# MATH203 Calculus

Dr. Bandar Al-Mohsin

School of Mathematics, KSU

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# Flux Integrals

## Flux Integral of $F$ over $S$

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS$$

This is called the flux integral of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ over a surface $S$, where $M, N$, and $P$ have continuous first partial derivatives on the surface, $\mathbf{n}$ is a unit normal vector to the surface $S$ at point $(x, y, z)$.

# Flux Integrals

There are two distinit sides for orientable surface

1-**Upward (upper)** unit normal for open surface Or **unit outer** normal for closed surface

$$\iint_S F\cdot \mathbf{n}ds = \iint_{R_{xy}} (-Mg_x - Ng_y + P)dA$$

2-**Downward (lower)** unit normal for open surface Or **unit inner** normal for closed surface

$$\iint_S F\cdot \mathbf{n}ds = \iint_{R_{xy}} (Mg_x + Ng_y - P)dA$$

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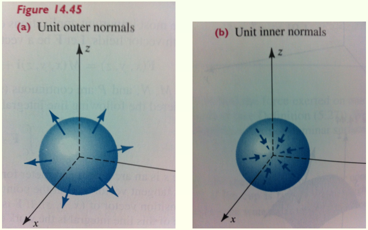

# Flux Integrals

**Examples:** Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ if $\mathbf{n}$ is a unit upper normal to $S$.

(1) $\mathbf{F}(x,y,z) = 3x\mathbf{i} + 3y\mathbf{j} + z\mathbf{k}$; $S$ is the part half of the graph of $z = 9 - x^2 - y^2$ with $z \geqslant 0$.

(2) $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $S$ is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

(3) $\mathbf{F}(x,y,z) = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$; $S$ is the portion of the cone $z = (x^2 + y^2)^{1/2}$ that is inside the cylinder $x^2 + y^2 = 1$.

(4) $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $S$ is the first-octant portion of the plane with equation $2x + 3y + z = 6$.

# The divergence theorem

## The divergence theorem

Let $Q$ be a region in three-dimensions bounded by a closed surface $S$, and let $\mathbf{n}$ denote the unit outer normal to $S$ at $(x, y, z)$. If $\mathbf{F}$ is a vector function that has continuous partial derivatives on $Q$, then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_Q \nabla \cdot \mathbf{F} \, dV = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV;$$

that is, the flux of $\mathbf{F}$ over $S$ equals the triple integral of the divergence of $\mathbf{F}$ over $Q$.

# The divergence theorem

**Examples:** Use the divergence theorem to find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

(1) Let $Q$ be the region bounded by the circular cylinder $x^2+y^2=4$ and planes $z=0$ and $z=3$, let $S$ denote the surface of $Q$ and $\mathbf{F}(x,y,z)=x^3\mathbf{i}+y^3\mathbf{j}+z^3\mathbf{k}$.

(2) Let $Q$ be the region bounded by the cylinder $z=4-x^2$ and plane $y+z=5$, and the $xy-$ and $xz-$planes, and let $S$ denote the surface of $Q$ and $\mathbf{F}(x,y,z)=(x^3+\sin z)\mathbf{i}+(x^2y+\cos z)\mathbf{j}+e^{x^2+y^2}\mathbf{k}$.

(3) $\mathbf{F}(x,y,z)=y^2z\mathbf{i}+y^3\mathbf{j}+xz\mathbf{k}$, where $S$ is the boundary of the cube defined by $-1\leqslant x\leqslant 1$, $-1\leqslant y\leqslant 1$ and $0\leqslant z\leqslant 2$.

(4) Let $Q$ be the region bounded by $z=x^2+y^2$ and the plane $z=1$, let $S$ denote the surface of $Q$ and $\mathbf{F}(x,y,z)=y\mathbf{i}+x\mathbf{j}+z^2\mathbf{k}$.

# Examples

(1) Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ if $\mathbf{n}$ is a unit upper normal to $S$.

(a) $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $S$ is the portion of the plane $3x + 2y + z = 12$ cut out by the planes $x = 0, y = 0, x = 1$ and $y = 2$.

(b) $\mathbf{F}(x,y,z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$; $S : x + y + z = 1$ in the first octant.

(2) Use the divergence theorem to find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

(a) $\mathbf{F}(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. $S$ is the surface of the region bounded by the cylinder $x^2 + y^2 = 4$ and planes $x + z = 2$ and $z = 0$.

(b) $\mathbf{F}(x,y,z) = 3x\mathbf{i} + xz\mathbf{j} + z^2\mathbf{k}$. $S$ is the surface of the region bounded by the parabolid $z = 4 - x^2 - y^2$ and $xy-$plane.