SYRIAN PRIVATE UNIVERSITY

FACULTY OF COMPUTER & INFORMATICS ENGINEERING

الجامعة السورية الخاصة

كلية هندسة الحاسوب والمعلوماتية

OPERATIONS RESEARCH

بحوث العمليات

الأستاذ الدكتور أنور اللحام

Prof. Dr. Anwar Al-Lahham

الجامعة السورية الخاصة

2017 م - 1438 هـ

SUMMARY

Linear Programming Definitions

A **linear programming problem** (LP) consists of three parts:

1 A linear function (the **objective function**) of decision variables (say, $x_1, x_2, \ldots, x_n$) that is to be maximized or minimized.

2 A set of **constraints** (each of which must be a linear equality or linear inequality) that restrict the values that may be assumed by the decision variables.

3 The **sign restrictions,** which specify for each decision variable x_j either (1) variable x_j must be nonnegative—$x_j \geq 0$; or (2) variable x_j may be positive, zero, or negative—x_j is **unrestricted in sign (urs).**

The coefficient of a variable in the objective function is the variable's **objective function coefficient.** The coefficient of a variable in a constraint is a **technological coefficient.** The right-hand side of each constraint is called a **right-hand side (rhs).**

A *point* is simply a specification of the values of each decision variable. The **feasible region** of an LP consists of all points satisfying the LP's constraints and sign restrictions. Any point in the feasible region that has the largest z-value of all points in the feasible region (for a max problem) is an **optimal solution** to the LP. An LP may have no optimal solution, one optimal solution, or an infinite number of optimal solutions.

A constraint in an LP is **binding** if the left-hand side and the right-hand side are equal when the values of the variables in the optimal solution are substituted into the constraint.

Graphical Solution of Linear Programming Problems

The feasible region for any LP is a **convex set.** If an LP has an optimal solution, there is an extreme (or corner) point of the feasible region that is an optimal solution to the LP.

We may graphically solve an LP (max problem) with two decision variables as follows:

Step 1 Graph the feasible region.

Step 2 Draw an isoprofit line.

Step 3 Move parallel to the isoprofit line in the direction of increasing z. The last point in the feasible region that contacts an isoprofit line is an optimal solution to the LP.

LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

Case 1 The LP has a unique solution.

Case 2 The LP has more than one (actually an infinite number of) optimal solutions. This is the case of **alternative optimal solutions.** Graphically, we recognize this case when the isoprofit line last hits an entire line segment before leaving the feasible region.

Case 3 The LP is **infeasible** (it has no feasible solution). This means that the feasible region contains no points.

Case 4 The LP is unbounded. This means (in a max problem) that there are points in the feasible region with arbitrarily large z-values. Graphically, we recognize this case by the fact that when we move parallel to an isoprofit line in the direction of increasing z, we never lose contact with the LP's feasible region.

Linear Programming: A Geometrical Approach Homework

1) MAXIMIZATION APPLICATIONS

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

Exercise (1)
A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

Exercise (2)
Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

Exercise (3)
A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize revenue?

Exercise (4)
The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?

2) MINIMIZATION APPLICATIONS

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

Exercise (5)
A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. If Food A costs $2 per unit and Food B cost $1 per unit, how many units of each food should be purchased to keep costs at a minimum?

Exercise (6)
A computer store sells two types of computers, desktops and laptops. The supplier demands that at least 150 computers be sold a month. In order to keep profits up, the number of desktops sold must be at least twice of laptops. The store pays its sales sta_ a $75 commission for each desk top, and a $50 commission for each lap top. How many of each type of computers must be sold to minimize commission to its sales people? What is the minimum commission?

Exercise (7)
An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil. How many days should each Refinery be operated to meet the goals at a minimum cost?

Exercise (8)
A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of $800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of $1000 per month. How many months should each of the two contractors be employed to minimize the cost?

3) REVIEW EXERCISES

Solve the following linear programming problems by the graphical method.

Exercise (9)
Mr. Shoemacher has $20,000 to invest in two types of mutual funds, Coleman High-yield Fund, and Coleman Equity Fund. The High-yield fund gives an annual yield of 12%, while the Equity fund earns 8%. Mr. Shoemacher would like to invest at least $3000 in the High-yield fund and at least $4000 in the Equity fund. How much money should he invest in each to maximize his annual yield, and what is the maximum yield?

Exercise (10)
Dr. Lum teaches part-time at two different community colleges, Hilltop College and Serra College. Dr. Lum can teach up to 5 classes per semester. For every class taught by him at Hilltop College, he needs to spend 3 hours per week preparing lessons and grading papers, and for each class at Serra College, he must do 4 hours of work per week. He has determined that he cannot spend more than 18 hours per week preparing lessons and grading papers. If he earns $4,000 per class at Hilltop College and $5,000 per class at Serra College, how many classes should he teach at each college to maximize his income, and what will be his income?

Exercise (11)
Mr. Shamir employs two part-time typists, Inna and Jim for his typing needs. Inna charges $10 an hour and can type 6 pages an hour, while Jim charges $12 an hour and can type 8 pages per hour. Each typist must be employed at least 8 hours per week to keep them on the payroll. If Mr. Shamir has at least 208 pages to be typed, how many hours per week should he employ each student to minimize his typing costs, and what will be the total cost?

Exercise (12)
Mr. Boutros wants to invest up to $20,000 in two stocks, Cal Computers and Texas Tools. The Cal Computers stock is expected to yield a 16% annual return, while the Texas Tools stock promises a 12% yield. Mr. Boutros would like to earn at least $2,880 this year. According to Value Line Magazine's safety index (1 highest to 5 lowest), Cal Computers has a safety number of 3 and Texas Tools has a safety number of 2. How much money should he invest in each to minimize the safety number? Note: A lower safety number means less risk.

Exercise (13)
A department store sells two types of televisions: Regular and Big Screen. The store can sell up to 90 sets a month. A Regular television requires 6 cubic feet of storage space, and a Big Screen television requires 18 cubic feet of space, and a maximum of 1080 cubic feet of storage space is available. The Regular and Big Screen televisions take up, respectively, 2 and 3 sales hours of labor, and a maximum of 198 hours of labor is available. If the profit made from each of these types is $60 and $80, respectively, how many of each type of television should be sold to maximize pro_t, and what is the maximum profit?

Exercise (14)
A company manufactures two types of printers, the Inkjet and the Laser. The Inkjet generates a profit of $100 per printer and the Laser a profit of $150. On the assembly line the Inkjet requires 7 hours, while the Laser takes 11 hours. Both printers require one hour for testing. The Inkjet requires one hour and the Laser needs 3 hours for finishing. On a particular production run the company has available 1,540 work hours on the assembly line, 200 work hours in the testing department, and 360 work hours for finishing. How many sets of each type should the company produce to maximize profit, and what is that maximum profit?

Exercise (15)
John wishes to choose a combination of two types of cereals for breakfast - Cereal A and Cereal B. A small box(one serving) of Cereal A costs $0.50 and contains 10 units of vitamins, 5 units of minerals, and 15 calories. A small box(one serving) of Cereal B costs $0.40 and contains 5 units of vitamins, 10 units of minerals, and 15 calories. John wants to buy enough boxes to have at least 500 units of vitamins, 600 units of minerals, and 1200 calories. How many boxes of each food should he buy to minimize his cost, and what is the minimum cost?

Exercise (16)
Jessica needs at least 60 units of vitamin A, 40 units of vitamin B, and 140 units of vitamin C each week. She can choose between Costless brand or Save more brand tablets. A Costless tablet costs 5 cents and contains 3 units of vitamin A, 1 unit of vitamin B, and 2 units of vitamin C, and a Save more tablet costs 7 cents and contains 1 unit of A, 1 of B, and 5 of C. How many tablets of each kind should she buy to minimize cost, and what is the minimum cost?

Exercise (17)
A small company manufactures two types of radios- regular and short-wave. The manufacturing of each radio requires three operations: Assembly, Finishing and Testing. The regular radios require 1 hour of Assembly, 3 hours of Finishing, and 1 hour of Testing. The short-wave radios require 3 hours of Assembly, 1 hour of Finishing, and 1 hour of Testing. The total work-hours available per week in the Assembly division is 60, in the Finishing division is 60, and in the Testing is 24. If a profit of $50 is realized for every regular radio, and $75 for every short-wave radio, how many of each should be manufactured to maximize profit, and what is the maximum profit?

Exercise (18)
A factory manufactures two products, A and B. Each product requires the use of three machines, Machine I, Machine II, and Machine III. The time requirements and total hours available on each machine are listed below.

	Machine I	Machine II	Machine III
Product A	1	2	4
Product B	2	2	2
Total hours	70	90	160

Table 6.1

If product A generates a profit of $60 per unit and product B a profit of $50 per unit, how many units of each product should be manufactured to maximize profit and what is the maximum profit?

Exercise (19)
A company produces three types of shoes, formal, casual, and athletic, at its two factories, Factory I and Factory II. Daily production of each factory for each type of shoe is listed below.

	Factory I	Factory II
Formal	100	100
Casual	100	200
Athletic	300	100

The company must produce at least 6000 pairs of formal shoes, 8000 pairs of casual shoes, and 9000 pairs of athletic shoes. If the cost of operating Factory I is $1500 per day and the cost of operating Factory II is $2000, how many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

Exercise (20)
A professor gives two types of quizzes, objective and recall. He is planning to give at least 15 quizzes this quarter. The student preparation time for an objective quiz is 15 minutes and for a recall quiz 30 minutes. The professor would like a student to spend at least 5 hours (300 minutes) preparing for these quizzes above and beyond the normal study time. The average score on an objective quiz is 7, and on a recall type 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type should he give in order to minimize his grading time?

Exercise (21)
A company makes two mixtures of nuts: Mixture A and Mixture B. Mixture A contains 30% peanuts, 30% almonds and 40% cashews and sells for $5 per pound. Mixture B contains 30% peanuts, 60% almonds and 10% cashews and sells for $3 a pound. The company has 540 pounds of peanuts, 900 pounds of almonds, 480 pounds of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit, and what is the maximum profit?

Solutions to Exercises

Solutions to Linear Programming: A Geometrical Approach: Homework

Solution to Exercise (1)
80 acres of wheat and 20 acres of corn should be planted to maximize profit to $8,400
Solution to Exercise (2)
(16000, 8000); $1600
Solution to Exercise (3)
10 chairs and 15 tables should be manufactured to maximize profit to $650.
Solution to Exercise (4)
(1320, 1120); $6880
Solution to Exercise (5)
30 units of Food A and 45 units of Food B should be purchased to keep costs at a minimum of $105.
Solution to Exercise (6)
(100, 50); $10000
Solution to Exercise (7)
Min: $Z = 12000.x1 + 10000.x2$
I. $200x1 + 100x2 \geq$ 800 high-grade oil
II. $300x1 + 100x2 \geq$ 900 medium-grade oil
III. $200x1+ 200x2 \geq$ 1000 low-grade oil
Refinery A should be operated for 3 days, while Refinery B should be operated for 2 days to keep a minimum cost of $56,000.
Solution to Exercise (8)
(6, 4); $8800

Solution to Exercise (9) (16000, 4000); $2240
Solution to Exercise (10) (2, 3); $23,000
Solution to Exercise (11) (8, 20); $320
Solution to Exercise (12) (12000, 8000)
Solution to Exercise (13) (72, 180); $5760
Solution to Exercise (14) (165, 35); $21,750
Solution to Exercise (15) (20, 60); $34
Solution to Exercise (16) (20, 20); $2.40
Solution to Exercise (17) (6, 18); $1650
Solution to Exercise (18) (35, 100); $2600
Solution to Exercise (19) (40, 20); $100,000
Solution to Exercise (20) (10, 5): 17.5 minutes
Solution to Exercise (21) (1000, 800); $7400

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

Formulating LPs

The most important step in formulating most LPs is to determine the decision variables correctly.

In any constraint, the terms must have the same units. For example, one term cannot have the units "pounds of raw material" while another term has the units "ounces of raw material."

SUMMARY Preparing an LP for Solution by the Simplex

An LP is in **standard form** if all constraints are equality constraints and all variables are nonnegative. To place an LP in standard form, we do the following:

Step 1 If the ith constraint is a $\leq$ constraint, then we convert it to an equality constraint by adding a slack variable s_i and the sign restriction $s_i \geq 0$.

Step 2 If the ith constraint is a $\geq$ constraint, then we convert it to an equality constraint by subtracting an excess variable e_i and adding the sign restriction $e_i \geq 0$.

Step 3 If the variable x_i is unrestricted in sign (urs), replace x_i in both the objective function and constraints by $x_i' - x_i''$, where $x_i' \geq 0$ and $x_i'' \geq 0$.

Suppose that once an LP is placed in standard form, it has m constraints and n variables.

A basic solution to $A\mathbf{x} = \mathbf{b}$ is obtained by setting $n - m$ variables equal to 0 and solving for the values of the remaining m variables. Any basic solution in which all variables are nonnegative is a **basic feasible solution** (bfs) to the LP.

For any LP, there is a unique extreme point of the LP's feasible region corresponding to each bfs. Also, at least one bfs corresponds to each extreme point of the feasible region.

If an LP has an optimal solution, then there is an extreme point that is optimal. Thus, in searching for an optimal solution to an LP, we may restrict our search to the LP's basic feasible solutions.

The Simplex Algorithm

1) Solving Maximization Problems

If the LP is in standard form and a bfs is readily apparent, then the simplex algorithm (for a max problem) proceeds as follows:

Step 1 If all nonbasic variables have nonnegative coefficients in row 0, then the current bfs is optimal. If any variables in row 0 have negative coefficients, then choose the variable with the most negative coefficient in row 0 to enter the basis.

Step 2 For each constraint in which the entering variable has a positive coefficient, compute the following ratio:

$$\frac{\text{Right-hand side of constraint}}{\text{Coefficient of entering variable in constraint}}$$

The Big M Method

Step 1 Modify the constraints so that the right-hand side of each constraint is nonnegative.

Step 1′ Identify each constraint that is now (after step 1) an = or ≥ constraint. In step 3, we will add an artificial variable to each of these constraints.

Step 2 Convert each inequality constraint to standard form.

Step 3 If (after step 1 has been completed) constraint i is a ≥ or = constraint, then add an artificial variable a_i and the sign restriction $a_i \geq 0$.

Step 4 Let M denote a very large positive number. If the LP is a min problem, then add (for each artificial variable) Ma_i to the objective function. For a max problem, add $-Ma_i$.

Step 5 Because each artificial variable will be in the starting basis, each must be eliminated from row 0 before beginning the simplex. If all artificial variables are equal to 0 in the optimal solution, then we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, then the original problem is infeasible.

2) Solving Minimization Problems

To solve a minimization problem by the simplex, choose as the entering variable the nonbasic variable in row 0 with the most positive coefficient. A tableau or canonical form is optimal if each variable in row 0 has a nonpositive coefficient.

Alternative Optimal Solutions

If a nonbasic variable has a zero coefficient in row 0 of an optimal tableau and the nonbasic variable can be pivoted into the basis, the LP may have **alternative optimal solutions.** If two basic feasible solutions are optimal, then any point on the line segment joining the two optimal basic feasible solutions is also an optimal solution to the LP.

Unrestricted-in-Sign Variables

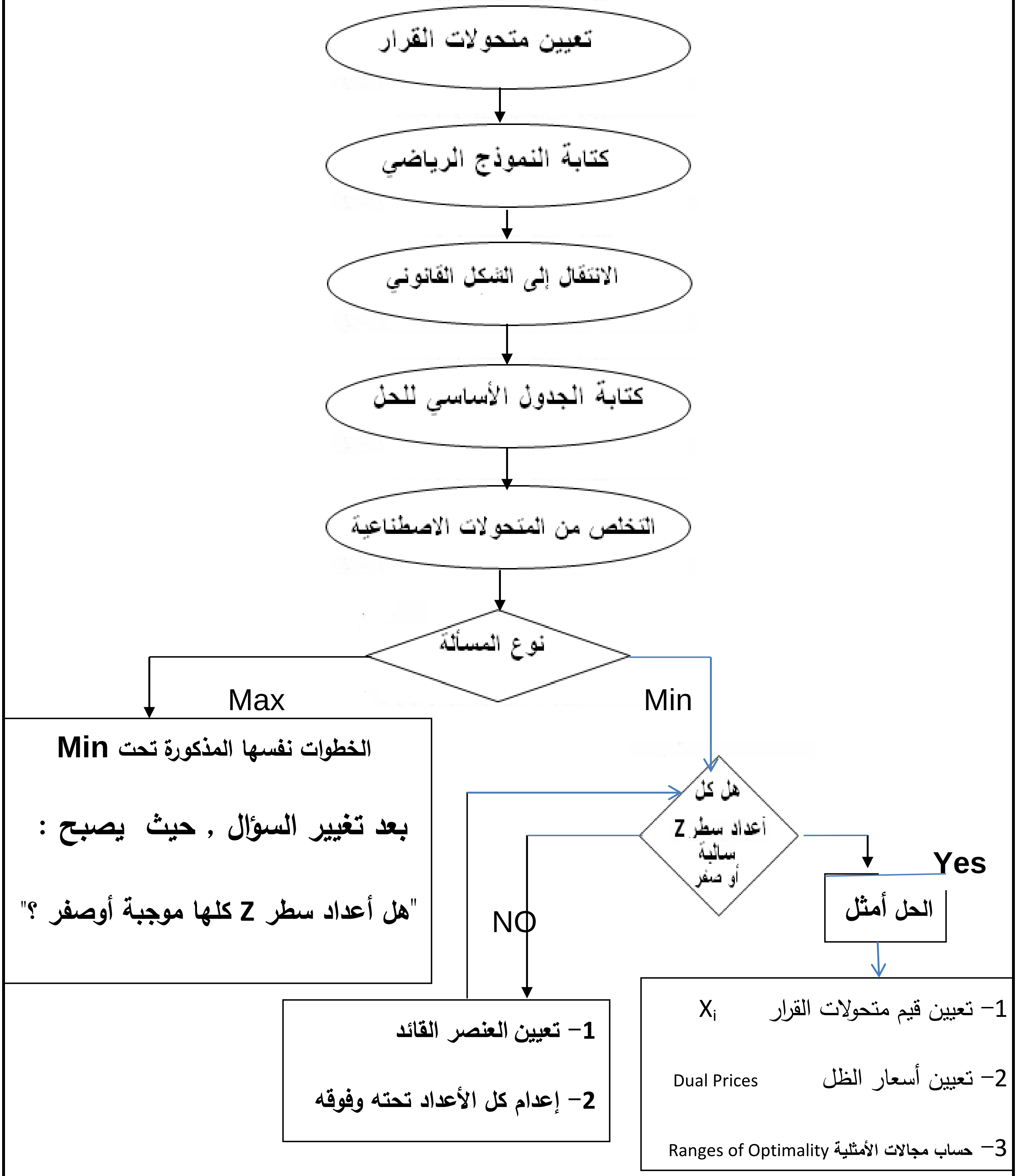
If we replace a urs variable x_i with $x_i' - x_i''$, the LP's optimal solution will have x_i', x_i'' or both x_i' and x_i'' equal to zero.

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

خوارزمية إيجاد الحل الأمثل

باستخدام طريقة السيمبلكس



تمارين محلولة

مثال محلول (1)

$$\begin{aligned} Max \quad & Z = x_1 + 2x_2 + x_3 \\ s.t. \quad & 2x_1 + 3x_2 - x_3 \le 5 \\ & x_1 \ - x_2 + 2x_3 \ \le 4 \\ & 3x_1 + 2x_2 + x_3 \le 7 \\ & x_1 \ , \ x_2 \ , x_3 \ge 0 \end{aligned}$$

The Optimal solution is:
X1 = 0 , X2= 2.4 , X3 = 2.2
The Optimal Value is: Z = 7
The Dual Prices Are:
0 0 1
The Ranges of C_i are:
$C1 \le 3$
$2 \le C2$
$-\frac{2}{3} \le C3 \le 1$

Basic V.	X1	X2	X3	S1	S2	S3	RHS
Z	-1	-2	-1	0	0	0	0
S1	2	3	-1	1	0	0	5
S2	1	-1	2	0	1	0	4
S3	3	2	1	0	0	1	7
Z	1/3	0	-5/3	2/3	0	0	10 / 3
X2	2/3	1	-1/3	1/3	0	0	5 / 3
S2	5/3	0	5/3	1/3	1	0	17 / 3
S3	5/3	0	5/3	-2/3	0	1	11/3
Z	2	0	0	0	0	1	7
X2	1	1	0	1/5	0	1/5	2.4
S2	0	0	0	1	1	-1	2
X3	1	0	1	-2/5	0	3/5	2.2

مثال محلول (2)

$$\begin{aligned} Max \quad & z = x_1 + 2x_2 + x_3 \\ s.t. \quad & \\ & 2x_1 + 3x_2 - x_3 \le 5 \\ & x_1 \ - x_2 + 2x_3 \ \ge 4 \\ & 3x_1 + 2x_2 + x_3 = 7 \\ & x_1 \ , \ x_2 \ , \ \ x_3 \ \ \ge 0 \end{aligned}$$

The Optimal solution is:
X1 = 0 , X2= 2 , X3 = 3
The Optimal Value is: Z = 7
The Dual Prices Are:
0 0 1
The Ranges of C_i are:
$C1 \le 3$
$2 \le C2$
$-1 \le C3 \le 1$

Basic V.	X1	X2	X3	S1	S2	A2	A3	RHS
Z	-1	-2	-1	0	0	M	M	0
S1	2	3	-1	1	0	0	0	5
A2	1	-1	2	0	-1	1	0	4
A3	3	2	1	0	0	0	1	7
Z	1/3	0	-5/3	2/3	0	M	M	10 / 3
X2	2 / 3	1	-1 / 3	1 / 3	0	0	0	5 / 3
A2	5 / 3	0	5 / 3	1 / 3	-1	1	0	17 / 3
A3	5/3	0	5/3	-2/3	0	0	1	11/3
Z	2	0	0	0	0	M	M+1	7
X2	1	1	0	1/5	0	0	1/5	12 / 5
A2	0	0	0	1	-1	1	-1	2
X3	1	0	1	-2/5	0	0	3/5	11/5
Z	2	0	0	0	0	M	M+1	7
X2	1	1	0	0	1/5	-1/5	2/5	2
S1	0	0	0	1	-1	1	-1	2
X3	1	1	1	0	-2/5	2/5	1/5	3

مثال محلول (3)

The Mathematical Model:

$$\begin{aligned} &Min \quad z = x_1 + 2x_2 + x_3 \\ &s.t. \\ &2x_1 + 3x_2 - x_3 \le 5 \\ &x_1 - x_2 + 2x_3 \ge 4 \\ &3x_1 + 2x_2 + x_3 = 7 \\ &x_1 \, , \; x_2 \; \ge 0 \end{aligned}$$

The Standard (Canonical) Form:

$$z - x_1 - 2x_2 - x_3 + 0S_1 + 0S_2 - MA_2 - MA_3 = 0$$

$$\begin{aligned} &2x_1 + 3x_2 - 1x_3 + 1S_1 = 5 \\ &1x_1 - x_2 + 2x_3 - 1S_2 + 1A_2 = 4 \\ &3x_1 + 2x_2 + 1x_3 + 1A_3 = 7 \\ &x_1 \, , x_2 , S_1 , S_2 , A_2 , A_3 \; \ge 0 \end{aligned}$$

Basic V.	X1	X2	X3	S1	S2	A2	A3	RHS
Z	-1	-2	-1	0	0	-M	-M	0
S1	2	3	-1	1	0	0	0	5
A2	1	-1	2	0	-1	1	0	4
A3	3	2	1	0	0	0	1	7
Z	2	0	0	0	0	-M	1-M	7
S1	5	5	0	1	0	0	1	12
A2	5	5	0	0	1	-1	2	10
X3	3	2	1	0	0	0	1	7
Z	0	-2	0	0	-2/5	2/5-M	1/5-M	3
S1	0	0	0	1	-1	1	-1	2
X1	1	1	0	0	1/5	-1/5	2/5	2
X3	0	-1	1	0	-3/5	3/5	-1/5	1

الحل الأمثل: The Optimal Solution $X3 = 1$ $X2 = 0$ $X1 = 2$

القيمة المثلى: The Optimal value $Z = 3$

الأسعار الثنوية : The Dual Prices للقيد الأول 0 للقيد الثاني -0.4 للقيد الثالث -0.2

مجالات أمثال دالة الهدف : The Optimal Ranges $1/3 \le C3$ $0 \le C2$ $C1 \le 3$

مثال محلول (4)

The Primal Problem:

$$\begin{aligned} Min \quad & Z = 3x_1 + 5x_2 \\ s.t. \quad & x_1 + x_2 \ \geq 6 \\ & x_1 + 2x_2 \leq 10 \\ & x_2 \ \geq 2 \\ & x_1 , x_2 \geq 0 \end{aligned}$$

The Dual Problem:

$$\begin{aligned} Max \quad & Z = 6y_1 - 10y_2 + 2y_3 \\ s.t. \quad & y_1 - y_2 \quad \leq 3 \\ & y_1 - 2y_2 + y_3 \ \leq 5 \\ & y_1 , y_2, y_3 \geq 0 \end{aligned}$$

B . V.	**X1**	**X2**	**S1**	**S2**	**S3**	**A1**	**A3**	**S**
Z	-3	-5	0	0	0	-M	-M	0
A1	1	1	-1	0	0	1	0	6
S2	1	2	0	1	0	0	0	10
A3	0	1	0	0	-1	0	1	2
Z	0	-2	-3	0	0	3 - M	-M	18
X1	1	1	-1	0	0	1	0	6
S2	0	1	1	1	0	-1	0	4
A3	0	1	0	0	-1	0	1	2
Z	0	0	-3	0	-2	3 - M	2 - M	22
X1	1	0	-1	0	1	1	-1	4
S2	0	0	1	1	1	-1	-1	2
X2	0	1	0	0	-1	0	1	2

B . V.	**Y1**	**Y2**	**Y3**	**S1**	**S2**	**HS**
Z	- 6	10	-2	0	0	0
S1	1	-1	0	1	0	3
S2	1	-2	1	0	1	5
Z	0	4	-2	6	0	18
Y1	1	-1	0	1	0	3
S2	0	-1	1	-1	1	2
Z	0	2	0	4	2	22
Y1	1	-1	0	1	0	3
Y3	0	-1	1	-1	1	2

The Optimal Solution Is:

X1 = 4 X2 = 2

The Optimal Value is:

Z = 22

The Optimal Ranges are:

$$0 \leq C1 \leq 5 \qquad 3 \leq C2$$

The Optimal Solution Is: Y1 = 3 Y2 = 0 Y3 = 2

The Optimal Value is: Z = 22

The Dual Prices are:

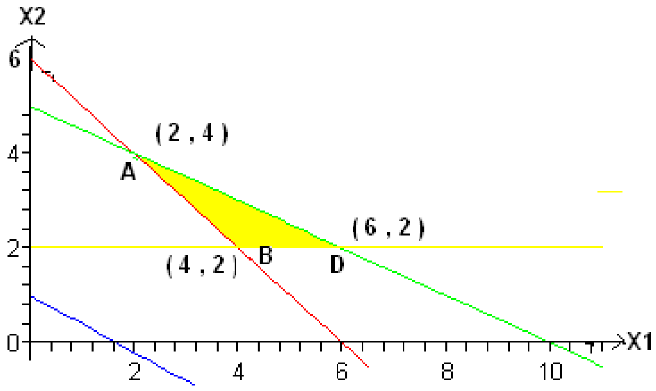
for the first constraint 4

for the Second constraint 2

The Optimal Ranges are:

$$2 \leq C1 \leq 8 \qquad C2 \leq - 8 \qquad 0 \leq C3 \leq 4$$

The Graphical Method:



Note :

1) The Optimal Solution of The Primal Problem is
The absolute value of the Dual Prices Of The Dual Problem.

2) The Optimal Value of The Primal Problem is The Optimal Value Of The Dual Problem

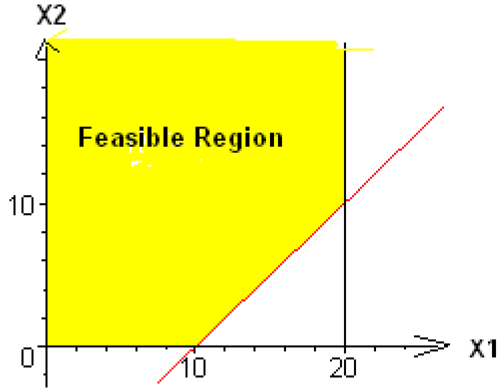
أمثلة محلولة (5 + 6)

(5)

$$\begin{aligned} Max \quad & Z = 2x_1 + 1x_2 \\ s.t. \quad & x_1 - x_2 \le 10 \\ & 2x_1 \le 40 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

B . V.	X1	X2	S1	S2	HS
Z	-2	-1	0	0	0
S1	1	-1	1	0	10
S2	2	0	0	1	40
Z	0	-3	2	0	20
X1	1	-1	1	0	10
S2	0	2	-2	1	20
Z	0	0	-1	3/2	50
X1	1	-1	0	1/2	20
X2	0	1	-1	1/2	10

UNBOUNDED



(6)

$$\begin{aligned} Max \quad & Z = 4x_1 + 1x_2 \\ s.t. \quad & 3x_1 + x_2 = 15 \\ & 4x_1 + 3x_2 \ge 30 \\ & 1x_1 + 2x_2 \le 20 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

B . V.	X1	X2	S2	S3	A1	A2	RHS
Z	- 4	-1	0	0	M	M	0
A1	3	1	0	0	1	0	15
A2	4	3	-1	0	0	1	30
S3	1	2	0	1	0	0	20
Z	0	1/3	0	0	M+4/3	M	20
X1	1	1/3	0	0	1/3	0	5
A2	0	5/3	-1	0	-4/3	1	10
S3	0	5/3	0	1	-1/3	0	15
Z	0	0	1/5	0	M+8/5	M+1/5	18
X1	1	0	1/5	0	3/5	1/5	3
X2	0	1	-3/5	0	-4/5	3/5	6
S3	0	0	1	1	1	-1	5

The Optimal Solution Is: X1 = 3 X2 = 6 Z = 18

The Dual Prices are: for the first constrain 1.6

for the Second constrain - 0.2

for the Third constrain 0

The Optimal Ranges are :

$$4 - 1 = 3 \le C_1 \qquad C_2 \le \frac{4}{3} = 1 - (-\frac{1}{3})$$

مثال محلول (7)

The Mathematical Model

$$\begin{aligned} Max\quad Z &= 6x_1+4x_2+3x_3+2x_4 \\ Subject\ to\quad & 2x_1+3x_2+1x_3+2x_4 \le 400 \\ & 1x_1+1x_2+2x_3+1x_4 \le 150 \\ & 2x_1+1x_2+1x_3+0.5x_4 \le 200 \\ & 3x_1+1x_2+\quad +1x_4 \le 250 \\ & x_1\ ,x_2\ ,x_3\ ,x_4 \ge 0 \end{aligned}$$

The Standard (canonical) Form is

$$\begin{aligned} Z-\ 6x_1-4x_2-3x_3-2x_4+0S_1+0S_2+0S_3+0S_4 &= 0 \\ 2x_1+3x_2+1x_3+2x_4+1S_1 &= 400 \\ 1x_1+1x_2+2x_3+1x_4\quad +1S_2 &= 150 \\ 2x_1+1x_2+1x_3+0.5x_4\quad +1S_3 &= 200 \\ 3x_1+1x_2+\quad +1x_4\quad +1S_4 &= 250 \\ x_1\ ,x_2\ ,x_3\ ,x_4,S_1\ ,S_2\ ,S_3\ ,S_4 &\ge 0 \end{aligned}$$

Basic V.	X1	X2	X3	X4	S1	S2	S3	S4	RHS
Z	-6	-4	-3	-2	0	0	0	0	0
S1	2	3	1	2	1	0	0	0	400
S2	1	1	2	1	0	1	0	0	150
S3	2	1	1	0.5	0	0	1	0	200
S4	3	1	0	1	0	0	0	1	250
Z	0	-2	-3	0	0	0	0	2	500
S1	0	$7/3$	1	$4/3$	1	0	0	$-2/3$	$700/3$
S2	0	$2/3$	2	$2/3$	0	1	0	$-1/3$	$200/3$
S3	0	$1/3$	1	$-1/6$	0	0	1	$-2/3$	$100/3$
X1	1	$1/3$	0	$1/3$	0	0	0	$1/3$	$250/3$
Z	0	-1	0	1	0	1.5	0	1.5	600
S1	0	2	0	1	1	$-1/2$	0	$-1/2$	200
X3	0	$1/3$	1	$1/3$	0	$1/2$	0	$-1/6$	100/3
S3	0	0	0	$-1/2$	0	$-1/2$	1	$-1/2$	0
X1	1	$1/3$	0	$1/3$	0	0	0	$1/3$	250/3
Z	0	0	0	1.5	0.5	1.25	0	1.25	700
X2	0	1	0	$1/2$	$1/2$	$-1/4$	0	$-1/4$	100
X3	0	0	1	$1/6$	$-1/6$	$7/12$	0	$-1/12$	0
S3	0	0	0	$-1/2$	0	$-1/2$	1	$-1/2$	0
X1	1	0	0	$1/6$	$-1/6$	$1/12$	0	$5/12$	50

The Optimal Solution is: X1 = 50 X2 = 100 X3 = 0 X4 = 0 The Optimal Value is: Z = 700

The Dual Prices are: For the First Constrain is 0.5 For the Second Constrain is 1.25

For the Third Constrain is 0 For the Fourth Constrain is 1.25

The Ranges of the CI (The Coefficients of the objective function) are: $3=6-3\le C_1 \le 6-(-3)=9$

$$3=4-1\le C_2 \le 4-(-5)=9\ \frac{6}{7}=3-\frac{15}{7}\le C_3 \le 3-(-3)=6\ C_4 \le 2+\frac{3}{2}=\frac{7}{2}$$

ملاحظة لتعيين حدود Ci : **أولاً:** إذا كان x_i متحولاً أساسيًا , نقسم سطر Z على سطر x_i ثم نكتب :

(قيمة C_i ناقص أكبر ناتج سالب) $\le C_i \le$ (قيمة C_i ناقص أصغر ناتج موجب)

ثانياً: إذا كان x_i متحولاً غير أساسي والمسألة (Max) فإن **$C_i \le$ (قيمة C_i + العدد الوارد تحت x_i على سطر Z)**

أما في حالة المسألة (Min) فتصبح **(قيمة C_i + العدد الوارد تحت x_i على سطر Z) $\le C_i$**

مثال محلول (8)

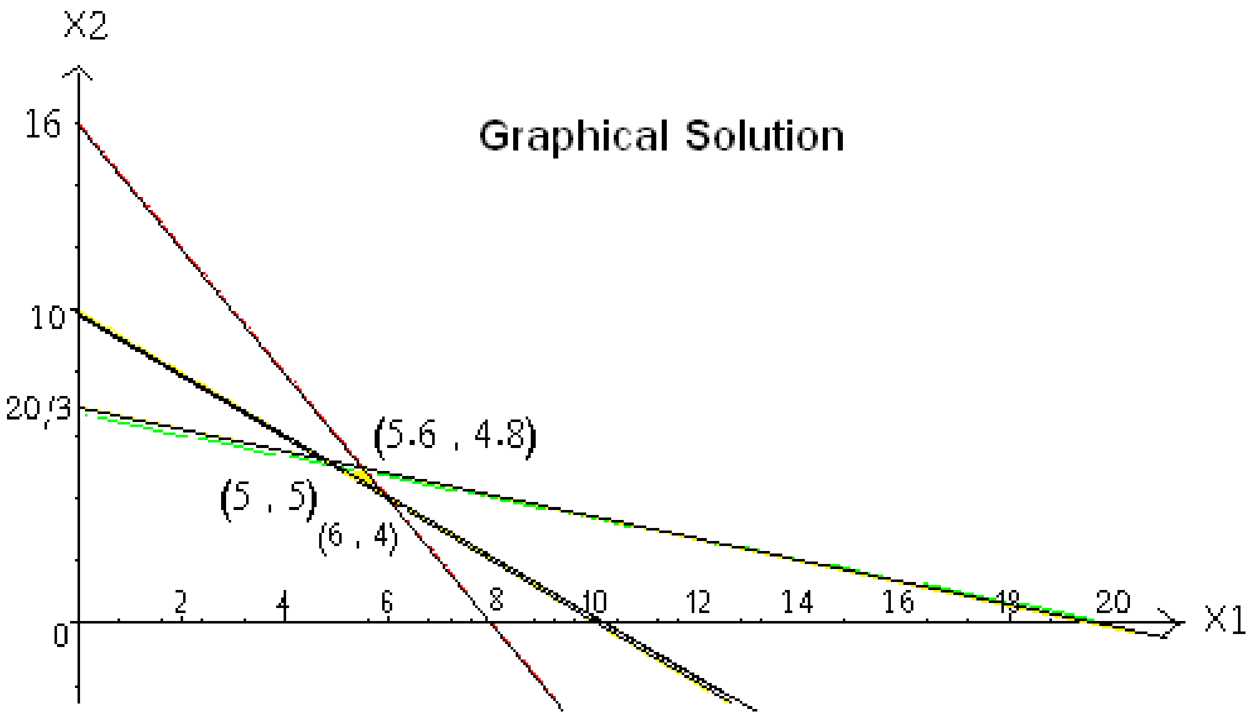
$Min \quad z = 2x_1 + 3x_2$

$s.t.$

$$\begin{aligned} 2x_1 + x_2 &\le 16 \\ x_1 + 3x_2 &\ge 20 \\ x_1 + x_2 &= 10 \\ x_1 ,\ x_2 &\ge 0 \end{aligned}$$

The Optimal solution is:
X1 = 5 , X2= 5
The Optimal Value is: Z = 25
The Dual Prices Are:
0 0.51.5
The Ranges of C_i are:
$C1 \le 3$
$2 \le C2$

Basic V.	X1	X2	S1	S2	A2	A3	RHS
Z	-2	-3	0	0	- M	- M	0
S1	2	1	1	0	0	0	16
A2	1	3	0	-1	1	0	20
A3	1	1	0	0	0	1	10
Z	- 1	0	0	- 1	1- M	- M	20
S1	5 / 3	0	1	1 / 3	-1 / 3	0	28/ 3
X2	1 / 3	1	0	-1 / 3	1 / 3	0	20 / 3
A3	2/3	0	0	1 / 3	-1 / 3	1	10/3
Z	0	0	0	- 1/2	1/2-M	3/2 - M	25
S1	0	0	1	-1 / 2	1 / 2	-5/2	1
X2	0	1	0	-1 / 2	1 / 2	-1/2	5
X1	1	0	0	1 / 2	- 1 / 2	3/2	5



مثال محلول (9)

$$\begin{aligned} Min \quad & z = 8x_1 + 12x_2 + 16x_3 \\ s.t. \quad & \\ & -1x_1 + 1x_2 + 2x_3 \ge 9 \\ & 2x_1 + 2x_2 + 1x_3 \ge 12 \\ & x_1 , \ x_2 , x_3 \ \ge 0 \end{aligned}$$

أولاً : أوجد الحل الأمثل لمسألة البرمجة الخطية

ثانياً: أوجد الأسعار الثنوية لها **The Dual Prices**

ثالثاً: أوجد مجالات الأمثلية لأمثال تابع الهدف (مجالات C_i)

رابعاً : أوجد المسألة الثنوية **The Dual Problem**

خامساً: أوجد الحل الأمثل لها بيانياً **The Opt. Sol. (Graphical)**

The Optimal solution is:
X1 = 0 , X2= 5, X3 = 2
The Optimal Value is:
Z = 92
The Dual Prices Are:

$$-\frac{20}{3} \qquad -\frac{8}{3}$$

The Ranges of C_i are:

$$-\frac{4}{3} \le C1$$

$8 \le C2 \le 17.6 \qquad 9 \le C3 \le 24$

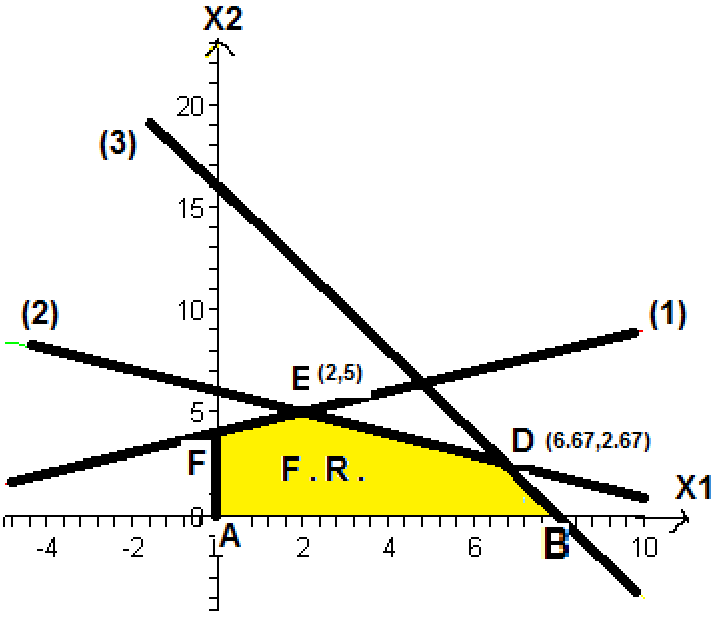
Basic V.	X1	X2	X3	S1	S2	A1	A2	RHS
Z	-8	-12	-16	0	0	-M	-M	0
A1	-1	1	2	-1	0	1	0	9
A2	2	2	1	0	-1	0	1	12
Z	-16	-4	0	-8	0	8-M	-M	72
X3	-1/2	1/2	1	-1/2	0	1/2	0	9/2
A2	5/2	3/2	0	1/2	-1	-1/2	1	15/2
Z	-28/3	0	0	-20/3	-8/3	20/3-M	8/3-M	92
X3	-4/3	0	1	-2/3	1/3	2/3	-1/3	2
X2	5/3	1	0	1/3	-2/3	-1/3	2/3	5

The Dual Problem is:

$$\begin{aligned} Max \quad & z = 9x_1 + 12x_2 \\ s.t. \quad & \\ & -1x_1 + 2x_2 \le 8 \\ & 1x_1 + 2x_2 \le 12 \\ & 2x_1 + 1x_2 \le 16 \\ & x_1 , \ x_2 \ \ge 0 \end{aligned}$$

A(0,0) B(8,0) D($\frac{20}{3}$,$\frac{8}{3}$) E(2,5)
F(0,4)
The Optimal Solution is:
$x_1 = \frac{20}{3}$ $x_2 = \frac{8}{3}$
The Optimal Value is:
Z = 92
The Ranges of C_i are:

$6 \le C2 \le 24 \qquad 4 \le C3 \le 18$



Graphical Solution

مثال محلول (10)

أولاً : أوجد الحل الأمثل لمسألة البرمجة الخطية

ثانياً: أوجد الأسعار الثنوية لها **The Dual Prices**

ثالثاً: أوجد مجالات الأمثلية لأمثال تابع الهدف (مجالات C_i)

رابعاً : أوجد المسألة الثنوية **The Dual Problem**

خامساً: أوجد الحل الأمثل لها بيانياً **The Opt. Sol. (Graphical)**

$$\begin{aligned} Max\quad & Z = 9x_1 + 2x_2 - 12x_3 \\ s.t.\quad & 1x_1 + 1x_2 - 1x_3 \le 5 \\ & -1x_1 + 1x_2 + 3x_3 \ge 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Basic V.	X1	X2	X3	S1	S2	A2	RHS
Z	-9	-2	12	0	0	M	0
S1	1	1	-1	1	0	0	5
A2	-1	1	3	0	-1	1	3
Z	0	7	3	9	0	M	45
X1	1	1	-1	1	0	0	5
A2	0	2	2	1	-1	1	8
Z	0	4	0	15/2	3/2	M-3/2	33
X1	1	2	0	3/2	-1/2	1/2	9
X3	0	1	1	1/2	-1/2	1/2	4

The Optimal solution is:

X1 = 9 , X2= 0, X3 = 4

The Optimal Value is:

Z = 33

The Dual Prices Are:

For the first constrain is: 7.5

For the second constrain is: 1.5

The Ranges of C_i are:

$$7 \le C1 \le 12$$

$$C2 \le 6 \qquad -16 \le C3 \le -9$$

The Dual Problem is:

$$\begin{aligned} Min\quad & z = 5x_1 - 3x_2 \\ s.t.\quad & 1x_1 + 1x_2 \ge 9 \\ & 1x_1 - 1x_2 \ge 2 \\ & 1x_1 + 3x_2 \le 12 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

A(9,0) B(12,0) D($\frac{15}{2}$,$\frac{3}{2}$)

The Optimal Solution is:

$x_1 = \frac{15}{2}$ $\qquad$ $x_2 = \frac{3}{2}$

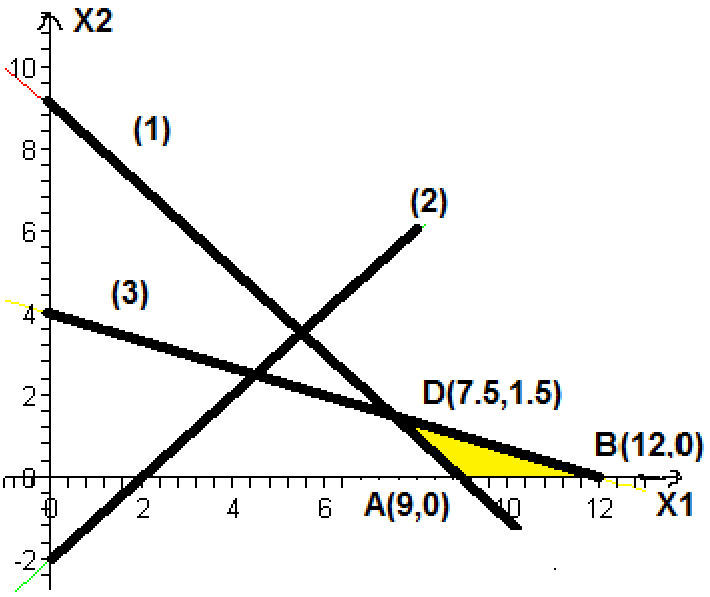
The Optimal Value is:

Z = 33

The Ranges of C_i are:

$$-1 \le C1$$

$$C2 \le 5$$



Graphical Solution

مثال محلول (11)

$$
\begin{aligned}
&Min \quad Z = 2x_1 + 3x_2 + 4x_3 \\
&s.t. \quad -1x_1 + 2x_2 + 1x_3 \ge 24 \\
&\qquad 1x_1 + 2x_2 - 2x_3 \le 18 \\
&\qquad x_1, x_2, x_3 \ge 0
\end{aligned}
$$

أولاً : أوجد الحل الأمثل لمسألة البرمجة الخطية

ثانياً: أوجد الأسعار الثنوية لها **The Dual Prices**

ثالثاً: أوجد مجالات الأمثلية لأمثال تابع الهدف (مجالات C_i)

رابعاً : أوجد المسألة الثنوية **The Dual Problem**

خامساً: أوجد الحل الأمثل لها بيانياً **The Opt. Sol. (Graphical)**

Basic V.	X1	X2	X3	S1	S2	A1	RHS
Z	-2	-3	-4	0	0	-M	0
A1	-1	2	1	-1	0	1	24
S2	1	2	-2	0	1	0	18
Z	-6	5	0	-4	0	4-M	96
X3	-1	2	1	-1	0	1	24
S2	-1	6	0	-2	1	2	66
Z	-31/6	0	0	-7/3	-5/6	7/3 - M	41
X3	-2/3	0	1	-1/3	-1/3	1/3	2
X2	-1/6	1	0	-1/3	1/6	1/3	11

The Optimal solution is:
X1 = 0 , X2= 11, X3 = 2
The Optimal Value is:
Z = 41
The Dual Prices Are:
For the first constrain is: $\frac{7}{3}$
For the second constrain is: $\frac{5}{6}$
The Ranges of C_i are:

$$-\frac{19}{6} \le C1$$

$$4 \le C2 \le 8 \qquad 1.5 \le C3$$

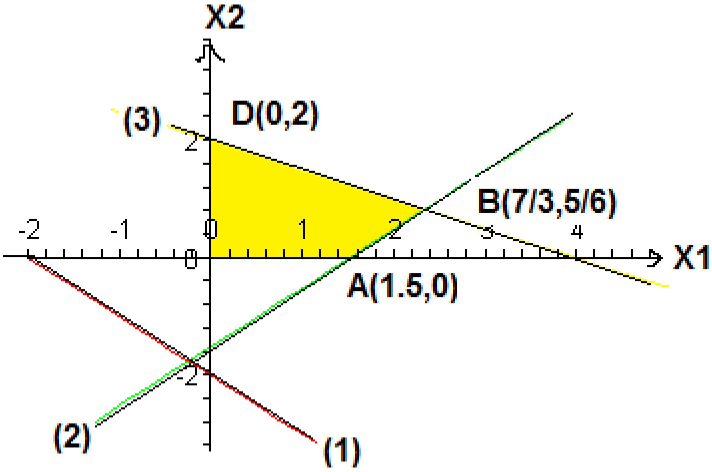
The Dual Problem is:

$$
\begin{aligned}
&Max \quad z = 24x_1 - 18x_2 \\
&s.t. \\
&\qquad 1x_1 + 1x_2 \ge 2 \\
&\qquad 2x_1 - 2x_2 \le 3 \\
&\qquad 1x_1 + 2x_2 \le 4 \\
&\qquad x_1 , x_2 \ge 0
\end{aligned}
$$

A(0,0) B(1.5,0) D($\frac{7}{3}$,$\frac{5}{6}$) E(0,2)

The Optimal Solution is:
$x_1 = \frac{7}{3}$ $x_2 = \frac{5}{6}$
The Optimal Value is:
Z = 41
The Ranges of C_i are:

$$18 \le C1 \quad , \quad -24 \le C2 \le 48$$



Graphical Solution

مثال محلول (12)

$$\begin{aligned} Min \quad & z = 3x_1 + 2x_2 + 1x_3 \\ s.t. \quad & 3x_1 + 1x_2 + 1x_3 \geq 3 \\ & -3x_1 + 3x_2 + 1x_3 \geq 6 \\ & 1x_1 + 1x_2 + 1x_3 \leq 3 \\ & x_1 ,\ x_2 , x_3 \geq 0 \end{aligned}$$

أولاً : أوجد الحل الأمثل لمسألة البرمجة الخطية

ثانياً: أوجد الأسعار الثنوية لها **The Dual Prices**

ثالثاً: أوجد مجالات الأمثلية لأمثال تابع الهدف (مجالات C_i)

رابعاً : أوجد المسألة الثنوية **The Dual Problem**

خامساً: أوجد الحل الأمثل لها **The Opt. Sol. (Graphical)**

The Optimal solution is:
X1 = 0 , X2= 1.5, X3 = 1.5
The Optimal Value is:
Z = 4.5

The Dual Prices Are:
-1/2 -1/2 0

The Ranges of C_i are:

$$0 \leq C1$$
$$2/3 \leq C2 \leq 1.5$$
$$9 \leq C3 \leq 24$$

Basic V.	X1	X2	X3	S1	S2	S3	A1	A2	RHS
Z	*-3*	*-2*	*-1*	*0*	*0*	*0*	*-M*	*-M*	*0*
A1	3	1	1	-1	0	0	1	0	3
A2	-3	3	1	0	-1	0	0	1	6
S3	1	1	1	0	0	1	0	0	3
Z	0	-1	0	-1	0	0	1-M	-M	3
X3	3	1	1	-1	0	0	1	0	3
A2	-6	2	0	1	-1	0	-1	1	3
S3	-2	0	0	1	0	1	-1	0	0
Z	-3	0	0	-1/2	-1/2	0	1/2-M	1/2-M	9/2
X3	6	0	1	-3/2	1/2	0	3/2	-1/2	3/2
X2	-3	1	0	1/2	-1/2	0	-1/2	1/2	3/2
S3	-2	0	0	1	0	1	-1	0	0

The Dual Problem is:

$$\begin{aligned} Max \quad & z = 3x_1 + 6x_2 - 3x_3 \\ s.t. \quad & 3x_1 - 3x_2 - 1x_3 \leq 3 \\ & 1x_1 + 3x_2 - 1x_3 \leq 2 \\ & 1x_1 + 1x_2 - 1x_3 \leq 1 \\ & x_1 ,\ x_2 , x_3 \geq 0 \end{aligned}$$

The Optimal solution is:
X1 = 1/2 , X2= 1/2, X3 = 0
The Optimal Value is:
Z = 4.5

The Dual Prices Are:
0 3/2 3/2

The Ranges of C_i are:

$$2 \leq C1 \leq 3$$
$$3 \leq C2 \leq 9$$
$$C3 \leq -3$$

Basic V.	X1	X2	X3	S1	S2	S3	RHS
Z	-3	-6	3	0	0	0	0
S1	3	-3	-1	1	0	0	3
S2	1	3	-1	0	1	0	2
S3	1	1	-1	0	0	1	1
Z	-1	0	1	0	2	0	4
S1	4	0	-2	1	1	0	5
X2	1/3	1	-1/3	0	1/3	0	2/3
S3	2/3	0	-2/3	0	-1/3	1	1/3
Z	0	0	0	0	3/2	3/2	9/2
S1	0	0	2	1	3	-6	3
X2	0	1	0	0	1/2	-1/2	1/2
X1	1	0	-1	0	-1/2	3/2	1/2

تمارين محلولة (13 – 14 – 15)

أولاً : أوجد الحل الأمثل لمسألة البرمجة الخطية

ثانياً: أوجد الأسعار الثنوية لها **The Dual Prices**

ثالثاً: أوجد مجالات الأمثلية لأمثال تابع الهدف (مجالات C_i)

رابعاً : أوجد المسألة الثنوية **The Dual Problem**

خامساً : حل المسألة الثنوية **Solve The Dual Problem**

$$
\begin{array}{llr}
Max & z = 1x_1 + 2x_2 + 1x_3 & \\
s.t. & 2x_1 + 3x_2 - 1x_3 & \le 25 \\
 & 1x_1 - 1x_2 + 3x_3 & \ge 20 \\
 & 3x_1 + 2x_2 + 1x_3 & \le 35 \\
 & x_1 , x_2 , x_3 & \ge 0
\end{array}
$$

Basic V.	X1	X2	X3	S1	S2	S3	A2	RHS
Z	-1	-2	-1	0	0	0	M	0
S1	2	3	-1	1	0	0	0	25
A2	1	-1	3	0	-1	0	1	20
S3	3	2	1	0	0	1	0	35
Z	1/3	0	-5/3	2/3	0	0	M	50/3
X2	2/3	1	-1/3	1/3	0	0	0	25/3
A2	5/3	0	8/3	1/3	-1	0	1	85/3
S3	5/3	0	5/3	-2/3	0	1	0	55/3
Z	11/8	0	0	7/8	-5/8	0	M +5/8	275/8
X2	7/8	1	0	3/8	-1/8	0	1/8	95/8
X3	5/8	0	1	1/8	-3/8	0	3/8	85/8
S3	5/8	0	0	-7/8	5/8	1	-5/8	5/8
Z	2	0	0	0	0	1	M	35
X2	1	1	0	1/5	0	1/5	0	12
X3	1	0	1	-2/5	0	3/5	0	11
S2	1	0	0	-7/5	1	8/5	-1	1

The Optimal solution is:
X1 = 0 , X2= 12, X3 = 11

The Optimal Value is:
Z = 35
The Dual Prices Are:

0 0 1

The Ranges of C_i are:

$$C1 \le 3 \;,$$
$$0 \le C2 \quad , \quad -2/3 \le C3$$

The Dual Problem is:

$$
\begin{array}{llr}
Min & z = 25x_1 - 20x_2 + 35x_3 & \\
s.t. & 2x_1 - 1x_2 + 3x_3 & \ge 1 \\
 & 3x_1 + 1x_2 + 2x_3 & \ge 2 \\
 & -1x_1 - 3x_2 + 1x_3 & \ge 1 \\
 & x_1 , x_2 , x_3 & \ge 0
\end{array}
$$

حل بديل (1)

Basic V.	X1	X2	X3	S1	S2	S3	A2	RHS
Z	-1	-2	-1	0	0	0	M	0
S1	2	3	-1	1	0	0	0	25
A2	1	-1	3	0	-1	0	1	20
S3	3	2	1	0	0	1	0	35
Z	2	0	0	0	0	1	M	35
S1	2.5	0	2.5	-1	0	1.5	0	27.5
A2	2.5	0	3.5	0	-1	0.5	1	37.5
X2	1.5	1	0.5	0	0	0.5	0	17.5
Z	2	0	0	0	0	1	M	35
S1	5/7	0	0	-1	5/7	8/7	-5/7	5/7
X3	5/7	0	1	0	-2/7	1/7	2/7	75/7
X2	8/7	1	0	0	1/7	8/7	-1/7	85/7

The Optimal solution is:
X1 = 0 , X2= 85/7, X3 = 75/7

The Optimal Value is:
Z = 35
The Dual Prices are:

0 0 1

The Ranges of C_i are:

$$C1 \le 3 \;,$$
$$\frac{9}{8} \le C2 \quad , \quad -9/5 \le C3$$

حل بديل (2)

The Optimal solution is:
X1 = 0 , X2= 0, X3 = 35

The Optimal Value is:
Z = 35
The Dual Prices Are:

0 0 1

The Ranges of C_i are:

$C1 \leq 3$,
$C2 \leq 2$, $C3 \leq 1$

Basic V.	X1	X2	X3	S1	S2	S3	A2	RHS
Z	-1	-2	-1	0	0	0	M	0
S1	2	3	-1	1	0	0	0	25
A2	1	-1	3	0	-1	0	1	20
S3	3	2	1	0	0	1	0	35
Z	2	0	0	0	0	1	M	35
S1	5	5	0	1	0	1	0	60
A2	8	7	0	0	1	3	-1	85
X3	3	2	1	0	0	1	0	35
Z	2	0	0	0	0	1	M	35
S1	5	5	0	1	0	1	0	60
S2	8	7	0	0	1	3	-1	85
X3	3	2	1	0	0	1	0	35

The Dual Problem

Basic V.	X1	X2	X3	S1	S2	S3	A1	A2	A3	RHS
Z	-25	20	-35	0	0	0	-M	-M	-M	0
A1	2	-1	3	-1	0	0	1	0	0	1
A2	3	1	2	0	-1	0	0	1	0	2
A3	-1	-3	1	0	0	-1	0	0	1	1
Z	-55	-85	0	0	0	-35	- M	- M	- M+35	35
A1	-5	-8	0	1	0	-3	-1	0	3	2
A2	-5	-7	0	0	1	-2	0	-1	2	4
X3	-1	-3	1	0	0	-1	0	0	1	1
Z	-55	-85	0	0	0	-35	- M	- M	- M+35	35
A1	-5	-8	0	1	0	-3	-1	0	3	2
S2	-5	-7	0	0	1	-2	0	-1	2	4
X3	-1	-3	1	0	0	-1	0	0	1	1
Z	-55	-85	0	0	0	-35	- M	- M	- M+35	35
S1	-5	-8	0	1	0	-3	-1	0	3	2
S2	-5	-7	0	0	1	-2	0	-1	2	4
X3	-1	-3	1	0	0	-1	0	0	1	1

The Optimal Solution is: X1 = 0 X2=0 X3=1 The Optimal Value is: Z =35

The Dual Prices are: 0 0 -35

The Optimal Ranges are: : $-35 \leq C1$, $105 \leq C2$, $-\frac{82}{3} \leq C3$

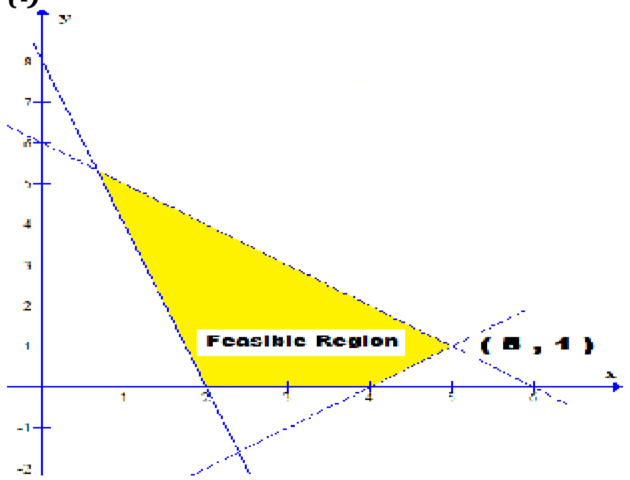
2(16)

$$Max \quad z = 5x_1 - 3x_2$$
$$s.t. \quad x_1 + x_2 \le 6$$
$$4x_1 + x_2 \ge 8$$
$$x_1 - x_2 \le 4$$
$$x_1 , x_2 \ge 0$$

(i) *Draw the feasible region.*
(ii) *Find the optimal solution.*
(iii) *Find the Dual Problem .*
(iv) *Solve the Dual Problem.*

The Solution

(i)



ii) The Optimal Solution is: X1 = 5 , X2 = 1
The Optimal Value is: Z = 22
iii) The Dual Problem is:

$$Min \quad Z = 6x_1 - 8x_2 + 4x_3$$

iii) $$s.t. \quad 1x_1 - 4x_2 + 1x_3 \ \ge 5$$
$$-1x_1 + 1x_2 + 1x_3 \ \le 3$$
$$x_1 , x_2 , x_3 \ge 0$$

B.V.	X1	X2	X3	S1	S2	A1	RHS
Z	-6	8	-4	0	0	-M	0
A1	1	-4	1	-1	0	1	5
S2	-1	1	1	0	1	0	3
Z	0	-16	2	-6	0	6-M	30
X1	1	-4	1	-1	0	1	5
S2	0	-3	2	-1	1	1	8
Z	0	-13	0	-5	-1	5-M	22
X1	1	-5/2	0	-1/2	-1/2	1/2	1
X3	0	-3/2	1	-1/2	1/2	1/2	4

The Optimal Solution is: X1 = 1 , X2 = 0 , X3 = 4
The Optimal Value is: Z = 22
The Dual Prices are: (-5 , -1)
The Optimal Ranges are:

$$4 \le C1 \quad , \quad -21 \le C2 \quad , \quad -\frac{14}{3} \le C3 \le 6$$

2(17)

$$Min \quad Z = 4x_1 + 6x_2 + 8x_3$$
$$s.t. \quad 1x_1 \quad + 1x_3 \quad \ge 2$$
$$1x_2 + 1x_3 \quad \ge 5$$
$$x_1 , x_2 , x_3 \ge 0$$

(i) *Find the optimal solution.*
(ii) *Find the Dual Problem .*
(iii) *Solve the Dual Problem (Graphical).*

The Solution

(i)

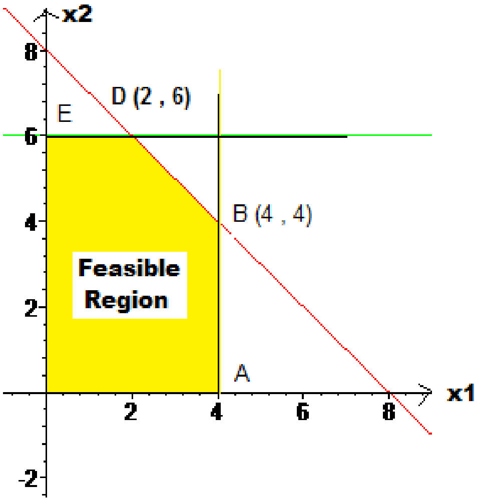
	X1	X2	X3	S1	S2	A1	A2	RHS
Z	-4	-6	-8	0	0	-M	-M	0
A1		0	1	-1	0	1	0	2
A2	0	1	1	0	-1	0	1	5
Z	-4	0	-2	0	-6	-M	6-M	30
A1	1	0	1	-1	0	1	0	2
X2	0	1	1	0	-1	0	1	5
Z	-2	0	0	-2	-6	2-M	6-M	34
X3	1	0	1	-1	0	1	0	2
X2	-1	1	0	1	-1	-1	1	3

The Optimal Solution is: X1 = 0 , X2 = 3 , X3 = 2
The Optimal Value is: Z = 34
The Dual Prices are: (-2 , -6)
The Optimal Ranges are:

$$2 \le C1 \quad ,4 \le C2 \le 8 \quad , \quad 6 \le C3 \le 10$$

(ii) **The Dual Problem is:**

$$Max \quad z = 2x_1 + 5x_2$$
$$s.t. \quad x_1 \quad \le 4$$
$$x_2 \le 6$$
$$x_1 + x_2 \le 8$$
$$x_1 , x_2 \ge 0$$



The Optimal Solution is: X1 = 2 , X2 = 6
The Optimal Value is: Z = 34

ملاحظات هامة

ملاحظة (1)

لتعيين المجالات الأمثلية (The Optimal Ranges) للأمثال C_i نكتب:

أولاً: إذا كان x_i متحولاً أساسيًا :

آ) نقسم سطر Z على سطر x_i .

ب) (قيمة C_i ناقص أكبر ناتج سالب) $\le C_i \le$ (قيمة C_i ناقص أصغر ناتج موجب)

ج) إذا لم نجد ناتجًا موجبًا تكون

(قيمة C_i ناقص أكبر ناتج سالب) $\le C_i$

إذا لم نجد ناتجًا سالبًا تكون

(قيمة C_i ناقص أصغر ناتج موجب) $\ge C_i$

ثانياً: إذا كان x_i متحولاً غير أساسي فإن :

1) (قيمة C_i + العدد الوارد تحت x_i على سطر Z) $\le C_i$ في حالة Max

2) (قيمة C_i + العدد الوارد تحت x_i على سطر Z) $\ge C_i$ في حالة Min

ملاحظة (2)

1) لكتابة الشكل المعياري (The Standard Form) لمسألة البرمجة الخطية :

آ) في حالة كون الشرط i "أصغر أو يساوي" نضيف متحولاً S_i

ب) في حالة كون الشرط i "أكبر أو يساوي" نطرح متحولاً S_i ثم نضيف متحولاً A_i

ج) في حالة كون الشرط i " يساوي " نضيف متحولاً A_i

ملاحظة (3)

1) نصل إلى الحل الأمثل (The Optimal Solution) في حل مسألة البرمجة الخطية بطريقة السيمبلكس:

آ) (في حالة Min) : إذا لم يكن هناك متحولاً رئسياً A_i وكانت كل الأعداد على السطر Z "سالبة أوصفر"

ب) (في حالة Max): إذا لم يكن هناك متحولاً رئسياً A_i وكانت كل الأعداد على السطر Z "موجبةأوصفر"

Operations Research Homework(1)

1) Minimize $z = 5x_1 + 2x_2$
subject to

$$x_1 - x_2 \geq 3$$
$$2x_1 + 3x_2 \geq 5$$
$$x_1, x_2 \geq 0$$

the opt. sol. (3 , 0) , Z=15, D.P. (-5,0) , 0 < C1 , -5 < C2

2) Maximize $z = x_1 + 5x_2 + 3x_3$
subject to

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 = 4$$
$$x_1, x_2, x_3 \geq 0$$

The Opt. Sol. (2 , 0 , 1), Z=5, D.P. (3,-1),C1<5, C2<7, 2.2<C3

3) Maximize $z = 3x_1 + 2x_2$
subject to

$$2x_1 + x_2 \leq 3$$
$$3x_1 + 4x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

The Opt. Sol. (0,3), Z=6, D.P (1.2,0.2), 1.5<C1<4 , 1.5<C2<4

5) $Max \quad Z = 3x_1 + 4x_2 + 5x_3$

s.t.

$$2x_1 + 3x_2 + 1x_3 \leq 5$$
$$1x_1 + 2x_2 + 3x_3 \leq 10$$
$$3x_1 + 1x_2 + 2x_3 \leq 15$$
$$x_1, x_2, x_3 \geq 0$$

The Opt. Sol. (1, 0, 3), Z=18, DP (0.8, 1.4, 0)
The Opt. Ran. 2.143 ≤ C1 ≤ 10, C2 ≤ 5, 1.5 ≤ C3 ≤ 9

6) Maximize $Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5,$

subject to

$$x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 \leq 6$$
$$4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 \leq 15$$
$$x_j \geq 0, \qquad \text{for } j = 1, 2, 3, 4, 5.$$

The Opt. Sol. (1.5, 1.5, 0, 0, 0) Z = 10.5 DP(4/3 ,0.167)
5/3 ≤ C1 ≤ 3 , 4 ≤ C2 ≤ 6, C3 ≤ 3.5, C4 ≤ 5.167, C5 ≤ 1.5

7) Maximize $z = -5x_1 + 2x_2$
subject to

$$-x_1 - x_2 \leq -2$$
$$2x_1 + 3x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

The Opt. Sol. (1, 1) , Z=-3, DP (-19, 7)
C1 ≤ 4/3, -5 ≤ C2

8) Minimize $z = 6x_1 + 3x_2$
subject to

$$6x_1 - 3x_2 + x_3 \geq 2$$
$$3x_1 + 4x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

The Opt. Sol. (0, 0, 5) , Z=0, DP (0, 0)
0 ≤ C1, 0 ≤ C2, 0 ≤ C1 ≤ 3/4

9) Minimize $z = 5x_1 + 2x_2 + 3x_3$
s.t.

$$x_1 + 5x_2 + 2x_3 = 30$$
$$x_1 - 5x_2 - 6x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$

The Opt. Sol. (0, 6, 0) , Z= 12, DP (-0.4, 0)
0.4 ≤ C1, C2 ≤ 7.5, 0.8 ≤ C3

10) Minimize $z = 3x_1 + 2x_2 + x_3$
s.t.

$$3x_1 + x_2 + x_3 \geq 3$$
$$-3x_1 + 3x_2 + x_3 \geq 6$$
$$x_1 + x_2 + x_3 \leq 3$$
$$x_1, x_2, x_3 \geq 0$$

The Opt. Sol. (0, 1.5, 1.5) , Z= 4.5, DP (-0.5,-0.5, 0)
0 ≤ C1, 1 ≤ C1 ≤ 3 , 2/3 ≤ C1 ≤ 3/2

12) Minimize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

subject to $5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12$

$x_2 - 5x_3 - 6x_4 \ge 10$

$2x_1 + 5x_2 + x_3 + x_4 \ge 8$

$x_1, x_2, x_3, x_4 \ge 0$

The Opt. Sol. (0, 10, 0,0)   Z = 70   DP(0 , -7, 0)
0 ≤ C1, 0 ≤ C2, C3 ≤ -35, C4 ≤ -42

13) Maximize $z = 30x_1 + 20x_2$

subject to $2x_1 + x_2 \le 8$ (Machine 1)

$x_1 + 3x_2 \le 8$ (Machine 2)

$x_1, x_2 \ge 0$

The Opt. Sol. (3.2, 1.6)   Z = 128   DP(14 , 2)
20/3 ≤ C1 ≤ 40, 15 ≤ C2 ≤ 90

14) Maximize $z = 3x_1 + 2x_2 + 5x_3$

s.t. $x_1 + 2x_2 + x_3 \le 430$ (Operation 1)

$3x_1 + 2x_3 \le 460$ (Operation 2)

$x_1 + 4x_2 \le 420$ (Operation 3)

$x_1, x_2, x_3 \ge 0$

The Opt. Sol. (0, 100, 230)   Z = 1350   DP(1 , 2, 0)
C1 ≤ 7, 0 ≤ C2 ≤ 10, 7/3 ≤ C3

15) Maximize $z = 3x_1 + 2x_2 + 3x_3$

s.t. $2x_1 + x_2 + x_3 = 2$

$x_1 + 3x_2 + x_3 = 6$

$3x_1 + 4x_2 + 2x_3 = 8$

$x_1, x_2, x_3 \ge 0$

The Opt. Sol. (0, 2, 0)   Z = 4   DP(3 , -1, 0.5)
C1 ≤ 6.5, C2 ≤ 9, 1.6 ≤ C3

16) Maximize $z = 3x_1 + 2x_2 + 3x_3$

s.t. $2x_1 + x_2 + x_3 \le 2$

$3x_1 + 4x_2 + 2x_3 \ge 8$

$x_1, x_2, x_3 \ge 0$

The Opt. Sol. (4, 2, 0)   Z = 4   DP(3 , -1, 1/2)
C1 ≤ 6.5, C2 ≤ 9, 1.6 ≤ C3

17) Minimize $z = 5x_1 + 6x_2$

subject to $x_1 + x_2 \ge 2$

$4x_1 + x_2 \ge 4$

$x_1, x_2 \ge 0$

The Opt. Sol. (2, 0)   Z = 10   DP(-5 , 0)
0 ≤ C1 ≤ 6, 5 ≤ C2

18) Minimize $z = 4x_1 + 2x_2$

subject to $x_1 + x_2 = 1$

$3x_1 - x_2 \ge 2$

$x_1, x_2 \ge 0$

The Opt. Sol. (3/4, 1/4)   Z = 7/2   DP(-5/2 , -1/2)
2 ≤ C1, C2 ≤ 4

19) Minimize $z = 2x_1 + 3x_2$

subject to $2x_1 + x_2 \ge 3$

$x_1 + x_2 = 2$

$x_1, x_2 \ge 0$

The Opt. Sol. (2, 0)   Z = 4   DP(0 , -2)
C1 ≤ 3, 2 ≤ C2

20) Maximize $z = 2x_3$

subject to $-x_1 + 2x_2 - 2x_3 \ge 8$

$-x_1 + x_2 + x_3 \le 4$

$2x_1 - x_2 + 4x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

The Opt. Sol. (14, 18, 0)   Z = 28   DP(0 , 2, 2)
0 ≤ C1, -1 ≤ C2, C3 ≤ 4

21) Maximize $z = x_1 - 3x_2$

subject to $x_1 - x_2 \le 2$

$x_1 + x_2 \ge 4$

$2x_1 - 2x_2 \ge 3$

$x_1, x_2 \ge 0$

The Opt. Sol. (3, 1)   Z = 0   DP(2, -1 , 0)
-3 ≤ C1 ≤ 3, C2 ≤ -1

22) Minimize $z = -x_1 + x_2$

subject to $x_1 - 4x_2 \ge 5$

$x_1 - 3x_2 \le 1$

$2x_1 - 5x_2 \ge 1$

$x_1, x_2 \ge 0$

No Feasible Solution

23) Maximize $z = 80x_1 + 100x_2$

subject to

$$4x_1 + 16x_2 \le 800$$
$$20x_1 + 40x_2 \le 2400$$
$$x_1 + x_2 \le 100$$
$$x_1, x_2 \ge 0$$

The Opt. Sol. (80, 20)   Z = 8400   DP(0, 1 ,6 0)

50 ≤ C1 ≤ 100,   80 ≤ C2 ≤ 160

24) Maximize $z = 2x_3$

subject to

$$-x_1 + 3x_2 - 7x_3 \ge 5$$
$$-x_1 + x_2 - x_3 \le 1$$
$$3x_1 + x_2 - 10x_3 \le 8$$
$$x_1, x_2, x_3 \ge 0$$

No Feasible Solution

25) Maximize $Z = 2x_1 + 6x_2 + 9x_3,$

subject to

$$x_1 + x_3 \le 3 \quad \text{(resource 1)}$$
$$x_2 + 2x_3 \le 5 \quad \text{(resource 2)}$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Unbounded

31)

$$\begin{aligned} Max \quad & z = 60x_1 + 30x_2 + 20x_3 \\ s.t. \quad & 8x_1 + 6x_2 + 1x_3 \le 48 \\ & 8x_1 + 4x_2 + 3x_3 \le 40 \\ & 4x_1 + 3x_2 + 1x_3 \le 16 \\ & x_2 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The Opt. Sol. (2, 0, 8)   Z = 280   DP(0 , 5, 5, 0)

56 ≤ C1 ≤ 80,   C2≤ 35 ,   15 ≤ C3 ≤ 22

26) Maximize $Z = x_1 - 3x_2 + 2x_3,$

subject to

$$2x_1 + 2x_2 - 2x_3 \le 6 \quad \text{(resource 1)}$$
$$-x_2 + 2x_3 \le 4 \quad \text{(resource 2)}$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

The Opt. Sol. (5, 0, 2)   Z = 9   DP(1/2, 3/2)

0 ≤ C1,   C2 ≤ -1/2,   -1 ≤ C3 ≤ 7

27) Maximize $Z = x_1 - 2x_2 + x_3.$

subject to

$$x_1 + x_2 + 2x_3 \le 12$$
$$x_1 + x_2 - x_3 \le 1$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

The Opt. Sol. (14/3, 0, 11/3)   Z = 25/3

DP(2/ 3 , 1/3),   1/2 ≤ C1,   C2 ≤ 1,   -1 ≤ C3 ≤ 2

28) Minimize $W = 5y_1 + 4y_2,$

subject to

$$4y_1 + 3y_2 \ge 4$$
$$2y_1 + y_2 \ge 3$$
$$y_1 + 2y_2 \ge 1$$
$$y_1 + y_2 \ge 2$$
$$y_1 \ge 0, \quad y_2 \ge 0.$$

The Opt. Sol. (1, 1)   Z = 9   DP(0, -1 , 0, -3)

4 ≤ C1 ≤ 8,   2 ≤ C2 ≤ 3

29) Maximize $Z = 2x_1 - x_2 + x_3,$

subject to

$$3x_1 + x_2 + x_3 \le 60$$
$$x_1 - x_2 + 2x_3 \le 10$$
$$x_1 + x_2 - x_3 \le 20$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

The Opt. Sol. (0, 110/3, 30/3)   Z = 100/3

DP(1/ 3 , 4/3, 0)

C1 ≤ 7/3,   -1.2 ≤ C2 ≤ 3,   11/4 ≤ C3

30) Maximize $Z = 2x_1 + 7x_2 + 4x_3,$

subject to

$$x_1 + 2x_2 + x_3 \le 10$$
$$3x_1 + 3x_2 + 2x_3 \le 10$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

The Opt. Sol. (0, 4/3, 0)   Z = 70/3   DP(0 , 7/3)

C1 ≤ 7,   6 ≤ C2 ,   C3 ≤ 14/3

(d) Minimize $z = 6x_1 + 7x_2 - 5\,x_3 + 5x_4$

subject to $5x_1 - 6x_2 - 3x_3 + 4x_4 \geq 15$

$x_2 - 5x_3 - 6x_4 \leq 12$

$2x_1 + 5x_2 + x_3 + x_4 = 16$

$x_1, x_2, x_3, x_4 \geq 0$

Max **The Opt. Sol. is** $(0,0,0,16)$ **Z=80**
Min **The Opt. Sol. is** $(0,0,7,9)$ **Z= 10**

(e) Minimize $z = 3x_1 + 2x_2 + x_3$

subject to $3x_1 + x_2 + x_3 \geq 3$

$-3x_1 + 3x_2 + x_3 \geq 6$

$x_1 + x_2 + x_3 \leq 3$

$x_1, x_2, x_3 \geq 0$

(f) Minimize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

subject to $5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$

$x_2 - 5x_3 - 6x_4 \geq 10$

$2x_1 + 5x_2 + x_3 + x_4 \geq 8$

$x_1, x_2, x_3, x_4 \geq 0$

(g) Maximize $z = 30x_1 + 20x_2$

subject to

$2x_1 + x_2 \leq 8$ (Machine 1)

$x_1 + 3x_2 \leq 8$ (Machine 2)

$x_1, x_2 \geq 0$

$$\begin{array}{ll} Max & z = 20\,x_1 + 30\,x_2 + 60x_3 \\ s.t. & 1x_1 + 6\,x_2 + 8x_3 \leq 48 \\ & 3\,x_1 + 4\,x_2 + 8x_3 \leq 40 \\ & 1x_1 + 3x_2 + 4x_3 \leq 16 \\ & x_2 \leq 10 \\ & x_1\,,\ x_2, x_3 \geq 0 \end{array}$$

The Opt. Sol. (8, 0, 2) Z = 280 DP(0 , 5, 5, 0)
15 ≤ C1 ≤ 22.5, C2≤ 35 , 56 ≤ C3 ≤ 80

(a) Maximize $z = -5x_1 + 2x_2$

subject to

$-x_1 + x_2 \leq -2$

$2x_1 + 3x_2 \leq 5$

$x_1, x_2 \geq 0$

The Opt. Sol. (2, 0) Z = -10 DP(-5 , 0)
C1 ≤ -2, C2 ≤ 5

(b) Minimize $z = 6x_1 + 3x_2$

subject to

$6x_1 - 3x_2 + x_3 \geq 2$

$3x_1 + 4x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

$$\begin{array}{ll} Max & Z = 6x_1 + 7x_2 + 9x_3 \\ s.t. & 3x_1 + 0\,x_2 + 3x_3 \leq 90 \\ & 0x_1 + 2x_2 + 1x_3 \leq 80 \\ & 2\,x_1 + 1x_2 + 0x_3 \leq 70 \\ & 1x_1 + 5x_2 + 1x_3 \leq 70 \\ & x_1\,,\ x_2\,, x_3 \geq 0 \end{array}$$

$Opt.Sol.is:\quad x_1 = 0 \quad , x_2 = 8 \quad , x_3 = 30$

$Tne\,Opt.Val\ is: \qquad Z = 326$

$Dual\ \Pr ices\ are: \quad [\ 2.533,\ \ 0,\ \ \ 0,\ \ \ 1.4\ ]$

$C_1 \leq 9 \qquad 0 \leq C_2 \leq 45 \qquad 6 \leq C_3$

(d) Maximize $z = 5x_1 + 2x_2 + 3x_3$

subject to $x_1 + 5x_2 + 2x_3 = 30$

$x_1 - 5x_2 - 6x_3 \leq 40$

$x_1, x_2, x_3 \geq 0$

The Opt. Sol. (30, 0, 0) Z = 150 DP(5, 0)
1.5 ≤ C1, C2≤ 25 , C3 ≤ 10

(a) Maximize $z = 2x_3$

subject to
$$\begin{aligned} -x_1 + 2x_2 - 2x_3 &\geq 8 \\ -x_1 + x_2 + x_3 &\leq 4 \\ 2x_1 - x_2 + 4x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(b) Maximize $z = x_1 - 3x_2$

subject to
$$\begin{aligned} x_1 - x_2 &\leq 2 \\ x_1 + x_2 &\geq 4 \\ 2x_1 - 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

***(c)** Minimize $z = -x_1 + x_2$

subject to
$$\begin{aligned} x_1 - 4x_2 &\geq 5 \\ x_1 - 3x_2 &\leq 1 \\ 2x_1 - 5x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(d) Maximize $z = 2x_3$

subject to
$$\begin{aligned} -x_1 + 3x_2 - 7x_3 &\geq 5 \\ -x_1 + x_2 - x_3 &\leq 1 \\ 3x_1 + x_2 - 10x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(e) Maximize $z = 3x_1 + 2x_2 + 5x_3$

subject to
$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 430 \text{ (Operation 1)} \\ 3x_1 \qquad + 2x_3 &\leq 460 \text{ (Operation 2)} \\ x_1 + 4x_2 \qquad &\leq 420 \text{ (Operation 3)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(g) Maximize $z = 2x_1 + 5x_2$

subject to
$$\begin{aligned} 3x_1 + 2x_2 &\geq 6 \\ 2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

***(a)** Minimize $z = 5x_1 + 2x_2$

subject to
$$\begin{aligned} x_1 - x_2 &\geq 3 \\ 2x_1 + 3x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Maximize $z = x_1 + 5x_2 + 3x_3$

subject to
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 \qquad &= 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(c) Maximize $z = 2x_1 + x_2$

subject tc
$$\begin{aligned} x_1 - x_2 &\leq 10 \\ 2x_1 \qquad &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(d) Maximize $z = 3x_1 + 2x_2$

subject to
$$\begin{aligned} 2x_1 + x_2 &\leq 3 \\ 3x_1 + 4x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(e) $Max \quad z = 60x_1 + 30x_2 + 20x_3$

$$\begin{aligned} s.t. \qquad 8x_1 + 6x_2 + 1x_3 &\leq 48 \\ 8x_1 + 4x_2 + 3x_3 &\leq 40 \\ 4x_1 + 3x_2 + 1x_3 &\leq 16 \\ x_2 \qquad &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The Opt. Sol. (2, 0, 8) Z = 280 DP(0 , 5, 5, 0)

56 ≤ C1 ≤ 80, C2≤ 35 , 15 ≤ C3 ≤ 22.5

(f) Maximize $Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5,$

subject to
$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &\leq 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &\leq 15 \end{aligned}$$

$x_j \geq 0, \quad \text{for } j = 1, 2, 3, 4, 5.$

$Max \quad z = 60x_1 + 30x_2 + 20x_3$

$$\begin{aligned} s.t. \qquad 8x_1 + 6x_2 + 1x_3 &\leq 48 \\ 8x_1 + 4x_2 + 3x_3 &\leq 40 \\ 4x_1 + 3x_2 + 1x_3 &\leq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The Opt. Sol. (2, 0, 8) Z = 280 DP(0 , 5, 5)

56 ≤ C1 ≤ 80, C2≤ 35 , 15 ≤ C3 ≤ 22.5

6.1-5. Consider the following problem.

Maximize $Z = -x_1 - 2x_2 - x_3$,

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 12 \\ x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

I **4.6-19.** Consider the following problem.

Maximize $Z = 4x_1 + 5x_2 + 3x_3$,

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\ge 20 \\ 15x_1 + 6x_2 - 5x_3 &\le 50 \\ x_1 + 3x_2 + 5x_3 &\le 30 \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

4.7-5. Consider the following problem.

Maximize $Z = x_1 - 7x_2 + 3x_3$,

subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 &\le 4 && \text{(resource 1)} \\ 4x_1 - 3x_2 &\le 2 && \text{(resource 2)} \\ -3x_1 + 2x_2 + x_3 &\le 3 && \text{(resource 3)} \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

4.6-10.* Consider the following problem.

(a) Minimize $Z = 3x_1 + 2x_2 + 4x_3$,

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 60 \\ 3x_1 + 3x_2 + 5x_3 &\ge 120 \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

6.1-6. Consider the following problem.

Maximize $Z = 2x_1 + 6x_2 + 9x_3$,

subject to

$$\begin{aligned} x_1 + x_3 &\le 3 && \text{(resource 1)} \\ x_2 + 2x_3 &\le 5 && \text{(resource 2)} \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

6.1-7. Follow the instructions of Prob. 6.1-6 for the following problem.

Maximize $Z = x_1 - 3x_2 + 2x_3$,

subject to

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\le 6 && \text{(resource 1)} \\ -x_2 + 2x_3 &\le 4 && \text{(resource 2)} \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

6.3-6. Consider the following problem.

Maximize $Z = 2x_1 + 7x_2 + 4x_3$,

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10 \\ 3x_1 + 3x_2 + 2x_3 &\le 10 \end{aligned}$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

D,I **6.6-3.** Consider the following problem.

Minimize $W = 5y_1 + 4y_2$,

subject to

$$\begin{aligned} 4y_1 + 3y_2 &\ge 4 \\ 2y_1 + y_2 &\ge 3 \\ y_1 + 2y_2 &\ge 1 \\ y_1 + y_2 &\ge 2 \end{aligned}$$

and

$y_1 \ge 0, \quad y_2 \ge 0.$

D,I **6.7-3.** Consider the following problem.

Maximize $Z = 2x_1 - x_2 + x_3$,

subject to
$$\begin{aligned} 3x_1 + x_2 + x_3 &\le 60 \\ x_1 - x_2 + 2x_3 &\le 10 \\ x_1 + x_2 - x_3 &\le 20 \end{aligned}$$

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

4.6-1.* Consider the following problem.

(a) Maximize $Z = 2x_1 + 3x_2$

subject to
$$\begin{aligned} x_1 + 2x_2 &\le 4 \\ x_1 + x_2 &= 3 \end{aligned}$$

$x_1 \ge 0, \quad x_2 \ge 0.$

(b) Minimize $Z = 3x_1 + 2x_2 + 7x_3,$

subject to
$$\begin{aligned} -x_1 + x_2 \qquad &= 10 \\ 2x_1 - x_2 + x_3 &\geq 10 \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

(C) $Min \quad Z = x_1 + x_2$

$s.t.$
$$\begin{aligned} 3x_1 + 2x_2 &\geq 10 \\ 2x_1 + 3x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6.1-5. Consider the following problem.

Maximize $Z = -x_1 - 2x_2 - x_3,$

subject to
$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 12 \\ x_1 + x_2 - x_3 &\leq 1 \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

6.1-6. Consider the following problem.

Maximize $Z = 2x_1 + 6x_2 + 9x_3,$

subject to
$$\begin{aligned} x_1 \qquad + x_3 &\leq 3 \qquad \text{(resource 1)} \\ x_2 + 2x_3 &\leq 5 \qquad \text{(resource 2)} \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

6.1-7. Follow the instructions of Prob. 6.1-6 for the following problem.

Maximize $Z = x_1 - 3x_2 + 2x_3,$

subject to
$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\leq 6 \qquad \text{(resource 1)} \\ -x_2 + 2x_3 &\leq 4 \qquad \text{(resource 2)} \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

6.3-6. Consider the following problem.

Maximize $Z = 2x_1 + 7x_2 + 4x_3,$

subject to
$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 10 \\ 3x_1 + 3x_2 + 2x_3 &\leq 10 \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **6.6-3.** Consider the following problem.

Minimize $W = 5y_1 + 4y_2,$

subject to
$$\begin{aligned} 4y_1 + 3y_2 &\geq 4 \\ 2y_1 + y_2 &\geq 3 \\ y_1 + 2y_2 &\geq 1 \\ y_1 + y_2 &\geq 2 \end{aligned}$$

$$y_1 \geq 0, \qquad y_2 \geq 0.$$

D,I **6.7-3.** Maximize $Z = 2x_1 - x_2 + x_3,$

subject to
$$\begin{aligned} 3x_1 + x_2 + x_3 &\leq 60 \\ x_1 - x_2 + 2x_3 &\leq 10 \\ x_1 + x_2 - x_3 &\leq 20 \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

(b) Minimize $Z = 3x_1 + 8x_2 + 5x_3,$

subject to
$$\begin{aligned} 3x_2 + 4x_3 &\geq 70 \\ 3x_1 + 5x_2 + 2x_3 &\geq 70 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

4.6-18. Consider the following problem.

(a) Maximize $Z = -2x_1 + x_2 - 4x_3 + 3x_4,$

subject to
$$\begin{aligned} x_1 + x_2 + 3x_3 + 2x_4 &\leq 4 \\ x_1 \qquad - x_3 + x_4 &\geq -1 \\ 2x_1 + x_2 \qquad &\leq 2 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 2 \end{aligned}$$

and

$$x_2 \geq 0, \qquad x_3 \geq 0, \qquad x_4 \geq 0$$

(b) Minimize $Z = 5x_1 + 7x_2,$

subject to
$$\begin{aligned} 2x_1 + 3x_2 &\geq 42 \\ 3x_1 + 4x_2 &\geq 60 \\ x_1 + x_2 &\geq 18 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

4.4-7. Consider the following problem.

Maximize $Z = 3x_1 + 5x_2 + 6x_3,$

subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 4 \\ x_1 + 2x_2 + x_3 &\leq 4 \\ x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + x_2 + x_3 &\leq 3 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

4.6-4.* Consider the following problem.

Minimize $Z = 2x_1 + 3x_2 + x_3,$

subject to
$$\begin{aligned} x_1 + 4x_2 + 2x_3 &\geq 8 \\ 3x_1 + 2x_2 \qquad &\geq 6 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

1) $Max \quad Z = 3x_1 + 2x_2 + 5x_3$

$$\begin{aligned} s.t. \quad & x_1 + 3x_2 + 2x_3 \le 345 \\ & 2x_2 - x_3 \ge 115 \\ & 2x_1 + x_2 - 5x_3 = 230 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$Opt.\ Sol.\ is: x_1 = 120,\ \ x_2 = 65,\ \ x_3 = 15,\quad Z = 565$

$-0.412 \le C_1 \quad -23.5 \le C_2 \le 8.44 \qquad -6.6 \le C_3$

1) $Max \quad Z = 3x_1 - 3x_2 + 5x_3$

$$\begin{aligned} s.t. \quad & x_1 + 3x_2 + 5x_3 \le 345 \\ & 5x_2 - 2x_3 \ge 115 \\ & x_1 + 3x_2 + 5x_3 = 230 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$Opt.\ Sol.\ is: x_1 = 161,\ \ x_2 = 23,\ \ x_3 = 0,\quad Z = 414$

$0.613 \le C_1 \qquad C_2 \le 9 \qquad C_3 \le 19.8$

3) $Max \quad Z = 5x_1 - 3x_2 + 5x_3$

$$\begin{aligned} s.t. \quad & 1x_1 + 3x_2 + 5x_3 \le 345 \\ & 5x_2 - 2x_3 \ge 115 \\ & 2x_1 + 4x_2 + 5x_3 = 230 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$Opt.\ Sol.\ is: x_1 = 69,\ \ x_2 = 23,\ \ x_3 = 0, Z = 276$

$1.152 \le C_1 \qquad C_2 \le 10 \qquad C_3 \le 17.7$

4) $Min \quad Z = 8x_1 - 3x_2 + 2x_3$

$$\begin{aligned} s.t. \quad & x_1 + 3x_2 + 5x_3 \le 345 \\ & 3x_1 + 5x_2 - 2x_3 \ge 115 \\ & 2x_1 + x_2 + 2x_3 = 230 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$Opt.\ Sol.\ is: x_1 = 69,\ \ x_2 = 92,\ \ x_3 = 0,\quad Z = 276$

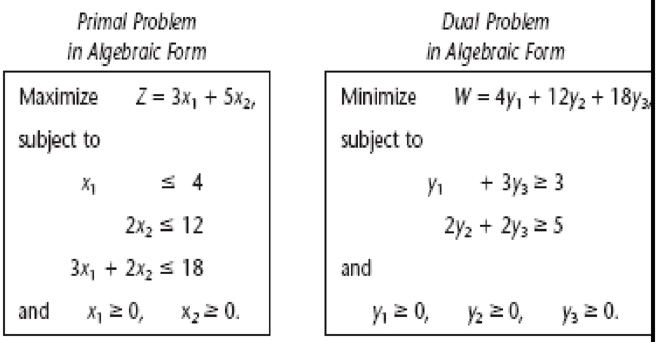
$-6 \le C_1 \le 34 \qquad C_2 \le 0.25 \qquad -3.2 \le C_3$

5) $Min \quad Z = 8x_1 - 3x_2 + 2x_3$

$$\begin{aligned} s.t. \quad & x_1 + 3x_2 + 5x_3 \ge 115 \\ & 3x_1 + 5x_2 - 2x_3 \le 325 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$Opt.\ Sol.\ is: x_1 = 0,\ \ x_2 = 69,\ \ x_3 = 0,\quad Z = -207$

$-1.8 \le C_1 \qquad -12.5 \le C_2 \le 0 \qquad 1.2 \le C_3$



Primal Problem in Algebraic Form

Maximize $Z = 3x_1 + 5x_2,$

subject to

$x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

and $x_1 \ge 0, \quad x_2 \ge 0.$

Dual Problem in Algebraic Form

Minimize $W = 4y_1 + 12y_2 + 18y_3$

subject to

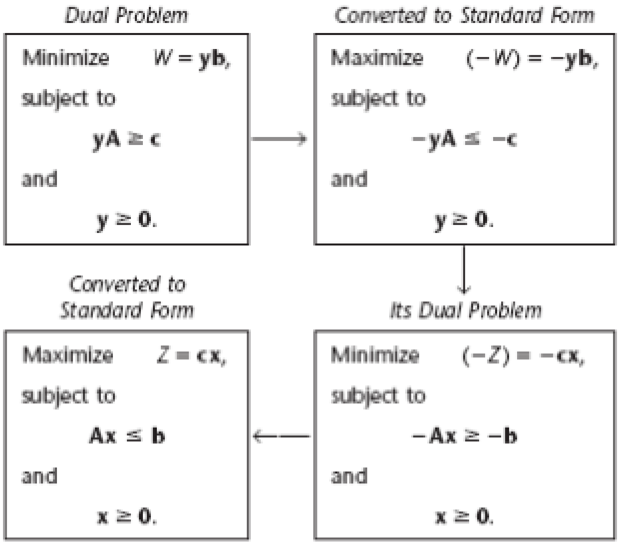
$y_1 + 3y_3 \ge 3$

$2y_2 + 2y_3 \ge 5$

and

$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

TABLE 6.13 Constructing the dual of the dual probler



Dual Problem

Minimize $W = \mathbf{yb},$

subject to

$\mathbf{yA} \ge \mathbf{c}$

and

$\mathbf{y} \ge \mathbf{0}.$

Converted to Standard Form

Maximize $(-W) = -\mathbf{yb},$

subject to

$-\mathbf{yA} \le -\mathbf{c}$

and

$\mathbf{y} \ge \mathbf{0}.$

Its Dual Problem

Minimize $(-Z) = -\mathbf{cx},$

subject to

$-\mathbf{Ax} \ge -\mathbf{b}$

and

$\mathbf{x} \ge \mathbf{0}.$

Converted to Standard Form

Maximize $Z = \mathbf{cx},$

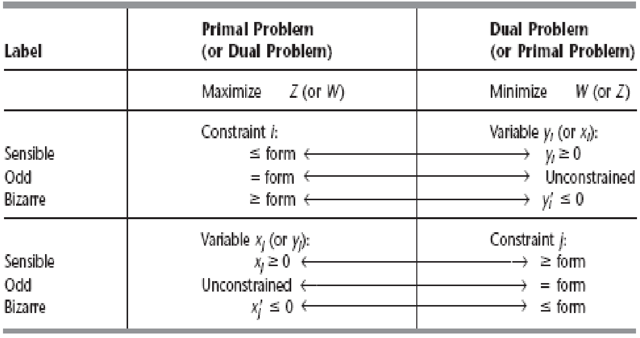
subject to

$\mathbf{Ax} \le \mathbf{b}$

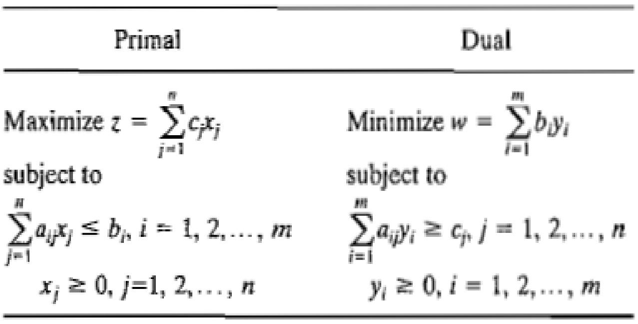
and

$\mathbf{x} \ge \mathbf{0}.$

TABLE 6.14 Corresponding primal-dual forms



Label	Primal Problem (or Dual Problem)	Dual Problem (or Primal Problem)
	Maximize Z (or W)	Minimize W (or Z)
	Constraint i:	Variable y_i (or x_i):
Sensible	≤ form	$y_i \ge 0$
Odd	= form	Unconstrained
Bizarre	≥ form	$y_i' \le 0$
	Variable x_j (or y_j):	Constraint j:
Sensible	$x_j \ge 0$	≥ form
Odd	Unconstrained	= form
Bizarre	$x_j' \le 0$	≤ form



Primal	Dual
Maximize $z = \sum_{j=1}^{n} c_j x_j$	Minimize $w = \sum_{i=1}^{m} b_i y_i$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \ge c_j,\ j = 1, 2, \ldots, n$
$x_j \ge 0,\ j=1, 2, \ldots, n$	$y_i \ge 0,\ i = 1, 2, \ldots, m$

مع أصدق تمنياتي لكم بالنجاح

$Max \quad Z = 2x_1 + 3x_2 + x_3 + 2x_4$

$s.t. \qquad 3x_1 + 2x_2 + x_3 + 2x_4 \le 17$

$\qquad 3x_1 + x_2 - x_3 + 3x_4 \le 10$

$\qquad x_1, x_2, x_3, x_4 \ge 0$

$Opt.\ Sol.\ is: x_1 = x_3 = x_4 = 0,\ \ x_2 = 8.5,\ \ Z = 25.5$

$C_1 \le 4.5 \qquad 2 \le C_2 \qquad C_3 \le 1.5 \qquad C_4 \le 3$

$Max \quad Z = x_1 - 2x_2 + x_3 \qquad\qquad Min$

$s.t. \qquad 2x_1 + 3x_2 - x_3 \quad \le 8 \qquad x_1 = x_3 = 0 \quad x_2 = 2$

$\qquad x_1 - x_2 + 2x_3 \quad \le 7 \qquad Z - 4$

$\qquad 3x_1 + 2x_2 + x_3 \quad \le 4 \quad -3 \le C_1 \ \ C_2 \le 0 \quad -1 \le C_3$

$\qquad x_1, x_2, x_3 \ge 0$

$Opt.\ Sol.\ is: x_1 = 0.2,\ \ x_2 = 0,\ \ \ x_3 = 3.4,\ \ \ Z = 3.6$

$0.5 \le C_1 \le 3 \qquad C_2 \le 0 \qquad \frac{1}{3} \le C_3 \le 2$

$Max \quad Z = x_1 + 2x_2 + x_3 \qquad\qquad Min$

$s.t. \qquad 2x_1 - x_2 + 3x_3 \quad \le 8 \qquad x_1 = x_3 = 0 \quad x_2 = 2$

$\qquad x_1 - x_2 + 2x_3 \quad \le 7 \qquad Z - 4$

$\qquad 3x_1 + 2x_2 + x_3 \quad \le 10 \quad -3 \le C_1 \ \ C_2 \le 0 \quad -1 \le C_3$

$\qquad x_1, x_2, x_3 \ge 0$

$Opt.\ Sol.\ is: x_1 = 0.2,\ \ x_2 = 0,\ \ \ x_3 = 3.4,\ \ \ Z = 3.6$

$0.5 \le C_1 \le 3 \qquad C_2 \le 0 \qquad \frac{1}{3} \le C_3 \le 2$

$Min \quad Z = 2x_1 - 5x_2 \qquad\qquad Max$

$s.t. \qquad 3x_1 + 8x_2 \quad \le 12 \qquad x_1 = 3 \quad x_2 = 0 \quad Z = 6$

$\qquad 2x_1 + 3x_2 \quad \le 6 \qquad 0 \le C_1 \qquad C_2 \le 3$

$\qquad x_1, x_2 \ge 0$

$Opt.\ Sol.\ is: x_1 = 0,\ \ x_2 = 1.5,\ \ Z = -7.5$

$-1.875 \le C_1 \qquad C_2 \le 0$

$Min \quad Z = -x_1 - x_2$

$s.t. \qquad x_1 - x_2 \quad \le 1$

$\qquad x_1 + x_2 \quad \le 2$

$\qquad x_1, x_2 \ge 0$

$Opt.\ Sol.\ is: x_1 = 1.5,\ \ x_2 = 0.5,\ \ Z = -2$

$C_1 \le -1 \qquad -1 \le C_2 \le 1$

$Max \quad Z = 5x_1 + 3x_2$

$s.t. \qquad 4x_1 + 2x_2 \le 12$

$\qquad 4x_1 + x_2 \le 10$

$\qquad x_1 + x_2 \le 4$

$\qquad x_1, x_2 \ge 0$

$Opt.\ Sol.\ is: x_1 = 2,\ \ x_2 = 2,\ \ Z = 16$

$3 \le C_1 \le 6 \qquad 2 \le C_2 \le 5$

$Max \quad Z = 60x_1 + 30x_2 + 20x_3$

$s.t. \qquad 8x_1 + 6x_2 + x_3 \qquad \le 48$

$\qquad 4x_1 + 2x_2 + 1.5x_3 \quad \le 20$

$\qquad 2x_1 + 1.5x_2 + 0.5x_3 \quad \le 8$

$\qquad x_2 \qquad\qquad \le 5$

$\qquad x_1, x_2, x_3 \ge 0$

$Opt.\ Sol.\ is: x_1 = 2,\ \ x_2 = 0,\ \ \ x_3 = 8,\ \ \ Z = 280$

$56 \le C_1 \le 80 \qquad C_2 \le 35 \qquad 15 \le C_3 \le 22.5$

$Min \quad Z = 4x_1 - x_2 \qquad\qquad Max$

$s.t. \qquad 2x_1 + x_2 \le 8 \qquad x_1 = 4 \quad x_2 = 0 \quad Z = 16$

$\qquad x_2 \le 5 \qquad 0 \le C_1 \qquad C_2 \le 3$

$\qquad x_1 - x_2 \le 4$

$\qquad x_1, x_2 \ge 0$

$Opt.\ Sol.\ is: x_1 = 0,\ \ x_2 = 5,\ \ Z = -5$

$0 \le C_1 \qquad C_2 \le 0$

$Min \quad Z = -3x_1 + 8x_2$

$s.t. \qquad 4x_1 + 2x_2 \quad \le 12$

$\qquad 2x_1 + 3x_2 \quad \le 6$

$\qquad x_1, x_2 \ge 0$

$Opt.\ Sol.\ is: x_1 = 3,\ \ x_2 = 0,\ \ Z = -9$

$C_1 \le 0 \qquad -1.5 \le C_2$

$$\begin{array}{ll} Max & Z = x_1 + 2x_2 + x_3 \\ s.t. & 2x_1 + x_2 + 3x_3 \leq 12 \\ & 3x_1 + 4x_2 + 2x_3 \geq 6 \\ & x_1 + 2x_2 + x_3 = 15 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$Opt. Sol. is : x_1 = 0, \quad x_2 = 7.5, \quad x_3 = 0, \quad Z = 15$

$C_1 \leq 1 \qquad 2 \leq C_2 \qquad C_3 \leq 1$

If the objective function is Min Then:

$x_1 = 0 \quad x_2 = 7.5 \quad x_3 = 0 \qquad Z = 15$

$1 \leq C_1 \qquad 2 \leq C_2 \qquad C_3 \leq 1$

$$\begin{array}{ll} Max & Z = x_1 + 2x_2 + x_3 \\ s.t. & 2x_1 + 3x_2 - x_3 \leq 5 \\ & x_1 - x_2 + 2x_3 \geq 4 \\ & 3x_1 + 2x_2 + x_3 = 7 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$Opt. Sol. is : x_1 = 0, \quad x_2 = 2, \quad x_3 = 3, \quad Z = 7$

$C_1 \leq 3 \qquad C_2 \geq 2 \qquad -1 \leq C_3 \leq 1$

If the objective function is Min Then:

$x_1 = 2 \quad x_2 = 8 \quad x_3 = 1 \qquad Z = 3$

$C_1 \leq 3 \qquad C_2 \geq 0 \qquad \frac{1}{3} \leq C_3$

$$\begin{array}{ll} Max & Z = 4x_1 + x_2 \\ s.t. & 10x_1 + 2x_2 \leq 40 \\ & 3x_1 + 2x_2 \geq 12 \\ & 2x_1 + 2x_2 = 10 \\ & x_1, x_2 \geq 0 \end{array}$$

$Opt. Sol. is : x_1 = 3.75, \quad x_2 = 1.25, \quad Z = 16.25$

$1 \leq C_1 \qquad C_2 \leq 4$

If the objective function is Min Then:

$x_1 = 2 \quad x_2 = 3 \quad Z = 11$

$1 \leq C_1 \qquad C_2 \leq 4$

$$\begin{array}{ll} Min & Z = x_1 - 8x_2 + 3x_3 \\ s.t. & x_1 + x_2 + x_3 = 7 \\ & 2x_1 - 5x_2 + x_3 \geq 10 \\ & 2x_1 + x_2 + 3x_3 \leq 30 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$Opt. Sol. is : x_1 = 6.429, \quad x_2 = 0.571, \quad x_3 = 0,$

$The\ Optimal\ Valueis : \quad Z = 1.857$

$-8 \leq C_1 \leq 4.833 \qquad C_2 \leq 1 \qquad -0.286 \leq C_3$

If the objective function is Max Then:

$x_1 = 3 \quad x_2 = 0 \quad x_3 = 4 \qquad Z = 15$

$C_1 \leq 3 \qquad C_2 \leq 15 \qquad 1 \leq C_3$

$$\begin{array}{ll} Min & Z = 4x_1 + 4x_2 + x_3 \\ s.t. & x_1 + x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 \leq 3 \\ & 2x_1 + 1x_2 + 3x_3 \geq 3 \\ & x_2 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$Opt. Sol. is : x_1 = 0, \quad x_2 = 0, \quad x_3 = 1, \quad Z = 1$

$\frac{2}{3} \leq C_1 \qquad \frac{1}{3} \leq C_2 \qquad 0 \leq C_3 \leq 6$

If the objective function is Max Then:

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 0 \qquad Z = 8$

$2.5 \leq C_1 \leq 7 \qquad 2.5 \leq C_2 \geq 7 \qquad C_3 \leq 4$

$$\begin{array}{ll} Max & Z = x_1 - 2x_2 - x_3 \\ s.t. & 2x_1 + x_2 - 3x_3 \leq 9 \\ & x_1 - x_2 + 2x_3 \geq 14 \\ & 3x_1 + 2x_2 + x = 27 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$Opt. Sol. is : x_1 = 8, \quad x_2 = 0, \quad x_3 = 3, \quad Z = 5$

$-3 \leq C_1 \qquad C_2 \leq 2 \qquad C_3 \leq \frac{1}{3}$

If the objective function is Min Then:

$x_1 = 0 \quad x_2 = 8 \quad x_3 = 11 \qquad Z = -27$

$-3 \leq C_1 \qquad C_2 \leq -2 \qquad -1 \leq C_3 \leq 3$

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

Finding the Dual of an LP

A normal max problem may be written as

$$
\begin{aligned}
&\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
&\qquad\ \ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
&\qquad\ \ \vdots \qquad\ \ \vdots \qquad\qquad\quad \vdots \\
&\qquad\ \ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
&\qquad\ \ x_j \ge 0 \quad (j = 1, 2, \ldots, n)
\end{aligned}
\qquad (1)
$$

The dual of a normal max problem such as (1) is defined to be

$$
\begin{aligned}
&\min w = b_1y_1 + b_2y_2 + \cdots + b_mx_m \\
&\text{s.t.} \quad a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\
&\qquad\ \ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\
&\qquad\ \ \vdots \qquad\ \ \vdots \qquad\qquad\quad \vdots \\
&\qquad\ \ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \\
&\qquad\ \ y_i \ge 0 \quad (i = 1, 2, \ldots, m)
\end{aligned}
\qquad (2)
$$

A min problem such as (2) that has all $\ge$ constraints and all variables nonnegative is called a **normal min problem.** If the primal is a normal min problem such as (2) then we define the dual of (2) to be (1).

For example

The Primal Prolem

$$
\begin{aligned}
&\max z = 60x_1 + 30x_2 + 20x_3 \\
&\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \le 48 \\
&\qquad\ \ 4x_1 + 2x_2 + 1.5x_3 \le 20 \\
&\qquad\ \ 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \\
&\qquad\qquad\qquad x_1, x_2, x_3 \ge 0
\end{aligned}
$$

The Dual Problem

$$
\begin{aligned}
&\min w = 48y_1 + 20y_2 + 8y_3 \\
&\text{s.t.} \quad 8y_1 + 4y_2 + 2y_3 \ge 60 \\
&\qquad\ \ 6y_1 + 2y_2 + 1.5y_3 \ge 30 \\
&\qquad\ \ y_1 + 1.5y_2 + 0.5y_3 \ge 20 \\
&\qquad\qquad\qquad y_1, y_2, y_3 \ge 0
\end{aligned}
$$

The Primal Prolem

$$
\begin{aligned}
&\min w = 50y_1 + 20y_2 + 30y_3 + 80y_4 \\
&\text{s.t.} \quad 400y_1 + 200y_2 + 150y_3 + 500y_4 \ge 500 \\
&\qquad\ \ 3y_1 + 2y_2 \ge 6 \\
&\qquad\ \ 2y_1 + 2y_2 + 4y_3 + 4y_4 \ge 10 \\
&\qquad\ \ 2y_1 + 4y_2 + y_3 + 5y_4 \ge 8 \\
&\qquad\qquad\qquad y_1, y_2, y_3, y_4 \ge 0
\end{aligned}
$$

The Dual Problem

$$
\begin{aligned}
&\max z = 500x_1 + 6x_2 + 10x_3 + 8x_4 \\
&\text{s.t.} \quad 400x_1 + 3x_2 + 2x_3 + 2x_4 \le 50 \\
&\qquad\ \ 200x_1 + 2x_2 + 2x_3 + 4x_4 \le 20 \\
&\qquad\ \ 150x_1 + 4x_3 + x_4 \le 30 \\
&\qquad\ \ 500x_1 + 4x_3 + 5x_4 \le 80 \\
&\qquad\qquad\qquad x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

The Primal Prolem

$$
\begin{aligned}
&\max z = 3x_1 + x_2 \\
&\text{s.t.} \quad x_1 + x_2 \le 1 \\
&\qquad\ \ -x_1 + x_2 \ge 2 \\
&\qquad\qquad x_1, x_2 \ge 0
\end{aligned}
$$

The Dual Problem

$$
\begin{aligned}
&\min \ z = x_1 - 2\,x_2 \\
&\text{s.t.} \quad x_1 + x_2 \ \ge \ 3 \\
&\qquad\ \ x_1 - x_2 \ \ge \ 1 \\
&\qquad\qquad x_1, x_2 \ge 0
\end{aligned}
$$

Review Problems

Find the duals of the following LPs

and find the optimal solutions of the two problems

1 $$\begin{aligned} \max z &= 2x_1 + x_2 \\ \text{s.t.} \quad -x_1 + x_2 &\le 1 \\ x_1 + x_2 &\le 3 \\ x_1 - 2x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

2 $$\begin{aligned} \min w &= y_1 - y_2 \\ \text{s.t.} \quad 2y_1 + y_2 &\ge 4 \\ y_1 + y_2 &\ge 1 \\ y_1 + 2y_2 &\ge 3 \\ y_1, y_2 &\ge 0 \end{aligned}$$

3 $$\begin{aligned} \max z &= 4x_1 - x_2 + 2x_3 \\ \text{s.t.} \quad x_1 + x_2 &\le 5 \\ 2x_1 + x_2 &\le 7 \\ 2x_2 + x_3 &\ge 6 \\ x_1 + x_3 &= 4 \\ x_1 \ge 0, x_2, x_3 &\text{ urs} \end{aligned}$$

4 $$\begin{aligned} \min w &= 4y_1 + 2y_2 - y_3 \\ \text{s.t.} \quad y_1 + 2y_2 &\le 6 \\ y_1 - y_2 + 2y_3 &= 8 \\ y_1, y_2 \ge 0, y_3 &\text{ urs} \end{aligned}$$

5) $$\begin{aligned} \max z &= -2x_1 - x_2 + x_3 \\ \text{s.t.} \quad x_1 + x_2 + x_3 &\le 3 \\ x_2 + x_3 &\ge 2 \\ x_1 + x_3 &= 1 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

6) $$\begin{aligned} \max z &= 3x_1 + 2x_2 \\ \text{s.t.} \quad 2x_1 + x_2 &\le 100 \\ x_1 + x_2 &\le 80 \\ x_1 &\le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

7) $$\begin{aligned} \max z &= 3x_1 + x_2 \\ \text{s.t.} \quad 2x_1 - x_2 &\le 2 \\ -x_1 + x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

9) $$\begin{aligned} \max z &= -x_1 + x_2 \\ \text{s.t.} \quad 2x_1 + x_2 &\le 4 \\ x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

10) $$\begin{aligned} \min z &= 2x_1 + 3x_2 \\ \text{s.t.} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &\le 4 \\ x_1 + 3x_2 &\ge 36 \\ x_1 + x_2 &= 10 \\ x_1, x_2 &\ge 0 \end{aligned}$$

11) $$\begin{aligned} \min z &= 3x_1 \\ \text{s.t.} \quad 2x_1 + x_2 &\ge 6 \\ 3x_1 + 2x_2 &= 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

12) $$\begin{aligned} \min z &= x_1 + x_2 \\ \text{s.t.} \quad 2x_1 + x_2 + x_3 &= 4 \\ x_1 + x_2 + 2x_3 &= 2 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

13) $$\begin{aligned} \min z &= x_1 + x_2 \\ \text{s.t.} \quad x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

14) $$\begin{aligned} \min z &= 4x_1 + 4x_2 + x_3 \\ \text{s.t.} \quad x_1 + x_2 + x_3 &\le 2 \\ 2x_1 + x_2 &\le 3 \\ 2x_1 + x_2 + 3x_3 &\ge 3 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

15) $$\begin{aligned} \min z &= 2x_1 + 3x_2 \\ \text{s.t.} \quad 2x_1 + x_2 &\ge 4 \\ x_1 - x_2 &\ge -1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

16) $$\begin{aligned} \max z &= 3x_1 + x_2 \\ \text{s.t.} \quad x_1 + x_2 &\ge 3 \\ 2x_1 + x_2 &\le 4 \\ x_1 + x_2 &= 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Transportation Problems مسائل النقل

EXAMPLE 1 **Powerco Formulation**

Powerco has three electric power plants that supply the needs of four cities.[†] Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1—35 million; plant 2—50 million; plant 3—40 million (see Table 1). The peak power demands in these cities, which occur at the same time (2 P.M.), are as follows (in kwh): city 1—45 million; city 2—20 million; city 3—30 million; city 4—30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

Solution To formulate Powerco's problem as an LP, we begin by defining a variable for each decision that Powerco must make. Because Powerco must determine how much power is sent from each plant to each city, we define (for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$)

$$x_{ij} = \text{number of (million) kwh produced at plant } i \text{ and sent to city } j$$

In terms of these variables, the total cost of supplying the peak power demands to cities 1–4 may be written as

$$\begin{aligned} &8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} && \text{(Cost of shipping power from plant 1)} \\ &+ 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} && \text{(Cost of shipping power from plant 2)} \\ &+ 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} && \text{(Cost of shipping power from plant 3)} \end{aligned}$$

Powerco faces two types of constraints. First, the total power supplied by each plant cannot exceed the plant's capacity. For example, the total amount of power sent from plant

Formulating Transportation Problems

We begin our discussion of transportation problems by formulating a linear programming model of the following situation.

TABLE 1

Shipping Costs, Supply, and Demand for Powerco

	To				
From	**City 1**	**City 2**	**City 3**	**City 4**	**Supply (million kwh)**
Plant 1	$8	$6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	40
Demand (million kwh)	45	20	30	30	

1 to the four cities cannot exceed 35 million kwh. Each variable with first subscript 1 represents a shipment of power from plant 1, so we may express this restriction by the LP constraint

$$x_{11} + x_{12} + x_{13} + x_{14} \le 35$$

In a similar fashion, we can find constraints that reflect plant 2's and plant 3's capacities. Because power is supplied by the power plants, each is a **supply point.** Analogously, a constraint that ensures that the total quantity shipped from a plant does not exceed plant capacity is a **supply constraint.** The LP formulation of Powerco's problem contains the following three supply constraints:

$$\begin{array}{ll} x_{11} + x_{12} + x_{13} + x_{14} \le 35 & \text{(Plant 1 supply constraint)} \\ x_{21} + x_{22} + x_{23} + x_{24} \le 50 & \text{(Plant 2 supply constraint)} \\ x_{31} + x_{32} + x_{33} + x_{34} \le 40 & \text{(Plant 3 supply constraint)} \end{array}$$

Second, we need constraints that ensure that each city will receive sufficient power to meet its peak demand. Each city demands power, so each is a **demand point.** For example, city 1 must receive at least 45 million kwh. Each variable with second subscript 1 represents a shipment of power to city 1, so we obtain the following constraint:

$$x_{11} + x_{21} + x_{31} \ge 45$$

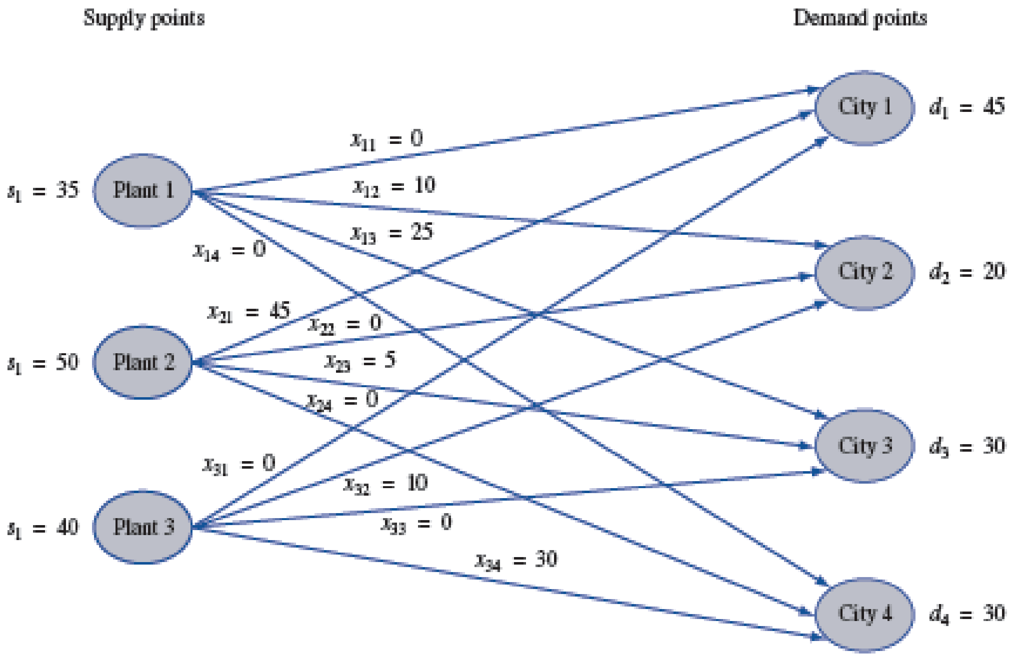
Similarly, we obtain a constraint for each of cities 2, 3, and 4. A constraint that ensures that a location receives its demand is a **demand constraint.** Powerco must satisfy the following four demand constraints:

$$\begin{array}{ll} x_{11} + x_{21} + x_{31} \ge 45 & \text{(City 1 demand constraint)} \\ x_{12} + x_{22} + x_{32} \ge 20 & \text{(City 2 demand constraint)} \\ x_{13} + x_{23} + x_{33} \ge 30 & \text{(City 3 demand constraint)} \\ x_{14} + x_{24} + x_{34} \ge 30 & \text{(City 4 demand constraint)} \end{array}$$

Because all the x_{ij}'s must be nonnegative, we add the sign restrictions $x_{ij} \ge 0$ ($i = 1, 2, 3; j = 1, 2, 3, 4$).

Combining the objective function, supply constraints, demand constraints, and sign restrictions yields the following LP formulation of Powerco's problem:

$$\begin{array}{lll} \min z = & 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} & \\ & \quad + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} & \\ \text{s.t.} & x_{11} + x_{12} + x_{13} + x_{14} \le 35 & \text{(Supply constraints)} \\ & x_{21} + x_{22} + x_{23} + x_{24} \le 50 & \\ & x_{31} + x_{32} + x_{33} + x_{34} \le 40 & \end{array}$$



$$\begin{aligned} x_{11} + x_{21} + x_{31} &\geq 45 \qquad \text{(Demand constraints)} \\ x_{12} + x_{22} + x_{32} &\geq 20 \\ x_{13} + x_{23} + x_{33} &\geq 30 \\ x_{14} + x_{24} + x_{34} &\geq 30 \\ x_{ij} \geq 0 \quad (i = 1, 2, 3;\ j &= 1, 2, 3, 4) \end{aligned}$$

General Description of a Transportation Problem

In general, a transportation problem is specified by the following information:

1 A set of m *supply points* from which a good is shipped. Supply point i can supply at most s_i units. In the Powerco example, $m = 3$, $s_1 = 35$, $s_2 = 50$, and $s_3 = 40$.

2 A set of n *demand points* to which the good is shipped. Demand point j must receive at least d_j units of the shipped good. In the Powerco example, $n = 4$, $d_1 = 45$, $d_2 = 20$, $d_3 = 30$, and $d_4 = 30$.

3 Each unit produced at supply point i and shipped to demand point j incurs a *variable cost* of c_{ij}. In the Powerco example, $c_{12} = 6$.

Let

$$x_{ij} = \text{number of units shipped from supply point } i \text{ to demand point } j$$

then the general formulation of a transportation problem is

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij}x_{ij}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)} \\ & \sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)} \\ & x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n) \end{aligned} \qquad (1)$$

if $$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j$$

then total supply equals total demand, and the problem is said to be a **balanced transportation problem.**

For a balanced transportation problem, (1) may be written as

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij}x_{ij}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)} \\ & \sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)} \\ & x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n) \end{aligned} \qquad (2)$$

8 35	6	10	9	35
9 10	12 20	13 20	7	50
14	9	16 10	5 30	40
45	20	30	30	

(5)

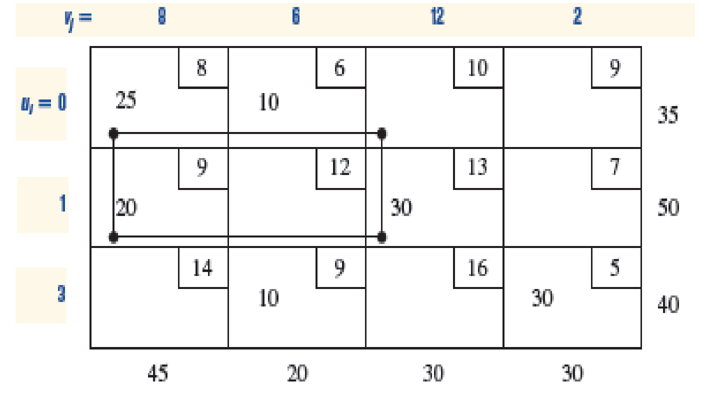
We find the u_i's and v_j's by solving

$$u_1 = 0 \tag{6}$$
$$u_1 + v_1 = 8 \tag{7}$$
$$u_2 + v_1 = 9 \tag{8}$$
$$u_2 + v_2 = 12 \tag{9}$$
$$u_2 + v_3 = 13 \tag{10}$$
$$u_3 + v_3 = 16 \tag{11}$$
$$u_3 + v_4 = 5 \tag{12}$$

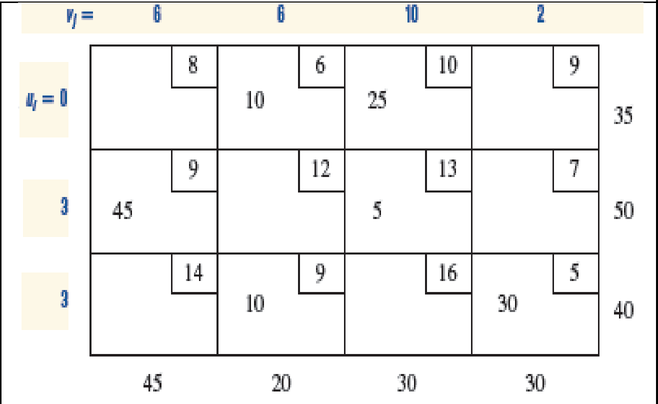
From (7), $v_1 = 8$. From (8), $u_2 = 1$. Then (9) yields $v_2 = 11$, and (10) yields $v_3 = 12$. From (11), $u_3 = 4$. Finally, (12) yields $v_4 = 1$. For each nonbasic variable, we now compute $\bar{c}_{ij} = u_i + v_j - c_{ij}$. We obtain

$$\bar{c}_{12} = 0 + 11 - 6 = 5 \qquad \bar{c}_{13} = 0 + 12 - 10 = 2$$
$$\bar{c}_{14} = 0 + 1 - 9 = -8 \qquad \bar{c}_{24} = 1 + 1 - 7 = -5$$
$$\bar{c}_{31} = 4 + 8 - 14 = -2 \qquad \bar{c}_{32} = 4 + 11 - 9 = 6$$

Because $\bar{c}_{32}$ is the most positive $\bar{c}_{ij}$, we would next enter x_{32} into the basis. Each unit of x_{32} that is entered into the basis will decrease Powerco's cost by \$6.



$v_j =$	8	6	12	2	
$u_i = 0$	8 25	6 10	10	9	35
1	9 20	12	13 30	7	50
3	14	9 10	16	5 30	40
	45	20	30	30	



$v_j =$	6	6	10	2	
$u_i = 0$	8	6 10	10 25	9	35
3	9 45	12	13 5	7	50
3	14	9 10	16	5 30	40
	45	20	30	30	

$$z = 6(10) + 10(25) + 9(45) + 13(5) + 9(10) + 5(30) = \$1{,}020$$

We can now summarize the procedure for using the transportation simplex to solve a transportation (min) problem.

Summary and Illustration of the Transportation Simplex Method

Step 1 If the problem is unbalanced, balance it.

Step 2 Use one of the methods described in Section 7.2 to find a bfs.

Step 3 Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the

$[u_1 \quad u_2 \quad \dots \quad u_m \quad v_1 \quad v_2 \quad \dots \quad v_n]$ for the current bfs.

Step 4 If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal.

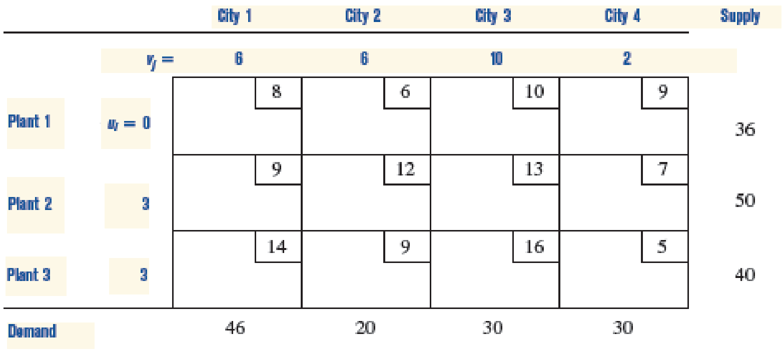
If this is not the case, then we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure. This yields a new bfs.

Step 5 Using the new bfs, return to steps 3 and 4.

For a maximization problem, proceed as stated, but replace step 4 by step 4′.

Step 4′ If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure described earlier.

SOLVE THE FOLLOWING TRANSPORTATION PROBLEMS



		City 1	City 2	City 3	City 4	Supply
	$v_j =$	6	6	10	2	
Plant 1	$u_i = 0$	8	6	10	9	36
Plant 2	3	9	12	13	7	50
Plant 3	3	14	9	16	5	40
Demand		46	20	30	30	

The Total Cost is: 1026 The Total Revenue is: 1508

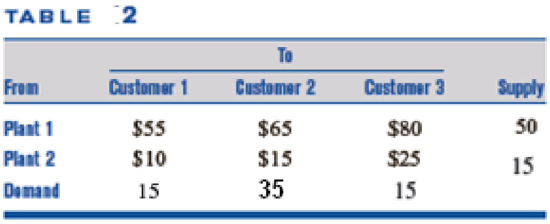


TABLE 2

	To			
From	Customer 1	Customer 2	Customer 3	Supply
Plant 1	$55	$65	$80	50
Plant 2	$10	$15	$25	15
Demand	15	35	15	

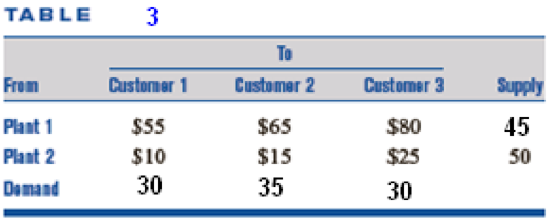


TABLE 3

	To			
From	Customer 1	Customer 2	Customer 3	Supply
Plant 1	$55	$65	$80	45
Plant 2	$10	$15	$25	50
Demand	30	35	30	

The T. Cost is: 3475 The T. Rev. is: 3625

The T. Cost is: 3675 The T. Rev. is: 3975

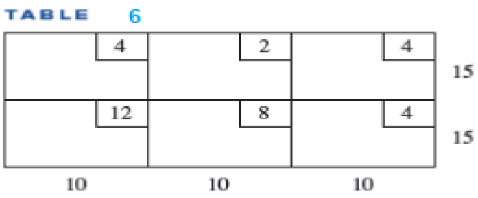


TABLE 6

4	2	4	15
12	8	4	15
10	10	10	

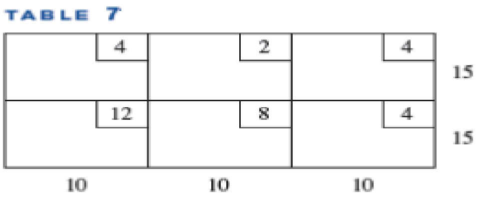


TABLE 7

4	2	4	15
12	8	4	15
10	10	10	

The T. Cost is: 130 The T. Rev. is: 210

The T. Cost is: 160 The T. Rev. is: 250

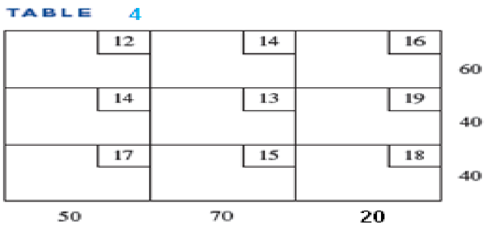


TABLE 4

12	14	16	60
14	13	19	40
17	15	18	40
50	70	20	

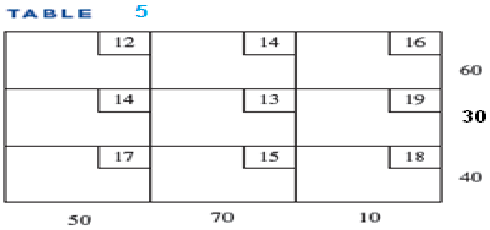


TABLE 5

12	14	16	60
14	13	19	30
17	15	18	40
50	70	10	

The T. Cost is: 1910 The T. Rev. is: 2170

The T. Cost is: 1750 The T. Rev. is: 1980

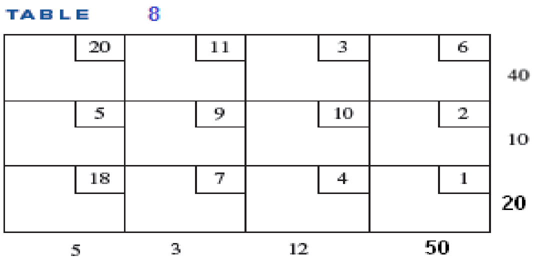


TABLE 8

20	11	3	6	40
5	9	10	2	10
18	7	4	1	20
5	3	12	50	

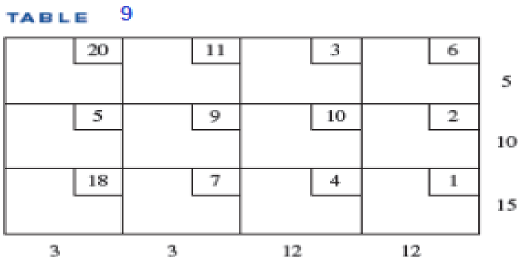


TABLE 9

20	11	3	6	5
5	9	10	2	10
18	7	4	1	15
3	3	12	12	

The T. Cost is: 274 The T. Rev. is: 469

The T. Cost is: 98 The T. Rev. is: 220

Customers

	1	2	3	4	Supply
Plant 1	8	6	10	9	36
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Demand	46	20	30	30	

Min

0	10	26	0
46	0	4	0
0	10	0	30

The Optimal cost is 1026

Max

6	0	0	30
0	20	30	0
40	0	0	0

The Optimal revenue is 1508

Customers

	1	2	3	4	Supply
Plant 1	55	65	80	69	30
Plant 2	10	15	24	45	20
Demand	10	10	10	20	

Min

10	0	0	20
0	10	10	0

The Optimal cost is 2330

Max

10	10	10	0
0	0	0	20

The Optimal revenue is 2900

Customers

	C1	C2	C3	C4	Supply
P1	3	2	7	6	5000
P2	7	5	2	3	6000
P3	2	5	4	5	2500
Demand	6000	4000	2000	1500	

Min

3500	1500	0	0
0	2500	2000	1500
2500	0	0	0

The Optimal cost is 39500

Max

0	1500	2000	1500
6000	0	0	0
0	2500	0	0

The Optimal revenue is 80500

TV Pictures Tubes

	C1	C2	C3	Supply
Plant 1	10	16	32	65
Plant 2	14	22	40	100
Plant 3	22	24	34	105
Demand	80	90	100	

Min

0	65	0
80	20	0
0	5	100

The Optimal cost is 6120

Max

0	65	0
0	0	100
80	25	0

The Optimal revenue is 7400

Customers

	C1	C2	C3	Supply
Plant 1	20	16	24	300
Plant 2	10	10	8	500
Plant 3	12	18	10	100
Demand	200	400	300	

Min

0	300	0
100	100	300
100	0	0

The Optimal cost is 10400

Max

0	0	300
200	300	0
0	100	0

The Optimal revenue is 14000

Customers

	C1	C2	C3	Supply
Plant 1	14	5	8	7
Plant 2	2	12	6	5
Plant 3	7	8	3	9
Demand	8	4	9	

Min

0	4	3
5	0	0
3	0	6

The Optimal cost is 93

Max

7	0	0
0	4	1
1	0	8

The Optimal revenue is 183

Customers

	C1	C2	C3	Supply
Plant 1	15	13	10	62
Plant 2	8	6	8	65
Plant 3	7	6	9	59
Demand	80	70	36	

Min

21	5	36
0	65	0
59	0	0

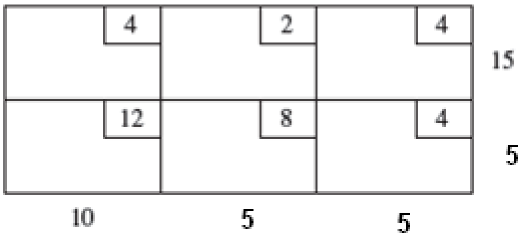
The Optimal cost is 1543

Max

15	47	0
65	0	0
0	23	36

The Optimal revenue is 1818

TABLE 1



4	2	4	15
12	8	4	5
10	5	5	

Min

10	5	0
0	0	5

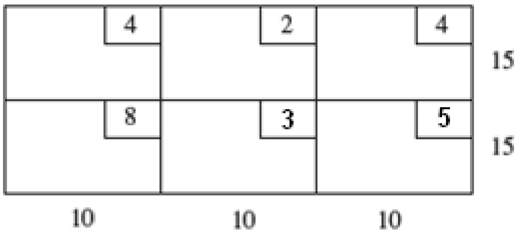
The Optimal cost is 70

Max

5	5	5
5	0	0

The Optimal revenue is 110

TABLE 2



4	2	4	15
8	3	5	15
10	10	10	

Min

10	0	5
0	10	5

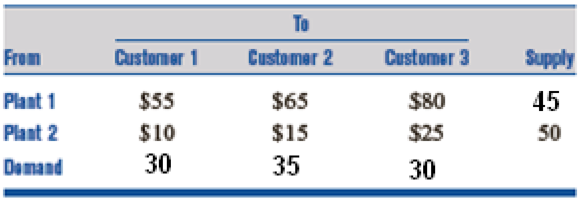
The Optimal cost is 115

Max

0	10	5
10	0	5

The Optimal revenue is 145

TABLE 3



	To			
From	Customer 1	Customer 2	Customer 3	Supply
Plant 1	$55	$65	$80	45
Plant 2	$10	$15	$25	50
Demand	30	35	30	

Min

30	15	0
0	20	30

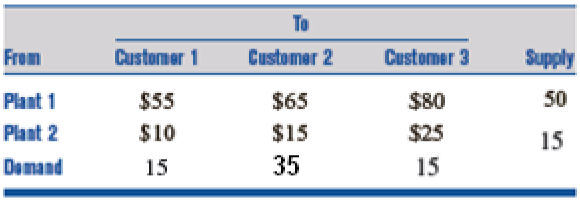
The Optimal cost is 3675

Max

0	15	30
30	20	0

The Optimal revenue is 3975

TABLE 4



	To			
From	Customer 1	Customer 2	Customer 3	Supply
Plant 1	$55	$65	$80	50
Plant 2	$10	$15	$25	15
Demand	15	35	15	

Min

15	35	0
0	0	15

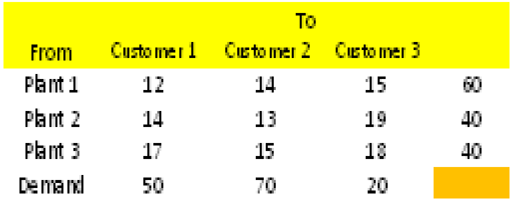
The Optimal cost is 3475

Max

0	35	15
15	0	0

The Optimal revenue is 3625

TABLE 5



		To		
From	Customer 1	Customer 2	Customer 3	
Plant 1	12	14	15	60
Plant 2	14	13	19	40
Plant 3	17	15	18	40
Demand	50	70	20	

Min

50	0	10
0	40	0
0	30	10

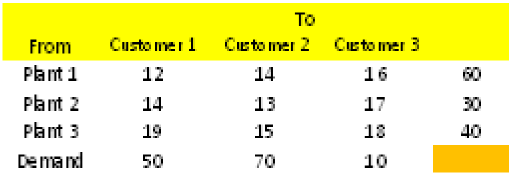
The Optimal cost is 1900

Max

0	60	0
10	10	20
40	0	0

The Optimal revenue is 2170

TABLE 6



		To		
From	Customer 1	Customer 2	Customer 3	
Plant 1	12	14	16	60
Plant 2	14	13	17	30
Plant 3	19	15	18	40
Demand	50	70	10	

Min

50	0	10
0	30	0
0	40	0

The Optimal cost is 1750

Max

0	60	0
10	10	10
40	0	0

The Optimal revenue is 2040

GOOD LUCK PROF. DR. ANWAR AL-LAHHAM

Balancing a Transportation Problem If Total Supply Exceeds Total Demand

If total supply exceeds total demand, we can balance a transportation problem by creating a **dummy demand point** that has a demand equal to the amount of excess supply. Because shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. Shipments to the dummy demand point indicate unused supply capacity. To understand the use of a dummy demand point, suppose that in the Powerco problem, the demand for city 1 were reduced to 40 million kwh. To balance the Powerco problem, we would add a dummy demand point (point 5) with a demand of $125 - 120 = 5$ million kwh. From each plant, the cost of shipping 1 million kwh to the dummy is 0. The optimal solution to this balanced transportation problem is $z = 975$, $x_{13} = 20$, $x_{12} = 15$, $x_{21} = 40$, $x_{23} = 10$, $x_{32} = 5$, $x_{34} = 30$, and $x_{35} = 5$. Because $x_{35} = 5$, 5 million kwh of plant 3 capacity will be unused (see Figure 2).

A transportation problem is specified by the supply, the demand, and the shipping costs, so the relevant data can be summarized in a **transportation tableau** (see Table 2). The square, or **cell**, in row i and column j of a transportation tableau corresponds to the variable x_{ij}. If x_{ij} is a basic variable, its value is placed in the lower left-hand corner of the ijth cell of the tableau. For example, the balanced Powerco problem and its optimal solution could be displayed as shown in Table 3. The tableau format implicitly expresses the supply and demand constraints through the fact that the sum of the variables in row i must equal s_i and the sum of the variables in column j must equal d_j.

Balancing a Transportation Problem If Total Supply Is Less Than Total Demand

If a transportation problem has a total supply that is strictly less than total demand, then the problem has no feasible solution. For example, if plant 1 had only 30 million kwh of capacity, then a total of only 120 million kwh would be available. This amount of power would be insufficient to meet the total demand of 125 million kwh, and the Powerco problem would no longer have a feasible solution.

When total supply is less than total demand, it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, a penalty is often associated with unmet demand. Example 2 illustrates how such a situation can yield a balanced transportation problem.

	الأسواق					
الكميات الموردة	M_4	M_3	M_2	M_1		
a_1	x_{14} c_{14}	x_{13} c_{13}	x_{12} c_{12}	x_{11} c_{11}	W_1	
a_2	x_{24} c_{24}	x_{23} c_{23}	x_{22} c_{22}	x_{21} c_{21}	W_2	المخازن
a_3	x_{34} c_{34}	x_{33} c_{33}	x_{32} c_{32}	x_{31} c_{31}	W_3	
	b_4	b_3	b_2	b_1	الطلبات	

طريقة u , v :

أولاً : في حالة المطلوب تصغير قيمة التابع المستهدف (الكلفة) , أي في الحالة Min:

تُحسب لكل حل نافذ أساسي الأعداد u_i للمخزن i، وv_j للسوق j بحيث:

$u_i + v_j = c_{ij}$ من أجل كل متحول أساسي x_{ij} (1.3)

ويمكن أن تكون هذه الأرقام موجبة أو سالبة أو معدومة، نكتب أيضاً:

$e_{ij} = (u_i + v_j) - c_{ij}$ من أجل كل متحول غير أساسي x_{ij} (2.3)

إذا كانت القيم e_{ij} غير موجبة فإن الحل النافذ الأساسي الحالي أمثل. وإلا، فهناك متحول غير أساسي x_{pq} يحقق ما يلي: $e_{pq} = \min(e_{ij}) > 0$: فنجعل ذاك المتحول x_{pq} متحولاً أساسياً لتحسين قيمة التابع المستهدف.

ثانيًا : في حالة المطلوب تعظيم قيمة التابع المستهدف (العائد) , أي في الحالة Max:

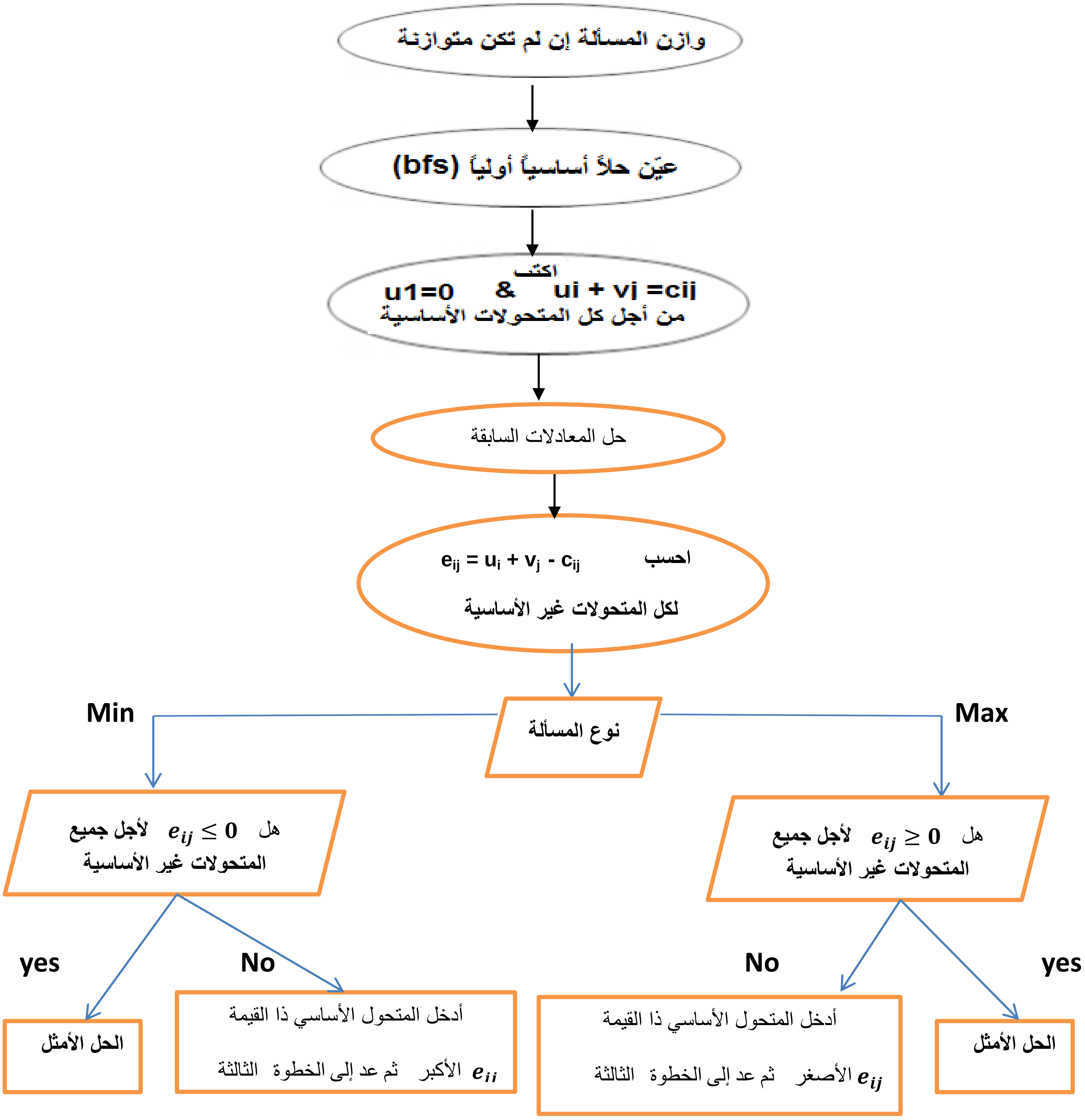
لما كان الهدف هو جعل الربح الكلي أعظمياً، فينبغي إذاً تعديل قاعدة اختيار المتحولات في خوارزميات النقل. يُسمح للمتحولات غير الأساسية بدخول المجموعة الأساسية بغية تحسين تابع الربح، إن تحققت العلاقة: $c_{ij} > u_i + v_j$ (أي: $e_{ij} < 0$)، وعندما تكون كل قيم الأمثال: $e_{ij} \geq 0$ يصبح الحل أمثلاً.

التردي Degeneracy

نقول عن حل نافذ أساسي لمسألة نقل إنه مترد إذا انعدم فيه قيمة متحول أساسي أو أكثر. ويمكن تردي الحل الابتدائي في كل مرة يُختار فيها متحول على أنه متحول أساسي وتتساوى من أجله قيمتا الطلب والتوريد المتبقيتين.

خوارزمية إيجاد الحل الأمثل

لمسائل النقل (Transportation Problems)



GOOD LUCK

Prof. Dr. Anwar Al-Lahham

Operations Research
Homework

Q1) **Consider the following linear program:**

$$
\begin{aligned}
&Max \quad z = 60x_1 + 30x_2 + 20x_3 \\
&s.t. \qquad 8x_1 + 6x_2 + 1x_3 \le 48 \\
&\qquad\qquad 8x_1 + 4x_2 + 3x_3 \le 40 \\
&\qquad\qquad 4x_1 + 3x_2 + 1x_3 \le 16 \\
&\qquad\qquad\qquad x_2 \le 10 \\
&\qquad\qquad x_1 , x_2 , x_3 \ge 0
\end{aligned}
$$

a • **Find the optimal solution** (x_1, x_2, x_3 *and* z).

b • **Find the Dual Prices.**

c • **Find the Optimal Ranges of C.**

d • **Find the Dual Problem.**

i) **Find the optimal solution.**
ii) **Find the Dual Prices.**
iii) **Find the Optimal Ranges of C.**

Q2) *consider the following two* ***transportation problems:***

1)

CUSTOMERS

PLANTS

	C1	*C2*	*C3*	*C3*	***Supply***
P1	*3*	*7*	*9*	*4*	***25***
P2	*9*	*4*	*7*	*5*	***45***
P3	*11*	*3*	*6*	*10*	***160***
Demand	***37***	***52***	***84***	***57***	

2)

CUSTOMERS

PLANTS

	C1	*C2*	*C3*	*C3*	***Supply***
P1	*3*	*7*	*9*	*4*	***70***
P2	*9*	*4*	*7*	*5*	***45***
P3	*11*	*3*	*6*	*10*	***115***
Demand	***37***	***52***	***84***	***57***	

i) Find the optimum solution (The Maximal Revenue) for each one.
ii) Find the optimum solution (The Minimal Cost) for each one.

GOOD LUCK *Prof. Dr. Anwar Al-Lahham*

Assignment Problems

Although the transportation simplex appears to be very efficient, there is a certain class of transportation problems, called assignment problems, for which the transportation simplex is often very inefficient. In this section, we define assignment problems and discuss an efficient method that can be used to solve them.

EXAMPLE 1 **Maximizing Wozac Yield**

Machineco has four machines and four jobs to be completed. Each machine must be assigned to complete one job. The time required to set up each machine for completing each job is shown in Table 43. Machineco wants to minimize the total setup time needed to complete the four jobs. Use linear programming to solve this problem.

Solution Machineco must determine which machine should be assigned to each job. We define (for $i, j = 1, 2, 3, 4$)

$$x_{ij} = 1 \text{ if machine } i \text{ is assigned to meet the demands of job } j$$
$$x_{ij} = 0 \text{ if machine } i \text{ is not assigned to meet the demands of job } j$$

Then Machineco's problem may be formulated as

$$\begin{aligned}
\min z = {} & 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\
& + 7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} = 1 \qquad \text{(Machine constraints)} \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 1 \\
& x_{41} + x_{42} + x_{43} + x_{44} = 1 \\
& x_{11} + x_{21} + x_{31} + x_{41} = 1 \qquad \text{(Job constraints)} \\
& x_{12} + x_{22} + x_{32} + x_{42} = 1 \\
& x_{13} + x_{23} + x_{33} + x_{43} = 1 \\
& x_{14} + x_{24} + x_{34} + x_{44} = 1 \\
& x_{ij} = 0 \quad \text{or} \quad x_{ij} = 1
\end{aligned} \tag{13}$$

The first four constraints in (13) ensure that each machine is assigned to a job, and the last four ensure that each job is completed. If $x_{ij} = 1$, then the objective function will pick up the time required to set up machine i for job j; if $x_{ij} = 0$, then the objective function will not pick up the time required.

Ignoring for the moment the $x_{ij} = 0$ or $x_{ij} = 1$ restrictions, we see that Machineco faces a balanced transportation problem in which each supply point has a supply of 1 and each

TABLE 43
Setup Times for Machineco

	Time (Hours)			
Machine	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

demand point has a demand of 1. In general, an **assignment problem** is a balanced transportation problem in which all supplies and demands are equal to 1. Thus, an assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point. The assignment problem's matrix of costs is its **cost matrix.**

All the supplies and demands for the Machineco problem (and for any assignment problem) are integers, so our discussion in Section 7.3 implies that all variables in Machineco's optimal solution must be integers. Because the right-hand side of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1. This means that we can ignore the restrictions that $x_{ij} = 0$ or 1 and solve (13) as a balanced transportation problem. By the minimum cost method, we obtain the bfs in Table 44. The current bfs is highly degenerate. (In any bfs to an $m \times m$ assignment problem, there will always be m basic variables that equal 1 and $m - 1$ basic variables that equal 0.)

We find that $\bar{c}_{43} = 1$ is the only positive $\bar{c}_{ij}$. We therefore enter x_{43} into the basis. The loop involving x_{43} and some of the basic variables is (4, 3)–(1, 3)–(1, 2)–(4, 2). The odd variables in the loop are x_{13} and x_{42}. Because $x_{13} = x_{42} = 0$, either x_{13} or x_{42} will leave

TABLE 44
Basic Feasible Solution for Machineco

		Job 1	Job 2	Job 3	Job 4	
	$v_j =$	3	4	8	7	
Machine 1	$u_i = 0$	$\boxed{14}$	1 $\boxed{5}$	0 $\boxed{8}$	0 $\boxed{7}$	1
Machine 2	−2	$\boxed{2}$	$\boxed{12}$	$\boxed{6}$	1 $\boxed{5}$	1
Machine 3	−5	$\boxed{7}$	$\boxed{8}$	1 $\boxed{3}$	$\boxed{9}$	1
Machine 4	−1	1 $\boxed{2}$	0 $\boxed{4}$	$\boxed{6}$	$\boxed{10}$	1
		1	1	1	1	

TABLE 45
x_{43} Has Entered the Basis

		Job 1	Job 2	Job 3	Job 4	
	$v_j =$	3	5	7	7	
Machine 1	$u_i = 0$	$\boxed{14}$	1 $\boxed{5}$	$\boxed{8}$	0 $\boxed{7}$	1
Machine 2	−2	$\boxed{2}$	$\boxed{12}$	$\boxed{6}$	1 $\boxed{5}$	1
Machine 3	−4	$\boxed{7}$	$\boxed{8}$	1 $\boxed{3}$	$\boxed{9}$	1
Machine 4	−1	1 $\boxed{2}$	0 $\boxed{4}$	0 $\boxed{6}$	$\boxed{10}$	1
		1	1	1	1	

the basis. We arbitrarily choose x_{13} to leave the basis. After performing the pivot, we obtain the bfs in Table 45. All $\bar{c}_{ij}$'s are now nonpositive, so we have obtained an optimal assignment: $x_{12} = 1$, $x_{24} = 1$, $x_{33} = 1$, and $x_{41} = 1$. Thus, machine 1 is assigned to job 2, machine 2 is assigned to job 4, machine 3 is assigned to job 3, and machine 4 is assigned to job 1. A total setup time of $5 + 5 + 3 + 2 = 15$ hours is required.

The Hungarian Method

Looking back at our initial bfs, we see that it was an optimal solution. We did not know that it was optimal, however, until performing one iteration of the transportation simplex. This suggests that the high degree of degeneracy in an assignment problem may cause the transportation simplex to be an inefficient way of solving assignment problems. For this reason (and the fact that the algorithm is even simpler than the transportation simplex), the Hungarian method is usually used to solve assignment (min) problems:

Step 1 Find the minimum element in each row of the $m \times m$ cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost the minimum cost in its column.

Step 2 Draw the minimum number of lines (horizontal, vertical, or both) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

Step 3 Find the smallest nonzero element (call its value k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to step 2.

REMARKS **1** To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by -1 and solve the problem as a minimization problem.
2 If the number of rows and columns in the cost matrix are unequal, then the assignment problem is unbalanced. The Hungarian method may yield an incorrect solution if the problem is unbalanced. Thus, any assignment problem should be balanced (by the addition of one or more dummy points) before it is solved by the Hungarian method.
3 In a large problem, it may not be easy to find the minimum number of lines needed to cover all zeros in the current cost matrix. For a discussion of how to find the minimum number of lines needed, see Gillett (1976). It can be shown that if j lines are required, then only j "jobs" can be assigned to zero costs in the current matrix. This explains why the algorithm terminates when m lines are required.

Solution of Machineco Example by the Hungarian Method

We illustrate the Hungarian method by solving the Machineco problem (see Table 46).

Step 1 For each row, we subtract the row minimum from each element in the row, obtaining Table 47. We now subtract 2 from each cost in column 4, obtaining Table 48.

Step 2 As shown, lines through row 1, row 3, and column 1 cover all the zeros in the reduced cost matrix. From remark 3, it follows that only three jobs can be assigned to zero costs in the current cost matrix. Fewer than four lines are required to cover all the zeros, so we proceed to step 3.

TABLE 46
Cost Matrix for Machineco

				Row Minimum
14	5	8	7	5
2	12	6	5	2
7	8	3	9	3
2	4	6	10	2

TABLE 47
Cost Matrix After Row Minimums Are Subtracted

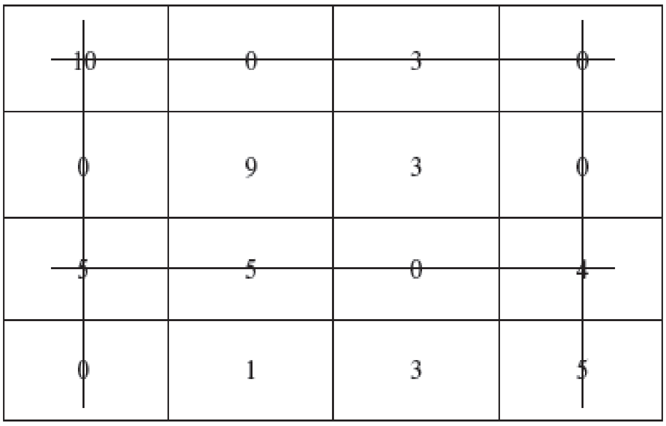
9	0	3	2
0	10	4	3
4	5	0	6
0	2	4	8
Column Minimum	0	0	2

TABLE 48
Cost Matrix After Column Minimums Are Subtracted

9	0	3	0
0	10	4	1
4	5	0	4
0	2	4	6

Step 3 The smallest uncovered element equals 1, so we now subtract 1 from each uncovered element in the reduced cost matrix and add 1 to each twice-covered element. The resulting matrix is Table 49. Four lines are now required to cover all the zeros. Thus, an optimal solution is available. To find an optimal assignment, observe that the only covered 0 in column 3 is x_{33}, so we must have $x_{33} = 1$. Also, the only available covered zero in column 2 is x_{12}, so we set $x_{12} = 1$ and observe that neither row 1 nor column 2 can be used again. Now the only available covered zero in column 4 is x_{24}. Thus, we choose $x_{24} = 1$ (which now excludes both row 2 and column 4 from further use). Finally, we choose $x_{41} = 1$.

TABLE 49
Four Lines Required; Optimal Solution Is Available



10	0	3	0
0	9	3	0
5	5	0	4
0	1	3	5

Thus, we have found the optimal assignment $x_{12} = 1$, $x_{24} = 1$, $x_{33} = 1$, and $x_{41} = 1$. Of course, this agrees with the result obtained by the transportation simplex.

Intuitive Justification of the Hungarian Method

To give an intuitive explanation of why the Hungarian algorithm works, we need to discuss the following result: *If a constant is added to each cost in a row (or column) of a balanced transportation problem, then the optimal solution to the problem is unchanged.* To show why the result is true, suppose we add k to each cost in the first row of the Machineco problem. Then

$$\text{New objective function} = \text{old objective function} + k(x_{11} + x_{12} + x_{13} + x_{14})$$

Because any feasible solution to the Machineco problem must have $x_{11} + x_{12} + x_{13} + x_{14} = 1$,

$$\text{New objective function} = \text{old objective function} + k$$

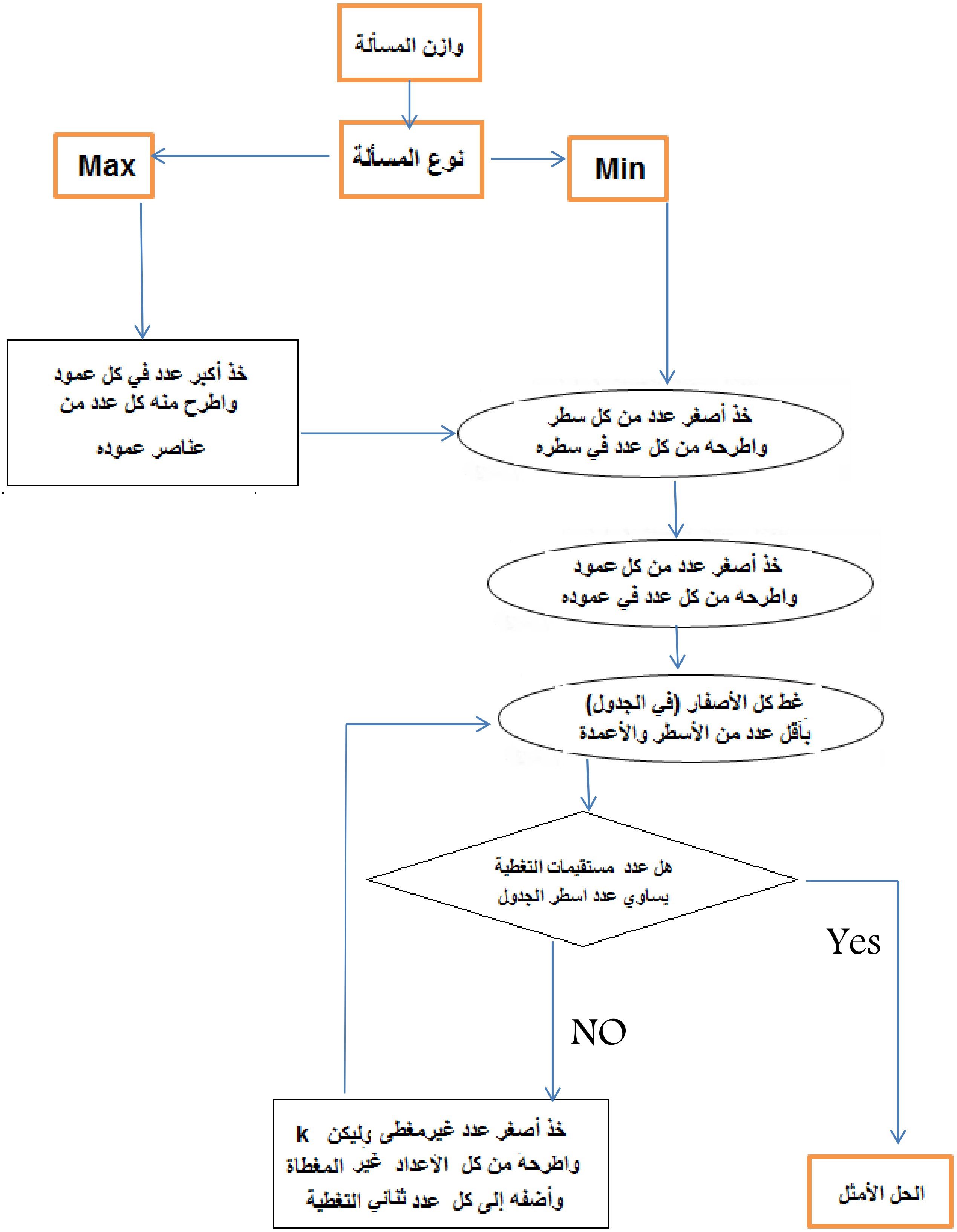
Thus, the optimal solution to the Machineco problem remains unchanged if a constant k is added to each cost in the first row. A similar argument applies to any other row or column.

Step 1 of the Hungarian method consists (for each row and column) of subtracting a constant from each element in the row or column. Thus, step 1 creates a new cost matrix having the same optimal solution as the original problem. Step 3 of the Hungarian method is equivalent (see Problem 7 at the end of this section) to adding k to each cost that lies in a covered row and subtracting k from each cost that lies in an uncovered column (or vice versa). Thus, step 3 creates a new cost matrix with the same optimal solution as the initial assignment problem. Each time step 3 is performed, at least one new zero is created in the cost matrix.

Steps 1 and 3 also ensure that all costs remain nonnegative. Thus, the net effect of steps 1 and 3 of the Hungarian method is to create a sequence of assignment problems (with nonnegative costs) that all have the same optimal solution as the original assignment problem. Now consider an assignment problem in which all costs are nonnegative. Any feasible assignment in which all the x_{ij}'s that equal 1 have zero costs must be optimal for such an assignment problem. Thus, when step 2 indicates that m lines are required to cover all the zeros in the cost matrix, an optimal solution to the original problem has been found.

خوارزمية إيجاد الحل الأمثل

باستخدام الطريقة الهنغارية



بسم الله الرحمن الرحيم

The Hungarian Method: (MIN Problem)

Step 1. For the original cost matrix, identify each row's minimum, and subtract it from all the entries of the row.

Step 2. For the matrix resulting from step 1, identify each column's minimum and subtract it from all the entries of the column.

Step 3. Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

Step 2a. If no feasible assignment (with all zero entries) can be secured from steps 1 and 2,
(i) Draw the *minimum* number of horizontal and vertical lines in the last reduced matrix that will cover *all* the zero entries.

Example(1): (MIN Problem)

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline A & 15 & 10 & 9 \\ M & 9 & 15 & 10 \\ K & 10 & 12 & 8 \\ \hline \end{array} \qquad \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline A & 6 & 1 & 0 \\ M & 0 & 6 & 1 \\ K & 2 & 4 & 0 \\ \hline \end{array} \qquad \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline A & 6 & \underline{0} & 0 \\ M & \underline{0} & 5 & 1 \\ K & 2 & 3 & \underline{0} \\ \hline \end{array}$$

(1) (2) (3)

The Optimal Solution is: A → 2 M → 1 K → 3

The Total Cost is : 10 + 9 + 8 =27

Example(1): (MIN Problem)

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline A & 1 & 4 & 3 & 3 \\ B & 9 & 7 & 7 & 9 \\ C & 4 & 5 & 8 & 7 \\ D & 8 & 7 & 5 & 5 \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline A & 0 & 3 & 2 & 2 \\ B & 2 & 0 & 0 & 2 \\ C & 0 & 1 & 3 & 2 \\ D & 3 & 2 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline A & 0 & 3 & 2 & 2 \\ B & 2 & 0 & 0 & 2 \\ C & 0 & 1 & 3 & 2 \\ D & 3 & 2 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline A & \underline{0} & 2 & 1 & 1 \\ B & 3 & 0 & \underline{0} & 2 \\ C & 0 & \underline{0} & 2 & 1 \\ D & 4 & 2 & 0 & \underline{0} \\ \hline \end{array}$$

(1) (2) (2) (3)

The Optimal Solution is: A → 1 B → 3 C → 2 D → 4

The Total Cost is : 1 + 7 + 5 + 5 = 18

The Hungarian Method: (MAX Problem)

Step (1): Identify each Column's <u>maximum</u> , and <u>subtract all the entries of the column from it</u>.

The problem becomes **(MIN Problem).** Go on as steps before . **(1)**

بسم الله الرحمن الرحيم

Assignment Problems

<table>
<tr><th>The Assignment Problem</th><th>Max</th><th>Min</th></tr>
<tr><td><table><tr><td>15</td><td>10</td><td>9</td></tr><tr><td>9</td><td>15</td><td>10</td></tr><tr><td>10</td><td>12</td><td>8</td></tr></table></td><td>1 → 1 15
2 → 2 15
3 → 3 8
38</td><td>2 → 1 9
1 → 2 10
3 → 3 8
27</td></tr>
<tr><td><table><tr><td>10</td><td>6</td><td>12</td><td>8</td></tr><tr><td>15</td><td>18</td><td>5</td><td>11</td></tr><tr><td>17</td><td>10</td><td>13</td><td>16</td></tr><tr><td>14</td><td>12</td><td>13</td><td>10</td></tr></table></td><td>4 → 1 14
2 → 2 18
1 → 3 12
3→ 4 16
60</td><td>1 → 1 10
2 → 3 5
3 → 2 10
4→ 4 10
35</td></tr>
<tr><td><table><tr><td>1</td><td>4</td><td>6</td><td>3</td></tr><tr><td>9</td><td>7</td><td>10</td><td>9</td></tr><tr><td>4</td><td>5</td><td>11</td><td>7</td></tr><tr><td>8</td><td>7</td><td>8</td><td>5</td></tr></table></td><td>4 → 1 8
1 → 2 4
3 → 3 11
2 → 4 9
32</td><td>1 → 1 1
2 → 3 10
3 → 2 5
4→ 4 5
21</td></tr>
<tr><td><table><tr><td>3</td><td>8</td><td>2</td><td>10</td><td>3</td></tr><tr><td>8</td><td>7</td><td>2</td><td>9</td><td>7</td></tr><tr><td>6</td><td>4</td><td>2</td><td>7</td><td>5</td></tr><tr><td>8</td><td>4</td><td>2</td><td>3</td><td>5</td></tr><tr><td>9</td><td>10</td><td>6</td><td>9</td><td>10</td></tr></table></td><td>4 → 1 8
2 → 2 7
3 → 3 2
1 → 4 10
5→5 10
37</td><td>5 → 1 9
3 → 2 4
2 → 3 2
4→ 4 3
1→ 5 3
21</td></tr>
<tr><td><table><tr><td>3</td><td>9</td><td>2</td><td>3</td><td>7</td></tr><tr><td>6</td><td>1</td><td>5</td><td>6</td><td>6</td></tr><tr><td>9</td><td>4</td><td>7</td><td>10</td><td>3</td></tr><tr><td>2</td><td>5</td><td>4</td><td>2</td><td>1</td></tr><tr><td>9</td><td>6</td><td>2</td><td>4</td><td>5</td></tr></table></td><td>5 → 1 9
1 → 2 9
4 → 3 4
3 → 4 10
2 → 5 6
38</td><td>4 → 1 2
2 → 2 1
5 → 3 2
1→ 4 3
3 → 5 3
11</td></tr>
<tr><td><table><tr><td>9</td><td>10</td><td>11</td><td>10</td></tr><tr><td>11</td><td>7</td><td>17</td><td>11</td></tr><tr><td>5</td><td>13</td><td>12</td><td>13</td></tr></table></td><td>1 → 2 10
2 → 3 17
3 → 4 13
40</td><td>3 → 1 5
2 → 2 7
1 → 4 10
22</td></tr>
<tr><td><table><tr><td>14</td><td>8</td><td>14</td><td>15</td><td>13</td></tr><tr><td>5</td><td>22</td><td>11</td><td>9</td><td>10</td></tr><tr><td>11</td><td>15</td><td>15</td><td>10</td><td>11</td></tr><tr><td>16</td><td>11</td><td>12</td><td>8</td><td>15</td></tr><tr><td>12</td><td>7</td><td>14</td><td>12</td><td>12</td></tr></table></td><td>4 → 1 16
2 → 2 22
3 → 3 15
1 → 4 15
5 → 5 12
80</td><td>2 → 1 5
5 → 2 7
1 → 3 14
4→ 4 8
3 → 5 11
45</td></tr>
<tr><td><table><tr><td>5</td><td>4</td><td>7</td><td>9</td><td>7</td></tr><tr><td>6</td><td>5</td><td>5</td><td>4</td><td>7</td></tr><tr><td>5</td><td>7</td><td>9</td><td>8</td><td>8</td></tr><tr><td>4</td><td>6</td><td>8</td><td>7</td><td>9</td></tr><tr><td>7</td><td>9</td><td>8</td><td>9</td><td>8</td></tr></table></td><td>2 → 1 6
5 → 2 9
3 → 3 9
1 → 4 9
4 → 5 9
42</td><td>4 → 1 4
1 → 2 4
5 → 3 8
2 → 4 4
3 → 5 8
28</td></tr>
<tr><td><table><tr><td>5</td><td>9</td><td>4</td><td>3</td><td>5</td></tr><tr><td>7</td><td>8</td><td>7</td><td>8</td><td>9</td></tr><tr><td>4</td><td>11</td><td>8</td><td>10</td><td>8</td></tr><tr><td>10</td><td>9</td><td>4</td><td>12</td><td>5</td></tr></table></td><td>4 → 4 12
1 → 2 9
3 → 3 8
2 → 5 9
38</td><td>3 → 1 4
2 → 2 8
4 → 3 4
1→ 4 3
19</td></tr>
</table>

Solve The Following Assignment Problems:

	Job		
Engineer	1	2	3
Ahmad	10	16	32
Walid	14	22	40
John	22	24	34

Min		Max	
1 → 2	16	1 → 2	16
2 → 1	14	2 → 3	40
3 → 3	34	3 → 1	22
	64		78

	Job			
Engineer	1	2	3	4
1	7	5	8	2
2	7	8	9	4
3	3	5	7	9
4	5	5	6	7

Min		Max	
3 → 1	3	4 → 1	5
1 → 2	5	2 → 2	8
4 → 3	6	1 → 3	8
2 → 4	4	3 → 4	9
	18		30

	Job		
Engineer	1	2	3
Ahmad	10	16	32
Walid	14	22	40
John	22	24	34
Samer	14	18	36

Min		Max	
1 → 1	10	3 → 1	22
3 → 3	34	4 → 2	18
4 → 2	18	2 → 3	40
	62		80
Walid unassigned		Ahmad unassigned	

	Time(Seconds)			
Swimmer	Freestyle	Breaststroke	Backstroke	Butterfly
1	54	54	51	53
2	51	57	52	52
3	50	53	54	56
4	56	54	55	53

Min		Max	
3 → 1	50	1 → 1	54
4 → 2	54	2 → 2	57
1 → 3	51	4 → 3	55
2 → 4	52	3 → 4	56
	207		222

	Job			
Engineer	1	2	3	4
1	22	18	30	18
2	18	20	27	22
3	26	22	24	28
4	21	27	25	28

Min		Max	
4 → 1	21	3 → 1	26
2 → 2	20	4 → 2	27
3 → 3	24	1 → 3	30
1 → 4	18	2 → 4	22
	83		105

	Job			
Engineer	1	2	3	4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	9	10

Min		Max	
1 → 2	2	1 → 1	14
2 → 4	5	2 → 2	12
3 → 3	3	4 → 3	6
4 → 1	5	3 → 4	9
	15		41

	Job		
Engineer	1	2	3
1	65	63	62
2	68	67	65
3	63	60	59
4	67	62	60

Min		Max	
1 → 1	65	1 → 3	62
3 → 2	60	2 → 2	67
4 → 3	60	4 → 1	67
	185		196
2 unassigned		Ahmad unassigned	

بسم الله الرحمن الرحيم

PROBLEMS

TABLE 51

	Time (seconds)			
Swimmer	Free	Breast	Fly	Back
Gary Hall	54	54	51	53
Mark Spitz	51	57	52	52
Jim Montgomery	50	53	54	56
Chet Jastremski	56	54	55	53

TABLE 52

	JA	CC	GP	JR
TC	7	5	8	2
FP	7	8	9	4
HF	3	5	7	9
ML	5	5	6	7

TABLE 55

	Ally	Georgia	Jane	Rene	Nell
Billie	8	6	4	7	5
John	5	7	6	4	9
Fish	10	6	5	2	10
Glen	1	7	5	8	11
Larry	5	7	9	8	6

TABLE 69

	Distance (Blocks)		
Car	Call 1	Call 2	Call 3
1	10	11	18
2	6	7	7
3	7	8	5

TABLE 65

	Time (Hours)			
Worker	Job 1	Job 2	Job 3	Job 4
1	10	15	10	15
2	12	8	20	16
3	12	9	12	18
4	6	12	15	18
5	16	12	8	12

TABLE 67

	Time (Hours)			
Maid	Vacuum	Clean Kitchen	Clean Bathroom	Straighten Up
1	6	5	2	1
2	9	8	7	3
3	8	5	9	4
4	7	7	8	3
5	5	5	6	4

TABLE 68

	Time (Minutes)		
Storage Medium	Word Processing	Packaged Program	Data
Hard disk	5	4	4
Memory	2	1	1
Tape	10	8	6

TABLE 89

Company	Fire 1	Fire 2	Fire 3
1	6	7	9
2	5	8	11
3	6	9	10

GOOD LUCK **PROF. DR. ANWAR AL-LAHHAM**

SUMMARY

Notation

m = number of supply points

n = number of demand points

x_{ij} = number of units shipped from supply point i to demand point j

c_{ij} = cost of shipping 1 unit from supply point i to demand point j

s_i = supply at supply point i

d_j = demand at demand point j

$\bar{c}_{ij}$ = coefficient of x_{ij} in row 0 of a given tableau

$\mathbf{a}_{ij}$ = column for x_{ij} in transportation constraints

A transportation problem is **balanced** if total supply equals total demand. To use the methods of this chapter to solve a transportation problem, the problem must first be balanced by use of a dummy supply or a dummy demand point. A balanced transportation problem may be written as

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

Finding Basic Feasible Solutions for Balanced Transportation Problems

We can find a bfs for a balanced transportation problem by the northwest corner method, the minimum-cost method, or Vogel's method. To find a bfs by the northwest corner method, begin in the upper left-hand (or northwest) corner of the transportation tableau and set x_{11} as large as possible. Clearly, x_{11} can be no larger than the smaller of s_1 and d_1. If $x_{11} = s_1$, then cross out the first row of the transportation tableau; this indicates that no more basic variables will come from row 1 of the tableau. Also change d_1 to $d_1 - s_1$. If $x_{11} = d_1$, then cross out the first column of the transportation tableau and change s_1 to $s_1 - d_1$. If $x_{11} = s_1 = d_1$, cross out either row 1 or column 1 (but not both) of the transportation tableau. If you cross out row 1, change d_1 to 0; if you cross out column 1, change s_1 to 0. Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed-out row or column. Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row and its column. A basic feasible solution has now been obtained.

Finding the Optimal Solution for a Transportation Problem

Step 1 If the problem is unbalanced, balance it.

Step 2 Use one of the methods described in Section 7.2 to find a bfs.

Step 3 Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1 \quad u_2 \ldots u_m \quad v_1 \quad v_2 \ldots v_n]$ for the current bfs.

Step 4 If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, then we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis. To do this, find the loop. Then, *counting only cells in the loop,* label the even cells. Also label the odd cells. Now find the odd cell whose variable assumes the smallest value, θ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by θ and increase the value of each even cell by θ. The values of variables not in the loop remain unchanged. The pivot is now complete. If $\theta = 0$, then the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case, a degenerate bfs will result. If more than one odd cell in the loop equals θ, you may arbitrarily choose one of these odd cells to leave the basis; again, a degenerate bfs will result. The pivoting yields a new bfs.

Step 5 Using the new bfs, return to steps 3 and 4.

For a maximization problem, proceed as stated, but replace step 4 by step 4′.

Step 4′ If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure.

Assignment Problems

An **assignment problem** is a balanced transportation problem in which all supplies and demands equal 1. An $m \times m$ assignment problem may be efficiently solved by the Hungarian method:

Step 1 Find the minimum element in each row of the cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column.

Step 2 Cover all the zeros in the reduced cost matrix using the minimum number of lines needed. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

Step 3 Find the smallest nonzero element (k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element and add k to each element that is covered by two lines. Return to step 2.

GOOD LUCK

Review Problems 1

Find : (i) The Optimal Solutions of the following Linear Programing Problems

(ii) The Dual Problems , and their Optimal Solutions.

$Min \quad z = 3x_1 + 1x_2$ $s.t. \quad 3x_1 + 2x_2 \ge 15$ $1x_1 + 3x_2 \le 12$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 3, \quad x_2 = 3 , \quad Z = 12$ The optimal Ranges are : $1.5 \le C_1 , \quad C_2 \le 2$	$Min \quad z = -11x_1 + 1x_2$ $s.t. \quad -3x_1 + 2x_2 \ge 3$ $5x_1 + 3x_2 \le 5$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 0.053 , \quad x_2 = 1.579 , \quad Z = 1$ The optimal Ranges are : $C_1 \le -1.5 , \quad -6.6 \le C_2 \le 22/3$	$Min \quad z = 3x_1 + 7x_2$ $s.t. \quad 2x_1 + 3x_2 \ge 13$ $5x_1 - 2x_2 \le 23$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 5 , \quad x_2 = 1 , \quad Z = 22$ The optimal Ranges are : $-17.5 \le C_1 \le 10/3, \quad 4.5 \le C_2$
$Max \quad z = 14x_1 - 9x_2$ $s.t. \quad 3x_1 - 2x_2 \le 1$ $2x_1 - 1x_2 \le 2$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 1 , \quad x_2 = 1 , \quad Z = 5$ The optimal Ranges are : $13.5 \le C_1 \le 18 , \quad -28/3 \le C_2 \le -7$	$Max \quad z = 3x_1 - 5x_2$ $s.t. \quad 3x_1 + 5x_2 \le 11$ $2x_1 - 3x_2 \le 1$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 0.5 , \quad x_2 = 0 , \quad Z = 1.5$ The optimal Ranges are : $0 \le C_1 \le 10/3 , \quad C_2 \le -4.5$	$Max \quad z = 3x_1 - 5x_2$ $s.t. \quad 3x_1 + 5x_2 \ge 11$ $2x_1 - 3x_2 \le 1$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 2 , \quad x_2 = 1 , \quad Z = 1$ The optimal Ranges are : $-3 \le C_1 \le 10/3 , \quad C_2 \le 4.5$
$Min \quad z = 3x_1 + 2x_2$ $s.t. \quad x_1 + x_2 \le 8$ $3x_1 + x_2 \ge 12$ $x_1 + 3x_2 \ge 12$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 3 , \quad x_2 = 3 , \quad Z = 15$ The optimal Ranges are : $2/3 \le C_1 \le 6 , \quad 1 \le C_2 \le 9$	$Min \quad z = 1x_1 + 2x_2$ $s.t. \quad -3x_1 + 2x_2 \ge 3$ $5x_1 + 3x_2 \le 20$ $2x_1 + 3x_2 \ge 5$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 1/13 , \quad x_2 = 21/13 , \quad Z = 43/13$ The optimal Ranges are : $-3 \le C_1 \le 4/3 , \quad C_2 \ge 3/2$	$Min \quad z = 10x_1 + 9x_2$ $s.t. \quad 3x_1 + x_2 \le 20$ $2x_1 + 5x_2 \ge 35$ $4x_1 + 3x_2 \le 75$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 0 , \quad x_2 = 7 , \quad Z = 63$ The optimal Ranges are : $C_1 \ge 3.6 , \quad C_2 \le 25$
$Min \quad z = 2x_1 + 5x_2$ $s.t. \quad 2x_1 + 1x_2 \le 15$ $3x_1 - 1x_2 \le 5$ $5x_1 + 2x_2 \ge 30$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 3.636 , \quad x_2 = 5.909 , \quad Z = 36.818$ The optimal Ranges are : $-15 \le C_1 \le 12.5 , \quad 0.8 \le C_2$	$Max \quad z = 5x_1 - 3x_2$ $s.t. \quad 3x_1 + x_2 \le 20$ $2x_1 + 5x_2 \ge 35$ $4x_1 + 3x_2 \le 75$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 5 , \quad x_2 = 5 , \quad Z = 10$ The optimal Ranges are : $-1.2 \le C_1 , \quad C_2 \le 5/3$	$Max \quad z = 5x_1 + 3x_2$ $s.t. \quad x_1 + x_2 \ge 1$ $3x_1 + 7x_2 \le 10$ $7x_1 + 3x_2 \le 10$ $x_1 , x_2 \ge 0$ The optimal solution is $x_1 = 1 , \quad x_2 = 1 , \quad Z = 8$ The optimal Ranges are : $1.286 \le C_1 \le 7 , \quad 2.143 \le C_2 \le 35/3$

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

$$\begin{aligned} Max \quad & z = 15x_1 - 12x_2 \\ s.t. \quad & 3x_1 - 2x_2 \le 3 \\ & 2x_1 - 3x_2 \le 1 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 1.4,\quad x_2 = 0.6,\quad Z = 13.8$

The optimal Ranges are :

$8 \le C_1 \le 18,\quad -22.5 \le C_2 \le -10$

$$\begin{aligned} Max \quad & z = 3x_1 - 5x_2 \\ s.t. \quad & 3x_1 + 5x_2 \ge 11 \\ & 2x_1 - 3x_2 \le 1 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 2,\quad x_2 = 1,\quad Z = 1$

The optimal Ranges are :

$-3 \le C_1 \le 10/3,\quad C_2 \le -4.5$

$$\begin{aligned} Max \quad & z = 27x_1 - 45x_2 \\ s.t. \quad & -2x_1 + 5x_2 \ge 2 \\ & 3x_1 - 2x_2 \le 7 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 6,\quad x_2 = 5,\quad Z = 23$

The optimal Ranges are :

$C_1 \le 14/3,\quad C_2 \ge -1.2$

$$\begin{aligned} Max \quad & z = 1x_1 + 5x_2 \\ s.t. \quad & 1x_1 + 2x_2 \le 30 \\ & 2x_1 - 1x_2 \ge 20 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 14,\quad x_2 = 8,\quad Z = 54$

The optimal Ranges are :

$-10 \le C_1 \le 2.5,\quad 2 \le C_2$

$$\begin{aligned} Max \quad & z = 3x_1 + 4x_2 \\ s.t. \quad & 3x_1 + 2x_2 \le 20 \\ & 2x_1 + 3x_2 \le 20 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 4,\quad x_2 = 4,\quad Z = 28$

The optimal Ranges are :

$2.667 \le C_1 \le 6,\quad 2 \le C_2 \le 4.5$

$$\begin{aligned} Max \quad & z = 18x_1 + 18x_2 \\ s.t. \quad & 1x_1 + 3x_2 \le 28 \\ & 3x_1 + 1x_2 \le 12 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 1,\quad x_2 = 9,\quad Z = 180$

The optimal Ranges are :

$6 \le C_1 \le 54,\quad 6 \le C_2 \le 54$

$$\begin{aligned} Max \quad & z = 3x_1 + 6x_2 \\ s.t. \quad & 4x_1 + 5x_2 \le 22 \\ & x_1 + x_2 \ge 1 \\ & x_1 - 2x_2 \le 2 \\ & x_1 \le 3 \quad , \quad x_2 \le 2 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is :

$x_1 = 3,\quad x_2 = 2,\quad Z = 21$

The optimal Ranges are :

$0 \le C_1 \le 4.8,\quad 3.75 \le C_2$

$$\begin{aligned} Min \quad & z = 2x_1 + 7x_2 \\ s.t. \quad & 2x_1 + 3x_2 \ge 24 \\ & 5x_1 + 2x_2 \ge 20 \\ & x_1 + x_2 \ge 15 \\ & -2x_1 + 3x_2 \le 6 \quad , \quad x_2 \ge 3 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is :

$x_1 = 12,\ x_2 = 3,\quad Z = 45$

The optimal Ranges are :

$0 \le C_1 \le 7,\quad C_2 \ge 2$

$$\begin{aligned} Min \quad & z = 3x_1 + 5x_2 \\ s.t. \quad & 2x_1 + 3x_2 \ge 8 \\ & 5x_1 + 2x_2 \ge 6 \\ & x_1 + x_2 \ge 5 \\ & -2x_1 + 3x_2 \le 2 \quad , \quad x_2 \ge 1 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is :

$x_1 = 4,\ x_2 = 1,\quad Z = 17$

The optimal Ranges are :

$0 \le C_1 \le 5,\quad C_2 \ge 3$

$$\begin{aligned} Max \quad & z = 3x_1 + 4x_2 \\ s.t. \quad & 2x_1 + 3x_2 \le 1200 \\ & 2x_1 + 1x_2 \le 1000 \\ & 4x_2 \le 800 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 450,\ x_2 = 100,\quad Z = 1750$

The optimal Ranges are :

$8/3 \le C_1 \le 8,\quad 1.5 \le C_2 \le 4.5$

$$\begin{aligned} Min \quad & z = 4x_1 + 1x_2 \\ s.t. \quad & 3x_1 + x_2 = 15 \\ & 4x_1 + 3x_2 \ge 30 \\ & 1x_1 + 2x_2 \le 20 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 2,\quad x_2 = 9,\quad Z = 17$

The optimal Ranges are :

$3 \le C_1,\quad C_2 \le 4/3$

$$\begin{aligned} Max \quad & z = 1x_1 + 2x_2 \\ s.t. \quad & x_1 + 2x_2 \le 10 \\ & 1x_1 + 1x_2 \ge 5 \\ & 1x_1 + 1x_2 \le 3 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

The optimal solution is

$x_1 = 2,\quad x_2 = 1,\quad Z = 4$

The optimal Ranges are :

$C_1 \le 2,\quad 1 \le C_2$

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

Review Problems 2

Find : (i) The Optimal Solutions of the following Linear Programing Problems.

(ii) The Dual Prices & The Optimal Ranges of C_i . (iii) The Dual Problems , and their Optimal Solutions.

$$\begin{aligned} Min \quad Z &= x_1 + 2x_2 + 4x_3 \\ s.t \quad 3x_1 + 2x_2 - 1x_3 &\geq 14 \\ 2x_1 + 1x_2 + 3x_3 &\leq 9 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 4$, $x_2 = 1$, $x_3 = 0$, $Z = 6$

The Dual Prices are; (-3 , 4 , -1)

The optimal Ranges are :

$C_1 \leq 3$, $\frac{2}{3} \leq C_2$, $-15 \leq C_3$

$$\begin{aligned} Max \quad Z &= 14x_1 - 9x_2 \\ s.t \quad 3x_1 - 2x_2 &\leq 1 \\ 2x_1 - 1x_2 &\leq 2 \\ 1x_1 + 3x_2 &\geq 12 \\ x_1 , x_2 &\geq 0 \end{aligned}$$

The optimal solution is

$x_1 = 3$, $x_2 = 4$, $Z = 6$

The Dual Prices are; (4 , 1 , 0)

The optimal Ranges are :

$13.5 \leq C_1 \leq 18$, $-\frac{28}{3} \leq C_2 \leq -7$

$$\begin{aligned} Min \quad Z &= 5x_1 - 1x_2 + 2x_3 \\ s.t \quad 3x_1 + 1x_2 + 2x_3 &\leq 9 \\ 1x_1 + 2x_2 + 3x_3 &\geq 10 \\ 2x_1 + 3x_2 + 1x_3 &= 8 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 0$, $x_2 = 2$, $x_3 = 2$, $Z = 2$

The Dual Prices are; (0 , -1 , 1)

The optimal Ranges are :

$-1 \leq C_1$, $C_2 \leq 6$, $-1/3 \leq C_3$

$$\begin{aligned} Max \quad Z &= 4x_1 - 1x_2 + 2x_3 \\ s.t \quad 1x_1 + 1x_2 + 0x_3 &\leq 15 \\ 0x_1 + 2x_2 + 1x_3 &\geq 18 \\ 1x_1 + 0x_2 + 1x_3 &= 12 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 8$, $x_2 = 7$, $x_3 = 4$, $Z = 33$

The Dual Prices are; (1 , -1 , 3)

The optimal Ranges are :

$2.5 \leq C_1$, $-4 \leq C_2 \leq 2$, $C_3 \leq 3.5$

$$\begin{aligned} Min \quad Z &= 3x_1 - 2x_2 + 5x_3 \\ s.t \quad 3x_1 + 2x_2 - 1x_3 &\leq 15 \\ 2x_1 + 1x_2 + 3x_3 &\geq 18 \\ 1x_1 + 1x_2 \quad\quad &= 10 \\ +2x_2 + 1x_3 &\geq 12 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 0$, $x_2 = 10$, $x_3 = 5$, $Z = 5$

The Dual Prices are; (5 , 0 , -8 , 0)

The optimal Ranges are :

$-7 \leq C_1$, $C_2 \leq 8$, $0 \leq C_3$

$$\begin{aligned} Max \quad Z &= 5x_1 - 2x_2 + 1x_3 \\ s.t \quad 3x_1 + 1x_2 + 2x_3 &\leq 15 \\ 1x_1 + 2x_2 + 1x_3 &\geq 12 \\ 2x_1 + 3x_2 + 1x_3 &= 18 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 2.25$, $x_2 = 3.75$, $x_3 = 2.25$, $Z = 6$

The Dual Prices are; (2 , -5 , 2)

The optimal Ranges are :

$1 \leq C_1$, $C_2 \leq 6$, $-7 \leq C_3 \leq 3.857$

$$\begin{aligned} Max \quad Z &= 2x_1 + 1x_2 - 1x_3 \\ s.t \quad 2x_1 + 1x_2 + 1x_3 &\leq 8 \\ 1x_1 + 2x_2 - 2x_3 &\leq 10 \\ 2x_1 + 1x_2 + 3x_3 &\geq 12 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 3$, $x_2 = 0$, $x_3 = 2$, $Z = 4$

The Dual Prices are; (2 , 0 , -1)

The optimal Ranges are :

$2 \leq C_1$, $C_2 \leq 1$, $C_3 \leq 1$

$$\begin{aligned} Max \quad Z &= 3x_1 + 4x_2 + 2x_3 \\ s.t \quad 1x_1 + 3x_2 + 2x_3 &\leq 42 \\ 3x_1 + 2x_2 + 1x_3 &\leq 36 \\ 2x_1 + 1x_2 + 3x_3 &\geq 48 \\ x_1 , x_2 , x_3 &\geq 0 \end{aligned}$$

The optimal solution is:

$x_1 = 5$, $x_2 = 5$, $x_3 = 11$, $Z = 57$

The Dual Prices are; (5/6 , 5/6 , -1/6)

The optimal Ranges are :

$0.857 \leq C_1 \leq 6$, $3.4 \leq C_2$, $-13 \leq C_3 \leq 2.429$

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

$Max \quad Z = 5x_1 - 2x_2 + 3x_3$
$s.t \quad 3x_1 + 2x_2 + 1x_3 \leq 125$
$4x_1 - 2x_2 + 1x_3 = 45$
$5x_1 + 1x_2 - 1x_3 \geq 75$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 16, \quad x_2 = 24, \quad x_3 = 29, \quad Z = 119$
The Dual Prices are; (0.829 , 1.486 , -0.686)
The optimal Ranges are :
$-24 \leq C_1 \leq 11, \quad -5.222 \leq C_2 \leq 22, \quad 1.286 \leq C_3$

$Max \quad Z = 5x_1 - 2x_2 + 1x_3$
$s.t \quad 3x_1 + 1x_2 + 2x_3 \leq 15$
$1x_1 + 2x_2 + 1x_3 \geq 12$
$2x_1 + 2x_2 + 1x_3 = 18$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 3, \quad x_2 = 6, \quad x_3 = 0, \quad Z = 3$
The Dual Prices are; (3.5 , 0 , - 0.2)
The optimal Ranges are :
$2/3 \leq C_1, \quad C_2 \leq 5, \quad C_3 \leq 4.25$

$Min \quad Z = 2x_1 - 3x_2 + 5x_3$
$s.t \quad 1x_1 + 1x_2 + 2x_3 = 20$
$1x_1 - 3x_2 + 1x_3 \leq 40$
$3x_1 + 1x_2 + 1x_3 \geq 50$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 15, \quad x_2 = 5, \quad x_3 = 0, \quad Z = 15$
The Dual Prices are; (5.5 , 0 , -2.5)
The optimal Ranges are :
$-3 \leq C_1, \quad C_2 \leq 2, \quad -8.5 \leq C_3$

$Max \quad Z = 2x_1 - 3x_2 + 5x_3$
$s.t \quad 1x_1 + 1x_2 + 2x_3 = 20$
$1x_1 - 3x_2 + 1x_3 \leq 40$
$3x_1 + 1x_2 + 1x_3 \geq 50$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 16, \quad x_2 = 0, \quad x_3 = 2, \quad Z = 42$
The Dual Prices are; (2.6 , 0 , -2.75)
The optimal Ranges are :
$-25 \leq C_1 \leq 2.5, \quad C_2 \leq 2.4, \quad 4 \leq C_3$

$Min \quad Z = 3x_1 - 2x_2 + 5x_3$
$s.t \quad 1x_1 + 1x_2 + 2x_3 = 30$
$1x_1 - 3x_2 + 1x_3 \leq 40$
$3x_1 + 1x_2 + 1x_3 \geq 50$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 10, \quad x_2 = 20, \quad x_3 = 0, \quad Z = -10$
The Dual Prices are; (4.5 , 0 , -2.5)
The optimal Ranges are :
$-2 \leq C_1, \quad C_2 \leq 2.6, \quad -6.5 \leq C_3$

$Max \quad Z = 3x_1 - 2x_2 + 5x_3$
$s.t \quad 1x_1 + 1x_2 + 2x_3 = 30$
$1x_1 - 3x_2 + 1x_3 \leq 40$
$3x_1 + 1x_2 + 1x_3 \geq 50$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 30, \quad x_2 = 0, \quad x_3 = 0, \quad Z = 90$
The Dual Prices are; (3 , 0 , 0)
The optimal Ranges are :
$2.5 \leq C_1, \quad C_2 \leq 3, \quad C_3 \leq 6$

$Max \quad Z = 5x_1 + 3x_2 + 2x_3$
$s.t \quad 3x_1 + 1x_2 - 1x_3 \leq 30$
$2x_1 + 4x_2 + 3x_3 \geq 45$
$4x_1 + 3x_2 + 5x_3 = 75$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 3, \quad x_2 = 21, \quad x_3 = 0, \quad Z = 78$
The Dual Prices are; (0.6 , 0 , 0.8)
The optimal Ranges are :
$4 \leq C_1 \leq 5.875, \quad 2.632 \leq C_2 \leq 3.75, \quad C_3 \leq 3.4$

$Max \quad Z = 5x_1 - 3x_2 + 2x_3$
$s.t \quad 3x_1 + 1x_2 - 1x_3 \leq 30$
$2x_1 + 0x_2 + 3x_3 \geq 45$
$4x_1 + 3x_2 + 5x_3 = 75$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 0, \quad x_2 = 0, \quad x_3 = 15, \quad Z = 30$
The Dual Prices are; (0 , -8.5 , 5.5)
The optimal Ranges are :
$1.6 \leq C_1, \quad C_2 \leq 16.5, \quad C_3 \leq 6.25$

$Max \quad Z = 5x_1 - 3x_2 + 2x_3$
$s.t \quad 3x_1 + 1x_2 - 1x_3 \leq 30$
$2x_1 + 4x_2 + 3x_3 \geq 45$
$4x_1 + 3x_2 + 5x_3 = 75$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 11, \quad x_2 = 2, \quad x_3 = 5, \quad Z = 59$
The Dual Prices are; (0.644 , -2.378 , 1.956)
The optimal Ranges are :
$2.364 \leq C_1, \quad -17.5 \leq C_2 \leq 2.632, -19.4 \leq C_3 \leq 4.9$

$Min \quad Z = 2x_1 + 3x_2 + 5x_3$
$s.t \quad 1x_2 - 1x_3 \leq 16$
$2x_1 + 4x_2 \geq 24$
$4x_1 + 5x_3 = 40$
$x_1, x_2, x_3 \geq 0$
The optimal solution is:
$x_1 = 10, \quad x_2 = 1, \quad x_3 = 0, \quad Z = 23$
The Dual Prices are; (0 , -3/4 , -1/8)
The optimal Ranges are :
$C_1 \leq 5.5, \quad C_2 \geq 0, \quad C_3 \geq 5/8$

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

$$\begin{aligned} Max \quad & Z = 5x_1 + 7x_2 - 4x_3 \\ s.t \quad & 2x_1 + 3x_2 - 1x_3 \le 75 \\ & 1x_1 - 1x_2 + 1x_3 = 15 \\ & 2x_1 + 1x_2 - 3x_3 \ge 50 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 24,\quad x_2 = 9,\quad x_3 = 0,\quad Z = 183$

The Dual Prices are; (2.4 , 0.2 , 0)

The optimal Ranges are :

$-0.5 \le C_1,\quad -5 \le C_2 \le 10,\quad C_3 \le\le -2.2$

$$\begin{aligned} Max \quad & Z = 5x_1 - 2x_2 + 1x_3 \\ s.t \quad & 3x_1 + 1x_2 + 2x_3 \le 15 \\ & 1x_1 + 2x_2 + 1x_3 \ge 12 \\ & 2x_1 + 2x_2 + 1x_3 = 18 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 3,\quad x_2 = 6,\quad x_3 = 0,\quad Z = 3$

The Dual Prices are; (3 , 0 , - 2.75)

The optimal Ranges are :

$2/3 \le C_1,\quad C_2 \le 5,\quad C_3 \le 4.25$

$$\begin{aligned} Min \quad & Z = 2x_1 - 3x_2 + 5x_3 \\ s.t \quad & 1x_1 + 1x_2 + 2x_3 = 20 \\ & 1x_1 - 3x_2 + 1x_3 \le 40 \\ & 3x_1 + 1x_2 + 1x_3 \ge 50 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 15,\quad x_2 = 5,\quad x_3 = 0,\quad Z = 15$

The Dual Prices are; (5.5 , 0 , -2.5)

The optimal Ranges are :

$-3 \le C_1,\quad C_2 \le 2,\quad -8.5 \le C_3$

$$\begin{aligned} Min \quad & Z = 5x_1 + 7x_2 + 4x_3 \\ s.t \quad & 2x_1 + 3x_2 - 1x_3 \le 45 \\ & 1x_1 - 1x_2 + 1x_3 = 9 \\ & 2x_1 + 1x_2 - 3x_3 \ge 30 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 13,\quad x_2 = 4,\quad x_3 = 0,\quad Z = 93$

The Dual Prices are; (0 , 3 , - 4)

The optimal Ranges are :

$-7 \le C_1,\quad -4 \le C_2,\quad -15 \le C_3$

$$\begin{aligned} Min \quad & Z = 3x_1 - 2x_2 + 5x_3 \\ s.t \quad & 1x_1 + 1x_2 + 2x_3 = 30 \\ & 1x_1 - 3x_2 + 1x_3 \le 40 \\ & 3x_1 + 1x_2 + 1x_3 \ge 50 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 10,\quad x_2 = 20,\quad x_3 = 0,\quad Z = -10$

The Dual Prices are; (4.5 , 0 , -2.5)

The optimal Ranges are :

$-2 \le C_1,\quad C_2 \le 2.6,\quad -6.5 \le C_3$

$$\begin{aligned} Max \quad & Z = 3x_1 - 2x_2 + 5x_3 \\ s.t \quad & 1x_1 + 1x_2 + 2x_3 = 30 \\ & 1x_1 - 3x_2 + 1x_3 \le 40 \\ & 3x_1 + 1x_2 + 1x_3 \ge 50 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 30,\quad x_2 = 0,\quad x_3 = 0,\quad Z = 90$

The Dual Prices are; (3 , 0 , 0)

The optimal Ranges are :

$2.5 \le C_1,\quad C_2 \le 3,\quad C_3 \le 6$

$$\begin{aligned} Max \quad & Z = 5x_1 + 3x_2 + 2x_3 \\ s.t \quad & 3x_1 + 1x_2 - 1x_3 \le 30 \\ & 2x_1 + 4x_2 + 3x_3 \ge 45 \\ & 4x_1 + 3x_2 + 5x_3 = 75 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 3,\quad x_2 = 21,\quad x_3 = 0,\quad Z = 78$

The Dual Prices are; (0.6 , 0 , 0.8)

The optimal Ranges are :

$4 \le C_1 \le 5.875,\quad 2.632 \le C_2 \le 3.75,\quad C_3 \le 3.4$

$$\begin{aligned} Max \quad & Z = 5x_1 - 3x_2 + 2x_3 \\ s.t \quad & 3x_1 + 1x_2 - 1x_3 \le 30 \\ & 2x_1 + 0x_2 + 3x_3 \ge 45 \\ & 4x_1 + 3x_2 + 5x_3 = 75 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 0,\quad x_2 = 0,\quad x_3 = 15,\quad Z = 30$

The Dual Prices are; (0 , -8.5 , 5.5)

The optimal Ranges are :

$1.6 \le C_1,\quad C_2 \le 16.5,\quad C_3 \le 6.25$

$$\begin{aligned} Max \quad & Z = 5x_1 - 3x_2 + 2x_3 \\ s.t \quad & 3x_1 + 1x_2 - 1x_3 \le 30 \\ & 2x_1 + 4x_2 + 3x_3 \ge 45 \\ & 4x_1 + 3x_2 + 5x_3 = 75 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 11,\quad x_2 = 2,\quad x_3 = 5,\quad Z = 59$

The Dual Prices are; (0.644 , -2.378 , 1.956)

The optimal Ranges are :

$2.364 \le C_1,\quad -17.5 \le C_2 \le 2.632, -19.4 \le C_3 \le 4.9$

$$\begin{aligned} Min \quad & Z = 2x_1 + 3x_2 + 5x_3 \\ s.t \quad & 1x_2 - 1x_3 \le 16 \\ & 2x_1 + 4x_2 \ge 24 \\ & 4x_1 + 5x_3 = 40 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The optimal solution is:

$x_1 = 10,\quad x_2 = 1,\quad x_3 = 0,\quad Z = 23$

The Dual Prices are; (0 , -3/4 , -1/4)

The optimal Ranges are :

$C_1 \le 5.5,\quad C_2 \ge 0,\quad C_3 \ge 5/8$

GOOD LUCK

Prof. Dr. Anwar Al-Lahham

تم بعونه تعالى

مع أصدق تمنياتي لكم

بالنجاح والتوفيق

أ.د. أنور اللحام