Full mark in math

Math 101

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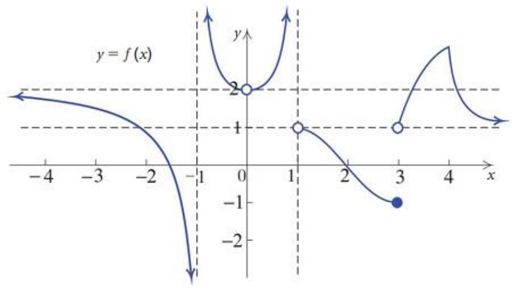
H.W.2. 1441

SETS OF NUBERS

Question 1 **6 Marks (1 each)**

Use the graph of $f(x)$ to answer the following questions (if any)



1. $\lim\limits_{x\to 3} f(x)$
2. $\lim\limits_{x\to 0} f(x)$
3. $2f(-2) + 3\lim\limits_{x\to 2} \dfrac{f(x)}{5x+1}$
4. Find all vertical asymptotes of f (if any).
5. Find all horizontal asymptotes of f (if any).
6. Discuss the continuity of f on its domain.

Solution:

1) $\lim\limits_{x\to 3^+} f(x) = 1$, $\lim\limits_{x\to 3^-} f(x) = -1$ *so* $\lim\limits_{x\to 3} f(x)$ *D.N.E*

2) $\lim\limits_{x\to 0^+} f(x) = 2$, $\lim\limits_{x\to 0^-} f(x) = 2$ *so* $\lim\limits_{x\to 0} f(x) = 2$

3) $\lim\limits_{x\to 2^+} \left(\dfrac{f(x)}{5x+1}\right) = \dfrac{\lim\limits_{x\to 2^+} f(x)}{5(2)+1} = \dfrac{0}{10+1} = 0$

$\lim\limits_{x\to 2^-} \left(\dfrac{f(x)}{5x+1}\right) = \dfrac{\lim\limits_{x\to 2^-} f(x)}{5(2)+1} = \dfrac{0}{10+1} = 0$ *so* $\lim\limits_{x\to 2} \left(\dfrac{f(x)}{5x+1}\right) = 0$

$2f(-2) + 3\lim\limits_{x\to 2} \left(\dfrac{f(x)}{5x+1}\right) = 2(1) + 3(0) = 2$

4) *V.A are* $x = -1$, $x = 1$

5) *H.A are* $y = 1$ *and* $y = 2$

5) $f(x)$ *is discotinuous at* $x = 3$ *on its domain*

Question 2 4 Marks (2 each)

Use the definition of limit to show the following:

1. $\lim_{x\to -2}(1-2x)=5$

2. $\lim_{x\to \frac{3}{2}^+}\sqrt{2x-3}=0$

Solution:

1 - $\boldsymbol{for\ any\ \ \varepsilon>0\ \ ther\ is\ \ \delta>0}$

Such that $0<\ |x+2|<0\quad then\quad |1-2x-5|<\varepsilon$

$$|-2x-4|<\varepsilon$$

$$|-2(x+2)|<\varepsilon$$

$$2|x+2|<\varepsilon$$

$$|x+2|<\frac{\varepsilon}{2}$$

$for\ agiven\ \ \varepsilon>0\ \ we\ can\ choose\ \ \ \delta=\frac{\varepsilon}{2}$

$Therefore\ by\ the\ definition\ of\ limit\ \lim_{x\to -2} 1-2x=5$

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2) $\ for\ any\ \ \varepsilon>0\ \ ther\ is\ \ \delta>0$

Such that $\frac{3}{2}<\ x<\frac{3}{2}+\delta\quad then\quad \left|\sqrt{2x-3}-0\right|=\left|\sqrt{2x-3}\right|<\varepsilon$

$\frac{3}{2}<\ x<\frac{3}{2}+\delta \Rightarrow 0<\ x-\frac{3}{2}<\delta \Rightarrow 0<\ 2x-3<2\delta$

$\sqrt{2x-3}<\sqrt{2\delta} \Rightarrow by\ choosing\ \ \sqrt{2\delta}\ =\ \varepsilon$

$$2\delta=\varepsilon^2 \Rightarrow \delta=\frac{\varepsilon^2}{2}$$

$for\ agiven\ \ \varepsilon>0\ \ there\ exist\ \delta>0\ we\ can\ choose\ \ \delta=\frac{\varepsilon^2}{2}$

$Therefore\ by\ the\ definition\ \ of\ \ limit\ \ \lim_{x\to 2^+}\sqrt{2x-3}=0$

Question 3 8 Marks (2 each)

A. Find all horizontal asymptotes for the following functions (if any)

1. $f(x) = \dfrac{2x-1}{\sqrt{9x^2+4x}-x}$ 2. $f(x) = \dfrac{x}{\sqrt{9-x^2}}$

B. Find all vertical asymptotes for the following functions (if any)

1. $f(x) = \dfrac{\sin x}{x}$ 2. $f(x) = \dfrac{x|x|-4}{x^2-2x}$

$A: 1: \quad f(x) = \dfrac{2x-1}{\sqrt{9x^2+4x}-x}$

Solution:

$$\lim_{x\to\infty} \frac{x(2-\frac{1}{x})}{\sqrt{x^2(9+\frac{4}{x})}-x}$$

$$= \lim_{x\to\infty} \frac{x(2-\frac{1}{x})}{|x|\sqrt{9+\frac{4}{x}}-x}$$

$$= \lim_{x\to\infty} \frac{x(2-\frac{1}{x})}{x\sqrt{9+\frac{4}{x}}-x}$$

$$= \lim_{x\to\infty} \frac{x(2-\frac{1}{x})}{x(\sqrt{9+\frac{4}{x}}-1)}$$

$$= \frac{2}{\sqrt{9}-1} = \frac{2}{2} = 1$$

$$\lim_{x\to-\infty} \frac{x(2-\frac{1}{x})}{\sqrt{x^2(9+\frac{4}{x})}-x}$$

$$= \lim_{x\to-\infty} \frac{x(2-\frac{1}{x})}{|x|\sqrt{9+\frac{4}{x}}-x}$$

$$= \lim_{x\to-\infty} \frac{x(2-\frac{1}{x})}{-x\sqrt{9+\frac{4}{x}}-x}$$

$$= \lim_{x\to\infty} \frac{x(2-\frac{1}{x})}{x(-\sqrt{9+\frac{4}{x}}-1)}$$

$$= \frac{2}{-\sqrt{9}-1} = \frac{2}{-4} = -\frac{1}{2}$$

So $y = 1$, $y = -\frac{1}{2}$ *are* $H.A$

$2: \quad f(x) = \dfrac{x}{\sqrt{9-x^2}} \quad D_f: 9-x^2 > 0 \Rightarrow x^2 < 9$

$|x| < 3 \Rightarrow -3 < x < 3 \Rightarrow D_f = (-3,3)$ *so* $f(x)$ *has not* $H.A$

$B: 1$ - $f(x) = \frac{\sin x}{x}$ zeroes of denominator is : $x = 0$

$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$ so $\lim_{x\to 0} \frac{\sin x}{x} = 1$

so $f(x)$ has not V.A

$2 -$ $f(x) = \frac{x|x|-4}{x^2-2x}$ zeroes of denominator is : $x^2 - 2x = 0$

$x(x-2) = 0 \Rightarrow x = 0$, $x = 2$

at $x = 0$

$$\lim_{x\to 0^+} \frac{x|x|-4}{x^2-2x} = \frac{-4}{0} = \infty \quad , \quad \lim_{x\to 0^-} \frac{x|x|-4}{x^2-2x} = \frac{-4}{0} = -\infty$$

at $x = 2$

$$\lim_{x\to 2^+} \frac{x|x|-4}{x^2-2x} = \lim_{x\to 2^+} \frac{x^2-4}{x^2-2x} = \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x(x-2)} = \frac{4}{2} = 2$$

$$\lim_{x\to 2^-} \frac{x|x|-4}{x^2-2x} = \lim_{x\to 2^-} \frac{x^2-4}{x^2-2x} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x(x-2)} = \frac{4}{2} = 2$$

so $f(x)$ has V.A at $x = 0$ only

Question 4 **3 Marks**

Find the value of a and b such that:

$$\lim_{x\to 0} \frac{a - \cos(bx)}{x^2} = 8$$

$f(x)$ is indeterminate form $\frac{0}{0}$ so numerator can be zero at

$x = 0$ so $a - \cos 0 = 0 \Rightarrow a - 1 = 0$ so $a = 1$

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$$\lim_{x\to 0} \frac{1-\cos bx}{x^2} \cdot \frac{1+\cos bx}{1+\cos bx} = \lim_{x\to 0} \frac{1-\cos^2 bx}{x^2(1+\cos bx)} = \lim_{x\to 0} \frac{\sin^2 bx}{x^2(1+\cos bx)}$$

$$\lim_{x\to 0} \frac{\sin^2 bx}{x^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos bx} = \lim_{x\to 0} \left(\frac{\sin bx}{x}\right)^2 \cdot \frac{1}{2} = b^2 \cdot \frac{1}{2}$$

$b^2 \cdot \frac{1}{2} = 8 \Rightarrow b^2 = 16 \Rightarrow b = \pm 4$

Question 5 20 Marks (2 each)

Find the following limits (if exists)

1. $\lim_{x\to -1}\frac{x^2-1}{2x+1}$
2. $\lim_{x\to 0}\frac{(x+2)^3-8}{x}$
3. $\lim_{x\to\infty}(\sqrt{x^2+2}-x)$
4. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$
5. $\lim_{x\to 3}\frac{x-3}{|x-3|}$
6. $\lim_{x\to\infty}\frac{2x+\sin x}{4x+1}$
7. $\lim_{x\to 0^+}\frac{\sqrt{1-\sin^2(\frac{\pi}{2}-x)}}{3x}$
8. $\lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{\sqrt{1+x}}-1\right)\right]$
9. $\lim_{x\to 2}\cos\left(\frac{x^2-4}{x+1}\right)$
10. $\lim_{x\to 0}\frac{x}{\tan(2x)+\sin(3x)}$

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1) $\lim_{x\to -1}\frac{x^2-1}{2x+1}=\frac{(-1)^2-1}{2(-1)+1}=\frac{1-1}{-2+1}=\frac{0}{-1}=0$

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2) $\lim_{x\to 0}\frac{(x+2)^3-8}{x}$ (*indeterminate form* $\frac{0}{0}$)

$\lim_{x\to 0}\frac{(x+2)^3-8}{x}=\lim_{x\to 0}\frac{(x+2-2)((x+2)^2+2(x+2)+4)}{x}$

$\lim_{x\to 0}(x+2)^2+2(x+2)+4=2^2+4+4=12$

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3) $\lim_{x\to\infty}\sqrt{x^2+2}-x$ $(\infty-\infty)$ *indeterminate form*

$\lim_{x\to\infty}\sqrt{x^2+2}-x\cdot\frac{\sqrt{x^2+2}+x}{\sqrt{x^2+2}+x}=\lim_{x\to\infty}\frac{x^2+2-x^2}{\sqrt{x^2+2}+x}=\lim_{x\to\infty}\frac{2}{\sqrt{x^2+2}+x}=0$

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4) $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\lim_{x\to 1}\left(\frac{x+1-2}{x^2-1}\right)=\lim_{x\to 1}\frac{x-1}{x^2-1}$ *indeterminate form* $\frac{0}{0}$

$\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$

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5) $\lim_{x\to 3}\frac{x-3}{|x-3|}$ *indeterminate form* $\frac{0}{0}$

$\lim_{x\to 3^+}\frac{x-3}{|x-3|}=\lim_{x\to 3^+}\frac{x-3}{x-3}=1$

$\lim_{x\to 3^-}\frac{x-3}{|x-3|}=\lim_{x\to 3^-}\frac{x-3}{-(x-3)}=-1$ SO $\lim_{x\to 3}\frac{x-3}{|x-3|}$ D.N.E

6) $\lim_{x\to\infty} \dfrac{2x + \sin x}{4x + 1}$ (*indeterminate form* $\dfrac{\infty}{\infty}$)

$= \lim_{x\to\infty} \dfrac{2+\frac{\sin x}{x}}{4+\frac{1}{x}}$

$= \dfrac{2+0}{4+0} = \dfrac{1}{2}$

$\lim_{x\to\infty} \dfrac{\sin x}{x}$

$-1 \le \sin x \le 1$

$-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$

$\lim_{x\to\infty} -\dfrac{1}{x} = 0\,, \qquad \lim_{x\to\infty} \dfrac{1}{x} = 0$

so by using sandwich theorem

$\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$

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7) $\lim_{x\to 0^+} \dfrac{\sqrt{1-\sin^2(\frac{\pi}{2}-x)}}{3x} = \lim_{x\to 0^+} \dfrac{\sqrt{\cos^2(\frac{\pi}{2}-x)}}{3x} = \lim_{x\to 0^+} \dfrac{\cos(\frac{\pi}{2}-x)}{3x} = \lim_{x\to 0^+} \dfrac{\sin x}{3x} = \dfrac{1}{3}$

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8) $\lim_{x\to 0} \left[\frac{1}{x}\left(\frac{1}{\sqrt{1+x}} - 1\right)\right] = \lim_{x\to 0} \left[\frac{1}{x}\left(\frac{1-\sqrt{1+x}}{\sqrt{1+x}}\right)\right] = \lim_{x\to 0} \dfrac{1-\sqrt{1+x}}{x\sqrt{1+x}} \cdot \dfrac{1+\sqrt{1+x}}{1+\sqrt{1+x}}$

$= \lim_{x\to 0} \dfrac{1-1-x}{x\sqrt{1+x}(1+\sqrt{1+x})} = \lim_{x\to 0} \dfrac{-x}{x\sqrt{1+x}(1+\sqrt{1+x})} = \lim_{x\to 0} \dfrac{-1}{\sqrt{1+x}(1+\sqrt{1+x})} = \dfrac{-1}{2}$

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9) $\lim_{x\to 2} \cos\left(\dfrac{x^2-4}{x+1}\right) = \cos \lim_{x\to 2}\left(\dfrac{x^2-4}{x+1}\right) = \cos\dfrac{0}{3} = \cos 0 = 1$

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10) $\lim_{x\to 0} \dfrac{x}{\tan(2x) + \sin(3x)}$ (*indeterminate form* $\dfrac{0}{0}$)

$\lim_{x\to 0} \dfrac{x}{\tan(2x)+\sin(3x)} = \lim_{x\to 0} \dfrac{\frac{x}{x}}{\frac{\tan(2x)}{x}+\frac{\sin(3x)}{x}} = \dfrac{1}{2+3} = \dfrac{1}{5}$

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Question 6 9 Marks (3 each)

A. Discuss the continuity of the function $f(x) = \cos(x^2 + 1)$.

B. Use the Intermediate Value Theorem to prove that the equation $\frac{x^5 + 1}{x + 3} = 3$ has at least a real solution.

C. Find the constants a and b such that the function

$$f(x) = \begin{cases} \sqrt{\frac{x+4}{x+b}}, & x > 0 \\ a + b, & x = 0 \\ \frac{\sin(2x)}{3x}, & x < 0 \end{cases}$$

is continuous on $\mathbb{R}$.

A) note that $f(x) = h(g(x))$ where $g(x) = x^2 + 1$

and $h(x) = \cos x$ since both h and g are continuous on R so

$f(x) = \cos(x^2 + 1)$ is continuous On R

B) rewrite the equation $\frac{x^5 + 1}{x + 3} - 3 = 0$

$$\text{so } f(x) = \frac{x^5 + 1 - 3x - 9}{x + 3} = \frac{x^5 - 3x - 8}{x + 3} \text{ so } D_f = R - \{-3\}$$

let $g(x) = x^5 - 3x - 8$ is apolynomial so it is continuous on R

let $h(x) = x + 3$ is apolynomial so it is continuous on R

so $f(x) = \frac{g(x)}{h(x)}$ is continuous on $R - \{-3\} = (-\infty, -3) \cup (-3, \infty)$

by trail and error $f(1) = \frac{1 - 3 - 8}{1 + 3} = -2.5$

and $f(2) = \frac{32 - 6 - 8}{5} = 3.6$ and we note that

$f(x)$ iscontinuous on $[1,2] \subseteq (-3, \infty)$ since $-2.5 < 0 < 3.6$

so by Using intermediate value theorem there is at least one

number of $c \in [1,2]$ such that $f(c) = 0$

so the equation has at least one real solution

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C) $f(x)$ *continuous on R so it is continuous at* $x = 0$ *so*

$$a + b = \lim_{x \to 0^+} \sqrt{\frac{x+4}{x+b}} = \sqrt{\frac{4}{b}} = \frac{2}{\sqrt{b}}$$

$$a + b = \lim_{x \to 0^-} \frac{\sin 2x}{3x} = \frac{2}{3}$$

$$\frac{2}{\sqrt{b}} = \frac{2}{3} \quad \rightarrow \sqrt{b} = 3 \quad \rightarrow b = 9$$

$$a + 9 = \frac{2}{3} \quad \rightarrow \quad a = \frac{-25}{3}$$