Maclaurin Series BMT-222 (September 2018)

Power series

A function can be written as a power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots\cdots\cdots a_n x^n + \cdots\cdots\cdots \qquad (1)$$

We need to determine the coefficients $a_0, a_1, a_3, \cdots\cdots a_n$. For this we differentiate both sides of equation (1) repeatedly and substitute zero for x. **Recalling that** $n! = n(n-1)(n-2)\cdots\cdots\cdots 2\cdot 1$ $\qquad (0! = 1)$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots\cdots\cdots$$

$$f(0) = a_0 \Rightarrow a_0 = f(0)$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots\cdots\cdots$$

$$f'(0) = a_1 \Rightarrow a_1 = f'(0)$$

$$f''(x) = 2\cdot 1a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_5 x^3 \cdots\cdots\cdots$$

$$f''(0) = 2\cdot 1a_2 \Rightarrow a_2 = \frac{f''(0)}{2\cdot 1} \Rightarrow a_2 = \frac{f''(0)}{2!}$$

$$f'''(x) = 3\cdot 2\cdot 1a_3 + 4\cdot 3\cdot 2a_4 x + 5\cdot 4\cdot 3a_5 x^2 + \cdots\cdots\cdots$$

$$f'''(0) = 3\cdot 2\cdot 1a_3 \Rightarrow a_3 = \frac{f'''(0)}{3\cdot 2\cdot 1} \Rightarrow a_3 = \frac{f'''(0)}{3!}$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2\cdot 1a_4 + 5\cdot 4\cdot 3\cdot 2a_5 x + \cdots\cdots\cdots$$

$$f^{(4)}(0) = 4\cdot 3\cdot 2\cdot 1a_4 \Rightarrow a_4 = \frac{f^{(4)}(0)}{4\cdot 3\cdot 2\cdot 1} \Rightarrow a_4 = \frac{f^{(4)}(0)}{4!}$$

$$f^{(5)}(x) = 5\cdot 4\cdot 3\cdot 2\cdot 1a_5 + \cdots\cdots\cdots$$

$$f^{(5)}(0) = 5\cdot 4\cdot 3\cdot 2\cdot 1a_5 \Rightarrow a_5 = \frac{f^{(5)}(0)}{5\cdot 4\cdot 3\cdot 2\cdot 1} \Rightarrow a_5 = \frac{f^{(5)}(0)}{5!}$$

$\vdots$

$\vdots$

we get $a_0 = f(0),\ a_1 = f'(0),\ a_2 = \dfrac{f''(0)}{2!},\ a_3 = \dfrac{f'''(0)}{3!},\ \cdots\cdots\ a_5 = \dfrac{f^{(5)}(0)}{5!}$

By substituting the values of the coefficients $a_0, a_1, a_3, \cdots\cdots a_n$, in equation (1) we get the desired form of the Maclaurin series of $f(x)$.

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots\cdots\cdots \qquad (2)$$

Maclaurin series ($\sum - form$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad where\ f^{(0)}(x) = f(x)\ \ and\ \ 0! = 1$$

Maclaurin series

- The Maclaurin (Scottish 1698 - 1746) series formula enables us to find the value of the function at a point, x, close to the origin.
- The Maclaurin series of a function is always unique.
- Maclaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots\cdots (2)$$

- This is an infinite series, although often we can approximate $f(x)$ by using just a finite number of the terms as we shall show.

Example 1.

Find the expansion of the function $f(x) = \cos 2x$.

Solution:

We make a list of derivatives and let x = 0:

$$f(x) = \cos 2x \qquad f(0) = 1$$
$$f'(x) = -2\sin 2x \qquad f'(0) = 0$$
$$f''(x) = -2^2 \cos 2x \qquad f''(0) = -2^2$$
$$f'''(x) = -2^3 \sin 2x \qquad f'''(0) = 0$$
$$f''''(x) = 2^4 \cos 2x \qquad f''''(0) = 2^4$$
$$f^{(5)}(x) = -2^5 \sin 2x \qquad f^{(5)}(0) = 0$$
$$f^{(6)}(x) = -2^6 \cos 2x \qquad f^{(6)}(0) = -2^6$$
$$\vdots \qquad \vdots$$

Problem # 1

$f(x) = \sin x$

$\boxed{f(0) = 0}$

$f'(x) = \cos x$

$\boxed{f'(0) = 1}$

$f''(x) = -\sin x$

$f''(0) = 0$

$f'''(x) = -\cos x$

$f'''(0) = -1$

$f''''(x) = \sin x$

$f''''(0) = 0$

$f^{(5)}(x) = \cos x$

$f^{5}(0) = 1$

As $f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\therefore \sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Substitution in equation (Maclaurin series)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots\cdots \text{ we get the desired series}$$

$$\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 \cdots\cdots\cdots$$

Verify the Maclaurin Series expansion for

Problem 1 $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$ (Example 4.2, page 822)

Problem 2 $\quad \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots\cdots \qquad \left(\sum_{n=0}^{\infty} 2^n x^n\right)$

Problem 3 $\quad ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

Problem 4 $\quad$ Verify the series expansion $\quad (1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$

Operations with series

- Certain operations with series can be used to find new series from those already known, as shown in the following examples.

Example 2.

Find Maclaurin series for $\sin x^2$.

Solution:

Consider the series $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$

If we replace x *by* x^2, we obtain $\quad \sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\cdots$

Example 3.

Find Maclaurin series for $x e^x$.

Solution:

From the series $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots$

We obtained from direct multiplication

$$x e^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots\cdots\cdots$$

Verify the Maclaurin Series for the functions

Problem 5 $\sin 2x = 2x - \dfrac{8x^3}{3!} + \dfrac{32x^5}{5!} - \dfrac{128x^7}{7!} + \cdots\cdots$

Problem 6 $e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \dfrac{x^8}{4!} - \cdots\cdots\cdots$

Problem 7 $\cos^2 x = 1 - x^2 + \dfrac{x^4}{3} - \dfrac{2x^6}{45} + \cdots\cdots\cdots$

Problem 8 $\dfrac{1-\cos x}{x^2} = \dfrac{1}{2!} - \dfrac{x^2}{4!} + \dfrac{x^4}{6!} - \dfrac{x^6}{8!} + \cdots\cdots\cdots$

Problem 9 Expand the function $\ln(1+x^2)^3$ in a Maclaurin series.

Example 4.

Show that $(d/dx)e^x = e^x$ by the use of Maclaurin series.

Solution:

$$\frac{d}{dx}e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\cdots\cdots\cdots\right) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!}\cdots\cdots\cdots = e^x$$

Binomial Series

The Maclaurin series of $(1+x)^k$ is called the binomial series

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots\cdots \; |x|\langle 1 \qquad (1)$$

Example 1.

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots\cdots \; |x|\langle 1$

Using equation (1) with k = -2

Example 2.

$\sqrt{1-x} = 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2 - \dfrac{1}{16}x^3 - \cdots\cdots \; |x|\langle 1$

Using equation (1) with k = $\dfrac{1}{2}$ and replaced x by -x

Example 3.

Show that

$\sqrt{x} = 1 + \dfrac{1}{2}(x-1) - \dfrac{1}{8}(x-1)^2 + \dfrac{1}{16}(x-1)^3 - \cdots\cdots \; |x|\langle 1$

Write $\sqrt{x} = \sqrt{1+(x-1)}$ and using equation (1) with k = $\dfrac{1}{2}$ and x –1 in replace of x.

Example 4.

$\dfrac{1}{\sqrt{1-x^2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{3}{8}x^4 + \dfrac{5}{16}x^6 + \cdots\cdots \; |x|\langle 1$

Using equation (1) with k = -$\dfrac{1}{2}$ and x replaced by $-x^2$.

Problem #2

~~Problem~~ Find the Maclaurin series for $f(x) = \frac{1}{1-2x}$

(First four non zero terms)

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$f(x) = \frac{1}{1-2x} = (1-2x)^{-1}$

$\boxed{f(0) = 1}$

$f'(x) = -1(1-2x)^{-2} \frac{d}{dx}(1-2x)$

$= -(1-2x)^{-2}(-2) = 2(1-2x)^{-2}$

$f'(x) = \frac{2}{(1-2x)^2}$

$\boxed{f'(0) = 2}$

$f''(x) = 2(-2)(1-2x)^{-3} \frac{d}{dx}(1-2x)$

$f''(x) = -4(-2)(1-2x)^{-3} = 8(1-2x)^{-3}$

$\boxed{f''(0) = 8}$

$f'''(x) = -24(1-2x)^{-4} \frac{d}{dx}(1-2x)$

$f'''(x) = 48(1-2x)^{-4}$

$\boxed{f'''(0) = 48}$

$\therefore \quad f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\frac{1}{1-2x} = 1 + 2x + \frac{8x^2}{2!} + \frac{48x^3}{3!} + \cdots$

$= 1 + 2x + \frac{8x^2}{2} + \frac{48x^3}{3 \times 2 \times 1} + \cdots$

$$\boxed{\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots}$$

$= \sum_{n=0}^{\infty} 2^n x^n$

Maclaurin Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \cdots \quad \text{①}$$

Problem # 3

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

L.H.S.

$$f(x) = \ln(1+x)$$

$$f(0) = \ln(1+0) = \ln(1) = 0$$

$$\therefore \boxed{f(0) = 0}$$

$$f'(x) = \frac{1}{1+x} \frac{d}{dx}(1+x) \Rightarrow f'(x) = \frac{1}{1+x} \Rightarrow$$

$$\Rightarrow f'(x) = (1+x)^{-1} \Rightarrow f'(0) = \frac{1}{1+0} \Rightarrow \boxed{f'(0) = 1}$$

$$f''(x) = (-1)(1+x)^{-1-1} \frac{d}{dx}(1+x)$$

$$\Rightarrow f''(x) = -(1+x)^{-2}$$

$$\Rightarrow \boxed{f''(0) = -1}$$

$$f'''(x) = (-1)(-2)(1+x)^{-2-1}$$

$$\Rightarrow f'''(x) = 2(1+x)^{-3}$$

$$\Rightarrow \boxed{f'''(0) = 2}$$

$$f''''(x) = 2(-3)(1+x)^{-3-1} \Rightarrow f''''(x) = -6(1+x)^{-4}$$

$$\Rightarrow \boxed{f''''(0) = -6}$$

Putting the values of $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$ in equation ①

$$\therefore \ln(1+x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \cdots$$

$$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{\cancel{2}x^3}{3\times\cancel{2}\times1} - \frac{6x^4}{4\times3\times2\times1} + \cdots$$

$$\Rightarrow \ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots = \text{R.H.S.}$$

Maclaurin Series

HW (Problem #2)

Problem #4

Verify the Maclaurin series expansion for:

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$$

R.H.S

$$\sum_{n=0}^{\infty} (n+1)x^n$$

$$= (0+1)x^0 + (1+1)x^1 + (2+1)x^2 + (3+1)x^3 + (4+1)x^4$$

$$\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \longrightarrow (1)$$

L.H.S. $(1-x)^{-2}$

$f(x) = (1-x)^{-2}$

$$\boxed{f(0) = (1)^{-2} = 1}$$

$f'(x) = -2(1-x)^{-3}\frac{d}{dx}(1-x)$

$\Rightarrow f'(x) = 2(1-x)^{-3}$

$$\boxed{\Rightarrow f'(0) = 2}$$

$f''(x) = +2(-3)(1-x)^{-4}\frac{d}{dx}(1-x) = 6(1-x)^{-4}$

$$\boxed{\Rightarrow f''(0) = +6}$$

$f'''(x) = 6(-4)(1-x)^{-5}\frac{d}{dx}(1-x)$

$\Rightarrow f'''(x) = 24(1-x)^{-5}$

$$\boxed{\Rightarrow f'''(0) = 24}$$

$f''''(x) = 24(-5)(1-x)^{-6}\frac{d}{dx}(1-x)$

$f''''(x) = 120(1-x)^{-6}$

$$\boxed{f^{(4)}(0) = 120}$$

$\therefore f(x) = (1-x)^{-2} = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \cdots$

$\Rightarrow (1-x)^{-2} = 1 + x(2) + \frac{x^2}{2}(6) + \frac{x^3}{6}\times 24 + \frac{x^4}{4\times3\times2}(120) + \cdots$

$\Rightarrow (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \longrightarrow (2)$

$= R.H.S$

$\therefore L.H.S = R.H.S$

Problem #7 Problem HW(1,2,4)

verify the MacLaurin Series for the function

$$\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots$$

L.H.S. $\cos^2 x$

As we know

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$(\cos x)(\cos x) = \cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\Rightarrow \cos^2 x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots - \frac{x^2}{2!} + \frac{x^4}{2!2!} - \frac{x^6}{2!4!} \cdots + \frac{x^4}{4!} - \frac{x^6}{4!2!} + \cdots - \frac{x^6}{6!} \cdots)$$

$$= 1 - x^2\left(\frac{1}{2!} + \frac{1}{2!}\right) + x^4\left(\frac{1}{4!} + \frac{1}{2!2!} + \frac{1}{4!}\right) - x^6\left(\frac{1}{6!} + \frac{1}{2!4!} + \frac{1}{4!2!} + \frac{1}{6!}\right) + \cdots$$

$$= 1 - x^2\left(\frac{1}{2} + \frac{1}{2}\right) + x^4\left(\frac{1}{4\times3\times2} + \frac{1}{2\times2} + \frac{1}{4\times3\times2}\right) - x^6\left(\frac{1}{6\times5\times4\times3\times2} + \frac{1}{2\times4\times3\times2} + \frac{1}{4\times3\times2\times2} + \frac{1}{6\times5\times4\times3\times2}\right)$$

$$\Rightarrow \cos^2 x = 1 - x^2 + x^4\left(\frac{1}{24} + \frac{1}{4} + \frac{1}{24}\right) - x^6\left(\frac{1}{720} + \frac{1}{48} + \frac{1}{48} + \frac{1}{720}\right) + \cdots$$

$$= 1 - x^2 + x^4\left(\frac{1+6+1}{24}\right) - x^6\left(\frac{1+15+15+1}{720}\right) + \cdots$$

$$= 1 - x^2 + x^4\left(\frac{8}{24}\right) - x^6\left(\frac{32}{720}\right) + \cdots$$

$$= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots$$

$= R.H.S.$

$\therefore L.H.S = R.H.S$

Problem #~~17~~ 9

Expand the function $\ln(1+x^2)^3$ in a Maclaurin series

$$f(x) = \ln(1+x^2)^3$$

$$\boxed{f(0) = 0}$$

$$f(x) = 3\ln(1+x^2)$$

$$f'(x) = \frac{3}{1+x^2}\frac{d}{dx}(x^2)$$

$$\Rightarrow f'(x) = \frac{6x}{1+x^2}$$

$$\boxed{f'(0) = 0}$$

$$f''(x) = 6\left[\frac{(1+x^2)\frac{dx}{dx} - x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow f''(x) = 6\left[\frac{1+x^2-2x^2}{(1+x^2)^2}\right]$$

$$\Rightarrow f''(x) = \frac{6(1-x^2)}{(1+x^2)^2}$$

$$\boxed{f''(0) = 6}$$

$$f'''(x) = 6\left[\frac{(1+x^2)^2\frac{d}{dx}(1-x^2) - (1-x^2)\frac{d}{dx}(1+x^2)^2}{((1+x^2)^2)^2}\right]$$

$$\Rightarrow f'''(x) = 6\left[\frac{(1+x^2)^2(-2x) - (1-x^2)(2(1+x^2)\frac{d}{dx}(1+x^2))}{(1+x^2)^4}\right]$$

$$\Rightarrow f'''(x) = 6\left\{\frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4}\right]$$

$$\boxed{f'''(0) = 0}$$

$$f'''(x) = 6\frac{[-2x(1+x^4+2x^2) - 4(1-x^4)]}{(1+x^2)^4}$$

$$f'''(x) = \frac{6[-2x-2x^5-4x^3-4+4x^4]}{(1+x^2)^4}$$

$$\Rightarrow f'''(x) = \frac{6[-2x^5+4x^4-4x^3-2x-4]}{(1+x^2)^4}$$

$$f^{4}(x) = 6\left[\frac{(1+x^2)^4\frac{d}{dx}(-2x^5+4x^4-4x^3-2x-4) - (-2x^5+4x^4-4x^3-2x-4)\frac{d}{dx}(1+x^2)^4}{((1+x^2)^4)^2}\right]$$

$$\Rightarrow f^{4}(x) = 6\left[\frac{(1+x^2)^4(-10x+16x^3-12x-2) - (-2x^5+4x^4-4x^3-2x-4)[4(1+x^2)^3(2x)]}{(1+x^2)^8}\right]$$

$$\Rightarrow f^{4}(0) = \frac{6(-2)}{1}$$

$$\Rightarrow f^{4}(0) = -12$$

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \cdots$$

$$\ln(1+x^2)^3 = 0 + x(0) + \frac{x^2}{2}(6) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-12) + \cdots$$

$$= 3x^2 - \frac{12x^4}{4\times3\times2} + \cdots$$

$$= 3x^2 - \frac{3x^4}{2} + \frac{3x^6}{3} - \frac{3x^8}{4} + \cdots$$

$$= 3\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right)$$

Derive binomial series $(1+x)^k$ using Maclaurin series

$f(x) = (1+x)^k$

$\boxed{f(0) = 1}$

$f'(x) = k(1+x)^{k-1}$

$f'(0) = k(0)^{k-1} \Rightarrow \boxed{f'(0) = k}$

$f''(x) = k(k-1)(1+x)^{k-2}$

$\boxed{f''(0) = k(k-1)}$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$

$\boxed{f'''(0) = k(k-1)(k-2)}$

$f''''(x) = k(k-1)(k-2)(k-3)(1+x)^{k-4}$

$\boxed{f''''(0) = k(k-1)(k-2)(k-3)}$

Putting the values of $f(0), f'(0), f''(0)$ etc in Maclaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \cdots$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \frac{k(k-1)(k-2)(k-3)}{4!} x^4 + \cdots$$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \cdots + \frac{n}{n^2+1} + \cdots$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots + \frac{1}{n^2+1} + \cdots$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$ 10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ 11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{3/2}}$ 12. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2-2n+2}$

Test the series in Exercises 13–20 for convergence or divergence by the comparison test.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ 14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ 15. $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n-5}$ 16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{n^3}$ 18. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^4-1}$ 19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ 20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}+1}$

Test the series in Exercises 21–29 for convergence by the ratio test; if the test fails, use another test.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ 22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ 23. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n!}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$ 25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{7^n}$ 26. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$ 28. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3}$ 29. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

26.3 Maclaurin Series

While the infinite series considered so far contained only constant terms, many useful series consist of variable terms. The most important of these are series representing known functions. The main purpose of this section is to study a method by which a function $f(x)$ can be written as a **power series:**

Power series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \tag{26.11}$$

(Series expansions other than power series will be taken up in Section 26.6.)

To express a function as a power series, we need to determine the coefficients in the form (26.11). This can be done by means of a simple trick: we differentiate both sides of (26.11) repeatedly, as if it were a regular polynomial, and substitute zero for x. Hence $f(x)$ must be differentiable near $x = 0$, to start with. Moreover, it is shown in many books on advanced calculus that a power series may be differentiated term by term, provided that it converges for all x in some interval. We now get

$$\begin{array}{rllllll}
f(x) = & a_0 + a_1x & + a_2x^2 & + a_3x^3 & + a_4x^4 & + a_5x^5 + \cdots \\
f'(x) = & a_1 & + 2a_2x & + 3a_3x^2 & + 4a_4x^3 & + 5a_5x^4 + \cdots \\
f''(x) = & & 2\cdot 1a_2 & + 3\cdot 2a_3x & + 4\cdot 3a_4x^2 & + 5\cdot 4a_5x^3 + \cdots \\
f'''(x) = & & & 3\cdot 2\cdot 1a_3 & + 4\cdot 3\cdot 2a_4x & + 5\cdot 4\cdot 3a_5x^2 + \cdots \\
f^{(4)}(x) = & & & & 4\cdot 3\cdot 2\cdot 1a_4 & + 5\cdot 4\cdot 3\cdot 2a_5x + \cdots \\
f^{(5)}(x) = & & & & & 5\cdot 4\cdot 3\cdot 2\cdot 1a_5 + \cdots
\end{array}$$

If we let $x = 0$, all the terms on the right collapse to zero, except for the first in each row. Thus $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2 \cdot 1 a_2$, $f'''(0) = 3 \cdot 2 \cdot 1 a_3$, $f^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot 1 a_4$, and $f^{(5)}(0) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 a_5$. Solving for the constants and recalling that

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

we have

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{f''(0)}{2!},\ \ldots,\ a_5 = \frac{f^{(5)}(0)}{5!}$$

The pattern is now clear:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Finally, after substituting in series (26.11) we get the desired form of the *Maclaurin series* of $f(x)$.

Maclaurin series of $f(x)$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \qquad (26.12)$$

Colin Maclaurin

The Maclaurin series is named after Colin Maclaurin (Scottish mathematician, 1698–1746). Maclaurin made many contributions to geometry, particularly to the development of higher algebraic curves. It is ironic that his name is now attached to a series which is only a special case of the *Taylor series* (Section 26.5). The latter series was published by Brook Taylor (English mathematician, 1685–1731) in 1715 (long before Maclaurin's work) but was known earlier to Johann Bernoulli.

The Maclaurin series can be written in particularly elegant form if we define $0! = 1$

Maclaurin series (Σ-form):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (26.13)$$

where $f^{(0)}(x) = f(x)$ and $0! = 1$.

(The Maclaurin series of a function is always unique.)

Example 1 Expand $f(x) = e^x$ in a Maclaurin series.

Solution. We differentiate first and let $x = 0$:

$$\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ f'(x) = e^x & f'(0) = 1 \\ f''(x) = e^x & f''(0) = 1 \\ \vdots & \vdots \\ \text{and so on} & \text{and so on} \end{array}$$

Direct substitution in series (26.12) yields

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Suppose we take a peek ahead to Section 26.5 and replace x by 1; then

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

Using the convention $0! = 1$, we now get the following beautiful representation of the number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Example 2 (Optional) Show that the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

is convergent for all x.

Solution. Convergence may be proved by the ratio test: Since

$$a_n = \frac{x^n}{n!} \qquad \text{and} \qquad a_{n+1} = \frac{x^{n+1}}{(n+1)!} = \frac{x^n \cdot x}{(n+1)\, n!}$$

we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^n \cdot x}{(n+1)\, n!} \cdot \frac{n!}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 = L < 1$$

Since $L < 1$ no matter what value we choose for x, the series is convergent for all x by the ratio test.

Example 3 Find the Maclaurin expansion of the function $f(x) = \cos 2x$.

Solution. As before, we make a list of derivatives and let $x = 0$:

$$\begin{array}{ll}
f(x) = \cos 2x & f(0) = 1 \\
f'(x) = -2 \sin 2x & f'(0) = 0 \\
f''(x) = -2^2 \cos 2x & f''(0) = -2^2 \\
f'''(x) = 2^3 \sin 2x & f'''(0) = 0 \\
f^{(4)}(x) = 2^4 \cos 2x & f^{(4)}(0) = 2^4 \\
f^{(5)}(x) = -2^5 \sin 2x & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -2^6 \cos 2x & f^{(6)}(0) = -2^6 \\
\vdots & \vdots \\
\text{and so on} & \text{and so on}
\end{array}$$

Substitution in (26.12) yields the desired series:

$$\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots$$

The following expansions are particularly important and are listed for later reference. (The first has already been obtained and the rest will be left as exercises.)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \qquad \text{for all } x \qquad (26.14)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \text{for all } x \qquad (26.15)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad \text{for all } x \qquad (26.16)$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \leq 1 \qquad (26.17)$$

Exercises / Section 26.3

Verify the Maclaurin series expansions in Exercises 1–9.

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

as $\cos x = \frac{d}{dx}(\sin x)$

$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$

$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!}$

$= 1 - \frac{x^2}{2!} + \frac{5 \cdot x^4}{5 \cdot 4!}$

3. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

4. $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

5. $\text{Arctan}\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

6. $\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

7. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

8. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

9. $\text{Arcsin}\, x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7x^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} - \cdots$

10. (Optional) Show that the series (26.15) and (26.16) converge for all x.

11. Verify the series expansion

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$$

by (a) using the binomial series; (b) finding the Maclaurin series expansion; (c) dividing out $1/(1-x)^2$.

26.4 Operations with Series

Certain operations with series can be used to find new series from those already known, as shown in the following examples.

Example 1 Find the Maclaurin series for $\sin x^2$.

Solution. Consider the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

from the last section. If we replace x by x^2, we obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

Since this series is a power series, it must be the Maclaurin series of $\sin x^2$, since such expansions are unique.

Example 2 Find the Maclaurin series for xe^x.

Solution. From

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we obtain by direct multiplication

$$xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$

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(26.14)

(26.15)

(26.16)

(26.17)

P# 11

L.H.S $(1-x)^{-2}$

$f(x) = (1-x)^{-2} \Rightarrow f(0) = 1$

$f'(x) = +2(1-x)^{-3} \Rightarrow f'(0) = 2$

$f''(x) = 6(1-x)^{-4} \Rightarrow f''(0) = 6$

$f'''(x) = 24(1-x)^{-5} \Rightarrow f'''(0) = 24$

$f''''(x) = 120(1-x)^{-6} \Rightarrow f''''(0) = 120$

$\therefore f(x) = (1-x)^{-2} = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \cdots$

$= 1 + 2x + \frac{x^2}{2!} \times 6 + \frac{x^3}{3.2.1} \times 24 + \frac{x^4}{4.3.2.1} \times 120 + \cdots$

$\boxed{\text{L.H.S} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots}$

$\text{R.H.S} = \sum_{n=0}^{\infty} (n+1)x^n = 1x^0 + 2x^1 + 3x^2 + 4x^3 + 5x^4$

$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

$\therefore$ L.H.S = R.H.S

It has already been noted that convergent power series can be differentiated termwise; the same is true of integration.

Example 3 Show that $(d/dx)\, e^x = e^x$ by the use of Maclaurin series.

Solution.

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots = e^x$$

Example 4 Find the Maclaurin series of Arctan x by integrating (d/dx) Arctan x termwise.

Solution. We recall that

$$\frac{d}{dx} \operatorname{Arctan} x = \frac{1}{1 + x^2}$$

This expression can be written as a geometric series: Let $r = -x^2$ and $a = 1$; then

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 - (-x^2)} = \frac{1}{1 - x^2}$$

Consequently,

$$\operatorname{Arctan} x = \int_0^x \frac{dx}{1 + x^2} = \int_0^x (1 - x^2 + x^4 - x^6 + \cdots)\, dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \Big|_0^x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

(It is actually poor practice to use x both for the variable of integration and the upper limit, but the steps are so much easier to see this way.)

Remark. Our main application of the integration of series will be discussed in the next section.

The four fundamental operations—addition, subtraction, multiplication, and division—can theoretically be carried out with series. Two of these operations are demonstrated in the following examples.

Example 5 Find the power series expansion of $e^x \sin x$ by multiplying the series for e^x and $\sin x$.

Solution. We first recall that

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

and

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

We may now multiply each term in the second series by each term in the first series in exactly the same way that we multiply polynomials. If we decide to carry only powers up to the fifth power, we obtain

$$e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

Example 6 Use the series in Exercises 3 and 4 of the last section to expand

$$\ln \frac{1 + x}{1 - x}$$

Solution. We have

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

and

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

Hence

$$\ln \frac{1 + x}{1 - x} = \ln(1 + x) - \ln(1 - x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$
$$- \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

Euler's Identity

As a final exercise we are going to uncover a relationship between three of our transcendental functions by making use of the basic imaginary unit $j =$

$\sqrt{-1}$. As a starting point, notice that the expansion of the sine function has only odd powers and that of the cosine function only even powers. However, all the powers occur in the expansion of e^x; so e^x comes very close to being the sum of the other two—if only the signs matched! Now, by introducing j formally, we find that

$$
\begin{aligned}
e^{jx} &= 1 + jx + \frac{j^2x^2}{2!} + \frac{j^3x^3}{3!} + \frac{j^4x^4}{4!} + \frac{j^5x^5}{5!} + \cdots \\
&= 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \frac{jx^5}{5!} - \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \cos x + j \sin x
\end{aligned}
$$

The resulting formula is known as *Euler's identity* after the Swiss mathematician Leonhard Euler (1707–1783).

Euler's identity:

$$e^{jx} = \cos x + j \sin x \tag{26.18}$$

Euler's identity arises in the study of differential equations and in the theory of electrical circuits.

Although there is some room for opinion, it can be argued that the most interesting numbers in mathematics are 0, 1, j, e, and π. By Euler's identity,

$$e^{j\pi} = \cos \pi + j \sin \pi = -1$$

or

$$e^{j\pi} + 1 = 0 \tag{26.19}$$

which involves all five of these numbers. This astounding relationship has been called the eutectic point of mathematics, for no matter how you try to analyze it, it seems to retain an air of mystery not easily explained away.

Exercises / Section 26.4

Use the method of Examples 1 and 2 to find the Maclaurin series of the functions in Exercises 1–10.

1. $\sin 3x$
2. $\cos 2x$
3. e^{-x}
4. e^{-x^2}
5. $\cos\sqrt{x}$
6. $\dfrac{\sin x^2}{x}$
7. $x \cos x$
8. x^2e^x
9. $\ln(1 + x^2)$
10. $\ln(1 - x)$ $\sin x = x - \frac{x^3}{3!} + x^5/5!$
11. Show that $(d/dx) \sin x = \cos x$ by use of the Maclaurin series. (See Example 3.)
12. Show that $(d/dx) \cos x = -\sin x$ by use of Maclaurin series.
13. Use the method of Example 4 to find the Maclaurin series of $\ln(1 + x)$.

$x = 3x$

Answers:

① $3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \cdots$

③ $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

⑤ $1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots$

#7 $x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots$

⑨ $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$

⑮ $1 - x + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$

⑰ $3\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right)$

⑲ $2x - \frac{x^2}{2} - \frac{x^4}{4} + \frac{2x^5}{5} - \frac{x^6}{6} - \frac{x^8}{8} + \frac{2x^9}{9} - \cdots$

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14. Expand $(\sin x - x)/x^2$ in a Maclaurin series.

15. Use the method of Example 5 to find the Maclaurin series of $e^{-x} \cos x$.

16. Use the method of Example 6 to find the Maclaurin series of $\ln(1 + x)^2$.

17. Expand the function $\ln(1 + x^2)^3$ in a Maclaurin series.

18. Find the Maclaurin series of $\frac{1}{2}(e^x + e^{-x})$ by addition of series.

19. Find the Maclaurin series of $\ln(1 + x) - \text{Arctan } x$.

A complex number $a + bj$ can be written in polar form $r(\cos\theta + j\sin\theta)$, which by Euler's identity becomes $re^{j\theta}$, known as the *exponential form*. Change the complex numbers in Exercises 20–25 to exponential form.

20. $1 + j$ **21.** $-\sqrt{3} + j$ **22.** $1 - \sqrt{3}j$ **23.** $3j$ **24.** $-4j$ **25.** $-2 + 2j$

26.5 Computations with Series

In this section we are going to do numerical computations by means of power series. By using a sufficiently large number of terms, we can obtain the value of a transcendental function to any desired degree of accuracy. A particularly important application of these numerical techniques is the evaluation of certain definite integrals.

Before we consider computations involving series, we need to make a few additional observations about series of constants. Suppose that $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence of positive numbers such that each number is less than the preceding one, that is, $a_{n+1} < a_n$ for all n, and consider the series

alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n - \cdots \tag{26.20}$$

called an **alternating series** since the signs alternate. If the series converges, then the sum may be obtained to any desired degree of accuracy by adding the first n terms and estimating the error made. To check this statement, suppose we add the first four terms of series (26.20) and estimate the error by writing the series as follows:

$$(a_1 - a_2 + a_3 - a_4) + a_5 - (a_6 - a_7) - (a_8 - a_9) - \cdots$$

Since the terms in the series are decreasing,

$$(a_6 - a_7) > 0, (a_8 - a_9) > 0 \qquad \text{and so forth}$$

Hence

$$a_5 - (a_6 - a_7) - (a_8 - a_9) - \cdots < a_5$$

So, by adding $a_1 - a_2 + a_3 - a_4$, the error made is less than a_5.

If we wish to add the first five terms, then the error is estimated from

$$(a_1 - a_2 + a_3 - a_4 + a_5) - a_6 + (a_7 - a_8) + (a_9 - a_{10}) + \cdots$$

Answers

(23) $3e^{\pi j/2}$

(25) $2\sqrt{2}\, e^{3\pi j/4}$

Again

$$(a_7 - a_8) > 0,\ (a_9 - a_{10}) > 0, \qquad \text{and so on}$$

so that the error is no worse than $-a_6$.

The **error** made by adding the first n terms of a **convergent alternating series**

$$a_1 - a_2 + a_3 - a_4 + \cdots \qquad (a_n > 0,\ a_{n+1} < a_n)$$

is numerically less than the first term omitted.

(We state without proof that an alternating series converges if $\lim_{n\to\infty} a_n = 0$.)

Example 1 Compute the value of $e^{-0.1}$ by using the first four terms of the expansion of e^x; estimate the error.

Solution. From

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

we get

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \cdots$$

Now let $x = 0.1$ and find the sum of the first four terms by using a calculator:

$$e^{-0.1} = 1 - 0.1 + \frac{(0.1)^2}{2!} - \frac{(0.1)^3}{3!} = 0.9048333$$

The error made is no worse than the next term.

$$\frac{(0.1)^4}{4!} = 0.0000042 \qquad \text{fifth term}$$

Based on these calculations, the correct value to six decimal places could be any one of the following: 0.904833, 0.904834, . . . , or 0.904838. Consequently,

$$e^{-0.1} = 0.9048, \qquad \text{accurate to four decimal places}$$

Example 2 Find an infinite-series representation of (a) e; (b) π.

Solution.

a. The representation of e was already obtained in Section 26.3:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

b. Since

$$\text{Arctan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

we let $x = 1$ and obtain

$$\text{Arctan } 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

or

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Although a striking relationship, series (b) does not provide us with a practical method of computing π, since the series converges too slowly. A better way is by means of the series for $f(x) = \text{Arcsin } x$ (Exercise 7).

As noted in Chapter 25, many functions do not possess elementary antiderivatives. Since a power series can be integrated termwise, many such integrals can be worked out by means of Maclaurin series, leading to *nonelementary functions*.

Example 3 Evaluate

$$\int_0^1 \frac{\sin x^2 \, dx}{x}$$

Solution. At $x = 0$ the integrand takes on the indeterminate form 0/0. Now, by L'Hospital's rule

$$\lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \frac{2x \cos x^2}{1} = 0$$

so that the function is bounded on (0, 1). (Otherwise we would be dealing with an improper integral.)

From

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

we have

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \qquad \text{replacing } x \text{ by } x^2$$