Math 101

Mada Altiary

Continuity

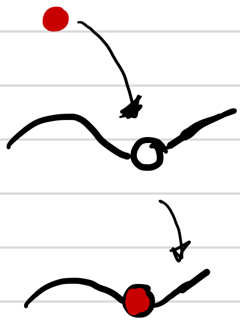
Continuity at a point:

Definition: A function f is continuous at a iff ✓

1)- $f(a)$ is defined

2) $\lim_{x\to a} f(x)$ exist

3)- $\lim_{x\to a} f(x) = f(a)$

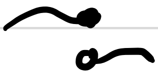


$f(a)$ undefined o hole

$\lim_{x\to a} f(x)$ DNE

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$

$\pm\infty$



Otherwise, the function is discontinuous.

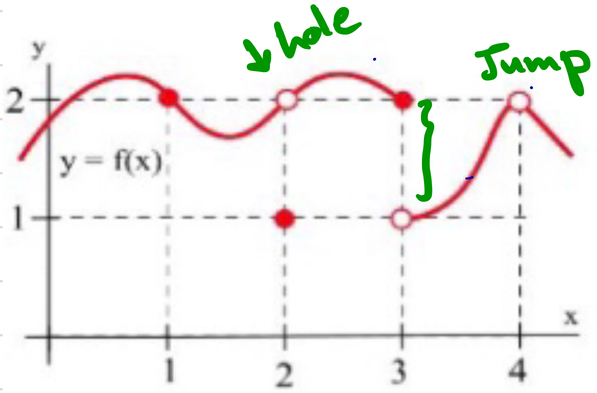


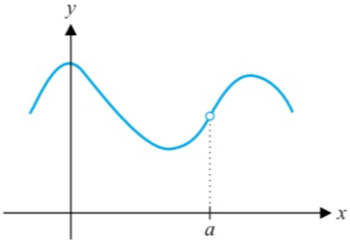
Fig. 1

a	f(a)	$\lim_{x\to a} f(x)$
1	2	2 ✓
2	1	2
3	2	does not exist
4	undefined	2

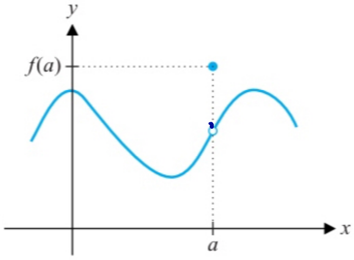
Types of Discontinuity

Removable Discontinuity

"hole"



f(a) is not defined

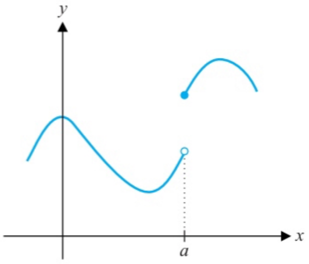


f(a) is defined and $\lim_{x \to a} f(x)$ exist but

$\lim_{x \to a} f(x) \neq f(a)$

Non-Removable Discontinuity

" Jump or AV "

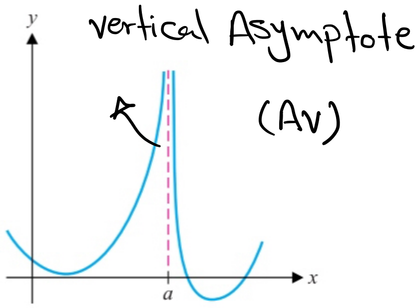


f(a) is defined but

$\lim_{x \to a} f(x)$ DNE

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

(Jump discontinuity)



f(a) is not defined

and

$\lim_{x \to a} f(x)$ DNE

$\pm\infty$

" Infinite discontinuity "

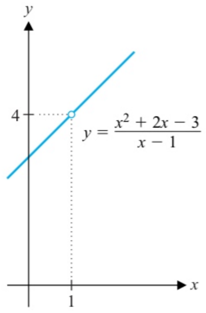
Example 1: Determine if the following function is continuous at $x = 1$

$$f(x) = \frac{x^2 + 2x - 3}{x-1}$$

$$f(1) = \frac{(1)^2 + 2(1) - 3}{1-1} = \frac{0}{0} \quad \text{undefined}$$

$\therefore$ f is not continuous at $x = 1$

f has removable discontinuity.



اعادة تعريف

أزلنا نقطة عدم الاتصال

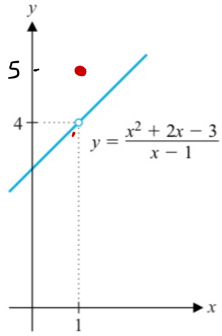
$$g(x) = \begin{cases} \frac{x^2 + 2x - 3}{x-1} & , x \neq 1 \\ a & , x = 1 \end{cases}$$

1)- $f(1) = a$

2)- $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 + 2x - 3}{x-1}$ $\left[\frac{0}{0} \text{ case}\right]$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+3)}{\cancel{x-1}} \rightarrow \text{حلل}$$

$$= \lim_{x \to 1} (x+3) = 4$$

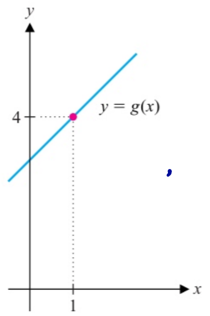


If we choose $a = 5$ then $f(1) \neq \lim_{x \to 1} f(x) = 4$

and f still is not continuous.

But if we choose $a=4$ then $\lim_{x\to a} f(x) = f(1) = 4$

and f becomes continuous.



Example: Determine if each function is continuous at the given number. If not continuous, classify each discontinuity.

$f(x) = 3x^2 + x - 1$ at $x=1$.

Solution:

$f(1) = 3(1)^2 + (1) - 1 = 3$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} 3x^2 + x - 1 = 3$

$\lim_{x\to 1} f(x) = f(1) = 3.$

$\therefore$ f is continuous at $x=1$

Remark:

دوال كثيرات الحدود

دائماً متصلة

$$g(x) = \begin{cases} x+1 & , x<2 \\ 2x-1 & , x>2 \end{cases}$$

لاتوجد إشارة مساواة

في المتراجحات

Solution:

$f(2)$ undefined.

$\therefore$ f is discontinuous at $x=2$.

$\therefore$ f has removable discontinuity

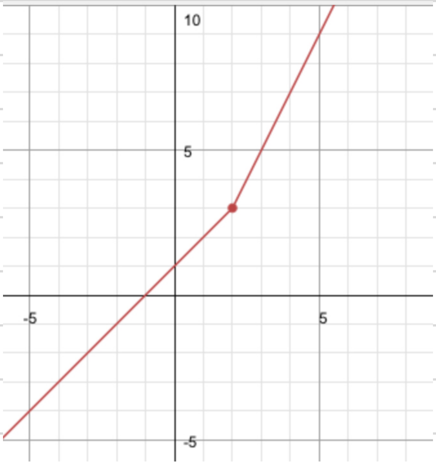
$$h(x) = \begin{cases} x+1 & , \quad x < 2 \\ 2x-1 & , \quad x \geq 2 \end{cases}$$

1)- $f(2) = 2(2)-1 = 3$

2)- $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x+1 = 3$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 2x-1 = 3$

} exsit and equals



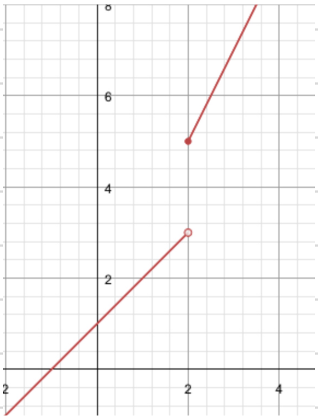
$\lim\limits_{x \to 2} f(x) = 3.$

3)- $\lim\limits_{x \to 2} f(x) = 3 = f(2)$

$\therefore$ f is continuous at $x = 2$.

$$f(x) = \begin{cases} x+1 & , x<2 \\ 2x+1 & , x \geq 2 \end{cases}$$

1)- $f(2) = 2.2+1 = 5.$



2) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+1 = 3.$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x-1 = 5$

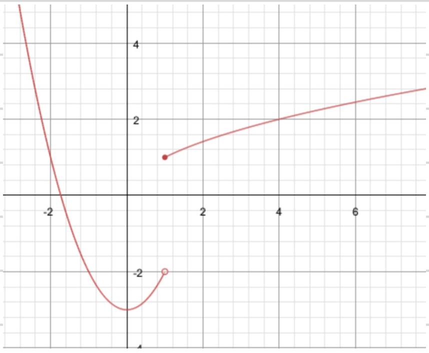
exsit but not equals

$\therefore \lim_{x\to 2} f(x)$ DNE.

$\therefore$ f has Jump discontinuity.

$$h(x) = \begin{cases} x^2-3 & \text{if } x<1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

1)- $(1) = \sqrt{1} = 1$



2)- $\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} x^2-3 = -2$

$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} \sqrt{x} = 1$

exsit but not equals

$\therefore \lim_{x\to 1} h(x)$ DNE

$\therefore$ f has Jump discontinuity.

$F(x) = \frac{x}{x-1}$ at $x = 1$.

1)_ $F(1) = \frac{1}{1-1} = \frac{1}{0}$ (undefined.)

2)_ $\lim_{x \to 1} F(x) = \lim_{x \to 1} \frac{x}{x-1}$ (limit form $\frac{1}{0}$)

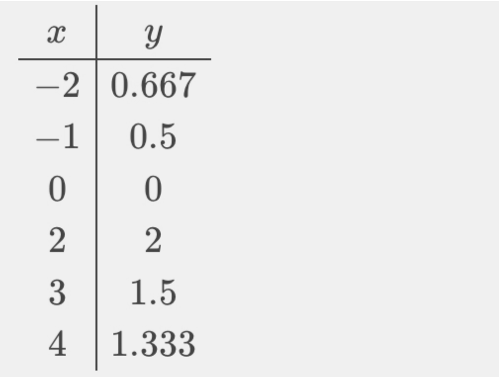
$= \infty$

Remember :-

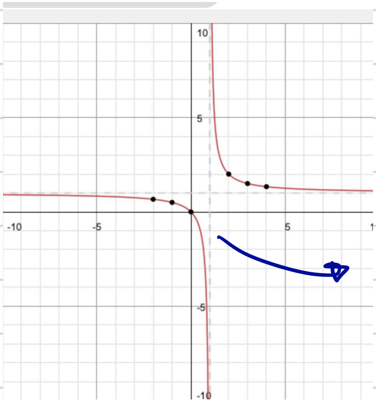
$\frac{x}{0} = \infty$

$\frac{x}{\infty} = 0$

$\therefore$ F has infinite discontinuity.



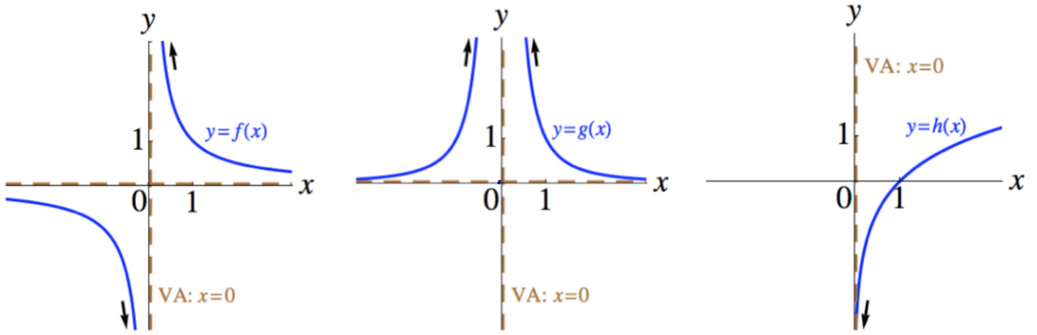
x	y
-2	0.667
-1	0.5
0	0
2	2
3	1.5
4	1.333



VA at $x = 1$

More Examples :-

$f(x) = \frac{1}{x}$ $\quad$ $g(x) = \frac{1}{x^2}$ $\quad$ $h(x) = \ln x$



$\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

} DNE

$\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0}$

$= \infty$

$\lim_{x \to 0} \ln(x) = \ln(0)$

undefined.

Continuity on open interval.

Definition: A function f is continuous on the (a,b) iff f is continuous at every point in (a,b).

Remark:- A function f that is continuous on the entire line $(-\infty, \infty)$ is everywhere continuous.

Theorem 2.4.2: ***[Basic Continuous Functions]***
The following types of function are continuous at every point in their domains.

1. Polynomial functions: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ $D = R$
2. Rational functions: $r(x) = \dfrac{p(x)}{q(x)}$, $p(x)$ and ,$q(x)$ are polynomials. $D = R - \{$أصفار المقام$\}$
3. Radical functions: $f(x) = \sqrt[n]{x}$ n → even → ما تحت الجذر ≥ 0; odd → $D = (-\infty, \infty) = R$
4. Trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, $\cot x$ $D = R$
5. Inverse Trigonometric functions: $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec\ ^{1}x$, $\csc^{-1}x$, $\cot^{-1}x$
6. Exponential functions: e^x, a^x $a > 0$ $D = R = (-\infty, \infty)$
7. Logarithmic functions: $\ln x$, $\log_a x$ ln ما بداخل > 0
8. Hyperbolic functions: $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{csch} x$, $\coth x$
9. Inverse Hyperbolic functions: $\sinh^{-1}x$, $\cosh^{-1}x$, $\tanh^{-1}x$, $\operatorname{sech}^{-1}x$, $\operatorname{csch}^{-1}x$, $\coth^{-1}x$

Example: Find the intervals in which each the following function is continuous.

$f(x) = x^2 + 2x + 1$

$f(x)$ is continnous everywhere. i.e

$f(x)$ is continnous on $R = (-\infty, \infty)$

$f(x) = \dfrac{x}{x^2 - 6x + 9}$

f is continuous on $R - \{$ أصفار المقام $\}$

$\therefore \quad x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0 \Rightarrow x = 3$

$\therefore \quad f$ is continuous on $R - \{3\} = (-\infty, 3) \cup (3, \infty)$

odd ←

$F(x) \quad \sqrt[5]{x+2}$

f is continous on $R = (-\infty, \infty)$

even

$f(x) = \sqrt{x(x-1)}$

f is continous if $x(x-1) \geq 0$

$x \geq 0$ or $x-1 \geq 0 \Rightarrow x \geq 1$

$\therefore$ f is continous on $(-\infty, 0] \cup [1, \infty)$

even

$f(x) = \sqrt[4]{x+7}$

f is continous if $x+7 \geq 0$

$\Rightarrow x \geq -7$

$\therefore$ f is cont on $[-7, \infty)$

$f(x) = \ln \underline{(x+4)}$

f is continuous $\Leftrightarrow x+4 > 0$

$\Rightarrow x > -4$

$\therefore$ f is continuous on $(-4, \infty)$

HW. $f(x) = \frac{1}{x^2+1}$

Continuity on Closed Interval

A function f is continuous on closed interval $[a,b]$ iff

1)- f is continuous on (a,b)

2)- $\lim\limits_{x \to a^+} f(x) = f(a)$,

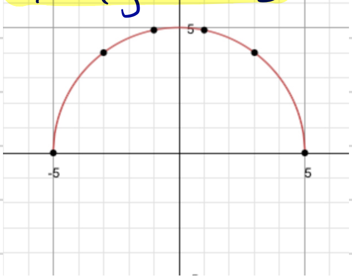
3)- $\lim\limits_{x \to b^-} f(x) = f(b)$.

Example: Discuss the continuity of $f(x) = \sqrt{25 - x^2}$

$y = \sqrt{25 - x^2}$

$y^2 = 25 - x^2$

$x^2 + y^2 = 25$



f is defined when $25 - x^2 \geq 0$

$\Rightarrow (5 - x)(5 + x) \geq 0$

$\Rightarrow 5 \geq 0$ or $x \geq -5$

$\Rightarrow -5 \leq x \leq 5$

$\therefore D(f) = [-5, 5]$.

1) f is continuous on $(-5, 5)$

2) $\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} \sqrt{25 - 25} = 0 = f(5)$

3)- $\lim\limits_{x \to -5^+} f(x) = \lim\limits_{x \to -5^+} \sqrt{25 - 25} = 0 = f(-5)$

H.W $f(x) = \sqrt{1 - x^2}$

One Sided Continuity

Right and left continuity:

- A function f is continuous from the right at a if

$$\lim_{x\to a^+} f(x) = f(a)$$

- A function f is continuous from the left at a if

$$\lim_{x\to a^-} f(x) = f(a)$$

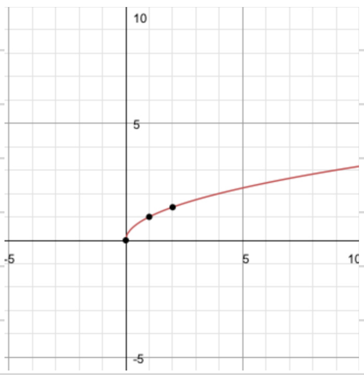
Example 1 :—

$f(x) = \sqrt{x}$

f is defined iff $x \geq 0$

$D(f) = [0,\infty)$

$$\left.\begin{array}{l} f(0)=0 \\ \lim\limits_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0 \end{array}\right\} \checkmark$$



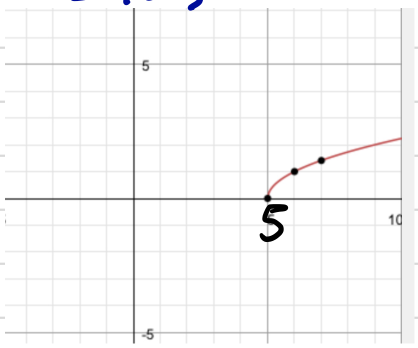
$f(x) = \sqrt{x-5}$

f is defined iff

$x-5 \geq 0 \Rightarrow x \geq 5$

$\therefore D(f) = [5,\infty)$

f is continuous from the right on $[5,\infty)$



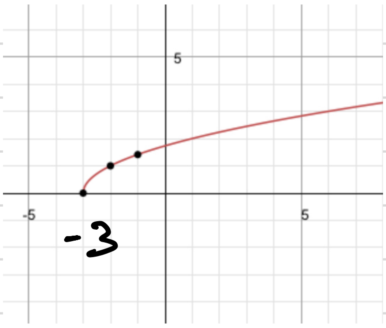
$f(x) = \sqrt{x+3}$

f is defined iff

$x+3 \geq 0 \Rightarrow x \geq -3$

$\therefore D(f) = [-3,\infty)$

f is continuou from the right on $[-3,\infty)$



Greatest integer function

The greatest integer function $[x]$ is the largest integer less than or equal to x.

$[x] = n \Leftrightarrow n \leq x < n+1$

$[2.9] = 2$	$[0] = 0$	$[1.4] = 1$	$[3] = 3$
$[-2.51] = -3$	$[-0.5] = -1$	$[-1.01] = -2$	$[-2] = -2$

$\lim_{x\to 1^-} [x] = 0$	$\lim_{x\to 1} [x] = \text{DNE}$	$\lim_{x\to 1^+} [x] = 1$
$\lim_{x\to 2^-} [x] = 1$	$\lim_{x\to 2} [x]$ DNE	$\lim_{x\to 2^+} [x] = 2$
$\lim_{x\to 3^-} [x] = 2$	$\lim_{x\to 3} [x]$ DNE	$\lim_{x\to 3^+} [x] = 3$
$\lim_{x\to n^-} [x] = n-1$	$\lim_{x\to n} [x]$ DNE	$\lim_{x\to n^+} [x] = n$

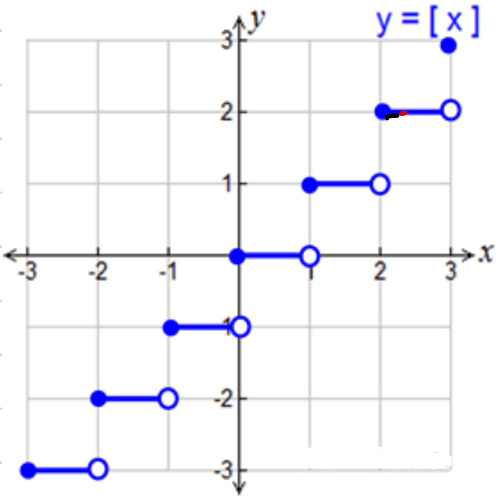
Discuss the continuity of $g(x) = [x]$ at $a = n$ ← عدد صحيح

① $g(n) = [n] = n$ ✓

$\lim_{x\to n^+} [x] = n$ ✓

$\lim_{x\to n^-} [x] = n-1$

} exsist but not equal



∴ g is discontinuous at $a = n$ or

g has jump discontinuity.

g is cont from the right

$\lim_{x\to n^+} [x] = n = g(n)$

Remark:-

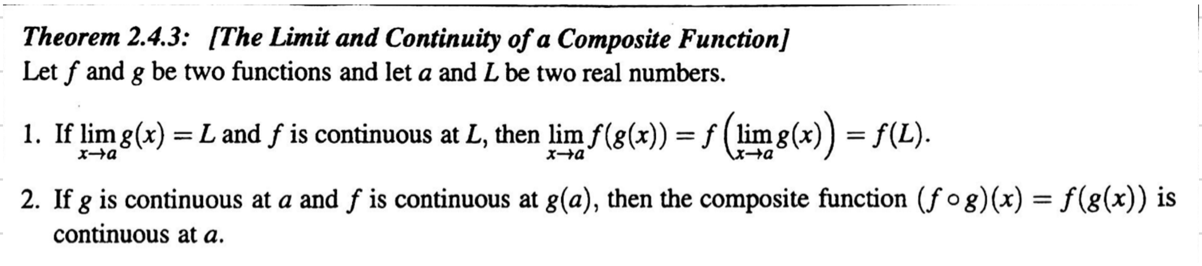
1)- There is a jump at each integer and so

$$\lim_{x \to n^+} [x] \neq \lim_{x \to n^-} [x]$$

2)- What about if a is not integer i.e $a = 1.5$

💡 Does $g(x) = [x]$ is continuous at $a = 1.5$.

Continuity of Composite of Function



***Theorem 2.4.3:** [The Limit and Continuity of a Composite Function]*
Let f and g be two functions and let a and L be two real numbers.

1. If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$.
2. If g is continuous at a and f is continuous at $g(a)$, then the composite function $(f\circ g)(x) = f(g(x))$ is continuous at a.

For Example:-

$f(x) = \sin\left(\frac{1}{x}\right)$ $\quad\quad$ $f = g(h(x))$

$g(x) = \sin x$ is continuous on $R = (-\infty,\infty)$

$h(x) = \frac{1}{x}$ is Continuous on $R-\{0\} = (-\infty,0)\cup(0,\infty)$

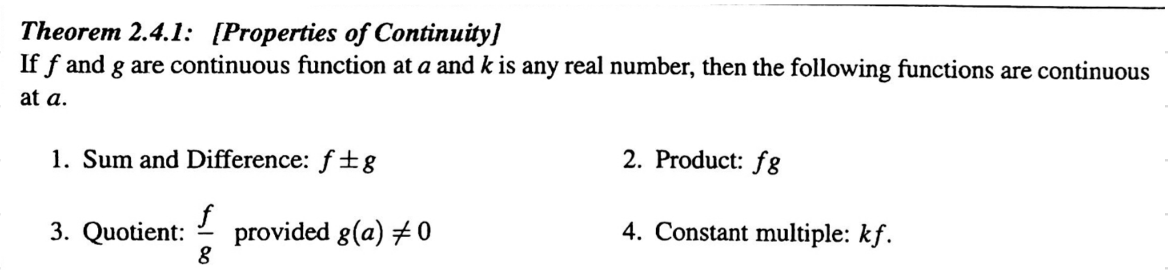
$\therefore$ f is continuous on $(-\infty,0)\cup(0,\infty)$.

$f(x) = \frac{\sin x}{x}$

$g(x) = \sin x$ is continuous on R.

$h(x) = \frac{1}{x}$ is continuous on $R-\{0\}$

$\therefore$ f is continuous on $R-\{0\} = (-\infty,0)\cup(0,\infty)$



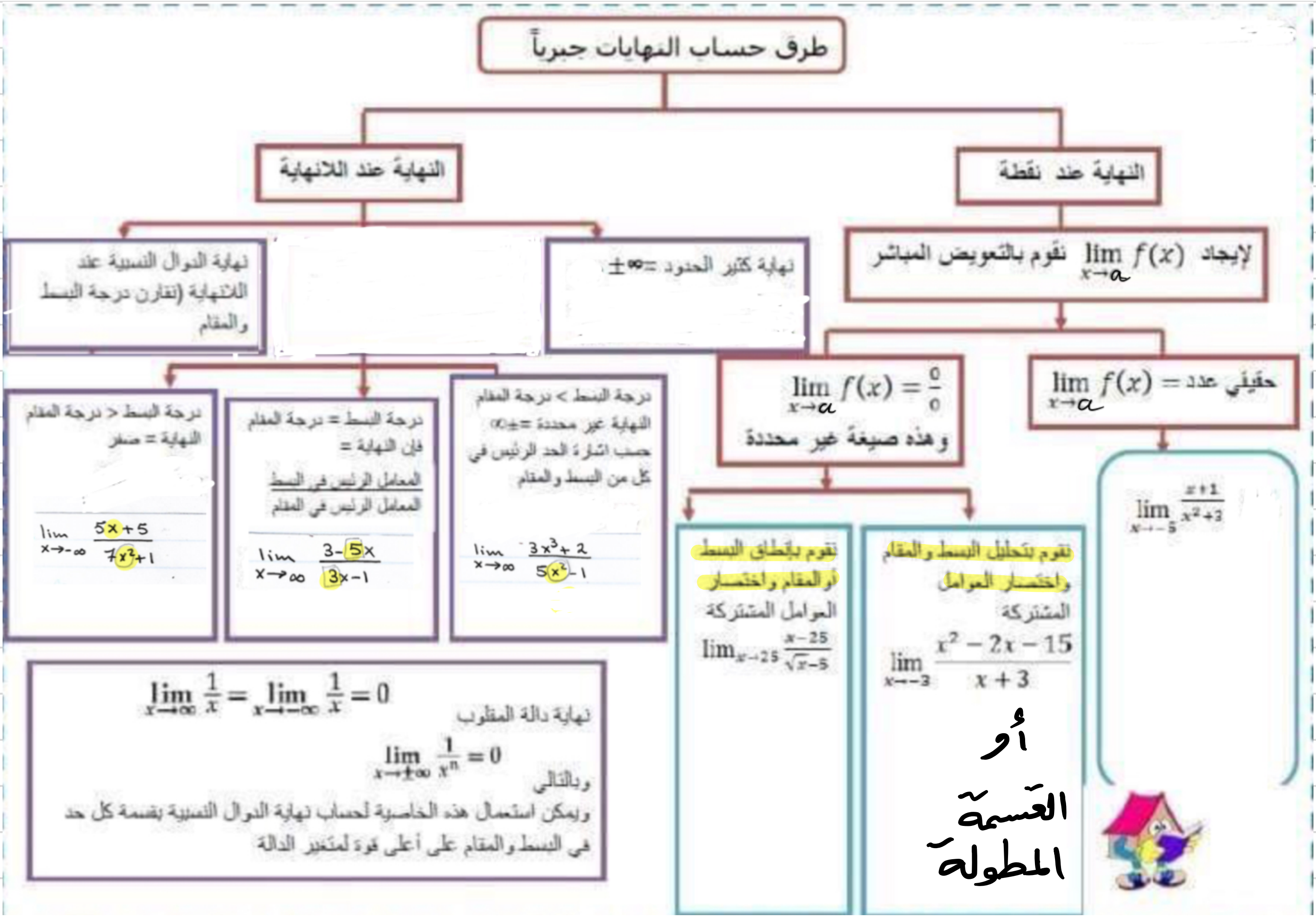
***Theorem 2.4.1:** [Properties of Continuity]*
If f and g are continuous function at a and k is any real number, then the following functions are continuous at a.

1. Sum and Difference: $f \pm g$
2. Product: fg
3. Quotient: $\frac{f}{g}$ provided $g(a) \neq 0$
4. Constant multiple: kf.

Some background

angle	0°	30°	45°	60°	90°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cos	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
tan	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{2}}$	$\sqrt{\frac{3}{1}}$	▪

angle	0°	30°	45°	60°	90°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\sqrt{\frac{1}{3}}$	1	$\sqrt{3}$	▪



طرق حساب النهايات جبرياً

النهاية عند نقطة

لإيجاد $\lim_{x\to a} f(x)$ نقوم بالتعويض المباشر

$\lim_{x\to a} f(x) =$ عدد حقيقي

$\lim_{x\to -5} \frac{x+1}{x^2+3}$

$\lim_{x\to a} f(x) = \frac{0}{0}$ وهذه صيغة غير محددة

نقوم بتحليل البسط والمقام واختصار العوامل المشتركة

$\lim_{x\to -3} \frac{x^2-2x-15}{x+3}$

أو القسمة المطولة

نقوم بإنطاق البسط أو المقام واختصار العوامل المشتركة

$\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}$

النهاية عند اللانهاية

نهاية كثير الحدود $=\pm\infty$

نهاية الدوال النسبية عند اللانهاية (تقارن درجة البسط والمقام

درجة البسط > درجة المقام النهاية غير محددة $=\pm\infty$ حسب اشارة الحد الرئيس في كل من البسط والمقام

$\lim_{x\to\infty} \frac{3x^3+2}{5x^2-1}$

درجة البسط = درجة المقام فإن النهاية = $\frac{\text{المعامل الرئيس في البسط}}{\text{المعامل الرئيس في المقام}}$

$\lim_{x\to\infty} \frac{3-5x}{3x-1}$

درجة البسط < درجة المقام النهاية = صفر

$\lim_{x\to-\infty} \frac{5x+5}{7x^2+1}$

نهاية دالة المقلوب

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$$

وبالتالي

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

ويمكن استعمال هذه الخاصية لحساب نهاية الدوال النسبية بقسمة كل حد في البسط والمقام على أعلى قوة لمتغير الدالة

هذه المعلومات تم دراستها في رياضيات ١ . ونحتاج هذه المعلومات لحل نهايات
الدوال المركبة في رياضيات 2 هذا ملخص لما تم دراسته و لمعلومات اكثر يمكنك
مراجعة المحتوى في ملف Limits على
https://t.me/MadaAltiary

Limits of composite function

Find the limits:

$$\lim_{x\to0^+}\cos\left(\frac{\pi}{3}e^{\sqrt{x}}\right) = \cos\left(\frac{\pi}{3}\lim_{x\to0^+}e^{\sqrt{x}}\right)$$

$$= \cos\left(\frac{\pi}{3}e^{\sqrt{0}}\right)$$

$$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\lim_{x\to1}\sin^{-1}\left(\frac{1-x}{1-x^2}\right) = \sin^{-1}\left(\lim_{x\to1}\frac{1-x}{1-x^2}\right)$$

$$= \sin^{-1}\left(\lim_{x\to1}\frac{1-x}{(1-x)(1+x)}\right)$$

$$= \sin^{-1}\left(\lim_{x\to1}\frac{1}{1+x}\right)$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\lim_{x\to\infty}\log_3(x^2-1) = \log_3\left(\lim_{x\to\infty}x^2-1\right)$$

$$= \log_3(\infty^2-1)$$

$$= \log_3(\infty) = \infty$$

$3^\infty = \infty$

$$\lim_{x\to\infty} \ln\left(\frac{4+x}{x-1}\right) = \ln\left(\lim_{x\to\infty} \frac{4+x}{x-1}\right)$$

درجة البسط = درجة المقام.

$$= \ln(1) = 0$$

$$\lim_{x\to\infty} \cos^{-1}\left(\frac{2+x}{2x+1}\right) = \cos^{-1}\left(\lim_{x\to\infty} \frac{2+x}{2x+1}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\lim_{x\to 2} \sin\left(\frac{\pi(x-2)}{x^2-4}\right) = \sin\left(\lim_{x\to 2} \frac{\pi(x-2)}{x^2-4}\right)$$

$$= \sin\left(\lim_{x\to 2} \frac{\pi\,\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}\right)$$

$$= \sin\left(\lim_{x\to 2} \frac{\pi}{x+2}\right)$$

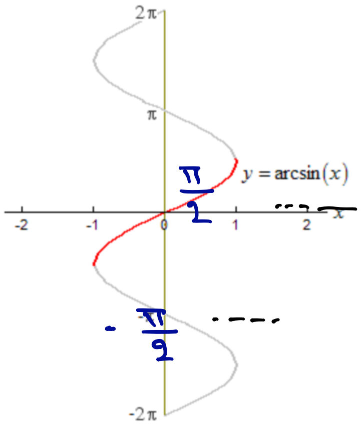
$$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\lim_{x\to\infty} \cos\left(\frac{\pi(2x^2-2)}{x^2-4}\right) = \cos\left(\lim_{x\to\infty} \frac{\pi(2x^2-2)}{x^2-4}\right)$$

$$= \cos(2\pi) = 1$$

$$\lim_{x \to 1^-} \sin^{-1} x = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\lim_{x \to -1^-} \sin^{-1}(x) = \sin^{-1}(-1) = -\frac{\pi}{2}.$$



$$\lim_{x \to -4^-} \tan^{-1} \sqrt[5]{\frac{x-3}{x+4}}$$

$$= \tan^{-1} \left(\lim_{x \to -4^-} \sqrt[5]{\frac{x-3}{x+4}} \right)$$

$$= \tan^{-1} \left(\sqrt[5]{\lim_{x \to -4^-} \frac{x-3}{x+4}} \right)$$

$$= \tan^{-1} \left(\sqrt[5]{\lim_{x \to -4^-} x-3 \cdot \lim_{x \to -4^-} \frac{1}{x+4}} \right.$$

$$= \tan^{-1} \left(\sqrt[5]{-7 \cdot -\infty} \right)$$

$$= \tan^{-1} \left(\sqrt[5]{\infty} \right)$$

$$= \tan^{-1} (\infty) = \frac{\pi}{2}.$$

$$\lim_{x\to 0} \cos\left(\frac{\pi}{\sqrt{17-\sec x}}\right)$$

$$= \cos\left(\lim_{x\to 0} \frac{\pi}{\sqrt{17-\sec x}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{17-\lim_{x\to 0} \sec x}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{17-1}}\right)$$

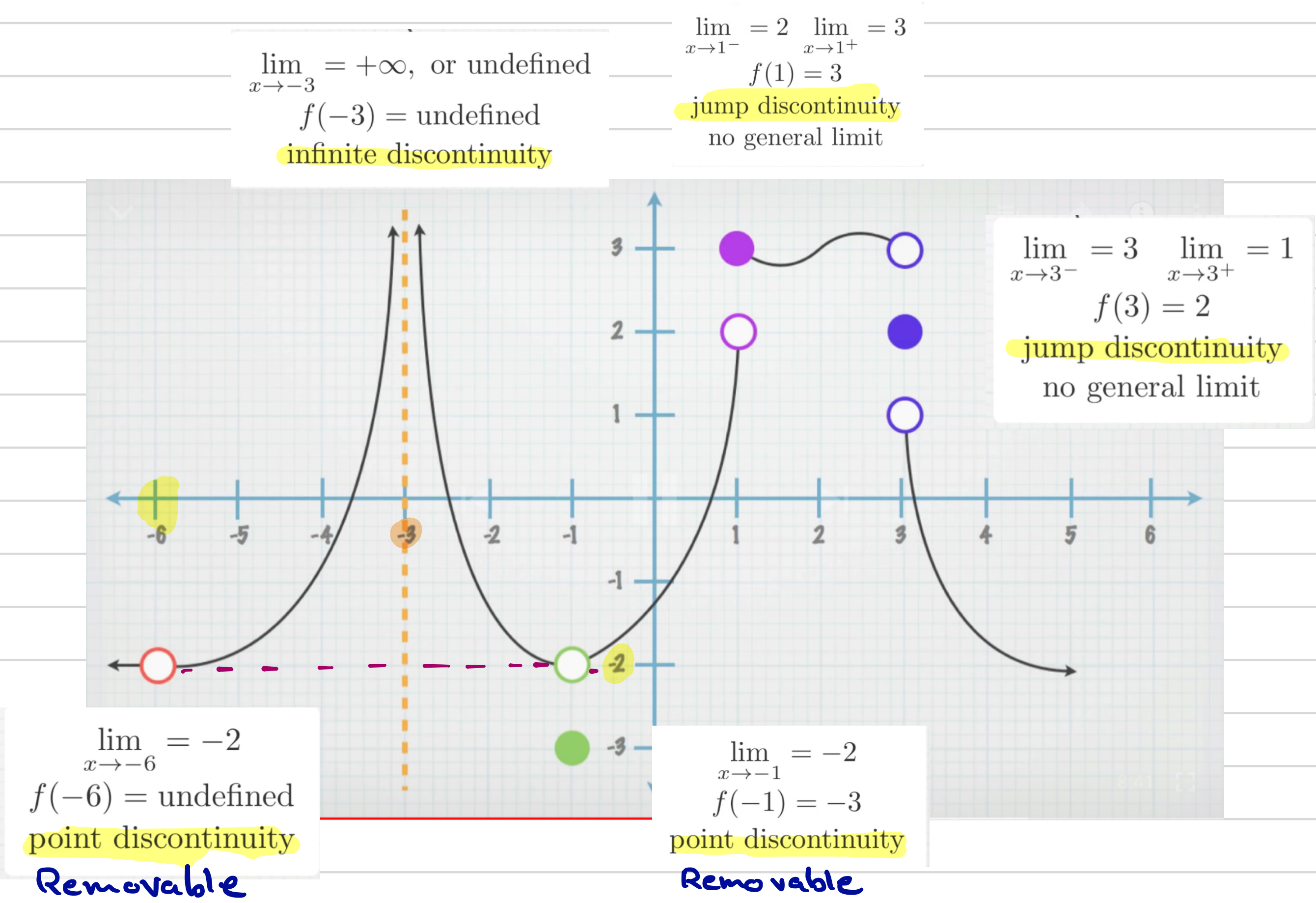
$$= \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Remember:

$$\sec x = \frac{1}{\cos x}$$

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

تمارين على الإتصال بيانيا



$\lim\limits_{x\to-3} = +\infty$, or undefined

$f(-3) = \text{undefined}$

infinite discontinuity

$\lim\limits_{x\to1^-} = 2 \quad \lim\limits_{x\to1^+} = 3$

$f(1) = 3$

jump discontinuity

no general limit

$\lim\limits_{x\to3^-} = 3 \quad \lim\limits_{x\to3^+} = 1$

$f(3) = 2$

jump discontinuity

no general limit

$\lim\limits_{x\to-6} = -2$

$f(-6) = \text{undefined}$

point discontinuity

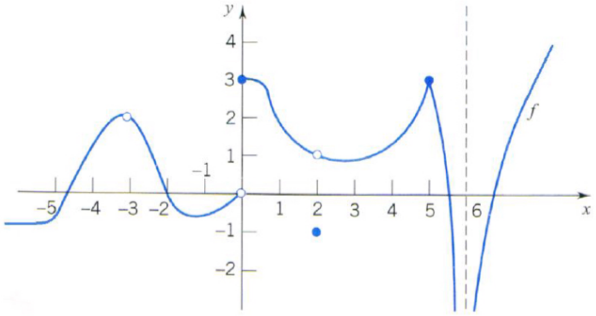
Removable

$\lim\limits_{x\to-1} = -2$

$f(-1) = -3$

point discontinuity

Removable



Let's investigate at the flowing points:

$x = -3$ Discontinuous at this point *The value is not defined at -3* "Removable discontinuity"	$x = 0$ Discontinuous at this point *The limit of the left is not equal to the limit from the right* "Jump discontinuity"	$x = 2$ Discontinuous at this point *The limit from the left is equal to the right, but is not equal to the value of the function* "Removable discontinuity"
$x = 4$ Continuous at this point *The limit from the left is equal to the limit from the right and equal to the value of the function*	$x = 5$ Continuous at this point *The limit from the left is equal to the limit from the right and equal to the value of the function*	$x = 6$ Discontinuous at this point *The value of the limit is equal to negative infinity and therefore not defined* "Infinite discontinuity"

Discuss the continuity of the following functions:

$$h(x) = \begin{cases} \dfrac{x^3-1}{x-1} & x<1 \\ -x^2+2x+2 & x \geq 1 \end{cases}$$

$$h(1) = (-1)^2 + 2(1) + 2 = 3$$

$$\lim_{x\to 1^+} h(x) = \lim_{x \to +1-} -x^2 + 2(1) + 2 = 3.$$

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} \frac{x^3-1}{x-1} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1^-} \frac{(x^2+x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x\to 1^-} x^2+x+1 = 3$$

$$\Rightarrow \lim_{x\to 1} h(x) = 3$$

$$\because h(1) = \lim_{x\to 1} h(x)$$

$\therefore$ f is continuous at $x = 1$

حل case 0/0
بالقسمة المطولة

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 + 1 \\ & -x^3 \pm x^2 \\ \hline & x^2 + 1 \\ & -x^2 \pm x \\ \hline & x + 1 \\ & -x \mp 1 \\ \hline & 0 \quad 0 \end{array}$$

$$\therefore x^3+1 = (x^2+x+1)(x-1)$$

or :- حل case 0/0 بطريقه أخرى تعرف بطريقة لوبيتال سوف ندرسها لاحقا

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\overset{\text{تفاضل البسط}}{3x^2}}{\underset{\text{تفاضل المقام}}{1}} = 3(1)^2 = 3$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$g(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

$g(0) = 1$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin x}{x} \qquad \left[\tfrac{0}{0} \text{ case}\right]$$

$$= \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1$$

Rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$\because g(0) = 1 = \lim_{x \to 0} g(x)$

$\therefore$ g is continuous.

$$F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$F(0) = 0$

undefined

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = DNE \qquad \sin\left(\frac{1}{0}\right)$$

$\therefore$ f is discontinuity at $x=0$.

$$h(x) = \begin{cases} \dfrac{x^2-a^2}{x-a} & , x \neq a \\ 6 & , x = a \end{cases}$$

$$h(a) = 6 = \lim_{x \to a} h(x)$$

$$= \lim_{x \to a} \frac{x^2-a^2}{x-a} \qquad \left[\tfrac{0}{0} \text{ , factor"}\right]$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{x-a}}$$

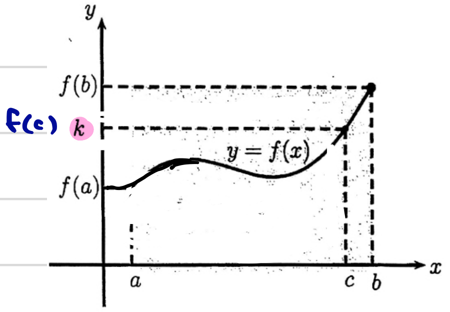
$$= \lim_{x \to a} x+a = 2a$$

$$\Rightarrow 6 = 2a \Rightarrow a = 3$$

The Intermediate Value Theorem

I.V.T

Theorem 2.4.4: **[Intermediate Value Theorem]** If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c)=k$.



Example: Show that the function $f(x)=x^3+2x^2-1$ has a zero in the interval $[0,1]$

$f(c)=0$

$f(0)=0^3+2(0)^2-1=-1$

$f(1)=1^3+2(1)-1=2$

اذا كانت f داله متصلة في الفترة [a,b] وكان k عدد
حقيقي محصور بين f(a) و f(b) فإنه يوجد على الأقل
عدد c واحد في الفترة [a,b] بحيث f(c)=k

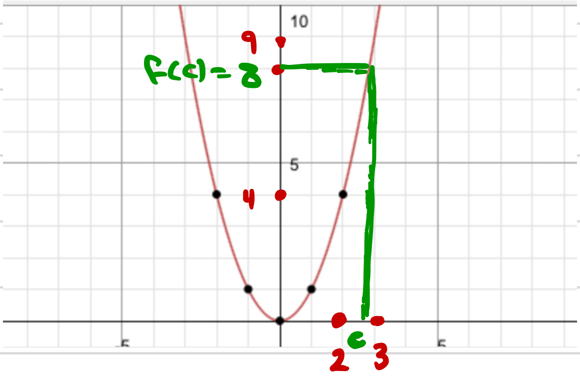
$f(0)=-1<0<2=f(1).$ ✅

Example: show that the function $f(x)=x^2$ has value 8 in $[2,3]$

$f(2)=2^2=4$

$f(3)=3^2=9$

$4<8<9$ ✅



ماهي قيمة c التي اذا عوضنا بها في
المعادلة f(c)=8

$f(c)=c^2$

$8=c^2$

$\sqrt{8}=c$

$2\sqrt{2}=c \in [2,3]$

$f(2\sqrt{2})=(2\sqrt{2})^2=4.2$

$=8$ ✓

ملاحظه نظرية القيمة المتوسطه تؤكد فقط
وجود حل للمعادلة f(c)=k دون الحاجه الى
تعيين قيمة c

$f(c) = 3$

💡 **What about** $g(x) = x^2$ has value 3 in $[2,3]$?

$$g(2) = 4$$
$$g(3) = 9$$

but

3 is not in the interval $[4, 9]$. ❎

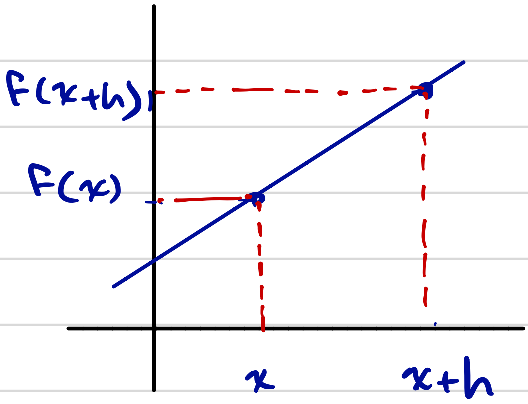
Example: Show that the function $f(x) = x^3 + x$ has value of 9 in the interval $[1, 2]$.

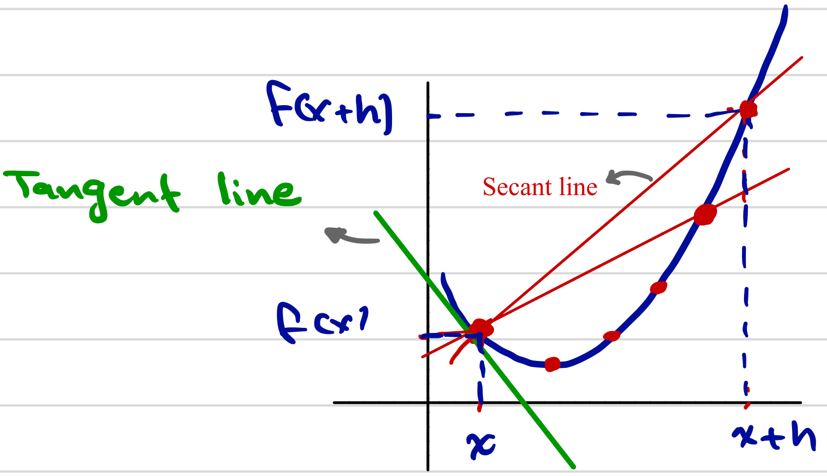
$f(c) = 9$

$$f(1) = 1^3 + 1 = 2$$
$$f(2) = 2^3 + 2 = 10$$

$$\underset{f(1)}{2} < \underset{f(c)}{9} < \underset{f(b)}{10}$$ ✅

Definition of the Derivative





Slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h)-f(x)}{\cancel{x}+h-\cancel{x}}$$

$$= \frac{f(x+h)-f(x)}{h}$$

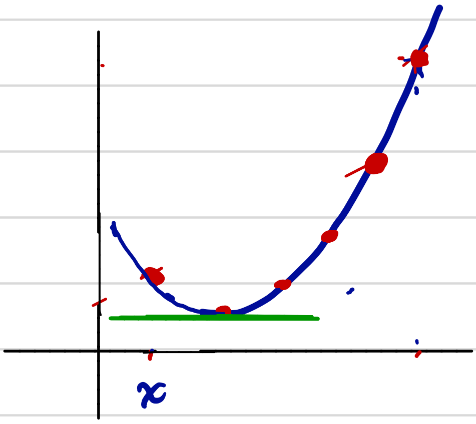
$$m = \underbrace{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

تعريف التفاضل

التفاضل = ميل المماس لمنحنى الداله عند نقطه

$$\therefore m = f'(a)$$

💡 What is happen if the tangent line horizontal



The tangent line is horizontal line

$$m = 0$$

Example 1: Let $f(x) = 2x^2$ and $a = 2$. Find $f'(a)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

✓ $f(x+h) = 2(x+h)^2 = 2(x^2+2xh+h^2)$

$$= 2x^2+4xh+2h^2$$

✓ $f(x) = 2x^2$

✓ $$\frac{f(x+h)-f(x)}{h} = \frac{\cancel{2x^2}+4xh+2h^2-\cancel{2x^2}}{h}$$

$$= \frac{4xh+2h^2}{h} = \frac{2\cancel{h}(2x+h)}{\cancel{h}} = 4x+h$$

✓ $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 4x+h = 4x.$$

✓ $f'(a) = f'(2) = 4.2 = 8.$

Example 2: let $f(x) = \frac{1}{x}$ and $a = 1$, find $f'(a)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$f(x+h) = \frac{1}{x+h}$ and $f(x) = \frac{1}{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \frac{\frac{\cancel{x}-\cancel{x}-h}{x(x+h)}}{h}$$

$$= \frac{-\cancel{h}}{(x+h)x} \cdot \frac{1}{\cancel{h}} = \frac{-1}{(x+h)x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

$$f'(a) = \frac{-1}{a^2} = \frac{-1}{(1)^2} = -1$$

Example 3: Let $f(x) = \sqrt{x}$. Find the equation of the tangent to the graph of $f(x)$ at $x = 4$.

$$y - y_1 = m(x - x_1)$$

m = slope

(x_1, y_1) points.

$m = f'(x)$ at $x = 4$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$\left[\frac{0}{0}\right] \rightarrow$ conjugat method

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} =$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$\therefore f'(4) = \frac{1}{\sqrt{4}+\sqrt{4}} = \frac{1}{2+2} = \frac{1}{4} \rightarrow m$$

$$\therefore y - y_1 = m(x - x_1)$$

$x_1 = 4$

$f(x_1) = f(4) = \sqrt{4}$

$= 2$

$\uparrow$
y_1

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}(x-4) + 2$$

$$= \frac{1}{4}x - \frac{1}{4}\cdot 4 + 2$$

$$= \frac{1}{4}x - 1 + 2$$

$$\therefore y = \frac{1}{4}x + 1$$

$f'(x)$, y' $f(x)$

Example 4: Find the derivative of $y = 2x^2 + 3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 2(x+h)^2 + 3 = 2(x^2 + 2xh + h^2) + 3$$

$$= 2x^2 + 4xh + 2h^2 + 3$$

$$f(x) = 2x^2 + 3$$

$$\therefore \frac{f(x+h) - f(x)}{h} = \frac{2x^2 + 4xh + 2h^2 + 3 - (2x^2 + 3)}{h}$$

$$= \frac{\cancel{2x^2}+4xh+2h^2\cancel{+3}\cancel{-2x^2}\cancel{-3}}{h}$$

$$= \frac{4xh+2h^2}{h} = \frac{2\cancel{h}(2x+h)}{\cancel{h}} = 2(2x+h).$$

$$\therefore f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 2(2x+h) = 4x.$$

Example5: Let $f(x)=\sqrt{x-1}$. Sowh that $f'(x) = \frac{1}{2\sqrt{x-1}}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h}$$

[$\frac{0}{0}$ conjugate method]

↓

$$= \lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h}\cdot\frac{\sqrt{x+h-1}+\sqrt{x-1}}{\sqrt{x+h-1}+\sqrt{x+1}}$$

$$= \lim_{h\to 0}\frac{x+h-1-(x-1)}{h(\sqrt{x+h-1}+\sqrt{x-1})}$$

$$= \lim_{h\to 0}\frac{\cancel{x}+h\cancel{-1}\cancel{-x}\cancel{+1}}{h(\sqrt{x+h-1}+\sqrt{x-1})} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h-1}+\sqrt{x-1}}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h-1}+\sqrt{x+1}} = \frac{1}{\sqrt{x-1}+\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

Basic Differentiation Rules

القوانين الأساسيه للتفاضل

For $y = f(x)$, all of the following are used to represent the derivative: $f'(x)$, y', $\frac{dy}{dx}$, $D_x y$, $\frac{d}{dx}[f(x)]$.

The constant Rule:

$f(x) = c$ then $f'(x) = 0$ Ex: $f(x) = e$, $f'(x) = 0$

Power Rule:

$f(x) = x^n$ then $f'(x) = n x^{n-1}$ Ex:

Radical Power Rule

$f(x) = x^{\frac{1}{n}}$ then $f'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$

Remark

$x^{1/n} = \sqrt[n]{x}$ and $x^{m/n} = \sqrt[n]{x^m}$

For example:

Let $f(x) = \sqrt{x}$. Find $f'(x)$.

$f(x) = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$

It is useful to know :

$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}\left[\sqrt{\text{مابداخل الجذر}}\right] = \frac{\text{تفاضل مابداخل الجذر}}{\text{الجذر نفسه* 2}}$

For example: $f(x) = \sqrt{x^2+1}$ then

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}}$$

Theorem 3.2.5: [The Constant Multiple, The Sum and Difference Rules]
Let c be a constant. If $f(x)$ and $g(x)$ are differentiable, then $cf(x)$ and $f(x)\pm g(x)$ are also differentiable, and

i) $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}(f(x))$

ii) $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

iii) $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$.

Example :- Let $f(x) = 2x^3 - \sqrt{x}$. Find $f'(x)$.

$$f'(x) = 6x^2 - \frac{1}{2\sqrt{x}}$$

Example : Let $f(x) = x^2 + 4^3$. Find $f'(x)$.

$\xrightarrow{}$ عدد ثابت

$$f'(x) = 2x + 0 = 2x$$

Example : let $y = 3x^4$. Find y'

$$y' = 3.4\,x^3 = 12x^3 .$$

Product Rule :

$$(f(x)\,g(x))' = f(x).g'(x) + g(x).f'(x)$$

For example: $f(x) = (3x - 2x^2)(5+4x)$

$$f'(x) = (3x-2x^2)(5+4x)' + (5+4x).(3x-2x^2)'$$

$$= (3x-2x^2)(4) + (5+4x)(3-4x)$$

$$= (12x - 8x^2) + (15 - 20x + 12x - 16x^2)$$

$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$

$$= -24x^2 + 4x + 15$$

Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{g^2(x)}$

For example:

$$f(x) = \frac{3x - 2x^2}{5 + 4x}$$

$$f'(x) = \frac{(3x - 2x^2)'(5+4x) - (5+4x)'(3x-2x^2)}{(5+4x)^2}$$

$$= \frac{(3 - 4x)(5+4x) - (4)(3x - 2x^2)}{(5+4x)^2}$$

$$= \frac{15 + \cancel{12x} - 20x - 16x^2 - \cancel{12x} + 8x^2}{(5+4x)^2}$$

$$= \frac{15 - 20x - 8x^2}{(5+4x)^2}$$

Derivative of Exponential, Trigonometric and logarithmic functions

Exponential function

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[a^x] = a^x \ln a$$

Properties:

$$\lim_{x\to\infty} e^x = \infty$$

$$\lim_{x\to-\infty} e^x = 0$$

$$e^x \cdot e^y = e^{x+y}$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$e^{-x} = \frac{1}{e^x}$$

Trig.function

$$\frac{d}{dx}[\cos x] = -\sin x$$

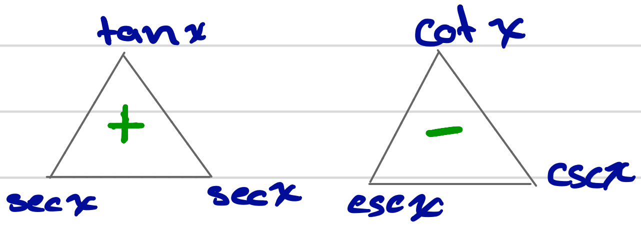
$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$



Log.function

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

In general :

$$\frac{d}{dx}[\log_a (g(x))] = \frac{g'(x)}{g(x)\ln a}$$

$$\frac{d}{dx}[\ln (g(x))] = \frac{g'(x)}{g(x)}$$

Exponential Function

Example: Find y' if $y = 5^x$

$$y' = 5^x \ln 5$$

Example: Find y' if $y = x^2 e^x$

الدالة عبارة عن حاصل
ضرب دالة أسيه بدالة
كثيرة حدود لذلك نطبق
قانون تفاضل حاصل
ضرب دالتين

$$y' = (x^2)' e^x + (x^2)(e^x)'$$

$$\frac{d}{dx} = \frac{d}{dx}(x^2)$$

$$= 2x\, e^x + x^2 e^x$$

$$= e^x(2x + x^2)$$

Example: Find y' if $y = \dfrac{3^x}{x+e^x}$

$$y' = \frac{(3^x)' \cdot (x+e^x) - (x+e^x)' \cdot 3^x}{(x+e^x)^2}$$

$$= \frac{3^x \ln 3\, (x+e^x) - (1+e^x) \cdot 3^x}{(x+e^x)^2}$$

$$= \frac{3^x \ln 3x + 3^x \ln 3\, e^x - 3^x - 3^x e^x}{(x+e^x)^2}$$

$$= \frac{3^x (\ln 3x - 1) + 3^x e^x (\ln 3 - 1)}{(x+e^x)^2}$$

$$= \frac{3^x (\ln 3x - 1) + (3e)^x (\ln 3 - 1)}{(x+e^x)^2}$$

(x,y)

Example: Find the points on the curve $y = x^2 e^x + 1$ at wich the tangent line is horizontal

$$y = x^2 e^x + 1$$

$$y' = (x^2)' e^x + x^2 (e^x)' + 0$$

$$= 2x e^x + x^2 e^x$$

$$= (x^2 + 2x) e^x$$

Horizontal tangents

$y' = 0$

$(x^2+2x)e^x = 0$

$x^2+2x = 0$

$x(x+2) = 0$

$x = 0$ or $x+2=0 \Rightarrow x=-2$

$\therefore$ Points: $(0, f(0))$ and $(-2, f(-2))$

$f(0) = 0^2 \cdot e^0 + 1 = 1$

$f(-2) = (-2)^2 e^{-2} + 1$

$= 4e^{-2} + 1$

$\therefore$ The curve has a horizontal line at $(0,1)$ and $(-2, 4e^{-2}+1)$

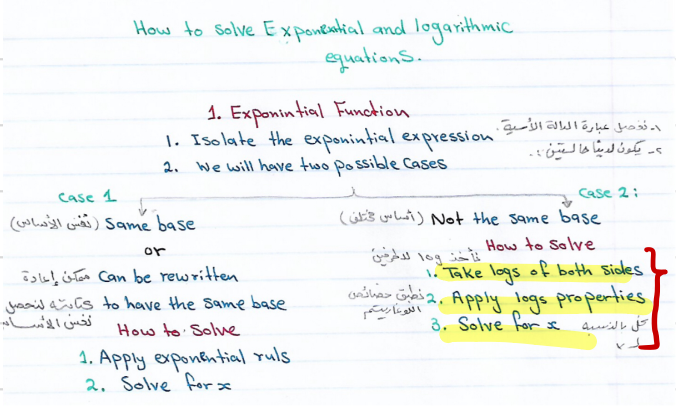
Example: For what value of x does the curve $f(x) = 2x - e^x$, have any horizontal tangents? Also for what value of x does the tangent line to the curve parallel to $y = -3x$.

$f'(x) = 0$ ← horizontal tangent

$2 - e^x = 0$

$2 = e^x$

$x = \ln 2$ → How → Remember from math 1



How to solve Exponential and logarithmic equations.

1. Exponintial Function

1. Isolate the exponintial expression ١- نفصل عبارة الدالة الأسية.
2. We will have two possible Cases ٢- يكون لدينا حالتين :

Case 1

(نفس الأساس) Same base

or

ممكن إعادة Can be rewritten

كتابته لنحصل to have the same base

نفس الأساس

How to Solve

1. Apply exponential ruls
2. Solve for x

Case 2:

(أساس مختلف) Not the same base

How to Solve

نأخذ log للطرفين

1. Take logs of both sides
2. Apply logs properties نطبق خصائص اللوغاريتم
3. Solve for x نحل بالنسبة لـ x

راجعي المعلومات في ملف Exp.F على الرابط التالي https://t.me/MadaAltiary

$f'(x) =$ slope of the given line $y = mx + b$ ← parallel tangent

$\because y = \overset{m}{-3}x$

$\therefore f'(x) = -3$

$2 - e^x = -3$

$2 + 3 = e^x$

$5 = e^x$

$\Rightarrow x = \ln 5$

Example: Find y' if

Remember:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sec x = \frac{1}{\cos x}$$

$$y = (\sin x + \cos x)\sec x$$

$$y = \sin x \sec x + \cos x \sec x$$

$$= \sin x \cdot \frac{1}{\cos x} + \cos x \cdot \frac{1}{\cos x}$$

$$= \frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}$$

$$= \tan x + 1$$

$$\because y = \tan x + 1 \Rightarrow y' = \sec^2 x$$

$$y = \tan x + \sqrt{x}$$

$$y' = \sec^2 x + \frac{1}{2\sqrt{x}}$$

$$y = x^2 \cos x - 2x \sin x$$

$$y' = 2x \cos x + x^2(-\sin x) - 2[1 . \sin x + x . \cos x]$$

$$= \cancel{2x \cos x} - x^2 \sin x - 2 \sin x - \cancel{2x \cos x}$$

$$= \sin x(-x^2 - 2)$$

$$y = \frac{\cot x}{1+\cot x}$$

$$y' = \frac{(-\csc^2 x)\cdot(1+\cot x) - (-\csc^2 x)(\cot x)}{(1+\cot x)^2}$$

$$= \frac{-\csc^2 x - \cancel{\csc^2 x \cot x} + \cancel{\csc^2 x \cot x}}{(1+\cot x)^2}$$

$$= \frac{-\csc^2 x}{(1+\cot x)^2}$$

$$y = \sin x \cos x$$

$$y' = (\sin x)'(\cos x) + (\sin x)(\cos x)'$$

$$= \cos x(\cos x) + \sin x(-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$y = \tan x + x^2 \cot x$$

$$y' = \sec^2 x + (2x\cot x + x^2(-\csc^2 x))$$

$$= \sec^2 x + 2x\cot x - x^2\csc^2 x$$

$$y = \frac{\sin x}{x}$$

$$y' = \frac{\cos x \cdot x - 1 \cdot \sin x}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

$$y = \sec x \tan x$$

$$y' = (\sec x)' \cdot (\tan x) + (\sec x) \cdot (\tan x)'$$

$$= (\sec x \tan x)(\tan x) + \sec x \cdot \sec^2 x$$

$$= \sec x \tan^2 x + \sec x \cdot \sec^2 x$$

$$= \sec x \, (\tan^2 x + \sec^2 x) .$$

$$y = \cos x \csc x$$

$$y = \cos x \cdot \frac{1}{\sin x}$$

$$y = \frac{\cos x}{\sin x} = \cot x$$

$$\therefore y' = -\csc^2 x$$

Remember

$$\csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$y = \sin x \csc x$

$y = \sin x \cdot \frac{1}{\sin x} = \frac{\sin x}{\sin x} = 1$

$\therefore y = 1$ and $y' = 0$

$y = \frac{\tan x}{\sec x}$

$y = \tan x \cdot \frac{1}{\sec x}$

$= \frac{\sin x}{\cos \cancel{x}} \cdot \cancel{\cos} x = \sin x$

$\therefore y' = \cos x .$

$y = \cos x \sec x$

$y = \cancel{\cos} x \cdot \frac{1}{\cancel{\cos x}} = 1$

$\therefore y' = 0$

Example: Find all points on the curve

$y = 3\tan x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

where the tangent line is parallel to the line $y = 6x$.

$f'(x) = 6$

$3\sec^2 x = 6$

$3 \frac{1}{\cos^2 x} = 6 \Rightarrow \frac{3}{\cos^2 x} = 6 \Rightarrow 6\cos^2 x = 3$

$\Rightarrow \cos^2 x = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \cos x = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \cos x = \frac{1}{\sqrt{2}}$ $[\cos x \geqslant 0 \text{ on } -\frac{\pi}{2} < x < \frac{\pi}{2}]$

$\Rightarrow x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \Rightarrow x = \pm \frac{\pi}{4}$

$\therefore f\left(\frac{\pi}{4}\right) = 3\tan\left(\frac{\pi}{4}\right) = 3 \cdot 1 = 3$

$f\left(-\frac{\pi}{4}\right) = 3\tan\left(-\frac{\pi}{4}\right) = 3 \cdot (-1) = -3$

$y - y_1 = m(x - x_1)$

$y - 3 = 6\left(x - \frac{\pi}{4}\right) \Rightarrow y = 6\left(x - \frac{\pi}{4}\right) + 3$

Derivatives Logarithmic Function

Example :

$y = \log_2(x)$

$y' = \frac{1}{x \ln 2}$

$y = \ln x$

$y' = \frac{1}{x}$

The Chain Rule

$$y = f(g(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad\qquad y' = f'(g(x)) \cdot g'(x)$$

Example: If $y = (x^2+1)^3$. Find y'

$\text{let } u = x^2+1 \quad , \quad y = u^3$

$\frac{du}{dx} = 2x \quad , \quad \frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3u^2 \cdot 2x$$

$$= 3(x^2+1) \cdot 2x$$

$$= 6x(x^2+1)$$

$$y' = 3(x^2+1) \cdot 2x$$

$$= 6x(x^2+1)$$

The General power Rule

Example: Find

$$y = \sqrt[3]{1+x^2}$$

$$y = (1+x^2)^{\frac{1}{3}}$$

$$\therefore y' = \frac{1}{3}(1+x^2)^{\frac{1}{3}-1} \cdot 2x$$

$$= \frac{1}{3}(1+x^2)^{-\frac{2}{3}} \cdot 2x$$

$$= \frac{2x}{3\sqrt[3]{(1+x^2)^2}}$$

$$y = \frac{1}{x^2-1}$$

$$y = (x^2-1)^{-1}$$

$$y' = -(x^2-1)^{-1-1} \cdot 2x$$

$$= -(x^2-1)^{-2} \cdot 2x$$

$$= \frac{-2x}{(x^2-1)^2}$$

Example: Let $g(x) = (3x+1)^6 \sqrt[3]{(2x-3)^5}$. Find $g'(x)$.

$$g(x) = (3x+1)^6 (2x-3)^{5/3}$$

$$g'(x) = [(3x+1)^6]' (2x-3)^{5/3} + (3x+1)^6 . [(2x-3)^{5/3}]'$$

$$= (6(3x+1)^5 \cdot 3)(2x-3)^{5/3} + (3x+1)^6 \cdot \frac{5}{3}(2x-3)^{\frac{5}{3}-1} \cdot 2$$

$$= 18(3x+1)^5 (2x-3)^{5/3} + (3x+1)^6 \cdot \frac{10}{3}(2x-3)^{2/3}$$

$$= 18(3x+1)^5 \sqrt[3]{(2x-3)^5} + (3x+1)^6 \cdot \frac{10}{3} \sqrt[3]{(2x-3)^2}$$

Example: Find all the points on the graph of

$$g(x) = \sqrt[3]{(x^2-4)^2}$$

for which $g'(x) = 0$ and those for which $g'(x)$ DNE.

$$g(x) = (x^2-4)^{\frac{2}{3}}$$

$$g'(x) = \frac{2}{3}(x^2-4)^{\frac{2}{3}-1} \cdot 2x$$

$$= \frac{4x}{3}(x^2-4)^{-\frac{1}{3}}$$

$$= \frac{4x}{3\sqrt[3]{x^2-4}}$$

$g'(x)=0 \iff \dfrac{4x}{3\sqrt[3]{x^2-4}}=0$ $\qquad \frac{a}{b}=0 \Rightarrow a=0$

$\iff 4x=0$

$\iff x=0$

$g'(x)$ DNE $\iff 3\sqrt[3]{x^2-4}=0$ $\qquad \frac{a}{b}$ DNE $\Rightarrow b=0$

3 مجرد عدد ثابت لايؤثر

$\iff x^2-4=0$

$\iff x=\pm 2$

Example : Find y' if $y=(3x-x^2+\sqrt{x})^6$

$$y'=6(3x-x^2+\sqrt{x})^5\cdot\left(3-2x+\frac{1}{2\sqrt{x}}\right)$$

Example : If $y=t^2$ and $x=\dfrac{t-1}{t+1}$ Find $\dfrac{dy}{dx}$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$y=t^2$ and $\dfrac{dy}{dt}=2t$

$x=\dfrac{t-1}{t+1}$ and

$$\frac{dx}{dt}=\frac{1\cdot(t+1)-1(t-1)}{(t+1)^2}=\frac{\cancel{t}+1-\cancel{t}+1}{(t+1)^2}=\frac{2}{(t+1)^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$-\cancel{2}t\cdot\frac{(t+1)^2}{\cancel{2}}=t(t+1)^2$$

قلبنا لناتج

Trig. Function and the chain Rule

Example:

Let $g(t) = 3t^2 - \cos(t)$ and $f(x) = \sec(x)$.

ⓘ Set $y = f(g(t))$. ② Find $\frac{dy}{dt}$

① $y = f(3t^2 - \cos t) = \sec(3t^2 - \cos t)$

② $$\frac{dy}{dt} = \underbrace{\sec(3t^2 - \cos t) . \tan(3t^2 - \cos t)}_{\text{تفاضل ال sec}} \cdot \underbrace{(6t + \sin t)}_{\text{تفاضل مابداخل ال sec}}$$

الحل بطريقه أخرى

$y = \sec(3t^2 - \cos t)$

let $u = 3t^2 - \cos t$ and $y = \sec(u)$

✓ $\frac{du}{dt} = 6t + \sin t$ and ✓ $\frac{dy}{du} = \sec(u) . \tan(u)$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

$$= \sec(u) . \tan(u) . (6t + \sin t)$$

$$= \sec(3t^2 - \cos t) . \tan(3t^2 - \cos t) . (6t + \sin t).$$

Example: Find the derivative of the following functions:

$$f(x) = \cos(3x)$$

$$f'(x) = -\sin(3x) \cdot 3 = -3\sin(3x).$$

$$f(x) = x^2 + \sin(x^3)$$

$$f'(x) = 2x + \cos(x^3) \cdot 3x^2$$

$$= 2x + 3x^2 \cos(x^3).$$

$$F(x) = \sec^2(\sqrt{x})$$

$$f(x) = \left(\sec(\sqrt{x})\right)^2$$

$$f'(x) = 2\left(\sec(\sqrt{x})\right)^1 \left(\sec(\sqrt{x}) \tan(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}\right)$$

$$= \cancel{2} \cdot \sec^2(\sqrt{x}) \tan(\sqrt{x}) \cdot \frac{1}{\cancel{2}\sqrt{x}}$$

$$= \frac{\sec^2(\sqrt{x}) \tan(\sqrt{x})}{\sqrt{x}}$$

$$g(x) = \tan(x + \sqrt{x})$$

$$g'(x) = \sec^2(x + \sqrt{x}) \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)$$

$f(x) = \csc(\cos x)$

$f'(x) = -\csc(\cos x) \cot(\cos x) \cdot (-\sin x)$

$= \sin x \csc(\cos x) \cot(\cos x)$.

$g(x) = \tan(\sin(\cos x))$

$g'(x) = \sec^2(\sin(\cos x)) \cdot \cos(\cos x) \cdot (-\sin x)$

$= -\sec^2(\sin(\cos x)) \cdot \cos(\cos x) \sin(x)$

Exp.Function and the Chain Rule

If $y = a^{f(x)}$ then $y' = a^{f(x)} \cdot f'(x) \cdot \ln a$

$y = e^{g(x)}$ then $y' = e^{g(x)} \cdot g'(x)$

Example : Find y' if

$y = 5^{\sqrt{x}}$

$y' = 5^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot \ln 5$

$= \frac{\ln(5) \cdot 5^{\sqrt{x}}}{2\sqrt{x}}$

Example: Find the derivative of the following:

$$y = e^{\sec(4x)}$$

$$y' = e^{\sec(4x)} \cdot \sec(4x) . \tan(4x) \cdot 4$$

$$= 4\sec(4x)\tan(4x)\, e^{\sec(4x)} .$$

$$f(x) = 2^{x + \csc x}$$

$$f'(x) = 2^{x+\csc x} \cdot (1 - \csc x \cot x) \cdot \ln 2$$

$$= \ln 2\,(1 - \csc x \cot x)\, 2^{x + \csc x}$$

Log.Function and the Chain Rule

$$\frac{d}{dx}\left[\log_a (g(x))\right] = \frac{g'(x)}{g(x)\ln a}$$

$$\frac{d}{dx}\left[\ln (g(x))\right] = \frac{g'(x)}{g(x)}$$

Example: Find y' if

$$y = \ln(\sin x)$$

$$y' = \frac{\cos x}{\sin x} = \cot x .$$

$y = \log_{10}(\sin x)$

$$y' = \frac{\cos x}{\sin x \ln 10} = \frac{1}{\ln 10} \cdot \frac{\cos x}{\sin x} = \frac{1}{\ln 10} \cdot \cot x$$

Example: Find where the tangent line to the graph $y = \ln(x^3 - x^2 + 4)$ is horizontal

$f'(x) = 0$ ← tangent line harizontal

$$f'(x) = \frac{3x^2 - 2x}{x^3 - x^2 + 4} = \frac{x(3x-2)}{x^3 - x^2 + 4} = 0$$

$\Rightarrow x(3x-2) = 0$

$\Rightarrow x = 0$ or $3x - 2 = 0$

$\Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}$

The graph has two horizontal tangent lines at $x = 0$ and $x = \frac{2}{3}$

Logarithmic Differentiation

If $y = [g(x)]^{f(x)}$

we will use Log. D to find y'

Taking natural logarithms for both sides.

Applying the properties of logarithms.

Differentiating with respect to x.

Solving for y'.

Replacing y by $f(x)$.

Example: Find y' if $y = x^{x+2}$

$$y = x^{x+2}$$

$$\ln y = \ln x^{x+2}$$

$$\ln y = (x+2)\ \ln(x)$$

$$\frac{y'}{y} = (1).\ln(x) + (x+2] \cdot \frac{1}{x}$$

$$\frac{y'}{y} = \ln(x) + \frac{x+2}{x}$$

$$y' = y\left[\ln(x) + \frac{x+2}{x}\right]$$

$$y' = x^{x+2}\left[\ln(x) + \frac{x+2}{x}\right].$$

Example: Find y' if $y = \left(\frac{3^x}{x+e^x}\right)$

لإيجاد تفاضل هذه الداله نحتاج الى تطبيق
قانون تفاضل الداله الكسرية لكي نوجد y.
ماذا لو طبقنا Log. Differentiation ؟
هل سنحصل عَلى نفس النتيجة.
Let's try ! and see Ex3 in the
lecture of Exp.function

$$\ln y = \ln\left[\frac{3^x}{x+e^x}\right]$$

$$\ln y = \ln(3^x) - \ln(x+e^x)$$

$$\frac{y'}{y} = \frac{\cancel{3^x}\ \ln 3}{\cancel{3^x}} - \frac{1+e^x}{x+e^x}$$

$$y' = y\left[\ln(3) - \frac{1+e^x}{x+e^x}\right]$$

$$y' = \frac{3^x}{x+e^x}\left[\frac{(x+e^x)\ln(3) - (1+e^x)}{x+e^x}\right]$$

$$= \frac{3^x(x+e^x)\ln(3) - 3^x(1+e^x)}{(x+e^x)^2}$$

$$= \frac{3^x \ln 3x + 3^x e^x \ln(3) - 3^x - 3^x e^x}{(x+e^x)^2}$$

$$= \frac{3^x(\ln 3x - 1) + 3^x e^x(\ln 3 - 1)}{(x+e^x)^2}$$

Same result

اذا من الممكن إيجاد تفاضل
الداله الكسرية بطريقة
Log.Differentiation

H.W: Find y' if $y = \dfrac{(2x-1)^2(x^2+1)^3}{\sqrt{x^4+1}}$

Implicit Differentiation and Higher Derivatives

$$\frac{d}{dx}[y^n] = ny^{n-1}\, y'$$

Example : If $x^2+y^2=5$, find the following

1) $$2x + 2yy' = 0$$

$$\Rightarrow 2yy' = -2x$$

$$\Rightarrow y' = \frac{-2x}{2y} = \frac{-x}{y}.$$

2) Equation of the tangent line to $x^2+y^2=5$ at the point $(3/\sqrt{5}, 4/\sqrt{5})$.

$$y - y_1 = m(x - x_1)$$

Remember

$m = f'(a)$

or

$m = y'\Big|_{x=a}$

$$\therefore m = y'(3/\sqrt{5}, 4/\sqrt{5}) = \frac{-3/\sqrt{5}}{4/\sqrt{5}}$$

$$= \frac{-3}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{\sqrt{5}}}{4} = \frac{-3}{4}$$

$$\therefore y - \frac{4}{\sqrt{5}} = \frac{-3}{4}\left(x - \frac{3}{\sqrt{5}}\right)$$

$$\Rightarrow y = \frac{-3}{4}\left(x - \frac{3}{\sqrt{5}}\right) + \frac{4}{\sqrt{5}}$$

$$= \frac{-3}{4}x + \frac{9}{4\sqrt{5}} + \frac{4}{\sqrt{5}}$$

$$= -\frac{3}{4}x + \frac{9\sqrt{5} + 16\sqrt{5}}{4 \cdot 5}$$

$$= \frac{-3}{4}x + \frac{\cancel{25}\sqrt{5}}{\cancel{20}}$$

$$= \frac{-3}{4}x + \frac{5\sqrt{5}}{4}$$

Example : If $y^3 + y^2 - 5y - x^2 = -4$. Find the following:

1)- y'

$$3y^2y' + 2yy' - 5y' - \overset{\curvearrowright}{2x} = 0$$

$$y'(3y^2 + 2y - 5) = 2x$$

$$y' = \frac{2x}{3y^2 + 2y - 5}$$

2)- Equation of the tangent line to $y^3 + y^2 - 5y - x^2 = -4$ at the point $(3,-1)$

$$y - y_1 = m(x - x_1)$$

$$m = y'(3,-1) = \frac{2.3}{3(-1)^2 + 2(-1) - 5} = \frac{6}{3-2-5} = \frac{6}{-4} = -\frac{3}{2}$$

$$\therefore \quad y - (-1) = -\frac{3}{2}(x - 3)$$

$$y = -\frac{3}{2}(x - 3) - 1$$

$$= -\frac{3}{2}x + \frac{9}{2} - 1$$

$$= -\frac{3}{2}x + \frac{7}{2}$$

Example: Compute the slope of the tangent line to the curve $\sin(xy) = x$ at the point $\left(\frac{1}{2}, \frac{\pi}{3}\right)$.

slope $= y'(x, y)$.

$$\frac{d}{dx}\left[\sin(xy)\right] = \cos(xy) \cdot (x \cdot y' + 1 \cdot y) = 1$$

$$\cos(xy) \cdot (xy' + y) = 1$$

$$xy' + y = \frac{1}{\cos(xy)}$$

$$\Rightarrow xy' = \frac{1}{\cos(xy)} - y$$

$$y' = \frac{1}{x}\left(\frac{1}{\cos(xy)} - y\right)$$

$$\therefore \text{slope} = y'\left(\frac{1}{2}, \frac{\pi}{3}\right) = \frac{1}{(1/2)}\left[\frac{1}{\cos\left(\frac{1}{2} \cdot \frac{\pi}{3}\right)} - \frac{\pi}{3}\right]$$

$$= 2\left[\frac{1}{\cos(\pi/6)} - \frac{\pi}{3}\right]$$

$1 \cdot \frac{2}{\sqrt{3}}$

$$= 2\left[\frac{1}{\sqrt{3}/2} - \frac{\pi}{3}\right]$$

$$= 2\left[\frac{2}{\sqrt{3}} - \frac{\pi}{3}\right]$$

$$= \frac{4}{\sqrt{3}} - \frac{2\pi}{3}$$

Example : Find the equation of the tangent line to the graph of $y = 2x^2y - 3y = x$ at the point $(1,-1)$.

$$y - y_1 = m(x - x_1)$$

$$m = y'(1,-1)$$

$$\frac{d}{dx}\left[2x^2y - 3y = x\right] = 4xy + 2x^2y' - 3y' = 1$$

$$\Rightarrow 2x^2y' - 3y' = 1 - 4xy$$

$$\Rightarrow y'(2x^2 - 3) = 1 - 4xy$$

$$\Rightarrow y' = \frac{1 - 4xy}{x^2 - 3}$$

2

$$\therefore m = y'(1,-1) = \frac{1 - 4(1)(-1)}{2(1)^2 - 3} = \frac{5}{-1} = -5$$

$$\therefore y - (-1) = -5(x - 1)$$

$$y + 1 = -5x + 5$$

$$y = -5x + 5 - 1$$

$$y = -5x + 4$$

Example: Find y' if $y = \cos(x-y) = xe^{x}$

تفاضل ال cos   تفاضل مابداخل ال cos   تفاضل ضرب دالتين

$$-\sin(x-y)(1-y') = 1.e^{x} + x.e^{x}$$

$$-\sin(x-y) + y'\sin(x-y) = e^{x}(1+x)$$

$$y'\sin(x-y) = e^{x}(1+x) + \sin(x-y)$$

$$y' = \frac{e^{x}(1+x)}{\sin(x-y)} + \frac{\sin(x-y)}{\sin(x-y)}$$

$$= \frac{e^{x}(1+x)}{\sin(x-y)} + 1$$

$$= e^{x}(1+x) \cdot \frac{1}{\sin(x-y)} + 1$$

$$= e^{x}(1+x) \cdot \csc(x-y) + 1$$

Remember

$$\frac{1}{\sin x} = \csc x$$

Higher Order Derivatives

The following notations for higher derivatives, with y= f(x) are usually used

$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x), \cdots f^{(n)}(x)$
$y', y'', y''', y^{(4)}, \cdots y^{(n)}$
$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \cdots, \frac{d^ny}{dx^n}$
$D_xy, D_x^2y, D_x^3y, D_x^4y, \ldots, D_x^ny$

Example : Find the third derivative of $f(x) = x^{\frac{1}{2}} + x^3$.

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} + 3x^2$$

$$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} + 3 \cdot 2x$$

$$= -\frac{1}{4} x^{-\frac{3}{2}} + 6x$$

$$f'''(x) = \left(-\frac{3}{2}\right)\left(-\frac{1}{4}\right) x^{-\frac{3}{2}-1} + 6$$

$$= \frac{3}{8} x^{-\frac{5}{2}} + 6$$

$$= \frac{3}{8} \cdot \frac{1}{\sqrt[2]{x^5}} + 6$$

$$= \frac{3}{8\sqrt{x^5}} + 6$$

Remember

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

ملاحظة : إذا كانت $n=2$ هذا يعني الجذر التربيعي وبالتالي على عدم كتابته

Example : Find y'' if $xy^3 = 2$

$$1 \cdot y^3 + x \cdot 3y^2 y' = 0$$

$$3xy^2 y' = -y^3$$

$$y' = \frac{-\cancel{y^3}}{3x\cancel{y^2}} = \frac{-y}{3x} = -\frac{1}{3}\left(\frac{y}{x}\right)$$

$$y'' = -\frac{1}{3}\left[\frac{y' \cdot x - 1 \cdot y}{x^2}\right]$$

$$= -\frac{1}{3}\left[\frac{\left(\frac{-y}{\cancel{3x}}\right) \cdot \cancel{x} - y}{x^2}\right]$$

$$= \frac{-1}{3}\left[\frac{\frac{-y}{3} - y}{x^2}\right]$$

$$\frac{-y}{3} - \frac{y}{1} = \frac{-y-3y}{3}$$

$$= \frac{-4y}{3}$$

$$= -\frac{1}{3}\left[\frac{\frac{-4y}{3}}{x^2}\right]$$

$$= \frac{\frac{4y}{3}}{3x^2} = \frac{4y}{3} \cdot \frac{1}{3x^2} = \frac{4y}{9x^2}$$

Example : Find the n^{th} derivatives of the function

$$f(x) = x^4 - x^3 + x^2 - \pi x + 4$$

$$f'(x) = 4x^3 - 3x^2 + 2x - \pi$$

$$f''(x) = 12x^2 - 6x + 2$$

$$f'''(x) = 24x - 6$$

$$f^{(4)}(x) = 24$$

$$f^{(5)}(x) = 0$$

Example : Find $D_x^{25}(\sin x)$

$$D_x^{1}(\sin x) = \cos x$$

$$D_x^{2}(\sin x) = -\sin x$$

$$D_x^{3}(\sin x) = -\cos x$$

$$D_x^{4}(\sin x) = \sin x$$

$$\begin{array}{r|l} & \ \ 6 \\ \hline 4 & \ \ 25 \\ & \ \ 24 \\ \hline & \ \ \ \ 1 \end{array}$$

$$\therefore \; D_x^{25}(\sin x) = D_x^{1}(\sin x) = \cos x.$$

L'Hopital's Rule

Indeterminate Forms (I.F) :-

The following expression are called (I.F)

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, ∞^0 , 0^0 , 1^∞ and $\infty - \infty$

L'Hopital Rule:

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where a can be real number.

Example:-

$$\lim_{x \to \infty} \frac{x^3+1}{x^2-1} = \frac{\infty^3+1}{\infty^2-1} = \frac{\infty}{\infty} \quad (I.F)$$

$$\hookrightarrow \overset{H}{=} \lim_{x \to \infty} \frac{3x^2}{2x} \qquad \left(\frac{\infty}{\infty} \text{ I.F again}\right)$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{6x}{2}$$

$$= \lim_{x \to \infty} 3x = \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \frac{\ln(\infty)}{\sqrt[3]{\infty}} = \frac{\infty}{\infty}$$ (I.F) ✓

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3x^{2/3}}}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x} \cdot \frac{3x^{\frac{2}{3}}}{1}\right)$$

$$= \lim_{x\to\infty} \frac{3x^{\frac{2}{3}}}{x}$$

$$x^{\frac{2}{3}-1} = x^{\frac{2-3}{3}} = x^{\frac{-1}{3}}$$

$$= \lim_{x\to\infty} \frac{3}{x^{1/3}} = \frac{3}{(\infty)^{1/3}} = 0$$

Example:

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1-1}{1+1} = \frac{0}{2} = 0$$ (· ⊥ I.F)

No need to use (L.R).

What happen if we use L.R ?

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \lim_{x\to 1} \frac{2x}{1} = 2$$

Example:

$$\lim_{x\to 0}\frac{\sin x}{x^2} = \frac{\sin(0)}{0^2} = \frac{0}{0} \quad (I.F) \checkmark$$

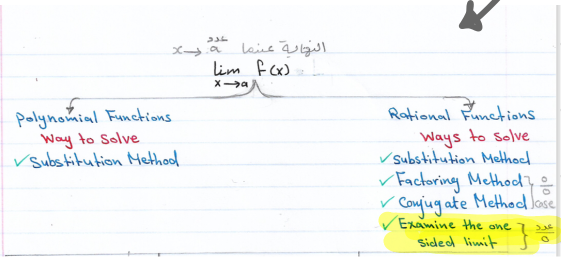
$$\overset{H}{=} \lim_{x\to 0}\frac{\cos x}{2x} = \frac{\cos(0)}{2 \frown} = \frac{1}{0}$$

Remember $\frac{\text{عدد}}{0}$

$$\lim_{x\to 0^+}\frac{\cos x}{2x} = +\infty \quad \text{and}$$

$$\lim_{x\to 0^-}\frac{\cos x}{2x} = -\infty$$

$$\therefore \lim_{x\to 0}\frac{\cos x}{2x} \quad \text{DNE}$$



$x \to \overset{\text{عدد}}{a}$ النهاية عندما

$\lim_{x\to a} f(x)$

Polynomial Functions

Way to Solve

✓ Substitution Method

Rational Functions

Ways to solve

✓ Substitution Method

✓ Factoring Method } $\frac{0}{0}$ case

✓ Conjugate Method

✓ Examine the one sided limit } $\frac{\text{عدد}}{0}$

From math (1)

لمزيد من المعلومات راجعي ملف Limits على

الرابط التالي https://t.me/MadaAltiary

$$\frac{\sin(\frac{\pi}{2})}{\cos(\frac{\pi}{2})} = \frac{1}{0} = \text{und}$$

Example:

$$\lim_{x \to (\frac{\pi}{2})^+} \frac{2\tan x}{1+\sec x} = \frac{2\tan(\frac{\pi}{2})}{1+\sec(\frac{\pi}{2})} = \frac{\infty}{\infty} \quad (\text{I.F}).$$

$$\overset{H}{=} \lim_{x \to (\pi/2)^+} \frac{2\sec^{\cancel{2}} x}{\cancel{\sec} x \tan x}$$

$$= \lim_{x \to (\pi/2)^+} \frac{2\sec x}{\tan x}$$

$$= \lim_{x \to (\pi/2)^+} \frac{2 \cdot \frac{1}{\cos x}}{\sin x}$$

$$= \lim_{x \to (\pi/2)^+} \left[\frac{2}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\sin x} \right]$$

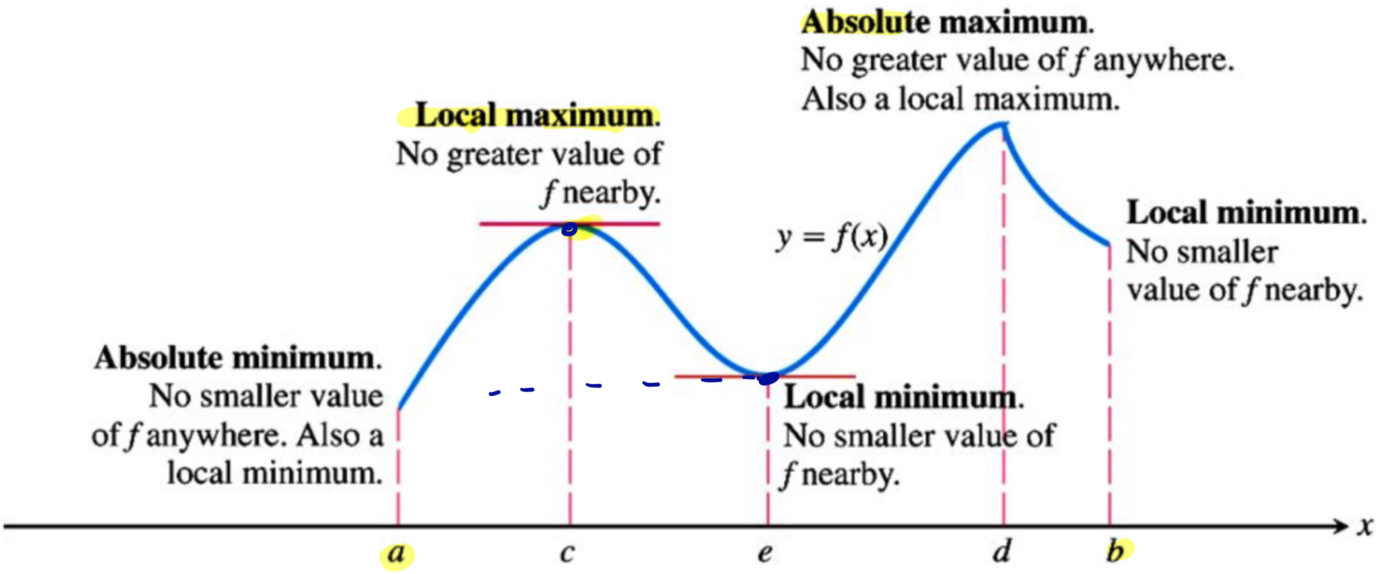
$$= \lim_{x \to (\pi/2)^+} \frac{2}{\sin x}$$

$$= \frac{2}{\sin(\pi/2)} = \frac{2}{1} = 2.$$

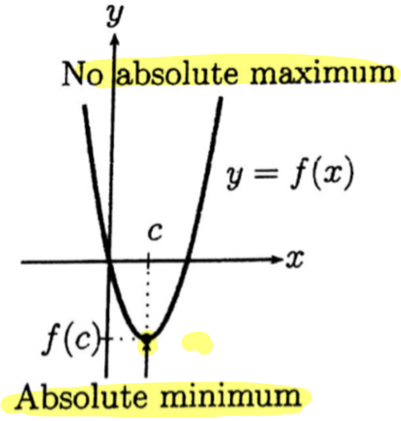
Maximum and Minimum Values

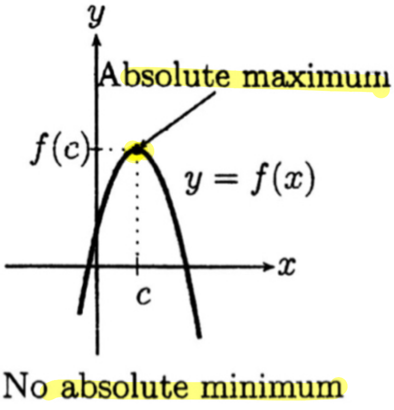
Extreme values:

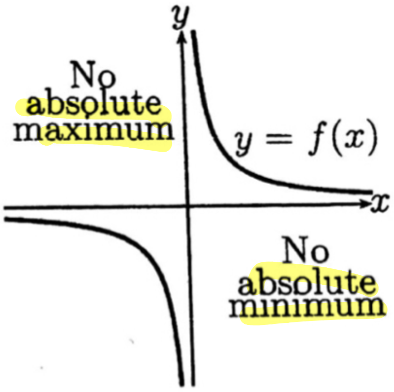
max and min values



Absolute Extreme Values	Local Extreme Values
$f(c)$ is an:	$f(c)$ is an
• Absolute min of f if: $f(c) \leq f(x) \ \forall x \in D(f)$	Local min of f $f(c) \leq f(x) \ \forall x$ in some open interval containing a.
• Absolute max of f if $f(c) \geq f(x) \ \forall x \in D(f)$	Local max of f $f(c) \geq f(x) \ \forall x$ in some open interval containing a.



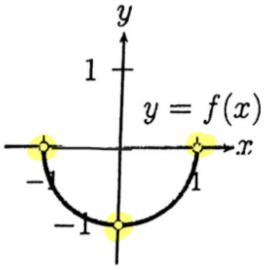




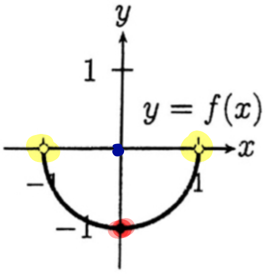
Remark:

Every absolute extremum is a local extremum but the converse is not true always.

Example 1: Determine the absolute extreme for the given graphs.

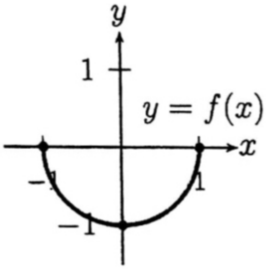


- f has no absolute max nor absolute min



- f has absolute min at $x=0$ with value $f(0) = -1$

$(0,-1)$

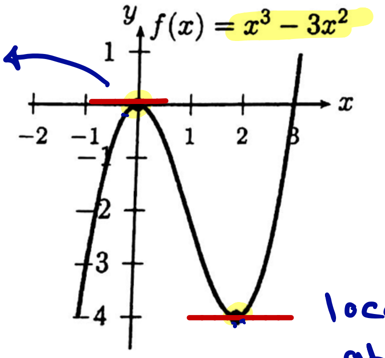


- f has absolute max at $x=\pm 1$ with value $f(\pm 1)=0$
- f has absolute min at $x=0$ with value $f(0)=-1$

Critical numbers :

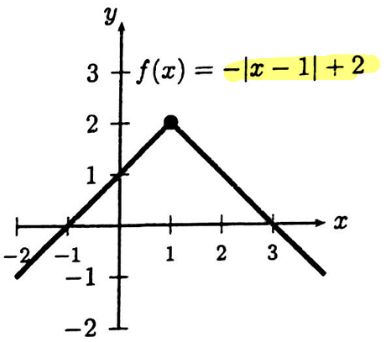
A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ DNE

Example : Find the value of the derivative at each of the local extremum shown in the following figures.



local max at $x=0$

local min at $x=2$



$F(x)=x^3-3x^2$

$F'(x)=3x^2-6x$

$F'(0)=3(0)^2-6(0)=0$

$F'(2)=3(2)^2-6(2)=0$

$\Rightarrow$ $x=0,2$ are critical numbers of F

$F(x)=-|x-1|+2$

$$F(x)=\begin{cases}-(x-1)+2, & x\geq 1\\ (x-1)+2, & x\leq 1\end{cases}$$

$$f'(x)=\begin{cases}-1, & x\geq 1\\ 1, & x\leq 1\end{cases}$$

$\Rightarrow$ $x=1$ is the critical number of F since $f'(1)=\pm 1$

$\Rightarrow$ $F'(1)$ DNE

Example: Find the critical numbers of $f(x) = x^3 - \frac{3}{2}x^2 + 1$

$$f'(x) = 3x^2 - \frac{3}{2} \cdot 2x$$

$$= 3x^2 - 3x$$

$$= 3x(x-1)$$

$$f'(x) = 0 \Rightarrow 3x(x-1) = 0$$

$$\Rightarrow 3x = 0 \Rightarrow x = 0 \text{ or } x-1=0 \Rightarrow x = 1$$

$\because D(f) = R \Rightarrow x = 0, 1$ are the critical numbers

Example: Find the critical numbers of

$$f(x) = 3x^{\frac{1}{3}} + \frac{3}{2} x^{\frac{4}{3}}$$

$$f'(x) = 3 \cdot \frac{1}{3} x^{-2/3} + \frac{\cancel{3}}{2} \cdot \frac{4}{\cancel{3}} x^{\frac{1}{3}}$$

$$= x^{-2/3} + 2x^{\frac{1}{3}}$$

$$= x^{-2/3}(1 + 2x)$$

$$= \frac{1+2x}{x^{2/3}}$$

$$x^{-\frac{2}{3}} \cdot x$$

$$= x^{-2/3+1}$$

$$= x^{\frac{1}{3}}$$

$f'(x) = 0 \Rightarrow 1 + 2x = 0$

$\Rightarrow 2x = -1 \Rightarrow x = \frac{-1}{2}$

$f'(x)$ undefined $\Rightarrow x^{2/3} = 0 \Rightarrow x = 0$

$\because D(f) = \mathbb{R} \Rightarrow x = 0, -\frac{1}{2}$ are the critical numbers.

Example: Find the critical numbers of $f(x) = \frac{x^2 - 1}{x^3}$

$$f'(x) = \frac{2x \cdot x^3 - 3x^2(x^2 - 1)}{(x^3)^2}$$

$$= \frac{2x^4 - 3x^4 + 3x^2}{x^6}$$

$$= \frac{3x^2 - x^4}{x^6}$$

$$= \frac{x^2(3 - x^2)}{x^6}$$

$$= \frac{3 - x^2}{x^4}$$

$f'(x)=0 \Rightarrow 3-x^2=0 \Rightarrow 3=x^2$

$\Rightarrow x=\pm\sqrt{3}$

$f'(x)$ undefined $\Rightarrow x^4=0 \Rightarrow x=0$.

$\because D(f)=\mathbb{R}-\{0\} \Rightarrow x=\pm\sqrt{3}$ are the only critical number of f

Format's theorem:-

If f has local extremum at c, then c is a critical number of f.

Example:- Find the absolute maximum and minimum of $f(x) = x^2 - 4x$ on $[0, 3]$

$$f'(x) = 2x - 4$$

$$f'(x) = 0 \Rightarrow 2x - 4 = 0$$

$$\Rightarrow 2x = 4 \Rightarrow x = 2$$

$f(0) = 0$ → Absolute max.

$f(3) = 3^2 - 4.3 = -3$

$f(2) = 2^2 - 4.2 = -4$ → Absolute min

Example: Find the absolute maximum and minimum of $f(x) = 3x^{2/3} - 2x$ on $[-1, 8]$.

$$f'(x) = \cancel{3} \cdot \frac{2}{\cancel{3}} x^{-\frac{1}{3}} - 2$$

$$= 2x^{-\frac{1}{3}} - 2$$

$$= x^{-1/3} (2 - 2x^{\frac{1}{3}})$$

$$= \frac{2 - 2x^{1/3}}{x^{1/3}}$$

$$f'(x) = 0 \Rightarrow 2 - 2x^{1/3} = 0$$

$$\Rightarrow 2x^{1/3} = 2$$

$\Rightarrow x^{1/3} = 1 \Rightarrow x = 1$

$f'(x)$ undefined $\Rightarrow x^{1/3} = 0 \Rightarrow x = 0$

$\because$ $0, 1 \in [-1, 8] \Rightarrow 0, 1$ are the critical numbers of f.

$f(-1) = 3(-1)^{2/3} - 2(-1) = 3+2 = 5$ → Absolute max

$f(1) = 3(1)^{2/3} - 2(1) = 3 - 2 = 1$

$f(0) = 3(0)^{2/3} - 2(0) = 0$

$f(8) = 3(8)^{2/3} - 2(8) = 12 - 16 = -4$ Absolute min

Rolle's Theorem and the Mean Value Theorem

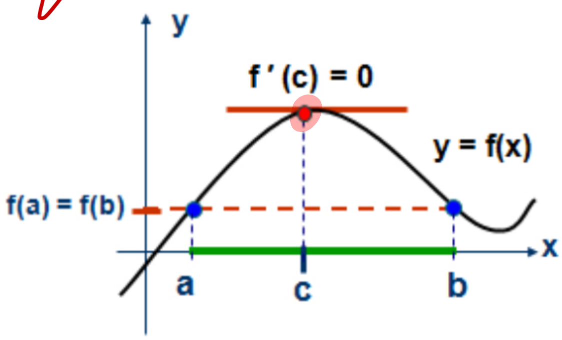
Rolle's Theorem :- Let f be

1)- continuous function on closed interval $[a,b]$

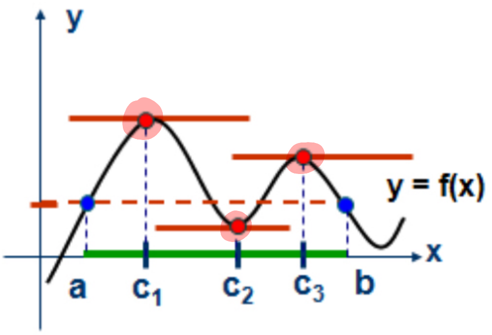
2)- differentiable on open interval (a,b), and

3)- $f(a) = f(b)$.

then there is a number $c \in (a,b)$ s.t $f'(c) = 0$



Example 1



Example 2

Example :- Let $f(x) = x^4 - 2x^2$. Find all value of c in the interval $[-2, 2]$ s.t $f'(c) = 0$.

1)- f is cont. on $[-2, 2]$

2)- f is diff on $(-2, 2)$

3)- $f(-2) = (-2)^4 - 2(-2)^2 = 16-8 = 8$

$f(2) = 2^4 - 2(2)^2 = 16 - 8 = 8$ $\Big]$ $f(-2) = f(2)$

$\therefore \exists\, c \in (-2, 2)$ s.t $f'(c) = 0$

$f'(x) = 4x^3 - 4x$

$f'(c) = 0 \Rightarrow 4c^3 - 4c = 0$

$\Rightarrow 4c(c - 1) = 0$

$\Rightarrow 4c = 0$ or $c^2 - 1 = 0$

$\Rightarrow c = 0$ or $c^2 = 1$

$\Rightarrow c = 0$ or $c = \pm 1$

Example: Let $f(x) = (1-x)^{2/3} + 1$. Show that $f(0) = f(2)$ but there is no $c \in (0, 2)$ s.t $f'(c) = 0$

$f(0) = (1-0)^{2/3} + 1 = 1+1 = 2$

$f(2) = (1-2)^{2/3} + 1 = (-1)^{2/3} + 1 = \sqrt[3]{(-1)^2} + 1 = \sqrt[3]{1} + 1 = 2$

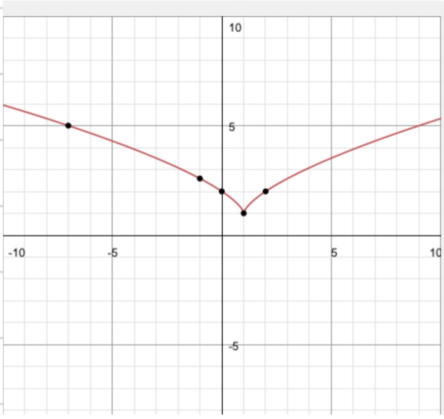
$\therefore f(0) = f(1)$.

$$f'(x) = \frac{2}{3} \cdot (1-x)^{2/3 - 1} \cdot (-1)$$ ✓

$$= -\frac{2}{3}(1-x)^{-\frac{1}{3}}$$

$$= \frac{-2}{3(1-x)^{\frac{1}{3}}}$$

$$= \frac{-2}{3\sqrt[3]{1-x}}$$



f is not differentiable at $x = 1$.
Since $f'(x)$ undefined at $x = 1$ → critical number

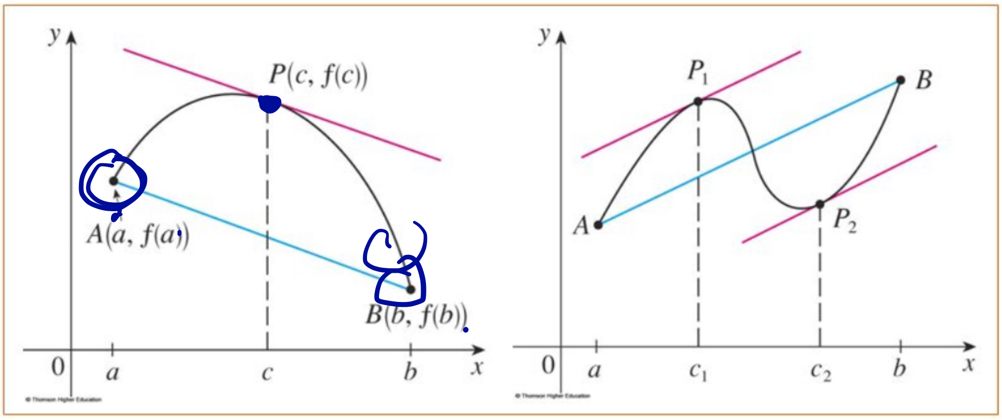
The Mean Value Theorem MVT

✓ Let f be

1)- continuous function on closed interval $[a, b]$

2) ✓ differentiable function on open interval (a, b)

then there is a number $c \in (a, b)$ s.t $f'(c) = \frac{f(b) - f(a)}{b - a}$



Example: Let $f(x) = 2 - \frac{3}{x}$ Find all values of c in the interval $(1, 3)$ s.t $f'(c) = \frac{f(3) - f(1)}{3-1}$

$\because D(f) = \mathbb{R} - \{0\} \Rightarrow f$ is continuous on $[1, 3]$.

f is differentiable on $(1, 3)$.

$\therefore \exists\, c \in (1, 3)$ s.t $f'(c) = \frac{f(3) - f(1)}{3-1}$

$$f(3) = 2 - \frac{3}{3} = 2 - 1 = 1$$

$$f(1) = 2 - \frac{3}{1} = 2 - 3 = -1$$

$$\therefore f'(c) = \frac{f(3) - f(1)}{3-1} = \frac{1-(-1)}{2} = \frac{2}{2} = 1$$

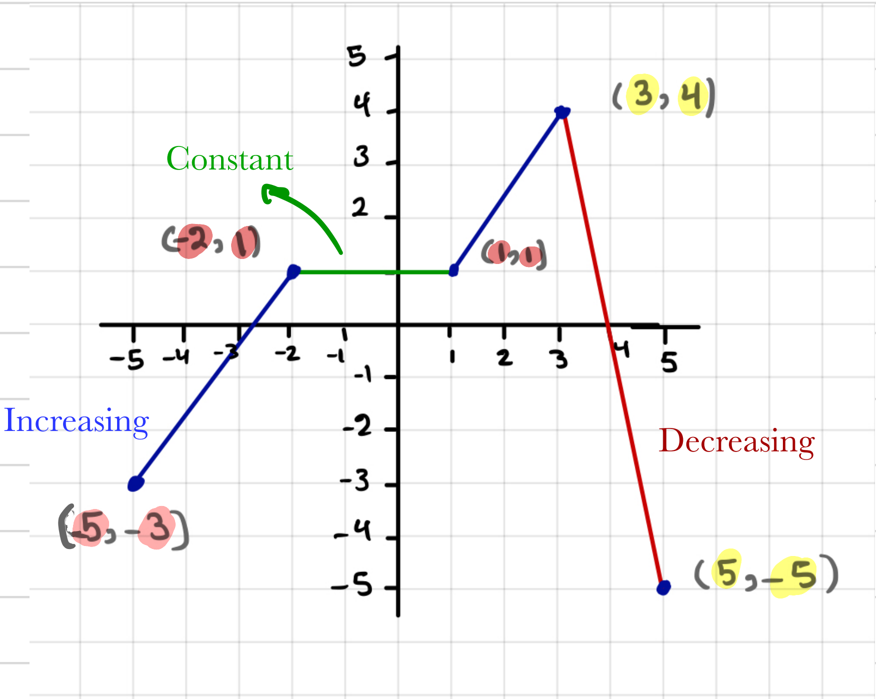
$$f'(x) = \frac{3}{x^2} \Rightarrow f'(c) = \frac{3}{c^2}$$

$$\Rightarrow \frac{3}{c^2} = 1 \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}$$

$$\because c = -\sqrt{3} \notin (1, 3) \Rightarrow c = \sqrt{3}.$$

Monotonocity and The First Derivative Test

Definition for Monotonic Function:



Increasing: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Decreasing: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Constant: $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$

Test for Monotonic Function:

Increasing: $f'(x) > 0$

Decreasing: $f'(x) < 0$

constant: $f'(x) = 0$

Example 1:

Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is increasing or decreasing.

① $f'(x) = 0$

✓ $12x^3 - 12x^2 - 24x = 0 \rightarrow$

$12x(x^2 - x - 2) = 0$

$12x(x-2)(x+1) = 0$

$\Rightarrow x = 0, \; x = 2, \; x = -1$

$\because D(f) = \mathbb{R} \Rightarrow -1, 0$ and 2 are critical numbers.

②

	$-\infty$ … -1	-1 … 0	0 … 2	2 … ∞
Test value	-2	-0.5	1	3
f'(Test value)	-96	7.5	-24	144
Sing of f'	$-$	$+$	$-$	$+$
	Dec	Inc	Dec	Inc

③ f increasing on the intervals $(-1, 0) \cup (2, \infty)$

decreasing on the intervals $(-\infty, -1) \cup (0, 2)$

Example 2:

Find the intervals on which $f(x) = (x^2-1)^{2/3}$ is increasing or decreasing

1)_ $f'(x) = \frac{2}{3}(x^2-1)^{-1/3}(2x)$

$= \frac{4x}{3(x^2-1)^{1/3}}$

$= \frac{4x}{3\sqrt{(x+1)(x-1)}}$

$f'(x)=0$

$4x=0$

$x=0$

$f'(x)$ undefined

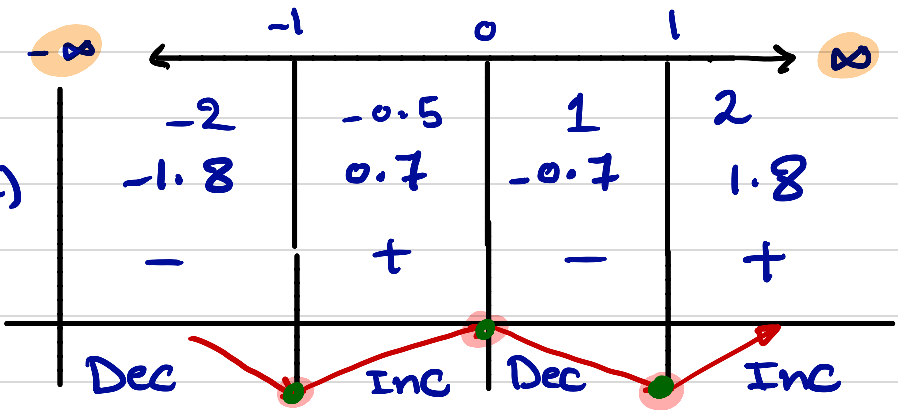
$\sqrt[3]{(x+1)(x-1)}=0$

$\Rightarrow x=\pm 1$

$\because D(f)=\mathbb{R}$

$\therefore -1, 0, 1$ are critical numbers of f.

2)_



Test value	-2	-0.5	1	2
f'(Test value)	-1.8	0.7	-0.7	1.8
Sing of f'	−	+	−	+
	Dec	Inc	Dec	Inc

3)_

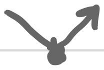
f is increasing on $(-1,0) \cup (1,\infty)$

Decreasing on $(-\infty,-1) \cup (0,1)$

First Derivative test and local Extremum:

1)_ If $f'(x)$ changes from + to - at c ⇒ f has a local max at c

2)_ If $f'(x)$ changes from - to + at c ⇒ f has a local min at c



3) _If $f'(x)$ does n't change at c ⇒ f has no local max or min at c

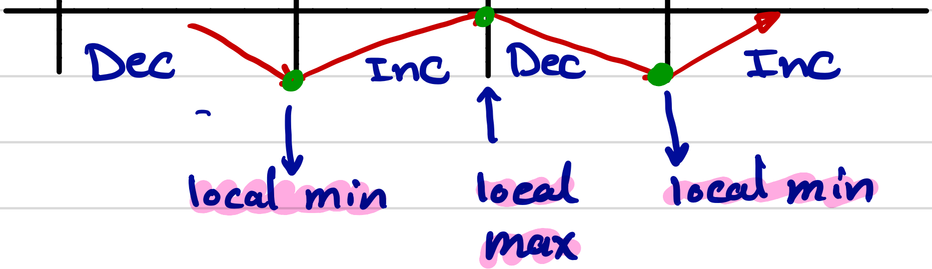
Example 3:

Find the local extreme for $f(x) = 3x^4 - 4x^3 - 12x^2 - 1$

From Example 1 we have:

$-\infty$ -1 0 2 ∞

Test value	-2	-0.5	1	3
f'(Test value)	-96	7.5	-24	144
Sing of f'	−	+	−	+



∴ f has local max at $x = 0$ with value $f(0) = 1$
local min at $x = -1$ with value $f(-1) = -4$
local min at $x = 2$ with value $f(2) = -31$

Find the local extreme for $f(x) = \frac{x}{2} + \sin x$ in the interval $(0, 2\pi)$.

$$f'(x) = \frac{1}{2} + \cos x$$

$$f'(x) = 0 \Rightarrow \frac{1}{2} + \cos x = 0$$

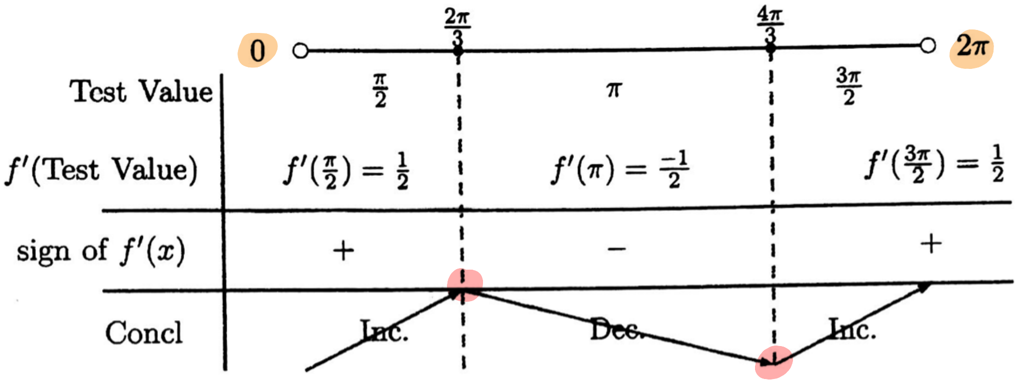
$$\Rightarrow \cos x = -\frac{1}{2}$$

$$\Rightarrow x = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow x = \frac{2\pi}{3}, \quad x = \frac{4\pi}{3}$$

$\because D(f) = (0, 2\pi)$

$\therefore \frac{2\pi}{3}, \frac{4\pi}{3}$ are the critical numbers.



Test Value	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
f'(Test Value)	$f'(\frac{\pi}{2}) = \frac{1}{2}$	$f'(\pi) = \frac{-1}{2}$	$f'(\frac{3\pi}{2}) = \frac{1}{2}$
sign of $f'(x)$	+	−	+
Concl	Inc.	Dec.	Inc.

local max

local min

$$F\left(\frac{2\pi}{3}\right) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\pi}{3} + \sin\left(\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$F\left(\frac{4\pi}{3}\right) = \frac{1}{2} \cdot \frac{4\pi}{3} + \sin\left(\frac{4\pi}{3}\right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

Example: Find the local extreme for $F(x) = \frac{x}{x^2+1}$ and Find the intervals on which f is increasing and decreasing.

$$F(x) = x(x^2+1)^{-1}$$

$$F'(x) = (1)(x^2+1)^{-1} - 2x(x^2+1)^{-2} \cdot (x)$$

$$= (x^2+1)^{-1} - 2x^2(x^2+1)^{-2}$$

$$= (x^2+1)\frac{(x^2+1)^{-1}}{(x^2+1)} - 2x^2(x^2+1)^{-2}$$

$$= (x^2+1)(x^2+1)^{-2} - 2x^2(x^2+1)^{-2}$$

$$= (x^2+1)^{-2}\left[x^2+1 - 2x^2\right]$$

$$= (x^2+1)^{-2}\left[-x^2+1\right]$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{(1-x)(1+x)}{(x^2+1)^2}$$

never zero

$f'(x)=0 \Rightarrow (1-x)(1+x)=0$

$\Rightarrow x=\pm 1$

$\because D(f)=\mathbb{R} \Rightarrow \pm 1$ are critical numbers for f.

	$-\infty$ … -1	-1 … 1	1 … ∞
Test value	-2	0	2
f'(Test value)	$-\frac{3}{25}$	0	$-\frac{1}{2}$
Sing of f'	$-$	$+$	$-$
	Dec	Inc	Dec

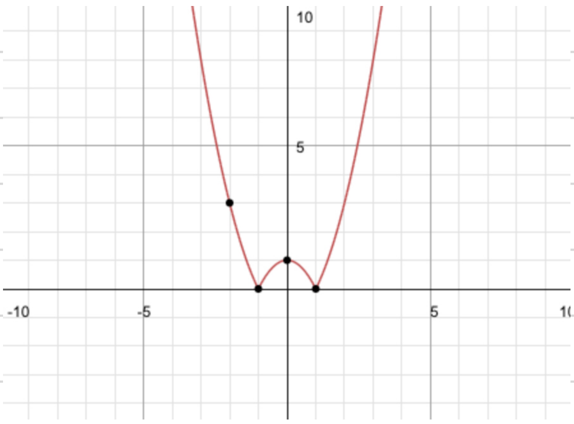
local min $f(-1)=-\frac{1}{2}$

local max $f(1)=\frac{1}{2}$

f is increasing on $(-1,1)$

is decreasing on $(-\infty,-1)\cup(1,\infty]$

Example: Find the local extreme for $f(x) = |x^2-1|$ and find the intervals on which f is increasing and decreasing.



$$f(x) = \begin{cases} x^2-1 & \text{if} \quad x \geq 1 \text{ or } x \leq -1 \\ -(x^2-1) & \text{if} \quad -1 \leq x \leq 1 \end{cases}$$

$$f'(x) = \begin{cases} 2x & \text{if} \quad x > 1 \text{ or } x < -1 \\ -2x & \text{if} \quad -1 < x < 1 \end{cases}$$

$f'(x) = 0$

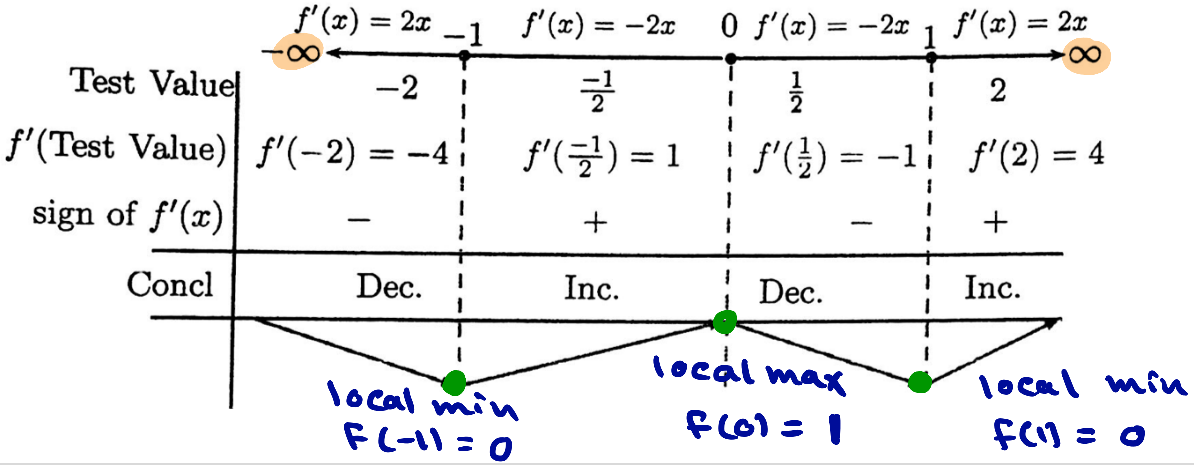
$-2x = 0$

$x = 0$

$f'(x)$ undefined

$x = \pm 1$

$\because D(f) = \mathbb{R} \Rightarrow -1, 0, 1$ are the critical numbers.



$-\infty$ ← $f'(x) = 2x$ -1 $f'(x) = -2x$ 0 $f'(x) = -2x$ 1 $f'(x) = 2x$ → ∞

Test Value	-2	$\frac{-1}{2}$	$\frac{1}{2}$	2
f'(Test Value)	$f'(-2) = -4$	$f'(\frac{-1}{2}) = 1$	$f'(\frac{1}{2}) = -1$	$f'(2) = 4$
sign of $f'(x)$	$-$	$+$	$-$	$+$
Concl	Dec.	Inc.	Dec.	Inc.

local min $f(-1) = 0$

local max $f(0) = 1$

local min $f(1) = 0$

f is increasing on $(-1, 0) \cup (1, \infty)$

Decreasing on $(-\infty, -1)$,

Example 8: Find the local extreme for $f(x) = \ln(9-x^2)$ and find the intervals on which f is increasing and decreasing.

$$f'(x) = \frac{-2x}{9-x^2}$$

$f'(x) = 0$

$-2x = 0$

$x = 0$ ✓

$f'(x)$ undefined

$9 - x^2 = 0$

$9 = x^2$

$\pm 3 = x$

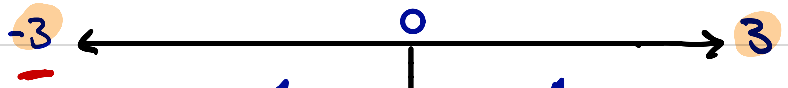
$\because D(f) = (-3, 3)$ because

$9 - x^2 > 0$

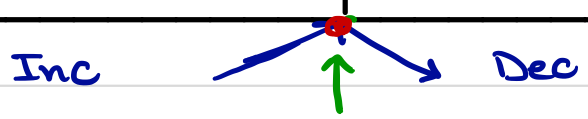
$9 > x$

$\pm 3 > x$

$\therefore x = 0$ is the only critical number.



Test value	-1	1
f'(Test value)	$f'(-1) = \frac{1}{4}$	$f'(1) = -\frac{1}{4}$
Sing of f'	$+$	$-$



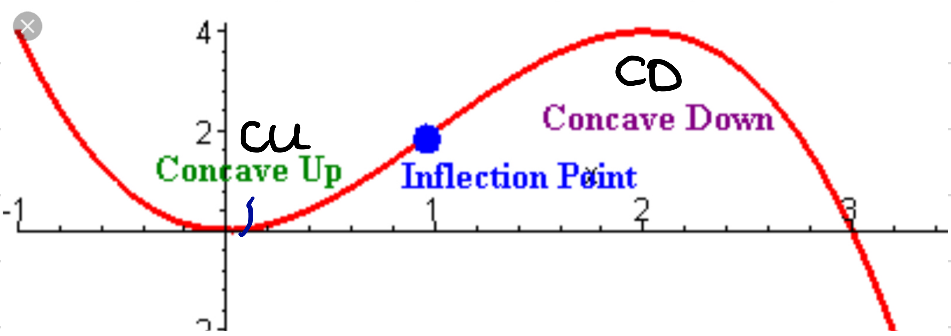
local max

$f(0) = \ln 9 = \ln 3^2 = 2\ln 3$

f is increasing on $(-3, 0)$

decreasing on $(0, 3)$

Concavity and the Second Derivative Test



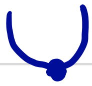
Definition and Test for Concavity

I: open interval
f: differentiable on I

By Defintion — By Test

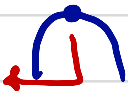
f' is increasin on I → CU
f' is decreasing on I → CD

$f''(x) > 0 \rightarrow$ CU



local min

$f''(x) < 0 \rightarrow$ CD



local max

Points of Inflection:

A point at which the graph of a function f changes concavity.

هي النقطة التي يتغير عندها تقعر منحنى الداله f

Example 1:

Find where the graph of $f(x) = \frac{1}{8}x^4 - \frac{1}{2}x^3 + \frac{1}{8}$ is concave up and concave down and points of inflection

$$f'(x) = \frac{1}{8} \cdot 4x^3 - 3 \cdot \frac{1}{2}x^2$$

$$= \frac{1}{2}x^3 - \frac{3}{2}x^2$$

$$f''(x) = \frac{3}{2}x^2 - \frac{6}{2}x$$

$$= \frac{3}{2}x^2 - \frac{3}{2}x \cdot 2$$

$$= \frac{3}{2}x(x-2)$$

$$f''(x) = 0 \Rightarrow \frac{3}{2}x(x-2) = 0$$

$$\Rightarrow \frac{3}{2}x = 0 \quad \text{or} \quad x-2=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x=2$$

$-\infty \quad 0 \quad 2 \quad \infty$

Test Value	-1	1	3
f''(Test Value)	$f''(-1) = \frac{9}{2}$	$f''(1) = \frac{-3}{2}$	$f''(3) = \frac{9}{2}$
sign of $f''(x)$	$+$	$-$	$+$
Concl	Up	Down	Up

$\therefore$ f is CU on $(-\infty, 0) \cup (2, \infty)$
is CD on $(0, 2)$

ملاحظه : نقط الإنقلاب هي
نفس النقاط التي يكون
التفاضل الثاني الداله يساوي
الصفر أو غير موجود
وتنتمي الى مجال الداله

The inflection points are 0 and 2.

Example 2:

Find where the graph of $f(x) = \frac{x^2+1}{x^2-1}$ is concave up and concave down.

$$f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{\cancel{2x^3} - 2x - \cancel{2x^3} - 2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

$$f''(x) = \frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4}$$

$$= \frac{4(3x^2+1)}{(x^2-1)^3}$$

$f''(x) = 0$

$3x^2+1 = 0$

but there is no such x

$f''(x)$ undefined

$(x^2-1)^3 = 0$

$\sqrt[3]{(x+1)(x-1)} = 0$

$x = \pm 1$

$\because D(f) = \mathbb{R} - \{\pm 1\}$

$\therefore$ f has no inflection points.

$-\infty \quad -1 \quad 1 \quad \infty$

Test Value	-2	0	2 ✓
f''(Test Value)	$f''(-2) = \frac{52}{27}$	$f''(0) = -4$	$f''(2) = \frac{52}{27}$ ✓
sign of $f''(x)$	+ ✓	− ✓	+ ✓
Concl	Up	Down	Up

$\therefore f$ is CU on $(-\infty,-1) \cup (1,\infty)$

is CD on $(-1,1)$

✓ **Example:** Find where the graph of $f(x) = \dfrac{x}{x^2-1}$ is concave up and concave down and points of inflection.

$$f'(x) = \frac{(x^2-1)(1) - x(2x)}{(x^2-1)^2}$$

$$= \frac{x^2-1-2x^2}{(x^2-1)^2}$$

$$= \frac{-x^2-1}{(x^2-1)^2}$$

$$= \frac{-(x^2+1)}{(x^2-1)^2}$$

$$f''(x) = \frac{(x^2-1)^2(-2x) + 2(x^2+1)\cdot 2(x)\,(x^2-1)}{(x^2-1)^4}$$

$$= \frac{2x\,(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0$

$2x\,(x^2+3) = 0$ → never zero

$2x = 0$

$x = 0$

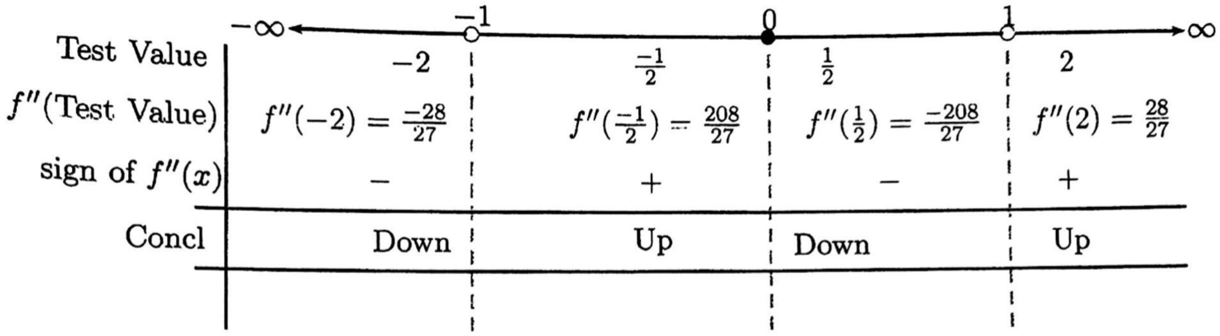
$f''(x)$ undefined

$(x^2-1)^3 = 0$

$\Rightarrow x = \pm 1$

$\because D(f) = \mathbb{R} - \{\pm 1\}$

$\therefore$ f has point of inflection at $x=0$



$-\infty$ ← −1 — 0 — 1 → ∞

Test Value	-2	$\frac{-1}{2}$	$\frac{1}{2}$	2
f''(Test Value)	$f''(-2) = \frac{-28}{27}$	$f''(\frac{-1}{2}) = \frac{208}{27}$	$f''(\frac{1}{2}) = \frac{-208}{27}$	$f''(2) = \frac{28}{27}$
sign of $f''(x)$	$-$	$+$	$-$	$+$
Concl	Down	Up	Down	Up

$\therefore$ f is CD on $(-\infty, -1) \cup (0, 1)$

is CU on $(-1, 0) \cup (1, \infty)$

Theorem 4.8.2: ***[The Second Derivative Test]***
Suppose that f'' is continuous on the open interval containing c such that $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a local minimum.
2. If $f''(c) < 0$, then $f(c)$ is a local maximum.

Example:

Find the local exterme for $f(x) = 2\sin x + \cos 2x$

$0 \leq x \leq 2\pi$.

$$f'(x) = 2\cos x - 2\sin 2x$$

$$= 2\cos x - 4\sin x \cos x$$

$$= 2\cos x(1 - 2\sin x)$$

$$f'(x) = 0 \Rightarrow 2\cos x(1 - 2\sin x) = 0$$

$$\Rightarrow 2\cos x = 0 \quad \text{or} \quad 1 - 2\sin x = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$f''(x) = -2\sin x - 4\cos(2x)$$

$f''(\pi/6) = -2\sin \pi/6) - 4\cos(2 \cdot \pi/6)$

$= -2 \cdot \frac{1}{2} - 4 \cdot \frac{1}{2}$

$= -1 - 2 = -3 < 0$

$F(\pi/6) = 2\sin(\pi/6) + \cos(2 \cdot \pi/6)$

$= 2 \cdot \frac{1}{2} + \frac{1}{2}$

$= 1 + \frac{1}{2} = \frac{3}{2}$ is a local max

$f''(5\pi/6) = -2\sin(5\pi/6) - 4\cos(2 \cdot 5\pi/6)$

$= -2 \cdot \frac{1}{2} - 4 \cdot \frac{1}{2}$

$= -1 - 2 = -3 < 0$

$F(5\pi/6) = 2\sin(5\pi/6) + \cos(2 \cdot 5\pi/6)$

$= 2 \cdot \frac{1}{2} + \frac{1}{2}$

$= 1 + \frac{1}{2} = \frac{3}{2}$ is a local max

$F''(\pi/2) = -2\sin(\pi/2) - 4\cos(2 \cdot \pi/2)$

$= -2(1) - 4(-1)$

$= -2 + 4 = 2 > 0$

$f(\pi/2) = 2\sin(\pi/2) + \cos(2 \cdot \pi/2)$

$= 2 \cdot (1) + (-1)$

$= 2 - 1 = 1$ is a local min

$F''(3\pi/2) = -2\sin(3\pi/2) - 4\cos(2 \cdot 3\pi/2)$

$= -2(-1) - 4(-1)$

$= 2 + 4 = 6 > 0$

$F(3\pi/2) = 2\sin(3\pi/2) + \cos(2 \cdot 3\frac{\pi}{2})$

$= 2 \cdot (-1) + (-1)$

$= -2 - 1 = -3$ is a local min

Antiderivative(integrals)

عكس التفاضل التكامل

Fundamental theorem of Calculus

For $F'(x) = f(x)$ then

Indefinite integral

$$\int f(x)\,dx = F(x) + c$$

Definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$F'(x) = \overset{f(x)}{2x}$$

what is $F(x)$ $\longrightarrow$ $\int 2x\,dx = x^2 + c$

باستخدام قوانين التكامل سنحصل على الدالة الاصيله F(x)

$$F(x) = x^2$$

سبب وجود ال c في التكامل

$$x^2 + 1 \xrightarrow{\text{مشتقه}} 2x$$

$$x^2 \longrightarrow 2x$$

∴ تكامل $2x$ قد تكون الدالة الأصلية له x^2 أو x^2+1 ولذلك نضع c

Properties of Integral

لوكان لدينا داخل التكامل دالتين بينهم عملية جمع او طرح فان التكامل يتوزع

(1)- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(2)- $\int c\,f(x)\,dx = c \int f(x)\,dx$

العدد الثابت يكون خارج التكامل

(3)- $\int_{-9}^{9} f(x)\,dx = 0$

اذا كان التكامل من سالب العدد الى موجب العدد دائما يكون صفر

$\int_{-3}^{3} x^2\,dx = 0$ check !!

Integral of Power Function

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad , n \neq -1$$

$$\int (f(x))^n \cdot f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Example 1: Evaluate $\int x^5 \, dx$

$$\int x^5 \, dx = \frac{x^6}{6} + c$$

Example 2: Evaluate $\int x^2 + 7 \, dx$

$$\int x^2 + 7 \, dx = \int x^2 \, dx + \int 7 \, dx$$

تطبيق الخاصيه ١ (توزيع التكامل)

$$= \int x^2 \, dx + 7 \int 1 \, dx$$

$$= \frac{x^3}{3} + 7x + c$$

دائماً :-

$$\int 1 \, dx = x$$

السبب: $x^0 =$ ١ و

$$\int x^0 \, dx$$

$$= \frac{x^{0+1}}{0+1} = x^1$$

Example 3: Evaluate $\int_0^1 x^2 \, dx$

$$\int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 =$$

$$= \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$$

داله مرفوعة لأس

مشتقتها

Example 4: Evaluate $\int \sin^2 x \cos x \, dx$

$$\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3} + C$$

نأخذ الداله المرفوعة للأس ونضيف عليه واحد ونقسم على الاس

دالة مرفوعة لأس

Example 5: Evaluate $\int_0^1 (3x-1)^3 \, dx$

$$\int_0^1 (3x-1) \, dx = \frac{1}{3} \int_0^1 (3x-1)^3 \cdot 3 \, dx$$

في هذا المثال لاتوجد مشتقه الداله ولكي اطبق القانون لابد ان اضرب في مشتقة الداله وأقسم عليه

$$= \frac{1}{3} \left. \frac{(3x-1)^4}{4} \right|_0^1$$

هنا طبقنا القانون لان التكامل اصبح على الصوره الداله في مشتقتها

$$= \frac{1}{3} \left[\frac{(3(1)-1)^4}{4} - \frac{(3(0)-1)^4}{4} \right]$$

$$= \frac{1}{3} \left[\frac{(2)^4}{4} - \frac{(-1)^4}{4} \right]$$

$$= \frac{1}{3} \left[\frac{16}{4} - \frac{1}{4} \right]$$

$$= \frac{1}{3} \left(\frac{15}{4} \right) = \frac{15}{12}$$

$$= \frac{5}{4}$$

Integral of Logarithmic Function

① $\int \frac{1}{x}\,dx = \ln|x| + c$

② $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

③ $\int \frac{f'(x)}{\sqrt{x}}\,dx = 2\sqrt{x} + c$

Example 1: compute $\int \frac{5}{x+1}\,dx$

تفاضل الداله

$$\int \frac{5}{x+1}\,dx = 5\int \frac{1}{x+1}\,dx$$

داله

نطبق القانون ٢ مباشره

$$= 5\ln|x+1| + c$$

Example 2: compute $\int \tan x\,dx$

مشتقة الداله

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

داله

في هذا المثال نطبق القانون 2

لأن مشتقة ال cosx هي
-sinx وإشارة السالب
غير موجودة لذلك نضرب
في السالب ونقسم عليه

$$= -\int \frac{-\sin x}{\cos x}\,dx$$

$$= -\ln|\cos x| + c$$

تم تطبيق القانون

$$= \ln|\cos x|^{-1} + c$$

تطبيق خواص الداله اللوغاريتميه

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c$$

Example 3: $\int_{1}^{e^2} \frac{3}{x}\,dx$

$$\int_{1}^{e^2} \frac{3}{x}dx = 3\int_{1}^{e^2} \frac{1}{x}\,dx$$ نطبق القانون ١ مباشره

$$= 3 \ln(x) \Big|_{1}^{e^2}$$

$$= 3\left[\ln(e^2) - \ln(1)\right]$$

$$= 3\left[2\ln(e) - \ln(1)\right]$$

$$= 3\,[2-0] = 6$$

Example 4: Compute $\int_{0}^{2} \frac{e^x}{e^x+1}\,dx$

مشتقة الدالة

داله

$$\int_{0}^{2} \frac{e^x}{e^x+1}\,dx = \ln\left|e^x+1\right| \Big|_{0}^{2}$$ نطبق القانون ٢ مباشره

$$= \ln\left|(e^2+1) - (e^0+1)\right|$$

$$= \ln\left|(e^2+1) - 2\right|$$

$$= \ln(e^2+1) - \ln(2)$$

Example 5 : Evaluate $\int \frac{1}{\cos^2 \sqrt{\tan x}} dx$

$$\frac{1}{\cos^2 x \sqrt{\tan x}} dx = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$$= 2\sqrt{\tan x} + c$$

تطبيق القانون ٣

Example 5 : Evaluate $\int_0^4 \frac{1}{\sqrt{2x+1}} dx$

لاتوجد مشتقه

داله تحت الجذر

$$\int_0^4 \frac{1}{\sqrt{2x+1}} dx = = \frac{1}{2} \int_0^4 \frac{2}{\sqrt{2x+1}} dx$$

مشتقة الداله

داله تحت الجذر

نضرب التكامل في ٢
و هو مشتقة الداله
ونقسم عليه لكي
نستطيع تطبيق قانون
التكامل

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{2x+1} \Big|_0^4$$

طبقنا القانون ٣ بعد تعديل
التكامل

$$= \sqrt{2(4)+1} - \sqrt{2(0)+1}$$

$$= \sqrt{9} - \sqrt{1}$$

$$= 3 - 1 = 2$$

Integral of Exponential function

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$

$$\int a^{x}\,dx = \frac{a^{x}}{\ln a} + c$$

Example 1 :- Compute $\int e^{2x}\,dx$

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$$

Example 2: Compute $\int_0^{\ln 5} 5e^{x}\,dx$

$$\int_0^{\ln 5} 5e^{x}\,dx = 5\int_0^{\ln 5} e^{x}\,dx$$

$$= 5e^{x}\Big|_0^{\ln 5}$$

$$= 5\left[e^{\ln 5} - e^{0}\right]$$

$$= 5\left[5-1\right]$$

$$= 5(4) = 20$$

Example 3: Compute $\int 2^x \, dx$

$$\int 2^x \, dx = \frac{2^x}{\ln 2} + c$$

Example 4: Evaluate $\int_0^1 2^x \, dx$

$$\int_0^1 2^x \, dx = \frac{2^x}{\ln 2} \Big|_0^1$$

$$= \frac{2^1}{\ln 2} - \frac{2^0}{\ln 2}$$

$$= \frac{2}{\ln 2} - \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2} .$$

Integral of Trigonometric Function

1) $\int \sin(kx)\,dx = \frac{-1}{k}\cos(kx) + c$

2) $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + c$

3) $\int \sec(kx)\tan(kx)\,dx = \frac{1}{k}\sec(kx) + c$

4) $\int \cos(kx)\,dx = -\frac{1}{k}\sin(kx) + c$

5) $\int \csc^2(kx)\,dx = -\frac{1}{k}\cot(kx) + c$

6) $\int \csc(kx)\cot(kx)\,dx = -\frac{1}{k}\csc(kx) + c$

Exampl 1: Evaluate $\int \sin(2x)\,dx$

$$\int \sin(2x)\,dx = \frac{-1}{2}\cos(2x) + c$$

Example 2: Evaluate $\int_0^{2\pi} \sin x\,dx$

$$\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi}$$

$$= -\cos(2\pi) + \cos(0)$$

$$= -1 + 1 = 0$$

Example 3: Evaluate $\int \cos(3x)\,dx$

$$\int \cos(3x)\,dx = \frac{1}{3}\sin x + C$$

Example 4: Evaluate $\int_0^{\pi/3} \cos(3x)\,dx$

$$\int_0^{\pi/3} \cos(3x)\,dx = \frac{1}{3}\sin(3x)\Big|_0^{\pi/3}$$

$$= \frac{1}{3}\left[\sin(3.\pi/3) - \sin(3.0)\right]$$

$$= \frac{1}{3}\left[\sin(\pi) - \sin(0)\right]$$

$$= \frac{1}{3}\left[0 - 0\right] = 0$$

Example 5: Evaluate $\int \sec^2(3x)\,dx$

$$\int \sec^2(5x)\,dx = \frac{1}{5}\tan(5x) + C$$

Example 6: Evaluate $\int_0^{\pi} \sec^2 \qquad dx$

$$\int_0^{\pi} \sec^2\left(\frac{x}{4}\right) dx = \int_0^{\pi} \sec^2\left(\frac{1}{4}x\right) dx$$

$$= \frac{1}{\frac{1}{4}} \tan\left(\frac{1}{4}x\right) \Big|_0^{\pi}$$

$$= 4 \tan\left(\frac{1}{4}x\right) \Big|_0^{\pi}$$

$$= 4 \left[\tan\left(\frac{1}{4}\cdot\pi\right) - \tan\left(\frac{1}{4}\cdot 0\right)\right]$$

$$= 4 \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right]$$

$$= 4 \left[1 - 0\right] = 4.$$

Example 7: Evaluate $\int \csc^2(3x)\, dx$

$$\int \csc^2(3x)\, dx = -\frac{1}{3} \cot(3x) + c$$

Example 8: Evaluate $\int \sec(4x) \tan(4x)\, dx$

$$\int \sec(4x) \tan(4x)\, dx = \frac{1}{4} \sec(4x) + C$$

Example 9: Evaluate $\int_{-\pi/4}^{\pi/4} \sec(4x)\tan(4x)\,dx$

$$\int \sec(4x)\tan(4x)\,dx = \frac{1}{4}\sec(4x)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\sec\left(4\cdot\frac{\pi}{4}\right) - \sec\left(4\cdot\frac{-\pi}{4}\right)\right]$$

$$= \frac{1}{4}\left[\sec(\pi) - \sec(-\pi)\right]$$

$$= \frac{1}{4}\left[-1 - (-1)\right]$$

$$= \frac{1}{4}\left[-1+1\right] = \frac{1}{4}(0) = 0.$$

Example 10: Evaluate $\int \csc(2x)\cot(2x)\,dx$

$$\int \csc(2x)\cot(2x)\,dx = -\frac{1}{2}\csc(2x) + C$$