2. The fundamental theorem of calculus

The fundamental theorem provides us with a much-needed shortcut for computing definite integral, and makes much stronger statements about the relationship between differentiation and integration.

THEOREM.2.1 (Fundamental Theorem of Calculus, part I)

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$,Then $\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$

EX.2.1 (Using the Fundamental Theorem of Calculus, part I)

- $\int_0^2 (x^2 - 2x)dx = (\frac{1}{3}x^3 - x^2)]_0^2 = (\frac{1}{3}.8 - 4) - 0 = \frac{-4}{3}$
- $\int_1^4 (\sqrt{x} - \frac{1}{x^2})dx = \int_1^4 (x^{\frac{1}{2}} - x^{-2})dx = (\frac{2}{3}x^{\frac{3}{2}} + x^{-1})]_1^4 = (\frac{2}{3}\sqrt[2]{4^3} + 4^{-1}) - (\frac{2}{3} + 1) = \frac{2}{3}(8) + \frac{1}{4} - \frac{5}{3} = \frac{11}{3} + \frac{1}{4} = \frac{47}{12}$

- A definite integral involving an Exponential function

$$\int_0^4 e^{-2x}\,dx = \left(\frac{1}{-2}e^{-2x}\right)\Big]_0^4 = -\frac{1}{2}e^{-8} + \frac{1}{2} \approx 0.49983$$

- A definite integral involving a Logarithm

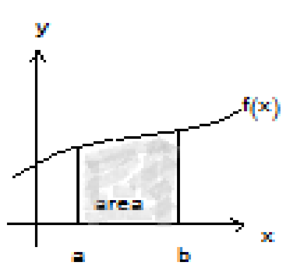
$$\int_{-3}^{-1} \frac{2}{x}\,dx = 2\int_{-3}^{-1} \frac{1}{x}\,dx = 2(\ln|x|)\Big]_{-3}^{-1}$$
$$= 2[\ln|-1| - \ln|-3|] = 2[\ln 1 - \ln 3] = -2\ln 3$$

EX.2.2 (Computing Areas)

Find the area under curve $f(x) = \sin x$ on the interval $[0, \pi]$

Solution

Recall that if $f(x) \geq 0$ on $[a, b]$, Then the integral $\int_a^b f(x)dx$ gives the area under the curve.



$$Area = \int_a^b f(x)dx$$

since $f(x) = \sin x \geq 0$ and $\sin x$ is continuous on $[0, \pi]$, we have that

$$Area = \int_0^{\pi} \sin x\ dx = -\cos x\Big]_0^{\pi} = -[\cos\pi - \cos 0] = [-1 - 1] = 2$$

The following Theorem gives us the form when the upper limit in a definite integral is unspecified value x.

THEOREM.2.2 (Fundamental Theorem of Calculus, part II)

If $f(x)$ is continuous on $[a,b]$ and $G(x)=\int_a^x f(t)dt$,

Then $G'(x)=f(x)$, on $[a,b]$

EX.2.3

For $G(x)=\int_1^x (t^2-2t+3)dt$, compute $G'(x)$

Solution Here, the integrand is $f(t)=t^2-2t+3$

By Fundamental theorem part (II), The derivative is

$$G'(x)=f(x)=x^2-2x+3$$

That is, $G'(x)$ is the function in the Integran

with t replaced by x.

Using the **Chain Rule** and Theorem2.2 ,we get the general form :

THEOREM.2.3 (An Integral with variable upper and lower limits)

(i) If $G(x) = \int_a^{u(x)} f(t)dt,\ then\ G'(x) = f\big(u(x)\big).u'(x)$

or $$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f\big(u(x)\big).u'(x)$$

(ii) $$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f\big(u(x)\big).u'(x) - f\big(v(x)\big).v'(x)$$

EX.2.4

If $G(x) = \int_2^{x^2} \cos t\ dt$, compute $G'(x)$

Solution Let $u(x) = x^2$, so that

$$G(x) = \int_2^{u(x)} \cos t\ dt$$

using the form (i), we get

$$G'(x) = \cos\big(u(x)\big)\,.\frac{d}{dx}\big(u(x)\big)$$

$$= \cos(x^2)\,.\frac{d}{dx}(x^2)$$

$$= \cos(x^2)\,.2x = 2x.\cos x^2$$

EX.2.5

If $F(x) = \int_{2x}^{x^2} \sqrt{t^2+1}\,dt$, compute $F'(x)$

Solution $F'(x) = \frac{d}{dx}\int_{2x}^{x^2} \sqrt{t^2+1}\,dt$

$$= \sqrt{(x^2)^2+1}\,\frac{d}{dx}(x^2) - \sqrt{(2x)^2+1}\,\frac{d}{dx}(2x)$$

$$= \sqrt{x^4+1}\,.\,2x - \sqrt{4x^2+1}\,.2$$

$$= 2x\sqrt{x^4+1} - 2\sqrt{4x^2+1}$$

EX.2.6 (Computing the distance fallen by an object)

Suppose the (downward) velocity of a skydiver is given by $v(t) = 30(1-e^{-t})$ ft/s for the first 5 seconds of a jump. Compute the distance fallen.

Solution Recall that the distance ***d*** is given by the definite integral (corresponding to area under the curve)

$$d = \int_0^5 v(t)\,dt$$

$$d = \int_0^5 (30 - 30e^{-t})\,dt = (30t + 30e^{-t})\Big]_0^5$$

$$= (150 + 30e^{-5}) - (0 + 30e^0)$$

$$= (150 + 30e^{-5}) - 30 = 120 + 30e^{-5} \approx 120.2 \text{ feet}$$

■

EX.2.7 (Rate of change and total change of volume of a tank)

Suppose the water can flow in and out of a storage Tank. The net rate of change (that is, the rate in minus the rate out) of water is $f(t) = 20(t^2 - 1)$ gallons per minute .

(a) For $0 \le t \le 3$, determine when the water level is increasing and when the water level is decreasing.

(b) If the tank has 200 gallons of water at time $t = 0$,

determine how many gallons are in the tank at time $t = 3$.

Solution Let $w(t)$ be the number of gallons in the tank at time t.

(a) Notice that the water level decreases if $w'(t) = f(t) < 0$ we have

$$f(t) = 20(t^2 - 1) < 0 \quad \text{if} \quad 0 \le t < 1$$

Alternatively, the water level increases if $w'(t) = f(t) > 0$ In this case, we have

$$f(t) = 20(t^2 - 1) > 0 \quad \text{if} \quad 1 < t \le 3$$

Diagram of sign $t^2 - 1 = (t-1)(t+1)$		-1		+1	
sign of (t+1)	−		+		+
sign of (t-1)	−		−		+
sign of $(t^2 - 1)$	+		−		+
Since $t \ge 0$ then			0	1	
sign of $f(t)$			−		+

(b) we start with $w'(t) = 20(t^2 - 1)$.

Integrating from $t = 0$ to $t = 3$, we have

$$\int_0^3 w'(t)\,dt = \int_0^3 20(t^2 - 1)\,dt$$

Evaluating the integral on both sides yields

$$w(3) - w(0) = 20\left(\frac{t^3}{3} - t\right)\Big]_{t=0}^{t=3}$$

Since $w(0) = 200$,we have

$$w(3) - 200 = 20(9 - 3) = 120$$

and hence

$$w(3) = 120 + 200 = 320$$

so that the tank will have 320 gallons at time $t = 3$ ■

EX.2.8 (Finding a tangent line for a function defined as an Integral)

For $F(x)=\int_4^{x^2}\ln(t^3+4)dt$, find an equation of the tangent line at $x=2$.

Solution Recall that the equation of the tangent line to $y=F(x)$ at $x=a$ is

$$y-F(a)=F'(a)(x-a)$$

From THEOREM.2.3, we have

$$F'(x)=\ln[(x^2)^3+4]\frac{d}{dx}(x^2)=[\ln(x^6+4)](2x).$$

so, the slope at x=2 is

$$F'(2)=(\ln 68)(4)\approx 16.878$$

But $F(2)=\int_4^{4}\ln(t^3+4)dt=0$ (since the upper limit equals the lower limit)

An equation of the tangent line to $y=F(x)$ at $x=2$ is

$$y-F(2)=F'(2).(x-2)$$

$$y-0=16.878\ (x-2)$$

or $$y=4\ln 6.8\ \ (x-2)$$ ∎

Exercises

1. Compute the following Integrals

- $I_1 = \int_1^2 (4x^3 - 2x)dx$
- $I_2 = \int_0^{\frac{\pi}{6}} \sin 3x \, dx$
- $I_3 = \int_1^2 \frac{3}{2}\sqrt{x} \, dx$

2. Compute the derivative of the following functions

- $G_1(x) = \int_5^{1+2x^2} \frac{t^4}{\sqrt{1+t^2}} dt$
- $G_2(x) = \int_{\sqrt{x}}^1 |\sin(1 + t^3)| \, dt$
- $G_3(x) = \int_{\sin x}^{\cos x} u\sqrt{1 + u^4} \, du$ at $x = \frac{\pi}{4}$

Recall the conclusion of part (I) and part (II) of the fundamental theorem:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

and $$\frac{d}{dx}\int_a^x f(t) \, dt = f(x)$$

THE END