King Saud University
College of Sciences
Department of Mathematics

M-106
INTEGRAL CALCULUS

CLASS NOTES
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Contents

ANTIDERIVATIVES

Definition (Antiderivative): A function G is called an antiderivative of the function f on the interval I if $G'(x) = f(x)$ for all $x \in I$.

Example : What is the antiderivative of the function $f(x) = 2x$?
Answer : The antiderivative is $G(x) = x^2 + c$, where c is a constant.

Note: If $G_1(x)$ and $G_2(x)$ are both antiderivatives of the function $f(x)$ then $G_1(x) - G_2(x) = constant$.

Definition (indefinite integral): If $G(x)$ is the antiderivative of $f(x)$ then $\int f(x)\,dx = G(x) + c$, $\int f(x)\,dx$ is called the indefinite integral of the function $f(x)$.

Basic Rules of integration :

1. $\int 1\,dx = x + c$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$, $n \in \mathbb{Q}$
3. $\int cosx\,dx = sinx + c$
4. $\int sinx\,dx = -cosx + c$
5. $\int sec^2x\,dx = tanx + c$
6. $\int csc^2x\,dx = -cotx + c$
7. $\int secx\ tanx\,dx = secx + c$
8. $\int cscx\ cotx\,dx = -cscx + c$

Properties of indefinite integral :

1. $\int a\ f(x)\,dx = a \int f(x)\,dx$, where $a \in \mathbb{R}$
2. $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Notes: If $G(x)$ is the antiderivative of the function $f(x)$ then

1. $\int f(x)\,dx = G(x) + c$

   $\int \frac{d}{dx} G(x)\,dx = G(x) + c$

2. $\frac{d}{dx} \int f(x)\,dx = f(x)$

Example (1): Solve $\int \left(\frac{3}{x^4} - 5x \right) dx$

Answer: $\int \left(\frac{3}{x^4} - 5x \right) dx = \int \left(3x^{-4} - 5x \right) dx = \int 3x^{-4}\,dx - \int 5x\,dx$

$= 3 \int x^{-4}\,dx - 5 \int x\,dx = 3\,\frac{x^{-3}}{-3} - 5\,\frac{x^2}{2} + c$

Example (2): Solve $\int \frac{2x^2+3}{\sqrt{x}}\,dx$

Answer: $\int \frac{2x^2+3}{\sqrt{x}}\,dx = \int \frac{2x^2+3}{x^{\frac{1}{2}}}\,dx$

$= \int x^{\frac{-1}{2}} \left(2x^2 + 3 \right) dx = \int \left(2x^{\frac{1}{2}} + 3x^{\frac{-1}{2}} \right) dx$

$= 2 \int x^{\frac{3}{2}}\,dx + 3 \int x^{\frac{-1}{2}}\,dx = 2\,\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

CHANGE OF VARIABLE

Example: Solve $\displaystyle\int (4x+1)^2\ dx$

Put $u = 4x+1$ then $du = 4\ dx$, hence $\frac{1}{4}\ du = dx$

$$\int (4x+1)^2\ dx = \int u^2\ \frac{1}{4}\ du = \frac{1}{4}\ \int u^2\ du = \frac{1}{4}\ \frac{u^3}{3} + c = \frac{1}{4}\ \frac{(4x+1)^3}{3} + c$$

Or we can use the form $\displaystyle\int [f(x)]^n\, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$, $(n \in \mathbb{Q}\ ,\ n \neq -1)$

$$\int (4x+1)^2\ dx = \frac{1}{4} \int (4x+1)^2\ \ 4\, dx = \frac{1}{4}\ \frac{(4x+1)^3}{3} + c$$

Where $f(x) = 4x+1$, $n = 2$ and $f'(x) = 4$.

Basic Rules :

1. $\displaystyle\int [f(x)]^n\, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$, $(n \in \mathbb{Q}\ ,\ n \neq -1)$
2. $\displaystyle\int \sin(f(x))\, f'(x)\, dx = -\cos(f(x)) + c$
3. $\displaystyle\int \cos(f(x))\, f'(x)\, dx = \sin(f(x)) + c$
4. $\displaystyle\int \sec^2(f(x))\, f'(x)\, dx = \tan(f(x)) + c$
5. $\displaystyle\int \csc^2(f(x))\, f'(x)\, dx = -\cot(f(x)) + c$
6. $\displaystyle\int \sec(f(x))\tan(f(x))\, f'(x)\, dx = \sec(f(x)) + c$
7. $\displaystyle\int \csc(f(x))\cot(f(x))\, f'(x)\, dx = -\csc(f(x)) + c$

Examples :

1. $\displaystyle\int \cos(3x+4)\ dx = \frac{1}{3}\int \cos(3x+4)\ \ 3\, dx = \frac{1}{3}\sin(3x+4) + c$
2. $\displaystyle\int \left(1+\frac{5}{x}\right)^3\ \frac{1}{x^2}\, dx = \frac{-1}{5}\int \left(1+\frac{5}{x}\right)^3\ \frac{-5}{x^2}\, dx = \frac{-1}{5}\,\frac{\left(1+\frac{5}{x}\right)^4}{4} + c$
3. $\displaystyle\int \sqrt{9-x^2}\ x\, dx = \frac{-1}{2}\int \left(9-x^2\right)^{\frac{1}{2}}\ \ (-2x)\, dx = \frac{-1}{2}\ \frac{\left(9-x^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + c$
4. $\displaystyle\int \frac{1}{\sqrt{x}\ \left(1+\sqrt{x}\right)^3}\, dx = 2\int \left(1+\sqrt{x}\right)^{-3}\ \frac{1}{2\sqrt{x}}\, dx = 2\ \frac{\left(1+\sqrt{x}\right)^{-2}}{-2} + c$
5. $\displaystyle\int \tan^2 x\ \ \sec^2 x\ \ dx = \int (\tan x)^2\ \ \sec^2 x\, dx = \frac{(\tan x)^3}{3} + c$

6. $\displaystyle\int \frac{1}{\cos^3 x \ \csc x}\, dx = \int (\cos x)^{-3} \ \ \sin x \, dx = -\int (\cos x)^{-3} (-\sin x) \ dx$

$\displaystyle = -\frac{(\cos x)^{-2}}{-2} + c$

7. $\displaystyle\int \frac{\sin(1+\sqrt{x})}{\sqrt{x}}\, dx = 2 \int \sin\left(1+\sqrt{x}\right) \ \frac{1}{2\sqrt{x}}\, dx = -2 \ \cos\left(1+\sqrt{x}\right) + c$

8. $\displaystyle\int \frac{\cos(\sqrt[3]{x})}{\sqrt[3]{x^2}}\, dx = 3 \int \cos\left(x^{\frac{1}{3}}\right) \ \frac{1}{3} x^{\frac{-2}{3}} \ dx = 3 \sin\left(x^{\frac{1}{3}}\right) + c$

9. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x} \ \sin^2\sqrt{x}}\, dx = 2 \int \left(\sin\sqrt{x}\right)^{-2} \ \ \cos\left(\sqrt{x}\right) \ \frac{1}{2\sqrt{x}}\, dx$

$\displaystyle = 2 \ \frac{(\sin\sqrt{x})^{-1}}{-1} + c$

Another Solution : $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x} \ \sin^2\sqrt{x}}\, dx = \int \frac{1}{\sin\sqrt{x}} \ \frac{\cos\sqrt{x}}{\sin\sqrt{x}} \ \frac{1}{\sqrt{x}}\, dx$

$\displaystyle = 2 \int \csc\sqrt{x} \cot\sqrt{x} \ \frac{1}{2\sqrt{x}}\, dx = -2\csc\sqrt{x} + c$

10. Find the value of k that satisfies $\displaystyle\int \sqrt{2x+3}\, dx = k\,(2x+3)^{\frac{3}{2}} + c$

$\displaystyle \frac{d}{dx}\left[k \ \ (2x+3)^{\frac{3}{2}} + c\right] = \sqrt{2x+3}$

$\displaystyle \frac{3}{2} k \ \ (2x+3)^{\frac{1}{2}} \ \ 2 = (2x+3)^{\frac{1}{2}}$

$3k = 1$, and hence $\displaystyle k = \frac{1}{3}$

SUMS AND SIGMA NOTATION

If $a_1, a_2, \cdots, a_n \in \mathbb{R}$ then $\displaystyle\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

Theorem : If $c, a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$ then

1. $\displaystyle\sum_{i=1}^{n} c = cn$.
2. $\displaystyle\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$.
3. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$.
4. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
5. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
6. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Examples :

1. $\displaystyle\sum_{k=1}^{4} \left(k^3 - k + 2\right) = \sum_{k=1}^{4} k^3 - \sum_{k=1}^{4} k + \sum_{k=1}^{4} 2$
   $\displaystyle= \left(\frac{4(4+1)}{2}\right)^2 - \frac{4(4+1)}{2} + 2(4) = 100 - 10 + 8 = 98.$
2. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5k}{n^2} = \lim_{n\to\infty} \frac{5}{n^2} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{5}{n^2}\ \frac{n(n+1)}{2} = \frac{5}{2}$.
3. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}\,(i-1)^2 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} \left(i^2 - 2i + 1\right)$
   $\displaystyle= \lim_{n\to\infty} \frac{1}{n^3} \left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right] = \lim_{n\to\infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$
   $\displaystyle\lim_{n\to\infty} \left[\frac{n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^3} + \frac{n}{n^3}\right] = \frac{2}{6} - 0 + 0 = \frac{1}{3}$

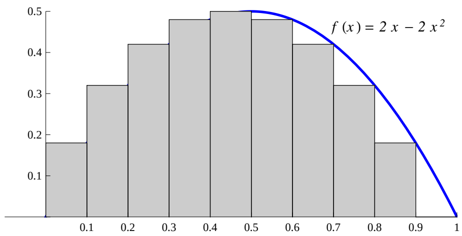
RIEMANN SUM

In this section we assume that the function $f(x) \geq 0$ on the interval $[a, b]$.

Definition (Regular Partition) : The set $\{x_0 = a, x_1, \cdots, x_n = b\}$ is called a regular partition of the interval $[a, b]$ if $x_i = x_0 + i\ \Delta x$ for every $i = 1, 2, \cdots, n$, and $\Delta x = \dfrac{b-a}{n}$.
This regular partition divides the interval $[a, b]$ into n subintervals of the form $[x_{i-1}, x_i]$ where $i = 1, 2, \cdots, n$

Area under the graph of a function :
If $f(x) \geq 0$ on the interval $[a, b]$ and $\{x_0 = a, x_1, \cdots, x_n = b\}$ is a regular partition of $[a, b]$, then the area under the graph of $f(x)$ can be approximated by n rectangles using the formula $A_n = \displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x$

Example : Approximate the area under the graph of $f(x) = 2x - 2x^2$ on the interval $[0, 1]$ using 10 rectangles .
Answer : $\Delta x = \dfrac{1-0}{10} = 0.1.$
$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $\cdots$, $x_9 = 0.9$, $x_{10} = 1$
$A_{10} = \displaystyle\sum_{i=1}^{10} f(x_i)\,\Delta x = \sum_{i=1}^{10} \left(2x_i - 2x_i^2\right)\ 0.1$
$A_{10} = 0.1\,[0.18 + 0.32 + 0.42 + 0.48 + 0.5 + 0.48 + 0.42 + 0.32 + 0.18 + 0]$
$A_{10} = 0.1(3.3) = 0.33$



Definition (Riemann Sum) :
Let $\{x_0 = a, x_1, \cdots, x_n = b\}$ be a regular partition of the interval $[a, b]$ with $\Delta x = \dfrac{b-a}{n}$. Pick points $c_1, c_2, \cdots, c_n$ where c_i is any point in the subintrval $[x_{i-1}, x_i]$, $i = 1, 2, \cdots, n$.
The Riemann sum is $R_n = \sum\limits_{i=1}^{n} f(c_i)\,\Delta x$.
The area under the curve of $f(x)$ is the limit of the Riemann sum .
$A = \lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(c_i)\,\Delta x$.

Example 1: Find the area under the curve of the function $f(x) = 3x+1$ on the interval $[1, 3]$ using Riemann sum and c_i is the middle point of the subinterval.
Answer: $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$
$x_0 = 1,\ x_i = x_0 + i\Delta x = 1 + \dfrac{2i}{n}$ for every $i = 1, 2, \cdots, n$.
For every $i = 1, 2, \cdots, n$, $c_i \in [x_{i-1}, x_i]$, $c_i = \dfrac{x_i + x_{i-1}}{2} = \dfrac{\left(1 + \frac{2i}{n}\right) + \left(1 + \frac{2(i-1)}{n}\right)}{2}$
$c_i = \dfrac{2 + (2i-1)\frac{2}{n}}{2} = 1 + \dfrac{2i-1}{n}$.

$$R_n = \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{n} \left[3\left(1 + \frac{2i-1}{n}\right) + 1\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{6i-3}{n} + 1\right] = \frac{2}{n}\sum_{i=1}^{n}\left[4 + \frac{6i}{n} - \frac{3}{n}\right]$$
$$= \frac{2}{n}\left[\sum_{i=1}^{n} 4 + \frac{6}{n}\sum_{i=1}^{n} i - \frac{1}{n}\sum_{i=1}^{n} 3\right] = \frac{2}{n}\left[4n + \frac{6}{n}\frac{n(n+1)}{2} - \frac{1}{n}3n\right]$$
$$= 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n} .$$
The desired area $= \lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty}\left[8 + 6\dfrac{n(n+1)}{n^2} - \dfrac{6}{n}\right] = 8 + 6 - 0 = 14$

Example 2: Do the last example where c_i is the end point of the subinterval.
Answer: For every $i = 1, 2, \cdots, n$, $c_i \in [x_{i-1}, x_i]$, $c_i = x_i = 1 + \frac{2i}{n}$

$$R_n = \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{n}\left[3\left(1 + \frac{2i}{n}\right) + 1\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{6i}{n} + 1\right] = \frac{2}{n}\sum_{i=1}^{n}\left[4 + \frac{6i}{n}\right]$$
$$= \frac{2}{n}\left[\sum_{i=1}^{n} 4 + \frac{6}{n}\sum_{i=1}^{n} i\right] = \frac{2}{n}\left[4n + \frac{6}{n}\frac{n(n+1)}{2}\right] = 8 + 6\frac{n(n+1)}{n^2}$$
The desired area $= \lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty}\left[8 + 6\dfrac{n(n+1)}{n^2}\right] = 8 + 6 = 14$

THE DEFINITE INTEGRAL

Definition (The definite Integral) : For any continuous function f defined on the interval $[a, b]$ the definite integral of f from a to b is
$\int_a^b f(x)\ dx = \underset{n\to\infty}{lim}\ R_n = \underset{n\to\infty}{lim} \sum_{i=1}^{n} f(c_i)\,\Delta x$, whenever the limit exists.
(where c_i is any point in the subintrval $[x_{i-1}, x_i]$, $i = 1, 2, \cdots, n$).

Notes :

1. Rieman Sum is the same for any choice of the points $c_1, c_2, \cdots, c_n$.
2. When the limit exists we say that the function f is integrable.

Notes : If the function f is continuous on $[a, b]$ and $f(x) \geq 0$ for every $x \in [a, b]$, then

1. $\int_a^b f(x)\ dx \geq 0$.
2. $\int_a^b f(x)\ dx =$ The area under the curve of f

Example 1:
$\int_1^3 (3x + 1)\ dx =$ Area under the curve of $f = \underset{n\to\infty}{lim}\ R_n = 14$.
(See the example on Riemann sum) .

Example 2: The definite integral representing $\underset{n\to\infty}{lim} \sum_{k=1}^{n} \sqrt{x_k + 1}\ \Delta x$ using regular partition of the interval $[1, 2]$ is $\int_1^2 \sqrt{x + 1}\ dx$.

Theorem: If the function f is continuous on the interval $[a, b]$ then f is integrable on $[a, b]$.

Properties of the definite integral : If the functions f and g are integrable on $[a, b]$ then :

1. $\int_a^b k\ f(x)\ dx = k \int_a^b f(x)\ dx$,   for every $k \in \mathbb{R}$.
2. $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$.
3. For every $c \in [a, b]$ $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$.

4. If $f(x) \leq g(x)$ for every $x \in [a, b]$ then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

Example 3:

1. $\int_2^7 (x^2 - 3)\ dx - \int_2^4 (x^2 - 3)\ dx = \int_4^7 (x^2 - 3)\ dx$.

2. Since $\cos x \geq \sin x$ for every $x \in \left[0, \frac{\pi}{4}\right]$ then $\int_0^{\frac{\pi}{4}} \cos x\ dx \geq \int_0^{\frac{\pi}{4}} \sin x\ dx$.

3. To show that $\int_{-1}^{1} \frac{x^2}{x^2+4}\ dx \leq \int_{-1}^{1} x^2\ dx$

   For every $x \in [-1, 1]$, $x^2 + 4 > 1 \Rightarrow \frac{1}{x^2+4} < 1 \Rightarrow \frac{x^2}{x^2+4} \leq x^2$

   Hence $\int_{-1}^{1} \frac{x^2}{x^2+4}\ dx \leq \int_{-1}^{1} x^2\ dx$.

FUNDAMENTAL THEOREM OF CALCULUS

Fundamental Theorem of Calculus (Part I) :
If f is a continuous function on the interval $[a, b]$ and $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$ then $\int_a^b f(x)\ dx = [G(x)]_a^b = G(b) - G(a)$.

Note : $\int_a^b \frac{d}{dx}G(x)\ dx = G(b) - G(a)$.

Examples :

1. $\int_0^2 \left(x^2 - 2x\right)\ dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - \left(\frac{0}{3} - 0\right) = -\frac{4}{3}$.
2. Find the area under the graph of $f(x) = \sin x$ on $[0, \pi]$
   Answer : The area $= \int_0^\pi \sin x\ dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = 2$

Fundamental Theorem of Calculus (Part II) :
If f is a continuous function on the interval $[a, b]$ and $G(x) = \int_a^x f(t)\ dt$ for every $x \in [a, b]$ then $G'(x) = f(x)$ for every $x \in [a, b]$

Note : $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$.

Examples :

1. $\frac{d}{dx}\int_0^x \sqrt{t^2+1}\ dt = \sqrt{x^2+1}$.
2. $\frac{d}{dx}\int_1^x \frac{1}{t^2+1}\ dt = \frac{1}{x^2+1}$.
3. $\frac{d}{dx}\int_3^x \left(2 + \frac{d}{dt}\cos t\right)\ dt = \frac{d}{dx}\int_3^x (2 - \sin t)\ dt = 2 - \sin x$
4. $\frac{d}{dt}\int_2^t \frac{1}{x^3+5}\ dx = \frac{1}{t^3+5}$

Theorem :
If f is acontinuous function , g and h are deifferentiable functions then
$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = f\left(h(x)\right)h'(x) - f\left(g(x)\right)g'(x)$.

Notes :

1. If $g(x) = a$ and $h(x) = b$ then $\dfrac{d}{dx}\displaystyle\int_a^b f(t)\ dt = f\left(b\right)\left(0\right) - f\left(a\right)\left(0\right) = 0$

2. If $g(x) = a$ and $h(x) = x$ then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\ dt = f\left(x\right)\left(1\right) - f\left(a\right)\left(0\right) = f(x)$

Examples :

1. Find $G'(x)$, if $G(x) = \displaystyle\int_{1-x}^{x^2} \frac{1}{4+3t^2}\ dt$.

   Answer : $G'(x) = \dfrac{d}{dx}\displaystyle\int_{1-x}^{x^2} \frac{1}{4+3t^2}\ dt = \frac{1}{4+3\left(x^2\right)^2}(2x) - \frac{1}{4+3\left(1-x\right)^2}(-1)$

   $G'(x) = \dfrac{2x}{4+3x^4} + \dfrac{1}{4+3\left(1-x\right)^2}$

2. $\dfrac{d}{dt}\left[\displaystyle\int_2^t \sqrt{x^2+1}\ dx + \int_t^{-1} \sqrt{x^2+1}\ dx\right] = \dfrac{d}{dt}\displaystyle\int_2^{-1} \sqrt{x^2+1}\ dx = 0$

3. Find $F'(2)$, if $F(x) = \displaystyle\int_1^{x^2} \frac{1}{t}\ dt$.

   Answer : $F'(x) = \dfrac{d}{dx}\displaystyle\int_1^{x^2} \frac{1}{t}\ dt = \frac{1}{x^2}(2x) - 0 = \frac{2x}{x^2} = \frac{2}{x}$.

   Hence $F'(2) = \frac{2}{2} = 1$.

4. Find $f(4)$, if $\displaystyle\int_0^x f(t)\ dt = x\cos\pi x$

   Answer : Differentiate both sides with respect to x

   $\dfrac{d}{dx}\displaystyle\int_0^x f(t)\ dt = \frac{d}{dx}\left[x\cos\pi x\right]$

   $f(x) = (1)\cos\pi x + x\left(-\sin\pi x\right)\pi = \cos\pi x - \pi x\sin\pi x$

   Hence $f(4) = \cos 4\pi - 4\pi\sin 4\pi = 1 - 4\pi(0) = 1$.

5. $\displaystyle\int_{-x}^{x} \frac{d}{dt} f(t)\ dt = f(x) - f(-x)$

   Here, we used $\displaystyle\int_a^b \frac{d}{dx} G(x)\ dx = G(b) - G(a)$

Exercises: Solve the following :

1. $\dfrac{d}{dx}\displaystyle\int_0^5 \sqrt{t^2+3}\ dt$.

2. $\dfrac{d}{dx}\displaystyle\int_x^1 u^2\cos u\ du$.

3. Find $F'(0)$, if $F(x) = \displaystyle\int_x^{x^2} \frac{1}{t-1}\ dt$.

AVERAGE VALUE OF A FUNCTION

Definition (Average value of a function) : Let f be a continuous function on $[a,b]$ then the average value of f on $[a,b]$ is $f_{av} = \dfrac{\int_a^b f(x)\ dx}{b-a}$.

Example : Find f_{av} of the following functions :

1. $f(x) = x^2 - 2x$ on the interval $[1,4]$
   $$\int_1^4 \left(x^2-2x\right)\ dx = \left[\frac{x^3}{3} - x^2\right]_1^4$$
   $$= \left(\frac{64}{3} - 16\right) - \left(\frac{1}{3} - 1\right) = \frac{63}{3} - 15 = \frac{63-45}{3} = \frac{18}{3} = 6$$
   Hence $f_{av} = \dfrac{\int_1^4 \left(x^2-2x\right)\ dx}{4-1} = \dfrac{6}{3} = 2$.

2. $f(x) = \sin^2 x \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$
   $$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\ dx = \int_0^{\frac{\pi}{2}} (\sin x)^2 \cos x\ dx = \left[\frac{(\sin x)^3}{3}\right]_0^{\frac{\pi}{2}}$$
   $$= \frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{(\sin 0)^3}{3} = \frac{1}{3} - 0 = \frac{1}{3}$$
   Hence $f_{av} = \dfrac{\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\ dx}{\frac{\pi}{2} - 0} = \dfrac{\frac{1}{3}}{\frac{\pi}{2}} = \dfrac{2}{3\pi}$.

Exercise : Find f_{av} of the function $f(x) = (2x+1)^2$ on the interval $[0,1]$

INTEGRAL MEAN VALUE THEOREM

Theorem (Integral Mean Value Theorem) :
If f is a contiuous function on the interval $[a, b]$ then there exists a number $c \in (a, b)$ for which $f(c) = \dfrac{\displaystyle\int_a^b f(x)\ dx}{b-a}$.

Example : Find the value that satisfies the integral Mean value theorem for the function $f(x) = 4x^3 - 1$ on the interval $[1, 2]$

Answer : $f(c) = \dfrac{\displaystyle\int_1^2 \left(4x^3 - 1\right)\ dx}{2-1}$

$4c^3 - 1 = \left[x^4 - x\right]_1^2$

$4c^3 - 1 = (16 - 2) - (1 - 1)$

$4c^3 - 1 = 14$

$c^3 = \dfrac{15}{4}$

$c = \sqrt[3]{\frac{15}{4}}$

Note that $c = \sqrt[3]{\frac{15}{4}} \in (1, 2)$.

NUMERICAL INTEGRATION

1. The Trapezoidal Rule :

It is used to approximate $\int_a^b f(x)\ dx$ with a regular partition of the interval $[a, b]$, where $\Delta x = \dfrac{b-a}{n}$, by using the formula

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}\left[f\left(x_0\right)+2f\left(x_1\right)+\cdots+2f\left(x_{n-1}\right)+f\left(x_n\right)\right]$$

Example : Approximate the integral $\int_0^1 \sqrt{x+x^2}\ dx$ using Trapezoidal rule with $n = 4$.

Answer : $[a, b] = [0, 1]$, $f(x) = \sqrt{x+x^2}$ and $\Delta x = \frac{1-0}{4} = 0.25$

n	x_n	$f\left(x_n\right)$	m	$mf\left(x_n\right)$
0	0	0	1	0
1	0.25	0.559017	2	1.11803
2	0.5	0.86625	2	1.73205
3	0.75	1.14564	2	2.29129
4	1	1.41421	1	1.41421
				6.55559

$$\int_0^1 \sqrt{x+x^2}\ dx \approx \frac{1-0}{2(4)}\left[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)\right]$$

$$\int_0^1 \sqrt{x+x^2}\ dx \approx \frac{1}{8}\left[6.55559\right] \approx 0.819448\ .$$

Exercise : Approximate the integral $\int_2^4 \dfrac{1}{x-1}\ dx$ using Trapezoidal rule with $n = 4$.

2. Simpson's Rule :

It is used to approximate $\int_a^b f(x)\ dx$ with a regular partition of the interval $[a, b]$, where $\Delta x = \frac{b-a}{n}$, and n is <u>even</u> , by using the formula

$$\begin{aligned}\int_a^b f(x)\ dx \quad &\approx \quad \frac{b-a}{3n}\ [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\ &\quad + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\end{aligned}$$

Example : Approximate the integral $\int_0^{10} \sqrt{10x - x^2}\ dx$ using Simpson's rule with $n = 4$.

Answer : $[a, b] = [0, 10]$, $f(x) = \sqrt{10x - x^2}$ and $\Delta x = \frac{10-0}{4} = 2.5$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	2.5	4.33013	4	17.3204
2	5	5	2	10
3	7.5	4.33013	4	17.3204
4	10	0	1	0
				44.6408

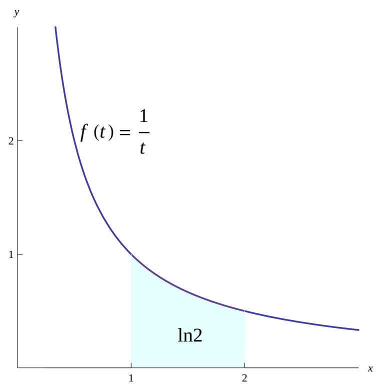
$$\int_0^{10} \sqrt{10x - x^2}\ dx \approx \frac{10-0}{3(4)}\,[f(0) + 4f(2.5) + 2f(5) + 4f(7.5) + f(10)]$$

$$\int_0^{1} \sqrt{10x - x^2}\ dx \approx \frac{10}{12}\,[44.6408] \approx 37.2007\ .$$

Exercise : Approximate the integral $\int_0^2 \frac{x}{x+1}\ dx$ using Simpson's rule with $n = 4$.

THE NATURAL LOGARITHMIC FUNCTION

Definition (The natural logarithmic function) :
For $x > 0$, the natural logarithmic function is defined by $\ln x = \displaystyle\int_1^x \frac{1}{t}\, dt$.



Note : The domain of the function $\ln x$ is the open interval $(0, \infty)$

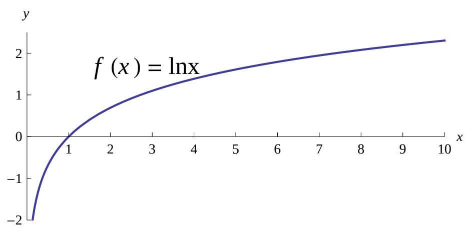
Example : What is the domain of the function $\ln(x - 2)$?
Answer : $x - 2 > 0 \Rightarrow x > 2 \Rightarrow$ the domain is $(2, \infty)$.

Notes :

1. If $x > 1$ then $\ln x > 0$.
2. $ln1 = 0$.
3. If $0 < x < 1$ then $\ln x < 0$.

The graph of $\ln x$:

1. First derivative test :
$\dfrac{d}{dx} \ln x = \dfrac{d}{dx} \displaystyle\int_1^x \frac{1}{t}\, dt = \frac{1}{x} > 0$ for every $x \in (o, \infty)$.
Hence $\ln x$ is an increasing function on $(0, \infty)$.
2. Second derivative test :
$\dfrac{d^2}{dx^2} \ln x = \dfrac{d}{dx} \dfrac{1}{x} = -\dfrac{1}{x^2} < 0$ for every $x \in (0, \infty)$.
Hence $\ln x$ is a convex function on $(0, \infty)$.



Notes :

1. The range of the function $\ln x$ is $\mathbb{R}$.
2. $\underset{x\to\infty}{lim} \ln x = \infty$.
3. $\underset{x\to 0^+}{lim} \ln x = -\infty$.

The derivative of $\ln|x|$:

1. $\dfrac{d}{dx} \ln|x| = \dfrac{1}{x}$.
2. $\dfrac{d}{dx} \ln|f(x)| = \dfrac{f'(x)}{f(x)}$.

Note : $\ln|x|$ is the antiderivative of $\dfrac{1}{x}$.

Integration :

1. $\displaystyle\int \frac{1}{x}\, dx = \ln|x| + c$.
2. $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$.

Some properties of $\ln|x|$: If $x, y > 0$ and $r \in \mathbb{R}$ then

1. $\ln(xy) = \ln x + \ln y$.
2. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$.
3. $\ln x^r = r\ln x$.

Examples :

1. Simplify $\dfrac{1}{5}\left[2\ln|x+1| + \ln|x| - \ln|x^2-2|\right]$
$$\frac{1}{5}\left[2\ln|x+1| + \ln|x| - \ln|x^2-2|\right] = \frac{1}{5}\left[\ln(x+1)^2 + \ln|x| - \ln|x^2-2|\right]$$
$$= \frac{1}{5}\left[\ln|x(x+1)^2| - \ln|x^2-2|\right] = \frac{1}{5}\ln\left|\frac{x(x+1)^2}{x^2-2}\right| = \ln\left|\left(\frac{x(x+1)^2}{x^2-2}\right)^{\frac{1}{5}}\right|$$

2. If $y = \sqrt{\dfrac{(x+1)^4(x+2)^3}{(x-1)^2}}$ then find y' .
$$\ln y = \ln\left|\sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}\right| = \frac{1}{2}\left[4\ln|x+1| + 3\ln|x+2| - 2\ln|x-1|\right]$$

Differentiate both sides

$\dfrac{y'}{y} = \dfrac{1}{2}\left[4\dfrac{1}{x+1} + 3\dfrac{1}{x+2} - 2\dfrac{1}{x-1}\right]$

Hence $y' = \dfrac{1}{2}\sqrt{\dfrac{(x+1)^4(x+2)^3}{(x-1)^2}}\left[\dfrac{4}{x+1} + \dfrac{3}{x+2} - \dfrac{2}{x-1}\right]$

Exercise : If $f(x) = \dfrac{x^2(2x-1)^3}{(x+5)^2}$ then find $f'(x)$?

More Basic Rules of Integration :

1. $\displaystyle\int \tan x\, dx = \ln|\sec x| + c$.
2. $\displaystyle\int \cot x\, dx = \ln|\sin x| + c$.
3. $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + c$.
4. $\displaystyle\int \csc x\, dx = \ln|\csc x - \cot x| + c$

Examples :

1. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x}\, dx = \frac{1}{3}\int \frac{3x^2+6x+9}{x^3+3x^2+9x}\, dx = \frac{1}{3}\ln\left|x^3+3x^2+9x\right| + c$.
2. $\displaystyle\int \frac{x^2+2x+3}{\left(x^3+3x^2+9x\right)^5}\, dx = \frac{1}{3}\int \left(x^3+3x^2+9x\right)^{-5}\left(3x^2+6x+9\right)\, dx$

   $\displaystyle = \frac{1}{3}\,\frac{\left(x^3+3x^2+9x\right)^{-4}}{-4} + c$.
3. $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\, dx = \int (\ln x)^{-\frac{1}{2}}\,\frac{1}{x}\, dx = \frac{(\ln x)^{\frac{1}{2}}}{\frac{1}{2}} + c$.
4. $\displaystyle\int \frac{1}{x\ln\sqrt{x}}\, dx = \int \frac{1}{x\frac{1}{2}\ln x}\, dx = 2\int \frac{\frac{1}{x}}{\ln x}\, dx = \ln|\ln x| + c$.
5. $\displaystyle\int \frac{x-1}{x+1}\, dx = \int \frac{(x+1)-2}{x+1}\, dx = \int \left(\frac{x+1}{x+1} - \frac{2}{x+1}\right) dx$

   $\displaystyle\int \left(1 - \frac{2}{x+1}\right) dx = \int 1\, dx - 2\int \frac{1}{x+1}\, dx = x - 2\ln|x+1| + c$.
6. Find $g(x)$ if $\displaystyle\int [\ln|x|]^2\, g(x)\, dx = \frac{2}{3}[\ln|x|]^3 + c$

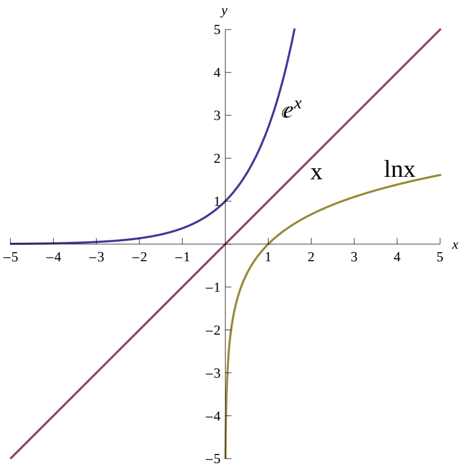
   $\displaystyle [\ln|x|]^2\, g(x) = \frac{d}{dx}\left(\frac{2}{3}[\ln|x|]^3 + c\right)$

   $\displaystyle [\ln|x|]^2\, g(x) = 2[\ln|x|]^2\,\frac{1}{x}$.

   Hence $g(x) = \dfrac{2}{x}$.

THE NATURAL EXPONENTIAL FUNCTION

Definition (The natural exponential function) :
The natural exponential function is the inverse of the natural logarithmic function , and it is denoted by e^x .



Notes :

1. The domain of the function e^x is $\mathbb{R}$.
2. The range of the function e^x is the open interval $(0, \infty)$.
3. $e^x > 0$ for every $x \in \mathbb{R}$.
4. $e^0 = 1$.
5. $e \approx 2.71828$ and $\ln(e) = 1$.
6. $\underset{x\to\infty}{lim} e^x = \infty$.
7. $\underset{x\to-\infty}{lim} e^x = 0$.
8. $\ln(e^x) = x$ and $e^{\ln x} = x$.

Some properties of the natural exponential function : If $x, y \in \mathbb{R}$ then

1. $e^x e^y = e^{x+y}$.
2. $\frac{e^x}{e^y} = e^{x-y}$.
3. $(e^x)^y = e^{xy}$.

Examples :

1. Find the value of x that satisfies the equation $\ln\dfrac{1}{x}=2$?
   Answer : $\ln\dfrac{1}{x}=2 \Rightarrow \ln x^{-1}=2 \Rightarrow -\ln x=2 \Rightarrow \ln x=-2$
   $\Rightarrow e^{\ln x}=e^{-2} \Rightarrow x=e^{-2}=\dfrac{1}{e^2}$.
2. Find the value of x that satisfies the equation $e^{5x+3}=4$?.
   Answer : $e^{5x+3}=4 \Rightarrow \ln e^{5x+3}=\ln 4 \Rightarrow 5x+3=\ln 4 \Rightarrow x=\dfrac{-3+\ln 4}{5}$.
3. Simplify $\ln\left(e^x\right)^2$?
   Answer : $\ln\left(e^x\right)^2=\ln\left(e^{2x}\right)=2x$.

Derivative of the natural exponential function:

1. $\dfrac{d}{dx}e^x=e^x$.
2. $\dfrac{d}{dx}e^{f(x)}=e^{f(x)}f'(x)$.

Integration :

1. $\displaystyle\int e^x\,dx=e^x+c$.
2. $\displaystyle\int e^{f(x)}f'(x)\,dx=e^{f(x)}+c$.

Example :

1. Find $f'(x)$ if $f(x)=e^{5x}+\dfrac{1}{e^x}$
   $f(x)=e^{5x}+\dfrac{1}{e^x}=e^{5x}+e^{-x}$
   $f'(x)=e^{5x}(5)+e^{-x}(-1)=5e^{5x}-e^{-x}$.
2. $\displaystyle\int\frac{e^{-x}}{\left(1-e^{-x}\right)^2}\,dx=\int\left(1-e^{-x}\right)^{-2}e^{-x}\,dx=\frac{\left(1-e^{-x}\right)^{-1}}{-1}+c$.
3. $\displaystyle\int\frac{e^{\frac{3}{x}}}{x^2}\,dx=-\frac{1}{3}\int e^{\frac{3}{x}}\frac{-3}{x^2}\,dx=-\frac{1}{3}e^{\frac{3}{x}}+c$.
4. $\displaystyle\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx=2\int e^{\sqrt{x}}\frac{1}{2\sqrt{x}}\,dx=2e^{\sqrt{x}}+c$.
5. $\displaystyle\int\frac{e^{\sin x}}{\sec x}\,dx=\int e^{\sin x}\cos x\,dx=e^{\sin x}+c$.
6. $\displaystyle\int_1^e\frac{\sqrt[3]{\ln x}}{x}\,dx=\int_1^e(\ln x)^{\frac{1}{3}}\,\frac{1}{x}\,dx=\left[\frac{(\ln x)^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^e=\frac{3}{4}(\ln e)^{\frac{4}{3}}-\frac{3}{4}(\ln 1)^{\frac{4}{3}}=\frac{3}{4}$

7. Find $g(x)$ if $\int e^{3x^2} g(x)\,dx = -e^{3x^2} + c$

   $\frac{d}{dx}\left[-e^{3x^2} + c\right] = e^{3x^2} g(x)$

   $-e^{3x^2}(6x) = e^{3x^2} g(x)$

   $-6xe^{3x^2} = e^{3x^2} g(x)$

   Hence $g(x) = -6x$

8. $\int e^{(x^2+\ln x)}\,dx = \int e^{x^2} e^{\ln x}\,dx = \int e^{x^2} x\,dx = \frac{1}{2}\int e^{x^2} 2x\,dx = \frac{1}{2}e^{x^2} + c$

THE GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Definition (The general exponential function) :
It has the form a^x where $a > 0$ and $a \neq 1$.

Note : $a^x = e^{x \ln a}$.

Derivative of the general exponential function :

1. $\dfrac{d}{dx} a^x = a^x \ln a.$
2. $\dfrac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a$.

Integration :

1. $\displaystyle\int a^x \, dx = \frac{a^x}{\ln a} + c.$
2. $\displaystyle\int a^{f(x)} f'(x) \, dx = \frac{a^{f(x)}}{\ln a} + c$.

Definition (The general logarithmic function) :
The general logarithmic function of base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x .

Notes :

1. $\log_a x = y \Leftrightarrow a^y = x$.
2. $\log_a x = \dfrac{\ln x}{\ln a}$.

Notations :

1. $\log x = \log_{10} x$.
2. $\ln x = \log_e x$.

Derivative of the general logarithmic function :

1. $\dfrac{d}{dx} \log_a |x| = \dfrac{1}{x} \dfrac{1}{\ln a}.$
2. $\dfrac{d}{dx} \log_a |f(x)| = \dfrac{f'(x)}{f(x)} \dfrac{1}{\ln a}$.

Examples :

1. Find the value of x if $\log_2 x = 3$?.
   $\log_2 x = 3 \Leftrightarrow x = 2^3 = 8$.

2. Find the value of a if $\log_a 125 = 3$?
   $\log_a 125 = 3 \Leftrightarrow 125 = a^3 \Leftrightarrow a = \sqrt[3]{125} = 5$.

3. Find the value of x if $2\log|x| = \log 2 + \log|3x-4|$? .
   $2\log|x| = \log 2 + \log|3x-4| \Rightarrow \log x^2 = \log|2(3x-4)|$
   $\Rightarrow x^2 = 2(3x-4) \Rightarrow x^2 = 6x-8 \Rightarrow x^2-6x+8=0$
   $(x-4)(x-2) = 0 \Rightarrow x = 4 or x = 2$.

4. Find y' if $2x = 4^y$?
   Differentiate both sides : $2 = 4^y y' \ln 4 \Rightarrow y' = \dfrac{2}{4^y \ln 4} = \dfrac{2}{2x\ln 4} = \dfrac{1}{x\ln 4}$.
   Another way : $2x = 4^y \Rightarrow \ln|2x| = \ln 4^y = y\ln 4 \Rightarrow y = \dfrac{\ln|2x|}{\ln 4}$
   Hence $y' = \dfrac{1}{\ln 4}\dfrac{2}{2x} = \dfrac{1}{x\ln 4}$.

5. Find $f'(x)$ if $f(x) = 7^{\sqrt[3]{x}}$?.
   $f'(x) = 7^{\sqrt[3]{x}}\dfrac{1}{3}x^{\frac{-2}{3}}\ln 7$.

6. Find $f'(x)$ if $f(x) = \pi^{3x}$?
   $f'(x) = \pi^{3x}(3)\ln\pi = 3\pi^{3x}\ln\pi$.

7. Find y' if $y = (\sin x)^x$?
   $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x\ln|\sin x|$
   Differentiate both sides : $\dfrac{y'}{y} = \ln|\sin x| + x\dfrac{\cos x}{\sin x} = \ln|\sin x| + x\cot x$
   $y' = y\left[\ln|\sin x| + x\cot x\right] = (\sin x)^x\left[\ln|\sin x| + x\cot x\right]$

8. Find y' if $y = \left(1+x^2\right)^{2x+1}$?
   $y = \left(1+x^2\right)^{2x+1} \Rightarrow \ln y = \ln\left(1+x^2\right)^{2x+1} = (2x+1)\ln(1+x^2)$
   Differentiate both sides : $\dfrac{y'}{y} = 2\ln(1+x^2) + (2x+1)\dfrac{2x}{1+x^2}$
   $y' = y\left[2\ln(1+x^2) + \dfrac{2x(2x+1)}{1+x^2}\right] = \left(1+x^2\right)^{2x+1}\left[2\ln(1+x^2) + \dfrac{2x(2x+1)}{1+x^2}\right]$

9. $\displaystyle\int x^2 6^{x^3}\,dx = \frac{1}{3\ln 6}\int 6^{x^3}(3x^2)\ln 6\,dx = \frac{6^{x^3}}{3\ln 6} + c$.

10. $\displaystyle\int \frac{2^x}{2^x+1}\,dx = \frac{1}{\ln 2}\int \frac{2^x\ln 2}{2^x+1}\,dx = \frac{\ln(2^x+1)}{\ln 2} + c$.

11. $\displaystyle\int \frac{3^{-\cot x}}{\sin^2 x}\,dx = \frac{1}{\ln 3}\int 3^{-\cot x}\csc^2 x\ln 3\,dx = \frac{3^{-\cot x}}{\ln 3} + c$

12. $\displaystyle\int 2^{x\ln x}(1+\ln|x|)\,dx = \frac{1}{\ln 2}\int 2^{x\ln x}(1+\ln|x|)\ln 2\,dx = \frac{2^{x\ln x}}{\ln 2} + c$

13. $\displaystyle\int 4^x 5^{4^x}\,dx = \frac{1}{\ln 4\ \ln 5}\int 5^{4^x}4^x\ln 4\ \ln 5\,dx = \frac{5^{4^x}}{\ln 4\ \ln 5} + c$

14. $\displaystyle\int 3^x\left(1+\sin 3^x\right)\,dx=\int\left(3^x+3^x\sin 3^x\right)\,dx=\int 3^x\,dx+\int 3^x\sin 3^x\,dx$

$$=\frac{1}{\ln 3}\int 3^x\ln 3\,dx+\frac{1}{\ln 3}\int \sin(3^x)3^x\ln 3\,dx=\frac{3^x}{\ln 3}-\frac{\cos 3^x}{\ln 3}+c$$

Exercises :

1. Find $f'(x)$ if $f(x)=\left(x^2+1\right)^x$?
2. Evaluate $\displaystyle\int \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx$?

THE INVERSE TRIGONOMETRIC FUNCTIONS

Definitions :

1. The inverse sine function is denoted by $\sin^{-1}$ and it is defined as $y = \sin^{-1} x \Leftrightarrow x = \sin y$, where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

   The domain of the inverse sine function is $[-1, 1]$

   The range of the inverse sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2. The inverse cosine function is denoted by $\cos^{-1}$ and it is defined as $y = \cos^{-1} x \Leftrightarrow x = \cos y$, where $x \in [-1, 1]$ and $y \in [0, \pi]$.

   The domain of the inverse cosine function is $[-1, 1]$

   The range of the inverse cosine function is $[0, \pi]$.

3. The inverse tangent function is denoted by $\tan^{-1}$ and it is defined as $y = \tan^{-1} x \Leftrightarrow x = \tan y$, where $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

   The domain of the inverse tangent function is $\mathbb{R}$

   The range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

4. The inverse cotangent function is denoted by $\cot^{-1}$ and it is defined as $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, where $x \in \mathbb{R}$.

   The domain of the inverse cotangent function is $\mathbb{R}$

   The range of the inverse cotangent function is $(0, \pi)$.

5. The inverse secant function is denoted by $\sec^{-1}$ and it is defined as $y = \sec^{-1} x \Leftrightarrow x = \sec y$, where $y \in \left[0, \frac{\pi}{2}\right)$ if $x \geq 1$,and $y \in \left[\pi, \frac{3\pi}{2}\right)$ if $x \leq -1$.

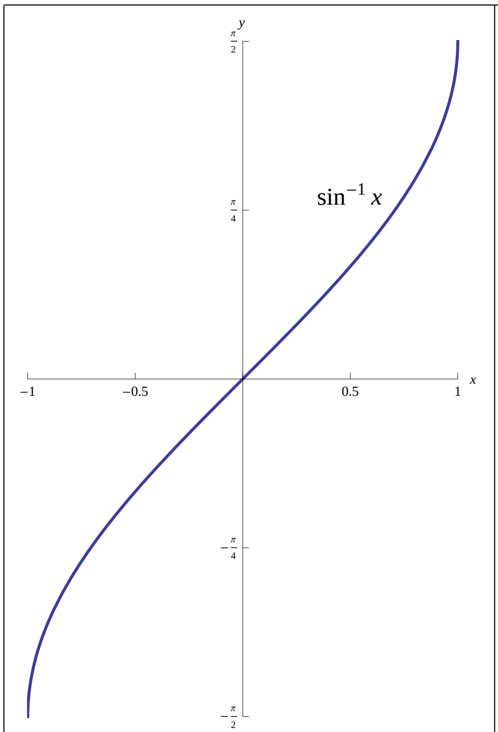
   The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

   The range of the inverse secant function is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

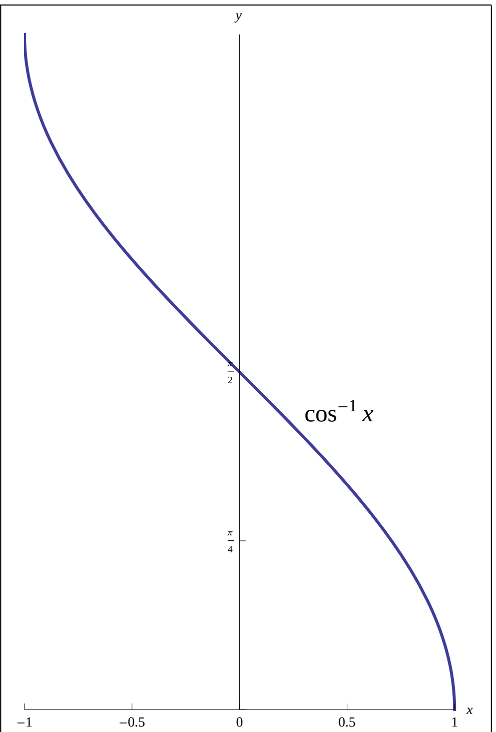
6. The inverse cosecant function is denoted by $\csc^{-1}$ and it is defined as $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$ where $|x| \geq 1$

   The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$

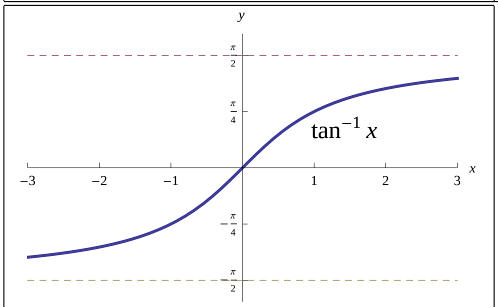
   The range of the inverse cosecant function is $\left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$.



Graph of $\sin^{-1} x$



Graph of $\cos^{-1} x$



Graph of $\tan^{-1} x$

Derivatives of the inverse trigonometric functions :

1. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.

2. $\dfrac{d}{dx}\cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$.

3. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$.

4. $\dfrac{d}{dx}\cot^{-1} x = \dfrac{-1}{1+x^2}$.

5. $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$, where $|x| > 1$.

6. $\dfrac{d}{dx}\csc^{-1} x = \dfrac{-1}{x\sqrt{x^2-1}}$, where $|x| > 1$.

Integration :

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c \;\;,\;\; (|x| < a)$

$\displaystyle\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\, dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c \;\;,\;\; (|f(x)| < a))$

2. $\displaystyle\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \;\;,\;\; (|x| > a)$

$\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right) + c \;\;,\;\; (|f(x)| > a))$

Examples :

1. $\displaystyle\int \frac{x^2}{5+x^6}\, dx = \frac{1}{3}\int \frac{3x^2}{\left(\sqrt{5}\right)^2+\left(x^3\right)^2}\, dx = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x^3}{\sqrt{5}}\right) + c\ .$

Here $a=\sqrt{5}$, $f(x) = x^3$ and $f'(x) = 3x^2$.

2. $\displaystyle\int \frac{3x}{\sqrt{9-x^4}}\, dx = \frac{3}{2}\int \frac{2x}{\sqrt{(3)^2-(x^2)^2}}\, dx = \frac{3}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c\ .$

Here $a=3$, $f(x) = x^2$ and $f'(x) = 2x$.

3. $\displaystyle\int \frac{3x}{\sqrt{9-x^2}}\, dx = \frac{3}{-2}\int \left(9-x^2\right)^{-\frac{1}{2}}(-2x)\, dx = -\frac{3}{2}\frac{\left(9-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c.$

4. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\, dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{(1)^2-(\ln x)^2}}\, dx = \sin^{-1}(\ln x) + c\ .$

Here $a=1$, $f(x) = \ln x$ and $f'(x) = \dfrac{1}{x}$.

5. $\displaystyle\int \frac{1}{1+3x^2}\, dx = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{(1)^2+\left(\sqrt{3}x\right)^2}\, dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}x\right) + c\ .$

Here $a=1$, $f(x) = \sqrt{3}x$ and $f'(x) = \sqrt{3}$.

6. $\displaystyle\int \frac{e^{2x}}{e^{4x}+16}\, dx = \frac{1}{2}\int \frac{2e^{2x}}{(4)^2+\left(e^{2x}\right)^2}\, dx = \frac{1}{2}\frac{1}{4}\tan^{-1}\left(\frac{e^{2x}}{4}\right) + c\ .$

Here $a = 4$, $f(x) = e^{2x}$ and $f'(x) = 2e^{2x}$.

7. $$\int \frac{1}{\sqrt{e^{2x} - 36}}\, dx = \int \frac{e^x}{e^x\sqrt{\left(e^x\right)^2 - (6)^2}}\, dx = \frac{1}{6}\sec^{-1}\left(\frac{e^x}{6}\right) + c\ .$$

Here $a = 6$, $f(x) = e^2$ and $f'(x) = e^x$.

8. $$\int \frac{\sin x}{\sqrt{25 - \cos^2 x}}\, dx = -\int \frac{-\sin x}{\sqrt{(5)^2 - \left(\cos x\right)^2}}\, dx = -\sin^{-1}\left(\frac{\cos x}{5}\right) + c\ .$$

Here $a = 5$, $f(x) = \cos x$ and $f'(x) = -\sin x$.

9. $$\int \frac{2^x}{\sqrt{4 - 4^x}}\, dx = \frac{1}{\ln 2}\int \frac{2^x \ln 2}{\sqrt{(2)^2 - \left(2^x\right)^2}}\, dx = \frac{1}{\ln 2}\sin^{-1}\left(\frac{2^x}{2}\right) + c\ .$$

Here $a = 2$, $f(x) = 2^x$ and $f'(x) = 2^x \ln 2$.

10. $$\int \frac{1}{x^2 + 6x + 25}\, dx = \int \frac{1}{(x^2 + 6x + 9) + 16}\, dx = \int \frac{1}{(x + 3)^2 + (4)^2}\, dx$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x + 3}{4}\right) + c\ .$$

Here $a = 4$, $f(x) = x + 3$ and $f'(x) = 1$.

11. $$\int \frac{x + 2}{\sqrt{4 - x^2}}\, dx = \int \left(\frac{x}{\sqrt{4 - x^2}} + \frac{2}{\sqrt{4 - x^2}}\right) dx$$

$$= \frac{1}{-2}\int \left(4 - x^2\right)^{-\frac{1}{2}} (-2x)\, dx + 2\int \frac{1}{\sqrt{(2)^2 - (x)^2}}\, dx$$

$$= -\frac{1}{2}\frac{\left(4 - x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + 2\sin^{-1}\left(\frac{x}{2}\right) + c\ .$$

12. $$\int \frac{x + \tan^{-1} x}{1 + x^2}\, dx = \int \left(\frac{x}{1 + x^2} + \frac{\tan^{-1} x}{1 + x^2}\right) dx$$

$$= \frac{1}{2}\int \frac{2x}{1 + x^2}\, dx + \int \left(\tan^{-1} x\right) \frac{1}{1 + x^2}\, dx$$

$$= \frac{1}{2}\ln(1 + x^2) + \frac{\left(\tan^{-1} x\right)^2}{2} + c\ .$$

Exercises : Solve the following integrals :

1. $$\int \frac{x + \sin^{-1} x}{\sqrt{1 - x^2}}\, dx\ .$$

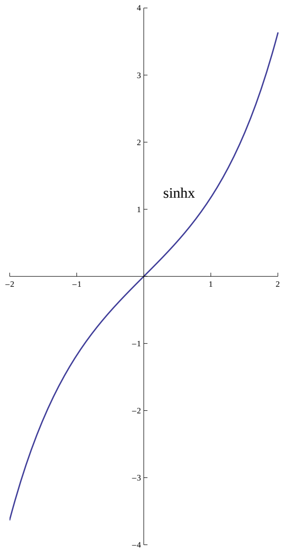
2. $$\int \frac{x + 1}{x^2 + 1}\, dx$$

HYPERBOLIC FUNCTIONS

Definition (The hyperbolic sine function):
It is denoted by $\sinh x$ and it is defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Notes :

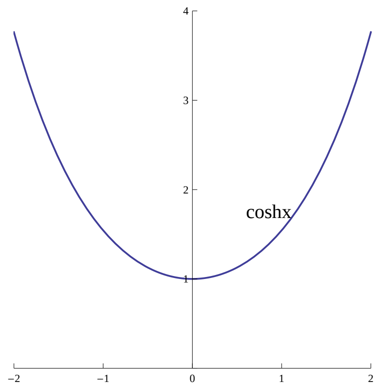
1. The domain of $\sinh x$ is $\mathbb{R}$.
2. The range of $\sinh x$ is $\mathbb{R}$.
3. It is an odd function and $\sinh(0) = 0$.
4. The graph of $\sinh x$



Definition (The hyperbolic cosine function):
It is denoted by $\cosh x$ and it is defined as $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Notes :

1. The domain of $\cosh x$ is $\mathbb{R}$.
2. The range of $\cosh x$ is $[1, \infty]$.
3. It is an even function and $\cosh(0) = 1$.
4. The graph of $\cosh x$



Definitions :

1. The hyperbolic tangent function is denoted by $\tanh x$ and it is defined as $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$.
2. The hyperbolic cotangent function is denoted by $\coth x$ and it is defined as $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$.
3. The hyperbolic secant function is denoted by $sechx$ and it is defined as $sechx = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$.
4. The hyperbolic cosecant function is denoted by $cschx$ and it is defined as $cschx = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$.

Notes :

1. $\cosh^2 x - \sinh^2 x = 1$ for every $x \in \mathbb{R}$.
2. $1 - \tanh^2 x = sech^2 x$ for every $x \in \mathbb{R}$.
3. $\coth^2 x - 1 = csch^2 x$ for every $x \in \mathbb{R} - \{0\}$.

Derivatives of the hyperbolic functions :

1. $\dfrac{d}{dx} \sinh x = \cosh x$

   $\dfrac{d}{dx} \sinh(f(x)) = \cosh(f(x))\ f'(x)$

2. $\dfrac{d}{dx} \cosh x = \sinh x$

   $\dfrac{d}{dx} \cosh(f(x)) = \sinh(f(x))\ f'(x)$

3. $\frac{d}{dx}\tanh x = sech^2 x$

   $\frac{d}{dx}\tanh(f(x)) = sech^2(f(x))\ f'(x)$

4. $\frac{d}{dx}\coth x = -csch^2 x$

   $\frac{d}{dx}\coth(f(x)) = -csch^2(f(x))\ f'(x)$

5. $\frac{d}{dx} sechx = -sechx\tanh x$

   $\frac{d}{dx} sech(f(x)) = -sech(f(x))\tanh(f(x))\ f'(x)$

6. $\frac{d}{dx} cschx = -cschx\coth x$

   $\frac{d}{dx} csch(f(x)) = -csch(f(x))\coth(f(x))\ f'(x)$

Examples :

1. Find the value of $f(0)$ if $f(x) = \ln[\cosh(3x)]$?

   $f(0) = \ln[\cosh(0)] = \ln(1) = 0$.

2. Find the value of $f'(0)$ if $f(x) = \ln|1+\sinh x|$?

   $$f'(x) = \frac{\cosh x}{1+\sinh x} \Rightarrow f'(0) = \frac{\cosh(0)}{1+\sinh(0)} = \frac{1}{1+0} = 1\ .$$

3. Find $f'(x)$ if $f(x) = e^{\sinh x}$?

   $f'(x) = e^{\sinh x}\cosh x$.

4. Find $f'(x)$ if $f(x) = sech\,(1+\sqrt{x})$?

   $$f'(x) = -sech\,(1+\sqrt{x})\tanh(1+\sqrt{x})\,\frac{1}{2\sqrt{x}}\ .$$

5. Find $f'(x)$ if $f(x) = \tan^{-1}(\sinh x)$?

   $$f'(x) = \frac{\cosh x}{1+(\sinh x)^2} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = sechx\ .$$

6. Find $f'(x)$ if $f(x) = \ln|\sinh(1-x^2)|$?

   $$f'(x) = \frac{\cosh(1-x^2)\ (-2x)}{\sinh(1-x^2)} = -2x\coth(1-x^2)\ .$$

7. Find $f'(x)$ if $f(x) = x^{\cosh x}$?

   $f(x) = x^{\cosh x} \Rightarrow \ln|f(x)| = \ln|x^{\cosh x}| = \cosh x\ln|x|$

Differentiate both sides

$$\frac{f'(x)}{f(x)} = \sinh x \ln|x| + \cosh x \ \left(\frac{1}{x}\right)$$

$$f'(x) = f(x)\left[\sinh x \ln|x| + \frac{\cosh x}{x}\right]$$

$$f'(x) = x^{\cosh x}\left[\sinh x \ln|x| + \frac{\cosh x}{x}\right].$$

Integration :

1. $\displaystyle\int \sinh x\, dx = \cosh x + c$

   $\displaystyle\int \sinh\left(f(x)\right) f'(x)\, dx = \cosh\left(f(x)\right) + c$

2. $\displaystyle\int \cosh x\, dx = \sinh x + c$

   $\displaystyle\int \cosh\left(f(x)\right) f'(x)\, dx = \sinh\left(f(x)\right) + c$

3. $\displaystyle\int sech^2 x\, dx = \tanh x + c$

   $\displaystyle\int sech^2\left(f(x)\right) f'(x)\, dx = \tanh\left(f(x)\right) + c$

4. $\displaystyle\int csch^2 x\, dx = -\coth x + c$

   $\displaystyle\int csch^2\left(f(x)\right) f'(x)\, dx = -\coth\left(f(x)\right) + c$

5. $\displaystyle\int sechx \tanh x\, dx = -sechx + c$

   $\displaystyle\int sech\left(f(x)\right) \tanh\left(f(x)\right) f'(x)\, dx = -sech\left(f(x)\right) + c$

6. $\displaystyle\int cschx \coth x\, dx = -cschx + c$

   $\displaystyle\int csch\left(f(x)\right) \coth\left(f(x)\right) f'(x)\, dx = -csch\left(f(x)\right) + c$

7. $\displaystyle\int \tanh x\, dx = \ln|\cosh x| + c$

   $\displaystyle\int \tanh\left(f(x)\right) f'(x)\, dx = \ln\left|\cosh\left(f(x)\right)\right| + c$

8. $\displaystyle\int \coth x\, dx = \ln|\sinh x| + c$

$$\int \coth(f(x))\, f'(x)\, dx = \ln|\sinh(f(x))| + c$$

Examples :

1. $\displaystyle\int x^2 \cosh x^3 \, dx = \frac{1}{3}\int \cosh x^3 (3x^2)\, dx = \frac{1}{3} \ \sinh x^3 + c$.

2. $\displaystyle\int \frac{csch\left(\frac{1}{x}\right)\coth\left(\frac{1}{x}\right)}{x^2}\, dx = \int -csch\left(\frac{1}{x}\right)\coth\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)\, dx$

   $\displaystyle= csch\left(\frac{1}{x}\right) + c$.

3. $\displaystyle\int \left(e^x - e^{-x}\right) sech^2\left(e^x + e^{-x}\right)\, dx = \tanh\left(e^x + e^{-x}\right) + c$.

4. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \ln|e^x + e^{-x}| + c$.

5. $\displaystyle\int \frac{\sinh x}{1+\sinh^2 x}\, dx = \int \frac{\sinh x}{\cosh^2 x}\, dx = \int \frac{1}{\cosh x}\frac{\sinh x}{\cosh x}\, dx$

   $\displaystyle= \int sechx \tanh x \, dx = -sechx + c$.

6. $\displaystyle\int \frac{\sinh x}{1+\cosh x}\, dx = \ln(1+\cosh x) + c$.

7. $\displaystyle\int \frac{\sinh x}{1+\cosh^2 x}\, dx = \int \frac{\sinh x}{(1)^2 + (\cosh x)^2}\, dx = \tan^{-1}(\cosh x) + c$.

8. $\displaystyle\int \frac{1}{sechx\sqrt{4-\sinh^2 x}}\, dx = \int \frac{\cosh x}{\sqrt{(2)^2 - (\sinh x)^2}}\, dx = \sin^{-1}\left(\frac{\sinh x}{2}\right) + c$

Exercises : Solve the following :

1. $\displaystyle\int \cosh 4x \, dx$

2. $\displaystyle\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\, dx$

THE INVERSE HYPERBOLIC FUNCTIONS

Definitions :

1. The inverse hyperbolic sine function is denoted by $\sinh^{-1}$ and it is defined as $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
2. The inverse hyperbolic cosine function is denoted by $\cosh^{-1}$ and it is defined as $y = \cosh^{-1} x \Leftrightarrow x = \cosh y$, where $x \in [1, \infty)$ and $y \in [0, \infty)$.
3. The inverse hyperbolic tangent function is denoted by $\tanh^{-1}$ and it is defined as $y = \tanh^{-1} x \Leftrightarrow x = \tanh y$, where $x \in [-1, 1]$ and $y \in \mathbb{R}$.
4. The inverse hyperbolic cotangent function is denoted by $\coth^{-1}$ and it is defined as $y = \coth^{-1} x \Leftrightarrow x = \coth y$, where $|x| > 1$ and $y \in \mathbb{R}$.
5. The inverse hyperbolic secant function is denoted by $sech^{-1}$ and it is defined as $y = sech^{-1}x \Leftrightarrow x = sechy$, where $x \in [0, 1]$ and $y \in [0, \infty)$.
6. The inverse hyperbolic cosecant function is denoted by $csch^{-1}$ and it is defined as $y = csch^{-1}x \Leftrightarrow x = cschy$, where $x \in \mathbb{R} - \{0\}$ and $y \in \mathbb{R} - \{0\}$

Derivatives of the inverse hyperbolic functions :

1. $\dfrac{d}{dx} \sinh^{-1} x = \dfrac{1}{\sqrt{1 + x^2}}$.

   $\dfrac{d}{dx} \sinh^{-1}(f(x)) = \dfrac{f'(x)}{\sqrt{1 + (f(x))^2}}$.

2. $\dfrac{d}{dx} \cosh^{-1} x = \dfrac{1}{\sqrt{x^2 - 1}}$, where $x > 1$.

   $\dfrac{d}{dx} \cosh^{-1}(f(x)) = \dfrac{f'(x)}{\sqrt{(f(x))^2 - 1}}$, where $f(x) > 1$.

3. $\dfrac{d}{dx} \tanh^{-1} x = \dfrac{1}{1 - x^2}$, where $|x| < 1$.

   $\dfrac{d}{dx} \tanh^{-1}(f(x)) = \dfrac{f'(x)}{1 - (f(x))^2}$, where $|f(x)| < 1$.

4. $\dfrac{d}{dx} \coth^{-1} x = \dfrac{1}{1 - x^2}$, where $|x| > 1$.

   $\dfrac{d}{dx} \coth^{-1}(f(x)) = \dfrac{f'(x)}{1 - (f(x))^2}$, where $|f(x)| > 1$.

5. $\dfrac{d}{dx} sech^{-1}x = \dfrac{-1}{x\sqrt{1 - x^2}}$, where $0 < x < 1$.

   $\dfrac{d}{dx} sech^{-1}(f(x)) = \dfrac{-f'(x)}{f(x)\sqrt{1 - (f(x))^2}}$, where $0 < f(x) < 1$.

6. $\dfrac{d}{dx}csch^{-1}x = \dfrac{-1}{|x|\sqrt{1+x^2}}$, where $x \neq 0$.

$\dfrac{d}{dx}csch^{-1}(f(x)) = \dfrac{-f'(x)}{|f(x)|\sqrt{1+(f(x))^2}}$, where $f(x) \neq 0$.

Examples :

1. Find $f'(x)$ if $f(x) = \tanh^{-1} 3x$?

$f'(x) = \dfrac{3}{1-(3x)^2} = \dfrac{3}{1-9x^2}$.

2. Find $f'(x)$ if $f(x) = \sinh^{-1} \sqrt{x}$?

$f'(x) = \dfrac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \dfrac{1}{2\sqrt{x}\sqrt{1+x}}$.

3. Find $f'(x)$ if $f(x) = sech^{-1}(\cos 2x)$?

$f'(x) = \dfrac{-(-2\sin 2x)}{\cos 2x\sqrt{1-(\cos 2x)^2}} = \dfrac{2\sin 2x}{\cos 2x\sqrt{1-\cos^2 2x}}$.

Integration :

1. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\, dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

$\displaystyle\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\, dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \cosh^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (x > a)$

$\displaystyle\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}\, dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (f(x) > a)$

3. $\displaystyle\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (|x| < a)$

$\displaystyle\int \frac{f'(x)}{a^2-[f(x)]^2}\, dx = \frac{1}{a}\tanh^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (|f(x)| < a))$

4. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = -\frac{1}{a}\ sech^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (0 < x < a)$

$\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-[f(x)]^2}}\, dx = -\frac{1}{a}\ sech^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (0 < f(x) < a))$

5. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = -\frac{1}{a}\ csch^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (x \neq 0)$

$$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2+a^2}}\,dx = -\frac{1}{a}\ csch^{-1}\left(\frac{f(x)}{a}\right)+c\ \ ,\ \ (f(x)\neq 0)$$

Examples :

1. $\displaystyle\int \frac{e^x}{1-e^{2x}}\,dx = \int \frac{e^x}{(1)^2-(e^x)^2}\,dx = \tanh^{-1}(e^x)+c.$

2. $\displaystyle\int \frac{e^x}{\sqrt{4e^{2x}+9}}\,dx = \frac{1}{2}\int \frac{2e^x}{\sqrt{(2e^x)^2+(3)^2}}\,dx = \frac{1}{2}\sinh^{-1}\left(\frac{2e^x}{3}\right)+c\ .$

3. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{4+x}}\,dx = 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(2)^2+(\sqrt{x})^2}}\,dx = 2\sinh^{-1}\left(\frac{\sqrt{x}}{2}\right)+c\ .$

4. $\displaystyle\int \frac{1}{\sqrt{16-e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{(4)^2-(e^x)^2}}\,dx = -\frac{1}{4}sech^{-1}\left(\frac{e^x}{4}\right)+c\ .$

5. $\displaystyle\int \frac{1}{\sqrt{1+e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{(1)^2+(e^x)^2}}\,dx = -csch^{-1}(e^x)+c\ .$

6. $\displaystyle\int \frac{1}{\sqrt{x^2+2x-8}}\,dx = \int \frac{1}{\sqrt{(x^2+2x+1)-9}}\,dx = \int \frac{1}{\sqrt{(x+1)^2-(3)^2}}\,dx$

   $\displaystyle= \cosh^{-1}\left(\frac{x+1}{3}\right)+c\ .$

7. $\displaystyle\int \frac{1}{(x-1)\sqrt{-x^2+2x+3}}\,dx = \int \frac{1}{(x-1)\sqrt{-(x^2-2x+1)+4}}\,dx$

   $\displaystyle= \int \frac{1}{(x-1)\sqrt{(2)^2-(x-1)^2}}\,dx = -\frac{1}{2}sech^{-1}\left(\frac{x-1}{2}\right)+c\ .$

INDETERMINATE FORMS

Theorem (L'Hôpital's Rule) :
Suppose that f and g are differentiable on the interval (a, b) , except possibly at a point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b) , except possibly at c.
Suppose further that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ has the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and that $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = L$ (or $\pm\infty$). Then , $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$.

Remark :
The conclusion of the theorem also holds if $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ is replaced with $\lim\limits_{x \to c^+} \dfrac{f(x)}{g(x)}$, $\lim\limits_{x \to c^-} \dfrac{f(x)}{g(x)}$, $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$ or $\lim\limits_{x \to -\infty} \dfrac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustment of the hypothesis.)

Types of indeterminate forms :

1. $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.
2. $\infty - \infty$ or $-\infty + \infty$.
3. $0\ \infty$ or $0\ (-\infty)$.
4. 0^0 , 1^∞ , $1^{-\infty}$ or ∞^0 .

Examples :

1. $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{\ln x} \qquad \left(\dfrac{0}{0}\right)$

   Apply L'Hôpital's rule

   $$\lim_{x \to 1} \frac{\sqrt{x} - 1}{\ln x} = \lim_{x \to 1} \frac{\left(\dfrac{1}{2\sqrt{x}}\right)}{\left(\dfrac{1}{x}\right)} = \lim_{x \to 1} \frac{x}{2\sqrt{x}} = \frac{1}{2} \ .$$

2. $\lim\limits_{x \to 0} \dfrac{\sin x \sqrt{1 - \sin x}}{x} \qquad \left(\dfrac{0}{0}\right)$

   $$\lim_{x \to 0} \frac{\sin x \sqrt{1 - \sin x}}{x} = \lim_{x \to 0} \frac{\sin x}{x} \sqrt{1 - \sin x} = 1\sqrt{1 - 0} = 1 \ .$$

3. $\lim\limits_{x \to 0} \dfrac{\int_0^x \sqrt{1 + \sin t}\ dt}{x} \qquad \left(\dfrac{0}{0}\right)$

   Apply L'Hôpital's rule

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \sqrt{1+\sin t}\; dt}{x} = \lim_{x\to 0} \frac{\sqrt{1+\sin x}}{1} = \frac{1+0}{1} = 1 \ .$$

4. $\displaystyle \lim_{x\to 1} \frac{\tan^{-1} x - \dfrac{\pi}{4}}{x-1} \qquad \left(\frac{0}{0}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to 1} \frac{\tan^{-1} x - \dfrac{\pi}{4}}{x-1} = \lim_{x\to 1} \frac{\left(\dfrac{1}{1+x^2}\right)}{1} = \lim_{x\to 1} \frac{1}{1+x^2} = \frac{1}{1+1} = \frac{1}{2} \ .$$

5. $\displaystyle \lim_{x\to 0} \frac{\tan x - x}{x^3} \qquad \left(\frac{0}{0}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{\tan^2 x}{3x^2}$$

$$= \frac{1}{3} \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2 = \frac{1}{3}(1)^2 = \frac{1}{3} \ .$$

6. $\displaystyle \lim_{x\to \infty} \frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to \infty} \frac{\ln x}{x} = \lim_{x\to \infty} \frac{\left(\dfrac{1}{x}\right)}{1} = \lim_{x\to \infty} \frac{1}{x} = 0 \ .$$

7. $\displaystyle \lim_{x\to \infty} \frac{x+e^x}{1+e^{3x}} \qquad \left(\frac{\infty}{\infty}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to \infty} \frac{x+e^x}{1+e^{3x}} = \lim_{x\to \infty} \frac{1+e^x}{3e^{3x}} \qquad \left(\frac{\infty}{\infty}\right)$$

Apply L'Hôpital's rule

$$\lim_{x\to \infty} \frac{1+e^x}{3e^{3x}} = \lim_{x\to \infty} \frac{e^x}{9e^{3x}} = \lim_{x\to \infty} \frac{1}{9e^{2x}} = 0 \ .$$

8. $\displaystyle \lim_{x\to \left(\frac{\pi}{2}\right)^-} \frac{2-\sec x}{3\tan x} \qquad \left(\frac{-\infty}{\infty}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to \left(\frac{\pi}{2}\right)^-} \frac{2-\sec x}{3\tan x} = \lim_{x\to \left(\frac{\pi}{2}\right)^-} \frac{-\sec x \tan x}{3\sec^2 x}$$

$$= \lim_{x\to \left(\frac{\pi}{2}\right)^-} \frac{-\tan x}{3\sec x} = \lim_{x\to \left(\frac{\pi}{2}\right)^-} \frac{-\sin x}{3} = -\frac{1}{3} \ .$$

9. $\lim\limits_{x\to 1^+}\left(\dfrac{3}{\ln x}-\dfrac{2}{x-1}\right) \qquad (\infty-\infty)$

$\lim\limits_{x\to 1^+}\left(\dfrac{3}{\ln x}-\dfrac{2}{x-1}\right)=\lim\limits_{x\to 1^+}\dfrac{3(x-1)-2\ln x}{(x-1)\ln x} \qquad \left(\dfrac{0}{0}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to 1^+}\frac{3(x-1)-2\ln x}{(x-1)\ln x}=\lim_{x\to 1^+}\frac{3-\frac{2}{x}}{\ln x+(x-1)\frac{1}{x}}=\lim_{x\to 1^+}\frac{3-\frac{2}{x}}{\ln x+1-\frac{1}{x}}=\infty$$

Note that $3-\dfrac{2}{x}\to 1$ and $\ln x+1-\dfrac{1}{x}\to 0^+$ as $x\to 1^+$

10. $\lim\limits_{x\to\infty}(x^2-1)e^{-x^2} \qquad (0\ \infty)$

$\lim\limits_{x\to\infty}(x^2-1)e^{-x^2}=\lim\limits_{x\to\infty}\dfrac{x^2-1}{e^{x^2}} \qquad \left(\dfrac{\infty}{\infty}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to\infty}\frac{x^2-1}{e^{x^2}}=\lim_{x\to\infty}\frac{2x}{2x\ e^{x^2}}=\lim_{x\to\infty}\frac{1}{e^{x^2}}=0$$

11. $\lim\limits_{x\to 0^+}x^x \qquad \left(0^0\right)$

Put $y=x^x\Leftrightarrow \ln y=\ln x^x=x\ln x$

$\lim\limits_{x\to 0^+}\ln y=\lim\limits_{x\to 0^+}x\ln x \qquad (0\ (-\infty))$

$\lim\limits_{x\to 0^+}x\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{x^{-1}} \qquad \left(\dfrac{-\infty}{\infty}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}=\lim_{x\to 0^+}\frac{\left(\frac{1}{x}\right)}{-x^{-2}}=\lim_{x\to 0^+}(-x)=0$$

Therefore, $\lim\limits_{x\to 0^+}x^x=\lim\limits_{x\to 0^+}y=e^0=1$.

12. $\lim\limits_{x\to\infty}\left(1+e^{2x}\right)^{\frac{1}{x}} \qquad \left(\infty^0\right)$

Put $y=\left(1+e^{2x}\right)^{\frac{1}{x}}\Leftrightarrow \ln y=\dfrac{1}{x}\ln\left(1+e^{2x}\right)=\dfrac{\ln\left(1+e^{2x}\right)}{x}$

$\lim\limits_{x\to\infty}\ln y=\lim\limits_{x\to\infty}\dfrac{\ln\left(1+e^{2x}\right)}{x} \qquad \left(\dfrac{\infty}{\infty}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to\infty}\frac{\ln\left(1+e^{2x}\right)}{x}=\lim_{x\to\infty}\frac{\left(\frac{2e^{2x}}{1+e^{2x}}\right)}{1}=\lim_{x\to\infty}\frac{2e^{2x}}{1+e^{2x}} \qquad \left(\frac{\infty}{\infty}\right)$$

Apply L'Hôpital's rule

$$\lim_{x\to\infty} \frac{4e^{2x}}{2e^{2x}} = 2$$

Therefore, $\lim_{x\to\infty} \left(1+e^{2x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} y = e^2$.

13. $\lim_{x\to\infty} \left(1+\frac{\ln 3}{x}\right)^x \qquad (1^\infty)$

Put $y = \left(1+\frac{\ln 3}{x}\right)^x \Leftrightarrow \ln y = x\ln\left(1+\frac{\ln 3}{x}\right)$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(1+\frac{\ln 3}{x}\right) \qquad (0\ \infty)$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{\ln 3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{\ln 3}{x}\right)}{x^{-1}} \qquad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{\ln 3}{x}\right)}{x^{-1}} = \lim_{x\to\infty} \frac{\left(\frac{-\ln 3x^{-2}}{1+\frac{\ln 3}{x}}\right)}{-x^{-2}}$$

$$= \lim_{x\to\infty} \frac{\ln 3}{\left(1+\frac{\ln 3}{x}\right)} = \frac{\ln 3}{1+0} = \ln 3$$

Therefore , $\lim_{x\to\infty} \left(1+\frac{\ln 3}{x}\right)^x = \lim_{x\to\infty} y = e^{\ln 3} = 3$

NOTE : $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$ where $a \neq 0$

14. $\lim_{x\to 0^+} (2x+1)^{\cot x} \qquad (1^\infty)$

Put $y = (2x+1)^{\cot x} \Leftrightarrow \ln y = \cot x \ln(2x+1) = \frac{\ln(2x+1)}{\tan x}$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(2x+1)}{\tan x} \qquad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule

$$\lim_{x\to 0^+} \frac{\ln(2x+1)}{\tan x} = \lim_{x\to 0^+} \frac{\left(\frac{2}{2x+1}\right)}{\sec^2 x} = \lim_{x\to 0^+} \frac{2}{(2x+1)\sec^2 x} = \frac{2}{2(1)^2} = 2$$

Therefore, $\lim_{x\to 0^+} (2x+1)^{\cot x} = \lim_{x\to 0^+} y = e^2$.

Exercises : Evaluate the following limits

1. $\displaystyle\lim_{x\to\infty} \frac{4e^x}{x^2}$.
2. $\displaystyle\lim_{x\to\infty} \frac{e^{2x}-1}{x}$.
3. $\displaystyle\lim_{x\to\infty} e^{-x}\sqrt{x}$.
4. $\displaystyle\lim_{x\to\infty} (1+4x)^{\frac{1}{x^2}}$.
5. $\displaystyle\lim_{x\to 0} \frac{x-\tan x}{1-\cos x}$.
6. $\displaystyle\lim_{x\to 0^+} (\sec x+\tan x)^{\csc x}$.

INTEGRATION BY PARTS

It is used to solve integration of a product of two functions using the formula

$$\int u\ dv = u\ v - \int v\ du\ .$$

Examples :

1. $\int xe^x\, dx$

   $u = x \qquad dv = e^x\ dx$
   $du = dx \qquad v = e^x$

   $$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + c = (x-1)e^x + c\ .$$

2. $\int x \sec^2 x\, dx$

   $u = x \qquad dv = \sec^2 x\ dx$
   $du = dx \qquad v = \tan x$

   $$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x - \ln|\sec x| + c\ .$$

3. $\int_0^{\pi} x \sin x\ dx$

   $u = x \qquad dv = \sin x\ dx$
   $du = dx \qquad v = -\cos x$

   $$\int_0^{\pi} x \sin x\ dx = [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x\ dx = [-x \cos x]_0^{\pi} + [\sin x]_0^{\pi}.$$

   $$= [(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = [-\pi(-1) - 0] + [0 - 0] = \pi$$

4. $\int x^2 \sin x\, dx$

   $u = x^2 \qquad dv = \sin x\ dx$
   $du = 2x\ dx \qquad v = -\cos x$

   $$\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx$$

   Now to solve $\int 2x \cos x\, dx$

   $u = 2x \qquad dv = \cos x\ dx$
   $du = 2\ dx \qquad v = \sin x$

   Therefore , $\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x - 2 \int \sin x\, dx$

   $$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

5. $\displaystyle\int e^x \cos x \, dx$

$u = \cos x \qquad dv = e^x \ dx$
$du = -\sin x \ dx \qquad v = e^x$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Now to solve $\displaystyle\int e^x \sin x \, dx$

$u = \sin x \qquad dv = e^x \ dx$
$du = \cos x \ dx \qquad v = e^x$

Therefore , $\displaystyle\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x \, dx = \frac{1}{2} \left[e^x \cos x + e^x \sin x\right] + c \ .$$

Another solution of $\displaystyle\int e^x \cos x \, dx$

$u = e^x \qquad dv = \cos x \ dx$
$du = e^x \ dx \qquad v = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Now to solve $\displaystyle\int e^x \sin x \, dx$

$u = e^x \qquad dv = \sin x \ dx$
$du = e^x \ dx \qquad v = -\cos x$

Therefore , $\displaystyle\int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx\right]$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2} \left[e^x \sin x + e^x \cos x\right] + c \ .$$

6. $\displaystyle\int \ln|x| \, dx$

$u = \ln|x| \qquad dv = dx$
$du = \frac{1}{x} \ dx \qquad v = x$

$$\int \ln|x| \, dx = x \ln|x| - \int x \ \frac{1}{x} \, dx = x \ln|x| - \int \ dx = x \ln|x| - x + c$$

7. $\displaystyle\int \tan^{-1} x\, dx$

$$u = \tan^{-1} x \qquad dv = dx$$
$$du = \frac{1}{1+x^2}\ dx \qquad v = x$$

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int x\ \frac{1}{1+x^2}\, dx$$

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c$$

8. $\displaystyle\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$

$$u = \sec x \qquad dv = \sec^2 x\ dx$$
$$du = \sec x \tan x\ dx \qquad v = \tan x$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\ dx$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2 \int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\, dx = \frac{1}{2} \left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$$

9. $\displaystyle\int \ln(1+x^2)\, dx$

$$u = \ln(1+x^2) \qquad dv = dx$$
$$du = \frac{2x}{1+x^2}\ dx \qquad v = x$$

$$\int \ln(1+x^2)\, dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2}\, dx$$

$$\int \ln(1+x^2)\, dx = x \ln(1+x^2) - \int \frac{(2x^2+2)-2}{1+x^2}\, dx$$

$$\int \ln(1+x^2)\, dx = x \ln(1+x^2) - \int \frac{2(x^2+1)}{1+x^2}\, dx + 2 \int \frac{1}{1+x^2}\, dx$$

$$\int \ln(1+x^2)\, dx = x \ln(1+x^2) - 2x + 2 \tan^{-1} x + c$$

10. $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\, dx = \int x^2\ \frac{x}{\sqrt{x^2+1}}\, dx$

$$u = x^2 \qquad dv = \frac{x}{\sqrt{x^2+1}}\ dx$$
$$du = 2x\ dx \qquad v = \sqrt{x^2+1}$$

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = x^2\sqrt{x^2+1} - \int 2x\sqrt{x^2+1}\,dx$$

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = x^2\sqrt{x^2+1} - \int \left(x^2+1\right)^{\frac{1}{2}} 2x\,dx$$

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = x^2\sqrt{x^2+1} - \frac{\left(x^2+1\right)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

11. $\int x^3 e^{x^2}\,dx = \int x^2\left(xe^{x^2}\right)\,dx$

$$u = x^2 \qquad dv = xe^{x^2}\,dx$$

$$du = 2x\,dx \qquad v = \frac{1}{2}e^{x^2}$$

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\int 2xe^{x^2}\,dx$$

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

Exercises : Solve the following integrals

1. $\int x\cos 2x\,dx$.
2. $\int x\cosh x\,dx$.
3. $\int \frac{x}{e^x}\,dx$.
4. $\int e^x \sin x\,dx$.
5. $\int \frac{1}{x^2}\ \ln|x|\,dx$.
6. $\int \sin^{-1} x\,dx$.

Notes :

1. $\displaystyle\int xe^x\,dx = (x-1)e^x + c$.

   $\displaystyle\int x^2e^x\,dx = (x^2-2x+2)e^x + c$.

   $\displaystyle\int x^3e^x\,dx = (x^3-3x^2+6x-6)e^x + c$.

2. $\displaystyle\int x\cos x\,dx = x\sin x + \cos x + c$

   $\displaystyle\int x^2\cos x\,dx = (x^2-2)\sin x + 2x\cos x + c$

   $\displaystyle\int x^3\cos x\,dx = (x^3-6x)\sin x + (3x^2-6)\cos x + c$

   $\displaystyle\int x^4\cos x\,dx = (x^4-12x^2+24)\sin x + (4x^3-24x)\cos x + c$

3. $\displaystyle\int x\sin x\,dx = -x\cos x + \sin x + c$

   $\displaystyle\int x^2\sin x\,dx = (-x^2+2)\cos x + 2x\sin x + c$

   $\displaystyle\int x^3\sin x\,dx = (-x^3+6x)\cos x + (3x^2-6)\sin x + c$

   $\displaystyle\int x^4\sin x\,dx = (-x^4+12x^2-24)\cos x + (4x^3-24x)\sin x + c$

INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

FIRST : Integrals of the forms

$$\int \sin ax \ \cos bx \, dx \quad , \quad \int \sin ax \ \sin bx \, dx \quad , \quad \int \cos ax \ \cos bx \, dx$$

Where $a, b \in \mathbb{Z}$.

1. The integral $\int \sin ax \ \cos bx \, dx$ can be solved using the formula

   $\sin ax \ \cos bx = \frac{1}{2} [\sin(ax + bx) + \sin(ax - bx)]$

2. The integral $\int \sin ax \ \sin bx \, dx$ can be solved using the formula

   $\sin ax \ \sin bx = \frac{1}{2} [\cos(ax - bx) - \cos(ax + bx)]$

3. The integral $\int \cos ax \ \cos bx \, dx$ can be solved using the formula

   $\cos ax \ \cos bx = \frac{1}{2} [\cos(ax + bx) + \cos(ax - bx)]$

Examples :

1. $\int \sin 3x \ \cos 2x \, dx = \int \frac{1}{2} [\sin(3x + 2x) + \sin(3x - 2x)] \ dx$

   $= \int \frac{1}{2} [\sin 5x + \sin x] \ dx = \frac{1}{2} \int \sin 5x \, dx + \frac{1}{2} \int \sin x \, dx$

   $= \frac{1}{2} \frac{1}{5}(-\cos 5x) + \frac{1}{2}(-\cos x) + c = -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + c$

2. $\int \sin x \ \sin 3x \, dx = \int \frac{1}{2} [\cos(3x - x) - \cos(3x + x)] \ dx$

   $= \int \frac{1}{2} [\cos 2x - \cos 4x] \ dx = \frac{1}{2} \int \cos 2x \, dx - \frac{1}{2} \int \cos 4x \, dx$

   $= \frac{1}{2} \frac{1}{2} \sin 2x - \frac{1}{2} \frac{1}{4} \sin 4x + c = \frac{1}{4} \sin 2x - \frac{1}{8} \sin 4x + c$

3. $\int \cos 5x \ \cos 2x \, dx = \int \frac{1}{2} [\cos(5x + 2x) + \cos(5x - 2x)] \ dx$

   $= \int \frac{1}{2} [\cos 7x + \cos 3x] \ dx = \frac{1}{2} \int \cos 7x \, dx + \frac{1}{2} \int \cos 3x \, dx$

   $= \frac{1}{2} \frac{1}{7} \sin 7x + \frac{1}{2} \frac{1}{3} \sin 3x + c = \frac{1}{14} \sin 7x + \frac{1}{6} \sin 3x + c$

SECOND : Integrals of the forms

$$\int \sin^n x \ \cos^m x \, dx \quad , \quad \int \sinh^n x \ \cosh^m x \, dx \quad , \quad \text{where } n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd .

1. If n is odd :

   The substitution $u = \cos x$ can be used to solve $\displaystyle\int \sin^n x \ \cos^m x \, dx$.

   The substitution $u = \cosh x$ can be used to solve $\displaystyle\int \sinh^n x \ \cosh^m x \, dx$.

2. If m is odd :

   The substitution $u = \sin x$ can be used to solve $\displaystyle\int \sin^n x \ \cos^m x \, dx$.

   The substitution $u = \sinh x$ can be used to solve $\displaystyle\int \sinh^n x \ \cosh^m x \, dx$.

Examples :

1. $\displaystyle\int \sin^5 x \ \cos^4 x \, dx = \int \sin^4 x \ \cos^4 x \ \sin x \, dx$

   $\displaystyle= \int \left(\sin^2 x\right)^2 \ \cos^4 x \ \sin x \, dx = \int \left(1 - \cos^2 x\right)^2 \ \cos^4 x \ \sin x \, dx$

   Put $u = \cos x \Rightarrow -du = \sin x \ dx$

   $\displaystyle\int \sin^5 x \ \cos^4 x \, dx = -\int (1-u^2)^2 u^4 \ du = -\int (1 - 2u^2 + u^4) u^4 \ du$

   $\displaystyle= -\int (u^4 - 2u^6 + u^8) \ du = -\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right] + c$

   $\displaystyle= -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c$

2. $\displaystyle\int \sqrt{\sin x} \ \cos^3 x \, dx = \int \sqrt{\sin x} \ \cos^2 x \ \cos x \, dx$

   $\displaystyle= \int (\sin x)^{\frac{1}{2}} \ \left(1 - \sin^2 x\right) \ \cos x \, dx$

   Put $u = \sin x \Rightarrow du = \cos x \ dx$

   $\displaystyle\int \sqrt{\sin x} \ \cos^3 x \, dx = \int u^{\frac{1}{2}} (1 - u^2) \ du = \int \left(u^{\frac{1}{2}} - u^{\frac{5}{2}}\right) \ du$

   $\displaystyle= \frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{7}{2}}}{7} + c = \frac{2\,(\sin x)^{\frac{3}{2}}}{3} - \frac{2\,(\sin x)^{\frac{7}{2}}}{7} + c$

3. $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \sin^2 x\ \cos^{-2} x\ \sin x\,dx = \int \left(1-\cos^2 x\right)\ \cos^{-2} x\ \sin x\,dx$

Put $u = \cos x \Rightarrow -du = \sin x\ dx$

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = -\int (1-u^2)\ u^{-2}\ du = -\int (u^{-2}-1)\ du$$

$$= -\frac{u^{-1}}{-1} + u + c = \frac{1}{u} + u + c = \sec x + \cos x + c$$

4. $\displaystyle\int \sinh^3 x\ \cosh^2 x\,dx = \int \sinh^2 x\ \cosh^2 x\ \sinh x\,dx$

$$= \int (\cosh^2 x - 1)\ \cosh^2 x\ \sinh x\,dx$$

Put $u = \cosh x \Rightarrow du = \sinh x\ dx$

$$\int \sinh^3 x\ \cosh^2 x\,dx = \int (u^2-1)\ u^2\ du = \int (u^4-u^2)\ du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$$

5. $\displaystyle\int \sin^7 x\ \cos^3 x\,dx = \int \sin^7 x\ \cos^2 x\ \cos x\,dx$

$$= \int \sin^7 x\ (1-\sin^2 x)\ \cos x\,dx$$

Put $u = \sin x \Rightarrow du = \cos x\ dx$

$$\int \sin^7 x\ \cos^3 x\,dx = \int u^7\ (1-u^2)\ du = \int (u^7-u^9)\ du$$

$$= \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + c$$

Special cases :

1. $\displaystyle\int \sin^2 x\,dx = \int \frac{1}{2}\,[1-\cos 2x]\ dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

2. $\displaystyle\int \cos^2 x\,dx = \int \frac{1}{2}\,[1+\cos 2x]\ dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

Exercises : Solve the following integrals

1. $\displaystyle\int \sin^3 x\,dx$

2. $\displaystyle\int \sin^2 x \cos^5 x\,dx$

THIRD : Integrals of the forms

$$\int \sec^n x \ \tan^m x \, dx \quad , \quad \int \csc^n x \ \cot^m x \, dx \quad ,$$
$$\int sech^n x \ \tanh^m x \, dx \quad , \quad \int csch^n x \ \coth^m x \, dx$$

The above four integrals can be solved by substitution if n is even or m is odd .

1. If n is even :

   The substitution $u = \tan x$ can be used to solve $\int \sec^n x \ \tan^m x \, dx$.

   The substitutions $u = \cot x$, $u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.

2. If m is odd :

   The substitution $u = \sec x$ can be used to solve $\int \sec^n x \ \tan^m x \, dx$.

   The substitutions $u = \csc x$, $u = sechx$ and $u = cschx$ can be used to solve the other theree integrals respictively.

Examples :

1. $\int \csc^4 x \ \cot^4 x \, dx$

   $= \int \csc^2 x \ \cot^4 x \ \csc^2 x \, dx = \int (1 + \cot^2 x) \cot^4 x \ \csc^2 x \, dx$

   Put $u = \cot x \Rightarrow -du = \csc^2 x \ dx$

   $\int \csc^4 x \ \cot^4 x \, dx = -\int (1 + u^2) u^4 \ du = -\int (u^4 + u^6) \ du$

   $= -\frac{u^5}{5} - \frac{u^7}{7} + c = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c$

2. $\int \tan^3 x \ \sec^3 x \, dx$

   $= \int \tan^2 x \ \sec^2 x \ \sec x \tan x \, dx = \int (\sec^2 x - 1) \ \sec^2 x \ \sec x \tan x \, dx$

   Put $u = \sec x \Rightarrow du = \sec x \tan x \ dx$

   $\int \tan^3 x \ \sec^3 x \, dx = \int (u^2 - 1) u^2 \ du = \int (u^4 - u^2) \ du$

   $= \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$

3. $\displaystyle\int \tanh^3 x \ sechx\, dx$

$\displaystyle= \int \tanh^2 x \ sechx \tanh x\, dx = \int (1 - sech^2 x) \ sechx \tanh x\, dx$

Put $u = sechx \Rightarrow -du = sechx \ \tanh x \ dx$

$\displaystyle\int \tanh^3 x \ sechx\, dx = -\int (1 - u^2) \ du$

$\displaystyle= -u + \frac{u^3}{3} + c = -sechx + \frac{sech^3 x}{3} + c$

4. $\displaystyle\int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx$

$\displaystyle\int \sec^2 x \ (\tan x)^{-\frac{1}{2}} \ \sec^2 x\, dx = \int (1 + \tan^2 x) \ (\tan x)^{-\frac{1}{2}} \ \sec^2 x\, dx$

Put $u = \tan x \Rightarrow du = \sec^2 x \ dx$

$\displaystyle\int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx = \int (1 + u^2) u^{-\frac{1}{2}} \ du = \int \left(u^{-\frac{1}{2}} + u^{\frac{3}{2}}\right) \ du$

$\displaystyle= 2u^{\frac{1}{2}} + \frac{2u^{\frac{5}{2}}}{5} + c = 2(\tan x)^{\frac{1}{2}} + \frac{2(\tan x)^{\frac{5}{2}}}{5} + c$

5. $\displaystyle\int \tan^4 x \sec^2 x\, dx = \int (\tan x)^4 \sec^2 x\, dx = \frac{\tan^5 x}{5} + c$

TRIGONOMETRIC SUBSTITUTIONS

If the integrand contains a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$, then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$: use the substitution $x = a \sin \theta$ where $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ to solve the integral .
2. An integral involving $\sqrt{a^2 + x^2}$: use the substitution $x = a \tan \theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ to solve the integral .
3. An integral involving $\sqrt{x^2 - a^2}$: use the substitution $x = a \sec \theta$ where $\theta \in \left[0, \dfrac{\pi}{2}\right)$ to solve the integral .

Examples :

1. To solve the integral $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x}\, dx$ we use the substitution :

   (a) $x = 3 \tan \theta$ (b) $x = 3 \sin \theta$ (c) $x = 3 \sec \theta$ (d) None of these

   Answer : We use the substitution $x = 3 \sec \theta$.

2. To solve the integral $\displaystyle\int \sqrt{1 + 4x^2}\, dx$ we use the substitution :

   (a) $2x = \cos \theta$ (b) $x = \dfrac{\tan \theta}{2}$ (c) $2x = \sin \theta$ (d) None of these

   Answer : $\sqrt{1 + 4x^2} = \sqrt{(1)^2 + (2x)^2}$

   So we use the substitution $2x = \tan \theta \Rightarrow x = \dfrac{\tan \theta}{2}$

3. $\displaystyle\int \frac{1}{x^2 \sqrt{16 - x^2}}\, dx = \int \frac{1}{x^2 \sqrt{(4)^2 - x^2}}\, dx$

   Put $x = 4 \sin \theta \Rightarrow \sin \theta = \dfrac{x}{4}$

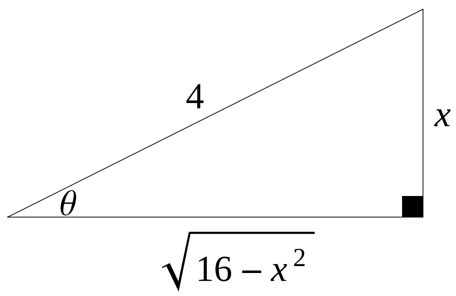
   $dx = 4 \cos \theta \; d\theta$

   $$\int \frac{1}{x^2 \sqrt{16 - x^2}}\, dx = \int \frac{4 \cos \theta}{(4 \sin \theta)^2 \sqrt{16 - (4 \sin \theta)^2}}\, d\theta$$

   $$= \int \frac{4 \cos \theta}{16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta}}\, d\theta = \int \frac{4 \cos \theta}{16 \sin^2 \theta \sqrt{16(1 - \sin^2 \theta)}}\, d\theta$$

   $$= \int \frac{4 \cos \theta}{16 \sin^2 \theta \; 4 \cos \theta}\, d\theta = \frac{1}{16} \int \frac{1}{\sin^2 \theta}\, d\theta = \frac{1}{16} \int \csc^2 \theta \; d\theta$$

   $$= -\frac{1}{16} \cot \theta + c$$



$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = -\frac{1}{16}\;\frac{\sqrt{16-x^2}}{x} + c$$

4. $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

Put $x = 2\sec\theta \Rightarrow \sec\theta = \dfrac{x}{2}$

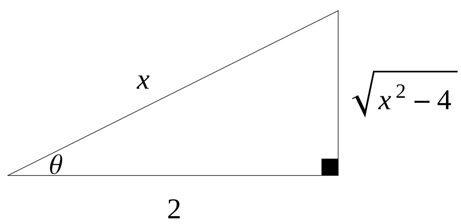
$dx = 2\sec\theta\ \tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \int \frac{\sqrt{4\sec^2\theta-4}\ 2\sec\theta\tan\theta}{4\sec^2\theta}\ d\theta$$

$$= \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{4\sec^2\theta}\ d\theta = \int \frac{\tan^2\theta}{\sec\theta}\ d\theta$$

$$= \int \frac{(\sec^2\theta-1)}{\sec\theta}\ d\theta = \int \frac{\sec^2\theta}{\sec\theta}\ d\theta - \int \frac{1}{\sec\theta}\ d\theta$$

$$= \int \sec\theta\ d\theta - \int \cos\theta\ d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + c$$



$$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c$$

5. $\displaystyle\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}\,dx$

$$= \int \frac{1}{[(x^2+8x+16)+9]^{\frac{3}{2}}}\,dx = \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}\,dx$$

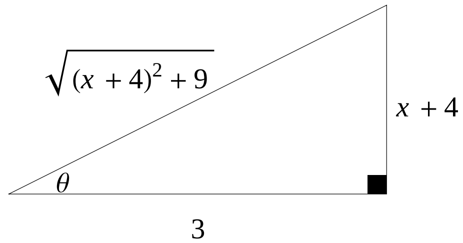
Put $x+4 = 3\tan\theta \Rightarrow \tan\theta = \dfrac{x+4}{3}$

$dx = 3\sec^2\theta \; d\theta$

$$\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}\,dx = \int \frac{3\sec^2\theta}{\left(9\tan^2\theta+9\right)^{\frac{3}{2}}}\;d\theta$$

$$= \int \frac{3\sec^2\theta}{(9\sec^2\theta)^{\frac{3}{2}}}\;d\theta = \int \frac{3\sec^2\theta}{27\sec^3\theta}\;d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\;d\theta = \frac{1}{9}\int \cos\theta\;d\theta = \frac{1}{9}\sin\theta + c$$



$$\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}\,dx = \frac{1}{9}\;\frac{x+4}{\sqrt{(x+4)^2+9}} + c = \frac{1}{9}\;\frac{x+4}{\sqrt{x^2+8x+25}} + c$$

6. $\displaystyle\int \frac{1}{(25-x^2)^{\frac{3}{2}}}\,dx$

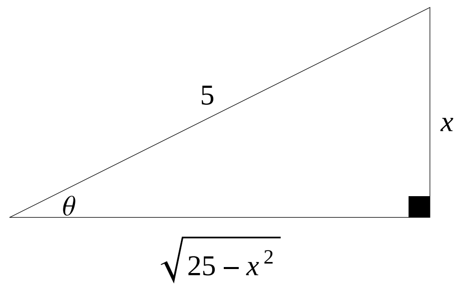
Put $x = 5\sin\theta \Rightarrow \sin\theta = \dfrac{x}{5}$

$dx = 5\cos\theta\;d\theta$

$$\int \frac{1}{(25-x^2)^{\frac{3}{2}}}\,dx = \int \frac{5\cos\theta}{(25-25\sin^2\theta)^{\frac{3}{2}}}\;d\theta$$

$$= \int \frac{5\cos\theta}{(25\cos^2\theta)^{\frac{3}{2}}}\;d\theta = \int \frac{5\cos\theta}{125\cos^3\theta}\;d\theta$$

$$= \frac{1}{25}\int \frac{1}{\cos^2\theta}\;d\theta = \frac{1}{25}\int \sec^2\theta\;d\theta = \frac{1}{25}\tan\theta + c$$



$$\int \frac{1}{(25-x^2)^{\frac{3}{2}}}\,dx = \frac{1}{25}\;\frac{x}{\sqrt{25-x^2}} + c$$

7. $\displaystyle\int \frac{x}{\sqrt{x^2-16}}\,dx = \frac{1}{2}\int (x^2-16)^{-\frac{1}{2}}2x\,dx = \sqrt{x^2-16}+c$

Notes :

1. $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

   Put $x = 3\sin\theta \Rightarrow \sin\theta = \dfrac{x}{3}$

   $dx = 3\cos\theta\ d\theta$

   $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\ d\theta$

   $\displaystyle= \int \frac{3\cos\theta}{3\cos\theta}\ d\theta = \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{3}\right) + c$

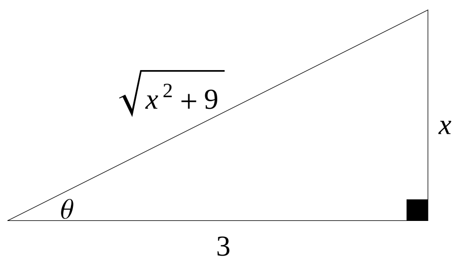
2. $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$

   Put $x = 3\tan\theta \Rightarrow \tan\theta = \dfrac{x}{3}$

   $dx = 3\sec^2\theta\ d\theta$

   $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx = \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\ d\theta$

   $\displaystyle= \int \frac{3\sec^2\theta}{3\sec\theta}\ d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta+\tan\theta| + c$



   $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + c$

Exercises : Solve the following integrals

1. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

   Hint : use $x = 2\sin\theta$

2. $\displaystyle\int x^3\sqrt{x^2-4}\,dx$

   Hint : use $x = 2\sec\theta$

3. $\displaystyle\int \sqrt{x^2+2x+2}\,dx$

   Hint : use $x+1 = \tan\theta$

4. $\displaystyle\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$

   Hint : use $x+1 = 2\tan\theta$

5. $\displaystyle\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$

   Hint : use $3x = 7\tan\theta$

INTEGRATION OF RATIONAL FUNCTIONS
(Method of Partial fractions)

Method of partial fractions is used to solve integrals of the form $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$, $Q(x)$ are polynomials and $degree\ P(x) < degree\ Q(x)$.
If $degree\ P(x) \geq degree\ Q(x)$ use long division of polynomials .

Definition (linear factor) :
A linear factor is a polynomial of degree 1.
It has the form $ax+b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

Examples :
x , $3x$, $2x-7$ are examples of linear factors .

Definition (irreducible quadratic) :
An irreducible quadratic is a polynomial of degree 2.
It has the form ax^2+bx+c where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $b^2-4ac < 0$.

Examples :

1. x^2+9 and x^2+x+1 are examples of irreducible quadratics.
2. $x^2 = x\ x$ and $x^2-1 = (x-1)(x+1)$ are reducible quadratics .

How to write $\dfrac{P(x)}{Q(x)}$ as partial fractions decomposition ?
Write $Q(x)$ as a product of linear factors and irreducible quadratics (if possible).
If $Q(x) = (a_1x+a_2)^m\ (b_1x^2+b_2x+b_3)^n$ where $m, n \in \mathbb{N}$ then

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{A_1}{a_1x+a_2} + \frac{A_2}{(a_1x+a_2)^2} + \cdots + \frac{A_m}{(a_1x+a_2)^m} \\ &+ \frac{B_1x+C_1}{b_1x^2+b_2x+b_3} + \frac{B_2x+C_2}{(b_1x^2+b_2x+b_3)^2} + \cdots + \frac{B_nx+C_n}{(b_1x^2+b_2x+b_3)^n}\end{aligned}$$

Where $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n, C_1, C_2, \cdots, C_n \in \mathbb{R}$.

Examples : Write the partial fractions decomposition of the follwoing

1. $\dfrac{2x+6}{x^2-2x-3} = \dfrac{2x+6}{(x-3)(x+1)} = \dfrac{A_1}{x-3} + \dfrac{A_2}{x+1}$
2. $\dfrac{x+5}{x^2+4x+4} = \dfrac{x+5}{(x+2)^2} = \dfrac{A_1}{x+2} + \dfrac{A_2}{(x+2)^2}$
3. $\dfrac{x^2+1}{x^4+4x^2} = \dfrac{x^2+1}{x^2(x^2+4)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{B_1x+C_1}{x^2+4}$
4. $\dfrac{2x+7}{(x+1)(x^2+9)^2} = \dfrac{A_1}{x+1} + \dfrac{B_1x+C_1}{x^2+9} + \dfrac{B_2x+C_2}{(x^2+9)^2}$
5. $\dfrac{x}{(x-1)(x^2-1)} = \dfrac{x}{(x+1)(x-1)^2} = \dfrac{A_1}{x+1} + \dfrac{A_2}{x-1} + \dfrac{A_3}{(x-1)^2}$

6. $\dfrac{x^4+2x^3+1}{x^4+x^3+x^2} = \dfrac{(x^4+x^3+x^2)+(x^3-x^2+1)}{x^4+x^3+x^2} = 1 + \dfrac{x^3-x^2+1}{x^4+x^3+x^2}$

$= 1 + \dfrac{x^3-x^2+1}{x^2(x^2+x+1)} = 1 + \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{B_1x+C_1}{x^2+x+1}$

Examples :

1. $\displaystyle\int \frac{x^2+12x+3}{x^3-4x}\,dx$

$\dfrac{x^2+12x+3}{x^3-4x} = \dfrac{x^2+12x+3}{x(x-2)(x+2)} = \dfrac{A_1}{x} + \dfrac{A_2}{x-2} + \dfrac{A_3}{x+2}$

$\dfrac{x^2+12x+3}{x^3-4x} = \dfrac{A_1(x-2)(x+2)}{x(x-2)(x+2)} + \dfrac{A_2\ x(x+2)}{x(x-2)(x+2)} + \dfrac{A_3\ x(x-2)}{x(x-2)(x+2)}$

$x^2+12x+3 = A_1(x-2)(x+2) + A_2\ x(x+2) + A_3\ x(x-2) \longrightarrow (*)$

Put $x=0$ in equation (*) : $(0)^2+12(0)+3 = A_1(0-2)(0+2)+0+0$

$3 = -4A_1 \Rightarrow A_1 = -\dfrac{3}{4}.$

Put $x=2$ in equation (*) : $(2)^2+12(2)+3 = 0 + A_2\ 2(2+2)+0$

$31 = 8A_2 \Rightarrow A_2 = \dfrac{31}{8}.$

Put $x=-2$ in equation (*) : $(-2)^2+12(-2)+3 = 0+0+A_3\ (-2)(-2-2)$

$-17 = 8A_3 \Rightarrow A_3 = -\dfrac{17}{8}.$

$\dfrac{x^2+12x+3}{x^3-4x} = \dfrac{-\frac{3}{4}}{x} + \dfrac{\frac{31}{8}}{x-2} + \dfrac{-\frac{17}{8}}{x+2}$

$\dfrac{x^2+12x+3}{x^3-4x} = -\dfrac{3}{4}\ \dfrac{1}{x} + \dfrac{31}{8}\ \dfrac{1}{x-2} - \dfrac{17}{8}\ \dfrac{1}{x+2}$

$\displaystyle\int \frac{x^2+12x+3}{x^3-4x}\,dx = \int \left[-\frac{3}{4}\ \frac{1}{x} + \frac{31}{8}\ \frac{1}{x-2} - \frac{17}{8}\ \frac{1}{x+2}\right] dx$

$\displaystyle\int \frac{x^2+12x+3}{x^3-4x}\,dx = -\frac{3}{4}\int \frac{1}{x}\,dx + \frac{31}{8}\int \frac{1}{x-2}\,dx - \frac{17}{8}\int \frac{1}{x+2}\,dx$

$\displaystyle\int \frac{x^2+12x+3}{x^3-4x}\,dx = -\frac{3}{4}\ln|x| + \frac{31}{8}\ln|x-2| - \frac{17}{8}\ln|x+2| + c$

2. $\displaystyle\int \frac{4}{x^4-x^3}\,dx$

$\dfrac{4}{x^4-x^3} = \dfrac{4}{x^3(x-1)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{A_3}{x^3} + \dfrac{A_4}{x-1}$

$\dfrac{4}{x^4-x^3} = \dfrac{A_1\ x^2(x-1)}{x^3(x-1)} + \dfrac{A_2\ x(x-1)}{x^3(x-1)} + \dfrac{A_3(x-1)}{x^3(x-1)} + \dfrac{A_4\ x^3}{x^3(x-1)}$

$4 = A_1\ x^2(x-1) + A_2\ x(x-1) + A_3(x-1) + A_4\ x^3$

$4 = A_1x^3 - A_1x^2 + A_2x^2 - A_2x + A_3x - A_3 + A_4x^3$

$4 = (A_1 + A_4)x^3 + (A_2 - A_1)x^2 + (A_3 - A_2)x - A_3$

By comparing the coefficients of both sides :

$$\begin{array}{rcll} A_1 + A_4 & = & 0 & \longrightarrow (1) \\ A_2 - A_1 & = & 0 & \longrightarrow (2) \\ A_3 - A_2 & = & 0 & \longrightarrow (3) \\ -A_3 & = & 4 & \longrightarrow (4) \end{array}$$

From equation (4) : $A_3 = -4$

From equation (3) : $A_2 = A_3 = -4$

From equation (2) : $A_1 = A_2 = -4$

From equation (1) : $A_4 = -A_1 = 4$

$$\frac{4}{x^4 - x^3} = \frac{-4}{x} + \frac{-4}{x^2} + \frac{-4}{x^3} + \frac{4}{x-1}$$

$$\int \frac{4}{x^4 - x^3}\,dx = -4\int \frac{1}{x}\,dx - 4\int x^{-2}\,dx - 4\int x^{-3}\,dx + 4\int \frac{1}{x-1}\,dx$$

$$\int \frac{4}{x^4 - x^3}\,dx = -4\ln|x| - 4\frac{x^{-1}}{-1} - 4\frac{x^{-2}}{-2} + 4\ln|x-1| + c$$

$$\int \frac{4}{x^4 - x^3}\,dx = -4\ln|x| + \frac{4}{x} + \frac{2}{x^2} + 4\ln|x-1| + c$$

3. $\displaystyle\int \frac{8}{(x^2+1)(x^2+9)}\,dx$

$$\frac{8}{(x^2+1)(x^2+9)} = \frac{B_1x + C_1}{x^2+1} + \frac{B_2x + C_2}{x^2+9}$$

$$\frac{8}{(x^2+1)(x^2+9)} = \frac{(B_1x + C_1)(x^2+9)}{(x^2+1)(x^2+9)} + \frac{(B_2x + C_2)(x^2+1)}{(x^2+1)(x^2+9)}$$

$8 = (B_1x + C_1)(x^2+9) + (B_2x + C_2)(x^2+1)$

$8 = B_1x^3 + 9B_1x + C_1x^2 + 9C_1 + B_2x^3 + B_2x + C2x^2 + C_2$

$8 = (B_1 + B_2)x^3 + (C_1 + C_2)x^2 + (9B_1 + B_2)x + (9C_1 + C_2)$

By comparing the coefficients of both sides :

$$\begin{array}{rcll} B_1 + B_2 & = & 0 & \longrightarrow (1) \\ C_1 + C_2 & = & 0 & \longrightarrow (2) \\ 9B_1 + B_2 & = & 0 & \longrightarrow (3) \\ 9C_1 + C_2 & = & 8 & \longrightarrow (4) \end{array}$$

Equation (3) - Equation (1) : $8B_1 = 0 \Rightarrow B_1 = 0$

From equation (1) : $B_2 = -B_1 = 0$

Equation (4) - Equation (2) : $8C_1 = 8 \Rightarrow C_1 = 1$

From equation (2) : $C_2 = -C_1 = -1$

$$\frac{8}{(x^2+1)(x^2+9)} = \frac{1}{x^2+1} + \frac{-1}{x^2+9}$$

$$\int \frac{8}{(x^2+1)(x^2+9)}\, dx = \int \frac{1}{x^2+1}\, dx - \int \frac{1}{x^2+9}\, dx$$

$$\int \frac{8}{(x^2+1)(x^2+9)}\, dx = \tan^{-1} x - \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + c$$

4. $\displaystyle\int \frac{8x^3+13x}{(x^2+2)^2}\, dx$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{B_1x+C_1}{x^2+2} + \frac{B_2x{+}_2}{(x^2+2)^2}$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{(B_1x+C_1)(x^2+2)}{(x^2+2)^2} + \frac{B_2x{+}_2}{(x^2+2)^2}$$

$$8x^3+13x = (B_1x+C_1)(x^2+2) + B_2x + C2$$

$$8x^3+13x = B_1x^3 + 2B_1x + C_1x^2 + 2C_1 + B_2x + C_2$$

$$8x^3+13x = B_1x^3 + C_1x^2 + (2B_1+B_2)x + (2C_1+C_2)$$

By comparing the coefficients of both sides :

$B_1 = 8$

$C_1 = 0$

$2B_1 + B_2 = 13 \Rightarrow B_2 = 13 - 2(8) = 13 - 16 = -3$

$2C_1 + C_2 = 0 \Rightarrow C_2 = 0 - 2(0) = 0$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}$$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\, dx = 4\int \frac{2x}{x^2+2}\, dx - \frac{3}{2}\int \frac{2x}{(x^2+2)^2}\, dx$$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\, dx = 4\ln(x^2+2) - \frac{3}{2}\,\frac{(x^2+2)^{-1}}{-1} + c$$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\, dx = 4\ln(x^2+2) + \frac{3}{2}\,\frac{1}{x^2+2} + c$$

5. $\displaystyle\int \frac{x^3+1}{x^3+4x}\, dx$

$$\frac{x^3+1}{x^3+4x} = \frac{(x^3+4x)+(1-4x)}{x^3+4x} = 1 + \frac{1-4x}{x^3+4x} = 1 + \frac{1-4x}{x(x^2+4)}$$

$$\frac{1-4x}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+c}{x^2+4}$$

$$\frac{1-4x}{x(x^2+4)} = \frac{A(x^2+4)}{x(x^2+4)} + \frac{(Bx+c)x}{x(x^2+4)}$$

$$1-4x = A(x^2+4) + (Bx+C)x = Ax^2 + 4A + Bx^2 + Cx$$

$$1-4x = (A+B)x^2 + Cx + 4A$$

By comparing the coefficients of both sides :

$$4A = 1 \Rightarrow A = \frac{1}{4}$$

$$C = -4$$

$$A + B = 0 \Rightarrow B = -A = -\frac{1}{4}$$

$$\frac{x^3+1}{x^3+4x} = 1 + \frac{\frac{1}{4}}{x} + \frac{-\frac{1}{4}x - 4}{x^2+4}$$

$$\frac{x^3+1}{x^3+4x} = 1 + \frac{1}{4}\ \frac{1}{x} - \frac{1}{4}\ \frac{x}{x^2+4} - 4\ \frac{1}{x^2+4}$$

$$\int \frac{x^3+1}{x^3+4x}\,dx = \int 1\,dx + \frac{1}{4}\int \frac{1}{x}\,dx - \frac{1}{8}\int \frac{2x}{x^2+4}\,dx - 4\int \frac{1}{x^2+4}\,dx$$

$$\int \frac{x^3+1}{x^3+4x}\,dx = x + \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) - 4\ \frac{1}{2}\ \tan^{-1}\left(\frac{x}{2}\right) + c$$

6. $\displaystyle\int \frac{3\cos x}{\sin^2 x + \sin x - 2}\,dx$

Put $u \sin x \Rightarrow du = \cos x\ dx$

$$\int \frac{3\cos x}{\sin^2 x + \sin x - 2}\,dx = \int \frac{3}{u^2+u-2}\ du$$

$$\frac{3}{u^2+u-2} = \frac{3}{(u-1)(u+2)} = \frac{1}{u-1} + \frac{-1}{u+2}$$

$$\int \frac{3}{u^2+u-2}\ du = \int \frac{1}{u-1}\ du - \int \frac{1}{u+2}\ du$$

$$\int \frac{3}{u^2+u-2}\ du = \ln|u-1| - \ln|u+2| + c$$

$$\int \frac{3\cos x}{\sin^2 x + \sin x - 2}\,dx = \ln|\sin x - 1| - \ln|\sin x + 2| + c$$

Exercises : Solve the following integrals

1. $\displaystyle\int \frac{1}{x^2 - 3x + 2}\, dx$
2. $\displaystyle\int \frac{3}{(x^2 + 1)(x^2 + 4)}\, dx$
3. $\displaystyle\int \frac{e^x}{(e^x - 1)(e^x + 4)}\, dx$

HALF-ANGLE SUBSTITUTION

It is used to solve integrals of rational functions involving $\sin x$ or $\cos x$, by putting $u = \tan\left(\frac{x}{2}\right)$, in this case $dx = \frac{2}{1+u^2}\ du$, $\sin x = \frac{2u}{1+u^2}$ and $\cos x = \frac{1-u^2}{1+u^2}$.

Examples :

1. $\displaystyle\int \frac{1}{2+\cos x}\,dx$

   Put $u = \tan\left(\frac{x}{2}\right)$

   $$dx = \frac{2}{1+u^2}\ du \quad \text{and} \quad \cos x = \frac{1-u^2}{1+u^2}$$

   $$\int \frac{1}{2+\cos x}\,dx = \int \frac{1}{2+\left(\frac{1-u^2}{1+u^2}\right)}\ \frac{2}{1+u^2}\ du$$

   $$= \int \frac{1}{\left(\frac{2(1+u^2)+(1-u^2)}{1+u^2}\right)}\ \frac{2}{1+u^2}\ du$$

   $$= \int \frac{1+u^2}{2+2u^2+1-u^2}\ \frac{2}{1+u^2}\ du = \int \frac{2}{3+u^2}\ du$$

   $$= 2\int \frac{1}{(\sqrt{3})^2+(u)^2}\ du = 2\ \frac{1}{\sqrt{3}}\ \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + c$$

   $$\int \frac{1}{2+\cos x}\,dx = \frac{2}{\sqrt{3}}\ \tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c$$

2. $\displaystyle\int \frac{1}{3\sin x+4\cos x}\,dx$

   Put $u = \tan\left(\frac{x}{2}\right)$

   $$dx = \frac{2}{1+u^2}\ du \ , \ \cos x = \frac{1-u^2}{1+u^2} \quad \text{and} \quad \sin x = \frac{2u}{1+u^2}$$

   $$\int \frac{1}{3\sin x+4\cos x}\,dx = \int \frac{1}{3\left(\frac{2u}{1+u^2}\right)+4\left(\frac{1-u^2}{1+u^2}\right)}\ \frac{2}{1+u^2}\ du$$

   $$= \int \frac{1}{\frac{3(2u)+4(1-u^2)}{1+u^2}}\ \frac{2}{1+u^2}\ du = \int \frac{1+u^2}{6u+4-4u^2}\ \frac{2}{1+u^2}\ du$$

   $$\int \frac{2}{-2(2u^2-3u-2)}\ du = -\int \frac{1}{(2u+1)(u-2)}\ du$$

$$\frac{1}{(2u+1)(u-2)} = \frac{A_1}{u-2} + \frac{A_2}{2u+1}$$

$$1 = A_1(2u+1) + A_2(u-2)$$

Put $u = 2$ then $1 = 5A_1 \Rightarrow A_1 = \frac{1}{5}$

Put $u = -\frac{1}{2}$ then $1 = -\frac{5}{2}A_2 \Rightarrow A_2 = -\frac{2}{5}$

$$\frac{1}{(2u+1)(u-2)} = \frac{\frac{1}{5}}{u-2} + \frac{-\frac{2}{5}}{2u+1} = \frac{1}{5}\ \frac{1}{u-2} - \frac{1}{5}\ \frac{2}{2u+1}$$

$$-\int \frac{1}{(2u+1)(u-2)}\ du = -\frac{1}{5}\int \frac{1}{u-2}\ du + \frac{1}{5}\int \frac{2}{2u+1}\ du$$

$$= -\frac{1}{5}\ln|u-2| + \frac{1}{5}\ln|2u+1| + c$$

$$\int \frac{1}{3\sin x + 4\cos x}\,dx = -\frac{1}{5}\ln\left|\tan\left(\frac{x}{2}\right) - 2\right| + \frac{1}{5}\ln\left|2\tan\left(\frac{x}{2}\right) + 1\right| + c$$

3. $\displaystyle\int \frac{1}{1-\sin x}\,dx$

$$= \int \frac{1}{1-\sin x}\ \frac{1+\sin x}{1+\sin x}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right)\,dx$$

$$= \int \sec^2 x\,dx + \int \sec x\tan x\,dx = \tan x + \sec x + c$$

4. $\displaystyle\int \frac{\sin x}{\sqrt{5 - 2\cos x + \cos^2 x}}\,dx$

Put $u = \cos x \Rightarrow -du = \sin x$

$$\int \frac{\sin x}{\sqrt{5-2\cos x+\cos^2 x}}\,dx = \int \frac{-1}{\sqrt{5-2u+u^2}}\ du$$

$$= -\int \frac{1}{\sqrt{(u^2-2u+1)+4}}\ du = -\int \frac{1}{\sqrt{(u-1)^2+(2)^2}}\ du$$

$$= -\sinh^{-1}\left(\frac{u-1}{2}\right) + c$$

$$\int \frac{\sin x}{\sqrt{5-2\cos x+\cos^2 x}}\,dx = -\sinh^{-1}\left(\frac{\cos x - 1}{2}\right) + c$$

Exercises : Solve the following integrals

1. $\displaystyle\int \frac{1}{5+3\cos x}\,dx$
2. $\displaystyle\int \frac{1}{\cos x+\sin x}\,dx$
3. $\displaystyle\int \frac{1}{\sin x-\cos x-1}\,dx$

MISCELLANEOUS SUBSTITUTIONS

1. Integrals involving fraction powers of x

Examples :

1. $\displaystyle\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\, dx$

   Put $u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5 \ du$

   Note that 6 is the least common multiple of 2 and 3

   $$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\, dx = \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{1}{3}}}\, du = \int \frac{6u^5}{u^3 + u^2}\, du$$

   $$= \int \frac{6u^5}{u^2(u+1)}\, du = \int \frac{6u^3}{u+1}\, du$$

   Use long division of polynomials

   $$\int \frac{6u^3}{u+1}\, du = \int \left(6u^2 - 6u + 6 - \frac{6}{u+1)}\right)\, du$$

   $$= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + c$$

   $$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\, dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$

2. $\displaystyle\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}} + 1}\, dx$

   Put $u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5 \ du$

   Note that 6 is the least common multiple of 3 and 6

   $$\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}} + 1}\, dx = \int \frac{u\ 6u^5}{u^2+1}\, du = \int \frac{6u^6}{u^2+1}\, du$$

   Use long division of polynomials

   $$\int \frac{6u^6}{u^2+1}\, du = \int \left(6u^4 - 6u^2 + 6 - \frac{6}{u^2+1}\right)\, du$$

   $$= \frac{6u^5}{5} - 2u^3 + 6u - 6\tan^{-1} u + c$$

   $$\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}} + 1}\, dx = \frac{6u^{\frac{5}{6}}}{5} - 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} - 6\tan^{-1}\left(x^{\frac{1}{6}}\right) + c$$

2. Integrals involving a square root of a linear factor

Examples :

1. $\displaystyle\int \frac{1}{(x+1)\sqrt{x-2}}\,dx$

   Put $u=\sqrt{x-2} \Rightarrow x=u^2+2 \Rightarrow dx=2u\ du$

   $$\int \frac{1}{(x+1)\sqrt{x-2}}\,dx=\int \frac{2u}{(u^2+3)\ u}\ du=\int \frac{2}{u^2+3}\ du$$

   $$=2\int \frac{1}{(u)^2+(\sqrt{3})^2}\ du=2\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c$$

   $$\int \frac{1}{(x+1)\sqrt{x-2}}\,dx=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{3}}\right)+c$$

2. $\displaystyle\int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$

   Put $u=\sqrt{1+\sqrt{x}} \Rightarrow \sqrt{x}=u^2-1 \Rightarrow x=(u^2-1)^2 \Rightarrow dx=4u(u^2-1)\ du$

   $$\int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx=\int \frac{4u(u^2-1)}{u}\ du=4\int (u^2-1)\ du=4\left[\frac{u^3}{3}-u\right]+c$$

   $$\int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx=4\left[\frac{(\sqrt{1+\sqrt{x}})^3}{3}-\sqrt{1+\sqrt{x}}\right]+c$$

3. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

   Put $u=\sqrt{x} \Rightarrow x=u^2 \Rightarrow dx=2u\ du$

   $$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx=\int \frac{(1-u)2u}{1+u}\ du=\int \frac{-2u^2+2u}{u+1}\ du$$

   Use long division of polynomials

   $$\int \frac{-2u^2+2u}{u+1}\ du=\int \left(-2u+4-\frac{4}{u+1}\right)\ du=-u^2+4u-4\ln|u+1|+c$$

   $$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx=-x+4\sqrt{x}-4\ln\left|1+\sqrt{x}\right|+c$$

4. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx=2\int \left(1+\sqrt{x}\right)^{\frac{1}{2}}\frac{1}{2\sqrt{x}}\,dx=\frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}}+c$

IMPROPER INTEGRALS

Definition (Improper Integrals with a discontinuous integrand):

1. If f is continuous on $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$ then

   $$\int_a^b f(x)\ dx = \underset{t \to b^-}{lim} \int_a^t f(x)\ dx$$

2. If f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$ then

   $$\int_a^b f(x)\ dx = \underset{t \to a^+}{lim} \int_t^b f(x)\ dx$$

   In either case, if the limit exists (and equals a value L) then the improper integral **converges** (to L) . If the limit does not exist then the improper integral **diverges** .

Remark :
If f is continuous on $[a, b]$ except at a point $c \in (a, b)$ and $|f(x)| \to \infty$ as $x \to c$ then $\int_a^b f(x)\ dx = \underset{t \to c^-}{lim} \int_a^t f(x)\ dx + \underset{t \to c^+}{lim} \int_t^b f(x)\ dx$
If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$). If at least one of the limits does not exist then the improper integral **diverges** .

Definition (Improper Integrals with an infinite limit of integration) :

1. If f is continuous on $[a, \infty)$ then $\displaystyle\int_a^\infty f(x)\ dx = \underset{t \to \infty}{lim} \int_a^t f(x)\ dx$

2. If f is continuous on $(-\infty, a]$ then $\displaystyle\int_{-\infty}^a f(x)\ dx = \underset{t \to -\infty}{lim} \int_t^a f(x)\ dx$

   In either case, if the limit exists (and equals a value L) then the improper integral **converges** (to L) . If the limit does not exist then the improper integral **diverges** .

Remark :
If f is continuous on $(-\infty, \infty)$ then for any constant a
$\displaystyle\int_{-\infty}^\infty f(x)\ dx = \underset{t \to -\infty}{lim} \int_t^a f(x)\ dx + \underset{t \to \infty}{lim} \int_a^t f(x)\ dx$
If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$). If at least one of the limits does not exist then the improper integral **diverges** .

Examples :

1. $\displaystyle\int_0^\infty xe^{-x}\ dx$

   The function xe^{-x} is continuous on $[0,\infty)$

   $$\int_0^\infty xe^{-x}\ dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx$$

   Using integration by parts

   $$\begin{array}{ll} u = x & dv = e^{-x}\ dx \\ du = dx & v = -e^{-x} \end{array}$$

   $$\int_0^\infty xe^{-x}\ dx = \lim_{t\to\infty}\left([-xe^{-x}]_0^t - \int_0^t -e^{-x}\ dx\right)$$

   $$= \lim_{t\to\infty}\left([-xe^{-x}]_0^t - [e^{-x}]_0^t\right)$$

   $$= \lim_{t\to\infty}\left([(-te^{-t}) - ((0)e^0)] - [(e^{-t} - e^0]\right) = \lim_{t\to\infty}\left(\frac{-t}{e^t} - e^{-t} + 1\right)$$

   Note that $\displaystyle\lim_{t\to\infty}\frac{-t}{e^t}\qquad\left(\frac{-\infty}{\infty}\right)$

   Apply L'Hôpital's rule

   $$\lim_{t\to\infty}\frac{-t}{e^t} = \lim_{t\to\infty}\frac{-1}{e^t} = 0$$

   Therefore, $\displaystyle\lim_{t\to\infty}\left(\frac{-t}{e^t} - e^{-t} + 1\right) = 0 - 0 + 1 = 1$

   Hence, $\displaystyle\int_0^\infty xe^{-x}\ dx$ converges to 1 .

2. $\displaystyle\int_1^\infty \frac{\ln x}{x}\ dx$

   The function $\dfrac{\ln x}{x}$ is continuous on $[1,\infty)$

   $$\int_1^\infty \frac{\ln x}{x}\ dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\ dx = \lim_{t\to\infty}\int_1^t \ln x\ \frac{1}{x}\ dx$$

   $$= \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\left[\frac{(\ln t)^2}{2} - \frac{\ln(1)}{2}\right] = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty$$

   Therefore, $\displaystyle\int_1^\infty \frac{\ln x}{x}\ dx$ diverges .

3. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\ dx$

   The function $\dfrac{1}{1+x^2}$ is continuous on $(-\infty,\infty)$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\ dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\ dx$$

$$= \lim_{t\to-\infty} \left[\tan^{-1} x\right]_t^0 + \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t$$

$$= \lim_{t\to-\infty} \left[\tan^{-1}(0) - \tan^{-1} t\right] + \lim_{t\to\infty} \left[\tan^{-1} t - \tan^{-1}(0)\right]$$

$$= \tan^{-1}(0) - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2} + \frac{\pi}{2} = \pi\ .$$

Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx$ convereges to π .

4. $\displaystyle\int_0^1 \frac{x}{(x^2-1)^3}\ dx$

The function $\dfrac{x}{(x^2-1)^3}$ is not continuous $x = 1$.

$$\int_0^1 \frac{x}{(x^2-1)^3}\ dx = \lim_{t\to1^-} \int_0^t \frac{x}{(x^2-1)^3}\ dx = \lim_{t\to1^-} \frac{1}{2} \int_0^t \frac{2x}{(x^2-1)^3}\ dx$$

$$= \lim_{t\to1^-} \frac{1}{2} \left[\frac{(x^2-1)^{-2}}{-2}\right]_0^t = \lim_{t\to1^-} -\frac{1}{4} \left[\frac{1}{(t^2-1)^2} - \frac{1}{(0-1)^2}\right]$$

$$= \lim_{t\to1^-} -\frac{1}{4} \left[\frac{1}{(t^2-1)^2} - 1\right] = -\infty$$

Therefore, $\displaystyle\int_0^1 \frac{x}{(x^2-1)^3}\ dx$ diverges .

5. $\displaystyle\int_1^e \frac{1}{x\sqrt{\ln x}}\ dx$

The function $\dfrac{1}{x\sqrt{\ln x}}$ is not continuous at $x = 1$

$$\int_1^e \frac{1}{x\sqrt{\ln x}}\ dx = \lim_{t\to1^+} \int_t^e \frac{1}{x\sqrt{\ln x}}\ dx = \lim_{t\to1^+} \int_t^e (\ln x)^{-\frac{1}{2}}\ \frac{1}{x}\ dx$$

$$= \lim_{t\to1^+} \left[2(\ln x)^{\frac{1}{2}}\right]_t^e = \lim_{t\to1^+} 2\left[\sqrt{\ln(e)} - \sqrt{\ln t}\right]$$

$$= \lim_{t\to1^+} 2\left[1 - \sqrt{\ln t}\right] = 2[1-0] = 2$$

Therefore, $\displaystyle\int_1^e \frac{1}{x\sqrt{\ln x}}\ dx$ converges to 2 .

6. $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\ dx$

The function $\dfrac{1}{x\sqrt{x^2-1}}$ is not continuous at $x = 1$.

$$\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= \lim_{t\to 1^+} \int_t^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= \lim_{t\to 1^+} \left[\sec^{-1} x\right]_t^2 + \lim_{t\to\infty} \left[\sec^{-1} x\right]_2^t$$

$$= \lim_{t\to 1^+} \left[\sec^{-1}(2) - \sec^{-1} t\right] + \lim_{t\to\infty} \left[\sec^{-1} t - \sec^{-1}(2)\right]$$

$$= \sec^{-1}(2) - 0 + \frac{\pi}{2} - \sec^{-1}(2) = \frac{\pi}{2}\ .$$

Therefore, $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$ converges to $\dfrac{\pi}{2}$.

Exercises :
Determine whether the following improper integrals convverge or diverge

1. $\displaystyle\int_{-\infty}^0 e^x dx$
2. $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$

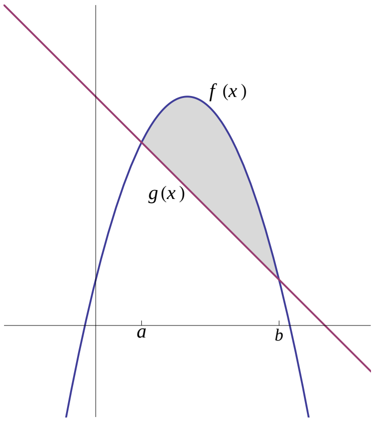
   Hint : $\dfrac{1}{\sqrt[3]{x}\ dx}$ is not continuous at $x = 0$
3. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$

   Hint : $\dfrac{\cos x}{\sqrt{\sin x}}$ is not continuous at $x = 0$
4. Show that the improper integral $\displaystyle\int_0^1 x\ln x\ dx$ converges .
5. $\displaystyle\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx$

   Hint : $\dfrac{1}{\sqrt{2x-x^2}}$ is not continuous at $x = 0$, complete the square
6. $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$

   Hint : $\dfrac{1}{x^2}$ is not continuous at $x = 0$

AREA BETWEEN CURVES



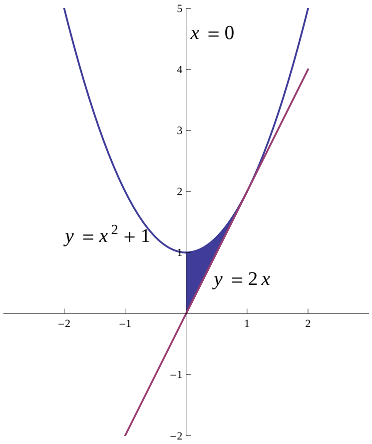
In the above figure the graphs of $f(x)$ and $g(x)$ intersect at the points $x = a$ and $x = b$.

The area bounded by the graphs of the curves of $f(x)$ and $g(x)$ equals

$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b [f(x) - g(x)]\ dx$$

Examples :

1. Find the area bounded by the graphs of the curves of $y = x^2 + 1$, $y = 2x$ and $x = 0$.



Note that $y = x^2+1$ is a parabola opens upward with vertex $(0, 1)$, $y = 2x$ is a straight line passing through the origin and $x = 0$ is the y-axis .

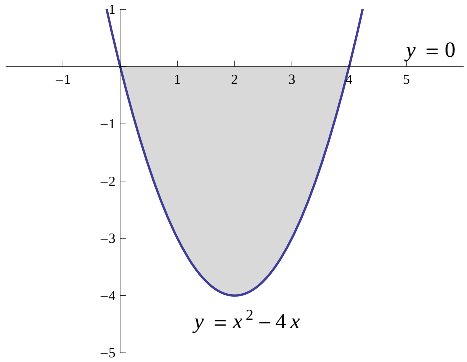
Points of intersetion between $y = x^2 + 1$ and $y = 2x$ is :

$x^2 + 1 = 2x \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow (x - 1)^2 = 0 \Rightarrow x = 1$

The desired area $= \int_0^1 [(x^2-1)-2x]\ dx = \int_0^1 (x-1)^2\ dx$

$= \left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{(1-1)^3}{3} - \frac{(0-1)^3}{3} = \frac{1}{3}$.

2. Find the area bounded by the graphs of the curves of $y = x^2 - 4x$ and $y = 0$



Note that $x^2 - 4x = (x^2 - 4x + 4) - 4 = (x-2)^2 + 4$ is a parabola opens upward with vertex $(2,-4)$ and $y = 0$ is the x-axis .

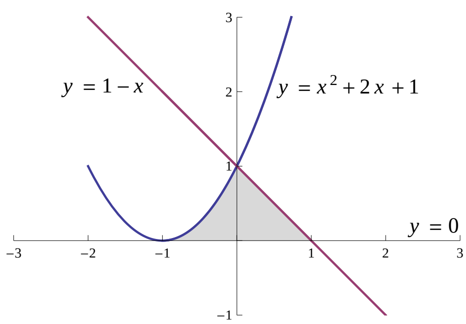
Points of intersection between $y = x^2 - 4x$ and $y = 0$

$x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x = 0 \quad , \quad x = 4.$

The desired area $= \int_0^4 [0-(x^2-4x)]\ dx = \int_0^4 (4x - x^2)\ dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left[\left(2(4)^2 - \frac{(4)^3}{3}\right) - 0\right] = 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$.

3. Find the area bounded by the graphs of the curves of $y = x^2 + 2x + 1$, $y = 1 - x$ and $y = 0$.



Note that $y = x^2 + 2x + 1 = (x+1)^2$ is a parabola opens upward with vertex (-1,0) , $y = 1 - x$ is a straight line and $y = 0$ is the x-axis .

Points of intersection between $y = x^2 + 2x + 1$ and $y = 1 - x$

$x^2 + 2x + 1 = 1 - x \Rightarrow x^2 + 3x = 0 \Rightarrow x(x+3) = 0 \Rightarrow x = 0 \quad , \quad x = -3$.

Points of intersection between $y = x^2 + 2x + 1$ and $y = 0$ is $x = -1$.

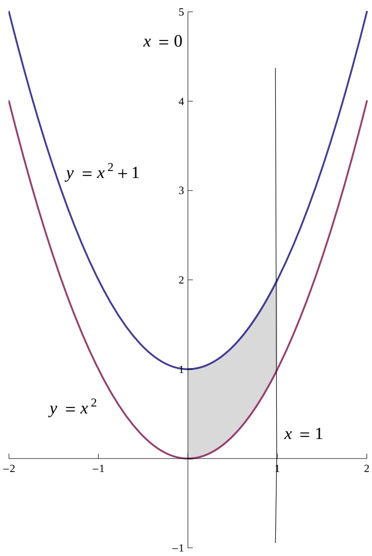
Points of intersection between $y = 1 - x$ and $y = 0$ is $x = 1$.

The desired area $= \displaystyle\int_{-1}^{0} (x^2 + 2x + 1)\ dx + \int_{0}^{1} (1 - x)\ dx$

$$= \left[\frac{(x+1)^3}{3}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1} = \left[\frac{(0+1)^3}{3} - \frac{(-1+1)^3}{3}\right] + \left[\left(1 - \frac{(1)^2}{2}\right) - 0\right]$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \ .$$

4. Find the area bounded by the graphs of the curves of $y = x^2$, $y = x^2 + 1$, $x = 0$ and $x = 1$.

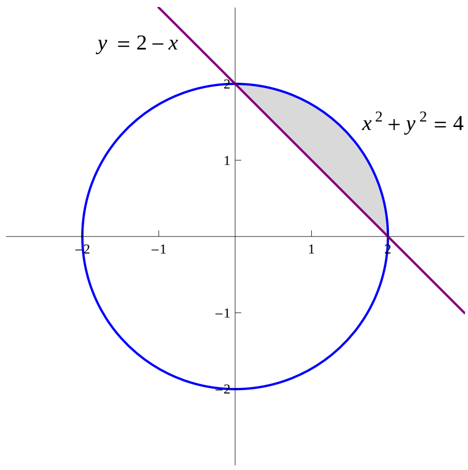


Note that $x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $y = x^2$ is another parabola opens upward with vertex $(0, 0)$, $x = 0$ is the y-axis and $x = 1$ is a straight line parallel to the y-axis and passing through the point $(1, 0)$.

Note also that $y = x^2 + 1$ and $y = x^2$ do not intersect .

The desired area $= \displaystyle\int_{0}^{1} [(x^2 + 1) - x^2]\ dx = \int_{0}^{1} dx = [x]_0^1 = 1 - 0 = 1$

5. Find the area inside the graph of the curve $x^2+y^2=4$ and above $x+y=2$.



NOTE : The desired area is one fourth of the area of the circle minus the area of the triangle which equals to $\pi - 2$

Note that $x^2+y^2=4$ is a circle with center $=(0,0)$ and radius $=2$ and $y=2-x$ is a straight line.

Points of intersection between $x^2+y^2=4$ and $y=2-x$

$x^2+(2-x)^2=4 \Rightarrow x^2+4-4x+x^2=4 \Rightarrow 2x^2-4x=0$

$\Rightarrow x^2-2x=0 \Rightarrow x(x-2)=0 \Rightarrow x=0\ ,\ x=2$

Note also that $x^2+y^2=4 \Rightarrow y=\pm\sqrt{4-x^2}$, where $\sqrt{4-x^2}$ represents the upper half of the circle and $-\sqrt{4-x^2}$ represents the lower half of the circle .

The desired area $= \displaystyle\int_0^2 \sqrt{4-x^2}\ dx - \int_0^2 (2-x)\ dx = I_1 - I_2$

$I_1 = \displaystyle\int_0^2 \sqrt{4-x^2}\ dx$

Put $x=2\sin\theta \Rightarrow dx = 2\cos\theta\ d\theta$

If $x=0 \Rightarrow 2\sin\theta=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$

If $x=2 \Rightarrow 2\sin\theta=2 \Rightarrow \sin\theta=1 \Rightarrow \theta=\frac{\pi}{2}$

$I_1 = \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2}\ 2\cos\theta\ d\theta = \int_0^{\frac{\pi}{2}} 4\cos^2\theta\ d\theta$

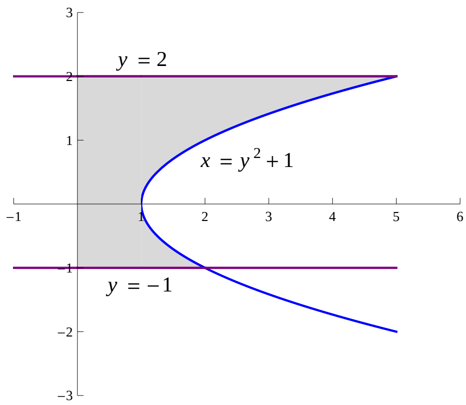
$= 4\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2\theta]\ d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= 2\left[\left(\dfrac{\pi}{2}+\dfrac{\sin\pi}{2}\right)-\left(0+\dfrac{\sin(0)}{2}\right)\right] = 2\left[\left(\dfrac{\pi}{2}+0\right)-(0+0)\right] = 2\ \dfrac{\pi}{2} = \pi$

$$I_2 = \int_0^2 (2-x)\ dx = \left[2x - \frac{x^2}{2}\right]_0^2 = \left[\left(2(2) - \frac{2^2}{2}\right) - (0-0)\right] = 4-2 = 2$$

Hence , The desired area $= I_1 - I_2 = \pi - 2$.

6. Find the area bounded by the graphs of the curves of $x = y^2 + 1$, $x = 0$, $y = -1$ and $y = 2$.



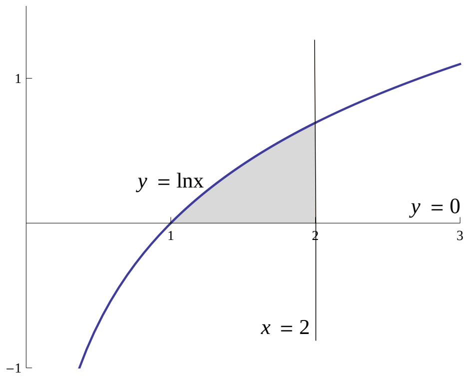
Note that $x = y^2 + 1$ is a parabola opens to the right with vertex $(1, 0)$, $x = 0$ is the y-axis , $y = 2$ is a straight line parallel to the x-axis and passing through the point $(0, 2)$ also $y = -1$ is another straight line parallel to the x-axis and passing through the point $(0, -1)$.

The desired area $= \displaystyle\int_{-1}^2 (y^2+1)\ dy = \left[\frac{y^3}{3} + y\right]_{-1}^2$

$$= \left[\left(\frac{(2)^3}{3} + 2\right) - \left(\frac{(-1)^3}{3} + (-1)\right)\right] = \frac{8}{3} + 2 + \frac{1}{3} + 1 = \frac{18}{3} = 6$$

Examples : Set up integrals to evaluate the areas bounded by the graphs of the curves of :

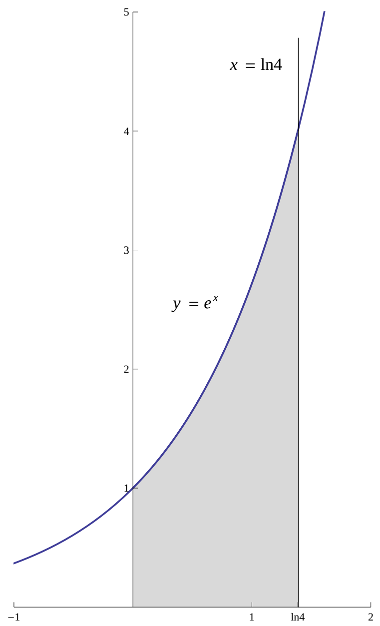
1. $y = \ln x$, $y = 0$ and $x = 2$.



Note that $y = \ln x$ intersects the x-axis at $x = 1$

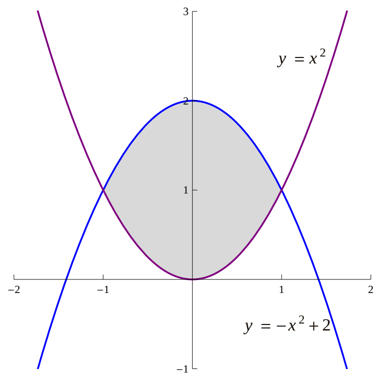
The desired area $= \displaystyle\int_1^2 \ln x \; dx$

2. $y = e^x$, $x = \ln 4$, $x = 0$ and $y = 0$.



The desired area $= \displaystyle\int_0^{\ln 4} e^x \; dx$

3. $y = x^2$ and $y = -x^2 + 2$



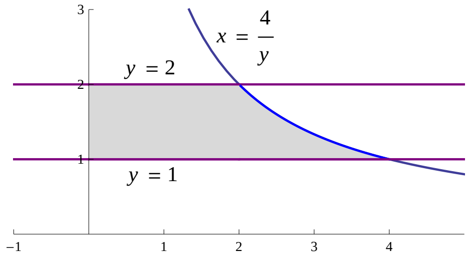
Note that $y = x^2$ is a parabola opens upward with vertex $(0, 0)$ and $y = -x^2 + 2$ is another parabola opens downward with vertex $(0, 2)$

Points of intersection between $y = x^2$ and $y = -x^2 + 2$

$x^2 = -x^2 + 2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

The desired area $= \int_{-1}^{1} [(-x^2+2) - x^2]\ dx$

4. $y = \dfrac{4}{x}$, $x = 0$, $y = 1$ and $y = 2$.

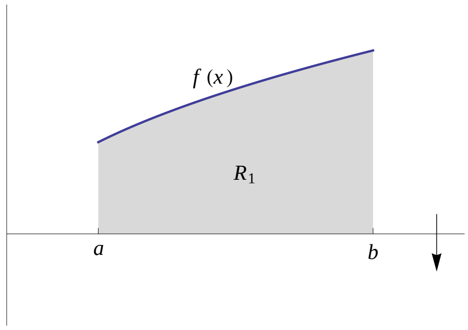


The desired area $= \int_{1}^{2} \dfrac{4}{y}\ dy$

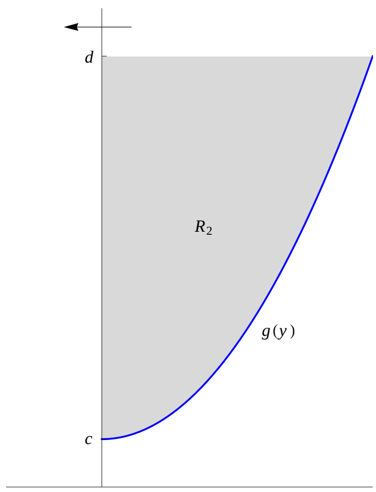
VOLUME OF A SOLID OF REVOLUTION
Disk or Washer method

1. Disk Method

Recall that the volume of a right circular cylinder equals $\pi r^2 h$ where r is the radius of the base (which is a circle) and h is the hight of the cylinder .



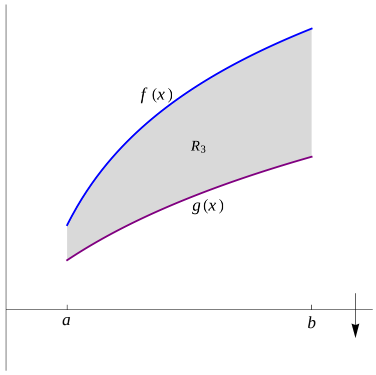
In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.
Using disk method , the volume of the solid of revolution generated by revolving the region R_1 around the x-axis is $V = \pi \int_a^b [f(x)]^2 \, dx$



In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.
Using disk method , the volume of the solid of revolution generated by revolving the region R_2 around the y-axis is $V = \pi \int_c^d [g(y)]^2 \, dy$

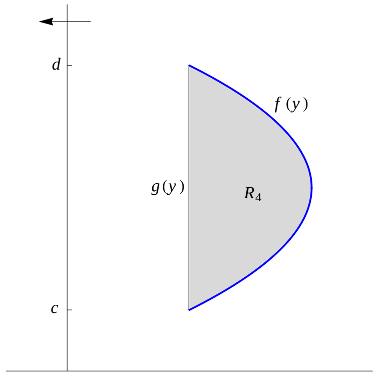
2. Washer Method

$Volume\ of\ a\ washer\ = \pi\left[(outer\ radius)^2 - (inner\ radius)^2\right]\ (thickness)$



In the above figure R_3 is the region bounded by the graphs of the curves of $f(x)$, $g(x)$, $x = a$ and $x = b$.

Using washer method , the volume of the solid of revolution generated by revolving the region R_3 around the x-axis is $V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right]\ dx$

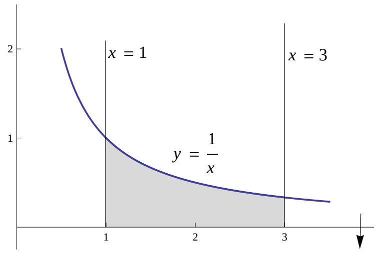


In the above figure R_4 is the region bounded by the graphs of the curves of $f(y)$ and $g(y)$, where $f(y)$ and $g(y)$ intersect at the points $y = c$ and $y = d$.

Using washer method , the volume of the solid of revolution generated by revolving the region R_4 around the y-axis is $V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2\right]\ dy$

Examples : Use disk or washer method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the curves of :

1. $y = \dfrac{1}{x}$, $x = 1$, $x = 3$ and $y = 0$, around the x-axis.

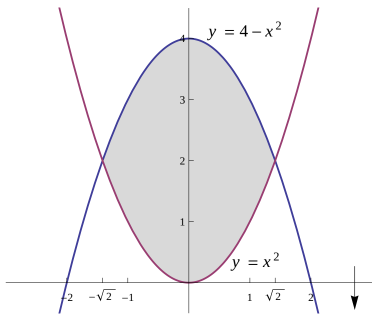


   Using Disk Method

   $$V = \pi \int_1^3 \left(\frac{1}{x}\right)^2 \, dx = \pi \int_1^3 x^{-2} \, dx.$$

   $$V = \pi \left[-\frac{1}{x}\right]_1^3 = \pi \left[-\frac{1}{3} + 1\right] = \frac{2}{3}\pi$$

2. $y = x^2$ and $y = 4 - x^2$, around the x-axis .



   Note that $y = x^2$ is a parabola opens upward with vertex $(0, 0)$ and $y = 4 - x^2$ is a parabola opens downward with vertex $(0, 4)$.

   Points of intersection between $y = x^2$ and $y = 4 - x^2$:

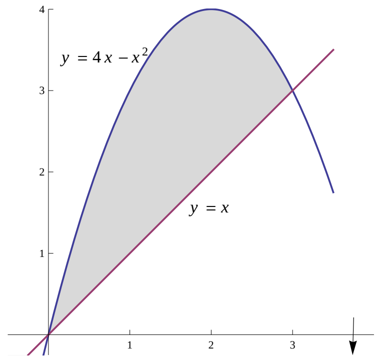
   $x^2 = 4 - x^2 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

   Using Washer Method

   $$V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[(4 - x^2)^2 - (x^2)^2\right] \, dx = 2\pi \int_0^{\sqrt{2}} [16 - 8x^2 + x^4 - x^4] \, dx$$

   $$= 2\pi \int_0^{\sqrt{2}} (16 - 8x^2) \, dx = 2\pi \left[16x - \frac{8}{3}x^3\right]_0^{\sqrt{2}} = \frac{64\sqrt{2}}{3}\pi$$

3. $y = 4x - x^2$ and $y = x$, around the x-axis .



$4x - x^2 = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2$ is a parabola opens downward with vertex (2,4) and $y = x$ is a straight line passing through the origin.

Points of intersection between $y = 4x - x^2$ and $y = x$

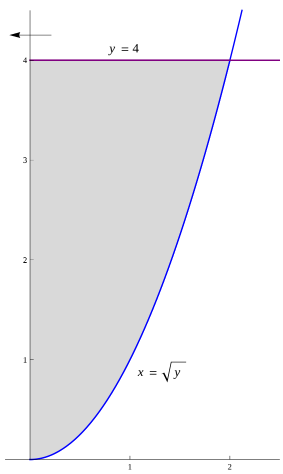
$x = 4x - x^2 x^2 - 3x = 0 \Rightarrow x(x-3) = 0 \Rightarrow x = 0$, $x = 3$

Using Washer Method

$$V = \pi \int_0^3 \left[(4x - x^2)^2 - (x)^2\right] \, dx = \pi \int_0^3 \left[16x^2 - 8x^3 + x^4 - x^2\right] \, dx$$

$$= \pi \int_0^3 \left[x^4 - 8x^3 + 15x^2\right] \, dx = \pi \left[\frac{x^5}{5} - 2x^4 + 5x^3\right]_0^3 = \frac{108}{5}\pi$$

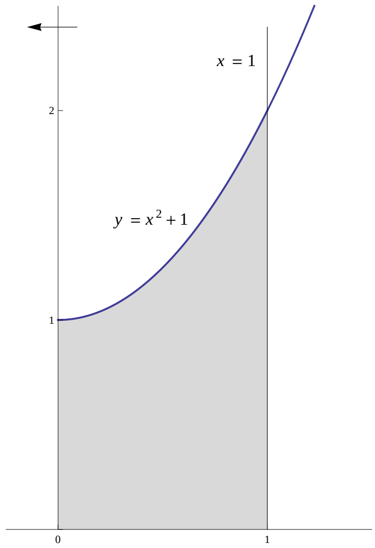
4. $x = \sqrt{y}$, $x = 0$ and $y = 4$, around the y-axis



Using Disk Method

$$V = \pi \int_0^4 (\sqrt{y})^2 \, dy = \pi \int_0^4 y \, dy = \pi \left[\frac{y^2}{2}\right]_0^4 = 8\pi$$

5. $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$, around the y-axis .



Note that $y = x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $x = 1$ is a straight line parallel to the y-axis and passing through the point $(1, 0)$

Point of intersection between $y = x^2 + 1$ and $x = 1$ is $(1, 2)$.

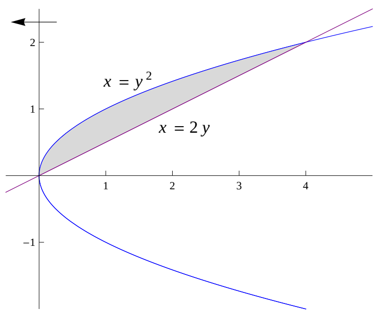
$y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \pm\sqrt{y-1}$, where $x = \sqrt{y-1}$ is the right half of the parabola and $y = -\sqrt{y-1}$ is the left half of the parabola .

Using Washer Method

$$V = \pi \int_0^2 (1)^2 \ dy - \pi \int_1^2 \left(\sqrt{y-1}\right)^2 \ dy$$

$$V = \pi [y]_0^2 - \pi \left[\frac{y^2}{2} - y\right]_1^2 = \frac{3}{2}\pi$$

6. $x = y^2$ and $x = 2y$, around the y-axis .



Note that $x = y^2$ is a parabola opens to the right with vertex $(0, 0)$ and $x = 2y$ is a straight line passing through the origin.

Points of intersection between $x = y^2$ and $x = 2y$

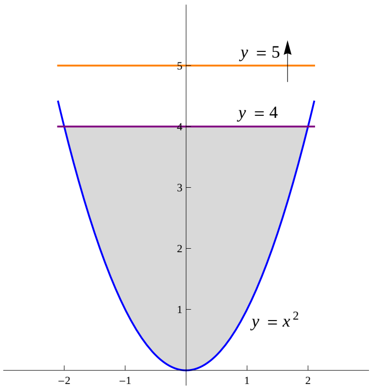
$y^2 = 2y \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0 \Rightarrow y = 0\ ,\ y = 2$

Using Washer Method

$$V = \pi \int_0^2 \left[(2y)^2 - (y^2)^2\right]\ dy = \pi \int_0^2 (4y^2 - y^4)\ dy$$

$$V = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5}\right]_0^2 = \frac{64}{15}\pi$$

7. $y = x^2$ and $y = 4$, around the line $y = 5$.



Note that $y = x^2$ is a parabola opens upward with vertex $(0,0)$ and $y = 4$ is a straight line parallel to the x-axis and passing through $(0,4)$.

Points of intersection between $y = x^2$ and $y = 4$

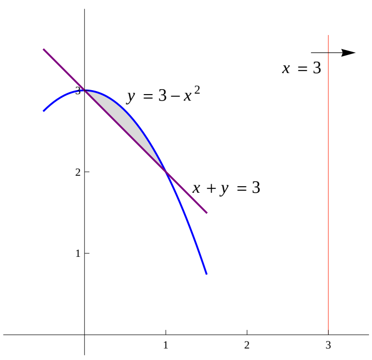
$x^2 = 4 \Rightarrow x = \pm 2$

Using Washer Method

$$V = \pi \int_{-2}^2 \left[(5 - x^2)^2 - (5-4)^2\right]\ dx = \pi \int_{-2}^2 (24 - 10x^2 + x^4)\ dx$$

$$V = \pi \left[24x - \frac{10x^3}{3} + \frac{x^5}{5}\right]_{-2}^2 = \frac{832}{15}\pi$$

8. $y + x^2 = 3$ and $y + x = 3$, around the line $x = 3$



Note that $y = 3 - x^2$ is a parabola opens downward with vertex $(0,3)$ and $x + y = 3$ is a straigh line.

Points of intersection between $y + x^2 = 3$ and $x + y = 3$

$y + x^2 = x + y \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0\ ,\ x = 1$

$\Rightarrow y = 2\ ,\ y = 3$

$y + x^2 = 3 \Rightarrow x^2 = 3 - y \Rightarrow x = \pm\sqrt{3-y}$, where $x = \sqrt{3-y}$ is the right half of the parabola and $x = -\sqrt{3-y}$ is the left half of the parabola .

Using Washer Method

$$V = \pi \int_2^3 \left[(3 - (3-y))^2 - (3 - \sqrt{3-y})^2 \right]\ dy$$

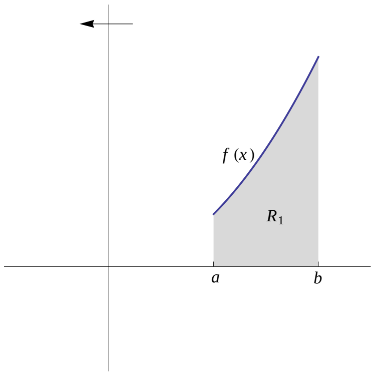
$$= \pi \int_2^3 \left[y^2 - \left(9 - 6\sqrt{3-y} + 3 - y \right) \right]\ dy$$

$$= \pi \int_2^3 (y^2 + y + 6\sqrt{3-y} - 12)\ dy$$

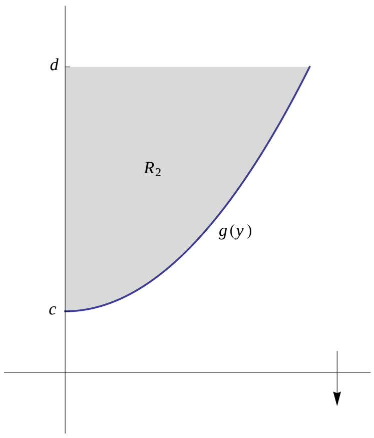
$$V = \pi \left[\frac{y^3}{3} + \frac{y^2}{2} - 4(3-y)^{\frac{3}{2}} - 12y \right]_2^3 = \frac{5}{6}\pi$$

VOLUME OF A SOLID OF REVOLUTION
Cylindrical shells method

$$Volume\ of\ a\ shell\ = 2\pi\ (average\ radius)\ (altitude)\ (thickness)$$



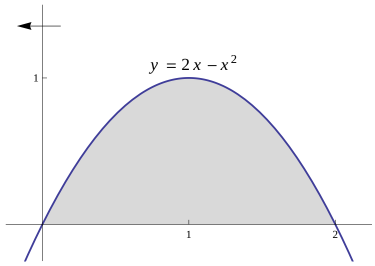
In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.
Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_1 around the y-axis is $V = 2\pi \int_a^b x\ f(x)\ dx$



In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.
Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_2 around the x-axis is $V = 2\pi \int_c^d y\ g(y)\ dy$

Examples : Use cylindrical shells method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the curves of :

1. $y = 2x - x^2$ and $y = 0$, around the y-axis .



   $y = 2x - x^2 = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$ is a parabola opens downward with vertex $(1, 1)$

   Points of intersection between $y = 2x - x^2$ and $y = 0$

   $2x - x^2 = 0 \Rightarrow x(2-x) = 0 \Rightarrow x = 0\ ,\ x = 2$

   Using Cylindrical shells method

   $$V = 2\pi \int_0^2 x(2x - x^2)\ dx = 2\pi \int_0^2 (2x^2 - x^3)\ dx$$

   $$V = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{8}{3}\pi$$

2. $y = \cos x$, $y = 2x + 1$ and $x = \frac{\pi}{2}$, around the y-axis .

   Recall that $\cos(0) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$.

   The line $y = 2x + 1$ passes through the point $(0, 1)$.

   The desired region is under the line $y = 2x + 1$ and above the curve of $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$

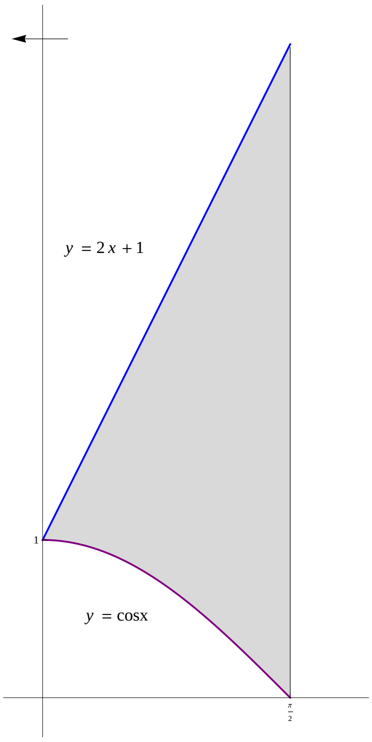
   Using Cylindricall shells method

   $$V = 2\pi \int_0^{\frac{\pi}{2}} x\left[(2x+1) - \cos x\right]\ dx$$

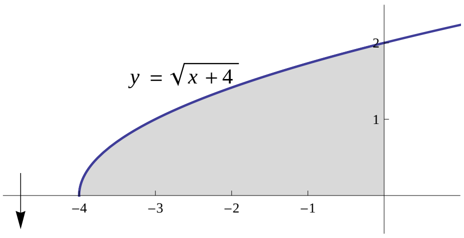
   $$V = 2\pi \int_0^{\frac{\pi}{2}} (2x^2 + x)\ dx - 2\pi \int_0^{\frac{\pi}{2}} (x\cos x)\ dx$$

   $$V = 2\pi \left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - 2\pi\left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}}$$

   $$V = 2\pi\left(\frac{\pi^3}{12} + \frac{\pi^2}{8}\right) - 2\pi\left(\frac{\pi}{2} - 1\right)$$



3. $y = \sqrt{x+4}$, $y = 0$ and $x = 0$, around the x-axis .



$y = \sqrt{x+4}$ is the upper half of the parabola $x = y^2 - 4$ which opens to the right with vertex $(-4, 0)$.

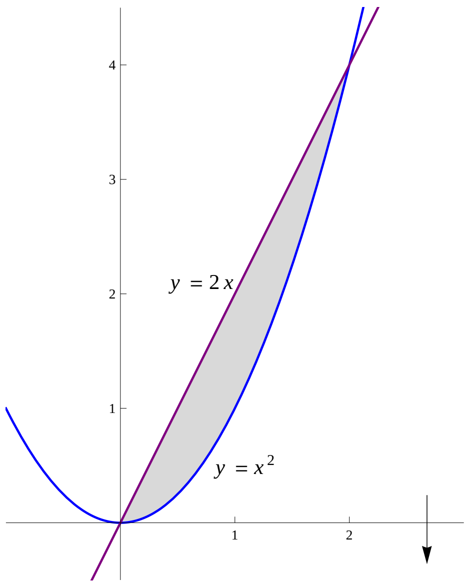
$y = \sqrt{x+4}$ intersects the x-axis at the point $(-4, 0)$ and intersects the y-axis at $(0, 2)$

Using Cylindricall shells method

$$V = 2\pi \int_0^2 y[-(y^2 - 4)]\ dy = 2 \int_0^2 (4y - y^3)\ dy$$

$$V = 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi$$

4. $y = x^2$ and $y = 2x$, around the x-axis .



$y = x^2$ is a parabola open upward with vertex $(0,0)$ and $y = 2x$ is a straight line passing through the origin.

Points of intersection between $y = x^2$ and $y = 2x$

$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0 \ , \ x = 2$

$\Rightarrow y = 0 \ , \ y = 4$

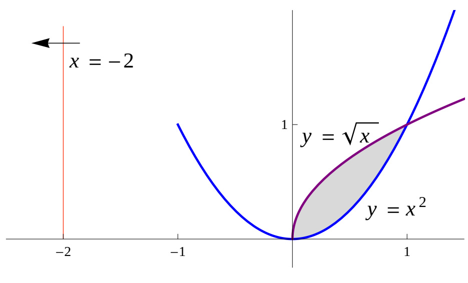
$y = x^2 \Rightarrow x = \pm\sqrt{y}$, where $x = \sqrt{y}$ is the right half of the parabola $y = x^2$ and $x = -\sqrt{y}$ is the left half of the parabola.

Using Cylindrical shells method

$$V = 2\pi \int_0^4 y\left(\sqrt{y} - \frac{y}{2}\right)\ dy = 2\pi \int_0^4 \left(y^{\frac{3}{2}} - \frac{y^2}{2}\right)\ dy$$

$$V = 2\pi \left[\frac{2y^{\frac{5}{2}}}{5} - \frac{y^3}{6}\right]_0^4 = \frac{64}{15}\pi$$

5. $y = \sqrt{x}$ and $y = x^2$, around the line $x = -2$.



$y = x^2$ is a parabola opens upward with vertex $(0,0)$, and $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$.

Points of intersection between $y = x^2$ and $y = \sqrt{x}$

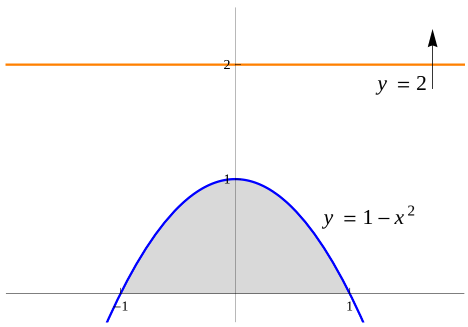
$x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0 \ , \ x = 1$

Using Cylindrical shells method

$$V = 2\pi \int_0^1 (x+2)(\sqrt{x} - x^2) \ dx = 2\pi \int_0^1 (-x^3 - 2x^2 + x^{\frac{3}{2}} + 2x^{\frac{1}{2}}) \ dx$$

$$V = 2\pi \left[-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{2x^{\frac{5}{2}}}{5} + \frac{x^{\frac{3}{2}}}{3} \right]_0^1 = \frac{49}{30}\pi$$

6. $y = 1 - x^2$ and $y = 0$, around the line $y = 2$.



$y = 1 - x^2$ is a parabola opens downward with vertex $(0,1)$ and $y = 0$ is the x-axis .

$y = 1 - x^2$ intersects $y = 0$ at $x = \pm 1$.

$y = 1 - x^2 \Rightarrow x^2 = 1 - y \Rightarrow x = \pm\sqrt{1-y}$, where $y = \sqrt{1-y}$ represents the right half of the parabola and $y = -\sqrt{1-y}$ represents the left half .

Note that the region is symmetric with respect to the y-axis.

Using Cylindrical shells method

$$V = 2\left(2\pi \int_0^1 (2-y)\sqrt{1-y} \ dy\right)$$

Put $u^2 = 1 - y$ then $2u \ du = - \ dy$

If $y = 0$ then $u = 1$, and if $y = 1$ then $u = 0$

$$V = 4\pi \int_1^0 (2 + u^2 - 1) \ u \ (-2u) \ du = 4\pi \int_0^1 (u^2+1)2u^2 \ du$$

$$V = 4\pi \int_0^1 (2u^4 + 2u^2) \ du = 4\pi \left[\frac{2u^5}{5} + \frac{2u^3}{3}\right]_0^1 = \frac{64}{15}\pi$$

ARC LENGTH

If $f(x)$ is continuous function on the interval $[a, b]$, then the arc length of $f(x)$ from $x = a$ to $x = b$ is $L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$

If $g(y)$ is continuous function on the interval $[c, d]$, then the arc length of $g(y)$ from $y = c$ to $y = d$ is $L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$

Examples : Find the arc length of the following :

1. $y = \dfrac{x^3}{12} + \dfrac{1}{x}$ from $A = \left(1, \frac{13}{12}\right)$ to $B = \left(2, \frac{7}{6}\right)$.

   $$f(x) = \frac{x^3}{12} + \frac{1}{x} \Rightarrow f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

   $$L = \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\ dx = \int_1^2 \sqrt{1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}}\ dx$$

   $$= \int_1^2 \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}\ dx = \int_1^2 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\ dx = \int_1^2 \left|\frac{x^2}{4} + \frac{1}{x^2}\right|\ dx$$

   $$L = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right)\ dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^2 = \frac{13}{12}$$

2. $y = \dfrac{1}{2}\left(e^x + e^{-x}\right)$, $x \in [0, 2]$

   $$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x \Rightarrow f'(x) = \sinh x$$

   $$L = \int_0^2 \sqrt{1 + \sinh^2 x}\ dx = \int_0^2 \sqrt{\cosh^2 x}\ dx$$

   $$= \int_0^2 |\cosh x|\ dx = \int_0^2 \cosh x\ dx$$

   $$L = [\sinh x]_0^2 = \sinh(2) - \sinh(0) = \frac{e^2 - e^{-2}}{2} - 0 = \frac{e^2 - e^{-2}}{2}$$

3. $x^2 + y^2 = 25$, $-5 \leq y \leq 5$

   Note : In this problem the arc length is equal to half of the perimeter of the circle $x^2 + y^2 = 25$, the arc length is equal to 5π .

   $x^2 + y^2 = 25 \Rightarrow x^2 = 25 - y^2 \Rightarrow x = \pm\sqrt{25 - y^2}$, in this problem $x = \sqrt{25 - y^2}$

   $$g(y) = \sqrt{25 - y^2} \Rightarrow g'(y) = \frac{-y}{\sqrt{25 - y^2}}$$

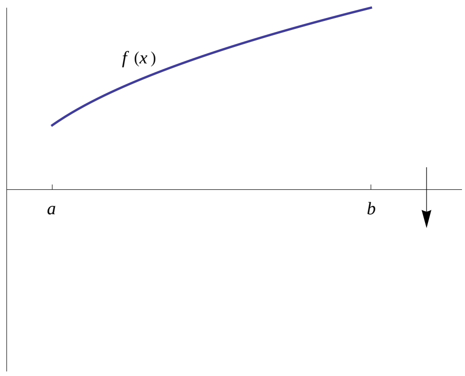
$$L=\int_{-5}^{5}\sqrt{1+\left(\frac{-y}{\sqrt{25-y^2}}\right)^2}\ dy=\int_{-5}^{5}\sqrt{1+\frac{y^2}{25-y^2}}\ dy$$

$$=\int_{-5}^{5}\sqrt{\frac{25-y^2+y^2}{25-y^2}}\ dy=5\int_{-5}^{5}\frac{1}{\sqrt{25-y^2}}\ dy$$

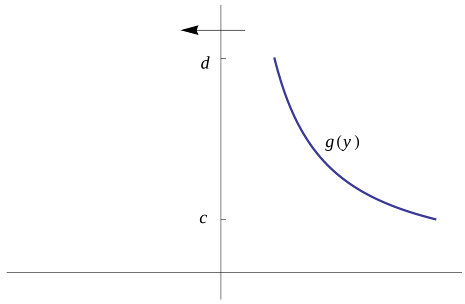
$$L=5\left[\sin^{-1}\left(\frac{y}{5}\right)\right]_{-5}^{5}=5\left[\sin^{-1}(1)-\sin^{-1}(-1)\right]$$

$$=5\left[\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)\right]=5\pi\ .$$

SURFACE AREA
(SURFACE OF REVOLUTION)



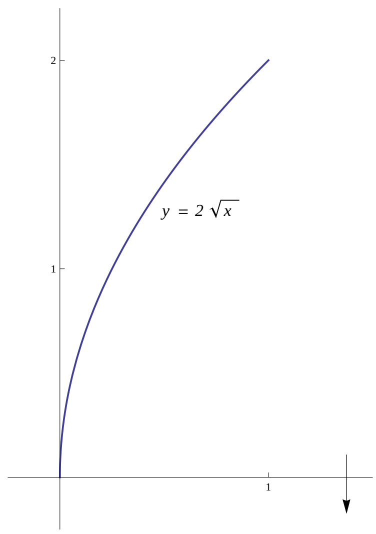
If $f(x)$ is a continuous function on the interval $[a,b]$, then the surface area generated by revolving the graph of the function $f(x)$ around the x-axis is
$SA = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\ dx$



If $g(y)$ is a continuous function on the interval $[c,d]$, then the surface area generated by revolving the graph of the function $g(y)$ around the y-axis is
$SA = 2\pi \int_c^d g(y)\sqrt{1+[g'(y)]^2}\ dy$

Examples : Find the surface area generated by revolving the following functions around the given axis :

1. $4x = y^2$, from $A = (0,0)$ to $B = (1,2)$, around the x-axis .



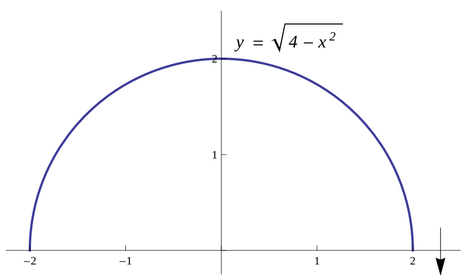
$4x = y^2 \Rightarrow y = \pm 2\sqrt{x}$

$f(x) = 2\sqrt{x} \Rightarrow f'(x) = \dfrac{1}{\sqrt{x}}$

$SA = 2\pi \displaystyle\int_0^1 2\sqrt{x}\sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2}\, dx = 4\pi \int_0^1 \sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$

$SA = 4\pi \displaystyle\int_0^1 \sqrt{x+1}\, dx = 4\pi \left[2\frac{(x+1)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{8\pi}{3}\left(2\sqrt{2} - 1\right)$

2. $y = \sqrt{4 - x^2}$, $x \in [-2, 2]$, around the x-axis .



Note : It is the surface area of the sphere with radius 2 , and it is equal to $4\pi(2)^2 = 16\pi$

$f(x) = \sqrt{4 - x^2} \Rightarrow f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$

$$SA = 2\pi \int_{-2}^{2} \sqrt{4-x^2} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2} \, dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{4-x^2} \sqrt{\frac{(4-x^2)+x^2}{4-x^2}} \, dx = 2\pi \int_{-2}^{2} \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} \, dx$$

$$SA = 4\pi \int_{-2}^{2} dx = 4\pi \left[x\right]_{-2}^{2} = 16\pi$$

3. $y = 2\sqrt[3]{x}$, from $A = (1,2)$ to $B = (8,4)$, around the y-axis .

$$y = 2\sqrt[3]{x} \Rightarrow \sqrt[3]{x} = \frac{y}{2} \Rightarrow x = \frac{y^3}{8}$$

$$g(y) = \frac{y^3}{8} \Rightarrow g'(y) = \frac{3}{8}y^2$$

$$SA = 2\pi \int_{2}^{4} \frac{y^3}{8} \sqrt{1+\left(\frac{3}{8}y^2\right)^2} \, dy = 2\pi \int_{2}^{4} \frac{y^3}{8} \sqrt{1+\frac{9}{64}y^4} \, dy$$

$$= 2\pi \frac{1}{8} \frac{16}{9} \int_{2}^{4} \left(1+\frac{9}{64}y^4\right)^{\frac{1}{2}} \left(\frac{9}{16}y^3\right) \, dy$$

$$SA = \frac{4\pi}{9} \left[2 \frac{\left(1+\frac{9}{64}y^4\right)^{\frac{3}{2}}}{3} \right]_{2}^{4}$$

4. $y = x^2$, $0 \leq x \leq 2$, around the y-axis .

$$y = x^2 \Rightarrow x = \pm\sqrt{y} \Rightarrow x = \sqrt{y} \ , \ \text{ since } 0 \leq x \leq 2$$

$$0 \leq x \leq 2 \Rightarrow 0 \leq y \leq 4$$

$$g(y) = \sqrt{y} \Rightarrow g'(y) = \frac{1}{2\sqrt{y}}$$

$$SA = 2\pi \int_{0}^{4} \sqrt{y} \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2} \, dy = 2\pi \int_{0}^{4} \sqrt{y} \sqrt{1+\frac{1}{4y}} \, dy$$

$$SA = 2\pi \int_{0}^{4} \sqrt{y+\frac{1}{4}} \, dy = 2\pi \left[\frac{2\left(y+\frac{1}{4}\right)^{\frac{3}{2}}}{3} \right]_{0}^{4}$$

PARAMETRIC EQUATIONS

Parametric equations are used to describe and represent plane curves.
The parameter "t" is used to write x and y as functions of t .
$C : x = x(t)$, $y = y(t)$; $a \leq t \leq b$ is the general form of a parametric curve , where $a, b \in \mathbb{R}$.
Any point on the parametric curve is represented by $P(t) = (x(t), y(t))$.

Notes :

1. If the parametric curve does not intersect itself then it is called a simple curve.
2. If $P(a) = P(b)$ then the parametric curve is called a closed curve.
3. Parametric equation of a curve indicates its orientation (direction of the path).

Examples : Sketch the graph of the following parametric curves :

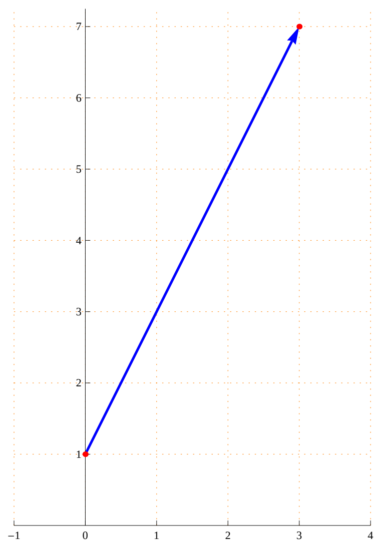
1. $C : \ x = t + 1$, $y = 2t + 3$; $-1 \leq t \leq 2$.

   $x = t + 1 \Rightarrow t = x - 1$

   $y = 2t + 3 \Rightarrow y = 2(x - 1) + 1 = 2x + 1$

t	-1	2
x	0	3
y	1	7

   The parametric equation represents a line segment from $(0, 1)$ to $(3, 7)$



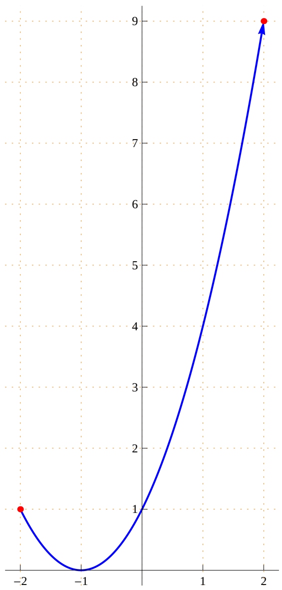
2. $C: \ x = t - 1 \ , \ y = t^2 \ ; -1 \leq t \leq 3$

$x = t - 1 \Rightarrow t = x + 1$

$y = t^2 \Rightarrow y = (x+1)^2$

t	-1	3
x	-2	2
y	1	9

The parametric equation represents a part of a parabola from $(-2, 1)$ to $(2, 9)$



3. $C: \ x = 1 + 3\cos t \ , \ y = -1 + 3\sin t \ ; 0 \leq t \leq 2\pi$

$x = 1 + 3\cos t \Rightarrow \cos t = \dfrac{x-1}{3}$

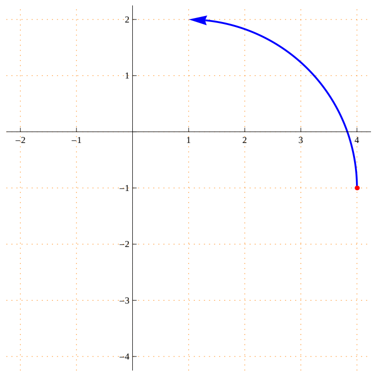
$y = -1 + 3\sin t \Rightarrow \sin t = \dfrac{y+1}{3}$

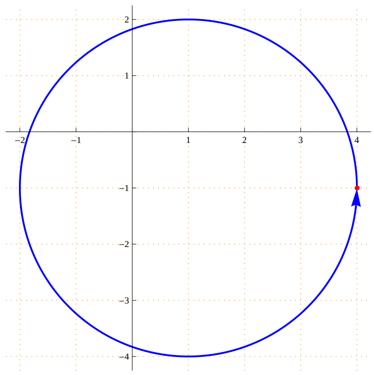
$\cos^2 t + \sin^2 t = 1 \Rightarrow \dfrac{(x-1)^2}{9} + \dfrac{(y+1)^2}{9} = 1 \Rightarrow (x-1)^2 + (y+1)^2 = 9$

t	0	$\frac{\pi}{2}$	2π
x	4	1	4
y	-1	2	-1

The parametric equation represents a circle with center $=(1, -1)$ and radius $= 3$.

It is a closed curve and its direction is counter-clockwise.





4. $C: \; x = 3 + 3\cos t \;,\; y = 2 + 2\sin t \;;\; 0 \le t \le 2\pi$

$$x = 3 + 3\cos t \Rightarrow \cos t = \frac{x-3}{3}$$

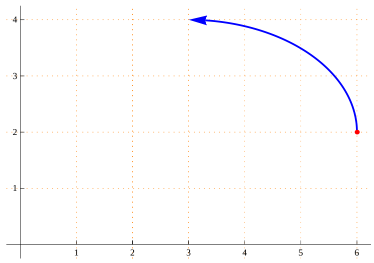
$$y = 2 + 2\sin t \Rightarrow \sin t = \frac{y-2}{2}$$

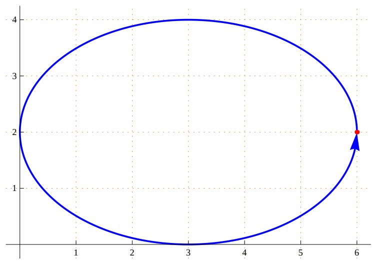
$$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

t	0	$\frac{\pi}{2}$	2π
x	6	3	6
y	2	4	2

The parametric equation represents an ellipse with center $=(3,2)$,the endpoints of the major axis are $(0,2)$, $(6,2)$ (its length is 6) and the endpoints of the minor axis are $(3,0)$, $(3,4)$ (its length is 4).

it is a closed curve and its direction is counter-clockwise.





The slope of the tangent line to a parametric curve

If $C: \ x = x(t) \ , \ y = y(t) \ ; \ a \le t \le b$ is a differentiable parametric curve then the slope of the tangent line to C at $t_0 \in [a, b]$ is

$$m = \frac{dy}{dx}|_{t=t_0} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}|_{t=t_0}$$

Notes :

1. The tangent line to the parametric curve is horizontal if the slope equals zero , which means that $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
2. The tangent line to the parametric curve is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)}$, where $y' = \frac{dy}{dx}$

Examples :

1. The slope of the tangent line to $C: \ x = t^3 + 1 \ , \ y = t^4 - 1$ at $t = 1$ is

   $(a)\ \frac{3}{4}$ $(b)\ 0$ $(c)\ \frac{4}{3}$ (d) None of these

   Answer : $m = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4t^3}{3t^2}$

   The slope at $t = 1$ is $m|_{t=1} = \frac{4}{3}$

   The right answer is (c) .

2. If $C: \ x = \sqrt{t} \ , \ y = \frac{1}{4}(t^2 - 1)$, find the first and second derivatives at $t = 4$.

   First derivative : $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{1}{2}t\right)}{\left(\frac{1}{2\sqrt{t}}\right)} = t^{\frac{3}{2}}$

   $\frac{dy}{dx}|_{t=4} = (4)^{\frac{3}{2}} = 8$.

   Second derivative : $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{3}{2}t^{\frac{1}{2}}\right)}{\left(\frac{1}{2\sqrt{t}}\right)} = 3t$

$\dfrac{d^2y}{dx^2}|_{t=4} = 3(4) = 12$.

3. If $C: \ x = 2\cos t$, $y = 2\sin t$, find the first and the second derivatives at $t = \dfrac{\pi}{4}$.

   First derivative : $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{2\cos t}{-2\sin t} = -\cot t$

   $\dfrac{dy}{dx}|_{t=\frac{\pi}{4}} = -\cot\left(\dfrac{\pi}{4}\right) = -1$.

   Second derivative : $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dx} = \dfrac{\left(\dfrac{dy'}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{\csc^2 t}{-2\sin t} = \dfrac{-1}{2\sin^3 t}$

   $\dfrac{d^2y}{dx^2}|_{t=\frac{\pi}{4}} = \dfrac{-1}{2\left(\frac{1}{\sqrt{2}}\right)^3} = \dfrac{-2\sqrt{2}}{2} = -\sqrt{2}$.

4. Find the equation of the tangent line to $C: \ x = t^3 - 3t$, $y = t^2 - 5t - 1$ at $t = 2$.

   $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{2t-5}{3t^2-3}$

   The slope of the tangent line is $\dfrac{dy}{dx}|_{t=2} = \dfrac{2(2)-5}{3(4)-3} = \dfrac{-1}{9}$

   At $t = 2: \ x = (2)^3 - 3(2) = 8 - 6 = 2$ and $y = (2)^2 - 5(2) - 1 = -7$

   The tangent line to C at $t = 2$ passes through the point $(2, -7)$ and its slope is $-\frac{1}{9}$, therefore its equation is $\dfrac{y+7}{x-2} = -\dfrac{1}{9}$

5. Find the points on $C: \ x = e^t$, $y = e^{-t}$ at which the slope of the tangent line to C equals $-e^{-2}$

   $m = \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{-e^{-t}}{e^t} = -e^{-2t}$

   $m = -e^{-2} \Rightarrow -e^{-2t} = -e^{-2} \Rightarrow t = 1$.

   At $t = 1: \ x = e^1 = e$ and $y = e^{-1} = \frac{1}{e}$.

   Hence, the point at which the slope of the tangent line to C equals $-e^{-2}$ is $\left(e, \frac{1}{e}\right)$.

6. Find the points on $C: \ x = 4 + 4\cos t \ , \ y = -1 + \sin t \ ; \ 0 \le t \le 2\pi$ at which the tangent line is : (a) Vertical , (b) Horizontal .

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-4\sin t}$$

(a) The tangent line is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$$\frac{dx}{dt} = 0 \Rightarrow -4\sin t = 0 \Rightarrow t = 0 \ , \ t = \pi$$

Note that $0, \pi \in [0, 2\pi]$ and $\frac{dy}{dt} \neq 0$ at $t = 0$ or $t = \pi$.

At $t = 0$: $x = 4 + 4(1) = 8$ and $y = -1 + 0 = -1$.

At $t = \pi$: $x = 4 + 4(-1) = 0$ and $y = -1 + 0 = -1$.

Hence, The tangent line to C is vertical at the points $(8, -1)$ and $(0, -1)$.

(b) The tangent line is horizontal if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$$\frac{dy}{dt} = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2} \ , \ t = \frac{3\pi}{2}$$

Note that $\frac{\pi}{2}, \frac{3\pi}{2} \in [0, 2\pi]$ and $\frac{dx}{dt} \neq 0$ at $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$.

At $t = \frac{\pi}{2}$: $x = 4 + 4(0) = 4$ and $y = -1 + 1 = 0$.

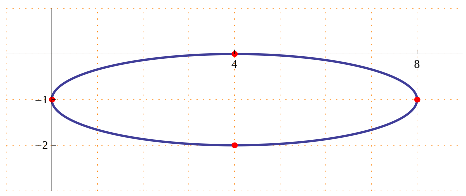
At $t = \frac{3\pi}{2}$: $x = 4 + 4(0) = 4$ and $y = -1 + (-1) = -2$.

Hence, The tangent line to C is horizontal at the points $(4, 0)$ and $(4, -2)$.

Note : $C: \ x = 4 + 4\cos t \ , \ y = -1 + \sin t \ ; \ 0 \le t \le 2\pi$ represents the ellipse $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{1} = 1$, with center $= (4 - 1)$, the endpoints of the major axis are $(0, -1)$ and $(8, -1)$, the endpoints of the minor axis are $(4, 0)$ and $(4, -2)$.

Clearly, there are two vertical tangent lines to C , one passes through $(-1, 0)$ and the other passes through $(8, -1)$.

Also, there are two horizontal tangent lines to C , one passes through $(4, 0)$ and the other passes through $(4, -2)$



Exercises :

1. If $C: \ x = t \ , \ y = t^2$, find the slope of the tangent line to C at $t = 1$.

2. The point at which the curve $C: \ x = 3\cos t \ , \ y = 3\sin t \ ; \ 0 \leq t \leq \pi$ has horizontal tangent line is

   (a) $(0, 3)$ (b) $(3, 3)$ (c) $(3, 0)$ (d) None of these

   (Hint : the parametric curve is the upper half of the circle with center $= (0, 0)$ and radius $= 3$) .

ARC LENGTH OF A PARAMETRIC CURVE

If $C: \ x = x(t) \ , \ y = y(t) \ ; \ a \leq t \leq b$ is a differentiable parametric curve ,then its arc length equals $L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$.

Examples : Find the arc length of the following parametric curves :

1. $C: \ x = \frac{1}{3}t^3 + 1 \ , \ y = \frac{1}{2}t^2 + 2 \ ; \ 0 \leq t \leq 2$

   $\dfrac{dx}{dt} = t^2$ and $\dfrac{dy}{dt} = t$

   $$L = \int_0^2 \sqrt{(t^2)^2 + (t)^2} \ dt = \int_0^2 \sqrt{t^4 + t^2} \ dt = \int_0^2 \sqrt{t^2(t^2+1)} \ dt$$

   $$L = \int_0^2 |t|\sqrt{t^2+1} \ dt = \frac{1}{2}\int_0^2 (t^2+1)^{\frac{1}{2}}(2t) \ dt$$

   $$L = \frac{1}{2}\left[\frac{2}{3}(t^2+1)^{\frac{3}{2}}\right]_0^2 = \frac{1}{3}\left(5\sqrt{5} - 1\right) .$$

2. $C: \ x = \sin t \ , \ y = \cos t \ ; \ 0 \leq t \leq \frac{\pi}{2}$

   $\dfrac{dx}{dt} = \cos t$ and $\dfrac{dy}{dt} = -\sin t$

   $$L = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t)^2 + (-\sin t)^2} \ dt = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t + \sin^2 t} \ dt$$

   $$L = \int_0^{\frac{\pi}{2}} dt = [t]_0^{\frac{\pi}{2}} = \frac{\pi}{2} .$$

   Note : The parametric curve represents the first quarter of the unit circle, therefore its arc length equals $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

3. $C: \ x = e^t \cos t \ , \ y = e^t \sin t \ ; \ 0 \leq t \leq \pi$

   $$\frac{dx}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$$

   $$\frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$$

   $$L = \int_0^{\pi} \sqrt{\left[e^t(\cos t - \sin t)\right]^2 + \left[e^t(\cos t + \sin t)\right]^2} \ dt$$

   $$L = \int_0^{\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} \ dt$$

   $$L = \int_0^{\pi} \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t)} \ dt$$

$$L = \int_0^\pi \sqrt{2e^{2t}}\ dt = \int_0^\pi \sqrt{2}|e^t|\ dt = \sqrt{2}\int_0^\pi e^t\ dt$$
$$L = \sqrt{2}\,[e^t]_0^\pi = \sqrt{2}(e^\pi - 1)\ .$$

SURFACE AREA GENERATED BY REVOLVING A PARAMETRIC CURVE

If $C:\ x = x(t)\ ,\ y = y(t)\ ;\ a \le t \le b$ is a differentiable parametric curve ,then the surface area generated by revolving C around the x-axis is

$$SA = 2\pi \int_a^b |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\ .$$

The surface area generated by revolving C around the y-axis is

$$SA = 2\pi \int_a^b |x(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\ .$$

Examples : Find the surface area generated by revolving the following parametric curves :

1. $C:\ x = t\ ,\ y = \frac{1}{3}t^3 + \frac{1}{4}t^{-1}\quad ;\quad 1 \le t \le 2$, around the x-axis .

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = t^2 - \frac{t^{-2}}{4}$$

$$SA = 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{(1)^2 + \left(t^2 - \frac{t^{-2}}{4}\right)^2}\ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{1 + \left(t^4 - \frac{1}{2} + \frac{t^{-4}}{16}\right)}\ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{t^4 + \frac{1}{2} + \frac{t^{-4}}{16}}\ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{\left(t^2 + \frac{t^{-2}}{4}\right)^2}\ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \left|t^2 + \frac{t^{-2}}{4}\right|\ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \left(t^2 + \frac{t^{-2}}{4}\right)\ dt$$

$$= 2\pi \int_1^2 \left(\frac{t^5}{3} + \frac{t}{2} + \frac{t^{-3}}{16}\right)\ dt$$

$$SA = 2\pi \left[\frac{t^6}{18} + \frac{t^2}{4} - \frac{t^{-2}}{32}\right]_1^2 = \frac{547\pi}{64}$$

2. $C:\ x = 4t^{\frac{1}{2}}\ ,\ y = \frac{1}{2}t^2 + t^{-1}\quad ;\quad 1 \le t \le 4$, around the y-axis .

$$\frac{dx}{dt} = 2t^{-\frac{1}{2}}$$

$$\frac{dy}{dt} = t - t^{-2}$$

$$SA = 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{\left(2t^{-\frac{1}{2}}\right)^2 + (t - t^{-2})^2}\ dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{4t^{-1} + (t^2 - 2t^{-1} + t^{-4})}\ dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{t^2 + 2t^{-1} + t^{-4}}\ dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{(t + t^{-2})^2}\ dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \left|t + t^{-2}\right|\ dt$$

$$= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \left(t + t^{-2}\right)\ dt$$

$$= 8\pi \int_1^4 \left(t^{\frac{3}{2}} + t^{-\frac{3}{2}}\right)\ dt$$

$$SA = 8\pi \left[\frac{2}{5}t^{\frac{5}{2}} - 2t^{-\frac{1}{2}}\right]_1^4 = \frac{536\pi}{5}$$

Exercises : Find the surface area generated by revolving the following parametric curves :

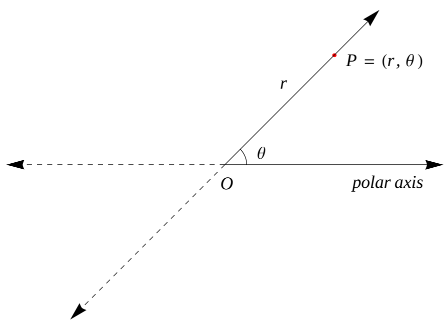
1. $C:\ x = 3t\ ,\ y = 4t\ \ ,0 \leq t \leq 2$, around the x-axis .
2. $C:\ x = t\ ,\ y = 2t\ \ ,0 \leq t \leq 4$, around the y-axis .

POLAR COORDINATES

In the recatangular coordinates system the ordered pair (a, b) represents a point , where "a" is the x-coordinat and "b" is the y-coordinate .

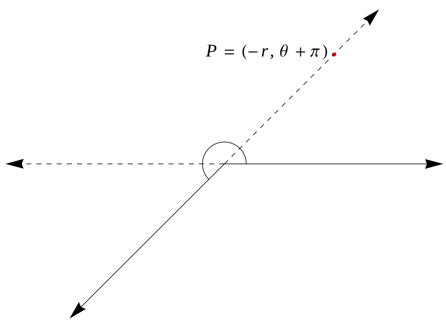
The polar coordinates system can be used also to represents points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system , and the **polar axis** is the directed half-line (the non-negative part of the x-axis).

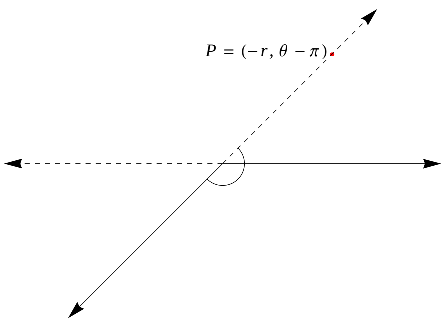
If P ia any point in the plane different from the origin, then its polar coordinates consists of two components r and θ , where r is the distance between P and the pole O , and θ is the measure of the angle determined by the polar axis and OP .



Note : The polar coordinates of a point is not unique , if $P = (r, \theta)$ then other representations are :

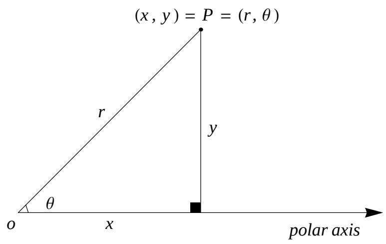
1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$.
2. $P = (-r, \theta + \pi)$.
3. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$.
4. $P = (-r, \theta - \pi)$
5. $P = (-r, \theta - \pi + 2n\pi)$, where $n \in \mathbb{Z}$.





Relationship between the polar and the rectangular coordinates
The polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point P are related as follows :

1. $x = r\cos\theta$ and $y = r\sin\theta$.
2. $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$.



Examples :

1. If $(r, \theta) = \left(2, \frac{\pi}{2}\right)$ then its other polar coordinates is
   a) $\left(-2, \frac{\pi}{2}\right)$ b) $\left(-2, \frac{3\pi}{2}\right)$ c) $\left(2, \frac{3\pi}{2}\right)$ d) $(2, \pi)$

   The answer : $(r, \theta) = \left(2, \frac{\pi}{2}\right) = \left(-2, \frac{\pi}{2} + \pi\right) = \left(-2, \frac{3\pi}{2}\right)$

   The right answer is (b) .

2. If $(r, \theta) = \left(-3, \frac{5\pi}{4}\right)$ then its other polar coordinates is
   a) $\left(-3, \frac{3\pi}{4}\right)$ b) $\left(3, \frac{7\pi}{4}\right)$ c) $\left(3, \frac{\pi}{4}\right)$ d) $\left(-3, \frac{\pi}{4}\right)$

   The answer : $(r, \theta) = \left(-3, \frac{5\pi}{4}\right) = \left(-(-3), \frac{5\pi}{4} - \pi\right) = \left(3, \frac{\pi}{4}\right)$

   The right answer is (c) .

3. If $(r, \theta) = (-5, \pi)$ then find its rectangular coordinates (x, y) .

   $x = -5\cos(\pi) = -5\ (-1) = 5$ and $y = -5\sin(\pi) = -5\ (0) = 0$

   $(x, y) = (5, 0)$.

4. If $(x, y) = \left(2\sqrt{3}, -2\right)$ then find its polar coordinates (r, θ) .

   $r^2 = \left(2\sqrt{3}\right)^2 + (-2)^2 = 12 + 4 = 16 \Rightarrow r = 4$

   $\tan\theta = \dfrac{-2}{2\sqrt{3}} = -\dfrac{1}{\sqrt{3}} \Rightarrow \theta = -\dfrac{\pi}{6}\ ,\ \theta = \dfrac{11\pi}{6}$

$$(r,\theta) = \left(4, -\frac{\pi}{6}\right) = \left(4, \frac{11\pi}{6}\right)$$

Exercises :

1. If $(r,\theta) = \left(2, \frac{\pi}{2}\right)$ then find its rectangular coordinates (x, y) .

   Answer : $(x, y) = (0, 2)$.

2. If $(x, y) = \left(\sqrt{2}, \sqrt{2}\right)$ then find its polar coordinates (r, θ) .

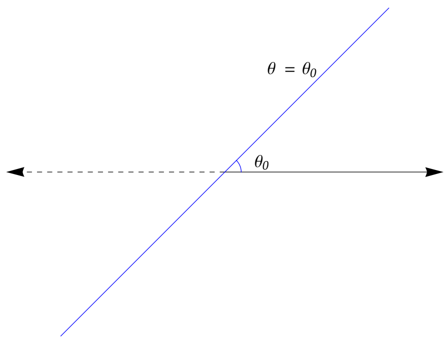
   Answer : $\left(2, \frac{\pi}{4}\right)$.

POLAR CURVES

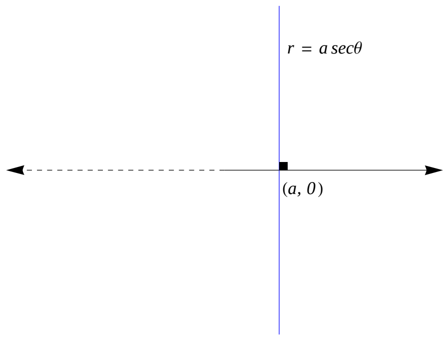
A polar curve is an equation in r and θ of the form $r = r(\theta)$.

First - Straight Lines :

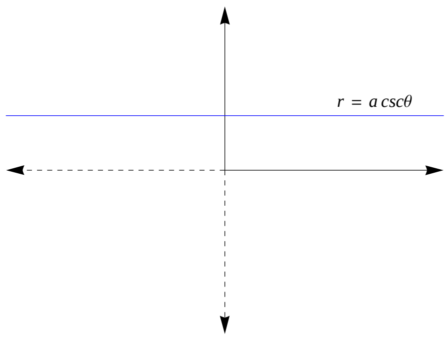
(1) Lines passing through the pole :
Any straight line passing through the pole has the form $\theta = \theta_0$, where θ_0 is the angle between the straight line and the polar axis .
$\theta = \theta_0 \Rightarrow \tan(\theta) = \tan(\theta_0) \Rightarrow \dfrac{y}{x} = \tan(\theta_0) \Rightarrow y = \tan(\theta_0)\ x$
The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan(\theta_0)$.



(2) Lines perpendicular to the polar axis :
Any straight line perpendicular to the polar axis has the form $r = a\ \sec\theta$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.
$r = a\ \sec\theta \Rightarrow r = \dfrac{a}{\cos\theta} \Rightarrow r\cos\theta = a \Rightarrow x = a$.
The straight line $r = a\ \sec\theta$ is perpendicular to the polar axis at the point $(r,\theta) = (a, 0)$



(3) Lines parallel to the polar axis :
Any straight line parallel to the polar axis has the form $r = a \ \csc\theta$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$..
$r = a \ \csc\theta \Rightarrow r = \dfrac{a}{\sin\theta} \Rightarrow r\sin\theta = a \Rightarrow y = a$.
The straight line $r = a \ \sec\theta$ is parallel to the polar axis and passing through the point $(r, \theta) = \left(a, \dfrac{\pi}{2}\right)$.



Examples :

1. $\theta = \dfrac{\pi}{4}$ is a straight line passing through the pole with a slope equals to $\tan\left(\frac{\pi}{4}\right) = 1$. Therefore its equation in $xy - form$ is $y = x$.
2. $r = 3\sec\theta$ is a straight line perpendicular to the polar axis and passing through the point $(r, \theta) = (3, 0)$. Therefore its equation in $xy - form$ is $x = 3$.
3. $r = -2\csc\theta$ is a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(-2, \frac{\pi}{2}\right)$. Therefore its equation in the $xy - form$ is $y = -2$.

Second - Circles :

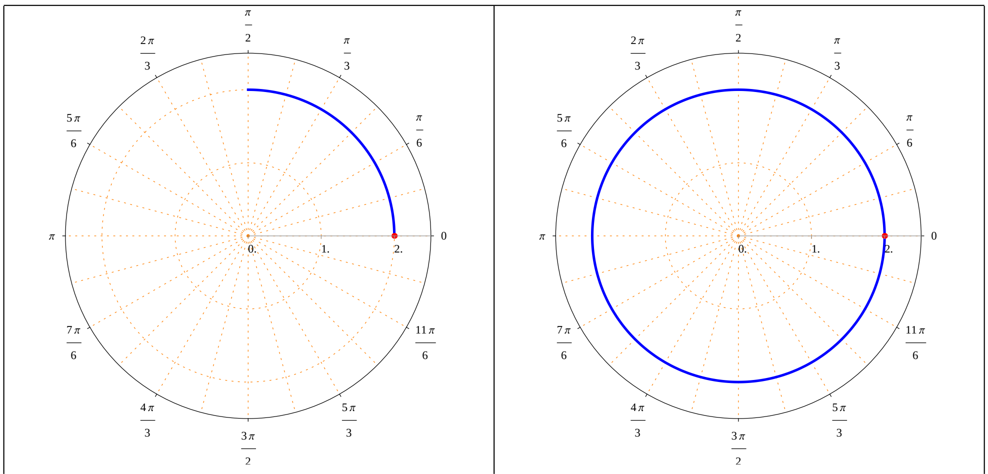
(1) Circles of the form $r = a$, where $a \in \mathbb{R}^*$

$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$

Therefore, $r = a$ represents a circle with center $= (0, 0)$ and radius equals $|a|$.

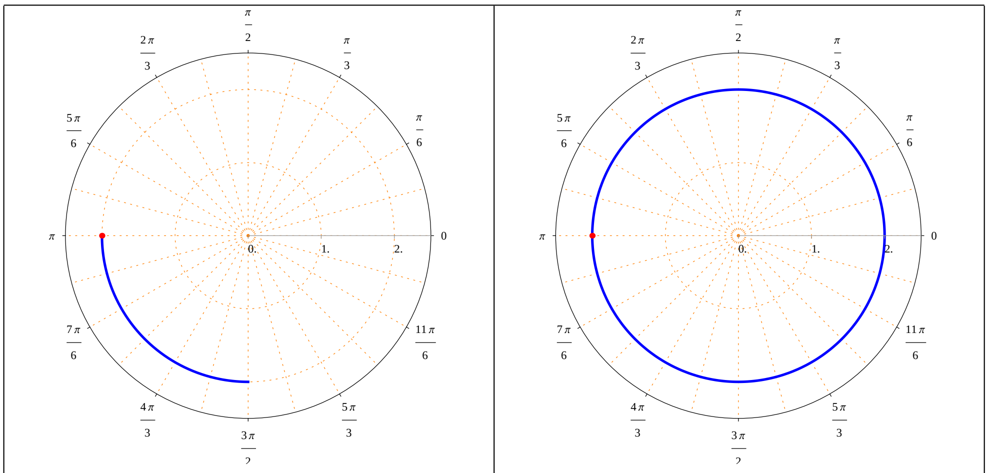
Example :

1. $r = 2$ represents a circle with center $= (0, 0)$ and radius equals to 2 .



$r = 2 \quad , \quad 0 \leq \theta \leq \frac{\pi}{2}$ $\qquad$ $r = 2 \quad , \quad 0 \leq \theta \leq 2\pi$

2. $r = -2$ represents a circle with center $= (0, 0)$ and radius equals to 2 .



$r = -2 \quad , \quad 0 \leq \theta \leq \frac{\pi}{2}$ $\qquad$ $r = -2 \quad , \quad 0 \leq \theta \leq 2\pi$

(2) Circles of the form $r = a\sin\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$

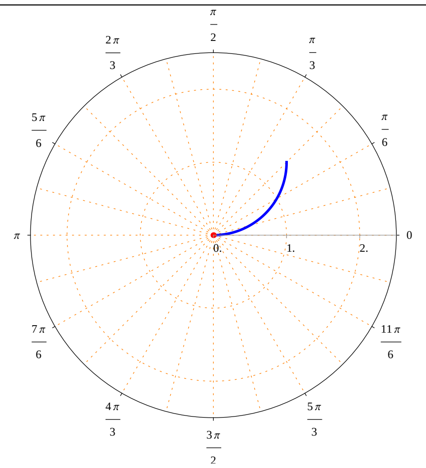
$r = a\sin\theta \Rightarrow r^2 = a\ r\sin\theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0$

$\Rightarrow x^2 + \left(y^2 - ay + \dfrac{a^2}{4}\right) = \dfrac{a^2}{4} \Rightarrow x^2 + \left(y - \dfrac{a}{2}\right)^2 = \dfrac{a^2}{4}$

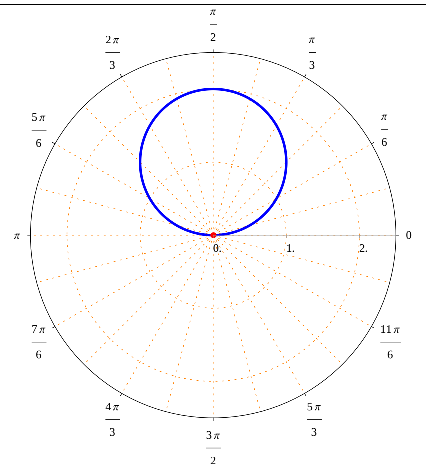
Therefore, $r = a\sin\theta$ represents a circle with center $= \left(0, \dfrac{a}{2}\right)$ and radius equals to $\dfrac{|a|}{2}$.

Examples :

1. $r = 2\sin\theta$ represents a circle with center $= (0, 1)$ and radius equals to 1

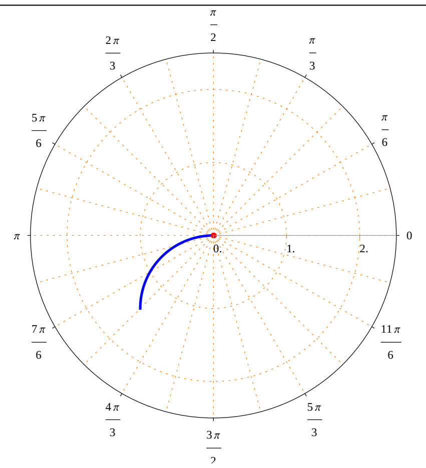


$r = 2\sin\theta \quad , \quad 0 \leq \theta \leq \dfrac{\pi}{4}$

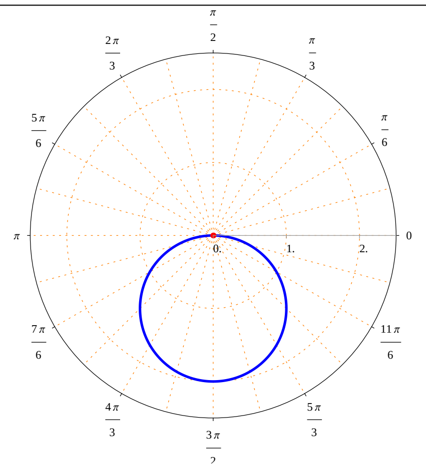


$r = 2\sin\theta \quad , \quad 0 \leq \theta \leq \pi$

2. $r = -2\sin\theta$ represents a circle with center $= (0, -1)$ and radius equals to 1



$r = -2\sin\theta \quad , \quad 0 \leq \theta \leq \dfrac{\pi}{4}$



$r = -2\sin\theta \quad , \quad 0 \leq \theta \leq \pi$

(3) Circles of the form $r = a\cos\theta$, where $a \in \mathbb{R}^*$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

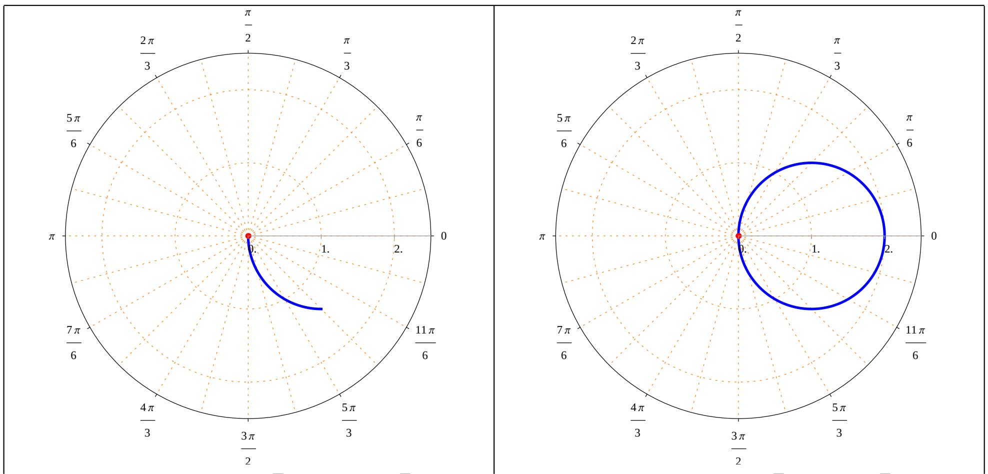
$r = a\cos\theta \Rightarrow r^2 = a\ r\cos\theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0$

$\Rightarrow \left(x^2 - ax + \dfrac{a^2}{4}\right) + y^2 = \dfrac{a^2}{4} \Rightarrow \left(x - \dfrac{a}{2}\right)^2 + y^2 = \dfrac{a^2}{4}$

Therefore, $r = a\cos\theta$ represents a circle with center $= \left(\dfrac{a}{2}, 0\right)$ and radius equals to $\dfrac{|a|}{2}$.

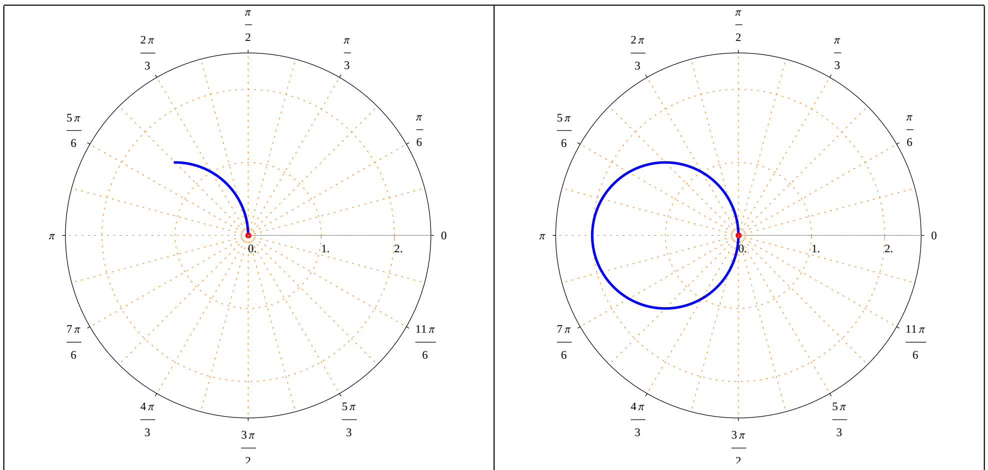
Examples :

1. $r = 2\cos\theta$ represents a circle with center $= (1, 0)$ and radius equals to 1



$r = 2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le -\dfrac{\pi}{4}$ $\qquad$ $r = 2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

2. $r = -2\cos\theta$ represents a circle with center $= (-1, 0)$ and radius equals to 1



$r = -2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le -\dfrac{\pi}{4}$ $\qquad$ $r = -2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

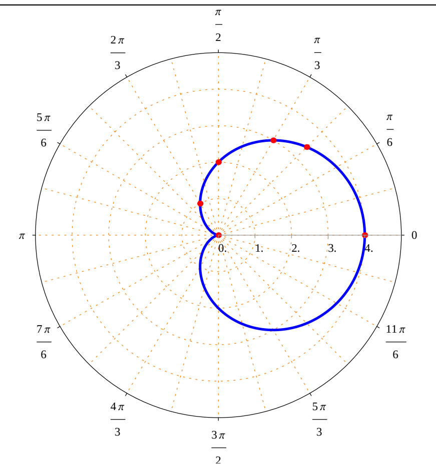
Third - Limaçon curves :
The general form of a Limaçon curve is
$r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

(1) Cardioid (Heart-shaped) :
It has the form $r(\theta) = a + a\sin\theta$ or $r(\theta) = a + a\cos\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

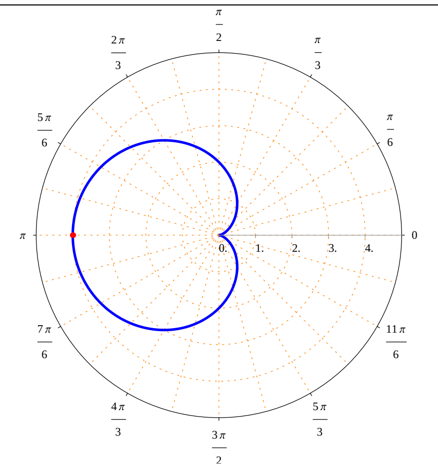
Examples :

1. $r(\theta) = 2 + 2\cos\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	$2+\sqrt{2}$	3	2	1	0

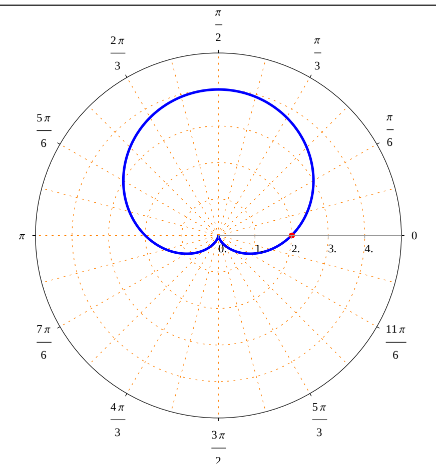


$r(\theta) = 2 + 2\cos\theta$, $0 \leq \theta \leq 2\pi$

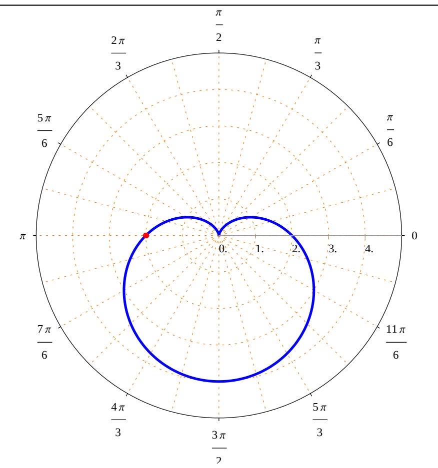


$r(\theta) = -2 - 2\cos\theta$, $0 \leq \theta \leq 2\pi$

2. $r(\theta) = 2 + 2\sin\theta$ and $r(\theta) = -2 - 2\sin\theta$



$r(\theta) = 2 + 2\sin\theta$, $0 \leq \theta \leq 2\pi$

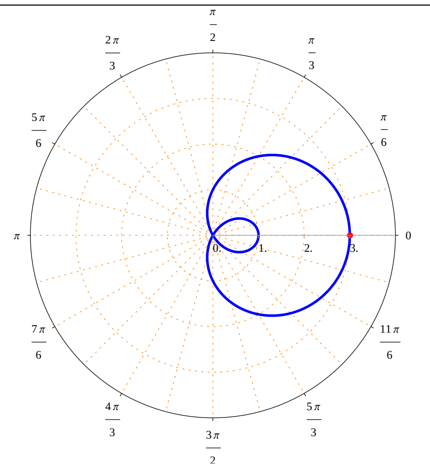


$r(\theta) = -2 - 2\sin\theta$, $0 \leq \theta \leq 2\pi$

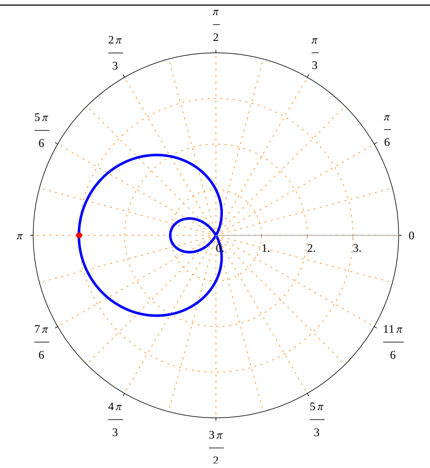
(2) Limaçon with inner loop :
It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \leq \theta \leq 2\pi$
Note : Note that $|a| < |b|$ in this case .

Examples :

1. $r(\theta) = 1 + 2\cos\theta$ and $r(\theta) = -1 - 2\cos\theta$

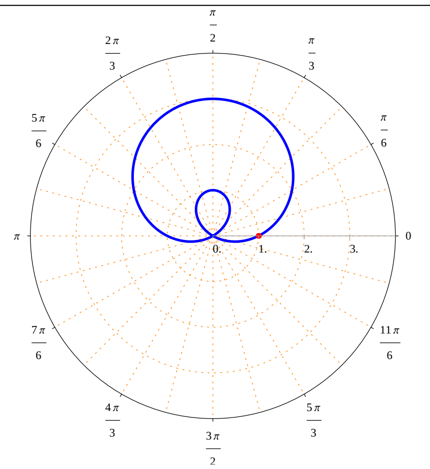


$r(\theta) = 1 + 2\cos\theta$, $0 \leq \theta \leq 2\pi$

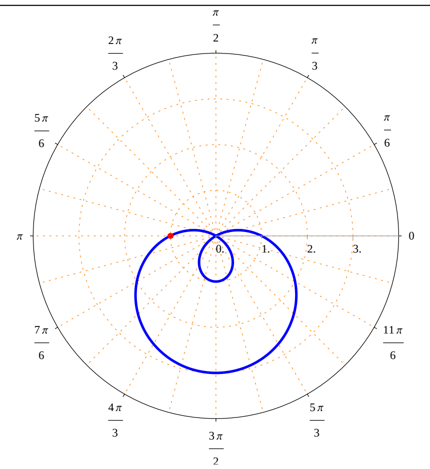


$r(\theta) = -1 - 2\cos\theta$, $0 \leq \theta \leq 2\pi$

2. $r(\theta) = 1 + 2\sin\theta$ and $r(\theta) = -1 - 2\sin\theta$



$r(\theta) = 1 + 2\sin\theta$, $0 \leq \theta \leq 2\pi$

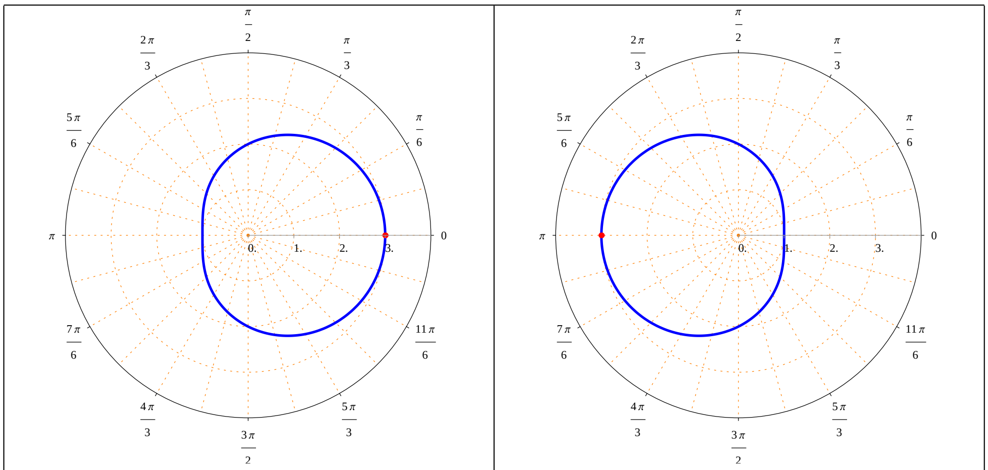


$r(\theta) = -1 - 2\sin\theta$, $0 \leq \theta \leq 2\pi$

(3) Dimpled Limaçon :
It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$
Note : Note that $|a| > |b|$ in this case .

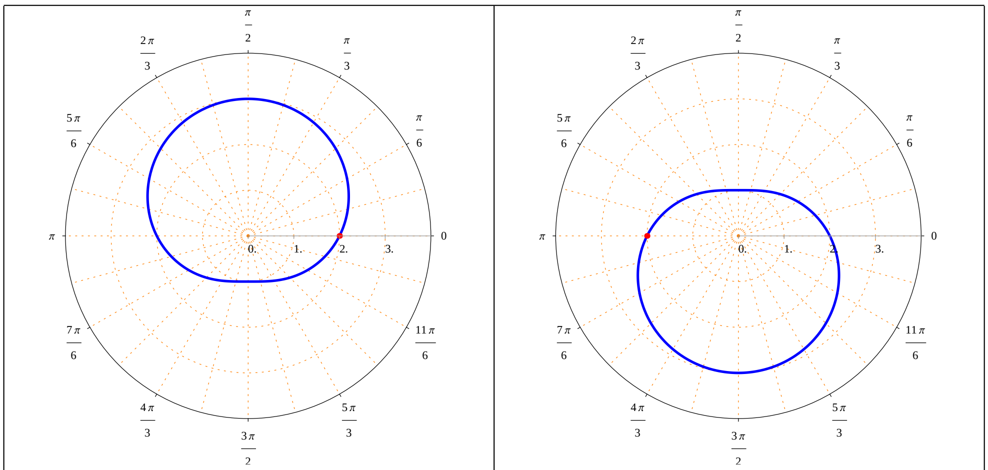
Examples :

1. $r(\theta) = 2 + \cos\theta$ and $r(\theta) = -2 - \cos\theta$



$r(\theta) = 2 + \cos\theta \quad , \quad 0 \leq \theta \leq 2\pi$ $\qquad$ $r(\theta) = -2 - \cos\theta \quad , \quad 0 \leq \theta \leq 2\pi$

2. $r(\theta) = 2 + \sin\theta$ and $r(\theta) = -2 - \sin\theta$



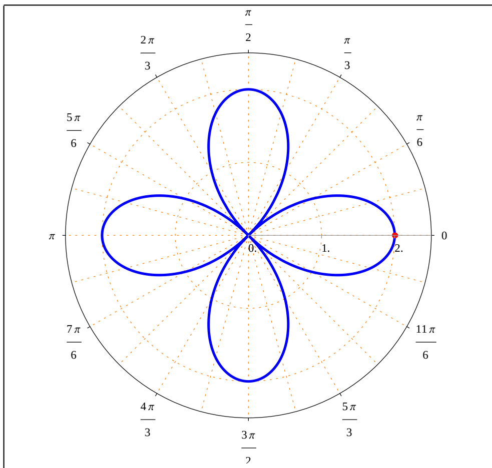
$r(\theta) = 2 + \sin\theta \quad , \quad 0 \leq \theta \leq 2\pi$ $\qquad$ $r(\theta) = -2 - \sin\theta \quad , \quad 0 \leq \theta \leq 2\pi$

Fourth - Rose curves :
It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*$, $n \in \mathbb{N}$ and $n \geq 2$

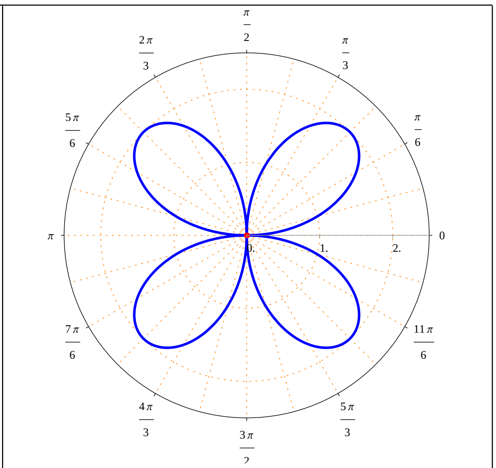
1. **n is even** : In this case the number of loops (or leaves) is $2n$.

   Examples : $r(\theta) = 2\cos(2\theta)$ or $r(\theta) = 2\sin(2\theta)$, $0 \leq \theta \leq 2\pi$

   The number of loops (or leaves) equals 4 .



$r(\theta) = 2\cos(2\theta)$, $0 \leq \theta \leq 2\pi$

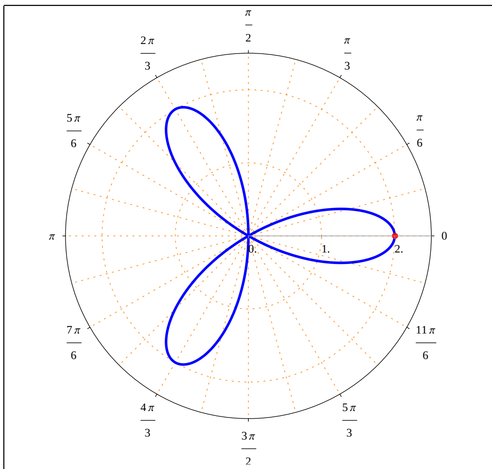


$r(\theta) = 2\sin(2\theta)$, $0 \leq \theta \leq 2\pi$

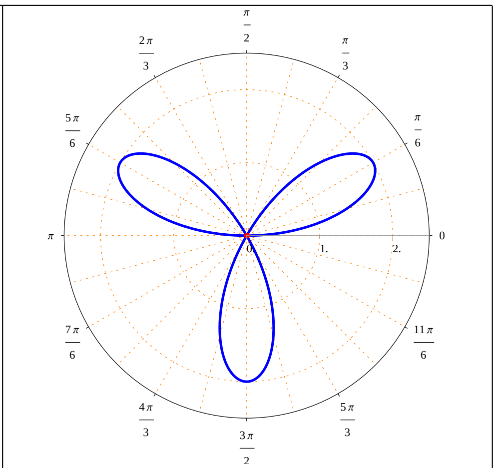
2. **n is odd** : In this case the number of loops (or leaves) is n .

   Examples : $r(\theta) = 2\cos(3\theta)$ or $r(\theta) = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$

   The number of loops (or leaves) equals 3 .



$r(\theta) = 2\cos(3\theta)$, $0 \leq \theta \leq \pi$



$r(\theta) = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$

Examples :

1. $r = \dfrac{2}{\cos\theta}$ represents
   a) a straight line b) a circle c) a cardioid d) a rose curve

   Answer : $r = \dfrac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2.$

   Hence , $r = \dfrac{2}{\cos\theta}$ represents a straigh line .

   The right answer is (a) .

2. The polar equation $r = 2\cos\theta - 2$ represents
   a) a straight line b) a circle c) a cardioid d) a rose curve

   $r = 2\cos\theta - 2$ is a Limaçon curve with $a = b = 2$.

   Therefore , $r = 2\cos\theta - 2$ represents a cardioid .

   The right answer is (c) .

3. The number of leaves in the rose curve $r = \sin 2\theta$ is
   a) 6 b) 4 c) 2 d) None of these

   Since $n = 2$ is an even number then the number of leaves in the rose curve $r = \sin 2\theta$ equals $2n = 2(2) = 4$

   The right answer is (b)

4. Write the polar equation $r = 2\cos\theta + 2\sin\theta$ in terms of x and y (or cartesian equation) .

   $r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2\ r\cos\theta + 2\ r\sin\theta \Rightarrow x^2 + y^2 = 2x + 2y$

   $\Rightarrow (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2 \Rightarrow (x - 1)^2 + (y - 1)^2 = 2$

   It is a circle with center $= (1, 1)$ and radius equals $\sqrt{2}$

Test of symmetry

1. The graph of $r = r(\theta)$ is symmetric with repect to the polar axis if

   $r(\theta) = r(-\theta)$

   Examples : The circle $r = 4\cos\theta$ and the cardioid $r = 2 + 2\cos\theta$ are both symmetric with respect to the polar axis .

2. The graph of $r = r(\theta)$ is symmetric with repect to the line $\theta = \frac{\pi}{2}$ if

   (a) $r(\theta) = -r(-\theta)$

   (b) $r(\theta) = r(\pi - \theta)$

   Examples : The circle $r = 4\sin\theta$ and the cardioid $r = 2 + 2\sin\theta$ are both symmetric with respect to the line $\theta = \frac{\pi}{2}$.

3. The graph of $r = r(\theta)$ is symmetric with repect to the pole if

   $r(\theta) = r(\pi + \theta)$

   Example : The rose curve $r = \sin 2\theta$ is symmetric with respect to the pole .

SLOPE OF THE TANGENT LINE TO A POLAR CURVE

If $r = r(\theta)$ is a smooth polar curve , then the slope of the tangent line to $r = r(\theta)$ is $m = \dfrac{dy}{dx}$, where $x = r(\theta)\cos\theta$ and $y = r(\theta)\sin\theta$.

More precisely , $m = \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$

Notes :

1. The slope of the tangent line to $r = r(\theta)$ is horizontal if $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$
2. The slope of the tangent line to $r = r(\theta)$ is vertical if $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$

Example :

1. Find the points on the polar curve $r(\theta) = 2\sin\theta$, $0 \leq \theta \leq \pi$ at which the tangent line to r is vertical .

   The answer :

   $x = r(\theta)\cos\theta \Rightarrow x = 2\sin\theta\cos\theta = \sin 2\theta \Rightarrow \dfrac{dx}{d\theta} = 2\cos 2\theta$

   $y = r(\theta)\sin\theta \Rightarrow y = 2\sin^2\theta \Rightarrow \dfrac{dy}{d\theta} = 4\sin\theta\cos\theta$

   The tangent line to $r = r(\theta)$ is vertical if $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$

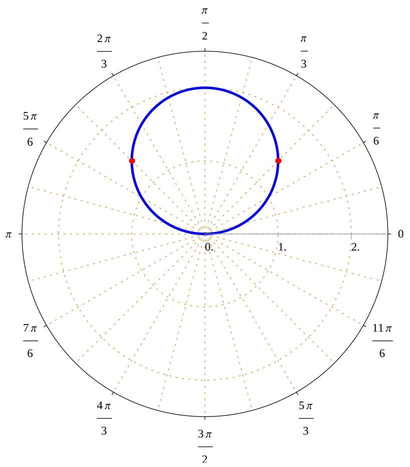
   $\dfrac{dx}{d\theta} = 0 \Rightarrow 2\cos 2\theta = 0 \Rightarrow 2\theta = \dfrac{\pi}{2}$, $2\theta = \dfrac{3\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}$, $\theta = \dfrac{3\pi}{4}$

   Note that $\theta = \dfrac{\pi}{4}$, $\theta = \dfrac{3\pi}{4} \in [0, \pi]$ and $\dfrac{dy}{d\theta} \neq 0$ when $\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$.

   At $\theta = \dfrac{\pi}{4}$: $r\left(\frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{4}\right) = 2\ \frac{1}{\sqrt{2}} = \sqrt{2}$

   At $\theta = \dfrac{3\pi}{4}$: $r\left(\frac{3\pi}{4}\right) = 2\sin\left(\frac{3\pi}{4}\right) = 2\ \frac{1}{\sqrt{2}} = \sqrt{2}$

   The points on $r(\theta) = 2\sin\theta$, $0 \leq \theta \leq \pi$ at which the tangent line to r is vertical are $\left(\sqrt{2}, \dfrac{\pi}{4}\right)$, $\left(\sqrt{2}, \dfrac{3\pi}{4}\right)$



2. Find the points on the polar curve $r(\theta) = 1 + \cos\theta$, $0 \le \theta \le 2\pi$ at which the tangent line to r is horizontal .

   The answer :

   $x = r(\theta)\cos\theta \Rightarrow x = \cos\theta(1+\cos\theta) = \cos\theta + \cos^2\theta$

   $y = r(\theta)\sin\theta \Rightarrow y = \sin\theta(1+\cos\theta) = \sin\theta + \sin\theta\cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$

   $\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta = -\sin\theta - \sin 2\theta$

   $\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$

   The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

   $\frac{dy}{d\theta} = 0 \Rightarrow \cos 2\theta + \cos\theta = 0 \Rightarrow 2\cos^2\theta - 1 + \cos\theta = 0$

   $\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = -1$ or $\cos\theta = \frac{1}{2}$

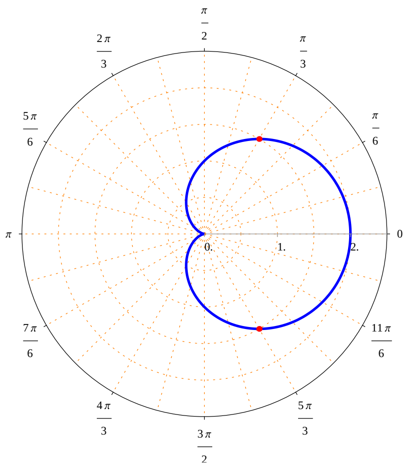
   $\Rightarrow \theta = \pi$ or $\theta = \frac{\pi}{3}$, $\theta = \frac{5\pi}{3}$

   Note that $\theta = \frac{\pi}{3}$, $\theta = \frac{5\pi}{3} \in [0, 2\pi]$ and $\frac{dx}{d\theta} \neq 0$ when $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$, but $\frac{dx}{d\theta} = 0$ when $\theta = \pi$.

   At $\theta = \frac{\pi}{3}$: $r\left(\frac{\pi}{3}\right) = 1 + \cos(\frac{\pi}{3}) = 1 + \frac{1}{2} = \frac{3}{2}$

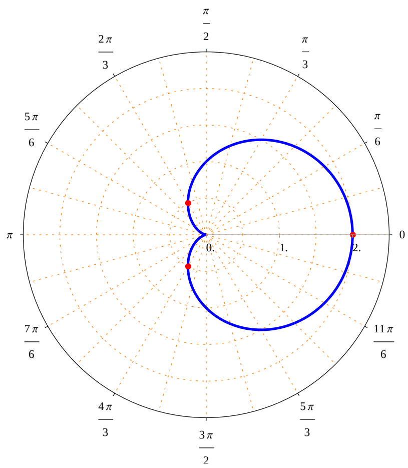
   At $\theta = \frac{5\pi}{3}$: $r\left(\frac{5\pi}{3}\right) = 1 + \cos(\frac{5\pi}{3}) = 1 + \frac{1}{2} = \frac{3}{2}$

   The points on $r(\theta) = 1 + \cos\theta$, $0 \le \theta \le 2\pi$ at which the tangent line to r is horizontal are $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

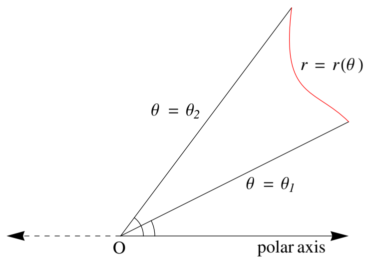


Exercise : Find the points on the polar curve $r(\theta) = 1 + \cos\theta$, $0 \le \theta \le 2\pi$ at which the tangent line to r is vertical .

The answer : $(2, 0)$, $\left(\dfrac{1}{2}, \dfrac{2\pi}{3}\right)$ and $\left(\dfrac{1}{2}, \dfrac{4\pi}{3}\right)$.



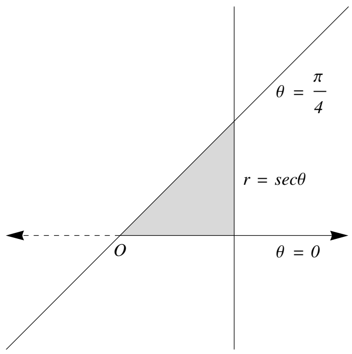
AREA INSIDE-BETWEEN POLAR CURVES



The area of the region bounded by the graphs of the polar curves $r = r(\theta)$, $\theta = \theta_1$ and $\theta = \theta_2$ is $A = \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$

Examples :

1. Find the area of the region bounded by the graph of the polar curves $r = \sec\theta$, $\theta = 0$ and $\theta = \frac{\pi}{4}$.

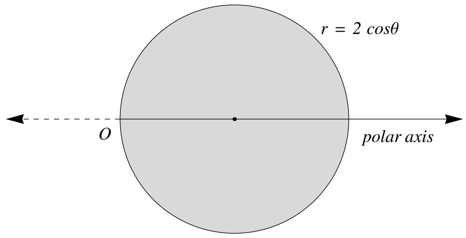


Note that $r = \sec\theta$ is a straight line perpendicular to the polar axis at the point $(r, \theta) = (1, 0)$, $\theta = 0$ is the polar axis and $\theta = \frac{\pi}{4}$ is a straight line passing the pole with a slope equals 1 (in fact it is the line $y = x$) .

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sec\theta)^2 d\theta = \frac{1}{2}\left[\tan\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{2}[1 - 0] = \frac{1}{2}$$

Note : In fact it is the area of the triangle of base equals 1 and height equals also 1 .

2. Find the area inside the polar curve $r = 2\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



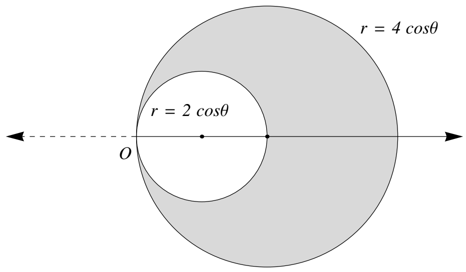
Note that $r = 2\cos\theta$ is a circle with center $= (1, 0)$ and radius equals 1 .

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2\theta \; d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1 + \cos 2\theta\right] \, d\theta$$

$$A = \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left[\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right)\right] = \pi \; .$$

Note : In fact it is the area of a circle of radius equals 1 and in this case $A = \pi(1)^2 = \pi$.

3. Find the area inside the polar curve $r = 4\cos\theta$ and outside the curve $r = 2\cos\theta$.



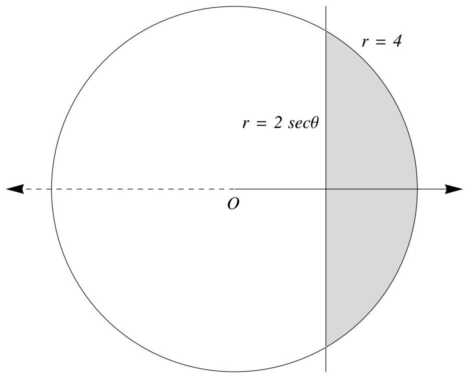
Note that $r = 4\cos\theta$ is a circel with center $= (2, 0)$ and radius equals to 2 , also $r = 2\cos\theta$ is another circle with center $= (1, 0)$ and radius equals 1 .

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12\cos^2\theta \; d\theta$$

$$A = 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1 + \cos 2\theta\right] \; d\theta = 3\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\pi$$

Note : In fact it is the difference between the area of a circle with radius 2 and the area of a circle of radius 1 , so the desired area is $A = \pi(2)^2 - \pi(1)^2 = 3\pi$.

4. Find the area inside $r = 4$ and to the right of $r = 2\sec\theta$



Note that $r = 4$ is a circle with center $= (0,0)$ and radius equals 4 , $r = 2\sec\theta$ is a straight line perpendicular to the polar axis (it is the line $x = 2$)

Angles of intersection between $r = 4$ and $r = 2\sec\theta$:

$2\sec\theta = 4 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$, $\theta = -\dfrac{\pi}{3}$

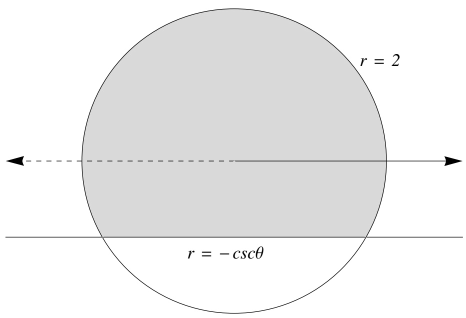
Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} (4)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} (2\sec\theta)^2 d\theta\right)$$

$$A = 16\int_0^{\frac{\pi}{3}} d\theta - 4\int_0^{\frac{\pi}{3}} \sec^2\theta \; d\theta$$

$$A = 16[\theta]_0^{\frac{\pi}{3}} - 4[\tan\theta]_0^{\frac{\pi}{3}} = 16\left(\frac{\pi}{3} - 0\right) - 4(\sqrt{3} - 0) = \frac{16\pi}{3} - 4\sqrt{3}$$

5. Find the area inside $r = 2$ and above $r = -\csc\theta$.



Note that $r = 2$ is a circle with center $= (0,0)$ and radius equals 2 , $r = -\csc\theta$ is a straight line parallel to the polar axis (it is the line $y = -1$)

Angles of intersection between $r = 2$ and $r = -\csc\theta$:

$$-\csc\theta = 2 \Rightarrow \csc\theta = -2 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}\ ,\ \theta = -\frac{5\pi}{6}$$

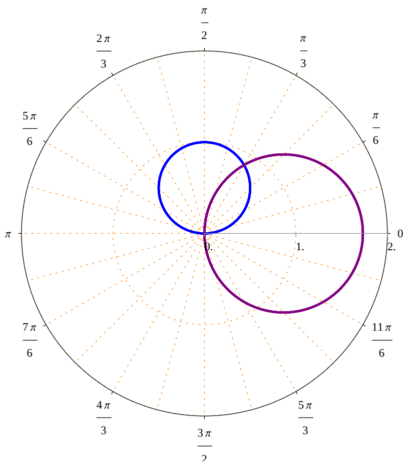
Since the desired area is symmetric with respect to the line $\theta = \frac{\pi}{2}$, then

$$A = 2\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} (-\csc\theta)^2 d\theta + \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (2)^2 d\theta\right)$$

$$A = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \csc^2\theta\ d\theta + 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta$$

$$A = [-\cot\theta]_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + 4[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3} + \frac{2\pi}{3}$$

6. Find the area of the common region between $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$



Note that $r = \sqrt{3}\cos\theta$ is a circle with center $= \left(\frac{\sqrt{3}}{2}, 0\right)$ and radius equals $\frac{\sqrt{3}}{2}$, also $r = \sin\theta$ is a circle with center $= \left(0, \frac{1}{2}\right)$ and radius equals $\frac{1}{2}$.

Angle of intersection between $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$

$$\sqrt{3}\cos\theta = \sin\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$$

$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{3}} (\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\theta)^2 d\theta$$

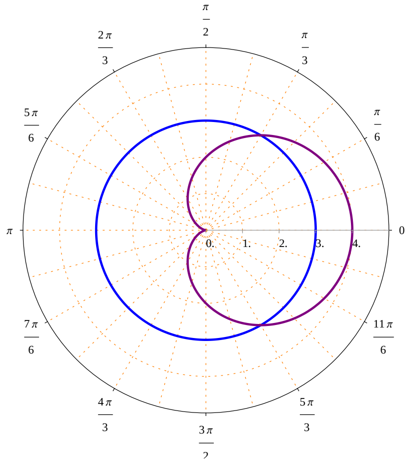
$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{3}} \frac{1}{2}[1 - \cos 2\theta]\ d\theta + \frac{3}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}[1 + \cos 2\theta]\ d\theta$$

$$A = \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_{0}^{\frac{\pi}{3}} + \frac{3}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$A = \frac{1}{4}\left(\frac{\pi}{3} - \frac{1}{2}\frac{\sqrt{3}}{2}\right) + \frac{3}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2}\right)\right]$$

$A = \dfrac{5\pi}{24} - \dfrac{\sqrt{3}}{4}$.

7. Find the area inside $r = 3$ and outside $r = 2 + 2\cos\theta$.



Note that $r = 3$ is a circle with center $= (0, 0)$ and radius equals 3 , $r = 2 + 2\cos\theta$ is a cardioid .

Angles of intersection between $r = 3$ and $r = 2 + 2\cos\theta$:

$2 + 2\cos\theta = 3 \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$, $\theta = \dfrac{5\pi}{3} = -\dfrac{\pi}{3}$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (3)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (2 + 2\cos\theta)^2 d\theta\right)$$

$$A = \int_{\frac{\pi}{3}}^{\pi} \left[9 - (4 + 8\cos\theta + 4\cos^2\theta)\right] d\theta$$

$$A = \int_{\frac{\pi}{3}}^{\pi} \left[5 - 8\cos\theta - 2(1 + \cos 2\theta)\right] d\theta$$

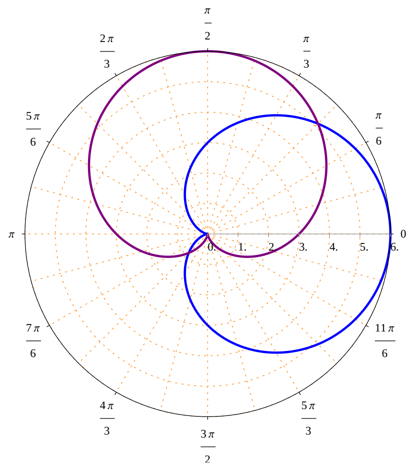
$$A = \int_{\frac{\pi}{3}}^{\pi} \left[3 - 8\cos\theta - 2\cos 2\theta\right] d\theta$$

$$A = \left[3\theta - 8\sin\theta - \sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$$

$$A = \left[(3\pi - 0 - 0) - \left(\pi - 8\,\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$$

$$A = 2\pi + \frac{9\sqrt{3}}{2}$$

8. Find the area inside $r = 3+3\cos\theta$, outside $r = 3+3\sin\theta$ and at the first quadrant.



Angles of intersection between $r = 3+3\cos\theta$ and $r = 3+3\sin\theta$:

$$3+3\cos\theta = 3+3\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \ , \ \theta = \frac{5\pi}{4}$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (3+3\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} (3+3\sin\theta)^2 d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[(9+18\cos\theta+9\cos^2\theta) - (9+18\sin\theta+9\sin^2\theta)\right] d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[18\cos\theta - 18\sin\theta + 9\cos^2\theta - 9\sin^2\theta\right] d\theta$$

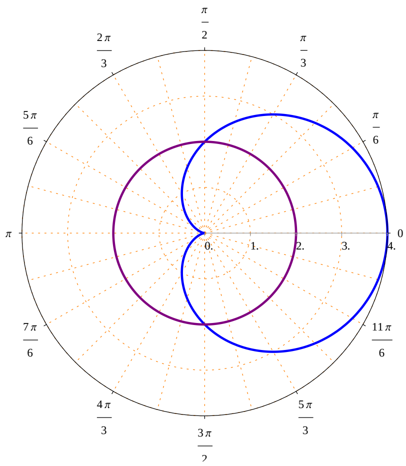
$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[18\cos\theta - 18\sin\theta + \frac{9}{2}(1+\cos 2\theta) - \frac{9}{2}(1-\cos 2\theta)\right] d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} [18\cos\theta - 18\sin\theta + 9\cos 2\theta]\, d\theta$$

$$A = \frac{1}{2}\left[18\sin\theta + 18\cos\theta + \frac{9}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$A = \frac{1}{2}\left[\left(\frac{18}{\sqrt{2}} + \frac{18}{\sqrt{2}} + \frac{9}{2}\right) - (0+18+0)\right] = \frac{18}{\sqrt{2}} - \frac{27}{4}$$

9. Find the area inside $r = 2 + 2\cos\theta$ and outside $r = 2$.



Note that $r = 2$ is a circle with center $= (0, 0)$ and radius equals 2 , $r = 2 + 2\cos\theta$ is a cardioid .

Angles of intersection between $r = 2$ and $r = 2 + 2\cos\theta$:

$2 + 2\cos\theta = 2 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$, $\theta = \dfrac{3\pi}{2}$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2 d\theta\right)$$

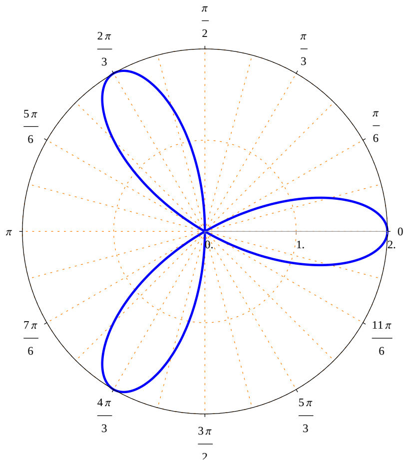
$$A = \int_0^{\frac{\pi}{2}} (4 + 8\cos\theta + 4\cos^2\theta - 4)\ d\theta$$

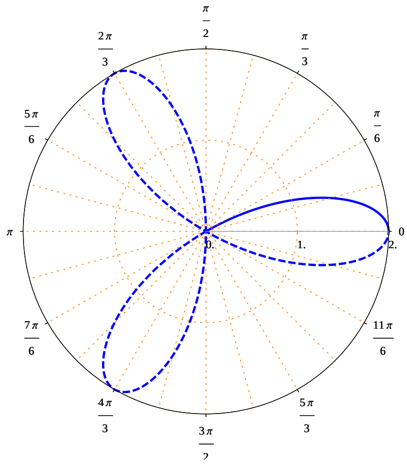
$$A = \int_0^{\frac{\pi}{2}} (8\cos\theta + 2(1 + \cos 2\theta))\ d\theta$$

$$A = \int_0^{\frac{\pi}{2}} (2 + 8\cos\theta + 2\cos 2\theta)\ d\theta)$$

$$A = [2\theta + 8\sin\theta + \sin 2\theta]_0^{\frac{\pi}{2}} = \pi + 8$$

10. Find the area inside one leaf of the rose curve $r = 2\cos 3\theta$.





The rose curve $r = 2\cos 3\theta$, $0 \le \theta \le \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$

$$r = 0 \Rightarrow 2\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$$

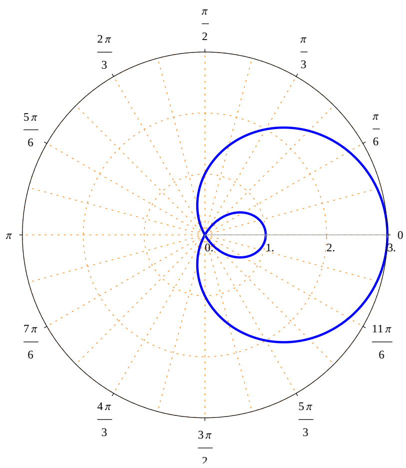
Since the desired area is symmetric with respect to the polar axis , then

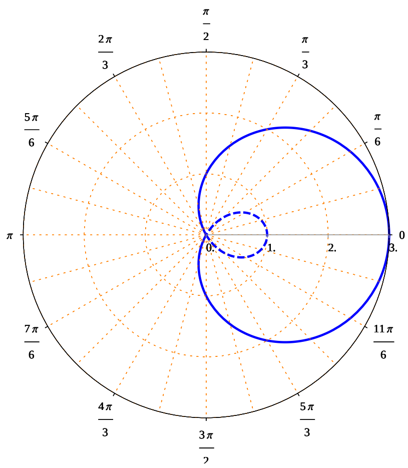
$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos 3\theta)^2 d\theta\right) = 4\int_0^{\frac{\pi}{6}} \cos^2 3\theta \ d\theta$$

$$A = 4\int_0^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos 6\theta) \ d\theta = 2\int_0^{\frac{\pi}{6}} (1 + \cos 6\theta) \ d\theta$$

$$A = 2\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{6}} = \frac{\pi}{3}$$

11. Find the area betwen the loops of the curve $r = 1 + 2\cos\theta$





$$r = 0 \Rightarrow 1 + 2\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \ , \ \theta = \frac{4\pi}{3}$$

The interior loop starts at $\theta = \dfrac{2\pi}{3}$ and ends at $\theta = \dfrac{4\pi}{3}$

$$A = \frac{1}{2}\int_{0}^{\frac{2\pi}{3}} (1+2\cos\theta)^2 d\theta + \int_{\frac{4\pi}{3}}^{2\pi} (1+2\cos\theta)^2 d\theta - \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+2\cos\theta)^2 d\theta$$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2\left(\frac{1}{2}\int_{0}^{\frac{2\pi}{3}} (1+2\cos\theta)^2 d\theta - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (1+2\cos\theta)^2 d\theta\right)$$

$$A = \int_{0}^{\frac{2\pi}{3}} \left(1+4\cos\theta+4\cos^2\theta\right)\, d\theta - \int_{\frac{2\pi}{3}}^{\pi} \left(1+4\cos\theta+4\cos^2\theta\right)\, d\theta$$

$$A = \int_{0}^{\frac{2\pi}{3}} \left(3+4\cos\theta+2\cos 2\theta\right)\, d\theta - \int_{\frac{2\pi}{3}}^{\pi} \left(3+4\cos\theta+2\cos 2\theta\right)\, d\theta$$

$$A = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{0}^{\frac{2\pi}{3}} - \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{\frac{2\pi}{3}}^{\pi}$$

$$A = \left[\left(2\pi + \tfrac{3\sqrt{3}}{2}\right) - 0\right] - \left[3\pi - \left(2\pi + \tfrac{3\sqrt{3}}{2}\right)\right] = \pi + 3\sqrt{3}$$

Exercises :

1. Find the area inside $r = \cos\theta$ and outside the curve $r = 1 - \cos\theta$
2. Find the area of the common region between the curves $r = 2\sin\theta$ and $r = 2\cos\theta$
3. Find the area inside the curve $r = 1$ and outside the curve $r = 1 - \cos\theta$

ARC LENGTH OF A POLAR CURVE

The arc length of the polar curve $r = r(\theta)$ from θ_1 to θ_2 is
$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Examples : Find the arc length of the following polar curves :

1. $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$

   $$\frac{dr}{d\theta} = -\sin\theta$$

   Since $r = 1 + \cos\theta$ is symmetric with respect to the polar axis then

   $$L = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\ d\theta$$

   $$L = 2\int_0^{\pi} \sqrt{(1 + 2\cos\theta + \cos^2\theta) + \sin^2\theta}\ d\theta$$

   $$L = 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\ d\theta$$

   $$L = 2\int_0^{\pi} \sqrt{2(1+\cos\theta)}\ d\theta$$

   Note that $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1+\cos\theta) \Rightarrow 2(1+\cos\theta) = 4\cos^2\left(\frac{\theta}{2}\right)$

   $$L = 2\int_0^{\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\ d\theta = 2\int_0^{\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right|\ d\theta$$

   $$L = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\ d\theta = 8\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 8(1-0) = 8$$

2. $r = 2\cos\theta$, $0 \leq \theta \leq 2\pi$

   $$\frac{dr}{d\theta} = -2\sin\theta$$

   $$L = \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\ d\theta$$

   $$L = \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

   $$L = \int_0^{2\pi} \sqrt{4}\ d\theta = \int_0^{2\pi} 2\ d\theta = [2\theta]_0^{2\pi} = 4\pi$$

   Note that $r = 2\cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is a circle with center $= (1,0)$ and radius equals 1 , therefore its circumference equals 2π , in this example $r = 2\cos\theta$, $0 \leq \theta \leq 2\pi$ which means that the curve is doubled , hence the circumference is also doubled .

3. $r = e^{-\theta}\ ,\ 0 \leq \theta \leq \pi$

$$\frac{dr}{d\theta} = -e^{-\theta}$$

$$L = \int_0^{\pi} \sqrt{\left(e^{-\theta}\right)^2 + \left(-e^{-\theta}\right)^2}\ d\theta$$

$$L = \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\ d\theta = \int_0^{\pi} \sqrt{2e^{-2\theta}} d\theta$$

$$L = \int_0^{\pi} \sqrt{2} \left|e^{-\theta}\right|\ d\theta = \sqrt{2} \int_0^{\pi} e^{-\theta} d\theta$$

$$L = \sqrt{2} \left[-e^{-\theta}\right]_0^{\pi} = \sqrt{2} \left[-e^{-\pi} + e^0\right] = \sqrt{2} \left(1 - e^{-\pi}\right)$$

SURFACE AREA GENERATED BY REVOLVING A POLAR CURVE

The surface area generated by revolving the polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the polar axis is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta$$

The surface area generated by revolving the polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the line $\theta = \frac{\pi}{2}$ is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta$$

Examples : Find the surface area generated by revolving the following polar curves :

1. $r = e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \pi$, around the polar axis .

$$\frac{dr}{d\theta} = \frac{1}{2}e^{\frac{\theta}{2}}$$

$$SA = 2\pi \int_0^{\pi} \left|e^{\frac{\theta}{2}}\sin\theta\right| \sqrt{\left(e^{\frac{\theta}{2}}\right)^2 + \left(\frac{1}{2}e^{\frac{\theta}{2}}\right)^2} \; d\theta$$

$$SA = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}}\sin\theta \sqrt{e^{\theta} + \frac{1}{4}e^{\theta}} \; d\theta = \int_0^{\pi} e^{\frac{\theta}{2}}\sin\theta \left|e^{\frac{\theta}{2}}\right| \sqrt{1 + \frac{1}{4}} \; d\theta$$

$$SA = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}}\sin\theta \; e^{\frac{\theta}{2}} \sqrt{\frac{5}{4}} \; d\theta = 2\pi\frac{\sqrt{5}}{2} \int_0^{\pi} e^{\theta}\sin\theta \; d\theta$$

Using integration by parts

$$SA = \sqrt{5}\pi \left[\frac{1}{2}e^{\theta}(\sin\theta - \cos\theta)\right]_0^{\pi} = \frac{\sqrt{5}\pi}{2}\left(e^{\pi} + 1\right)$$

2. $r = 2 + 2\cos\theta$, $0 \leq \theta \leq \frac{\pi}{2}$, around the polar axis .

$$\frac{dr}{d\theta} = -2\sin\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |(2 + 2\cos\theta)\sin\theta| \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2} \; d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)\sin\theta \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta} \; d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)\sin\theta \sqrt{8 + 8\cos\theta} \; d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)\sin\theta \sqrt{4(2 + 2\cos\theta)} \; d\theta$$

$$SA = 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)\sin\theta\sqrt{2+2\cos\theta}\ d\theta$$

$$SA = 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)^{\frac{3}{2}}\sin\theta\ d\theta$$

$$SA = -2\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)^{\frac{3}{2}}(-2\sin\theta)\ d\theta$$

$$SA = -2\pi\left[\frac{2}{5}(2+2\cos\theta)^{\frac{5}{2}}\right]_0^{\frac{\pi}{2}} = -2\pi\frac{2}{5}\left[4\sqrt{2}-32\right] = \frac{16\pi}{5}(8-\sqrt{2})$$

3. $r = \cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, around the line $\theta = \frac{\pi}{2}$

$$\frac{dr}{d\theta} = -\sin\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta\ \cos\theta|\sqrt{(\cos\theta)^2+(-\sin\theta)^2}\ d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\cos^2\theta\right|\sqrt{\cos^2\theta+\sin^2\theta}\ d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\ d\theta$$

$$SA = \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi\left[\left(\frac{\pi}{2}+0\right)-\left(-\frac{\pi}{2}+0\right)\right] = \pi^2$$

4. $r = 2\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, around the line $\theta = \frac{\pi}{2}$

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |2\sin\theta\ \cos\theta|\sqrt{(2\sin\theta)^2+(2\cos\theta)^2}\ d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |\sin 2\theta|\sqrt{4\sin^2\theta+4\cos^2\theta}\ d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} \sin 2\theta\ \sqrt{4}\ d\theta$$

$$SA = 4\pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 4\pi$$

Note : it is the surface area of a sphere of radius 1.