1. Antiderivatives

Given a function $f(x)$, we would like to find another function $F(x)$ such that

$$F'(x) = f(x)\,, \qquad \forall\, x \in I = [a\,, b] \subseteq \Re$$

we call such a function **$F(x)$ an antiderivative of $f(x)$**.

EX.1.1 (Finding several Antiderivatives of a given function)

Find an antiderivative of $f(x) = x^2$

Solution Notice that

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 5\right) = x^2$$

In fact, for any constant c, we have

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2$$

Thus, $F(x) = \frac{1}{3}x^3\,,\ H(x) = \frac{1}{3}x^3 + 5\,,\ G(x) = \frac{1}{3}x^3 + c$

are all antiderivatives of $f(x)$ ▪

In general, observe that if $F(x)$ is any antiderivative of $f(x)$ and c is any constant, then

$$\frac{d}{dx}[F(x)+c] = F'(x) + 0 = f(x)$$

Thus, $G(x) = F(x) + c$ is also an antiderivative of $f(x)$,

for any constant c. At this point, you might ask if there any other antiderivatives of $f(x)$ besides $G(x) = F(x) + c$ The answer, as provided in the following theorem, is no.

THEOREM 1.1 Suppose that F and G both antiderivatives of f on an interval $I = [a, b]$, then

$G(x) = F(x) + c, \forall\, x \in I$ for some constant c.

Proof setting $H(x) = G(x) - F(x)\,;\ \forall\, x \in I$, we get

$H'(x) = G'(x) - F'(x) = f - f = 0 \ \Rightarrow\ \exists\, c\,;\ H = c$

$$G - F = c \ \Rightarrow G = F + c$$

Definition1.1 (Indefinite Integral)

Let F be any antiderivative of f. The indefinite integral of $f(x)$ w.r.t.x (with respect to x), is defined by

$$\int f(x)dx = F(x) + c$$

Where c is an arbitrary constant (the constant of integration)

EX.1.2 Evaluate $\int x^5 dx$

Solution we know that

$$\frac{d}{dx}(x^6) = 6x^5$$

and so,

$$\frac{d}{dx}\underbrace{\left(\frac{1}{6}x^6\right)}_{F(x)} = \underbrace{x^5}_{f(x)}$$

Therefore

$$\int x^5 dx = \frac{1}{6}x^6 + c \qquad \blacksquare$$

We should point out that every differentiation rule gives rise to corresponding integration.

For instance, recall that for every number, r,

$$\frac{d}{dx}\, x^r = r\, x^{r-1}$$

Likewise, we have

$$\frac{d}{dx}\, x^{r+1} = (r+1)\, x^r$$

This proves the following result.

THEOREM 1.2 (Power Rule)

For any real number $r \neq -1$,

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c$$

EX.1.3 (Using the Power Rule)

- $\int x^{17}\,dx = \frac{x^{17+1}}{17+1} + c = \frac{x^{18}}{18} + c$
- $\int \frac{1}{t^3}dt = \int t^{-3}\,dt = \frac{t^{-3+1}}{-3+1} + c = \frac{t^{-2}}{-2} + c = -\frac{1}{2}t^{-2} + c$
- $\int \sqrt{y}\,dy = \int y^{\frac{1}{2}}dy = \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\,y^{\frac{3}{2}} + c$
- $\int \frac{1}{3\sqrt{x}}dx = \int x^{-\frac{1}{3}}\,dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}\,x^{\frac{2}{3}} + c$

Notice that since $\frac{d}{dx}(\sin x) = \cos x$, we have

$$\boxed{\int \cos x\,dx = \sin x + c}$$

Again, by reversing any derivative formula, we get a corresponding integration formula.

The following tables contain Review of Differentiation, and Brief Table of Integrals0

Rule **If** $\frac{d}{dx}(F(x)) = f(x)$ **(or** $F'(x) = f(x)$ **) ,then**

$$\int f(x)dx = F(x) + c$$

where c is the integral constant

The derivative formula	The integration formula
Power rule: $\frac{d}{dx}(x^{r+1}) = (r+1)x^r$	$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + c\,; r \neq -1$
Six trigonometric functions: $\frac{d}{dx}(sinx) = cosx$	$\int cosx = \sin x + c$
$\frac{d}{dx}(cosx) = -sinx$	$\int sinx\,dx = -\ cosx + c$
$\frac{d}{dx}(tanx) = sec^2x$	$\int sec^2x = \tan x + c$
$\frac{d}{dx}(\cot x) = -csc^2x$	$\int csc^2x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x\ .\tan x$	$\int \sec x\ .\tan x\ dx = \sec x + c$
$\frac{d}{dx}(\csc x) = -\csc x\ .\cot x$	$\int \csc x\ .\cot x\ dx = -\csc x + c$
Exponential functions: For any $a > 0 : \frac{d}{dx}(a^x) = a^x.\ln a$	$\int a^x\,dx = \frac{1}{\ln a}.a^x + c$
$\frac{d}{dx}(e^x) = e^x.\ln e$ $\frac{d}{dx}(e^x) = e^x\ ;\ \ \ln e = 1$	$\int e^x\,dx = e^x + c$

Logarithmic Functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln x + c$$

Generally:

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Inverse Functions:

$$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}x + c$$

$$\frac{d}{dx}(cos^{-1}) = -\frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{-1}{\sqrt{1-x^2}}dx = cos^{-1}x + c$$

$$\frac{d}{dx}(tan^{-1}) = \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2}dx = tan^{-1}x + c$$

$$\frac{d}{dx}(cot^{-1}) = -\frac{1}{1+x^2}$$

$$\int \frac{-1}{1+x^2}dx = cot^{-1}x + c$$

$$\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|.\sqrt{x^2-1}} \;;\; |x| > 1$$

$$\int \frac{1}{|x|.\sqrt{x^2-1}}dx = sec^{-1}x + c$$

$$\frac{d}{dx}(csc^{-1}x) = \frac{-1}{|x|.\sqrt{x^2-1}} \;;\; |x| > 1$$

$$\int \frac{-1}{|x|.\sqrt{x^2-1}}dx = csc^{-1}x + c$$

The following Theorem says that we can easily compute integrals of sums, differences and constant multiples of functions. However, it turns out that the integral of a product (or a quotient) is not generally the product (or quotient) of the integrals.

THEOREM.1.3 Suppose that $f(x)$ and $g(x)$ have antiderivatives, then for any constants a and b we have:

$$\int [af(x) \pm bg(x)]dx = a\int f(x)dx \pm b\int g(x)dx$$

Proof we have

$$\frac{d}{dx}\int f(x)dx = f(x)$$

$$\frac{d}{dx}\int g(x)dx = g(x)$$

It then follows that

$$\frac{d}{dx}\left[a\int f(x)dx \pm b\int g(x)dx\right] =$$

$$a\frac{d}{dx}\int f(x)dx \pm b\frac{d}{dx}\int g(x)dx = af(x) \pm b\,g(x)$$

$$\Rightarrow \int [af(x) \pm bg(x)]\,dx = a\int f(x)dx \pm b\int g(x)dx \quad \blacksquare$$

Note: $\int f(x).g(x)dx \neq \int f(x)dx.\int g(x)dx$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

EX.1.4

Find $\int (3\cos x + 4x^8)dx$

Solution

$$\int (3\cos x + 4x^8)dx = 3\int \cos x\,dx + 4\int x^8 dx$$

$$= 3(\sin x + c_1) + 4(\frac{x^9}{9} + c_2)$$

$$= 3\sin x + \frac{4}{9}x^9 + c$$

$$;\ c = 3c_1 + 4c_2$$

EX.1.5

compute $\int (3e^x - \frac{2}{1+x^2})dx$

Solution

$$\int (3e^x - \frac{2}{1+x^2})dx = 3\int e^x dx - 2\int \frac{1}{1+x^2}dx$$

$$= 3(e^x + c_1) - 2(tan^{-1}x + c_2)$$

$$= 3e^x - 2tan^{-1}x + c\ ; \quad c = 3c_1 - 2c_2$$

By the chain rule, for any constant $a \neq 0$, we have

$$\frac{d}{dx}\sin(a\,x) = a\cos(a\,x)$$

or $$\frac{d}{dx}\left[\frac{1}{a}\sin(ax)\right] = \cos(ax)$$

Therefore $$\int \cos(ax)dx = \frac{1}{a}\sin(ax)$$

In fact, we have the general result

THEOREM.1.4

If $\int f(x)dx = F(x) + c$ then for any constant $a \neq 0$

(i) $\boxed{\int f(a\,x)dx = \frac{1}{a}F(a\,x) + c}$

(ii) $\boxed{\int f(a\,x + b)dx = \frac{1}{a}F(a\,x + b) + c}$; b is constant with practice, working problems like those in the following example will become automatic.

EX.1.6

(Indefinite Integrals of Functions of the form $f(ax)$)

Evaluate

(a) $\int \sin(3x)\, dx$, (b) $\int 5e^{4x} dx$, (c) $\int 8\, sec^2\,(5x)dx$,

(d) $\int \frac{dx}{\sqrt{2x+5}}$, (e) $\int \frac{dx}{5+4x^2}$

Solution

(a) $\int \sin(3x)\, dx = -\frac{1}{3}\cos(3x) + c$

(b) $\int 5e^{4x} dx = 5\int e^{4x}\, dx = 5\left(\frac{1}{4}e^{4x} + c\right) = \frac{5}{4}\, e^{4x} + c$

(c) $\int 8\, sec^2(5x)dx = 8\int sec^2(5x)dx = 8.\frac{1}{5}\tan 5x + c$

(d) $\int \frac{dx}{\sqrt{2x+5}} = \int (2x+5)^{-\frac{1}{2}} dx = \frac{1}{2}\left[\frac{(2x+5)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c =$

$(2x+5)^{\frac{1}{2}} + c$

(e) $\int \frac{dx}{5+4x^2} = \frac{1}{4}\int \frac{dx}{\left(\frac{5}{4}\right)+x^2} = \frac{1}{4}\int \frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2+x^2} = \frac{1}{4}\left(\frac{2}{\sqrt{5}}\right) tan^{-1}\left(\frac{2\,x}{\sqrt{5}}\right) + c$

Notice that $\int \frac{1}{a^2+u^2} du = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + c$

• The indefinite Integral of a Fraction of the form $\frac{f'(x)}{f(x)}$

THEOREM.1.5 $$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Provided $f(x) \neq 0$

EX.1.7

- $\int \frac{sec^2 x}{\tan x} dx = \int \frac{(\tan x)'}{\tan x} dx = \ln|\tan x| + c$
- $\int \frac{2\,x}{x^2+1} dx = \int \frac{(x^2+1)'}{x^2+1} dx = \ln(x^2+1) + c$

Where we can remove the absolute value signs since $(x^2+1) > 0$ for all x.

- $\int \frac{3\,x}{4\,x^2-3} dx = \frac{3}{8}\int \frac{8\,x}{4\,x^2-3} dx = \frac{3}{8}\int \frac{(4x^2-3)'}{4x^2-3} dx = \frac{3}{8}\ln|4x^2-3| + c$
- $\int \frac{dx}{x\ln x} = \frac{3}{8}\int \frac{\frac{1}{x}}{\ln x} dx = \int \frac{(\ln x)'}{\ln x} dx = \ln|\ln x| + c$
- $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(\cos x)'}{\cos x} dx = -\,ln|\cos x| + c$

$$= ln|\sec x| + c$$

- Evaluate $\int \sec x \, dx$

Notice that

$$\sec x = \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x} = \frac{sec^2 x + \sec x \tan x}{(\sec x + \tan x)}$$

But $$(\sec x)' = \sec x \tan x$$

$$(\tan x)' = sec^2 x$$

Thus $\int \sec x \, dx = \int \frac{(\sec x + \tan x)'}{\sec x + \tan x} dx = \ln|\tan x + \sec x| + c$

THE END