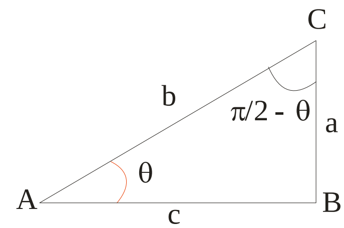
Trigonometric identities with $\frac{\pi}{2}$



Consider a right triangle with an angle of θ radians. Because the angles of a triangle add up to π radians, the triangle's other acute angle is $\frac{\pi}{2}-\theta$ radians, as shown in the figure. If we were working in degrees rather than radians, then we would be stating that a right triangle with an angle of θ^o also has an angle of $(90-\theta)^o$.

Focusing on the angle θ: $\cos\theta = \frac{c}{b}$, $\quad \sin\theta = \frac{a}{b}$

Now focusing instead on the angle $\left(\frac{\pi}{2}-\theta\right)$ in the triangle above,

$$\cos\left(\frac{\pi}{2}-\theta\right)=\frac{a}{b}, \qquad \sin\left(\frac{\pi}{2}-\theta\right)=\frac{c}{b}$$

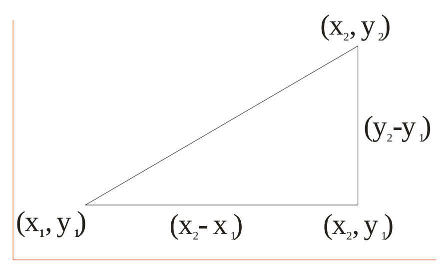
Comparing the last two sets of displayed equations, we get the following identities:

Trigonometric identities with $\frac{\pi}{2}$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta, \qquad \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

Distance between two points

More generally, to find the formula for the distance between two points (x_1, y_1) *and* (x_2, y_2), consider the right triangle in the figure below:



Starting with the points (x_1, y_1) *and* (x_2, y_2) in the figure, the horizontal side of the triangle has length $(x_2 - x_1)$ and the vertical side of the triangle has length $(y_2 - y_1)$. The Pythagorean Theorem then gives the length of the hypotenuse, leading to the following formula:

The distance between the points (x_1, y_1) *and* (x_2, y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

Using the formula above, we can now find the distance between two points without drawing a figure.

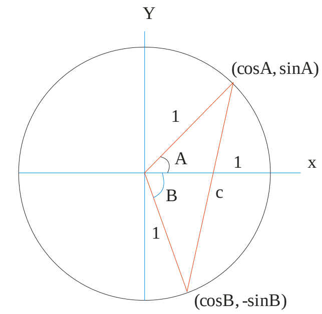
Example Find the distance between the points $(3,1)$ *and* $(-4,-99)$.

solution The distance between these two points is

$$\sqrt{(3-(-4))^2+(1-(-99))^2}=\sqrt{(7)^2+(100)^2}=\sqrt{10049}$$

The cosine of a sum and difference

Consider the figure below, which shows the unit circle along with the radius corresponding to A and the radius corresponding to $-B$.



We defined the cosine and sine so that the endpoint of the radius corresponding to A has coordinates $(\cos A, \sin A)$ The endpoint of the radius corresponding to $-B$ has coordinates equals $\left(\cos(-B), \sin(-B)\right)$, which we know equals $\left(\cos B, -\sin B\right)$, as shown above.

The large triangle in the figure above has two sides that are radii of the unit circle and thus have length 1. The angle between these two sides is $A+B$. The length of the third side of this triangle has been labeled c. We can compute c^2 in two different ways: first by using the formula for the distance between two points, and second by using the law of cosines. We will then set these two computed values of c^2 equal to each other, obtaining a formula for $\cos(A+B)$.

To carry out the plan discussed in the paragraph above, note that one end point of the line segment above with length c has coordinates $(\cos A, \sin A)$ and the other endpoint has coordinates $\left(\cos B, -\sin B\right)$. The distance between two points is the square root of the sum of the squares of the differences of the coordinates. Thus

$$c = \sqrt{(\cos A - \cos B)^2 + (\sin A + \sin B)^2}.$$

Squaring both sides of this equation, we have

$c^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2 = \cos^2 A + \cos^2 B - 2\cos A\cos B + \sin^2 A + \sin^2 B + 2\sin A\sin B$

$\left(\cos^2 A + \sin^2 A = 1, \quad \cos^2 B + \sin^2 B = 1\right)$

$$c^2 = 2 - 2\cos A\cos B + 2\sin A\sin B \qquad (1)$$

To compute c^2 by another method, apply the law of cosines to the large triangle in the figure above, getting $c^2 = 1^2 + 1^2 - 2\times 1\times 1\times \cos(A+B)$

$$c^2 = 2 - 2\cos(A+B) \qquad (2)$$

From equation (1) and (2)

$2 - 2\cos A\cos B + 2\sin A\sin B = 2 - 2\cos(A+B)$

$$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$$

This is the addition formula for cosine

***Never, ever, make the mistake of thinking that $\cos(A+B) = \cos A + \cos B$.

We can find a formula for the cosine of the difference of two angles. In the formula for $\cos(A+B)$, replace B by $-B$ on both sides of the equation and using $\cos(-B)=\cos B$ and $\sin(-B)=-\sin B$ to get

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

This is the subtraction formula for cosine.

The sine of a sum and difference

To find the formula for the sine of the sum of two angles, we will make use of the identities $\sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right)$ *and* $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ (**Trigonometric identities with $\frac{\pi}{2}$**)

We begin by converting the sine into a cosine and then we use the identity just derived above:

$$\sin(A+B)=\cos\left(\frac{\pi}{2}-(A+B)\right)=\cos\left(\frac{\pi}{2}-A-B\right)=\cos\left(\left(\frac{\pi}{2}-A\right)-B\right)$$

$$\Rightarrow \sin(A+B)=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$$

The equation above and the identities above now imply the following result:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

This is the addition formula for sine

***Never, ever, make the mistake of thinking that $\sin(A+B)=\sin A+\sin B$.

We can find a formula for the sine of the difference of two angles. In the formula for $\sin(A+B)$, replace B by $-B$ on both sides of the equation and using $\cos(-B)=\cos B$ and $\sin(-B)=-\sin B$ to get

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

This is the subtraction formula for sine.

Exercise 16.1 Page(493)

Change the expression to an equivalent expression involving sines and cosines also simplify if possible

Q#7 $\cot x + \frac{1}{\sin x}$

$$\Rightarrow = \frac{\cos x}{\sin x} + \frac{1}{\sin x}$$

$$= \frac{1+\cos x}{\sin x}$$

Q#13

$$1 - \sec^2 x$$

$$= 1 - \frac{1}{\cos^2 x}$$

$$= \frac{\cos^2 x - 1}{\cos^2 x}$$

$$= \frac{-1(1-\cos^2 x)}{\cos^2 x}$$

$$= -\frac{\sin^2 x}{\cos^2 x}$$

Q#17 Simplify the expression and convert to sines and cosines

$$\frac{1}{1-\cos^2 x}$$

$$= \frac{1}{\sin^2 x}$$

$$= \operatorname{cosec}^2 x$$

Q#21

$$\frac{\tan\theta \operatorname{cosec}\theta}{\sec\theta}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta}}{\frac{1}{\cos\theta}}$$

$$= \frac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}} = \frac{1}{\cos\theta} \times \cos\theta$$

$$= 1$$

Q#29

$$\frac{1+\tan^2 x}{\cos x}$$

$$= \frac{1+\frac{\sin^2 x}{\cos^2 x}}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \times \frac{1}{\cos x}$$

$$= \frac{1}{\cos^3 x}$$

$$= \sec^3 x$$

Exercise 16.2 Page 497-499

Q#5 Prove the identity

$$\frac{\cos^2\beta}{\sin\beta} + \sin\beta = \operatorname{cosec}\beta$$

L.H.S $\frac{\cos^2\beta}{\sin\beta} + \sin\beta$

$$= \frac{\cos^2\beta + \sin^2\beta}{\sin\beta}$$

$$= \frac{1}{\sin\beta}$$

$$= \operatorname{cosec}\beta = R.H.S$$

$\therefore$ L.H.S = R.H.S

Q#15

$$\frac{\sin\beta + \tan\beta}{1+\cos\beta} = \tan\beta$$

L.H.S $\frac{\sin\beta + \tan\beta}{1+\cos\beta}$

$$= \frac{\sin\beta + \frac{\sin\beta}{\cos\beta}}{1+\cos\beta} = \frac{\sin\beta\cos\beta + \sin\beta}{\cos\beta\,(1+\cos\beta)}$$

$$= \frac{\sin\beta(1+\cos\beta)}{\cos\beta(1+\cos\beta)}$$

$$= \sin\beta/\cos\beta = \tan\beta = R.H.S.$$

$\therefore$ L.H.S = R.H.S.

Q#35

$$\cos^4 x - \sin^4 x = 2\cos^2 x - 1$$

L.H.S $\cos^4 x - \sin^4 x$

$$= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$$

$$= 1(\cos^2 x - \sin^2 x)$$

$$= \cos^2 x - (1-\cos^2 x)$$

$$= \cos^2 x - 1 + \cos^2 x$$

$$= 2\cos^2 x - 1 = R.H.S$$

$\therefore$ L.H.S = R.H.S

#47 In some problems on the motion of a pendulum, the expression $\frac{1}{\sqrt{1-\cos x}}$ arises. Show that this expression is equivalent to $\frac{\sqrt{1+\cos x}}{\sin x}$

$$\frac{1}{\sqrt{1-\cos x}} = \frac{1}{\sqrt{1-\cos x}} \times \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}$$

$$= \frac{\sqrt{1+\cos x}}{\sqrt{(1-\cos x)(1+\cos x)}} = \frac{\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}$$

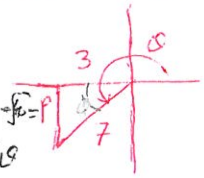
$$= \frac{\sqrt{1+\cos x}}{\sqrt{\sin^2 x}} = \frac{\sqrt{1+\cos x}}{\sin x}$$

$$\therefore \frac{1}{\sqrt{1-\cos x}} = \frac{\sqrt{1+\cos x}}{\sin x}$$

Exercise 16.4 Page 509-510

Q # 11

Find $\cos 2\theta$, given that $\cos\theta = -3/7$, θ in quadrant III



$\cos 2\theta = ?$

$\cos\theta = -3/7$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

For $\sin\theta = ?$

$7^2 = P^2 + 3^2$

$\Rightarrow P^2 = 49 - 9$

$\Rightarrow P = \sqrt{40}$

$\therefore \sin\theta = -\dfrac{\sqrt{40}}{7}$

$\cos\phi = \frac{3}{7}$

$\phi = 64.62$

$\theta = 180 + 64.62$

$\theta = 244.62$

$2\theta = 489.25$

-360

$= 129$

$\therefore \cos 2\theta = \left(-\frac{3}{7}\right)^2 - \left(\frac{-\sqrt{40}}{7}\right)^2$

$= \frac{9}{49} - \frac{40}{49}$

$\cos 2\theta = \frac{9-40}{49}$

$\therefore \cos 2\theta = -\frac{31}{49}$

Q # 33

Prove the given identity

$\dfrac{1 + \cos 2w}{\sin 2w} = \cot w$

L.H.S $\dfrac{1+\cos 2w}{\sin 2w}$

$= \dfrac{1 + \cos^2 w - \sin^2 w}{2\sin w \cos w}$

$= \dfrac{1 + \cos^2 w - (1 - \cos^2 w)}{2\sin w \cos w}$

$= \dfrac{\cancel{1} + \cos^2 w - \cancel{1} + \cos^2 w}{2\sin w \cos w}$

$= \dfrac{\cancel{2}\cos^{\cancel{2}} w}{\cancel{2}\sin w \cancel{\cos w}}$

$= \dfrac{\cos w}{\sin w} = \cot w = $ R.H.S

$\therefore$ L.H.S = R.H.S

Q # 39 $T = k\omega^2 \sin\alpha\cos\alpha$

$\Rightarrow T = \dfrac{2k\omega^2 \sin\alpha\cos\alpha}{2}$

$= \dfrac{k\omega^2 (2\sin\alpha\cos\alpha)}{2}$

$T = \dfrac{k\omega^2 \sin 2\alpha}{2}$

Exercise 16.3 Page 503-505

Q # 19 Write the expression in single term

$\sin(x+y)\cos y - \cos(x+y)\sin y$

$= \cos y(\sin x\cos y + \cos x \sin y) - \sin y(\cos x\cos y - \sin x \sin y)$

$= \sin x\cos^2 y + \cancel{\cos x \cos y \sin y} - \cancel{\cos x \cos y \sin y} + \sin x \sin^2 y$

$= \sin x(\sin^2 y + \cos^2 y)$

$= \sin x$

Q # 37 Write expression in terms of x

$\tan(x + \frac{\pi}{4})$

As $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\therefore \tan(x + \frac{\pi}{4}) = \dfrac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}$ $\tan\frac{\pi}{4} = 1$

$\therefore \tan(x + \frac{\pi}{4}) = \dfrac{\tan x + 1}{1 - \tan x}$

Q # 55

If a force $F_0 \cos\omega t$ is applied to a weight oscillating on a spring, then the energy supplied to the system can be written in the form

$E = A\omega F_0 \cos(\omega t - \gamma)\cos\omega t$

Show that

$E = A\omega F_0(\cos^2\omega t \cos\gamma + \cos\omega t \sin\omega t \sin\gamma)$

$E = A\omega F_0 \cos(\omega t - \gamma)\cos\omega t$

$= A\omega F_0[\cos\omega t\cos\gamma + \sin\omega t \sin\gamma]\cos\omega t$

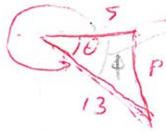
$= A\omega F_0[\cos^2\omega t\cos\gamma + \cos\omega t \sin\omega t \sin\gamma]$

Exercise 16.5 Page 514

Q# 9

Find $\sin\left(\frac{\theta}{2}\right)$, given that $\cos\theta = \frac{5}{13}$, θ in quadrant $\overline{IV}$

$\cos\theta = 5/13$, $\sin\theta/2 = ?$



$\phi = 67.38$

$\theta = 360 - 67.38$

$= 292.62^\circ$

$\frac{\theta}{2} = 146.31$

2nd coordinate

$\overline{IV}$

As $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2$

$\Rightarrow \cos\theta = 1 - \sin^2\frac{\theta}{2} - \sin^2\theta/2$

$\Rightarrow \cos\theta = 1 - 2\sin^2\theta/2$

$\Rightarrow 2\sin^2\theta/2 = 1 - \cos\theta \Rightarrow \sin^2\theta/2 = \frac{1-\cos\theta}{2}$

$\Rightarrow \sin\theta/2 = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$\Rightarrow \sin\theta/2 = \pm\sqrt{\frac{1-\frac{5}{13}}{2}} \Rightarrow \sin\frac{\theta}{2} = \pm\sqrt{\frac{13-5}{2\times 13}}$

$\Rightarrow \sin\theta/2 = \pm\sqrt{\frac{8\,4}{2\times 13}}$

$\Rightarrow \sin\frac{\theta}{2} = +\frac{2}{\sqrt{13}} \Rightarrow \sin\frac{\theta}{2} = \frac{2}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$

$\therefore \sin\theta/2 = \frac{2\sqrt{13}}{13}$

Q# 21

Eliminate the exponent

$2\sin^2 3x$

We know

$\cos 2x = \cos^2 x - \sin^2 x$

$\Rightarrow \cos 2x = 1 - \sin^2 x - \sin^2 x$

$\Rightarrow \cos 2x = 1 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x = 1 - \cos 2x$

put $x = 3x$

$\Rightarrow 2\sin^2 3x = 1 - \cos 2(3x)$

$\Rightarrow \boxed{2\sin^2 3x = 1 - \cos 6x}$

Q# 27 $\quad \operatorname{cosec}^2\theta = \frac{2}{1-\cos 2\theta}$

L.H.S $\operatorname{cosec}^2\theta$

$= \frac{1}{\sin^2\theta}$

$= \frac{2}{2\sin^2\theta}$

$= \frac{2}{1-\cos 2\theta} = R.H.S$

$\therefore L.H.S = R.H.S$

OR R.H.S.

$\frac{2}{1-\cos 2\theta}$

$= \frac{2}{1-(1-2\sin^2\theta)}$

$= \frac{2}{\cancel{1}-\cancel{1}+2\sin^2\theta}$

$= \frac{2}{2\sin^2\theta} = \frac{1}{\sin^2\theta} = \operatorname{cosec}^2\theta = L.H.S.$

CHAPTER 16

Additional Topics in Trigonometry

488

Objectives Upon completion of this chapter, you should be able to:

1. State the fundamental trigonometric identities.
2. Use the fundamental trigonometric identities to
   a. Simplify certain trigonometric expressions.
   b. Prove additional elementary identities.
3. State the sum, difference, half-angle, and double-angle formulas.
4. Use the identities in objective (3) to:
   a. Find certain function values.
   b. Transform certain given trigonometric expressions.
   c. Prove other identities.
5. Solve trigonometric equations.
6. Evaluate inverse trigonometric relations and functions.

16.1 Fundamental Identities

We saw in earlier chapters that solving triangles is an integral part of trigonometry. Another branch, called **analytic trigonometry,** deals mainly with **identities.** This aspect of the subject plays a major role in more advanced areas of mathematics, especially calculus.

Most of this chapter is devoted to the study of trigonometric identities. Identities are then used in Section 16.6 to help solve trigonometric equations. The chapter ends with a brief study of inverse trigonometric functions.

Identity

Recall that an **identity** is an equation that is satisfied for every value of the variable. For example, $x^2 - 1 = (x - 1)(x + 1)$ is an identity. In

trigonometry, identities arise almost as soon as the basic definitions are given. For example, $\sin\theta = 1/\csc\theta$ is valid for every $\theta \neq 0 \pm n\pi$. To obtain other identities, let us recall the basic definitions of the trigonometric functions.

Definitions of trigonometric functions:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

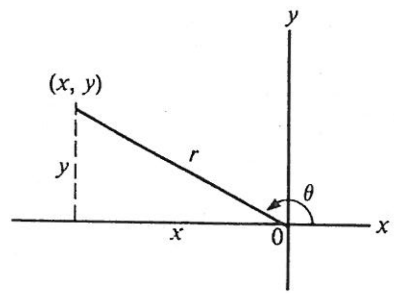


Figure 16.1

From these definitions we get the following reciprocal relations:

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \tan\theta = \frac{1}{\cot\theta} \tag{16.1}$$

Since $\tan\theta = y/x = (y/r)/(x/r)$, we get from the definitions of sine and cosine the identity

$$\tan\theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin\theta}{\cos\theta} \tag{16.2}$$

Finally, since $\cot\theta = 1/\tan\theta$, we have

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{16.3}$$

Note especially that the secant, cosecant, tangent, and cotangent functions can be expressed in terms of sines and cosines:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

490

It follows that any combination of these functions can be expressed in terms of sines and cosines. For example,

$$2 \tan \theta + \frac{1}{2} \cot \theta = \frac{2 \sin \theta}{\cos \theta} + \frac{\cos \theta}{2 \sin \theta}$$

Consider another example.

Example #1

e 1 Change the following expressions to equivalent expressions involving sines and cosines:

a. $\sec \theta + \tan \theta$ **b.** $\sin \theta + \dfrac{1}{\csc \theta}$

Solution. **a.** $\sec \theta + \tan \theta = \dfrac{1}{\cos \theta} + \dfrac{\sin \theta}{\cos \theta} = \dfrac{1 + \sin \theta}{\cos \theta}$

by identities (16.1) and (16.2), respectively.

b. $\sin \theta + \dfrac{1}{\csc \theta} = \sin \theta + \sin \theta = 2 \sin \theta$

since $1/\csc \theta = \sin \theta$.

The definitions of the trigonometric functions yield other basic relationships, but the derivations can be carried out in a more interesting way. Consider the unit circle ($r = 1$) in Figure 16.2 and any point (a, b) on the circle. Note that

$$\sin \theta = \frac{a}{1} \quad \text{and} \quad \cos \theta = \frac{b}{1}$$

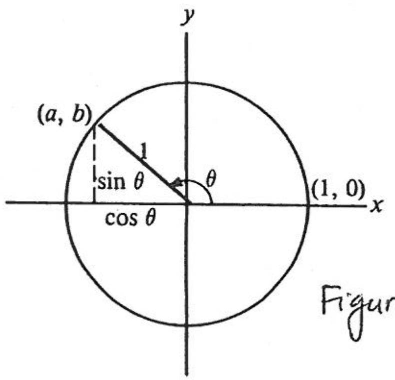


Figure 16.2

Figure 16.2

491

or

$$a = \sin \theta \quad \text{and} \quad b = \cos \theta$$

Since $a^2 + b^2 = 1$, it follows that

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{16.4}$$

Similar identities hold for the remaining trigonometric functions. In Figure 16.3, the y-coordinate of P is equal to $\tan \theta$ and the length of PO is numerically equal to $\sec \theta$. By the Pythagorean theorem

$$1 + \tan^2 \theta = \sec^2 \theta \tag{16.5}$$

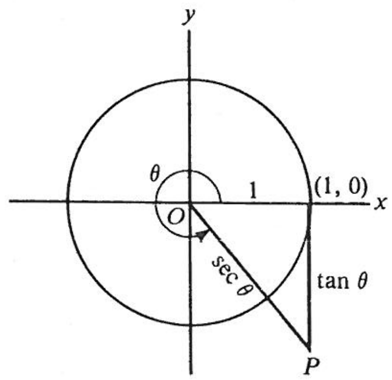


Figure 16.3

Of particular interest in Figure 16.3 is that the names *tangent* and *secant* suggest themselves quite naturally. (Recall that a secant line intersects a circle at two points and a tangent line at one point.)

The derivation of the remaining identity is similar and will be left as an exercise:

$$1 + \cot^2 \theta = \csc^2 \theta \tag{16.6}$$

Example #2

2 Use the fundamental identities to simplify the given expressions. Write the expressions in terms of sines and cosines, if necessary.

a. $1 - \sin^2 \alpha$ **b.** $\dfrac{\cot \beta}{\csc \beta}$ **c.** $\csc x(1 - \cos^2 x)$

d. $\dfrac{\sec^2 \theta - \tan^2 \theta}{\cot \theta}$

Solution.

a. $1 - \sin^2 \alpha = \cos^2 \alpha$

by identity (16.4).

b. $\dfrac{\cot\beta}{\csc\beta} = \dfrac{\cos\beta}{\sin\beta} \cdot \dfrac{\sin\beta}{1} = \cos\beta$

by identities (16.3) and (16.1).

c. $\csc x(1 - \cos^2 x) = \csc x \sin^2 x$

by identity (16.4), and

$$\csc x \sin^2 x = \frac{1}{\sin x} \cdot \frac{\sin^2 x}{1} = \sin x \quad \left(\text{replacing } \csc x \text{ by } \frac{1}{\sin x}\right)$$

d. $\dfrac{\sec^2\theta - \tan^2\theta}{\cot\theta} = \dfrac{1}{\cot\theta}$

by identity (16.5), and

$$\frac{1}{\cot\theta} = \tan\theta$$

It follows that

$$\frac{\sec^2\theta - \tan^2\theta}{\cot\theta} = \tan\theta$$

For easy reference, the basic identities are given in the box below.

Fundamental trigonometric identities:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Alternate forms of the identities can be obtained by rearranging the terms. The following sets of identities are equivalent:

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 - \cos^2\theta = \sin^2\theta, \quad 1 - \sin^2\theta = \cos^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta, \quad \sec^2\theta - \tan^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta, \quad \csc^2\theta - \cot^2\theta = 1$$

Exercises / Section 16.1

In Exercises 1–14, change each expression to an equivalent expression involving sines and cosines. Simplify if possible.

1. $\cot\beta$ **2.** $\sin\alpha\cot\alpha$ **3.** $\cos\theta\tan\theta$

4. $\sec\theta$ **5.** $\cos\gamma\sec\gamma$ **6.** $1 - \tan\beta\cot\beta$

7. $\cot x + \dfrac{1}{\sin x}$ **8.** $\dfrac{1}{\sec\omega} + \cos\omega$ **9.** $\tan\theta\cos\theta\cot\theta$

10. $\cot^2 t\sin^2 t$ **11.** $\cot^2 s(1 + \tan^2 s)$ **12.** $\tan^2 x - \sec^2 x$

13. $1 - \sec^2\theta$ **14.** $\dfrac{1 + \tan^2 x}{\sec^2 x}$

In Exercises 15–31, use the fundamental identities to simplify each given expression. Convert to an expression involving sines and cosines if necessary.

15. $\dfrac{\cos^2 x + \sin^2 x}{\sin x}$ **16.** $\dfrac{\tan\beta}{\sec\beta}$ **17.** $\dfrac{1}{1 - \cos^2\theta}$

18. $\sin^2\gamma(1 + \tan^2\gamma)$ **19.** $\sin\theta(1 + \cot^2\theta)$ **20.** $\csc\beta(1 - \cos^2\beta)$

21. $\dfrac{\tan\theta\csc\theta}{\sec\theta}$ **22.** $\dfrac{\sec y}{\csc y}$ **23.** $\dfrac{\csc t}{\sec t}$

24. $\sin^2 x\sec^2 x$ **25.** $\dfrac{\sin\theta}{\tan\theta}$ **26.** $\dfrac{\tan^2\omega - \sec^2\omega}{\sec\omega}$

27. $\csc^2\alpha - \cot^2\alpha$ **28.** $\tan^2 y - \dfrac{\sec^2 y}{\csc^2 y}$ **29.** $\dfrac{1 + \tan^2 x}{\cos x}$

30. $\tan\theta\sin^2\theta + \tan\theta\cos^2\theta$ **31.** $\cot\theta\cos^2\theta + \cot\theta\sin^2\theta$

16.2 Proving Identities

In this section we shall use the fundamental identities to verify more complicated identities. Writing trigonometric expressions in alternate form is a skill required in more advanced work in mathematics.

In one respect, proving identities is similar to solving word problems: Each identity has its own features and must be verified in its own way. A facility for verifying identities can be developed only through practice. Although no general method can be given, the guidelines that follow will help you decide what approach to take.

Guidelines for proving identities

1. Memorize the fundamental identities and use them whenever possible.
2. Start with the more complicated side of the identity and try to reduce it to the simpler side.
3. Perform any algebraic operation indicated. For example, it may help to multiply out the terms in an expression, to factor an expression, to add fractions, and so on.
4. If everything else fails, try expressing all functions in terms of sines and cosines.
5. When working on one side of the identity, always keep in mind the other side for possible clues on how to proceed.

Caution. When proving an identity, the given relationship may not be treated as an equation—establishing equality is the very purpose of the verification. For this reason, it is not permissible to transpose terms, to multiply both sides by an expression, and so on. Instead, work on one side of the identity until the other side is obtained.

To see how to use the guidelines, consider the identity

$$\cot\theta\sin\theta = \cos\theta$$

In accordance with Guideline 2, start with the left side of the equation, since it is more complicated. While no algebraic operations come to mind (Guideline 3), the identity $\cot\theta = \cos\theta/\sin\theta$ may be useful (Guideline 1). This identity converts the left side to sines and cosines (Guideline 4), so that

$$\cot\theta\sin\theta = \frac{\cos\theta}{\sin\theta}\sin\theta = \cos\theta$$

The identity is thereby verified.

The examples below further illustrate these guidelines.

Example #1

Prove the identity

$$\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$$

Solution. The left side, which is the more complicated side (Guideline 2), is factorable as a difference of two squares (Guideline 3). Thus

$$\begin{aligned} \cos^4\theta - \sin^4\theta &= (\cos^2\theta)^2 - (\sin^2\theta)^2 \\ &= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) && \text{difference of two squares} \\ &= (\cos^2\theta - \sin^2\theta)(1) && \text{replacing } \cos^2\theta + \sin^2\theta \text{ by } 1 \\ &= \cos^2\theta - \sin^2\theta \end{aligned}$$

which is the right side. Note that Guideline 1 was also used.

Example #2

ple 2 Prove the identity

$$\sin^3\beta\cos\beta + \cos^3\beta\sin\beta = \sin\beta\cos\beta$$

Solution. The left side, which is more complicated, contains a common factor $\sin\beta\cos\beta$ (Guideline 3). Thus

$$\begin{aligned} &\sin^3\beta\cos\beta + \cos^3\beta\sin\beta \\ &\quad = \sin\beta\cos\beta(\sin^2\beta + \cos^2\beta) && \text{common factor} \\ &\quad = \sin\beta\cos\beta(1) && \text{from Guideline 1} \\ &\quad = \sin\beta\cos\beta \end{aligned}$$

Example #3

ple 3 Show that

$$\left(\csc\gamma + \frac{\cos^2\gamma}{\sin^3\gamma}\right)\sin\gamma = \csc^2\gamma$$

Solution. We multiply the expression on the left side (Guideline 3) to obtain

$$\begin{aligned} \left(\csc\gamma + \frac{\cos^2\gamma}{\sin^3\gamma}\right)\sin\gamma &= \csc\gamma\sin\gamma + \frac{\cos^2\gamma}{\sin^2\gamma} \\ &= \frac{1}{\sin\gamma}\sin\gamma + \left(\frac{\cos\gamma}{\sin\gamma}\right)^2 && \csc\gamma = 1/\sin\gamma \\ &= 1 + \cot^2\gamma && \cos\gamma/\sin\gamma = \cot\gamma \\ &= \csc^2\gamma && \text{identity (16.6)} \end{aligned}$$

Example #4

ple 4 Show that

$$\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = 2\sec^2 x$$

Solution. The left side is more complicated and contains two fractions, which should be combined:

$$\begin{aligned} &\frac{\mathbf{1+\sin x}}{(1-\sin x)(\mathbf{1+\sin x})} + \frac{\mathbf{1-\sin x}}{(1+\sin x)(\mathbf{1-\sin x})} \\ &\quad = \frac{(1+\sin x)+(1-\sin x)}{(1-\sin x)(1+\sin x)} \\ &\quad = \frac{2}{1-\sin^2 x} \\ &\quad = \frac{2}{\cos^2 x} = 2\sec^2 x && \text{since } \sin^2 x + \cos^2 x = 1 \\ &\quad = 2\sec^2 x && \text{replacing } 1/\cos x \text{ by } \sec x \end{aligned}$$

replacing $1/\cos x$ by $\sec x$

Example #5

496

5 Show that

$$\tan\theta + \cot\theta = \sec\theta\csc\theta$$

Solution. Since the left side contains two terms, it must be considered the more complicated side. Since no algebraic operations and no fundamental identities come to mind, we write both terms as expressions involving sines and cosines (Guideline 4). Thus

$$\begin{aligned}\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin\theta\,\sin\theta}{\cos\theta\,\sin\theta} + \frac{\cos\theta\,\cos\theta}{\sin\theta\,\cos\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta}\\ &= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} = \sec\theta\csc\theta\end{aligned}$$

Example #6

6 Show that

$$\frac{\sin^2\theta}{1+\cos\theta} = 1 - \cos\theta$$

Solution. Apart from the fact that the left side is more complicated than the right side, none of the guidelines we have used so far seem to apply. Thinking of Guideline 5, see if the right side offers a clue. It does suggest one possibility: Multiply the numerator and denominator of the left side by $1 - \cos\theta$, not only to introduce this expression but also to reduce the denominator. Thus

$$\begin{aligned}\frac{\sin^2\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} &= \frac{\sin^2\theta(1-\cos\theta)}{1-\cos^2\theta}\\ &= \frac{\cancel{\sin^2\theta}(1-\cos\theta)}{\cancel{\sin^2\theta}} = 1-\cos\theta\end{aligned}$$

Guidelines 1 and 3 could actually be used to prove this identity: since $\sin^2\theta + \cos^2\theta = 1$, the numerator of the left side can be written $\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta)$, so that the factor $1+\cos\theta$ cancels. However, the above technique using Guideline 5 is needed in Exercises 32 and 33.

Example #7

7 Show that

$$(\sec\beta - \tan\beta)^2 = \frac{1-\sin\beta}{1+\sin\beta}$$

497

Solution. The most promising approach is to multiply out the expression on the left side (Guideline 3):

$$\begin{aligned}&(\sec\beta - \tan\beta)^2\\ &\quad= \sec^2\beta - 2\sec\beta\tan\beta + \tan^2\beta && (a-b)^2 = a^2 - 2ab + b^2\\ &\quad= \frac{1}{\cos^2\beta} - \frac{2\sin\beta}{\cos^2\beta} + \frac{\sin^2\beta}{\cos^2\beta} && \text{converting to sines and cosines}\\ &\quad= \frac{1 - 2\sin\beta + \sin^2\beta}{\cos^2\beta} && \text{combining fractions}\\ &\quad= \frac{(1-\sin\beta)^2}{1-\sin^2\beta} && \text{factoring and identity (16.4)}\\ &\quad= \frac{(1-\sin\beta)^{\cancel{2}}}{\cancel{(1-\sin\beta)}(1+\sin\beta)} && \text{difference of two squares}\\ &\quad= \frac{1-\sin\beta}{1+\sin\beta}\end{aligned}$$

Example 8 The current in a certain circuit as a function of time is given by

$$i = \sqrt{0.04\cos^2\omega t - 0.04 + 2.0\sin^2\omega t}$$

Simplify this expression.

Solution.

$$\begin{aligned}i &= \sqrt{0.04\cos^2\omega t - 0.04 + 2.0\sin^2\omega t}\\ &= \sqrt{-0.04(1-\cos^2\omega t) + 2.0\sin^2\omega t} && \text{common factor } -0.04\\ &= \sqrt{-0.04\sin^2\omega t + 2.0\sin^2\omega t} && \text{replacing } 1-\cos^2\omega t \text{ by } \sin^2\omega t\\ &= \sqrt{1.96\sin^2\omega t}\\ &= 1.4\sin\omega t\end{aligned}$$

Exercises / Section 16.2

In Exercises 1–40, prove the given identities.

1. $\cot\theta\sin\theta = \cos\theta$
2. $\sin\theta\cot\theta\sec\theta = 1$
3. $\tan\theta\csc\theta = \sec\theta$
4. $\cos\theta + \sin\theta\tan\theta = \sec\theta$
5. $\dfrac{\cos^2\beta}{\sin\beta} + \sin\beta = \csc\beta$
6. $\sin^3 x + \sin x\cos^2 x = \sin x$
7. $\dfrac{1-\cos^2\gamma}{\cos^2\gamma} = \tan^2\gamma$
8. $\sin^2 y + \tan^2 y + \cos^2 y = \sec^2 y$
9. $\dfrac{1+\tan^2\omega}{1+\cot^2\omega} = \tan^2\omega$
10. $\tan\theta + \cot\theta = \dfrac{\tan\theta}{\sin^2\theta}$

11. $(1 + \tan^2 x)\cos^2 x = 1$

12. $\dfrac{\csc\theta}{\tan\theta + \cot\theta} = \cos\theta$

13. $\dfrac{1}{\cot\gamma + \tan\gamma} = \sin\gamma\cos\gamma$

14. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$

15. $\dfrac{\sin\beta + \tan\beta}{1 + \cos\beta} = \tan\beta$

16. $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{\sin\theta}{1 - \cos\theta} = 2\csc\theta$

17. $\dfrac{\sin\theta}{1 - \cos\theta} - \dfrac{1 - \cos\theta}{\sin\theta} = 2\cot\theta$

18. $\sec^2 x + \csc^2 x = \sec^2 x\csc^2 x$

19. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$

20. $\tan^4\alpha + \tan^2\alpha = \sin^2\alpha\sec^4\alpha$

21. $\dfrac{1 + \sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1 + \sin\theta} = 2\sec\theta$

22. $\dfrac{\cos\theta}{\tan\theta + \sec\theta} + \dfrac{\cos\theta}{\tan\theta - \sec\theta} = -2$ $= -2\sin\theta$

23. $2\csc x - \cot x\cos x = \sin x + \csc x$

24. $(1 - \cos\beta)(1 + \cos\beta) = \dfrac{1}{1 + \cot^2\beta}$

25. $\dfrac{1 - \tan\gamma}{1 + \tan\gamma} = \dfrac{\cot\gamma - 1}{\cot\gamma + 1}$

26. $\dfrac{\cos\omega}{\cos\omega - \sin\omega} = \dfrac{1}{1 - \tan\omega}$

27. $\dfrac{\tan\theta}{\csc\theta - \cot\theta} - \dfrac{\sin\theta}{\csc\theta + \cot\theta} = \sec\theta + \cos\theta$

28. $\dfrac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = 1 - 2\sin^2\alpha$

29. $\dfrac{1 + \tan^2\theta}{\csc^2\theta} = \tan^2\theta$

30. $\dfrac{\cot\theta + \tan\theta}{\sec\theta} = \csc\theta$

31. $\dfrac{1 + \cos\gamma - \sin^2\gamma}{\sin\gamma(1 + \cos\gamma)} = \cot\gamma$

32. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$ (See Example (See Example 6.)

33. $\sec\theta + \tan\theta = \dfrac{\cos\theta}{1 - \sin\theta}$ (See Example 6.)

34. $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

35. $\cos^4 x - \sin^4 x = 2\cos^2 x - 1$

36. $\dfrac{\sin^4\alpha - \cos^4\alpha}{\sin^2\alpha - \cos^2\alpha} = 1$

37. $\dfrac{\tan\theta + \sec^2\theta - 1}{\tan\theta - \sec^2\theta + 1} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

38. $\left(\dfrac{\sec\beta + \csc\beta}{1 + \tan\beta}\right)^2 = \csc^2\beta$

39. $\dfrac{\tan x - \sin x}{\sin^3 x} = \dfrac{\sec x}{1 + \cos x}$

40. $\dfrac{\tan\alpha + \cot\alpha}{\cos^2\alpha} - \sin\alpha\sec^3\alpha = \sec\alpha$ $= \sec\alpha\csc\alpha$

41. An object traveling along a circle of radius a (in feet) at an angular velocity of $\omega/2\pi$ rev/sec has linear velocity velocity

$$\sqrt{(a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2}$$

Simplify this expression. (See Example 8.)

42. Suppose a particle moves along a line with velocity $v = 2\cos t + 2\sin t$ (in meters per second). m/s (in meter per second). The methods of calculus show that the acceleration is given by $a = -2\sin t + 2\cos t$ (in meters per second pe (in meter per second (~~...~~ per second). m/s2 Show that $a = 0$ whenever $\tan t = 1$.

43. Neglecting air resistance, the equation of the path of a missile projected at velocity v_0 at an angle, * * angle α with the horizontal is horizontal is

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

Write as a single fraction.

44. In the study of the motion of a pendulum, the expression $\sqrt{1 - k^2\sin^2\theta}$ arises. Show that

$$1 - k^2\sin^2\theta = k^2\cos^2\theta + 1 - k^2$$

45. A beam of length L (in inches) weighing w (pounds per inch) and clamped at the left end is subjected subjected to a a compressive force P at the free end. The minimum deflection is given by

$$y_{\min} = \frac{wEI}{P^2}\left(1 - \frac{1}{2}\theta^2 - \sec\theta + \theta\tan\theta\right)$$

where $\theta = L\sqrt{P/EI}$. Show that

$$y_{\min} = \frac{wEI}{P^2}\,\frac{2\cos\theta - \theta^2\cos\theta - 2 + 2\theta\sin\theta}{2\cos\theta}$$

46. If a vertical plate is partly submerged in a liquid, then the capillarity will cause the liquid to rise on the a rise on the plate to a height of

$$h = c\sqrt{\frac{1 - \sin\theta}{2}}$$

where θ is the contact angle between the liquid and the plate and c a constant that depends on the su on the surface tension and specific gravity of the liquid. Show that

$$h = \frac{c\cos\theta}{\sqrt{2(1 + \sin\theta)}}$$

47. In some problems on the motion of a pendulum, the expression $1/\sqrt{1 - \cos x}$ arises. Show that Show that this this expression is equivalent to $\sqrt{1 + \cos x}/\sin x$.

16.3 The Sum and Difference Formulas

It is sometimes useful to write a trigonometric function of the sum of of two angles in terms of trigonometric functions of each angle. For exan For example, $\sin(A + B)$ can be expressed in terms of $\sin A$, $\cos A$, $\sin B$, and cos and $\cos B$. To do so, let A and B be two acute angles. Then $A + B$ may be e be either acute (Figure 16.4) or obtuse (Figure 16.5). In both figures, PQ and M and MN are

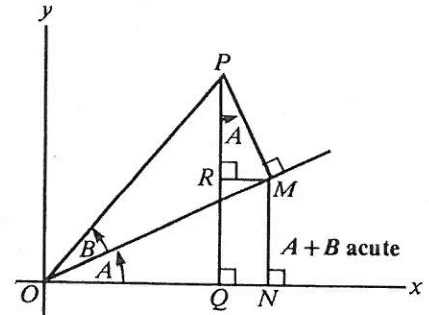


Figure 16.4

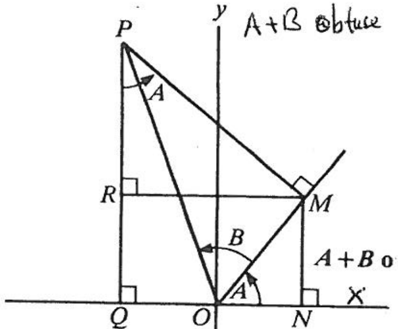


Figure 16.5

500

perpendicular to the x-axis, PM is perpendicular to OM, and MR is perpendicular to PQ. Note that $\angle MPQ = \angle A$, since the two angles have their sides perpendicular, right side to right side and left side to left side.

In both figures we have

$$\sin (A + B) = \frac{PQ}{OP} = \frac{PR + RQ}{OP} = \frac{PR}{OP} + \frac{RQ}{OP} = \frac{PR}{OP} + \frac{MN}{OP}$$

The last two fractions do not define functions of either A or B. However, if we multiply numerator and denominator of the first fraction by PM and the second by OM, each of the resulting ratios is a function of A or B:

$$\frac{PR}{OP} \cdot \frac{PM}{PM} + \frac{MN}{OP} \cdot \frac{OM}{OM}$$
$$= \frac{PR}{PM} \cdot \frac{PM}{OP} + \frac{MN}{OM} \cdot \frac{OM}{OP} = \cos A \sin B + \sin A \cos B$$

or

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \qquad (16.7)$$

For the corresponding identity involving $\cos (A + B)$, we get

$$\cos (A + B) = \frac{OQ}{OP} = \frac{ON - QN}{OP} = \frac{ON}{OP} - \frac{QN}{OP} = \frac{ON}{OP} - \frac{RM}{OP}$$
$$= \frac{ON}{OM} \cdot \frac{OM}{OP} - \frac{RM}{PM} \cdot \frac{PM}{OP}$$

or

$$\cos (A + B) = \cos A \cos B - \sin A \sin B \qquad (16.8)$$

Since $\sin (-B) = -\sin B$ and $\cos (-B) = \cos B$, we also get

$$\sin (A - B) = \sin A \cos B - \cos A \sin B \qquad (16.9)$$

and

$$\cos (A - B) = \cos A \cos B + \sin A \sin B \qquad (16.10)$$

These four formulas can be written more compactly in the forms given below. (The combination "$\pm$" and "$\mp$" in formula (16.12) indicates that the terms have opposite signs.)

Sum and difference formulas:

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad (16.11)$$
$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad (16.12)$$

As noted earlier, these identities enable us to express a function of a sum of two angles in terms of functions of the angles themselves. To illustrate these identities, let us find the values of certain trigonometric functions without tables or calculators.

Example #1

1 Find the exact value of cos 75° by means of the sum and difference formulas.

Solution. Since 75° is not a special angle, cos 75° cannot be found from a diagram. However, 75° = 30° + 45°, a sum of two special angles having known function values. So it follows from identity (16.8) that

$$\cos 75^\circ = \cos (30^\circ + 45^\circ)$$
$$= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example #2

2 Find the exact value of

$$\sin 25^\circ \cos 20^\circ + \cos 25^\circ \sin 20^\circ$$

Solution. By identity (16.7)

$$\sin 25^\circ \cos 20^\circ + \cos 25^\circ \sin 20^\circ = \sin (25^\circ + 20^\circ)$$
$$= \sin 45^\circ = \frac{\sqrt{2}}{2}$$

The sum and difference identities are sometimes used to combine certain expressions, as shown in the next example.

Example #3

3 Combine

$$\sin 3x \cos 2x - \cos 3x \sin 2x$$

into a single term.

Solution. By identity (16.9) we get directly

$$\sin (3x - 2x) = \sin x$$

between the two functions can be obtained readily by means of the sum and difference identities.

In our study of the graphs of sinusoidal functions, we found that the graph of $y = \sin (x \pm c)$ can be obtained from the graph of $y = \sin x$ by translating the latter graph by c units. If c is a special angle, the relationship

Example #4 502

4 Simplify $\sin(x + \pi/2)$.

Solution. By identity (16.7)

$$\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}$$
$$= (\sin x)(0) + (\cos x)(1)$$
$$= \cos x$$

Example # 5

5 Simplify $\cos(2x - \pi)$.

Solution. By identity (16.10)

$$\cos(2x - \pi) = \cos 2x \cos \pi + \sin 2x \sin \pi$$
$$= (\cos 2x)(-1) + (\sin 2x)(0)$$
$$= -\cos 2x$$

The sum and difference identities for the tangent occur less frequently and are listed mainly for completeness.

By identities (16.7) and (16.8),

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing numerator and denominator by $\cos A \cos B$, we get

$$\tan(A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

or

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \rightarrow (16.13) \qquad (16.13)$$

Similarly,

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \rightarrow (16.14) \qquad (16.14)$$

Example #6

6 Simplify $\tan(2x + \pi/4)$.

Solution. By identity (16.13)

$$\tan\left(2x + \frac{\pi}{4}\right) = \frac{\tan 2x + \tan\frac{\pi}{4}}{1 - \tan 2x \tan\frac{\pi}{4}} = \frac{1 + \tan 2x}{1 - \tan 2x}$$

Remark. The identities $\tan\theta = \sin\theta/\cos\theta$ and $\cot\theta = \cos\theta/\sin\theta$ were known to the Arabs. The Hindus knew the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, while the formula for $\sin(A + B)$ was discovered by the Belgian mathematician Romanus (1561–1615).

Common error Writing

$$\sin(A + B) \quad \text{as} \quad \sin A + \sin B$$

Instead, $\sin(A + B)$ should be written as

$$\sin A \cos B + \cos A \sin B$$

Similarly,

$\cos(A + B)$ should not be written $\cos A + \cos B$.

Example 7 If $i = 3\sin(\omega t - \pi/2)$ is the current in a circuit and $e = 5\sin\omega t$ the voltage, find an expression for the power $P = ei$ as a function of time and simplify the result.

Solution. The power is given by

$$P = ei = (5\sin\omega t)\left[3\sin\left(\omega t - \frac{\pi}{2}\right)\right]$$
$$= (5\sin\omega t)\cdot 3\left[\left(\sin\omega t\cos\frac{\pi}{2} - \cos\omega t\sin\frac{\pi}{2}\right)\right]$$
$$= 15\sin\omega t(\sin\omega t\cdot 0 - \cos\omega t\cdot 1)$$
$$= -15\sin\omega t\cos\omega t$$

Exercises / Section 16.3

In Exercises 1–10, use the sum and difference identities to find each given value without using a table or a calculator. (See Examples 1 and 2.)

1. $\cos 15°$
2. $\sin 105°$
3. $\cos(-105°)$
4. $\sin 285°$
5. $\cos 16° \cos 29° - \sin 16° \sin 29°$
6. $\sin 50° \cos 10° + \cos 50° \sin 10°$
7. $\cos 55° \cos 10° + \sin 55° \sin 10°$
8. $\sin 76° \cos 16° - \cos 76° \sin 16°$
9. $\sin 39° \cos 6° + \cos 39° \sin 6°$
10. $\cos 18° \cos 12° - \sin 18° \sin 12°$

In Exercises 11–20, write each expression as a single term. (See Example 3.)

11. $\sin 4x \cos 2x - \cos 4x \sin 2x$
12. $\sin 6x \cos 3x - \sin 6x \cos 3x$
13. $\sin x \cos 2x + \cos x \sin 2x$
14. $\sin 3x \cos x + \cos 3x \sin x$
15. $\cos 3x \cos x + \sin 3x \sin x$
16. $\cos 5x \cos 3x + \sin 5x \sin 3x$
17. $\cos 5x \cos 4x - \sin 5x \sin 4x$
18. $\cos 2x \cos 3x - \sin 2x \sin 3x$
19. $\sin(x+y)\cos y - \cos(x+y)\sin y$
20. $\cos(2x-y)\cos(y-x) - \sin(2x-y)\sin(y-x)$

504

In
1 Exercises 21–38, write each expression as a function of x or $2x$. (See Examples 4 and 5.)

21 1. $\cos(x + 30^\circ)$ — 22. $\sin(x - 30^\circ)$

23 3. $\cos\left(2x + \frac{\pi}{2}\right)$ — 24. $\sin(x - \pi)$

25 5. $\cos(x + \pi)$ — 26. $\cos(x - \pi)$

27 7. $\sin(2x - 2\pi)$ — 28. $\cos\left(x - \frac{\pi}{2}\right)$

29 9. $\sin\left(2x + \frac{\pi}{2}\right)$ — 30. $\cos\left(x - \frac{\pi}{3}\right)$

31 1. $\sin\left(x - \frac{\pi}{3}\right)$ — 32. $\sin\left(x + \frac{\pi}{6}\right)$

33 3. $\cos\left(x + \frac{\pi}{6}\right)$ — 34. $\cos\left(2x - \frac{\pi}{6}\right)$

35 5. $\sin\left(x + \frac{\pi}{4}\right)$ — 36. $\cos\left(x - \frac{\pi}{4}\right)$

37 7. $\tan\left(x + \frac{\pi}{4}\right)$ — 38. $\tan\left(2x - \frac{\pi}{4}\right)$

In Exercises 39–48, prove the given identities.

39 9. $\sin\left(\theta - \frac{\pi}{4}\right) = -\cos\left(\theta + \frac{\pi}{4}\right)$ — 40. $\sin 2\theta = \sin(\theta + \theta) = 2\sin\theta\cos\theta$

41 1. $\cos 2\theta = \cos(\theta + \theta) = \cos^2\theta - \sin^2\theta$ — 42. $\sin(x + y) + \sin(x - y) = 2\sin x\cos y$

43 3. $\sin(x + y) - \sin(x - y) = 2\cos x\sin y$ — 44. $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{3}\right) = \cos x$

5. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$ — 46. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

7. $\tan(x - y) - \tan(y - x) = \dfrac{2(\tan x - \tan y)}{1 + \tan x\tan y}$ — 48. $\dfrac{\tan(x + y) - \tan y}{1 + \tan(x + y)\tan y} = \tan x$

In Exercises 49–51, we shall obtain a few other standard identities. The first four are known as the produc*t-to-*sum formulas and the last four as the *sum-to-product formulas*.

the Product-to-

49. Derive the following **product-to-sum formulas** by adding and subtracting formulas (16.7) and (16.9):

$$\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)] \rightarrow (16.15) \qquad (16$$

$$\cos A\sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)] \rightarrow (16.16) \qquad (16$$

50. Derive the following **product-to-sum formulas** by adding and subtracting formulas (16.8) and (16.10):

$$\cos A\cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)] \rightarrow (16.17) \qquad (16$$

$$\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)] \rightarrow (16.18) \qquad (16$$

51. The **sum-to-product formulas** can be obtained from identities (16.15) through (16.18) by letting $A + B = x$ and $A - B = y$. Thus

$$A = \frac{x + y}{2} \quad\text{and}\quad B = \frac{x - y}{2}$$

505

By substituting show that

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right) \rightarrow (16.19)$$

$$\sin x - \sin y = 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right) \rightarrow (16.20)$$

$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right) \rightarrow (16.21)$$

$$\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right) \rightarrow (16.22)$$

52. Prove the following identity occurring in the study of alpha particle scattering:

$$\sin\frac{1}{2}(\pi - \theta) = \cos\frac{\theta}{2}$$

* of two waves of equal

53. The equation of a standing wave may be obtained by adding the displacements of two* amplitude and wavelength but traveling in opposite directions. Given that at some partic

Some particular instant

$$y_1 = A\sin 2\left(x - \frac{\pi}{4}\right)$$

is the equation of a wave traveling in the positive x-direction and

$$y_2 = A\sin 2\left(x + \frac{\pi}{4}\right)$$

, show that $y_1 + y_2 = 0$, that is,

is the equation of the corresponding wave traveling in the negative x-direction, show that y_1 + the waves cancel each other at the instant in question.

54. The current in a certain electric circuit is given by

$$i = A\sin\left(\omega t - \frac{\pi}{4}\right) + B\cos\left(\omega t + \frac{\pi}{4}\right)$$

Simplify this expression.

* to the system can

55. If a force $F_0\cos\omega t$ is applied to a weight oscillating on a spring, then the energy supplied to* can be written in the form

$$E = A\omega F_0\cos(\omega t - \gamma)\cos\omega t$$

Show that

$$E = A\omega F_0(\cos^2\omega t\cos\gamma + \cos\omega t\sin\omega t\sin\gamma)$$

* of refraction within

56. A light ray strikes a glass plate of thickness a at an angle of incidence ϕ. If ϕ' is the angle of r* the glass, then the lateral displacement D of the emerging beam is given by

$$D = \frac{a\sin(\phi - \phi')}{\cos\phi'}$$

Show that $D = a(\sin\phi - \cos\phi\tan\phi')$.

57. Given that

506

is the equation of a wave traveling in the positive x-direction and

$$y_2 = A \cos 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

is the equation of the corresponding wave traveling in the negative x-direction, find $y = y_1 + y_2$, the equation of a standing wave. (Refer to Exercise 53.)

58. In the development of the theory of Fourier series (see Section 8.5) the product

$$\cos \frac{m\pi t}{p} \cos \frac{n\pi t}{p}$$

has to be written as a sum. Carry out this operation.

59. Show that the product of two complex numbers r_1 cis θ_1 and r_2 cis θ_2 is

$$r_1r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + j(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$$

Simplify this expression to obtain the standard form r_1r_2 cis $(\theta_1 + \theta_2)$

16.4 Double-Angle Formulas

Some special cases of the sum and difference formulas occur often enough to warrant separate classification. One such classification includes the **double-angle formulas.**

Let $A = B$ in the identity

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

Then

$$\sin (A + A) = \sin A \cos A + \cos A \sin A$$

or

$$\sin 2A = 2 \sin A \cos A$$

If $A = B$ in the identity

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

then

$$\cos (A + A) = \cos A \cos A - \sin A \sin A$$

or

$$\cos 2A = \cos^2 A - \sin^2 A$$

If we let $\cos^2 A = 1 - \sin^2 A$, then $\cos 2A = 1 - \sin^2 A - \sin^2 A = 1 - 2 \sin^2 A$. Similarly, $\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1$.

Double-angle formulas:

$$\sin 2A = 2 \sin A \cos A \quad (16.23)$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad (16.24)$$

$$= 2 \cos^2 A - 1 \quad (16.25)$$

$$= 1 - 2 \sin^2 A \quad (16.26)$$

The double-angle formulas can be used to express the sine or cosine of twice an angle in terms of functions of a single angle. In particular, if $\sin \theta$ or $\cos \theta$ are known, we can use the identities to find $\sin 2\theta$ and $\cos 2\theta$. Consider the examples below.

Example #1

1 Use the double-angle formulas to find $\sin 2\theta$ and $\cos 2\theta$, given that $\sin \theta = \frac{5}{13}$, θ in quadrant II.

Solution. Since $\sin \theta = \frac{5}{13}$, θ in quadrant II, we obtain $\cos \theta = -\frac{12}{13}$ (Figure 16.6). Thus

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = -\frac{120}{169}$$

and

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2$$

$$= \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$$

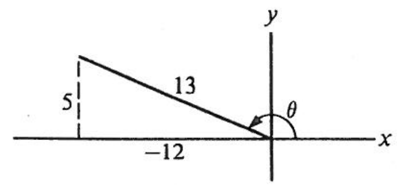


Figure 16.6

Example #2

2 Find $\sin 2\theta$ and $\cos 2\theta$, given that $\cos \theta = -\frac{2}{5}$, θ in quadrant III.

Solution. From the diagram (Figure 16.7 on page 508), we obtain $\sin \theta = -\sqrt{21}/5$. Hence

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2\left(-\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right) = \frac{4\sqrt{21}}{25}$$

508

and

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{4}{25} - \frac{21}{25} = -\frac{17}{25}$$

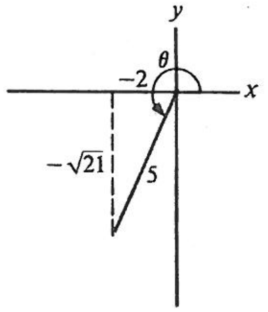


Figure 16.7

The double-angle formulas are also applicable to trigonometric functions of multiple angles. For example, from the identity $\sin 2A = 2 \sin A \cos A$, it follows that

$$\sin 16\theta = 2 \sin 8\theta \cos 8\theta$$

Similarly, since $\cos 2A = 2\cos^2 A - 1$, we have

$$2\cos^2 6x - 1 = \cos 12x$$

Example #3

3 Change $\cos^2 4x - \sin^2 4x$ to a single term.

Solution. By formula (16.24), $\cos 2A = \cos^2 A - \sin^2 A$, we get

$$\cos^2 4x - \sin^2 4x = \cos 8x \qquad A = 4x \text{ and } 2A = 8x$$

Example #4

4 Prove the identity

$$\cos^4 2\theta - \sin^4 2\theta = \cos 4\theta$$

Solution. Factoring the left side, we get

$$\begin{aligned}\cos^4 2\theta - \sin^4 2\theta &= (\cos^2\theta)^2 - (\sin^2\theta)^2 \\ &= (\cos^2 2\theta - \sin^2 2\theta)(\cos^2 2\theta + \sin^2 2\theta) \\ &= \cos^2 2\theta - \sin^2 2\theta = \cos 4\theta \qquad \cos^2 2\theta + \sin^2 2\theta = 1\end{aligned}$$

by formula (16.24).

509

Common error Equating $\sin 2A$ with $2 \sin A$ and $\cos 2A$ with $2 \cos A$. As we have seen,

$$\sin 2A = 2 \sin A \cos A$$

and

$$\cos 2A = \cos^2 A - \sin^2 A$$

Example 5 The range R of a projectile fired with velocity v at an angle θ with the ground is given by

$$R = \frac{2v^2}{g} \sin\theta \cos\theta$$

Write R as a single trigonometric function of θ.

Solution.

$$\begin{aligned}R &= \frac{2v^2}{g} \sin\theta\cos\theta \\ &= \frac{v^2}{g}(2\sin\theta\cos\theta) \\ &= \frac{v^2}{g}\sin 2\theta\end{aligned}$$

by the double-angle formula (16.23).

Exercises / Section 16.4

1. Find $\sin 2\theta$, given that $\sin\theta = \frac{3}{5}$, θ in quadrant I.
2. Find $\sin 2\theta$, given that $\sin\theta = \frac{4}{5}$, θ in quadrant II.
3. Find $\cos 2\theta$, given that $\sin\theta = -\frac{3}{5}$, θ in quadrant III.
4. Find $\cos 2\theta$, given that $\cos\theta = \frac{5}{13}$, θ in quadrant I.
5. Find $\sin 2\theta$, given that $\cos\theta = -\frac{12}{13}$, θ in quadrant II.
6. Find $\cos 2\theta$, given that $\sin\theta = -\frac{5}{13}$, θ in quadrant III.
7. Find $\sin 2\theta$, given that $\cos\theta = \frac{1}{2}$, θ in quadrant IV.
8. Find $\cos 2\theta$, given that $\sin\theta = -\frac{1}{2}$, θ in quadrant IV.
9. Find $\cos 2\theta$, given that $\sin\theta = \frac{2}{5}$, θ in quadrant II.
10. Find $\sin 2\theta$, given that $\sin\theta = \frac{1}{3}$, θ in quadrant I.
11. Find $\cos 2\theta$, given that $\cos\theta = -\frac{3}{7}$, θ in quadrant III.
12. Find $\sin 2\theta$, given that $\cos\theta = -\frac{2}{3}$, θ in quadrant II.

In Exercises 13–24, write each expression as a single trigonometric function. (See Examples 3 and 5.)

13. $\cos^2 3y - \sin^2 3y$
14. $\sin^2 x - \cos^2 x$
15. $2 \sin 3\theta \cos 3\theta$
16. $1 - 2\sin^2 5x$

17. $2 \cos^2 2\beta - 1$

18. $\sin 2x \cos 2x$

19. $1 - 2 \cos^2 4y$

20. $2 \sin^2 A - 1$

21. $\sin 4\omega \cos 4\omega$

22. $\sin 3\theta \cos 3\theta$

23. $4 \sin 2x \cos 2x$

24. $6 \sin 5x \cos 5x$

In Exercises 25–35, prove the given identities.

25. $\cos^4 x - \sin^4 x = \cos 2x$

26. $\sin 2\theta = \tan \theta \, (1 + \cos 2\theta)$

27. $1 - \cos 2\beta = \tan \beta \sin 2\beta$

28. $\sin 2\beta = \dfrac{2 \tan \beta}{1 + \tan^2\beta}$

29. $\dfrac{\cos 2\theta + \cos \theta + 1}{\sin 2\theta + \sin \theta} = \cot \theta$

30. $\sin 4x = 4 \sin x \cos x \cos 2x$

31. $\dfrac{\cos^2 \gamma + 1}{2 \cos^4 \gamma + \cos^2 \gamma - 1} = \sec 2\gamma$

32. $(\cos x + \sin x)^2 = 1 + \sin 2x$

33. $\dfrac{1 + \cos 2\omega}{\sin 2\omega} = \cot \omega$

34. $\dfrac{\cot^2 y - 1}{2 \cot y} = \cot 2y$

35. $\dfrac{\csc^2 \theta - 2}{\csc^2 \theta} = \cos 2\theta$

36. By letting $A = B$ in identity (16.13), show that

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

which is the *double-angle formula for the tangent.*

37. Suppose a particle is traveling along a line according to the equation $s = 4 \sin^2 t$, where s is measured in meters and t in seconds. Calculus shows that the velocity is given by $v = 8 \sin t \cos t$. Write v as a single trigonometric function of t.

38. Prove the following identity from the derivation of Rutherford's scattering formula:

$$2\pi r^2 \sin \theta = 4\pi r^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

39. An axle is placed through the center of a circular disk at an angle α. The magnitude T of the torque on the bearings holding the axle has the form $T = k\omega^2 \sin \alpha \cos \alpha$, where ω is the angular velocity. Show that

$$T = \frac{1}{2} k\omega^2 \sin 2\alpha$$

40. The equation of the path of a missile projected at velocity v at an angle θ with the ground is

$$y = x \tan \theta - \frac{gx^2}{2v^2 \cos^2 \theta}$$

Show that

$$y = \frac{x(v^2 \sin 2\theta - gx)}{2v^2 \cos^2 \theta}$$

16.5

Half-Angle Formulas

The identities in the previous section allow us to write a function of $2A$ in terms of functions of A. In this section we shall study the **half-angle formulas,** which enable us to express a function of $\frac{1}{2}A$ in terms of functions of A.

The half-angle formulas can be obtained from the double-angle formulas by properly rearranging the terms. If we start with

$$\cos 2x = 1 - 2 \sin^2 x$$

we get

$$2 \sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}$$

Letting $x = A/2$, we have

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$$

Similarly, from $\cos 2x = 2 \cos^2 x - 1$, we obtain

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$$

The algebraic sign depends on the quadrant in which the terminal side of $A/2$ lies.

Half-angle formulas:

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}} \qquad (16.27)$$

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \qquad (16.28)$$

The half-angle identities can be used to express the sine and cosine of a given half-angle in terms of the cosine of the angle, as shown in the first two examples.

Example#1

512

1 Use the appropriate half-angle formula to find the exact value of cos 165°.

Solution. By identity (16.28)

$$\cos 165° = \pm\sqrt{\frac{1+\cos[(2)(165°)]}{2}} = \pm\sqrt{\frac{1+\cos 330°}{2}}$$
$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}}$$
$$= \pm\sqrt{\frac{2+\sqrt{3}}{4}} = \pm\frac{\sqrt{2+\sqrt{3}}}{2}$$

Since 165° is in the second quadrant, cos 165° is negative. So

$$\cos 165° = -\frac{\sqrt{2+\sqrt{3}}}{2}$$

Example #2

2 Find $\cos \theta/2$, given that $\sin\theta = -\frac{12}{13}$, θ in quadrant IV.

Solution. If $\sin\theta = -\frac{12}{13}$, then $\cos\theta = \frac{5}{13}$ for θ in quadrant IV. Moreover, since $270° < \theta < 360°$, it follows that $135° < \theta/2 < 180°$. Thus $\cos\theta/2 < 0$ and

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{5}{13}}{2}} = -\sqrt{\frac{1+\frac{5}{13}}{2}\cdot\frac{13}{13}}$$
$$= -\sqrt{\frac{13+5}{26}} = -\sqrt{\frac{9}{13}}$$
$$= -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

Sometimes the half-angle formulas are used to simplify certain radical expressions.

Example#3

3 Simplify $\sqrt{6-6\cos 4\theta}$.

Solution. The expression resembles the right side of identity (16.27). To get the proper form, we need to remove 6 from the radical and obtain a 2 in the denominator. Thus

$$\sqrt{6-6\cos 4\theta} = \sqrt{6(1-\cos 4\theta)} = \sqrt{6}\sqrt{1-\cos 4\theta}$$
$$= \sqrt{6}\sqrt{\frac{2(1-\cos 4\theta)}{2}}$$

513

$$= \sqrt{6}\sqrt{2}\sqrt{\frac{1-\cos 4\theta}{2}} = \sqrt{12}\sin 2\theta$$
$$= 2\sqrt{3}\sin 2\theta$$

In the study of calculus, the forms of the half-angle formulas stated below are sometimes more useful.

$$\sin^2 A = \frac{1-\cos 2A}{2} \qquad (16.29)$$
$$\cos^2 A = \frac{1+\cos 2A}{2} \qquad (16.30)$$

Note that these formulas are really the double-angle identities (16.25) and (16.26) slightly rewritten.

Example #4

4 Write $4\sin^2 3x$ without the square.

Solution. By identity (16.29) with $A = 3x$,

$$4\sin^2 3x = 4\left(\frac{1-\cos 6x}{2}\right) = 2(1-\cos 6x)$$

Example #5

5 Write $6\cos^2 2x$ without the square.

Solution. By identity (16.30) with $A = 2x$,

$$6\cos^2 2x = 6\left(\frac{1+\cos 4x}{2}\right) = 3(1+\cos 4x)$$

Example #6

6 Prove the identity

$$2\cos^2\frac{\theta}{2} = \frac{\sin^2\theta}{1-\cos\theta}$$

Solution. Since $\sin^2\theta = 1-\cos^2\theta$,

$$\frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} \qquad \text{difference of two squares}$$
$$= 1+\cos\theta = \frac{2(1+\cos\theta)}{2}$$
$$= 2\cos^2\frac{\theta}{2} \qquad \text{by (16.30)}$$

514

Example 7 The motion of a planet or comet about the sun can be described by an equation of the form

$$r = \frac{c}{a - b\cos\theta}$$

*this equation represents

where a, b, and c are constants. (See Figure 16.8.) This equ[ation represents] a conic section. For example, the equation of the elliptic pa[th of Mercury is]

Path of Mercury's

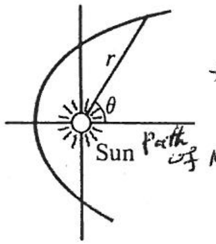


Figure 16.8

$$r = \frac{3.442 \times 10^7}{1 - 0.206\cos\theta}$$

where r is measured in miles.

Some comets follow a path that is nearly parabolic:

$$r = \frac{A}{1 - \cos\theta}$$

Show that

$$r = \frac{A}{2}\csc^2\frac{\theta}{2}$$

Solution. $$r = \frac{A}{1-\cos\theta} = \frac{A}{2}\cdot\frac{2}{1-\cos\theta} = \frac{A}{2}\cdot\frac{1}{\frac{1-\cos\theta}{2}}$$

$$= \frac{A}{2}\cdot\frac{1}{\frac{1-\cos\theta}{2}}$$

$$= \frac{A}{2}\cdot\frac{1}{\sin^2\frac{\theta}{2}} = \frac{A}{2}\csc^2\frac{\theta}{2}$$

Exercises / Section 16.5

half-angle formula

In Exercises 1–5, find the exact value of each trigonometric function by means of the half-an[gle formulas] of the ~~half~~ half-angle formulas.

1. sin 15° 2. cos 75° 3. cos 22.5°
4. cos 105° 5. sin 112.5°
6. Find $\sin(\theta/2)$, given that $\cos\theta = \frac{4}{5}$, θ in quadrant I.
7. Find $\sin(\theta/2)$, given that $\cos\theta = -\frac{24}{25}$, θ in quadrant III.
8. Find $\cos(\theta/2)$, given that $\sin\theta = \frac{3}{5}$, θ in quadrant II.
9. Find $\sin(\theta/2)$, given that $\cos\theta = \frac{5}{13}$, θ in quadrant IV.
10. Find $\cos(\theta/2)$, given that $\sin\theta = -\frac{7}{25}$, θ in quadrant IV.

In Exercises 11–16, simplify each given expression. (See Example 3.)

11. $\sqrt{\frac{1-\cos 4\theta}{2}}$ 12. $\sqrt{\frac{1+\cos 6\theta}{2}}$ 13. $\sqrt{1+\cos 6\theta}$
14. $\sqrt{4-4\cos 8\theta}$ 15. $\sqrt{5-5\cos 4\theta}$ 16. $\sqrt{6+6\cos 8\theta}$

6cos8θ

515

In Exercises 17–24, eliminate the exponent. (See Examples 4 and 5.)

17. $\sin^2 4x$ 18. $\sin^2 3x$ 19. $\cos^2 2x$
20. $\cos^2 3x$ 21. $2\sin^2 3x$ 22. $4\cos^2 4x$
23. $12\sin^2 x$ 24. $2\cos^2 x$

In Exercises 25–28, prove the given identities.

25. $\frac{\sin 2\theta}{2\sin\theta} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$ 26. $\cos^2\frac{x}{4} = \frac{1+\cos\frac{x}{2}}{2}$ 27. $\csc^2\theta = \frac{2}{1-\cos 2\theta}$

28. $2\cos\frac{\beta}{2} = (1+\cos\beta)\sec\frac{\beta}{2}$

29. Simplify the following expression from a problem in the study of the pendulum:

$$\frac{1}{\sqrt{1-\cos x}}$$

30. In the study of the motion of a pendulum, the expression

$$A = \sqrt{\frac{1-\cos\theta}{1-\cos\alpha}}$$

needs to be simplified. Show that

$$A = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$$

31. In determining the length of the path along which a particle will slide from a higher to a lower point in minimum time, the expression $\sqrt{2-2\cos\theta}$ needs to be simplified. Carry out this simplification.

32. A common exercise in calculus is determining the area under a curve. To find the area under one arch of the curve $y = \sin^2 x$, the equation must be written without the exponent. Rewrite this equation.

33. The index of refraction n of a prism with apex angle A whose minimum angle of refraction is δ is given by

$$n = \frac{\sin\frac{A+\delta}{2}}{\sin\frac{A}{2}}, \quad n \geq 0$$

Show that the expression is equivalent to

$$n = \sqrt{\frac{1 + \sin A\sin\delta - \cos A\cos\delta}{1-\cos A}}$$

16.6 Trigonometric Equations

So far we have concentrated only on identities, equations that are valid for all values of the variable. Now we shall turn to **conditional equations,** which are valid only for certain values of the angle.

For example, the equation

$$\sin \theta = 0$$

is not an identity, since equality holds only if

$$\theta = 0°, \pm 180°, \pm 360°, \quad \text{and so on}$$

To solve an equation containing a single trigonometric function, we solve the equation for this function and then determine the values of the angle for which equality holds. Consider the next example.

Example #1

1 Solve the equation $2 \cos x - 1 = 0$, $0 \le x < 2\pi$.

Solution. The first step is to solve the given equation for $\cos x$. Thus

$$2 \cos x - 1 = 0 \quad \text{given equation}$$
$$2 \cos x = 1 \quad \text{transposing } -1$$
$$\cos x = \frac{1}{2} \quad \text{dividing by 2}$$

The angles between 0 and 2π whose cosine is $\frac{1}{2}$ are

$$x = \frac{\pi}{3} \quad \text{and} \quad x = \frac{5\pi}{3}$$

Substituting into the given equation shows that the solutions check.

If an equation involves more than one function, we can often use the identities to convert it to an equation involving only one function, as shown in the next example.

Example #2

2 Solve the equation $\sec^2 x - 4 \tan^2 x = 0$, $0 \le x < 2\pi$.

Solution. Since the equation involves two different functions, no direct solution is possible. However, if we recall that $1 + \tan^2 x = \sec^2 x$, we can convert one of the functions. Thus

$$\sec^2 x - 4 \tan^2 x = 0$$
$$1 + \tan^2 x - 4 \tan^2 x = 0$$
$$1 - 3 \tan^2 x = 0$$
$$\tan^2 x = \frac{1}{3}$$
$$\sqrt{\tan^2 x} = \pm \sqrt{\frac{1}{3}}$$

$$\tan x = \pm \frac{\sqrt{1}}{\sqrt{3}}$$
$$\tan x = \pm \frac{1}{\sqrt{3}}$$

It follows that $x = 30°, 150°, 210°$, and $330°$. In radian measure

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

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ns

Some trigonometric equations are actually in quadratic form, as shown in the next example.

Example #3

3 Solve the equation $2 \csc^2 x + 3 \csc x - 2 = 0$, $0 \le x < 2\pi$.

Solution. Let $y = \csc x$. Then the equation becomes

$$2y^2 + 3y - 2 = 0$$
$$(2y - 1)(y + 2) = 0$$
$$y = -2, \frac{1}{2}$$

It follows from $y = \csc x$ that

$$\csc x = -2 \quad \text{and} \quad \csc x = \frac{1}{2}$$

Since a value of $\csc x$ cannot be less than unity, the equation $\csc x = \frac{1}{2}$ has no solution. From $\csc x = -2$, we obtain $x = 210°$ and $330°$. In radian measure

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Example #4

4 Solve the equation $\sin 2x = 0$, $0 \le x < 2\pi$.

Solution. Since $\sin 2x = 0$, we have $2x = 0°, 180°$, so that $x = 0°, 90°$. Because of the double angle, these are not the only solutions in the range $0 \le x < 360°$. From $2x = 360°, 540°$, we have $x = 180°, 270°$. In other words,

$$\sin 2x = 0$$

whenever

$$2x = 0, \pi, 2\pi, 3\pi$$

and

$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Note that the largest of the roots, $3\pi/2$, is still less than 2π, so that there are four solutions to the equation.

In most cases, equations involving functions of multiple angles should be solved by using an appropriate identity, as shown in the next example.

Example #5

5 Solve the equation $\cos 2x - \cos x = 0$, $0 \le x < 2\pi$.

Solution. Because of the double angle, $\cos 2x$ must first be changed to $2\cos^2 x - 1$ by one of the double-angle formulas for the cosine function (reminder: $\cos 2x \neq 2\cos x$). Then we obtain

$$\begin{aligned} \cos 2x - \cos x &= 0 \\ 2\cos^2 x - 1 - \cos x &= 0 \\ 2\cos^2 x - \cos x - 1 &= 0 \\ (2\cos x + 1)(\cos x - 1) &= 0 \qquad 2z^2 - z - 1 = (2z+1)(z-1) \\ \cos x &= -\frac{1}{2}, 1 \end{aligned}$$

It follows that $x = 120°, 240°$, and $0°$. In radians

$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Example #6

6 Use a calculator to solve the equation

$$2\sin 2x - 3\sin x = 0$$

to the nearest tenth of a degree ($0° \le x < 360°$).

Solution. By the double-angle formula for the sine function, $\sin 2x = 2\sin x\cos x$, we get

$$\begin{aligned} 2\sin 2x - 3\sin x &= 0 \qquad \text{given equation} \\ 2(2\sin x\cos x) - 3\sin x &= 0 \\ 4\sin x\cos x - 3\sin x &= 0 \\ \sin x(4\cos x - 3) &= 0 \qquad \text{common factor } \sin x \end{aligned}$$

$\sin x = 0 \qquad 4\cos x - 3 = 0$

$\cos x = \frac{3}{4}$

From $\sin x = 0$, we get $x = 0°, 180°$. Using a calculator, $\cos x = \frac{3}{4}$ yields $x = 41.4°, 318.6°$.

Example 7

The range R (in feet) of a projectile fired at an angle θ with the horizontal at velocity v (in feet per second) is given by

$$R = \frac{2v^2 \cos\theta \sin\theta}{g}$$

where $g = 32$ ft/sec². (See Figure 16.9.) If $v = 40$ ft/sec, determine the angle θ at which the projectile has to be aimed to hit an object 45 ft away.

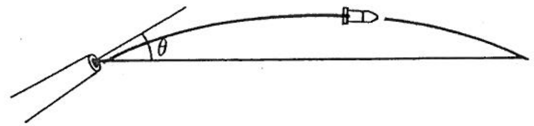


Figure 16.9

Solution. Substituting into the given equation, we get

$$\begin{aligned} \frac{2(40)^2\cos\theta\sin\theta}{32} &= 45 \\ \frac{(40)^2}{32}(2\sin\theta\cos\theta) &= 45 \\ 2\sin\theta\cos\theta &= \frac{(45)(32)}{(40)^2} \\ \sin 2\theta &= 0.9 \\ 2\theta &= 64°, 116° \\ \theta &= 32°, 58° \end{aligned}$$

So the projectile can be aimed at either 32° or 58° to land 45 ft away.

Exercises / Section 16.6

In Exercises 1–31, solve the given equations for x, $0 \le x < 2\pi$.

1. $2\sin x - 1 = 0$
2. $3\sin x + 3 = 0$
3. $4\tan x + 4 = 8$
4. $2\sec x + 4 = 0$
5. $\cos^2 x - 1 = 0$
6. $2\sin^2 x - 1 = 0$
7. $\sin^2 x - \sin x = 0$
8. $\tan x(\csc x + 1) = 0$
9. $4\cos^2 x - 3 = 0$
10. $(\sec x - 1)(\tan x + 1) = 0$
11. $(\cot x - 1)(\cos x + 1) = 0$
12. $(2\cos x - 1)(\csc x - 2) = 0$
13. $2\sin^2 x - \sin x - 1 = 0$
14. $2\sin^2 x - \sin x - 3 = 0$
15. $3\cos^2 x - 7\cos x + 4 = 0$
16. $2\sec^2 x + 3\sec x - 2 = 0$
17. $\sin x - \cos x = 0$
18. $\cot^2 x - \tan^2 x = 0$
19. $2\sin^2 x + \cos x + 1 = 0$
20. $2\tan^2 x - \sec^2 x = 0$
21. $\sin 2x = 1$

520

22. $\cos 2x = 1$
23. $\cos 2x = 0$
24. $\sin x + \sin 2x = 0$
25. $\sin^2 x + \cos 2x = 0$
26. $\sin 2x + \cos 2x = 0$
27. $\cos 2x - \cos x = 0$
28. $\cos 2x - \sin x = 0$
29. $\sin \frac{x}{2} = \cos \frac{x}{2}$
30. $1 - \sin x \cos x = 1$
31. $\sin x \cos x - \sin 2x = 0$

In Exercises 32–40, use a calculator to solve the given equations to the nearest tenth of a degree ($0 \le x < 360°$).

32. $3 \sin x \cos x - \cos x = 0$
33. $\tan^2 x - 2 = 0$
34. $2 \cos^2 x = 1 + \sin^2 x$
35. $2 \sin 2x + \cos x = 0$
36. $\sec^2 x - 2 \tan x - 4 = 0$
37. $2 \sin 2x = 3 \sin x$
38. $\cos^2 x - 2 \sin^2 x = 0$
39. $\csc^2 x - 3 \cot^2 x = 0$
40. $5 \sin^2 x + 8 \sin x - 4 = 0$

41. Certain problems in mechanics are simplified by rotating the coordinate axes. In the process, the following equation has to be solved:

$$2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) = 0$$

Solve this equation for θ ($0 \le \theta < 180°$), given that $A = B = 1$ and $C = 0$.

42. Suppose a projectile fired at a velocity of 80 ft/sec is to hit a target 100 ft away. At what angle with respect to the ground does the projectile have to be fired? (See Example 7.)

43. The current in a certain circuit is given by $i = e^{-5t}(\cos 4.0t - \sqrt{3} \sin 4.0t)$. Find the smallest positive value of t (in seconds) for which the current is zero.

44. For a certain mass oscillating on a spring, the vertical displacement is given by

$$x = 2.0 \cos 2t - 1.0 \sin 2t, \quad t \ge 0$$

where x is measured in centimeters and t in seconds. Find the smallest value of t for which the displacement is zero. (Set your calculator in the radian mode.)

45. Starting at $t = 0$, the current in a circuit is

$$i = 2 \sin^2 \omega t + 3 \sin \omega t$$

Find the smallest value of t (in seconds) for which $i = 2$ A.

16.7 Inverse Trigonometric Relations

We know from our study of equations that it is often desirable to solve a given equation for one of the variables in terms of the other variables. To solve a trigonometric equation $y = f(x)$ for the variable x, we need the concept of an *inverse trigonometric relation*.

Consider, for example, the function $y = \sin x$. To solve this equation for x in terms of y, we introduce the following notation:

$$x = \arcsin y$$

Expressed verbally, "x is a number (or angle measure) whose sine is y." Following the usual convention of placing y on the left side of the equal sign, we write

$$y = \arcsin x \qquad (16.31)$$

The expression says that y is a number (or angle measure) whose sine is x. The equation $y = \arcsin x$ is called an **inverse trigonometric relation.**

To illustrate the meaning of this kind of notation, let us evaluate $y = \arcsin x$ for certain values of x.

Example #1

1 Find y if $y = \arcsin \frac{1}{2}$, $0 \le y < 2\pi$.

Solution. By the definition of $\arcsin \frac{1}{2}$ we need to find a number whose sine is $\frac{1}{2}$. Two such numbers exist between 0 and 2π, namely,

$$y = \frac{\pi}{6} \quad \text{and} \quad y = \frac{5\pi}{6}$$

Example #2

2 Find y if $y = \arcsin 0$.

Solution. Since $\sin 0 = 0$ and $\sin \pi = 0$, it follows that

$$\sin(0 + k \cdot 2\pi) = 0 \quad \text{and} \quad \sin(\pi + k \cdot 2\pi) = 0, \quad k = 0, \pm 1, \pm 2, \ldots$$

or

$$\sin k\pi = 0, \quad k = 0, \pm 1, \pm 2, \ldots$$

So

$$y = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

Remark. Recall that a relation between two variables x and y is called a **function** if for every value of x there exists a unique value of y, denoted by $y = f(x)$. By this definition, $y = \arcsin x$ is *not* a function, as we can see from Example 2: The value $x = 0$ does not yield a unique value for y. To obtain a function, the values of y must be suitably restricted. That is the topic of the next section.

The other trigonometric functions have similar inverse relations, as shown in the next two examples.

Example #3

3 Find y if $y = \arccos(\sqrt{3}/2)$, $0 \le y < 2\pi$.

Solution. The notation $\arccos(\sqrt{3}/2)$ has an analogous meaning as an angle whose cosine is $\sqrt{3}/2$. For $0 \le y < 2\pi$, we have

$$y = \frac{\pi}{6} \quad \text{and} \quad y = \frac{11\pi}{6}$$

Example #4

4 Find y if $y = \arctan(-1)$, $0 \le y < 2\pi$.

tangents are equal to

Solution. For y between 0 and 2π, the only angles whose tangents are equal to -1 are

tangents are equal

$$y = \frac{3\pi}{4} \quad \text{and} \quad y = \frac{7\pi}{4}$$

As noted at the beginning of this section, the notation for the inverse relationship enables us to solve a trigonometric equation for x in terms of y, as shown in the next example.

Example #5

5 Solve the equation $y = 1 + \sin 2x$ for x in terms of y.

Solution. The equation $y = 1 + \sin 2x$ can also be written

$$\sin 2x = y - 1$$

Using the inverse relationship,

$$2x = \arcsin(y - 1)$$

we get

$$x = \frac{1}{2}\arcsin(y - 1)$$

Exercises / Section 16.7

In Exercises 1–16, find y $(0 \le y < 2\pi)$ without using a table or calculator.

1. $y = \arcsin \frac{\sqrt{3}}{2}$ **2.** $y = \arcsin(-1)$ **3.** $y = \arctan 1$

4. $y = \text{arccot}(-1)$ **5.** $y = \text{arccsc } 1$ **6.** $y = \arccos(-1)$

7. $y = \arcsin 0$ **8.** $y = \arcsin\left(-\frac{1}{2}\right)$ **9.** $y = \arccos \frac{1}{2}$

10. $y = \arcsin\left(-\frac{\sqrt{2}}{2}\right)$ **11.** $y = \arccos 0$ **12.** $y = \text{arcsec } 1$

13. $y = \text{arccsc}(-2)$ **14.** $y = \arctan\left(-\frac{1}{\sqrt{3}}\right)$ **15.** $y = \text{arccot} \frac{1}{\sqrt{3}}$

16. $y = \arctan 0$

In Exercises 17–30, solve each equation for x in terms of y.

17. $y = \arctan x$ **18.** $y = \arccos 3x$ **19.** $y = 1 - \arcsin x$

20. $y = 2 + \text{arcsec } x$ **21.** $y = \arcsin 2x - 1$ **22.** $y = \arccos(x - 2)$

23. $y = \text{arccsc}(x + 1)$ **24.** $y = \text{arccot } 2x$ **25.** $y = \text{arcsec } 3x + 1$

26. $y = \arcsin 2(x - 2)$ **27.** $y = 3 \text{ arccot } 3x$ **28.** $y = 2 \arctan 5x + 1$

29. $y = 2 \arcsin(x + 1) + 3$ **30.** $y = 3 \arccos(x - 2) - 2$

16.8 Inverse Trigonometric Functions

We learned in our study of logarithms that the equation $y = b^x$ can be written $x = \log_b y$. While the two equations mean the same thing, the first expresses y as a function of x and the second expresses x as a function of y; $y = b^x$ and $y = \log_b x$ are called **inverse functions.** An analogous situation exists in trigonometry in the sense that every trigonometric function has an inverse function.

In the last section we introduced the customary notation for inverse trigonometric relations. We also noted that a relation such as $y = \arcsin x$ does not represent a function. Given the importance of the function concept, this state of affairs is unsatisfactory. The variable y must be suitably restricted so that every value of x yields a unique value of y. This restriction leads to the definition of an **inverse trigonometric function.**

First we need to consider the graph of the relation $y = \arcsin x$. Writing this equation in the form $x = \sin y$, we get the graph of the sine function with x and y interchanged, as shown in Figure 16.10. This graph shows why the

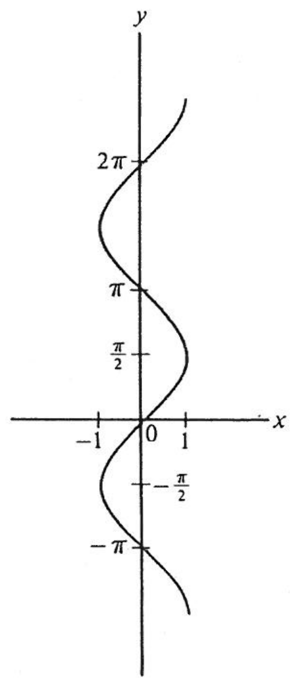


Figure 16.10

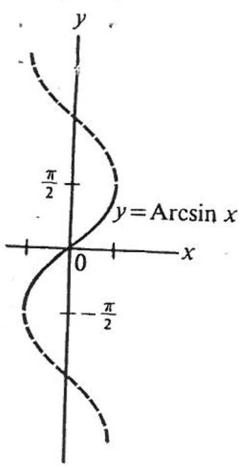


Figure 16.11

relation $y = \arcsin x$ is not a function: For every x such that $-1 \le x \le 1$, we get infinitely many values for y.

We can also see from the graph that y becomes unique if all but a small section of the graph is eliminated. This elimination can be done in several ways. The restriction that has become standard is $-\pi/2 \le y \le \pi/2$, which corresponds to the portion of the graph through the origin, drawn as the solid curve in Figure 16.11. To distinguish between the solid curve and the dashed curve, the equation of the solid curve is written

$$y = \text{Arcsin } x$$

using the capital letter A. Note especially that

$$\boxed{y = \text{Arcsin } x \text{ is a function.}}$$

Example 1 Find the exact value of y in each case:

a. $y = \arcsin \frac{1}{2}$ $(0 \le y < 2\pi)$ **b.** $y = \text{Arcsin } \frac{1}{2}$

Solution. a. As we saw in the previous section,

$$y = \frac{\pi}{6} \quad \text{and} \quad y = \frac{5\pi}{6}$$

b. Since $-\pi/2 \le y \le \pi/2$, the only permissible value is

$$y = \frac{\pi}{6}$$

Thus Arcsin $\frac{1}{2} = \pi/6$, a unique value.

The inverse functions corresponding to $y = \arctan x$ and $y = \arccos x$ are obtained from the graphs of $x = \tan y$ and $x = \cos y$, shown in Figure 16.12 and Figure 16.13, respectively.

Following the usual conventions, $y = \text{Arctan } x$ is the solid curve in Figure 16.12 and $y = \text{Arccos } x$ the solid curve in Figure 16.13. Note that in all cases the angle y is in the first quadrant whenever x is positive. The different cases are summarized next.

Inverse trigonometric functions:

$$y = \text{Arcsin } x, \quad -\pi/2 \le y \le \pi/2 \tag{16.32}$$
$$y = \text{Arctan } x, \quad -\pi/2 < y < \pi/2 \tag{16.33}$$
$$y = \text{Arccos } x, \quad 0 \le y \le \pi \tag{16.34}$$

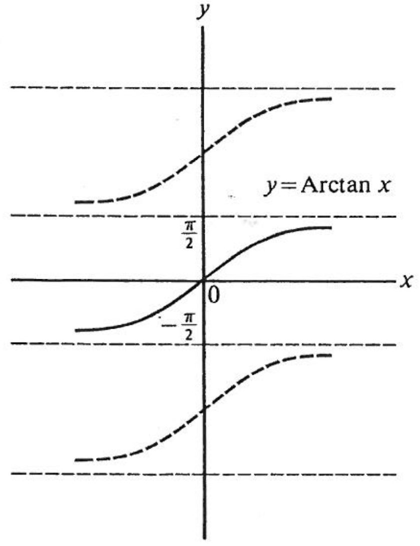


Figure 16.12

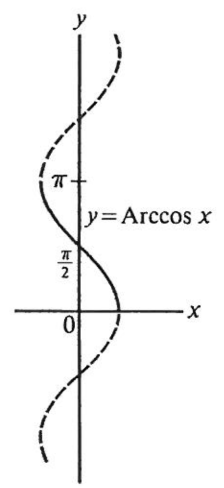


Figure 16.13

Although inverse trigonometric functions exist for the remaining functions, we shall confine ourselves to the cases already presented. One reason is that different authors define the other functions in different ways. For example, $y = \text{Arcsec } x$ is sometimes defined by using the restriction $0 \le y < \pi/2$, $\pi/2 < y \le \pi$ and sometimes by the restriction $0 \le y < \pi/2$, $-\pi \le y < -\pi/2$.

Example #2

2 Find the exact values of

a. Arccos $\frac{1}{2}$ **b.** Arccos $\left(-\frac{1}{2}\right)$

Solution. a. Since x is positive, Arccos $\frac{1}{2}$ is in the first quadrant. Thus

$$\text{Arccos } \frac{1}{2} = \frac{\pi}{3}$$

(Don't forget that Arccos $\frac{1}{2}$ is an angle!)

b. Since x is negative, the angle cannot be in the first quadrant. To find the proper quadrant, we must refer to the definition of Arccos x. By agreement, *the angle must lie between 0 and π.* Thus

$$\text{Arccos}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

E Example #3

3 Find the exact value of Arctan (-1).

Solution. This is a problem that many students find troublesome. If we were looking merely for some angles between 0 and 2π, we would choose 135° and 315° ($3\pi/4$ and $7\pi/4$). Knowing that the value has to be unique, some students proceed to drop one of the values and keep only $7\pi/4$. Now, while this angle does lie in the fourth quadrant, this choice still violates the convention in statement (16.33). Since $-\pi/2 < y < \pi/2$, the angle chosen must be negative. Thus

$$\text{Arctan}\,(-1) = -\frac{\pi}{4}$$

Example # 4

4 Find the exact value of Arcsin $\left(-\frac{\sqrt{3}}{2}\right)$.

Solution. Since x is negative, the angle cannot lie in the first quadrant. By the definition of Arcsin x, we have $-\pi/2 \le y \le \pi/2$, so that

$$\text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

OR
NT

To show how strictly these conventions must be observed, let us find some of the values of the inverse trigonometric functions by using a calculator. (If the angles are not special angles, a calculator should be used anyway.)

Example #5

5 Use a calculator to find Arcsin (0.4278).

ode

Solution. First set the calculator in the radian mode. Now enter 0.4278 and press the $\boxed{\text{INV}}$ key, followed by the $\boxed{\text{SIN}}$ key, to obtain 0.4421. As expected, the angle is in the first quadrant.

ode

By setting the calculator in the degree mode, the same sequence yields Arcsin (0.4278) = 25.33°.

Example #6

6 Evaluate Arcsin (-0.6845).

Solution. Set the calculator in the radian mode and proceed as in Example 5. We obtain

$$\text{Arcsin}\,(-0.6845) = -0.7539$$

The result agrees with the convention in statement (16.32).

Inverse trigonometric functions can be used to solve certain trigonometric equations.

Example #7

7 Solve the equation $y = 2 \tan 3x - 1$ for x in terms of y using the proper inverse function.

Solution.

$$\begin{aligned} y &= 2 \tan 3x - 1 && \text{given equation} \\ y + 1 &= 2 \tan 3x && \text{transposing } -1 \\ \tan 3x &= \frac{1}{2}(y+1) && \text{dividing by 2} \\ 3x &= \text{Arctan}\,\frac{1}{2}(y+1) && \text{inverse function} \\ x &= \frac{1}{3}\,\text{Arctan}\,\frac{1}{2}(y+1) && \text{dividing by 3} \end{aligned}$$

Example #8

8 Solve the equation $y = 3$ Arccos $2x$ for x in terms of y.

Solution.

$$\begin{aligned} y &= 3\,\text{Arccos}\,2x \\ \frac{y}{3} &= \text{Arccos}\,2x \\ 2x &= \cos\frac{y}{3} \\ x &= \frac{1}{2}\cos\frac{y}{3} \end{aligned}$$

The remaining examples involve trigonometric functions in a way that is particularly useful in calculus.

Example #9

9 Find the exact value of sin [Arctan $(-\frac{1}{4})$].

Solution. Recall that Arctan $(-\frac{1}{4})$ is an angle whose tangent is $-\frac{1}{4}$. Let $\theta =$ Arctan $(-\frac{1}{4})$. To find sin θ, we draw the diagram in Figure 16.14. It follows that

$$\sin\left[\text{Arctan}\left(-\frac{1}{4}\right)\right] = \sin\theta = \frac{-1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$$

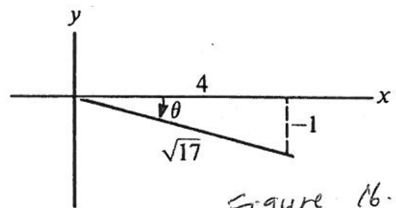


Figure 16.14

Example # 10

10 Find an algebraic expression equivalent to tan (Arccos x).

Solution. Let θ = Arccos x. Thus θ is an angle whose cosine is $x/1$. Draw a right triangle with x on the adjacent side and 1 on the hypotenuse. (See Figure 16.15.) By the Pythagorean theorem, the length of the opposite side is $\sqrt{1-x^2}$. It follows that

$$\tan(\text{Arccos } x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$$

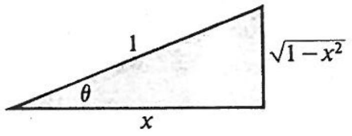


Figure 16.15

Example # 11

11 The width w of a laser beam at a distance d from the source is given by

$$w = 2d\tan\frac{\alpha}{2}$$

where α is the angle of the beam. Solve this equation for α.

Solution.
$$w = 2d\tan\frac{\alpha}{2}$$
$$\frac{w}{2d} = \tan\frac{\alpha}{2}$$
$$\frac{\alpha}{2} = \text{Arctan}\,\frac{w}{2d}$$
$$\alpha = 2\,\text{Arctan}\,\frac{w}{2d}$$

Exercises / Section 16.8

In Exercises 1–17, find the exact value (in radian measure) of each expression without using* a table Using a table or a calculator.

1. Arcsin $\frac{\sqrt{3}}{2}$
2. Arcsin (-1)
3. Arctan 1
4. Arccos (-1)
5. Arcsin 0
6. Arcsin $\left(-\frac{1}{2}\right)$
7. Arccos 0
8. Arctan $\left(-\frac{1}{\sqrt{3}}\right)$
9. Arctan 0

10. Arcsin 1
11. Arcsin $\left(-\frac{1}{\sqrt{2}}\right)$
12. Arcsin $\left(\frac{1}{\sqrt{2}}\right)$
13. Arctan $(-\sqrt{3})$
14. Arccos $\left(-\frac{1}{\sqrt{2}}\right)$
15. Arctan $\sqrt{3}$
16. Arccos $\frac{1}{\sqrt{2}}$
17. Arccos $\left(-\frac{\sqrt{3}}{2}\right)$

In Exercises 18–40, evaluate the given expressions without a table or a calculator. (See Example 9.)

18. sin (Arctan 2)
19. tan $\left[\text{Arccos}\left(-\frac{1}{3}\right)\right]$
20. tan $\left[\text{Arcsin}\left(-\frac{1}{3}\right)\right]$
21. csc $\left[\text{Arcsin}\left(-\frac{3}{4}\right)\right]$
22. csc $\left[\text{Arccos}\left(-\frac{3}{4}\right)\right]$
23. cos [Arctan (-2)]
24. sec (Arctan 3)
25. cos $\left(\text{Arcsin}\,\frac{2}{3}\right)$
26. sec $\left(\text{Arcsin}\,\frac{4}{5}\right)$
27. csc $\left[\text{Arctan}\left(-\frac{3}{4}\right)\right]$
28. tan $\left[\text{Arcsin}\left(-\frac{12}{13}\right)\right]$
29. cot $\left[\text{Arccos}\left(-\frac{5}{13}\right)\right]$
30. cot $\left[\text{Arctan}\left(-\frac{5}{6}\right)\right]$
31. sec $\left(\text{Arccos}\,\frac{1}{4}\right)$
32. csc $\left(\text{Arcsin}\,\frac{2}{5}\right)$
33. cot $\left[\text{Arcsin}\left(-\frac{1}{4}\right)\right]$
34. sec $\left[\text{Arcsin}\left(-\frac{3}{7}\right)\right]$
35. sin $\left[\text{Arccos}\left(-\frac{2}{5}\right)\right]$
36. csc (Arctan $\sqrt{5}$)
37. sin $\left(\text{Arcsin}\,\frac{1}{5}\right)$
38. tan (Arctan 4)
39. cot (Arctan 4)
40. cos $\left(\text{Arccos}\,\frac{2}{5}\right)$

Use the positive square root in square ro

In Exercises 41–50, for each expression find an equivalent algebraic expression. Use the positive sq each case. (See Example 10.)

41. tan (Arcsin x)
42. cos (Arctan x)
43. sec (Arctan x)
44. sin (Arccos x)
45. cot (Arcsin $2x$)
46. sin (Arccos $2x$)
47. csc (Arctan $3x$)
48. tan (Arccos $3x$)
49. sin (Arccos $2x$)
50. tan (Arcsin $3x$)

In Exercises 51–58, use a calculator to evaluate each inverse function. (Set your calculator in the r in the radian mode

51. Arctan 2
52. Arctan (-2)
53. Arcsin $\left(-\frac{1}{3}\right)$
54. Arccos $\left(-\frac{2}{3}\right)$
55. Arctan (1.3142)
56. Arcsin (-0.7418)
57. Arccos (-0.4915)
58. Arctan (2.672)

In Exercises 59–68, solve the equations for x.

59. $y = 2$ Arcsin x
60. $y = 3$ Arccos x
61. $y = 2\sin x$
62. $y = 3\cos x$
63. $y =$ Arctan $x + 3$
64. $y =$ Arctan $(x + 3)$

530

65. $y = 2 \sin 3x$

66. $y = 4 \text{ Arcsin } (x + 4)$

67. $y = 4 \tan (x - 2)$

68. $y = \frac{1}{2} \text{ Arccos } (x + 1)$

of the building,

69. A woman is walking toward a building 100 ft high. Show that when she is x feet from the base of the building, the angle of elevation of the top is given by $\theta = \text{Arctan } (100/x)$.

70. Show that angle $A = \text{Arctan } [(b/a) \tan B]$ in Figure 16.16.

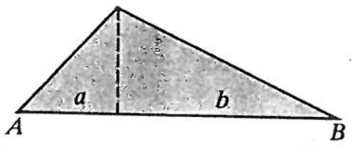


Figure 16.16

71. The formula $\phi = \text{Arctan } (X/R)$ arises in the study of alternating current. Solve this formula for R.

72. A small body is revolving in a horizontal circle at the end of a cord of length L making an angle θ with the vertical (Figure 16.17). The time for one complete revolution is

$$T = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

Solve this equation for θ.

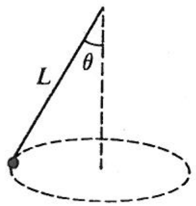


Figure 16.17

73. Recall that the equation of simple harmonic motion is

$$x = A \cos \sqrt{\frac{k}{m}}\, t$$

Find the formula for the time t required for the particle to move from its starting position $x = A$ (when $t = 0$) to a new position $(0 \le t \le \pi\sqrt{m/k})$.

74. The formula for *magnetic intensity* is

$$B = \frac{F}{qv \sin \phi}$$

where q is the magnitude of the charge, v its velocity, ϕ the angle between the direction of motion and the direction of the magnetic field, and F the force acting on the moving charge. Solve this formula for ϕ.

Review Exercises / Chapter 16

In Exercises 1–4, use the appropriate identity to evaluate each function without using a table or a calculator.

1. $\sin 22.5°$

2. $\cos 112.5°$

3. $\cos 12° \cos 18° - \sin 12° \sin 18°$

4. $\sin 110° \cos 20° - \cos 110° \sin 20°$

531

In Exercises 5–8, combine each expression into a single term.

5. $\sin 2x \cos 4x + \cos 2x \sin 4x$

6. $\cos 6x \cos x + \sin 6x \sin x$

7. $\sin 4x \cos x - \cos 4x \sin x$

8. $\cos (x - y) \cos y - \sin (x - y) \sin$

$\cos(x-y) \cos y - \sin(x-y) \sin y$

In Exercises 9–14, write each expression as a function of x or $2x$.

9. $\cos (2x - \pi)$

10. $\sin \left(x - \frac{\pi}{2}\right)$

11. $\cos \left(2x + \frac{\pi}{2}\right)$

12. $\cos (x - 2\pi)$

13. $\sin \left(x - \frac{\pi}{6}\right)$

14. $\cos \left(x - \frac{\pi}{4}\right)$

15. Find $\sin 2\theta$, given that $\cos \theta = -\frac{4}{5}$, θ in quadrant II.

16. Find $\sin 2\theta$, given that $\sin \theta = -\frac{1}{3}$, θ in quadrant III.

17. Find $\cos 2\theta$, given that $\sin \theta = -\frac{5}{13}$, θ in quadrant IV.

18. Find $\sin (\theta/2)$, given that $\cos \theta = -\frac{4}{5}$, θ in quadrant II.

19. Find $\cos (\theta/2)$, given that $\sin \theta = -\frac{24}{25}$, θ in quadrant III.

20. Find $\cos (\theta/2)$, given that $\cos \theta = \frac{12}{13}$, θ in quadrant IV.

In Exercises 21–26, write each expression as a single trigonometric function.

21. $\cos^2 3x - \sin^2 3x$

22. $\sin^2 2x - \cos^2 2x$

23. $1 - 2 \sin^2 4x$

24. $2 \cos^2 3\beta - 1$

25. $2 \sin 3x \cos 3x$

26. $\sin 4x \cos 4x$

In Exercises 27–30, simplify the given expressions.

27. $\sqrt{\frac{1 - \cos 4\theta}{2}}$

28. $\sqrt{\frac{1 + \cos 4\theta}{2}}$

29. $\sqrt{1 + \cos 4\theta}$

30. $\sqrt{2 - 2 \cos 8\theta}$

In Exercises 31–34, eliminate the exponent.

31. $\sin^2 3x$

32. $\cos^2 4x$

33. $2 \cos^2 3x$

34. $4 \sin^2 4x$

In Exercises 35–56, prove the given identities.

35. $\frac{\cos \beta \tan \beta + \sin \beta}{\tan \beta} = 2 \cos \beta$

36. $(\sec \alpha - \tan \alpha)(\csc \alpha + 1) = $ cc

$= \cot \alpha$

$\simeq \cot \alpha$

37. $\frac{1}{1 + \sin x} - \frac{1}{1 - \sin x} = -2 \tan x \sec x$

38. $\frac{\sec \theta}{\cot \theta + \tan \theta} = \sin \theta$

39. $\frac{1 + \sin^2 \theta \sec^2 \theta}{1 + \cos^2 \theta \csc^2 \theta} = \tan^2 \theta$

40. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$

41. $\frac{\cos^4 x - \sin^4 x}{1 - \tan^4 x} = \cos^4 x$

42. $\frac{1}{\csc x + \cot x} = \frac{1 - \cos x}{\sin x}$

532

43. $\cos y \sin (x - y) + \sin y \cos (x - y) = \sin x$

44. $\cos\left(x - \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{6}\right) = \sin x$

45. $\cos (x + y) + \cos (x - y) = 2 \cos x \cos y$

46. $\cos 2x + 2 \sin^2 x = 1$

47. $\frac{2 - \sec^2 y}{\sec^2 y} = \cos 2y$

48. $\frac{2 \tan \theta}{\sin 2\theta} = \sec^2 \theta$

49. $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta$

50. $\cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta}$

51. $\frac{1 - \cos 2\gamma + \sin 2\gamma}{1 + \cos 2\gamma + \sin 2\gamma} = \tan \gamma$

52. $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

53. $\csc^2 \theta = \frac{2}{1 - \cos 2\theta}$

54. $2 \sec^2 \theta = \frac{1}{1 + \cos 2\theta}$

55. $\left(\sin \frac{\theta}{2} - \cos \frac{\theta}{2}\right)^2 = 1 - \sin \theta$

56. $\sin^2 \alpha = \frac{\sin^2 2\alpha}{2(1 + \cos 2\alpha)}$

In Exercises 57–62, solve the given equations ($0 \le x < 2\pi$).

57. $\cos^2 x + 2 \sin x = 1$

58. $\sin 2x - 3 \sin x = 0$

59. $2 \cos^2 x + \cos x = 1$

60. $\tan x = \tan^2 x$

61. $\cos \frac{x}{2} + \cos x + 1 = 0$

62. $3 \cos^2 x - 14 \cos x + 8 = 0$

63. Suppose a projectile is fired along an inclined plane making an angle α to the horizontal from a gun making an angle θ to the horizontal. Calculus shows that the range of the projectile along the inclined plane is a maximum when θ satisfies the relation

$$\cos \theta \cos (\theta - \alpha) - \sin \theta \sin (\theta - \alpha) = 0$$

Find angle θ.

64. A body of weight W is dragged along a horizontal plane by a force whose line of action makes an angle θ with the plane. Calculus shows that the pull is least when θ satisfies the equation

$$\mu \cos \theta - \sin \theta = 0$$

where μ is the coefficient of friction. Show that the pull is least when $\theta = \text{Arctan } \mu$.

In Exercises 65–68, find y ($0 \le y < 2\pi$) in each case without using a table or a calculator.

65. $y = \arccos \frac{1}{2}$

66. $y = \text{arccsc } 1$

67. $y = \text{arccot}(-\sqrt{3})$

68. $y = \arctan \frac{1}{\sqrt{3}}$

In Exercises 69–72, evaluate each expression without using a table or a calculator.

69. $\text{Arccos}\left(-\frac{1}{2}\right)$

70. $\text{Arctan}\left(-\frac{1}{\sqrt{3}}\right)$

71. $\cot\left[\text{Arcsin}\left(-\frac{1}{6}\right)\right]$

72. $\sin [\text{Arctan}(-2)]$

533

73. Find an algebraic expression for $\sin (\text{Arccos } 2x)$

74. Solve for x: $y = 2 \text{ Arctan } (x + 2)$.

75. Solve for x: $y = 2 \sin 4x$.

76. The expression $a \sin \theta + b \cos \theta$ can be written in simpler form by noting that

$$\sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \sin \theta + \frac{b}{\sqrt{a^2 + b^2}} \cos \theta\right) = a \sin \theta + b \cos \theta$$

and that

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

where α is an angle determined by a and b. (See Figure 16.18.) Use the identity for $\sin (A + B)$
sin(A+B) to Show that

$$a \sin \theta + b \cos \theta = k \sin (\theta + \alpha)$$

where $k = \sqrt{a^2 + b^2}$ and α is any angle for which

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

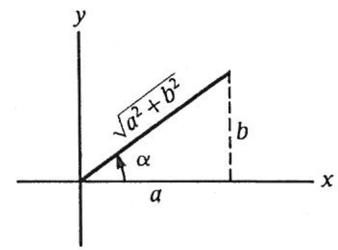


Figure 16.18

23. $\frac{1}{3}\begin{bmatrix} -11 & -15 & 5 \\ 6 & 9 & -3 \\ -14 & -21 & 8 \end{bmatrix}$ **25.** $\frac{1}{25}\begin{bmatrix} 10 & 5 & -15 \\ 1 & -2 & -14 \\ 2 & -4 & -3 \end{bmatrix}$ **27.** $\frac{1}{3}\begin{bmatrix} 5 & -6 & 11 & -3 \\ 4 & -6 & 13 & -3 \\ -7 & 9 & -19 & 6 \\ -4 & 6 & -10 & 3 \end{bmatrix}$ **29.** $\begin{bmatrix} -\frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$

31. $\begin{bmatrix} \frac{7}{5} \\ \frac{11}{25} \\ \frac{3}{25} \end{bmatrix}$ **33.** $\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ **35.** $3, -\frac{4}{3}, -\frac{7}{3}, 4$ (in amps)

Cumulative Review Exercises for Chapters 13–15 (page 486)

1. $(0.6, 0.4), (-1.6, 2.6)$ **2.** $(2, 4), (4, 2)$ **3.** $(\sqrt{14}, \sqrt{2}), (\sqrt{14}, -\sqrt{2}), (-\sqrt{14}, \sqrt{2}), (-\sqrt{14}, -\sqrt{2})$
4. $\pm\frac{1}{2}, \pm j$ **5.** $x = 6$ **6.** no **7.** yes **8.** 5 **9.** 1, 1, 2, 4 **10.** $1, 1, \frac{4}{3}, -\frac{3}{2}$ **11.** 0.75
12. 394 **13.** $(1, 2, -2, 3)$ **14.** $\begin{bmatrix} 2 & -3 & -2 \\ -1 & 7 & 5 \\ -6 & 6 & 11 \end{bmatrix}$ **15.** $\begin{bmatrix} 0 & 8 \\ 10 & -17 \end{bmatrix}$ **16.** $\begin{bmatrix} -3 & 2 & 2 \\ 5 & -3 & -3 \\ -1 & 1 & 0 \end{bmatrix}$
17. $(-1, 1, 0)$ **18.** $3.10\ \Omega, 6.90\ \Omega$

Chapter 16

Section 16.1 (page 493)

1. $\frac{\cos\beta}{\sin\beta}$ **3.** $\sin\theta$ **5.** 1 **7.** $\frac{\cos x + 1}{\sin x}$ **9.** $\cos\theta$ **11.** $\frac{1}{\sin^2 s}$ **13.** $-\frac{\sin^2\theta}{\cos^2\theta}$
15. $\csc x$ **17.** $\csc^2\theta$ **19.** $\csc\theta$ **21.** 1 **23.** $\cot t$ **25.** $\cos\theta$ **27.** 1 **29.** $\sec^3 x$
31. $\cot\theta$

Section 16.2 (page 497)

41. $a\omega$ **43.** $y = \frac{2v_0^2 x \sin\alpha\cos\alpha - gx^2}{2v_0^2\cos^2\alpha}$

Section 16.3 (page 503)

1. $\frac{\sqrt{6}+\sqrt{2}}{4}$ **3.** $\frac{\sqrt{2}-\sqrt{6}}{4}$ **5.** $\frac{\sqrt{2}}{2}$ **7.** $\frac{\sqrt{2}}{2}$ **9.** $\frac{\sqrt{2}}{2}$ **11.** $\sin 2x$ **13.** $\sin 3x$

15. $\cos 2x$ **17.** $\cos 9x$ **19.** $\sin x$ **21.** $\frac{1}{2}(\sqrt{3}\cos x - \sin x)$ **23.** $-\sin 2x$ **25.** $-\cos x$

27. $\sin 2x$ **29.** $\cos 2x$ **31.** $\frac{1}{2}(\sin x - \sqrt{3}\cos x)$ **33.** $\frac{1}{2}(\sqrt{3}\cos x - \sin x)$

35. $\frac{\sqrt{2}}{2}(\sin x + \cos x)$ **37.** $\frac{1+\tan x}{1-\tan x}$ **57.** $y = 2A\cos\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi x}{\lambda}\right)$

Section 16.4 (page 509)

1. $\frac{24}{25}$ **3.** $\frac{7}{25}$ **5.** $-\frac{120}{169}$ **7.** $-\frac{\sqrt{3}}{2}$ **9.** $\frac{17}{25}$ **11.** $-\frac{31}{49}$ **13.** $\cos 6y$ **15.** $\sin 6\theta$

17. $\cos 4\beta$ **19.** $-\cos 8y$ **21.** $\frac{1}{2}\sin 8\omega$ **23.** $2\sin 4x$ **37.** $v = 4\sin 2t$

Section 16.5 (page 514)

1. $\frac{\sqrt{2-\sqrt{3}}}{2} = 0.2588$ **3.** $\frac{\sqrt{2+\sqrt{2}}}{2} = 0.9239$ **5.** $\frac{\sqrt{2+\sqrt{2}}}{2} = 0.9239$ **7.** $\frac{7\sqrt{2}}{10}$ **9.** $\frac{2\sqrt{13}}{13}$

11. $\sin 2\theta$ **13.** $\sqrt{2}\cos 3\theta$ **15.** $\sqrt{10}\sin 2\theta$ **17.** $\frac{1}{2}(1-\cos 8x)$ **19.** $\frac{1}{2}(1+\cos 4x)$

21. $1-\cos 6x$ **23.** $6(1-\cos 2x)$ **29.** $\frac{\sqrt{2}}{2}\csc\frac{x}{2}$ **31.** $2\sin\frac{\theta}{2}$

Section 16.6 (page 519)

1. $\frac{\pi}{6}, \frac{5\pi}{6}$ **3.** $\frac{\pi}{4}, \frac{5\pi}{4}$ **5.** $0, \pi$ **7.** $0, \frac{\pi}{2}, \pi$ **9.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **11.** $\frac{\pi}{4}, \pi, \frac{5\pi}{4}$

13. $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **15.** 0 **17.** $\frac{\pi}{4}, \frac{5\pi}{4}$ **19.** π **21.** $\frac{\pi}{4}, \frac{5\pi}{4}$ **23.** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **25.** $\frac{\pi}{2}, \frac{3\pi}{2}$

27. $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ **29.** $\frac{\pi}{2}$ **31.** $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ **33.** $54.7°, 125.3°, 234.7°, 305.3°$

35. $90°, 194.5°, 270°, 345.5°$ **37.** $0°, 41.4°, 180°, 318.6°$ **39.** $54.7°, 125.3°, 234.7°, 305.3°$

41. $22.5°, 112.5°$ **43.** $t = \frac{\pi}{24} = 0.13$ sec **45.** $t = \frac{\pi}{6\omega}$ sec

Section 16.7 (page 522)

1. $\frac{\pi}{3}, \frac{2\pi}{3}$ **3.** $\frac{\pi}{4}, \frac{5\pi}{4}$ **5.** $\frac{\pi}{2}$ **7.** $0, \pi$ **9.** $\frac{\pi}{3}, \frac{5\pi}{3}$ **11.** $\frac{\pi}{2}, \frac{3\pi}{2}$ **13.** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **15.** $\frac{\pi}{3}, \frac{4\pi}{3}$

17. $x = \tan y$ **19.** $x = \sin(1-y)$ **21.** $x = \frac{1}{2}\sin(y+1)$ **23.** $x = \csc y - 1$

25. $x = \frac{1}{3}\sec(y-1)$ **27.** $x = \frac{1}{3}\cot\frac{y}{3}$ **29.** $x = \sin\frac{1}{2}(y-3) - 1$

Section 16.8 (page 528)

1. $\frac{\pi}{3}$ **3.** $\frac{\pi}{4}$ **5.** 0 **7.** $\frac{\pi}{2}$ **9.** 0 **11.** $-\frac{\pi}{4}$ **13.** $-\frac{\pi}{3}$ **15.** $\frac{\pi}{3}$ **17.** $\frac{5\pi}{6}$

19. $-2\sqrt{2}$ **21.** $-\frac{4}{3}$ **23.** $\frac{\sqrt{5}}{5}$ **25.** $\frac{\sqrt{5}}{3}$ **27.** $-\frac{5}{3}$ **29.** $-\frac{5}{12}$ **31.** 4 **33.** $-\sqrt{15}$

35. $\frac{\sqrt{21}}{5}$ **37.** $\frac{1}{5}$ **39.** $\frac{1}{4}$ **41.** $\frac{x}{\sqrt{1-x^2}}$ **43.** $\sqrt{1+x^2}$ **45.** $\frac{\sqrt{1-4x^2}}{2x}$ **47.** $\frac{\sqrt{9x^2+1}}{3x}$

49. $\sqrt{1-4x^2}$ **51.** 1.1071 **53.** -0.3398 **55.** 0.9203 **57.** 2.0846 **59.** $x = \sin\frac{y}{2}$

61. $x = \text{Arcsin}\,\frac{y}{2}$ **63.** $x = \tan(y-3)$ **65.** $x = \frac{1}{3}\text{Arcsin}\,\frac{y}{2}$ **67.** $x = \text{Arctan}\,\frac{y}{4} + 2$

71. $R = X\cot\theta$ **73.** $t = \sqrt{\frac{m}{k}}\,\text{Arccos}\,\frac{x}{A}$

Review Exercises for Chapter 16 (page 530)

1. $\sqrt{\frac{2-\sqrt{2}}{2}}$ **3.** $\frac{\sqrt{3}}{2}$ **5.** $\sin 6x$ **7.** $\sin 3x$ **9.** $-\cos 2x$ **11.** $-\sin 2x$

13. $\frac{1}{2}(\sqrt{3}\sin x - \cos x)$ **15.** $-\frac{24}{25}$ **17.** $\frac{119}{169}$ **19.** $-\frac{3}{5}$ **21.** $\cos 6x$ **23.** $\cos 8x$

25. $\sin 6x$ **27.** $\sin 2\theta$ **29.** $\sqrt{2}\cos 2\theta$ **31.** $\frac{1}{2}(1-\cos 6x)$ **33.** $1+\cos 6x$ **57.** $0, \pi$

59. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **61.** $\pi, \frac{4\pi}{3}$ **63.** $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$ **65.** $\frac{\pi}{3}, \frac{5\pi}{3}$ **67.** $\frac{5\pi}{6}, \frac{11\pi}{6}$ **69.** $\frac{2\pi}{3}$

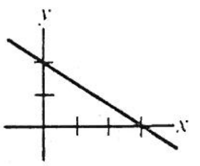
71. $-\sqrt{35}$ **73.** $\sqrt{1 - 4x^2}$ **75.** $x = \frac{1}{4} \text{ Arcsin } \frac{y}{2}$

Chapter 17

Section 17.1 (page 538)

1. $x < 3$ **3.** $x \le 2$ **5.** $x < -6$ **7.** $x \ge -5$ **9.** $x \le \frac{4}{3}$ **11.** $x > -\frac{3}{2}$ **13.** $x \ge \frac{7}{3}$

15. $x > \frac{5}{2}$ **17.**



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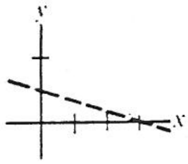

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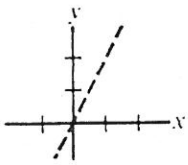

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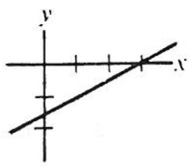

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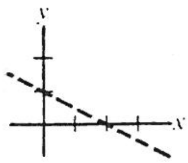


27. 7.6 hr **29.** $0.0 \le F \le 80$ (pounds)

Section 17.2 (page 543)

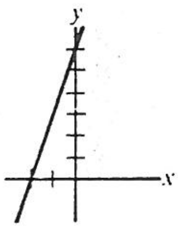
1. $x > -3$ **3.** $-2 < x < 2$ **5.** $x \le -3, x \ge -2$ **7.** $x < -2, 0 < x < 3$
9. $x \le -5, -2 \le x \le 3$ **11.** $-8 < x < 6, x > 7$ **13.** $1 < x < 2$ **15.** $x \le 6, x > 7$
17. $3 < x < 4$ **19.** $x \le -4, 1 < x \le 3$ **21.** $-3 < x < 3, x > 6$ **23.** $x \le -6, 1 \le x < 2, x > 3$
25. $x > 4$ **27.** all x **29.** $x < -1$ **31.** $x < -4, 1 < x \le 4$ **33.** $0 \le t \le \frac{4}{3}$ (seconds)
35. $x \le 5, x \ge 15$ (feet) **37.** $1 \le x < 20$

Section 17.3 (page 547)

1. $-1 < x < 3$ **3.** $-4 \le x \le -2$ **5.** $-1 < x < 0$ **7.** $x \le \frac{1}{2}, x \ge \frac{7}{2}$ **9.** $-\frac{13}{2} < x < -\frac{1}{2}$
11. $x < -1, x > 5$ **13.** $x < -3, -2 < x < 3, x > 4$ **15.** $x < 0, x > 4$
17. $-2 < x < 1 - \sqrt{7}, 1 + \sqrt{7} < x < 4$ **19.** $-3 - \sqrt{10} < x < -5, -1 < x < -3 + \sqrt{10}$
21. $|d - 2.5550| \le 0.0001$

Section 17.4 (page 552)

1.



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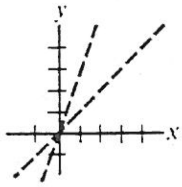

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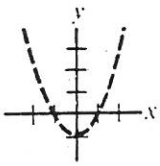

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