CHAPTER (7)

Limits and Continuity النهايات و الاتصال

7.1 LIMITS OF FUNCTIONS: Page (200)

Objectives:

*One-Sided Limits

*Rules for Calculating Limits

* The Squeeze Theorem.

INTRODUCTION TO LIMITS

Example: 1

Describe the behavior of the function

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ near } x=1.$$

Solution:

* As an illustration , consider

$$f(x) = \frac{x^2 - 1}{x - 1}$$

* Note that 1 is not in the domain of f , since substituting $x = 1$ gives us the undefined expression $\frac{1^2 - 1}{1 - 1} = \frac{0}{0}$.

x	$f(x)$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	$f(x)$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

0.9999	1.9999		1.00001	2.00001
0.999999	1.999999		1.000001	2.000001

* *It appears that the closer x to 1, the closer $f(x)$ to 2.*

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = 2 .$$

* *In general*

$$f(x) = \frac{x^2-1}{x-1}$$

* *The number 1 is not in the domain of f since the meaningless expression $\frac{(1)^2-1}{1-1} = \frac{0}{0}$ is obtained if 1 is substituted for x.*

* *Factoring the numerator and denominator*

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)}$$

* *Since $x \neq 1$, we may cancel the common factor $(x-1)$*

$$= \lim_{x\to 1} (x+1) = 1+1 = 2 .$$

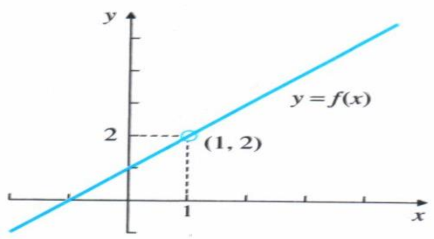

Figure 1 The graph of $f(x) = \frac{x^2-1}{x-1}$

Homework 1:

What happens to the function $g(x)=(1+x^2)^{\frac{1}{x^2}}$ *as x approaches zero.*

Solution:

x	$f(x)$
0.1	2.7048138294
0.01	2.7181459268
0.001	2.7182804691
0.0001	2.718287983
0.00001	2.7182820532
0.000001	2.718523496

x	$f(x)$
-0.1	2.7048138294
-0.01	2.7181459268
-0.001	2.7182804691
-0.0001	2.718287983
-0.00001	2.7182820532
-0.000001	2.718523496

* *It appears that the closer* x *to* 0, *the closer* $f(x)$ *to* 2.7182818285 .

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+x^2)^{\frac{1}{x^2}} = e = 2.7182818285$$

* *In general*

$$f(x) = (1+x^2)^{\frac{1}{x^2}}$$

* *The number* 0 *is not in the domain of* f *since the meaningless expression* $\frac{1}{0}$ *is obtained if* 0 *is substituted for* x .

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} (1+x^2)^{\frac{1}{x^2}} = e = 2.7182818285$$

Definition 1 An informal definition of Limits of a function :
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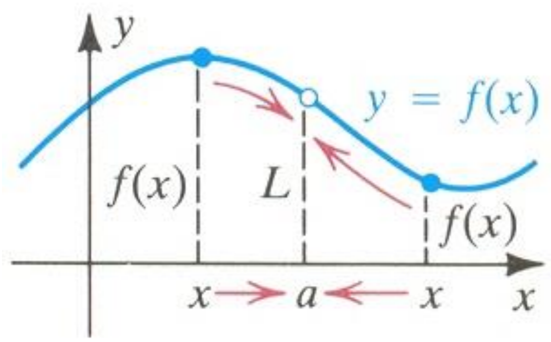
NOTATION	INTUITIVE MEANING	GRAPHICAL INTERPRETATION
$\lim_{x \to a} f(x) = L$	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a, and $x \neq a$.	

Example (2) : ***Page (201)***

Find (i) $\lim_{x \to a} x$ (ii) $\lim_{x \to a} c$ (where c is a constant)

Solution :

(i) $\lim_{x \to a} x = a$. (ii) $\lim_{x \to a} c = c$.

HOMEWORK 2

Evaluate:

$$(a)\ \lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6} \qquad (b)\ \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \qquad (c)\ \lim_{x \to 4} \frac{\sqrt{x} - 2}{x^2 - 16}$$

Solution:

a) $f(x) = \dfrac{x^2 + x - 2}{x^2 + 5x + 6}$

* The number -2 is not in the domain of f since the meaningless expression $\frac{(-2)^2+(-2)-2}{(-2)^2+5(-2)+6}=\frac{0}{0}$ is obtained if -2 is substituted for x.

* Factoring the numerator and denominator

$$\lim_{x\to -2}\frac{x^2+x-2}{x^2+5x+6}=\lim_{x\to -2}\frac{(x+2)(x-1)}{(x+2)(x+3)}$$

* Since $x\neq -2$, we may cancel the common factor $(x+2)$

$$\lim_{x\to -2} f(x)=\lim_{x\to -2}\frac{x-1}{x+3}=\frac{(-2)-1}{(-2)+3}=\frac{-3}{1}=-3\ .$$

b)

$$f(x)=\frac{\frac{1}{x}-\frac{1}{a}}{x-a}$$

* The number a is not in the domain of f since the meaningless expression $\frac{\frac{1}{a}-\frac{1}{a}}{a-a}=\frac{0}{0}$ is obtained if a is substituted for x.

* Factoring the numerator and denominator

$$\lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a}=\lim_{x\to a}\frac{\frac{(a-x)}{xa}}{(x-a)}=\lim_{x\to a}\frac{(a-x)}{xa}\,\frac{1}{(x-a)}$$

* Since $x\neq a$, we may cancel the common factor $(x-a)$

$$=\lim_{x\to a}\frac{-(x-a)}{xa}\,\frac{1}{(x-a)}=\lim_{x\to a}\frac{-1}{xa}=\frac{-1}{a^2}$$

$$c)\ f(x)=\frac{\sqrt{x}-2}{x^2-16}$$

* *The number 4 is not in the domain of f since the meaningless expression* $\frac{\sqrt{4}-2}{4^2-16}=\frac{2-2}{16-16}=\frac{\ }{\ }$ *is obtained if 4 is substituted for x.*

* *Factoring the numerator and denominator*

$$\lim_{x\to 4} f(x)=\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}=\lim_{x\to 4}\frac{(\sqrt{x}-2)}{(x-4)(x+4)}$$

$$=\lim_{x\to 4}\frac{(\sqrt{x}-2)}{(\sqrt{x}-2)(\sqrt{x}+2)(x+4)}$$

* *Since $x\neq 4$, we may cancel the common factor $(\sqrt{x}-2)$*

$$=\lim_{x\to 4}\frac{1}{(\sqrt{x}+2)(x+4)}=\frac{1}{(\sqrt{4}+2)(4+4)}=\frac{1}{32}$$

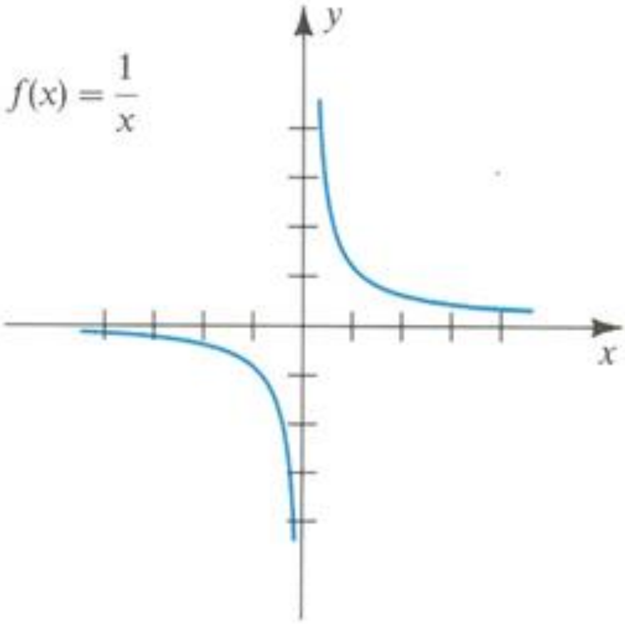
Example (3) : ***Page (201)***

a) Show that $\lim_{x\to 0}\frac{1}{x}$ *does not exist.*

Solution

* *The graph of* $f(x)=\frac{1}{x}$ *is sketched in Figure 2.*

Figure 2



* Note that we can make $|f(x)|$ as large as desired by choosing x sufficiently close to 0 (but $x \neq 0$) .

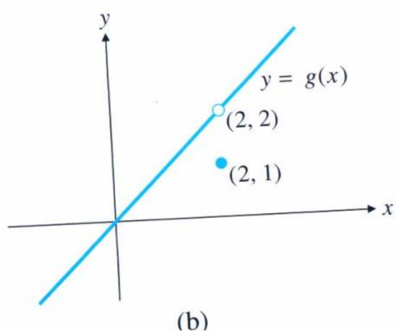
* For example , if we choose $x = -0.000001$, we obtain $f(x) = -1{,}000{,}000$ and if we choose $x = 0.000001$, we obtain $f(x) = 1{,}000{,}000$.

* Since $f(x)$ does not approach a specific number as x approaches 0 , the lim it does not exist

$$\lim_{x \to 0} \frac{1}{x} = \text{does not exists} .$$

Example 3 b)

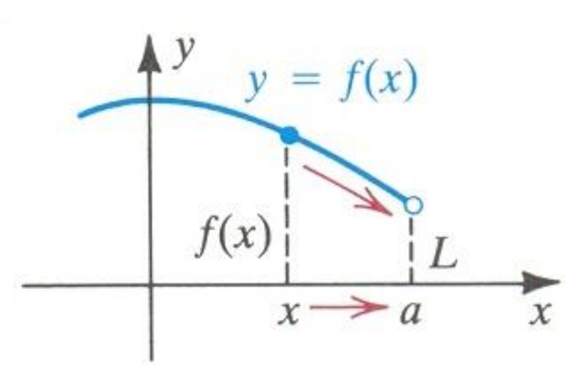
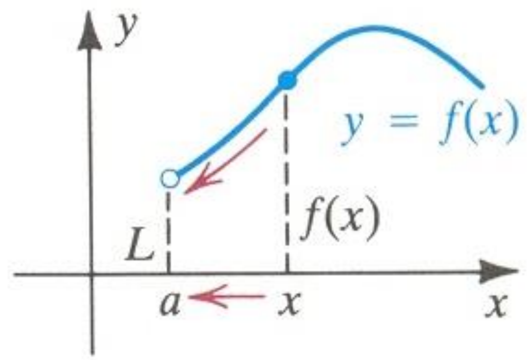
Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



(b)

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x = 2, \quad \text{Although } g(2)=1$$

One- sided Limits of a function : Page (208)

NOTATION	INTUITIVE MEANING	GRAPHICAL INTERPRETATION
$\lim_{x \to a^-} f(x) = L$ (left –hand limit)	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a, and $x < a$.	
$\lim_{x \to a^+} f(x) = L$ (right –hand limit)	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a, and $x > a$.	

Homework 3 : Page (202)

The signum function $sgn(x) = \dfrac{x}{|x|}$, sketch the graph of f and find, if possible,

(a) $\lim_{x \to 0^-} sgn(x)$. (b) $\lim_{x \to 0^+} sgn(x)$. (c) $\lim_{x \to 0} sgn(x)$.

Solution

$$f(x) = sgn(x) = \frac{x}{|x|}$$

* Since $|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$

* f is undefined , $\frac{0}{0}$, at $x = 0$.

* If $x > 0$, the $|x| = x$ and $f(x) = \frac{x}{|x|} = \frac{x}{x} = 1$.

* If $x < 0$, the $|x| = -x$ and $f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1$.

Figure 2.7

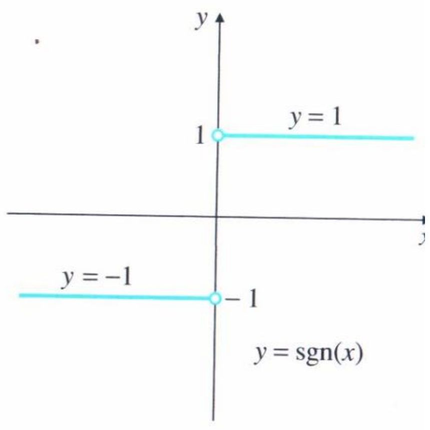


Figure 4 $\lim_{x\to 0} sgn(x)$ does not exist, because $\lim_{x\to 0-} sgn(x) = -1$, $\lim_{x\to 0+} sgn(x) = 1$

(a) $\lim_{x\to 0^-} sgn(x) = \boxed{-1}$.

(b) $\lim_{x\to 0^+} sgn(x) = \boxed{1}$.

(c) Since $\lim_{x\to 0^-} sgn(x) \neq \lim_{x\to 0^+} sgn(x)$, then

$$\boxed{\lim_{x\to 0} sgn(x) \text{ does not exist}} \ .$$

Theorem (1) : Relationship between one-sided and two-sided limits ***Page (202)***

A function f(x) has limit L at x=a if and only if it has both left and right limits there and these one-sided limits are both equal to L

$$\lim_{x\to a} f(x)=L \Leftrightarrow \lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$$

Example (4) : ***Page (202)***

$$If: f(x)=\frac{|x-2|}{x^2+x-6}$$

Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, *and* $\lim_{x\to 2} f(x)$.

Solution

$$f(x)=\begin{cases}\dfrac{x-2}{x^2+x-6}, & for\ \ x>2\\[2ex] \dfrac{-(x-2)}{x^2+x-6}, & for\ \ x<2\end{cases}$$

* *If* $x>2$ *, then*

$$\lim_{x\to 2^+}\frac{x-2}{x^2+x-6}=\lim_{x\to 2^+}\frac{(x-2)}{(x-2)(x+3)}=\lim_{x\to 2^+}\frac{1}{(x+3)}$$

$$=\frac{1}{5}.$$

* *If* $x<2$ *, then*

$$\lim_{x\to 2^-}\frac{-(x-2)}{x^2+x-6}=\lim_{x\to 2^-}\frac{-(x-2)}{(x-2)(x+3)}=\lim_{x\to 2^-}\frac{-1}{(x+3)}=\boxed{-\frac{1}{5}}.$$

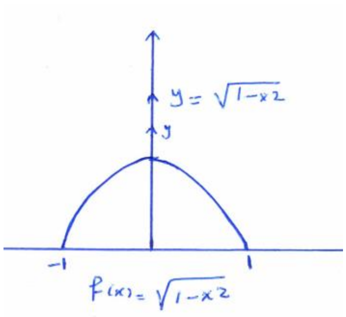
* Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, then

$$\boxed{\lim_{x\to 2} f(x) = \text{does not exist.}}\ .$$

Homework 4:

What one-sided limits does $\boxed{g(x)=\sqrt{1-x^2}}$ **have at x=-1 and x=1?**

Solution:



* *If* $x>-1$ *, then*

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}\sqrt{1-x^2}=\lim_{x\to -1^+}\sqrt{1-(-1)^2}\boxed{=0}.$$

* *If* $x<1$ *, then*

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\sqrt{1-x^2}=\sqrt{1-(1)^2}=\boxed{0}.$$

Rules for Calculating LIMITS : Page (209)

Theorem (2) :Limits Rules

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist , then

(i) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$.

Example: find

$\lim_{x \to -2} (3x^2 + 5x - 9)$

$= \lim_{x \to -2} (3x^2) + \lim_{x \to -2} (5x) + \lim_{x \to -2} (-9)$

$= 3(-2)^2 + 5(-2) - 9 = 12 - 10 - 9 = -7$

(ii) $\lim_{x \to a} [f(x) . g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x) = L.M$.

$\lim_{x \to 2} 20x \cdot (x+5) = \lim_{x \to 2} (20x) \cdot \lim_{x \to 2} (x+5) = (20 \cdot 2) \cdot (2+5)$

$= 40 \cdot 7 = 280$

(iii) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$, provided $M \neq 0$.

$$\lim_{x \to 4} \frac{(x^2 - 1)}{\sqrt{4x}} = \frac{\lim_{x \to 4} (x^2 - 1)}{\lim_{x \to 4} \sqrt{4x}} = \frac{16 - 1}{\sqrt{4 \cdot 4}} = \frac{15}{4}$$

(iv) $\lim_{x \to a} [k f(x)] = k \left[\lim_{x \to a} f(x) \right] = kL$.

Example : $\lim_{x \to 8} 9x = 9 \lim_{x \to 8} x = 9 \cdot 8 = 72$

(v) $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)=L-M$.

Example: find

$$\lim_{x\to 4}(3x-5)=\lim_{x\to 4}(3x)-\lim_{x\to 4}(5)=3\cdot 4-5=12-5=7$$

(vi) $\lim_{x\to a}[f(x)]^{\frac{m}{n}}=\left[\lim_{x\to a} f(x)\right]^{\frac{m}{n}}=L^{\frac{m}{n}}$, provided L>0 if n is even, and $L\neq 0$ if $m<0$.

Example: find $\lim_{x\to 4}(3x-5)^2=[\lim_{x\to 4}(3x-5)]^2=7^2=49$

(vii) If $f(x)\leq g(x)$ on an interval containing a in its interior, then

$$L\leq M$$

Example 5

Find a) $\lim_{x\to a}\dfrac{x^2+x+4}{x^3-2x^2+7}$ and b) $\lim_{x\to 2}\sqrt{2x+1}$

Solution:

a) $\lim_{x\to a}\dfrac{x^2+x+4}{x^3-2x^2+7}=\dfrac{a^2+a+4}{a^3-2a^2+7}$, provide $a^3-2a^2+7\neq 0$

b) $\lim_{x\to 2}\sqrt{2x+1}=\sqrt{2.2+1}=\sqrt{5}$

Theorem (3) Limits of Polynomials and Rational Functions

1-If P(x) is a polynomial and a is any real number, then

$$\lim_{x\to a}P(x)=P(a)$$

2-If P(x) and Q(x) are any polynomials and $Q(a) \neq 0$, then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

3-If m , b , and a are real numbers , then

$$\lim_{x \to a} (m\,x + b) = m\,a + b$$

Quiz (1)

Find a) $\lim_{x \to 4} (x^2 - 4x + 1)$ **b)** $\lim_{x \to 3} \frac{x+3}{x+6}$ **, c)** $\lim_{x \to 2} \frac{3x+4}{5x+7}$

Solution

a) $\lim_{x \to 4} (x^2 - 4x + 1) = (4)^2 - 4(4) + 1 = 1$

b) $\lim_{x \to 3} \frac{x+3}{x+6} = \frac{3+3}{3+6} = \frac{6}{9} = \frac{2}{3}$

c) $\lim_{x \to 2} \frac{3x+4}{5x+7} = \frac{\lim_{x \to 2}(3x+4)}{\lim_{x \to 2}(5x+7)} = \frac{3(2)+4}{5(2)+7} = \boxed{\frac{10}{17}}$.

Quiz (2) :

Prove that $\lim_{x \to a} x^3 = a^3$.

Solution

* Since $\lim_{x \to a} x = a$,

$$\lim_{x \to a} x^3 = \lim_{x \to a} (x \cdot x \cdot x)$$

$$= \left(\lim_{x \to a} x\right) \cdot \left(\lim_{x \to a} x\right) \cdot \left(\lim_{x \to a} x\right).$$

$$= a \cdot a \cdot a = \boxed{a^3}.$$

Theorem 4. The Squeeze(sandwich) theorem Page (211)

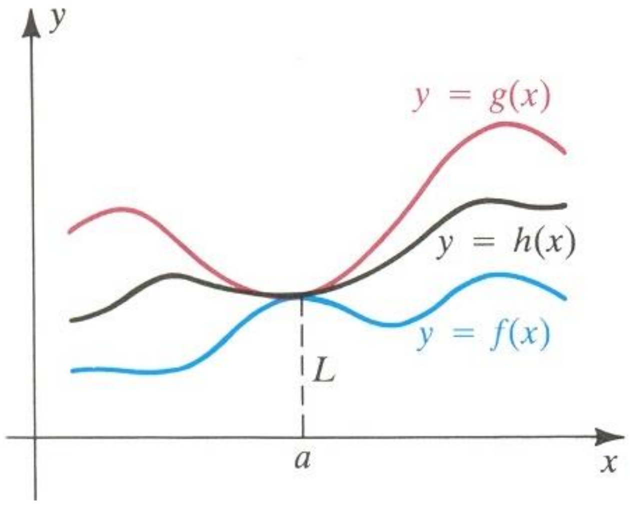
Suppose $f(x) \le h(x) \le g(x)$ holds for all x in some open interval containing a, except possibly at a, suppose also that

$$\text{If} \quad \lim_{x \to a} f(x) = L = \lim_{x \to a} g(x),$$

$$\text{Then} \quad \lim_{x \to a} h(x) = L.$$

Similar statements hold for left and right limits.

Figure 5



Homework (5) : Page (211)

Given that $3 - x^2 \le u(x) \le 3 + x^2$ for all $x \ne 0$, find $\lim_{x \to 0} u(x)$

Solution :

*** Since** $\lim_{x \to 0} \left(3 - x^2\right) = 3$, $\lim_{x \to 0} \left(3 + x^2\right) = 3$ **, then**

$$\lim_{x \to 0} u(x) = 3$$

End

More examples

QUIZ1:

If $f(x) = \dfrac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$, find $\lim_{x \to 2} f(x)$.

Solution

$$f(x) = \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$$

* The number 2 is not in the domain of f since the meaningless expression $\dfrac{0}{0}$ is obtained if 2 is substituted for x .

* Factoring the numerator and denominator

$$\lim_{x \to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6} = \lim_{x \to 2} \frac{(x-2)(2x-1)}{(x-2)(5x+3)}$$

* Since $x \neq 2$, we may cancel the common factor $(x-2)$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x-1}{5x+3} = \frac{2(2)-1}{5(2)+3} = \boxed{\frac{3}{13}} .$$

QUIZ (2) :

If $f(x) = \dfrac{x-9}{\sqrt{x} - 3}$.

(a) find $\lim_{x \to 9} f(x)$.

(b) Sketch the graph of f and illustrate the limit in part (a) graphically .

(a) $$f(x) = \frac{x-9}{\sqrt{x}-3}$$

* *The number 9 is not in the domain of f since the meaningless expression $\frac{0}{0}$ is obtained if 9 is substituted for x .*

* *Rationalizing the denominator by multiplying the numerator and denominator by $\sqrt{x}+3$*

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9}\left(\frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \right)$$

$$= \lim_{x\to 9} \frac{(x-9)\left(\sqrt{x}+3\right)}{x-9}$$

* *Since $x \neq 9$, we may cancel the common factor $(x-9)$*

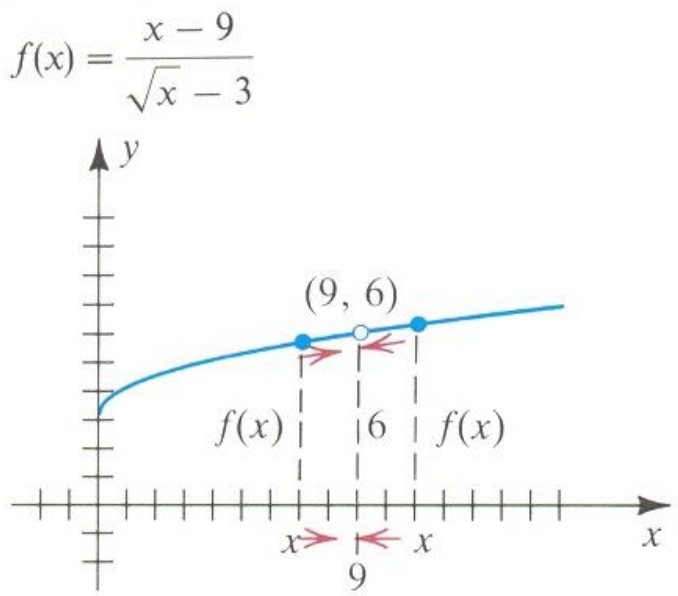
$$\lim_{x\to 9} f(x) = \lim_{x\to 9} \left(\sqrt{x}+3\right) = \sqrt{9}+3 = \boxed{6} .$$

(b) The graph of f is the same as the graph of the equation $y = \sqrt{x}+3$, except for the point $(9,6)$, as illustrated in Figure 2.3 .

* *As x gets closer to 9 , the point $(x, f(x))$ on the graph of f gets closer to the point $(9,6)$.*

* *Note that $f(x)$ never actually attains the value 6 ; however , $f(x)$ can be made as close to 6 as desired by choosing x sufficiently close to 9 .*

Figure 2.3



Use the sandwich theorem to prove that $\lim_{x \to 0} x^2 \sin\frac{1}{x^2} = 0$.

Solution

* Since $-1 \le \sin t \le 1$ for every real number t,

$$-1 \le \sin\frac{1}{x^2} \le 1 \ , \quad \text{for every } x \ne 0$$

* Multiplying by x^2 (which is positive if $x \ne 0$) , we obtain

$$-x^2 \le x^2 \sin\frac{1}{x^2} \le x^2$$

* This inequality implies that the graph of $y = x^2 \sin\frac{1}{x^2}$ lies between the parabolas $y = -x^2$ and $y = x^2$.

* Since $\lim_{x \to 0} \left(-x^2\right) = 0$, $\lim_{x \to 0} \left(x^2\right) = 0$, then

$$\boxed{\lim_{x \to 0} x^2 \sin\frac{1}{x^2} = 0} \ .$$

Theorem (2.10) : Page (62)

If n is a positive integer, then

(i) $\lim_{x \to a} x^n = a^n$.

(ii) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$, provided $\lim_{x \to a} f(x)$ exists .

Example (3) : Page (62)

Find $\lim_{x \to 2} (3x+4)^5$.

Solution

* $\lim_{x \to 2} (3x+4)^5 = \left[\lim_{x \to 2} (3x+4)\right]^5$

$= [3(2)+4]^5 = 10^5 = \boxed{100{,}000}$.

Example (4) : Page (62)

Find $\lim_{x \to 2} \left(5x^3+3x^2-6\right)$.

Solution

$\lim_{x \to -2} \left(5x^3+3x^2-6\right) = \lim_{x \to -2} \left(5x^3\right) + \lim_{x \to -2} \left(3x^2\right) - \lim_{x \to -2} (6)$

$= 5 \lim_{x \to -2} \left(x^3\right) + 3 \lim_{x \to -2} \left(x^2\right) - 6$

$= 5(-2)^3 + 3(-2)^2 - 6$

$$= 5(-8) + 3(4) - 6 = \boxed{-34}.$$

Theorem (2.11) : Page (62)

If f is a polynomial function and a is a real number, then

$$\lim_{x \to a} f(x) = f(a).$$

Corollary (2.12) : Page (63)

If q is a rational function and a is in the domain of q, then

$$\lim_{x \to a} q(x) = q(a).$$

Example (5) : Page (63)

Find $\lim_{x \to 3} \dfrac{5x^2 - 2x + 1}{4x^3 - 7}$.

Solution

$$* \quad \lim_{x \to 3} \frac{5x^2 - 2x + 1}{4x^3 - 7} = \frac{5(3)^2 - 2(3) + 1}{4(3)^3 - 7}$$

$$= \frac{45 - 6 + 1}{108 - 7} = \boxed{\frac{40}{101}}.$$

Theorem (2.13) : Page (63)

1-If $a > 0$ and n is a positive integer , or if $a \le 0$ and n is an odd positive integer , then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \; .$$

2-If m and n are positive integers and $a > 0$, then

$$\lim_{x \to a} \left(\sqrt[n]{x}\right)^m = \left(\lim_{x \to a} \sqrt[n]{x}\right)^m = \left(\sqrt[n]{a}\right)^m$$

$$\text{or } \lim_{x \to a} x^{m/n} = a^{m/n}$$

Example (6) : ***Page (64)***

Find $\lim_{x \to 8} \dfrac{x^{2/3} + 3\sqrt{x}}{4 - (16 / x)}$.

Solution

$$* \lim_{x \to 8} \frac{x^{2/3} + 3\sqrt{x}}{4 - (16 / x)} = \frac{\lim_{x \to 8} \left(x^{2/3} + 3\sqrt{x}\right)}{\lim_{x \to 8} \left[4 - (16 / x)\right]}$$

$$= \frac{\lim_{x \to 8} x^{2/3} + \lim_{x \to 8} 3\sqrt{x}}{\lim_{x \to 8} 4 - \lim_{x \to 8} (16 / x)}$$

$$= \frac{8^{2/3} + 3\sqrt{8}}{4 - (16 / 8)}$$

$$= \frac{4 + 6\sqrt{2}}{4 - 2} = \frac{4 + 6\sqrt{2}}{2} = \boxed{2 + 3\sqrt{2}} \; .$$

Theorem (2.14) : **Page (64)**

If a function f has a limit as x approaches a, then

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \, ,$$

provided either n is an odd positive integer or n is an even positive integer and $\lim_{x \to a} f(x) > 0$.

Example (7) : **Page (64)**

Find $\lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9}$.

Solution

$$* \quad \lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9} = \sqrt[3]{\lim_{x \to 5} \left(3x^2 - 4x + 9\right)}$$

$$= \sqrt[3]{3(5)^2 - 4(5) + 9}$$

$$= \sqrt[3]{64} = \boxed{4} \, .$$

7.2 LIMITS at INFINITY and INFINITE LIMITS:

Page (212)

Objectives:

**Limits at infinity*

**Limits at infinity for Rational Functions*

** Infinite Limits.*

Limit at Infinity:

Consider the function:

$$f(x)=\frac{x}{\sqrt{x^2+1}}$$

Find $\lim_{x\to\infty} f(x),\ \lim_{x\to-\infty} f(x)$.

Solution:

able 1

x	$f(x)=x/\sqrt{x^2+1}$
−1,000	−0.9999995
−100	−0.9999500
−10	−0.9950372
−1	−0.7071068
0	0.0000000
1	0.7071068
10	0.9950372
100	0.9999500
1,000	0.9999995

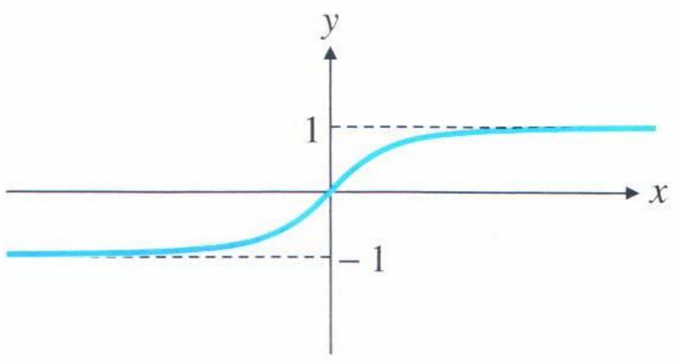


Figure 7 The graph of $x/\sqrt{x^2+1}$

$$\lim_{x\to\infty} f(x)=1,\qquad \lim_{x\to-\infty} f(x)=-1$$

Definition (3) : ***Limits at infinity and negative infinity (informal definition).*** ***Page (213)***

If the function f is defined on an interval (a, ∞) and if we can ensure that f(x) is as close as we want to the number L by taking x large enough, then we say that f(x) approaches the limit L as x approaches infinity, and we write

$$\lim_{x \to \infty} f(x) = L$$

If the function f is defined on an interval (- ∞,b) and if we can ensure that f(x) is as close as we want to the number M by taking x negative and large enough in absolute value, then we say that f(x) approaches the limit L as x approaches negative infinity, and we write

$$\lim_{x \to -\infty} f(x) = M$$

Example 1: Find $\lim_{x \to \pm\infty} \frac{1}{x}$

Solution:

In figure 8, we can see that $\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$**, the x-axis is a horizontal asymptote of the graph** $y = \frac{1}{x}$**, then** $\lim_{x \to \pm\infty} \frac{1}{x} = 0$

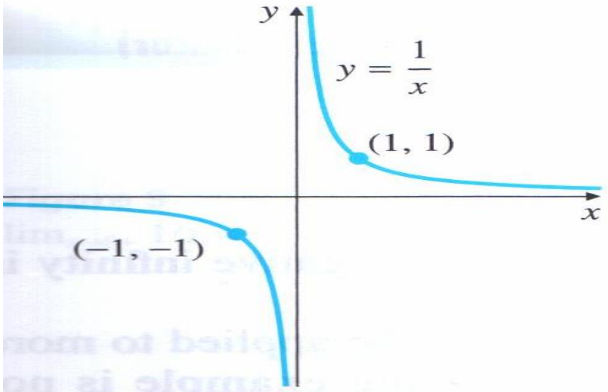


Figure 8 $\lim_{x \to \pm\infty} \frac{1}{x} = 0$

Definition ***Page (214)***

If n is a positive rational number and c is any number , then

$$\lim_{x \to \infty} \frac{c}{x^n} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{c}{x^n} = 0 \ ,$$

provided x^n is always defined .

For example:

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0 , \ \lim_{x \to \pm\infty} \frac{1}{x^2} = 0 , \ \lim_{x \to \pm\infty} \frac{1}{x^3} = 0 , \ldots$$

Example (2):

Evaluate $\lim_{x \to \pm\infty} \frac{x}{\sqrt{x^2 + 1}}$

Solution:

Rewrite the expression for f(x) as follows:

$$\lim_{x\to\pm\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to\pm\infty}\frac{x}{\sqrt{x^2(1+\frac{1}{x^2})}}=\lim_{x\to\pm\infty}\frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}$$

$$=\lim_{x\to\pm\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to\pm\infty}\frac{\operatorname{sgn} x}{\sqrt{1+\frac{1}{x^2}}},\qquad Remeber\ \sqrt{x^2}=|x|$$

Where $\operatorname{sgn}(x)=\frac{x}{|x|}=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$

Then

$$:\ \lim_{x\to\pm\infty}\frac{\operatorname{sgn} x}{\sqrt{1+\frac{1}{x^2}}}=\begin{cases}\frac{1}{\sqrt{1+0}}=1 & \text{if } x\to\infty\\ \frac{-1}{\sqrt{1+0}}=-1 & \text{if } x\to-\infty\end{cases}$$

Then

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=1,\quad \text{and}\quad \lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=-1$$

Limits at infinity for rational functions:

Let

$P_m(x)=a_m x^m+\dots+a_0$, and $Q_n(x)=b_n x^n+\dots+b_0$

be polynomials of degree m and n, respectively, so that

$a_m\neq 0, b_n\neq 0$. Then

$$\lim_{x\to\pm\infty}\frac{P_m(x)}{Q_n(x)}=\begin{cases}0 \quad \text{if } m<n\\ \dfrac{a_m}{b_n} \quad \text{if } m=n\\ (\pm\infty)\text{does not exist if } m>n\end{cases}$$

Homework (1) : Page (214)

Evaluate $\displaystyle\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5}$

Solution:

$$\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5}=\frac{2}{3}, \quad m=n=2$$

Homework (2) : Page (214)

Evaluate $\displaystyle\lim_{x\to\pm\infty}\frac{5x+2}{2x^3-1}$

Solution:

$$\lim_{x\to\pm\infty}\frac{5x+2}{2x^3-1}=0, m=1, n=3, m<n$$

Quiz (1) :

Find $\displaystyle\lim_{x\to-\infty}\frac{2x^2-5}{3x^2+x+2}$.

Quiz (2)

$$\text{Find } \lim_{x \to \infty} \frac{2x^2 - 5}{3x^4 + x + 2} \ .$$

Example (5) : Page (216)

$$\lim_{x \to \infty} \frac{x^3 + 1}{x^2 + 1}$$

Solution:

$$\lim_{x \to \infty} \frac{x^3 + 1}{x^2 + 1} = \infty, \quad m = 3, n = 2, m > n$$

Quiz (3)

$$\text{Find } \lim_{x \to \infty} \frac{2x^3 - 5}{3x^2 + x + 2} \ .$$

Example (3) : Page (215)

If $f(x) = \sqrt{x^2 + x} - x$, find

(a) $\lim_{x \to \infty} f(x)$.

(b) $\lim_{x \to -\infty} f(x)$.

Solution

$$f(x) = \sqrt{x^2 + x} - x$$

(a)

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)}$$

$$=\lim_{x\to\infty}\frac{x^2+x-x^2}{(\sqrt{x^2(1+\frac{1}{x})}+x)}$$

$$=\lim_{x\to\infty}\frac{x}{(x\sqrt{(1+\frac{1}{x})}+x)}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{(1+\frac{1}{x})}+1}=\frac{1}{2}$$

b) $\lim_{x\to-\infty}\sqrt{x^2+x}-x=\infty$ (does not exists)

Infinite Limits:

A function whose values grow arbitrary large can sometimes be said to have an infinite limit.

$$\lim_{x\to a} f(x)=\pm\infty$$

Homework 3

Describe the behavior of the function $f(x)=\frac{1}{x^2}$ *near x=0*

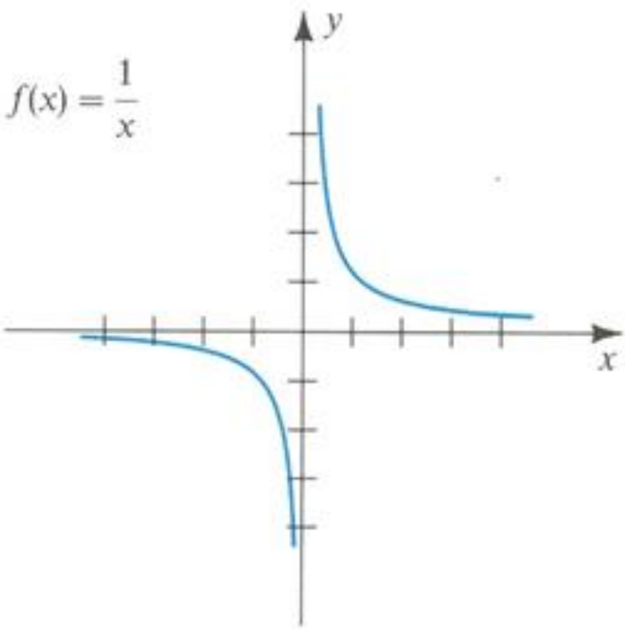
Solution :

$$\lim_{x\to 0^+}\frac{1}{x^2}=\infty,\quad \text{and}\quad \lim_{x\to 0^-}\frac{1}{x^2}=\infty$$

Example 4:

Describe the behavior of the function $f(x) = \frac{1}{x}$ near x=0

Solution :



$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0} \frac{1}{x} = \text{does not exists}$$

Homework 4

$$a)\ \lim_{x \to \infty} (3x^3 - x^2 + 2) = \infty \qquad b)\ \lim_{x \to -\infty} (3x^3 - x^2 + 2) = -\infty$$

$$c)\ \lim_{x \to \infty} (x^4 - 5x^3 - x) = \infty \qquad c)\ \lim_{x \to -\infty} (x^4 - 5x^3 - x) = \infty$$

Example (5) : ***Page (216)***

$$\lim_{x \to \infty} \frac{x^3 + 1}{x^2 + 1}$$

Solution:

$$\lim_{x \to \infty} \frac{x^3+1}{x^2+1} = \infty, \quad m=3, n=2, m>n$$

End

More examples.

Example (2) : Page (70)

Find each limit , if it exists .

(a) $\lim_{x \to 4^-} \frac{1}{(x-4)^3}$. (b) $\lim_{x \to 4^+} \frac{1}{(x-4)^3}$. (c) $\lim_{x \to 4} \frac{1}{(x-4)^3}$.

Solution

(a) If x is close to 4 and $x<4$, then $x-4$ is close to 0 and negative , and

$$\lim_{x \to 4^-} \frac{1}{(x-4)^3} = \boxed{-\infty} .$$

(a) If x is close to 4 and $x > 4$, then $x - 4$ is close to 0 and positive , and

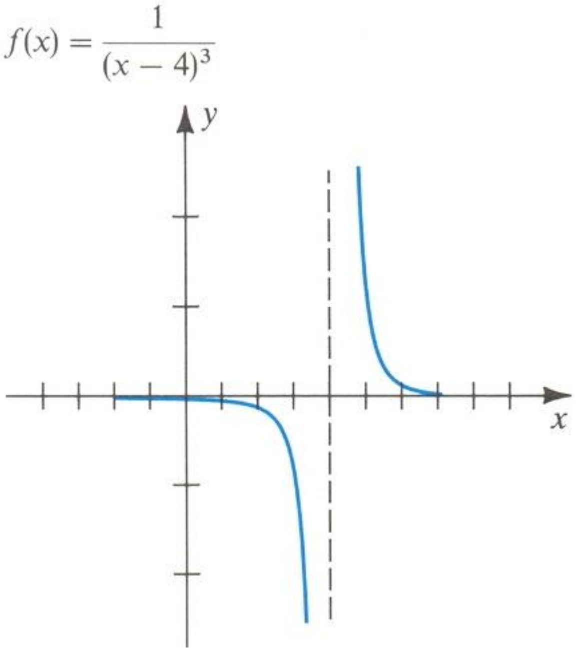
$$\lim_{x \to 4^+} \frac{1}{(x-4)^3} = \boxed{\infty} .$$

(c) Since $\lim_{x \to 4^-} \frac{1}{(x-4)^3} \neq \lim_{x \to 4^+} \frac{1}{(x-4)^3}$, then

$$\boxed{\lim_{x \to 4} \frac{1}{(x-4)^3} \text{ does not exist}} .$$

* The graph of $y = \frac{1}{(x-4)^3}$ is sketched in Figure 2.29 . The line $x = 4$ is a vertical asymptote .

Figure 2.29



Quiz (3)

Find $\lim_{x \to \infty} \frac{2x^3 - 5}{3x^2 + x + 2}$.

Solution

* Since the highest power of x in the denominator is 2, we first divide numerator and denominator by x^2, obtaining

$$\lim_{x\to\infty}\frac{2x^3-5}{3x^2+x+2}=\lim_{x\to\infty}\frac{2x-\dfrac{5}{x^2}}{3+\dfrac{1}{x}+\dfrac{2}{x^2}}$$

$$=\frac{\infty-0}{3+0+0}=\frac{\infty}{3}=\boxed{\infty}.$$

Example (6) : Page (74)

If $f(x)=\dfrac{\sqrt{9x^2+2}}{4x+3}$, find

(a) $\lim\limits_{x\to\infty} f(x)$.

(b) $\lim\limits_{x\to-\infty} f(x)$.

Solution

$$f(x)=\frac{\sqrt{9x^2+2}}{4x+3}$$

(a) If x is large and positive, then

$$\sqrt{9x^2+2}\approx\sqrt{9x^2}=3x \quad \text{and} \quad 4x+3\approx 4x$$

and hence

$$f(x)=\frac{\sqrt{9x^2+2}}{4x+3}\approx\frac{3x}{4x}=\frac{3}{4}$$

this suggests that $\boxed{\lim\limits_{x\to\infty} f(x)=\dfrac{3}{4}}$.

* To give a rigorous proof we may write

$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{4x+3}=\lim_{x\to\infty}\frac{\sqrt{x^2\left(9+\frac{2}{x^2}\right)}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2}\,\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

If x *is* ***positive*** *, then* $\sqrt{x^2}=x$ *, and dividing numerator and denominator of the last fraction by* x *gives us*

$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{4x+3}=\lim_{x\to\infty}\frac{\sqrt{x^2}\,\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{x\,\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{\sqrt{9+\frac{2}{x^2}}}{4+\frac{3}{x}}$$

$$=\frac{\sqrt{9+0}}{4+0}=\boxed{\frac{3}{4}}.$$

(b) *If* x *is large* ***negative*** *, then* $\sqrt{x^2}=-x$ *. If we use the same steps as in part* ***(a)*** *, we obtain*

$$\lim_{x\to-\infty}\frac{\sqrt{9x^2+2}}{4x+3}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\,\sqrt{x^2\left(9+\frac{2}{x^2}\right)}}{4x+3}$$

$$= \lim_{x \to -\infty} \frac{(-x)\sqrt{9 + \frac{2}{x^2}}}{4x + 3}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{9 + \frac{2}{x^2}}}{4 + \frac{3}{x}}$$

$$= \frac{-\sqrt{9 + 0}}{4 + 0} = \boxed{-\frac{3}{4}}.$$

7.3 CONTINUTY: Page (218)

Objectives:

- Continuity at a Point
- Continuity on an interval
- There are lots of continuous functions.
- Continuous Extension and Removable Discontinuities.للاطلاع

Continuity at a point.

Definition (4) : Continuity at an interior point. Page (219)

We say that a function f is continuous at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

If either $\lim_{x \to c} f(x)$ fails to exists or it exists but is not equal to $f(c)$, then we say that f is discontinuous at c.

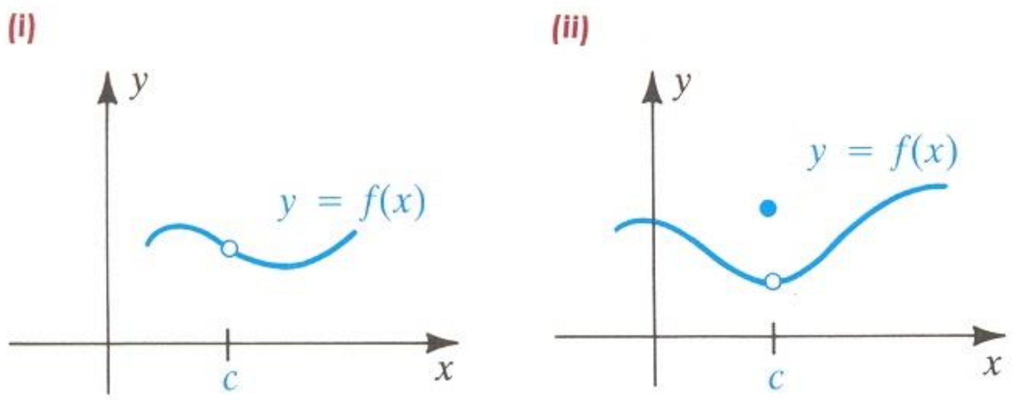
Which equivalent to the following conditions :

(i) $f(c)$ is defined .

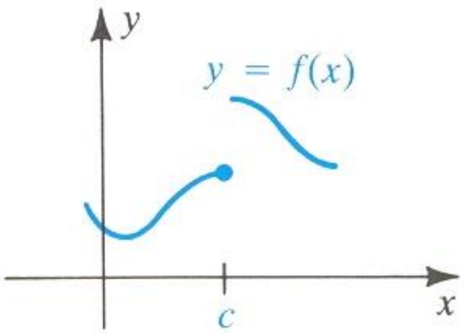
(ii) $\lim_{x \to c} f(x)$ exists .

(iii) $\lim_{x \to c} f(x) = f(c)$.

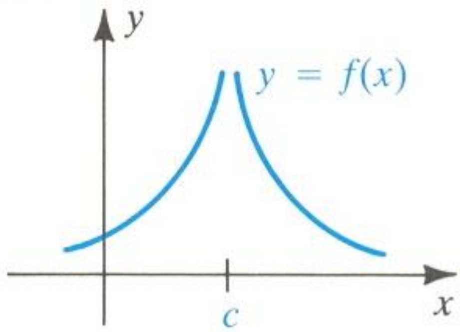
Figure 10



(iii)



(iv)



*** Not that :**

In (i) of the Figure 10 , $f(c)$ is not defined .

In (ii) , $f(c)$ is defined ;however , $\lim_{x\to c} f(x) \neq f(c)$.

In (iii) , $\lim_{x\to c} f(x)$ does not exist, $f(c)$ is defined.

In (iv) , $f(c)$ is undefined and , in addition , $\lim_{x\to c} f(x) = \infty$

Definition (5) : ***Right and left continuity*** ***Page (219)***

We say that f is right continuous at c if $\lim_{x\to c^+} f(x) = f(c)$

We say that f is left continuous at c if $\lim_{x\to c^-} f(x) = f(c)$

.

Example 1:

The Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x<0 \end{cases}$$

$$\lim_{x\to 0^+} H(x) = \lim_{x\to 0^+} 1 = 1, H(0) = 1$$

$$\therefore \lim_{x\to 0^+} H(x) = H(0)$$

Then H(x) is right continuous at 0.

$$\lim_{x \to 0^-} H(x) = \lim_{x \to 0^-} -1 = -1 , H(0) = 1$$

$$\therefore \lim_{x \to 0^-} H(x) \neq H(0)$$

Then H(x) is not left continuous at 0.

Theorem (5) : Page (219)

Function f is continuous at c if and only if I it is both right and left continuous at c.

Definition (6) Continuity at endpoint Page (220)

We say that f is continuous at a left end point c if it is right continuous there. $\lim_{x \to c^+} f(x) = f(c)$

We say that f is continuous at a right end point c if it is left continuous there. $\lim_{x \to c^-} f(x) = f(c)$.

Homework (1) : Page (220)

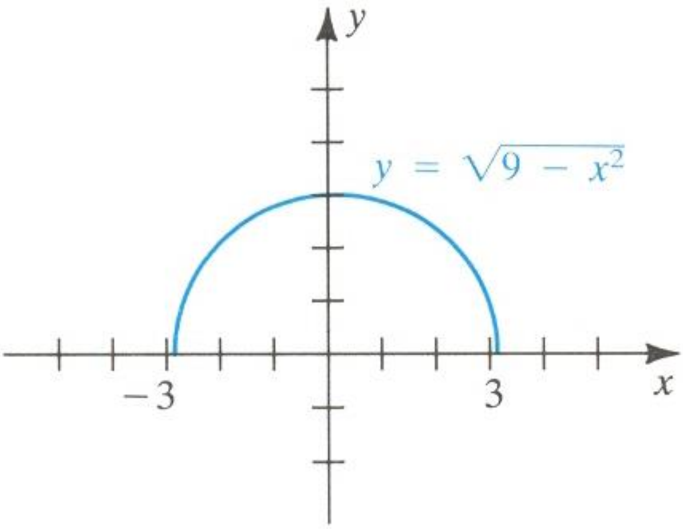
If $f(x) = \sqrt{9 - x^2}$, sketch the graph of f and prove that f is continuous on the closed interval $[-3, 3]$.

Solution

$$f(x) = \sqrt{9 - x^2}$$

* The graph of $x^2 + y^2 = 9$ is a circle with center at the origin and radius 3 . Solving for y gives us $y = \pm\sqrt{9 - x^2}$, and hence the graph of $y = \sqrt{9 - x^2}$ is the upper half of that circle .

Figure 2.37



* *If* $-3 < c < 3$ *, then*

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{9 - c^2} = f(c) .$$

Hence f *is continuous at* c *.*

* *All that remains is to check the endpoint of the interval* $[-3, 3]$ *using one-sided limits as follows :*

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{9 - 3^2} = 0 = f(-3)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{9 - 3^2} = 0 = f(3)$$

Thus , f *is continuous from the right at* -3 *and from the left at* 3 *.*

* *Then ,* **f *is continuous on*** $[-3, 3]$ *.*

Continuity on an interval

Definition 7 Continuity on an interval:

Page (220)

We say that a function f is continuous on the interval I if it is continuous at each point of I. In particular, we will say that f is a continuous function if f is continuous at every point in its domain.

Let a function f be defined on a closed interval $[a,b]$. The function f is continuous on $[a,b]$ if it is continuous on (a,b) and if, in addition,

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

Example 2

Show that the function $f(x) = \sqrt{x}$ is a continuous function.

Solution:

The Domain is $[0,\infty)$. The function is continuous at the left endpoint 0 because it is right continuous there.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = 0 = f(0)$$

Also, f is continuous at every number c>0 since:

$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{x} = \sqrt{c} = f(c), c > 0$$

Homework 2:

Show that the function $f(x) = \frac{1}{x}$ is a continuous function.

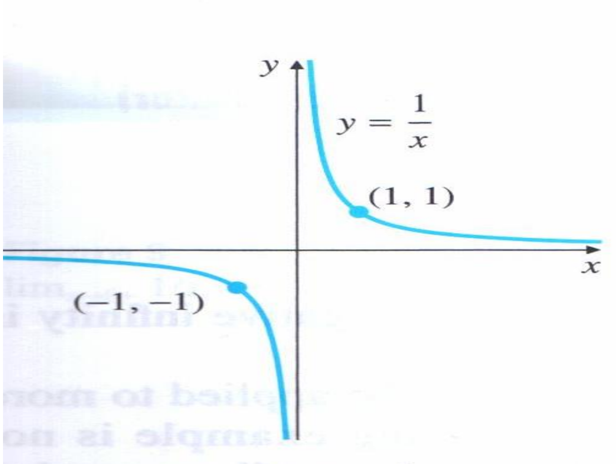


Figure 8 $\lim_{x \to \pm\infty} \frac{1}{x} = 0$

The function f is continuous on its domain $(-\infty, 0) \cup (0, \infty)$, the point 0 is not in its domain.

Example 3:

The greatest integer function: $f(x) = [x] = n$ if $x \in [n, n+1)$, ***n is an integer.***

Solution : examples: [2]=2,[2,5]=2,[-1]=-1,[-1.5]=-2,…

This function is continuous on every interval $[n, n+1)$, n is an integer. It is right continuous at each integer n but it is not left continuous there, so it is discontinuous at the integers.

$$\lim_{x \to n^+} [x] = n = [n],$$

$$\lim_{x \to n^-} [x] = n - 1 \neq n = [n].$$

There are Lots of Continuous Functions:

The following functions are continuous wherever they are defined

a) All polynomials

b) A rational function $q = \frac{f}{g}$ ***is continuous at every number except the numbers*** c ***such that*** $g(c) = 0$.

c) All rational powers $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

d) The sine, cosine, tangent, secant, cosecant and cotangent functions and

e) The absolute values function $|x|$.

Theorem 6: Combining continuous functions Page (221)

If f and g are continuous at c , then the following are also continuous at c :

(i) the sum $f+g$ and the difference $f-g$.

(ii) the product $f\,g$.

(iii) the constant multiple kf, where k is any number.

(iv) the quotient f/g , provided $g(c)\neq 0$.

(vi) the nth root $(f(x))^{\frac{1}{n}}$, provided f(c)>0 if n is even.

Theorem7 : Page (221)

If $f(g(x))$ is defined on an interval containing c, and if $\lim_{x\to c} g(x)=L$ and if f is continuous at L, then

$$\lim_{x\to c} f(g(x))=f(L)=f\left(\lim_{x\to c} g(x)\right).$$

In particular, if g is continuous at c and if f is continuous at $L=g(c)$, then

(i) $\lim_{x\to c} f(g(x))=f\left(\lim_{x\to c} g(x)\right)=f(g(c))$.

(ii) the composite function $f\circ g$ is continuous at c .

Homework 3

The following functions are continuous everywhere on their respective domains.

a) $3x^2 - 2x$ b) $\frac{x-2}{x^2-4}$ c) $|x^2 - 1|$

d) $\sqrt{x}$ e) $\sqrt{x^2 - 2x - 5}$ f) $\frac{|x|}{\sqrt{|x+2|}}$

Quiz (1) :

If $f(x) = \frac{x^2 - 1}{x^3 + x^2 - 2x}$, find the discontinuities of f.

Solution

$$f(x) = \frac{x^2 - 1}{x^3 + x^2 - 2x}$$

* Since f is a rational function, it follows that the only discontinuities at the zeros of the denominator $x^3 + x^2 - 2x$.

* By factoring we obtain

$$x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1) = 0$$

* Setting each factor equal to zero , we see that the discontinuities of f are at $0, -2$, and 1 .

Continuous Extension and Removable Discontinuities

First: Continuous Extension

If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists, we can define a new function $F(x)$ by

$$F(x) = \begin{cases} f(x), \text{ if } x \text{ is in the domain of } f \\ L \text{ if } x = c. \end{cases}$$

$F(x)$ is continuous at $x = c$. It is called the continuous extension of $f(x)$ to $x = c$. For rational functions $f(x)$, continuous extensions are usually found by cancelling common factors.

Example 4:

Show that $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ has a continuous extension to x=1 and find that extension.

Solution :

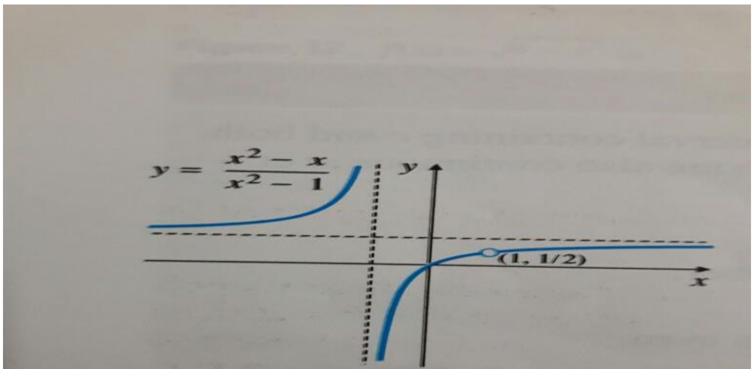


Figure 14 This function has a continuous extension to $x = 1$

Although $f(1) = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ is not defined, if $x \neq 1$ we have

$$f(x)=\frac{x^2-x}{x^2-1}=\frac{x(x-1)}{(x+1)(x-1)}=\frac{x}{x+1}$$

The function

$$F(x)=\frac{x}{x+1}$$

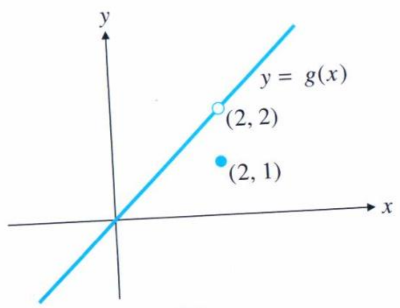
is equal to $f(x)$ for $x\neq 1$ but it is also continuous at $x=1$, having there the value $\frac{1}{2}$. The graph of $f(x)$ is shown in figure 14. The continuous extension of $f(x)$ to $x=1$ is $F(x)$. It has the same graph as $f(x)$ except with no hole at $(1,\frac{1}{2})$.

Second: Removable Discontinuities

If a function $f(x)$ is undefined or discontinuous at a point a but can be redefined at the single point so that it becomes continuous there, then we say that f has a removable discontinuity at a . The function f in the above example has a removable discontinuity at x=a. To remove it, define $f(1)=\frac{1}{2}$

Homework 4:

The function $g(x)=\begin{cases} x & \text{if } x\neq 2 \\ 1 & \text{if } x=2 \end{cases}$



(b)

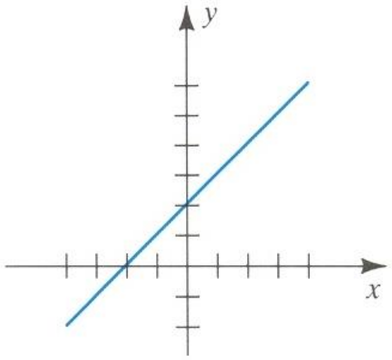
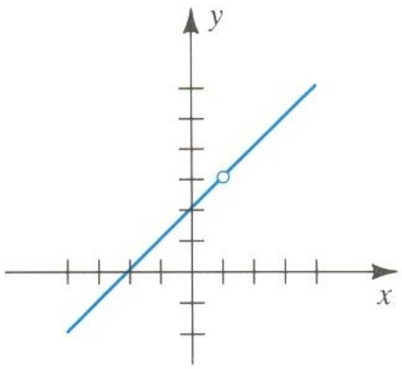
$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x = 2, \quad \text{Although } g(2)=1$$

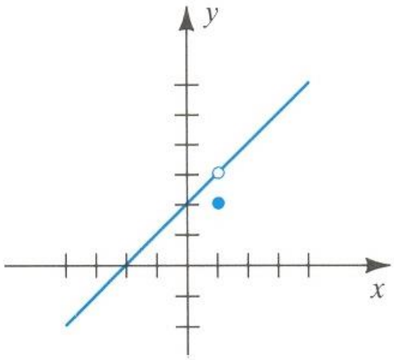
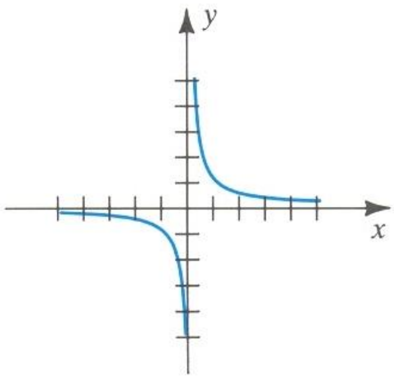
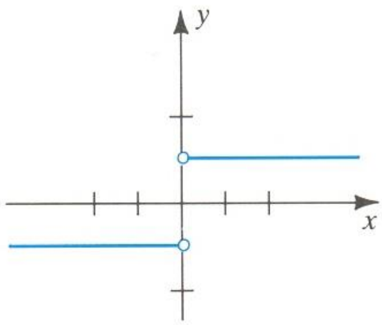
Has a removable discontinuity at x=2. To remove it redefine g(2)=2.

$$G(x) = \begin{cases} x & \text{if } x \neq 2 \\ 2 & \text{if } x=2 \end{cases}$$

End

ILLUSTRATION :

FUNCTION VALUE	*GRAPH*	*DISCONTINUITIES*
$f(x) = x + 2$		*None , since for every* c , $\lim_{x \to c} f(x) = c + 2 = f(c)$
$g(x) = \dfrac{x^2 + x + 2}{x - 1}$		$c = 1$ *since* $g(1)$ *is undefined (removable discontinuity).* $G(x) = \begin{cases} \dfrac{x^2 + x + 2}{x - 1}; & x \neq 1 \\ 3; & x = 1 \end{cases}$

$h(x)=\begin{cases}\dfrac{x^2+x-2}{x-1} & \text{if } x\neq 1\\ 2 & \text{if } x=1\end{cases}$		$c=1$ since $\lim\limits_{x\to 1} h(x)=3$ $\neq h(1)$ (removable discontinuity). $H(x)=\begin{cases}\dfrac{x^2+x-2}{x-1} & \text{if } x\neq 1\\ 3 & \text{if } x=1\end{cases}$
$h(x)=\dfrac{1}{x}$		$c=0$ since $h(0)$ does not exist and also $\lim\limits_{x\to 0} h(x)$ does not exist (Infinite discontinuity). (non-removable discontinuity)
$p(x)=\dfrac{\lvert x\rvert}{x}$		$c=0$ since $p(0)$ is undefined and also $\lim\limits_{x\to 0} p(x)$ does not exist (jump discontinuity). (non-removable discontinuity)

Quiz (2) :

If $k(x)=\dfrac{\sqrt{9-x^2}}{3x^4+5x^2+1}$ *,prove that k is continuous on the closed interval* $[-3,3]$.

Solution

Let $f(x)=\sqrt{9-x^2}$ and $g(x)=3x^4+5x^2+1$ *from example 3, f is continuous on the closed interval* $[-3,3]$ *and* *from Theorem (2.21), g is continuous at every real number. Moreover for every c in $[-3,3]$, $g(c)\neq 0$ Hence by Theorem (2.23) (iv), the quotient $k=\frac{f}{g}$ is continuous on the closed interval $[-3,3]$.*

Quiz(3):

If $k(x)=\left|3x^2-7x-12\right|$, show that k is continuous at every number.

Solution

If we let $f(x)=\left|x\right|$ and $g(x)=3x^2-7x-12$

Then

$$k(x)=f(g(x))=(f\circ g)(x)$$

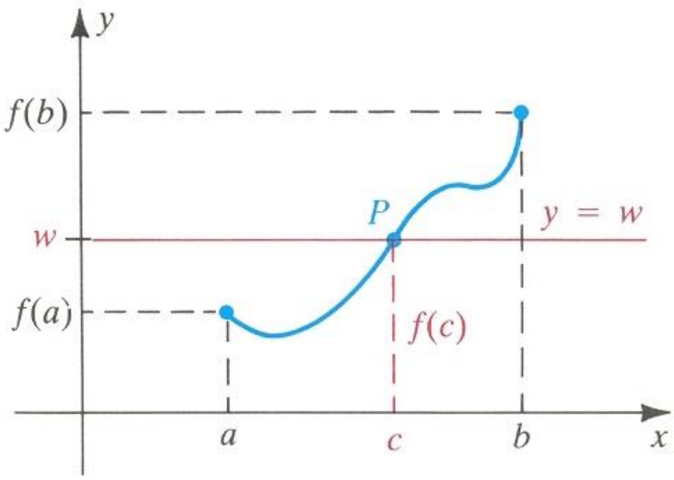
Since both f and g are continuous function (see example 1 and (i) of Theorem (6)), it follows from (ii) of Theorem (7) that the composite function $k=f\circ g$ is continuous at c.

Intermediate value theorem (2.26) : ***Page (84)***

If f is continuous on a closed interval $[a,b]$ and if w is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that

$$f(c)=w.$$

Figure 2.38



If $f(a)$ and $f(b)$ have opposite signs, then there is a number c (zero or root) in $[a,b]$ such that

$$f(c)=0.$$

Example (6) : **Page (84)**

Show that $f(x)=x^5+2x^4-6x^3+2x-3$ has a zero between 1 and 2 .

Solution

$$f(x)=x^5+2x^4-6x^3+2x-3$$

*** Substituting 1 and 2 for x gives us the function values :**

$$f(1)=1+2-6+2-3=-4$$

$$f(2)=32+32-48+4-3=17$$

*** Since $f(1)$ and $f(2)$ have opposite signs , it follows from the intermediate value theorem that $f(c)=0$ for at least one real number c between 1 and 2 .**

Theorem [2.27].

If a function f is continuous and has no zeroes on an interval ,then either $f(x) > 0$ or and $f(x) < 0$ for every x in the interval.

Exercises 2.5, pages 85-87, from 1-60.

Example (2) : *Page (70)*

Find each limit , if it exists .

(a) $\lim_{x \to 4^-} \frac{1}{(x-4)^3}$. (b) $\lim_{x \to 4^+} \frac{1}{(x-4)^3}$. (c) $\lim_{x \to 4} \frac{1}{(x-4)^3}$.

Solution

(a) If x is close to 4 and $x < 4$, then $x - 4$ is close to 0 and negative , and

$$\lim_{x \to 4^-} \frac{1}{(x-4)^3} = \boxed{-\infty} .$$

(a) If x is close to 4 and $x > 4$, then $x - 4$ is close to 0 and positive , and

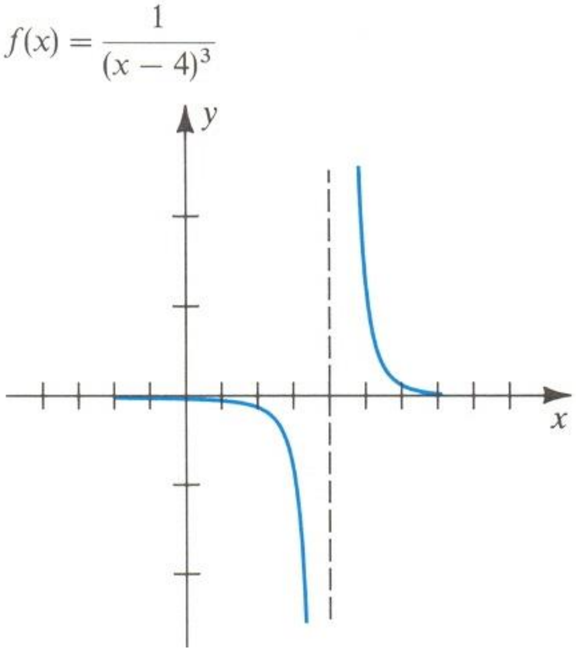
$$\lim_{x \to 4^+} \frac{1}{(x-4)^3} = \boxed{\infty} .$$

(c) Since $\lim_{x \to 4^-} \frac{1}{(x-4)^3} \neq \lim_{x \to 4^+} \frac{1}{(x-4)^3}$, then

$$\boxed{\lim_{x \to 4} \frac{1}{(x-4)^3} \text{ does not exist}} .$$

* The graph of $y = \frac{1}{(x-4)^3}$ is sketched in Figure 2.29 . The line $x = 4$ is a vertical asymptote .

Figure 2.29



Quiz (3)

Find $\lim_{x \to \infty} \frac{2x^3 - 5}{3x^2 + x + 2}$.

Solution

* ***Since the highest power of x in the denominator is 2 , we first divide numerator and denominator by*** x^2 ***, obtaining***

$$\lim_{x \to \infty} \frac{2x^3 - 5}{3x^2 + x + 2} = \lim_{x \to \infty} \frac{2x - \frac{5}{x^2}}{3 + \frac{1}{x} + \frac{2}{x^2}}$$

$$= \frac{\infty - 0}{3 + 0 + 0} = \frac{\infty}{3} = \boxed{\infty} .$$

Example (6) : ***Page (74)***

If $f(x) = \frac{\sqrt{9x^2 + 2}}{4x + 3}$ ***, find***

(a) $\lim_{x \to \infty} f(x)$. (b) $\lim_{x \to -\infty} f(x)$.

Solution

$$f(x) = \frac{\sqrt{9x^2 + 2}}{4x + 3}$$

(a) If x is large and positive, then

$$\sqrt{9x^2 + 2} \approx \sqrt{9x^2} = 3x \quad \text{and} \quad 4x + 3 \approx 4x$$

and hence

$$f(x) = \frac{\sqrt{9x^2 + 2}}{4x + 3} \approx \frac{3x}{4x} = \frac{3}{4}$$

this suggests that $\lim_{x \to \infty} f(x) = \frac{3}{4}$.

* To give a rigorous proof we may write

$$\lim_{x \to \infty} \frac{\sqrt{9x^2 + 2}}{4x + 3} = \lim_{x \to \infty} \frac{\sqrt{x^2\left(9 + \frac{2}{x^2}\right)}}{4x + 3}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{9 + \frac{2}{x^2}}}{4x + 3}$$

If x is positive, then $\sqrt{x^2} = x$, and dividing numerator and denominator of the last fraction by x gives us

$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{4x+3}=\lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{x\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{\sqrt{9+\frac{2}{x^2}}}{4+\frac{3}{x}}$$

$$=\frac{\sqrt{9+0}}{4+0}=\boxed{\frac{3}{4}}.$$

***(b)** If x is large **negative**, then $\sqrt{x^2}=-x$. If we use the same steps as in part **(a)**, we obtain*

$$\lim_{x\to-\infty}\frac{\sqrt{9x^2+2}}{4x+3}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{x^2\left(9+\frac{2}{x^2}\right)}}{4x+3}$$

$$=\lim_{x\to-\infty}\frac{(-x)\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

$$=\lim_{x\to-\infty}\frac{-\sqrt{9+\frac{2}{x^2}}}{4+\frac{3}{x}}$$

$$=\frac{-\sqrt{9+0}}{4+0}=\boxed{-\frac{3}{4}}.$$