2.1 Linear Equations

- Basic Terminology of Equations
- Solving Linear Equations
- Identities, Conditional Equations, and Contradictions
- Solving for a Specified Variable (Literal Equations)

Equations

An equation is a statement that two expressions are equal.

$x + 2 = 9 \qquad 11x = 5x + 6x \qquad x^2 - 2x - 1 = 0$

To solve an equation means to find all numbers that make the equation a true statement. These numbers are the **solutions,** or **roots,** of the equation. A number that is a solution of an equation is said to *satisfy* the equation, and the solutions of an equation make up its **solution set.** Equations with the same solution set are **equivalent equations.**

Addition and Multiplication Properties of Equality

Let a, b, and c represent real numbers.

If $a = b$, then $a + c = b + c$.

That is, the same number may be added to each side of an equation without changing the solution set.

Addition and Multiplication Properties of Equality

Let a, b, and c represent real numbers.

If $a = b$ and $c \neq 0$, then $ac = bc$.

That is, each side of an equation may be multiplied by the same nonzero number without changing the solution set.

Linear Equation in One Variable

A **linear equation in one variable** is an equation that can be written in the form

$$\boldsymbol{ax + b = 0},$$

where a and b are real numbers with $a \neq 0$.

Linear Equations

A linear equation is also called a **first-degree equation** since the greatest degree of the variable is 1.

$$3x + \sqrt{2} = 0 \qquad \frac{3}{4}x = 12 \qquad 0.5(x+3) = 2x - 6$$

Linear equations

$$\sqrt{x} + 2 = 5 \qquad \frac{1}{x} = -8 \qquad x^2 + 3x + 0.2 = 0$$

Nonlinear equations

Example 1 SOLVING A LINEAR EQUATION

Solve $3(2x-4)=7-(x+5)$.

Solution $3(2x-4)=7-(x+5)$

Be careful with signs.

$$6x-12=7-x-5$$ Distributive property

$$6x-12=2-x$$ Combine like terms.

$$6x-12+x=2-x+x$$ Add x to each side.

$$7x-12=2$$ Combine like terms.

$$7x-12+12=2+12$$ Add 12 to each side.

$$7x=14$$ Combine like terms.

$$\frac{7x}{7}=\frac{14}{7},\quad x=2$$ Divide each side by 7.

Example 1 **SOLVING A LINEAR EQUATION**

Check

$$3(2x-4)=7-(x+5) \quad \text{Original equation}$$

$$3(2\times 2-4)\overset{?}{=}7-(2+5) \quad \text{Let } x = 2.$$

$$3(4-4)\overset{?}{=}7-(7)$$

$$0=0 \quad \text{True}$$

A check of the solution is recommended.

The solution set is {2}.

Homework 2 SOLVING A LINEAR EQUATION WITH FRACTIONS

Solve $\dfrac{2x+4}{3}+\dfrac{1}{2}x=\dfrac{1}{4}x-\dfrac{7}{3}$.

Solution

$$12\left(\frac{2x+4}{3}+\frac{1}{2}x\right)=12\left(\frac{1}{4}x-\frac{7}{3}\right)$$

Multiply by 12, the LCD of the fractions.

Distribute the 12 to *all* terms within parentheses.

$$12\left(\frac{2x+4}{3}\right)+12\left(\frac{1}{2}x\right)=12\left(\frac{1}{4}x\right)-12\left(\frac{7}{3}\right)$$

Distributive property

$$4(2x+4)+6x=3x-28$$

Multiply.

Example 2 SOLVING A LINEAR EQUATION WITH FRACTIONS

Solve $\dfrac{2x+4}{3}+\dfrac{1}{2}x=\dfrac{1}{4}x-\dfrac{7}{3}$.

Solution

$$8x+16+6x=3x-28$$ Distributive property

$$14x+16=3x-28$$ Combine like terms.

$$11x=-44$$ Subtract $3x$; subtract 16.

$$x=-4$$ Divide each side by 11.

Example 2 **SOLVING A LINEAR EQUATION WITH FRACTIONS**

Check

$$\frac{2(-4)+4}{3}+\frac{1}{2}(-4) \stackrel{?}{=} \frac{1}{4}(-4)-\frac{7}{3}$$ Let $x = -4$.

$$\frac{-4}{3}+(-2) \stackrel{?}{=} -1-\frac{7}{3}$$ Simplify.

$$-\frac{10}{3} = -\frac{10}{3}$$ True

The solution set is {- 4}.

Identities, Conditional Equations, and Contradictions

An equation satisfied by every number that is a meaningful replacement for the variable is an **identity.**

$$3(x+1) = 3x+3$$

Identities, Conditional Equations, and Contradictions

An equation that is satisfied by some numbers but not others is a **conditional equation.**

$$2x = 4$$

Identities, Conditional Equations, and Contradictions

An equation that has no solution is a **contradiction**.

$$x = x + 1$$

Example 3 IDENTIFYING TYPES OF EQUATIONS

Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction*.

(a) $-2(x+4)+3x=x-8$

Solution

$$-2(x+4)+3x=x-8$$

$$-2x-8+3x=x-8 \quad \text{Distributive property}$$

$$x-8=x-8 \quad \text{Combine like terms.}$$

$$0=0 \quad \text{Subtract } x \text{ and add 8.}$$

Example 3 IDENTIFYING TYPES OF EQUATIONS

Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction*.

(a) $-2(x+4)+3x=x-8$

Solution

$$0=0$$ Subtract x and add 8.

When a *true* statement such as 0 = 0 results, the equation is an identity, and the solution set is **{all real numbers}**.

Example 3 IDENTIFYING TYPES OF EQUATIONS

Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction*.

(b) $5x - 4 = 11$

Solution

$$5x - 4 = 11$$

$$5x = 15 \quad \text{Add 4 to each side.}$$

$$x = 3 \quad \text{Divide each side by 5.}$$

This is a conditional equation, and its solution set is {3}.

Example 3 IDENTIFYING TYPES OF EQUATIONS

Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction*.

(c) $3(3x-1)=9x+7$

Solution $3(3x-1)=9x+7$

$9x-3=9x+7$ Distributive property

$-3=7$ Subtract 9*x*.

When a *false* statement such as −3 = 7 results, the equation is a contradiction, and the solution set is the **empty set** or **null set**, symbolized by $\emptyset$.

Identifying Types of Linear Equations

1. If solving a linear equation leads to a true statement such as $0 = 0$, the equation is an **identity**. Its solution set is **{all real numbers}.**
2. If solving a linear equation leads to a single solution such as $x = 3$, the equation is **conditional**. Its solution set consists of a single element.
3. If solving a linear equation leads to a false statement such as $-3 = 7$, then the equation is a **contradiction**. Its solution set is $\emptyset$.

Solving for a Specified Variable (Literal Equations)

A formula is an example of a **linear equation** (an equation involving letters). This is the formula for **simple interest.**

I is the variable for simple interest

$$I = Prt$$

t is the variable for years

P is the variable for dollars

r is the variable for annual interest rate

SOLVING FOR A SPECIFIED VARIABLE

Solve for *t.*

(a) $I = Prt$

Solution $I = Prt$

Goal: Isolate *t* on one side.

$$\frac{I}{Pr} = \frac{Prt}{Pr}$$

Divide each side by *Pr.*

$$\frac{I}{Pr} = t \quad \text{or} \quad t = \frac{I}{Pr}$$

Solving for a Specified Variable (Literal Equations)

This formula gives the **future value,** or **maturity value,** *A* of *P* dollars invested for *t* years at an annual simple interest rate *r*.

A is the variable for future or maturity value

$$A = P(1 + r\,t)$$

t is the variable for years

P is the variable for dollars

r is the variable for annual simple interest rate

Example 4 SOLVING FOR A SPECIFIED VARIABLE

Solve for *P*.

(b) $A - P = Prt$

Goal: Isolate *P*, the specified variable.

Solution $A - P = Prt$

$$A = P + Prt$$

Transform so that all terms involving *P* are on one side.

$$A = P(1 + rt)$$

Factor out *P*.

$$\frac{A}{1+rt} = P \quad \text{or} \quad P = \frac{A}{1+rt}$$

Divide by $1 + rt$.

Example 4 SOLVING FOR A SPECIFIED VARIABLE

Solve for *x.*

(c) $3(2x-5a)+4b=4x-2$

Solution $3(2x-5a)+4b=4x-2$ Solve for *x*.

$6x-15a+4b=4x-2$ Distributive property

$6x-4x=15a-4b-2$ Isolate the *x*-terms on one side.

Combine like terms. $2x=15a-4b-2$

Divide each side by 2. $x=\dfrac{15a-4b-2}{2}$

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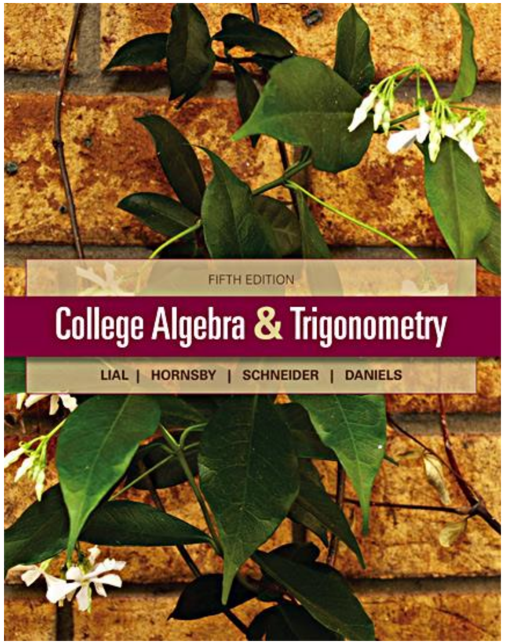
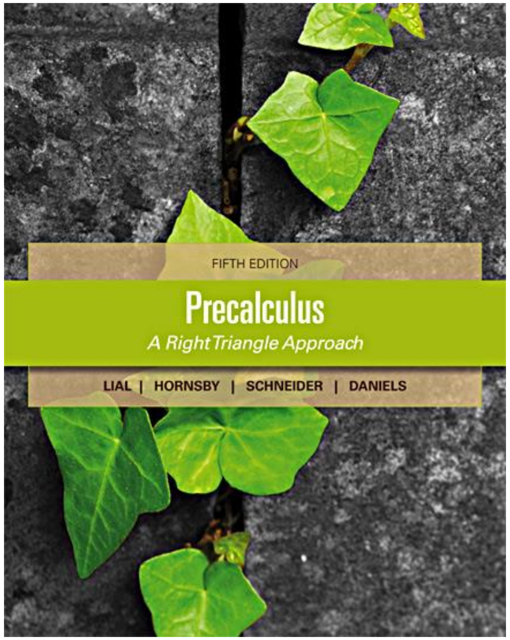
Equations and Inequalities





2.2

Complex Numbers

- Basic Concepts of Complex Numbers
- Operations on Complex Numbers

Basic Concepts of Complex Numbers

There is no real number solution of the equation

$$x^2 = -1.$$

since no real number, when squared, gives -1. To extend the real number system to include solutions of equations of this type, the number *i* is defined to have the following property.

$$i = \sqrt{-1}, \text{ and therefore, } i^2 = -1.$$

Basic Concepts of Complex Numbers

If a and b are real numbers, then any number of the form $\boldsymbol{a + bi}$ is a **complex number.**

In the complex number $a + bi$, a is the **real part** and b is the **imaginary part**.

Basic Concepts of Complex Numbers

Two complex numbers $a + bi$ and $c + di$ are equal provided that their real parts are equal and their imaginary parts are equal; that is

$$a + bi = c + di \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d.$$

Basic Concepts of Complex Numbers

For complex number $a + bi$, if $b = 0$, then

$$a + bi = a.$$

Thus, the set of real numbers is a subset of the set of complex numbers.

Basic Concepts of Complex Numbers

If $a = 0$ and $b \neq 0$, the complex number is said to be a **pure imaginary number.**

A pure imaginary number, or a number like $7 + 2i$ with $a \neq 0$ and $b \neq 0$, is a **nonreal complex number.**

A complex number written in the form $a + bi$ (or $a + ib$) is in **standard form.**

Complex Numbers $a + bi$, for a and b Real

Real numbers

$a + bi,\ b = 0$

Rational numbers

$\frac{4}{9},\ -\frac{5}{8},\ \frac{11}{7}$

Integers

$-11,\ -6,\ -3,\ -2,\ -1$

Whole numbers

0

Natural numbers

$1, 2, 3, 4, 5, 37, 40$

Irrational numbers

$\sqrt{2}$

$\sqrt{15}$

$-\sqrt{8}$

π

$\frac{\pi}{4}$

Nonreal complex numbers

$a + bi,\ b \neq 0$

$7 + 2i,\ 5 - i\sqrt{3},\ -\frac{1}{2} + \frac{3}{2}i$

Pure imaginary numbers

$a + bi,\ a = 0$ and $b \neq 0$

$3i,\ -i,\ -\frac{2}{3}i,\ i\sqrt{5}$

THE EXPRESSION $\sqrt{-a}$

If $a > 0$, then $\sqrt{-a} = i\sqrt{a}$.

Example 1 WRITING $\sqrt{-a}$ AS $i\sqrt{a}$

Write as the product of a real number and *i*, using the definition of $\sqrt{-a}$.

(a) $\sqrt{-16}$

Solution

$$\sqrt{-16} = i\sqrt{16} = 4i$$

Example 1 WRITING $\sqrt{-a}$ AS $i\sqrt{a}$

Write as the product of a real number and *i*, using the definition of $\sqrt{-a}$.

(b) $\sqrt{-70}$

Solution

$$\sqrt{-70} = i\sqrt{70}$$

Example 1 WRITING $\sqrt{-a}$ AS $i\sqrt{a}$

Write as the product of a real number and *i*, using the definition of $\sqrt{-a}$.

(c) $\sqrt{-48}$

Solution

$$\sqrt{-48} = i\sqrt{48} = i\sqrt{16 \cdot 3} = 4i\sqrt{3}$$

Product rule for radicals

Operations on Complex Numbers

Products or quotients with negative radicands are simplified by first rewriting $\sqrt{-a}$ as $i\sqrt{a}$ for a positive number a.

Then the properties of real numbers and the fact that $i^2 = -1$ are applied

Operations on Complex Numbers

Caution ***When working with negative radicands, use the definition***

$$\sqrt{-a} = i\sqrt{a}$$

before using any of the other rules for radicals.

Operations on Complex Numbers

Caution In particular, the rule

$$\sqrt{c} \cdot \sqrt{d} = \sqrt{cd}$$

is valid only when c and d are not both negative.

$$\sqrt{-4} \cdot \sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$$ is correct,

while

$$\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6$$ is incorrect.

Example 2 FINDING PRODUCTS AND QUOTIENTS INVOLVING $\sqrt{-a}$

Multiply or divide, as indicated. Simplify each answer.

(a) $\sqrt{-7} \cdot \sqrt{-7}$

Solution

$$\sqrt{-7} \cdot \sqrt{-7} = i\sqrt{7} \cdot i\sqrt{7}$$

First write all square roots in terms of *i*.

$$= i^2 \cdot \left(\sqrt{7}\right)^2$$

$$= -1 \cdot 7$$

$i^2 = -1$

$$= -7$$

Example 2 FINDING PRODUCTS AND QUOTIENTS INVOLVING $\sqrt{-a}$

Multiply or divide, as indicated. Simplify each answer.

(b) $\sqrt{-6} \cdot \sqrt{-10}$

Solution

$$\sqrt{-6} \cdot \sqrt{-10} = i\sqrt{6} \cdot i\sqrt{10}$$

$$= i^2 \cdot \sqrt{60}$$

$$= -1\sqrt{4 \cdot 15}$$

$$= -1 \cdot 2\sqrt{15}$$

$$= -2\sqrt{15}$$

Example 2 **FINDING PRODUCTS AND QUOTIENTS INVOLVING $\sqrt{-a}$**

Multiply or divide, as indicated. Simplify each answer.

(c) $\dfrac{\sqrt{-20}}{\sqrt{-2}}$

Solution

$$\frac{\sqrt{-20}}{\sqrt{-2}} = \frac{i\sqrt{20}}{i\sqrt{2}} = \sqrt{\frac{20}{2}} = \sqrt{10}$$

Quotient rule for radicals

Example 2 **FINDING PRODUCTS AND QUOTIENTS INVOLVING $\sqrt{-a}$**

Multiply or divide, as indicated. Simplify each answer.

(d) $\dfrac{\sqrt{-48}}{\sqrt{24}}$

Solution

$$\frac{\sqrt{-48}}{\sqrt{-24}} = \frac{i\sqrt{48}}{i\sqrt{24}} = i\sqrt{\frac{48}{24}} = i\sqrt{2}$$

Example 3 **SIMPLIFYING A QUOTIENT INVOLVING $\sqrt{-a}$**

Write $\dfrac{-8+\sqrt{-128}}{4}$ in standard form $a + bi$.

Solution

$$\frac{-8+\sqrt{-128}}{4} = \frac{-8+\sqrt{-64 \cdot 2}}{4}$$

$$= \frac{-8+8i\sqrt{2}}{4}$$

$\sqrt{-64} = 8i$

Example 3

SIMPLIFYING A QUOTIENT INVOLVING $\sqrt{-a}$

Write $\dfrac{-8+\sqrt{-128}}{4}$ in standard form $a + bi$.

Solution

$$= \frac{-8+8i\sqrt{2}}{4}$$

Be sure to factor before simplifying

$$= \frac{4\left(-2+2i\sqrt{2}\right)}{4} \qquad \text{Factor.}$$

$$= -2+2i\sqrt{2} \qquad \text{Lowest terms}$$

Addition and Subtraction of Complex Numbers

For complex numbers $a + bi$ and $c + di$,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

and $(a+bi)-(c+di)=(a-c)+(b-d)i.$

Example 4 ADDING AND SUBTRACTING COMPLEX NUMBERS

Find each sum or difference.

(a) $(3-4i)+(-2+6i)$

Solution

$$(3-4i)+(-2+6i)=\underbrace{[3+(-2)]}_{\text{Add real parts.}}+\underbrace{[-4+6]}_{\text{Add imaginary parts.}}i$$

Commutative, associative, distributive properties

$$=1+2i$$

Example 4 ADDING AND SUBTRACTING COMPLEX NUMBERS

Find each sum or difference.

(b) $(-4+3i)-(6-7i)$

Solution

$$
\begin{aligned}
(-4+3i)-(6-7i) &= (-4-6)+\left[3-(-7)\right]i \\
&= -10+10i
\end{aligned}
$$

Multiplication of Complex Numbers

The product of two complex numbers is found by multiplying as though the numbers were binomials and using the fact that $i^2 = -1$, as follows.

$$(a+bi)(c+di) = ac + adi + bic + bidi$$

FOIL

$$= ac + adi + bci + bdi^2$$

Distributive property; $i^2 = -1$

$$= ac + (ad+bc)i + bd(-1)$$

$$= (ac - bd) + (ad + bc)i$$

Multiplication of Complex Numbers

For complex numbers $a + bi$ and $c + di$,

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i.$$

Example 5 MULTIPLYING COMPLEX NUMBERS

Find each product.

(a) $(2-3i)(3+4i)$

Solution

$$(2-3i)(3+4i) = 2(3)+2(4i)-3i(3)-3i(4i) \quad \text{FOIL}$$

$$= 6+8i-9i-12i^2 \quad \text{Multiply.}$$

$$= 6-i-12(-1) \quad \text{Combine like terms; } i^2 = -1$$

$$= 18-i \quad \text{Standard form}$$

Example 5 MULTIPLYING COMPLEX NUMBERS

Find each product.

(b) $(4+3i)^2$

Solution

Remember to add twice the product of the two terms.

$$(4+3i)^2 = 4^2 + 2(4)(3i) + (3i)^2$$ Square of a binomial

$$= 16 + 24i + 9i^2$$ Multiply.

$$= 16 + 24i + 9(-1)$$ $i^2 = -1$

$$= 7 + 24i$$ Standard form

Example 5 MULTIPLYING COMPLEX NUMBERS

Find each product.

(c) $(6+5i)(6-5i)$

Solution

$$(6+5i)(6-5i) = 6^2 - (5i)^2$$ Product of the sum and difference of two terms

$$= 36 - 25(-1)$$ $i^2 = -1$

$$= 36 + 25$$ Multiply.

$$= 61, \text{ or } 61 + 0i$$ Standard form

Example 5(c) showed that

$$(6+5i)(6-5i)=61$$

The numbers 6 + 5*i* and 6 – 5*i* differ only in the sign of their imaginary parts and are called **complex conjugates**. ***The product of a complex number and its conjugate is always a real number.*** This product is the sum of the squares of real and imaginary parts.

Property of Complex Conjugates

For real numbers *a* and *b*,

$$(a+bi)(a-bi)=a^2+b^2.$$

Example 6 DIVIDING COMPLEX NUMBERS

Write each quotient in standard form $a + bi$.

(a) $\dfrac{3+2i}{5-i}$

Solution

$$\frac{3+2i}{5-i}=\frac{(3+2i)(5+i)}{(5-i)(5+i)}$$

Multiply by the complex conjugate of the denominator in both the numerator and the denominator.

$$=\frac{15+3i+10i+2i^2}{25-i^2}$$

Multiply.

Example 6 **DIVIDING COMPLEX NUMBERS**

Write each quotient in standard form $a + bi$.

(a) $\dfrac{3+2i}{5-i}$

Solution

$$=\frac{13+13i}{26}$$

Combine like terms; $i^2 = -1$

$$=\frac{13}{26}+\frac{13i}{26}$$

$$\frac{a+bi}{c}=\frac{a}{c}+\frac{bi}{c}$$

Example 6 DIVIDING COMPLEX NUMBERS

Write each quotient in standard form $a + bi$.

(a) $\dfrac{3+2i}{5-i}$

Solution

$$= \frac{1}{2} + \frac{1}{2}i$$

Write in lowest terms and standard form.

Check

$$\left(\frac{1}{2} + \frac{1}{2}i\right)(5-i) = 3+2i$$

Example 6 DIVIDING COMPLEX NUMBERS

Write each quotient in standard form $a + bi$.

(b) $\dfrac{3}{i}$

Solution

$$\frac{3}{i} = \frac{3(-i)}{i(-i)}$$

$-i$ is the conjugate of i.

$$= \frac{-3i}{-i^2}$$

Multiply.

Example 6 DIVIDING COMPLEX NUMBERS

Write each quotient in standard form $a + bi$.

(b) $\dfrac{3}{i}$

Solution

$$= \frac{-3i}{1} \qquad i^2 = -1(-1) = 1$$

$$= -3i, \text{ or } 0 - 3i \qquad \text{Standard form}$$

Simplifying Powers of *i*

Powers of *i* can be simplified using the facts

$$i^2 = -1 \quad \text{and} \quad i^4 = (i^2)^2 = (-1)^2 = 1.$$

Powers of i

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

$i^8 = 1$ and so on.

Powers of i cycle through the same four outcomes (i, –1, –i, and 1) since i^4 has the same multiplicative property as 1. Also, any power of i with an exponent that is a multiple of 4 has value 1. As with real numbers, $i^0 = 1$.

Example 7 SIMPLIFYING POWERS OF *i*

Simplify each power of *i*.

(a) i^{15}

Solution

Since $i^4 = 1$, write the given power as a product involving i^4.

$$i^{15} = i^{12} \cdot i^3 = \left(i^4\right)^3 \cdot i^3 = 1^3(-i) = -i$$

Example 7 SIMPLIFYING POWERS OF *i*

Simplify each power of *i*.

(b) i^{-3}

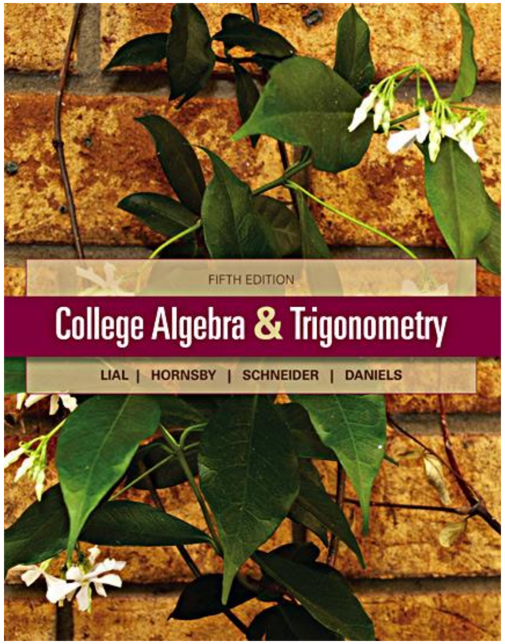
Solution

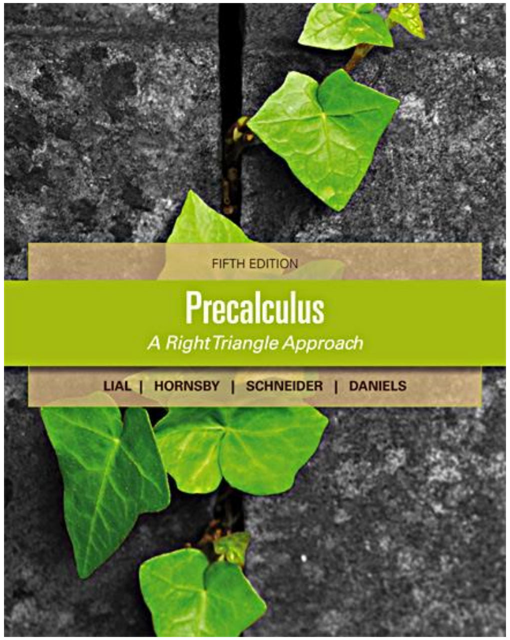
Multiply i^{-3} by 1 in the form of i^4 to create the least positive exponent for *i*.

$$i^{-3} = i^{-3} \cdot 1 = i^{-3} \cdot i^4 = i$$

2

Equations and Inequalities





2.3 Quadratic Equations

- Solving a Quadratic Equation
- Completing the Square
- The Quadratic Formula
- Solving for a Specified Variable
- The Discriminant

Quadratic Equation in One Variable

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$, is a **quadratic equation.** The given form is called **standard form.**

Second-degree Equation

A quadratic equation is a **second-degree equation**—that is, an equation with a squared variable term and no terms of greater degree.

$$x^2 = 25, \quad 4x^2 + 4x - 5 = 0, \quad 3x^2 = 4x - 8$$

Zero-Factor Property

If a and b are complex numbers with $ab = 0$, then $a = 0$ or $b = 0$ or both equal zero.

Example 1 USING THE ZERO-FACTOR PROPERTY

Solve $6x^2 + 7x = 3.$

Solution

$$6x^2 + 7x = 3$$

$$6x^2 + 7x - 3 = 0 \qquad \text{Standard form}$$

$$(3x - 1)(2x + 3) = 0 \qquad \text{Factor.}$$

$$3x - 1 = 0 \quad \text{or} \quad 2x + 3 = 0 \qquad \text{Zero-factor property}$$

$$3x = 1 \quad \text{or} \quad 2x = -3 \qquad \text{Solve each equation.}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -\frac{3}{2}$$

Square Root Property

A quadratic equation of the form $x^2 = k$ can also be solved by factoring.

$$x^2 = k$$

$$x^2 - k = 0 \quad \text{Subtract } k.$$

$$\left(x - \sqrt{k}\right)\left(x + \sqrt{k}\right) = 0 \quad \text{Factor.}$$

$$x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} = 0 \quad \text{Zero-factor property.}$$

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{Solve each equation.}$$

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad or \quad x = -\sqrt{k}.$$

Square-Root Property

That is, the solution set of $x^2 = k$ is

$\left\{\sqrt{k},\ -\sqrt{k}\right\}$, which may be abbreviated $\left\{\pm\sqrt{k}\right\}$.

Both solutions are real if $k > 0$, and both are pure imaginary if $k < 0$. If $k < 0$, we write the solution set as $\left\{\pm i\sqrt{|k|}\right\}$.

If $k = 0$, then there is only one distinct solution, 0, sometimes called a **double solution**.

Example 2 USING THE SQUARE ROOT PROPERTY

Solve each quadratic equation.

(a) $x^2 = 17$

Solution

By the square root property, the solution set of $x^2 = 17$ is $\left\{\pm\sqrt{17}\right\}$.

Example 2 USING THE SQUARE ROOT PROPERTY

Solve each quadratic equation.

(b) $x^2 = -25$

Solution

Since $\sqrt{-1} = i,$

the solution set of $x^2 = -25$

is $\left\{\pm 5i\right\}$.

Example 2 USING THE SQUARE ROOT PROPERTY

Solve each quadratic equation.

(c) $(x-4)^2 = 12$

Solution

$$(x-4)^2 = 12$$

$$x - 4 = \pm\sqrt{12}$$ Generalized square root property

$$x = 4 \pm \sqrt{12}$$ Add 4.

$$x = 4 \pm 2\sqrt{3}$$ $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

Solving a Quadratic Equation by Completing the Square

To solve $ax^2 + bx + c = 0$, where $a \neq 0$, by completing the square, use these steps.

Step 1 If $a \neq 1$, divide both sides of the equation by a.

Step 2 Rewrite the equation so that the constant term is alone on one side of the equality symbol.

Step 3 Square half the coefficient of x, and add this square to each side of the equation.

Step 4 Factor the resulting trinomial as a perfect square and combine like terms on the other side.

Step 5 Use the square root property to complete the solution.

Example 3 USING COMPLETING THE SQUARE ($a = 1$)

Solve $x^2 - 4x - 14 = 0.$

Solution $x^2 - 4x - 14 = 0$

Step 1 This step is not necessary since $a = 1$.

Step 2 $x^2 - 4x = 14$ Add 14 to each side.

Step 3 $x^2 - 4x + 4 = 14 + 4$ $\left[\frac{1}{2}(-4)\right]^2 = 4;$ add 4 to each side.

Step 4 $(x - 2)^2 = 18$ Factor. Combine like terms.

Example 3 USING COMPLETING THE SQUARE ($a = 1$)

Solve $x^2 - 4x - 14 = 0.$

Solution

Step 5 $x - 2 = \pm\sqrt{18}$ Square root property.

Take *both* roots.

$x = 2 \pm \sqrt{18}$ Add 2 to each side.

$x = 2 \pm 3\sqrt{2}$ Simplify the radical.

The solution set is $\left\{2 \pm 3\sqrt{2}\right\}$.

Example 4 USING COMPLETING THE SQUARE ($a \neq 1$)

Solve $9x^2 - 12x + 9 = 0$.

Solution

$$9x^2 - 12x + 9 = 0$$

$$x^2 - \frac{4}{3}x + 1 = 0$$

Divide by 9. (Step 1)

$$x^2 - \frac{4}{3}x = -1$$

Subtract 1 from each side. (Step 2)

$$x^2 - \frac{4}{3}x + \frac{4}{9} = -1 + \frac{4}{9}$$

$\left[\frac{1}{2}\left(-\frac{4}{3}\right)\right]^2 = \frac{4}{9}$; Add $\frac{4}{9}$ to each side. (Step 3)

Example 4 USING COMPLETING THE SQUARE ($a \neq 1$)

Solve $9x^2 - 12x + 9 = 0.$

Solution

$$\left(x - \frac{2}{3}\right)^2 = -\frac{5}{9}$$

Factor. Combine like terms. (Step 4)

$$x - \frac{2}{3} = \pm\sqrt{-\frac{5}{9}}$$

Square root property (Step 5)

Example 4 USING COMPLETING THE SQUARE ($a \neq 1$)

Solve $9x^2 - 12x + 9 = 0$ by completing the square.

Solution

$$x - \frac{2}{3} = \pm\frac{\sqrt{5}}{3}i \qquad \sqrt{-\frac{5}{9}} = \frac{\sqrt{-5}}{\sqrt{9}} = \frac{i\sqrt{5}}{3}, \text{ or } \frac{\sqrt{5}}{3}i$$

$$x = \frac{2}{3} \pm \frac{\sqrt{5}}{3}i \qquad \text{Add } \frac{2}{3} \text{ to each side.}$$

The solution set is $\left\{\frac{2}{3} \pm \frac{\sqrt{5}}{3}i\right\}$.

The Quadratic Formula

If we start with the equation $ax^2 + bx + c = 0$, for $a > 0$, and complete the square to solve for x in terms of the constants a, b, and c, the result is a general formula for solving any quadratic equation.

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Caution ***Remember to extend the fraction bar in the quadratic formula extends under the –b term in the numerator.***

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5 USING THE QUADRATIC FORMULA (REAL SOLUTIONS)

Solve $x^2 - 4x = -2$.

Solution

$$x^2 - 4x + 2 = 0$$

Write in standard form. Here $a = 1$, $b = -4$, $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

The fraction bar extends under *–b.*

Example 5 USING THE QUADRATIC FORMULA (REAL SOLUTIONS)

Solve $x^2 - 4x = -2$

Solution

$$x = \frac{4 \pm \sqrt{16 - 8}}{2}$$

Simplify.

$$x = \frac{4 \pm 2\sqrt{2}}{2}$$

$\sqrt{16 - 8} = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$

$$x = \frac{2\left(2 \pm \sqrt{2}\right)}{2}$$

Factor out 2 in the numerator.

Factor first, then divide.

$$x = 2 \pm \sqrt{2}$$

Lowest terms

The solution set is $\left\{2 \pm \sqrt{2}\right\}$.

Example 6 USING THE QUADRATIC FORMULA (NONREAL COMPLEX SOLUTIONS)

Solve $2x^2 = x - 4.$

Solution

$$2x^2 - x + 4 = 0$$

Write in standard form.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(4)}}{2(2)}$$

Quadratic formula with $a = 2$, $b = -1$, $c = 4$

Use parentheses and substitute carefully to avoid errors.

$$x = \frac{1 \pm \sqrt{1 - 32}}{4}$$

Example 6 USING THE QUADRATIC FORMULA (NONREAL COMPLEX SOLUTIONS)

Solve $2x^2 = x - 4.$

Solution

$$x = \frac{1 \pm \sqrt{-31}}{4}$$ Simplify.

$$x = \frac{1 \pm i\sqrt{31}}{4}$$ $\sqrt{-1} = i$

The solution set is $\left\{ \frac{1}{4} \pm \frac{\sqrt{31}}{4} i \right\}.$

Cubic Equation

The equation $x^3 + 8 = 0$ is called a **cubic equation** because the greatest degree of the terms is 3.

Example 7 SOLVING A CUBIC EQUATION

Solve $x^3 + 8 = 0.$

Solution

$$x^3 + 8 = 0$$

$$(x+2)\left(x^2 - 2x + 4\right) = 0$$ Factor as a sum of cubes.

$$x + 2 = 0 \quad \text{or} \quad x^2 - 2x + 4 = 0$$ Zero-factor property

$$x = -2 \quad \text{or} \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

Quadratic formula with $a = 1$, $b = -2$, $c = 4$

Example 7 SOLVING A CUBIC EQUATION

Solve $x^3 + 8 = 0.$

Solution

$$x = \frac{2 \pm \sqrt{-12}}{2}$$ Simplify.

$$x = \frac{2 \pm 2i\sqrt{3}}{2}$$ Simplify the radical.

$$x = \frac{2\left(1 \pm i\sqrt{3}\right)}{2}$$ Factor out 2 in the numerator.

Example 7 SOLVING A CUBIC EQUATION

Solve $x^3 + 8 = 0.$

Solution

$$x = 1 \pm i\sqrt{3}$$ Lowest terms

The solution set is $\left\{-2, 1 \pm i\sqrt{3}\right\}.$

Example 8 SOLVING FOR A QUADRATIC VARIABLE IN A FORMULA

Solve for the specified variable. Use ± when taking square roots.

(a) $A = \frac{\pi d^2}{4}$, for d

Solution

$$A = \frac{\pi d^2}{4}$$

Goal: Isolate *d*, the specified variable.

$$4A = \pi d^2$$ Multiply each side by 4.

$$\frac{4A}{\pi} = d^2$$ Divide each side by π.

Example 8 SOLVING FOR A QUADRATIC VARIABLE IN A FORMULA

Solve for the specified variable. Use ± when taking square roots.

(a) $A = \dfrac{\pi d^2}{4}$, for d

Solution

$$d = \pm\sqrt{\frac{4A}{\pi}}$$

Square root property

See the Note following this example.

$$d = \frac{\pm\sqrt{4A}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

Multiply by $\dfrac{\sqrt{\pi}}{\sqrt{\pi}}$.

Example 8 SOLVING FOR A QUADRATIC VARIABLE IN A FORMULA

Solve for the specified variable. Use $\pm$ when taking square roots.

(a) $A = \dfrac{\pi d^2}{4}$, for d

Solution

$$d = \frac{\pm\sqrt{4A\pi}}{\pi}$$

Multiply numerators.
Multiply denominators.

$$d = \frac{\pm 2\sqrt{A\pi}}{\pi}$$

Simplify the radical.

Example 8 SOLVING FOR A QUADRATIC VARIABLE IN A FORMULA

Solve for the specified variable. Use ± when taking square roots.

(b) $rt^2 - st = k \quad (r \neq 0)$, for t

Solution

Because $rt^2 - st = k$ has terms with t^2 and t, use the quadratic formula.

$$rt^2 - st - k = 0$$ Write in standard form.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

Example 8 SOLVING FOR A QUADRATIC VARIABLE IN A FORMULA

Solve for the specified variable. Use $\pm$ when taking square roots.

(b) $rt^2 - st = k \quad (r \neq 0)$, for t

Solution

$$t = \frac{-(-s) \pm \sqrt{(-s)^2 - 4(r)(-k)}}{2r}$$

Here, $a = r$, $b = -s$, and $c = -k$.

$$t = \frac{s \pm \sqrt{s^2 + 4rk}}{2r}$$

Simplify.

Solving for a Specified Variable

Note In **Example 8,** we took both positive and negative square roots. However, if the variable represents time or length in an application, we consider only the *positive* square root.

The Discriminant

The Discriminant The quantity under the radical in the quadratic formula, $b^2 - 4ac$, is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \longleftarrow \text{Discriminant}$$

The Discriminant

Discriminant	Number of Solutions	Type of Solutions
Positive, perfect square	Two	Rational
Positive, but not a perfect square	Two	Irrational
Zero	One (a double solution)	Rational
Negative	Two	Nonreal complex

Caution ***The restriction on a, b, and c is important.*** For example,

$$x^2 - \sqrt{5x} - 1 = 0$$

has discriminant $b^2 - 4ac = 5 + 4 = 9$, which would indicate two rational solutions *if the coefficients were integers*. By the quadratic formula, the two solutions

$$\frac{\sqrt{5} \pm 3}{2}$$

are *irrational* numbers.

Example 9 USING THE DISCRIMINANT

Determine the number of distinct solutions, and tell whether they are *rational*, *irrational*, or *nonreal complex* numbers.

(a) $5x^2 + 2x - 4 = 0$

Solution

For $5x^2 + 2x - 4 = 0$, use $a = 5$, $b = 2$, and $c = -4$.

$$b^2 - 4ac = 2^2 - 4(5)(-4) = 84$$

The discriminant 84 is positive and not a perfect square, so there are two distinct irrational solutions.

Example 9 USING THE DISCRIMINANT

Determine the number of distinct solutions, and tell whether they are *rational*, *irrational*, or *nonreal complex* numbers.

(b) $x^2 - 10x = -25$

Solution

First, write the equation in standard form as $x^2 - 10x + 25 = 0$. Thus, $a = 1$, $b = -10$, and $c = 25$.

$$b^2 - 4ac = (-10)^2 - 4(1)(25) = 0$$

There is one distinct rational solution, a double solution.

Example 9 USING THE DISCRIMINANT

Determine the number of distinct solutions, and tell whether they are *rational*, *irrational*, or *nonreal complex* numbers.

(c) $2x^2 - x + 1 = 0$

Solution

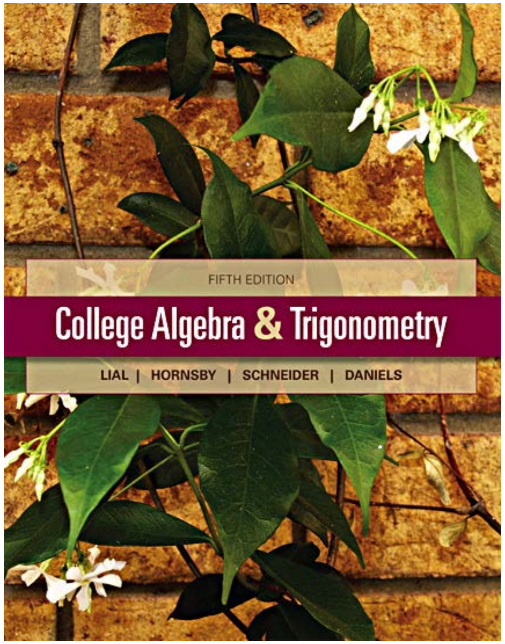
For $2x^2 - x + 1 = 0$, use $a = 2$, $b = -1$, and $c = 1$.

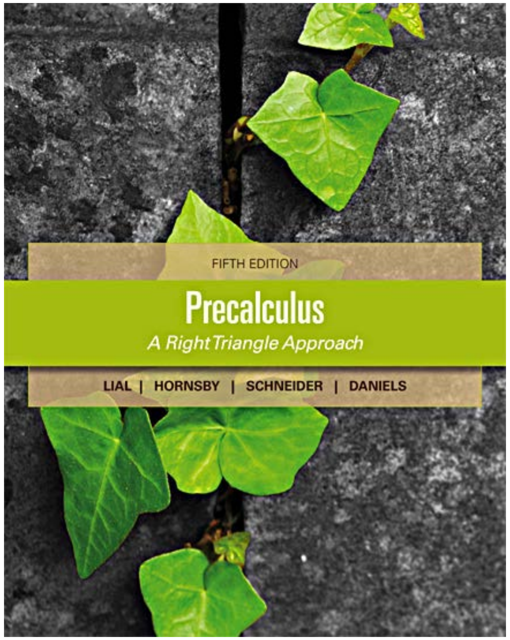
$$b^2 - 4ac = (-1)^2 - 4(2)(1) = -7.$$

There are two distinct nonreal complex solutions. (They are complex conjugates.)

2

Equations and Inequalities





2.4 Inequalities

- Linear Inequalities
- Three-Part Inequalities
- Quadratic Inequalities
- Rational Inequalities

Properties of Inequality

Let *a*, *b* and *c* represent real numbers.

1. If $a < b$, then $a \pm c < b \pm c$.

2. If $a < b$ and if $c > 0$, then $ac < bc$.

3. If $a < b$ and if $c < 0$, then $ac > bc$.

4.If $a < b$ and if $c > 0$, then $a/c < b/c$.

5. If $a < b$ and if $c < 0$, then $a/c > b/c$.

Replacing < with >, ≤ , or ≥ results in similar properties.

Example:

- $2<5$ then $2+3<5+3$ i.e. $5<8$
- $2<5$, then $2-1<5-1$, i.e. $1<4$
- $2<5$, $c=3>0$, $2.3<5.3$, i.e. $6<15$
- $2<5$, $c=-3<0$, $2.-3>5.-3$, i.e. $-6>-15$
- $4<6$, $c=2>0$, $4/2<6/2$, i.e. $2<3$
- $4<6$, $c=-2<0$, $4/-2>6/-2$, i.e. $-2>-3$

Motion Problems

Note Multiplication may be replaced by division in Properties 2 and 3.

Always remember to reverse the direction of the inequality symbol when multiplying or dividing by a negative number.

Linear Inequality in One Variable

A **linear inequality in one variable** is an inequality that can be written in the form

$$ax + b > 0,$$

where a and b are real numbers, with $a \neq 0$. (Any of the symbols $\geq$, $<$, and $\leq$ may also be used.)

Example 1 SOLVING A LINEAR INEQUALITY

Solve $-3x + 5 > -7$.

Solution

$$-3x + 5 > -7$$

$$-3x + 5 - 5 > -7 - 5 \qquad \text{Subtract 5.}$$

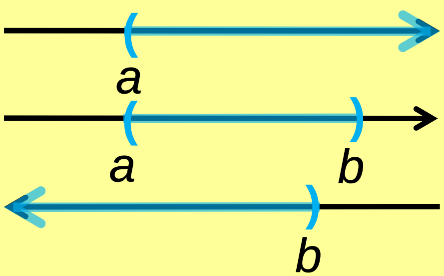
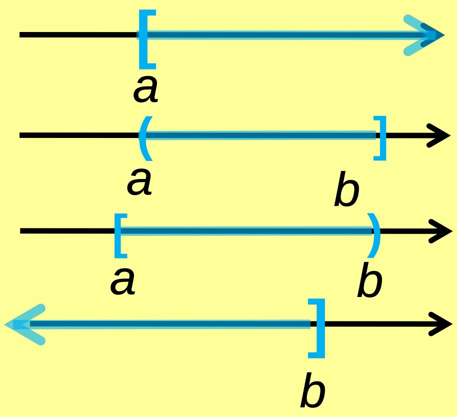
$$-3x > -12 \qquad \text{Combine like terms.}$$

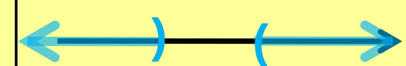
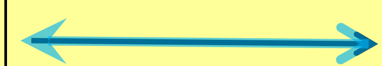
$$\frac{-3x}{-3} < \frac{-12}{-3}$$

Divide by –3. Reverse the direction of the inequality symbol when multiplying or dividing by a negative number.

Don't forget to reverse the symbol here.

$$x < 4 = (-\infty, 4)$$

Type of Interval	Set	Interval Notation	Graph
Open interval	$\{x \mid x > a\}$ $\{x \mid a < x < b\}$ $\{x \mid x < b\}$	(a, ∞) (a, b) $(-\infty, b)$	
Other intervals	$\{x \mid x \geq a\}$ $\{x \mid a < x \leq b\}$ $\{x \mid a \leq x < b\}$ $\{x \mid x \leq b\}$	$[a, \infty)$ $(a, b]$ $[a, b)$ $(-\infty, b]$	

Type of Interval	Set	Interval Notation	Graph
Closed interval	$\{x \mid a \le x \le b\}$	$[a, b]$	
Disjoint interval	$\{x \mid x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$	
All real numbers	$\{x \mid x \text{ is a real number}\}$	$(-\infty, \infty)$	

Homework 1 SOLVING A LINEAR INEQUALITY

Solve $4 - 3x \le 7 + 2x$. Give the solution set in interval notation.

Solution

$$4 - 3x \le 7 + 2x$$

$$4 - 3x - 4 \le 7 + 2x - 4$$ Subtract 4.

$$-3x \le 3 + 2x$$ Combine like terms.

$$-3x - 2x \le 3 + 2x - 2x$$ Subtract $2x$.

$$-5x \le 3$$ Combine like terms.

Homework 1 **SOLVING A LINEAR INEQUALITY**

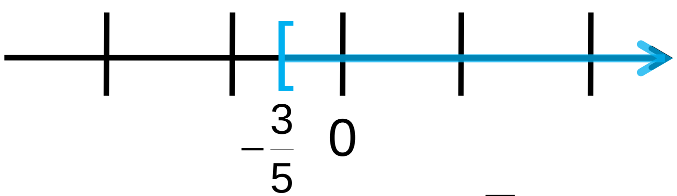
Solve $4 - 3x \le 7 + 2x$. Give the solution set in interval notation.

Solution

$$\frac{-5x}{-5} \ge \frac{3}{-5}$$

Divide by –5. Reverse the direction of the inequality symbol.

$$x \ge -\frac{3}{5}$$



In interval notation the solution set is $\left[-\frac{3}{5}, \infty\right)$.

Example 2 FINDING THE BREAK-EVEN POINT غير مطلوب

If the revenue and cost of a certain product are given by

$$R = 4x \qquad \text{and} \qquad C = 2x + 1000,$$

where x is the number of units produced and sold, at what production level does *R at least equal C*?

Example 2 FINDING THE BREAK-EVEN POINT غير مطلوب

Solution Set $R \geq C$ and solve for x.

$$R \geq C$$

$$4x \geq 2x + 1000 \qquad \text{Substitute.}$$

$$2x \geq 1000 \qquad \text{Subtract } 2x.$$

$$x \geq 500 \qquad \text{Divide by 2.}$$

The break-even point is at $x = 500$. This product will at least break even if the number of units produced and sold is in the interval $[500, \infty)$.

Homework 2 SOLVING A THREE-PART INEQUALITY

Solve $-2 < 5 + 3x < 20$.

Solution

$$-2 < 5 + 3x < 20$$

$$-2 - 5 < 5 + 3x - 5 < 20 - 5$$

Subtract 5 from each part.

$$-7 < 3x < 15$$

$$\frac{-7}{3} < \frac{3x}{3} < \frac{15}{3}$$

Divide each part by 3.

$$-\frac{7}{3} < x < 5$$

The solution set is the interval $\left(-\frac{7}{3}, 5\right)$.

Quadratic Inequalities

A **quadratic inequality** is an inequality that can be written in the form

$$ax^2 + bx + c < 0$$

for real numbers *a*, *b*, and *c*, with $a \neq 0$. (The symbol < can be replaced with >, ≤, or ≥.)

Solving a Quadratic Inequality

Step 1 Solve the corresponding quadratic equation.

Step 2 Identify the intervals determined by the solutions of the equation.

Step 3 Use a test value from each interval to determine which intervals form the solution set.

Example 3 SOLVING A QUADRATIC INEQULITY

Solve $x^2 - x - 12 < 0.$

Solution

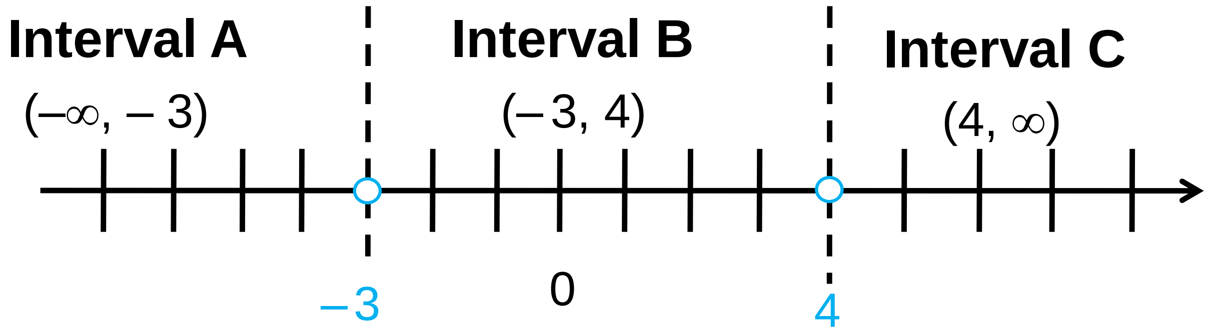
Step 1 Find the values of x that satisfy

$$x^2 - x - 12 = 0.$$

$x^2 - x - 12 = 0$	Corresponding quadratic equation
$(x+3)(x-4) = 0$	Factor.
$x+3=0$ or $x-4=0$	Zero-factor property
$x=-3$ or $x=4$	Solve each equation.

Example 5 **SOLVING A QUADRATIC INEQULITY**

Step 2 The two numbers –3 and 4 divide the number line into three intervals. The expression $x^2 - x - 12$ will take on a value that is either ***less than*** zero or ***greater than*** zero on each of these intervals.



Example 5 SOLVING A QUADRATIC INEQULITY

Step 3 Choose a test value from each interval.

Interval	Test Value	Is $x^2 - x - 12 < 0$ True or False?	
A: $(-\infty, -3)$	-4	$(-4)^2 - (-4) - 12 < 0$ $8 < 0$	? False
B: $(-3, 4)$	0	$0^2 - 0 - 12 < 0$ $-12 < 0$	? True
C: $(4, \infty)$	5	$5^2 - 5 - 12 < 0$ $8 < 0$	? False

Since the values in Interval B make the inequality true, the solution set is $(-3, 4)$.

Homework 3 **SOLVING A QUADRATIC INEQUALITY**

Solve $2x^2 + 5x - 12 \geq 0.$

Solution

Step 1 Find the values of x that satisfy

$$2x^2 + 5x - 12 = 0.$$

$$2x^2 + 5x - 12 = 0$$ Corresponding quadratic equation

$$(2x - 3)(x + 4) = 0$$ Factor.

$$2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$ Zero-factor property

Homework 3 SOLVING A QUADRATIC INEQUALITY

Solve $2x^2 + 5x - 12 \geq 0$.

Solution

Step 1

$$2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$

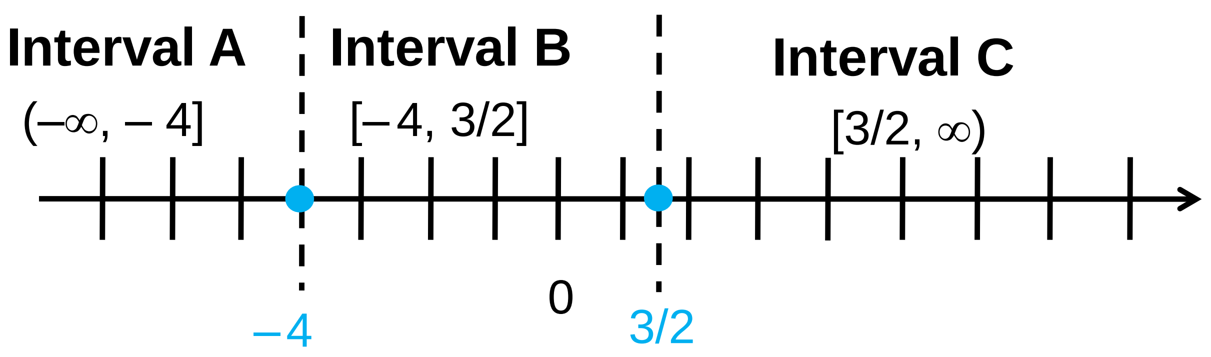
$$x = \frac{3}{2} \quad \text{or} \quad x = -4$$

SOLVING A QUADRATIC INEQUALITY

Solve $2x^2 + 5x - 12 \geq 0.$

Solution

Step 2 The values form the intervals on the number line.



Homework 3 SOLVING A QUADRATIC INEQUALITY

Step3 Choose a test value from each interval.

Interval	Test Value	Is $2x^2 + 5x - 12 \geq 0$ True or False?
A: $(-\infty, -4]$	-5	$2(-5)^2 + 5(-5) - 12 \geq 0$? $13 \geq 0$ True
B: $[-4, 3/2]$	0	$2(0)^2 + 5(0) - 12 \geq 0$? $-12 \geq 0$ False
C: $[3/2, \infty)$	2	$2(2)^2 + 5(2) - 12 \geq 0$? $6 \geq 0$ True

The values in Intervals A and C make the inequality true, so the solution set is the union of the intervals $(-\infty, -4] \cup \left[\frac{3}{2}, \infty\right)$.

Inequalities

Note Inequalities that use the symbols < and > are **strict inequalities**; ≤ and ≥ are used in **nonstrict inequalities.** The solutions of the equation in **Example 5** were not included in the solution set since the inequality was a *strict* inequality. In **Example 6,** the solutions of the equation *were* included in the solution set because of the nonstrict inequality.

Example 4 FINDING PROJECTILE HEIGHT

غير مطلوب

If a projectile is launched from ground level with an initial velocity of 96 ft per sec, its height *s* in feet *t* seconds after launching is given by the following equation.

$$s = -16t^2 + 96t$$

When will the projectile be greater than 80 ft above ground level?

Example 4 FINDING PROJECTILE HEIGHT غير مطلوب

Solution

$$-16t^2 + 96t > 80$$ Set s greater than 80.

$$-16t^2 + 96t - 80 > 0$$ Subtract 80.

$$t^2 - 6t + 5 < 0$$ Divide by – 16.

Reverse the direction of the inequality symbol.

Now solve the corresponding *equation*.

Example 4 FINDING PROJECTILE HEIGHT غير مطلوب

Solution

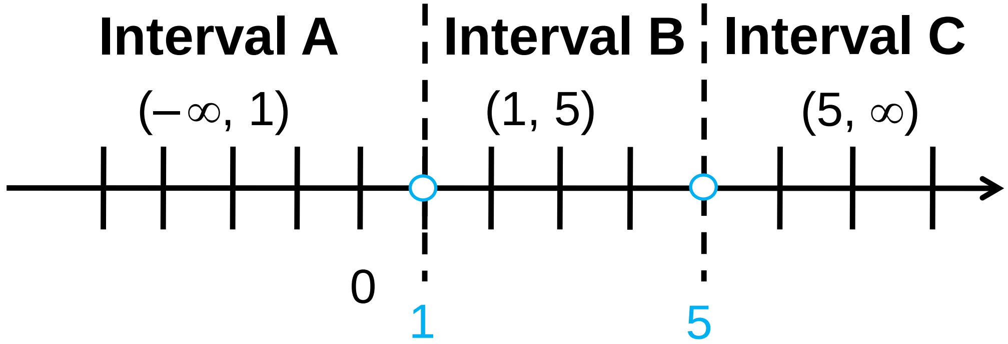
$$t^2 - 6t + 5 = 0$$

$$(t-1)(t-5) = 0 \quad \text{Factor.}$$

$$t-1=0 \quad \text{or} \quad t-5=0 \quad \text{Zero-factor property}$$

$$t=1 \quad \text{or} \quad t=5 \quad \text{Solve each equation.}$$

Solution



Use the procedure of **Examples 5 and 6** to determine that values in Interval B, (1, 5) , satisfy the inequality. The projectile is greater than 80 ft above ground level between 1 and 5 sec after it is launched.

Solving a Rational Inequality

Step 1 Rewrite the inequality, if necessary, so that 0 is on one side and there is a single fraction on the other side.

Step 2 Determine the values that will cause either the numerator or the denominator of the rational expression to equal 0. These values determine the intervals of the number line to consider.

Solving a Rational Inequality

Step 3 Use a test value from each interval to determine which intervals form the solution set.

A value causing the denominator to equal zero will never be included in the solution set. If the inequality is strict, any value causing the numerator to equal zero will be excluded. If the inequality is nonstrict, any such value will be included.

Caution Solving a rational inequality such as

$$\frac{5}{x+4} \geq 1$$

by multiplying each side by $x + 4$ to obtain $5 \geq x + 4$ requires considering *two cases*, since the sign of $x + 4$ depends on the value of x. If $x + 4$ is negative, then the inequality symbol must be reversed. The procedure used in the next two examples eliminates the need for considering separate cases.

Homework 4 **SOLVING A RATIONAL INEQUALITY**

Solve $\dfrac{5}{x+4} \geq 1.$

Solution

Step 1

$$\frac{5}{x+4} - 1 \geq 0$$

Subtract 1 so that 0 is on one side.

$$\frac{5}{x+4} - \frac{x+4}{x+4} \geq 0$$

Use x + 4 as the common denominator.

$$\frac{5-(x+4)}{x+4} \geq 0$$

Write as a single fraction.

Note the careful use of parentheses.

Homework 4 SOLVING A RATIONAL INEQUALITY

Solve $\dfrac{5}{x+4} \geq 1.$

Solution

Step 1 $\dfrac{1-x}{x+4} \geq 0$ Combine terms in the numerator, being careful with signs.

Step 2 The quotient possibly changes sign only where *x*-values make the numerator or denominator 0. This occurs at

$$1 - x = 0 \quad \text{or} \quad x + 4 = 0$$

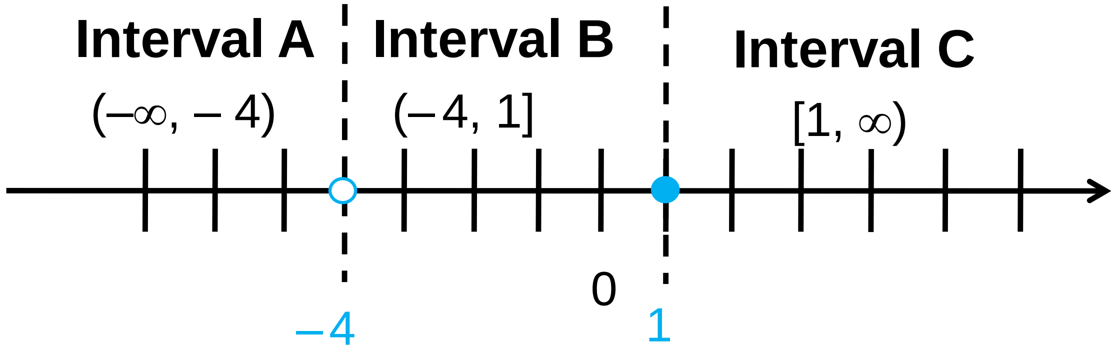
$$x = 1 \quad \text{or} \quad x = -4$$

SOLVING A RATIONAL INEQUALITY

Solve $\dfrac{5}{x+4} \geq 1.$

Solution

Step 2



SOLVING A RATIONAL INEQUALITY

Step 3 Choose test values.

Interval	Test Value	Is $\frac{5}{x+4} \geq 1$ True or False?
A: $(-\infty, -4)$	-5	$\frac{5}{-5+4} \geq 1$? $\quad -5 \geq 1$ False
B: $(-4, 1]$	0	$\frac{5}{0+4} \geq 1$? $\quad \frac{5}{4} \geq 1$ True
C: $[1, \infty)$	2	$\frac{5}{2+4} \geq 1$? $\quad \frac{5}{6} \geq 1$ False

Example 8 SOLVING A RATIONAL INEQUALITY

Step 3

The values in the interval (–4, 1) satisfy the original inequality. The value 1 makes the nonstrict inequality true, so it must be included in the solution set. Since –4 makes the denominator 0, it must be excluded. The solution set is (–4, 1].

► Caution ***Be careful with the endpoints of the intervals when solving rational inequalities***.

Example 5 SOLVING A RATIONAL INEQUALITY

Solve $\dfrac{2x-1}{3x+4} < 5.$

Solution

$$\frac{2x-1}{3x+4} - 5 < 0$$

Subtract 5.

$$\frac{2x-1}{3x+4} - \frac{5(3x+4)}{3x+4} < 0$$

Common denominator is $3x + 4$.

$$\frac{2x-1-5(3x+4)}{3x+4} < 0$$

Write as a single fraction.

Example 5 SOLVING A RATIONAL INEQUALITY

Solve $\dfrac{2x-1}{3x+4} < 5.$

Solution

$$\frac{2x-1-15x-20}{3x+4} < 0$$

Distributive property

Be careful with signs.

$$\frac{-13x-21}{3x+4} < 0$$

Combine like terms in the numerator.

Example 5 SOLVING A RATIONAL INEQUALITY

Solve $\dfrac{2x-1}{3x+4} < 5.$

Solution

Set the numerator and denominator equal to 0 and solve the resulting equations to find the values of *x* where sign changes may occur.

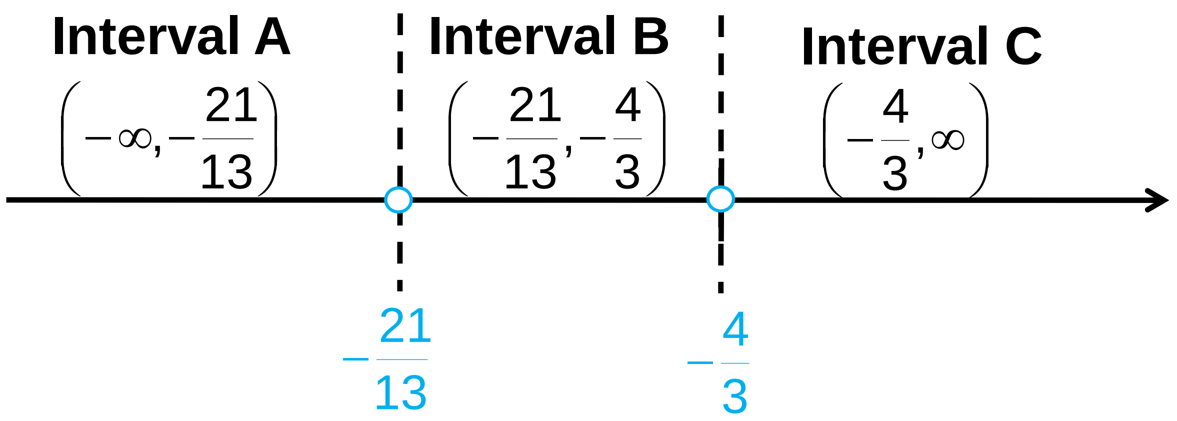
$$-13x - 21 = 0 \quad \text{or} \quad 3x + 4 = 0$$

$$x = -\frac{21}{13} \quad \text{or} \quad x = -\frac{4}{3}$$

Example 5 SOLVING A RATIONAL INEQUALITY

Solve $\dfrac{2x-1}{3x+4} < 5.$

Solution



Example 5 SOLVING A RATIONAL INEQUALITY

Solution

Now choose test values from the intervals and verify that:

–2 from Interval A makes the inequality true;

–1.5 from Interval B makes the inequality false;

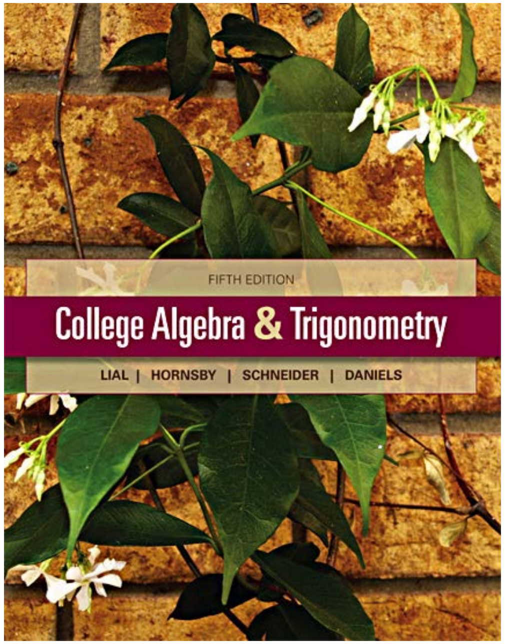
0 from Interval C makes the inequality true.

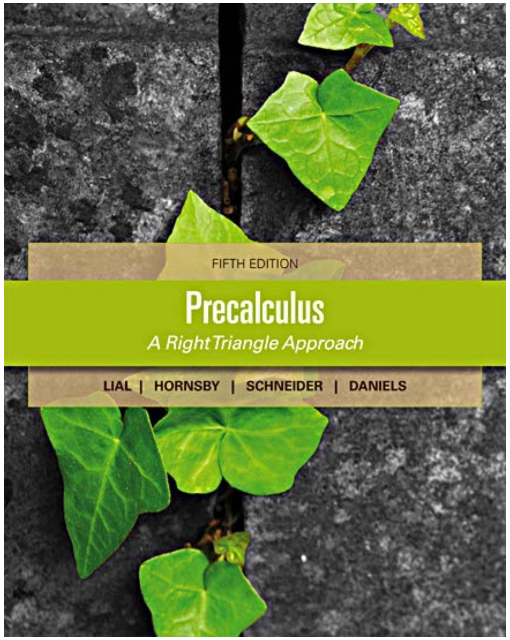
Because of the < symbol, neither endpoint satisfies the inequality, so the solution set is

$$\left(-\infty, -\frac{21}{13}\right) \cup \left(-\frac{4}{3}, \infty\right).$$

2

Equations and Inequalities

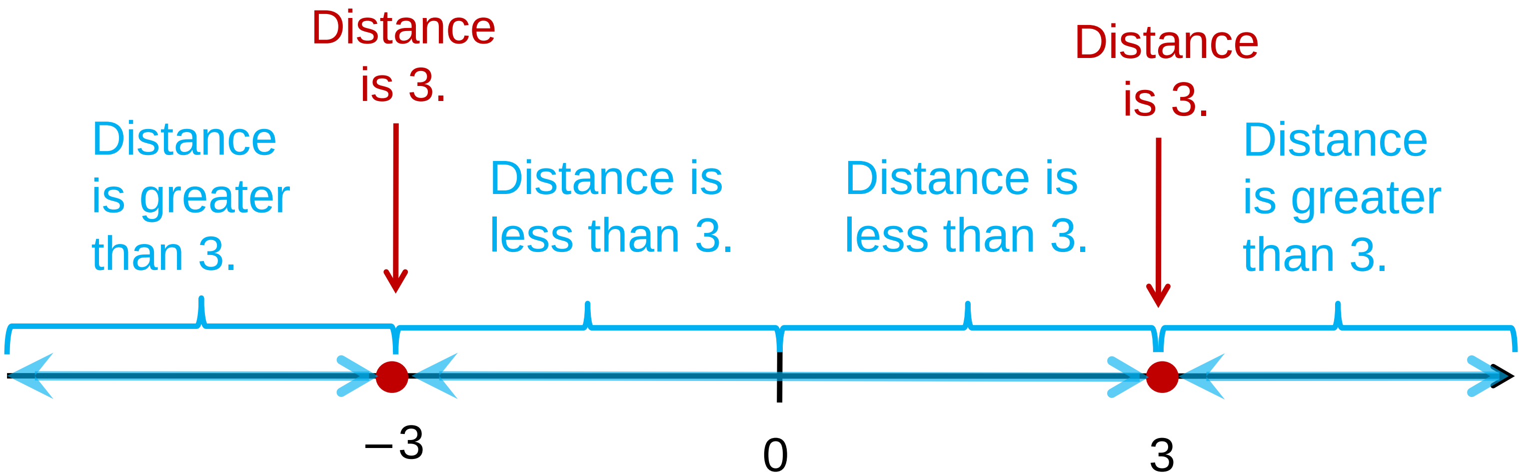




2.5 Absolute Value Equations and Inequalities

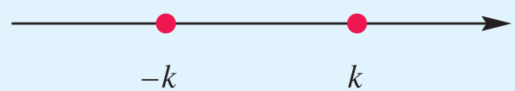
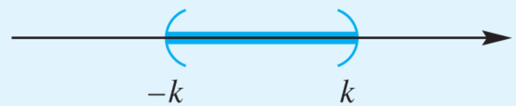
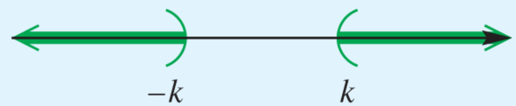
- Basic Concepts
- Absolute Value Equations
- Absolute Value Inequalities
- Special Cases
- Absolute Value Models for Distance and Tolerance

The **absolute value** of a number *a* gives the distance from *a* to 0 on a number line.



By this definition, the equation $|x| = 3$ can be solved by finding all real numbers at a distance of 3 units from 0. Two numbers satisfy this equation, 3 and –3. So the solution set is $\{-3, 3\}$.

For each equation or inequality in Cases 1-3 in the table, assume that $k > 0$.

Absolute Value Equation or Inequality	Equivalent Form	Graph of the Solution Set	Solution Set
Case 1: $\lvert x \rvert = k$	$x = k$ or $x = -k$		$\{-k, k\}$
Case 2: $\lvert x \rvert < k$	$-k < x < k$		$(-k, k)$
Case 3: $\lvert x \rvert > k$	$x < -k$ or $x > k$		$(-\infty, -k) \cup (k, \infty)$

In Cases 2 and 3, the strict inequality may be replaced by its nonstrict form. Additionally, if an absolute value equation takes the form $\lvert a \rvert = \lvert b \rvert$, then *a* and *b* must be equal in value or opposite in value.

Thus, the equivalent form of $\boldsymbol{\lvert a \rvert = \lvert b \rvert}$ ***is a = b or a = –b.***

Example 1 SOLVING ABSOLUTE VALUE EQUATIONS

Solve each equation.

(a) $|5 - 3x| = 12$

Solution

For the given expression 5 – 3*x* to have absolute value 12, it must represent either 12 or –12. This equation fits the form of Case 1.

Example 1 SOLVING ABSOLUTE VALUE EQUATIONS

Solve each equation.

(a) $|5-3x|=12$

Solution

$$|5-3x|=12$$

$$5-3x=12 \quad \text{or} \quad 5-3x=-12 \qquad \text{Case 1}$$

$$-3x=7 \quad \text{or} \quad -3x=-17 \qquad \text{Subtract 5.}$$

$$x=-\frac{7}{3} \quad \text{or} \quad x=\frac{17}{3} \qquad \text{Divide by } -3.$$

Example 1 SOLVING ABSOLUTE VALUE EQUATIONS

Solve each equation.

(a) $|5-3x|=12$

Solution

$$x=-\frac{7}{3} \quad \text{or} \quad x=\frac{17}{3}$$

Check the solutions by substituting them in the original absolute value equation. The solution set is $\left\{-\frac{7}{3},\frac{17}{3}\right\}$.

Example 1 SOLVING ABSOLUTE VALUE EQUATIONS

Solve each equation.

(b) $|4x-3| = |x+6|$

Solution $|4x-3| = |x+6|$

$$4x-3 = x+6 \quad \text{or} \quad 4x-3 = -(x+6)$$

$$3x = 9 \quad \text{or} \quad 4x-3 = -x-6$$

$$x = 3 \quad \text{or} \quad 5x = -3$$

$$x = -\frac{3}{5}$$

The solution set is $\left\{-\frac{3}{5}, 3\right\}$.

Solve each inequality.

(a) $|2x+1|<7$

Solution This inequality fits Case 2.

$$|2x+1|<7$$

$$-7<2x+1<7 \quad \text{Case 2}$$

$$-8<2x<6 \quad \text{Subtract 1 from each part.}$$

$$-4<x<3 \quad \text{Divide each part by 2.}$$

The final inequality gives the solution set (–4, 3).

SOLVING ABSOLUTE VALUE INEQUALITIES

Solve each inequality.

(b) $|2x+1| > 7$

Solution This inequality fits Case 3.

$$|2x+1| > 7$$

$2x+1 < -7$ or $2x+1 > 7$ Case 3

$2x < -8$ or $2x > 6$ Subtract 1 from each side.

$x < -4$ or $x > 3$ Divide each part by 2.

The solution set is $(-\infty, -4) \cup (3, \infty)$.

Example 2

SOLVING AN ABSOLUTE VALUE INEQUALITY

Solve $|2-7x|-1>4.$

Solution

$$|2-7x|-1>4$$

$$|2-7x|>5 \qquad \text{Add 1 to each side.}$$

$$2-7x<-5 \quad \text{or} \quad 2-7x>5 \qquad \text{Case 3}$$

$$-7x<-7 \quad \text{or} \quad -7x>3 \qquad \text{Subtract 2.}$$

$$x>1 \quad \text{or} \quad x<-\frac{3}{7}$$

Divide by –7. Reverse the direction of each inequality.

The solution set is $\left(-\infty,-\frac{3}{7}\right)\cup(1,\infty).$

Homework 2 **SOLVING SPECIAL CASES**

Solve each equation or inequality.

(a) $|2-5x| \geq -4$

Solution Since the absolute value of a number is always nonnegative, the inequality is always true. The solution set includes all real numbers, written $(-\infty, \infty)$.

(b) $|4x-7| < -3$

Solution There is no number whose absolute value is less than –3 (or less than *any* negative number). The solution set is $\emptyset$.

Homework 2 SOLVING SPECIAL CASES

Solve each equation or inequality.

(c) $|5x + 15| = 0$

Solution The absolute value of a number will be 0 only if that number is 0. Therefore,

$$|5x + 15| = 0 \text{ is equivalent to } 5x + 15 = 0$$

which has solution set {–3}. Check by substituting into the original equation.

Example 3 USING ABSOLUTE INEQUALITIES TO DESCRIBE DISTANCESغير مطلوب

Write each statement using an absolute value inequality.

(a) *k* is no less than 5 units from 8.

Solution

Since the distance from *k* to 8, written $|k - 8|$ or $|8 - k|$, is no less than 5, the distance is greater than or equal to 5. This can be written as

$$|k-8| \geq 5, \text{ or equivalently } |8-k| \geq 5.$$

Example 3 USING ABSOLUTE INEQUALITIES TO DESCRIBE DISTANCESغير مطلوب

Write each statement using an absolute value inequality.

(b) *n* is within 0.001 unit of 6.

Solution

This statement indicates that the distance between *n* and 6 is less than 0.001.

$$|n-6|<0.001 \quad \text{or, equivalently} \quad |6-n|<0.001$$

Example 6 USING ABSOLUTE VALUE TO MODEL TOLERANCE غير مطلوب

In quality control and other applications, we often wish to keep the difference between two quantities within some predetermined amount, called the **tolerance.** Suppose $y = 2x + 1$ and we want y to be within 0.01 unit of 4. For what values of x will this be true?

Solution

$$|y - 4| < 0.01$$ Write an absolute value inequality.

$$|2x + 1 - 4| < 0.01$$ Substitute $2x + 1$ for y.

$$|2x - 3| < 0.01$$ Combine like terms.

Example 6 **USING ABSOLUTE VALUE TO MODEL TOLERANCE** غير مطلوب

Suppose $y = 2x + 1$ and we want y to be within 0.01 unit of 4. For what values of x will this be true?

Solution

$$-0.01 < 2x - 3 < 0.01 \quad \text{Case 2}$$

$$2.99 < 2x < 3.01 \quad \text{Add 3 to each part.}$$

$$1.495 < x < 1.505 \quad \text{Divide each part by 2.}$$

Reversing these steps shows that keeping x in the interval (1.495,1.505) ensures that the difference between y and 4 is within 0.01 unit.