CHAPTER 6: Using Sample Data to Make Estimations About Population Parameters

6.1 Introduction:

Statistical Inferences: (Estimation and Hypotheses Testing)
It is the procedure by which we reach a conclusion about a population on the basis of the information contained in a sample drawn from that population.

There are two main purposes of statistics;

- Descriptive Statistics: (Chapter 1 & 2): Organization & summarization of the data
- Statistical Inference: (Chapter 6 and 7): Answering research questions about some unknown population parameters.

(1) Estimation: (chapter 6)
Approximating (or estimating) the actual values of the unknown parameters:

- **Point Estimate:** A point estimate is single value used to estimate the corresponding population parameter.
- **Interval Estimate (or Confidence Interval):** An interval estimate consists of two numerical values defining a range of values that most likely includes the parameter being estimated with a specified degree of confidence.

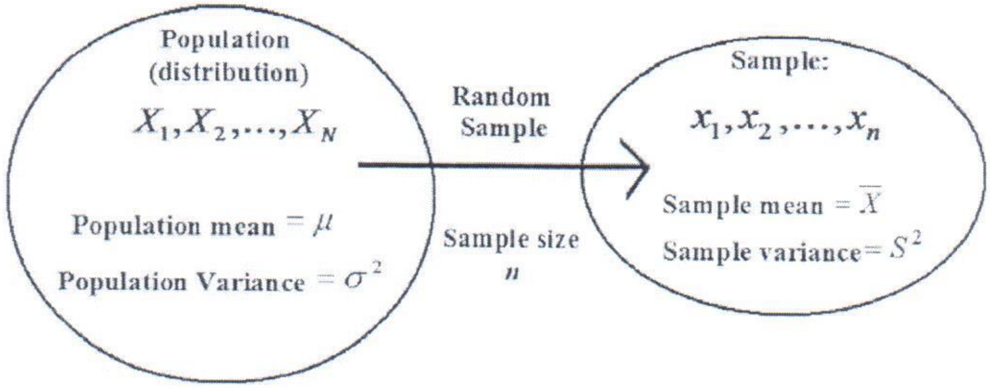
(2) Hypothesis Testing: (chapter 7)
Answering research questions about the unknown parameters of the population (confirming or denying some conjectures or statements about the unknown parameters).

6.1: The Point Estimates of the Population Parameters:

	Population Parameters	Point estimator
Mean	μ	$\overline{X}$
Variance	σ^2	S^2
Standard Deviation	σ	S
Proportion	P	$\widehat{p}$
The Difference between Two Means	$\mu_1 - \mu_2$	$\overline{X_1} - \overline{X_2}$
The Difference between Two Proportion	$P_1 - P_2$	$\widehat{P_1} - \widehat{P_2}$

6.2 Confidence Interval for a Population Mean (μ) :

In this section we are interested in estimating the mean of a certain population (μ).



Population:

Population Size = N

Population Values: $X_1, X_2, \ldots, X_N$

Population Mean: $\mu = \frac{\sum_{i=1}^{N} X_i}{N}$

Population Variance: $\sigma^2 = \frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}$

Sample:

Sample Size = n

Sample values: $x_1, x_2, \ldots, x_n$

Sample Mean: $\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$

Sample Variance: $S^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n-1}$

(i) Point Estimation of μ:

A point estimate of the mean is a single number used to estimate (or approximate) the true value of μ .

- Draw a random sample of size n from the population:

  - $x_1, x_2, \ldots, x_n$

- Compute the sample mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Result:

The sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a "good" point estimator of the population mean (μ).

(1-α) % confident level

- How to get α when confidence level (1-α) % known

Example1 :

If we are 95% confident ,find α ?

$$\alpha = \frac{5}{100} = 0.05$$

Example2 :

If we are 99% confident ,find α ?

$$\alpha = \frac{1}{100} = 0.01$$

Example3 :

If we are 80% confident ,find α ?

$$\alpha = \frac{20}{100} = 0.20$$

Example4 :

If we are 92% confident ,find α ?

$$\alpha = \frac{8}{100} = 0.08$$

(ii) Confidence Interval (Interval Estimate) of μ:

An interval estimate of μ is an interval (L,U) containing the true value of μ "with a probability of $1-\alpha$".

- ⋆ $1-\alpha$ = is called the confidence coefficient (level)
- ⋆ L = lower limit of the confidence interval
- ⋆ U = upper limit of the confidence interval

Result: (For the case when σ is known)

(a) If $X_1, X_2 \ldots, X_n$ is a random sample of size n from a normal distribution with mean μ and known variance σ^2, then:
A $(1-\alpha)100\%$ confidence interval for μ is:

$$\overline{X} \pm Z_{1-\frac{\alpha}{2}}\,\sigma_{\overline{X}}$$

$$\overline{X} \pm Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

$$\left(\overline{X} - Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$\overline{X} - Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

(b) If $X_1, X_2 \ldots, X_n$ is a random sample of size n from a non-normal distribution with mean μ and known variance σ^2, and if the sample size n is large $(n \geq 30)$, then:
An approximate $(1-\alpha)100\%$ confidence interval for μ is:

$$\overline{X} \pm Z_{1-\frac{\alpha}{2}}\,\sigma_{\overline{X}}$$

$$\overline{X} \pm Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

$$\left(\overline{X} - Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$\overline{X} - Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

Note that:

1. We are $(1-\alpha)100\%$ confident that the true value of μ belongs to the interval $(\bar{X}-Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\ ,\ \bar{X}+Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$.

2. Upper limit of the confidence interval = $\bar{X}+Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$

3. Lower limit of the confidence interval = $\bar{X}-Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$

4. $Z_{1-\frac{\alpha}{2}}$ = Reliability Coefficient

5. $Z_{1-\frac{\alpha}{2}}\times\frac{\sigma}{\sqrt{n}}$ = margin of error = precision of the estimate

6. In general the interval estimate (confidence interval) may be expressed as follows:

$$\bar{X}\pm Z_{1-\frac{\alpha}{2}}\ \sigma_{\bar{X}}$$

estimator ± (reliability coefficient) × (standard Error)

estimator ± margin of error

6.3 The t Distribution: (Confidence Interval Using t)

We have already introduced and discussed the t distribution.

Result: (For the case when σ is unknown + normal population) If $X_1, X_2 \ldots, X_n$ is a random sample of size n from a normal distribution with mean μ and unknown variance σ^2 , then:

A $(1-\alpha)100\%$ confidence interval for μ is:

$$\bar{X}\pm t_{1-\frac{\alpha}{2}}\ \hat{\sigma}_{\bar{X}}$$

$$\bar{X}\pm t_{1-\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\left(\bar{X}-t_{1-\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\ ,\ \bar{X}+t_{1-\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right)$$

where the degrees of freedom is:

$$df = \nu = n-1.$$

Note that:

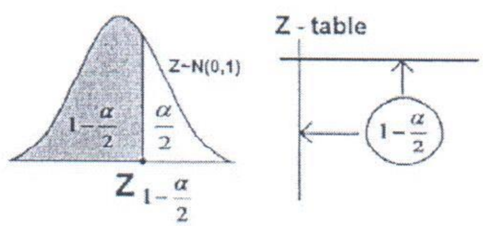
1. We are $(1-\alpha)100\%$ confident that the true value of μ belongs to the interval $\left(\overline{X} - t_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}}, \overline{X} + t_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}}\right)$.
2. $\hat{\sigma}_{\overline{X}} = \frac{S}{\sqrt{n}}$ (estimate of the standard error of $\overline{X}$)
3. $t_{1-\frac{\alpha}{2}}$ = Reliability Coefficient
4. In this case, we replace σ by S and Z by t.
5. In general the interval estimate (confidence interval) may be expressed as follows:

Estimator ± (Reliability Coefficient) × (Estimate of the Standard Error)

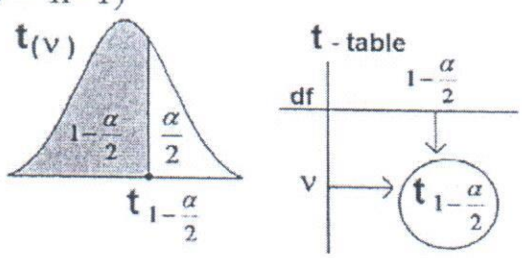
$$\overline{X} \pm t_{1-\frac{\alpha}{2}} \hat{\sigma}_{\overline{X}}$$

Notes: (Finding Reliability Coefficient)

(1) We find the reliability coefficient $Z_{1-\frac{\alpha}{2}}$ from the Z-table as follows:



(2) We find the reliability coefficient $t_{1-\frac{\alpha}{2}}$ from the t-table as follows: $(df = \nu = n-1)$



Example:

Suppose that Z ~ N(0,1). Find $Z_{1-\frac{\alpha}{2}}$ for the following cases:

(1) α =0.1 (2) α =0.05 (3) α =0.01

Solution:

(1) For α =0.1:

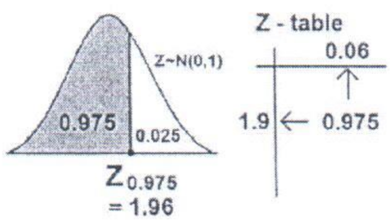
$$1-\frac{\alpha}{2}=1-\frac{0.1}{2}=0.95 \quad \Rightarrow \quad Z_{1-\frac{\alpha}{2}}=Z_{0.95}=1.645$$

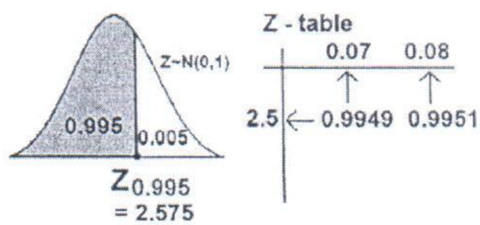
(2) For α =0.05:

$$1-\frac{\alpha}{2}=1-\frac{0.05}{2}=0.975 \quad \Rightarrow \quad Z_{1-\frac{\alpha}{2}}=Z_{0.975}=1.96.$$

(3) For α =0.01:

$$1-\frac{\alpha}{2}=1-\frac{0.01}{2}=0.995 \quad \Rightarrow \quad Z_{1-\frac{\alpha}{2}}=Z_{0.995}=2.575.$$





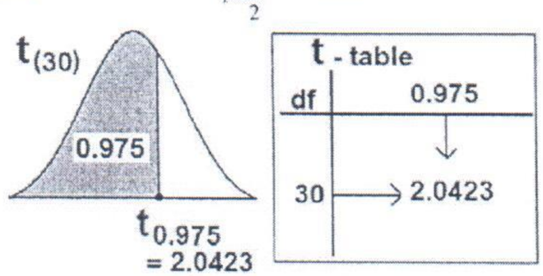
Example:

Suppose that t ~ t(30). Find $t_{1-\frac{\alpha}{2}}$ for α =0.05.

Solution:

df = ν = 30

$$1-\frac{\alpha}{2}=1-\frac{0.05}{2}=0.975 \quad \Rightarrow \quad t_{1-\frac{\alpha}{2}}=t_{0.975}=2.0423$$



The Confident Interval (C.I) for the Population Mean (μ)

Confident Interval (C.I) for the Population Mean (μ)

σ^2 known + Normal or non-normal(n large)

$$\bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

σ^2 unknown + Normal n small ($n \leq 30$)

$$\bar{X} \pm t_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$t_{1-\frac{\alpha}{2}}$ at $df = n - 1$

Example: (The case where σ^2 is known)

Diabetic ketoacidosis is a potential fatal complication of diabetes mellitus throughout the world and is characterized in part by very high blood glucose levels. In a study on 123 patients living in Saudi Arabia of age 15 or more who were admitted for diabetic ketoacidosis, the mean blood glucose level was 26.2 mmol/l. Suppose that the blood glucose levels for such patients have a normal distribution with a standard deviation of 3.3 mmol/l.

(1) Find a point estimate for the mean blood glucose level of such diabetic ketoacidosis patients.

(2) Find a 90% confidence interval for the mean blood glucose level of such diabetic ketoacidosis patients.

Solution:

Variable = X = blood glucose level (quantitative variable).

Population = diabetic ketoacidosis patients in Saudi Arabia of age 15 or more.

Parameter of interest is: μ= the mean blood glucose level.

Distribution is normal with standard deviation $\sigma = 3.3$.

σ^2 is known ($\sigma^2 = 10.89$)

X ~ Normal(μ , 10.89)

μ = ?? (unknown- we need to estimate μ)

Sample size: $n = 123$ (large)

Sample mean: $\overline{X} = 26.2$

(1) Point Estimation:

We need to find a point estimate for μ.

$\overline{X} = 26.2$ is a point estimate for μ.

$\mu \approx 26.2$

(2) Interval Estimation (Confidence Interval = C. I.):

We need to find 90% C. I. for μ.

90% = $(1-\alpha)100\%$

$$1-\alpha = 0.9 \Leftrightarrow \alpha = 0.1 \Leftrightarrow \frac{\alpha}{2} = 0.05 \quad \Leftrightarrow \quad 1-\frac{\alpha}{2} = 0.95$$

The reliability coefficient is: $Z_{1-\frac{\alpha}{2}} = Z_{0.95} = 1.645$

90% confidence interval for μ is:

$$\left(\overline{X}-Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\,,\,\overline{X}+Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$\left(26.2-(1.645)\frac{3.3}{\sqrt{123}}\,,\,26.2+(1.645)\frac{3.3}{\sqrt{123}}\right)$$

$$(26.2-0.4894714\,,\,26.2+0.4894714)$$

$$(25.710529\,,\,26.689471)$$

We are 90% confident that the true value of the mean μ lies in the interval (25.71 , 26.69), that is:

$$25.71<\mu<26.69$$

Note: for this example even if the distribution is not normal, we may use the same solution because the sample size n=123 is large.

Example: **(The case where σ^2 is unknown)**
A study was conducted to study the age characteristics of Saudi women having breast lump. A sample of 21 Saudi women gave a mean of 37 years with a standard deviation of 10 years. Assume that the ages of Saudi women having breast lumps are normally distributed.

(a) Find a point estimate for the mean age of Saudi women having breast lumps.

(b) Construct a 99% confidence interval for the mean age of Saudi women having breast lumps

Solution:

X = Variable = age of Saudi women having breast lumps (quantitative variable).

Population = All S

X = Variable = age of Saudi women having breast lumps (quantitative variable).

Population = All Saudi women having breast lumps.

Parameter of interest is:

μ = the age mean of Saudi women having breast lumps.

X ~ Normal(μ , σ^2)

μ = ?? (unknown- we need to estimate μ)

σ^2 = ?? (unknown)

Sample size:

n = 21 (n small)

$\bar{X} = 37$

(a)Point estimation :We need to find point estimate for μ .

$\overline{X}$ = 37 is a "good " point estimate for μ

μ ≈ 37 years

(b) Interval Estimation (Confident Interval = C.I) :

We need to find 99% C.I for μ

$(1-\alpha)\ \% = 99\% \leftrightarrow \alpha = \frac{1}{100} = 0.01 \leftrightarrow \frac{\alpha}{2} = 0.005 \leftrightarrow 1 - \frac{\alpha}{2} = 0.995$

The reliability coefficient is

$$\leftrightarrow t_{0.995} = 2.845 \quad (at\ df = n - 1 = 20)$$

99% Confident Interval for μ is:

$$\overline{X} \pm t_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$37 \pm 2.845\ \frac{10}{\sqrt{21}} = 37 \pm 6.208$$

$$(37 - 6.208\ ,\ 37 + 6.208)$$

$$(30.792\ ,43.208)$$

We are 99% confident that the true value of the mean μ lies in the interval $(30.792\ ,43.208)$, that is

$$30.792 < \mu < 42.208$$

$$(34.62\ ,\ 39.38)$$

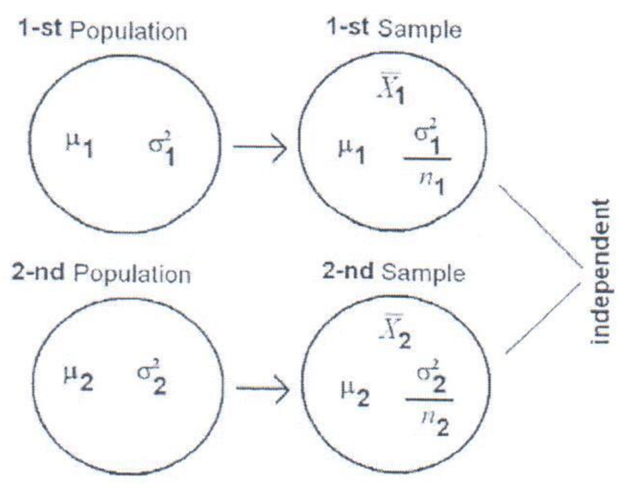
We are 99% confident that the true value of the mean μ lies in the interval $(34.61\ ,\ 39.39)$, that is:

$$34.62 < \mu < 39.38$$

6.4 Confidence Interval for the Difference between Two Population Means $(\mu_1-\mu_2)$:

Suppose that we have two populations:

- 1-st population with mean μ_1 and variance σ_1^2
- 2-nd population with mean μ_2 and variance σ_2^2
- We are interested in comparing μ_1 and μ_2, or equivalently, making inferences about the difference between the means $(\mu_1-\mu_2)$.
- We independently select a random sample of size n_1 from the 1-st population and another random sample of size n_2 from the 2-nd population:
- Let $\overline{X}_1$ and S_1^2 be the sample mean and the sample variance of the 1-st sample.
- Let $\overline{X}_2$ and S_2^2 be the sample mean and the sample variance of the 2-nd sample.
- The sampling distribution of $\overline{X}_1-\overline{X}_2$ is used to make inferences about $\mu_1-\mu_2$.



6.4.A Point Estimation of $\mu_1 - \mu_2$

Result:

$$\overline{X_1} - \overline{X_2} \text{ is a good estimate for } \mu_1 - \mu_2$$

6.4.B. Interval Estimation (Confidence Estimation) of $\mu_1 - \mu_2$

We will consider two cases

(i) First case : $\sigma_1^2,\ \sigma_2^2\ are\ knowm$

If $\sigma_1^2,\ \sigma_2^2\ are\ knowm$, we use the following result to find an Interval estimate for $\mu_1 - \mu_2$

Result :

A(1- α) 100% confidence interval for $\mu_1 - \mu_2\ \ is$

$$(\overline{X_1} - \overline{X_2}) \pm Z_{1-\frac{\alpha}{2}}\ \sigma_{\overline{X_1}-\overline{X_2}}$$

$$(\bar{X}_1-\bar{X}_2)\pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

$$\left((\bar{X}_1-\bar{X}_2)-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\ ,\ (\bar{X}_1-\bar{X}_2)+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$$

$$(\bar{X}_1-\bar{X}_2)-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}<\mu_1-\mu_2<(\bar{X}_1-\bar{X}_2)+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

Estimator ± (Reliability Coefficient) × (Standard Error)

(ii) Second Case:

Unknown equal Variances: ($\sigma_1^2=\sigma_2^2=\sigma^2$ is unknown):

If σ_1^2 and σ_2^2 are equal but unknown ($\sigma_1^2=\sigma_2^2=\sigma^2$), then the pooled estimate of the common variance σ^2 is

$$S_p^2=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$

where S_1^2 is the variance of the 1-st sample and S_2^2 is the variance of the 2-nd sample. The degrees of freedom of S_p^2 is

$$df=\nu=n_1+n_2-2.$$

We use the following result to find an interval estimate for $\mu_1-\mu_2$ when we have normal populations with unknown and equal variances.

Result:

A $(1-\alpha)100\%$ confidence interval for $\mu_1-\mu_2$ is:

$$(\bar{X}_1-\bar{X}_2)\pm t_{1-\frac{\alpha}{2}}\sqrt{\frac{S_p^2}{n_1}+\frac{S_p^2}{n_2}}$$

$$\left((\bar{X}_1-\bar{X}_2)-t_{1-\frac{\alpha}{2}}\sqrt{\frac{S_p^2}{n_1}+\frac{S_p^2}{n_2}}\ ,\ (\bar{X}_1-\bar{X}_2)+t_{1-\frac{\alpha}{2}}\sqrt{\frac{S_p^2}{n_1}+\frac{S_p^2}{n_2}}\right)$$

where reliability coefficient $t_{1-\frac{\alpha}{2}}$ is the t-value with $df=\nu=n_1+n_2-2$ degrees of freedom.

The Confident Interval (C.I) for the Difference between two Population Means ($\mu_1 - \mu_2$)

Confident Interval (C.I) for the Difference between two Population Means ($\mu_1 - \mu_2$)

σ_1^2, σ_2^2 *are knowm*

$$(\overline{X_1} - \overline{X_2}) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$\sigma_1^2 = \sigma_2^2 = \sigma^2$ *are unkown* **but equal** , normal distribution

n_1, n_2 **small ($n_1 \leq 30, n_2 \leq 30$)**

$$(\overline{X_1} - \overline{X_2}) \pm t_{1-\frac{\alpha}{2}} \sqrt{\frac{S_P^2}{n_1} + \frac{S_P^2}{n_2}}$$

$t_{1-\frac{\alpha}{2}}$ at df = $n_1 + n_2 - 2$

Pooled variance:

$$S_P^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Example: An Experiment was Conducted to Compare time length (duration time) of two types of surgeries (A) and (B). 75 surgeries of type (A) and 50 surgeries of type (B) were performed. The average time length for (A) was 42 minutes and the average for (B) was 36 minutes.

(1) Find a point estimate for $\mu_A - \mu_B$, where μ_A and μ_B are population means of the time length of surgeries of type (A) and (B), respectively.

(2) Find a 96% confidence interval for $\mu_A - \mu_B$. Assume that the population standard deviations are 8 and 6 for type (A) and (B), respectively.

Solution:

Surgery	Type (A)	Type (B)
Sample Size	$n_A = 75$	$n_B = 50$
Sample Mean	$\bar{X}_A = 42$	$\bar{X}_B = 36$
Population Standard Deviation	$\sigma_A = 8$	$\sigma_B = 6$

(1) A point estimate for $\mu_A - \mu_B$ is:

$$\bar{X}_A - \bar{X}_B = 42 - 36 = 6.$$

(2) Finding a 96% confidence interval for $\mu_A - \mu_B$:

$\alpha = ??$

$96\% = (1-\alpha)100\% \Leftrightarrow 0.96 = (1-\alpha) \Leftrightarrow \alpha = 0.04 \Leftrightarrow \alpha/2 = 0.02$

Reliability Coefficient: $Z_{1-\frac{\alpha}{2}} = Z_{0.98} = 2.055$

A 96% C.I. for $\mu_A - \mu_B$ is:

$$(\bar{X}_A - \bar{X}_B) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}$$

$$6 \pm Z_{0.98} \sqrt{\frac{8^2}{75} + \frac{6^2}{50}}$$

$$6 \pm (2.055) \sqrt{\frac{64}{75} + \frac{36}{50}}$$

$$6 \pm 2.578$$

$$3.422 < \mu_A - \mu_B < 8.58$$

We are 96% confident that $\mu_A - \mu_B \in (3.42, 8.58)$.

Note: Since the confidence interval does not include zero, we conclude that the two population means are not equal ($\mu_A - \mu_B \neq 0 \Leftrightarrow \mu_A \neq \mu_B$). Therefore, we may conclude that the mean time length is not the same for the two types of surgeries.

Example: (2nd Case: $\sigma_1^2 = \sigma_2^2$ unknown)

To compare the time length (duration time) of two types of surgeries (A) and (B), an experiment shows the following results based on two independent samples:

Type A: 140, 138, 143, 142, 144, 137

Type B: 135, 140, 136, 142, 138, 140

(1) Find a point estimate for $\mu_A - \mu_B$, where μ_A (μ_B) is the mean time length of type A (B).

(2) Assuming normal populations with equal variances, find a 95% confidence interval for $\mu_A - \mu_B$.

Solution:

First we calculate the mean and the variances of the two samples, and we get:

Surgery	Type (A)	Type (B)
Sample Size	$n_A = 6$	$n_B = 6$
Sample Mean	$\overline{X}_A = 140.67$	$\overline{X}_B = 138.50$
Sample Variance	$S^2_A = 7.87$	$S^2_B = 7.10$

(1) A point estimate for $\mu_A - \mu_B$ is:

$$\overline{X}_A - \overline{X}_B = 140.67 - 138.50 = 2.17.$$

(2) Finding 95% Confidence interval for $\mu_A - \mu_B$:

$95\% = (1-\alpha)100\% \Leftrightarrow 0.95 = (1-\alpha) \Leftrightarrow \alpha = 0.05 \Leftrightarrow \alpha/2 = 0.025$

.df $= \nu = n_A + n_B - 2 = 10$

Reliability Coefficient: $t_{1-\frac{\alpha}{2}} = t_{0.975} = 2.228$

The pooled estimate of the common variance is:

$$S_p^2 = \frac{(n_A-1)S_A^2+(n_B-1)S_B^2}{n_A+n_B-2}$$

$$= \frac{(6-1)(7.87)+(6-1)(7.1)}{6+6-2} = 7.485$$

A 95% C.I. for $\mu_A - \mu_B$ is:

$$(\bar{X}_A - \bar{X}_B) \pm t_{1-\frac{\alpha}{2}} \sqrt{\frac{S_p^2}{n_A}+\frac{S_p^2}{n_B}}$$

$$2.17 \pm (2.228)\sqrt{\frac{7.485}{6}+\frac{7.485}{6}}$$

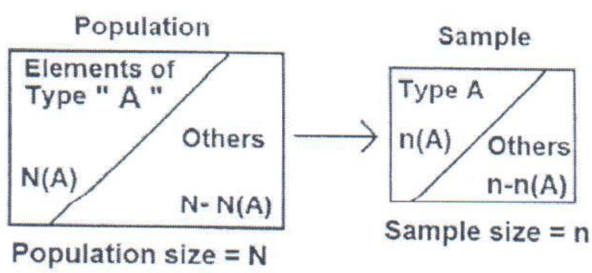
$$2.17 \pm 3.519$$

$$-1.35 < \mu_A - \mu_B < 5.69$$

We are 95% confident that $\mu_A - \mu_B \in (-1.35, 5.69)$.

Note: Since the confidence interval includes zero, we conclude that the two population means may be equal ($\mu_A - \mu_B = 0 \Leftrightarrow \mu_A = \mu_B$). Therefore, we may conclude that the mean time length is the same for both types of surgeries.

6.5 Confidence Interval for a Population Proportion (p):



Recall:

1. For the population:

   $N(A)$ = number of elements in the population with a specified characteristic "A"

   N = total number of elements in the population (population size)

   The population proportion is:

   $$p = \frac{N(A)}{N} \qquad \text{(p is a parameter)}$$

2. For the sample:

.n(A) = number of elements in the sample with the same characteristic "A"

.n = sample size

The sample proportion is

$$\hat{P} = \frac{n(A)}{n} \quad , \hat{P} \text{ is a statistic}$$

6.5.A Point Estimation of (P)

Result:

$\hat{P}$ **is a good estimate for population proportion** (P)

6.5.B. Interval Estimation (Confidence Estimation) of

For large sample size ($n \geq 30, np > 5, n(1-p) > 5$), an approximate $(1-\alpha)$ 100%confidence interval for (p) is:

$$\hat{p} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\left(\hat{p} - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \; , \; \hat{p} + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

Estimator ± (Reliability Coefficient) × (Standard Error)

Example:

In a study on the obesity of Saudi women, a random sample of 950 Saudi women was taken. It was found that 611 of these women were obese (overweight by a certain percentage).

(1) Find a point estimate for the true proportion of Saudi women who are obese.

(2) Find a 95% confidence interval for the true proportion of Saudi women who are obese.

Solution:

Variable: whether or not a women is obese (qualitative variable)

Population: all Saudi women

Parameter: p =the proportion of women who are obese.

Sample:

$n = 950$ (950 women in the sample)

$n(A) = 611$ (611 women in the sample who are obese)

The sample proportion (the proportion of women who are obese in the sample.) is:

$$\hat{p} = \frac{n(A)}{n} = \frac{611}{950} = 0.643$$

(1) A point estimate for p is: $\hat{p} = 0.643$.

(2) We need to construct 95% C.I. for the proportion (p).

$$95\% = (1-\alpha)100\% \Leftrightarrow 0.95 = 1-\alpha \Leftrightarrow \alpha = 0.05 \Leftrightarrow \frac{\alpha}{2} = 0.025 \Leftrightarrow 1-\frac{\alpha}{2} = 0.975$$

The reliability coefficient: $Z_{1-\frac{\alpha}{2}} = z_{0.975} = 1.96$.

A 95% C.I. for the proportion (p) is:

$$\hat{p} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.643 \pm (1.96)\sqrt{\frac{(0.643)(1-0.643)}{950}}$$

$$0.643 \pm (1.96)(0.01554)$$

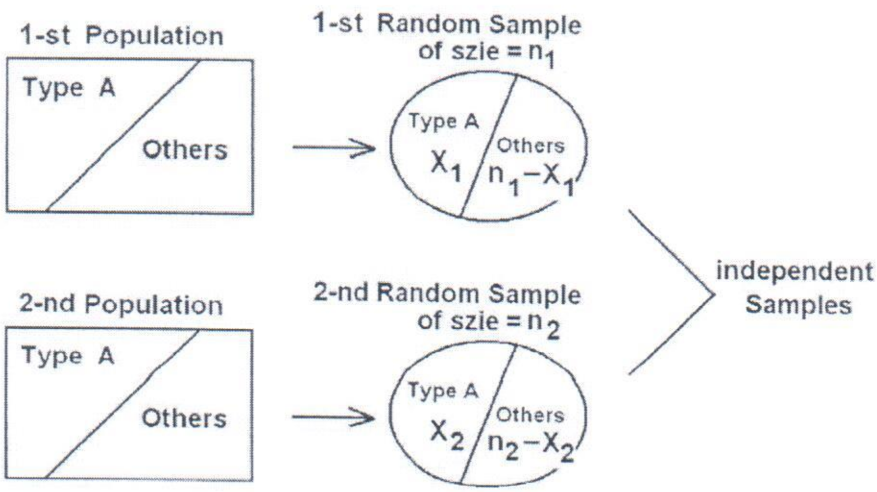
$$0.643 \pm 0.0305$$

$$(0.6127\ ,\ 0.6735)$$

We are 95% confident that the true value of the population proportion of obese women, p, lies in the interval $(0.61, 0.67)$, that is:

$$0.61 < p < 0.67$$

6.6 Confidence Interval for the Difference Between Two Population Proportions ($p_1 - p_2$):



Suppose that we have two populations with:

- p_1 = population proportion of elements of type (A) in the 1-st population.
- p_2 = population proportion of elements of type (A) in the 2-nd population.
- We are interested in comparing p_1 and p_2, or equivalently, making inferences about $p_1 - p_2$.
- We independently select a random sample of size n_1 from the 1-st population and another random sample of size n_2 from the 2-nd population:

- Let X_1 = no. of elements of type (A) in the 1-st sample.
- Let X_2 = no. of elements of type (A) in the 2-nd sample.

$\widehat{P_1} = \frac{X_1}{n_1}$ = the sample proportion of the 1-st sample

$\widehat{P_2} = \frac{X_2}{n_2}$ = the sample proportion of the 2-nd sample

- The sampling distribution of $\widehat{P_1} - \widehat{P_2}$ is used to make inferences about $P_1 - P_2$

Point Estimation for $P_1 - P_2$:

Result:

A good point estimator for the difference between the two proportions, $P_1 - P_2$, is:

$$\widehat{P_1} - \widehat{P_2}$$

$$\hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$$

Interval Estimation (Confidence Interval) for $p_1 - p_2$:

Result:

For large n_1 and n_2, an approximate $(1-\alpha)100\%$ confidence interval for $p_1 - p_2$ is:

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

$$\left((\hat{p}_1 - \hat{p}_2) - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}},\ (\hat{p}_1 - \hat{p}_2) + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} \right)$$

Estimator ± (Reliability Coefficient) × (Standard Error)

Example:

A researcher was interested in comparing the proportion of people having cancer disease in two cities (A) and (B). A random sample of 1500 people was taken from the first city (A), and another independent random sample of 2000 people was taken from the second city (B). It was found that 75 people in the first sample and 80 people in the second sample have cancer disease.

(1) Find a point estimate for the difference between the proportions of people having cancer disease in the two cities.

(2) Find a 90% confidence interval for the difference between the two proportions.

Solution:

p_1 = population proportion of people having cancer disease in the first city (A)

p_2 = population proportion of people having cancer disease in the second city (B)

$\hat{p}_1$ = sample proportion of the first sample

$\hat{p}_2$ = sample proportion of the second sample

X_1 = number of people with cancer in the first sample

X_2 = number of people with cancer in the second sample

For the first sample we have:

$n_1 = 1500$, $X_1 = 75$

$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{75}{1500} = 0.05 \quad , \quad \hat{q}_1 = 1 - 0.05 = 0.95$$

For the second sample we have:

$n_2 = 2000$, $X_2 = 80$

$$\hat{p}_2 = \frac{X_2}{n_2} = \frac{80}{2000} = 0.04 \quad , \quad \hat{q}_2 = 1 - 0.04 = 0.96$$

(1) Point Estimation for $p_1 - p_2$:

A good point estimate for the difference between the two proportions, $p_1 - p_2$, is:

$$\hat{p}_1 - \hat{p}_2 = 0.05 - 0.04$$
$$= 0.01$$

(2) Finding 90% Confidence Interval for $p_1 - p_2$:

$90\% = (1-\alpha)100\% \Leftrightarrow 0.90 = (1-\alpha) \Leftrightarrow \alpha = 0.1 \Leftrightarrow \alpha/2 = 0.05$

The reliability coefficient: $Z_{1-\frac{\alpha}{2}} = z_{0.95} = 1.645$

A 90% confidence interval for $p_1 - p_2$ is:

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$$

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{0.95} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$$

$$0.01 \pm 1.645 \sqrt{\frac{(0.05)(0.95)}{1500} + \frac{(0.04)(0.96)}{2000}}$$

$$0.01 \pm 0.01173$$

$$-0.0017 < p_1 - p_2 < 0.0217$$

We are 90% confident that $p_1 - p_2 \in (-0.0017, 0.0217)$.

Note: Since the confidence interval includes zero, we may conclude that the two population proportions are equal ($p_1 - p_2 = 0 \Leftrightarrow p_1 = p_2$). Therefore, we may conclude that the proportion of people having cancer is the same in both cities.