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Artificial Intelligence
Lecture III

First-Order Logic

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Limitations of propositional logic

- So far we studied propositional logic
- Some English statements are hard to model in propositional logic:
- "If your roommate is wet because of rain, your roommate must not be carrying **any** umbrella"
- Pathetic attempt at modeling this:
- RoommateWetBecauseOfRain => (NOT(RoommateCarryingUmbrella0) AND NOT(RoommateCarryingUmbrella1) AND NOT(RoommateCarryingUmbrella2) AND …)

Problems with propositional logic

- No notion of objects
- No notion of relations among objects
- RoommateCarryingUmbrella0 is instructive **to us**, suggesting
  - there is an object we call Roommate,
  - there is an object we call Umbrella0,
  - there is a relationship Carrying between these two objects
- Formally, none of this meaning is there
  - Might as well have replaced RoommateCarryingUmbrella0 by P

Elements of first-order logic

- Objects: can give these names such as Umbrella0, Person0, John, Earth, …
- Relations: Carrying(., .), IsAnUmbrella(.)
  - Carrying(Person0, Umbrella0), IsUmbrella(Umbrella0)
  - Relations with one object = unary relations = properties
- Functions: Roommate(.)
  - Roommate(Person0)
- Equality: Roommate(Person0) = Person1

Things to note about functions

- It could be that we have a separate name for Roommate(Person0)
- E.g., Roommate(Person0) = Person1
- … but we do not **need** to have such a name

- A function can be applied to any object
- E.g., Roommate(Umbrella0)

Reasoning about many objects at once

- Variables: x, y, z, … can refer to multiple objects
- New operators "for all" and "there exists"
  - Universal quantifier and existential quantifier
- for all x: CompletelyWhite(x) => NOT(PartiallyBlack(x))
  - Completely white objects are never partially black
- there exists x: PartiallyWhite(x) AND PartiallyBlack(x)
  - There exists some object in the world that is partially white and partially black

Practice converting English to first-order logic

- "John has Jane's umbrella"
- Has(John, Umbrella(Jane))
- "John has an umbrella"
- there exists y: (Has(John, y) AND IsUmbrella(y))
- "Anything that has an umbrella is not wet"
- for all x: ((there exists y: (Has(x, y) AND IsUmbrella(y))) => NOT(IsWet(x)))
- "Any person who has an umbrella is not wet"
- for all x: (IsPerson(x) => ((there exists y: (Has(x, y) AND IsUmbrella(y))) => NOT(IsWet(x))))

More practice converting English to first-order logic

- "John has at least two umbrellas"
- there exists x: (there exists y: (Has(John, x) AND IsUmbrella(x) AND Has(John, y) AND IsUmbrella(y) AND NOT(x=y))
- "John has at most two umbrellas"
- for all x, y, z: ((Has(John, x) AND IsUmbrella(x) AND Has(John, y) AND IsUmbrella(y) AND Has(John, z) AND IsUmbrella(z)) => (x=y OR x=z OR y=z))

Even more practice converting English to first-order logic…

- "Duke's basketball team defeats any other basketball team"
- for all x: ((IsBasketballTeam(x) AND NOT(x=BasketballTeamOf(Duke))) => Defeats(BasketballTeamOf(Duke), x))
- "Every team defeats some other team"
- for all x: (IsTeam(x) => (there exists y: (IsTeam(y) AND NOT(x=y) AND Defeats(x,y))))

More realistically…

- “Any basketball team that defeats Duke’s basketball team in one year will be defeated by Duke’s basketball team in a future year”
- for all x,y: (IsBasketballTeam(x) AND IsYear(y) AND DefeatsIn(x, BasketballTeamOf(Duke), y)) => there exists z: (IsYear(z) AND IsLaterThan(z,y) AND DefeatsIn(BasketballTeamOf(Duke), x, z))

Is this a tautology?

- “Property P implies property Q, or propery Q implies property P ”
- for all x: ((P(x) => Q(x)) OR (Q(x) => P(x)))
- (for all x: (P(x) => Q(x)) OR (for all x: (Q(x) => P(x)))

Relationship between universal and existential

- for all x: a
- is equivalent to
- NOT(there exists x: NOT(a))

Something we can**not** do in first-order logic

- We are **not** allowed to reason in general about relations and functions
- The following would correspond to higher-order logic (which is more powerful):
- "If John is Jack's roommate, then any property of John is also a property of Jack's roommate"
- (John=Roommate(Jack)) => for all p: (p(John) => p(Roommate(Jack)))
- "If a property is inherited by children, then for any thing, if that property is true of it, it must also be true for any child of it"
- for all p: (IsInheritedByChildren(p) => (for all x, y: ((IsChildOf(x,y) AND p(y)) => p(x))))

Axioms and theorems

- Axioms: basic facts about the domain, our "initial" knowledge base
- Theorems: statements that are logically derived from axioms

SUBST

- SUBST replaces one or more variables with something else
- For example:
  - SUBST({x/John}, IsHealthy(x) => NOT(HasACold(x))) gives us
  - IsHealthy(John) => NOT(HasACold(John))

Instantiating quantifiers

- From

for all x: a

we can obtain

SUBST({x/g}, a)

- From

there exists x: a

we can obtain

SUBST({x/k}, a)

where k is a constant that does not appear elsewhere in the knowledge base (Skolem constant)

- Don't need original sentence anymore

Instantiating existentials after universals

- for all x: there exists y: IsParentOf(y,x)
- WRONG: for all x: IsParentOf(k, x)
- RIGHT: for all x: IsParentOf(k(x), x)
- Introduces a new function (Skolem function)
- … again, assuming k has not been used previously

Generalized modus ponens

- for all x: Loves(John, x)
  - John loves every thing
- for all y: (Loves(y, Jane) => FeelsAppreciatedBy(Jane, y))
  - Jane feels appreciated by every thing that loves her
- Can infer from this:
- FeelsAppreciatedBy(Jane, John)

- Here, we used the substitution {x/Jane, y/John}
  - Note we used different variables for the different sentences
- General UNIFY algorithms for finding a good substitution

Keeping things as general as possible in unification

- Consider EdibleByWith
  - e.g., EdibleByWith(Soup, John, Spoon) – John can eat soup with a spoon
- for all x: for all y: EdibleByWith(Bread, x, y)
  - Anything can eat bread with anything
- for all u: for all v: (EdibleByWith(u, v, Spoon) => CanBeServedInBowlTo(u,v))
  - Anything that is edible with a spoon by something can be served in a bowl to that something
- Substitution: {x/z, y/Spoon, u/Bread, v/z}
- Gives: for all z: CanBeServedInBowlTo(Bread, z)
- Alternative substitution {x/John, y/Spoon, u/Bread, v/John} would only have given CanBeServedInBowlTo(Bread, John), which is not as general

Resolution for first-order logic

- for all x: (NOT(Knows(John, x)) OR IsMean(x) OR Loves(John, x))
  - John loves everything he knows, with the possible exception of mean things
- for all y: (Loves(Jane, y) OR Knows(y, Jane))
  - Jane loves everything that does not know her
- What can we unify? What can we conclude?
- Use the substitution: {x/Jane, y/John}
- Get: IsMean(Jane) OR Loves(John, Jane) OR Loves(Jane, John)
- Complete (i.e., if not satisfiable, will find a proof of this), **if** we can remove literals that are duplicates after unification
  - Also need to put everything in canonical form first

Notes on inference in first-order logic

- Deciding whether a sentence is entailed is semidecidable: there are algorithms that will eventually produce a proof of any entailed sentence
- It is not decidable: we cannot always conclude that a sentence is not entailed

(Extremely informal statement of) Gödel's Incompleteness Theorem

- First-order logic is not rich enough to model basic arithmetic
- For any consistent system of axioms that is rich enough to capture basic arithmetic (in particular, mathematical induction), there exist true sentences that cannot be proved from those axioms

A more challenging exercise

- Suppose:
  - There are exactly 3 objects in the world,
  - If x is the spouse of y, then y is the spouse of x (spouse is a function, i.e., everything has a spouse)
- Prove:
  - Something is its own spouse

More challenging exercise

- there exist x, y, z: (NOT(x=y) AND NOT(x=z) AND NOT (y=z))
- for all w, x, y, z: (w=x OR w=y OR w=z OR x=y OR x=z OR y=z)
- for all x, y: ((Spouse(x)=y) => (Spouse(y)=x))
- for all x, y: ((Spouse(x)=y) => NOT(x=y)) *(for the sake of contradiction)*

Umbrellas in first-order logic

- You know the following things:
  - You have exactly one other person living in your house, who is wet
  - If a person is wet, it is because of the rain, the sprinklers, or both
  - If a person is wet because of the sprinklers, the sprinklers must be on
  - If a person is wet because of rain, that person must not be carrying any umbrella
  - There is an umbrella that "lives in" your house, which is not in its house
  - An umbrella that is not in its house must be carried by some person who lives in that house
  - You are not carrying any umbrella
- Can you conclude that the sprinklers are on?

Theorem prover on the web

- http://www.spass-prover.org/webspass/index.html (use -DocProof option)
- begin_problem(TinyProblem).
- list_of_descriptions.
- name({*TinyProblem*}).
- author({*Vincent Conitzer*}).
- status(unknown).
- description({*Just a test*}).
- end_of_list.
- list_of_symbols.
-   predicates[(F,1),(G,1)].
- end_of_list.
- list_of_formulae(axioms).
- **formula(exists([U],F(U))).**
- **formula(forall([V],implies(F(V),G(V)))).**
- end_of_list.
- list_of_formulae(conjectures).
- **formula(exists([W],G(W))).**
- end_of_list.
- end_problem.

Theorem prover on the web…

- begin_problem(ThreeSpouses).
- list_of_descriptions.
- name({*ThreeSpouses*}).
- author({*Vincent Conitzer*}).
- status(unknown).
- description({*Three Spouses*}).
- end_of_list.
- list_of_symbols.
- functions[spouse].
- end_of_list.
- list_of_formulae(axioms).
- **formula(exists([X],exists([Y],exists([Z],and(not(equal(X,Y)),and(not(equal(X,Z)),not(equal(Y,Z)))))))).**
- **formula(forall([W],forall([X],forall([Y],forall([Z],or(equal(W,X),or(equal(W,Y),or(equal(W,Z),or(equal(X,Y), or(equal(X,Z),equal(Y,Z)))))))))).**
- **formula(forall([X],forall([Y],implies(equal(spouse(X),Y),equal(spouse(Y),X))))).**
- end_of_list.
- list_of_formulae(conjectures).
- **formula(exists([X],equal(spouse(X),X))).**
- end_of_list.
- end_problem.

Theorem prover on the web…

- begin_problem(TwoOrThreeSpouses).
- list_of_descriptions.
- name({*TwoOrThreeSpouses*}).
- author({*Vincent Conitzer*}).
- status(unknown).
- description({*TwoOrThreeSpouses*}).
- end_of_list.
- list_of_symbols.
- functions[spouse].
- end_of_list.
- list_of_formulae(axioms).
- **formula(exists([X],exists([Y],not(equal(X,Y))))).**
- **formula(forall([W],forall([X],forall([Y],forall([Z],or(equal(W,X),or(equal(W,Y),or(equal(W,Z),or(equal(X,Y), or(equal(X,Z),equal(Y,Z)))))))))))).**
- **formula(forall([X],forall([Y],implies(equal(spouse(X),Y),equal(spouse(Y),X))))).**
- end_of_list.
- list_of_formulae(conjectures).
- **formula(exists([X],equal(spouse(X),X))).**
- end_of_list.
- end_problem.

Theorem prover on the web…

- begin_problem(Umbrellas).
- list_of_descriptions.
- name({*Umbrellas*}).
- author({*CPS270*}).
- status(unknown).
- description({*Umbrellas*}).
- end_of_list.
- list_of_symbols.
- functions[(House,1),(You,0)].
- predicates[(Person,1),(Wet,1),(WetDueToR,1),(WetDueToS,1),(SprinklersOn,0),(Umbrella,1),(Carrying,2),(NotAtHome,1)].
- end_of_list.
- list_of_formulae(axioms).
- formula(forall([X],forall([Y],implies(and(Person(X),and(Person(Y),and(not(equal(X,You)),and(not(equal(Y,You)),and(equal(House(X),House(You)),equal(House(Y),House(You))))))),equal(X,Y))))).
- formula(exists([X],and(Person(X),and(equal(House(You),House(X)),and(not(equal(X,You)),Wet(X)))))).
- formula(forall([X],implies(and(Person(X),Wet(X)),or(WetDueToR(X),WetDueToS(X))))).
- formula(forall([X],implies(and(Person(X),WetDueToS(X)),SprinklersOn))).
- formula(forall([X],implies(and(Person(X),WetDueToR(X)),forall([Y],implies(Umbrella(Y),not(Carrying(X,Y))))))).
- formula(exists([X],and(Umbrella(X),and(equal(House(X),House(You)),NotAtHome(X))))).
- formula(forall([X],implies(and(Umbrella(X),NotAtHome(X)),exists([Y],and(Person(Y),and(equal(House(X),House(Y)),Carrying(Y,X))))))).
- formula(forall([X],implies(Umbrella(X),not(Carrying(You,X))))).
- end_of_list.
- list_of_formulae(conjectures).
- formula(SprinklersOn).
- end_of_list.

Applications

- Some serious novel mathematical results proved
- Verification of hardware and software
  - Prove outputs satisfy required properties for all inputs
- Synthesis of hardware and software
  - Try to prove that there exists a program satisfying such and such properties, in a constructive way
- Also: contributions to planning (up next)