(R + 2r)(R + r - 1)}$
29. $\dfrac{2(2L - 7)}{(L + 4)(C - 5)}$ **31.** $\dfrac{2}{c - d}$ **33.** 2 **35.** $\dfrac{2w_1w_2}{w_1 - w_2}$ **37.** $\dfrac{1}{8}w(s - 2H)$

Section 5.8 (page 183)

1. 1 **3.** $-\dfrac{5x + 1}{9}$ **5.** $\dfrac{3x - 3}{4x}$ **7.** $\dfrac{4a - 1}{12b}$ **9.** $\dfrac{8xy}{x^2 - y^2}$ **11.** $\dfrac{1}{y - x}$ **13.** $\dfrac{2x^2 + 2xy - y^2}{(x + 2y)(x - y)}$
15. $\dfrac{1}{x - 3}$ **17.** $\dfrac{1}{x - y}$ **19.** $\dfrac{4x^2 - 2y^2}{(3x - y)(2x - 3y)}$ **21.** $-\dfrac{x}{y(x + y)}$ **23.** $\dfrac{4a + 2b}{a + 2b}$ **25.** $\dfrac{A + B}{A}$
27. 1 **29.** $\dfrac{1}{x^2 - y^2}$ **31.** $\dfrac{4}{x^2 - 4}$ **33.** $\dfrac{c^2 - cd}{(c - 2d)(2c - d)}$ **35.** $-\dfrac{3y}{(x + y)(2x + y)(x + 3y)}$
37. $\dfrac{k(2np - p^2)}{n^2(n - p)^2}$ **39.** $\dfrac{k^2}{k^2 + L^2}$

Chapter 7 (Partial fractions Page 54-59)

7.1 Introduction to partial fractions

In order to resolve an algebraic expression into partial fractions:

(i) The denominator must factorise

(ii) The numerator must be at least one degree less than the denominator. When the degree of numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator (see problem 3 and 4).

Table 7.1

Type	Denominator containing	Expression	Form of partial fraction
	Linear factors (see problem 1 to 4)	$\dfrac{f(x)}{(x+a)(x-b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x-b)}+\dfrac{C}{(x+c)}$
2	Repeated linear factors (see problem 5 to 7)	$\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
3	Quadratic factors (see problem 8 and 9)	$\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{(x+d)}$

7.2 Worked Problems on partial fractions with linear factors

Problem 1. . Resolve $\dfrac{11-3x}{x^2+2x-3}$ into the sum of three partial fractions

Problem 2. Convert $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$ into the sum of three partial fractions

Problem 3. Resolve $\dfrac{x^2+1}{x^2-3x+2}$ into partial fractions

Problem 4. Express $\dfrac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions

7.3 Worked Problems on partial fractions with repeated linear factors

Problem 5. Resolve $\dfrac{2x+3}{(x-2)^2}$ into partial fractions

Problem 6. Express $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ as the sum of three partial fractions

Problem 7. Resolve $\dfrac{3x^2+16x+15}{(x+3)^3}$ into partial fractions

7.4 Worked problems on partial fraction with quadratic factors

Problem 8. Express $\dfrac{7x^2+5x+13}{(x^2+2)(x+1)}$ in partial fractions **Problem 9.** Resolve $\dfrac{3+6x+4x^2-2x^3}{x^2(x^2+3)}$ into partial fractions

Chapter 7

Partial fractions

7.1 Introduction to partial fractions

By algebraic addition,

$$\frac{1}{x-2}+\frac{3}{x+1}=\frac{(x+1)+3(x-2)}{(x-2)(x+1)}$$
$$=\frac{4x-5}{x^2-x-2}$$

The reverse process of moving from $\frac{4x-5}{x^2-x-2}$ to $\frac{1}{x-2}+\frac{3}{x+1}$ is called resolving into **partial fractions**.

In order to resolve an algebraic expression into partial fractions:

(i) the denominator must factorise (in the above example, x^2-x-2 factorises as $(x-2)(x+1)$, and

(ii) the numerator must be at least one degree less than the denominator (in the above example $(4x-5)$ is of degree 1 since the highest powered x term is x^1 and (x^2-x-2) is of degree 2)

When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator (see Problems 3 and 4).

There are basically three types of partial fraction and the form of partial fraction used is summarised in Table 7.1 where $f(x)$ is assumed to be of less degree than the relevant denominator and A, B and C are constants to be determined.

(In the latter type in Table 7.1, ax^2+bx+c is a quadratic expression which does not factorise without containing surds or imaginary terms.)

Resolving an algebraic expression into partial fractions is used as a preliminary to integrating certain functions (see Chapter 51).

7.2 Worked problems on partial fractions with linear factors

Problem 1. Resolve $\frac{11-3x}{x^2+2x-3}$ into partial fractions

The denominator factorises as $(x-1)(x+3)$ and the numerator is of less degree than the denominator.

Table 7.1

Type	Denominator containing	Expression	Form of partial fraction
1	Linear factors (see Problems 1 to 4)	$\frac{f(x)}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)}+\frac{B}{(x-b)}+\frac{C}{(x+c)}$
2	Repeated linear factors (see Problems 5 to 7)	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
3	Quadratic factors (see Problems 8 and 9)	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{C}{(x+d)}$

Thus $\frac{11-3x}{x^2+2x-3}$ may be resolved into partial fractions.
Let

$$\frac{11-3x}{x^2+2x-3} \equiv \frac{11-3x}{(x-1)(x+3)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+3)},$$

where A and B are constants to be determined,

$$\text{i.e.} \quad \frac{11-3x}{(x-1)(x+3)} \equiv \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

by algebraic addition.

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

Thus, $\quad 11-3x \equiv A(x+3)+B(x-1)$

To determine constants A and B, values of x are chosen to make the term in A or B equal to zero.

When $x=1$, then $11-3(1) \equiv A(1+3)+B(0)$

i.e. $\quad 8=4A$

i.e. $\quad \mathbf{A=2}$

When $x=-3$, then $11-3(-3) \equiv A(0)+B(-3-1)$

i.e. $\quad 20=-4B$

i.e. $\quad \mathbf{B=-5}$

$$\text{Thus} \quad \frac{\mathbf{11-3x}}{\mathbf{x^2+2x-3}} \equiv \frac{2}{(x-1)} + \frac{-5}{(x+3)}$$

$$\equiv \frac{\mathbf{2}}{\mathbf{(x-1)}} - \frac{\mathbf{5}}{\mathbf{(x+3)}}$$

$$\left[\text{Check:} \quad \frac{2}{(x-1)} - \frac{5}{(x+3)} = \frac{2(x+3)-5(x-1)}{(x-1)(x+3)} = \frac{11-3x}{x^2+2x-3}\right]$$

Problem 2. Convert $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$ into the sum of three partial fractions

Let

$$\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x+3)}$$

$$\equiv \frac{A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)}{(x+1)(x-2)(x+3)}$$

by algebraic addition

Equating the numerators gives:

$$2x^2-9x-35 \equiv A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)$$

Let $x=-1$. Then

$$2(-1)^2-9(-1)-35 \equiv A(-3)(2)+B(0)(2)+C(0)(-3)$$

i.e. $\quad -24=-6A$

i.e. $\quad \mathbf{A} = \frac{-24}{-6} = \mathbf{4}$

Let $x=2$. Then

$$2(2)^2-9(2)-35 \equiv A(0)(5)+B(3)(5)+C(3)(0)$$

i.e. $\quad -45=15B$

i.e. $\quad \mathbf{B} = \frac{-45}{15} = \mathbf{-3}$

Let $x=-3$. Then

$$2(-3)^2-9(-3)-35 \equiv A(-5)(0)+B(-2)(0)+C(-2)(-5)$$

i.e. $\quad 10=10C$

i.e. $\quad \mathbf{C=1}$

$$\text{Thus} \quad \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{\mathbf{4}}{\mathbf{(x+1)}} - \frac{\mathbf{3}}{\mathbf{(x-2)}} + \frac{\mathbf{1}}{\mathbf{(x+3)}}$$

Problem 3. Resolve $\frac{x^2+1}{x^2-3x+2}$ into partial fractions

Section 1

The denominator is of the same degree as the numerator. Thus dividing out gives:

$$\begin{array}{r|l} & 1 \\ x^2-3x+2 & x^2 \quad\quad +1 \\ & x^2-3x+2 \\ \hline & 3x-1 \end{array}$$

For more on polynomial division, see Section 6.1, page 48.

Hence $\dfrac{x^2+1}{x^2-3x+2} \equiv 1+\dfrac{3x-1}{x^2-3x+2}$

$$\equiv 1+\frac{3x-1}{(x-1)(x-2)}$$

Let $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{A}{(x-1)}+\dfrac{B}{(x-2)}$

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Equating numerators gives:

$$3x-1 \equiv A(x-2)+B(x-1)$$

Let $x=1$. Then $2=-A$

i.e. $\mathbf{A=-2}$

Let $x=2$. Then $\mathbf{5=B}$

Hence $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{-2}{(x-1)}+\dfrac{5}{(x-2)}$

Thus $\boldsymbol{\dfrac{x^2+1}{x^2-3x+2} \equiv 1-\dfrac{2}{(x-1)}+\dfrac{5}{(x-2)}}$

Problem 4. Express $\dfrac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions

The numerator is of higher degree than the denominator. Thus dividing out gives:

$$\begin{array}{r|l} & x-3 \\ x^2+x-2 & x^3-2x^2-4x-4 \\ & x^3+x^2-2x \\ \hline & -3x^2-2x-4 \\ & -3x^2-3x+6 \\ \hline & x-10 \end{array}$$

Thus $\dfrac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\dfrac{x-10}{x^2+x-2}$

$$\equiv x-3+\frac{x-10}{(x+2)(x-1)}$$

Let $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{A}{(x+2)}+\dfrac{B}{(x-1)}$

$$\equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Equating the numerators gives:

$$x-10 \equiv A(x-1)+B(x+2)$$

Let $x=-2$. Then $-12=-3A$

i.e. $\mathbf{A=4}$

Let $x=1$. Then $-9=3B$

i.e. $\mathbf{B=-3}$

Hence $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{4}{(x+2)}-\dfrac{3}{(x-1)}$

Thus $\boldsymbol{\dfrac{x^3-2x^2-4x-4}{x^2+x-2}}$

$$\boldsymbol{\equiv x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}}$$

Now try the following exercise

Exercise 26 Further problems on partial fractions with linear factors

Resolve the following into partial fractions:

1. $\dfrac{12}{x^2-9}$ $\left[\dfrac{2}{(x-3)}-\dfrac{2}{(x+3)}\right]$
2. $\dfrac{4(x-4)}{x^2-2x-3}$ $\left[\dfrac{5}{(x+1)}-\dfrac{1}{(x-3)}\right]$
3. $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$ $\left[\dfrac{3}{x}+\dfrac{2}{(x-2)}-\dfrac{4}{(x-1)}\right]$
4. $\dfrac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}$ $\left[\dfrac{7}{(x+4)}-\dfrac{3}{(x+1)}-\dfrac{2}{(2x-1)}\right]$
5. $\dfrac{x^2+9x+8}{x^2+x-6}$ $\left[1+\dfrac{2}{(x+3)}+\dfrac{6}{(x-2)}\right]$

6. $\dfrac{x^2-x-14}{x^2-2x-3}$ $\left[1-\dfrac{2}{(x-3)}+\dfrac{3}{(x+1)}\right]$

7. $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$

$$\left[3x-2+\frac{1}{(x-2)}-\frac{5}{(x+2)}\right]$$

7.3 Worked problems on partial fractions with repeated linear factors

Problem 5. Resolve $\dfrac{2x+3}{(x-2)^2}$ into partial fractions

The denominator contains a repeated linear factor, $(x-2)^2$

$$\text{Let } \frac{2x+3}{(x-2)^2} \equiv \frac{A}{(x-2)}+\frac{B}{(x-2)^2} \equiv \frac{A(x-2)+B}{(x-2)^2}$$

Equating the numerators gives:

$$2x+3 \equiv A(x-2)+B$$

Let $x=2$. Then $7=A(0)+B$

i.e. $\mathbf{B=7}$

$$2x+3 \equiv A(x-2)+B \equiv Ax-2A+B$$

Since an identity is true for all values of the unknown, the coefficients of similar terms may be equated.

Hence, equating the coefficients of x gives: $\mathbf{2=A}$
[Also, as a check, equating the constant terms gives: $3=-2A+B$. When $A=2$ and $B=7$, $\text{RHS}=-2(2)+7=3=\text{LHS}$]

$$\text{Hence } \mathbf{\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)}+\frac{7}{(x-2)^2}}$$

Problem 6. Express $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ as the sum of three partial fractions

The denominator is a combination of a linear factor and a repeated linear factor.

$$\text{Let } \frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{A}{(x+3)}+\frac{B}{(x-1)}+\frac{C}{x-1)^2} \equiv \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$$

by algebraic addition

Equating the numerators gives:

$$5x^2-2x-19 \equiv A(x-1)^2+B(x+3)(x-1) + C(x+3) \quad (1)$$

Let $x=-3$. Then

$$5(-3)^2-2(-3)-19 \equiv A(-4)^2+B(0)(-4)+C(0)$$

i.e. $32=16A$

i.e. $\mathbf{A=2}$

Let $x=1$. Then

$$5(1)^2-2(1)-19 \equiv A(0)^2+B(4)(0)+C(4)$$

i.e. $-16=4C$

i.e. $\mathbf{C=-4}$

Without expanding the RHS of equation (1) it can be seen that equating the coefficients of x^2 gives: $5=A+B$, and since $A=2$, $\mathbf{B=3}$
[Check: Identity (1) may be expressed as:

$$5x^2-2x-19 \equiv A(x^2-2x+1) + B(x^2+2x-3)+C(x+3)$$

i.e. $$5x^2-2x-19 \equiv Ax^2-2Ax+A+Bx^2 + 2Bx-3B+Cx+3C$$

Equating the x term coefficients gives:

$$-2 \equiv -2A+2B+C$$

When $A=2$, $B=3$ and $C=-4$ then $-2A+2B+C=-2(2)+2(3)-4=-2=\text{LHS}$
Equating the constant term gives:

$$-19 \equiv A-3B+3C$$

$$\text{RHS}=2-3(3)+3(-4)=2-9-12 = -19=\text{LHS}]$$

Section 1

Hence $\dfrac{\mathbf{5x^2 - 2x - 19}}{\mathbf{(x+3)(x-1)^2}}$

$$\equiv \frac{\mathbf{2}}{\mathbf{(x+3)}} + \frac{\mathbf{3}}{\mathbf{(x-1)}} - \frac{\mathbf{4}}{\mathbf{(x-1)^2}}$$

Problem 7. Resolve $\dfrac{3x^2+16x+15}{(x+3)^3}$ into partial fractions

Let

$$\frac{3x^2+16x+15}{(x+3)^3} \equiv \frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$
$$\equiv \frac{A(x+3)^2 + B(x+3) + C}{(x+3)^3}$$

Equating the numerators gives:

$$3x^2 + 16x + 15 \equiv A(x+3)^2 + B(x+3) + C \quad (1)$$

Let $x = -3$. Then

$$3(-3)^2 + 16(-3) + 15 \equiv A(0)^2 + B(0) + C$$

i.e. $\mathbf{-6 = C}$

Identity (1) may be expanded as:

$$3x^2 + 16x + 15 \equiv A(x^2 + 6x + 9) + B(x+3) + C$$

i.e. $3x^2 + 16x + 15 \equiv Ax^2 + 6Ax + 9A + Bx + 3B + C$

Equating the coefficients of x^2 terms gives:

$$\mathbf{3 = A}$$

Equating the coefficients of x terms gives:

$$16 = 6A + B$$

Since $A = 3$, $\mathbf{B = -2}$

[Check: equating the constant terms gives:

$$15 = 9A + 3B + C$$

When $A = 3$, $B = -2$ and $C = -6$,

$$9A + 3B + C = 9(3) + 3(-2) + (-6)$$
$$= 27 - 6 - 6 = 15 = \text{LHS}]$$

Thus $\dfrac{\mathbf{3x^2 + 16x + 15}}{\mathbf{(x+3)^3}}$

$$\equiv \frac{\mathbf{3}}{\mathbf{(x+3)}} - \frac{\mathbf{2}}{\mathbf{(x+3)^2}} - \frac{\mathbf{6}}{\mathbf{(x+3)^3}}$$

Now try the following exercise

Exercise 27 **Further problems on partial fractions with repeated linear factors**

1. $\dfrac{4x-3}{(x+1)^2}$ $\quad \left[\dfrac{4}{(x+1)} - \dfrac{7}{(x+1)^2}\right]$

2. $\dfrac{x^2+7x+3}{x^2(x+3)}$ $\quad \left[\dfrac{1}{x^2} + \dfrac{2}{x} - \dfrac{1}{(x+3)}\right]$

3. $\dfrac{5x^2-30x+44}{(x-2)^3}$ $\quad \left[\dfrac{5}{(x-2)} - \dfrac{10}{(x-2)^2} + \dfrac{4}{(x-2)^3}\right]$

4. $\dfrac{18+21x-x^2}{(x-5)(x+2)^2}$ $\quad \left[\dfrac{2}{(x-5)} - \dfrac{3}{(x+2)} + \dfrac{4}{(x+2)^2}\right]$

7.4 Worked problems on partial fractions with quadratic factors

Problem 8. Express $\dfrac{7x^2+5x+13}{(x^2+2)(x+1)}$ in partial fractions

The denominator is a combination of a quadratic factor, (x^2+2), which does not factorise without introducing imaginary surd terms, and a linear factor, $(x+1)$. Let

$$\frac{7x^2+5x+13}{(x^2+2)(x+1)} \equiv \frac{Ax+B}{(x^2+2)} + \frac{C}{(x+1)}$$
$$\equiv \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$

Equating numerators gives:

$$7x^2 + 5x + 13 \equiv (Ax+B)(x+1) + C(x^2+2) \quad (1)$$

Let $x = -1$. Then

$$7(-1)^2 + 5(-1) + 13 \equiv (Ax+B)(0) + C(1+2)$$

i.e. $15 = 3C$

i.e. $\mathbf{C = 5}$

Identity (1) may be expanded as:

$$7x^2 + 5x + 13 \equiv Ax^2 + Ax + Bx + B + Cx^2 + 2C$$

Equating the coefficients of x^2 terms gives:

$$7 = A + C, \text{ and since } C = 5, \mathbf{A = 2}$$

Equating the coefficients of x terms gives:

$$5 = A + B, \text{ and since } A = 2, \mathbf{B = 3}$$

[Check: equating the constant terms gives:

$$13 = B + 2C$$

When $B = 3$ and $C = 5$, $B + 2C = 3 + 10 = 13 = \text{LHS}$]

Hence $$\frac{7x^2 + 5x + 13}{(x^2 + 2)(x + 1)} \equiv \frac{2x + 3}{(x^2 + 2)} + \frac{5}{(x + 1)}$$

Problem 9. Resolve $\dfrac{3 + 6x + 4x^2 - 2x^3}{x^2(x^2 + 3)}$ into partial fractions

Terms such as x^2 may be treated as $(x + 0)^2$, i.e. they are repeated linear factors

Let $$\frac{3 + 6x + 4x^2 - 2x^3}{x^2(x^2 + 3)}$$

$$\equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{(x^2 + 3)}$$

$$\equiv \frac{Ax(x^2 + 3) + B(x^2 + 3) + (Cx + D)x^2}{x^2(x^2 + 3)}$$

Equating the numerators gives:

$$\begin{aligned} 3 + 6x + 4x^2 - 2x^3 &\equiv Ax(x^2 + 3) \\ &\quad + B(x^2 + 3) + (Cx + D)x^2 \\ &\equiv Ax^3 + 3Ax + Bx^2 + 3B \\ &\quad + Cx^3 + Dx^2 \end{aligned}$$

Let $x = 0$. Then $3 = 3B$

i.e. $\mathbf{B = 1}$

Equating the coefficients of x^3 terms gives:

$$-2 = A + C \qquad (1)$$

Equating the coefficients of x^2 terms gives:

$$4 = B + D$$

Since $B = 1$, $\mathbf{D = 3}$

Equating the coefficients of x terms gives:

$$6 = 3A$$

i.e. $\mathbf{A = 2}$

From equation (1), since $A = 2$, $\mathbf{C = -4}$

Hence $$\frac{\mathbf{3 + 6x + 4x^2 - 2x^3}}{\mathbf{x^2(x^2 + 3)}}$$

$$\equiv \frac{2}{x} + \frac{1}{x^2} + \frac{-4x + 3}{x^2 + 3}$$

$$\equiv \frac{\mathbf{2}}{\mathbf{x}} + \frac{\mathbf{1}}{\mathbf{x^2}} + \frac{\mathbf{3 - 4x}}{\mathbf{x^2 + 3}}$$

Now try the following exercise

Exercise 28 Further problems on partial fractions with quadratic factors

1. $\dfrac{x^2 - x - 13}{(x^2 + 7)(x - 2)}$ $\quad\left[\dfrac{2x + 3}{(x^2 + 7)} - \dfrac{1}{(x - 2)}\right]$

2. $\dfrac{6x - 5}{(x - 4)(x^2 + 3)}$ $\quad\left[\dfrac{1}{(x - 4)} + \dfrac{2 - x}{(x^2 + 3)}\right]$

3. $\dfrac{15 + 5x + 5x^2 - 4x^3}{x^2(x^2 + 5)}$ $\quad\left[\dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{2 - 5x}{(x^2 + 5)}\right]$

4. $\dfrac{x^3 + 4x^2 + 20x - 7}{(x - 1)^2(x^2 + 8)}$

   $\left[\dfrac{3}{(x - 1)} + \dfrac{2}{(x - 1)^2} + \dfrac{1 - 2x}{(x^2 + 8)}\right]$

5. When solving the differential equation $\dfrac{d^2\theta}{dt^2} - 6\dfrac{d\theta}{dt} - 10\theta = 20 - e^{2t}$ by Laplace transforms, for given boundary conditions, the following expression for $\mathcal{L}\{\theta\}$ results:

   $$\mathcal{L}\{\theta\} = \frac{4s^3 - \frac{39}{2}s^2 + 42s - 40}{s(s - 2)(s^2 - 6s + 10)}$$

   Show that the expression can be resolved into partial fractions to give:

   $$\mathcal{L}\{\theta\} = \frac{2}{s} - \frac{1}{2(s - 2)} + \frac{5s - 3}{2(s^2 - 6s + 10)}$$

Chapter #6

Exercise #6.1 Page (203)
Exercise #6.2 Page (208)

Q # 1

$x^2 - 1 = 0$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

or $x^2 - 1 = 0$

$\Rightarrow (x+1)(x-1) = 0$

$\Rightarrow$ either $x+1 = 0 \Rightarrow x = -1$

or $x - 1 = 0 \Rightarrow x = 1$

Solution $\{1, -1\}$

Q # 5

$x^2 - 9 = 0$

$\Rightarrow (x+3)(x-3) = 0$

$\Rightarrow x+3 = 0 \Rightarrow x = -3$

or $x - 3 = 0 \Rightarrow x = 3$

Solution $\{3, -3\}$

or $x^2 - 9 = 0$

$\Rightarrow x^2 = 9$

$\Rightarrow \boxed{x = \pm 3}$

Q # 15

$x^2 + 2x - 24 = 0$

$\Rightarrow x^2 + 6x - 4x - 24 = 0$

$\Rightarrow x(x+6) - 4(x+6) = 0$

$\Rightarrow (x+6)(x-4) = 0$

$\Rightarrow x+6 = 0 \Rightarrow x = -6$

or $x - 4 = 0 \Rightarrow x = 4$

Solution $\{4, -6\}$

2	24
2	12
2	6
	3

4×6

Q # 21

$4x^2 + 5x - 6 = 0$

$\Rightarrow 4x^2 + 8x - 3x - 6 = 0$

$\Rightarrow 4x(x+2) - 3(x+2) = 0$

$\Rightarrow (x+2)(4x-3) = 0$

$\Rightarrow x+2 = 0 \Rightarrow x = -2$

~~and~~ or $4x - 3 = 0 \Rightarrow x = \frac{3}{4}$

Solution $\{3/4, -2\}$

2	24
2	12
2	6
	3

8×3

Q # 45

when the object land

$y = 0$

$\therefore x - \frac{32x^2}{v_0^2} = 0$

$\Rightarrow x\left(1 - \frac{32x}{v_0^2}\right) = 0$

$\Rightarrow x = 0$ (which is not)

$\therefore 1 - \frac{32x}{v_0^2} = 0$

$\Rightarrow \frac{32x}{v_0^2} = 1$

$\Rightarrow 32x = v_0^2$

$\Rightarrow x = \frac{v_0^2}{32}$ feet

Exercise 6.2 (Page 208)

Q # 3

$x^2 + 4x - 12 = 0$

$\Rightarrow x^2 + 4x = 12$

$\Rightarrow x^2 + 4x + 2^2 = 12 + 2^2$

$\Rightarrow (x+2)^2 = 12 + 4$

$\Rightarrow (x+2)^2 = 16$

$\Rightarrow x + 2 = \pm 4$

$\Rightarrow x = -2 \pm 4$

$\Rightarrow x = -2 + 4 \Rightarrow x = 2$

or $x = -2 - 4 \Rightarrow x = -6$

Solution : $\{2, -6\}$

Exercise 6.2 Page (208)

Q # 23

$6x^2 + x + 2 = 0$

$\Rightarrow 6x^2 + x = -2$

$\Rightarrow x^2 + \frac{x}{6} = -\frac{1}{3}$

$\Rightarrow x^2 + \frac{x}{6} + \left(\frac{1}{12}\right)^2 = -\frac{1}{3} + \left(\frac{1}{12}\right)^2$

$\Rightarrow \left(x + \frac{1}{12}\right)^2 = -\frac{1}{3} + \frac{1}{144}$

$\Rightarrow \left(x + \frac{1}{12}\right)^2 = \frac{-144 + 3}{3(144)}$

$\Rightarrow \left(x + \frac{1}{12}\right)^2 = \frac{-\cancel{141}\,47}{\cancel{3} \times 144} = -\frac{47}{144}$

$\Rightarrow x + \frac{1}{12} = \pm\sqrt{-\frac{47}{144}}$

$\Rightarrow x + \frac{1}{12} = \frac{\pm\sqrt{47}\,i}{12}$

$\Rightarrow x = -\frac{1}{12} \pm \frac{\sqrt{47}\,i}{12}$

$\Rightarrow x = \frac{1}{12}\left(-1 \pm \sqrt{47}\,i\right)$

Q # 39

$ax^2 + 5x - 1 = 0$

$\Rightarrow ax^2 + 5x = 1$

$\Rightarrow x^2 + \frac{5}{a}x = \frac{1}{a}$

$\Rightarrow x^2 + \frac{5}{a}x + \left(\frac{5}{2a}\right)^2 = \frac{1}{a} + \left(\frac{5}{2a}\right)^2$

$\Rightarrow \left(x + \frac{5}{2a}\right)^2 = \frac{1}{a} + \frac{25}{4a^2}$

$\Rightarrow \left(x + \frac{5}{2a}\right)^2 = \frac{4a + 25}{4a^2}$

$\Rightarrow x + \frac{5}{2a} = \pm\sqrt{\frac{4a+25}{4a^2}}$

$\Rightarrow x = -\frac{5}{2a} \pm \frac{\sqrt{4a+25}}{2a}$

$\Rightarrow x = \frac{1}{2a}\left(-5 \pm \sqrt{4a+25}\right)$

Q#25

Exercise #6.3 Page 213-214

Q#5

$$2x^2 = 5x - 2$$

$$\Rightarrow 2x^2 - 5x + 2 = 0$$

$a = 2,\ b = -5,\ c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$$

$$\Rightarrow x = \frac{5 \pm \sqrt{25 - 16}}{4}$$

$$\Rightarrow x = \frac{5 \pm \sqrt{9}}{4}$$

$$\Rightarrow x = \frac{5 \pm 3}{4}$$

$$\Rightarrow x = \frac{5+3}{4} = \frac{8}{4} \Rightarrow x = 2$$

or $x = \frac{5-3}{4} = \frac{2}{4} \Rightarrow x = \frac{1}{2}$

Solution $\left\{\frac{1}{2}, 2\right\}$

Q#25

$$2x^2 + 3x = 0$$

$a = 2,\ b = 3,\ c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9 - 0}}{2(2)}$$

$$\Rightarrow x = \frac{-3 \pm 3}{4}$$

$$\Rightarrow x = \frac{-3+3}{4} \Rightarrow x = 0$$

and $x = \frac{-3-3}{4} \Rightarrow x = -\frac{3}{2}$

Solution $\{0, -3/2\}$

Q#33

$$4x^2 - 20x + 25 = 0$$

$a = 4,\ b = -20,\ c = 25$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}$$

$$\Rightarrow x = \frac{20 \pm \sqrt{400 - 400}}{8}$$

$$\Rightarrow x = \frac{20}{8}$$

$$\Rightarrow x = 5/2$$

Solution $\{5/2\}$

Q#53

$$\frac{1}{x} + \frac{1}{x-4} = \frac{3}{8}$$

$$\Rightarrow \frac{x-4+x}{x(x-4)} = \frac{3}{8}$$

$$\Rightarrow \frac{2x-4}{x^2-4x} = \frac{3}{8}$$

$$\Rightarrow 8(2x-4) = 3(x^2-4x)$$

$$\Rightarrow 16x - 32 = 3x^2 - 12x$$

$$\Rightarrow 3x^2 - 12x - 16x + 32 = 0$$

$$\Rightarrow 3x^2 - 28x + 32 = 0$$

$a = 3,\ b = -28,\ c = 32$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{28 \pm \sqrt{(-28)^2 - 4(3)(32)}}{2(3)}$$

$$\Rightarrow x = \frac{28 \pm \sqrt{784 - 384}}{6}$$

$$\Rightarrow x = \frac{28 \pm \sqrt{400}}{6}$$

$$\Rightarrow x = \frac{28 \pm 20}{6}$$

$$\Rightarrow x = \frac{28+20}{6} \Rightarrow x = \frac{48}{6} \Rightarrow x = 8$$

and $x = \frac{28-20}{6} \Rightarrow x = \frac{8}{6} \Rightarrow x = \frac{4}{3}$

Solution $\therefore \left\{8, \frac{4}{3}\right\}$

Exercise # 6.4 Page (217-218)

Q#5

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$f = 2m,\ p = q + 3$

$$\frac{1}{2} = \frac{1}{q+3} + \frac{1}{q}$$

$$\Rightarrow \frac{1}{2} = \frac{q + q + 3}{q(q+3)}$$

$$\Rightarrow \frac{1}{2} = \frac{2q+3}{q^2+3q}$$

$$\Rightarrow q^2 + 3q = 4q + 6$$

$$\Rightarrow q^2 - q - 6 = 0$$

$$\Rightarrow q^2 - 3q + 2q - 6 = 0$$

$$\Rightarrow q(q-3) + 2(q-3) = 0$$

$$\Rightarrow (q-3)(q+2) = 0$$

$\Rightarrow q - 3 = 0 \Rightarrow q = 3$

or $q + 2 = 0 \Rightarrow q = -2$

Solution $\{3, -2\}$

$p = q + 3 \Rightarrow p = 6 \ldots p = 1$

Q#17

Let x be the bike rate

Then $x + 15$ be for the car

$$\frac{50}{x} - \frac{50}{x+15} = 3$$

$$\Rightarrow \frac{50(x+15) - 50x}{x(x+15)} = 3$$

$$\Rightarrow \frac{50x + 750 - 50x}{x^2 + 15x} = 3$$

$$\Rightarrow 750 = 3x^2 + 45x$$

$$\Rightarrow 3x^2 + 45x - 750 = 0$$

$$\Rightarrow x^2 + 15x - 250 = 0$$

$a = 1,\ b = 15,\ c = -250$

or let x bike, y car

$y - x = 15$

$\frac{50}{x} - \frac{50}{y} = 3$

Q#23

Let low share cost $= x$

high " " $= x + 10$

$$\frac{240}{x} - \frac{240}{x+10} = 4$$

$$\Rightarrow \frac{240(x+10) - 240x}{x(x+10)} = 4$$

$$\Rightarrow \frac{240x + 2400 - 240x}{x^2 + 10x} = 4$$

$$\Rightarrow 2400 = 4x^2 + 40x$$

$$\Rightarrow x^2 + 10x - 600 = 0$$

$a = 1,\ b = 10,\ c = -600$

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Basic Technical Mathematics with Calculus
Peter K. F. Kuhfitting

CHAPTER 6 198

Quadratic Equations

Objectives Upon completion of this chapter, you should be able to:

1. Solve a given quadratic equation by:
   a. Factoring.
   b. Completing the square.
   c. Using the quadratic formula.
2. Solve stated problems leading to quadratic equations.

6.1 Solution by Factoring and Pure Quadratic Equations

All the equations introduced in the earlier chapters were of first degree (involving only the first power of the unknown). Many equations arising in technical problems are of second or higher degree. For example, if an object is tossed upward from the ground with initial velocity v_0, then the distance above the ground as a function of time is given by $s = v_0t - 16t^2$, where s is measured in feet and t in seconds. This is an example of an equation of second degree. The equation $x^2 - 3x + 4 = 0$ is also of second degree, while $x^3 - 2x^2 + 6x - 1 = 0$ is of third degree. In this chapter we shall study only **quadratic equations**, which are equations of second degree.

> A **quadratic** equation has the form
>
> $$ax^2 + bx + c = 0, \quad a \neq 0 \qquad (6.1)$$

The equation $ax^2 + bx + c = 0$ is called the **standard form** of the quadratic equation.

In this section we shall confine ourselves to those cases in which the left side of equation (6.1) is factorable. The method of solution depends on the following property of real numbers:

> $ab = 0$ if, and only if, $a = 0$ or $b = 0$.

Consider, for example, the equation

$$x^2 = 8 - 2x$$

For the equation to fit the standard form (6.1), all the terms have to be collected on the left side and written in descending powers of x. Adding $-8 + 2x$ to both sides, we get

Step 1. $x^2 + 2x - 8 = 0$ (6.2)

If we now factor the left side, the equation becomes

Step 2. $(x + 4)(x - 2) = 0$ (6.3)

Next, we set each factor equal to 0:

Step 3. $x + 4 = 0 \quad x - 2 = 0$

Finally, we solve each of the resulting linear equations:

Step 4. $x = -4 \quad x = 2$

The solution is therefore given by two values, $x = -4$ and $x = 2$.

As a check, let us substitute both values into the original equation $x^2 = 8 - 2x$:

$$\begin{aligned} (-4)^2 &= 8 - 2(-4) \qquad & 2^2 &= 8 - 2(2) \\ 16 &= 8 + 8 & 4 &= 8 - 4 \\ 16 &= 16 & 4 &= 4 \end{aligned}$$

Since both values check, we see that the equation has two distinct roots.

Finally, note that the roots are unique:

$$(x + 4)(x - 2) = 0$$

if, and only if, $x + 4 = 0$ or $x - 2 = 0$. But $x + 4 = 0$ if, and only if, $x = -4$; and $x - 2 = 0$ if, and only if, $x = 2$.

To illustrate the solution of the equation $x^2 + 2x - 8 = 0$ in (6.2) graphically, consider the function

$$y = x^2 + 2x - 8$$

whose graph appears in Figure 6.1. The solution of $x^2 + 2x - 8 = 0$ consists of the x-intercepts (where $y = 0$).

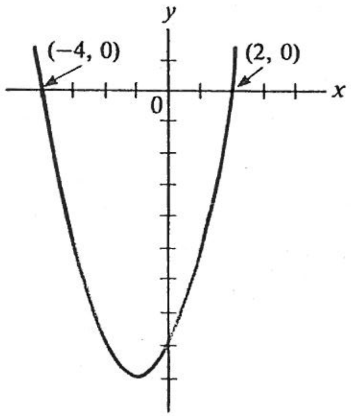

Figure 6.1

Solution by factoring:

1. Write the equation in standard form by collecting all the terms on the left side.
2. Factor the expression on the left side.
3. Set each of the factors equal to zero.
4. Solve the resulting two linear equations.

Example 1

Solve the equation $2x^2 + 15 = 13x$.

Solution.

$2x^2 + 15 = 13x$		given equation
$2x^2 - 13x + 15 = 0$		collecting terms on left side
$(2x - 3)(x - 5) = 0$		factoring left side
$2x - 3 = 0$	$x - 5 = 0$	setting factors equal to 0
$2x = 3$	$x = 5$	solving the resulting linear equations
$x = \frac{3}{2}$	$x = 5$	

Check:

	Left Side	Right Side
$x = \frac{3}{2}$:	$2\left(\frac{3}{2}\right)^2 + 15 = \frac{39}{2}$	$13\left(\frac{3}{2}\right) = \frac{39}{2}$
$x = 5$:	$2(5)^2 + 15 = 65$	$13(5) = 65$

Example 2

Solve the equation $x^2 + 7x = 0$.

Solution. In this equation $c = 0$. As a result, the terms on the left have a common factor x.

$x^2 + 7x = 0$		given equation
$x(x + 7) = 0$		common factor x
$x = 0$	$x + 7 = 0$	setting each factor equal to zero
$x = 0$	$x = -7$	

(See Figure 6.2.) The solution can be checked as in Example 1.

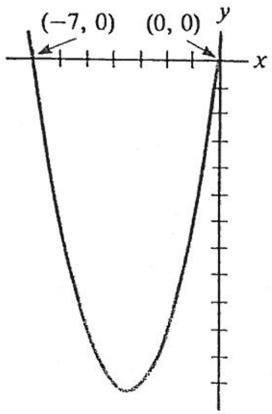

Figure 6.2

Pure quadratic equations

An equation for which $b = 0$ is called a **pure quadratic equation** and may be solved by a different method. For example, to solve the equation

$$x^2 - 4 = 0$$

we first solve for x^2 to obtain

$$x^2 = 4$$

Taking the square root of both sides, we find that $x = \pm\sqrt{4} = \pm 2$, where $\pm$ means *plus or minus*. So the equation has two roots, $x = 2$ and $x = -2$.

Alternatively,

$$x^2 - 4 = (x - 2)(x + 2) = 0$$

leads to the same solution. However, factoring does not work for the equation in the next example.

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le 3 Solve the equation $x^2 - 5 = 0$.

Solution. Adding 5 to both sides of the equation, we get

$$x^2 - 5 + 5 = 0 + 5$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

Remark: The left side of the equation $x^2 + 4x + 4 = 0$ is a perfect-square trinomial leading to identical factors:

ouble

$$(x + 2)(x + 2) = 0$$

e root

The solution $x = -2$ and $x = -2$ is called a **repeating root** or a **double root.**

non

errors

mmon

(1) Forgetting the negative root when solving $x^2 - a^2 = 0$. The roots are $x = \pm a$.

(2) Attempting to solve by factoring when the right side is not zero. For example, if

$$(x - 3)(x + 2) = 6$$

we may *not* conclude that $x - 3 = 6$ and $x + 2 = 6$. Instead, we need to write the equation in the form $ax^2 + bx + c = 0$:

$$x^2 - x - 6 = 6$$
$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4, -3$$

Historical note. As mentioned in Chapter 1, the first systematic study of second-degree equations was undertaken by al-Khowarizmi in Baghdad. He gave an exhaustive exposition of various cases using ingenious geometric arguments, in the manner of the ancient Greeks. Consequently, al-Khowarizmi's algebra was rhetorical, using words and drawings instead of symbols. Further progress in algebra was slow until algebraic notation was introduced. This far-reaching innovation was due to the French lawyer Franciscus Vieta (1540–1603), who recognized the advantage of using letters to denote both known and unknown quantities.

Other now-familiar symbols were introduced only gradually. In medieval times the letters p and m were widely used to denote addition and subtraction, and the Latin word *cosa* for the unknown. The signs + and − first appeared in print in a book published in 1489 by Johann Widman, a lecturer in Leipzig. The English mathematician William Oughtred (1574–1660) popularized the symbol × for multiplication, and Johan Rahn of Switzerland first used the sign ÷ for division in 1659. The French philosopher

Francis

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René Descartes (1596–1650) introduced the exponential notation, and the Englishman Robert Recorde (1510–1558) used the symbol = for equality because, as he put it, "noe 2. thynges can be moare equalle."

the

equality

Exercises / Section 6.1

In Exercises 1–12, solve the given pure quadratic equations.

1. $x^2 - 1 = 0$ **2.** $x^2 - 25 = 0$ **3.** $x^2 - 36 = 0$

4. $x^2 - 121 = 0$ **5.** $x^2 - 9 = 0$ **6.** $x^2 - 16 = 0$

7. $x^2 - 10 = 0$ **8.** $x^2 - 12 = 0$ **9.** $2x^2 - 32 = 0$

10. $4x^2 - 1 = 0$ **11.** $36x^2 - 25 = 0$ **12.** $49x^2 - 16 = 0$

In Exercises 13–44, solve the given quadratic equations by the method of factoring.

13. $x^2 + x - 2 = 0$ **14.** $x^2 + 7x + 10 = 0$ **15.** $x^2 + 2x - 24 = 0$

16. $x^2 + 4x - 21 = 0$ **17.** $2x^2 - 5x - 3 = 0$ **18.** $2x^2 - 7x - 15 = 0$

19. $3x^2 + 7x + 2 = 0$ **20.** $6x^2 + 11x + 4 = 0$ **21.** $4x^2 + 5x - 6 = 0$

22. $5x^2 + 16x + 12 = 0$ **23.** $5x^2 + 8x - 21 = 0$ **24.** $4x^2 + 29x - 24 = 0$

25. $4x^2 + 4x = 15$ **26.** $6x^2 + 7x = 5$ **27.** $30x^2 = 7x + 15$

28. $12x^2 = 7x + 10$ **29.** $8x^2 = 2x + 45$ **30.** $40x^2 = 67x - 28$

31. $7x^2 + 4x = 11$ **32.** $14x^2 + 53x + 45 = 0$ **33.** $18x^2 + 17x = 15$

34. $3x^2 = 10x + 13$ **35.** $11x^2 = 76x + 7$ **36.** $4x^2 = 49x - 90$

37. $72x^2 + 13x = 15$ **38.** $45x^2 + 52x + 15 = 0$ **39.** $18x^2 = 93x - 110$

40. $21x^2 + 10x = 91$ **41.** $9x^2 + 24x + 16 = 0$ **42.** $25x^2 - 10x + 1 = 0$

43. $16x^2 - 8x + 1 = 0$ **44.** $4x^2 - 20x + 25 = 0$

45. The path of an object tossed at an angle of 45° to the ground is $y = x - 32x^2/v_0^2$, where v_0 is the initial velocity in feet per second and y is the distance in feet above the ground. How far from the starting point ($x = 0$) will the object land?

46. The weekly profit P of a company is $P = x^4 - 30x^3$, $x \geq 1$, where x is the week in the year. During what week is the profit equal to zero? (*Hint:* Factor out x^3.)

47. The load on a beam of length L is such that the deflection is given by $d = 3x^4 - 4Lx^3 + L^2x^2$, where x is the distance from one end. Determine where the deflection is zero. (*Hint:* Factor out x^2.)

48. The formula for the output of a battery is $P = VI - RI^2$. For what values of I is the output equal to zero?

6.2 Solution by Completing the Square

In the last section we restricted our attention to factorable quadratic equations. In this section we shall take up a general method for solving any given quadratic equation. The procedure is referred to as **completing the square.**

Completing the square depends on the fact that any quadratic equation can be written in the form

$$(x + b)^2 = a \qquad (6.4)$$

To understand this, recall that the square of a binomial is given by

$$(x+b)^2 = x^2 + 2bx + b^2$$

Perfect ct-square trinomial

Looking at the right side, observe that *a trinomial in x with a coefficient of x^2 equal to 1 is a perfect square if the square of one-half the coefficient of x is equal to the third term.* For example,

$$x^2 + 6x + 9$$

is a perfect square, since $(\frac{1}{2} \cdot 6)^2 = 9$. Similarly,

$$x^2 - \frac{3}{2}x + \frac{9}{16}$$

is a perfect square, since

$$\frac{9}{16} = \left[\frac{1}{2} \cdot \left(-\frac{3}{2}\right)\right]^2$$

so that

$$x^2 - \frac{3}{2}x + \frac{9}{16} = \left(x - \frac{3}{4}\right)^2 \qquad \left(x - \frac{3}{4}\right)^2 = x^2 + 2\left(-\frac{3}{4}x\right) + \left(-\frac{3}{4}\right)^2$$

> The method of completing the square consists of rewriting one side of the equation so that it forms a perfect square.

Consider the next example.

Example #1

Example 1 Solve the equation $x^2 - 6x + 8 = 0$ by completing the square.

Solution. The first step is to transpose the 8 (or add −8 to both sides) in order to retain only the x^2 and x terms on the left side. Thus

$$x^2 - 6x = -8$$

The critical step is to complete the square on the left side by adding to both sides the square of one-half the coefficient of x, or $[\frac{1}{2} \cdot (-6)]^2 = 9$. We then get

$$x^2 - 6x + 9 = -8 + 9$$

The left side is now a perfect square, so that the equation can be written

$$(x-3)^2 = 1 \qquad (x-3)^2 = x^2 + 2(-3x) + (-3)^2$$

The resulting pure quadratic form can be solved by taking the square root of both sides, yielding the two linear equations

$$\sqrt{(x-3)^2} = \pm\sqrt{1}$$
$$x - 3 = \pm 1$$

Solving, we get 205

$$x = 3 \pm 1$$

which gives $x = 4$ and $x = 2$.

Check:

$$x = 4:\quad (4)^2 - 6(4) + 8 = 0$$
$$x = 2:\quad (2)^2 - 6(2) + 8 = 0$$

Let us now summarize the procedure for completing the square.

> **Solution by completing the square:**
>
> 1. Write the equation in the form $ax^2 + bx = -c$.
> 2. Multiply each side by $1/a$.
> 3. Complete the square on the left side by adding the square of one-half the coefficient of x to both sides.
> 4. Write the left side as a square; simplify the right side.
> 5. Take the square root of both sides.
> 6. Solve the resulting two linear equations.

Example #2

Solve the equation $2x^2 + 6x - 3 = 0$ by completing the square.

Solution. Following the procedure for completing the square, we get

	$2x^2 + 6x - 3 = 0$	given equation
Step 1.	$2x^2 + 6x = 3$	transposing −3
Step 2.	$x^2 + 3x = \frac{3}{2}$	dividing by 2

Note that the square of one-half the coefficient of x is

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

This number has to be added to both sides to complete the square. It follows that

Step 3. $x^2 + 3x + \frac{9}{4} = \frac{3}{2} + \frac{9}{4}$

and

Step 4. $\left(x + \frac{3}{2}\right)^2 = \frac{6}{4} + \frac{9}{4} = \frac{15}{4}$ factoring the left side and simplifying the right side

Taking the square root of both sides, we get

Step 5. $x + \frac{3}{2} = \pm\sqrt{\frac{15}{4}} = \pm\frac{\sqrt{15}}{2}$

Solving for x,

Step 6. $x = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}$

or

$$x = \frac{-3 \pm \sqrt{15}}{2}$$

The roots can also be written separately as

$$x = \frac{-3 + \sqrt{15}}{2} \quad \text{and} \quad x = \frac{-3 - \sqrt{15}}{2}$$

(Although highly desirable, checking the solution is difficult at this point, since we do not discuss the multiplication of radical expressions of this complexity until Chapter 10.)

Complex Roots

In Chapter 1 we mentioned that numbers fall into two categories, real and complex. Complex numbers will be studied in detail in Chapter 11. In this chapter we need only to understand the basic concepts and notations.

Consider the pure quadratic equation

$$x^2 + 4 = 0$$

Solving for x, we get $x = \pm\sqrt{-4}$. Since we cannot find a (real) number whose square root is -4, $\sqrt{-4}$ is called a **pure imaginary number.**

For the past two hundred years the letter i (for "imaginary") has been used to denote the imaginary number $\sqrt{-1}$. When imaginary numbers were first introduced in electrical circuit theory, the convention of using i for instantaneous current had already become well established. Consequently, using the letter j to denote $\sqrt{-1}$ became standard in physics and technology, and we shall observe this convention here.

Returning now to $\sqrt{-4}$, observe that

$$\sqrt{-4} = \sqrt{(4)(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} = 2j,$$

Thus $\sqrt{-4} = 2j$, where $j = \sqrt{-1}$. Imaginary numbers are always written in this form.

> **Basic imaginary unit:**
>
> $j = \sqrt{-1}$ or $j^2 = -1$
>
> $\sqrt{-a}$, $a > 0$, is written $\sqrt{a}\,j$

If a and b are real numbers, then $a + bj$ is called a **complex number;** a is called the **real part** of the complex number, and b is called the **imaginary part.** Thus a pure imaginary number is a complex number whose real part is zero; a real number is a complex number whose imaginary part is zero.

> **Complex number:** A **complex number** has the form $a + bj$, where a and b are real numbers and $j = \sqrt{-1}$.

Some quadratic equations lead to complex roots, as shown in the remaining examples.

Example 3 Solve the equation $x^2 + 4x + 16 = 0$ by completing the square.

Solution.

$x^2 + 4x + 16 = 0$	given equation
$x^2 + 4x = -16$	transposing
$x^2 + 4x + 4 = -16 + 4$	adding $\left(\frac{1}{2}\cdot 4\right)^2$ to both sides
$(x + 2)^2 = -12$	factoring the left side
$x + 2 = \pm\sqrt{-12}$	taking the square root of each side

Recall that $\sqrt{-12} = \sqrt{(-1)\cdot 4 \cdot 3} = 2\sqrt{3}\sqrt{-1} = 2\sqrt{3}j$. It follows that $x = -2 \pm 2\sqrt{3}j$.

Example 4 Solve the equation $2x^2 - 3x + 4 = 0$ by completing the square.

Solution.

$2x^2 - 3x + 4 = 0$	given equation
$x^2 - \frac{3}{2}x + 2 = 0$	dividing by 2
$x^2 - \frac{3}{2}x = -2$	transposing the 2
$x^2 - \frac{3}{2}x + \frac{9}{16} = -2 + \frac{9}{16}$	adding $\left[\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right]^2$ to both sides
$\left(x - \frac{3}{4}\right)^2 = -\frac{32}{16} + \frac{9}{16} = -\frac{23}{16}$	factoring the left side
$x - \frac{3}{4} = \pm\sqrt{-\frac{23}{16}} = \pm\frac{\sqrt{-23}}{4} = \pm\frac{\sqrt{23}\,j}{4}$	taking the square root of each side
$x = \frac{3}{4} \pm \frac{\sqrt{23}}{4}j$	

Remark. We shall see in the next section that a nonfactorable quadratic equation can be solved directly by a formula. Once you learn this formula, you may feel that completing the square is a waste of time. However, *completing the square is an algebraic technique that arises in contexts other than solving equations.* In fact, solving quadratic equations is merely a convenient way to introduce this technique. It is therefore very important for you to practice solving equations by completing the square in the next exercise set.

Exercises / Section 6.2

Solve each equation by completing the square.

1. $x^2 - 6x + 8 = 0$	**2.** $x^2 - 6x + 5 = 0$	**3.** $x^2 + 4x - 12 = 0$
4. $x^2 + 2x - 15 = 0$	**5.** $x^2 + x - 12 = 0$	**6.** $x^2 + 3x - 28 = 0$
7. $x^2 + 7x + 10 = 0$	**8.** $x^2 - 9x + 20 = 0$	**9.** $x^2 + 5x + 2 = 0$
10. $x^2 - 4x - 6 = 0$	**11.** $x^2 + 6x + 6 = 0$	**12.** $x^2 + 3x + 1 = 0$
13. $2x^2 - 6x + 1 = 0$	**14.** $2x^2 + 5x + 2 = 0$	**15.** $2x^2 + 3x - 3 = 0$
16. $2x^2 - 3x - 5 = 0$	**17.** $3x^2 + 2x - 1 = 0$	**18.** $3x^2 - 2x - 3 = 0$
19. $3x^2 - 4x - 5 = 0$	**20.** $2x^2 + 5x - 2 = 0$	**21.** $4x^2 - x - 3 = 0$
22. $5x^2 - 2x - 1 = 0$	**23.** $6x^2 + x + 2 = 0$	**24.** $5x^2 + 9x + 1 = 0$
25. $x^2 - 4x + 5 = 0$	**26.** $3x^2 - 2x + 1 = 0$	**27.** $4x^2 - 5x + 3 = 0$
28. $2x^2 + 3x + 2 = 0$	**29.** $7x^2 + 2x - 1 = 0$	**30.** $8x^2 + 3x + 1 = 0$
31. $6x^2 - 5x - 2 = 0$	**32.** $7x^2 - 19x - 6 = 0$	**33.** $6x^2 + 5x - 50 = 0$
34. $8x^2 - 7x + 2 = 0$	**35.** $5x^2 - x + 1 = 0$	**36.** $5x^2 + 2x - 3 = 0$
37. $x^2 - bx + 2 = 0$	**38.** $x^2 - x + c = 0$	**39.** $ax^2 + 5x - 1 = 0$
40. $x^2 - 3bx + 5 = 0$		

6.3 The Quadratic Formula

Completing the square can be used to obtain a general formula for solving any quadratic equation. We start with the standard form:

$$ax^2 + bx + c = 0 \qquad (6.5)$$

$$ax^2 + bx = -c \qquad \text{transposing}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{dividing by } a$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{adding } \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 \text{ to each side}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{factoring the left side and simplifying the right side}$$

$$= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{taking the square root of each side}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \qquad (6.6)$$

Formula (6.6), known as the **quadratic formula**, should be carefully memorized.

> **Quadratic formula:** The roots of the quadratic equation
>
> $$ax^2 + bx + c = 0, \quad a \neq 0$$
>
> are given by
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By using the quadratic formula, the solutions of a quadratic equation can be written directly, but they usually have to be simplified. The first three examples illustrate the technique.

Example #1

Solve the equation $6x^2 = 2x + 1$ by means of the quadratic formula.

Solution. The equation is first written in the standard form

$$6x^2 - 2x - 1 = 0$$

By equation (6.5), $a = 6$, $b = -2$, and $c = -1$. So by the quadratic formula (6.6), the solution is

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(-1)}}{2\cdot 6}$$

$$= \frac{2 \pm \sqrt{28}}{12}$$

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To simplify the radical, note that $\sqrt{28} = \sqrt{4 \cdot 7} = 2\sqrt{7}$. Thus

$$x = \frac{2 \pm 2\sqrt{7}}{12} = \frac{2(1 \pm \sqrt{7})}{12}$$

$$x = \frac{1 \pm \sqrt{7}}{6}$$

Example #2

Solve the equation $5x^2 + 2x + 4 = 0$ by the quadratic formula.

Solution. From the standard form (6.5), we see that $a = 5$, $b = 2$, and $c = 4$. So by the quadratic formula

$$x = \frac{-2 \pm \sqrt{2^2 - 4(5)(4)}}{2 \cdot 5} = \frac{-2 \pm \sqrt{4 - 80}}{10} = \frac{-2 \pm \sqrt{-76}}{10}$$

Since $\sqrt{-76} = \sqrt{(-1)(4)(19)} = 2\sqrt{19}j$, we get

$$x = \frac{-2 \pm 2\sqrt{19}j}{10} = \frac{-1 \pm \sqrt{19}j}{5}$$

Example #3

Solve the equation $4x^2 - 12x + 9 = 0$.

Solution. Since $a = 4$, $b = -12$, and $c = 9$, we get

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2 \cdot 4} = \frac{12 \pm \sqrt{144 - 144}}{8} = \frac{12 \pm 0}{8} = \frac{3 \pm 0}{2}$$

Hence $x = \frac{3}{2}, \frac{3}{2}$. (Whenever the radical is 0, we get a double root.)

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These examples show that the radical in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

determines whether the roots are real or complex. The expression $b^2 - 4ac$ under the radical sign is called the **discriminant.** We have seen that a given equation has two distinct real roots if $b^2 - 4ac > 0$ (Example 1) and complex roots if $b^2 - 4ac < 0$ (Example 2). If $b^2 - 4ac = 0$, then the equation has a double root (Example 3).

Example 4 Use a calculator to solve the equation

CALCULATOR COMMENT

$$3.17x^2 - 1.98x - 6.83 = 0$$

Solution. Since $a = 3.17$, $b = -1.98$, and $c = -6.83$, we get

$$x = \frac{1.98 \pm \sqrt{(1.98)^2 - 4(3.17)(-6.83)}}{2(3.17)}$$

The simplest way to carry out this calculation is to find the value of the radical and store it in the memory. Now add 1.98 to the positive value of the radical and divide the sum by $(2 \times 3.17) = 6.34$ to get

$$x = 1.81$$

to two decimal places. Next, transfer the content of the memory to the register, change the sign to minus, and proceed as before. The second root is

$$x = -1.19$$

again to two decimal places.

The sequences are

1.98 $\boxed{x^2}$ $\boxed{-}$ 4 $\boxed{\times}$ 3.17 $\boxed{\times}$ 6.83 $\boxed{+/-}$ $\boxed{=}$
$\boxed{\sqrt{\ }}$ $\boxed{\text{STO}}$ $\boxed{+}$ 1.98 $\boxed{=}$ $\boxed{\div}$ 2 $\boxed{\div}$ 3.17 $\boxed{=}$

Display: 1.8130051

$\boxed{\text{MR}}$ $\boxed{+/-}$ $\boxed{+}$ 1.98 $\boxed{=}$ $\boxed{\div}$ 2 $\boxed{\div}$ 3.17 $\boxed{=}$

Display: −1.1883994

The quadratic formula can also be used to solve equations containing two different variables, say x and y. If the equation is to be solved for x in terms of y, then y has to be treated as if it were just another constant. Conversely, to solve for y, x is treated as a constant.

Example 5 Solve the equation

$$x^2 - 3x + xy + 2 - 3y - 2y^2 = 0$$ For x in terms of y

Solution. We first write the equation in standard form. Noting the common factor x, we get

$$x^2 + (-3 + y)x + (2 - 3y - 2y^2) = 0$$

From this equation we see that

$$a = 1, \quad b = -3 + y, \quad \text{and} \quad c = 2 - 3y - 2y^2$$

By the quadratic formula

$$x = \frac{-(-3 + y) \pm \sqrt{(-3 + y)^2 - 4(2 - 3y - 2y^2)}}{2}$$

$$= \frac{3 - y \pm \sqrt{9 - 6y + y^2 - 8 + 12y + 8y^2}}{2}$$

$$= \frac{3 - y \pm \sqrt{9y^2 + 6y + 1}}{2}$$

The radical can be simplified by noting that

$$9y^2 + 6y + 1 = (3y + 1)^2$$

and

$$\sqrt{(3y + 1)^2} = 3y + 1$$

It follows that

$$x = \frac{3 - y \pm (3y + 1)}{2}$$

Thus

$$x = \frac{3 - y + 3y + 1}{2} \quad \text{and} \quad x = \frac{3 - y - 3y - 1}{2}$$

These fractions simplify to

$$x = 2 + y \quad \text{and} \quad x = 1 - 2y$$

Fractional equations may also lead to quadratic equations, as illustrated in the next example.

xample #6

Solve the equation

$$\frac{1}{x} - \frac{1}{x + 1} = \frac{1}{20}$$

Solution. Recall that the simplest way to solve a fractional equation is to clear the fractions by multiplying both sides by the LCD—in this case

$20x(x + 1)$. Then

$$20x(x + 1)\left(\frac{1}{x} - \frac{1}{x + 1}\right) = 20x(x + 1)\,\frac{1}{20}$$

$$20(x + 1) - 20x = x(x + 1)$$

$$20x + 20 - 20x = x^2 + x$$

$$20 = x^2 + x$$

$$0 = x^2 + x - 20$$

$$(x + 5)(x - 4) = 0$$

$$x = 4, -5$$

Exercises / Section 6.3

In Exercises 1–34, solve each equation by the quadratic formula.

1. $x^2 + x - 6 = 0$
2. $x^2 - 3x - 4 = 0$
3. $x^2 - 9x + 20 = 0$
4. $x^2 + 8x + 15 = 0$
5. $2x^2 = 5x - 2$
6. $3x^2 = 13x + 10$
7. $6x^2 - x = 2$
8. $5x^2 - 7x = 6$
9. $2x^2 = 3x + 1$
10. $2x^2 + 2x = 1$
11. $3x^2 + 2x = 2$
12. $x^2 = 3x + 2$
13. $x^2 - 2x + 2 = 0$
14. $x^2 + 3x + 3 = 0$
15. $x^2 + 2x + 4 = 0$
16. $x^2 + 3x + 5 = 0$
17. $3x^2 + 3x + 1 = 0$
18. $3x^2 + 5x + 2 = 0$
19. $4x^2 + 2x + 1 = 0$
20. $4x^2 + 3x - 2 = 0$
21. $4x^2 + 5x = 3$
22. $5x^2 + 3x = 3$
23. $5x^2 + 1 = 0$
24. $3x^2 + 4 = 0$
25. $2x^2 + 3x = 0$
26. $3x^2 - 5x = 0$
27. $2x^2 - 3cx + 1 = 0$
28. $3x^2 + 3x - 2a = 0$
29. $bx^2 + 3x + 1 = 0$
30. $cx^2 + bx - 4 = 0$
31. $4x^2 - 12x + 9 = 0$
32. $9x^2 + 12x + 4 = 0$
33. $4x^2 - 20x + 25 = 0$
34. $9x^2 + 42x + 49 = 0$

In Exercises 35–40, solve the given equations using a calculator. (Find the roots to two decimal places.)

35. $2.00x^2 + 3.12x - 3.19 = 0$
36. $4.12x^2 - 1.30x - 12.1 = 0$
37. $1.79x^2 - 10.0x - 1.91 = 0$
38. $7.179x^2 + 2.862x - 1.998 = 0$
39. $10.103x^2 - 1.701x - 3.28 = 0$
40. $1.738x^2 - 10.162x - 11.773 = 0$

In Exercises 41–48, solve the given equations for x in terms of y.

41. $x^2 - 2x + 1 - y^2 = 0$
42. $x^2 - 2xy + y^2 - 9 = 0$

43. $x^2 - 4xy + 4y^2 - 1 = 0$

44. $x^2 + 4xy + 4y^2 - 9 = 0$

45. $x^2 - 2xy + 3x + y^2 - 3y + 2 = 0$

46. $x^2 - 2xy + 2x + y^2 - 2y - 3 = 0$

47. $x^2 - 3x + 2 - y - y^2 = 0$

48. $x^2 - 3x - xy + 2 + 3y - 2y^2 = 0$

In Exercises 49–56, solve each equation for x.

49. $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{20}$

50. $\frac{1}{x} - \frac{1}{x+2} = \frac{1}{4}$

51. $\frac{1}{x+2} + \frac{1}{x} = \frac{5}{12}$

52. $\frac{1}{x} + \frac{1}{x+8} = \frac{1}{3}$

53. $\frac{1}{x} + \frac{1}{x-4} = \frac{3}{8}$

54. $\frac{1}{x} - \frac{1}{x+3} = 1$

55. $\frac{1}{x} + \frac{2}{x-4} = 1$

56. $\frac{2}{x} - \frac{3}{x+1} = 2$

6.4 Applications of Quadratic Equations

Many physical problems lead quite naturally to quadratic equations. One such case was already mentioned at the beginning of the chapter. Now we will consider a similar example.

Example 1 A rock is hurled upward at the rate of 24 m/sec from a height of 5 m. The distance s (in meters) above the ground as a function of time t (in seconds) is given by $s = -5t^2 + 24t + 5$. (The instant at which the rock is hurled upward corresponds to $t = 0$ sec.) When will the rock strike the ground?

Solution. Since s is the distance above the ground, the problem is to find the value of t for which $s = 0$. Thus

$$-5t^2 + 24t + 5 = 0$$
$$5t^2 - 24t - 5 = 0$$
$$(5t + 1)(t - 5) = 0$$
$$t = -\frac{1}{5}, 5$$

Since $t = 0$ corresponds to the instant when the motion begins, the root $t = -\frac{1}{5}$ has no meaning here. We conclude that the rock hits the ground in 5 sec.

Example 2 If the length of a square is increased by 6.0 in., the area becomes 4 times as large. Find the original length of the side.

Solution. Let x be the length of the original side. Then $x + 6.0$ in. is the length of the side when increased. Since the new area is equal to 4 times the old area, we get (in square inches, omitting final zeros)

$$(x + 6)^2 = 4x^2$$
$$x^2 + 12x + 36 = 4x^2 \quad \text{expanding the left side}$$
$$-3x^2 + 12x + 36 = 0 \quad \text{subtracting } 4x^2$$
$$x^2 - 4x - 12 = 0 \quad \text{dividing by } -3$$
$$(x - 6)(x + 2) = 0 \quad \text{factoring the left side}$$
$$x = 6, -2$$

Since the root $x = -2$ has no meaning here, the original side is 6.0 in. long (using two significant figures).

Example #3

Two resistors connected in parallel have a combined resistance of 4 Ω, and the resistance of one resistor is 6 Ω more than that of the other. Find the resistance of each.

Solution. Let

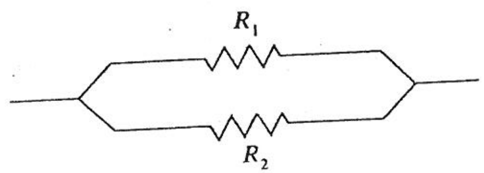
R = resistance of first resistor

Then

$R + 6$ = resistance of second resistor

Since the resistors are connected in parallel (Figure 6.3), we have

$$\frac{1}{R} + \frac{1}{R+6} = \frac{1}{4}$$



If R_T is the combined resistance, then

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

Figure 6.3

This is an example of a fractional equation that reduces to a quadratic equation after clearing fractions. Multiplying both sides of the equation by the LCD $= 4R(R + 6)$, we get

$$\begin{aligned} 4(R + 6) + 4R &= R(R + 6) \\ 4R + 24 + 4R &= R^2 + 6R \\ 8R + 24 &= R^2 + 6R \\ R^2 - 2R - 24 &= 0 \\ (R - 6)(R + 4) &= 0 \\ R &= -4, 6 \end{aligned}$$

Since the negative root has no meaning here, we conclude that $R = 6\ \Omega$. The resistance of the other resistor is therefore $12\ \Omega$.

ple 4 Example #4

A tank can be filled by two inlet valves in 2 hr. One inlet valve requires $7\frac{1}{2}$ hr longer to fill the tank than the other. How long does it take for each valve alone to fill the tank?

Solution. Let x equal the time taken for the faster valve to fill the tank. Then $x + 7\frac{1}{2} = x + \frac{15}{2}$ is the time required for the slower valve to fill the tank.

Now recall from Section 2.3 that an equation can be readily obtained from this information by finding expressions for the fractional part of the tank that can be filled in one time unit, in this case 1 hr. So

$$\frac{1}{x} + \frac{1}{x + \frac{15}{2}} = \frac{1}{2}$$

To clear fractions, note that the LCD equals $2x(x + \frac{15}{2})$. We now get

$$2\left(x + \frac{15}{2}\right) + 2x = x\left(x + \frac{15}{2}\right)$$

$$2x + 15 + 2x = x^2 + \frac{15}{2}x$$

Multiplying both sides by 2, we then have

$$\begin{aligned} 8x + 30 &= 2x^2 + 15x \\ 2x^2 + 7x - 30 &= 0 \\ (2x - 5)(x + 6) &= 0 \\ x &= -6, \frac{5}{2} \end{aligned}$$

Again, the negative root has no meaning, so we conclude that the times are $2\frac{1}{2}$ hr and 10 hr, respectively.

Example #5

An executive drives to a conference early in the day. Due to heavy morning traffic, her average speed for the first 120 mi is 10 mi/hr less than for the second 120 mi and requires 1 hr more time. Find the two average speeds.

Solution. Recall from Section 2.3 that the basic relationship is distance = rate × time.

If we let x equal the slower rate, then $x + 10$ equals the faster rate. Since the distance is the same in both cases, it follows that $120/x$ is the time required to cover the first 120 mi, and $120/(x + 10)$ the time required to cover the second 120 mi. The difference in the two times is 1 hr. Hence

$$\frac{120}{x} - \frac{120}{x + 10} = 1$$

Clearing fractions,

$$\begin{aligned} 120(x + 10) - 120x &= x(x + 10) \\ 120x + 1{,}200 - 120x &= x^2 + 10x \\ x^2 + 10x - 1{,}200 &= 0 \\ (x - 30)(x + 40) &= 0 \\ x &= 30, -40 \end{aligned}$$

Taking the positive root again, we conclude that 30 mi/hr is the slower rate and $x + 10 = 40$ mi/hr the faster rate.

Exercises / Section 6.4

1. The current i (in amperes) in a certain circuit at any time t (in seconds) is given by $i = 9.5t^2 - 4.7t$. At what time is the current equal to zero?
2. The sum of two electric currents is 35 A and their product 294 A^2. Find the two currents.
3. A certain resistance is 2.00 Ω more than another. Their product is 84.0 Ω^2. Find the two resistances.
4. If an object is hurled vertically downward with velocity v_0, then the distance s that the object falls at any time is $s = v_0t + \frac{1}{2}gt^2$. Find an expression for t.
5. Recall that the relationship of the focal length f of a lens to the object distance q and the image distance p is

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

If $f = 2.0$ cm and p is 3.0 cm longer than q, find p.

6. A metal shop has an order for a rectangular metal plate of area 84 in.^{2}. Find its dimensions if the length exceeds the width by 5.0 in.
7. A parallelogram has an area of 149.0 in.^{2}, and the base exceeds the height by 10.00 in. Find the base and height.
8. The difference between a positive integer and its reciprocal is $2\frac{2}{3}$. Find the number.
9. A rectangular casting 0.500 in. thick is to be made from 44.0 in. of forming (Figure 6.4 on page 218). If 42.5 in.^{3} are poured into the form, find the dimensions of the casting.
10. Suppose the casting in Exercise 9 is 3.00 in. thick and its length is 2.00 in. more than its width. If the volume is 72.0 in.^{3}, find its dimensions.
11. To cover the floor of a new storage area, 100 square tiles of a certain size are needed. If square tiles 2 in. longer on each side are used, only 64 tiles are needed. What is the size of the smaller tile?

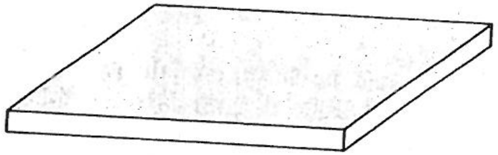


Figure 6.4

#12
A rectangular metal plate is twice as long as it is wide. When heated, each side is increased by 2 mm and the area is multiplied by $\frac{248}{225}$. Find its original dimensions.

#13
If both inlets of a tank are open, then the tank can be filled in 2 hr. One of the inlets alone requires $5\frac{1}{3}$ hr more than the other to fill the tank. How long does each one take?

#14
Two machines are used to print labels for a large mailing; the job normally takes 2 hr. One day the faster machine breaks down and the slower machine, which takes 3 hr longer than the faster machine to do the task alone, has to be used. How long will it take to complete the job?

#15
Two card sorters used simultaneously can sort a set of cards in 24 min. If only one machine is used, then the slower one requires 20 min more than the faster one. How long does each one take?

#16
A technician has to order a frame meeting the following specifications: It has the shape of a right triangle with a hypotenuse 26.0 cm long, while the sum of the lengths of the other two sides is 34.0 cm. Find the dimensions.

#17
In city traffic, a car travels 15 mi/hr faster than a bicycle. The car can travel 50 mi in 3 hr less time than the bicycle. Find the rate of each.

#18
A heavy machine is delivered by truck to a factory 200 mi away. The empty truck makes the return trip 5 mi/hr faster and gets back in 1 hr less time. Find the rate each way.

19
A rectangular enclosure is to be fenced along four sides and divided into two parts by a fence parallel to one of the sides. (See Figure 6.5.) If 170 ft of fence are available and the total area is 1,200 ft², what are the dimensions? (There are two possible solutions.)

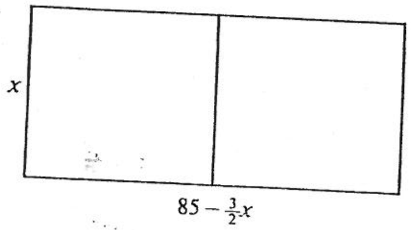


Figure 6.5

20
A tool shed is adjacent to a building, which serves as the back wall of the tool shed. The total length of the other three walls is 39 ft, and the area of the floor is 180 ft². Find the dimensions.

The cost of carpeting an office at \$10/ft² was \$1,500. If the length exceeds the width by 5 ft, what are the dimensions of the office?

Maggie's car gets 5 mi/gal less in the city than on the highway. Driving 300 mi in the city requires 2 gal more than driving the same distance on the highway. Determine the gas mileage in the city.

23. An engineer wants to buy \$240 worth of stock. One stock costs \$10 more per share than another; if she decides to buy the cheaper stock, she can afford four more shares. How many shares of the more expensive stock can she buy?

24. An investor purchases a number of shares of stock for \$600. If the investor had paid \$2 less per share, the number of shares would have been increased by 10. How many shares of the cheaper stock can he buy?

Review Exercises / Chapter 6

In Exercises 1–8, solve the equations by factoring.

1. $x^2 - 3x - 10 = 0$
2. $2x^2 - 7x - 4 = 0$
3. $5x^2 - 7x - 6 = 0$
4. $2x^2 + 15x - 27 = 0$
5. $6x^2 + 7x + 2 = 0$
6. $4x^2 - 17x + 15 = 0$
7. $6x^2 = 11x - 5$
8. $10x^2 + 3x = 18$

In Exercises 9–16, solve the equations by completing the square.

9. $x^2 - 2x - 8 = 0$
10. $2x^2 + 3x - 5 = 0$
11. $x^2 + 5x + 3 = 0$
12. $2x^2 - 4x + 1 = 0$
13. $2x^2 - 5x + 4 = 0$
14. $3x^2 - 2x + 3 = 0$
15. $4x^2 - x + 3 = 0$
16. $2x^2 + 5x + 2 = 0$

In Exercises 17–24, solve the equations by applying the quadratic formula.

17. $x^2 - 6x + 8 = 0$
18. $6x^2 + 7x - 3 = 0$
19. $3x^2 - 3x + 1 = 0$
20. $x^2 - 8x + 6 = 0$
21. $2x^2 - 4x - 3 = 0$
22. $2x^2 - 4x + 3 = 0$
23. $6x^2 - 8x - 3 = 0$
24. $5x^2 + x + 1 = 0$

In Exercises 25–30, solve the equations using any method.

25. $1.72x^2 + 1.89x - 2.64 = 0$
26. $3.98x^2 + 0.46x - 0.42 = 0$
27. $x^2 - 4x + 4 - y^2 = 0$
28. $x^2 - 2xy + x + y^2 - y - 2 = 0$
29. $\frac{1}{x} + \frac{1}{x+1} = \frac{9}{20}$
30. $\frac{1}{x-1} + \frac{1}{x+2} = 1$

31. Two currents differ by 2.0 A, while their product is 288 A². Find the two currents.

32. A rectangular enclosure is to be fenced along four sides and divided into three parts by two fences that are parallel to one side. If 80 ft of fence are available and the total area is 200 ft², what are the dimensions of the enclosure?

33. Working together, two men can unload a boxcar in 4 hr. Working alone, one man requires 6 hr more than the other. How long does it take for each man to do the job alone?

34. The cost of tiling a kitchen is \$5/ft². If the length of the kitchen exceeds the width by 4 ft and the total cost of tiling comes to \$960, what are the dimensions of the kitchen?

35. Early one morning a shop assistant delivers a motor to a garage 70 mi from downtown. Because of lighter traffic, his average speed on the return trip is 15 mi/hr more than on the delivery run, and he returns in 40 min less time. Find his average speed each way.

Section 5.9 (page 188)

1. $\frac{1}{2}$ **3.** $\frac{8}{11}$ **5.** $\frac{x}{3x + 1}$ **7.** $\frac{x - 4}{x}$ **9.** $C_1 + C_2$ **11.** $\frac{x + 5}{x - 2}$ **13.** $\frac{h - 5}{h + 4}$ **15.** $\frac{w}{w +}$
17. $\frac{1}{\beta + 1}$ **19.** $\frac{2E - 3}{E^2 - 2E + 1}$ **21.** $\frac{k + 3}{k + 4}$ **23.** $\frac{x + 1}{x - 4}$ **25.** $\frac{(t + 1)(t - 4)}{(t - 3)(2t + 1)}$ **27.** $\frac{R_1R_2}{R_1 + R_2}$
29. $\frac{pf}{p - f}$ **31.** $\frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$

Section 5.10 (page 193)

1. 3 **3.** -2 **5.** 10 **7.** 1 **9.** -8 **11.** no solution **13.** -2 **15.** 12
17. no solution **19.** $\frac{3}{2}$ **21.** 0 **23.** 10 **25.** -5 **27.** 9 **29.** $d = \frac{c}{3c - 1}$
31. $R_1 = \frac{3R}{3 - R}$ **33.** $f = \frac{pq}{p + q}$ **35.** $R = \frac{R_1R_2}{R_1 + R_2}$ **37.** $10\ \Omega$

Review Exercises for Chapter 5 (page 195)

1. $8s^3t^3 - 12s^3t^5 + 20s^4t^4$ **3.** $4i_1^2 - 12i_1i_2 - 9i_2^2$ **5.** $4s^2 - 25t^2$ **7.** $2c^2 + cd - 15d^2$
9. $x^3y - 9xy^3$ **11.** $v^2 + 4vw + 4w^2 - 2v - 4w + 1$ **13.** $4x(a - b)$ **15.** $pq(3p^2q^3 + 1)$
17. $(2\beta - \gamma)(2\beta + \gamma)$ **19.** $16(L - 2C)(L - 2C)$ **21.** $(3h + g)(9h^2 - 3hg + g^2)$
23. $(a + 2b - 1)(a^2 + 4ab + 4b^2 + a + 2b + 1)$
25. $(a - 1)(a + 1)(a^4 + a^2 + 1)$ or $(a - 1)(a - 1)(a^2 - a - 1)(a^2 + a + 1)$
27. $(v_0 - 4)(v_0 + 3)$ **29.** $(3v_1 - v_2)(v_1 + v_2)$ **31.** $(C_1 - 5C_2)(C_1 + 3C_2)$ **33.** not factorable
35. $(x - y - 1)(x - y + 1)$ **37.** $(s + t)(s - 2t + 1)$ **39.** $(a + b)(x - y)$
41. $(2x + y - a)(2x + y + a)$ **43.** $x + 3y$ **45.** $\frac{q + 7r}{q + 9r}$ **47.** $\frac{2(a + d)}{a^2d}$ **49.** $x + y$
51. $\frac{x + 4y}{x + 2y}$ **53.** $\frac{w^2}{y^2 - w^2}$ **55.** 1 **57.** $\frac{a}{(a - b)(a - b)(a + 2b)}$ **59.** $\frac{2vw - w^2}{(v - 2w)(v + w)}$
61. $\frac{1}{\omega - 2}$ **63.** $\frac{i}{i - 2}$ **65.** $\frac{1}{r_1}$ **67.** 1 **69.** 3 **71.** $-\frac{12}{5}$ **73.** no solution **75.** 6
77. 1 **79.** $a = \frac{S(1 - r)}{1 - r^n}$ **81.** $t = 4.5$ sec, $t = 5$ sec **83.** $q = 12$ cm **85.** $\frac{m_2}{m_1 + m_2}$

Chapter 6

Section 6.1 (page 203)

1. $1, -1$ **3.** $6, -6$ **5.** $3, -3$ **7.** $\sqrt{10}, -\sqrt{10}$ **9.** $4, -4$ **11.** $\frac{5}{6}, -\frac{5}{6}$ **13.** $1, -2$
15. $4, -6$ **17.** $3, -\frac{1}{2}$ **19.** $-2, -\frac{1}{3}$ **21.** $\frac{3}{4}, -2$ **23.** $-3, \frac{7}{5}$ **25.** $-\frac{5}{2}, \frac{3}{2}$ **27.** $-\frac{3}{5}, \frac{5}{6}$
29. $-\frac{9}{4}, \frac{5}{2}$ **31.** $-\frac{11}{7}, 1$ **33.** $-\frac{3}{2}, \frac{5}{9}$ **35.** $-\frac{1}{11}, 7$ **37.** $-\frac{5}{9}, \frac{3}{8}$ **39.** $\frac{11}{6}, \frac{10}{3}$
41. $-\frac{4}{3}, -\frac{4}{3}$ **43.** $\frac{1}{4}, \frac{1}{4}$ **45.** $\frac{v_0^2}{32}$ ft **47.** $x = L, \frac{1}{3}L, 0$

Section 6.2 (page 208)

1. $2, 4$ **3.** $-6, 2$ **5.** $-4, 3$ **7.** $-5, -2$ **9.** $\frac{1}{2}(-5 \pm \sqrt{17})$ **11.** $-3 \pm \sqrt{3}$
13. $\frac{1}{2}(3 \pm \sqrt{7})$ **15.** $\frac{1}{4}(-3 \pm \sqrt{33})$ **17.** $-1, \frac{1}{3}$ **19.** $\frac{1}{3}(2 \pm \sqrt{19})$ **21.** $-\frac{3}{4}, 1$

23. $\frac{1}{12}(-1 \pm \sqrt{47}j)$ **25.** $2 \pm j$ **27.** $\frac{5}{8} \pm \frac{\sqrt{23}}{8}j$ **29.** $\frac{1}{7}(-1 \pm 2\sqrt{2})$ **31.** $\frac{1}{12}(5 \pm \sqrt{73})$
33. $-\frac{10}{3}, \frac{5}{2}$ **35.** $\frac{1}{10} \pm \frac{\sqrt{19}}{10}j$ **37.** $\frac{1}{2}(b \pm \sqrt{b^2 - 8})$ **39.** $\frac{1}{2a}(-5 \pm \sqrt{4a + 25})$

Section 6.3 (page 213)

1. $-3, 2$ **3.** $4, 5$ **5.** $\frac{1}{2}, 2$ **7.** $-\frac{1}{2}, \frac{2}{3}$ **9.** $\frac{3 \pm \sqrt{17}}{4}$ **11.** $\frac{-1 \pm \sqrt{7}}{3}$ **13.** $1 \pm j$
15. $-1 \pm \sqrt{3}j$ **17.** $\frac{-3 \pm \sqrt{3}j}{6}$ **19.** $\frac{-1 \pm \sqrt{3}j}{4}$ **21.** $\frac{-5 \pm \sqrt{73}}{8}$ **23.** $\pm \frac{\sqrt{5}}{5}j$ **25.** $-\frac{3}{2}, 0$
27. $\frac{3c \pm \sqrt{9c^2 - 8}}{4}$ **29.** $\frac{-3 \pm \sqrt{9 - 4b}}{2b}$ **31.** $\frac{3}{2}, \frac{3}{2}$ **33.** $\frac{5}{2}, \frac{5}{2}$ **35.** $0.70, -2.26$
37. $5.77, -0.18$ **39.** $0.66, -0.49$ **41.** $x = 1 + y, x = 1 - y$ **43.** $x = 2y - 1, x = 2y + 1$
45. $x = y - 1, x = y - 2$ **47.** $x = 1 - y, x = 2 + y$ **49.** $-5, 4$ **51.** $-\frac{6}{5}, 4$ **53.** $\frac{4}{3}, 8$
55. $\frac{7 \pm \sqrt{33}}{2}$

Section 6.4 (page 217)

1. $t = 0$ sec, $t = 0.49$ sec **3.** 8.22 Ω, 10.22 Ω **5.** 6.0 cm **7.** 18.19 in. for base
9. 5.00 in. × 17.0 in. **11.** 8 in. × 8 in. **13.** $2\frac{2}{3}$ hr, 8 hr **15.** 60 min, 40 min
17. 25 mi/hr, 10 mi/hr **19.** 30 ft × 40 ft or $\frac{80}{3}$ ft × 45 ft **21.** 10 ft × 15 ft **23.** 8 shares

Review Exercises for Chapter 6 (page 219)

1. $-2, 5$ **3.** $-\frac{3}{5}, 2$ **5.** $-\frac{2}{3}, -\frac{1}{2}$ **7.** $\frac{5}{6}, 1$ **9.** $-2, 4$ **11.** $\frac{-5 \pm \sqrt{13}}{2}$ **13.** $\frac{5}{4} \pm \frac{\sqrt{7}}{4}j$
15. $\frac{1}{8} \pm \frac{\sqrt{47}}{8}j$ **17.** $2, 4$ **19.** $\frac{1}{2} \pm \frac{\sqrt{3}}{6}j$ **21.** $\frac{2 \pm \sqrt{10}}{2}$ **23.** $\frac{4 \pm \sqrt{34}}{6}$ **25.** $0.81, -1.90$
27. $x = 2 + y, x = 2 - y$ **29.** $-\frac{5}{9}, 4$ **31.** 16.0 A, 18.0 A **33.** 6 hr, 12 hr
35. 20 mi/hr, 35 mi/hr

Cumulative Review Exercises for Chapters 4–6 (page 220)

1. 122°19′ **2.** $\sin\theta = \frac{\sqrt{3}}{4}$, $\cos\theta = \frac{\sqrt{13}}{4}$, $\tan\theta = \frac{\sqrt{39}}{13}$, $\csc\theta = \frac{4\sqrt{3}}{3}$, $\sec\theta = \frac{4\sqrt{13}}{13}$, $\cot\theta = \frac{\sqrt{39}}{3}$
3. $\frac{3\sqrt{7}}{7}$ **4.** $\frac{\sqrt{3}}{3}$ **5.** 1.688 **6.** 56°59′ **7.** 20°31′ **8.** $(V_a - V_b)(s - 1)$
9. $\frac{1}{2}(L^2 + LC + C^2)$ **10.** $\frac{x + y}{2(x + 3y)}$ **11.** $\frac{2a - 1}{a(2x + y)}$ **12.** $\frac{3st - t^2 + 1}{s^2 - t^2}$ **13.** $\frac{L}{L + 4}$
14. $x = 4, -2$ **15.** $x = \frac{1}{4}, -3$ **16.** $x = 1, 3$ **17.** $x = 1 \pm j$ **18.** $-0.975, 0.427$
19. 0.433 A **20.** 1.55 in. **21.** $\frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$ **22.** 3.4 in. by 5.4 in.