Error propagation

Differentiation

Terminology and notation

The process of finding the rate of change of given function is called differentiation. The function is said to be differentiated. If y is a function of the independent variable x, we say that y is differentiated with respect to (w.r.t.) x.

Ordinary differentiation (denoted by $\frac{d}{dx}$)

- There is a notation for writing down the derivative of a function. If the function is $y(x)$, we denote the derivative of y by $\frac{dy}{dx}$, pronounced 'dee y by dee x'.
- Another notation for the derivative is simply y', pronounced y dash.
- Similarly if the function is $y(t)$ we write the derivative as $\frac{dy}{dt}$ *or* $\dot{y}$, pronounced "y dot".

If we have a function f(x) derivative of f with respect to x is written as $\frac{df}{dx}$. $\left(Rule: \frac{d}{dx}\left(x^n\right) = nx^{n-1} \right)$

For example:

(1). $f\left(x\right) = x^3$, $\frac{df}{dx} = 3x^2$

(2). $f\left(x\right) = 5x^2 + 8x + 9$, $\frac{df}{dx} = 10x + 8$

(3). $f\left(x, y, z\right) = 3x^4 + 3y + z$

$\frac{df}{dx} = 12x^3 + 3\frac{dy}{dx} + \frac{dz}{dx}$, $\frac{df}{dy} = 12x^3\frac{dx}{dy} + 3 + \frac{dz}{dy}$ $\frac{df}{dz} = 12x^3\frac{dx}{dz} + 3\frac{dy}{dz} + 1$

(4). If $f\left(x, y\right) = \sin\left(x - y\right)$, Find $\frac{df}{dx} = ?, and$ $\frac{df}{dy} = ?$

$$\frac{df}{dx} = \cos\left(x - y\right)\frac{d}{dx}\left(x - y\right) \Rightarrow \frac{df}{dx} = \cos\left(x - y\right)\left(\frac{dx}{dx} - \frac{dy}{dx}\right) \Rightarrow \frac{df}{dx} = \cos\left(x - y\right)\left(1 - \frac{dy}{dx}\right)$$

$$\frac{df}{dy} = \cos\left(x - y\right)\frac{d}{dy}\left(x - y\right) \Rightarrow \frac{df}{dy} = \cos\left(x - y\right)\left(\frac{dx}{dy} - \frac{dy}{dy}\right) \Rightarrow \frac{df}{dy} = \cos\left(x - y\right)\left(\frac{dx}{dy} - 1\right)$$

Partial differentiation (denoted by $\frac{\partial}{\partial x}$)

Suppose we have function of several variables f(x, y, z). Then the partial derivative of f(x, y, z) with respect to x $\frac{\partial f}{\partial x}$ is defined to be the ordinary derivative of f(x, y, z) with respect to x with all other variables treated as constant. For example:

(1). $f\left(x, y, z\right) = 3x^4 + 3y + z$

$\frac{\partial f}{\partial x} = 12x^3$, $\frac{\partial f}{\partial y} = 3$ $\frac{\partial f}{\partial z} = 1$

(2) $f\left(x, y, z\right) = 3x^2 + 4xz + yz$

$\frac{\partial f}{\partial x} = 6x + 4z$ $\frac{\partial f}{\partial y} = z$ $\frac{\partial f}{\partial z} = 4x + y$

(3) $f(x,y,z)=z\,x\,e^{2y}$

$$\frac{\partial f}{\partial x} = z\,e^{2y} \qquad \frac{\partial f}{\partial y} = z\,x\,2e^{2y} \qquad \frac{\partial f}{\partial z} = x\,e^{2y}$$

(4) If $f(x,y)=\sin(x-y)$, Find $\frac{\partial f}{\partial x}=?, \textit{ and } \frac{\partial f}{\partial y}=?$

Solution: $\frac{\partial f}{\partial x}=\cos(x-y)\frac{\partial}{\partial x}(x-y)\Rightarrow\frac{\partial f}{\partial x}=\cos(x-y)$, $\frac{\partial f}{\partial y}=\cos(x-y)\frac{\partial}{\partial y}(x-y)\Rightarrow\frac{\partial f}{\partial y}=-\cos(x-y)$

(5). If $f(x,y)=x^2-2xy+3y^2$, Find $\frac{\partial f}{\partial x}=?, \textit{ and } \frac{\partial f}{\partial y}=?$ Solution: $\frac{\partial f}{\partial x}=2x-2y, \textit{ and } \frac{\partial f}{\partial y}=-2x+6y$

(6) $f(x,y,z)=3x^2+4xz+yz$ Solution: $\frac{\partial f}{\partial x} = 6x + 4z$ $\frac{\partial f}{\partial y} = z$ $\frac{\partial f}{\partial z} = 4x + y$

(7) $f(x,y,z)=z\,x\,e^{2y}$,

Solution: $\frac{\partial f}{\partial x} = z\,e^{2y}$ $\frac{\partial f}{\partial y} = z\,x\,2e^{2y}$ $\frac{\partial f}{\partial z} = x\,e^{2y}$

(8) $f(x,y)=xe^{xy}$, Find $\frac{\partial f}{\partial x}=?, \textit{ and } \frac{\partial f}{\partial y}=?$

Solution:

$$\frac{\partial f}{\partial x}=xe^{xy}\frac{\partial}{\partial x}(xy)+e^{xy}\frac{\partial}{\partial x}(x)\Rightarrow\frac{\partial f}{\partial x}=xe^{xy}(y)+e^{xy}\Rightarrow\frac{\partial f}{\partial x}=e^{xy}(xy+1)$$

$$\frac{\partial f}{\partial y}=xe^{xy}\frac{\partial}{\partial y}(xy)\Rightarrow\frac{\partial f}{\partial y}=xe^{xy}(x)\Rightarrow\frac{\partial f}{\partial y}=x^2e^{xy}$$

Derivative rules

$\frac{d}{dx}(k)=0$, where k is a constant, $\qquad \frac{d}{dx}(x)=1$

$\frac{d}{dx}(au)=a\frac{du}{dx} \qquad \frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$

$\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}(e^u)=e^u\frac{du}{dx}$

$\frac{d}{dx}(\sin u)=\cos u\frac{du}{dx}$, $\qquad \frac{d}{dx}(\cos u)=-\sin u\frac{du}{dx}$

$\frac{d}{dx}(\tan u)=\sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}(\cot u)=-\csc^2 u\frac{du}{dx}$

- The derivative of $f(x)\pm g(x)$ *is* $\frac{df}{dx}\pm\frac{dg}{dx}$
- The derivative of $k\,f(x)$ *is* $k\frac{df}{dx}$

Error propagation

Let us start by considering a simple experiment, to measure the time it takes to a metal ball to fall through a given distance. The switch that releases the ball also starts the clock. The micro-switch stops the clock when the ball strikes it.

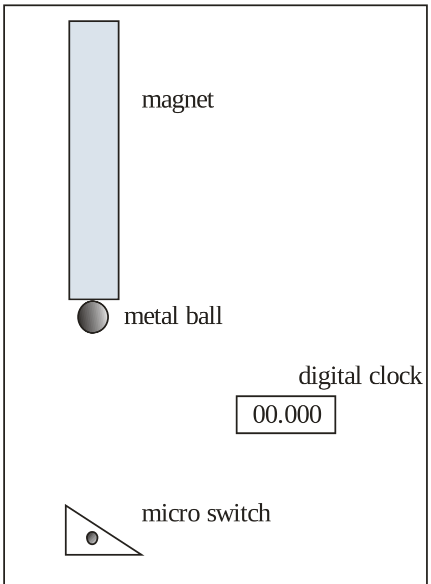

Figure 1.

- Suppose we make few measurements. Some typical numbers might be: 3.467 s. 3.459 s, 3.463 s, 3.465 s etc. All the numbers are different. This is known as phenomenon of errors. Whenever, we repeat a measurement, using sufficiently precise apparatus, the results will not always be the same. Always a small spread in the values. This spread can never be reduced to zero.
- We can never know the true value. If we use a clock that only counts seconds (instead of 0.001 s) all values are equal to 3 s. No spread but not very useful. We can use better equipment. Lasers and photocells could be used to start and stop the clock, and everything could be mounted in a vacuum. We might get results like 3.46275 s. 3.46266 s, 3.46270 s, etc. The spread of number is much smaller, but it still exists.

Frequency diagram

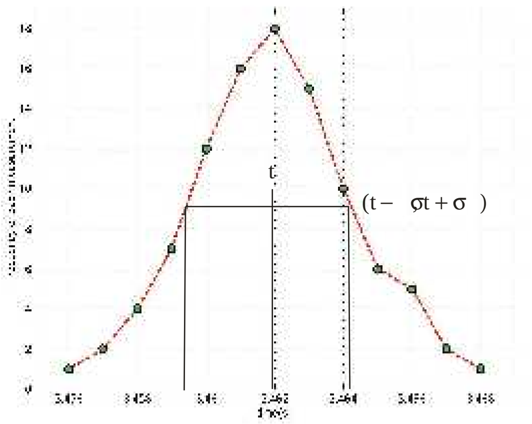

Figure 2.

The plot of measured values (table 1) is shown in figure 2. X-axis coves all possible measurements. Y-axis records the number of times each measurement occurred. For example, 3.461 occurred 16 times, so x-y coordinates (3.461 16).

- The frequency diagram gives a visual picture of how the measurements are distributed. We can easily see that the center of the distribution is around 3.462 s. Figure 2 is called frequency distribution, distribution curve, or histogram etc. The point at 3.476 s can be ignored (bad data).

Mean value

The mean value (or average) of any set of measurements is just the sum of the measurements divided by the number of measurement.

$$\prec x \succ = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad (1)$$

$$\prec x \succ = \bar{x} = \frac{x_1 + x_2 + x_3 + \cdots\cdots + x_n}{n}$$

From the measurements (table 1) excluding 3.976 s.

$$\prec t \succ = \frac{3.461 + 3.463 + \cdots\cdots + 3.462}{99}$$

It is easy to group the values together using the frequency diagram.

$$\prec t \succ = \frac{1\times(3.456) + 2\times(3.457) + 4\times(3.458) + \cdots\cdots}{99}$$

3.456 occurs once, 3.457 twice, 3.458 four times etc.
If f_l be the frequency that any particular value "t" occurs, we can write

$$\prec t \succ = \frac{\sum_l f_l \cdot t}{n} \qquad (2)$$

Where $n = \sum_l f_l$, Using equation (2), $\prec t \succ$ = 3.4619 s

Standard deviation

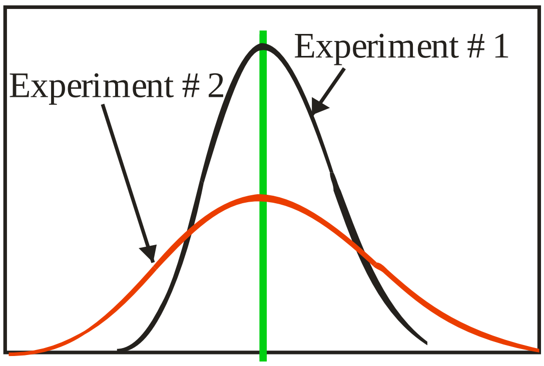

Figure 3.

- Suppose that two separate experiments are done to measure the same quantity. The frequency diagrams from the two experiments might look like figure 3.
- Both have the same mean but second experiment has larger spread than the first experiment.
- Experiment # 1 is better than the experiment # 2. We would like a method by which we could calculate a number, which express this fact. The method is to calculate the **standard deviation (±)**.

Here we will just state how to calculate it and use it for our purposes. It is represented by sigma (σ).

$$\pm_x = \sqrt{\frac{\sum_{i=1}^{n} (\bar{x} - x_i)^2}{n-1}} \qquad (3)$$

σ_x ≡ means standard deviation for the distribution of x-values. It is the width for the distribution since it depends on the distances of the individual measurements fro the mean.

- From the table 1, using $\prec t \succ$ = 3.462 s.

$$\pm_l = \left[\frac{1(3.462-3.456)^2 + 2(3.462-3.457)^2 + \cdots}{98}\right]^{1/2} \qquad \Rightarrow \pm_l = 0.00239 \text{ s}$$

It has the same units as the units on the measurements.

- The meaning of "$\pm$ " is that if we move a distance $\pm_l$ on either side of the mean value, we will form an interval that contains about $\frac{2}{3}$ of the measurements. ($\prec t \succ$ - $\pm_l$, $\prec t \succ$ + $\pm_l$) = (3.4596, 3.4644).

So the standard deviation is a measure of the precision of the experiment.

General formula for error propagation

We want to calculate the error on some quantity Z, which is a function of several variables, Z = Z (x, y, …..) then

$$\pm_z = \sqrt{\left(\frac{\partial Z}{\partial x}\pm_x\right)^2 + \left(\frac{\partial Z}{\partial y}\pm_y\right)^2 + \cdots} \qquad (1)$$

If Z is a function of only one variable, x

$$\pm_z = \left|\frac{dZ}{dx}\right|\pm_x \qquad (2)$$

Absolute value is used to ensure that σ_Z is always positive. Equation (1) is valid when

(1) $\frac{\pm_x}{x} \prec\prec 1$, $\frac{\pm_y}{y} \prec\prec 1$

(2) x, y are independent.

The majority of calculations will fall into two categories (a) addition and subtraction (b) multiplication and division

Addition and subtraction

Let $Z = ax + by + cu \cdots\cdots$

Where x, y, u, …. are variables and a, b, c, …. are constants. Using equation (1)

$$\pm_z = \sqrt{(a\pm_x)^2 + (b\pm_y)^2 + (c\pm_u)^2 + \cdots} \qquad (3)$$

This equation is the error formula to use adding and subtracting quantities. We can say the errors are added in quadrature, which means add the squares, then, take the square root.

Examples

Example 1. $z = x - y$

Where x = 3.67 ± 0.03 and y = 2.14 ± 0.02

z = 3.67 – 2.14 = 1.53

$$\sigma_Z = \sqrt{\pm_x^2 + \pm_y^2} \Rightarrow \sigma_Z = \sqrt{(0.03)^2 + (0.02)^2} \Rightarrow \sigma_Z = 0.036$$

$\therefore$ z = 1.53 ± 0.036

Example 2. R = 3L + 4M

Where L = 63.6 ± 0.2 and M = 19.7 ± 0.3

R = 3(63.6) + 4(19.7) = 269.6

$$\sigma_R = \sqrt{(3\pm_L)^2 + (4\pm_M)^2} \Rightarrow \sigma_R = \sqrt{(3\times 0.2)^2 + (4\times 0.3)^2} \Rightarrow \sigma_R = 1.3$$

$\therefore$ R = 269.6 ± 1.3

Multiplication, division or raised to power

Consider $Z = Ax^n y^{-m}$

$$\frac{\partial Z}{\partial x} = A\,n\,x^{n-1} y^{-m} \qquad\qquad \frac{\partial Z}{\partial y} = A\,x^n (-m) y^{-m-1}$$

As equation (1)

$$\sigma_Z = \left[\left(\frac{\partial Z}{\partial x} \right)^2 \cdot \pm_x{}^2 + \left(\frac{\partial Z}{\partial y} \right)^2 \cdot \pm_y{}^2 + \cdots \right]^{1/2}$$

$$\sigma_Z = \left[\left(Anx^{n-1} y^{-m} \right)^2 \cdot \pm_x{}^2 + \left(Ax^n (-m) y^{-m-1} \right)^2 \cdot \pm_y{}^2 \right]^{1/2}$$

It will be complicated when there are more than two variables, to simplify, divide it's both sides by Z.

$$\frac{\pm_z}{Z} = \left[\left(\frac{Anx^{n-1} y^{-m}}{Ax^n y^{-m}} \right)^2 \cdot \pm_x{}^2 + \left(\frac{Ax^n (-m) y^{-m-1}}{Ax^n y^{-m}} \right)^2 \cdot \pm_y{}^2 \right]^{1/2}$$

$$\Rightarrow \sigma_Z = Z \left[\left(\frac{n}{x} \right)^2 \cdot \pm_x{}^2 + \left(\frac{m}{y} \right)^2 \cdot \pm_y{}^2 \right]^{1/2}$$

If we have more variables like $Z = Ax^n y^m u^{-p}$

$$\sigma_Z = Z \left[\left(\frac{n}{x} \right)^2 \cdot \pm_x{}^2 + \left(\frac{m}{y} \right)^2 \cdot \pm_y{}^2 + \left(\frac{p}{u} \right)^2 \cdot \pm_u{}^2 + \cdots \right]^{1/2} \qquad (3)$$

For single variable

$$\sigma_Z = Z \left[\left(\frac{n}{x} \right)^2 \cdot \pm_x{}^2 \right]^{1/2} \qquad \Rightarrow \sigma_Z = Z \left(\frac{n}{x} \right) \cdot \pm_x \qquad (4)$$

Example 1. $A = LW$

$$\sigma_A = A \left[\left(\frac{1}{L} \right)^2 \cdot \pm_L{}^2 + \left(\frac{1}{W} \right)^2 \cdot \pm_W{}^2 \right]^{1/2} \qquad \Rightarrow \sigma_A = A \left[\left(\frac{\pm_L}{L} \right)^2 + \left(\frac{\pm_W}{W} \right)^2 \right]^{1/2}$$

Example 2. $Z = x^2$

$$\sigma_Z = Z \left[\left(\frac{2}{x} \cdot \pm_x \right)^2 \right]^{1/2} \Rightarrow \sigma_Z = Z\, \frac{2}{x} \cdot \pm_x \Rightarrow \sigma_Z = 2x\, \sigma_x$$

Example 3. $Z = \sqrt{y}$

$$\pm_z = Z \left[\left(\frac{1}{2} \cdot \frac{1}{y} \cdot \pm_y \right)^2 \right]^{1/2} \Rightarrow \sigma_Z = Z\, \frac{1}{2y} \cdot \pm_y \Rightarrow \sigma_Z = \frac{1}{2\sqrt{y}} \cdot \pm_y$$

Example 4. $V = \frac{1}{3} ® r^3$

$$\pm_V = V \left[\left(\frac{3}{r} \cdot \pm_r \right)^2 \right]^{1/2} \qquad \Rightarrow \pm_V = \frac{3 \cdot V}{r} \cdot \pm_r$$

Example 5. Given that $f = \dfrac{\sqrt{g}}{2s\sqrt{\mu}}$

g = 980.6 cm / s^2 s = (0.76 ± 0.02) cm / $g^{1/2}$ μ = (0.040 ± 0.001) g / cm

(1) Calculate f and its error

(2) Which variable contributes most of the error, s or μ.

$$f = \frac{\sqrt{980.6}}{2\times 0.76\times\sqrt{0.040}} \Rightarrow f = 103.01 / s$$

For σ_f, as f = $\dfrac{\sqrt{g}}{2}$ s^{-1} $\mu^{-1/2}$

$$\sigma_f = f\left[(-1)^2\cdot\left(\frac{\sigma_s}{s}\right)^2 + (-\frac{1}{2})^2\cdot\left(\frac{\sigma_\mu}{\mu}\right)^2\right]^{1/2}$$

$$\Rightarrow \sigma_f = 103.01\left[\left(\frac{0.02}{0.76}\right)^2 + \frac{1}{4}\cdot\left(\frac{0.001}{0.04}\right)^2\right]^{1/2} = 103.01\left[0.000692 + \frac{0.000625}{4}\right]^{\frac{1}{2}}$$

$$\Rightarrow \sigma_f = 103.01\left[6.9252\times10^{-4} + 1.5625\times10^{-4}\right]^{1/2} \qquad (B)$$

$$\Rightarrow \sigma_f = 103.01\times10^{-2}\left[8.4877\right]^{1/2} \Rightarrow \sigma_f = 103.01\times2.9133\times10^{-2} \Rightarrow \sigma_f = 300.105\times10^{-2} \Rightarrow \sigma_f = 3.00105\,s^{-1}$$

∴ f = (103.01 ± 3.00) s^{-1}

(2) s contribute more than μ.(clear from equation (B))

Example 6. $Z = A\cos\theta$, where A is a constant and θ has error.

$$\sigma_Z = \left[\left(\frac{\partial Z}{\partial\theta}\right)^2\cdot\sigma_\theta^{\,2}\right]^{1/2} \qquad \Rightarrow \sigma_Z = \left|\frac{dZ}{d\theta}\right|\sigma_\theta$$

σ_z = A sinθ σ_θ (Note σ_θ must be in radians (π radians = 180°))

Let A = 16, θ = (36.3 ± 0.2) degree

Z = 16 cos(36.3) ⇒ Z =12.895

σ_θ = 0.2 x $\dfrac{\pi}{180}$ = 0.0035 radian

∴ σ_z = 16 sin(36.3) (0.0035) ⇒ σ_z = 0.0332 ∴ Z =12.895 ± 0.0332

Example 7. $Z = x\,y + w$

Let x = 3.6 ± 0.1 y = 2.7 ± 0.2 w = 10.2 ± 0.8

Z = (3.6)x(2.7) + 10.2 ⇒ Z = 19.92

$$\sigma_z = \left[\left(y\cdot\sigma_x\right)^2 + \left(x\cdot\sigma_y\right)^2 + \left(\sigma_w\right)^2\right]^{1/2} \qquad \Rightarrow \sigma_z = \left[\left(2.7\times0.1\right)^2 + \left(3.6\times0.2\right)^2 + \left(0.8\right)^2\right]^{1/2}$$

$\Rightarrow \sigma_z = \left[0.0729 + 0.5184 + 0.64\right]^{1/2}$ ⇒ σ_z = 1.1 ∴ Z =19.92 ± 1.1

Table 1. Measurements of fall time taken for the falling ball experiment. A histogram of the data is shown in figure 2.

Time (s)	**Time (s)**	**Time (s)**	**Time (s)**	**Time (s)**	**Time (s)**
3.461	3.460	3.461	3.456	3.461	3.461
3.463	3.464	3.465	3.464	3.465	3.463
3.460	3.461	3.466	3.462	3.463	3.462
3.465	3.462	3.462	3.458	3.460	3.457
3.463	3.464	3.463	*3.476*	3.464	3.465
3.461	3.461	3.460	3.461	3.461	3.460
3.463	3.459	3.466	3.465	3.465	3.461
3.464	3.460	3.461	3.462	3.462	3.463
3.460	3.457	3.462	3.459	3.459	3.458
3.458	3.462	3.460	3.458	3.463	3.462
3.462	3.467	3.463	3.462	3.464	3.459
3.468	3.460	3.462	3.463	3.461	3.461
3.463	3.463	3.459	3.461	3.459	3.460
3.459	3.461	3.462	3.464	3.462	3.467
3.464	3.464	3.461	3.460	3.463	3.462
3.462	3.462	3.463	3.466	3.460	
3.466	3.463	3.466	3.462	3.464	

Table 2.

time	frequency	time	frequency
3.460	12	3.467	2
3.461	16	3.468	1
3.462	18	3.476	1 ****
3.463	15	3.459	7
3.464	10	3.458	4
3.465	6	3.457	2
3.466	5	3.456	1