Differentiation

Exercise page 629

Q # 1 (c) $y = 6x^3$

$\frac{dy}{dx} = 6 \times 3 x^{3-1}$

$y' = \frac{dy}{dx} = 18x^2$

$\therefore \boxed{y' = 18x^2}$

(d) $y = -3x^2$

$\frac{dy}{dx} = -3 \times 2 x^{2-1}$

$\Rightarrow y' = \frac{dy}{dx} = -6x$

$\therefore \boxed{y' = -6x}$

(e) $y = \ln 3t$

$\frac{dy}{dt} = \frac{1}{3t} \frac{d}{dt}(3t)$

$\Rightarrow y' = \frac{3}{3t}$

$\Rightarrow \boxed{y' = \frac{1}{t}}$

Q # 2 (a)

$z = \frac{4}{t^3}$

$\frac{dz}{dt} = 4 t^{-3}$

$\Rightarrow \frac{dz}{dt} = 4(-3) t^{-3-1}$

$\Rightarrow z' = -12 t^{-4}$

$\Rightarrow \boxed{z' = \frac{-12}{t^4}}$

Q # 2 (b)

$z = \sqrt{t^3}$

$\Rightarrow z = t^{3/2}$

$\Rightarrow \frac{dz}{dt} = \frac{3}{2} t^{3/2 - 1}$

$\Rightarrow \boxed{z' = \frac{3}{2} t^{\frac{1}{2}}}$

$\frac{1}{2}(t^3)^{-1/2} \times 3t^2$

$\frac{3}{2} \cdot \frac{t^2}{t^{3/2}}$

$= \frac{3}{2} t^{2-\frac{3}{2}}$

$= \frac{3}{2} t^{\frac{1}{2}}$

Q # 3 (b)

$y = \cos 4t$

$\frac{dy}{dt} = -\sin 4t \frac{d}{dt}(4t)$

$\Rightarrow y' = -\sin 4t (4)$

$\boxed{y' = -4 \sin 4t}$

(c) $y = \tan 3r$

$\frac{dy}{dr} = \sec^2 3r \frac{d}{dr}(3r)$

$\Rightarrow \frac{dy}{dr} = \sec^2 3r (3)$

$\Rightarrow \boxed{y' = 3 \sec^2 3r}$

(e) $\frac{1}{e^{3t}}$

$y = \frac{1}{e^{3t}}$

$\Rightarrow y = e^{-3t}$

$\Rightarrow \frac{dy}{dt} = e^{-3t} \frac{d}{dt}(-3t)$

$\Rightarrow y' = e^{-3t}(-3)$

$\Rightarrow \boxed{y' = -3e^{-3t}}$

Q#4

(a) $\cos\frac{2x}{3}$

$y = \cos\frac{2x}{3}$

$\frac{dy}{dx} = -\sin\frac{2x}{3}\frac{d}{dx}\left(\frac{2x}{3}\right)$

$= -\sin\frac{2x}{3}\cdot\left(\frac{2}{3}\right)$

$y' = -\frac{2}{3}\sin\frac{2x}{3}$

(c) $\tan\pi x$

$y = \tan\pi x$

$\frac{dy}{dx} = \sec^2\pi x \frac{d}{dx}(\pi x)$

$\Rightarrow y' = \sec^2\pi x\cdot(\pi)$

$y' = \pi\sec^2\pi x$

~~(e)~~

Exercise page 631

Q#1 (b)

~~$\sin 3x \times \tan x$~~

$-3x^4 + 2x^{1.5}$

$y = -3x^4 + 2x^{1.5}$

$\frac{dy}{dx} = -3\times 4x^{4-1} + 2\times 1.5x^{1.5-1}$

$= -12x^3 + 3x^{0.5}$

$\therefore y' = -12x^3 + 3x^{0.5}$

(d) $\frac{3+2x}{4}$

$y = \frac{3+2x}{4}$

$y = \frac{3}{4} + \frac{x}{2}$

$dy/dx = 0 + \frac{1}{2}$

$\therefore y' = \frac{1}{2}$

Q#1 (c) $(2+3x)^2$

$y = (2+3x)^2$

$\Rightarrow \frac{dy}{dx} = 2(2+3x)^{2-1}\frac{d}{dx}(2+3x)$

$\Rightarrow \frac{dy}{dx} = 2(2+3x)(3)$

$\Rightarrow y' = 6(2+3x)$

$\Rightarrow y' = 12+18x$

Q#2

(b) $h(v) = 3\cos 2v - 6\sin\frac{v}{2}$

$\Rightarrow \frac{dh}{dv} = 3\left(-\sin 2v\,\frac{d}{dv}(2v)\right) - 6\cos\frac{v}{2}\frac{d}{dv}\left(\frac{v}{2}\right)$

$\Rightarrow \frac{dh}{dv} = -3\sin 2v(2) - 6\cos\frac{v}{2}\times\left(\frac{1}{2}\right)$

$\Rightarrow \frac{dh}{dv} = -6\sin 2v - 3\cos\frac{v}{2}$

(d) $H(t) = \frac{e^{3t}}{2} + 2\tan 2t$

$\frac{dH}{dt} = \frac{1}{2}\left[e^{3t}\frac{d}{dt}(3t)\right] + 2\left[\sec^2 2t\,\frac{d}{dt}(2t)\right]$

$\Rightarrow \frac{dH}{dt} = \frac{1}{2}e^{3t}\times(3) + 2\sec^2 2t\times(2)$

$\Rightarrow \frac{dH}{dt} = \frac{3}{2}e^{3t} + 4\sec^2 2t$

Exercise Page 631

Q #3

(a) $A(t) = (3+e^{t})^2$

$\Rightarrow \frac{dA}{dt} = 2(3+e^{t})^{2-1} \frac{d}{dt}(3+e^{t})$

$\Rightarrow \frac{dA}{dt} = 2(3+e^{t}) \times (e^{t})$

$\Rightarrow \mathring{A} = 2e^{t}(3+e^{t})$

$\Rightarrow \boxed{\mathring{A} = 6e^{t} + 2e^{2t}}$

(c) $V(r) = \left(1+\frac{1}{r}\right)^2 + (r+1)^2$

$\Rightarrow \frac{dV}{dr} = 2\left(1+\frac{1}{r}\right)^{2-1} \frac{d}{dr}\left(1+\frac{1}{r}\right) + 2(r+1)^{2-1} \frac{d}{dr}(r+1)$

$\Rightarrow \frac{dV}{dr} = 2\left(1+\frac{1}{r}\right) \times \frac{d}{dr}(r^{-1}) + 2(r+1)$

$\Rightarrow \frac{dV}{dr} = 2\left(1+\frac{1}{r}\right)(-1r^{-2}) + 2(r+1)$

$\Rightarrow \frac{dV}{dr} = -\frac{2}{r^2}\left(1+\frac{1}{r}\right) + 2(r+1)$

$\Rightarrow \boxed{\frac{dV}{dr} = -\frac{2}{r^2} - \frac{2}{r^3} + 2r + 2}$

(d) ~~$\frac{dH}{dt} = 2\left[e^{3t} \frac{d}{dt}(3t)\right] + 2\left[\sec^2 2t \frac{d}{dt}(2t)\right]$~~

~~$\Rightarrow \frac{dH}{dt} = 2e^{3t} \times (3) + 2\sec^2 2t \times 2$~~

~~$\Rightarrow H' = 6e^{3t} + 4\sec^2 2t$~~

Page 631

Q # 3

(d) $M(\theta) = 6\sin 2\theta - 2\cos\left(\frac{\theta}{4}\right) + 2\theta^2$

$\frac{dM}{d\theta} = 6\left[\cos 2\theta \frac{d}{d\theta}(2\theta)\right] - 2\left[-\sin\frac{\theta}{4} \frac{d}{d\theta}\left(\frac{\theta}{4}\right)\right] + 2 \times 2\theta^{2-1}$

$\Rightarrow \frac{dM}{d\theta} = 6\cos 2\theta \times (2) + 2\sin\frac{\theta}{4} \times \frac{1}{4} + 4\theta$

$\Rightarrow \boxed{\frac{dM}{d\theta} = 12\cos 2\theta + \frac{1}{2}\sin\frac{\theta}{4} + 4\theta}$

Exercise Page 633

Q # 1

$$y = 3x^2 + e^x$$

when $x = 0.5$

$$y = 3x^2 + e^x$$

$$\frac{dy}{dx} = 3 \times 2x^{2-1} + e^x$$

$$\Rightarrow y' = 6x + e^x$$

~~$\Rightarrow y'(0.5) = 6(0.5) + e^{0.5}$~~

~~$\Rightarrow y'(0.5$~~

at $x = 0.5$

$$y' = 6(0.5) + e^{0.5}$$

$$\Rightarrow y' = 3 + (1.6487)$$

$$\Rightarrow y' = 4.6487$$

Problem #3

$$H(t) = 5\sin t - 3\cos 2t$$

at (a) $t = 0$ (b) $t = 1.3$

$$\frac{dH}{dt} = 5\cos t - 3\left[-\sin 2t \frac{d}{dt}(2t)\right]$$

$$\Rightarrow \frac{dH}{dt} = 5\cos t + 3\sin 2t \times 2$$

$$\Rightarrow \frac{dH}{dt} = 5\cos t + 6\sin 2t$$

(a) at $t = 0$

$$\frac{dH}{dt} = 5\cos(0) + 6\sin(0)$$

$$= 5(1) + 6(0)$$

$$\therefore \frac{dH}{dt} = 5$$

(b) at $t = 1.3$

[Note:- 1.3 is radian either convert it to degree or ~~you~~ you can put your calculator in radian mode]

π radian $= 180°$

$1 = \frac{180°}{\pi}$

$1.3 = \frac{180°}{\pi} \times 1.3$

and $2.6 = \frac{180}{\pi} \times 2.6$

$$\therefore \frac{dH}{dt} = 5\cos(74.485) + 6\sin(148.97)$$

$$= 5(0.2675) + 6(0.5155)$$

$$= 1.3375 + 3.093$$

$$\frac{dH}{dt} = 4.4305$$

Exercise page 637

Q#1

(b)

$y(x) = \sin 3x + \cos x$

$\frac{dy}{dx} = \cos 3x \frac{d}{dx}(3x) - \sin x$

$\Rightarrow \frac{dy}{dx} = 3\cos 3x - \sin x$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3\cos 3x - \sin x)$

$\Rightarrow \frac{d^2y}{dx^2} = 3\left[-\sin 3x \frac{d}{dx}(3x)\right] - \cos x$

$\Rightarrow y'' = -3\sin 3x\,(3) - \cos x$

$\Rightarrow y'' = -9\sin 3x - \cos x$

(c) $y = \sqrt{x}$

$y = x^{1/2}$

$\frac{dy}{dx} = \frac{1}{2} x^{\frac{1}{2}-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2}$

$\frac{d^2y}{dx^2} = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1}$

$\Rightarrow y'' = -\frac{1}{4} x^{-3/2}$

(d) $y = e^{x} + e^{-x}$

$\frac{dy}{dx} = e^{x} + e^{-x}\frac{d}{dx}(-x)$

$\Rightarrow \frac{dy}{dx} = e^{x} - e^{-x}$

$\frac{d^2y}{dx^2} = e^{x} - e^{-x}\frac{d}{dx}(-x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{x} + e^{-x}$

Q#2

(c) $y = \sqrt{x}$

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} x^{\frac{1}{2}-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2}$

and $\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\left(-\frac{1}{2} x^{-\frac{1}{2}-1}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{4} x^{-3/2}$

and $\frac{d^3y}{dx^3} = -\frac{1}{4}\left(\frac{-3}{2} x^{-\frac{3}{2}-1}\right)$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{3}{8} x^{-5/2}$

(d) $y = e^{x} + e^{-x}$

$\frac{dy}{dx} = e^{x} + e^{-x}\frac{d}{dx}(-x)$

$\frac{dy}{dx} = e^{x} - e^{-x}$

and $\frac{d^2y}{dx^2} = e^{x} - e^{-x}\frac{d}{dx}(-x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{x} + e^{-x}$

and $\frac{d^3y}{dx^3} = e^{x} + e^{-x}\frac{d}{dx}(-x)$

$\Rightarrow \frac{d^3y}{dx^3} = e^{x} - e^{-x}$

Exercise Page 637

Q #3

(a) $y(t) = t(t^2+1)$

$y = t(t^2+1)$

$\Rightarrow y = t^3 + t$

$\frac{dy}{dt} = 3t^{3-1} + 1$

$\Rightarrow \frac{dy}{dt} = \dot{y} = 3t^2 + 1$

and $\frac{d^2y}{dt^2} = 3(2t^{2-1})$

$\Rightarrow \frac{d^2y}{dt^2} = 6t$

$\therefore \ddot{y}(t) = 6t$

at $t = 1$

$\ddot{y}(1) = 6 \times 1$

$\Rightarrow \ddot{y}(1) = 6$

(c) $y = 2e^t + e^{2t}$

$\frac{dy}{dt} = 2e^t \frac{d(t)}{dt} + e^{2t} \frac{d}{dt}(2t)$

$\frac{dy}{dt} = 2e^t + 2e^{2t}$

$\frac{d^2y}{dt^2} = 2e^t + 2e^{2t} \frac{d}{dt}(2t)$

$\Rightarrow \frac{d^2y}{dt^2} = 2e^t + 4e^{2t}$

$\therefore \ddot{y}(t) = 2e^t + 4e^{2t}$

at $t = 1$

$\ddot{y}(1) = 2e + 4e^2$

$= 2(2.7183) + 4(7.389)$

$5.436 + 29.556$

$\ddot{y}(1) = 34.9926$

(d) $y = \frac{1}{t}$

$y = t^{-1}$

$\frac{dy}{dt} = -1t^{-1-1}$

$\Rightarrow \frac{dy}{dt} = -t^{-2}$

$\frac{d}{dt}\left(\frac{dy}{dt}\right) = -\frac{d}{dt}(t^{-2})$

$\Rightarrow \frac{d^2y}{dt^2} = -(-2t^{-2-1})$

$\Rightarrow \frac{d^2y}{dt^2} = 2t^{-3}$

$\Rightarrow \ddot{y}(t) = \frac{2}{t^3}$

at $t = 1$

$\ddot{y}(1) = \frac{2}{1}$

$\Rightarrow \ddot{y}(1) = 2$

Q #4

(a) $y = t(t^2+1)$

$\Rightarrow y = t^3 + t$

$\Rightarrow \frac{dy}{dt} = 3t^2 + 1$

$\Rightarrow \frac{d^2y}{dt^2} = 6t$

$\Rightarrow \frac{d^3y}{dt^3} = 6$

$\therefore \dddot{y}(t) = 6$

$\dddot{y}(-1) = 6$

Q #4 Page 637

(c)

$$y = 2e^{t} + e^{2t}$$

$$\frac{dy}{dt} = 2e^{t} + e^{2t}\frac{d}{dt}(2t)$$

$$\Rightarrow \frac{dy}{dt} = 2e^{t} + 2e^{2t}$$

$$\Rightarrow \frac{d^2y}{dt^2} = 2e^{t} + 2e^{2t}\frac{d}{dt}(2t)$$

$$\Rightarrow \frac{d^2y}{dt^2} = 2e^{t} + 4e^{2t}$$

$$\frac{d^3y}{dt^3} = 2e^{t} + 4e^{2t}\frac{d}{dt}(2t)$$

$$\Rightarrow \frac{d^3y}{dt^3} = 2e^{t} + 8e^{2t}$$

$$\Rightarrow \dddot{y}(t) = 2e^{t} + 8e^{2t}$$

at $t = -1$

$$\Rightarrow \dddot{y}(-1) = 2e^{-1} + 8e^{-2}$$

$$= 2(0.36788) + 8(0.13533)$$

$$= ~~0.7357~~ \quad 1.0827$$

$$\dddot{y}(-1) = 1.8185$$

Q #4

$$y(t) = \frac{1}{t}$$

$$y = t^{-1}$$

$$\frac{dy}{dt} = -t^{-2}$$

$$\frac{d^2y}{dt^2} = 2t^{-3}$$

$$\frac{d^3y}{dt^3} = -6t^{-4}$$

$$\therefore \dddot{y}(t) = \frac{-6}{t^4}$$

$$\Rightarrow \dddot{y}(-1) = \frac{-6}{(-1)^4}$$

$$\Rightarrow \dddot{y}(-1) = -6$$

End of Block Exercise
Page (637-638)

Q #1 (a)

$$y = \cos 2t - \sin 2t$$

$$\frac{dy}{dt} = -\sin 2t\frac{d}{dt}(2t) - \cos 2t\frac{d}{dt}(2t)$$

$$\Rightarrow \frac{dy}{dt} = -2\sin 2t - 2\cos 2t$$

$$\frac{d^2y}{dt^2} = -2\left[\cos 2t\frac{d}{dt}(2t)\right] - 2\left[-\sin 2t\frac{d}{dt}(2t)\right]$$

$$\Rightarrow \frac{d^2y}{dt^2} = -2\times 2\cos 2t + 2\times 2\sin 2t$$

$$\Rightarrow \ddot{y} = -4\cos 2t + 4\sin 2t$$

(b)

$$y = e^{2x} - e^{x}$$

$$\frac{dy}{dx} = e^{2x}\frac{d}{dx}(2x) - e^{x}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = 2e^{2x} - e^{x}$$

$$\frac{d^2y}{dx^2} = 2\left[e^{2x}\frac{d}{dx}(2x)\right] - e^{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4e^{2x} - e^{x}$$

$$\therefore y'' = 4e^{2x} - e^{x}$$

(d)

$$y = -x^3 + 3x^2$$

$$\frac{dy}{dx} = -3x^{3-1} + 3\times 2x^{2-1}$$

$$\Rightarrow \frac{dy}{dx} = -3x^2 + 6x$$

and

$$\frac{d^2y}{dx^2} = -3\times 2x + 6$$

$$\Rightarrow y'' = -6x + 6$$

Page 637-638 (8)

Q#2 (a)

$y = e^{3t}$

$\frac{dy}{dt} = e^{3t} \frac{d}{dt}(3t)$

$\Rightarrow \frac{dy}{dt} = 3e^{3t}$

and $\frac{d^2y}{dt^2} = 3[e^{3t} \frac{d}{dt}(3t)]$

$\Rightarrow \frac{d^2y}{dt^2} = 3e^{3t} \times 3$

$\Rightarrow \frac{d^2y}{dt^2} = 9e^{3t}$

and $\frac{d^3y}{dt^3} = 9[e^{3t} \frac{d}{dt}(3t)]$

$\Rightarrow \frac{d^3y}{dt^3} = 27e^{3t}$

and $\frac{d^4y}{dt^4} = 27e^{3t} \frac{d}{dt}(3t)$

$\Rightarrow \frac{d^4y}{dt^4} = 27 \times 3 e^{3t}$

$\Rightarrow \frac{d^4y}{dt^4} = 81 e^{3t}$

(d) $y = \sin kt$

$\frac{dy}{dt} = \cos kt \frac{d}{dt}(kt)$

$\Rightarrow \frac{dy}{dt} = \cos kt \times k$

$\Rightarrow dy/dt = k \cos kt$

and $\frac{d^2y}{dt^2} = k[-\sin kt \frac{d}{dt}(kt)]$

$\Rightarrow \frac{d^2y}{dt^2} = -k \sin kt \, (k)$

$\Rightarrow \frac{d^2y}{dt^2} = -k^2 \sin kt$

and $\frac{d^3y}{dt^3} = -k^2[\cos kt \frac{d}{dt}(kt)]$

$\Rightarrow \frac{d^3y}{dt^3} = -k^3 \cos kt$

and $\frac{d^4y}{dt^4} = -k^4[-\sin kt \frac{d}{dt}(kt)]$

$\therefore \ddddot{y} = k^4 \sin kt$

Q#3

$y = e^x + 2x$

$y'' - y' - 2y = -2 - 2x - e^x \rightarrow (1)$

$y = e^x + 2x$

$\frac{dy}{dx} = y' = e^x + 2$

$\frac{d^2y}{dx^2} = y'' = e^x$

L.H.S of equation (1)

$y'' - y' - 2y$

Putting values of y, y' and y''

$e^x - (e^x + 2) - 2(e^x + 2x)$

$= e^x - e^x - 2 - 2e^x - 2x$

$= -2 - 2x - e^x =$ R.H.S

$\therefore$ L.H.S = R.H.S

Q#4 (a)

$y(t) = \sin 3t + t^3$

$\frac{dy}{dt} = \cos 3t \frac{d}{dt}(3t) + 3t^{3-1}$

$\Rightarrow \frac{dy}{dt} = 3\cos 3t + 3t^2$

and $\frac{d^2y}{dt^2} = 3[-\sin 3t \frac{d}{dt}(3t)] + 3 \times 2t$

$= -3\sin 3t \, (3) + 6t$

$\Rightarrow \frac{d^2y}{dt^2} = -9\sin 3t + 6t$

$\frac{d^3y}{dt^3} = -9[\cos 3t \frac{d}{dt}(3t)] + 6$

$\Rightarrow d^3y/dt^3 = -9 \times 3 \cos 3t + 6$

$\Rightarrow \dddot{y}(t) = -27 \cos 3t + 6$

at $t = 0$

$\Rightarrow \dddot{y}(0) = -27 \cos 0 + 6$

$\Rightarrow \dddot{y}(0) = -27 + 6$

$\Rightarrow \dddot{y}(0) = -21$

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Q#4 (c)

$y(x) = e^{-x}(e^{x}+1)$

$\Rightarrow y(x) = e^{-x}e^{x} + e^{-x}$

$\Rightarrow y(x) = e^{-x+x} + e^{-x}$

$\Rightarrow y = e^{0} + e^{-x}$

$\Rightarrow y = 1 + e^{-x}$

$\frac{dy}{dx} = 0 + e^{-x}\frac{d}{dx}(-x)$

$\Rightarrow \frac{dy}{dx} = -e^{-x}$

and $\frac{d^2y}{dx} = -[e^{-x}\frac{d}{dx}(-x)]$

$\Rightarrow \frac{d^2y}{dx^2} = +e^{-x}$

and $\frac{d^3y}{dx^3} = e^{-x}\frac{d}{dx}(-x)$

$\Rightarrow y'''(x) = -e^{-x}$

$\Rightarrow$ at $x=0$

$y'''(0) = -e^{0}$

$\Rightarrow y'''(0) = -1$

Q#4 (d)

$y = 3 - 3t^{4}$

$\frac{dy}{dt} = 0 - 3\times4t^{4-1}$

$\Rightarrow \frac{dy}{dt} = -12t^{3}$

and $\frac{d^2y}{dt^2} = -12\times3t^{3-1}$

$\Rightarrow \frac{d^2y}{dt^2} = -36t^{2}$

and $\frac{d^3y}{dt^3} = -36\times2t$

$\Rightarrow \frac{d^3y}{dt^3} = -72t$

$\Rightarrow \dddot{y}(t) = -72t$

at $t=0$

$\dddot{y}(0) = -72\times0$

$\Rightarrow \dddot{y}(0) = 0$

Q#5

$y(x) = x^{4} - 3x^{3} + 3x^{2} + 1$

$\frac{dy}{dx} = 4x^{4-1} - 3\times3x^{3-1} + 3\times2x^{2-1} + 0$

$\Rightarrow \frac{dy}{dx} = 4x^{3} - 9x^{2} + 6x$

and $\frac{d^2y}{dx^2} = 4\times3x^{3-1} - 9\times2x^{2-1} + 6$

$\Rightarrow \frac{d^2y}{dx^2} = 12x^{2} - 18x + 6$

$\Rightarrow y''(x) = 12x^{2} - 18x + 6$

at $x=0$

i.e. $y''(0) = 12(0) - 18(0) + 6$

$\therefore y''(0) = 6$

when $y'' = 0$

$\Rightarrow 12x^{2} - 18x + 6 = 0$

$\Rightarrow 6(2x^{2} - 3x + 1) = 0$ $\quad 6 \neq 0$

so $2x^{2} - 3x + 1 = 0$

$\Rightarrow 2x^{2} - 2x - x + 1 = 0$

$\Rightarrow 2x(x-1) - 1(x-1) = 0$

$\Rightarrow (x-1)(2x-1) = 0$

either $x-1=0$ or $2x-1=0$

$\Rightarrow x=1$ $\quad \Rightarrow 2x=1$

$\Rightarrow x=\frac{1}{2}$

$\therefore x = \{1, \frac{1}{2}\}$

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11 Differentiation

Differentiation is one of the most important processes in engineering mathematics. It is the study of the way in which functions change. The function may represent pressure, stress, volume or some other physical variable. For example, pressure of a vessel may depend upon temperature. As the temperature of the vessel increases, then so does the pressure. Engineers often need to know the rate at which such a variable changes.

Block 1 explains how to calculate the rate of change of a function, $y(x)$, across a range of values of the input variable, x. The rate of change of $y(x)$ at a single point is then developed. This requires the introduction of the idea of taking limits which is also important in Chapter 14 on the applications of integration.

In practice, most people use a standard table to differentiate functions and how this is done is explained in Block 2. The chapter closes with a study of repeated differentiation.

Chapter 11 contents

BLOCK

1 Interpretation of a derivative

1.1 Introduction

Engineers are often interested in the rate at which some variable is changing. For example, an engineer needs to know the rate at which the pressure in a vessel is changing, the rate at which the voltage across a capacitor is changing and the rate at which the temperature is changing in a chemical reaction. Rapid rates of change of a variable may indicate that a system is not operating normally and approaching critical values. Alarms may be triggered.

Rates of change may be positive, negative or zero. A positive rate of change means that the variable is increasing; a negative rate of change means that the variable is decreasing. A zero rate of change means that the variable is not changing.

Consider Figure 1.1 which illustrates a variable, $y(x)$.

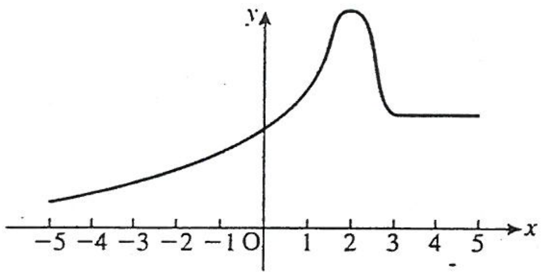

Figure 1.1 The function $y(x)$ changes at different rates for different values of x.

Between $x = -5$ and $x = -3$, y is increasing slowly. Across this interval the rate of change of y is small and positive. Between $x = -3$ and $x = 1$, y is increasing more rapidly; the rate of change of y is positive and fairly large. Between $x = 1$ and $x = 2$, y is increasing very rapidly and so the rate of change is positive and large. From $x = 2$ to $x = 3$, y decreases rapidly; the rate of change is large and negative. From $x = 3$ to $x = 5$, y is constant and so the rate of change on this interval is zero.

The technique for calculating rate of change is called **differentiation**. Often it is not sufficient to describe a rate of change as, for example, 'positive and large' or 'negative and quite small'. A precise value is needed. Use of differentiation provides a precise value or expression for the rate of change of a function.

1.2 Average rate of change across an interval

We see from Figure 1.1 that a function can have different rates of change at different points on its graph. We begin by defining and then calculating the

average rate of change of a function across an interval. Figure 1.2 shows a function, $y(x)$, and values x_1, x_2, $y(x_1)$ and $y(x_2)$.

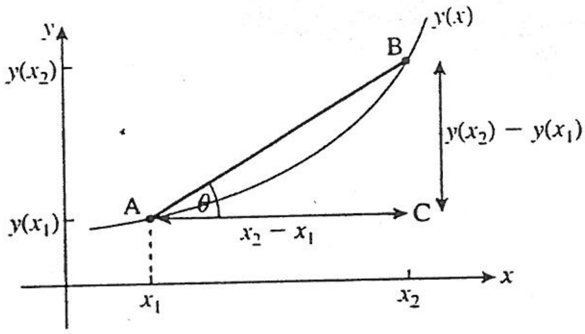

Figure 1.2 Average rate of change $= \dfrac{y(x_2) - y(x_1)}{x_2 - x_1}$.

Consider x increasing from x_1 to x_2. The change in x is $x_2 - x_1$. As x increases from x_1 to x_2, then y increases from $y(x_1)$ to $y(x_2)$. The change in y is $y(x_2) - y(x_1)$. Then the average rate of change of y across the interval is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y(x_2) - y(x_1)}{x_2 - x_1} = \frac{\text{BC}}{\text{AC}}$$

From Figure 1.2 we see that $\frac{\text{BC}}{\text{AC}} = \tan\theta$ which is also the gradient of the straight line or **chord** AB. Hence we see that the average rate of change across an interval is identical to the gradient of the chord across that interval.

KEY POINT

$$\text{average rate of change of } y = \frac{\text{change in } y}{\text{change in } x} = \text{gradient of chord}$$

Example 1.1

Calculate the average rate of change of $y = x^2$ across the interval

(a) $x = 1$ to $x = 4$
(b) $x = -2$ to $x = 0$

Solution

(a) Change in $x = 4 - 1 = 3$.

When $x = 1$, $y = 1^2 = 1$. When $x = 4$, $y = 4^2 = 16$. Hence the change in y is $16 - 1 = 15$. So

$$\text{average rate of change across interval } [1, 4] = \tfrac{15}{3} = 5$$

This means that across the interval [1, 4], on average the y value increases by 5 for every 1 unit increase in x.

(b) Change in $x = 0 - (-2) = 2$. We have $y(-2) = 4$ and $y(0) = 0$ so the change in y is $0 - 4 = -4$. Hence

$$\text{average rate of change} = \frac{-4}{2} = -2$$

On average, across the interval $x = -2$ to $x = 0$, y decreases by 2 units for every 1 unit increase in x.

Example 1.2

The voltage, $v(t)$, across a capacitor varies with time, t, according to

$$v(t) = 3 + 2e^{-t}$$

Find the average rate of change of voltage as time varies

(a) from $t = 0$ to $t = 2$
(b) from $t = 1$ to $t = 3$

Solution

(a) Change in $t = 2 - 0 = 2$.

When $t = 0$, $v = 3 + 2e^0 = 5$.

When $t = 2$, $v =$ ☐ — $3 + 2e^{-2} = 3.2707$

So

average rate of change of $v(t) =$ ☐ — $\dfrac{3.2707 - 5}{2} = -0.8647$

(b) Change in $t =$ ☐ — $3 - 1 = 2$

$v(1) = 3 + 2e^{-1} = 3.7358$

$v(3) =$ ☐ — $3 + 2e^{-3} = 3.0996$

So

average rate of change of $v(t)$ across $[1, 3] =$ ☐ — $\dfrac{3.0996 - 3.7358}{2} = -0.3181$

Across the interval from $t = 1$ to $t = 3$, the voltage is decreasing but at a slower rate than across the interval from $t = 0$ to $t = 2$.

Exercises

1 Calculate the average rate of change of $y = x^2 + 2x$ from $x = 1$ to $x = 4$.

2 Calculate the average rate of change of $h(t) = 2t^2 - 2t + 1$ from $t = 0$ to $t = 2$.

3 Calculate the average rate of change of $i(t) = 50 \sin t$ from $t = 0$ to $t = \pi$.

4 Calculate the average rate of change of $r(x) = \dfrac{1}{x+1}$ from $x = -3$ to $x = -2$.

5 Calculate the average rate of change of $z(t) = 4 + 2t^2$ across (a) $t = 1$ to $t = 3$, (b) $t = -1$ to $t = 0$.

6 The temperature, T, of a vessel varies with time, t, according to

$$T(t) = 320 + \frac{65}{t^2}$$

Calculate the average rate of change of T from $t = 2$ to $t = 4$.

Solutions to exercises

1 7

2 2

3 0

4 −0.5

5 (a) 8 (b) −2

6 −6.094

1.3 Rate of change at a point

As mentioned earlier, we often need to know the rate of change of a function at a point, and not simply an average rate of change across an interval.

Refer again to Figure 1.2. Suppose we wish to find the rate of change of y at the point A. The average rate of change across the interval from $x = x_1$ to $x = x_2$ is given by the gradient of the chord AB. This provides an approximation to the rate of change at A.

Suppose the chord AB is extended on both sides as shown in Figure 1.3. As B is moved closer to A, the gradient of the chord provides a better approximation to the rate of change at A.

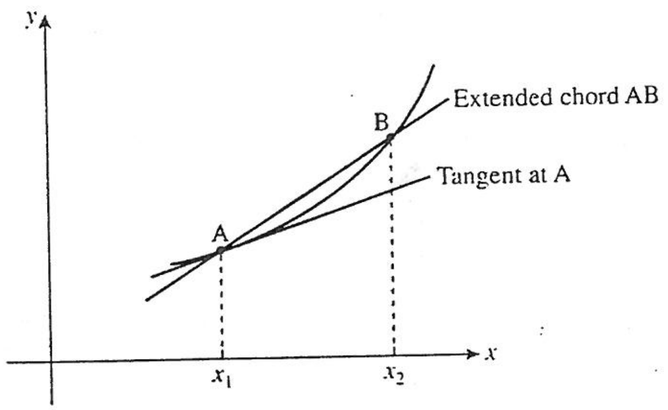

Figure 1.3 The extended chord ultimately becomes the tangent at A.

Ultimately B is made coincident with A and then the chord AB becomes a tangent to the curve at A. The gradient of this tangent gives the rate of change of y at A:

KEY POINT

rate of change at a point = gradient of tangent to the curve at that point

Calculating the rate of change of a function at a point by measuring the gradient of a tangent is usually not an accurate method. Consequently we develop an exact, algebraic way of finding rates of change.

Consider the function $y(x)$ as shown in Figure 1.4.

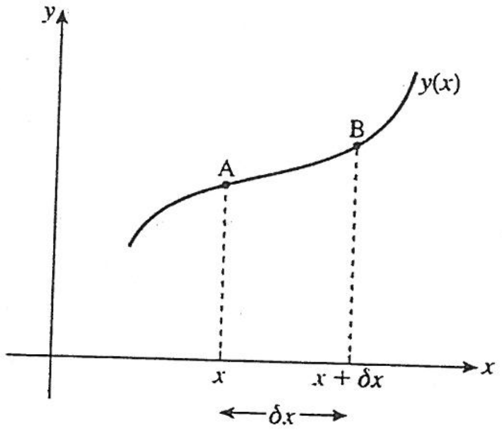

Figure 1.4 As B approaches A, $\delta x \to 0$.

Let A be a point on the curve with coordinates $(x, y(x))$. B is a point on the curve near to A. The x coordinate of B is $x + \delta x$. The term δx is pronounced 'delta x'. It represents a small change in the x direction. The y coordinate of B is $y(x + \delta x)$. We calculate the gradient of the chord AB:

$$\text{gradient of AB} = \frac{\text{change in } y}{\text{change in } x} = \frac{y(x+\delta x) - y(x)}{x + \delta x - x} = \frac{y(x+\delta x) - y(x)}{\delta x}$$

The change in y, that is $y(x + \delta x) - y(x)$, is also written as δy. So

$$\text{gradient of AB} = \frac{y(x+\delta x) - y(x)}{\delta x} = \frac{\delta y}{\delta x}$$

The gradient of AB gives the average rate of change of $y(x)$ across the small interval from x to $x + \delta x$. To calculate the rate of change of $y(x)$ at A we require the gradient of the tangent at A.

Consider A as a fixed point and let B move along the curve towards A. At each position of B we can calculate the gradient of the chord AB. As B gets closer to A, the chord AB approximates more closely to the tangent at A. Also, as B approaches A, the distance δx decreases. To find the gradient of the tangent at A we calculate the gradient of the chord AB and let δx get smaller and smaller. We say δx tends to zero and write this as $\delta x \to 0$.

As B approaches A, the x difference between A and B gets smaller, that is $\delta x \to 0$, and likewise the y difference, δy, also gets smaller, so $\delta y \to 0$. However, the gradient of AB, given by the ratio $\frac{\delta y}{\delta x}$, approaches a definite value, called a limit. So we seek the limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$. We write this as

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Note that 'limit' has been shortened to 'lim'.

In summary we have

KEY POINT

$$\text{rate of change of } y = \text{gradient of tangent} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Let us see this applied to an example.

Example 1.3

Find the rate of change of $y(x) = x^2$.

Solution

Suppose A is the fixed point with coordinates (x, x^2) as shown in Figure 1.5. B is a point on the curve near to A with coordinates $(x + \delta x, (x + \delta x)^2)$. We calculate the gradient of the chord AB.

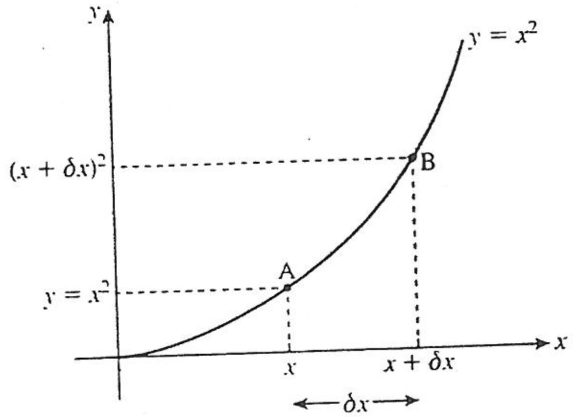

Figure 1.5 The gradient of the tangent at A is approximated by the gradient of the chord AB.

$$\begin{aligned}
\text{change in } x &= \delta x \\
\text{change in } y &= \delta y \\
&= (x+\delta x)^2 - x^2 \\
&= x^2 + 2x\delta x + (\delta x)^2 - x^2 \\
&= 2x(\delta x) + (\delta x)^2
\end{aligned}$$

$$\begin{aligned}
\text{gradient of chord AB} &= \frac{\delta y}{\delta x} \\
&= \frac{2x(\delta x) + (\delta x)^2}{\delta x} \\
&= 2x + \delta x
\end{aligned}$$

This is the average rate of change of $y(x)$ across the small interval from x to $x + \delta x$. To obtain the gradient of the tangent at A, we let $\delta x \to 0$.

$$\begin{aligned}
\text{gradient of tangent at A} &= \lim_{\delta x \to 0} (2x + \delta x) \\
&= 2x
\end{aligned}$$

Hence the rate of change of x^2 is $2x$.

For example, if $x = 3$, then A is the point (3, 9) and the rate of change of y at this point is 6. Similarly if $x = -1$, A is the point $(-1, 1)$ and the rate of change of y is -2.

Exercises

1 Find the rate of change of $y(x) = x^2 + 1$. Calculate the rate of change of y when x is (a) 6 (b) 3 (c) -2 (d) 0

2 Find the rate of change of $y(x) = x^2 + 2x$. Calculate the rate of change of y when x is (a) 6 (b) -5 (c) 0

Solutions to exercises

1 $2x$, (a) 12 (b) 6 (c) -4 (d) 0

2 $2x + 2$, (a) 14 (b) -8 (c) 2

1.4 Terminology and notation

The process of finding the rate of change of a given function is called **differentiation**. The function is said to be **differentiated**. If y is a function of the independent variable x, we say that y is differentiated with respect to (w.r.t.) x. The rate of change of a function is also known as the **derivative** of the function.

There is a notation for writing down the derivative of a function. If the function is $y(x)$, we denote the derivative of y by

$$\frac{dy}{dx}$$

pronounced 'dee y by dee x'. Hence

KEY POINT

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Another notation for the derivative is simply y', pronounced y dash. Similarly if the function is $z(t)$ we write the derivative as $\frac{dz}{dt}$ or z'. When the independent variable is t, the derivative may also be denoted using the dot notation. Thus, for example, $\frac{dz}{dt}$ may be written as $\dot{z}$, pronounced 'z dot'. Sometimes, instead of writing y, a function is written in full; for example, to show the derivative of $\sin 5x$ we write

$$\frac{d(\sin 5x)}{dx}$$

Exercises

1 If x is a function of the independent variable t, write down two ways in which the derivative can be written.

2 If f is a function of x, write down two ways in which the derivative can be written.

Solutions

1 $\frac{dx}{dt}$ or $\dot{x}$

2 $\frac{df}{dx}$ or f'

End of block exercises

1 Calculate the average rate of change of $y = x^3 - 1$ from
(a) $x = 1$ to $x = 3$
(b) $x = 0$ to $x = 2$
(c) $x = -2$ to $x = 2$

2 The pressure, P atmospheres, in a vessel varies with temperature, T (Celsius), according to

$$P(T) = 120 - 20e^{-T/20}$$

Calculate the average rate of change of pressure as T varies from 10°C to 100°C.

3 The current, $i(t)$, in a circuit decays exponentially with time, t, according to the equation

$$i(t) = 5 + 2e^{-t}$$

Calculate the average rate of change of current as t varies from 0 to 3.

4 Explain the meaning of the expression $\frac{dy}{dx}$.

5 (a) Calculate the rate of change of $y(x) = 5 - x^2$.
(b) Calculate the rate of change of y when $x = -4$.

6 (a) Calculate $\frac{dR}{dx}$ when $R(x) = 2x^2$.

(b) Calculate $\frac{dR}{dx}$ when $x = 0.5$.

Solutions to exercises

1 (a) 13 (b) 4 (c) 4

2 0.13

3 -0.63

5 (a) $-2x$ (b) 8

6 (a) $4x$ (b) 2

BLOCK

2 Using a table of derivatives

2.1 Introduction

Block 1 gave a brief introduction to the meaning of a derivative. A derivative is the rate of change of a function. Geometrically we saw that this is given by the gradient of a tangent. If we consider a typical function, as illustrated in Figure 2.1 it is clear that the gradient of a tangent depends upon where the tangent is drawn. For example, tangent A, drawn where x has a value x_1, has a different gradient to tangent B, drawn where x has a value x_2. In other words, the gradient of the tangent is itself a function of x. This was seen in Example 1.3 of Block 1 where the gradient was found to be $2x$.

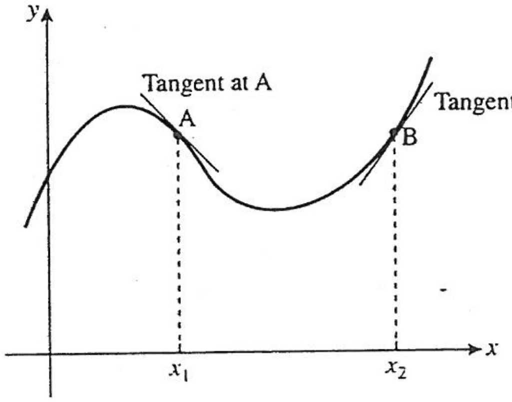

Figure 2.1 The gradient of a tangent varies along the curve.

Rather than calculate the derivative of a function as explained in Block 1, it is common practice to use a table of derivatives. This block shows how to use such a table.

2.2 Table of derivatives

Table 2.1 lists some of the common functions used in engineering and their corresponding derivatives

Example 2.1

Use Table 2.1 to find $\frac{dy}{dx}$ when y is given by

(a) $3x$ (b) 3 (c) $3x^2$ (d) $4x^7$

Table 2.1 *Common functions and their derivatives.*

Function	Derivative
constant	0
x	1
kx	k k constant
x^n	nx^{n-1} n constant
kx^n	knx^{n-1} k, n constants
e^x	e^x
e^{kx}	ke^{kx} k constant
$\ln x$	$\frac{1}{x}$
$\ln kx$	$\frac{1}{x}$
$\sin x$	$\cos x$ x in radians
$\sin kx$	$k\cos kx$ k constant, kx in radians
$\sin(kx+\alpha)$	$k\cos(kx+\alpha)$ k, α constants, $kx+\alpha$ in radians
$\cos x$	$-\sin x$ x in radians
$\cos kx$	$-k\sin kx$ k constant, kx in radians
$\cos(kx+\alpha)$	$-k\sin(kx+\alpha)$ k, α constants, $kx+\alpha$ in radians
$\tan x$	$\sec^2 x$ x in radians
$\tan kx$	$k\sec^2 kx$ k constant, kx in radians
$\tan(kx+\alpha)$	$k\sec^2(kx+\alpha)$ k, α constant, $kx+\alpha$ in radians

Solution

(a) We note that $3x$ is of the form kx where $k=3$. Using Table 2.1 we then have $\frac{dy}{dx}=3$.

(b) Noting that 3 is a constant we see that $\frac{dy}{dx}=0$.

(c) We see that $3x^2$ is of the form kx^n, with $k=3$ and $n=2$. The derivative, knx^{n-1}, is then $6x^1$, or more simply, $6x$. So if $y=3x^2$, then $\frac{dy}{dx}=6x$.

(d) We see that $4x^7$ is of the form kx^n, with $k=4$ and $n=7$. Hence the derivative, $\frac{dy}{dx}$, is given by $28x^6$.

Example 2.2

Find $\frac{dy}{dx}$ when y is (a) $\sqrt{x}$ (b) $\frac{3}{x^2}$ (c) $\frac{2}{x}$.

Solution

(a) We write $\sqrt{x}$ as $x^{\frac{1}{2}}$, and use the result for x^n with $n = \frac{1}{2}$. So

$$\frac{dy}{dx} = nx^{n-1}$$
$$= \tfrac{1}{2}x^{\frac{1}{2}-1}$$
$$= \tfrac{1}{2}x^{-\frac{1}{2}}$$

This may be written as $\dfrac{1}{2\sqrt{x}}$.

(b) We write $\frac{3}{x^2}$ as $3x^{-2}$. Using the result of kx^n we see that

$$\frac{dy}{dx} = knx^{n-1}$$
$$= \boxed{}$$

$3(-2)x^{-2-1} = -6x^{-3}$

(c) We write $\frac{2}{x}$ as $2x^{-1}$. Then we see that

$$\frac{dy}{dx} = \boxed{}$$

$2(-1)x^{-1-1} = -2x^{-2}$

Example 2.3

Use Table 2.1 to find $\dfrac{dz}{dt}$ given z is (a) e^t (b) e^{3t} (c) e^{-5t}.

Solution

Although Table 2.1 is written using x as the independent variable, it can be used for any variable.

(a) From Table 2.1, if $y = e^x$, then $\dfrac{dy}{dx} = e^x$. Hence if $z = e^t$ then $\dfrac{dz}{dt} = e^t$.

(b) From Table 2.1 we see that when $y = e^{3x}$ then

$$\frac{dy}{dx} = \boxed{}$$

$3e^{3x}$

Hence $\dfrac{dz}{dt} = \boxed{}$

$3e^{3t}$

(c) Using the result for e^{kx} in Table 2.1 we see that when $z = e^{-5t}$,

$$\frac{dz}{dt} = \boxed{}$$

$-5e^{-5t}$

Example 2.4

Find the derivative, $\dfrac{dy}{dx}$, when y is (a) $\sin 3x$ (b) $\cos\dfrac{x}{2}$ (c) $\tan 2x$.

Solution

(a) Using the result for $\sin kx$, and taking $k = 3$, we see that

$$\frac{dy}{dx} = 3\cos 3x$$

(b) From the result for $\cos kx$, and taking $k = \frac{1}{2}$ we see that

$$\frac{dy}{dx} = \boxed{}$$

$$-\frac{1}{2}\sin\frac{x}{2}$$

(c) From the result for $\tan kx$, we see that

$$\frac{dy}{dx} = \boxed{}$$

$$2\sec^2 2x$$

Exercises

1 Find the derivative of the following functions:
(a) $9x$ (b) $4x$ (c) $6x^3$ (d) $-3x^2$ (e) $\ln 3t$

2 Find $\frac{dz}{dt}$ when z is given by
(a) $\frac{4}{t^3}$ (b) $\sqrt{t^3}$ (c) $5t^{-2}$ (d) $-\frac{3}{2}t^{3/2}$

3 Find the derivative of each of the following functions:
(a) $\sin 5x$ (b) $\cos 4t$ (c) $\tan 3r$ (d) e^{2v} (e) $\frac{1}{e^{3t}}$

4 Find the derivative of the following:
(a) $\cos\frac{2x}{3}$ (b) $\sin(-2x)$ (c) $\tan \pi x$ (d) $e^{x/2}$

Solutions to exercises

1 (a) 9 (b) 4 (c) $18x^2$ (d) $-6x$ (e) $\frac{1}{t}$

2 (a) $-12t^{-4}$ (b) $\frac{3}{2}t^{1/2}$ (c) $-10t^{-3}$ (d) $-\frac{9}{4}t^{1/2}$

3 (a) $5\cos 5x$ (b) $-4\sin 4t$ (c) $3\sec^2 3r$
(d) $2e^{2v}$ (e) $-3e^{-3t}$

4 (a) $-\frac{2}{3}\sin\frac{2x}{3}$ (b) $-2\cos(-2x)$ (c) $\pi\sec^2 \pi x$
(d) $\frac{1}{2}e^{x/2}$

2.3 Extending the table of derivatives

We introduce two simple rules which enable us to extend the range of functions which we can differentiate.

KEY POINT

> The derivative of $f(x) \pm g(x)$ is
>
> $$\frac{df}{dx} \pm \frac{dg}{dx}$$

This rule says that to find the derivative of the sum (difference) of two functions, we simply calculate the sum (difference) of the derivatives of each function.

KEY POINT

The derivative of $kf(x)$ is

$$k\frac{df}{dx}$$

This rule tells us that if a function is multiplied by a constant, k, then the derivative is likewise multiplied by the same constant, k.

Example 2.5

Find the derivative of each of the following functions:

(a) $y = 6\sin 2x$ (b) $y = 6\sin 2x + 3x^2$ (c) $y = 6\sin 2x + 3x^2 - 5e^{3x}$

Solution

(a) From Table 2.1, the derivative of $\sin 2x$ is $2\cos 2x$. Hence the derivative of $6\sin 2x$ is $6(2\cos 2x)$, that is $12\cos 2x$.

$$y = 6\sin 2x, \quad \frac{dy}{dx} = 6(2\cos 2x) = 12\cos 2x$$

(b) The function comprises two parts: $6\sin 2x$ and $3x^2$. We have already differentiated $6\sin 2x$ in part (a), so we consider the derivative of $3x^2$. The derivative of x^2 is $2x$ and so the derivative of $3x^2$ is $3(2x)$, that is $6x$. These derivatives are now summed.

$$y = 6\sin 2x + 3x^2, \quad \frac{dy}{dx} = 12\cos 2x + 6x$$

(c) We differentiate each part of the function in turn.

$$y = 6\sin 2x + 3x^2 - 5e^{3x}$$

$$\frac{dy}{dx} = 6(2\cos 2x) + 3(2x) - 5(3e^{3x}) = 12\cos 2x + 6x - 15e^{3x}$$

Example 2.6

Find $\frac{dy}{dx}$ where y is defined by

(a) $\frac{x^6}{2} - 3e^{-2x}$ (b) $4\cos\frac{x}{2} + 9 - 9x^3$

Solution

(a) The derivative of x^6 is $6x^5$. Hence the derivative of $\frac{x^6}{2}$ is [ ]

$\frac{6x^5}{2} = 3x^5$

The derivative of e^{-2x} is [ ]

$-2e^{-2x}$

Hence the derivative of $3e^{-2x}$ is

[ ] $3(-2)e^{-2x} = -6e^{-2x}$

So given

$$y = \frac{x^6}{2} - 3e^{-2x}$$

then

$$\frac{dy}{dx} = \boxed{}$$ $3x^5 + 6e^{-2x}$

(b) The derivative of $\cos\frac{x}{2}$ is

[ ] $-\frac{1}{2}\sin\frac{x}{2}$

The derivative of 9 is zero. The derivative of $9x^3$ is

[ ] $27x^2$

So given

$$y = 4\cos\frac{x}{2} + 9 - 9x^3$$

then

$$\frac{dy}{dx} = \boxed{}$$ $-2\sin\frac{x}{2} - 27x^2$

Exercises

1 Find $\frac{dy}{dx}$ when y is given by
(a) $4x^6 + 8x^3$ (b) $-3x^4 + 2x^{1.5}$
(c) $\frac{9}{x^2} + \frac{14}{x} - 3x$ (d) $\frac{3+2x}{4}$ (e) $(2+3x)^2$

2 Find the derivative of each of the following functions:
(a) $z(t) = 5\sin t + \sin 5t$
(b) $h(v) = 3\cos 2v - 6\sin\frac{v}{2}$
(c) $m(n) = 4e^{2n} + \frac{2}{e^{2n}} + \frac{n^2}{2}$
(d) $H(t) = \frac{e^{3t}}{2} + 2\tan 2t$
(e) $S(r) = (r^2+1)^2 - 4e^{-2r}$

3 Differentiate the following functions:
(a) $A(t) = (3 + e^t)^2$
(b) $B(s) = \pi e^{2s} + \frac{1}{s} + 2\sin\pi s$
(c) $V(r) = \left(1 + \frac{1}{r}\right)^2 + (r+1)^2$
(d) $M(\theta) = 6\sin 2\theta - 2\cos\frac{\theta}{4} + 2\theta^2$
(e) $H(t) = 4\tan 3t + 3\sin 2t - 2\cos 4t$

Solutions to exercises

1 (a) $24x^5 + 24x^2$ (b) $-12x^3 + 3x^{0.5}$
(c) $-\frac{18}{x^3} - \frac{14}{x^2} - 3$ (d) $\frac{1}{2}$ (e) $12 + 18x$

2 (a) $5\cos t + 5\cos 5t$ (b) $-6\sin 2v - 3\cos\frac{v}{2}$
(c) $8e^{2n} - 4e^{-2n} + n$ (d) $\frac{3e^{3t}}{2} + 4\sec^2 2t$
(e) $4r^3 + 4r + 8e^{-2r}$

3 (a) $6e^t + 2e^{2t}$ (b) $2\pi e^{2s} - \frac{1}{s^2} + 2\pi\cos(\pi s)$
(c) $-\frac{2}{r^2} - \frac{2}{r^3} + 2r + 2$
(d) $12\cos 2\theta + \frac{1}{2}\sin\frac{\theta}{4} + 4\theta$
(e) $12\sec^2 3t + 6\cos 2t + 8\sin 4t$

2.4 Evaluating a derivative

Engineers may need to find the rate of change of a function at a particular point; that is, find the derivative of a function at a specific point. We do this by finding the derivative of the function, and then evaluating the derivative at the given value of x. When evaluating, all angles are in radians. Consider a function, $y(x)$. We use the notation $\frac{dy}{dx}(0.7)$ or $y'(0.7)$ to denote the derivative of y evaluated at $x = 0.7$.

Example 2.7

Find the value of the derivative of $y = 3x^2$ where $x = 4$. Interpret your result.

Solution

We have $y = 3x^2$ and so $\frac{dy}{dx} = 6x$. We now evaluate the derivative.

When $x = 4$, $\frac{dy}{dx} = 6(4) = 24$, that is

$$\frac{dy}{dx}(4) = 24$$

The derivative is positive when $x = 4$ and so y is increasing at this point. Thus when $x = 4$, y is increasing at a rate of 24 vertical units per horizontal unit.

Example 2.8

Find the rate of change of current, $i(t)$, given by

$$i(t) = 3e^{-t} + 2 \quad t \geq 0$$

when $t = 0.7$ seconds.

Solution

The rate of change of a function is the same as the derivative of the function, that is $\frac{di}{dt}$.

$$\frac{di}{dt} = \square$$

$-3e^{-t}$

When $t = 0.7$

$$\frac{di}{dt} = \boxed{}$$

$-3e^{-0.7} = -1.4898$

The derivative is negative and so we know that $i(t)$ is decreasing when $t = 0.7$. Thus, when $t = 0.7$, the current is decreasing at a rate of $1.49\ \mathrm{A\,s^{-1}}$.

Exercises

1 Calculate the derivative of $y = 3x^2 + e^x$ when $x = 0.5$.

2 Calculate the rate of change of $i(t) = 4\sin 2t + 3t$ when (a) $t = \frac{\pi}{3}$ (b) $t = 0.6$.

3 Evaluate the rate of change of $H(t) = 5\sin t - 3\cos 2t$ when (a) $t = 0$ (b) $t = 1.3$.

Solutions to exercises

1 4.6487

2 (a) −1 (b) 5.8989

3 (a) 5 (b) 4.4305

End of block exercises

1 Find $\frac{dy}{dx}$ when y is given by

(a) $7x^5 + 6x^{-2} + \sin 2x$
(b) $3\cos 4x - 6\sin 5x$
(c) $e^{3x} + e^{-3x} + 2e^x + 1$
(d) $4\tan\frac{x}{2} + \frac{1}{\sqrt{x}}$
(e) $3\sqrt{x} + \frac{9}{x} + \frac{1}{2}\sin 6x + \ln x$

2 Find the rate of change of the following functions:
(a) $e^t + e^{-t}$ (b) $2\sin 3t + \ln 2t$ (c) $-3\cos x$
(d) $\sqrt{r} + 2r^2$ (e) $2e^{-0.5r} + r^3$

3 Find $\dot{x}$ when x is given by
(a) $2t^4 - 3t + 1 + 2\ln t$ (b) $\sin \pi t - 2\cos \pi t$
(c) $3\tan 2t - e^t$ (d) $e^{2t} - e^{-2t} + t$
(e) $t^{3/2} - t^{2/3}$

4 Find the rate of change of each function when $t = 1.2$:

(a) $3t^2 - 2t^3$ (b) $\frac{e^{4t}}{2} + 3e^{-t}$

(c) $6\sin\frac{t}{2} + 3\cos\frac{t}{2}$ (d) $2\tan t - \tan 2t$

(e) $\frac{4}{t} + 4\ln t$

5 Find the derivative of the following:

(a) $e^{2t}(e^t + e^{-t})$ (b) $\frac{2t^2 + 1}{t}$
(c) $(2t + 1)(2t - 1)$
(d) $\sin^2 2x + \cos^2 2x + \sin 2x + \cos 2x$
(e) $\frac{1}{e^x}$ (f) $\frac{\sin x}{\cos x}$

Solutions to exercises

1 (a) $35x^4 - 12x^{-3} + 2\cos 2x$
(b) $-12\sin 4x - 30\cos 5x$
(c) $3e^{3x} - 3e^{-3x} + 2e^x$
(d) $2\sec^2\frac{x}{2} - \frac{1}{2}x^{-3/2}$
(e) $\frac{3}{2}x^{-1/2} - \frac{9}{x^2} + 3\cos 6x + \frac{1}{x}$

2 (a) $e^t - e^{-t}$ (b) $6\cos 3t + \frac{1}{t}$ (c) $3\sin x$

(d) $\frac{1}{2}r^{-1/2} + 4r$ (e) $-e^{-0.5v} + 3v^2$

3 (a) $8t^3 - 3 + \frac{2}{t}$ (b) $\pi\cos\pi t + 2\pi\sin\pi t$

(c) $6\sec^2 2t - e^t$

(d) $2e^{2t} + 2e^{-2t} + 1$ (e) $\frac{3}{2}t^{1/2} - \frac{2}{3}t^{-1/3}$

4 (a) -1.44 (b) 242.12 (c) 1.6290 (d) 11.5538
(e) 0.5556

5 (a) $3e^{3t} + e^t$ (b) $2 - \frac{1}{t^2}$ (c) $8t$

(d) $2\cos 2x - 2\sin 2x$ (e) $-e^{-x}$ (f) $\sec^2 x$

BLOCK

3 Higher derivatives

3.1 Introduction

Block 2 showed how to calculate the derivative of a function using a table of derivatives. By differentiating the function, $y(x)$, we obtain the derivative, $\frac{dy}{dx}$.

The function, $\frac{dy}{dx}$, is more correctly called the **first derivative** of y. By differentiating the first derivative, we obtain the **second derivative**; by differentiating the second derivative we obtain the **third derivative** and so on. The second and subsequent derivatives are known as higher derivatives.

Example 3.1

Calculate the first, second and third derivatives of $y = e^{2x} + x^4$.

Solution

The first derivative is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2e^{2x} + 4x^3$$

To obtain the second derivative we differentiate the first derivative:

$$\text{second derivative} = 4e^{2x} + 12x^2$$

The third derivative is found by differentiating the second derivative:

$$\text{third derivative} = 8e^{2x} + 24x$$

3.2 Notation

Just as there is a notation for the first derivative so there is a similar notation for higher derivatives.

Consider the function, $y(x)$. We know that the first derivative is denoted by $\frac{dy}{dx}$ or y'. The second derivative is calculated by differentiating the first derivative, that is

$$\text{second derivative} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

So, the second derivative is denoted by $\frac{d^2y}{dx^2}$. This is often written more concisely as y''.

The third derivative is denoted by $\frac{d^3y}{dx^3}$ or y''' and so on. So, referring to Example 3.1 we could have written

$$\frac{dy}{dx} = 2e^{2x} + 4x^3$$

$$\frac{d^2y}{dx^2} = 4e^{2x} + 12x^2$$

$$\frac{d^3y}{dx^3} = 8e^{2x} + 24x$$

KEY POINT

If $y = y(x)$

$$\text{first derivative} = \frac{dy}{dx}$$

$$\text{second derivative} = \frac{d^2y}{dx^2}$$

$$\text{third derivative} = \frac{d^3y}{dx^3}$$

Derivatives with respect to t are often indicated using a dot notation, so $\frac{dx}{dt}$ can be written as $\dot{x}$. Similarly, a second derivative with respect to t can be written as $\ddot{x}$, pronounced x double dot.

Example 3.2

Calculate $\frac{d^2y}{dt^2}$ and $\frac{d^3y}{dt^3}$ given $y = \sin t + \cos t$.

Solution

$$\frac{dy}{dt} = \cos t - \sin t$$

$$\frac{d^2y}{dt^2} = \boxed{}$$

$-\sin t - \cos t$

$$\frac{d^3y}{dt^3} = \boxed{}$$

$-\cos t + \sin t$

We could have used the dot notation and written $\dot{y} = \cos t - \sin t$, and $\ddot{y} = -\sin t - \cos t$.

We may need to evaluate higher derivatives at specific points. We use an obvious notation. The second derivative of $y(x)$, evaluated at, say, $x = 2$, is written as $\frac{d^2y}{dx^2}(2)$, or more simply as $y''(2)$. The third derivative evaluated at $x = -1$ is written as $\frac{d^3y}{dx^3}(-1)$ or $y'''(-1)$.

Example 3.3

Given

$$y(x) = 2\sin x + 3x^2$$

find (a) $y'(1)$ (b) $y''(-1)$ (c) $y'''(0)$

Solution

We have

$$y = 2\sin x + 3x^2$$
$$y' = 2\cos x + 6x$$
$$y'' = \boxed{}$$

$-2\sin x + 6$

$$y''' = -2\cos x$$

(a) $y'(1) = 2\cos 1 + 6(1) = 7.0806.$

(b) $y''(-1) = \boxed{}$

$-2\sin(-1) + 6 = 7.6829$

(c) $y'''(0) = -2\cos 0 = -2.$

Exercises

1 Find $\dfrac{d^2y}{dx^2}$ where $y(x)$ is defined by
(a) $3x^2 - e^{2x}$ (b) $\sin 3x + \cos x$ (c) $\sqrt{x}$
(d) $e^x + e^{-x}$ (e) $1 + x + x^2 + \ln x$

2 Find $\dfrac{d^3y}{dx^3}$ where y is given in question 1.

3 Calculate $y''(1)$ where $y(t)$ is given by
(a) $t(t^2 + 1)$ (b) $\sin(-2t)$ (c) $2e^t + e^{2t}$ (d) $\dfrac{1}{t}$
(e) $\cos\dfrac{t}{2}$

4 Calculate $y'''(-1)$ of the functions given in question 3.

Solutions to exercises

1 (a) $6 - 4e^{2x}$ (b) $-9\sin 3x - \cos x$
(c) $-\dfrac{1}{4}x^{-3/2}$ (d) $e^x + e^{-x}$ (e) $2 - \dfrac{1}{x^2}$

2 (a) $-8e^{2x}$ (b) $-27\cos 3x + \sin x$ (c) $\frac{3}{8}x^{-5/2}$
(d) $e^x - e^{-x}$ (e) $\dfrac{2}{x^3}$

3 (a) 6 (b) 3.6372 (c) 34.9928 (d) 2
(e) −0.2194

4 (a) 6 (b) −3.3292 (c) 1.8184 (d) −6
(e) −0.0599

End of block exercises

1 Calculate y'' where y is given by
(a) $\cos 2t - \sin 2t$ (b) $e^{2x} - e^x$ (c) $2x^6 - 3x^7$
(d) $-x^3 + 3x^2$ (e) $9 - \dfrac{9}{x}$

2 Find the fourth derivative of the following functions:
(a) e^{3t} (b) e^{kt}, k constant (c) $\sin 2t$
(d) $\sin kt$, k constant (e) $\cos kt$, k constant

3 Show that $y = e^x + 2x$ satisfies the equation

$$y'' - y' - y = -2 - 2x - e^x$$

4 Evaluate $y'''(0)$ where y is given by
(a) $\sin 3t + t^3$ (b) $2\cos t + \cos 2t$
(c) $e^{-x}(e^x + 1)$ (d) $3 - 3t^4$ (e) $\dfrac{e^{2x}+1}{e^x}$

5 The function $y(x)$ is defined by

$$y(x) = x^4 - 3x^3 + 3x^2 + 1$$

Calculate the values of x where $y'' = 0$.

Solutions to exercises

1 (a) $-4\cos 2t + 4\sin 2t$ (b) $4e^{2x} - e^x$
(c) $60x^4 - 126x^5$ (d) $-6x + 6$ (e) $-18x^{-3}$

2 (a) $81e^{3t}$ (b) $k^4 e^{kt}$ (c) $16\sin 2t$ (d) $k^4 \sin kt$
(e) $k^4 \cos kt$

4 (a) -21 (b) 0 (c) -1 (d) 0 (e) 0

5 $\frac{1}{2}$, 1

End of chapter exercises

1 Calculate $\dfrac{dy}{dx}$ where y is given by
(a) $3x^4 - 2x + \ln x$ (b) $\sin 5x - 5\cos x$
(c) $(x+1)^2$ (d) $e^{3x} + 2e^{-2x} + 1$
(e) $5 + 5x + \dfrac{5}{x} + 5\ln x$

2 Find the second derivatives of the functions in question 1.

3 Find $y'(1)$ of the functions in question 1.

4 Find $y''(1)$ of the functions in question 1.

5 Find the rate of change of the following functions:

(a) $\dfrac{3t^3 - t^2}{2t}$ (b) $\ln\sqrt{x}$ (c) $(t+2)(2t-1)$
(d) $e^{3v}(1 - e^v)$ (e) $\sqrt{x}(\sqrt{x} - 1)$

6 Find the third and fourth derivatives of y given the second derivative of y is

(a) $\dfrac{2}{e^{3x}}$ (b) $\dfrac{1+x}{x^2}$ (c) $3\ln x^2$
(d) $\sin x + \sin(-2x)$ (e) $\dfrac{\cos^2 x + \cos x}{\cos x}$

7 Differentiate $(\sin x + \cos x)^2$. (Hint: use the trigonometrical identities on p. 279.)

8 Verify that

$y = A\sin kx + B\cos kx \quad A, B, k$ constants

is a solution of

$$y'' + k^2 y = 0$$

9 The function $y(x)$ is given by $y(x) = 1 - \cos x$. Find the values of x where (a) $y' = 0$, (b) $y'' = 0$.

10 The function $y(x)$ is given by $y(x) = x^3 - 3x$. Calculate the intervals on which y is (a) increasing, (b) decreasing.

11 The function $y(x)$ is given by

$$y(x) = 2x^3 - 9x^2 + 1$$

(a) Calculate the values of x for which $y' = 0$.
(b) Calculate the values of x for which $y'' = 0$.
(c) State the interval(s) on which y is increasing.
(d) State the interval(s) on which y is decreasing.
(e) State the interval(s) on which y' is increasing.
(f) State the interval(s) on which y' is decreasing.

Solutions to exercises

1 (a) $12x^3 - 2 + \frac{1}{x}$ (b) $5\cos 5x + 5\sin x$ (c) $2x + 2$ (d) $3e^{3x} - 4e^{-2x}$ (e) $5 - \frac{5}{x^2} + \frac{5}{x}$

2 (a) $36x^2 - \frac{1}{x^2}$ (b) $-25\sin 5x + 5\cos x$ (c) 2 (d) $9e^{3x} + 8e^{-2x}$ (e) $\frac{10}{x^3} - \frac{5}{x^2}$

3 (a) 11 (b) 5.6257 (c) 4 (d) 59.7153 (e) 5

4 (a) 35 (b) 26.6746 (c) 2 (d) 181.8525 (e) 5

5 (a) $3t - 0.5$ (b) $\frac{1}{2x}$ (c) $4t + 3$ (d) $3e^{3r} - 4e^{4r}$ (e) $1 - \frac{1}{2\sqrt{x}}$

6 (a) $-6e^{-3x}$, $18e^{-3x}$ (b) $-2x^{-3} - x^{-1}$, $6x^{-4} + x^{-2}$ (c) $\frac{6}{x}, \frac{-6}{x^2}$ (d) $\cos x - 2\cos(-2x)$, $-\sin x - 4\sin(-2x)$ (e) $-\sin x, -\cos x$

7 $2\cos 2x$

9 (a) $\pm n\pi,\ n = 0, 1, 2, 3, \ldots$ (b) $\frac{\pi}{2} \pm n\pi,\ n = 0, 1, 2, 3, \ldots$

10 (a) $(-\infty, -1)$ and $(1, \infty)$ (b) $(-1, 1)$

11 (a) 0, 3 (b) 1.5 (c) $(-\infty, 0)$ and $(3, \infty)$ (d) $(0, 3)$ (e) $(1.5, \infty)$ (f) $(-\infty, 1.5)$