Chapter One

Systems of Linear Equations

1.1 Introduction to System of Linear Equations.

Linear equations in two dimensional has the form

$a_1x_1 + a_2x_2 = b$, for a_1, a_2, and b are constant real numbers

Also, the above equation is called a linear equation in two variables x_1 and x_2.

Definition 1. (Linear Equation in $n-$variables)

A linear equation in $n-$variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \qquad (*)$$

Where $a_1, a_2, \ldots, a_n$, and b are real numbers.

We call $a_1, a_2, \ldots, a_n$ the coefficients, and b the constant term. Also, a_1 is called the leading coefficient and x_1 is called the leading variable.

Remarks. Linear equations have no products or roots of variable, and no variables involved in trigonometric, logarithmic or exponential functions, that is variables appear only to the first order (power).

Example 1. The following examples are linear equations.

1) $2x + 3y = 17$

2) $4x_1 - 3x_2 = 9$

3) $4x_1 - 3x_2 + 5x_2 = \sqrt{5}$

4) $\sin(2)x_1 - 3x_2 = 14$

5) $e^3x_1 - 3x_2 = 11$

6) $e^3 x_1 - \log_3(2) x_2 = 11$

Example 2. The following examples are non-linear equations.

1) $2xy + 3x = 5$
2) $e^{x_1} - 2x_2 + x_3 = 1$
3) $\sin x - 2y + 3z = 9$
4) $x^2 - 2y^3 + 3z = 4$
5) $\log_2 x - 2y = 3$
6) $\frac{1}{x} - \frac{2}{y} = 3$

Home Work. (page 11).

Ex: 1, 2, 3, and 5.

Definition 2. A solution of a linear equation in $n-$variables is a sequence of n real numbers $s_1, s_2, \ldots, s_n$ such that $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ such that the equation (*) is satisfied by $s_1, s_2, \ldots, s_n$.

For example, $x = 2$, and $y = -1$ satisfied the linear equation $2x - y = 5$. Thus, $x = 2$, and $y = -1$ is a solution for $2x - y = 5$.

The solution set is the set of all solutions of the linear equation.

Example 2. (Parametric solution)

Consider the linear equation $x_1 + 2x_2 = 4$, To get the solution of the above equation, we solve the equation for one of the variables in term of the others. Thus $x_1 = 4 - 2x_2$. Then x_2 is called the free variable (independent), and x_1 is called the dependent variable (not free). Thus, we have infinite number of solutions for the equation $x_1 + 2x_2 = 4$. Hence, we could have $t \in \mathbb{R}$ called the parameter, and by letting $x_2 = t$. Then the solution set is $x_2 = t$, and $x_1 = 4 - 2t$.

If $x_2 = t = 2$, then $x_1 = 4 - 2t = 4 - 2(2) = 0$. The solution set is $\{x_1 = 0, x_2 = 2\}$.

If $x_2 = t = 3$, then $x_1 = 4 - 2t = 4 - 2(3) = 4 - 6 = -2$. The solution set is $\{x_1 = -2, x_2 = 3\}$.

Example 3. (Parametric solution)

Consider the linear equation $3x + 2y - z = 3$, To get the solution of the above equation, we let the variables y, and z be the free variables.. Thus, we have infinite number of solutions for the equation $3x + 2y - z = 3$. Hence, we could have $t, s \in \mathbb{R}$ called the parameter, and by letting $y = t \in \mathbb{R}$, and $z = s \in \mathbb{R}$. Then the solution set is $y = t$, $z = s$, and $x = \frac{1}{3}s - \frac{2}{3}t + 1$.

Home Work. (page 11).

Ex. 8, 9, 10, and 12.

Systems of Linear Equations

A system of m linear equations in n variables is a set of equations, each of which is linear in the same variables:

$$\begin{array}{ccccc} a_{11}x_1 & +a_{12}x_2 + & \cdots & +a_{1n}x_n = & b_1 \\ a_{21}x_1 & +a_{22}x_2 + & \cdots & +a_{2n}x_n = & b_2 \\ a_{31}x_1 & +a_{32}x_2 + & \cdots & +a_{3n}x_n = & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2 + & \cdots & +a_{mn}x_n = & b_m \end{array}$$

Remark. *The double-subscript notation indicates* a_{ij} *is the coefficient of the variable* x_j *in the equation.*

Definition 3. A **solution** of a system of linear equations is a sequence of n real numbers $s_1, s_2, \ldots, s_n$ such that

$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ that is a solution of each linear equation in the system.

For example, the system

$$\begin{aligned} 3x_1 + 2x_2 &= 3 \\ -x_1 + \ x_2 &= 4 \end{aligned}$$

has $x_1 = -1$, and $x_2 = 3$ as a solution because *both* equations are satisfied when $x_1 = -1$, and $x_2 = 3$. On the other hand, $x_1 = 1$, and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

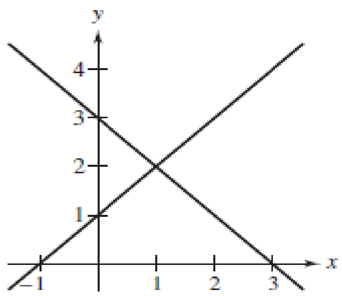
Remarks. *It is possible for a system of linear equations to have exactly one solution, an infinite number of solutions, or no solution. A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.*

Example 4.

(a) Consider the system

$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

Solving the system, we get $x = 1$, and $y = 2$. Hence, the system has only one solution (unique solution). Hence, the lines are intersected.

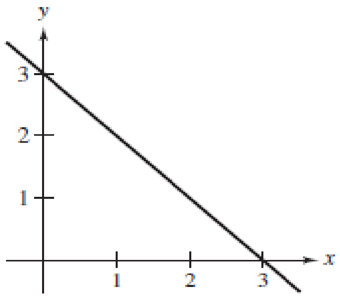


(a) Two intersecting lines: $x + y = 3$, $x - y = -1$

(b) Consider the system

$$\begin{aligned} x + \ y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

Solving the system, we get an infinite number of solutions. Hence, Let $y = t \in \mathbb{R}$, then $x = 3 - 2y$, and the solution set is $x = 3 - 2t$, and $y = t \in \mathbb{R}$ is the parametric solution. Hence, the lines are coincide.



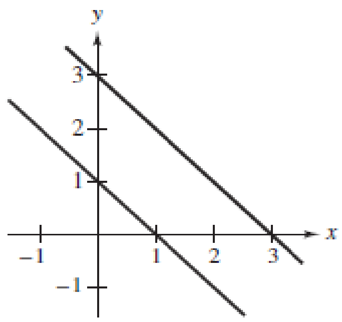
(b) Two coincident lines:
$x + y = 3$
$2x + 2y = 6$

(c) Consider the system

$$x + y = 3$$

$$x + y = 1$$

Then the system has no solution. Hence, the lines are parallel.



(c) Two parallel lines:
$x + y = 3$
$x + y = 1$

Theorem 1. (Number of Solutions of a System of Linear Equations)

For a system of linear equations in n variables, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has an infinite number of solutions (consistent system).
3. The system has no solution (inconsistent system).

Back-Substitution Method and Row-Echelon Form

Example 5. Consider the system

$$x - 2y = 5 \qquad (E_1)$$

$$y = -2 \qquad (E_2)$$

We use back-substation method to solve the system. From (E_2), we have $y = -2$. By substitution $y = -2$ in (E_1), we get

$$x - 2y = 5 \Rightarrow x - 2(-2) = 5$$

$$\Rightarrow x + 4 = 5 \Rightarrow x = 5 - 4 \Rightarrow x = 1$$

The system has exactly one solution: $x = 1$, and $y = -2$.

Example 6. Consider the system

$$x - 2y + 3z = 9 \qquad (E_1)$$

$$y + 3z = 5 \qquad (E_2)$$

$$z = 2 \qquad (E_3)$$

We use back-substation method to solve the system. From (E_3), we have $z = 2$. Put $z = 2$ in (E_2), we have

$$y + 3z = 5 \Rightarrow y + 3(2) = 5 \Rightarrow y + 6 = 5 \Rightarrow y = 5 - 6 = -1$$

Finally, substitute $y = -1$, and $z = 2$ in (E_1), we get

$$x - 2y + 3z = 9 \Rightarrow x - 2(-1) + 3(2) = 9 \Rightarrow x + 2 + 6 = 9 \Rightarrow x = 9 - 8 = 1$$

Hence, the solution is $x = 1, y = -1$ and z=2.

Definition 4. Two systems of linear equations are called **<u>equivalent</u>** if they have precisely the same solution set.

To solve a system that is not in row-echelon form, first change it to an *equivalent* system that is in row-echelon form by using the operations listed below.

Operations That Lead to Equivalent Systems of Equations

Each of the following operations on a system of linear equations produces an ***equivalent*** system.

1. Interchange two equations. $(E_i \leftrightarrow E_j)$
2. Multiply an equation by a nonzero constant. $(E_i \to kE_i ; 0 \neq k \in \mathbb{R})$
3. Add a multiple of an equation to another equation. $(E_i \to E_i + kE_j ; 0 \neq k \in \mathbb{R})$

Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination**.

Example 7. Consider the system

$$\begin{aligned} x - 2y + 3z &= 9 & (E_1) \\ -x + 3y &= -4 & (E_2) \\ 2x - 5y + 5z &= 17 & (E_3) \end{aligned}$$

Adding the first equation to the second equation produces a new second equation. The operation is $E_2' \Rightarrow E_2 + E_1$

$$\begin{aligned} x - 2y + 3z &= 9 & (E_1') \\ y + 3z &= 5 & (E_2') \\ 2x - 5y + 5z &= 17 & (E_3') \end{aligned}$$

Adding the second equation to the third equation produces a new third equation. The operation is $E_2'' \Rightarrow E_2' + (-2)E_1'$

$$\begin{aligned} x - 2y + 3z &= 9 & (E_1'') \\ y + 3z &= 5 & (E_2'') \\ -y - z &= -1 & (E_3'') \end{aligned}$$

Adding the second equation to the third equation produces a new third equation. The operation is $E_3''' \Rightarrow E_3'' + E_2''$

$$\begin{aligned} x - 2y + 3z &= 9 & (E_1''') \\ y + 3z &= 5 & (E_2''') \\ 2z &= 4 & (E_3''') \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation. The operation is $E_3^{(4)} \Rightarrow \frac{1}{2} E_3'''$

$$\begin{aligned} x - 2y + 3z &= 9 & (E_1^{(4)}) \\ y + 3z &= 5 & (E_2^{(4)}) \\ z &= 2 & (E_3^{(4)}) \end{aligned}$$

This is the same system you solved in Example 6, and, as in that example, the solution is Hence, the solution of the system is $x=1, y=-1$ and z=2.

Example 8. Consider the system

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 & (E_1) \\ 2x_1 - x_2 - 2x_3 &= 2 & (E_2) \\ x_1 + 2x_2 - 3x_3 &= -1 & (E_3) \end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation. The operation is $E_2' \Rightarrow E_2 + (-2)E_1$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 & (E_1') \\ 5x_2 - 4x_3 &= 0 & (E_2') \\ x_1 + 2x_2 - 3x_3 &= -1 & (E_3') \end{aligned}$$

Adding -1 times the first equation to the third equation produces a new third equation. The operation is $E_3'' \Rightarrow E_3' + (-1)E_1'$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 & (E_1'') \\ 5x_2 - 4x_3 &= 0 & (E_2'') \\ 5x_2 - 4x_3 &= -2 & (E_3'') \end{aligned}$$

Now, continuing the elimination process, add -1 times the second equation to the third equation to produce a new third equation. The operation is $E_3''' \Rightarrow E_3'' + (-1)E_2''$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 & (E_1''') \\ 5x_2 - 4x_3 &= 0 & (E_2''') \\ 0 &= -2 & (E_3''') \end{aligned}$$

Because the third "equation" is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution. The system is inconsistent.

Example 9. Consider the system

$$\begin{aligned} x_2 - x_3 &= 0 & (E_1) \\ x_1 \qquad - 3x_3 &= -1 & (E_2) \\ -x_1 + 3x_2 \qquad &= 1 & (E_3) \end{aligned}$$

The first two equations are interchanged. Thus $E_1 \leftrightarrow E_2$

$$\begin{aligned} x_1 \quad -3x_3 &= -1 \qquad (E_1) \\ x_2 - x_3 &= 0 \qquad (E_2) \\ -x_1 + 3x_2 \quad &= 1 \qquad (E_3) \end{aligned}$$

Adding the first equation to the third equation produces a new third equation. Hence, $E_3 \to E_3 + E_1$

$$\begin{aligned} x_1 \quad -3x_3 &= -1 \qquad (E_1) \\ x_2 - x_3 &= 0 \qquad (E_2) \\ 3x_2 - 3x_3 &= 0 \qquad (E_3) \end{aligned}$$

Adding –3 times the second equation to the third equation eliminates the third equation. Thus $E_3 \to E_3 + (-3)E_2$

$$\begin{aligned} x_1 \quad -3x_3 &= -1 \qquad (E_1) \\ x_2 - x_3 &= 0 \qquad (E_2) \\ 0 &= 0 \qquad (E_3) \end{aligned}$$

Then, we have the system

$$\begin{aligned} x_1 \quad -3x_3 &= -1 \qquad (E_1) \\ x_2 - x_3 &= 0 \qquad (E_2) \end{aligned}$$

Hence, we have $x_2 - x_3 = 0$, and $x_2 = x_3$. Put $x_2 = x_3 = t \in \mathbb{R}$. Then $x_1 - 3x_3 = -1$, and $x_1 = 3x_3 - 1 = 3t - 1$. The solution set is $x_1 = 3t - 1$, and $x_2 = x_3 = t \in \mathbb{R}$.

In Exercises 1–6, determine whether the equation is linear in the variables x and y.

1. $2x - 3y = 4$

2. $3x - 4xy = 0$

3. $\frac{3}{y} + \frac{2}{x} - 1 = 0$

4. $x^2 + y^2 = 4$

5. $2 \sin x - y = 14$

6. $(\sin 2)x - y = 14$

Answers:

1) Linear equation.

2) Non-linear equation.

3) Non-linear equation.

4) Non-linear equation.

5) Non-linear equation.

6) Linear equation.

In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

7. $2x - 4y = 0$ **8.** $3x - \frac{1}{2}y = 9$

9. $x + y + z = 1$ **10.** $13x_1 - 26x_2 + 39x_3 = 13$

7) Let $2x - 4y = 0$, and let $y = t \in \mathbb{R}$. Then $x = 2y = 2t$.
Hence, the solution is $x = 2y = 2t$, and $y = t \in \mathbb{R}$.

8) Let $3x - \frac{1}{2}y = 9$, and let $y = t \in \mathbb{R}$. $3x = 9 + \frac{1}{2}y ; y = t$

Then $x = 3 + \frac{1}{6}y = 3 + \frac{1}{6}t$. Then the solution is

$$x = 3 + \frac{1}{6}t, y = t$$

9) Let $x + y + z = 0$, and let $y = t, z = s \in \mathbb{R}$. Then
$x = -y - z = -t - s = -(t + s)$
Hence, the solution set is $x = -(t + s), y = t, z = s$.

10) We have $13x_1 - 26x_2 + 39x_3 = 13$. Then

$$x_1 - 2x_2 + 3x_3 = 1$$

Let $x_1 = t, x_2 = s \in \mathbb{R}$. Then, the solution set is

$$x_1 = 1 + 2t - 3s, x_2 = t, x_3 = s$$

In Exercises 11–16, use back-substitution to solve the system.

11. $x_1 - x_2 = 2$; $x_2 = 3$

12. $2x_1 - 4x_2 = 6$; $3x_2 = 9$

13. $-x + y - z = 0$; $2y + z = 3$; $\frac{1}{2}z = 0$

14. $x - y = 4$; $2y + z = 6$; $3z = 6$

15. $5x_1 + 2x_2 + x_3 = 0$; $2x_1 + x_2 = 0$

16. $x_1 + x_2 + x_3 = 0$; $x_2 = 0$

11) Let

$$x_1 - x_2 = 2 \qquad (E_1)$$

$$x_2 = 3 \qquad (E_2)$$

Then the solution set is $x_1 = x_2 + 3 + 2 = 5$, and $x_2 = 3$.

15) Let

$$5x_1 + 2x_2 + x_3 = 0 \qquad (E_1)$$
$$2x_1 + x_2 \qquad = 0 \qquad (E_2)$$

Then from $\left(E_2\right)$, we have $x_2 = -2x_1$. Let $x_1 = t \in \mathbb{R}$. Then $x_2 = -2t$, and

$5x_1 + 2x_2 + x_3 = 0$

$\Rightarrow x_3 = -5x_1 - 2x_2 = -5t - 2(-2t) = -5t + 4t = -t$

The solution set is $x_1 = t, x_2 = -2t$, and $x_3 = -t$.

16) Let

$$x_1 + x_2 + x_3 = 0 \qquad (E_1)$$
$$x_2 \qquad = 0 \qquad (E_2)$$

Then $x_1 + x_3 = 0$, and $x_1 = -x_3 = -t; x_3 = t \in \mathbb{R}$. The solution set is $x_1 = -t; x_2 = 0$, and $x_3 = t$.

1.2 Gaussian Elimination and Gaussian Jordan Elimination

In Section 1.1, Gaussian elimination was introduced as procedure for solving a system of linear equations. In this section we will study this procedure more thoroughly, beginning with some definitions.

Definition 1. Let m and n be positive integers. Then an $m \times n$ **matrix** is a rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in which each **entry** a_{ij} of the matrix is a number, and $1 \le i \le m; 1 \le j \le n$ where m is the number of rows and n is the number of columns.

Remark. 1) If each entry of a matrix is a *real* number, then the matrix is called a **real matrix.**
2) The order of the matrix with m rows and n columns is $m \times n$.
3) If $m = n$, then the matrix is called a **square matrix** of order n.
4) For a square matrix of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **main diagonal** entries.

Example 1. (Examples of matrices)

$$1)\ [3]_{1\times 1} \quad 2)\ \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}_{1\times 3} \quad 3)\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2} \quad 4)\ \begin{bmatrix} 1 & -2 & \sqrt{2} \\ e & -1 & 1 \\ -1 & 2 & 0 \end{bmatrix}_{3}$$

Definition 2. The matrix derived from the coefficients and constant terms of a system of linear equations is called the **augmented matrix** of the system. The matrix containing only the coefficients of the system is called the **coefficient matrix** of the system.

Let consider the system

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

The coefficient matrix is

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}$$

Remark. When forming either the coefficient matrix or the augmented matrix of a system, you should begin by aligning the variables in the equations vertically.

Elementary Row Operations

1. Interchange two rows. ($R_i \leftrightarrow R_j$)
2. Multiply a row by a nonzero constant. ($R_i \to kR_i\ ; 0 \neq k \in \mathbb{R}$)
3. Add a multiple of a row to another row. ($R_i \to R_i + kR_j$ $; 0 \neq k \in \mathbb{R}$)

An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are said to be **row-equivalent** if one can be obtained from the other by a finite sequence of elementary row operations.

Example 2. (Elementary row operations)

1) $$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 4 \\ 1 & 3 & 0 & -1 \\ 2 & 4 & -7 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 4 \\ 2 & 4 & -7 & 0 \end{array}\right]$$

2) $$\left[\begin{array}{ccc|c} 2 & -4 & 8 & -6 \\ 3 & -3 & 0 & -1 \\ -1 & 4 & -7 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & -3 \\ 3 & -3 & 0 & -1 \\ -1 & 4 & -7 & 0 \end{array}\right]$$

3) $$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-2)R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{array}\right]$$

Example 3. Let consider the system

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$R_2 \to R_2 + R_1$$
$$R_3 \to R_3 + (-2)R_1$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$
$$R_3 \to R_3 + R_2$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$
$$R_3 \to \frac{1}{2} R_3$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Then we have the corresponding of linear equations system is

$$x - 2y + 3z = 9 \qquad (E_1)$$
$$y + 3z = 5 \qquad (E_2)$$
$$z = 2 \qquad (E_3)$$

We use back-substation method to solve the system. From (E_3), we have $z = 2$. Put $z = 2$ in (E_2), we have

$$y + 3z = 5 \Rightarrow y + 3(2) = 5 \Rightarrow y + 6 = 5 \Rightarrow y = 5 - 6 = -1$$

Finally, substitute $y = -1$, and $z = 2$ in (E_1), we get

$$x - 2y + 3z = 9 \Rightarrow x - 2(-1) + 3(2) = 9 \Rightarrow x + 2 + 6 = 9 \Rightarrow x = 9 - 8 = 1$$

Hence, the solution is $x = 1, y = -1$ and z=2.

Definition 3. (Row-Echelon Form of a Matrix)

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Definition 4. (reduced row-echelon form)

A matrix in row-echelon form is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

Example 4. (Row-Echelon Form)

The matrices below are in **row-echelon** form.

$$\text{(a)} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrices shown in parts (b) and (d) are in ***reduced* row echelon form**. The matrices listed below are not in row-echelon form.

$$\text{(e)} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \qquad \text{(f)} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.

3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Example 5. (Gaussian Elimination with Back-Substitution)
Consider the system

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

$$R_3 \to R_3 + (-2)R_1$$

$$R_4 \to R_4 + (-1)R_2$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right]$$

$$R_4 \to R_4 + 6R_2$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$

$$R_3 \to \frac{1}{3}R_3$$

$$R_4 \to -\frac{1}{13}R_4$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Then we have the corresponding of linear equations system is

$$\begin{aligned} x_1 + 2x_2 - x_3 \quad &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Using back-substitution, you can determine that the solution is

$x_1 = -1, x_2 = 2, x_3 = 1,$ and $x_4 = 3$

Example 6. (Gaussian Elimination with Back-Substitution- A system with no solution)

Consider the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \quad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{array}\right]$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 + (-2)R_1$$
$$R_4 \to R_4 + (-3)R_1$$
$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{array}\right]$$

$$R_3 \to R_3 + R_2$$
$$R_4 \to R_4 + (-5)R_2$$
$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Note that the third row of this matrix consists of all zeros except for the last entry. This means that the original system of linear equations is ***inconsistent***. You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ -2x_3 &= -1 \end{aligned}$$

Because the third "equation" is a false statement, the system has no solution.

Gauss-Jordan Elimination

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in the next example.

Example 7. (Gaussian-Jordan elimination)

Consider the system

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad\quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$R_2 \to R_2 + R_1$$

$$R_3 \to R_3 + (-2)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$\Downarrow$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$R_3 \to \frac{1}{2}R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2 \to R_2 + (-3)R_3$$

$$R_1 \to R_1 + (-3)R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 \to R_1 + (2)R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Now, converting back to a system of linear equations, you have the solution is $x = 1, y = -1$ and z=2.

Example 8. (Gaussian-Jordan elimination. A system with an infinite number of solutions)

Consider the system

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

The augmented matrix of the system of linear equations is

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$R_1 \to \frac{1}{2}R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 + (-3)R_1$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right]$$
$$R_2 \to -R_2$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{array}\right]$$
$$R_2 \to R_2 + (-3)R_1$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + \quad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

Now, using the parameter to represent the *non-leading* variable , we have $x_1 = 2 - 5t, x_2 = -1 + 3t$, and $x_3 = t \in \mathbb{R}$.

Homogeneous Systems of Linear Equations

We will look at systems of linear equations in which each of the constant terms is zero. We call such systems **homogeneous.** For example, a homogeneous system of m equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

It is easy to see that a homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called **trivial** (or **obvious**).

Example 9. (Gaussian-Jordan elimination. A system with an infinite number of solutions)
Consider the system

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 0 \end{aligned}$$

The augmented matrix of the system of linear equations is

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 0 \end{array}\right]$$

$$R_1 \to \frac{1}{2} R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 0 \end{array}\right]$$

$$R_2 \to R_2 + (-3)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$$R_2 \to -R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 \to R_2 + (-3)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + \qquad + 5x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Now, using the parameter to represent the *non-leading* variable , we have $x_1 = -5t, x_2 = 3t$, and $x_3 = t \in \mathbb{R}$.

Theorem 1.(The Number of Solutions of a Homogeneous System)

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have an infinite number of solutions.