BMT-222 (September 2018)

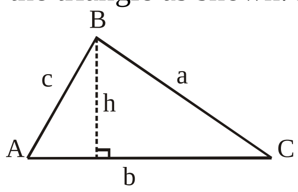
Chapter 9 The law of sines and cosines (Summary)

The Law of Sines

- The law of sines enables us to solve many oblique triangles (triangles not containing right angle).
- Solving triangles means finding the measures of all sides and angles of the triangle.

Derivation of Law of Sines

Consider the triangle as shown. Draw a perpendicular from vertex B to the opposite side.



$$\sin A = \frac{h}{c} \quad \Rightarrow h = c\sin A \qquad (1)$$

$$\sin C = \frac{h}{a} \quad \Rightarrow h = a\sin C \qquad (2)$$

From equation (1) and (2) $\quad a\sin C = c\sin A$

Dividing both side $\sin A \times \sin C$ $\quad \dfrac{a\sin C}{\sin A \times \sin C} = \dfrac{c\sin A}{\sin A \times \sin C}$

$$\Rightarrow \frac{a}{\sin A} = \frac{c}{\sin C} \qquad (3)$$

If we drop a perpendicular from C to c, we get by the same argument

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad (4)$$

Combining equations (3) and (4), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (5)$$

called the law of sines or simply the sine law.

To solve an oblique triangle by the law of sines, we need either:

1. Two angles and one side, or
2. Two sides and the angle opposite one of them

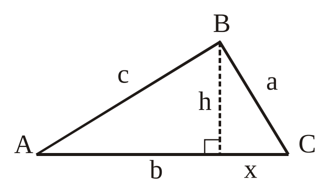
The Law of Cosines

Use the law of cosines to solve an oblique triangle given

1. Two sides and the included angle, or
2. Three side.

Derivation of Law of Cosines

Consider the triangle as shown. Draw a perpendicular from vertex B to the opposite side and let x denote the distance from C to the foot of the perpendicular.



Since, $\sin A = \dfrac{h}{c} \quad \Rightarrow h = c\sin A$, From the Pythagorean theorem: $\quad a^2 = c^2\sin^2 A + x^2 \qquad (1)$

From the above figure $\cos A = \dfrac{b-x}{c} \quad \Rightarrow b - x = c\cos A$ or $\quad x = b - c\cos A$

Substituting x in equation (1), $a^2 = c^2\sin^2 A + (b - c\cos A)^2 \quad \Rightarrow a^2 = c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$

$$\Rightarrow a^2 = c^2\left(\sin^2 A + \cos^2 A\right) + b^2 - 2bc\cos A \quad \Rightarrow a^2 = c^2 + b^2 - 2bc\cos A \qquad (2)$$

This formula is known as law of cosines or simply the cosine law. From dropping the perpendicular from the other vertices, we get two other forms of the cosine law, as summarized below.

$$a^2 = c^2 + b^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

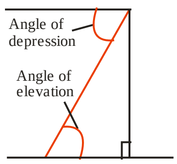
Law of cosines (verbal form): The square of any side of a triangle equals the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of angle between them.

Note: If angle $A = 90^0$, then the equation (2) reduces to $a^2 = c^2 + b^2 - 2bc \times 0 \Rightarrow a^2 = c^2 + b^2$

It means that the cosine law is a generalization of the Pythagorean theorem.

Angle of elevation and depression

- If an object is located above a horizontal plane, then the angle between the horizontal and line of sight is called angle of elevation as shown.
- If an observer is looking down at an object then the angle between the horizontal and line of sight is called angle of depression as shown.
- The angle of elevation is equal to the angle of depression as shown.



Exercises / Section 9.4 (page 289-290)

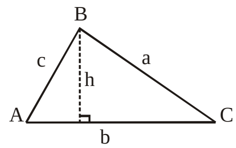


Figure 9.22

Problem # 9: Solve the triangle from the given information (Figure 9.22). $A = 29.3^o$ $a = 71.6,$ $c = 136$; angle C acute.

Problem # 21: A plane maintaining an air speed of $560\, mi/hr$ is heading 10^o west of north. A north wind causes the actual course to be 11^o west of north. Find the velocity of the plane with respect to the ground.

Problem # 25: From a point on the ground, the angle of elevation of a balloon is 49^o. From a second point $1250\ ft$ away on the opposite side of the balloon and in the same vertical plane as the balloon and the first point, the angle of elevation is 33^o. Find the distance from the second point to the balloon.

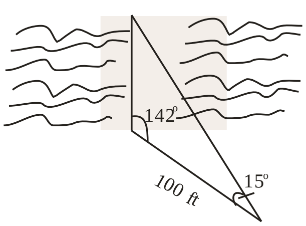
(Problems solved in class # 1, 7, 17, 23) **Home Work** (Problem # 9, 21, 25)

Problem # 1: Solve the triangle from the given information (Figure 9.22). $A = 20^o\,10', C = 50^o\,40',\ b = 4.00$

Problem # 7: Solve the triangle from the given information (Figure 9.22). $C = 31.5^o,$ $a = 13.3,$ $c = 6.82$

Problem # 17: Solve the triangle from the given information (Figure 9.22). $B = 33^o\,45',$ $a = 1.146,$ $b = 1.00$ angle C obtuse.

Problem # 23: A surveyor wants to find the width of a river from a certain point on the bank. Since no other points on the bank nearby are accessible, he takes the measurements shown in the figure. Find the width of the river.



Exercises / Section 9.5 (page 295-296)

(Problems solved in class # 5, 13, 19) **Home Work** (Problem # 1, 9, 15, 17)

Problem # 1 Solve the given triangles. $A = 46.3^o$, $b = 1.00$, $c = 2.30$

Problem # 9 Solve the given triangles. $C = 39.4^o$, $a = 126$, $b = 80.1$

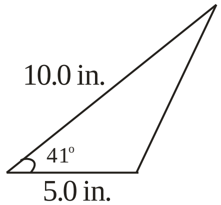
Problem # 15 A ship sails $16.0\ km$ due east, turns 20^o north of east, and then continues for another $11.5\ km$. Find its distance from the starting point.

Problem # 17 A small plane is heading 5^o north of east with an air speed of $151\, mi/hr$. The wind is from the south at $35.3\, mi/hr$. Find the actual course and the velocity with respect to the ground.

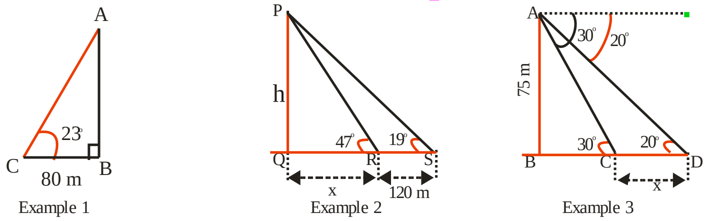
Problem # 5 Solve the given triangles. $a = 20.1$, $b = 30.3$, $c = 25.7$

Problem # 13 A civil engineer wants to find the length of a proposed tunnel. From a distant point he observes that the respective distance to the ends of the tunnel are 585 ft and 624 ft. The angle between the lines of sight is 33.4^o. Find the length of the proposed tunnel.

Problem # 19 Find the perimeter (the border or outer boundary of a two-dimensional figure) of the triangle shown below.



Solved Examples



Example # 1: An electricity pylon stands on a horizontal ground. At a point $80\,m$ from the base of the pylon, the angle of elevation of the top of the pylon is 23^o. Calculate the height of the pylon to the nearest meter.

Solution: Figure show the pylon AB and the angle of elevation of A from point C is

$$\tan 23^0 = \frac{AB}{BC} \Rightarrow \tan 23^0 = \frac{AB}{80} \Rightarrow AB = 80\tan 23^0 \Rightarrow AB = 80(0.4245) \Rightarrow AB = 33.96\,m$$

Example # 2: A surveyor measures the angle of elevation of the top of a perpendicular building as 19^0. He moves $120\,m$ nearer to the building and find the angle of elevation is now 47^0. Determine the height of the building?

Solution: The building PQ and the angles of elevations are shown in the figure. In triangle PQS,

$$\tan 19^0 = \frac{h}{x+120} \Rightarrow h = \tan 19^0 (x+120) \Rightarrow h = 0.3443(x+120) \qquad (1)$$

In triangle PQR, $\tan 47^0 = \frac{h}{x} \Rightarrow h = \tan 47^0 (x) \Rightarrow h = 1.0724(x)$ (2)

Equating equation (1) and (2):

$$0.3443(x+120) = 1.0724\,x \Rightarrow 1.0724\,x - 0.3443x = 41.316 \Rightarrow 0.7281\,x = 41.316 \Rightarrow x = 56.74\,m$$

From equation (2) $h = 1.0724\,x \Rightarrow h = 1.0724(56.74) \Rightarrow h = 60.85\,m$

Example # 3: The angle of depression of a ship viewed at a particular instant from the top of a $75m$ vertical cliff is 30^0. Find the distance of the ship from the base of the cliff at this instant. The ship is sailing away from the cliff at a constant speed and 1 minute later its angle of depression from the top of the cliff is 20^0. Determine the speed of the ship in km/h.

Solution: Cliff AB, the initial position of the ship is at C and the final position at D, $\angle ACB = 30^o$

$$\tan 30^0 = \frac{AB}{BC} \Rightarrow \tan 20^0 = \frac{75}{BC} \Rightarrow 0.5773 = \frac{75}{BC} \Rightarrow BC = \frac{75}{0.5773} \Rightarrow BC = 129.9\,m$$

Hence the initial position of the ship from the base of the cliff is $129.9\,m$.

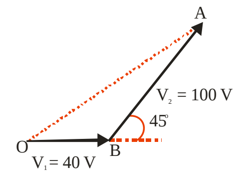
In triangle ABD,

$$\tan 20^0 = \frac{AB}{BD} \Rightarrow \tan 20^0 = \frac{75}{BC+CD} \Rightarrow \tan 20^0 = \frac{75}{129.9+x} \Rightarrow 0.3639(129.9+x) = 75 \Rightarrow x = 76.16\,m$$

The ship sails $76.16\,m$ in 1 minute (60 s),

$$speed = \frac{76.16}{60}\,m/s \Rightarrow speed = \frac{76.16\times60\times60}{60\times1000}\,km/h \Rightarrow speed = 4.57\,km/h$$

Example # 4: Two voltage phasors are shown in the figure. If $V_1 = 40V$ and $V_2 = 100V$, determine the value of their resultant (that is length OA) and the angle the resultant makes with V_1.



Solution: $\angle OBA = 180^o - 45^o = 135^o$

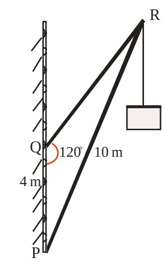
Applying law of cosines $(OA)^2 = V_1^2 + V_2^2 - 2V_1V_2\cos135^o \Rightarrow (OA)^2 = (40)^2 + (100)^2 - 2\times40\times100(-0.707)$

$$\Rightarrow (OA)^2 = 1600 + 10000 + 5657 \Rightarrow (OA)^2 = 17257 \Rightarrow OA = 131.4V$$

law of sines $\dfrac{131.4}{\sin 135} = \dfrac{100}{\sin\angle AOB} \Rightarrow \sin\angle AOB = 0.5381 \Rightarrow \angle AOB = \sin^{-1}(0.5381) \Rightarrow \angle AOB = 32.55^o$

(or 147.45^o which is not possible). Hence the resultant voltage is $131.4V$ and 32.55^o to V_1.

Example # 5 In the figure PR represents the inclined jib of a crane and is $10\,m$ long. PQ is $4\,m$ long. Determine the inclination of the jib to the vertical and length of the QR.



Solution: sine rule: $\dfrac{PR}{\sin 120} = \dfrac{PQ}{\sin R} \Rightarrow \sin R = \dfrac{4\sin 120}{10} \Rightarrow \sin R = 0.3464$

$$\Rightarrow R = \sin^{-1}(0.3464) \Rightarrow R = 20.27^o\ (or\ 159.73^o\ which\ is\ not\ possible)$$

$\therefore\ P = 180^o - 120^o - 20.27^o \Rightarrow P = 39.73^o$ which is the inclination of the jib to the vertical

Again applying the sine rule, $\dfrac{10}{\sin 120} = \dfrac{QR}{\sin 39.73} = 7.38\,m$