Set Theory and Mathematical Logic

Math 251

Wada Altiary

Chapter 1

Sets and basic operations on sets

Introduction: Sets and Elements

Sets

A set is a collection of well defined objects, called the elements of set.

Notation:

- We use capital letters A,B,C,...... to denote sets
- We use small letters a, b, c, ... to denote elements of sets
- If a is an element of a set A,we write $a \in A$
- If a is not an element of a set A,we write $a \notin A$

Method of representing sets:

List form:

ذكر جميع العناصر

In this method a set is represented by listing all elements, separating these by commas and enclosing these in curly bracket

Example

If V be the set of vowels of English alphabet, it can be written in Roster form as :

$V = \{a, e, i, o, u\}$

(ii) If A be the set of natural numbers less than 7.

then $A=\{1, 2, 3, 4, 5, 6\}$, is in the Roster form.

Set-builder form

ذكر الخاصية المميزة

In this form elements are represented by some common property.

Example

Let V be the set of vowels of English alphabet then V can be written in the set builder form as:

$V = \{x : x \text{ is a vowel of English alphabet}\}$

(ii) Let A be the set of natural numbers less than 7.

then $A = \{x : x \in N \text{ and } 1 \leq x < 7\}$

Example Write the following in set-builder form

(a) $A = \{-3, -2, -1, 0, 1, 2, 3\}$

(b) $B = \{3, 6, 9, 12\}$

Solution

$A = \{x : x \in \mathbb{Z} \text{ and } -3 \leqslant x \leqslant 3\}$

$B = \{x : x = 3n, n \in \mathbb{N} \text{ and } n \leqslant 4\}$

Example Write the following in list form

(a) $A = \{x : x \in \mathbb{N} \text{ and } 50 \leqslant x \leqslant 60\}$

(b) $B = \{x : x \in \mathbb{R} \text{ and } x^2 - 5x + 6 = 0\}$

(c) $C = \{x : x \in P, x \text{ is even and } x < 15\}$

(d) $D = \{x : x \text{ is a multiple of } 5\}$

Solution

(a) $A = \{50, 51, 52, 53, 54, 55, 56, 57, 85, 95, 60\}$

(b)

$x^2 - 5x + 6 = 0$

$(x - 3)(x - 2) = 0$

$\Rightarrow x = 3$ or $x = 2$

$\therefore B = \{2, 3\}$

(c) $\{2, 4, 6, 8, 10, 12, 14\}$

(d) $\{0, 5, 10, 15, 20, \text{----}\}$

Classification of Sets
أنواع المجموعات

1. Finite and infinite set

Finite set: A set containing finite number of elements.

The number of elements in a given finite set is called

Cardinal Number of a finite set denoted by d(n) or |n|
عدد عناصر المجموعة المنتهية

Infinite Set A set containing infinite number of elements

Example

If A is the set of odd integer less than 10. Then

$A = \{1, 3, 5, 9\} \longrightarrow$ Finite d(n)=4

If B is the set of natural number. Then

$B = \{1, 2, 3, 4, \ldots\} \longrightarrow$ Infinite

2. Empty set

A set containing no elements, it is denoted by 0 or { }

Example

Wich of the following set

$A = \{ x : x \text{ is irrational and } x^2+1 = 0 \}$

$B = \{ x : x \in \mathbb{Z} \text{ and } -2 \leq x \leq 2 \}$ are empty.

Solution:

Set A:

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

$\therefore A = \{\emptyset\}$

Set B.

$B = \{-2, -1, 0, 1, 2\}$

not empty

3. Equal Sets:

Tow sets are equal if they both have the same elements. and we write A=B

Note that

الترتيب غير مهم

The order is irrelevant. يتم تجاهل التكرار

Any repetition of an element is ignored.

Notes

Example

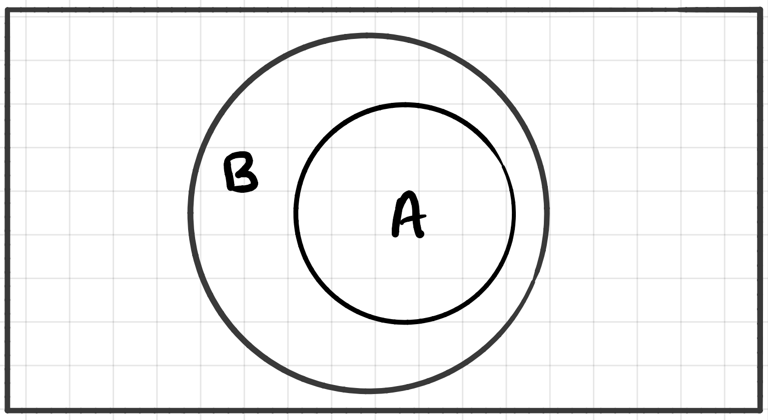
$\{2,3,5,7\} = \{3,2,7,5\}$ since a set is *unordered*.

Also, $\{2,3,5,7\} = \{2,2,3,3,5,7\}$ since a set contains *unique* elements.

However, $\{2,3,5,7\} \neq \{2,3\}$.

4. subset

Let A and B be two sets. If every element of A is an element of B then A is called subset of B.



We say that A is contained in B or B contains A. $A \subseteq B$ **Or** $B \supseteq A$

Example

A={2,4,6}

B={6,4,8,2}

$A \subseteq B$

كل عنصر في A موجود في B

5. proper subset

If A is a subset of B and $A \neq B$ then A is called proper subset of B and we write $A \subset B$

Notes

في المثال أعلاه جاز مكان

كتابة AcB لأن كل عنصر في

A موجود في B و المجموعتين

A و B غير متساويتين.

Example

If $A = \{x : x$ is a prime number less than $5\}$ and

$B = \{y : y$ is an even prime number$\}$

then is B a proper subset of A ?

yes since $A = \{2, 3\}$, $B = \{2\}$

$\therefore B \subset A$ and $B \neq A$

Let $A = \{2, 4, 6\}$

$B = \{x : x$ is an even natural number less than $8\}$

$A \subseteq B$ and $B \subseteq A$. Hence $A = B$.

Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$. Then

$A \not\subset B$ and $B \not\subset A$.

Remarks

- $A \subseteq A$ for every set A. Every set is a subset of itself.
- The empty set is a subset of every set: $\emptyset \subseteq A$ for any set A.
- If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*
- If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

6. power set

power set of a set A is the set of all subset of the given set and it is denoted by P(A).

هي مجموعة كل المجموعات الجزئية

Example If $A = \{1, 2\}$ then all the subset of A will be

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

$$n(P(A)) = 4.$$

Remarks Number of elements of p (A) = 2^n → The number of elements in set A.

عدد عناصر المجموعة A

Example

write the power set of each of the following sets:

1. $A = \{x : x \in R \text{ and } x^2 + 7 = 0\}$
2. $B = \{y : y \in N \text{ and } 1 \leq y \leq 3\}$

Solution:

1. $x^2 + 7 = 0 \Rightarrow x = \pm\sqrt{-7} \notin R.$

$\therefore A = \emptyset$

$\therefore P(A) = \{\emptyset\}.$

2. $B = \{1, 2, 3\}$

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2,3\}, \{1, 2, 3\}\}$$

7. Universal set

A set which contains all the elements of other given sets is Called a universal set.

Example If

$A = \{2, 4, 5, 6\}$

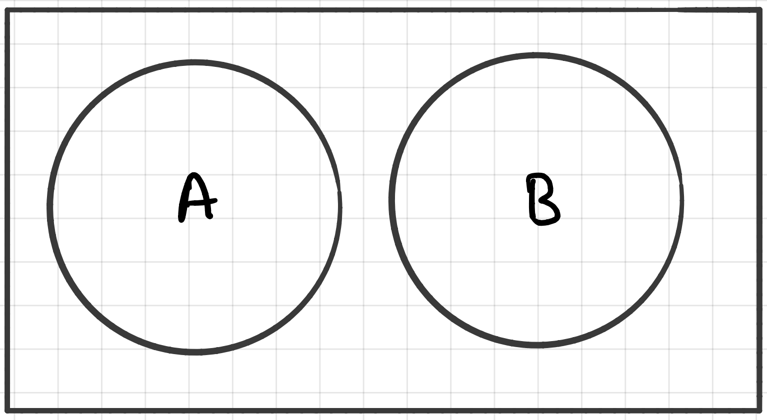
$B = \{1, 3, 7\}$

Then $U = \{1, 2, 3, 4, 5, 6, 7\}$

المجموعة التي تشمل على
جميع عناصر المجموعات
المعطاه(في مسألة معينه)
تسمى المجموعة الشاملة

8. Disjoint sets:

Two sets A and B are said to be disjoint if there is no element in common.



we call this shap veen diagram

Example

$A = \{2, 4, 5, 6\}$

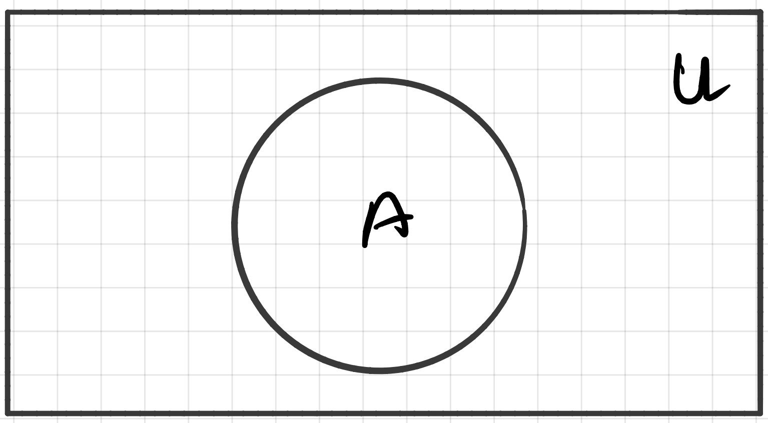
$B = \{1, 3, 7, 8\}$

A and B are disjoint sets

نقول عن مجموعتين انها مفصولتين اذا كان
لايوجد عناصر مشتركة بينهم

Venn Diagram

In Venn diagram, the universal set is represented by rectangular region and a set is represented by circle or a closed geometrical figure inside the universal set.



المجموعة الشاملة تمثل
بمستطيل و المجموعات تمثل
بواسطة دائرة او شكل هندسي
مغلق داخل المجموعة الشاملة

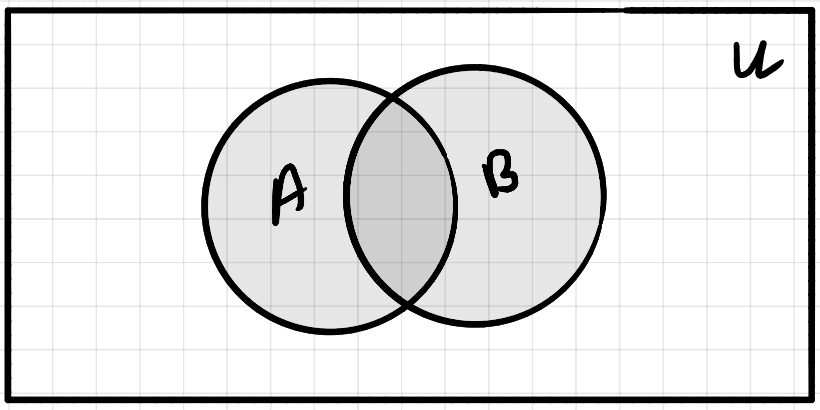
Operation on Sets

1. Union of sets

$A \cup B$ denotes the set of all elements which belong to A or to B:

$$A \cup B = \{\, x \;:\; x \in A \textbf{ or } x \in B \,\}.$$

$A \cup B$ is called the *union* of A and B.*



Example

$$A = \{2, 4, 5, 6\}$$

$$B = \{4, 6, 7, 8\}$$

$$A \cup B = \{2, 4, 5, 6, 7, 8\}$$

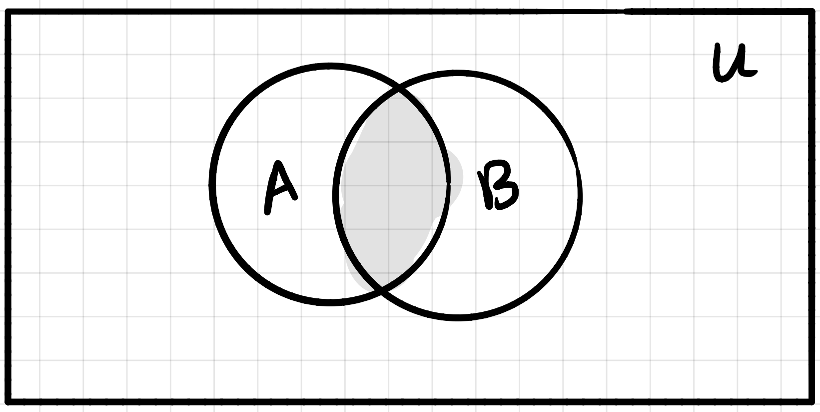
جميع عناصر المجموعتين
A و B من دون تكرار

2. Intersection of sets

Suppose A and B are sets. Then $A \cap B$ denotes the set of all elements which belong to both A and B:

$$A \cap B = \{\, x \;:\; x \in A \text{ and } x \in B \,\}.$$

$A \cap B$ is called the *intersection* of A and B.*



Example

$$A = \{2, 4, 5, 6\}$$

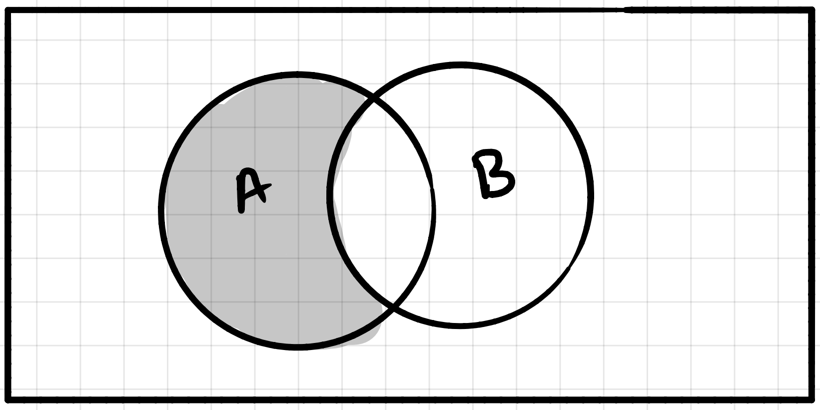
$$B = \{4, 6, 7, 8\}$$

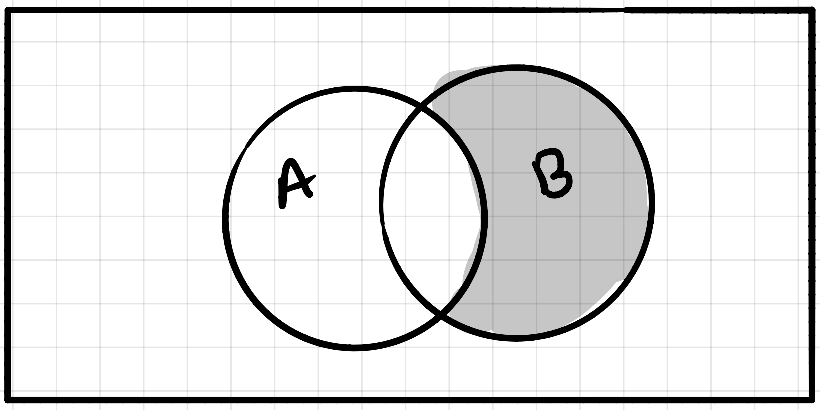
$$A \cap B = \{4, 6\}$$

3. Difference of sets

The *difference* of A and B is the set of elements of A that do **not** belong to B:

$$
\begin{aligned}
A \setminus B &= \{x \mid (x \in A) \wedge (x \notin B)\} \\
&= \{x \mid (x \in A) \wedge (x \in B^c)\} \text{ by definition of complement} \\
&= A \cap B^c \text{ by definition of intersection.}
\end{aligned}
$$





Example

$A = \{2, 4, 5, 6\}$

$B = \{4, 6, 7, 8\}$

$A \backslash B = \{2, 5\}$

Example

$A = \{2, 4, 5, 6\}$

$B = \{4, 6, 7, 8\}$

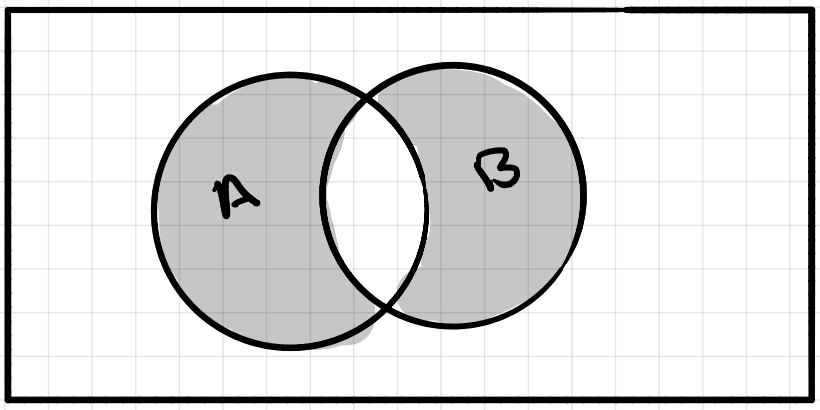
$B \backslash A = \{7, 8\}$

في الفرق نحذف
العناصر المشتركة
بين المجموعتين
ونكتب عناصر A
فقط لنحصل على
A/B او عناصر B
فقط لنحصل على
B/A

4. Symmetric difference

The *symmetric difference* of A and B is the set of elements that belong to A or B **but not both**:

$$
\begin{aligned}
A \triangle B &= \{x \mid (x \in A \ \vee \ x \in B) \wedge (x \notin (A \cap B))\} = (A \cup B) \setminus (A \cap B) \\
&= (A \cup B) \cap (A \cap B)^c \text{ by above result on difference.}
\end{aligned}
$$



Example

$A = \{2, \cancel{4}, 5, \cancel{6}\}$

$B = \{\cancel{4}, \cancel{6}, 7, 8\}$

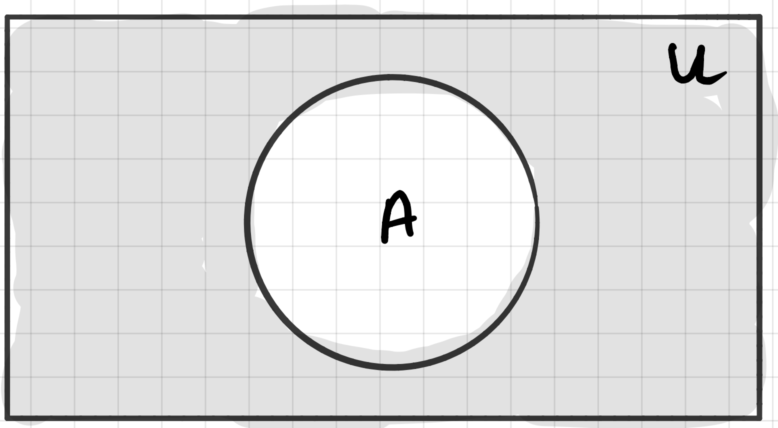
$A \Delta B = \{2, 5, 7, 8\}$

في الفرق المتكافئ نحذف العناصر المشتركة
بين المجموعتين ونكتب العناصر المتبقية من
المجموعتين A و B لنحصل على $A \Delta B$

Complement of sets

If U is a universal set and A is a subset of U, then the set of all elements in U that are not in A is called the complement of A and is denoted by A' or A^c

$$A' = \{x \; : \; x \in U \text{ and } x \notin A\}.$$



Example

$$U = \{1, 2, 3, 4, 5, 6, 7\}$$

$$A = \{2, 4, 5, 6\}$$

$$A^c = \{1, 3, 7\}$$

Example Let U be the universal set

$$U = \{x : x \in N \text{ and } x \leq 10\}$$

A={y: y is a prime number less than 10}

Find A^c

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{2, 3, 5, 7\}$$

$$A^c = U - A = \{1, 4, 6, 8, 9, 10\}$$

Notes

ممكن نكتب :

$A^c = U - A$

Law of Algebra of sets

$$\left.\begin{aligned} A \cap B &= B \cap A \\ A \cup B &= B \cup A \end{aligned}\right\} \quad \text{commutative laws} \qquad (1)$$

$$\left.\begin{aligned} A \cap A &= A \\ A \cup A &= A \end{aligned}\right\} \quad \text{idempotent laws} \qquad (2)$$

$$\left.\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C \\ A \cup (B \cup C) &= (A \cup B) \cup C \end{aligned}\right\} \quad \text{associative laws} \qquad (3)$$

$$\left.\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}\right\} \quad \text{distributive laws} \qquad (4)$$

$$\left.\begin{aligned} A \cap (A \cup B) &= A \\ A \cup (A \cap B) &= A \end{aligned}\right\} \quad \text{absorbtion laws} \qquad (5)$$

identity laws:

$$\begin{matrix} A \cap U = A & \qquad & A \cup U = U \\ A \cup \emptyset = A & \qquad & A \cap \emptyset = \emptyset \end{matrix} \qquad (6)$$

complement laws:

$$\begin{matrix} (A')' = A & \qquad & A \cap A' = \emptyset & \qquad & U' = \emptyset \\ & & A \cup A' = U & & \emptyset' = U \end{matrix} \qquad (7)$$

$$\left.\begin{aligned} (A \cap B)' &= A' \cup B' \\ (A \cup B)' &= A' \cap B' \end{aligned}\right\} \quad \text{De Morgan's laws} \qquad (8)$$

Results on Number of elements in sets

Theorem: Suppose that A and B are finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If $A \cap B = \phi$

$$n(A \cup B) = n(A) + n(B)$$

$A \cap B \neq \phi$

Example

$A = \{2, 4, 5, 6\}$ $n(A) = 4$

$B = \{4, 6, 7, 8\}$ $n(B) = 4$

$A \cap B = \{4, 6\}$, $n(A \cap B) = 2$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$4 + 4 - 2 = 6$

Check

$A \cup B = \{2, 4, 5, 6, 7, 8\}$, $n(A \cup B) = 6$

$A \cap B = \phi$

Example

$A = \{2, 4, 5, 6\}$ $n(A) = 4$

$B = \{1, 3, 7, 8\}$ $n(B) = 4$

$n(A \cup B) = n(A) + n(B)$

$= 4 + 4 = 8$

Check

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$n(A \cup B) = 8$

Remark

$$n(U) = n(A) + n(A^c)$$

We call U disjoint union since

$A \cap A^c = \emptyset$

$A \cup A^c = U$

Example

$U = \{1, 2, 3, 4, 5, 6, 7\}$

$A = \{2, 4, 5, 6\}$

$A^c = \{1, 3, 7\}$

$n(U) = n(A) + n(A^c)$

$= 4 + 3 = 7$

Let A be any set in finite universal set U. Then

$$n(A^c) = n(U) - n(A)$$

$$n(A \setminus B) = n(A) - n(A \cap B)$$

Example

$A = \{2, 4, 5, 6\}$ $\quad n(A) = 4$

$B = \{4, 6, 7, 8\}$

$A \cap B = \{4, 6\}, n(A \cap B) = 2$

$$n(A \setminus B) = n(A) - n(A \cap B)$$
$$= 4 - 2 = 2$$

Check

$A \setminus B = \{2, 5\}$, $\quad n(A \setminus B) = 2$

Can we say A is the disjoint union of $A \setminus B$ and $A \cap B$?

Notes

Example

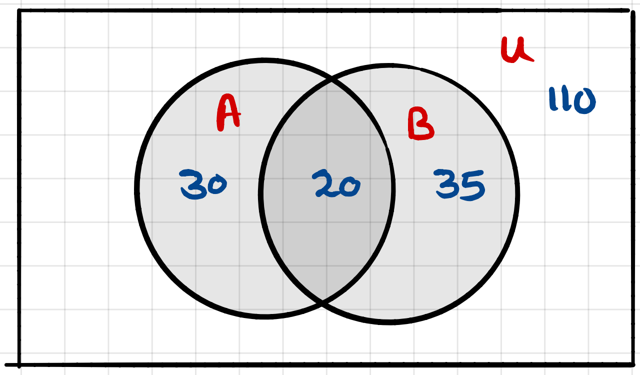
Consider the following data among 110 students in a college

30 students are on list A (taking Accounting)
35 Students are on a list B (taking biology)
20 Students are on both lists.

Find the number of students

(a) on list A or B

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 30 + 35 - 20 = 45$$



(b) on exactly one of the two lists

$$n(A | B) = n(A) - n(A \cap B)$$
$$= 30 - 20 = 10$$

$$n(B | A) = n(B) - n(A \cap B)$$
$$= 35 - 20 = 15$$

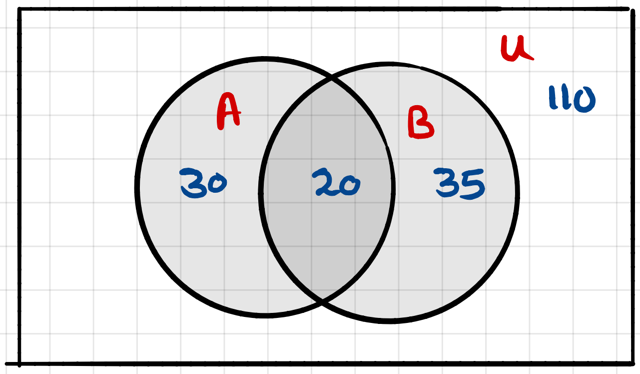
Thus $10 + 15 = 25$ students

(c) The students on nither the A list nor B list

$n(A \cup B)^c = n(U) - n(A \cup B)$

$= 110 - 45$

$= 65$



Example

Let $n(U) = 70$, $n(A) = 30$, $n(B) = 45$ and $n(A \cap B) = 10$.

Find the following

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 30 + 45 - 10 = 65$

(b) $n(A^c) = n(U) - n(A)$

$= 70 - 30 = 40$

(c) $n(A \Delta B) = n(A \cup B) - n(A \cap B)$

$= 65 - 10 = 55$

Mathematical Induction

Definition: let A(n) be a subset of the positive integer p
that is A(n) is true or false for each integer
such that

1) -The statement is true for n = I

2) -Assume the statement is true for n = K.

3)-We prove that the statement is true for n = K+1

Example

Show that $A(n): 1+3+5+\cdots+(2n-1) = n^2$

Solution:

1) A(n) is true for n=1 since $1 = 1^2$ then

2) Assume that A(n) is true for n=K $\forall K \in N$.

$$1+3+5+\cdots+(2K-1) = K^2$$

3) We show that A(n) is true for n=K+1

L.H.S R.H.S

$$1+3+5+\cdots+(2K-1)+(2(K+1)-1) = (K+1)^2$$

$$L.H.S = 1+3+5+\cdots+(2K-1)+(2K+1)$$

$$= K + 2K + 1$$

$$= (K+1)^2 = R.H.S$$