Integral Calculus PYP

التكامل

MAT 1060

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تم تنسيق هذا الملف وترتيبه حسب الكتاب المعتمد للتكامل

لجامعة الأمير سطام بن عبد العزيز

تنسيق/ عبد الولي الشميري

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5 INTEGRATION

If a dragster moves with varying velocity over a certain time interval, it is possible to find the distance it travels during that time interval using techniques of calculus.

In this chapter we will begin with an overview of the problem of finding areas—we will discuss what the term "area" means, and we will outline two approaches to defining and calculating areas. Following this overview, we will discuss the Fundamental Theorem of Calculus, which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. We will then use the ideas in this chapter to define the average value of a function, to continue our study of rectilinear motion, and to examine some consequences of the chain rule in integral calculus. We conclude the chapter by studying functions defined by integrals, with a focus on the natural logarithm function.

5.1 AN OVERVIEW OF THE AREA PROBLEM

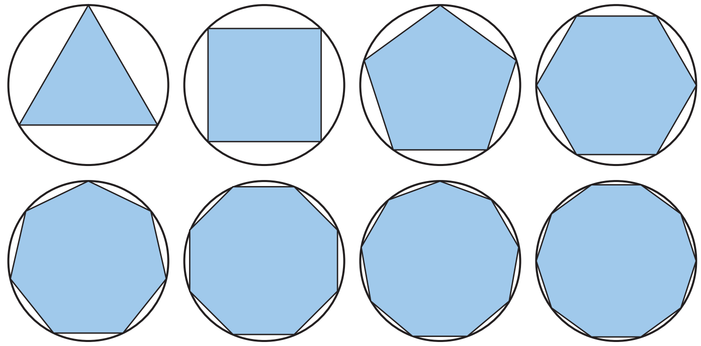
In this introductory section we will consider the problem of calculating areas of plane regions with curvilinear boundaries. All of the results in this section will be reexamined in more detail later in this chapter. Our purpose here is simply to introduce and motivate the fundamental concepts.

THE AREA PROBLEM

Formulas for the areas of polygons, such as squares, rectangles, triangles, and trapezoids, were well known in many early civilizations. However, the problem of finding formulas for regions with curved boundaries (a circle being the simplest example) caused difficulties for early mathematicians.

The first real progress in dealing with the general area problem was made by the Greek mathematician Archimedes, who obtained areas of regions bounded by circular arcs, parabolas, spirals, and various other curves using an ingenious procedure that was later called the *method of exhaustion*. The method, when applied to a circle, consists of inscribing a succession of regular polygons in the circle and allowing the number of sides to increase indefinitely (Figure 5.1.1). As the number of sides increases, the polygons tend to "exhaust" the region inside the circle, and the areas of the polygons become better and better approximations of the exact area of the circle.

To see how this works numerically, let $A(n)$ denote the area of a regular n-sided polygon inscribed in a circle of radius 1. Table 5.1.1 shows the values of $A(n)$ for various choices of n. Note that for large values of n the area $A(n)$ appears to be close to π (square units),



▲ Figure 5.1.1

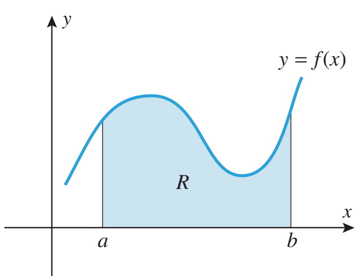
Table 5.1.1

n	$A(n)$
100	3.13952597647
200	3.14107590781
300	3.14136298250
400	3.14146346236
500	3.14150997084
1000	3.14157198278
2000	3.14158748588
3000	3.14159035683
4000	3.14159136166
5000	3.14159182676
10,000	3.14159244688

as one would expect. This suggests that for a circle of radius 1, the method of exhaustion is equivalent to an equation of the form

$$\lim_{n\to\infty} A(n) = \pi$$

Since Greek mathematicians were suspicious of the concept of "infinity," they avoided its use in mathematical arguments. As a result, computation of area using the method of exhaustion was a very cumbersome procedure. It remained for Newton and Leibniz to obtain a general method for finding areas that explicitly used the notion of a limit. We will discuss their method in the context of the following problem.



▲ Figure 5.1.2

5.1.1 THE AREA PROBLEM Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 5.1.2).

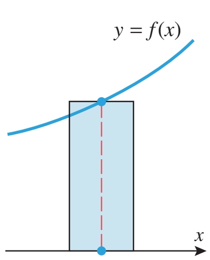
THE RECTANGLE METHOD FOR FINDING AREAS

One approach to the area problem is to use Archimedes' method of exhaustion in the following way:

- Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter—it can be above the center, above an endpoint, or above any other point in the subinterval. In Figure 5.1.3 it is above the center.
- For each n, the total area of the rectangles can be viewed as an *approximation* to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit (Figure 5.1.4). That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

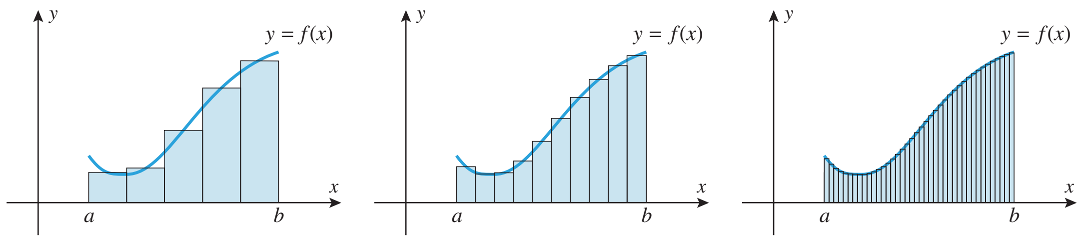
$$A = \lim_{n\to+\infty} A_n$$

We will call this the ***rectangle method*** for computing A.



▲ Figure 5.1.3

Logically speaking, we cannot really talk about computing areas without a precise mathematical definition of the term "area." Later in this chapter we will give such a definition, but for now we will treat the concept intuitively.



▲ **Figure 5.1.4**

To illustrate this idea, we will use the rectangle method to approximate the area under the curve $y = x^2$ over the interval [0, 1] (Figure 5.1.5). We will begin by dividing the interval [0, 1] into n equal subintervals, from which it follows that each subinterval has length $1/n$; the endpoints of the subintervals occur at

$$0, \quad \frac{1}{n}, \quad \frac{2}{n}, \quad \frac{3}{n}, \ldots, \quad \frac{n-1}{n}, \quad 1$$

Archimedes **(287 B.C.–212 B.C.)** Greek mathematician and scientist. Born in Syracuse, Sicily, Archimedes was the son of the astronomer Pheidias and possibly related to Heiron II, king of Syracuse. Most of the facts about his life come from the Roman biographer, Plutarch, who inserted a few tantalizing pages about him in the massive biography of the Roman soldier, Marcellus. In the words of one writer, "the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich."

Archimedes ranks with Newton and Gauss as one of the three greatest mathematicians who ever lived, and he is certainly the greatest mathematician of antiquity. His mathematical work is so modern in spirit and technique that it is barely distinguishable from that of a seventeenth-century mathematician, yet it was all done without benefit of algebra or a convenient number system. Among his mathematical achievements, Archimedes developed a general method (exhaustion) for finding areas and volumes, and he used the method to find areas bounded by parabolas and spirals and to find volumes of cylinders, paraboloids, and segments of spheres. He gave a procedure for approximating π and bounded its value between $3\frac{10}{71}$ and $3\frac{1}{7}$. In spite of the limitations of the Greek numbering system, he devised methods for finding square roots and invented a method based on the Greek myriad (10,000) for representing numbers as large as 1 followed by 80 million billion zeros.

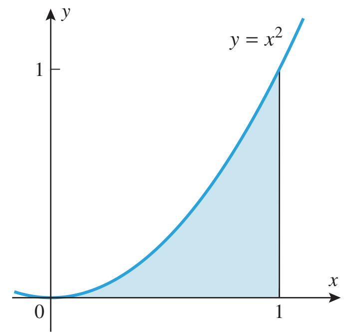
Of all his mathematical work, Archimedes was most proud of his discovery of a method for finding the volume of a sphere—he showed that the volume of a sphere is two-thirds the volume of the smallest cylinder that can contain it. At his request, the figure of a sphere and cylinder was engraved on his tombstone.

In addition to mathematics, Archimedes worked extensively in mechanics and hydrostatics. Nearly every schoolchild knows Archimedes as the absent-minded scientist who, on realizing that a floating object displaces its weight of liquid, leaped from his bath and ran naked through the streets of Syracuse shouting, "Eureka, Eureka!"—(meaning, "I have found it!"). Archimedes actually created the discipline of hydrostatics and used it to find equilibrium positions for various floating bodies. He laid down the fundamental postulates of mechanics, discovered the laws of levers, and calculated centers of gravity for various flat surfaces and solids. In the excitement of discovering the mathematical laws of the lever, he is said to have declared, "Give me a place to stand and I will move the earth."

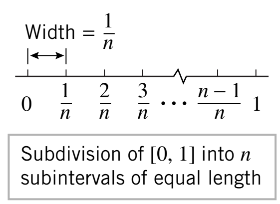
Although Archimedes was apparently more interested in pure mathematics than its applications, he was an engineering genius. During the second Punic war, when Syracuse was attacked by the Roman fleet under the command of Marcellus, it was reported by Plutarch that Archimedes' military inventions held the fleet at bay for three years. He invented super catapults that showered the Romans with rocks weighing a quarter ton or more, and fearsome mechanical devices with iron "beaks and claws" that reached over the city walls, grasped the ships, and spun them against the rocks. After the first repulse, Marcellus called Archimedes a "geometrical Briareus (a hundred-armed mythological monster) who uses our ships like cups to ladle water from the sea."

Eventually the Roman army was victorious and contrary to Marcellus' specific orders the 75-year-old Archimedes was killed by a Roman soldier. According to one report of the incident, the soldier cast a shadow across the sand in which Archimedes was working on a mathematical problem. When the annoyed Archimedes yelled, "Don't disturb my circles," the soldier flew into a rage and cut the old man down.

Although there is no known likeness or statue of this great man, nine works of Archimedes have survived to the present day. Especially important is his treatise, *The Method of Mechanical Theorems*, which was part of a palimpsest found in Constantinople in 1906. In this treatise Archimedes explains how he made some of his discoveries, using reasoning that anticipated ideas of the integral calculus. Thought to be lost, the Archimedes palimpsest later resurfaced in 1998, when it was purchased by an anonymous private collector for two million dollars.



▲ **Figure 5.1.5**



▲ **Figure 5.1.6**

(Figure 5.1.6). We want to construct a rectangle over each of these subintervals whose height is the value of the function $f(x) = x^2$ at some point in the subinterval. To be specific, let us use the right endpoints, in which case the heights of our rectangles will be

$$\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, 1^2$$

and since each rectangle has a base of width $1/n$, the total area A_n of the n rectangles will be

$$A_n = \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + 1^2\right]\left(\frac{1}{n}\right) \tag{1}$$

For example, if $n = 4$, then the total area of the four approximating rectangles would be

$$A_4 = \left[\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 + 1^2\right]\left(\tfrac{1}{4}\right) = \tfrac{15}{32} = 0.46875$$

Table 5.1.2 shows the result of evaluating (1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to $\frac{1}{3}$. Later in this chapter we will prove that this area is exactly $\frac{1}{3}$ by showing that

$$\lim_{n\to\infty} A_n = \tfrac{1}{3}$$

TECHNOLOGY MASTERY

Use a calculating utility to compute the value of A_{10} in Table 5.1.2. Some calculating utilities have special commands for computing sums such as that in (1) for any specified value of n. If your utility has this feature, use it to compute A_{100} as well.

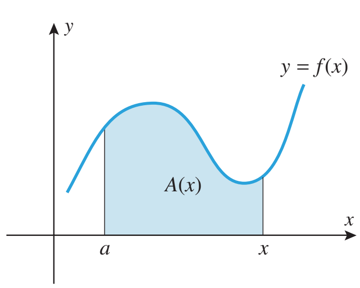
Table 5.1.2

n	4	10	100	1000	10,000	100,000
A_n	0.468750	0.385000	0.338350	0.333834	0.333383	0.333338

THE ANTIDERIVATIVE METHOD FOR FINDING AREAS

Although the rectangle method is appealing intuitively, the limits that result can only be evaluated in certain cases. For this reason, progress on the area problem remained at a rudimentary level until the latter part of the seventeenth century when Isaac Newton and Gottfried Leibniz independently discovered a fundamental relationship between areas and derivatives. Briefly stated, they showed that if f is a nonnegative continuous function on the interval $[a, b]$, and if $A(x)$ denotes the area under the graph of f over the interval $[a, x]$, where x is any point in the interval $[a, b]$ (Figure 5.1.7), then

$$A'(x) = f(x) \tag{2}$$



▲ **Figure 5.1.7**

The following example confirms Formula (2) in some cases where a formula for $A(x)$ can be found using elementary geometry.

▶ **Example 1** For each of the functions f, find the area $A(x)$ between the graph of f and the interval $[a, x] = [-1, x]$, and find the derivative $A'(x)$ of this area function.

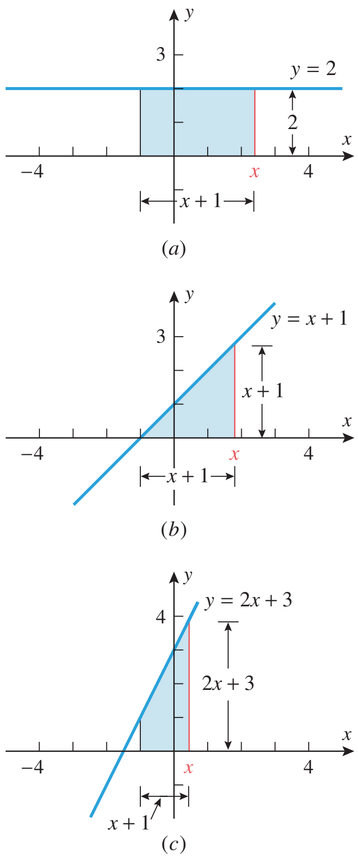
(a) $f(x) = 2$ (b) $f(x) = x + 1$ (c) $f(x) = 2x + 3$

Solution (a). From Figure 5.1.8a we see that

$$A(x) = 2(x - (-1)) = 2(x + 1) = 2x + 2$$

is the area of a rectangle of height 2 and base $x + 1$. For this area function,

$$A'(x) = 2 = f(x)$$



▲ **Figure 5.1.8**

How does the solution to Example 2 change if the interval $[0, 1]$ is replaced by the interval $[-1, 1]$?

Solution (b). From Figure 5.1.8*b* we see that

$$A(x) = \frac{1}{2}(x+1)(x+1) = \frac{x^2}{2} + x + \frac{1}{2}$$

is the area of an isosceles right triangle with base and height equal to $x + 1$. For this area function,

$$A'(x) = x + 1 = f(x)$$

Solution (c). Recall that the formula for the area of a trapezoid is $A = \frac{1}{2}(b + b')h$, where b and b' denote the lengths of the parallel sides of the trapezoid, and the altitude h denotes the distance between the parallel sides. From Figure 5.1.8*c* we see that

$$A(x) = \tfrac{1}{2}((2x+3)+1)(x-(-1)) = x^2 + 3x + 2$$

is the area of a trapezoid with parallel sides of lengths 1 and $2x + 3$ and with altitude $x - (-1) = x + 1$. For this area function,

$$A'(x) = 2x + 3 = f(x) \quad ◀$$

Formula (2) is important because it relates the area function A and the region-bounding function f. Although a formula for $A(x)$ may be difficult to obtain directly, its derivative, $f(x)$, is given. If a formula for $A(x)$ can be recovered from the given formula for $A'(x)$, then the area under the graph of f over the interval $[a, b]$ can be obtained by computing $A(b)$.

The process of finding a function from its derivative is called ***antidifferentiation***, and a procedure for finding areas via antidifferentiation is called the ***antiderivative method***. To illustrate this method, let us revisit the problem of finding the area in Figure 5.1.5.

▶ **Example 2** Use the antiderivative method to find the area under the graph of $y = x^2$ over the interval $[0, 1]$.

Solution. Let x be any point in the interval $[0, 1]$, and let $A(x)$ denote the area under the graph of $f(x) = x^2$ over the interval $[0, x]$. It follows from (2) that

$$A'(x) = x^2 \tag{3}$$

To find $A(x)$ we must look for a function whose derivative is x^2. By guessing, we see that one such function is $\frac{1}{3}x^3$, so by Theorem 4.8.3

$$A(x) = \tfrac{1}{3}x^3 + C \tag{4}$$

for some real constant C. We can determine the specific value for C by considering the case where $x = 0$. In this case (4) implies that

$$A(0) = C \tag{5}$$

But if $x = 0$, then the interval $[0, x]$ reduces to a single point. If we agree that the area above a single point should be taken as zero, then $A(0) = 0$ and (5) implies that $C = 0$. Thus, it follows from (4) that

$$A(x) = \tfrac{1}{3}x^3$$

is the area function we are seeking. This implies that the area under the graph of $y = x^2$ over the interval $[0, 1]$ is

$$A(1) = \tfrac{1}{3}(1^3) = \tfrac{1}{3}$$

This is consistent with the result that we previously obtained numerically. ◀

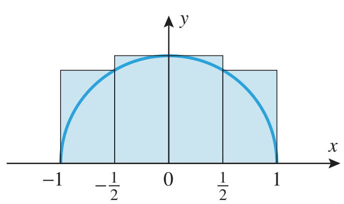
As Example 2 illustrates, antidifferentiation is a process in which one tries to "undo" a differentiation. One of the objectives in this chapter is to develop efficient antidifferentiation procedures.

THE RECTANGLE METHOD AND THE ANTIDERIVATIVE METHOD COMPARED

The rectangle method and the antiderivative method provide two very different approaches to the area problem, each of which is important. The antiderivative method is usually the more efficient way to *compute* areas, but it is the rectangle method that is used to formally *define* the notion of area, thereby allowing us to prove mathematical results about areas. The underlying idea of the rectangle approach is also important because it can be adapted readily to such diverse problems as finding the volume of a solid, the length of a curve, the mass of an object, and the work done in pumping water out of a tank, to name a few.

✔QUICK CHECK EXERCISES 5.1 *(See page 322 for answers.)*

1. Let R denote the region below the graph of $f(x) = \sqrt{1-x^2}$ and above the interval $[-1, 1]$.
   (a) Use a geometric argument to find the area of R.
   (b) What estimate results if the area of R is approximated by the total area within the rectangles of the accompanying figure?



◀ **Figure Ex-1**

2. Suppose that when the area A between the graph of a function $y = f(x)$ and an interval $[a, b]$ is approximated by the areas of n rectangles, the total area of the rectangles is $A_n = 2 + (2/n)$, $n = 1, 2, \ldots$. Then, $A =$ ________.
3. The area under the graph of $y = x^2$ over the interval $[0, 3]$ is ________.
4. Find a formula for the area $A(x)$ between the graph of the function $f(x) = x$ and the interval $[0, x]$, and verify that $A'(x) = f(x)$.
5. The area under the graph of $y = f(x)$ over the interval $[0, x]$ is $A(x) = x + e^x - 1$. It follows that $f(x) =$ ________.

EXERCISE SET 5.1

1–12 Estimate the area between the graph of the function f and the interval $[a, b]$. Use an approximation scheme with n rectangles similar to our treatment of $f(x) = x^2$ in this section. If your calculating utility will perform automatic summations, estimate the specified area using $n = 10$, 50, and 100 rectangles. Otherwise, estimate this area using $n = 2$, 5, and 10 rectangles. ■

1. $f(x) = \sqrt{x}$; $[a, b] = [0, 1]$
2. $f(x) = \dfrac{1}{x+1}$; $[a, b] = [0, 1]$
3. $f(x) = \sin x$; $[a, b] = [0, \pi]$
4. $f(x) = \cos x$; $[a, b] = [0, \pi/2]$
5. $f(x) = \dfrac{1}{x}$; $[a, b] = [1, 2]$
6. $f(x) = \cos x$; $[a, b] = [-\pi/2, \pi/2]$
7. $f(x) = \sqrt{1-x^2}$; $[a, b] = [0, 1]$
8. $f(x) = \sqrt{1-x^2}$; $[a, b] = [-1, 1]$
9. $f(x) = e^x$; $[a, b] = [-1, 1]$
10. $f(x) = \ln x$; $[a, b] = [1, 2]$
11. $f(x) = \sin^{-1} x$; $[a, b] = [0, 1]$
12. $f(x) = \tan^{-1} x$; $[a, b] = [0, 1]$

13–18 Graph each function over the specified interval. Then use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function f and the interval $[a, x]$. Confirm that $A'(x) = f(x)$ in every case. ■

13. $f(x) = 3$; $[a, x] = [1, x]$
14. $f(x) = 5$; $[a, x] = [2, x]$
15. $f(x) = 2x + 2$; $[a, x] = [0, x]$
16. $f(x) = 3x - 3$; $[a, x] = [1, x]$
17. $f(x) = 2x + 2$; $[a, x] = [1, x]$
18. $f(x) = 3x - 3$; $[a, x] = [2, x]$

19–22 True–False Determine whether the statement is true or false. Explain your answer. ■

19. If $A(n)$ denotes the area of a regular n-sided polygon inscribed in a circle of radius 2, then $\lim_{n\to+\infty} A(n) = 2\pi$.
20. If the area under the curve $y = x^2$ over an interval is approximated by the total area of a collection of rectangles, the approximation will be too large.
21. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A'(x) = f(x)$.

22. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A(x)$ will be a continuous function.

FOCUS ON CONCEPTS

23. Explain how to use the formula for $A(x)$ found in the solution to Example 2 to determine the area between the graph of $y = x^2$ and the interval $[3, 6]$.

24. Repeat Exercise 23 for the interval $[-3, 9]$.

25. Let A denote the area between the graph of $f(x) = \sqrt{x}$ and the interval $[0, 1]$, and let B denote the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. Explain geometrically why $A + B = 1$.

26. Let A denote the area between the graph of $f(x) = 1/x$ and the interval $[1, 2]$, and let B denote the area between the graph of f and the interval $\left[\frac{1}{2}, 1\right]$. Explain geometrically why $A = B$.

27–28 The area $A(x)$ under the graph of f and over the interval $[a, x]$ is given. Find the function f and the value of a.

27. $A(x) = x^2 - 4$

28. $A(x) = x^2 - x$

29. **Writing** Compare and contrast the rectangle method and the antiderivative method.

30. **Writing** Suppose that f is a nonnegative continuous function on an interval $[a, b]$ and that $g(x) = f(x) + C$, where C is a positive constant. What will be the area of the region *between* the graphs of f and g?

✔QUICK CHECK ANSWERS 5.1

1. (a) $\dfrac{\pi}{2}$ (b) $1 + \dfrac{\sqrt{3}}{2}$ 2. 2 3. 9 4. $A(x) = \dfrac{x^2}{2}$; $A'(x) = \dfrac{2x}{2} = x = f(x)$ 5. $e^x + 1$

5.2 THE INDEFINITE INTEGRAL

In the last section we saw how antidifferentiation could be used to find exact areas. In this section we will develop some fundamental results about antidifferentiation.

ANTIDERIVATIVES

5.2.1 DEFINITION A function F is called an ***antiderivative*** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

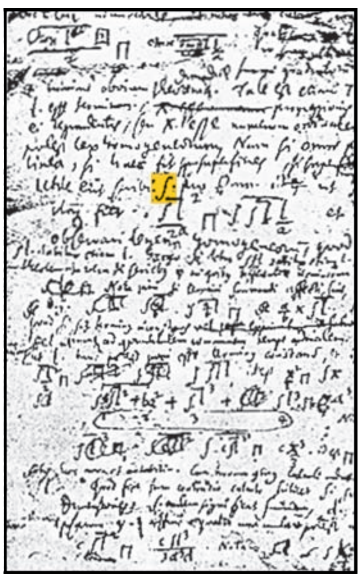
$$\tfrac{1}{3}x^3, \quad \tfrac{1}{3}x^3 + 2, \quad \tfrac{1}{3}x^3 - 5, \quad \tfrac{1}{3}x^3 + \sqrt{2}$$

are all antiderivatives of $f(x) = x^2$.

It is reasonable to ask if there are antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F. The answer is *no*—once a single antiderivative of f on an open interval is known, all other antiderivatives on that interval are obtainable by adding constants to the known antiderivative. This is so because Theorem 4.8.3 tells us that if two functions have the same derivative on an open interval, then the functions differ by a constant on the interval. The following theorem summarizes these observations.

5.2.2 THEOREM *If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x)+C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x)+C$ by choosing the constant C appropriately.*

THE INDEFINITE INTEGRAL



Reproduced from C. I. Gerhardt's "Briefwechsel von G. W. Leibniz mit Mathematikern (1899)."

Extract from the manuscript of Leibniz dated October 29, 1675 in which the integral sign first appeared (see yellow highlight).

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x)+C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

$$\int f(x)\,dx = F(x) + C \tag{2}$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \tag{3}$$

The expression $\int f(x)\,dx$ is called an ***indefinite integral***. The adjective "indefinite" emphasizes that the result of antidifferentiation is a "generic" function, described only up to a constant term. The "elongated s" that appears on the left side of (2) is called an ***integral sign***,* the function $f(x)$ is called the ***integrand***, and the constant C is called the ***constant of integration***. Equation (2) should be read as:

The integral of $f(x)$ with respect to x is equal to $F(x)$ plus a constant.

The differential symbol, dx, in the differentiation and antidifferentiation operations

$$\frac{d}{dx}[\ \ ] \quad \text{and} \quad \int [\ \ ]\,dx$$

*This notation was devised by Leibniz. In his early papers Leibniz used the notation "omn." (an abbreviation for the Latin word "omnes") to denote integration. Then on October 29, 1675 he wrote, "It will be useful to write $\int$ for omn., thus $\int l$ for omn. l ...." Two or three weeks later he refined the notation further and wrote $\int [\ \ ]\,dx$ rather than $\int$ alone. This notation is so useful and so powerful that its development by Leibniz must be regarded as a major milestone in the history of mathematics and science.

serves to identify the independent variable. If an independent variable other than x is used, say t, then the notation must be adjusted appropriately. Thus,

$$\frac{d}{dt}[F(t)] = f(t) \quad \text{and} \quad \int f(t)\,dt = F(t) + C$$

are equivalent statements. Here are some examples of derivative formulas and their equivalent integration formulas:

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\,dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$

For simplicity, the dx is sometimes absorbed into the integrand. For example,

$$\int 1\,dx \quad \text{can be written as} \quad \int dx$$

$$\int \frac{1}{x^2}\,dx \quad \text{can be written as} \quad \int \frac{dx}{x^2}$$

INTEGRATION FORMULAS

Integration is essentially educated guesswork—given the derivative f of a function F, one tries to guess what the function F is. However, many basic integration formulas can be obtained directly from their companion differentiation formulas. Some of the most important are given in Table 5.2.1.

Table 5.2.1
INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$	9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b,\ b \neq 1)$	$\int b^x\,dx = \frac{b^x}{\ln b} + C \quad (0 < b,\ b \neq 1)$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$	11. $\frac{d}{dx}[\ln\|x\|] = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\|x\| + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	14. $\frac{d}{dx}[\sec^{-1}\|x\|] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\|x\| + C$

See Exercise 72 for a justification of Formula 14 in Table 5.2.1.

Example 1 The second integration formula in Table 5.2.1 will be easier to remember if you express it in words:

To integrate a power of x (other than -1), add 1 to the exponent and divide by the new exponent.

Although Formula 2 in Table 5.2.1 is not applicable to integrating x^{-1}, this function can be integrated by rewriting the integral in Formula 11 as

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C$$

Here are some examples:

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \boxed{r=2}$$

$$\int x^3\,dx = \frac{x^4}{4} + C \qquad \boxed{r=3}$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C \qquad \boxed{r=-5}$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C = \tfrac{2}{3}(\sqrt{x})^3 + C \qquad \boxed{r=\tfrac{1}{2}}$$

PROPERTIES OF THE INDEFINITE INTEGRAL

Our first properties of antiderivatives follow directly from the simple constant factor, sum, and difference rules for derivatives.

5.2.3 THEOREM *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(*a*) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(*b*) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(*c*) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

PROOF In general, to establish the validity of an equation of the form

$$\int h(x)\,dx = H(x) + C$$

one must show that

$$\frac{d}{dx}[H(x)] = h(x)$$

We are given that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, so we know that

$$\frac{d}{dx}[F(x)] = f(x) \quad \text{and} \quad \frac{d}{dx}[G(x)] = g(x)$$

Thus,

$$\frac{d}{dx}[cF(x)] = c\frac{d}{dx}[F(x)] = cf(x)$$

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[G(x)] = f(x) + g(x)$$

$$\frac{d}{dx}[F(x) - G(x)] = \frac{d}{dx}[F(x)] - \frac{d}{dx}[G(x)] = f(x) - g(x)$$

which proves the three statements of the theorem. ■

The statements in Theorem 5.2.3 can be summarized by the following formulas:

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{4}$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{5}$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx \tag{6}$$

However, these equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration. For example, if you use (4) to integrate $2x$ by writing

$$\int 2x\,dx = 2\int x\,dx = 2\left(\frac{x^2}{2} + C\right) = x^2 + 2C$$

then you will have an unnecessarily complicated form of the arbitrary constant. This kind of problem can be avoided by inserting the constant of integration in the final result rather than in intermediate calculations. Exercises 65 and 66 explore how careless application of these formulas can lead to errors.

▶ **Example 2** Evaluate

$$\text{(a)}\quad \int 4\cos x\,dx \qquad \text{(b)}\quad \int (x + x^2)\,dx$$

Solution (a). Since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ (Table 5.2.1), we obtain

$$\int 4\cos x\,dx \underset{\boxed{(4)}}{=} 4\int \cos x\,dx = 4\sin x + C$$

Solution (b). From Table 5.2.1 we obtain

$$\int (x + x^2)\,dx \underset{\boxed{(5)}}{=} \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$$ ◀

Parts (b) and (c) of Theorem 5.2.3 can be extended to more than two functions, which in combination with part (a) results in the following general formula:

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\,dx$$
$$= c_1\int f_1(x)\,dx + c_2\int f_2(x)\,dx + \cdots + c_n\int f_n(x)\,dx \tag{7}$$

▶ Example 3

$$\int (3x^6 - 2x^2 + 7x + 1)\, dx = 3\int x^6\, dx - 2\int x^2\, dx + 7\int x\, dx + \int 1\, dx$$
$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C \blacktriangleleft$$

Sometimes it is useful to rewrite an integrand in a different form before performing the integration. This is illustrated in the following example.

▶ Example 4 Evaluate

$$\text{(a)}\ \int \frac{\cos x}{\sin^2 x}\, dx \qquad \text{(b)}\ \int \frac{t^2 - 2t^4}{t^4}\, dt \qquad \text{(c)}\ \int \frac{x^2}{x^2+1}\, dx$$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\, dx = \int \csc x \cot x\, dx = \underbrace{-\csc x + C}_{\text{Formula 8 in Table 5.2.1}}$$

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\, dt = \int \left(\frac{1}{t^2} - 2\right) dt = \int (t^{-2} - 2)\, dt$$
$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

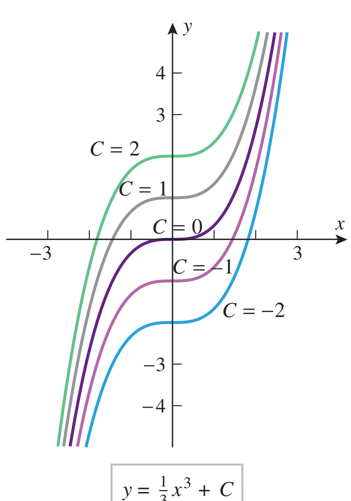
Perform the integration in part (c) by first performing a long division on the integrand.

Solution (c). By adding and subtracting 1 from the numerator of the integrand, we can rewrite the integral in a form in which Formulas 1 and 12 of Table 5.2.1 can be applied:

$$\int \frac{x^2}{x^2+1}\, dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right) dx$$
$$= \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C \blacktriangleleft$$

INTEGRAL CURVES

Graphs of antiderivatives of a function f are called ***integral curves*** of f. We know from Theorem 5.2.2 that if $y = F(x)$ is any integral curve of $f(x)$, then all other integral curves are vertical translations of this curve, since they have equations of the form $y = F(x) + C$. For example, $y = \frac{1}{3}x^3$ is one integral curve for $f(x) = x^2$, so all the other integral curves have equations of the form $y = \frac{1}{3}x^3 + C$; conversely, the graph of any equation of this form is an integral curve (Figure 5.2.1).



▲ **Figure 5.2.1**

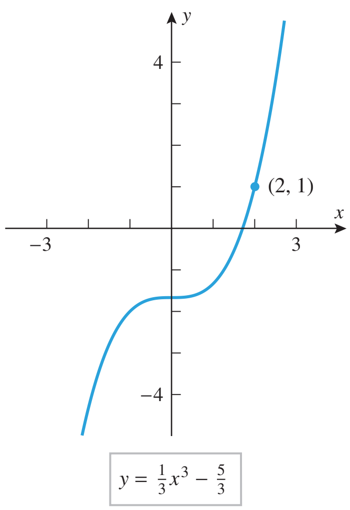
In many problems one is interested in finding a function whose derivative satisfies specified conditions. The following example illustrates a geometric problem of this type.

▶ Example 5 Suppose that a curve $y = f(x)$ in the xy-plane has the property that at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve given that it passes through the point $(2, 1)$.

Solution. Since the slope of the line tangent to $y = f(x)$ is dy/dx, we have $dy/dx = x^2$, and

$$y = \int x^2\, dx = \tfrac{1}{3}x^3 + C$$

In Example 5, the requirement that the graph of f pass through the point $(2, 1)$ selects the single integral curve $y = \frac{1}{3}x^3 - \frac{5}{3}$ from the family of curves $y = \frac{1}{3}x^3 + C$ (Figure 5.2.2).



▲ **Figure 5.2.2**

Since the curve passes through (2, 1), a specific value for C can be found by using the fact that $y = 1$ if $x = 2$. Substituting these values in the above equation yields

$$1 = \tfrac{1}{3}(2^3) + C \quad \text{or} \quad C = -\tfrac{5}{3}$$

so an equation of the curve is

$$y = \tfrac{1}{3}x^3 - \tfrac{5}{3}$$

(Figure 5.2.2). ◀

INTEGRATION FROM THE VIEWPOINT OF DIFFERENTIAL EQUATIONS

We will now consider another way of looking at integration that will be useful in our later work. Suppose that $f(x)$ is a known function and we are interested in finding a function $F(x)$ such that $y = F(x)$ satisfies the equation

$$\frac{dy}{dx} = f(x) \tag{8}$$

The solutions of this equation are the antiderivatives of $f(x)$, and we know that these can be obtained by integrating $f(x)$. For example, the solutions of the equation

$$\frac{dy}{dx} = x^2 \tag{9}$$

are

$$y = \int x^2\,dx = \frac{x^3}{3} + C$$

Equation (8) is called a ***differential equation*** because it involves a derivative of an unknown function. Differential equations are different from the kinds of equations we have encountered so far in that the unknown is a *function* and not a *number* as in an equation such as $x^2 + 5x - 6 = 0$.

Sometimes we will not be interested in finding all of the solutions of (8), but rather we will want only the solution whose graph passes through a specified point (x_0, y_0). For example, in Example 5 we solved (9) for the integral curve that passed through the point (2, 1).

For simplicity, it is common in the study of differential equations to denote a solution of $dy/dx = f(x)$ as $y(x)$ rather than $F(x)$, as earlier. With this notation, the problem of finding a function $y(x)$ whose derivative is $f(x)$ and whose graph passes through the point (x_0, y_0) is expressed as

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{10}$$

This is called an ***initial-value problem***, and the requirement that $y(x_0) = y_0$ is called the ***initial condition*** for the problem.

▶ **Example 6** Solve the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

Solution. The solution of the differential equation is

$$y = \int \cos x\,dx = \sin x + C \tag{11}$$

The initial condition $y(0) = 1$ implies that $y = 1$ if $x = 0$; substituting these values in (11) yields

$$1 = \sin(0) + C \quad \text{or} \quad C = 1$$

Thus, the solution of the initial-value problem is $y = \sin x + 1$. ◀

SLOPE FIELDS

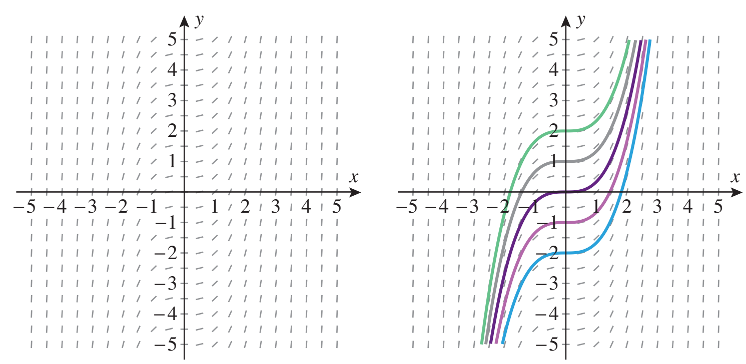
If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of the equation $dy/dx = f(x)$, the slope of the tangent line is $f(x)$. What is interesting about this is that the slopes of the tangent lines to the integral curves can be obtained without actually solving the differential equation. For example, if

$$\frac{dy}{dx} = \sqrt{x^2+1}$$

then we know without solving the equation that at the point where $x = 1$ the tangent line to an integral curve has slope $\sqrt{1^2+1} = \sqrt{2}$; and more generally, at a point where $x = a$, the tangent line to an integral curve has slope $\sqrt{a^2+1}$.

A geometric description of the integral curves of a differential equation $dy/dx = f(x)$ can be obtained by choosing a rectangular grid of points in the xy-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small portions of the tangent lines through those points. The resulting picture, which is called a ***slope field*** or ***direction field*** for the equation, shows the "direction" of the integral curves at the gridpoints. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves; for example, Figure 5.2.3*a* shows a slope field for the differential equation $dy/dx = x^2$, and Figure 5.2.3*b* shows that same field with the integral curves imposed on it—the more gridpoints that are used, the more completely the slope field reveals the shape of the integral curves. However, the amount of computation can be considerable, so computers are usually used when slope fields with many gridpoints are needed.

Slope fields will be studied in more detail later in the text.



Slope field for $dy/dx = x^2$ (*a*)

Slope field with integral curves (*b*)

Figure 5.2.3

✔ QUICK CHECK EXERCISES 5.2 *(See page 332 for answers.)*

1. A function F is an antiderivative of a function f on an interval if ________ for all x in the interval.

2. Write an equivalent integration formula for each given derivative formula.

(a) $\dfrac{d}{dx}[\sqrt{x}] = \dfrac{1}{2\sqrt{x}}$ (b) $\dfrac{d}{dx}[e^{4x}] = 4e^{4x}$

3. Evaluate the integrals.

(a) $\displaystyle\int [x^3 + x + 5]\,dx$ (b) $\displaystyle\int [\sec^2 x - \csc x \cot x]\,dx$

4. The graph of $y = x^2 + x$ is an integral curve for the function $f(x) =$ ________. If G is a function whose graph is also an integral curve for f, and if $G(1) = 5$, then $G(x) =$ ________.

5. A slope field for the differential equation

$$\frac{dy}{dx} = \frac{2x}{x^2-4}$$

has a line segment with slope ________ through the point $(0, 5)$ and has a line segment with slope ________ through the point $(-4, 1)$.

EXERCISE SET 5.2

Graphing Utility   C CAS

1. In each part, confirm that the formula is correct, and state a corresponding integration formula.
   (a) $\frac{d}{dx}[\sqrt{1+x^2}\,] = \frac{x}{\sqrt{1+x^2}}$
   (b) $\frac{d}{dx}[xe^x] = (x+1)e^x$
2. In each part, confirm that the stated formula is correct by differentiating.
   (a) $\int x \sin x\, dx = \sin x - x\cos x + C$
   (b) $\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

FOCUS ON CONCEPTS

3. What is a *constant of integration*? Why does an answer to an integration problem involve a constant of integration?
4. What is an *integral curve* of a function f? How are two integral curves of a function f related?

5–8 Find the derivative and state a corresponding integration formula. ■

5. $\frac{d}{dx}[\sqrt{x^3+5}\,]$
6. $\frac{d}{dx}\left[\frac{x}{x^2+3}\right]$
7. $\frac{d}{dx}[\sin(2\sqrt{x})]$
8. $\frac{d}{dx}[\sin x - x\cos x]$

9–10 Evaluate the integral by rewriting the integrand appropriately, if required, and applying the power rule (Formula 2 in Table 5.2.1). ■

9. (a) $\int x^8\, dx$ (b) $\int x^{5/7}\, dx$ (c) $\int x^3\sqrt{x}\, dx$
10. (a) $\int \sqrt[3]{x^2}\, dx$ (b) $\int \frac{1}{x^6}\, dx$ (c) $\int x^{-7/8}\, dx$

11–14 Evaluate each integral by applying Theorem 5.2.3 and Formula 2 in Table 5.2.1 appropriately. ■

11. $\int \left[5x + \frac{2}{3x^5}\right] dx$
12. $\int \left[x^{-1/2} - 3x^{7/5} + \frac{1}{9}\right] dx$
13. $\int [x^{-3} - 3x^{1/4} + 8x^2]\, dx$
14. $\int \left[\frac{10}{y^{3/4}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy$

15–34 Evaluate the integral and check your answer by differentiating. ■

15. $\int x(1+x^3)\, dx$
16. $\int (2+y^2)^2\, dy$
17. $\int x^{1/3}(2-x)^2\, dx$
18. $\int (1+x^2)(2-x)\, dx$
19. $\int \frac{x^5+2x^2-1}{x^4}\, dx$
20. $\int \frac{1-2t^3}{t^3}\, dt$
21. $\int \left[\frac{2}{x} + 3e^x\right] dx$
22. $\int \left[\frac{1}{2t} - \sqrt{2}e^t\right] dt$
23. $\int [3\sin x - 2\sec^2 x]\, dx$
24. $\int [\csc^2 t - \sec t \tan t]\, dt$
25. $\int \sec x(\sec x + \tan x)\, dx$
26. $\int \csc x(\sin x + \cot x)\, dx$
27. $\int \frac{\sec\theta}{\cos\theta}\, d\theta$
28. $\int \frac{dy}{\csc y}$
29. $\int \frac{\sin x}{\cos^2 x}\, dx$
30. $\int \left[\phi + \frac{2}{\sin^2\phi}\right] d\phi$
31. $\int [1 + \sin^2\theta \csc\theta]\, d\theta$
32. $\int \frac{\sec x + \cos x}{2\cos x}\, dx$
33. $\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx$
34. $\int \left[\frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2}\right] dx$
35. Evaluate the integral
$$\int \frac{1}{1+\sin x}\, dx$$
by multiplying the numerator and denominator by an appropriate expression.
36. Use the double-angle formula $\cos 2x = 2\cos^2 x - 1$ to evaluate the integral
$$\int \frac{1}{1+\cos 2x}\, dx$$

37–40 True–False Determine whether the statement is true or false. Explain your answer. ■

37. If $F(x)$ is an antiderivative of $f(x)$, then
$$\int f(x)\, dx = F(x) + C$$
38. If C denotes a constant of integration, the two formulas
$$\int \cos x\, dx = \sin x + C$$
$$\int \cos x\, dx = (\sin x + \pi) + C$$
are both correct equations.
39. The function $f(x) = e^{-x} + 1$ is a solution to the initial-value problem
$$\frac{dy}{dx} = -\frac{1}{e^x}, \quad y(0) = 1$$
40. Every integral curve of the slope field
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$
is the graph of an increasing function of x.

41. Use a graphing utility to generate some representative integral curves of the function $f(x) = 5x^4 - \sec^2 x$ over the interval $(-\pi/2, \pi/2)$.

42. Use a graphing utility to generate some representative integral curves of the function $f(x) = (x - 1)/x$ over the interval $(0, 5)$.

43–46 Solve the initial-value problems.

43. (a) $\dfrac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\dfrac{dy}{dt} = \sin t + 1,\ y\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

(c) $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}},\ y(1) = 0$

44. (a) $\dfrac{dy}{dx} = \dfrac{1}{(2x)^3},\ y(1) = 0$

(b) $\dfrac{dy}{dt} = \sec^2 t - \sin t,\ y\left(\dfrac{\pi}{4}\right) = 1$

(c) $\dfrac{dy}{dx} = x^2\sqrt{x^3},\ y(0) = 0$

45. (a) $\dfrac{dy}{dx} = 4e^x,\ y(0) = 1$ (b) $\dfrac{dy}{dt} = \dfrac{1}{t},\ y(-1) = 5$

46. (a) $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{1-t^2}},\ y\left(\dfrac{\sqrt{3}}{2}\right) = 0$

(b) $\dfrac{dy}{dx} = \dfrac{x^2-1}{x^2+1},\ y(1) = \dfrac{\pi}{2}$

47–50 A particle moves along an s-axis with position function $s = s(t)$ and velocity function $v(t) = s'(t)$. Use the given information to find $s(t)$.

47. $v(t) = 32t;\ \ s(0) = 20$

48. $v(t) = \cos t;\ \ s(0) = 2$

49. $v(t) = 3\sqrt{t};\ \ s(4) = 1$

50. $v(t) = 3e^t;\ \ s(1) = 0$

51. Find the general form of a function whose second derivative is $\sqrt{x}$. [*Hint:* Solve the equation $f''(x) = \sqrt{x}$ for $f(x)$ by integrating both sides twice.]

52. Find a function f such that $f''(x) = x + \cos x$ and such that $f(0) = 1$ and $f'(0) = 2$. [*Hint:* Integrate both sides of the equation twice.]

53–57 Find an equation of the curve that satisfies the given conditions.

53. At each point (x, y) on the curve the slope is $2x + 1$; the curve passes through the point $(-3, 0)$.

54. At each point (x, y) on the curve the slope is $(x + 1)^2$; the curve passes through the point $(-2, 8)$.

55. At each point (x, y) on the curve the slope is $-\sin x$; the curve passes through the point $(0, 2)$.

56. At each point (x, y) on the curve the slope equals the square of the distance between the point and the y-axis; the point $(-1, 2)$ is on the curve.

57. At each point (x, y) on the curve, y satisfies the condition $d^2y/dx^2 = 6x$; the line $y = 5 - 3x$ is tangent to the curve at the point where $x = 1$.

58. In each part, use a CAS to solve the initial-value problem.

(a) $\dfrac{dy}{dx} = x^2 \cos 3x,\ y(\pi/2) = -1$

(b) $\dfrac{dy}{dx} = \dfrac{x^3}{(4 + x^2)^{3/2}},\ y(0) = -2$

59. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = x$ in the region $-5 \le x \le 5$ and $-5 \le y \le 5$.

(b) Graph some representative integral curves of the function $f(x) = x$.

(c) Find an equation for the integral curve that passes through the point $(2, 1)$.

60. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = e^x/2$ in the region $-1 \le x \le 4$ and $-1 \le y \le 4$.

(b) Graph some representative integral curves of the function $f(x) = e^x/2$.

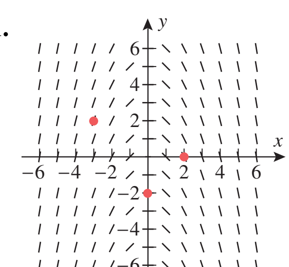
(c) Find an equation for the integral curve that passes through the point $(0, 1)$.

61–64 The given slope field figure corresponds to one of the differential equations below. Identify the differential equation that matches the figure, and sketch solution curves through the highlighted points.

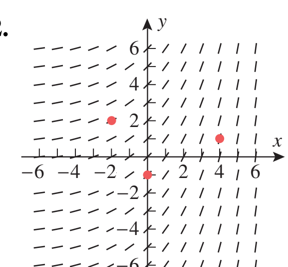
(a) $\dfrac{dy}{dx} = 2$ (b) $\dfrac{dy}{dx} = -x$

(c) $\dfrac{dy}{dx} = x^2 - 4$ (d) $\dfrac{dy}{dx} = e^{x/3}$

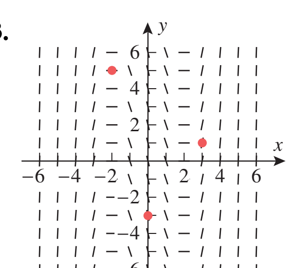
61.



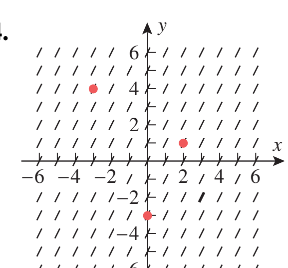
62.



63.



64.



FOCUS ON CONCEPTS

65. Critique the following "proof" that an arbitrary constant must be zero:

$$C = \int 0\, dx = \int 0 \cdot 0\, dx = 0 \int 0\, dx = 0$$

66. Critique the following "proof" that an arbitrary constant must be zero:

$$0 = \left(\int x\,dx\right) - \left(\int x\,dx\right)$$
$$= \int (x - x)\,dx = \int 0\,dx = C$$

67. (a) Show that

$$F(x) = \tan^{-1} x \quad \text{and} \quad G(x) = -\tan^{-1}(1/x)$$

differ by a constant on the interval $(0, +\infty)$ by showing that they are antiderivatives of the same function.

(b) Find the constant C such that $F(x) - G(x) = C$ by evaluating the functions $F(x)$ and $G(x)$ at a particular value of x.

(c) Check your answer to part (b) by using trigonometric identities.

68. Let F and G be the functions defined by

$$F(x) = \frac{x^2 + 3x}{x} \quad \text{and} \quad G(x) = \begin{cases} x + 3, & x > 0 \\ x, & x < 0 \end{cases}$$

(a) Show that F and G have the same derivative.

(b) Show that $G(x) \neq F(x) + C$ for any constant C.

(c) Do parts (a) and (b) contradict Theorem 5.2.2? Explain.

69–70 Use a trigonometric identity to evaluate the integral.

69. $\displaystyle\int \tan^2 x\,dx$ **70.** $\displaystyle\int \cot^2 x\,dx$

71. Use the identities $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$ to help evaluate the integrals

(a) $\displaystyle\int \sin^2(x/2)\,dx$ (b) $\displaystyle\int \cos^2(x/2)\,dx$

72. Recall that

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Use this to verify Formula 14 in Table 5.2.1.

73. The speed of sound in air at 0°C (or 273 K on the Kelvin scale) is 1087 ft/s, but the speed v increases as the temperature T rises. Experimentation has shown that the rate of change of v with respect to T is

$$\frac{dv}{dT} = \frac{1087}{2\sqrt{273}}T^{-1/2}$$

where v is in feet per second and T is in kelvins (K). Find a formula that expresses v as a function of T.

74. Suppose that a uniform metal rod 50 cm long is insulated laterally, and the temperatures at the exposed ends are maintained at 25°C and 85°C, respectively. Assume that an x-axis is chosen as in the accompanying figure and that the temperature $T(x)$ satisfies the equation

$$\frac{d^2T}{dx^2} = 0$$

Find $T(x)$ for $0 \le x \le 50$.

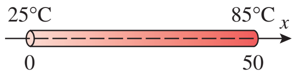


Figure Ex-74

75. Writing What is an *initial-value problem*? Describe the sequence of steps for solving an initial-value problem.

76. Writing What is a *slope field*? How are slope fields and integral curves related?

✔ QUICK CHECK ANSWERS 5.2

1. $F'(x) = f(x)$ **2.** (a) $\displaystyle\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$ (b) $\displaystyle\int 4e^{4x}\,dx = e^{4x} + C$

3. (a) $\frac{1}{4}x^4 + \frac{1}{2}x^2 + 5x + C$ (b) $\tan x + \csc x + C$ **4.** $2x + 1$; $x^2 + x + 3$ **5.** 0; $-\frac{2}{3}$

5.3 INTEGRATION BY SUBSTITUTION

In this section we will study a technique, called ***substitution****, that can often be used to transform complicated integration problems into simpler ones.*

u-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \tag{1}$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \tag{2}$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (2) can be expressed as

$$\boxed{\int f(u)\,du = F(u) + C} \tag{3}$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the ***method of u-substitution***. Here our emphasis is *not* on the interpretation of the expression $du = g'(x)\,dx$. Rather, the differential notation serves primarily as a useful "bookkeeping" device for the method of u-substitution. The following example illustrates how the method works.

▶ Example 1 Evaluate $\int (x^2 + 1)^{50} \cdot 2x\,dx$.

Solution. If we let $u = x^2 + 1$, then $du/dx = 2x$, which implies that $du = 2x\,dx$. Thus, the given integral can be written as

$$\int (x^2 + 1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C \blacktriangleleft$$

It is important to realize that in the method of u-substitution you have control over the choice of u, but once you make that choice you have no control over the resulting expression for du. Thus, in the last example we *chose* $u = x^2 + 1$ but $du = 2x\,dx$ was *computed*. Fortunately, our choice of u, combined with the computed du, worked out perfectly to produce an integral involving u that was easy to evaluate. However, in general, the method of u-substitution will fail if the chosen u and the computed du cannot be used to produce an integrand in which no expressions involving x remain, or if you cannot evaluate the resulting integral. Thus, for example, the substitution $u = x^2$, $du = 2x\,dx$ will not work for the integral

$$\int 2x \sin x^4\,dx$$

because this substitution results in the integral

$$\int \sin u^2\,du$$

which still cannot be evaluated in terms of familiar functions.

In general, there are no hard and fast rules for choosing u, and in some problems no choice of u will work. In such cases other methods need to be used, some of which will be discussed later. Making appropriate choices for u will come with experience, but you may find the following guidelines, combined with a mastery of the basic integrals in Table 5.2.1, helpful.

Guidelines for u-Substitution

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)\,dx$$

produces an integral that is expressed entirely in terms of u and its differential du. This may or may not be possible.

Step 2. If you are successful in Step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step 3. If you are successful in Step 2, then replace u by $g(x)$ to express your final answer in terms of x.

EASY TO RECOGNIZE SUBSTITUTIONS

The easiest substitutions occur when the integrand is the derivative of a known function, except for a constant added to or subtracted from the independent variable.

Example 2

$$\int \sin(x+9)\,dx = \int \sin u\,du = -\cos u + C = -\cos(x+9) + C$$

$$\boxed{\begin{gathered} u = x+9 \\ du = 1 \cdot dx = dx \end{gathered}}$$

$$\int (x-8)^{23}\,dx = \int u^{23}\,du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

$$\boxed{\begin{gathered} u = x-8 \\ du = 1 \cdot dx = dx \end{gathered}}$$

Another easy u-substitution occurs when the integrand is the derivative of a known function, except for a constant that multiplies or divides the independent variable. The following example illustrates two ways to evaluate such integrals.

Example 3 Evaluate $\displaystyle\int \cos 5x\,dx$.

Solution.

$$\int \cos 5x\,dx = \int (\cos u) \cdot \frac{1}{5}\,du = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

$$\boxed{\begin{gathered} u = 5x \\ du = 5\,dx \text{ or } dx = \tfrac{1}{5}\,du \end{gathered}}$$

Alternative Solution. There is a variation of the preceding method that some people prefer. The substitution $u = 5x$ requires $du = 5\,dx$. If there were a factor of 5 in the integrand, then we could group the 5 and dx together to form the du required by the substitution. Since there is no factor of 5, we will insert one and compensate by putting a factor of $\frac{1}{5}$ in front of the integral. The computations are as follows:

$$\int \cos 5x\,dx = \frac{1}{5}\int \cos 5x \cdot 5\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

$$\boxed{\begin{gathered} u = 5x \\ du = 5\,dx \end{gathered}}$$

More generally, if the integrand is a composition of the form $f(ax+b)$, where $f(x)$ is an easy to integrate function, then the substitution $u = ax+b$, $du = a\,dx$ will work.

▶ Example 4

$$\int \frac{dx}{\left(\frac{1}{3}x-8\right)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4}+C = -\frac{3}{4}\left(\frac{1}{3}x-8\right)^{-4}+C \quad ◀$$

$u = \frac{1}{3}x - 8$
$du = \frac{1}{3}\,dx$ or $dx = 3\,du$

▶ Example 5 Evaluate $\displaystyle\int \frac{dx}{1+3x^2}$.

Solution. Substituting

$$u = \sqrt{3}x, \quad du = \sqrt{3}\,dx$$

yields

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}}\tan^{-1}u + C = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x)+C \quad ◀$$

With the help of Theorem 5.2.3, a complicated integral can sometimes be computed by expressing it as a sum of simpler integrals.

▶ Example 6

$$\begin{aligned}\int\left(\frac{1}{x}+\sec^2\pi x\right)dx &= \int\frac{dx}{x}+\int\sec^2\pi x\,dx\\ &= \ln|x| + \int \sec^2 \pi x\,dx\\ &= \ln|x| + \frac{1}{\pi}\int \sec^2 u\,du\\ &= \ln|x| + \frac{1}{\pi}\tan u + C = \ln|x| + \frac{1}{\pi}\tan \pi x + C \quad ◀\end{aligned}$$

$u = \pi x$
$du = \pi\,dx$ or $dx = \frac{1}{\pi}\,du$

The next four examples illustrate a substitution $u = g(x)$ where $g(x)$ is a nonlinear function.

▶ Example 7 Evaluate $\displaystyle\int \sin^2 x \cos x\,dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x\,dx$$

Thus,

$$\int \sin^2 x\cos x\,dx = \int u^2\,du = \frac{u^3}{3}+C = \frac{\sin^3 x}{3}+C \quad ◀$$

Example 8 Evaluate $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}}\,dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}}\,dx$$

Thus,

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2e^u\,du = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C \blacktriangleleft$$

Example 9 Evaluate $\displaystyle\int t^4\sqrt[3]{3-5t^5}\,dt$.

Solution.

$$\int t^4\sqrt[3]{3-5t^5}\,dt = -\frac{1}{25}\int \sqrt[3]{u}\,du = -\frac{1}{25}\int u^{1/3}\,du$$

$$\boxed{\begin{array}{c} u = 3-5t^5 \\ du = -25t^4\,dt \text{ or } -\frac{1}{25}\,du = t^4\,dt \end{array}}$$

$$= -\frac{1}{25}\frac{u^{4/3}}{4/3} + C = -\frac{3}{100}\left(3-5t^5\right)^{4/3} + C \blacktriangleleft$$

Example 10 Evaluate $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

Solution. Substituting

$$u = e^x, \quad du = e^x\,dx$$

yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C \blacktriangleleft$$

LESS APPARENT SUBSTITUTIONS

The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f(g(x))$ and the remainder of the integrand is a constant multiple of $g'(x)$. If this is not the case, the method may still apply but may require more computation.

Example 11 Evaluate $\displaystyle\int x^2\sqrt{x-1}\,dx$.

Solution. The composition $\sqrt{x-1}$ suggests the substitution

$$u = x - 1 \quad \text{so that} \quad du = dx \tag{4}$$

From the first equality in (4)

$$x^2 = (u+1)^2 = u^2 + 2u + 1$$

so that

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+2u+1)\sqrt{u}\,du = \int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du$$

$$= \tfrac{2}{7}u^{7/2} + \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{7}(x-1)^{7/2} + \tfrac{4}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C \blacktriangleleft$$

Example 12 Evaluate $\int \cos^3 x\, dx$.

Solution. The only compositions in the integrand that suggest themselves are

$$\cos^3 x = (\cos x)^3 \quad \text{and} \quad \cos^2 x = (\cos x)^2$$

However, neither the substitution $u = \cos x$ nor the substitution $u = \cos^2 x$ work (verify). In this case, an appropriate substitution is not suggested by the composition contained in the integrand. On the other hand, note from Equation (2) that the derivative $g'(x)$ appears as a factor in the integrand. This suggests that we write

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$$

and solve the equation $du = \cos x\, dx$ for $u = \sin x$. Since $\sin^2 x + \cos^2 x = 1$, we then have

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx = \int (1 - u^2)\, du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

Example 13 Evaluate $\int \frac{dx}{a^2 + x^2}\, dx$, where $a \neq 0$ is a constant.

Solution. Some simple algebra and an appropriate u-substitution will allow us to use Formula 12 in Table 5.2.1.

$$\int \frac{dx}{a^2 + x^2} = \int \frac{a(dx/a)}{a^2(1 + (x/a)^2)} = \frac{1}{a}\int \frac{dx/a}{1 + (x/a)^2} \qquad \boxed{\begin{array}{c} u = x/a \\ du = dx/a \end{array}}$$

$$= \frac{1}{a}\int \frac{du}{1 + u^2} = \frac{1}{a}\tan^{-1} u + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

The method of Example 13 leads to the following generalizations of Formulas 12, 13, and 14 in Table 5.2.1 for $a > 0$:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{5}$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \tag{6}$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \tag{7}$$

Example 14 Evaluate $\int \frac{dx}{\sqrt{2 - x^2}}$.

Solution. Applying (6) with $u = x$ and $a = \sqrt{2}$ yields

$$\int \frac{dx}{\sqrt{2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + C$$

TECHNOLOGY MASTERY

If you have a CAS, use it to calculate the integrals in the examples in this section. If your CAS produces an answer that is different from the one in the text, then confirm algebraically that the two answers agree. Also, explore the effect of using the CAS to simplify the expressions it produces for the integrals.

INTEGRATION USING COMPUTER ALGEBRA SYSTEMS

The advent of computer algebra systems has made it possible to evaluate many kinds of integrals that would be laborious to evaluate by hand. For example, a handheld calculator evaluated the integral

$$\int \frac{5x^2}{(1+x)^{1/3}}\,dx = \frac{3(x+1)^{2/3}(5x^2-6x+9)}{8} + C$$

in about a second. The computer algebra system *Mathematica*, running on a personal computer, required even less time to evaluate this same integral. However, just as one would not want to rely on a calculator to compute $2+2$, so one would not want to use a CAS to integrate a simple function such as $f(x) = x^2$. Thus, even if you have a CAS, you will want to develop a reasonable level of competence in evaluating basic integrals. Moreover, the mathematical techniques that we will introduce for evaluating basic integrals are precisely the techniques that computer algebra systems use to evaluate more complicated integrals.

✔ QUICK CHECK EXERCISES 5.3 (See page 340 for answers.)

1. Indicate the u-substitution.

(a) $\displaystyle\int 3x^2(1+x^3)^{25}\,dx = \int u^{25}\,du$ if $u =$ ______ and $du =$ ______.

(b) $\displaystyle\int 2x\sin x^2\,dx = \int \sin u\,du$ if $u =$ ______ and $du =$ ______.

(c) $\displaystyle\int \frac{18x}{1+9x^2}\,dx = \int \frac{1}{u}\,du$ if $u =$ ______ and $du =$ ______.

(d) $\displaystyle\int \frac{3}{1+9x^2}\,dx = \int \frac{1}{1+u^2}\,du$ if $u =$ ______ and $du =$ ______.

2. Supply the missing integrand corresponding to the indicated u-substitution.

(a) $\displaystyle\int 5(5x-3)^{-1/3}\,dx = \int$ ______ du; $u = 5x-3$

(b) $\displaystyle\int (3-\tan x)\sec^2 x\,dx = \int$ ______ du; $u = 3-\tan x$

(c) $\displaystyle\int \frac{\sqrt[3]{8+\sqrt{x}}}{\sqrt{x}}\,dx = \int$ ______ du; $u = 8+\sqrt{x}$

(d) $\displaystyle\int e^{3x}\,dx = \int$ ______ du; $u = 3x$

EXERCISE SET 5.3 Graphing Utility C CAS

1–12 Evaluate the integrals using the indicated substitutions. ■

1. (a) $\displaystyle\int 2x(x^2+1)^{23}\,dx$; $u = x^2+1$

(b) $\displaystyle\int \cos^3 x\sin x\,dx$; $u = \cos x$

2. (a) $\displaystyle\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$; $u = \sqrt{x}$

(b) $\displaystyle\int \frac{3x\,dx}{\sqrt{4x^2+5}}$; $u = 4x^2+5$

3. (a) $\displaystyle\int \sec^2(4x+1)\,dx$; $u = 4x+1$

(b) $\displaystyle\int y\sqrt{1+2y^2}\,dy$; $u = 1+2y^2$

4. (a) $\displaystyle\int \sqrt{\sin\pi\theta}\cos\pi\theta\,d\theta$; $u = \sin\pi\theta$

(b) $\displaystyle\int (2x+7)(x^2+7x+3)^{4/5}\,dx$; $u = x^2+7x+3$

5. (a) $\displaystyle\int \cot x\csc^2 x\,dx$; $u = \cot x$

(b) $\displaystyle\int (1+\sin t)^9\cos t\,dt$; $u = 1+\sin t$

6. (a) $\displaystyle\int \cos 2x\,dx$; $u = 2x$ (b) $\displaystyle\int x\sec^2 x^2\,dx$; $u = x^2$

7. (a) $\displaystyle\int x^2\sqrt{1+x}\,dx$; $u = 1+x$

(b) $\displaystyle\int [\csc(\sin x)]^2\cos x\,dx$; $u = \sin x$

8. (a) $\displaystyle\int \sin(x-\pi)\,dx$; $u = x-\pi$

(b) $\displaystyle\int \frac{5x^4}{(x^5+1)^2}\,dx$; $u = x^5+1$

9. (a) $\displaystyle\int \frac{dx}{x\ln x}$; $u = \ln x$

(b) $\displaystyle\int e^{-5x}\,dx$; $u = -5x$

10. (a) $\int \frac{\sin 3\theta}{1+\cos 3\theta}\,d\theta;\ u = 1+\cos 3\theta$
(b) $\int \frac{e^x}{1+e^x}\,dx;\ u = 1+e^x$

11. (a) $\int \frac{x^2\,dx}{1+x^6};\ u = x^3$
(b) $\int \frac{dx}{x\sqrt{1-(\ln x)^2}};\ u = \ln x$

12. (a) $\int \frac{dx}{x\sqrt{9x^2-1}};\ u = 3x$
(b) $\int \frac{dx}{\sqrt{x}(1+x)};\ u = \sqrt{x}$

FOCUS ON CONCEPTS

13. Explain the connection between the chain rule for differentiation and the method of u-substitution for integration.

14. Explain how the substitution $u = ax + b$ helps to perform an integration in which the integrand is $f(ax+b)$, where $f(x)$ is an easy to integrate function.

15–56 Evaluate the integrals using appropriate substitutions. ■

15. $\int (4x-3)^9\,dx$
16. $\int x^3\sqrt{5+x^4}\,dx$
17. $\int \sin 7x\,dx$
18. $\int \cos\frac{x}{3}\,dx$
19. $\int \sec 4x \tan 4x\,dx$
20. $\int \sec^2 5x\,dx$
21. $\int e^{2x}\,dx$
22. $\int \frac{dx}{2x}$
23. $\int \frac{dx}{\sqrt{1-4x^2}}$
24. $\int \frac{dx}{1+16x^2}$
25. $\int t\sqrt{7t^2+12}\,dt$
26. $\int \frac{x}{\sqrt{4-5x^2}}\,dx$
27. $\int \frac{6}{(1-2x)^3}\,dx$
28. $\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$
29. $\int \frac{x^3}{(5x^4+2)^3}\,dx$
30. $\int \frac{\sin(1/x)}{3x^2}\,dx$
31. $\int e^{\sin x}\cos x\,dx$
32. $\int x^3 e^{x^4}\,dx$
33. $\int x^2 e^{-2x^3}\,dx$
34. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$
35. $\int \frac{e^x}{1+e^{2x}}\,dx$
36. $\int \frac{t}{t^4+1}\,dt$
37. $\int \frac{\sin(5/x)}{x^2}\,dx$
38. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$
39. $\int \cos^4 3t \sin 3t\,dt$
40. $\int \cos 2t \sin^5 2t\,dt$
41. $\int x\sec^2(x^2)\,dx$
42. $\int \frac{\cos 4\theta}{(1+2\sin 4\theta)^4}\,d\theta$
43. $\int \cos 4\theta\sqrt{2-\sin 4\theta}\,d\theta$
44. $\int \tan^3 5x \sec^2 5x\,dx$
45. $\int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$
46. $\int \frac{\sin\theta}{\cos^2\theta+1}\,d\theta$
47. $\int \sec^3 2x \tan 2x\,dx$
48. $\int [\sin(\sin\theta)]\cos\theta\,d\theta$
49. $\int \frac{dx}{e^x}$
50. $\int \sqrt{e^x}\,dx$
51. $\int \frac{dx}{\sqrt{x}\,e^{(2\sqrt{x})}}$
52. $\int \frac{e^{\sqrt{2y+1}}}{\sqrt{2y+1}}\,dy$
53. $\int \frac{y}{\sqrt{2y+1}}\,dy$
54. $\int x\sqrt{4-x}\,dx$
55. $\int \sin^3 2\theta\,d\theta$
56. $\int \sec^4 3\theta\,d\theta$ [*Hint:* Apply a trigonometric identity.]

57–60 Evaluate each integral by first modifying the form of the integrand and then making an appropriate substitution, if needed. ■

57. $\int \frac{t+1}{t}\,dt$
58. $\int e^{2\ln x}\,dx$
59. $\int [\ln(e^x)+\ln(e^{-x})]\,dx$
60. $\int \cot x\,dx$

61–62 Evaluate the integrals with the aid of Formulas (5), (6), and (7). ■

61. (a) $\int \frac{dx}{\sqrt{9-x^2}}$ (b) $\int \frac{dx}{5+x^2}$ (c) $\int \frac{dx}{x\sqrt{x^2-\pi}}$

62. (a) $\int \frac{e^x}{4+e^{2x}}\,dx$ (b) $\int \frac{dx}{\sqrt{9-4x^2}}$ (c) $\int \frac{dy}{y\sqrt{5y^2-3}}$

63–65 Evaluate the integrals assuming that n is a positive integer and $b \neq 0$. ■

63. $\int (a+bx)^n\,dx$
64. $\int \sqrt[n]{a+bx}\,dx$
65. $\int \sin^n(a+bx)\cos(a+bx)\,dx$

C 66. Use a CAS to check the answers you obtained in Exercises 63–65. If the answer produced by the CAS does not match yours, show that the two answers are equivalent. [*Suggestion: Mathematica* users may find it helpful to apply the Simplify command to the answer.]

FOCUS ON CONCEPTS

67. (a) Evaluate the integral $\int \sin x \cos x\,dx$ by two methods: first by letting $u = \sin x$, and then by letting $u = \cos x$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

68. (a) Evaluate the integral $\int (5x-1)^2\,dx$ by two methods: first square and integrate, then let $u = 5x - 1$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

69–72 Solve the initial-value problems. ■

69. $\dfrac{dy}{dx} = \sqrt{5x+1},\ y(3) = -2$

70. $\dfrac{dy}{dx} = 2 + \sin 3x,\ y(\pi/3) = 0$

71. $\dfrac{dy}{dt} = -e^{2t},\ y(0) = 6$

72. $\dfrac{dy}{dt} = \dfrac{1}{25+9t^2},\ y\left(-\dfrac{5}{3}\right) = \dfrac{\pi}{30}$

73. (a) Evaluate $\int [x/\sqrt{x^2+1}]\,dx$.
(b) Use a graphing utility to generate some typical integral curves of $f(x) = x/\sqrt{x^2+1}$ over the interval $(-5, 5)$.

74. (a) Evaluate $\int [x/(x^2+1)]\,dx$.
(b) Use a graphing utility to generate some typical integral curves of $f(x) = x/(x^2+1)$ over the interval $(-5, 5)$.

75. Find a function f such that the slope of the tangent line at a point (x, y) on the curve $y = f(x)$ is $\sqrt{3x+1}$ and the curve passes through the point $(0, 1)$.

76. A population of minnows in a lake is estimated to be 100,000 at the beginning of the year 2005. Suppose that t years after the beginning of 2005 the rate of growth of the population $p(t)$ (in thousands) is given by $p'(t) = (3 + 0.12t)^{3/2}$. Estimate the projected population at the beginning of the year 2010.

77. Derive integration Formula (6).

78. Derive integration Formula (7).

79. Writing If you want to evaluate an integral by u-substitution, how do you decide what part of the integrand to choose for u?

80. Writing The evaluation of an integral can sometimes result in apparently different answers (Exercises 67 and 68). Explain why this occurs and give an example. How might you show that two apparently different answers are actually equivalent?

✔ QUICK CHECK ANSWERS 5.3

1. (a) $1 + x^3$; $3x^2\,dx$ (b) x^2; $2x\,dx$ (c) $1 + 9x^2$; $18x\,dx$ (d) $3x$; $3\,dx$ **2.** (a) $u^{-1/3}$ (b) $-u$ (c) $2\sqrt[3]{u}$ (d) $\frac{1}{3}e^u$

5.4 THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

Our main goal in this section is to use the rectangle method to give a precise mathematical definition of the "area under a curve."

■ SIGMA NOTATION

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called ***sigma notation*** or ***summation notation*** because it uses the uppercase Greek letter Σ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form k^2, where k is one of the integers from 1 to 5. In sigma notation this sum can be written as

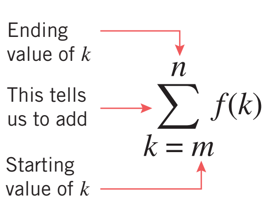
$$\sum_{k=1}^{5} k^2$$

which is read "the summation of k^2, where k runs from 1 to 5." The notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression k^2, starting with $k = 1$ and ending with $k = 5$.

More generally, if $f(k)$ is a function of k, and if m and n are integers such that $m \le n$, then

$$\sum_{k=m}^{n} f(k) \tag{1}$$

denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$ (Figure 5.4.1).



▲ **Figure 5.4.1**

▶ **Example 1**

$$\sum_{k=4}^{8} k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

$$\sum_{k=1}^{5} 2k = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=0}^{5} (2k+1) = 1 + 3 + 5 + 7 + 9 + 11$$

$$\sum_{k=0}^{5} (-1)^k (2k+1) = 1 - 3 + 5 - 7 + 9 - 11$$

$$\sum_{k=-3}^{1} k^3 = (-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3 = -27 - 8 - 1 + 0 + 1$$

$$\sum_{k=1}^{3} k \sin\left(\frac{k\pi}{5}\right) = \sin\frac{\pi}{5} + 2\sin\frac{2\pi}{5} + 3\sin\frac{3\pi}{5}$$ ◀

The numbers m and n in (1) are called, respectively, the ***lower*** and ***upper limits of summation***; and the letter k is called the ***index of summation***. It is not essential to use k as the index of summation; any letter not reserved for another purpose will do. For example,

$$\sum_{i=1}^{6} \frac{1}{i}, \quad \sum_{j=1}^{6} \frac{1}{j}, \quad \text{and} \quad \sum_{n=1}^{6} \frac{1}{n}$$

all denote the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

If the upper and lower limits of summation are the same, then the "sum" in (1) reduces to a single term. For example,

$$\sum_{k=2}^{2} k^3 = 2^3 \quad \text{and} \quad \sum_{i=1}^{1} \frac{1}{i+2} = \frac{1}{1+2} = \frac{1}{3}$$

In the sums

$$\sum_{i=1}^{5} 2 \quad \text{and} \quad \sum_{j=0}^{2} x^3$$

the expression to the right of the Σ sign does not involve the index of summation. In such cases, we take all the terms in the sum to be the same, with one term for each allowable value of the summation index. Thus,

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 \quad \text{and} \quad \sum_{j=0}^{2} x^3 = x^3 + x^3 + x^3$$

CHANGING THE LIMITS OF SUMMATION

A sum can be written in more than one way using sigma notation with different limits of summation and correspondingly different summands. For example,

$$\sum_{i=1}^{5} 2i = 2 + 4 + 6 + 8 + 10 = \sum_{j=0}^{4} (2j+2) = \sum_{k=3}^{7} (2k-4)$$

On occasion we will want to change the sigma notation for a given sum to a sigma notation with different limits of summation.

PROPERTIES OF SUMS

When stating general properties of sums it is often convenient to use a subscripted letter such as a_k in place of the function notation $f(k)$. For example,

$$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{j=1}^{5} a_j = \sum_{k=-1}^{3} a_{k+2}$$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \sum_{k=-1}^{n-2} a_{k+2}$$

Our first properties provide some basic rules for manipulating sums.

5.4.1 THEOREM

(a) $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ (*if c does not depend on k*)

(b) $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

We will prove parts (*a*) and (*b*) and leave part (*c*) as an exercise.

PROOF (*a*)

$$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k$$

PROOF (*b*)

$$\begin{aligned}\sum_{k=1}^{n} (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \quad \blacksquare\end{aligned}$$

Restating Theorem 5.4.1 in words:

(*a*) *A constant factor can be moved through a sigma sign.*

(*b*) *Sigma distributes across sums.*

(*c*) *Sigma distributes across differences.*

SUMMATION FORMULAS

The following theorem lists some useful formulas for sums of powers of integers. The derivations of these formulas are given in Appendix D.

TECHNOLOGY MASTERY

If you have access to a CAS, it will provide a method for finding closed forms such as those in Theorem 5.4.2. Use your CAS to confirm the formulas in that theorem, and then find closed forms for

$$\sum_{k=1}^{n} k^4 \quad \text{and} \quad \sum_{k=1}^{n} k^5$$

5.4.2 THEOREM

(a) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

TECHNOLOGY MASTERY

Many calculating utilities provide some way of evaluating sums expressed in sigma notation. If your utility has this capability, use it to confirm that the result in Example 2 is correct.

▶ **Example 2** Evaluate $\sum_{k=1}^{30} k(k+1)$.

Solution.

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30} (k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{30(31)(61)}{6} + \frac{30(31)}{2} = 9920 \qquad \boxed{\text{Theorem 5.4.2}(a), (b)} \quad ◀$$

In formulas such as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{or} \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

the left side of the equality is said to express the sum in ***open form*** and the right side is said to express it in ***closed form***. The open form indicates the summands and the closed form is an explicit formula for the sum.

▶ **Example 3** Express $\sum_{k=1}^{n} (3+k)^2$ in closed form.

Solution.

$$\sum_{k=1}^{n} (3+k)^2 = 4^2 + 5^2 + \cdots + (3+n)^2$$

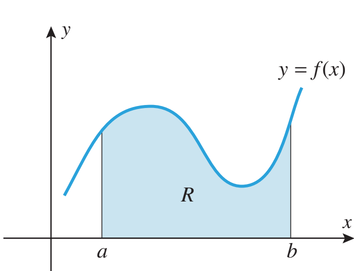
$$= [1^2 + 2^2 + 3^3 + 4^2 + 5^2 + \cdots + (3+n)^2] - [1^2 + 2^2 + 3^2]$$

$$= \left(\sum_{k=1}^{3+n} k^2\right) - 14$$

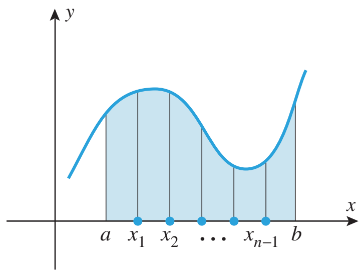
$$= \frac{(3+n)(4+n)(7+2n)}{6} - 14 = \frac{1}{6}(73n + 21n^2 + 2n^3) \quad ◀$$

A DEFINITION OF AREA

We now turn to the problem of giving a precise definition of what is meant by the "area under a curve." Specifically, suppose that the function f is continuous and nonnegative on the interval $[a, b]$, and let R denote the region bounded below by the x-axis, bounded on the sides by the vertical lines $x = a$ and $x = b$, and bounded above by the curve $y = f(x)$ (Figure 5.4.2). Using the rectangle method of Section 5.1, we can motivate a definition for the area of R as follows:



▲ **Figure 5.4.2**



▲ **Figure 5.4.3**

- Divide the interval $[a, b]$ into n equal subintervals by inserting $n - 1$ equally spaced points between a and b, and denote those points by

$$x_1, x_2, \ldots, x_{n-1}$$

(Figure 5.4.3). Each of these subintervals has width $(b - a)/n$, which is customarily denoted by

$$\Delta x = \frac{b-a}{n}$$

- Over each subinterval construct a rectangle whose height is the value of f at an arbitrarily selected point in the subinterval. Thus, if

$$x_1^*, x_2^*, \ldots, x_n^*$$

denote the points selected in the subintervals, then the rectangles will have heights $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ and areas

$$f(x_1^*)\Delta x, \quad f(x_2^*)\Delta x, \ldots, \quad f(x_n^*)\Delta x$$

(Figure 5.4.4).

- The union of the rectangles forms a region R_n whose area can be regarded as an approximation to the area A of the region R; that is,

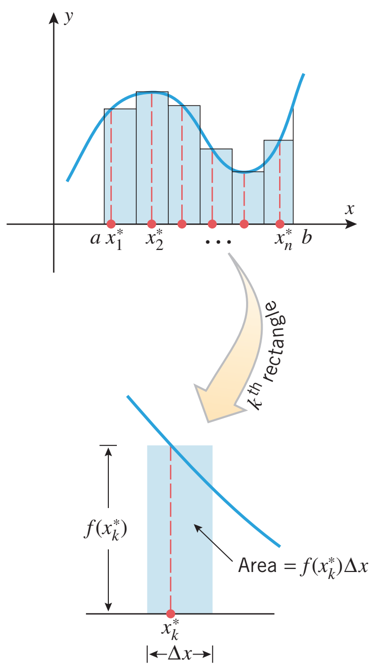
$$A = \text{area}(R) \approx \text{area}(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

(Figure 5.4.5). This can be expressed more compactly in sigma notation as

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

- Repeat the process using more and more subdivisions, and define the area of R to be the "limit" of the areas of the approximating regions R_n as n increases without bound. That is, we define the area A as

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$



▲ **Figure 5.4.4**

In summary, we make the following definition.

The limit in (2) is interpreted to mean that given any number $\epsilon > 0$ the inequality

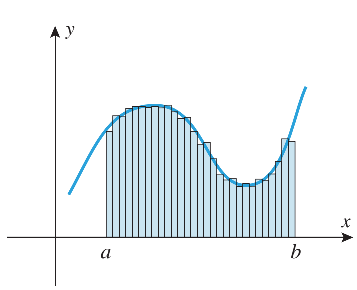
$$\left| A - \sum_{k=1}^{n} f(x_k^*)\Delta x \right| < \epsilon$$

holds when n is sufficiently large, no matter how the points x_k^* are selected.

5.4.3 DEFINITION ***(Area Under a Curve)*** If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{2}$$

REMARK There is a difference in interpretation between $\lim_{n\to+\infty}$ and $\lim_{x\to+\infty}$, where n represents a positive integer and x represents a real number. Later we will study limits of the type $\lim_{n\to+\infty}$ in detail, but for now suffice it to say that the computational techniques we have used for limits of type $\lim_{x\to+\infty}$ will also work for $\lim_{n\to+\infty}$.



▲ **Figure 5.4.5** area$(R_n) \approx$ area(R)

The values of $x_1^*, x_2^*, \ldots, x_n^*$ in (2) can be chosen arbitrarily, so it is conceivable that different choices of these values might produce different values of A. Were this to happen, then Definition 5.4.3 would not be an acceptable definition of area. Fortunately, this does not happen; it is proved in advanced courses that if f is continuous (as we have assumed), then the same value of A results no matter how the x_k^* are chosen. In practice they are chosen in some systematic fashion, some common choices being

- the left endpoint of each subinterval
- the right endpoint of each subinterval
- the midpoint of each subinterval

To be more specific, suppose that the interval $[a, b]$ is divided into n equal parts of length $\Delta x = (b - a)/n$ by the points $x_1, x_2, \ldots, x_{n-1}$, and let $x_0 = a$ and $x_n = b$ (Figure 5.4.6). Then,

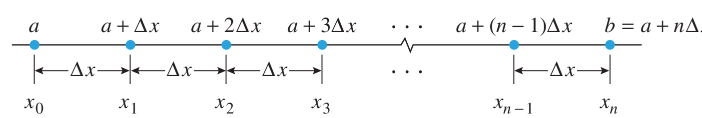
$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, 2, \ldots, n$$

Thus, the left endpoint, right endpoint, and midpoint choices for $x_1^*, x_2^*, \ldots, x_n^*$ are given by

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \boxed{\text{Left endpoint}} \tag{3}$$

$$x_k^* = x_k = a + k\Delta x \qquad \boxed{\text{Right endpoint}} \tag{4}$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \boxed{\text{Midpoint}} \tag{5}$$



▶ **Figure 5.4.6**

When applicable, the antiderivative method will be the method of choice for finding exact areas. However, the following examples will help to reinforce the ideas that we have just discussed.

▶ **Example 4** Use Definition 5.4.3 with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval [0, 1].

Solution. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so it follows from (4) that

$$x_k^* = a + k\Delta x = \frac{k}{n}$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^*)^2 \Delta x &= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \boxed{\text{Part } (b) \text{ of Theorem 5.4.2}} \\
&= \frac{1}{6}\left(\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right) = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)
\end{aligned}$$

from which it follows that

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{3}$$

Observe that this is consistent with the results in Table 5.1.2 and the related discussion in Section 5.1. ◀

In the solution to Example 4 we made use of one of the "closed form" summation formulas from Theorem 5.4.2. The next result collects some consequences of Theorem 5.4.2 that can facilitate computations of area using Definition 5.4.3.

What pattern is revealed by parts (*b*)–(*d*) of Theorem 5.4.4? Does part (*a*) also fit this pattern? What would you conjecture to be the value of

$$\lim_{n\to+\infty} \frac{1}{n^m}\sum_{k=1}^{n} k^{m-1}$$

5.4.4 THEOREM

(*a*) $\displaystyle\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} 1 = 1$ (*b*) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2}$

(*c*) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{3}$ (*d*) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$

The proof of Theorem 5.4.4 is left as an exercise for the reader.

▶ **Example 5** Use Definition 5.4.3 with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval $[0, 3]$.

Solution. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so it follows from (5) that

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned} f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\ &= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \end{aligned}$$

from which it follows that

$$\begin{aligned} A &= \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= \lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right) \\ &= \lim_{n\to+\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right] \\ &= 27\left[1 - \frac{1}{3} + 0\cdot\frac{1}{2} - 0\cdot 1\right] = 18 \qquad \boxed{\text{Theorem 5.4.4}} \end{aligned}$$

◀

NUMERICAL APPROXIMATIONS OF AREA

The antiderivative method discussed in Section 5.1 (and to be studied in more detail later) is an appropriate tool for finding the exact area under a curve when an antiderivative of the integrand can be found. However, if an antiderivative cannot be found, then we must resort to *approximating* the area. Definition 5.4.3 provides a way of doing this. It follows from this definition that if n is large, then

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k^*) = \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)] \tag{6}$$

will be a good approximation to the area A. If one of Formulas (3), (4), or (5) is used to choose the x_k^* in (6), then the result is called the ***left endpoint approximation***, the ***right endpoint approximation***, or the ***midpoint approximation***, respectively (Figure 5.4.7).

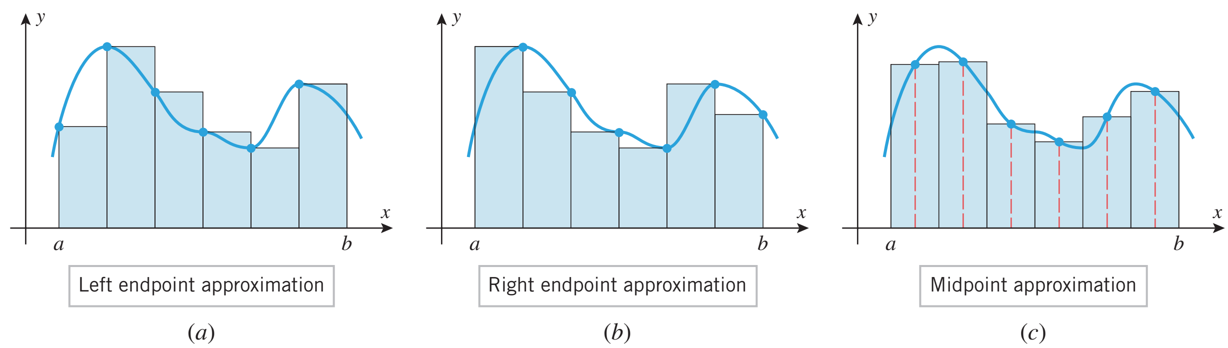


Figure 5.4.7

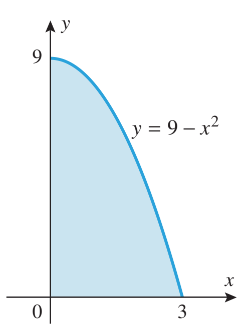


Figure 5.4.8

Example 6 Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y = 9 - x^2$ over the interval $[0, 3]$ with $n = 10$, $n = 20$, and $n = 50$ (Figure 5.4.8). Compare the accuracies of these three methods.

Solution. Details of the computations for the case $n = 10$ are shown to six decimal places in Table 5.4.1 and the results of all the computations are given in Table 5.4.2. We showed in Example 5 that the exact area is 18 (i.e., 18 square units), so in this case the midpoint approximation is more accurate than the endpoint approximations. This is also evident geometrically from Figure 5.4.9. You can also see from the figure that in this case the left endpoint approximation overestimates the area and the right endpoint approximation underestimates it. Later in the text we will investigate the error that results when an area is approximated by the midpoint rule. ◀

NET SIGNED AREA

In Definition 5.4.3 we assumed that f is continuous and nonnegative on the interval $[a, b]$. If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{7}$$

no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas—the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the ***net signed area***

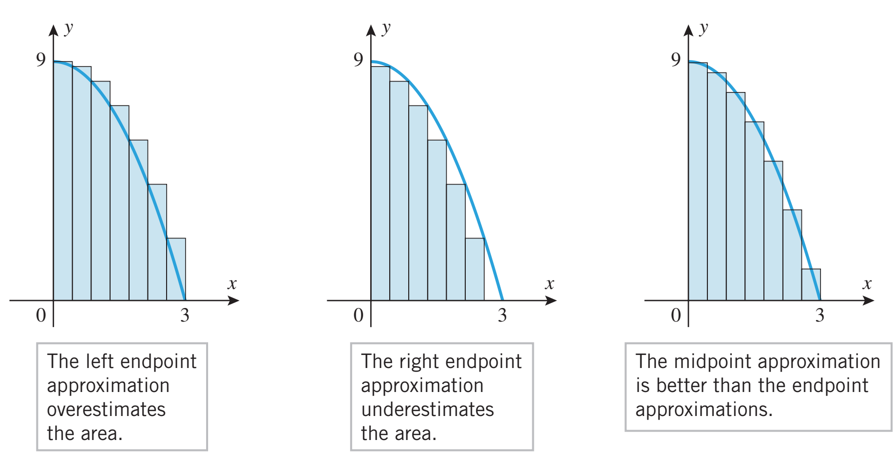
Table 5.4.1

$n = 10,\ \Delta x = (b-a)/n = (3-0)/10 = 0.3$

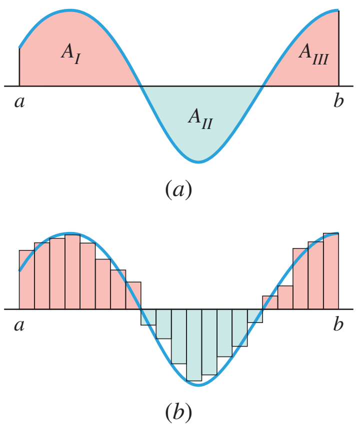
	LEFT ENDPOINT APPROXIMATION		RIGHT ENDPOINT APPROXIMATION		MIDPOINT APPROXIMATION	
k	x_k^*	$9-(x_k^*)^2$	x_k^*	$9-(x_k^*)^2$	x_k^*	$9-(x_k^*)^2$
1	0.0	9.000000	0.3	8.910000	0.15	8.977500
2	0.3	8.910000	0.6	8.640000	0.45	8.797500
3	0.6	8.640000	0.9	8.190000	0.75	8.437500
4	0.9	8.190000	1.2	7.560000	1.05	7.897500
5	1.2	7.560000	1.5	6.750000	1.35	7.177500
6	1.5	6.750000	1.8	5.760000	1.65	6.277500
7	1.8	5.760000	2.1	4.590000	1.95	5.197500
8	2.1	4.590000	2.4	3.240000	2.25	3.937500
9	2.4	3.240000	2.7	1.710000	2.55	2.497500
10	2.7	1.710000	3.0	0.000000	2.85	0.877500
		64.350000		55.350000		60.075000
$\Delta x \sum_{k=1}^{n} f(x_k^*)$		(0.3)(64.350000) = 19.305000		(0.3)(55.350000) = 16.605000		(0.3)(60.075000) = 18.022500

Table 5.4.2

n	LEFT ENDPOINT APPROXIMATION	RIGHT ENDPOINT APPROXIMATION	MIDPOINT APPROXIMATION
10	19.305000	16.605000	18.022500
20	18.663750	17.313750	18.005625
50	18.268200	17.728200	18.000900



▲ **Figure 5.4.9**



▲ **Figure 5.4.10**

between the graph of $y = f(x)$ and the interval $[a, b]$. For example, in Figure 5.4.10a, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(A_I + A_{III}) - A_{II} = \big[\text{area above } [a, b]\big] - \big[\text{area below } [a, b]\big]$$

To explain why the limit in (7) represents this net signed area, let us subdivide the interval $[a, b]$ in Figure 5.4.10a into n equal subintervals and examine the terms in the sum

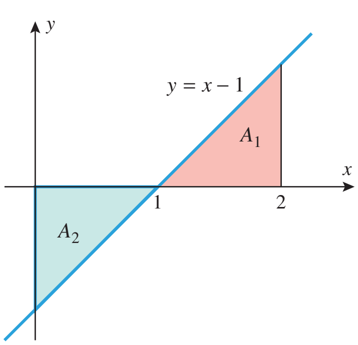
$$\sum_{k=1}^{n} f(x_k^*)\Delta x \tag{8}$$

If $f(x_k^*)$ is positive, then the product $f(x_k^*)\Delta x$ represents the area of the rectangle with height $f(x_k^*)$ and base Δx (the pink rectangles in Figure 5.4.10b). However, if $f(x_k^*)$ is negative, then the product $f(x_k^*)\Delta x$ is the *negative* of the area of the rectangle with height $|f(x_k^*)|$ and base Δx (the green rectangles in Figure 5.4.10b). Thus, (8) represents the total area of the pink rectangles minus the total area of the green rectangles. As n increases, the pink rectangles fill out the regions with areas A_I and A_{III} and the green rectangles fill out the region with area A_{II}, which explains why the limit in (7) represents the signed area between $y = f(x)$ and the interval $[a, b]$. We formalize this in the following definition.

As with Definition 5.4.3, it can be proved that the limit in (9) always exists and that the same value of A results no matter how the points in the subintervals are chosen.

5.4.5 DEFINITION (*Net Signed Area*) If the function f is continuous on $[a, b]$, then the ***net signed area*** A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{9}$$



▲ **Figure 5.4.11**

Figure 5.4.11 shows the graph of $f(x) = x - 1$ over the interval [0, 2]. It is geometrically evident that the areas A_1 and A_2 in that figure are equal, so we expect the net signed area between the graph of f and the interval [0, 2] to be zero.

▶ **Example 7** Confirm that the net signed area between the graph of $f(x) = x - 1$ and the interval [0, 2] is zero by using Definition 5.4.5 with x_k^* chosen to be the left endpoint of each subinterval.

Solution. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

so it follows from (3) that

$$x_k^* = a + (k-1)\Delta x = (k-1)\left(\frac{2}{n}\right)$$

Thus,

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^* - 1)\Delta x &= \sum_{k=1}^{n} \left[(k-1)\left(\frac{2}{n}\right) - 1\right]\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^{n} \left[\left(\frac{4}{n^2}\right)k - \frac{4}{n^2} - \frac{2}{n}\right]\end{aligned}$$

from which it follows that

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} \left[4\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{4}{n}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 2\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right]$$

$$= 4\left(\frac{1}{2}\right) - 0\cdot 1 - 2\cdot 1 = 0 \quad \boxed{\text{Theorem 5.4.4}}$$

This confirms that the net signed area is zero. ◀

✔QUICK CHECK EXERCISES 5.4 *(See page 352 for answers.)*

(See page 352 for answers.)

1. (a) Write the sum in two ways:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \sum_{k=1}^{4} \_\_\_\_\_\_ = \sum_{j=0}^{3} \_\_\_\_\_\_$$

(b) Express the sum $10 + 10^2 + 10^3 + 10^4 + 10^5$ using sigma notation.

2. Express the sums in closed form.

(a) $\sum_{k=1}^{n} k$ (b) $\sum_{k=1}^{n} (6k+1)$ (c) $\sum_{k=1}^{n} k^2$

3. Divide the interval $[1, 3]$ into $n = 4$ subintervals of equal length.

(a) Each subinterval has width ______.

(b) The left endpoints of the subintervals are ______.

(c) The midpoints of the subintervals are ______.

(d) The right endpoints of the subintervals are ______.

4. Find the left endpoint approximation for the area between the curve $y = x^2$ and the interval $[1, 3]$ using $n = 4$ equal subdivisions of the interval.

5. The right endpoint approximation for the net signed area between $y = f(x)$ and an interval $[a, b]$ is given by

$$\sum_{k=1}^{n} \frac{6k+1}{n^2}$$

Find the exact value of this net signed area.

EXERCISE SET 5.4 [C] CAS

1. Evaluate.

(a) $\sum_{k=1}^{3} k^3$ (b) $\sum_{j=2}^{6} (3j-1)$ (c) $\sum_{i=-4}^{1} (i^2 - i)$

(d) $\sum_{n=0}^{5} 1$ (e) $\sum_{k=0}^{4} (-2)^k$ (f) $\sum_{n=1}^{6} \sin n\pi$

2. Evaluate.

(a) $\sum_{k=1}^{4} k \sin \frac{k\pi}{2}$ (b) $\sum_{j=0}^{5} (-1)^j$ (c) $\sum_{i=7}^{20} \pi^2$

(d) $\sum_{m=3}^{5} 2^{m+1}$ (e) $\sum_{n=1}^{6} \sqrt{n}$ (f) $\sum_{k=0}^{10} \cos k\pi$

3–8 Write each expression in sigma notation but do not evaluate. ■

3. $1 + 2 + 3 + \cdots + 10$

4. $3\cdot 1 + 3\cdot 2 + 3\cdot 3 + \cdots + 3\cdot 20$

5. $2 + 4 + 6 + 8 + \cdots + 20$ **6.** $1 + 3 + 5 + 7 + \cdots + 15$

7. $1 - 3 + 5 - 7 + 9 - 11$ **8.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

9. (a) Express the sum of the even integers from 2 to 100 in sigma notation.

(b) Express the sum of the odd integers from 1 to 99 in sigma notation.

10. Express in sigma notation.

(a) $a_1 - a_2 + a_3 - a_4 + a_5$

(b) $-b_0 + b_1 - b_2 + b_3 - b_4 + b_5$

(c) $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

(d) $a^5 + a^4 b + a^3 b^2 + a^2 b^3 + a b^4 + b^5$

11–16 Use Theorem 5.4.2 to evaluate the sums. Check your answers using the summation feature of a calculating utility. ■

11. $\sum_{k=1}^{100} k$ **12.** $\sum_{k=1}^{100} (7k+1)$ **13.** $\sum_{k=1}^{20} k^2$

14. $\sum_{k=4}^{20} k^2$ **15.** $\sum_{k=1}^{30} k(k-2)(k+2)$

16. $\sum_{k=1}^{6} (k - k^3)$

17–20 Express the sums in closed form. ■

17. $\sum_{k=1}^{n} \frac{3k}{n}$ **18.** $\sum_{k=1}^{n-1} \frac{k^2}{n}$ **19.** $\sum_{k=1}^{n-1} \frac{k^3}{n^2}$

20. $\sum_{k=1}^{n} \left(\frac{5}{n} - \frac{2k}{n}\right)$

21–24 True–False Determine whether the statement is true or false. Explain your answer. ■

21. For all positive integers n
$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

22. The midpoint approximation is the average of the left endpoint approximation and the right endpoint approximation.

23. Every right endpoint approximation for the area under the graph of $y = x^2$ over an interval $[a, b]$ will be an overestimate.

24. For any continuous function f, the area between the graph of f and an interval $[a, b]$ (on which f is defined) is equal to the absolute value of the net signed area between the graph of f and the interval $[a, b]$.

FOCUS ON CONCEPTS

25. (a) Write the first three and final two summands in the sum
$$\sum_{k=1}^{n}\left(2 + k \cdot \frac{3}{n}\right)^4 \frac{3}{n}$$
Explain why this sum gives the right endpoint approximation for the area under the curve $y = x^4$ over the interval $[2, 5]$.

(b) Show that a change in the index range of the sum in part (a) can produce the left endpoint approximation for the area under the curve $y = x^4$ over the interval $[2, 5]$.

26. For a function f that is continuous on $[a, b]$, Definition 5.4.5 says that the net signed area A between $y = f(x)$ and the interval $[a, b]$ is
$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$
Give geometric interpretations for the symbols n, x_k^*, and Δx. Explain how to interpret the limit in this definition.

27–30 Divide the specified interval into $n = 4$ subintervals of equal length and then compute
$$\sum_{k=1}^{4} f(x_k^*)\Delta x$$
with x_k^* as (a) the left endpoint of each subinterval, (b) the midpoint of each subinterval, and (c) the right endpoint of each subinterval. Illustrate each part with a graph of f that includes the rectangles whose areas are represented in the sum. ■

27. $f(x) = 3x + 1;\ [2, 6]$ **28.** $f(x) = 1/x;\ [1, 9]$

29. $f(x) = \cos x;\ [0, \pi]$ **30.** $f(x) = 2x - x^2;\ [-1, 3]$

C **31–34** Use a calculating utility with summation capabilities or a CAS to obtain an approximate value for the area between the curve $y = f(x)$ and the specified interval with $n = 10$, 20, and 50 subintervals using the (a) left endpoint, (b) midpoint, and (c) right endpoint approximations. ■

31. $f(x) = 1/x;\ [1, 2]$ **32.** $f(x) = 1/x^2;\ [1, 3]$

33. $f(x) = \sqrt{x};\ [0, 4]$ **34.** $f(x) = \sin x;\ [0, \pi/2]$

35–40 Use Definition 5.4.3 with x_k^* as the *right* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

35. $f(x) = x/2;\ [1, 4]$ **36.** $f(x) = 5 - x;\ [0, 5]$

37. $f(x) = 9 - x^2;\ [0, 3]$ **38.** $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

39. $f(x) = x^3;\ [2, 6]$ **40.** $f(x) = 1 - x^3;\ [-3, -1]$

41–44 Use Definition 5.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

41. $f(x) = x/2;\ [1, 4]$ **42.** $f(x) = 5 - x;\ [0, 5]$

43. $f(x) = 9 - x^2;\ [0, 3]$ **44.** $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

45–48 Use Definition 5.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

45. $f(x) = 2x;\ [0, 4]$ **46.** $f(x) = 6 - x;\ [1, 5]$

47. $f(x) = x^2;\ [0, 1]$ **48.** $f(x) = x^2;\ [-1, 1]$

49–52 Use Definition 5.4.5 with x_k^* as the *right* endpoint of each subinterval to find the net signed area between the curve $y = f(x)$ and the specified interval. ■

49. $f(x) = x;\ [-1, 1]$. Verify your answer with a simple geometric argument.

50. $f(x) = x;\ [-1, 2]$. Verify your answer with a simple geometric argument.

51. $f(x) = x^2 - 1;\ [0, 2]$ **52.** $f(x) = x^3;\ [-1, 1]$

53. (a) Show that the area under the graph of $y = x^3$ and over the interval $[0, b]$ is $b^4/4$.
(b) Find a formula for the area under $y = x^3$ over the interval $[a, b]$, where $a \ge 0$.

54. Find the area between the graph of $y = \sqrt{x}$ and the interval $[0, 1]$. [*Hint:* Use the result of Exercise 25 of Section 5.1.]

55. An artist wants to create a rough triangular design using uniform square tiles glued edge to edge. She places n tiles in a row to form the base of the triangle and then makes each successive row two tiles shorter than the preceding row. Find a formula for the number of tiles used in the design. [*Hint:* Your answer will depend on whether n is even or odd.]

56. An artist wants to create a sculpture by gluing together uniform spheres. She creates a rough rectangular base that has 50 spheres along one edge and 30 spheres along the other. She then creates successive layers by gluing spheres in the grooves of the preceding layer. How many spheres will there be in the sculpture?

57–60 Consider the sum

$$\sum_{k=1}^{4}[(k+1)^3 - k^3] = [5^3 - 4^3] + [4^3 - 3^3] + [3^3 - 2^3] + [2^3 - 1^3] = 5^3 - 1^3 = 124$$

For convenience, the terms are listed in reverse order. Note how cancellation allows the entire sum to collapse like a telescope. A sum is said to ***telescope*** when part of each term cancels part of an adjacent term, leaving only portions of the first and last terms uncanceled. Evaluate the telescoping sums in these exercises.

57. $\displaystyle\sum_{k=5}^{17}(3^k - 3^{k-1})$

58. $\displaystyle\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

59. $\displaystyle\sum_{k=2}^{20}\left(\frac{1}{k^2} - \frac{1}{(k-1)^2}\right)$

60. $\displaystyle\sum_{k=1}^{100}(2^{k+1} - 2^k)$

61. (a) Show that

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$$\left[\textit{Hint: } \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right]$$

(b) Use the result in part (a) to find

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}$$

62. (a) Show that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

$$\left[\textit{Hint: } \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\right]$$

(b) Use the result in part (a) to find

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}$$

63. Let $\bar{x}$ denote the arithmetic average of the n numbers $x_1, x_2, \ldots, x_n$. Use Theorem 5.4.1 to prove that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0$$

64. Let

$$S = \sum_{k=0}^{n} ar^k$$

Show that $S - rS = a - ar^{n+1}$ and hence that

$$\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1-r} \quad (r \neq 1)$$

(A sum of this form is called a ***geometric sum***.)

65. By writing out the sums, determine whether the following are valid identities.

(a) $\displaystyle\int\left[\sum_{i=1}^{n} f_i(x)\right]dx = \sum_{i=1}^{n}\left[\int f_i(x)\,dx\right]$

(b) $\displaystyle\frac{d}{dx}\left[\sum_{i=1}^{n} f_i(x)\right] = \sum_{i=1}^{n}\left[\frac{d}{dx}[f_i(x)]\right]$

66. Which of the following are valid identities?

(a) $\displaystyle\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$ (b) $\displaystyle\sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i\right)^2$

(c) $\displaystyle\sum_{i=1}^{n}\frac{a_i}{b_i} = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$ (d) $\displaystyle\sum_{i=1}^{n} a_i = \sum_{j=0}^{n-1} a_{j+1}$

67. Prove part (*c*) of Theorem 5.4.1.

68. Prove Theorem 5.4.4.

69. Writing What is *net signed area*? How does this concept expand our application of the rectangle method?

70. Writing Based on Example 6, one might conjecture that the midpoint approximation always provides a better approximation than either endpoint approximation. Discuss the merits of this conjecture.

✔ QUICK CHECK ANSWERS 5.4

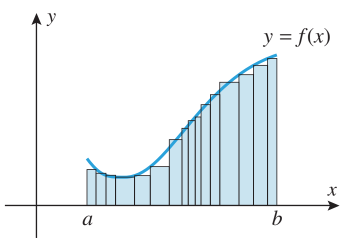
1. (a) $\dfrac{1}{2k}$; $\dfrac{1}{2(j+1)}$ (b) $\displaystyle\sum_{k=1}^{5} 10^k$ **2.** (a) $\dfrac{n(n+1)}{2}$ (b) $3n(n+1)+n$ (c) $\dfrac{n(n+1)(2n+1)}{6}$ **3.** (a) 0.5 (b) 1, 1.5, 2, 2.5 (c) 1.25, 1.75, 2.25, 2.75 (d) 1.5, 2, 2.5, 3 **4.** 6.75 **5.** $\displaystyle\lim_{n\to+\infty}\frac{3n^2+4n}{n^2} = 3$

5.5 THE DEFINITE INTEGRAL

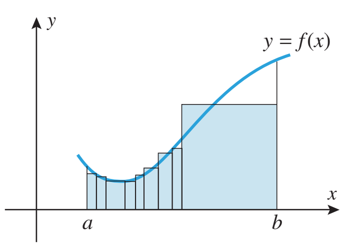
In this section we will introduce the concept of a "definite integral," which will link the concept of area to other important concepts such as length, volume, density, probability, and work.

RIEMANN SUMS AND THE DEFINITE INTEGRAL

In our definition of net signed area (Definition 5.4.5), we assumed that for each positive number n, the interval $[a, b]$ was subdivided into n subintervals of equal length to create bases for the approximating rectangles. For some functions it may be more convenient to use rectangles with different widths (see Making Connections Exercises 2 and 3); however, if we are to "exhaust" an area with rectangles of different widths, then it is important that successive subdivisions be constructed in such a way that the widths of all the rectangles approach zero as n increases (Figure 5.5.1). Thus, we must preclude the kind of situation that occurs in Figure 5.5.2 in which the right half of the interval is never subdivided. If this kind of subdivision were allowed, the error in the approximation would not approach zero as n increased.



▲ **Figure 5.5.1**



▲ **Figure 5.5.2**

A ***partition*** of the interval $[a, b]$ is a collection of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

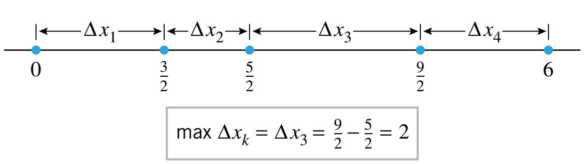
that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \Delta x_3 = x_3 - x_2, \ldots, \quad \Delta x_n = x_n - x_{n-1}$$

The partition is said to be ***regular*** provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

For a regular partition, the widths of the approximating rectangles approach zero as n is made large. Since this need not be the case for a general partition, we need some way to measure the "size" of these widths. One approach is to let $\max \Delta x_k$ denote the largest of the subinterval widths. The magnitude $\max \Delta x_k$ is called the ***mesh size*** of the partition. For example, Figure 5.5.3 shows a partition of the interval $[0, 6]$ into four subintervals with a mesh size of 2.



▶ **Figure 5.5.3**

If we are to generalize Definition 5.4.5 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \quad \text{is replaced by} \quad \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Some writers use the symbol $\|\Delta\|$ rather than $\max \Delta x_k$ for the mesh size of the partition, in which case $\max \Delta x_k \to 0$ would be replaced by $\|\Delta\| \to 0$.

We also need to replace the expression $n \to +\infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose. Based on our intuitive concept of area, we would then expect the net signed area A between the graph of f and the interval $[a, b]$ to satisfy the equation

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

(We will see in a moment that this is the case.) The limit that appears in this expression is one of the fundamental concepts of integral calculus and forms the basis for the following definition.

5.5.1 DEFINITION A function f is said to be ***integrable*** on a finite closed interval $[a, b]$ if the limit

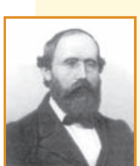
$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the ***definite integral*** of f from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $f(x)$ is called the ***integrand***.

The notation used for the definite integral deserves some comment. Historically, the expression "$f(x)\,dx$" was interpreted to be the "infinitesimal area" of a rectangle with height $f(x)$ and "infinitesimal" width dx. By "summing" these infinitesimal areas, the entire area under the curve was obtained. The integral symbol "$\int$" is an "elongated s" that was used to indicate this summation. For us, the integral symbol "$\int$" and the symbol "dx" can serve as reminders that the definite integral is actually a limit of a *summation* as $\Delta x_k \to 0$. The sum that appears in Definition 5.5.1 is called a ***Riemann sum***, and the definite integral is sometimes called the ***Riemann integral*** in honor of the German mathematician Bernhard Riemann who formulated many of the basic concepts of integral calculus. (The reason for the similarity in notation between the definite integral and the indefinite integral will become clear in the next section, where we will establish a link between the two types of "integration.")



Georg Friedrich Bernhard Riemann (1826–1866) German mathematician. Bernhard Riemann, as he is commonly known, was the son of a Protestant minister. He received his elementary education from his father and showed brilliance in arithmetic at an early age. In 1846 he enrolled at Göttingen University to study theology and philology, but he soon transferred to mathematics. He studied physics under W. E. Weber and mathematics under Carl Friedrich Gauss, whom some people consider to be the greatest mathematician who ever lived. In 1851 Riemann received his Ph.D. under Gauss, after which he remained at Göttingen to teach. In 1862, one month after his marriage, Riemann suffered an attack of pleuritis, and for the remainder of his life was an extremely sick man. He finally succumbed to tuberculosis in 1866 at age 39.

An interesting story surrounds Riemann's work in geometry. For his introductory lecture prior to becoming an associate professor, Riemann submitted three possible topics to Gauss. Gauss surprised Riemann by choosing the topic Riemann liked the least, the foundations of geometry. The lecture was like a scene from a movie. The old and failing Gauss, a giant in his day, watching intently as his brilliant and youthful protégé skillfully pieced together portions of the old man's own work into a complete and beautiful system. Gauss is said to have gasped with delight as the lecture neared its end, and on the way home he marveled at his student's brilliance. Gauss died shortly thereafter. The results presented by Riemann that day eventually evolved into a fundamental tool that Einstein used some 50 years later to develop relativity theory.

In addition to his work in geometry, Riemann made major contributions to the theory of complex functions and mathematical physics. The notion of the definite integral, as it is presented in most basic calculus courses, is due to him. Riemann's early death was a great loss to mathematics, for his mathematical work was brilliant and of fundamental importance.

The limit that appears in Definition 5.5.1 is somewhat different from the kinds of limits discussed in Chapter 1. Loosely phrased, the expression

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = L$$

is intended to convey the idea that we can force the Riemann sums to be as close as we please to L, regardless of how the values of x_k^* are chosen, by making the mesh size of the partition sufficiently small. While it is possible to give a more formal definition of this limit, we will simply rely on intuitive arguments when applying Definition 5.5.1.

Although a function need not be continuous on an interval to be integrable on that interval (Exercise 42), we will be interested primarily in definite integrals of continuous functions. The following theorem, which we will state without proof, says that if a function is continuous on a finite closed interval, then it is integrable on that interval, and its definite integral is the net signed area between the graph of the function and the interval.

5.5.2 THEOREM *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is*

$$A = \int_a^b f(x)\,dx \tag{1}$$

REMARK Formula (1) follows from the integrability of f, since the integrability allows us to use any partitions to evaluate the integral. In particular, if we use *regular* partitions of $[a, b]$, then

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

for all values of k. This implies that $\max \Delta x_k = (b-a)/n$, from which it follows that $\max \Delta x_k \to 0$ if and only if $n \to +\infty$. Thus,

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = A$$

In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute signed areas.

In Example 1, it is understood that the units of area are the squared units of length, even though we have not stated the units of length explicitly.

► **Example 1** Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

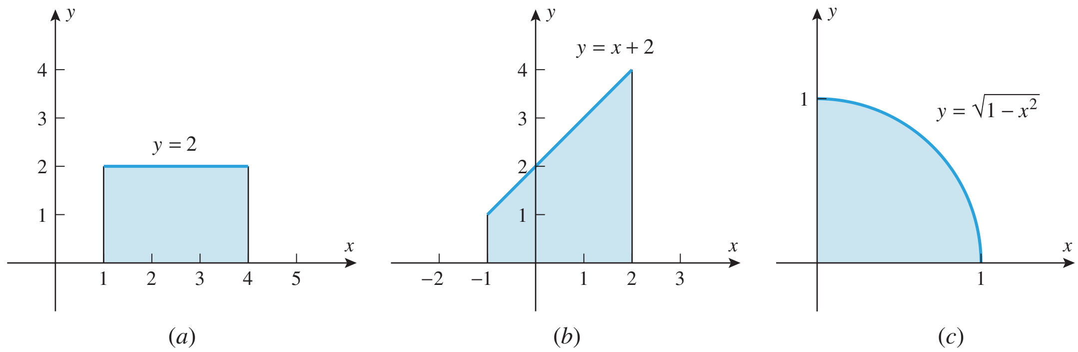
$$\text{(a)} \int_1^4 2\,dx \qquad \text{(b)} \int_{-1}^2 (x+2)\,dx \qquad \text{(c)} \int_0^1 \sqrt{1-x^2}\,dx$$

Solution (a). The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 5.5.4*a*). Thus,

$$\int_1^4 2\,dx = (\text{area of rectangle}) = 2(3) = 6$$

Solution (b). The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$ (Figure 5.5.4*b*). Thus,

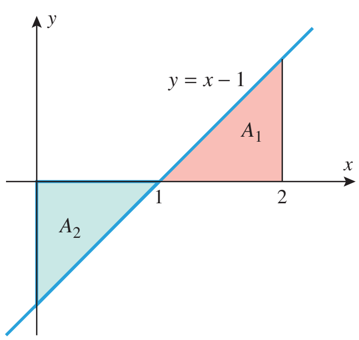
$$\int_{-1}^2 (x+2)\,dx = (\text{area of trapezoid}) = \frac{1}{2}(1+4)(3) = \frac{15}{2}$$



▲ Figure 5.5.4

Solution (c). The graph of $y = \sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 5.5.4*c*). Thus,

$$\int_0^1 \sqrt{1 - x^2}\, dx = (\text{area of quarter-circle}) = \frac{1}{4}\pi(1^2) = \frac{\pi}{4} \blacktriangleleft$$



▲ Figure 5.5.5

▶ Example 2 Evaluate

$$\text{(a)} \int_0^2 (x - 1)\, dx \qquad \text{(b)} \int_0^1 (x - 1)\, dx$$

Solution. The graph of $y = x - 1$ is shown in Figure 5.5.5, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval [0, 2] the net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$, and over the interval [0, 1] the net signed area is $-A_2 = -\frac{1}{2}$. Thus,

$$\int_0^2 (x - 1)\, dx = 0 \quad \text{and} \quad \int_0^1 (x - 1)\, dx = -\tfrac{1}{2}$$

(Recall that in Example 7 of Section 5.4, we used Definition 5.4.5 to show that the net signed area between the graph of $y = x - 1$ and the interval [0, 2] is zero.) ◀

PROPERTIES OF THE DEFINITE INTEGRAL

It is assumed in Definition 5.5.1 that $[a, b]$ is a finite closed interval with $a < b$, and hence the upper limit of integration in the definite integral is greater than the lower limit of integration. However, it will be convenient to extend this definition to allow for cases in which the upper and lower limits of integration are equal or the lower limit of integration is greater than the upper limit of integration. For this purpose we make the following special definitions.

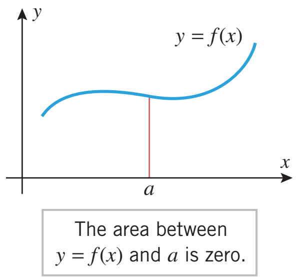
5.5.3 DEFINITION

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\, dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$



▲ **Figure 5.5.6**

Part (a) of this definition is consistent with the intuitive idea that the area between a point on the x-axis and a curve $y = f(x)$ should be zero (Figure 5.5.6). Part (b) of the definition is simply a useful convention; it states that interchanging the limits of integration reverses the sign of the integral.

▶ Example 3

(a) $\displaystyle\int_1^1 x^2\,dx = 0$

(b) $\displaystyle\int_1^0 \sqrt{1-x^2}\,dx = -\underbrace{\int_0^1 \sqrt{1-x^2}\,dx}_{\text{Example 1(c)}} = -\frac{\pi}{4}$ ◀

Because definite integrals are defined as limits, they inherit many of the properties of limits. For example, we know that constants can be moved through limit signs and that the limit of a sum or difference is the sum or difference of the limits. Thus, you should not be surprised by the following theorem, which we state without formal proof.

> **5.5.4 THEOREM** *If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f + g$, and $f - g$ are integrable on $[a, b]$ and*
>
> (*a*) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
>
> (*b*) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
>
> (*c*) $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

▶ Example 4

Evaluate

$$\int_0^1 (5 - 3\sqrt{1-x^2})\,dx$$

Solution. From parts (*a*) and (*c*) of Theorem 5.5.4 we can write

$$\int_0^1 (5 - 3\sqrt{1-x^2})\,dx = \int_0^1 5\,dx - \int_0^1 3\sqrt{1-x^2}\,dx = \int_0^1 5\,dx - 3\int_0^1 \sqrt{1-x^2}\,dx$$

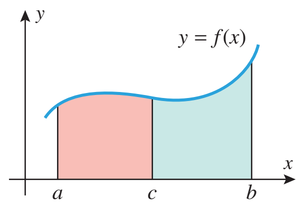
The first integral in this difference can be interpreted as the area of a rectangle of height 5 and base 1, so its value is 5, and from Example 1 the value of the second integral is $\pi/4$. Thus,

$$\int_0^1 (5 - 3\sqrt{1-x^2})\,dx = 5 - 3\left(\frac{\pi}{4}\right) = 5 - \frac{3\pi}{4}$$ ◀

Part (*b*) of Theorem 5.5.4 can be extended to more than two functions. More precisely,

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)]\,dx$$
$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx$$

Some properties of definite integrals can be motivated by interpreting the integral as an area. For example, if f is continuous and nonnegative on the interval $[a, b]$, and if c is a point between a and b, then the area under $y = f(x)$ over the interval $[a, b]$ can be split into two parts and expressed as the area under the graph from a to c plus the area under the graph from c to b (Figure 5.5.7), that is,



▲ **Figure 5.5.7**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This is a special case of the following theorem about definite integrals, which we state without proof.

5.5.5 THEOREM *If f is integrable on a closed interval containing the three points a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.

The following theorem, which we state without formal proof, can also be motivated by interpreting definite integrals as areas.

Part (b) of Theorem 5.5.6 states that the direction (sometimes called the *sense*) of the inequality $f(x) \geq g(x)$ is unchanged if one integrates both sides. Moreover, if $b > a$, then both parts of the theorem remain true if $\geq$ is replaced by $\leq$, $>$, or $<$ throughout.

5.5.6 THEOREM

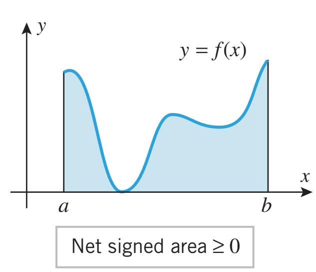
(a) *If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq 0$$

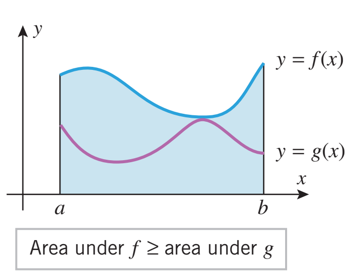
(b) *If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Geometrically, part (a) of this theorem states the obvious fact that if f is nonnegative on $[a, b]$, then the net signed area between the graph of f and the interval $[a, b]$ is also nonnegative (Figure 5.5.8). Part (b) has its simplest interpretation when f and g are nonnegative on $[a, b]$, in which case the theorem states that if the graph of f does not go below the graph of g, then the area under the graph of f is at least as large as the area under the graph of g (Figure 5.5.9).



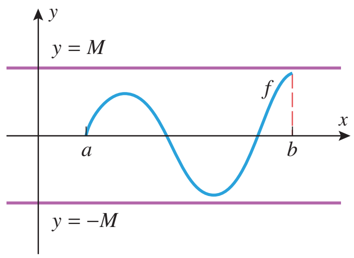
▲ **Figure 5.5.8**



▲ **Figure 5.5.9**

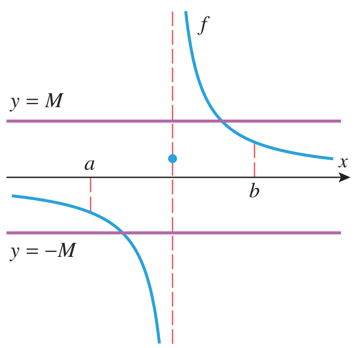
DISCONTINUITIES AND INTEGRABILITY

In the late nineteenth and early twentieth centuries, mathematicians began to investigate conditions under which the limit that defines an integral fails to exist, that is, conditions under which a function fails to be integrable. The matter is quite complex and beyond the scope of this text. However, there are a few basic results about integrability that are important to know; we begin with a definition.



f is bounded on $[a, b]$.

▲ **Figure 5.5.10**



f is not bounded on $[a, b]$.

▲ **Figure 5.5.11**

> **5.5.7 DEFINITION** A function f that is defined on an interval is said to be ***bounded*** on the interval if there is a positive number M such that
>
> $$-M \leq f(x) \leq M$$
>
> for all x in the interval. Geometrically, this means that the graph of f over the interval lies between the lines $y = -M$ and $y = M$.

For example, a continuous function f is bounded on *every* finite closed interval because the Extreme-Value Theorem (4.4.2) implies that f has an absolute maximum and an absolute minimum on the interval; hence, its graph will lie between the lines $y = -M$ and $y = M$, provided we make M large enough (Figure 5.5.10). In contrast, a function that has a vertical asymptote inside of an interval is not bounded on that interval because its graph over the interval cannot be made to lie between the lines $y = -M$ and $y = M$, no matter how large we make the value of M (Figure 5.5.11).

The following theorem, which we state without proof, provides some facts about integrability for functions with discontinuities. In the exercises we have included some problems that are concerned with this theorem (Exercises 42, 43, and 44).

> **5.5.8 THEOREM** *Let f be a function that is defined on the finite closed interval $[a, b]$.*
>
> (*a*) *If f has finitely many discontinuities in $[a, b]$ but is bounded on $[a, b]$, then f is integrable on $[a, b]$.*
>
> (*b*) *If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.*

✔ QUICK CHECK EXERCISES 5.5

(See page 362 for answers.)

1. In each part, use the partition of [2, 7] in the accompanying figure.

▲ **Figure Ex-1**

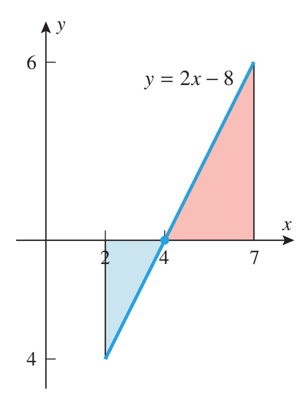
(a) What is n, the number of subintervals in this partition?
(b) $x_0 =$ ________; $x_1 =$ ________; $x_2 =$ ________;
$x_3 =$ ________; $x_4 =$ ________
(c) $\Delta x_1 =$ ________; $\Delta x_2 =$ ________; $\Delta x_3 =$ ________;
$\Delta x_4 =$ ________
(d) The mesh of this partition is ________.

2. Let $f(x) = 2x - 8$. Use the partition of [2, 7] in Quick Check Exercise 1 and the choices $x_1^* = 2$, $x_2^* = 4$, $x_3^* = 5$, and $x_4^* = 7$ to evaluate the Riemann sum

$$\sum_{k=1}^{4} f(x_k^*)\Delta x_k$$

3. Use the accompanying figure to evaluate

$$\int_2^7 (2x - 8)\, dx$$



◀ **Figure Ex-3**

4. Suppose that $g(x)$ is a function for which

$$\int_{-2}^{1} g(x)\, dx = 5 \quad \text{and} \quad \int_{1}^{2} g(x)\, dx = -2$$

Use this information to evaluate the definite integrals.

(a) $\int_1^2 5g(x)\, dx$ (b) $\int_{-2}^{2} g(x)\, dx$

(c) $\int_1^1 [g(x)]^2\, dx$ (d) $\int_2^{-2} 4g(x)\, dx$

EXERCISE SET 5.5

1–4 Find the value of

(a) $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (b) $\max \Delta x_k$.

1. $f(x) = x + 1;\ a = 0, b = 4;\ n = 3;$
$\Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2;$
$x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$

2. $f(x) = \cos x;\ a = 0, b = 2\pi;\ n = 4;$
$\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4, \Delta x_3 = \pi/2, \Delta x_4 = \pi/4;$
$x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2, x_4^* = 7\pi/4$

3. $f(x) = 4 - x^2;\ a = -3, b = 4;\ n = 4;$
$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 1, \Delta x_4 = 3;$
$x_1^* = -\frac{5}{2}, x_2^* = -1, x_3^* = \frac{1}{4}, x_4^* = 3$

4. $f(x) = x^3;\ a = -3, b = 3;\ n = 4;$
$\Delta x_1 = 2, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = 2;$
$x_1^* = -2, x_2^* = 0, x_3^* = 0, x_4^* = 2$

5–8 Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.)

5. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k;\ a = -1, b = 2$

6. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k;\ a = 1, b = 2$

7. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3x_k^*)\Delta x_k;\ a = -3, b = 3$

8. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\sin^2 x_k^*)\Delta x_k;\ a = 0, b = \pi/2$

9–10 Use Definition 5.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)

9. (a) $\int_1^2 2x\,dx$ (b) $\int_0^1 \frac{x}{x+1}\,dx$

10. (a) $\int_1^2 \sqrt{x}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

FOCUS ON CONCEPTS

11. Explain informally why Theorem 5.5.4(a) follows from Definition 5.5.1.

12. Explain informally why Theorem 5.5.6(a) follows from Definition 5.5.1.

13–16 Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.

13. (a) $\int_0^3 x\,dx$ (b) $\int_{-2}^{-1} x\,dx$
(c) $\int_{-1}^4 x\,dx$ (d) $\int_{-5}^5 x\,dx$

14. (a) $\int_0^2 \left(1 - \frac{1}{2}x\right) dx$ (b) $\int_{-1}^1 \left(1 - \frac{1}{2}x\right) dx$
(c) $\int_2^3 \left(1 - \frac{1}{2}x\right) dx$ (d) $\int_0^3 \left(1 - \frac{1}{2}x\right) dx$

15. (a) $\int_0^5 2\,dx$ (b) $\int_0^\pi \cos x\,dx$
(c) $\int_{-1}^2 |2x - 3|\,dx$ (d) $\int_{-1}^1 \sqrt{1 - x^2}\,dx$

16. (a) $\int_{-10}^{-5} 6\,dx$ (b) $\int_{-\pi/3}^{\pi/3} \sin x\,dx$
(c) $\int_0^3 |x - 2|\,dx$ (d) $\int_0^2 \sqrt{4 - x^2}\,dx$

17. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} |x - 2|, & x \geq 0 \\ x + 2, & x < 0 \end{cases}$$

(a) $\int_{-2}^0 f(x)\,dx$ (b) $\int_{-2}^2 f(x)\,dx$
(c) $\int_0^6 f(x)\,dx$ (d) $\int_{-4}^6 f(x)\,dx$

18. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} 2x, & x \leq 1 \\ 2, & x > 1 \end{cases}$$

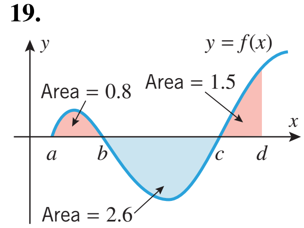
(a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^1 f(x)\,dx$
(c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^5 f(x)\,dx$

FOCUS ON CONCEPTS

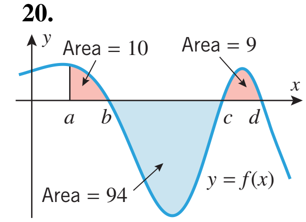
19–20 Use the areas shown in the figure to find

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx$.

19.



20.



21. Find $\int_{-1}^{2}[f(x)+2g(x)]\,dx$ if

$$\int_{-1}^{2} f(x)\,dx = 5 \quad \text{and} \quad \int_{-1}^{2} g(x)\,dx = -3$$

22. Find $\int_{1}^{4}[3f(x)-g(x)]\,dx$ if

$$\int_{1}^{4} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{4} g(x)\,dx = 10$$

23. Find $\int_{1}^{5} f(x)\,dx$ if

$$\int_{0}^{1} f(x)\,dx = -2 \quad \text{and} \quad \int_{0}^{5} f(x)\,dx = 1$$

24. Find $\int_{3}^{-2} f(x)\,dx$ if

$$\int_{-2}^{1} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{3} f(x)\,dx = -6$$

25–28 Use Theorem 5.5.4 and appropriate formulas from geometry to evaluate the integrals. ■

25. $\int_{-1}^{3}(4-5x)\,dx$ **26.** $\int_{-2}^{2}(1-3|x|)\,dx$

27. $\int_{0}^{1}(x+2\sqrt{1-x^2})\,dx$ **28.** $\int_{-3}^{0}(2+\sqrt{9-x^2})\,dx$

29–32 True–False Determine whether the statement is true or false. Explain your answer. ■

29. If $f(x)$ is integrable on $[a,b]$, then $f(x)$ is continuous on $[a,b]$.

30. It is the case that

$$0 < \int_{-1}^{1} \frac{\cos x}{\sqrt{1+x^2}}\,dx$$

31. If the integral of $f(x)$ over the interval $[a,b]$ is negative, then $f(x) \le 0$ for $a \le x \le b$.

32. The function

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2, & x > 0 \end{cases}$$

is integrable over every closed interval $[a,b]$.

33–34 Use Theorem 5.5.6 to determine whether the value of the integral is positive or negative. ■

33. (a) $\int_{2}^{3} \frac{\sqrt{x}}{1-x}\,dx$ (b) $\int_{0}^{4} \frac{x^2}{3-\cos x}\,dx$

34. (a) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3-x}}\,dx$ (b) $\int_{-2}^{2} \frac{x^3-9}{|x|+1}\,dx$

35. Prove that if f is continuous and if $m \le f(x) \le M$ on $[a,b]$, then

$$m(b-a) \le \int_{a}^{b} f(x)\,dx \le M(b-a)$$

36. Find the maximum and minimum values of $\sqrt{x^3+2}$ for $0 \le x \le 3$. Use these values, and the inequalities in Exercise 35, to find bounds on the value of the integral

$$\int_{0}^{3} \sqrt{x^3+2}\,dx$$

37–38 Evaluate the integrals by completing the square and applying appropriate formulas from geometry. ■

37. $\int_{0}^{10} \sqrt{10x-x^2}\,dx$ **38.** $\int_{0}^{3} \sqrt{6x-x^2}\,dx$

39–40 Evaluate the limit by expressing it as a definite integral over the interval $[a,b]$ and applying appropriate formulas from geometry. ■

39. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^*+1)\Delta x_k;\ a=0, b=1$

40. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4-(x_k^*)^2}\,\Delta x_k;\ a=-2, b=2$

FOCUS ON CONCEPTS

41. Let $f(x) = C$ be a constant function.

(a) Use a formula from geometry to show that

$$\int_{a}^{b} f(x)\,dx = C(b-a)$$

(b) Show that any Riemann sum for $f(x)$ over $[a,b]$ evaluates to $C(b-a)$. Use Definition 5.5.1 to show that

$$\int_{a}^{b} f(x)\,dx = C(b-a)$$

42. Define a function f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Use Definition 5.5.1 to show that

$$\int_{0}^{1} f(x)\,dx = 1$$

43. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is integrable on a closed interval $[a,b]$? Explain your reasoning.

44. Define the function f by

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

It follows from Theorem 5.5.8(*b*) that f is not integrable on the interval $[0, 1]$. Prove this to be the case by applying Definition 5.5.1. [*Hint:* Argue that no matter how small the mesh size is for a partition of $[0, 1]$, there will always be a choice of x_1^* that will make the Riemann sum in Definition 5.5.1 as large as we like.]

45. In each part, use Theorems 5.5.2 and 5.5.8 to determine whether the function f is integrable on the interval $[-1, 1]$.

(a) $f(x) = \cos x$

(b) $f(x) = \begin{cases} x/|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(d) $f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

46. **Writing** Write a short paragraph that discusses the similarities and differences between indefinite integrals and definite integrals.

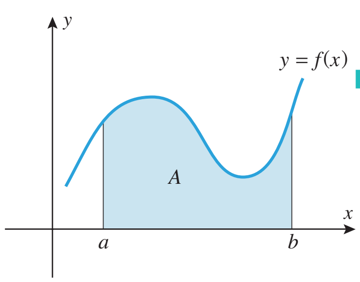
47. **Writing** Write a paragraph that explains informally what it means for a function to be "integrable."

✔QUICK CHECK ANSWERS 5.5

1. (a) $n = 4$ (b) 2, 3, 4.5, 6.5, 7 (c) 1, 1.5, 2, 0.5 (d) 2 **2.** 3 **3.** 5 **4.** (a) -10 (b) 3 (c) 0 (d) -12

5.6 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the "Fundamental Theorem of Calculus." One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.



▲ Figure 5.6.1

THE FUNDAMENTAL THEOREM OF CALCULUS

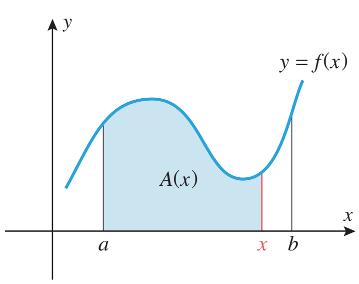
As in earlier sections, let us begin by assuming that f is nonnegative and continuous on an interval $[a, b]$, in which case the area A under the graph of f over the interval $[a, b]$ is represented by the definite integral

$$A = \int_a^b f(x)\,dx \tag{1}$$

(Figure 5.6.1).

Recall that our discussion of the antiderivative method in Section 5.1 suggested that if $A(x)$ is the area under the graph of f from a to x (Figure 5.6.2), then

- $A'(x) = f(x)$
- $A(a) = 0$ — The area under the curve from a to a is the area above the single point a, and hence is zero.
- $A(b) = A$ — The area under the curve from a to b is A.



▲ Figure 5.6.2

The formula $A'(x) = f(x)$ states that $A(x)$ is an antiderivative of $f(x)$, which implies that every other antiderivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let

$$F(x) = A(x) + C$$

be any antiderivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a) = A - 0 = A$$

Hence (1) can be expressed as

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In words, this equation states:

> *The definite integral can be evaluated by finding any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.*

Although our evidence for this result assumed that f is nonnegative on $[a, b]$, this assumption is not essential.

5.6.1 THEOREM (*The Fundamental Theorem of Calculus, Part 1*) *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{2}$$

PROOF Let $x_1, x_2, \ldots, x_{n-1}$ be any points in $[a, b]$ such that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

These values divide $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b] \tag{3}$$

whose lengths, as usual, we denote by

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_n$$

(see Figure 5.6.3). By hypothesis, $F'(x) = f(x)$ for all x in $[a, b]$, so F satisfies the hypotheses of the Mean-Value Theorem (4.8.2) on each subinterval in (3). Hence, we can find points $x_1^*, x_2^*, \ldots, x_n^*$ in the respective subintervals in (3) such that

$$\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1\\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2\\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3\\
&\ \ \vdots\\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n
\end{aligned}$$

Adding the preceding equations yields

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{4}$$

Let us now increase n in such a way that max $\Delta x_k \to 0$. Since f is assumed to be continuous, the right side of (4) approaches $\int_a^b f(x)\,dx$ by Theorem 5.5.2 and Definition 5.5.1. However,

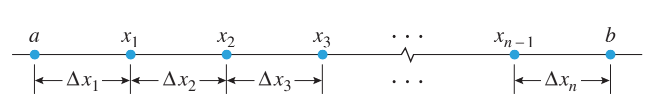


Figure 5.6.3

the left side of (4) is independent of n; that is, the left side of (4) remains constant as n increases. Thus,

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx \quad \blacksquare$$

It is standard to denote the difference $F(b) - F(a)$ as

$$F(x)\Big]_a^b = F(b) - F(a) \quad \text{or} \quad \Big[F(x)\Big]_a^b = F(b) - F(a)$$

For example, using the first of these notations we can express (2) as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \tag{5}$$

We will sometimes write

$$F(x)\Big]_{x=a}^b = F(b) - F(a)$$

when it is important to emphasize that a and b are values for the variable x.

The integral in Example 1 represents the area of a certain trapezoid. Sketch the trapezoid, and find its area using geometry.

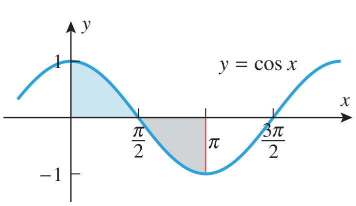
▶ Example 1 Evaluate $\displaystyle\int_1^2 x\,dx$.

Solution. The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus, from (2)

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2} \quad ◀$$

▶ Example 2 In Example 5 of Section 5.4 we used the definition of area to show that the area under the graph of $y = 9 - x^2$ over the interval [0, 3] is 18 (square units). We can now solve that problem much more easily using the Fundamental Theorem of Calculus:

$$A = \int_0^3 (9 - x^2)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3 = \left(27 - \frac{27}{3}\right) - 0 = 18 \quad ◀$$



▲ Figure 5.6.4

▶ Example 3

(a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$ (Figure 5.6.4).

(b) Make a conjecture about the value of the integral

$$\int_0^\pi \cos x\,dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.

Solution (a). Since $\cos x \geq 0$ over the interval $[0, \pi/2]$, the area A under the curve is

$$A = \int_0^{\pi/2} \cos x\,dx = \sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Solution (b). The given integral can be interpreted as the signed area between the graph of $y = \cos x$ and the interval $[0, \pi]$. The graph in Figure 5.6.4 suggests that over the interval $[0, \pi]$ the portion of area above the x-axis is the same as the portion of area below the x-axis, so we conjecture that the signed area is zero; this implies that the value of the integral is zero. This is confirmed by the computations

$$\int_0^{\pi} \cos x\, dx = \sin x\Big]_0^{\pi} = \sin\pi - \sin 0 = 0 \blacktriangleleft$$

THE RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

Observe that in the preceding examples we did not include a constant of integration in the antiderivatives. In general, when applying the Fundamental Theorem of Calculus there is no need to include a constant of integration because it will drop out anyway. To see that this is so, let F be any antiderivative of the integrand on $[a, b]$, and let C be any constant; then

$$\int_a^b f(x)\, dx = \big[F(x) + C\big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Thus, for purposes of evaluating a definite integral we can omit the constant of integration in

$$\int_a^b f(x)\, dx = \big[F(x) + C\big]_a^b$$

and express (5) as

$$\int_a^b f(x)\, dx = \int f(x)\, dx\Big]_a^b \tag{6}$$

which relates the definite and indefinite integrals.

Example 4

$$\int_1^9 \sqrt{x}\, dx = \int x^{1/2}\, dx\Big]_1^9 = \frac{2}{3}x^{3/2}\Big]_1^9 = \frac{2}{3}(27-1) = \frac{52}{3} \blacktriangleleft$$

Example 5 Table 5.2.1 will be helpful for the following computations.

$$\int_4^9 x^2\sqrt{x}\, dx = \int_4^9 x^{5/2}\, dx = \frac{2}{7}x^{7/2}\Big]_4^9 = \frac{2}{7}(2187 - 128) = \frac{4118}{7} = 588\frac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\, dx = -\frac{1}{5}\cos x\Big]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0-1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\, dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$\int_0^{\ln 3} 5e^x\, dx = 5e^x\Big]_0^{\ln 3} = 5[e^{\ln 3} - e^0] = 5[3-1] = 10$$

$$\int_{-e}^{-1} \frac{1}{x}\, dx = \ln|x|\Big]_{-e}^{-1} = \ln|-1| - \ln|-e| = 0 - 1 = -1$$

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x\Big]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} \blacktriangleleft$$

TECHNOLOGY MASTERY

If you have a CAS, read the documentation on evaluating definite integrals and then check the results in Example 5.

WARNING The requirements in the Fundamental Theorem of Calculus that f be continuous on $[a, b]$ and that F be an antiderivative for f over the entire interval $[a, b]$ are important to keep in mind. Disregarding these assumptions will likely lead to incorrect results. For example, the function $f(x) = 1/x^2$ fails on two counts to be continuous at $x = 0$: $f(x)$ is not defined at $x = 0$ and $\lim_{x\to 0} f(x)$ does not exist. Thus, the Fundamental Theorem of Calculus should not be used to integrate f on any interval that contains $x = 0$. However, if we ignore this and mistakenly apply Formula (2) over the interval $[-1, 1]$, we might *incorrectly* compute $\int_{-1}^{1}(1/x^2)\,dx$ by evaluating an antiderivative, $-1/x$, at the endpoints, arriving at the answer

$$-\frac{1}{x}\bigg]_{-1}^{1} = -[1-(-1)] = -2$$

But $f(x) = 1/x^2$ is a nonnegative function, so clearly a negative value for the definite integral is impossible.

The Fundamental Theorem of Calculus can be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

Example 6

$$\int_1^1 x^2\,dx = \frac{x^3}{3}\bigg]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_4^0 x\,dx = \frac{x^2}{2}\bigg]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$$

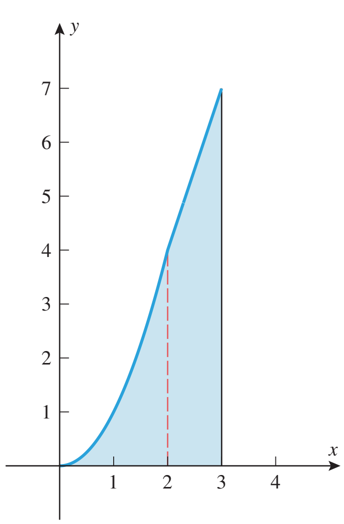
The latter result is consistent with the result that would be obtained by first reversing the limits of integration in accordance with Definition 5.5.3(b):

$$\int_4^0 x\,dx = -\int_0^4 x\,dx = -\frac{x^2}{2}\bigg]_0^4 = -\left[\frac{16}{2} - \frac{0}{2}\right] = -8 \blacktriangleleft$$

To integrate a continuous function that is defined piecewise on an interval $[a, b]$, split this interval into subintervals at the breakpoints of the function, and integrate separately over each subinterval in accordance with Theorem 5.5.5.

Example 7 Evaluate $\displaystyle\int_0^3 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \ge 2 \end{cases}$$



▲ Figure 5.6.5

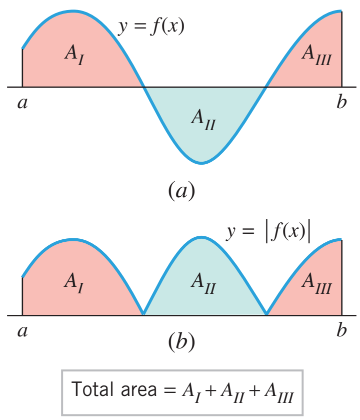
Solution. See Figure 5.6.5. From Theorem 5.5.5 we can integrate from 0 to 2 and from 2 to 3 separately and add the results. This yields

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$

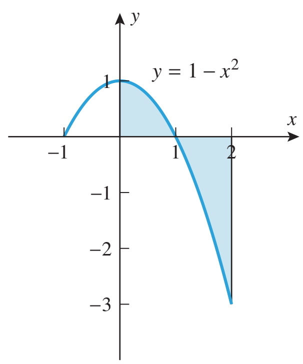
$$= \frac{x^3}{3}\bigg]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6} \blacktriangleleft$$

If f is a continuous function on the interval $[a, b]$, then we define the ***total area*** between the curve $y = f(x)$ and the interval $[a, b]$ to be

$$\text{total area} = \int_a^b |f(x)|\,dx \tag{7}$$



▲ **Figure 5.6.6**



▲ **Figure 5.6.7**

(Figure 5.6.6). To compute total area using Formula (7), begin by dividing the interval of integration into subintervals on which $f(x)$ does not change sign. On the subintervals for which $0 \le f(x)$ replace $|f(x)|$ by $f(x)$, and on the subintervals for which $f(x) \le 0$ replace $|f(x)|$ by $-f(x)$. Adding the resulting integrals then yields the total area.

▶ **Example 8** Find the total area between the curve $y = 1 - x^2$ and the x-axis over the interval $[0, 2]$ (Figure 5.6.7).

Solution. The area A is given by

$$A = \int_0^2 |1 - x^2|\, dx = \int_0^1 (1 - x^2)\, dx + \int_1^2 -(1 - x^2)\, dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2 \blacktriangleleft$$

DUMMY VARIABLES

To evaluate a definite integral using the Fundamental Theorem of Calculus, one needs to be able to find an antiderivative of the integrand; thus, it is important to know what kinds of functions have antiderivatives. It is our next objective to show that all continuous functions have antiderivatives, but to do this we will need some preliminary results.

Formula (6) shows that there is a close relationship between the integrals

$$\int_a^b f(x)\, dx \quad \text{and} \quad \int f(x)\, dx$$

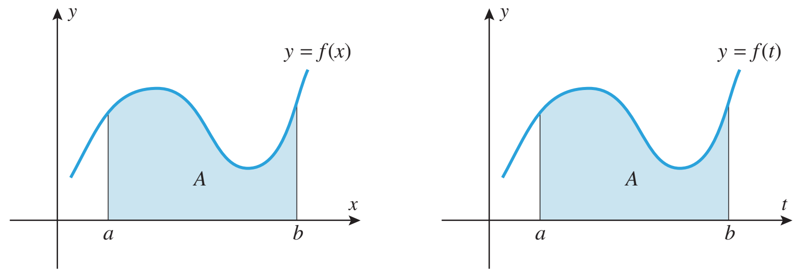
However, the definite and indefinite integrals differ in some important ways. For one thing, the two integrals are different kinds of objects—the definite integral is a *number* (the net signed area between the graph of $y = f(x)$ and the interval $[a, b]$), whereas the indefinite integral is a *function*, or more accurately a family of functions [the antiderivatives of $f(x)$]. However, the two types of integrals also differ in the role played by the variable of integration. In an indefinite integral, the variable of integration is "passed through" to the antiderivative in the sense that integrating a function of x produces a function of x, integrating a function of t produces a function of t, and so forth. For example,

$$\int x^2\, dx = \frac{x^3}{3} + C \quad \text{and} \quad \int t^2\, dt = \frac{t^3}{3} + C$$

In contrast, the variable of integration in a definite integral is not passed through to the end result, since the end result is a number. Thus, integrating a function of x over an interval and integrating the same function of t over the same interval of integration produce the same value for the integral. For example,

$$\int_1^3 x^2\, dx = \frac{x^3}{3}\bigg]_{x=1}^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} \quad \text{and} \quad \int_1^3 t^2\, dt = \frac{t^3}{3}\bigg]_{t=1}^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

However, this latter result should not be surprising, since the area under the graph of the curve $y = f(x)$ over an interval $[a, b]$ on the x-axis is the same as the area under the graph of the curve $y = f(t)$ over the interval $[a, b]$ on the t-axis (Figure 5.6.8).



$$A = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

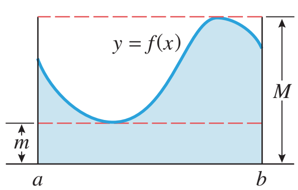
▶ **Figure 5.6.8**

Because the variable of integration in a definite integral plays no role in the end result, it is often referred to as a ***dummy variable***. In summary:

Whenever you find it convenient to change the letter used for the variable of integration in a definite integral, you can do so without changing the value of the integral.

THE MEAN-VALUE THEOREM FOR INTEGRALS

To reach our goal of showing that continuous functions have antiderivatives, we will need to develop a basic property of definite integrals, known as the *Mean-Value Theorem for Integrals*. In Section 5.8 we will interpret this theorem using the concept of the "average value" of a continuous function over an interval. Here we will need it as a tool for developing other results.



▲ **Figure 5.6.9**

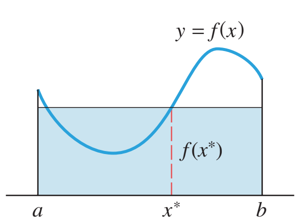
Let f be a continuous nonnegative function on $[a, b]$, and let m and M be the minimum and maximum values of $f(x)$ on this interval. Consider the rectangles of heights m and M over the interval $[a, b]$ (Figure 5.6.9). It is clear geometrically from this figure that the area

$$A = \int_a^b f(x)\,dx$$

under $y = f(x)$ is at least as large as the area of the rectangle of height m and no larger than the area of the rectangle of height M. It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height $f(x^*)$ between m and M whose area is precisely A; that is,

$$\int_a^b f(x)\,dx = f(x^*)(b - a)$$

(Figure 5.6.10). This is a special case of the following result.



The area of the shaded rectangle is equal to the area of the shaded region in Figure 5.6.9.

▲ **Figure 5.6.10**

5.6.2 THEOREM (*The Mean-Value Theorem for Integrals*) *If f is continuous on a closed interval $[a, b]$, then there is at least one point x^* in $[a, b]$ such that*

$$\int_a^b f(x)\,dx = f(x^*)(b - a) \tag{8}$$

PROOF By the Extreme-Value Theorem (4.4.2), f assumes a maximum value M and a minimum value m on $[a, b]$. Thus, for all x in $[a, b]$,

$$m \le f(x) \le M$$

and from Theorem 5.5.6(b)

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

or

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \tag{9}$$

or

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

This implies that

$$\frac{1}{b-a}\int_a^b f(x)\,dx \tag{10}$$

is a number between m and M, and since $f(x)$ assumes the values m and M on $[a, b]$, it follows from the Intermediate-Value Theorem (1.5.7) that $f(x)$ must assume the value (10) at some x^* in $[a, b]$; that is,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(x^*) \quad \text{or} \quad \int_a^b f(x)\,dx = f(x^*)(b-a) \quad \blacksquare$$

▶ **Example 9** Since $f(x) = x^2$ is continuous on the interval $[1, 4]$, the Mean-Value Theorem for Integrals guarantees that there is a point x^* in $[1, 4]$ such that

$$\int_1^4 x^2\,dx = f(x^*)(4-1) = (x^*)^2(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\bigg]_1^4 = 21$$

so that

$$3(x^*)^2 = 21 \quad \text{or} \quad (x^*)^2 = 7 \quad \text{or} \quad x^* = \pm\sqrt{7}$$

Thus, $x^* = \sqrt{7} \approx 2.65$ is the point in the interval $[1, 4]$ whose existence is guaranteed by the Mean-Value Theorem for Integrals. ◀

PART 2 OF THE FUNDAMENTAL THEOREM OF CALCULUS

In Section 5.1 we suggested that if f is continuous and nonnegative on $[a, b]$, and if $A(x)$ is the area under the graph of $y = f(x)$ over the interval $[a, x]$ (Figure 5.6.2), then $A'(x) = f(x)$. But $A(x)$ can be expressed as the definite integral

$$A(x) = \int_a^x f(t)\,dt$$

(where we have used t rather than x as the variable of integration to avoid confusion with the x that appears as the upper limit of integration). Thus, the relationship $A'(x) = f(x)$ can be expressed as

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

This is a special case of the following more general result, which applies even if f has negative values.

5.6.3 THEOREM (*The Fundamental Theorem of Calculus, Part 2*) *If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \tag{11}$$

PROOF We will show first that $F(x)$ is defined at each x in the interval. If $x > a$ and x is in the interval, then Theorem 5.5.2 applied to the interval $[a, x]$ and the continuity of f ensure that $F(x)$ is defined; and if x is in the interval and $x \le a$, then Definition 5.5.3 combined with Theorem 5.5.2 ensures that $F(x)$ is defined. Thus, $F(x)$ is defined for all x in the interval.

Next we will show that $F'(x) = f(x)$ for each x in the interval. If x is not an endpoint, then it follows from the definition of a derivative that

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \qquad \boxed{\text{Theorem 5.5.5}} \end{aligned} \tag{12}$$

Applying the Mean-Value Theorem for Integrals (5.6.2) to the integral in (12) we obtain

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{h}[f(t^*)\cdot h] = f(t^*) \tag{13}$$

where t^* is some number between x and $x+h$. Because t^* is trapped between x and $x+h$, it follows that $t^* \to x$ as $h \to 0$. Thus, the continuity of f at x implies that $f(t^*) \to f(x)$ as $h \to 0$. Therefore, it follows from (12) and (13) that

$$F'(x) = \lim_{h\to 0}\left(\frac{1}{h}\int_x^{x+h} f(t)\,dt\right) = \lim_{h\to 0} f(t^*) = f(x)$$

If x is an endpoint of the interval, then the two-sided limits in the proof must be replaced by the appropriate one-sided limits, but otherwise the arguments are identical. ■

In words, Formula (11) states:

If a definite integral has a variable upper limit of integration, a constant lower limit of integration, and a continuous integrand, then the derivative of the integral with respect to its upper limit is equal to the integrand evaluated at the upper limit.

▶ **Example 10** Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

by applying Part 2 of the Fundamental Theorem of Calculus, and then confirm the result by performing the integration and then differentiating.

Solution. The integrand is a continuous function, so from (11)

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$$

Alternatively, evaluating the integral and then differentiating yields

$$\int_1^x t^3\,dt = \frac{t^4}{4}\bigg]_{t=1}^{x} = \frac{x^4}{4} - \frac{1}{4}, \quad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

so the two methods for differentiating the integral agree. ◀

▶ Example 11 Since

$$f(x) = \frac{\sin x}{x}$$

is continuous on any interval that does not contain the origin, it follows from (11) that on the interval $(0, +\infty)$ we have

$$\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right] = \frac{\sin x}{x}$$

Unlike the preceding example, there is no way to evaluate the integral in terms of familiar functions, so Formula (11) provides the only simple method for finding the derivative. ◀

DIFFERENTIATION AND INTEGRATION ARE INVERSE PROCESSES

The two parts of the Fundamental Theorem of Calculus, when taken together, tell us that differentiation and integration are inverse processes in the sense that each undoes the effect of the other. To see why this is so, note that Part 1 of the Fundamental Theorem of Calculus (5.6.1) implies that

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

which tells us that if the value of $f(a)$ is known, then the function f can be recovered from its derivative f' by integrating. Conversely, Part 2 of the Fundamental Theorem of Calculus (5.6.3) states that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

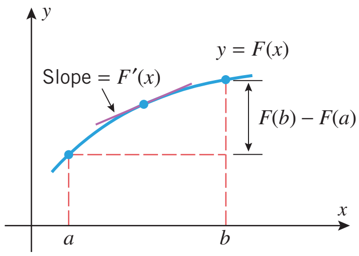
which tells us that the function f can be recovered from its integral by differentiating. Thus, differentiation and integration can be viewed as inverse processes.

It is common to treat parts 1 and 2 of the Fundamental Theorem of Calculus as a single theorem and refer to it simply as the *Fundamental Theorem of Calculus*. This theorem ranks as one of the greatest discoveries in the history of science, and its formulation by Newton and Leibniz is generally regarded to be the "discovery of calculus."

INTEGRATING RATES OF CHANGE

The Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{14}$$



Integrating the slope of $y = F(x)$ over the interval $[a, b]$ produces the change $F(b) - F(a)$ in the value of $F(x)$.

▲ **Figure 5.6.11**

has a useful interpretation that can be seen by rewriting it in a slightly different form. Since F is an antiderivative of f on the interval $[a, b]$, we can use the relationship $F'(x) = f(x)$ to rewrite (14) as

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{15}$$

In this formula we can view $F'(x)$ as the rate of change of $F(x)$ with respect to x, and we can view $F(b) - F(a)$ as the *change* in the value of $F(x)$ as x increases from a to b (Figure 5.6.11). Thus, we have the following useful principle.

5.6.4 INTEGRATING A RATE OF CHANGE Integrating the rate of change of $F(x)$ with respect to x over an interval $[a, b]$ produces the change in the value of $F(x)$ that occurs as x increases from a to b.

Here are some examples of this idea:

- If $s(t)$ is the position of a particle in rectilinear motion, then $s'(t)$ is the instantaneous velocity of the particle at time t, and
$$\int_{t_1}^{t_2} s'(t)\,dt = s(t_2) - s(t_1)$$
is the displacement (or the change in the position) of the particle between the times t_1 and t_2.
- If $P(t)$ is a population (e.g., plants, animals, or people) at time t, then $P'(t)$ is the rate at which the population is changing at time t, and
$$\int_{t_1}^{t_2} P'(t)\,dt = P(t_2) - P(t_1)$$
is the change in the population between times t_1 and t_2.
- If $A(t)$ is the area of an oil spill at time t, then $A'(t)$ is the rate at which the area of the spill is changing at time t, and
$$\int_{t_1}^{t_2} A'(t)\,dt = A(t_2) - A(t_1)$$
is the change in the area of the spill between times t_1 and t_2.
- If $P'(x)$ is the marginal profit that results from producing and selling x units of a product (see Section 4.5), then
$$\int_{x_1}^{x_2} P'(x)\,dx = P(x_2) - P(x_1)$$
is the change in the profit that results when the production level increases from x_1 units to x_2 units.

Mitchell Funk/Getty Images

Mathematical analysis plays an important role in understanding human population growth.

✔ QUICK CHECK EXERCISES 5.6 *(See page 376 for answers.)*

1. (a) If $F(x)$ is an antiderivative for $f(x)$, then
$$\int_a^b f(x)\,dx = \underline{\qquad}$$
(b) $\displaystyle\int_a^b F'(x)\,dx = \underline{\qquad}$

(c) $\displaystyle\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = \underline{\qquad}$

2. (a) $\int_0^2 (3x^2 - 2x)\,dx =$ ______

(b) $\int_{-\pi}^{\pi} \cos x\,dx =$ ______

(c) $\int_0^{\frac{1}{2}\ln 5} e^x\,dx =$ ______

(d) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx =$ ______

3. For the function $f(x) = 3x^2 - 2x$ and an interval $[a, b]$, the point x^* guaranteed by the Mean-Value Theorem for Integrals is $x^* = \frac{2}{3}$. It follows that

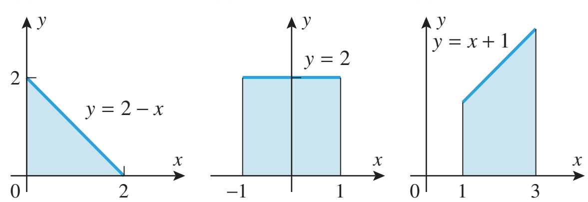
$$\int_a^b (3x^2 - 2x)\,dx = \underline{\qquad\qquad}$$

4. The area of an oil spill is increasing at a rate of $25t$ ft^2/s t seconds after the spill. Between times $t = 2$ and $t = 4$ the area of the spill increases by ______ ft^2.

EXERCISE SET 5.6 Graphing Utility C CAS

1. In each part, use a definite integral to find the area of the region, and check your answer using an appropriate formula from geometry.

(a) (b) (c)



2. In each part, use a definite integral to find the area under the curve $y = f(x)$ over the stated interval, and check your answer using an appropriate formula from geometry.
(a) $f(x) = x$; $[0, 5]$
(b) $f(x) = 5$; $[3, 9]$
(c) $f(x) = x + 3$; $[-1, 2]$

3. In each part, sketch the analogue of Figure 5.6.10 for the specified region. [Let $y = f(x)$ denote the upper boundary of the region. If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
(a) The region in part (a) of Exercise 1.
(b) The region in part (b) of Exercise 1.
(c) The region in part (c) of Exercise 1.

4. In each part, sketch the analogue of Figure 5.6.10 for the function and interval specified. [If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
(a) The function and interval in part (a) of Exercise 2.
(b) The function and interval in part (b) of Exercise 2.
(c) The function and interval in part (c) of Exercise 2.

5–10 Find the area under the curve $y = f(x)$ over the stated interval.

5. $f(x) = x^3$; $[2, 3]$ **6.** $f(x) = x^4$; $[-1, 1]$

7. $f(x) = 3\sqrt{x}$; $[1, 4]$ **8.** $f(x) = x^{-2/3}$; $[1, 27]$

9. $f(x) = e^{2x}$; $[0, \ln 2]$ **10.** $f(x) = \dfrac{1}{x}$; $[1, 5]$

11–12 Find all values of x^* in the stated interval that satisfy Equation (8) in the Mean-Value Theorem for Integrals (5.6.2), and explain what these numbers represent.

11. (a) $f(x) = \sqrt{x}$; $[0, 3]$
(b) $f(x) = x^2 + x$; $[-12, 0]$

12. (a) $f(x) = \sin x$; $[-\pi, \pi]$ (b) $f(x) = 1/x^2$; $[1, 3]$

13–30 Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.

13. $\int_{-2}^{1} (x^2 - 6x + 12)\,dx$ **14.** $\int_{-1}^{2} 4x(1 - x^2)\,dx$

15. $\int_1^4 \frac{4}{x^2}\,dx$ **16.** $\int_1^2 \frac{1}{x^6}\,dx$

17. $\int_4^9 2x\sqrt{x}\,dx$ **18.** $\int_1^4 \frac{1}{x\sqrt{x}}\,dx$

19. $\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta$ **20.** $\int_0^{\pi/4} \sec^2\theta\,d\theta$

21. $\int_{-\pi/4}^{\pi/4} \cos x\,dx$ **22.** $\int_0^{\pi/3} (2x - \sec x \tan x)\,dx$

23. $\int_{\ln 2}^{3} 5e^x\,dx$ **24.** $\int_{1/2}^{1} \frac{1}{2x}\,dx$

25. $\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$ **26.** $\int_{-1}^{1} \frac{dx}{1+x^2}$

27. $\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$ **28.** $\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$

29. $\int_1^4 \left(\frac{1}{\sqrt{t}} - 3\sqrt{t}\right)dt$ **30.** $\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right)dx$

31–34 Use Theorem 5.5.5 to evaluate the given integrals.

31. (a) $\int_{-1}^{1} |2x - 1|\,dx$ (b) $\int_0^{3\pi/4} |\cos x|\,dx$

32. (a) $\int_{-1}^{2} \sqrt{2 + |x|}\,dx$ (b) $\int_0^{\pi/2} \left|\frac{1}{2} - \cos x\right|\,dx$

33. (a) $\displaystyle\int_{-1}^{1} |e^x - 1|\, dx$ (b) $\displaystyle\int_{1}^{4} \frac{|2-x|}{x}\, dx$

34. (a) $\displaystyle\int_{-3}^{3} \left| x^2 - 1 - \frac{15}{x^2+1} \right| dx$

(b) $\displaystyle\int_{0}^{\sqrt{3}/2} \left| \frac{1}{\sqrt{1-x^2}} - \sqrt{2} \right| dx$

35–36 A function $f(x)$ is defined piecewise on an interval. In these exercises: (a) Use Theorem 5.5.5 to find the integral of $f(x)$ over the interval. (b) Find an antiderivative of $f(x)$ on the interval. (c) Use parts (a) and (b) to verify Part 1 of the Fundamental Theorem of Calculus.

35. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x^2, & 1 < x \le 2 \end{cases}$

36. $f(x) = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1/x^2, & 1 \le x \le 4 \end{cases}$

37–40 True–False Determine whether the statement is true or false. Explain your answer.

37. There does not exist a differentiable function $F(x)$ such that $F'(x) = |x|$.

38. If $f(x)$ is continuous on the interval $[a, b]$, and if the definite integral of $f(x)$ over this interval has value 0, then the equation $f(x) = 0$ has at least one solution in the interval $[a, b]$.

39. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$$

if and only if

$$G(a) + F(b) = F(a) + G(b)$$

40. If $f(x)$ is continuous everywhere and

$$F(x) = \int_0^x f(t)\, dt$$

then the equation $F(x) = 0$ has at least one solution.

41–44 Use a calculating utility to find the midpoint approximation of the integral using $n = 20$ subintervals, and then find the exact value of the integral using Part 1 of the Fundamental Theorem of Calculus.

41. $\displaystyle\int_1^3 \frac{1}{x^2}\, dx$ **42.** $\displaystyle\int_0^{\pi/2} \sin x\, dx$

43. $\displaystyle\int_{-1}^{1} \sec^2 x\, dx$ **44.** $\displaystyle\int_1^3 \frac{1}{x}\, dx$

45–48 Sketch the region described and find its area.

45. The region under the curve $y = x^2 + 1$ and over the interval $[0, 3]$.

46. The region below the curve $y = x - x^2$ and above the x-axis.

47. The region under the curve $y = 3 \sin x$ and over the interval $[0, 2\pi/3]$.

48. The region below the interval $[-2, -1]$ and above the curve $y = x^3$.

49–52 Sketch the curve and find the total area between the curve and the given interval on the x-axis.

49. $y = x^2 - x$; $[0, 2]$ **50.** $y = \sin x$; $[0, 3\pi/2]$

51. $y = e^x - 1$; $[-1, 1]$ **52.** $y = \dfrac{x^2-1}{x^2}$; $\left[\frac{1}{2}, 2\right]$

53. A student wants to find the area enclosed by the graphs of $y = 1/\sqrt{1-x^2}$, $y = 0$, $x = 0$, and $x = 0.8$.
(a) Show that the exact area is $\sin^{-1} 0.8$.
(b) The student uses a calculator to approximate the result in part (a) to two decimal places and obtains an incorrect answer of 53.13. What was the student's error? Find the correct approximation.

FOCUS ON CONCEPTS

54. Explain why the Fundamental Theorem of Calculus may be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

55. (a) If $h'(t)$ is the rate of change of a child's height measured in inches per year, what does the integral $\int_0^{10} h'(t)\, dt$ represent, and what are its units?
(b) If $r'(t)$ is the rate of change of the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_1^2 r'(t)\, dt$ represent, and what are its units?
(c) If $H(t)$ is the rate of change of the speed of sound with respect to temperature measured in ft/s per °F, what does the integral $\int_{32}^{100} H(t)\, dt$ represent, and what are its units?
(d) If $v(t)$ is the velocity of a particle in rectilinear motion, measured in cm/h, what does the integral $\int_{t_1}^{t_2} v(t)\, dt$ represent, and what are its units?

56. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{1}{100}(x+2)(x+1)(x-3)(x-5)$$

and use the graph to make a conjecture about the sign of the integral

$$\int_{-2}^{5} f(x)\, dx$$

(b) Check your conjecture by evaluating the integral.

57. Define $F(x)$ by

$$F(x) = \int_1^x (3t^2 - 3)\, dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.

58. Define $F(x)$ by

$$F(x) = \int_{\pi/4}^{x} \cos 2t \, dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.

59–62 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives. ■

59. (a) $\dfrac{d}{dx}\displaystyle\int_1^x \sin(t^2)\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_0^x e^{\sqrt{t}}\,dt$

60. (a) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{dt}{1+\sqrt{t}}$ (b) $\dfrac{d}{dx}\displaystyle\int_1^x \ln t\,dt$

61. $\dfrac{d}{dx}\displaystyle\int_x^0 t\sec t\,dt$ [*Hint:* Use Definition 5.5.3(b).]

62. $\dfrac{d}{du}\displaystyle\int_0^u |x|\,dx$

63. Let $F(x) = \displaystyle\int_4^x \sqrt{t^2+9}\,dt$. Find
(a) $F(4)$ (b) $F'(4)$ (c) $F''(4)$.

64. Let $F(x) = \displaystyle\int_{\sqrt{3}}^x \tan^{-1} t\,dt$. Find
(a) $F(\sqrt{3})$ (b) $F'(\sqrt{3})$ (c) $F''(\sqrt{3})$.

65. Let $F(x) = \displaystyle\int_0^x \frac{t-3}{t^2+7}\,dt$ for $-\infty < x < +\infty$.
(a) Find the value of x where F attains its minimum value.
(b) Find intervals over which F is only increasing or only decreasing.
(c) Find open intervals over which F is only concave up or only concave down.

C **66.** Use the plotting and numerical integration commands of a CAS to generate the graph of the function F in Exercise 65 over the interval $-20 \le x \le 20$, and confirm that the graph is consistent with the results obtained in that exercise.

67. (a) Over what open interval does the formula

$$F(x) = \int_1^x \frac{dt}{t}$$

represent an antiderivative of $f(x) = 1/x$?
(b) Find a point where the graph of F crosses the x-axis.

68. (a) Over what open interval does the formula

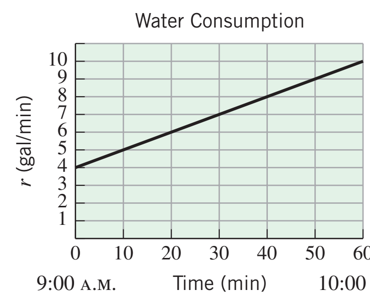
$$F(x) = \int_1^x \frac{1}{t^2-9}\,dt$$

represent an antiderivative of

$$f(x) = \frac{1}{x^2-9}?$$

(b) Find a point where the graph of F crosses the x-axis.

69. (a) Suppose that a reservoir supplies water to an industrial park at a constant rate of $r = 4$ gallons per minute (gal/min) between 8:30 A.M. and 9:00 A.M. How much water does the reservoir supply during that time period?
(b) Suppose that one of the industrial plants increases its water consumption between 9:00 A.M. and 10:00 A.M. and that the rate at which the reservoir supplies water increases linearly, as shown in the accompanying figure. How much water does the reservoir supply during that 1-hour time period?
(c) Suppose that from 10:00 A.M. to 12 noon the rate at which the reservoir supplies water is given by the formula $r(t) = 10 + \sqrt{t}$ gal/min, where t is the time (in minutes) since 10:00 A.M. How much water does the reservoir supply during that 2-hour time period?



◀ **Figure Ex-69**

70. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M.
(a) When does the peak traffic flow into the highway occur?
(b) Estimate the number of cars that enter the highway during the rush hour.

71–72 Evaluate each limit by interpreting it as a Riemann sum in which the given interval is divided into n subintervals of equal width. ■

71. $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n} \frac{\pi}{4n}\sec^2\left(\frac{\pi k}{4n}\right)$; $\left[0, \dfrac{\pi}{4}\right]$

72. $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n} \frac{n}{n^2+k^2}$; $[0, 1]$

73. Prove the Mean-Value Theorem for Integrals (Theorem 5.6.2) by applying the Mean-Value Theorem (4.8.2) to an antiderivative F for f.

74. **Writing** Write a short paragraph that describes the various ways in which integration and differentiation may be viewed as inverse processes. (Be sure to discuss both definite and indefinite integrals.)

75. **Writing** Let f denote a function that is continuous on an interval $[a, b]$, and let x^* denote the point guaranteed by the Mean-Value Theorem for Integrals. Explain geometrically why $f(x^*)$ may be interpreted as a "mean" or average value of $f(x)$ over $[a, b]$. (In Section 5.8 we will discuss the concept of "average value" in more detail.)

✔ QUICK CHECK ANSWERS 5.6

1. (a) $F(b) - F(a)$ (b) $F(b) - F(a)$ (c) $f(x)$ **2.** (a) 4 (b) 0 (c) $\sqrt{5} - 1$ (d) $\pi/3$ **3.** 0 **4.** 150 ft^2

5.7 RECTILINEAR MOTION REVISITED USING INTEGRATION

In Section 4.6 we used the derivative to define the notions of instantaneous velocity and acceleration for a particle in rectilinear motion. In this section we will resume the study of such motion using the tools of integration.

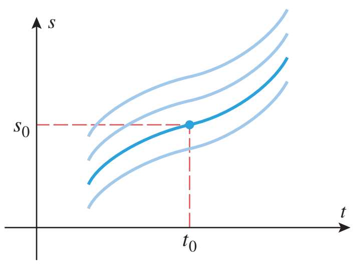
FINDING POSITION AND VELOCITY BY INTEGRATION

Recall from Formulas (1) and (3) of Section 4.6 that if a particle in rectilinear motion has position function $s(t)$, then its instantaneous velocity and acceleration are given by the formulas

$$v(t) = s'(t) \quad \text{and} \quad a(t) = v'(t)$$

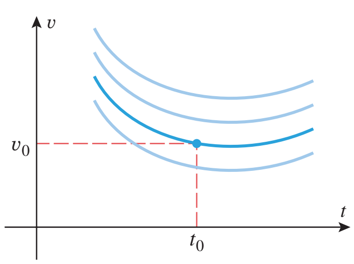
It follows from these formulas that $s(t)$ is an antiderivative of $v(t)$ and $v(t)$ is an antiderivative of $a(t)$; that is,

$$s(t) = \int v(t)\,dt \quad \text{and} \quad v(t) = \int a(t)\,dt \tag{1–2}$$



There is a unique position function such that $s(t_0) = s_0$.

Figure 5.7.1



There is a unique velocity function such that $v(t_0) = v_0$.

Figure 5.7.2

By Formula (1), if we know the velocity function $v(t)$ of a particle in rectilinear motion, then by integrating $v(t)$ we can produce a family of position functions with that velocity function. If, in addition, we know the position s_0 of the particle at any time t_0, then we have sufficient information to find the constant of integration and determine a unique position function (Figure 5.7.1). Similarly, if we know the acceleration function $a(t)$ of the particle, then by integrating $a(t)$ we can produce a family of velocity functions with that acceleration function. If, in addition, we know the velocity v_0 of the particle at any time t_0, then we have sufficient information to find the constant of integration and determine a unique velocity function (Figure 5.7.2).

▶ **Example 1** Suppose that a particle moves with velocity $v(t) = \cos \pi t$ along a coordinate line. Assuming that the particle has coordinate $s = 4$ at time $t = 0$, find its position function.

Solution. The position function is

$$s(t) = \int v(t)\,dt = \int \cos \pi t\,dt = \frac{1}{\pi}\sin \pi t + C$$

Since $s = 4$ when $t = 0$, it follows that

$$4 = s(0) = \frac{1}{\pi}\sin 0 + C = C$$

Thus,

$$s(t) = \frac{1}{\pi}\sin \pi t + 4 \blacktriangleleft$$

COMPUTING DISPLACEMENT AND DISTANCE TRAVELED BY INTEGRATION

Recall that the displacement over a time interval of a particle in rectilinear motion is its final coordinate minus its initial coordinate. Thus, if the position function of the particle is $s(t)$, then its displacement (or change in position) over the time interval $[t_0, t_1]$ is $s(t_1) - s(t_0)$. This can be written in integral form as

$$\begin{bmatrix} \text{displacement} \\ \text{over the time} \\ \text{interval } [t_0, t_1] \end{bmatrix} = \int_{t_0}^{t_1} v(t)\, dt = \int_{t_0}^{t_1} s'(t)\, dt = s(t_1) - s(t_0) \tag{3}$$

Recall that Formula (3) is a special case of the formula

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

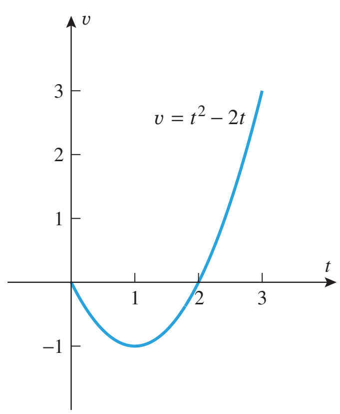
for integrating a rate of change.

In contrast, to find the distance traveled by the particle over the time interval $[t_0, t_1]$ (distance traveled in the positive direction plus the distance traveled in the negative direction), we must integrate the absolute value of the velocity function; that is,

$$\begin{bmatrix} \text{distance traveled} \\ \text{during time} \\ \text{interval } [t_0, t_1] \end{bmatrix} = \int_{t_0}^{t_1} |v(t)|\, dt \tag{4}$$

Since the absolute value of velocity is speed, Formulas (3) and (4) can be summarized informally as follows:

Integrating velocity over a time interval produces displacement, and integrating speed over a time interval produces distance traveled.



▲ **Figure 5.7.3**

▶ **Example 2** Suppose that a particle moves on a coordinate line so that its velocity at time t is $v(t) = t^2 - 2t$ m/s (Figure 5.7.3).

(a) Find the displacement of the particle during the time interval $0 \le t \le 3$.

(b) Find the distance traveled by the particle during the time interval $0 \le t \le 3$.

Solution (a). From (3) the displacement is

$$\int_0^3 v(t)\, dt = \int_0^3 (t^2 - 2t)\, dt = \left[\frac{t^3}{3} - t^2\right]_0^3 = 0$$

Thus, the particle is at the same position at time $t = 3$ as at $t = 0$.

Solution (b). The velocity can be written as $v(t) = t^2 - 2t = t(t-2)$, from which we see that $v(t) \le 0$ for $0 \le t \le 2$ and $v(t) \ge 0$ for $2 \le t \le 3$. Thus, it follows from (4) that the distance traveled is

$$\begin{aligned} \int_0^3 |v(t)|\, dt &= \int_0^2 -v(t)\, dt + \int_2^3 v(t)\, dt \\ &= \int_0^2 -(t^2 - 2t)\, dt + \int_2^3 (t^2 - 2t)\, dt \\ &= -\left[\frac{t^3}{3} - t^2\right]_0^2 + \left[\frac{t^3}{3} - t^2\right]_2^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \text{ m} \end{aligned}$$ ◀

In physical problems it is important to associate correct units with definite integrals. In general, the units for

$$\int_a^b f(x)\, dx$$

are units of $f(x)$ times units of x, since the integral is the limit of Riemann sums, each of whose terms has these units. For example, if $v(t)$ is in meters per second (m/s) and t is in seconds (s), then

$$\int_a^b v(t)\, dt$$

is in meters since

$$(\text{m/s}) \times \text{s} = \text{m}$$

ANALYZING THE VELOCITY VERSUS TIME CURVE

In Section 4.6 we showed how to use the position versus time curve to obtain information about the behavior of a particle in rectilinear motion (Table 4.6.1). Similarly, there is valuable information that can be obtained from the velocity versus time curve. For example, the integral in (3) can be interpreted geometrically as the *net signed area* between the graph of $v(t)$ and the interval $[t_0, t_1]$, and the integral in (4) can be interpreted as the *total area* between the graph of $v(t)$ and the interval $[t_0, t_1]$. Thus we have the following result.

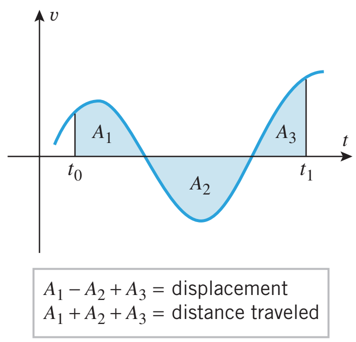


Figure 5.7.4

5.7.1 FINDING DISPLACEMENT AND DISTANCE TRAVELED FROM THE VELOCITY VERSUS TIME CURVE For a particle in rectilinear motion, the net signed area between the velocity versus time curve and the interval $[t_0, t_1]$ on the t-axis represents the displacement of the particle over that time interval, and the total area between the velocity versus time curve and the interval $[t_0, t_1]$ on the t-axis represents the distance traveled by the particle over that time interval (Figure 5.7.4).

Example 3 Figure 5.7.5 shows three velocity versus time curves for a particle in rectilinear motion along a horizontal line with the positive direction to the right. In each case find the displacement and the distance traveled over the time interval $0 \le t \le 4$, and explain what that information tells you about the motion of the particle.

Solution (a). In part (a) of the figure the area and the net signed area over the interval are both 2. Thus, at the end of the time period the particle is 2 units to the right of its starting point and has traveled a distance of 2 units.

Solution (b). In part (b) of the figure the net signed area is -2, and the total area is 2. Thus, at the end of the time period the particle is 2 units to the left of its starting point and has traveled a distance of 2 units.

Solution (c). In part (c) of the figure the net signed area is 0, and the total area is 2. Thus, at the end of the time period the particle is back at its starting point and has traveled a distance of 2 units. More specifically, it traveled 1 unit to the right over the time interval $0 \le t \le 1$ and then 1 unit to the left over the time interval $1 \le t \le 2$ (why?). ◀

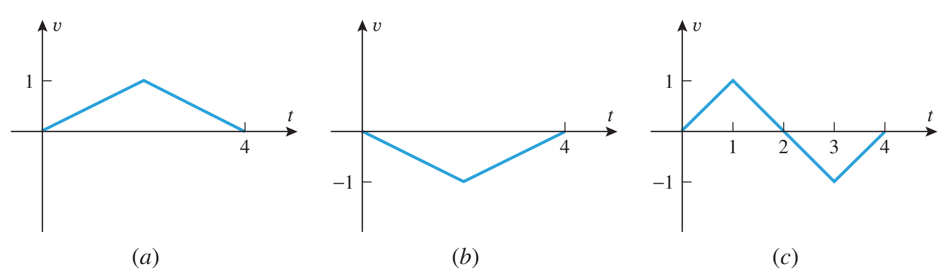


Figure 5.7.5

CONSTANT ACCELERATION

One of the most important cases of rectilinear motion occurs when a particle has ***constant acceleration***. We will show that if a particle moves with constant acceleration along an s-axis, and if the position and velocity of the particle are known at some point in time, say when $t = 0$, then it is possible to derive formulas for the position $s(t)$ and the velocity $v(t)$ at any time t. To see how this can be done, suppose that the particle has constant acceleration

$$a(t) = a \tag{5}$$

and

$$s = s_0 \quad \text{when} \quad t = 0 \tag{6}$$

$$v = v_0 \quad \text{when} \quad t = 0 \tag{7}$$

where s_0 and v_0 are known. We call (6) and (7) the ***initial conditions***.

With (5) as a starting point, we can integrate $a(t)$ to obtain $v(t)$, and we can integrate $v(t)$ to obtain $s(t)$, using an initial condition in each case to determine the constant of integration. The computations are as follows:

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1 \tag{8}$$

To determine the constant of integration C_1 we apply initial condition (7) to this equation to obtain

$$v_0 = v(0) = a \cdot 0 + C_1 = C_1$$

Substituting this in (8) and putting the constant term first yields

$$v(t) = v_0 + at$$

Since v_0 is constant, it follows that

$$s(t) = \int v(t)\,dt = \int (v_0 + at)\,dt = v_0 t + \tfrac{1}{2}at^2 + C_2 \tag{9}$$

To determine the constant C_2 we apply initial condition (6) to this equation to obtain

$$s_0 = s(0) = v_0 \cdot 0 + \tfrac{1}{2}a \cdot 0 + C_2 = C_2$$

Substituting this in (9) and putting the constant term first yields

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2$$

In summary, we have the following result.

> **5.7.2 CONSTANT ACCELERATION** If a particle moves with constant acceleration a along an s-axis, and if the position and velocity at time $t = 0$ are s_0 and v_0, respectively, then the position and velocity functions of the particle are
>
> $$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2 \tag{10}$$
>
> $$v(t) = v_0 + at \tag{11}$$

How can you tell from the graph of the velocity versus time curve whether a particle moving along a line has constant acceleration?

Example 4 Suppose that an intergalactic spacecraft uses a sail and the "solar wind" to produce a constant acceleration of 0.032 m/s^2. Assuming that the spacecraft has a velocity of 10,000 m/s when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of this hour?

Solution. In this problem the choice of a coordinate axis is at our discretion, so we will choose it to make the computations as simple as possible. Accordingly, let us introduce an s-axis whose positive direction is in the direction of motion, and let us take the origin to coincide with the position of the spacecraft at the time $t = 0$ when the sail is raised. Thus, Formulas (10) and (11) apply with

$$s_0 = s(0) = 0, \quad v_0 = v(0) = 10{,}000, \quad \text{and} \quad a = 0.032$$

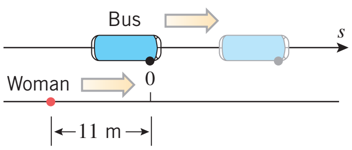
Since 1 hour corresponds to $t = 3600$ s, it follows from (10) that in 1 hour the spacecraft travels a distance of

$$s(3600) = 10{,}000(3600) + \tfrac{1}{2}(0.032)(3600)^2 \approx 36{,}200{,}000 \text{ m}$$

and it follows from (11) that after 1 hour its velocity is

$$v(3600) = 10{,}000 + (0.032)(3600) \approx 10{,}100 \text{ m/s} \ ◀$$

▶ **Example 5** A bus has stopped to pick up riders, and a woman is running at a constant velocity of 5 m/s to catch it. When she is 11 m behind the front door the bus pulls away with a constant acceleration of 1 m/s^2. From that point in time, how long will it take for the woman to reach the front door of the bus if she keeps running with a velocity of 5 m/s?



▲ **Figure 5.7.6**

Solution. As shown in Figure 5.7.6, choose the s-axis so that the bus and the woman are moving in the positive direction, and the front door of the bus is at the origin at the time $t = 0$ when the bus begins to pull away. To catch the bus at some later time t, the woman will have to cover a distance $s_w(t)$ that is equal to 11 m plus the distance $s_b(t)$ traveled by the bus; that is, the woman will catch the bus when

$$s_w(t) = s_b(t) + 11 \tag{12}$$

Since the woman has a constant velocity of 5 m/s, the distance she travels in t seconds is $s_w(t) = 5t$. Thus, (12) can be written as

$$s_b(t) = 5t - 11 \tag{13}$$

Since the bus has a constant acceleration of $a = 1\ \text{m/s}^2$, and since $s_0 = v_0 = 0$ at time $t = 0$ (why?), it follows from (10) that

$$s_b(t) = \tfrac{1}{2}t^2$$

Substituting this equation into (13) and reorganizing the terms yields the quadratic equation

$$\tfrac{1}{2}t^2 - 5t + 11 = 0 \quad \text{or} \quad t^2 - 10t + 22 = 0$$

Solving this equation for t using the quadratic formula yields two solutions:

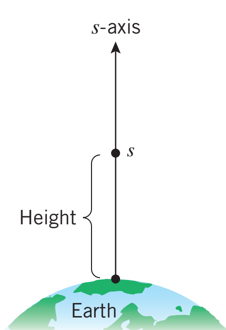
$$t = 5 - \sqrt{3} \approx 3.3 \quad \text{and} \quad t = 5 + \sqrt{3} \approx 6.7$$

(verify). Thus, the woman can reach the door at two different times, $t = 3.3$ s and $t = 6.7$ s. The reason that there are two solutions can be explained as follows: When the woman first reaches the door, she is running faster than the bus and can run past it if the driver does not see her. However, as the bus speeds up, it eventually catches up to her, and she has another chance to flag it down. ◀

FREE-FALL MODEL

Motion that occurs when an object near the Earth is imparted some initial velocity (up or down) and thereafter moves along a vertical line is called ***free-fall motion***. In modeling free-fall motion we assume that the only force acting on the object is the Earth's gravity and that the object stays sufficiently close to the Earth that the gravitational force is constant. In particular, air resistance and the gravitational pull of other celestial bodies are neglected.

In our model we will ignore the physical size of the object by treating it as a particle, and we will assume that it moves along an s-axis whose origin is at the surface of the Earth and whose positive direction is up. With this convention, the s-coordinate of the particle is the height of the particle above the surface of the Earth (Figure 5.7.7).



▲ **Figure 5.7.7**

It is a fact of physics that a particle with free-fall motion has constant acceleration. The magnitude of this constant, denoted by the letter g, is called the ***acceleration due to gravity*** and is approximately 9.8 m/s^2 or 32 ft/s^2, depending on whether distance is measured in meters or feet.*

Recall that a particle is speeding up when its velocity and acceleration have the same sign and is slowing down when they have opposite signs. Thus, because we have chosen

*Strictly speaking, the constant g varies with the latitude and the distance from the Earth's center. However, for motion at a fixed latitude and near the surface of the Earth, the assumption of a constant g is satisfactory for many applications.

the positive direction to be up, it follows that the acceleration $a(t)$ of a particle in free fall is negative for all values of t. To see that this is so, observe that an upward-moving particle (positive velocity) is slowing down, so its acceleration must be negative; and a downward-moving particle (negative velocity) is speeding up, so its acceleration must also be negative. Thus, we conclude that

$$a(t) = -g \tag{14}$$

It now follows from this and Formulas (10) and (11) that the position and velocity functions for a particle in free-fall motion are

How would Formulas (14), (15), and (16) change if we choose the direction of the positive s-axis to be down?

$$s(t) = s_0 + v_0 t - \tfrac{1}{2}gt^2 \tag{15}$$

$$v(t) = v_0 - gt \tag{16}$$

Corbis.Bettmann

Nolan Ryan's rookie baseball card.

Example 6 Nolan Ryan, a member of the Baseball Hall of Fame and one of the fastest baseball pitchers of all time, was able to throw a baseball 150 ft/s (over 102 mi/h). During his career, he had the opportunity to pitch in the Houston Astrodome, home to the Houston Astros Baseball Team from 1965 to 1999. The Astrodome was an indoor stadium with a ceiling 208 ft high. Could Nolan Ryan have hit the ceiling of the Astrodome if he were capable of giving a baseball an upward velocity of 100 ft/s from a height of 7 ft?

Solution. Since distance is in feet, we take $g = 32$ ft/s^2. Initially, we have $s_0 = 7$ ft and $v_0 = 100$ ft/s, so from (15) and (16) we have

$$s(t) = 7 + 100t - 16t^2$$
$$v(t) = 100 - 32t$$

The ball will rise until $v(t) = 0$, that is, until $100 - 32t = 0$. Solving this equation we see that the ball is at its maximum height at time $t = \frac{25}{8}$. To find the height of the ball at this instant we substitute this value of t into the position function to obtain

$$s\left(\tfrac{25}{8}\right) = 7 + 100\left(\tfrac{25}{8}\right) - 16\left(\tfrac{25}{8}\right)^2 = 163.25 \text{ ft}$$

which is roughly 45 ft short of hitting the ceiling. ◀

In Example 6 the ball is moving up when the velocity is positive and is moving down when the velocity is negative, so it makes sense physically that the velocity is zero when the ball reaches its peak.

Example 7 A penny is released from rest near the top of the Empire State Building at a point that is 1250 ft above the ground (Figure 5.7.8). Assuming that the free-fall model applies, how long does it take for the penny to hit the ground, and what is its speed at the time of impact?

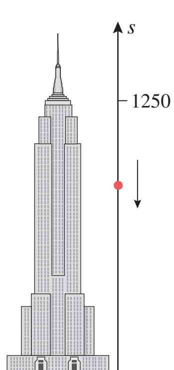


Figure 5.7.8

Solution. Since distance is in feet, we take $g = 32$ ft/s^2. Initially, we have $s_0 = 1250$ and $v_0 = 0$, so from (15)

$$s(t) = 1250 - 16t^2 \tag{17}$$

Impact occurs when $s(t) = 0$. Solving this equation for t, we obtain

$$1250 - 16t^2 = 0$$
$$t^2 = \frac{1250}{16} = \frac{625}{8}$$
$$t = \pm\frac{25}{\sqrt{8}} \approx \pm 8.8 \text{ s}$$

Since $t \geq 0$, we can discard the negative solution and conclude that it takes $25/\sqrt{8} \approx 8.8$ s for the penny to hit the ground. To obtain the velocity at the time of impact, we substitute $t = 25/\sqrt{8}$, $v_0 = 0$, and $g = 32$ in (16) to obtain

$$v\left(\frac{25}{\sqrt{8}}\right) = 0 - 32\left(\frac{25}{\sqrt{8}}\right) = -200\sqrt{2} \approx -282.8 \text{ ft/s}$$

Thus, the speed at the time of impact is

$$\left|v\left(\frac{25}{\sqrt{8}}\right)\right| = 200\sqrt{2} \approx 282.8 \text{ ft/s}$$

which is more than 192 mi/h. ◄

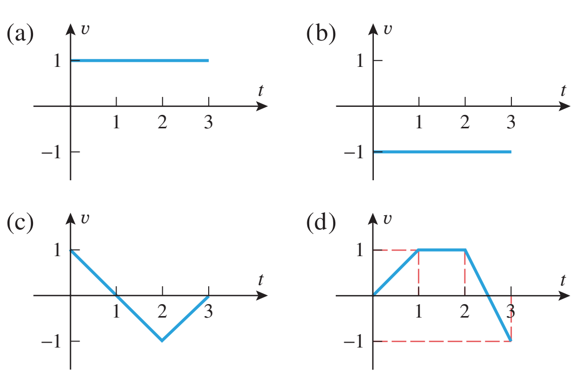
✔ QUICK CHECK EXERCISES 5.7 (See page 385 for answers.)

1. Suppose that a particle is moving along an s-axis with velocity $v(t) = 2t + 1$. If at time $t = 0$ the particle is at position $s = 2$, the position function of the particle is $s(t) =$ ______.
2. Let $v(t)$ denote the velocity function of a particle that is moving along an s-axis with constant acceleration $a = -2$. If $v(1) = 4$, then $v(t) =$ ______.
3. Let $v(t)$ denote the velocity function of a particle in rectilinear motion. Suppose that $v(0) = -1$, $v(3) = 2$, and the velocity versus time curve is a straight line. The displacement of the particle between times $t = 0$ and $t = 3$ is ______, and the distance traveled by the particle over this period of time is ______.
4. Based on the free-fall model, from what height must a coin be dropped so that it strikes the ground with speed 48 ft/s?

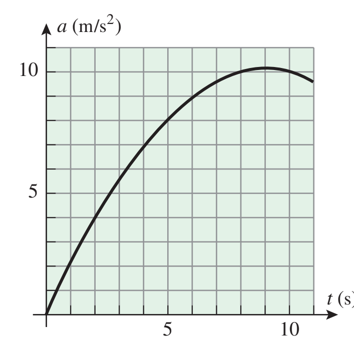
EXERCISE SET 5.7 Graphing Utility C CAS

FOCUS ON CONCEPTS

1. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the displacement and the distance traveled by the particle over the time interval $0 \le t \le 3$.

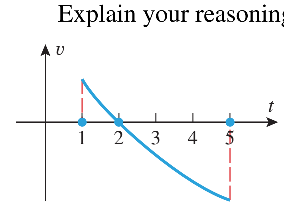


2. Sketch a velocity versus time curve for a particle that travels a distance of 5 units along a coordinate line during the time interval $0 \le t \le 10$ and has a displacement of 0 units.
3. The accompanying figure shows the acceleration versus time curve for a particle moving along a coordinate line. If the initial velocity of the particle is 20 m/s, estimate
   (a) the velocity at time $t = 4$ s
   (b) the velocity at time $t = 6$ s.



◄ **Figure Ex-3**

4. The accompanying figure shows the velocity versus time curve over the time interval $1 \le t \le 5$ for a particle moving along a horizontal coordinate line.
   (a) What can you say about the sign of the acceleration over the time interval?
   (b) When is the particle speeding up? Slowing down?
   (c) What can you say about the location of the particle at time $t = 5$ relative to its location at time $t = 1$? Explain your reasoning.



◄ **Figure Ex-4**

5–8 A particle moves along an s-axis. Use the given information to find the position function of the particle.

5. (a) $v(t) = 3t^2 - 2t;\ s(0) = 1$
(b) $a(t) = 3\sin 3t;\ v(0) = 3;\ s(0) = 3$

6. (a) $v(t) = 1 + \sin t;\ s(0) = -3$
(b) $a(t) = t^2 - 3t + 1;\ v(0) = 0;\ s(0) = 0$

7. (a) $v(t) = 3t + 1;\ s(2) = 4$
(b) $a(t) = t^{-2};\ v(1) = 0;\ s(1) = 2$

8. (a) $v(t) = t^{2/3};\ s(8) = 0$
(b) $a(t) = \sqrt{t};\ v(4) = 1;\ s(4) = -5$

9–12 A particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

9. (a) $v(t) = \sin t;\ 0 \le t \le \pi/2$
(b) $v(t) = \cos t;\ \pi/2 \le t \le 2\pi$

10. (a) $v(t) = 3t - 2;\ 0 \le t \le 2$
(b) $v(t) = |1 - 2t|;\ 0 \le t \le 2$

11. (a) $v(t) = t^3 - 3t^2 + 2t;\ 0 \le t \le 3$
(b) $v(t) = \sqrt{t} - 2;\ 0 \le t \le 3$

12. (a) $v(t) = t - \sqrt{t};\ 0 \le t \le 4$
(b) $v(t) = \dfrac{1}{\sqrt{t+1}};\ 0 \le t \le 3$

13–16 A particle moves with acceleration $a(t)$ m/s^2 along an s-axis and has velocity v_0 m/s at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

13. $a(t) = 3;\ v_0 = -1;\ 0 \le t \le 2$

14. $a(t) = t - 2;\ v_0 = 0;\ 1 \le t \le 5$

15. $a(t) = 1/\sqrt{3t+1};\ v_0 = \frac{4}{3};\ 1 \le t \le 5$

16. $a(t) = \sin t;\ v_0 = 1;\ \pi/4 \le t \le \pi/2$

17. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t = 1$.
(a) $v = \sin \frac{1}{2}\pi t;\ s = 0$ when $t = 0$
(b) $a = -3t;\ s = 1$ and $v = 0$ when $t = 0$

18. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t = 1$.
(a) $v = \cos \frac{1}{3}\pi t;\ s = 0$ when $t = \frac{3}{2}$
(b) $a = 4e^{2t-2};\ s = 1/e^2$ and $v = (2/e^2) - 3$ when $t = 0$

19. Suppose that a particle moves along a line so that its velocity v at time t is given by

$$v(t) = \begin{cases} 5t, & 0 \le t < 1 \\ 6\sqrt{t} - \dfrac{1}{t}, & 1 \le t \end{cases}$$

where t is in seconds and v is in centimeters per second (cm/s). Estimate the time(s) at which the particle is 4 cm from its starting position.

20. Suppose that a particle moves along a line so that its velocity v at time t is given by

$$v(t) = \frac{3}{t^2+1} - 0.5t, \quad t \ge 0$$

where t is in seconds and v is in centimeters per second (cm/s). Estimate the time(s) at which the particle is 2 cm from its starting position.

21. Suppose that the velocity function of a particle moving along an s-axis is $v(t) = 20t^2 - 110t + 120$ ft/s and that the particle is at the origin at time $t = 0$. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 6 s of motion.

22. Suppose that the acceleration function of a particle moving along an s-axis is $a(t) = 4t - 30$ m/s^2 and that the position and velocity at time $t = 0$ are $s_0 = -5$ m and $v_0 = 3$ m/s. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 25 s of motion.

23–26 True–False Determine whether the statement is true or false. Explain your answer. Each question refers to a particle in rectilinear motion.

23. If the particle has constant acceleration, the velocity versus time graph will be a straight line.

24. If the particle has constant nonzero acceleration, its position versus time curve will be a parabola.

25. If the total area between the velocity versus time curve and a time interval $[a, b]$ is positive, then the displacement of the particle over this time interval will be nonzero.

26. If $D(t)$ denotes the distance traveled by the particle over the time interval $[0, t]$, then $D(t)$ is an antiderivative for the speed of the particle.

C **27–30** For the given velocity function $v(t)$:
(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the given time interval.
(b) Use a CAS to find the displacement.

27. $v(t) = 0.5 - t\sin t;\ 0 \le t \le 5$

28. $v(t) = 0.5 - t\cos \pi t;\ 0 \le t \le 1$

29. $v(t) = 0.5 - te^{-t};\ 0 \le t \le 5$

30. $v(t) = t\ln(t + 0.1);\ 0 \le t \le 1$

31. Suppose that at time $t = 0$ a particle is at the origin of an x-axis and has a velocity of $v_0 = 25$ cm/s. For the first 4 s thereafter it has no acceleration, and then it is acted on by a retarding force that produces a constant negative acceleration of $a = -10$ cm/s^2.
(a) Sketch the acceleration versus time curve over the interval $0 \le t \le 12$.
(b) Sketch the velocity versus time curve over the time interval $0 \le t \le 12$.
(c) Find the x-coordinate of the particle at times $t = 8$ s and $t = 12$ s.

(cont.)

(d) What is the maximum x-coordinate of the particle over the time interval $0 \leq t \leq 12$?

32–36 In these exercises assume that the object is moving with constant acceleration in the positive direction of a coordinate line, and apply Formulas (10) and (11) as appropriate. In some of these problems you will need the fact that 88 ft/s = 60 mi/h.

32. A car traveling 60 mi/h along a straight road decelerates at a constant rate of 11 ft/s^2.
(a) How long will it take until the speed is 45 mi/h?
(b) How far will the car travel before coming to a stop?

33. Spotting a police car, you hit the brakes on your new Porsche to reduce your speed from 90 mi/h to 60 mi/h at a constant rate over a distance of 200 ft.
(a) Find the acceleration in ft/s^2.
(b) How long does it take for you to reduce your speed to 55 mi/h?
(c) At the acceleration obtained in part (a), how long would it take for you to bring your Porsche to a complete stop from 90 mi/h?

34. A particle moving along a straight line is accelerating at a constant rate of 5 m/s^2. Find the initial velocity if the particle moves 60 m in the first 4 s.

35. A car that has stopped at a toll booth leaves the booth with a constant acceleration of 4 ft/s^2. At the time the car leaves the booth it is 2500 ft behind a truck traveling with a constant velocity of 50 ft/s. How long will it take for the car to catch the truck, and how far will the car be from the toll booth at that time?

36. In the final sprint of a rowing race the challenger is rowing at a constant speed of 12 m/s. At the point where the leader is 100 m from the finish line and the challenger is 15 m behind, the leader is rowing at 8 m/s but starts accelerating at a constant 0.5 m/s^2. Who wins?

37–46 Assume that a free-fall model applies. Solve these exercises by applying Formulas (15) and (16). In these exercises take $g = 32$ ft/s^2 or $g = 9.8$ m/s^2, depending on the units.

37. A projectile is launched vertically upward from ground level with an initial velocity of 112 ft/s.
(a) Find the velocity at $t = 3$ s and $t = 5$ s.
(b) How high will the projectile rise?
(c) Find the speed of the projectile when it hits the ground.

38. A projectile fired downward from a height of 112 ft reaches the ground in 2 s. What is its initial velocity?

39. A projectile is fired vertically upward from ground level with an initial velocity of 16 ft/s.
(a) How long will it take for the projectile to hit the ground?
(b) How long will the projectile be moving upward?

40. In 1939, Joe Sprinz of the San Francisco Seals Baseball Club attempted to catch a ball dropped from a blimp at a height of 800 ft (for the purpose of breaking the record for catching a ball dropped from the greatest height set the preceding year by members of the Cleveland Indians).
(a) How long does it take for a ball to drop 800 ft?
(b) What is the velocity of a ball in miles per hour after an 800 ft drop (88 ft/s = 60 mi/h)?
[*Note:* As a practical matter, it is unrealistic to ignore wind resistance in this problem; however, even with the slowing effect of wind resistance, the impact of the ball slammed Sprinz's glove hand into his face, fractured his upper jaw in 12 places, broke five teeth, and knocked him unconscious. He dropped the ball!]

41. A projectile is launched upward from ground level with an initial speed of 60 m/s.
(a) How long does it take for the projectile to reach its highest point?
(b) How high does the projectile go?
(c) How long does it take for the projectile to drop back to the ground from its highest point?
(d) What is the speed of the projectile when it hits the ground?

42. (a) Use the results in Exercise 41 to make a conjecture about the relationship between the initial and final speeds of a projectile that is launched upward from ground level and returns to ground level.
(b) Prove your conjecture.

43. A projectile is fired vertically upward with an initial velocity of 49 m/s from a tower 150 m high.
(a) How long will it take for the projectile to reach its maximum height?
(b) What is the maximum height?
(c) How long will it take for the projectile to pass its starting point on the way down?
(d) What is the velocity when it passes the starting point on the way down?
(e) How long will it take for the projectile to hit the ground?
(f) What will be its speed at impact?

44. A man drops a stone from a bridge. What is the height of the bridge if
(a) the stone hits the water 4 s later
(b) the sound of the splash reaches the man 4 s later? [Take 1080 ft/s as the speed of sound.]

45. In Example 6, how fast would Nolan Ryan have to throw a ball upward from a height of 7 ft in order to hit the ceiling of the Astrodome?

46. A rock thrown downward with an unknown initial velocity from a height of 1000 ft reaches the ground in 5 s. Find the velocity of the rock when it hits the ground.

47. **Writing** Make a list of important features of a velocity versus time curve, and interpret each feature in terms of the motion.

48. **Writing** Use Riemann sums to argue informally that integrating speed over a time interval produces the distance traveled.

✔ QUICK CHECK ANSWERS 5.7

1. $t^2 + t + 2$ **2.** $6 - 2t$ **3.** $\frac{3}{2}; \frac{5}{2}$ **4.** 36 ft

5.8 AVERAGE VALUE OF A FUNCTION AND ITS APPLICATIONS

In this section we will define the notion of the "average value" of a function, and we will give various applications of this idea.

AVERAGE VELOCITY REVISITED

Let $s = s(t)$ denote the position function of a particle in rectilinear motion. In Section 2.1 we defined the average velocity v_{ave} of the particle over the time interval $[t_0, t_1]$ to be

$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Let $v(t) = s'(t)$ denote the velocity function of the particle. We saw in Section 5.7 that integrating $s'(t)$ over a time interval gives the displacement of the particle over that interval. Thus,

$$\int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0)$$

It follows that

$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} v(t)\,dt \qquad (1)$$

▶ **Example 1** Suppose that a particle moves along a coordinate line so that its velocity at time t is $v(t) = 2 + \cos t$. Find the average velocity of the particle during the time interval $0 \le t \le \pi$.

Solution. From (1) the average velocity is

$$\frac{1}{\pi - 0}\int_0^{\pi} (2 + \cos t)\,dt = \frac{1}{\pi}\Big[2t + \sin t\Big]_0^{\pi} = \frac{1}{\pi}(2\pi) = 2 \blacktriangleleft$$

We will see that Formula (1) is a special case of a formula for what we will call the *average value* of a continuous function over a given interval.

AVERAGE VALUE OF A CONTINUOUS FUNCTION

In scientific work, numerical information is often summarized by an *average value* or *mean value* of the observed data. There are various kinds of averages, but the most common is the ***arithmetic mean*** or ***arithmetic average***, which is formed by adding the data and dividing by the number of data points. Thus, the arithmetic average $\overline{a}$ of n numbers $a_1, a_2, \ldots, a_n$ is

$$\overline{a} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \frac{1}{n}\sum_{k=1}^{n} a_k$$

In the case where the a_k's are values of a function f, say,

$$a_1 = f(x_1),\ a_2 = f(x_2), \ldots, a_n = f(x_n)$$

then the arithmetic average $\overline{a}$ of these function values is

$$\overline{a} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

We will now show how to extend this concept so that we can compute not only the arithmetic average of finitely many function values but an average of *all* values of $f(x)$ as x varies over a closed interval $[a, b]$. For this purpose recall the Mean-Value Theorem for Integrals (5.6.2), which states that if f is continuous on the interval $[a, b]$, then there is at least one point x^* in this interval such that

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

The quantity

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

will be our candidate for the average value of f over the interval $[a, b]$. To explain what motivates this, divide the interval $[a, b]$ into n subintervals of equal length

$$\Delta x = \frac{b-a}{n} \tag{2}$$

and choose arbitrary points $x_1^*, x_2^*, \ldots, x_n^*$ in successive subintervals. Then the arithmetic average of the values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\text{ave} = \frac{1}{n}[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

or from (2)

$$\text{ave} = \frac{1}{b-a}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Taking the limit as $n \to +\infty$ yields

$$\lim_{n\to+\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since this equation describes what happens when we compute the average of "more and more" values of $f(x)$, we are led to the following definition.

Note that the Mean-Value Theorem for Integrals, when expressed in form (3), ensures that there is always at least one point x^* in $[a, b]$ at which the value of f is equal to the average value of f over the interval.

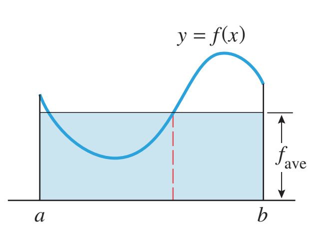
5.8.1 DEFINITION If f is continuous on $[a, b]$, then the ***average value*** (or ***mean value***) of f on $[a, b]$ is defined to be

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{3}$$

REMARK When f is nonnegative on $[a, b]$, the quantity f_{ave} has a simple geometric interpretation, which can be seen by writing (3) as

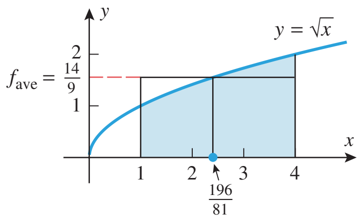
$$f_{\text{ave}} \cdot (b-a) = \int_a^b f(x)\,dx$$

The left side of this equation is the area of a rectangle with a height of f_{ave} and base of length $b-a$, and the right side is the area under $y = f(x)$ over $[a, b]$. Thus, f_{ave} is the height of a rectangle constructed over the interval $[a, b]$, whose area is the same as the area under the graph of f over that interval (Figure 5.8.1).



▲ Figure 5.8.1

▶ **Example 2** Find the average value of the function $f(x) = \sqrt{x}$ over the interval $[1, 4]$, and find all points in the interval at which the value of f is the same as the average.



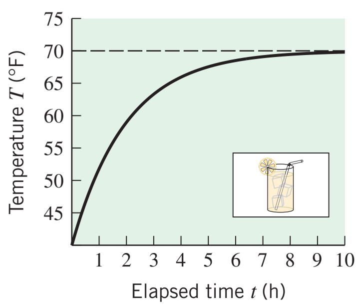
▲ **Figure 5.8.2**

Solution.

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2x^{3/2}}{3}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{14}{9} \approx 1.6$$

The x-values at which $f(x) = \sqrt{x}$ is the same as this average satisfy $\sqrt{x} = 14/9$, from which we obtain $x = 196/81 \approx 2.4$ (Figure 5.8.2). ◀



▲ **Figure 5.8.3**

In Example 3, the temperature T of the lemonade rises from an initial temperature of 40°F toward the room temperature of 70°F. Explain why the formula

$$T = 70 - 30e^{-0.5t}$$

is a good model for this situation.

▶ **Example 3** A glass of lemonade with a temperature of 40°F is left to sit in a room whose temperature is a constant 70°F. Using a principle of physics called ***Newton's Law of Cooling***, one can show that if the temperature of the lemonade reaches 52°F in 1 hour, then the temperature T of the lemonade as a function of the elapsed time t is modeled by the equation

$$T = 70 - 30e^{-0.5t}$$

where T is in degrees Fahrenheit and t is in hours. The graph of this equation, shown in Figure 5.8.3, conforms to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room. Find the average temperature T_{ave} of the lemonade over the first 5 hours.

Solution. From Definition 5.8.1 the average value of T over the time interval [0, 5] is

$$T_{\text{ave}} = \frac{1}{5}\int_0^5 (70 - 30e^{-0.5t})\,dt \tag{4}$$

To evaluate the definite integral, we first find the indefinite integral

$$\int (70 - 30e^{-0.5t})\,dt$$

by making the substitution

$$u = -0.5t \quad \text{so that} \quad du = -0.5\,dt \quad (\text{or } dt = -2\,du)$$

Thus,

$$\int (70 - 30e^{-0.5t})\,dt = \int (70 - 30e^{u})(-2)\,du = -2(70u - 30e^{u}) + C$$

$$= -2[70(-0.5t) - 30e^{-0.5t}] + C = 70t + 60e^{-0.5t} + C$$

and (4) can be expressed as

$$T_{\text{ave}} = \frac{1}{5}\left[70t + 60e^{-0.5t}\right]_0^5 = \frac{1}{5}\left[(350 + 60e^{-2.5}) - 60\right]$$

$$= 58 + 12e^{-2.5} \approx 59°\text{F}$$ ◀

AVERAGE VALUE AND AVERAGE VELOCITY

We now have two ways to calculate the average velocity of a particle in rectilinear motion, since

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} v(t)\,dt \tag{5}$$

and both of these expressions are equal to the average velocity. The left side of (5) gives the average rate of change of s over $[t_0, t_1]$, while the right side gives the average value of

$v = s'$ over the interval $[t_0, t_1]$. That is, *the average velocity of the particle over the time interval* $[t_0, t_1]$ *is the same as the average value of the velocity function over that interval.*

Since velocity functions are generally continuous, it follows from the marginal note associated with Definition 5.8.1 that a particle's average velocity over a time interval matches the particle's velocity at some time in the interval.

Example 4 Show that if a body released from rest (initial velocity zero) is in free fall, then its average velocity over a time interval $[0, T]$ during its fall is its velocity at time $t = T/2$.

Solution. It follows from Formula (16) of Section 5.7 with $v_0 = 0$ that the velocity function of the body is $v(t) = -gt$. Thus, its average velocity over a time interval $[0, T]$ is

$$\begin{aligned} v_{\text{ave}} &= \frac{1}{T-0}\int_0^T v(t)\,dt \\ &= \frac{1}{T}\int_0^T -gt\,dt \\ &= -\frac{g}{T}\left[\frac{1}{2}t^2\right]_0^T = -g\cdot\frac{T}{2} = v\left(\frac{T}{2}\right) \end{aligned}$$ ◄

The result of Example 4 can be generalized to show that the average velocity of a particle with constant acceleration during a time interval $[a, b]$ is the velocity at time $t = (a+b)/2$. (See Exercise 18.)

✔QUICK CHECK EXERCISES 5.8 *(See page 390 for answers.)*

1. The arithmetic average of n numbers, $a_1, a_2, \ldots, a_n$ is ________.
2. If f is continuous on $[a, b]$, then the average value of f on $[a, b]$ is ________.
3. If f is continuous on $[a, b]$, then the Mean-Value Theorem for Integrals guarantees that for at least one point x^* in $[a, b]$ ________ equals the average value of f on $[a, b]$.
4. The average value of $f(x) = 4x^3$ on $[1, 3]$ is ________.

EXERCISE SET 5.8 [C] CAS

1. (a) Find f_{ave} of $f(x) = 2x$ over $[0, 4]$.
(b) Find a point x^* in $[0, 4]$ such that $f(x^*) = f_{\text{ave}}$.
(c) Sketch a graph of $f(x) = 2x$ over $[0, 4]$, and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.
2. (a) Find f_{ave} of $f(x) = x^2$ over $[0, 2]$.
(b) Find a point x^* in $[0, 2]$ such that $f(x^*) = f_{\text{ave}}$.
(c) Sketch a graph of $f(x) = x^2$ over $[0, 2]$, and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

3–12 Find the average value of the function over the given interval.

3. $f(x) = 3x$; $[1, 3]$
4. $f(x) = \sqrt[3]{x}$; $[-1, 8]$
5. $f(x) = \sin x$; $[0, \pi]$
6. $f(x) = \sec x \tan x$; $[0, \pi/3]$
7. $f(x) = 1/x$; $[1, e]$
8. $f(x) = e^x$; $[-1, \ln 5]$
9. $f(x) = \dfrac{1}{1+x^2}$; $[1, \sqrt{3}]$
10. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$; $\left[-\frac{1}{2}, 0\right]$
11. $f(x) = e^{-2x}$; $[0, 4]$
12. $f(x) = \sec^2 x$; $[-\pi/4, \pi/4]$

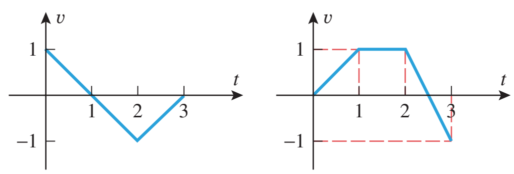
FOCUS ON CONCEPTS

13. Let $f(x) = 3x^2$.
(a) Find the arithmetic average of the values $f(0.4)$, $f(0.8)$, $f(1.2)$, $f(1.6)$, and $f(2.0)$.
(b) Find the arithmetic average of the values $f(0.1)$, $f(0.2)$, $f(0.3), \ldots, f(2.0)$.
(c) Find the average value of f on $[0, 2]$.
(d) Explain why the answer to part (c) is less than the answers to parts (a) and (b).
14. In parts (a)–(d), let $f(x) = 1 + (1/x)$.
(a) Find the arithmetic average of the values $f\left(\frac{6}{5}\right)$, $f\left(\frac{7}{5}\right)$, $f\left(\frac{8}{5}\right)$, $f\left(\frac{9}{5}\right)$, and $f(2)$.
(b) Find the arithmetic average of the values $f(1.1)$, $f(1.2)$, $f(1.3), \ldots, f(2)$. *(cont.)*

(c) Find the average value of f on $[1, 2]$.
(d) Explain why the answer to part (c) is greater than the answers to parts (a) and (b).

15. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the average velocity of the particle over the time interval $0 \le t \le 3$.

(a) (b)



16. Suppose that a particle moving along a line starts from rest and has an average velocity of 2 ft/s over the time interval $0 \le t \le 5$. Sketch a velocity versus time curve for the particle assuming that the particle is also at rest at time $t = 5$. Explain how your curve satisfies the required properties.

17. Suppose that f is a linear function. Using the graph of f, explain why the average value of f on $[a, b]$ is

$$f\left(\frac{a+b}{2}\right)$$

18. Suppose that a particle moves along a coordinate line with constant acceleration. Show that the average velocity of the particle during a time interval $[a, b]$ matches the velocity of the particle at the midpoint of the interval.

19–22 True–False Determine whether the statement is true or false. Explain your answer. (Assume that f and g denote continuous functions on an interval $[a, b]$ and that f_{ave} and g_{ave} denote the respective average values of f and g on $[a, b]$.)

19. If $g_{\text{ave}} < f_{\text{ave}}$, then $g(x) \le f(x)$ on $[a, b]$.

20. The average value of a constant multiple of f is the same multiple of f_{ave}; that is, if c is any constant,

$$(c \cdot f)_{\text{ave}} = c \cdot f_{\text{ave}}$$

21. The average of the sum of two functions on an interval is the sum of the average values of the two functions on the interval; that is,

$$(f + g)_{\text{ave}} = f_{\text{ave}} + g_{\text{ave}}$$

22. The average of the product of two functions on an interval is the product of the average values of the two functions on the interval; that is

$$(f \cdot g)_{\text{ave}} = f_{\text{ave}} \cdot g_{\text{ave}}$$

23. (a) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ by integrating.
(b) Suppose that the position function of a particle moving along a coordinate line is $s(t) = 6t^2 + t$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ algebraically.

24. (a) Suppose that the acceleration function of a particle moving along a coordinate line is $a(t) = t + 1$. Find the average acceleration of the particle over the time interval $0 \le t \le 5$ by integrating.
(b) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = \cos t$. Find the average acceleration of the particle over the time interval $0 \le t \le \pi/4$ algebraically.

25. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3.]

26. (a) The temperature of a 10 m long metal bar is 15°C at one end and 30°C at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?
(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.

27. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M. Find the average rate, in cars per minute, at which cars enter the highway during the first half-hour of rush hour.

28. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000e^{-0.17t}$. What is the average value of the yacht over its first 10 years of use?

29. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.

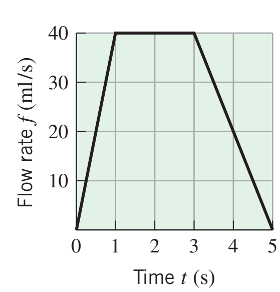


Figure Ex-29

C **30.** The function J_0 defined by

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\,dt$$

is called the ***Bessel function of order zero***.

(a) Find a function f and an interval $[a, b]$ for which $J_0(1)$ is the average value of f over $[a, b]$.

(b) Estimate $J_0(1)$.

(c) Use a CAS to graph the equation $y = J_0(x)$ over the interval $0 \le x \le 8$.

(d) Estimate the smallest positive zero of J_0.

31. Find a positive value of k such that the average value of $f(x) = \sqrt{3x}$ over the interval $[0, k]$ is 6.

32. Suppose that a tumor grows at the rate of $r(t) = kt$ grams per week for some positive constant k, where t is the number of weeks since the tumor appeared. When, during the second 26 weeks of growth, is the mass of the tumor the same as its average mass during that period?

33. Writing Consider the following statement: *The average value of the rate of change of a function over an interval is equal to the average rate of change of the function over that interval.* Write a short paragraph that explains why this statement may be interpreted as a rewording of Part 1 of the Fundamental Theorem of Calculus.

34. Writing If an automobile gets an average of 25 miles per gallon of gasoline, then it is also the case that on average the automobile expends 1/25 gallon of gasoline per mile. Interpret this statement using the concept of the average value of a function over an interval.

✔ QUICK CHECK ANSWERS 5.8

1. $\frac{1}{n}\sum_{k=1}^{n} a_k$ **2.** $\frac{1}{b-a}\int_a^b f(x)\,dx$ **3.** $f(x^*)$ **4.** 40

5.9 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

In this section we will discuss two methods for evaluating definite integrals in which a substitution is required.

TWO METHODS FOR MAKING SUBSTITUTIONS IN DEFINITE INTEGRALS

Recall from Section 5.3 that indefinite integrals of the form

$$\int f(g(x))g'(x)\,dx$$

can sometimes be evaluated by making the u-substitution

$$u = g(x), \quad du = g'(x)\,dx \tag{1}$$

which converts the integral to the form

$$\int f(u)\,du$$

To apply this method to a definite integral of the form

$$\int_a^b f(g(x))g'(x)\,dx$$

we need to account for the effect that the substitution has on the x-limits of integration. There are two ways of doing this.

Method 1.

First evaluate the indefinite integral

$$\int f(g(x))g'(x)\,dx$$

by substitution, and then use the relationship

$$\int_a^b f(g(x))g'(x)\,dx = \left[\int f(g(x))g'(x)\,dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x-limits of integration.

Method 2.

Make the substitution (1) directly in the definite integral, and then use the relationship $u = g(x)$ to replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u = g(a)$ and $u = g(b)$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)\,du$$

that is expressed entirely in terms of u.

▶ **Example 1** Use the two methods above to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution by Method 1. If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx \tag{2}$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_{x=0}^2$$

$$= \left.\frac{(x^2+1)^4}{8}\right]_{x=0}^2 = \frac{625}{8} - \frac{1}{8} = 78$$

Solution by Method 2. If we make the substitution $u = x^2 + 1$ in (2), then

$$\text{if} \quad x = 0, \quad u = 1$$
$$\text{if} \quad x = 2, \quad u = 5$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du$$

$$= \left.\frac{u^4}{8}\right]_{u=1}^5 = \frac{625}{8} - \frac{1}{8} = 78$$

which agrees with the result obtained by Method 1. ◀

The following theorem states precise conditions under which Method 2 can be used.

5.9.1 THEOREM *If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

PROOF Since f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, it follows that f has an antiderivative F on that interval. If we let $u = g(x)$, then the chain rule implies that

$$\frac{d}{dx}F(g(x)) = \frac{d}{dx}F(u) = \frac{dF}{du}\frac{du}{dx} = f(u)\frac{du}{dx} = f(g(x))g'(x)$$

for each x in $[a, b]$. Thus, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a, b]$. Therefore, by Part 1 of the Fundamental Theorem of Calculus (5.6.1)

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du \quad \blacksquare$$

The choice of methods for evaluating definite integrals by substitution is generally a matter of taste, but in the following examples we will use the second method, since the idea is new.

▶ Example 2 Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x\,dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9\,dx$$

Solution (a). Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x\,dx \quad \left(\text{or } \tfrac{1}{2}\,du = \cos 2x\,dx\right)$$

With this substitution,

$$\text{if} \quad x = 0, \quad u = \sin(0) = 0$$
$$\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2}$$

so

$$\int_0^{\pi/8} \sin^5 2x \cos 2x\,dx = \frac{1}{2}\int_0^{1/\sqrt{2}} u^5\,du$$

$$= \frac{1}{2}\cdot\frac{u^6}{6}\bigg]_{u=0}^{1/\sqrt{2}} = \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96}$$

Solution (b). Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x - 5$ unresolved in the integrand. However,

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u+3) - 5 = 2u + 1$$

With this substitution,

$$\text{if} \quad x = 2, \quad u = 2 - 3 = -1$$
$$\text{if} \quad x = 5, \quad u = 5 - 3 = 2$$

so

$$\int_2^5 (2x-5)(x-3)^9\,dx = \int_{-1}^2 (2u+1)u^9\,du = \int_{-1}^2 (2u^{10} + u^9)\,du$$

$$= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^2 = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right)$$

$$= \frac{52{,}233}{110} \approx 474.8 \quad ◀$$

▶ **Example 3** Evaluate

$$\text{(a)} \int_0^{3/4} \frac{dx}{1-x} \qquad \text{(b)} \int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx$$

Solution (a). Let

$$u = 1 - x \quad \text{so that} \quad du = -dx$$

With this substitution,

$$\text{if} \quad x = 0, \quad u = 1$$
$$\text{if} \quad x = \tfrac{3}{4}, \quad u = \tfrac{1}{4}$$

Thus,

$$\int_0^{3/4} \frac{dx}{1-x} = -\int_1^{1/4} \frac{du}{u}$$
$$= -\ln|u|\Big]_{u=1}^{1/4} = -\left[\ln\left(\frac{1}{4}\right) - \ln(1)\right] = \ln 4$$

Solution (b). Make the u-substitution

$$u = 1 + e^x, \quad du = e^x\,dx$$

and change the x-limits of integration ($x = 0$, $x = \ln 3$) to the u-limits

$$u = 1 + e^0 = 2, \quad u = 1 + e^{\ln 3} = 1 + 3 = 4$$

This yields

$$\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx = \int_2^4 u^{1/2}\,du$$
$$= \frac{2}{3}u^{3/2}\Big]_{u=2}^4 = \frac{2}{3}[4^{3/2} - 2^{3/2}] = \frac{16 - 4\sqrt{2}}{3} \blacktriangleleft$$

The u-substitution in Example 3(a) produces an integral in which the upper u-limit is smaller than the lower u-limit. Use Definition 5.5.3(b) to convert this integral to one whose lower limit is smaller than the upper limit and verify that it produces an integral with the same value as that in the example.

✔ QUICK CHECK EXERCISES 5.9 *(See page 396 for answers.)*

1. Assume that g' is continuous on $[a, b]$ and that f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$. If F is an antiderivative for f, then
$$\int_a^b f(g(x))g'(x)\,dx = \underline{\hspace{2cm}}$$

2. In each part, use the substitution to replace the given integral with an integral involving the variable u. (Do not evaluate the integral.)
   (a) $\int_0^2 3x^2(1+x^3)^3\,dx;\ u = 1 + x^3$
   (b) $\int_0^2 \frac{x}{\sqrt{5-x^2}}\,dx;\ u = 5 - x^2$
   (c) $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx;\ u = \sqrt{x}$

3. Evaluate the integral by making an appropriate substitution.
   (a) $\int_{-\pi}^0 \sin(3x - \pi)\,dx = \underline{\hspace{2cm}}$
   (b) $\int_2^3 \frac{x}{x^2 - 2}\,dx = \underline{\hspace{2cm}}$
   (c) $\int_0^{\pi/2} \sqrt[3]{\sin x}\cos x\,dx = \underline{\hspace{2cm}}$

EXERCISE SET 5.9 Graphing Utility C CAS

1–4 Express the integral in terms of the variable u, but do not evaluate it.

1. (a) $\int_1^3 (2x-1)^3\,dx;\ u=2x-1$

(b) $\int_0^4 3x\sqrt{25-x^2}\,dx;\ u=25-x^2$

(c) $\int_{-1/2}^{1/2} \cos(\pi\theta)\,d\theta;\ u=\pi\theta$

(d) $\int_0^1 (x+2)(x+1)^5\,dx;\ u=x+1$

2. (a) $\int_{-1}^4 (5-2x)^8\,dx;\ u=5-2x$

(b) $\int_{-\pi/3}^{2\pi/3} \frac{\sin x}{\sqrt{2+\cos x}}\,dx;\ u=2+\cos x$

(c) $\int_0^{\pi/4} \tan^2 x\sec^2 x\,dx;\ u=\tan x$

(d) $\int_0^1 x^3\sqrt{x^2+3}\,dx;\ u=x^2+3$

3. (a) $\int_0^1 e^{2x-1}\,dx;\ u=2x-1$

(b) $\int_e^{e^2} \frac{\ln x}{x}\,dx;\ u=\ln x$

4. (a) $\int_1^{\sqrt{3}} \frac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx;\ u=\tan^{-1}x$

(b) $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln x)^2}};\ u=\ln x$

5–18 Evaluate the definite integral two ways: first by a u-substitution in the definite integral and then by a u-substitution in the corresponding indefinite integral.

5. $\int_0^1 (2x+1)^3\,dx$

6. $\int_1^2 (4x-2)^3\,dx$

7. $\int_0^1 (2x-1)^3\,dx$

8. $\int_1^2 (4-3x)^8\,dx$

9. $\int_0^8 x\sqrt{1+x}\,dx$

10. $\int_{-3}^0 x\sqrt{1-x}\,dx$

11. $\int_0^{\pi/2} 4\sin(x/2)\,dx$

12. $\int_0^{\pi/6} 2\cos 3x\,dx$

13. $\int_{-2}^{-1} \frac{x}{(x^2+2)^3}\,dx$

14. $\int_{1-\pi}^{1+\pi} \sec^2\left(\frac{1}{4}x-\frac{1}{4}\right)\,dx$

15. $\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4}\,dx$

16. $\int_0^{\ln 5} e^x(3-4e^x)\,dx$

17. $\int_1^3 \frac{dx}{\sqrt{x}\,(x+1)}$

18. $\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}\,dx}{\sqrt{1-e^{-2x}}}$

19–22 Evaluate the definite integral by expressing it in terms of u and evaluating the resulting integral using a formula from geometry.

19. $\int_{-5/3}^{5/3} \sqrt{25-9x^2}\,dx;\ u=3x$

20. $\int_0^2 x\sqrt{16-x^4}\,dx;\ u=x^2$

21. $\int_{\pi/3}^{\pi/2} \sin\theta\sqrt{1-4\cos^2\theta}\,d\theta;\ u=2\cos\theta$

22. $\int_{e^{-3}}^{e^3} \frac{\sqrt{9-(\ln x)^2}}{x}\,dx;\ u=\ln x$

23. A particle moves with a velocity of $v(t)=\sin\pi t$ m/s along an s-axis. Find the distance traveled by the particle over the time interval $0\le t\le 1$.

24. A particle moves with a velocity of $v(t)=3\cos 2t$ m/s along an s-axis. Find the distance traveled by the particle over the time interval $0\le t\le \pi/8$.

25. Find the area under the curve $y=9/(x+2)^2$ over the interval $[-1, 1]$.

26. Find the area under the curve $y=1/(3x+1)^2$ over the interval $[0, 1]$.

27. Find the area of the region enclosed by the graphs of $y=1/\sqrt{1-9x^2}$, $y=0$, $x=0$, and $x=\frac{1}{6}$.

28. Find the area of the region enclosed by the graphs of $y=\sin^{-1}x$, $x=0$, and $y=\pi/2$.

29–48 Evaluate the integrals by any method.

29. $\int_1^5 \frac{dx}{\sqrt{2x-1}}$

30. $\int_1^2 \sqrt{5x-1}\,dx$

31. $\int_{-1}^1 \frac{x^2\,dx}{\sqrt{x^3+9}}$

32. $\int_{\pi/2}^{\pi} 6\sin x(\cos x+1)^5\,dx$

33. $\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$

34. $\int_1^2 \frac{dx}{x^2-6x+9}$

35. $\int_0^{\pi/4} 4\sin x\cos x\,dx$

36. $\int_0^{\pi/4} \sqrt{\tan x}\sec^2 x\,dx$

37. $\int_0^{\sqrt{\pi}} 5x\cos(x^2)\,dx$

38. $\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

39. $\int_{\pi/12}^{\pi/9} \sec^2 3\theta\,d\theta$

40. $\int_0^{\pi/6} \tan 2\theta\,d\theta$

41. $\int_0^1 \frac{y^2\,dy}{\sqrt{4-3y}}$

42. $\int_{-1}^4 \frac{x\,dx}{\sqrt{5+x}}$

43. $\int_0^e \frac{dx}{2x+e}$

44. $\int_1^{\sqrt{2}} xe^{-x^2}\,dx$

45. $\int_0^1 \frac{x}{\sqrt{4-3x^4}}\,dx$

46. $\int_1^2 \frac{1}{\sqrt{x}\sqrt{4-x}}\,dx$

47. $\displaystyle\int_0^{1/\sqrt{3}} \frac{1}{1+9x^2}\,dx$ **48.** $\displaystyle\int_1^{\sqrt{2}} \frac{x}{3+x^4}\,dx$

C **49.** (a) Use a CAS to find the exact value of the integral

$$\int_0^{\pi/6} \sin^4 x \cos^3 x\,dx$$

(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $\cos^2 x = 1 - \sin^2 x$.]

C **50.** (a) Use a CAS to find the exact value of the integral

$$\int_{-\pi/4}^{\pi/4} \tan^4 x\,dx$$

(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $1 + \tan^2 x = \sec^2 x$.]

51. (a) Find $\displaystyle\int_0^1 f(3x+1)\,dx$ if $\displaystyle\int_1^4 f(x)\,dx = 5$.

(b) Find $\displaystyle\int_0^3 f(3x)\,dx$ if $\displaystyle\int_0^9 f(x)\,dx = 5$.

(c) Find $\displaystyle\int_{-2}^0 x f(x^2)\,dx$ if $\displaystyle\int_0^4 f(x)\,dx = 1$.

52. Given that m and n are positive integers, show that

$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx$$

by making a substitution. Do not attempt to evaluate the integrals.

53. Given that n is a positive integer, show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

by using a trigonometric identity and making a substitution. Do not attempt to evaluate the integrals.

54. Given that n is a positive integer, evaluate the integral

$$\int_0^1 x(1-x)^n\,dx$$

55. Suppose that at time $t = 0$ there are 750 bacteria in a growth medium and the bacteria population $y(t)$ grows at the rate $y'(t) = 802.137e^{1.528t}$ bacteria per hour. How many bacteria will there be in 12 hours?

56. Suppose that a particle moving along a coordinate line has velocity $v(t) = 25 + 10e^{-0.05t}$ ft/s.

(a) What is the distance traveled by the particle from time $t = 0$ to time $t = 10$?

(b) Does the term $10e^{-0.05t}$ have much effect on the distance traveled by the particle over that time interval? Explain your reasoning.

57. (a) The accompanying table shows the fraction of the Moon that is illuminated (as seen from Earth) at midnight (Eastern Standard Time) for the first week of 2005. Find the average fraction of the Moon illuminated during the first week of 2005.

Source: Data from the U.S Naval Observatory Astronomical Applications Department.

(b) The function $f(x) = 0.5 + 0.5\sin(0.213x + 2.481)$ models data for illumination of the Moon for the first 60 days of 2005. Find the average value of this illumination function over the interval [0, 7].

DAY	1	2	3	4	5	6	7
ILLUMINATION	0.74	0.65	0.56	0.45	0.35	0.25	0.16

▲ **Table Ex-57**

58. Electricity is supplied to homes in the form of ***alternating current***, which means that the voltage has a sinusoidal waveform described by an equation of the form

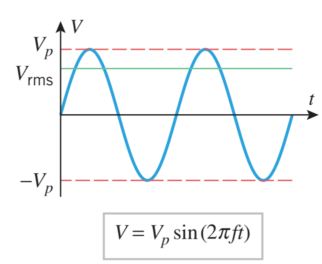
$$V = V_p \sin(2\pi f t)$$

(see the accompanying figure). In this equation, V_p is called the ***peak voltage*** or ***amplitude*** of the current, f is called its ***frequency***, and $1/f$ is called its ***period***. The voltages V and V_p are measured in volts (V), the time t is measured in seconds (s), and the frequency is measured in hertz (Hz). (1 Hz = 1 cycle per second; a ***cycle*** is the electrical term for one period of the waveform.) Most alternating-current voltmeters read what is called the ***rms*** or ***root-mean-square*** value of V. By definition, this is the square root of the average value of V^2 over one period.

(a) Show that $$V_{\text{rms}} = \frac{V_p}{\sqrt{2}}$$

[*Hint:* Compute the average over the cycle from $t = 0$ to $t = 1/f$, and use the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to help evaluate the integral.]

(b) In the United States, electrical outlets supply alternating current with an rms voltage of 120 V at a frequency of 60 Hz. What is the peak voltage at such an outlet?



◀ **Figure Ex-58**

59. Find a positive value of k such that the area under the graph of $y = e^{2x}$ over the interval $[0, k]$ is 3 square units.

60. Use a graphing utility to estimate the value of k $(k > 0)$ so that the region enclosed by $y = 1/(1 + kx^2)$, $y = 0$, $x = 0$, and $x = 2$ has an area of 0.6 square unit.

C **61.** (a) Find the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{\sin(k\pi/n)}{n}$$

by evaluating an appropriate definite integral over the interval [0, 1].

(b) Check your answer to part (a) by evaluating the limit directly with a CAS.

FOCUS ON CONCEPTS

62. Let
$$I = \int_{-1}^{1} \frac{1}{1+x^2}\,dx$$
(a) Explain why $I > 0$.
(b) Show that the substitution $x = 1/u$ results in
$$I = -\int_{-1}^{1} \frac{1}{1+x^2}\,dx = -I$$
Thus, $2I = 0$, which implies that $I = 0$. But this contradicts part (a). What is the error?

63. (a) Prove that if f is an odd function, then
$$\int_{-a}^{a} f(x)\,dx = 0$$
and give a geometric explanation of this result. [*Hint:* One way to prove that a quantity q is zero is to show that $q = -q$.]
(b) Prove that if f is an even function, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$
and give a geometric explanation of this result. [*Hint:* Split the interval of integration from $-a$ to a into two parts at 0.]

64. Show that if f and g are continuous functions, then
$$\int_{0}^{t} f(t-x)g(x)\,dx = \int_{0}^{t} f(x)g(t-x)\,dx$$

65. (a) Let
$$I = \int_{0}^{a} \frac{f(x)}{f(x)+f(a-x)}\,dx$$
Show that $I = a/2$.
[*Hint:* Let $u = a - x$, and then note the difference between the resulting integrand and 1.]
(b) Use the result of part (a) to find
$$\int_{0}^{3} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\,dx$$
(c) Use the result of part (a) to find
$$\int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$$

66. Evaluate
(a) $\displaystyle\int_{-1}^{1} x\sqrt{\cos(x^2)}\,dx$
(b) $\displaystyle\int_{0}^{\pi} \sin^8 x \cos^5 x\,dx.$
[*Hint:* Use the substitution $u = x - (\pi/2)$.]

67. **Writing** The two substitution methods discussed in this section yield the same result when used to evaluate a definite integral. Write a short paragraph that carefully explains why this is the case.

68. **Writing** In some cases, the second method for the evaluation of definite integrals has distinct advantages over the first. Provide some illustrations, and write a short paragraph that discusses the advantages of the second method in each case. [*Hint:* To get started, consider the results in Exercises 52–54, 63, and 65.]

✔ QUICK CHECK ANSWERS 5.9

1. $F(g(b)) - F(g(a))$ **2.** (a) $\displaystyle\int_{1}^{9} u^3\,du$ (b) $\displaystyle\int_{1}^{5} \frac{1}{2\sqrt{u}}\,du$ (c) $\displaystyle\int_{0}^{1} 2e^u\,du$ **3.** (a) $\frac{2}{3}$ (b) $\frac{1}{2}\ln\left(\frac{7}{2}\right)$ (c) $\frac{3}{4}$

5.10 LOGARITHMIC AND OTHER FUNCTIONS DEFINED BY INTEGRALS

In Section 0.5 we defined the natural logarithm function $\ln x$ *to be the inverse of* e^x. *Although this was convenient and enabled us to deduce many properties of* $\ln x$, *the mathematical foundation was shaky in that we accepted the continuity of* e^x *and of all exponential functions without proof. In this section we will show that* $\ln x$ *can be defined as a certain integral, and we will use this new definition to prove that exponential functions are continuous. This integral definition is also important in applications because it provides a way of recognizing when integrals that appear in solutions of problems can be expressed as natural logarithms.*

THE CONNECTION BETWEEN NATURAL LOGARITHMS AND INTEGRALS

The connection between natural logarithms and integrals was made in the middle of the seventeenth century in the course of investigating areas under the curve $y = 1/t$. The problem being considered was to find values of $t_1, t_2, t_3, \ldots, t_n, \ldots$ for which the areas $A_1, A_2, A_3, \ldots, A_n, \ldots$ in Figure 5.10.1a would be equal. Through the combined work of Isaac Newton, the Belgian Jesuit priest Gregory of St. Vincent (1584–1667), and Gregory's student Alfons A. de Sarasa (1618–1667), it was shown that by taking the points to be

$$t_1 = e, \quad t_2 = e^2, \quad t_3 = e^3, \ldots, \quad t_n = e^n, \ldots$$

each of the areas would be 1 (Figure 5.10.1b). Thus, in modern integral notation

$$\int_1^{e^n} \frac{1}{t}\,dt = n$$

which can be expressed as

$$\int_1^{e^n} \frac{1}{t}\,dt = \ln(e^n)$$

By comparing the upper limit of the integral and the expression inside the logarithm, it is a natural leap to the more general result

$$\int_1^{x} \frac{1}{t}\,dt = \ln x$$

which today we take as the formal definition of the natural logarithm.

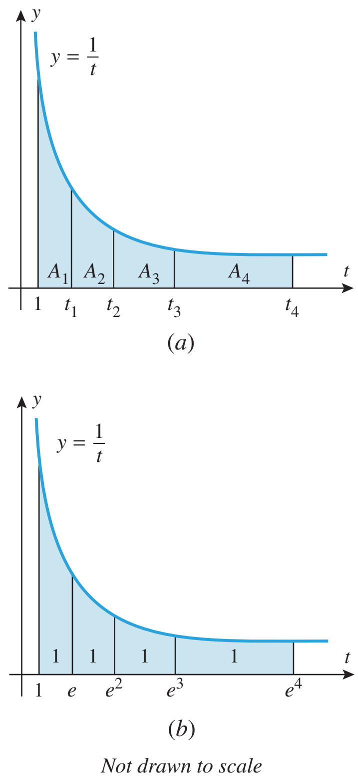


Figure 5.10.1

5.10.1 DEFINITION The ***natural logarithm*** of x is denoted by $\ln x$ and is defined by the integral

$$\ln x = \int_1^{x} \frac{1}{t}\,dt, \quad x > 0 \tag{1}$$

Review Theorem 5.5.8 and then explain why x is required to be positive in Definition 5.10.1.

None of the properties of $\ln x$ obtained in this section should be new, but now, for the first time, we give them a sound mathematical footing.

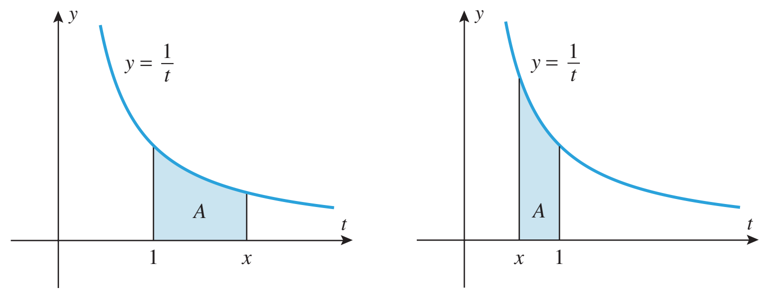
Our strategy for putting the study of logarithmic and exponential functions on a sound mathematical footing is to use (1) as a starting point and then define e^x as the inverse of $\ln x$. This is the exact opposite of our previous approach in which we defined $\ln x$ to be the inverse of e^x. However, whereas previously we had to *assume* that e^x is continuous, the continuity of e^x will now follow from our definitions as a *theorem*. Our first challenge is to demonstrate that the properties of $\ln x$ resulting from Definition 5.10.1 are consistent with those obtained earlier. To start, observe that Part 2 of the Fundamental Theorem of Calculus (5.6.3) implies that $\ln x$ is differentiable and

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^{x} \frac{1}{t}\,dt\right] = \frac{1}{x} \quad (x > 0) \tag{2}$$

This is consistent with the derivative formula for $\ln x$ that we obtained previously. Moreover, because differentiability implies continuity, it follows that $\ln x$ is a continuous function on the interval $(0, +\infty)$.

Other properties of $\ln x$ can be obtained by interpreting the integral in (1) geometrically: In the case where $x > 1$, this integral represents the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 5.10.2a); in the case where $0 < x < 1$, the integral represents the negative of the area under the curve $y = 1/t$ from $t = x$ to $t = 1$ (Figure 5.10.2b); and in the case where $x = 1$, the integral has value 0 because its upper and lower limits of integration are the same. These geometric observations imply that

$$\begin{aligned} &\ln x > 0 \quad \text{if} \quad x > 1 \\ &\ln x < 0 \quad \text{if} \quad 0 < x < 1 \\ &\ln x = 0 \quad \text{if} \quad x = 1 \end{aligned}$$



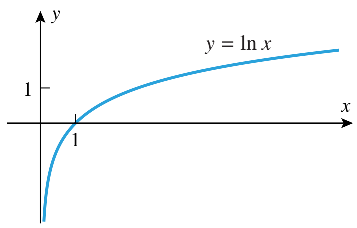
$$\ln x = \int_1^x \frac{1}{t}\,dt = A$$

(a)

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt = -A$$

(b)

▶ **Figure 5.10.2**



▲ **Figure 5.10.3**

Also, since $1/x$ is positive for $x > 0$, it follows from (2) that $\ln x$ is an increasing function on the interval $(0, +\infty)$. This is all consistent with the graph of $\ln x$ in Figure 5.10.3.

ALGEBRAIC PROPERTIES OF ln *x*

We can use (1) to show that Definition 5.10.1 produces the standard algebraic properties of logarithms.

> **5.10.2 THEOREM** *For any positive numbers a and c and any rational number r:*
>
> (a) $\ln ac = \ln a + \ln c$ (b) $\ln \dfrac{1}{c} = -\ln c$
>
> (c) $\ln \dfrac{a}{c} = \ln a - \ln c$ (d) $\ln a^r = r \ln a$

PROOF (*a*) Treating a as a constant, consider the function $f(x) = \ln(ax)$. Then

$$f'(x) = \frac{1}{ax} \cdot \frac{d}{dx}(ax) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

Thus, $\ln ax$ and $\ln x$ have the same derivative on $(0, +\infty)$, so these functions must differ by a constant on this interval. That is, there is a constant k such that

$$\ln ax - \ln x = k \tag{3}$$

on $(0, +\infty)$. Substituting $x = 1$ into this equation we conclude that $\ln a = k$ (verify). Thus, (3) can be written as

$$\ln ax - \ln x = \ln a$$

Setting $x = c$ establishes that

$$\ln ac - \ln c = \ln a \quad \text{or} \quad \ln ac = \ln a + \ln c$$

PROOFS (*b*) AND (*c*) Part (*b*) follows immediately from part (*a*) by substituting $1/c$ for a (verify). Then

$$\ln \frac{a}{c} = \ln\left(a \cdot \frac{1}{c}\right) = \ln a + \ln \frac{1}{c} = \ln a - \ln c$$

PROOF (*d*) First, we will argue that part (*d*) is satisfied if r is any nonnegative integer. If $r = 1$, then (*d*) is clearly satisfied; if $r = 0$, then (*d*) follows from the fact that $\ln 1 = 0$. Suppose that we know (*d*) is satisfied for r equal to some integer n. It then follows from part (*a*) that

$$\ln a^{n+1} = \ln[a \cdot a^n] = \ln a + \ln a^n = \ln a + n \ln a = (n+1)\ln a$$

How is the proof of Theorem 5.10.2(d) for the case where r is a nonnegative integer analogous to a row of falling dominos? (This "domino" argument uses an informal version of a property of the integers known as the *principle of mathematical induction.*)

That is, if (d) is valid for r equal to some integer n, then it is also valid for $r = n + 1$. However, since we know (d) is satisfied if $r = 1$, it follows that (d) is valid for $r = 2$. But this implies that (d) is satisfied for $r = 3$, which in turn implies that (d) is valid for $r = 4$, and so forth. We conclude that (d) is satisfied if r is any nonnegative integer.

Next, suppose that $r = -m$ is a negative integer. Then

$$\begin{aligned} \ln a^r = \ln a^{-m} = \ln \frac{1}{a^m} &= -\ln a^m && \boxed{\text{By part } (b)} \\ &= -m \ln a && \boxed{\text{Part } (d) \text{ is valid for positive powers.}} \\ &= r \ln a \end{aligned}$$

which shows that (d) is valid for any negative integer r. Combining this result with our previous conclusion that (d) is satisfied for a nonnegative integer r shows that (d) is valid if r is *any* integer.

Finally, suppose that $r = m/n$ is any rational number, where $m \neq 0$ and $n \neq 0$ are integers. Then

$$\begin{aligned} \ln a^r = \frac{n \ln a^r}{n} &= \frac{\ln[(a^r)^n]}{n} && \boxed{\text{Part } (d) \text{ is valid for integer powers.}} \\ &= \frac{\ln a^{rn}}{n} && \boxed{\text{Property of exponents}} \\ &= \frac{\ln a^m}{n} && \boxed{\text{Definition of } r} \\ &= \frac{m \ln a}{n} && \boxed{\text{Part } (d) \text{ is valid for integer powers.}} \\ &= \frac{m}{n} \ln a = r \ln a \end{aligned}$$

which shows that (d) is valid for any rational number r. ■

APPROXIMATING ln x NUMERICALLY

For specific values of x, the value of $\ln x$ can be approximated numerically by approximating the definite integral in (1), say by using the midpoint approximation that was discussed in Section 5.4.

Table 5.10.1

$n = 10$
$\Delta t = (b - a)/n = (2 - 1)/10 = 0.1$

k	t_k^*	$1/t_k^*$
1	1.05	0.952381
2	1.15	0.869565
3	1.25	0.800000
4	1.35	0.740741
5	1.45	0.689655
6	1.55	0.645161
7	1.65	0.606061
8	1.75	0.571429
9	1.85	0.540541
10	1.95	0.512821
		6.928355

$$\Delta t \sum_{k=1}^{n} f(t_k^*) \approx (0.1)(6.928355) \approx 0.692836$$

▶ Example 1 Approximate $\ln 2$ using the midpoint approximation with $n = 10$.

Solution. From (1), the exact value of $\ln 2$ is represented by the integral

$$\ln 2 = \int_1^2 \frac{1}{t}\, dt$$

The midpoint rule is given in Formulas (5) and (6) of Section 5.4. Expressed in terms of t, the latter formula is

$$\int_a^b f(t)\, dt \approx \Delta t \sum_{k=1}^{n} f(t_k^*)$$

where Δt is the common width of the subintervals and $t_1^*, t_2^*, \ldots, t_n^*$ are the midpoints. In this case we have 10 subintervals, so $\Delta t = (2 - 1)/10 = 0.1$. The computations to six decimal places are shown in Table 5.10.1. By comparison, a calculator set to display six decimal places gives $\ln 2 \approx 0.693147$, so the magnitude of the error in the midpoint approximation is about 0.000311. Greater accuracy in the midpoint approximation can be obtained by increasing n. For example, the midpoint approximation with $n = 100$ yields $\ln 2 \approx 0.693144$, which is correct to five decimal places. ◀

DOMAIN, RANGE, AND END BEHAVIOR OF ln x

5.10.3 THEOREM

(*a*) *The domain of* $\ln x$ *is* $(0, +\infty)$.

(*b*) $\lim_{x \to 0^+} \ln x = -\infty$ *and* $\lim_{x \to +\infty} \ln x = +\infty$

(*c*) *The range of* $\ln x$ *is* $(-\infty, +\infty)$.

PROOFS (*a*) AND (*b*) We have already shown that $\ln x$ is defined and increasing on the interval $(0, +\infty)$. To prove that $\ln x \to +\infty$ as $x \to +\infty$, we must show that given any number $M > 0$, the value of $\ln x$ exceeds M for sufficiently large values of x. To do this, let N be any integer. If $x > 2^N$, then

$$\ln x > \ln 2^N = N \ln 2 \tag{4}$$

by Theorem 5.10.2(*d*). Since

$$\ln 2 = \int_1^2 \frac{1}{t}\, dt > 0$$

it follows that $N \ln 2$ can be made arbitrarily large by choosing N sufficiently large. In particular, we can choose N so that $N \ln 2 > M$. It now follows from (4) that if $x > 2^N$, then $\ln x > M$, and this proves that

$$\lim_{x \to +\infty} \ln x = +\infty$$

Furthermore, by observing that $v = 1/x \to +\infty$ as $x \to 0^+$, we can use the preceding limit and Theorem 5.10.2(*b*) to conclude that

$$\lim_{x \to 0^+} \ln x = \lim_{v \to +\infty} \ln \frac{1}{v} = \lim_{v \to +\infty} (-\ln v) = -\infty$$

PROOF (*c*) It follows from part (*a*), the continuity of $\ln x$, and the Intermediate-Value Theorem (1.5.7) that $\ln x$ assumes every real value as x varies over the interval $(0, +\infty)$ (why?). ■

DEFINITION OF e^x

In Chapter 0 we defined $\ln x$ to be the inverse of the natural exponential function e^x. Now that we have a formal definition of $\ln x$ in terms of an integral, we will define the natural exponential function to be the inverse of $\ln x$.

Since $\ln x$ is increasing and continuous on $(0, +\infty)$ with range $(-\infty, +\infty)$, there is exactly one (positive) solution to the equation $\ln x = 1$. We *define* e to be the unique solution to $\ln x = 1$, so

$$\ln e = 1 \tag{5}$$

Furthermore, if x is any real number, there is a unique positive solution y to $\ln y = x$, so for irrational values of x we *define* e^x to be this solution. That is, when x is irrational, e^x is defined by

$$\ln e^x = x \tag{6}$$

Note that for rational values of x, we also have $\ln e^x = x \ln e = x$ from Theorem 5.10.2(*d*). Moreover, it follows immediately that $e^{\ln x} = x$ for any $x > 0$. Thus, (6) defines the exponential function for all real values of x as the inverse of the natural logarithm function.

5.10.4 DEFINITION The inverse of the natural logarithm function $\ln x$ is denoted by e^x and is called the ***natural exponential function***.

We can now establish the differentiability of e^x and confirm that

$$\frac{d}{dx}[e^x] = e^x$$

5.10.5 THEOREM *The natural exponential function e^x is differentiable, and hence continuous, on $(-\infty, +\infty)$, and its derivative is*

$$\frac{d}{dx}[e^x] = e^x$$

PROOF Because $\ln x$ is differentiable and

$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0$$

for all x in $(0, +\infty)$, it follows from Theorem 3.3.1, with $f(x) = \ln x$ and $f^{-1}(x) = e^x$, that e^x is differentiable on $(-\infty, +\infty)$ and its derivative is

$$\frac{d}{dx}\underbrace{[e^x]}_{f^{-1}(x)} = \underbrace{\frac{1}{1/e^x}}_{f'(f^{-1}(x))} = e^x \quad \blacksquare$$

IRRATIONAL EXPONENTS

Recall from Theorem 5.10.2(*d*) that if $a > 0$ and r is a rational number, then $\ln a^r = r \ln a$. Then $a^r = e^{\ln a^r} = e^{r \ln a}$ for any positive value of a and any rational number r. But the expression $e^{r \ln a}$ makes sense for *any* real number r, whether rational or irrational, so it is a good candidate to give meaning to a^r for any real number r.

Use Definition 5.10.6 to prove that if $a > 0$ and r is a real number, then $\ln a^r = r \ln a$.

5.10.6 DEFINITION If $a > 0$ and r is a real number, a^r is defined by

$$a^r = e^{r \ln a} \tag{7}$$

With this definition it can be shown that the standard algebraic properties of exponents, such as

$$a^p a^q = a^{p+q}, \quad \frac{a^p}{a^q} = a^{p-q}, \quad (a^p)^q = a^{pq}, \quad (a^p)(b^p) = (ab)^p$$

hold for any real values of a, b, p, and q, where a and b are positive. In addition, using (7) for a real exponent r, we can define the power function x^r whose domain consists of all positive real numbers, and for a positive base b we can define the ***base b exponential function b^x*** whose domain consists of all real numbers.

5.10.7 THEOREM

(*a*) *For any real number r, the power function x^r is differentiable on $(0, +\infty)$ and its derivative is*

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

(*b*) *For $b > 0$ and $b \neq 1$, the base b exponential function b^x is differentiable on $(-\infty, +\infty)$ and its derivative is*

$$\frac{d}{dx}[b^x] = b^x \ln b$$

PROOF The differentiability of $x^r = e^{r \ln x}$ and $b^x = e^{x \ln b}$ on their domains follows from the differentiability of $\ln x$ on $(0, +\infty)$ and of e^x on $(-\infty, +\infty)$:

$$\frac{d}{dx}[x^r] = \frac{d}{dx}[e^{r \ln x}] = e^{r \ln x} \cdot \frac{d}{dx}[r \ln x] = x^r \cdot \frac{r}{x} = rx^{r-1}$$

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x \ln b}] = e^{x \ln b} \cdot \frac{d}{dx}[x \ln b] = b^x \ln b \quad ■$$

We expressed e as the value of a limit in Formulas (7) and (8) of Section 1.3 and in Formula (1) of Section 3.2. We now have the mathematical tools necessary to prove the existence of these limits.

5.10.8 THEOREM

(a) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (b) $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ (c) $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$

PROOF We will prove part (a); the proofs of parts (b) and (c) follow from this limit and are left as exercises. We first observe that

$$\frac{d}{dx}[\ln(x+1)]\bigg|_{x=0} = \frac{1}{x+1} \cdot 1\bigg|_{x=0} = 1$$

However, using the definition of the derivative, we obtain

$$1 = \frac{d}{dx}[\ln(x+1)]\bigg|_{x=0} = \lim_{h \to 0} \frac{\ln(0+h+1) - \ln(0+1)}{h}$$
$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \ln(1+h)\right]$$

or, equivalently,

$$\lim_{x \to 0} \frac{1}{x} \cdot \ln(1+x) = 1 \tag{8}$$

Now

$$\begin{aligned} \lim_{x \to 0} (1+x)^{1/x} &= \lim_{x \to 0} e^{(\ln(1+x))/x} && \boxed{\text{Definition 5.10.6}} \\ &= e^{\lim_{x \to 0}[(\ln(1+x))/x]} && \boxed{\text{Theorem 1.5.5}} \\ &= e^1 && \boxed{\text{Equation (8)}} \\ &= e \quad ■ \end{aligned}$$

GENERAL LOGARITHMS

We note that for $b > 0$ and $b \neq 1$, the function b^x is one-to-one and so has an inverse function. Using the definition of b^x, we can solve $y = b^x$ for x as a function of y:

$$y = b^x = e^{x \ln b}$$
$$\ln y = \ln(e^{x \ln b}) = x \ln b$$
$$\frac{\ln y}{\ln b} = x$$

Thus, the inverse function for b^x is $(\ln x)/(\ln b)$.

5.10.9 DEFINITION For $b > 0$ and $b \neq 1$, the ***base b logarithm*** function, denoted $\log_b x$, is defined by

$$\log_b x = \frac{\ln x}{\ln b} \tag{9}$$

It follows immediately from this definition that $\log_b x$ is the inverse function for b^x and satisfies the properties in Table 0.5.3. Furthermore, $\log_b x$ is differentiable, and hence continuous, on $(0, +\infty)$, and its derivative is

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$$

As a final note of consistency, we observe that $\log_e x = \ln x$.

FUNCTIONS DEFINED BY INTEGRALS

The functions we have dealt with thus far in this text are called ***elementary functions***; they include polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric functions, and all other functions that can be obtained from these by addition, subtraction, multiplication, division, root extraction, and composition.

However, there are many important functions that do not fall into this category. Such functions occur in many ways, but they commonly arise in the course of solving initial-value problems of the form

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{10}$$

Recall from Example 6 of Section 5.2 and the discussion preceding it that the basic method for solving (10) is to integrate $f(x)$, and then use the initial condition to determine the constant of integration. It can be proved that if f is continuous, then (10) has a unique solution and that this procedure produces it. However, there is another approach: Instead of solving each initial-value problem individually, we can find a general formula for the solution of (10), and then apply that formula to solve specific problems. We will now show that

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\, dt \tag{11}$$

is a formula for the solution of (10). To confirm this we must show that $dy/dx = f(x)$ and that $y(x_0) = y_0$. The computations are as follows:

$$\frac{dy}{dx} = \frac{d}{dx}\left[y_0 + \int_{x_0}^{x} f(t)\, dt\right] = 0 + f(x) = f(x)$$

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\, dt = y_0 + 0 = y_0$$

Example 2 In Example 6 of Section 5.2 we showed that the solution of the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

is $y(x) = 1 + \sin x$. This initial-value problem can also be solved by applying Formula (11) with $f(x) = \cos x$, $x_0 = 0$, and $y_0 = 1$. This yields

$$y(x) = 1 + \int_0^x \cos t\, dt = 1 + \big[\sin t\big]_{t=0}^{x} = 1 + \sin x \blacktriangleleft$$

In the last example we were able to perform the integration in Formula (11) and express the solution of the initial-value problem as an elementary function. However, sometimes this will not be possible, in which case the solution of the initial-value problem must be left in terms of an "unevaluated" integral. For example, from (11), the solution of the initial-value problem

$$\frac{dy}{dx} = e^{-x^2}, \quad y(0) = 1$$

is

$$y(x) = 1 + \int_0^x e^{-t^2}\,dt$$

However, it can be shown that there is no way to express the integral in this solution as an elementary function. Thus, we have encountered a *new* function, which we regard to be *defined* by the integral. A close relative of this function, known as the ***error function***, plays an important role in probability and statistics; it is denoted by $\operatorname{erf}(x)$ and is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \tag{12}$$

Indeed, many of the most important functions in science and engineering are defined as integrals that have special names and notations associated with them. For example, the functions defined by

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \tag{13–14}$$

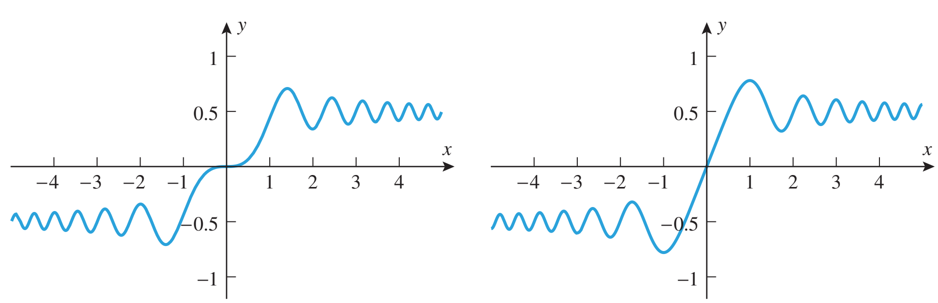
are called the ***Fresnel sine and cosine functions***, respectively, in honor of the French physicist Augustin Fresnel (1788–1827), who first encountered them in his study of diffraction of light waves.

EVALUATING AND GRAPHING FUNCTIONS DEFINED BY INTEGRALS

The following values of $S(1)$ and $C(1)$ were produced by a CAS that has a built-in algorithm for approximating definite integrals:

$$S(1) = \int_0^1 \sin\left(\frac{\pi t^2}{2}\right) dt \approx 0.438259, \qquad C(1) = \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt \approx 0.779893$$

To generate graphs of functions defined by integrals, computer programs choose a set of x-values in the domain, approximate the integral for each of those values, and then plot the resulting points. Thus, there is a lot of computation involved in generating such graphs, since each plotted point requires the approximation of an integral. The graphs of the Fresnel functions in Figure 5.10.4 were generated in this way using a CAS.



Fresnel sine function   Fresnel cosine function

▶ **Figure 5.10.4**

REMARK Although it required a considerable amount of computation to generate the graphs of the Fresnel functions, the derivatives of $S(x)$ and $C(x)$ are easy to obtain using Part 2 of the Fundamental Theorem of Calculus (5.6.3); they are

$$S'(x) = \sin\left(\frac{\pi x^2}{2}\right) \quad \text{and} \quad C'(x) = \cos\left(\frac{\pi x^2}{2}\right) \tag{15–16}$$

These derivatives can be used to determine the locations of the relative extrema and inflection points and to investigate other properties of $S(x)$ and $C(x)$.

INTEGRALS WITH FUNCTIONS AS LIMITS OF INTEGRATION

Various applications can lead to integrals in which at least one of the limits of integration is a function of x. Some examples are

$$\int_{x}^{1} \sqrt{\sin t}\, dt, \qquad \int_{x^2}^{\sin x} \sqrt{t^3+1}\, dt, \qquad \int_{\ln x}^{\pi} \frac{dt}{t^7-8}$$

We will complete this section by showing how to differentiate integrals of the form

$$\int_{a}^{g(x)} f(t)\, dt \tag{17}$$

where a is constant. Derivatives of other kinds of integrals with functions as limits of integration will be discussed in the exercises.

To differentiate (17) we can view the integral as a composition $F(g(x))$, where

$$F(x) = \int_{a}^{x} f(t)\, dt$$

If we now apply the chain rule, we obtain

$$\frac{d}{dx}\left[\int_{a}^{g(x)} f(t)\, dt\right] = \frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) \underset{\text{Theorem 5.6.3}}{=} f(g(x))g'(x)$$

Thus,

$$\boxed{\frac{d}{dx}\left[\int_{a}^{g(x)} f(t)\, dt\right] = f(g(x))g'(x)} \tag{18}$$

In words:

To differentiate an integral with a constant lower limit and a function as the upper limit, substitute the upper limit into the integrand, and multiply by the derivative of the upper limit.

Example 3

$$\frac{d}{dx}\left[\int_{1}^{\sin x} (1-t^2)\, dt\right] = (1-\sin^2 x)\cos x = \cos^3 x$$

✔ QUICK CHECK EXERCISES 5.10

(See page 408 for answers.)

1. $\displaystyle\int_{1}^{1/e} \frac{1}{t}\, dt =$ ________

2. Estimate $\ln 2$ using Definition 5.10.1 and
   (a) a left endpoint approximation with $n = 2$
   (b) a right endpoint approximation with $n = 2$.

3. $\pi^{1/(\ln \pi)} =$ ________

4. A solution to the initial-value problem
$$\frac{dy}{dx} = \cos x^3, \quad y(0) = 2$$
that is defined by an integral is $y =$ ________.

5. $\displaystyle\frac{d}{dx}\left[\int_{0}^{e^{-x}} \frac{1}{1+t^4}\, dt\right] =$ ________

EXERCISE SET 5.10 Graphing Utility C CAS

1. Sketch the curve $y = 1/t$, and shade a region under the curve whose area is
(a) $\ln 2$ (b) $-\ln 0.5$ (c) 2.

2. Sketch the curve $y = 1/t$, and shade two different regions under the curve whose areas are $\ln 1.5$.

3. Given that $\ln a = 2$ and $\ln c = 5$, find
(a) $\displaystyle\int_1^{ac} \frac{1}{t}\,dt$ (b) $\displaystyle\int_1^{1/c} \frac{1}{t}\,dt$
(c) $\displaystyle\int_1^{a/c} \frac{1}{t}\,dt$ (d) $\displaystyle\int_1^{a^3} \frac{1}{t}\,dt$.

4. Given that $\ln a = 9$, find
(a) $\displaystyle\int_1^{\sqrt{a}} \frac{1}{t}\,dt$ (b) $\displaystyle\int_1^{2a} \frac{1}{t}\,dt$
(c) $\displaystyle\int_1^{2/a} \frac{1}{t}\,dt$ (d) $\displaystyle\int_2^{a} \frac{1}{t}\,dt$.

5. Approximate $\ln 5$ using the midpoint rule with $n = 10$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.

6. Approximate $\ln 3$ using the midpoint rule with $n = 20$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.

7. Simplify the expression and state the values of x for which your simplification is valid.
(a) $e^{-\ln x}$ (b) $e^{\ln x^2}$
(c) $\ln\left(e^{-x^2}\right)$ (d) $\ln(1/e^x)$
(e) $\exp(3\ln x)$ (f) $\ln(xe^x)$
(g) $\ln\left(e^{x-\sqrt[3]{x}}\right)$ (h) $e^{x-\ln x}$

8. (a) Let $f(x) = e^{-2x}$. Find the simplest exact value of the function $f(\ln 3)$.
(b) Let $f(x) = e^x + 3e^{-x}$. Find the simplest exact value of the function $f(\ln 2)$.

9–10 Express the given quantity as a power of e.

9. (a) 3^π (b) $2^{\sqrt{2}}$

10. (a) π^{-x} (b) x^{2x}, $x > 0$

11–12 Find the limits by making appropriate substitutions in the limits given in Theorem 5.10.8.

11. (a) $\displaystyle\lim_{x\to+\infty}\left(1 + \frac{1}{2x}\right)^x$ (b) $\displaystyle\lim_{x\to 0}(1 + 2x)^{1/x}$

12. (a) $\displaystyle\lim_{x\to+\infty}\left(1 + \frac{3}{x}\right)^x$ (b) $\displaystyle\lim_{x\to 0}(1 + x)^{1/(3x)}$

13–14 Find $g'(x)$ using Part 2 of the Fundamental Theorem of Calculus, and check your answer by evaluating the integral and then differentiating.

13. $\displaystyle g(x) = \int_1^x (t^2 - t)\,dt$

14. $\displaystyle g(x) = \int_\pi^x (1 - \cos t)\,dt$

15–16 Find the derivative using Formula (18), and check your answer by evaluating the integral and then differentiating the result.

15. (a) $\displaystyle\frac{d}{dx}\int_1^{x^3} \frac{1}{t}\,dt$ (b) $\displaystyle\frac{d}{dx}\int_1^{\ln x} e^t\,dt$

16. (a) $\displaystyle\frac{d}{dx}\int_{-1}^{x^2} \sqrt{t+1}\,dt$ (b) $\displaystyle\frac{d}{dx}\int_\pi^{1/x} \sin t\,dt$

17. Let $\displaystyle F(x) = \int_0^x \frac{\sin t}{t^2+1}\,dt$. Find
(a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

18. Let $\displaystyle F(x) = \int_2^x \sqrt{3t^2+1}\,dt$. Find
(a) $F(2)$ (b) $F'(2)$ (c) $F''(2)$.

19–22 True–False Determine whether the equation is true or false. Explain your answer.

19. $\displaystyle\int_1^{1/a} \frac{1}{t}\,dt = -\int_1^a \frac{1}{t}\,dt$, for $0 < a$

20. $\displaystyle\int_1^{\sqrt{a}} \frac{1}{t}\,dt = \frac{1}{2}\int_1^a \frac{1}{t}\,dt$, for $0 < a$

21. $\displaystyle\int_{-1}^{e} \frac{1}{t}\,dt = 1$

22. $\displaystyle\int \frac{2x}{1+x^2}\,dx = \int_1^{1+x^2} \frac{1}{t}\,dt + C$

C 23. (a) Use Formula (18) to find
$$\frac{d}{dx}\int_1^{x^2} t\sqrt{1+t}\,dt$$
(b) Use a CAS to evaluate the integral and differentiate the resulting function.
(c) Use the simplification command of the CAS, if necessary, to confirm that the answers in parts (a) and (b) are the same.

24. Show that
(a) $\displaystyle\frac{d}{dx}\left[\int_x^a f(t)\,dt\right] = -f(x)$
(b) $\displaystyle\frac{d}{dx}\left[\int_{g(x)}^a f(t)\,dt\right] = -f(g(x))g'(x)$.

25–26 Use the results in Exercise 24 to find the derivative.

25. (a) $\displaystyle\frac{d}{dx}\int_x^\pi \cos(t^3)\,dt$ (b) $\displaystyle\frac{d}{dx}\int_{\tan x}^3 \frac{t^2}{1+t^2}\,dt$

26. (a) $\displaystyle\frac{d}{dx}\int_x^0 \frac{1}{(t^2+1)^2}\,dt$ (b) $\displaystyle\frac{d}{dx}\int_{1/x}^\pi \cos^3 t\,dt$

27. Find
$$\frac{d}{dx}\left[\int_{3x}^{x^2} \frac{t-1}{t^2+1}\,dt\right]$$
by writing
$$\int_{3x}^{x^2} \frac{t-1}{t^2+1}\,dt = \int_{3x}^{0} \frac{t-1}{t^2+1}\,dt + \int_0^{x^2} \frac{t-1}{t^2+1}\,dt$$

28. Use Exercise 24(b) and the idea in Exercise 27 to show that

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(h(x))h'(x)$$

29. Use the result obtained in Exercise 28 to perform the following differentiations:

(a) $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sin^2 t\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} \frac{1}{1+t}\,dt.$

30. Prove that the function

$$F(x) = \int_x^{5x} \frac{1}{t}\,dt$$

is constant on the interval $(0, +\infty)$ by using Exercise 28 to find $F'(x)$. What is that constant?

FOCUS ON CONCEPTS

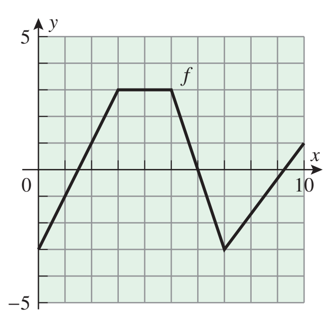
31. Let $F(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the accompanying figure.

(a) Find $F(0)$, $F(3)$, $F(5)$, $F(7)$, and $F(10)$.

(b) On what subintervals of the interval $[0, 10]$ is F increasing? Decreasing?

(c) Where does F have its maximum value? Its minimum value?

(d) Sketch the graph of F.



◀ **Figure Ex-31**

32. Determine the inflection point(s) for the graph of F in Exercise 31.

33–34 Express $F(x)$ in a piecewise form that does not involve an integral. ■

33. $F(x) = \displaystyle\int_{-1}^{x} |t|\,dt$

34. $F(x) = \displaystyle\int_0^x f(t)\,dt$, where $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 2, & x > 2 \end{cases}$

35–38 Use Formula (11) to solve the initial-value problem. ■

35. $\dfrac{dy}{dx} = \dfrac{2x^2+1}{x}$, $y(1) = 2$ **36.** $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}}$, $y(1) = 0$

37. $\dfrac{dy}{dx} = \sec^2 x - \sin x$, $y(\pi/4) = 1$

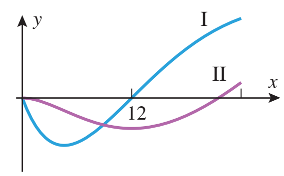
38. $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$, $y(e) = 1$

39. Suppose that at time $t = 0$ there are P_0 individuals who have disease X, and suppose that a certain model for the spread of the disease predicts that the disease will spread at the rate of $r(t)$ individuals per day. Write a formula for the number of individuals who will have disease X after x days.

40. Suppose that $v(t)$ is the velocity function of a particle moving along an s-axis. Write a formula for the coordinate of the particle at time T if the particle is at s_1 at time $t = 1$.

FOCUS ON CONCEPTS

41. The accompanying figure shows the graphs of $y = f(x)$ and $y = \int_0^x f(t)\,dt$. Determine which graph is which, and explain your reasoning.



◀ **Figure Ex-41**

42. (a) Make a conjecture about the value of the limit

$$\lim_{k\to 0}\int_1^b t^{k-1}\,dt \quad (b > 0)$$

(b) Check your conjecture by evaluating the integral and finding the limit. [*Hint:* Interpret the limit as the definition of the derivative of an exponential function.]

43. Let $F(x) = \int_0^x f(t)\,dt$, where f is the function graphed in the accompanying figure.

(a) Where do the relative minima of F occur?

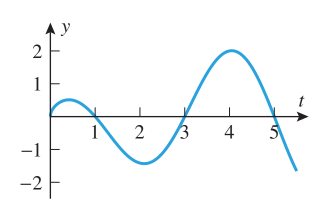
(b) Where do the relative maxima of F occur?

(c) Where does the absolute maximum of F on the interval $[0, 5]$ occur?

(d) Where does the absolute minimum of F on the interval $[0, 5]$ occur?

(e) Where is F concave up? Concave down?

(f) Sketch the graph of F.



◀ **Figure Ex-43**

C **44.** CAS programs have commands for working with most of the important nonelementary functions. Check your CAS documentation for information about the error function $\operatorname{erf}(x)$ [see Formula (12)], and then complete the following.

(a) Generate the graph of $\operatorname{erf}(x)$.

(b) Use the graph to make a conjecture about the existence and location of any relative maxima and minima of $\operatorname{erf}(x)$.

(c) Check your conjecture in part (b) using the derivative of $\operatorname{erf}(x)$.

(cont.)

(d) Use the graph to make a conjecture about the existence and location of any inflection points of $\operatorname{erf}(x)$.
(e) Check your conjecture in part (d) using the second derivative of $\operatorname{erf}(x)$.
(f) Use the graph to make a conjecture about the existence of horizontal asymptotes of $\operatorname{erf}(x)$.
(g) Check your conjecture in part (f) by using the CAS to find the limits of $\operatorname{erf}(x)$ as $x \to \pm\infty$.

45. The Fresnel sine and cosine functions $S(x)$ and $C(x)$ were defined in Formulas (13) and (14) and graphed in Figure 5.10.4. Their derivatives were given in Formulas (15) and (16).
(a) At what points does $C(x)$ have relative minima? Relative maxima?
(b) Where do the inflection points of $C(x)$ occur?
(c) Confirm that your answers in parts (a) and (b) are consistent with the graph of $C(x)$.

46. Find the limit
$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} \ln t \, dt$$

47. Find a function f and a number a such that
$$4 + \int_a^x f(t)\, dt = e^{2x}$$

48. (a) Give a geometric argument to show that
$$\frac{1}{x+1} < \int_x^{x+1} \frac{1}{t}\, dt < \frac{1}{x}, \quad x > 0$$
(b) Use the result in part (a) to prove that
$$\frac{1}{x+1} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}, \quad x > 0$$
(c) Use the result in part (b) to prove that
$$e^{x/(x+1)} < \left(1 + \frac{1}{x}\right)^x < e, \quad x > 0$$
and hence that
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
(d) Use the result in part (b) to prove that
$$\left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}, \quad x > 0$$

49. Use a graphing utility to generate the graph of
$$y = \left(1 + \frac{1}{x}\right)^{x+1} - \left(1 + \frac{1}{x}\right)^x$$
in the window $[0, 100] \times [0, 0.2]$, and use that graph and part (d) of Exercise 48 to make a rough estimate of the error in the approximation
$$e \approx \left(1 + \frac{1}{50}\right)^{50}$$

50. Prove: If f is continuous on an open interval and a is any point in that interval, then
$$F(x) = \int_a^x f(t)\, dt$$
is continuous on the interval.

51. **Writing** A student objects that it is circular reasoning to make the definition
$$\ln x = \int_1^x \frac{1}{t}\, dt$$
since to evaluate the integral we need to know the value of $\ln x$. Write a short paragraph that answers this student's objection.

52. **Writing** Write a short paragraph that compares Definition 5.10.1 with the definition of the natural logarithm function given in Chapter 0. Be sure to discuss the issues surrounding continuity and differentiability.

✔ QUICK CHECK ANSWERS 5.10

1. -1 **2.** (a) $\frac{5}{6}$ (b) $\frac{7}{12}$ **3.** e **4.** $y = 2 + \int_0^x \cos t^3\, dt$ **5.** $-\dfrac{e^{-x}}{1 + e^{-4x}}$

CHAPTER 5 REVIEW EXERCISES

Graphing Utility C CAS

1–8 Evaluate the integrals. ■

1. $\displaystyle\int \left[\frac{1}{2x^3} + 4\sqrt{x}\right] dx$

2. $\displaystyle\int [u^3 - 2u + 7]\, du$

3. $\displaystyle\int [4\sin x + 2\cos x]\, dx$

4. $\displaystyle\int \sec x(\tan x + \cos x)\, dx$

5. $\displaystyle\int [x^{-2/3} - 5e^x]\, dx$

6. $\displaystyle\int \left[\frac{3}{4x} - \sec^2 x\right] dx$

7. $\displaystyle\int \left[\frac{1}{1+x^2} + \frac{2}{\sqrt{1-x^2}}\right] dx$

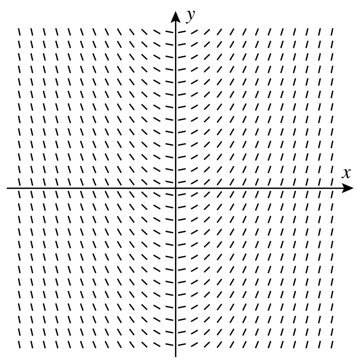
8. $\displaystyle\int \left[\frac{12}{x\sqrt{x^2-1}} + \frac{1-x^4}{1+x^2}\right] dx$

9. Solve the initial-value problems.
(a) $\dfrac{dy}{dx} = \dfrac{1-x}{\sqrt{x}}, \; y(1) = 0$
(b) $\dfrac{dy}{dx} = \cos x - 5e^x, \; y(0) = 0$
(c) $\dfrac{dy}{dx} = \sqrt[3]{x}, \; y(1) = 2$
(d) $\dfrac{dy}{dx} = xe^{x^2}, \; y(0) = 0$

10. The accompanying figure shows the slope field for a differential equation $dy/dx = f(x)$. Which of the following functions is most likely to be $f(x)$?

$$\sqrt{x}, \quad \sin x, \quad x^4, \quad x$$

Explain your reasoning.



◀ **Figure Ex-10**

11. (a) Show that the substitutions $u = \sec x$ and $u = \tan x$ produce different values for the integral

$$\int \sec^2 x \tan x \, dx$$

(b) Explain why both are correct.

12. Use the two substitutions in Exercise 11 to evaluate the definite integral

$$\int_0^{\pi/4} \sec^2 x \tan x \, dx$$

and confirm that they produce the same result.

13. Evaluate the integral

$$\int \frac{x}{(x^2-1)\sqrt{x^4-2x^2}} \, dx$$

by making the substitution $u = x^2 - 1$.

14. Evaluate the integral

$$\int \sqrt{1 + x^{-2/3}} \, dx$$

by making the substitution $u = 1 + x^{2/3}$.

C **15–18** Evaluate the integrals by hand, and check your answers with a CAS if you have one. ■

15. $\displaystyle\int \frac{\cos 3x}{\sqrt{5 + 2\sin 3x}} \, dx$

16. $\displaystyle\int \frac{\sqrt{3 + \sqrt{x}}}{\sqrt{x}} \, dx$

17. $\displaystyle\int \frac{x^2}{(ax^3 + b)^2} \, dx$

18. $\displaystyle\int x \sec^2(ax^2) \, dx$

19. Express

$$\sum_{k=4}^{18} k(k-3)$$

in sigma notation with
(a) $k = 0$ as the lower limit of summation
(b) $k = 5$ as the lower limit of summation.

20. (a) Fill in the blank:

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{k=1}^{n} \_\_\_\_\_\_$$

(b) Use part (a) to prove that the sum of the first n consecutive odd integers is a perfect square.

21. Find the area under the graph of $f(x) = 4x - x^2$ over the interval $[0, 4]$ using Definition 5.4.3 with x_k^* as the *right* endpoint of each subinterval.

22. Find the area under the graph of $f(x) = 5x - x^2$ over the interval $[0, 5]$ using Definition 5.4.3 with x_k^* as the *left* endpoint of each subinterval.

23–24 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals. ■

23. $y = \ln x$; $[1, 2]$

24. $y = e^x$; $[0, 1]$

25. The *definite integral* of f over the interval $[a, b]$ is defined as the limit

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Explain what the various symbols on the right side of this equation mean.

26. Use a geometric argument to evaluate

$$\int_0^1 |2x - 1| \, dx$$

27. Suppose that

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}, \quad \int_1^2 f(x)\,dx = \tfrac{1}{4},$$

$$\int_0^3 f(x)\,dx = -1, \quad \int_0^1 g(x)\,dx = 2$$

In each part, use this information to evaluate the given integral, if possible. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_1^3 f(x)\,dx$ (c) $\displaystyle\int_2^3 5f(x)\,dx$

(d) $\displaystyle\int_1^0 g(x)\,dx$ (e) $\displaystyle\int_0^1 g(2x)\,dx$ (f) $\displaystyle\int_0^1 [g(x)]^2\,dx$

28. In parts (a)–(d), use the information in Exercise 27 to evaluate the given integral. If there is not enough information to evaluate the integral, then say so.

(cont.)

(a) $\int_0^1 [f(x) + g(x)]\,dx$ (b) $\int_0^1 f(x)g(x)\,dx$

(c) $\int_0^1 \frac{f(x)}{g(x)}\,dx$ (d) $\int_0^1 [4g(x) - 3f(x)]\,dx$

29. In each part, evaluate the integral. Where appropriate, you may use a geometric formula.

(a) $\int_{-1}^1 (1 + \sqrt{1 - x^2})\,dx$

(b) $\int_0^3 (x\sqrt{x^2 + 1} - \sqrt{9 - x^2})\,dx$

(c) $\int_0^1 x\sqrt{1 - x^4}\,dx$

30. In each part, find the limit by interpreting it as a limit of Riemann sums in which the interval [0, 1] is divided into n subintervals of equal length.

(a) $\lim_{n\to+\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n^{3/2}}$

(b) $\lim_{n\to+\infty} \frac{1^4 + 2^4 + 3^4 + \cdots + n^4}{n^5}$

(c) $\lim_{n\to+\infty} \frac{e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n}}{n}$

31–38 Evaluate the integrals using the Fundamental Theorem of Calculus and (if necessary) properties of the definite integral. ■

31. $\int_{-3}^0 (x^2 - 4x + 7)\,dx$ **32.** $\int_{-1}^2 x(1 + x^3)\,dx$

33. $\int_1^3 \frac{1}{x^2}\,dx$ **34.** $\int_1^8 (5x^{2/3} - 4x^{-2})\,dx$

35. $\int_0^1 (x - \sec x \tan x)\,dx$

36. $\int_1^4 \left(\frac{3}{\sqrt{t}} - 5\sqrt{t} - t^{-3/2}\right) dt$

37. $\int_0^2 |2x - 3|\,dx$ **38.** $\int_0^{\pi/2} \left|\frac{1}{2} - \sin x\right| dx$

39–42 Find the area under the curve $y = f(x)$ over the stated interval. ■

39. $f(x) = \sqrt{x}$; [1, 9] **40.** $f(x) = x^{-3/5}$; [1, 4]

41. $f(x) = e^x$; [1, 3] **42.** $f(x) = \frac{1}{x}$; $[1, e^3]$

43. Find the area that is above the x-axis but below the curve $y = (1 - x)(x - 2)$. Make a sketch of the region.

C **44.** Use a CAS to find the area of the region in the first quadrant that lies below the curve $y = x + x^2 - x^3$ and above the x-axis.

45–46 Sketch the curve and find the total area between the curve and the given interval on the x-axis. ■

45. $y = x^2 - 1$; [0, 3] **46.** $y = \sqrt{x + 1} - 1$; [−1, 1]

47. Define $F(x)$ by

$$F(x) = \int_1^x (t^3 + 1)\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

48. Define $F(x)$ by

$$F(x) = \int_4^x \frac{1}{\sqrt{t}}\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

49–54 Use Part 2 of the Fundamental Theorem of Calculus and (where necessary) Formula (18) of Section 5.10 to find the derivatives. ■

49. $\frac{d}{dx}\left[\int_0^x e^{t^2}\,dt\right]$ **50.** $\frac{d}{dx}\left[\int_0^x \frac{t}{\cos t^2}\,dt\right]$

51. $\frac{d}{dx}\left[\int_0^x |t - 1|\,dt\right]$ **52.** $\frac{d}{dx}\left[\int_\pi^x \cos\sqrt{t}\,dt\right]$

53. $\frac{d}{dx}\left[\int_2^{\sin x} \frac{1}{1 + t^3}\,dt\right]$ **54.** $\frac{d}{dx}\left[\int_e^{\sqrt{x}} (\ln t)^2\,dt\right]$

55. State the two parts of the Fundamental Theorem of Calculus, and explain what is meant by the statement "Differentiation and integration are inverse processes."

C **56.** Let $F(x) = \int_0^x \frac{t^2 - 3}{t^4 + 7}\,dt$.

(a) Find the intervals on which F is increasing and those on which F is decreasing.

(b) Find the open intervals on which F is concave up and those on which F is concave down.

(c) Find the x-values, if any, at which the function F has absolute extrema.

(d) Use a CAS to graph F, and confirm that the results in parts (a), (b), and (c) are consistent with the graph.

57. (a) Use differentiation to prove that the function

$$F(x) = \int_0^x \frac{1}{1 + t^2}\,dt + \int_0^{1/x} \frac{1}{1 + t^2}\,dt$$

is constant on the interval $(0, +\infty)$.

(b) Determine the constant value of the function in part (a) and then interpret (a) as an identity involving the inverse tangent function.

58. What is the natural domain of the function

$$F(x) = \int_1^x \frac{1}{t^2 - 9}\,dt?$$

Explain your reasoning.

59. In each part, determine the values of x for which $F(x)$ is positive, negative, or zero without performing the integration; explain your reasoning.

(a) $F(x) = \int_1^x \frac{t^4}{t^2+3}\,dt$ (b) $F(x) = \int_{-1}^x \sqrt{4-t^2}\,dt$

C **60.** Use a CAS to approximate the largest and smallest values of the integral

$$\int_{-1}^x \frac{t}{\sqrt{2+t^3}}\,dt$$

for $1 \le x \le 3$.

61. Find all values of x^* in the stated interval that are guaranteed to exist by the Mean-Value Theorem for Integrals, and explain what these numbers represent.

(a) $f(x) = \sqrt{x}$; $[0, 3]$ (b) $f(x) = 1/x$; $[1, e]$

62. A 10-gram tumor is discovered in a laboratory rat on March 1. The tumor is growing at a rate of $r(t) = t/7$ grams per week, where t denotes the number of weeks since March 1. What will be the mass of the tumor on June 7?

63. Use the graph of f shown in the accompanying figure to find the average value of f on the interval $[0, 10]$.

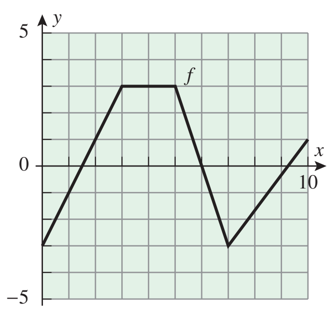


Figure Ex-63

64. Find the average value of $f(x) = e^x + e^{-x}$ over the interval $\left[\ln \frac{1}{2}, \ln 2\right]$.

65. Derive the formulas for the position and velocity functions of a particle that moves with constant acceleration along a coordinate line.

66. The velocity of a particle moving along an s-axis is measured at 5 s intervals for 40 s, and the velocity function is modeled by a smooth curve. (The curve and the data points are shown in the accompanying figure.) Use this model in each part.

(a) Does the particle have constant acceleration? Explain your reasoning.

(b) Is there any 15 s time interval during which the acceleration is constant? Explain your reasoning.

(c) Estimate the distance traveled by the particle from time $t = 0$ to time $t = 40$.

(d) Estimate the average velocity of the particle over the 40 s time period.

(e) Is the particle ever slowing down during the 40 s time period? Explain your reasoning.

(f) Is there sufficient information for you to determine the s-coordinate of the particle at time $t = 10$? If so, find it. If not, explain what additional information you need.

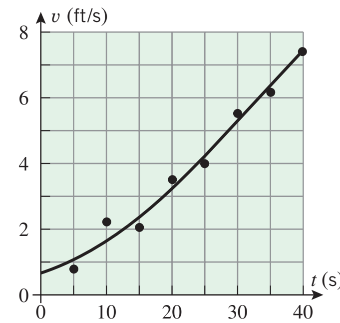


Figure Ex-66

67–70 A particle moves along an s-axis. Use the given information to find the position function of the particle.

67. $v(t) = t^3 - 2t^2 + 1$; $s(0) = 1$

68. $a(t) = 4\cos 2t$; $v(0) = -1, s(0) = -3$

69. $v(t) = 2t - 3$; $s(1) = 5$

70. $a(t) = \cos t - 2t$; $v(0) = 0, s(0) = 0$

71–74 A particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

71. $v(t) = 2t - 4$; $0 \le t \le 6$

72. $v(t) = |t - 3|$; $0 \le t \le 5$

73. $v(t) = \frac{1}{2} - \frac{1}{t^2}$; $1 \le t \le 3$

74. $v(t) = \frac{3}{\sqrt{t}}$; $4 \le t \le 9$

75–76 A particle moves with acceleration $a(t)$ m/s^2 along an s-axis and has velocity v_0 m/s at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

75. $a(t) = -2$; $v_0 = 3$; $1 \le t \le 4$

76. $a(t) = \frac{1}{\sqrt{5t+1}}$; $v_0 = 2$; $0 \le t \le 3$

77. A car traveling 60 mi/h (= 88 ft/s) along a straight road decelerates at a constant rate of 10 ft/s^2.

(a) How long will it take until the speed is 45 mi/h?

(b) How far will the car travel before coming to a stop?

78. Suppose that the velocity function of a particle moving along an s-axis is $v(t) = 20t^2 - 100t + 50$ ft/s and that the particle is at the origin at time $t = 0$. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 6 s of motion.

79. A ball is thrown vertically upward from a height of s_0 ft with an initial velocity of v_0 ft/s. If the ball is caught at height s_0, determine its average speed through the air using the free-fall model.

80. A rock, dropped from an unknown height, strikes the ground with a speed of 24 m/s. Find the height from which the rock was dropped.

81–88 Evaluate the integrals by making an appropriate substitution. ■

81. $\int_0^1 (2x+1)^4\, dx$

82. $\int_{-5}^0 x\sqrt{4-x}\, dx$

83. $\int_0^1 \frac{dx}{\sqrt{3x+1}}$

84. $\int_0^{\sqrt{\pi}} x \sin x^2\, dx$

85. $\int_0^1 \sin^2(\pi x)\cos(\pi x)\, dx$

86. $\int_e^{e^2} \frac{dx}{x \ln x}$

87. $\int_0^1 \frac{dx}{\sqrt{e^x}}$

88. $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\, dx$

89. Evaluate the limits.

(a) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{2x}$ (b) $\lim_{x\to+\infty}\left(1+\frac{1}{3x}\right)^{x}$

90. Find a function f and a number a such that

$$2 + \int_a^x f(t)\, dt = e^{3x}$$

CHAPTER 5 MAKING CONNECTIONS

1. Consider a Riemann sum

$$\sum_{k=1}^{n} 2x_k^* \Delta x_k$$

for the integral of $f(x) = 2x$ over an interval $[a, b]$.

(a) Show that if x_k^* is the midpoint of the kth subinterval, the Riemann sum is a telescoping sum. (See Exercises 57–60 of Section 5.4 for other examples of telescoping sums.)

(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to evaluate the definite integral of $f(x) = 2x$ over $[a, b]$.

2. The function $f(x) = \sqrt{x}$ is continuous on $[0, 4]$ and therefore integrable on this interval. Evaluate

$$\int_0^4 \sqrt{x}\, dx$$

by using Definition 5.5.1. Use subintervals of unequal length given by the partition

$$0 < 4(1)^2/n^2 < 4(2)^2/n^2 < \cdots < 4(n-1)^2/n^2 < 4$$

and let x_k^* be the right endpoint of the kth subinterval.

3. Make appropriate modifications and repeat Exercise 2 for

$$\int_0^8 \sqrt[3]{x}\, dx$$

4. Given a continuous function f and a positive real number m, let g denote the function defined by the composition $g(x) = f(mx)$.

(a) Suppose that

$$\sum_{k=1}^{n} g(x_k^*)\Delta x_k$$

is any Riemann sum for the integral of g over $[0, 1]$. Use the correspondence $u_k = mx_k$, $u_k^* = mx_k^*$ to create a Riemann sum for the integral of f over $[0, m]$. How are the values of the two Riemann sums related?

(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to find an equation that relates the integral of g over $[0, 1]$ with the integral of f over $[0, m]$.

(c) How is your answer to part (b) related to Theorem 5.9.1?

5. Given a continuous function f, let g denote the function defined by $g(x) = 2xf(x^2)$.

(a) Suppose that

$$\sum_{k=1}^{n} g(x_k^*)\Delta x_k$$

is any Riemann sum for the integral of g over $[2, 3]$, with $x_k^* = (x_k + x_{k-1})/2$ the midpoint of the kth subinterval. Use the correspondence $u_k = x_k^2$, $u_k^* = (x_k^*)^2$ to create a Riemann sum for the integral of f over $[4, 9]$. How are the values of the two Riemann sums related?

(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to find an equation that relates the integral of g over $[2, 3]$ with the integral of f over $[4, 9]$.

(c) How is your answer to part (b) related to Theorem 5.9.1?

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The floating roof on the Stade de France sports complex is an ellipse. Finding the arc length of an ellipse involves numerical integration techniques introduced in this chapter.

7 PRINCIPLES OF INTEGRAL EVALUATION

In earlier chapters we obtained many basic integration formulas as an immediate consequence of the corresponding differentiation formulas. For example, knowing that the derivative of sin x is cos x enabled us to deduce that the integral of cos x is sin x. Subsequently, we expanded our integration repertoire by introducing the method of u-substitution. That method enabled us to integrate many functions by transforming the integrand of an unfamiliar integral into a familiar form. However, u-substitution alone is not adequate to handle the wide variety of integrals that arise in applications, so additional integration techniques are still needed. In this chapter we will discuss some of those techniques, and we will provide a more systematic procedure for attacking unfamiliar integrals. We will talk more about numerical approximations of definite integrals, and we will explore the idea of integrating over infinite intervals.

7.1 AN OVERVIEW OF INTEGRATION METHODS

In this section we will give a brief overview of methods for evaluating integrals, and we will review the integration formulas that were discussed in earlier sections.

METHODS FOR APPROACHING INTEGRATION PROBLEMS

There are three basic approaches for evaluating unfamiliar integrals:

- **Technology**—CAS programs such as *Mathematica*, *Maple*, and the open source program *Sage* are capable of evaluating extremely complicated integrals, and such programs are increasingly available for both computers and handheld calculators.
- **Tables**—Prior to the development of CAS programs, scientists relied heavily on tables to evaluate difficult integrals arising in applications. Such tables were compiled over many years, incorporating the skills and experience of many people. One such table appears in the endpapers of this text, but more comprehensive tables appear in various reference books such as the *CRC Standard Mathematical Tables and Formulae*, CRC Press, Inc., 2002.
- **Transformation Methods**—Transformation methods are methods for converting unfamiliar integrals into familiar integrals. These include u-substitution, algebraic manipulation of the integrand, and other methods that we will discuss in this chapter.

None of the three methods is perfect; for example, CAS programs often encounter integrals that they cannot evaluate and they sometimes produce answers that are unnecessarily complicated, tables are not exhaustive and may not include a particular integral of interest, and transformation methods rely on human ingenuity that may prove to be inadequate in difficult problems.

In this chapter we will focus on transformation methods and tables, so it will *not be necessary* to have a CAS such as *Mathematica*, *Maple*, or *Sage*. However, if you have a CAS, then you can use it to confirm the results in the examples, and there are exercises that are designed to be solved with a CAS. If you have a CAS, keep in mind that many of the algorithms that it uses are based on the methods we will discuss here, so an understanding of these methods will help you to use your technology in a more informed way.

A REVIEW OF FAMILIAR INTEGRATION FORMULAS

The following is a list of basic integrals that we have encountered thus far:

CONSTANTS, POWERS, EXPONENTIALS

1. $\displaystyle\int du = u + C$

2. $\displaystyle\int a\,du = a\int du = au + C$

3. $\displaystyle\int u^r\,du = \frac{u^{r+1}}{r+1} + C,\ r \neq -1$

4. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

5. $\displaystyle\int e^u\,du = e^u + C$

6. $\displaystyle\int b^u\,du = \frac{b^u}{\ln b} + C,\ b > 0, b \neq 1$

TRIGONOMETRIC FUNCTIONS

7. $\displaystyle\int \sin u\,du = -\cos u + C$

8. $\displaystyle\int \cos u\,du = \sin u + C$

9. $\displaystyle\int \sec^2 u\,du = \tan u + C$

10. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

11. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

12. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

13. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$

14. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

HYPERBOLIC FUNCTIONS

15. $\displaystyle\int \sinh u\,du = \cosh u + C$

16. $\displaystyle\int \cosh u\,du = \sinh u + C$

17. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

18. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

19. $\displaystyle\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

20. $\displaystyle\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

ALGEBRAIC FUNCTIONS ($a > 0$)

21. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \qquad (|u| < a)$

22. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

23. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad (0 < a < |u|)$

24. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{u^2+a^2})+C$

25. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right|+C \qquad (0<a<|u|)$

26. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$

27. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C \qquad (0<|u|<a)$

28. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right|+C$

REMARK Formula 25 is a generalization of a result in Theorem 6.9.6. Readers who did not cover Section 6.9 can ignore Formulas 24–28 for now, since we will develop other methods for obtaining them in this chapter.

✔QUICK CHECK EXERCISES 7.1 *(See page 491 for answers.)*

1. Use algebraic manipulation and (if necessary) u-substitution to integrate the function.

(a) $\displaystyle\int \frac{x+1}{x}\,dx =$ ________

(b) $\displaystyle\int \frac{x+2}{x+1}\,dx =$ ________

(c) $\displaystyle\int \frac{2x+1}{x^2+1}\,dx =$ ________

(d) $\displaystyle\int xe^{3\ln x}\,dx =$ ________

2. Use trigonometric identities and (if necessary) u-substitution to integrate the function.

(a) $\displaystyle\int \frac{1}{\csc x}\,dx =$ ________

(b) $\displaystyle\int \frac{1}{\cos^2 x}\,dx =$ ________

(c) $\displaystyle\int (\cot^2 x+1)\,dx =$ ________

(d) $\displaystyle\int \frac{1}{\sec x+\tan x}\,dx =$ ________

3. Integrate the function.

(a) $\displaystyle\int \sqrt{x-1}\,dx =$ ________

(b) $\displaystyle\int e^{2x+1}\,dx =$ ________

(c) $\displaystyle\int (\sin^3 x\cos x+\sin x\cos^3 x)\,dx =$ ________

(d) $\displaystyle\int \frac{1}{(e^x+e^{-x})^2}\,dx =$ ________

EXERCISE SET 7.1

1–30 Evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section. ■

1. $\displaystyle\int (4-2x)^3\,dx$

2. $\displaystyle\int 3\sqrt{4+2x}\,dx$

3. $\displaystyle\int x\sec^2(x^2)\,dx$

4. $\displaystyle\int 4x\tan(x^2)\,dx$

5. $\displaystyle\int \frac{\sin 3x}{2+\cos 3x}\,dx$

6. $\displaystyle\int \frac{1}{9+4x^2}\,dx$

7. $\displaystyle\int e^x\sinh(e^x)\,dx$

8. $\displaystyle\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

9. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

10. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

11. $\displaystyle\int \cos^5 5x\sin 5x\,dx$

12. $\displaystyle\int \frac{\cos x}{\sin x\sqrt{\sin^2 x+1}}\,dx$

13. $\displaystyle\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$

14. $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

15. $\displaystyle\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$

16. $\displaystyle\int (x+1)\cot(x^2+2x)\,dx$

17. $\displaystyle\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$

18. $\displaystyle\int \frac{dx}{x(\ln x)^2}$

19. $\displaystyle\int \frac{dx}{\sqrt{x}\,3^{\sqrt{x}}}$

20. $\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta\, d\theta$

21. $\int \frac{\operatorname{csch}^2(2/x)}{x^2}\,dx$

22. $\int \frac{dx}{\sqrt{x^2-4}}$

23. $\int \frac{e^{-x}}{4-e^{-2x}}\,dx$

24. $\int \frac{\cos(\ln x)}{x}\,dx$

25. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

26. $\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\,dx$

27. $\int \frac{x}{\csc(x^2)}\,dx$

28. $\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

29. $\int x4^{-x^2}\,dx$

30. $\int 2^{\pi x}\,dx$

FOCUS ON CONCEPTS

31. (a) Evaluate the integral $\int \sin x\cos x\,dx$ using the substitution $u = \sin x$.
(b) Evaluate the integral $\int \sin x\cos x\,dx$ using the identity $\sin 2x = 2\sin x\cos x$.
(c) Explain why your answers to parts (a) and (b) are consistent.

32. (a) Derive the identity

$$\frac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$$

(b) Use the result in part (a) to evaluate $\int \operatorname{sech} x\,dx$.
(c) Derive the identity

$$\operatorname{sech} x = \frac{2e^x}{e^{2x}+1}$$

(d) Use the result in part (c) to evaluate $\int \operatorname{sech} x\,dx$.
(e) Explain why your answers to parts (b) and (d) are consistent.

33. (a) Derive the identity

$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x\cos x}$$

(b) Use the identity $\sin 2x = 2\sin x\cos x$ along with the result in part (a) to evaluate $\int \csc x\,dx$.
(c) Use the identity $\cos x = \sin[(\pi/2)-x]$ along with your answer to part (a) to evaluate $\int \sec x\,dx$.

✔ QUICK CHECK ANSWERS 7.1

1. (a) $x + \ln|x| + C$ (b) $x + \ln|x+1| + C$ (c) $\ln(x^2+1) + \tan^{-1}x + C$ (d) $\frac{x^5}{5} + C$ 2. (a) $-\cos x + C$ (b) $\tan x + C$ (c) $-\cot x + C$ (d) $\ln(1+\sin x) + C$ 3. (a) $\frac{2}{3}(x-1)^{3/2} + C$ (b) $\frac{1}{2}e^{2x+1} + C$ (c) $\frac{1}{2}\sin^2 x + C$ (d) $\frac{1}{4}\tanh x + C$

7.2 INTEGRATION BY PARTS

In this section we will discuss an integration technique that is essentially an antiderivative formulation of the formula for differentiating a product of two functions.

THE PRODUCT RULE AND INTEGRATION BY PARTS

Our primary goal in this section is to develop a general method for attacking integrals of the form

$$\int f(x)g(x)\,dx$$

As a first step, let $G(x)$ be *any* antiderivative of $g(x)$. In this case $G'(x) = g(x)$, so the product rule for differentiating $f(x)G(x)$ can be expressed as

$$\frac{d}{dx}[f(x)G(x)] = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x) \tag{1}$$

This implies that $f(x)G(x)$ is an antiderivative of the function on the right side of (1), so we can express (1) in integral form as

$$\int [f(x)g(x) + f'(x)G(x)]\,dx = f(x)G(x)$$

or, equivalently, as

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx \tag{2}$$

This formula allows us to integrate $f(x)g(x)$ by integrating $f'(x)G(x)$ instead, and in many cases the net effect is to replace a difficult integration with an easier one. The application of this formula is called ***integration by parts***.

In practice, we usually rewrite (2) by letting

$$u = f(x), \quad du = f'(x)\,dx$$
$$v = G(x), \quad dv = G'(x)\,dx = g(x)\,dx$$

This yields the following alternative form for (2):

$$\int u\,dv = uv - \int v\,du \tag{3}$$

Note that in Example 1 we omitted the constant of integration in calculating v from dv. Had we included a constant of integration, it would have eventually dropped out. This is always the case in integration by parts [Exercise 68(b)], so it is common to omit the constant at this stage of the computation. However, there are certain cases in which making a clever choice of a constant of integration to include with v can simplify the computation of $\int v\,du$ (Exercises 69–71).

Example 1 Use integration by parts to evaluate $\int x\cos x\,dx$.

Solution. We will apply Formula (3). The first step is to make a choice for u and dv to put the given integral in the form $\int u\,dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x\,dx$$

(Other possibilities will be considered later.) The second step is to compute du from u and v from dv. This yields

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x\,dx = \sin x$$

The third step is to apply Formula (3). This yields

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$
$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$

GUIDELINES FOR INTEGRATION BY PARTS

The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original. In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice. A strategy that often works is to choose u and dv so that u becomes "simpler" when differentiated, while leaving a dv that can be readily integrated to obtain v. Thus, for the integral $\int x\cos x\,dx$ in Example 1, both goals were achieved by letting $u = x$ and $dv = \cos x\,dx$. In contrast, $u = \cos x$ would not have been a good first choice in that example, since $du/dx = -\sin x$ is no simpler than u. Indeed, had we chosen

$$u = \cos x \qquad dv = x\,dx$$
$$du = -\sin x\,dx \qquad v = \int x\,dx = \frac{x^2}{2}$$

then we would have obtained

$$\int x\cos x\,dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx = \frac{x^2}{2}\cos x + \frac{1}{2}\int x^2\sin x\,dx$$

For this choice of u and dv, the new integral is actually more complicated than the original.

The LIATE method is discussed in the article "A Technique for Integration by Parts," *American Mathematical Monthly*, Vol. 90, 1983, pp. 210–211, by Herbert Kasube.

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

Note, for example, that the integrand in Example 1 consists of the product of the *algebraic* function x and the *trigonometric* function $\cos x$. Thus, the LIATE method suggests that we should let $u = x$ and $dv = \cos x\,dx$, which proved to be a successful choice.

Example 2 Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C \blacktriangleleft$$

Example 3 Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C \blacktriangleleft$$

REPEATED INTEGRATION BY PARTS

It is sometimes necessary to use integration by parts more than once in the same problem.

Example 4 Evaluate $\int x^2e^{-x}\,dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that from (3)

$$\int x^2e^{-x}\,dx = \int u\,dv = uv - \int v\,du$$
$$= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx$$
$$= -x^2e^{-x} + 2\int xe^{-x}\,dx \tag{4}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C$$
$$= -(x^2 + 2x + 2)e^{-x} + C \;◂$$

The LIATE method suggests that integrals of the form

$$\int e^{ax}\sin bx\,dx \quad \text{and} \quad \int e^{ax}\cos bx\,dx$$

can be evaluated by letting $u = \sin bx$ or $u = \cos bx$ and $dv = e^{ax}\,dx$. However, this will require a technique that deserves special attention.

▶ Example 5 Evaluate $\int e^x \cos x\,dx$.

Solution. Let

$$u = \cos x, \quad dv = e^x\,dx, \quad du = -\sin x\,dx, \quad v = \int e^x\,dx = e^x$$

Thus,

$$\int e^x\cos x\,dx = \int u\,dv = uv - \int v\,du = e^x\cos x + \int e^x\sin x\,dx \tag{5}$$

Since the integral $\int e^x\sin x\,dx$ is similar in form to the original integral $\int e^x\cos x\,dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x\,dx, \quad du = \cos x\,dx, \quad v = \int e^x\,dx = e^x$$

Thus,

$$\int e^x\sin x\,dx = \int u\,dv = uv - \int v\,du = e^x\sin x - \int e^x\cos x\,dx$$

Together with Equation (5) this yields

$$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx \tag{6}$$

which is an equation we can solve for the unknown integral. We obtain

$$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x\,dx = \tfrac{1}{2}e^x \cos x + \tfrac{1}{2}e^x \sin x + C \blacktriangleleft$$

A TABULAR METHOD FOR REPEATED INTEGRATION BY PARTS

Integrals of the form

$$\int p(x)f(x)\,dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts in which u is taken to be $p(x)$ or one of its derivatives at each stage. Since du is computed by differentiating u, the repeated differentiation of $p(x)$ will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called ***tabular integration by parts***.

More information on tabular integration by parts can be found in the articles "Tabular Integration by Parts," *College Mathematics Journal*, Vol. 21, 1990, pp. 307–311, by David Horowitz and "More on Tabular Integration by Parts," *College Mathematics Journal*, Vol. 22, 1991, pp. 407–410, by Leonard Gillman.

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x)\cos x\,dx$.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\boxed{\begin{aligned}\int (x^2 - x)\cos x\,dx &= (x^2 - x)\sin x + (2x - 1)\cos x - 2\sin x + C\\ &= (x^2 - x - 2)\sin x + (2x - 1)\cos x + C\end{aligned}}$$

▶ **Figure 7.2.1**

▶ **Example 6** In Example 11 of Section 5.3 we evaluated $\int x^2\sqrt{x-1}\,dx$ using u-substitution. Evaluate this integral using tabular integration by parts.

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

The result obtained in Example 6 looks quite different from that obtained in Example 11 of Section 5.3. Show that the two answers are equivalent.

Thus, it follows that

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C \blacktriangleleft$$

INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

For definite integrals the formula corresponding to (3) is

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du \tag{7}$$

REMARK It is important to keep in mind that the variables u and v in this formula are functions of x and that the limits of integration in (7) are limits on the variable x. Sometimes it is helpful to emphasize this by writing (7) as

$$\int_{x=a}^b u\,dv = uv\Big]_{x=a}^b - \int_{x=a}^b v\,du \tag{8}$$

The next example illustrates how integration by parts can be used to integrate the inverse trigonometric functions.

Example 7 Evaluate $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

Solution. Let

$$u = \tan^{-1} x, \quad dv = dx, \quad du = \frac{1}{1+x^2}\,dx, \quad v = x$$

Thus,

$$\int_0^1 \tan^{-1} x\,dx = \int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du$$

The limits of integration refer to x; that is, $x = 0$ and $x = 1$.

$$= x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

But

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{1}{2}\ln 2$$

so

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big]_0^1 - \frac{1}{2}\ln 2 = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2} \blacktriangleleft$$

REDUCTION FORMULAS

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{9}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{10}$$

To illustrate how such formulas can be obtained, let us derive (10). We begin by writing $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$ and letting

$$\begin{aligned} u &= \cos^{n-1} x & dv &= \cos x\,dx \\ du &= (n-1)\cos^{n-2} x(-\sin x)\,dx & v &= \sin x \\ &= -(n-1)\cos^{n-2} x\sin x\,dx \end{aligned}$$

so that

$$\begin{aligned} \int \cos^n x\,dx &= \int \cos^{n-1} x\cos x\,dx = \int u\,dv = uv - \int v\,du \\ &= \cos^{n-1} x\sin x + (n-1)\int \sin^2 x\cos^{n-2} x\,dx \\ &= \cos^{n-1} x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \end{aligned}$$

Moving the last term on the right to the left side yields

$$n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$$

from which (10) follows. The derivation of reduction formula (9) is similar (Exercise 63).

Reduction formulas (9) and (10) reduce the exponent of sine (or cosine) by 2. Thus, if the formulas are applied repeatedly, the exponent can eventually be reduced to 0 if n is even or 1 if n is odd, at which point the integration can be completed. We will discuss this method in more detail in the next section, but for now, here is an example that illustrates how reduction formulas work.

▶ **Example 8** Evaluate $\int \cos^4 x\,dx$.

Solution. From (10) with $n = 4$

$$\begin{aligned} \int \cos^4 x\,dx &= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{4}\int \cos^2 x\,dx \qquad \boxed{\text{Now apply (10) with } n = 2.} \\ &= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{4}\left(\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx\right) \\ &= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{8}\cos x\sin x + \tfrac{3}{8}x + C \end{aligned}$$ ◀

✔ QUICK CHECK EXERCISES 7.2 (See page 500 for answers.)

1. (a) If $G'(x) = g(x)$, then

$$\int f(x)g(x)\,dx = f(x)G(x) - \_\_\_\_\_\_$$

(b) If $u = f(x)$ and $v = G(x)$, then the formula in part (a) can be written in the form $\int u\,dv = \_\_\_\_\_\_$.

2. Find an appropriate choice of u and dv for integration by parts of each integral. Do not evaluate the integral.

(a) $\int x\ln x\,dx$; $u = \_\_\_\_\_\_$, $dv = \_\_\_\_\_\_$

(b) $\int (x-2)\sin x\,dx$; $u = \_\_\_\_\_\_$, $dv = \_\_\_\_\_\_$

(c) $\int \sin^{-1} x\,dx$; $u = \_\_\_\_\_\_$, $dv = \_\_\_\_\_\_$

(d) $\int \dfrac{x}{\sqrt{x-1}}dx$; $u = \_\_\_\_\_\_$, $dv = \_\_\_\_\_\_$

3. Use integration by parts to evaluate the integral.

(a) $\int xe^{2x}\,dx$ (b) $\int \ln(x-1)\,dx$

(c) $\int_0^{\pi/6} x\sin 3x\,dx$

4. Use a reduction formula to evaluate $\int \sin^3 x\,dx$.

EXERCISE SET 7.2

1–38 Evaluate the integral. ■

1. $\int xe^{-2x}\,dx$
2. $\int xe^{3x}\,dx$
3. $\int x^2e^x\,dx$
4. $\int x^2e^{-2x}\,dx$
5. $\int x\sin 3x\,dx$
6. $\int x\cos 2x\,dx$
7. $\int x^2\cos x\,dx$
8. $\int x^2\sin x\,dx$
9. $\int x\ln x\,dx$
10. $\int \sqrt{x}\ln x\,dx$
11. $\int (\ln x)^2\,dx$
12. $\int \dfrac{\ln x}{\sqrt{x}}\,dx$
13. $\int \ln(3x-2)\,dx$
14. $\int \ln(x^2+4)\,dx$
15. $\int \sin^{-1} x\,dx$
16. $\int \cos^{-1}(2x)\,dx$
17. $\int \tan^{-1}(3x)\,dx$
18. $\int x\tan^{-1} x\,dx$
19. $\int e^x\sin x\,dx$
20. $\int e^{3x}\cos 2x\,dx$
21. $\int \sin(\ln x)\,dx$
22. $\int \cos(\ln x)\,dx$
23. $\int x\sec^2 x\,dx$
24. $\int x\tan^2 x\,dx$
25. $\int x^3e^{x^2}\,dx$
26. $\int \dfrac{xe^x}{(x+1)^2}\,dx$
27. $\int_0^2 xe^{2x}\,dx$
28. $\int_0^1 xe^{-5x}\,dx$
29. $\int_1^e x^2\ln x\,dx$
30. $\int_{\sqrt{e}}^e \dfrac{\ln x}{x^2}\,dx$
31. $\int_{-1}^1 \ln(x+2)\,dx$
32. $\int_0^{\sqrt{3}/2} \sin^{-1} x\,dx$
33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
34. $\int_1^2 x\sec^{-1} x\,dx$
35. $\int_0^{\pi} x\sin 2x\,dx$
36. $\int_0^{\pi} (x + x\cos x)\,dx$
37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$
38. $\int_0^2 \ln(x^2+1)\,dx$

39–42 True–False Determine whether the statement is true or false. Explain your answer. ■

39. The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original.

40. Applying the LIATE strategy to evaluate $\int x^3\ln x\,dx$, we should choose $u = x^3$ and $dv = \ln x\,dx$.

41. To evaluate $\int \ln e^x\,dx$ using integration by parts, choose $dv = e^x\,dx$.

42. Tabular integration by parts is useful for integrals of the form $\int p(x)f(x)\,dx$, where $p(x)$ is a polynomial and $f(x)$ can be repeatedly integrated.

43–44 Evaluate the integral by making a u-substitution and then integrating by parts. ■

43. $\int e^{\sqrt{x}}\,dx$
44. $\int \cos\sqrt{x}\,dx$

45. Prove that tabular integration by parts gives the correct answer for

$$\int p(x)f(x)\,dx$$

where $p(x)$ is any *quadratic polynomial* and $f(x)$ is any function that can be repeatedly integrated.

46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate $\int e^x\cos x\,dx$ in

two ways: first by repeated differentiation of $\cos x$ (compare Example 5), and then by repeated differentiation of e^x.

47–52 Evaluate the integral using tabular integration by parts. ■

47. $\int (3x^2 - x + 2)e^{-x}\,dx$ **48.** $\int (x^2 + x + 1)\sin x\,dx$

49. $\int 4x^4 \sin 2x\,dx$ **50.** $\int x^3\sqrt{2x+1}\,dx$

51. $\int e^{ax}\sin bx\,dx$ **52.** $\int e^{-3\theta}\sin 5\theta\,d\theta$

53. Consider the integral $\int \sin x \cos x\,dx$.
(a) Evaluate the integral two ways: first using integration by parts, and then using the substitution $u = \sin x$.
(b) Show that the results of part (a) are equivalent.
(c) Which of the two methods do you prefer? Discuss the reasons for your preference.

54. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

using
(a) integration by parts
(b) the substitution $u = \sqrt{x^2+1}$.

55. (a) Find the area of the region enclosed by $y = \ln x$, the line $x = e$, and the x-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

56. Find the area of the region between $y = x\sin x$ and $y = x$ for $0 \le x \le \pi/2$.

57. Find the volume of the solid generated when the region between $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

58. Find the volume of the solid generated when the region enclosed between $y = \cos x$ and $y = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

59. A particle moving along the x-axis has velocity function $v(t) = t^3 \sin t$. How far does the particle travel from time $t = 0$ to $t = \pi$?

60. The study of sawtooth waves in electrical engineering leads to integrals of the form

$$\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)\,dt$$

where k is an integer and ω is a nonzero constant. Evaluate the integral.

61. Use reduction formula (9) to evaluate
(a) $\int \sin^4 x\,dx$ (b) $\int_0^{\pi/2} \sin^5 x\,dx$.

62. Use reduction formula (10) to evaluate
(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

63. Derive reduction formula (9).

64. In each part, use integration by parts or other methods to derive the reduction formula.
(a) $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x\tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$
(b) $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$
(c) $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

65–66 Use the reduction formulas in Exercise 64 to evaluate the integrals. ■

65. (a) $\int \tan^4 x\,dx$ (b) $\int \sec^4 x\,dx$ (c) $\int x^3 e^x\,dx$

66. (a) $\int x^2 e^{3x}\,dx$ (b) $\int_0^1 xe^{-\sqrt{x}}\,dx$
[*Hint:* First make a substitution.]

67. Let f be a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^{1} xf''(x)\,dx = f'(1) + f'(-1) - f(1) + f(-1)$$

FOCUS ON CONCEPTS

68. (a) In the integral $\int x\cos x\,dx$, let

$$u = x, \quad dv = \cos x\,dx,$$
$$du = dx, \quad v = \sin x + C_1$$

Show that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1.
(b) Show that in general

$$uv - \int v\,du = u(v + C_1) - \int (v + C_1)\,du$$

thereby justifying the omission of the constant of integration when calculating v in integration by parts.

69. Evaluate $\int \ln(x+1)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = 1$ when going from dv to v.

70. Evaluate $\int \ln(3x-2)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = -\frac{2}{3}$ when going from dv to v. Compare your solution with your answer to Exercise 13.

71. Evaluate $\int x\tan^{-1} x\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.

72. What equation results if integration by parts is applied to the integral

$$\int \frac{1}{x\ln x}\,dx$$

with the choices

$$u = \frac{1}{\ln x} \quad \text{and} \quad dv = \frac{1}{x}\,dx?$$

In what sense is this equation true? In what sense is it false?

73. Writing Explain how the product rule for derivatives and the technique of integration by parts are related.

74. Writing For what sort of problems are the integration techniques of substitution and integration by parts "competing" techniques? Describe situations, with examples, where each of these techniques would be preferred over the other.

✔ QUICK CHECK ANSWERS 7.2

1. (a) $\int f'(x)G(x)\,dx$ (b) $uv - \int v\,du$ **2.** (a) $\ln x;\ x\,dx$ (b) $x-2;\ \sin x\,dx$ (c) $\sin^{-1} x;\ dx$ (d) $x;\ \dfrac{1}{\sqrt{x-1}}\,dx$
3. (a) $\left(\dfrac{x}{2}-\dfrac{1}{4}\right)e^{2x}+C$ (b) $(x-1)\ln(x-1)-x+C$ (c) $\frac{1}{9}$ **4.** $-\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C$

7.3 INTEGRATING TRIGONOMETRIC FUNCTIONS

In the last section we derived reduction formulas for integrating positive integer powers of sine, cosine, tangent, and secant. In this section we will show how to work with those reduction formulas, and we will discuss methods for integrating other kinds of integrals that involve trigonometric functions.

INTEGRATING POWERS OF SINE AND COSINE

We begin by recalling two reduction formulas from the preceding section.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \tag{1}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx \tag{2}$$

In the case where $n=2$, these formulas yield

$$\int \sin^2 x\,dx = -\tfrac{1}{2}\sin x\cos x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x + C \tag{3}$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x\cos x + C \tag{4}$$

Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \text{and} \quad \cos^2 x = \tfrac{1}{2}(1+\cos 2x) \tag{5–6}$$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2\sin^2 x \quad \text{and} \quad \cos 2x = 2\cos^2 x - 1$$

These identities yield

$$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1-\cos 2x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C \tag{7}$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C \tag{8}$$

Observe that the antiderivatives in Formulas (3) and (4) involve both sines and cosines, whereas those in (7) and (8) involve sines alone. However, the apparent discrepancy is easy to resolve by using the identity

$$\sin 2x = 2 \sin x \cos x$$

to rewrite (7) and (8) in forms (3) and (4), or conversely.

In the case where $n = 3$, the reduction formulas for integrating $\sin^3 x$ and $\cos^3 x$ yield

$$\int \sin^3 x\, dx = -\tfrac{1}{3} \sin^2 x \cos x + \tfrac{2}{3} \int \sin x\, dx = -\tfrac{1}{3} \sin^2 x \cos x - \tfrac{2}{3} \cos x + C \qquad (9)$$

$$\int \cos^3 x\, dx = \tfrac{1}{3} \cos^2 x \sin x + \tfrac{2}{3} \int \cos x\, dx = \tfrac{1}{3} \cos^2 x \sin x + \tfrac{2}{3} \sin x + C \qquad (10)$$

If desired, Formula (9) can be expressed in terms of cosines alone by using the identity $\sin^2 x = 1 - \cos^2 x$, and Formula (10) can be expressed in terms of sines alone by using the identity $\cos^2 x = 1 - \sin^2 x$. We leave it for you to do this and confirm that

TECHNOLOGY MASTERY

The *Maple* CAS produces forms (11) and (12) when asked to integrate $\sin^3 x$ and $\cos^3 x$, but *Mathematica* produces

$$\int \sin^3 x\, dx = -\tfrac{3}{4} \cos x + \tfrac{1}{12} \cos 3x + C$$

$$\int \cos^3 x\, dx = \tfrac{3}{4} \sin x + \tfrac{1}{12} \sin 3x + C$$

Use trigonometric identities to reconcile the results of the two programs.

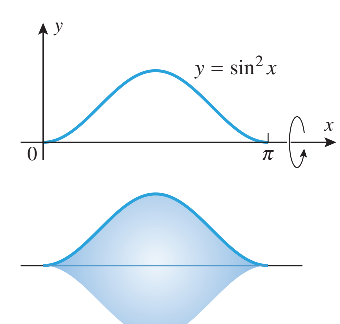
$$\int \sin^3 x\, dx = \tfrac{1}{3} \cos^3 x - \cos x + C \qquad (11)$$

$$\int \cos^3 x\, dx = \sin x - \tfrac{1}{3} \sin^3 x + C \qquad (12)$$

We leave it as an exercise to obtain the following formulas by first applying the reduction formulas, and then using appropriate trigonometric identities.

$$\int \sin^4 x\, dx = \tfrac{3}{8}x - \tfrac{1}{4} \sin 2x + \tfrac{1}{32} \sin 4x + C \qquad (13)$$

$$\int \cos^4 x\, dx = \tfrac{3}{8}x + \tfrac{1}{4} \sin 2x + \tfrac{1}{32} \sin 4x + C \qquad (14)$$



▲ **Figure 7.3.1**

► **Example 1** Find the volume V of the solid that is obtained when the region under the curve $y = \sin^2 x$ over the interval $[0, \pi]$ is revolved about the x-axis (Figure 7.3.1).

Solution. Using the method of disks, Formula (5) of Section 6.2, and Formula (13) above yields

$$V = \int_0^{\pi} \pi \sin^4 x\, dx = \pi \left[\tfrac{3}{8}x - \tfrac{1}{4} \sin 2x + \tfrac{1}{32} \sin 4x \right]_0^{\pi} = \tfrac{3}{8}\pi^2 \; ◄$$

INTEGRATING PRODUCTS OF SINES AND COSINES

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x\, dx$$

can be evaluated by one of the three procedures stated in Table 7.3.1, depending on whether m and n are odd or even.

► **Example 2** Evaluate

$$\text{(a)} \int \sin^4 x \cos^5 x\, dx \qquad \text{(b)} \int \sin^4 x \cos^4 x\, dx$$

Table 7.3.1
INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Solution (a). Since $n = 5$ is odd, we will follow the first procedure in Table 7.3.1:

$$\begin{aligned}
\int \sin^4 x \cos^5 x\, dx &= \int \sin^4 x \cos^4 x \cos x\, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x\, dx \\
&= \int u^4 (1 - u^2)^2\, du \\
&= \int (u^4 - 2u^6 + u^8)\, du \\
&= \tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9 + C \\
&= \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C
\end{aligned}$$

Solution (b). Since $m = n = 4$, both exponents are even, so we will follow the third procedure in Table 7.3.1:

$$\begin{aligned}
\int \sin^4 x \cos^4 x\, dx &= \int (\sin^2 x)^2 (\cos^2 x)^2\, dx \\
&= \int \left(\tfrac{1}{2}[1 - \cos 2x]\right)^2 \left(\tfrac{1}{2}[1 + \cos 2x]\right)^2\, dx \\
&= \tfrac{1}{16}\int (1 - \cos^2 2x)^2\, dx \\
&= \tfrac{1}{16}\int \sin^4 2x\, dx \\
&= \tfrac{1}{32}\int \sin^4 u\, du \\
&= \tfrac{1}{32}\left(\tfrac{3}{8}u - \tfrac{1}{4}\sin 2u + \tfrac{1}{32}\sin 4u\right) + C \\
&= \tfrac{3}{128}x - \tfrac{1}{128}\sin 4x + \tfrac{1}{1024}\sin 8x + C
\end{aligned}$$

Note that this can be obtained more directly from the original integral using the identity $\sin x \cos x = \frac{1}{2}\sin 2x$.

$u = 2x$
$du = 2\,dx$ or $dx = \frac{1}{2}\,du$

Formula (13)

Integrals of the form

$$\int \sin mx \cos nx\,dx, \quad \int \sin mx \sin nx\,dx, \quad \int \cos mx \cos nx\,dx \tag{15}$$

can be found by using the trigonometric identities

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \tag{16}$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{17}$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{18}$$

to express the integrand as a sum or difference of sines and cosines.

Example 3 Evaluate $\int \sin 7x \cos 3x\,dx$.

Solution. Using (16) yields

$$\int \sin 7x\cos 3x\,dx = \frac{1}{2}\int (\sin 4x + \sin 10x)\,dx = -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$

INTEGRATING POWERS OF TANGENT AND SECANT

The procedures for integrating powers of tangent and secant closely parallel those for sine and cosine. The idea is to use the following reduction formulas (which were derived in Exercise 64 of Section 7.2) to reduce the exponent in the integrand until the resulting integral can be evaluated:

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \tag{19}$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx \tag{20}$$

In the case where n is odd, the exponent can be reduced to 1, leaving us with the problem of integrating $\tan x$ or $\sec x$. These integrals are given by

$$\int \tan x\,dx = \ln|\sec x| + C \tag{21}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \tag{22}$$

Formula (21) can be obtained by writing

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\ln|\cos x| + C && \boxed{\begin{aligned}u &= \cos x\\ du &= -\sin x\,dx\end{aligned}} \\ &= \ln|\sec x| + C && \boxed{\ln|\cos x| = -\ln\frac{1}{|\cos x|}}\end{aligned}$$

To obtain Formula (22) we write

$$\begin{aligned}\int \sec x\,dx &= \int \sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x+\tan x}\,dx \\ &= \ln|\sec x + \tan x| + C && \boxed{\begin{aligned}u &= \sec x + \tan x\\ du &= (\sec^2 x + \sec x\tan x)\,dx\end{aligned}}\end{aligned}$$

The following basic integrals occur frequently and are worth noting:

$$\int \tan^2 x\, dx = \tan x - x + C \tag{23}$$

$$\int \sec^2 x\, dx = \tan x + C \tag{24}$$

Formula (24) is already known to us, since the derivative of $\tan x$ is $\sec^2 x$. Formula (23) can be obtained by applying reduction formula (19) with $n = 2$ (verify) or, alternatively, by using the identity

$$1 + \tan^2 x = \sec^2 x$$

to write

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$$

The formulas

$$\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C \tag{25}$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C \tag{26}$$

can be deduced from (21), (22), and reduction formulas (19) and (20) as follows:

$$\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x - \int \tan x\, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x\, dx$$

can be evaluated by one of the three procedures stated in Table 7.3.2, depending on whether m and n are odd or even.

Table 7.3.2

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Example 4 Evaluate

(a) $\int \tan^2 x \sec^4 x\, dx$ (b) $\int \tan^3 x \sec^3 x\, dx$ (c) $\int \tan^2 x \sec x\, dx$

Solution (a). Since $n = 4$ is even, we will follow the first procedure in Table 7.3.2:

$$\begin{aligned}\int \tan^2 x \sec^4 x\, dx &= \int \tan^2 x \sec^2 x \sec^2 x\, dx\\ &= \int \tan^2 x(\tan^2 x + 1)\sec^2 x\, dx\\ &= \int u^2(u^2+1)\, du\\ &= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\tan^5 x + \tfrac{1}{3}\tan^3 x + C\end{aligned}$$

Solution (b). Since $m = 3$ is odd, we will follow the second procedure in Table 7.3.2:

$$\begin{aligned}\int \tan^3 x \sec^3 x\, dx &= \int \tan^2 x \sec^2 x(\sec x \tan x)\, dx\\ &= \int (\sec^2 x - 1)\sec^2 x(\sec x \tan x)\, dx\\ &= \int (u^2-1)u^2\, du\\ &= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

Solution (c). Since $m = 2$ is even and $n = 1$ is odd, we will follow the third procedure in Table 7.3.2:

$$\begin{aligned}\int \tan^2 x \sec x\, dx &= \int (\sec^2 x - 1)\sec x\, dx\\ &= \int \sec^3 x\, dx - \int \sec x\, dx \qquad \boxed{\text{See (26) and (22).}}\\ &= \tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C\\ &= \tfrac{1}{2}\sec x\tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$ ◀

AN ALTERNATIVE METHOD FOR INTEGRATING POWERS OF SINE, COSINE, TANGENT, AND SECANT

The methods in Tables 7.3.1 and 7.3.2 can sometimes be applied if $m = 0$ or $n = 0$ to integrate positive integer powers of sine, cosine, tangent, and secant without reduction formulas. For example, instead of using the reduction formula to integrate $\sin^3 x$, we can apply the second procedure in Table 7.3.1:

$$\begin{aligned}\int \sin^3 x\, dx &= \int (\sin^2 x)\sin x\, dx\\ &= \int (1-\cos^2 x)\sin x\, dx \qquad \boxed{\begin{aligned}u &= \cos x\\ du &= -\sin x\, dx\end{aligned}}\\ &= -\int (1-u^2)\, du\\ &= \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\cos^3 x - \cos x + C\end{aligned}$$

which agrees with (11).

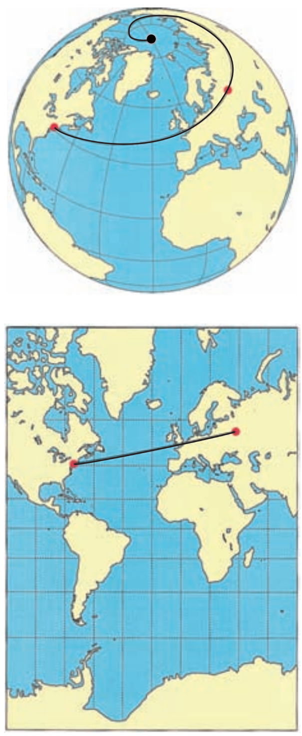
With the aid of the identity

$$1 + \cot^2 x = \csc^2 x$$

the techniques in Table 7.3.2 can be adapted to evaluate integrals of the form

$$\int \cot^m x \csc^n x\, dx$$

It is also possible to derive reduction formulas for powers of cot and csc that are analogous to Formulas (19) and (20).



▲ **Figure 7.3.2** A flight path with constant compass heading from New York City to Moscow follows a spiral toward the North Pole but is a straight line segment on a Mercator projection.

MERCATOR'S MAP OF THE WORLD

The integral of $\sec x$ plays an important role in the design of navigational maps for charting nautical and aeronautical courses. Sailors and pilots usually chart their courses along paths with constant compass headings; for example, the course might be 30° northeast or 135° southeast. Except for courses that are parallel to the equator or run due north or south, a course with constant compass heading spirals around the Earth toward one of the poles (as in the top part of Figure 7.3.2). In 1569 the Flemish mathematician and geographer Gerhard Kramer (1512–1594) (better known by the Latin name Mercator) devised a world map, called the *Mercator projection*, in which spirals of constant compass headings appear as straight lines. This was extremely important because it enabled sailors to determine compass headings between two points by connecting them with a straight line on a map (as in the bottom part of Figure 7.3.2).

If the Earth is assumed to be a sphere of radius 4000 mi, then the lines of latitude at 1° increments are equally spaced about 70 mi apart (why?). However, in the Mercator projection, the lines of latitude become wider apart toward the poles, so that two widely spaced latitude lines near the poles may be actually the same distance apart on the Earth as two closely spaced latitude lines near the equator. It can be proved that on a Mercator map in which the equatorial line has length L, the vertical distance D_β on the map between the equator (latitude 0°) and the line of latitude $\beta°$ is

$$D_\beta = \frac{L}{2\pi}\int_0^{\beta\pi/180} \sec x \, dx \tag{27}$$

✔ QUICK CHECK EXERCISES 7.3 (See page 508 for answers.)

1. Complete each trigonometric identity with an expression involving $\cos 2x$.
   (a) $\sin^2 x =$ ________ (b) $\cos^2 x =$ ________
   (c) $\cos^2 x - \sin^2 x =$ ________
2. Evaluate the integral.
   (a) $\int \sec^2 x \, dx =$ ________
   (b) $\int \tan^2 x \, dx =$ ________
   (c) $\int \sec x \, dx =$ ________
   (d) $\int \tan x \, dx =$ ________
3. Use the indicated substitution to rewrite the integral in terms of u. Do not evaluate the integral.
   (a) $\int \sin^2 x \cos x \, dx;\ u = \sin x$
   (b) $\int \sin^3 x \cos^2 x \, dx;\ u = \cos x$
   (c) $\int \tan^3 x \sec^2 x \, dx;\ u = \tan x$
   (d) $\int \tan^3 x \sec x \, dx;\ u = \sec x$

EXERCISE SET 7.3

1–52 Evaluate the integral.

1. $\int \cos^3 x \sin x \, dx$
2. $\int \sin^5 3x \cos 3x \, dx$
3. $\int \sin^2 5\theta \, d\theta$
4. $\int \cos^2 3x \, dx$
5. $\int \sin^3 a\theta \, d\theta$
6. $\int \cos^3 at \, dt$
7. $\int \sin ax \cos ax \, dx$
8. $\int \sin^3 x \cos^3 x \, dx$
9. $\int \sin^2 t \cos^3 t \, dt$
10. $\int \sin^3 x \cos^2 x \, dx$

11. $\int \sin^2 x \cos^2 x\, dx$

12. $\int \sin^2 x \cos^4 x\, dx$

13. $\int \sin 2x \cos 3x\, dx$

14. $\int \sin 3\theta \cos 2\theta\, d\theta$

15. $\int \sin x \cos(x/2)\, dx$

16. $\int \cos^{1/3} x \sin x\, dx$

17. $\int_0^{\pi/2} \cos^3 x\, dx$

18. $\int_0^{\pi/2} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx$

19. $\int_0^{\pi/3} \sin^4 3x \cos^3 3x\, dx$

20. $\int_{-\pi}^{\pi} \cos^2 5\theta\, d\theta$

21. $\int_0^{\pi/6} \sin 4x \cos 2x\, dx$

22. $\int_0^{2\pi} \sin^2 kx\, dx$

23. $\int \sec^2(2x-1)\, dx$

24. $\int \tan 5x\, dx$

25. $\int e^{-x} \tan(e^{-x})\, dx$

26. $\int \cot 3x\, dx$

27. $\int \sec 4x\, dx$

28. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}\, dx$

29. $\int \tan^2 x \sec^2 x\, dx$

30. $\int \tan^5 x \sec^4 x\, dx$

31. $\int \tan 4x \sec^4 4x\, dx$

32. $\int \tan^4 \theta \sec^4 \theta\, d\theta$

33. $\int \sec^5 x \tan^3 x\, dx$

34. $\int \tan^5 \theta \sec \theta\, d\theta$

35. $\int \tan^4 x \sec x\, dx$

36. $\int \tan^2 x \sec^3 x\, dx$

37. $\int \tan t \sec^3 t\, dt$

38. $\int \tan x \sec^5 x\, dx$

39. $\int \sec^4 x\, dx$

40. $\int \sec^5 x\, dx$

41. $\int \tan^3 4x\, dx$

42. $\int \tan^4 x\, dx$

43. $\int \sqrt{\tan x} \sec^4 x\, dx$

44. $\int \tan x \sec^{3/2} x\, dx$

45. $\int_0^{\pi/8} \tan^2 2x\, dx$

46. $\int_0^{\pi/6} \sec^3 2\theta \tan 2\theta\, d\theta$

47. $\int_0^{\pi/2} \tan^5 \frac{x}{2}\, dx$

48. $\int_0^{1/4} \sec \pi x \tan \pi x\, dx$

49. $\int \cot^3 x \csc^3 x\, dx$

50. $\int \cot^2 3t \sec 3t\, dt$

51. $\int \cot^3 x\, dx$

52. $\int \csc^4 x\, dx$

53–56 True–False Determine whether the statement is true or false. Explain your answer.

53. To evaluate $\int \sin^5 x \cos^8 x\, dx$, use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

54. To evaluate $\int \sin^8 x \cos^5 x\, dx$, use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

55. The trigonometric identity
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$
is often useful for evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$.

56. The integral $\int \tan^4 x \sec^5 x\, dx$ is equivalent to one whose integrand is a polynomial in $\sec x$.

57. Let m, n be distinct nonnegative integers. Use Formulas (16)–(18) to prove:
(a) $\int_0^{2\pi} \sin mx \cos nx\, dx = 0$
(b) $\int_0^{2\pi} \cos mx \cos nx\, dx = 0$
(c) $\int_0^{2\pi} \sin mx \sin nx\, dx = 0.$

58. Evaluate the integrals in Exercise 57 when m and n denote the *same* nonnegative integer.

59. Find the arc length of the curve $y = \ln(\cos x)$ over the interval $[0, \pi/4]$.

60. Find the volume of the solid generated when the region enclosed by $y = \tan x$, $y = 1$, and $x = 0$ is revolved about the x-axis.

61. Find the volume of the solid that results when the region enclosed by $y = \cos x$, $y = \sin x$, $x = 0$, and $x = \pi/4$ is revolved about the x-axis.

62. The region bounded below by the x-axis and above by the portion of $y = \sin x$ from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the volume of the resulting solid.

63. Use Formula (27) to show that if the length of the equatorial line on a Mercator projection is L, then the vertical distance D between the latitude lines at $\alpha°$ and $\beta°$ on the same side of the equator (where $\alpha < \beta$) is
$$D = \frac{L}{2\pi}\ln\left|\frac{\sec\beta° + \tan\beta°}{\sec\alpha° + \tan\alpha°}\right|$$

64. Suppose that the equator has a length of 100 cm on a Mercator projection. In each part, use the result in Exercise 63 to answer the question.
(a) What is the vertical distance on the map between the equator and the line at 25° north latitude?
(b) What is the vertical distance on the map between New Orleans, Louisiana, at 30° north latitude and Winnipeg, Canada, at 50° north latitude?

FOCUS ON CONCEPTS

65. (a) Show that
$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$
(b) Show that the result in part (a) can also be written as
$$\int \csc x\, dx = \ln|\csc x - \cot x| + C$$
and
$$\int \csc x\, dx = \ln\left|\tan \tfrac{1}{2}x\right| + C$$

66. Rewrite $\sin x + \cos x$ in the form
$$A \sin(x + \phi)$$
and use your result together with Exercise 65 to evaluate
$$\int \frac{dx}{\sin x + \cos x}$$

67. Use the method of Exercise 66 to evaluate
$$\int \frac{dx}{a \sin x + b \cos x} \quad (a, b \text{ not both zero})$$

68. (a) Use Formula (9) in Section 7.2 to show that
$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\,dx \quad (n \geq 2)$$
(b) Use this result to derive the ***Wallis sine formulas***:
$$\int_0^{\pi/2} \sin^n x\,dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad \left(\begin{matrix} n \text{ even} \\ \text{and} \geq 2 \end{matrix}\right)$$
$$\int_0^{\pi/2} \sin^n x\,dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \quad \left(\begin{matrix} n \text{ odd} \\ \text{and} \geq 3 \end{matrix}\right)$$

69. Use the Wallis formulas in Exercise 68 to evaluate

(a) $\displaystyle\int_0^{\pi/2} \sin^3 x\,dx$ (b) $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx$

(c) $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx$ (d) $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx$.

70. Use Formula (10) in Section 7.2 and the method of Exercise 68 to derive the ***Wallis cosine formulas***:
$$\int_0^{\pi/2} \cos^n x\,dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad \left(\begin{matrix} n \text{ even} \\ \text{and} \geq 2 \end{matrix}\right)$$
$$\int_0^{\pi/2} \cos^n x\,dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \quad \left(\begin{matrix} n \text{ odd} \\ \text{and} \geq 3 \end{matrix}\right)$$

71. **Writing** Describe the various approaches for evaluating integrals of the form
$$\int \sin^m x \cos^n x\,dx$$
Into what cases do these types of integrals fall? What procedures and identities are used in each case?

72. **Writing** Describe the various approaches for evaluating integrals of the form
$$\int \tan^m x \sec^n x\,dx$$
Into what cases do these types of integrals fall? What procedures and identities are used in each case?

✔ QUICK CHECK ANSWERS 7.3

1. (a) $\dfrac{1-\cos 2x}{2}$ (b) $\dfrac{1+\cos 2x}{2}$ (c) $\cos 2x$ **2.** (a) $\tan x + C$ (b) $\tan x - x + C$ (c) $\ln|\sec x + \tan x| + C$ (d) $\ln|\sec x| + C$
3. (a) $\int u^2\,du$ (b) $\int (u^2-1)u^2\,du$ (c) $\int u^3\,du$ (d) $\int (u^2-1)\,du$

7.4 TRIGONOMETRIC SUBSTITUTIONS

In this section we will discuss a method for evaluating integrals containing radicals by making substitutions involving trigonometric functions. We will also show how integrals containing quadratic polynomials can sometimes be evaluated by completing the square.

THE METHOD OF TRIGONOMETRIC SUBSTITUTION

To start, we will be concerned with integrals that contain expressions of the form
$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}$$
in which a is a positive constant. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical. For example, to eliminate the radical in the expression $\sqrt{a^2 - x^2}$, we can make the substitution
$$x = a \sin\theta, \quad -\pi/2 \leq \theta \leq \pi/2 \tag{1}$$
which yields
$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)}$$
$$= a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta \qquad \boxed{\cos\theta \geq 0 \text{ since } -\pi/2 \leq \theta \leq \pi/2}$$

The restriction on θ in (1) serves two purposes—it enables us to replace $|\cos\theta|$ by $\cos\theta$ to simplify the calculations, and it also ensures that the substitutions can be rewritten as $\theta = \sin^{-1}(x/a)$, if needed.

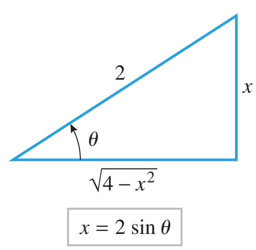
▶ Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. To eliminate the radical we make the substitution

$$x = 2\sin\theta, \quad dx = 2\cos\theta\, d\theta$$

This yields

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}\\ &= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}\\ &= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C \qquad (2)\end{aligned}$$



▲ **Figure 7.4.1**

At this point we have completed the integration; however, because the original integral was expressed in terms of x, it is desirable to express $\cot\theta$ in terms of x as well. This can be done using trigonometric identities, but the expression can also be obtained by writing the substitution $x = 2\sin\theta$ as $\sin\theta = x/2$ and representing it geometrically as in Figure 7.4.1. From that figure we obtain

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

Substituting this in (2) yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C \quad ◀$$

▶ Example 2 Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. There are two possible approaches: we can make the substitution in the indefinite integral (as in Example 1) and then evaluate the definite integral using the x-limits of integration, or we can make the substitution in the definite integral and convert the x-limits to the corresponding θ-limits.

Method 1.

Using the result from Example 1 with the x-limits of integration yields

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}$$

Method 2.

The substitution $x = 2\sin\theta$ can be expressed as $x/2 = \sin\theta$ or $\theta = \sin^{-1}(x/2)$, so the θ-limits that correspond to $x = 1$ and $x = \sqrt{2}$ are

$$\begin{aligned} x = 1&: \quad \theta = \sin^{-1}(1/2) = \pi/6\\ x = \sqrt{2}&: \quad \theta = \sin^{-1}(\sqrt{2}/2) = \pi/4\end{aligned}$$

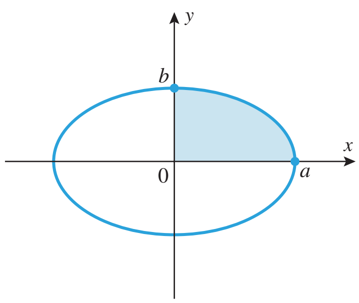
Thus, from (2) in Example 1 we obtain

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta \qquad \boxed{\text{Convert } x\text{-limits to } \theta\text{-limits.}}$$

$$= -\frac{1}{4}\Big[\cot\theta\Big]_{\pi/6}^{\pi/4} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4} \blacktriangleleft$$

▶ **Example 3** Find the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



▲ **Figure 7.4.2**

Solution. Because the ellipse is symmetric about both axes, its area A is four times the area in the first quadrant (Figure 7.4.2). If we solve the equation of the ellipse for y in terms of x, we obtain

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

where the positive square root gives the equation of the upper half. Thus, the area A is given by

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

To evaluate this integral, we will make the substitution $x = a\sin\theta$ (so $dx = a\cos\theta\,d\theta$) and convert the x-limits of integration to θ-limits. Since the substitution can be expressed as $\theta = \sin^{-1}(x/a)$, the θ-limits of integration are

$$x = 0: \quad \theta = \sin^{-1}(0) = 0$$

$$x = a: \quad \theta = \sin^{-1}(1) = \pi/2$$

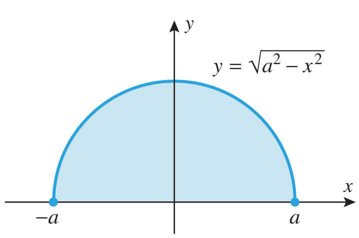
Thus, we obtain

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{4b}{a}\int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$$

$$= \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left[\frac{\pi}{2}-0\right] = \pi ab \blacktriangleleft$$



▲ **Figure 7.4.3**

REMARK In the special case where $a = b$, the ellipse becomes a circle of radius a, and the area formula becomes $A = \pi a^2$, as expected. It is worth noting that

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\pi a^2 \tag{3}$$

since this integral represents the area of the upper semicircle (Figure 7.4.3).

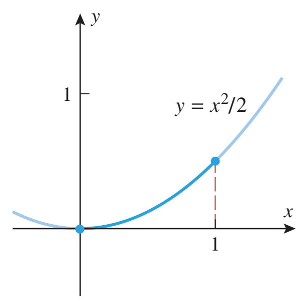
TECHNOLOGY MASTERY

If you have a calculating utility with a numerical integration capability, use it and Formula (3) to approximate π to three decimal places.

Thus far, we have focused on using the substitution $x = a\sin\theta$ to evaluate integrals involving radicals of the form $\sqrt{a^2-x^2}$. Table 7.4.1 summarizes this method and describes some other substitutions of this type.

Table 7.4.1
TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2-x^2 = a^2-a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2+x^2 = a^2+a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$



▲ **Figure 7.4.4**

▶ **Example 4** Find the arc length of the curve $y = x^2/2$ from $x = 0$ to $x = 1$ (Figure 7.4.4).

Solution. From Formula (4) of Section 6.4 the arc length L of the curve is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+x^2}\, dx$$

The integrand involves a radical of the form $\sqrt{a^2+x^2}$ with $a = 1$, so from Table 7.4.1 we make the substitution

$$x = \tan\theta, \quad -\pi/2 < \theta < \pi/2$$

$$\frac{dx}{d\theta} = \sec^2\theta \quad \text{or} \quad dx = \sec^2\theta\, d\theta$$

Since this substitution can be expressed as $\theta = \tan^{-1} x$, the θ-limits of integration that correspond to the x-limits, $x = 0$ and $x = 1$, are

$$x = 0: \quad \theta = \tan^{-1} 0 = 0$$
$$x = 1: \quad \theta = \tan^{-1} 1 = \pi/4$$

Thus,

$$\begin{aligned}
L = \int_0^1 \sqrt{1+x^2}\, dx &= \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\, \sec^2\theta\, d\theta \\
&= \int_0^{\pi/4} \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta && \boxed{1+\tan^2\theta = \sec^2\theta} \\
&= \int_0^{\pi/4} |\sec\theta| \sec^2\theta\, d\theta \\
&= \int_0^{\pi/4} \sec^3\theta\, d\theta && \boxed{\sec\theta > 0 \text{ since } -\pi/2 < \theta < \pi/2} \\
&= \left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} && \boxed{\text{Formula (26) of Section 7.3}} \\
&= \tfrac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right] \approx 1.148 \blacktriangleleft
\end{aligned}$$

▶ **Example 5** Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\, dx$, assuming that $x \ge 5$.

Solution. The integrand involves a radical of the form $\sqrt{x^2 - a^2}$ with $a = 5$, so from Table 7.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \le \theta < \pi/2$$

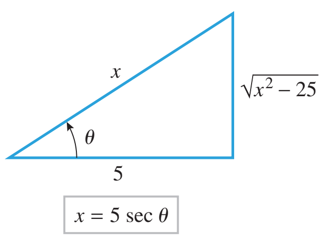
$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\, d\theta$$

Thus,

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\
&= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \qquad \boxed{\tan\theta \ge 0 \text{ since } 0 \le \theta < \pi/2} \\
&= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C
\end{aligned}$$

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 7.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$



▲ **Figure 7.4.5**

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C \quad ◀$$

INTEGRALS INVOLVING $ax^2 + bx + c$

Integrals that involve a quadratic expression $ax^2 + bx + c$, where $a \ne 0$ and $b \ne 0$, can often be evaluated by first completing the square, then making an appropriate substitution. The following example illustrates this idea.

▶ **Example 6** Evaluate $\displaystyle\int \frac{x}{x^2 - 4x + 8}\,dx$.

Solution. Completing the square yields

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4 = (x-2)^2 + 4$$

Thus, the substitution

$$u = x - 2, \quad du = dx$$

yields

$$\begin{aligned}
\int \frac{x}{x^2-4x+8}\,dx &= \int \frac{x}{(x-2)^2+4}\,dx = \int \frac{u+2}{u^2+4}\,du \\
&= \int \frac{u}{u^2+4}\,du + 2\int \frac{du}{u^2+4} \\
&= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{du}{u^2+4} \\
&= \frac{1}{2}\ln(u^2+4) + 2\left(\frac{1}{2}\right)\tan^{-1}\frac{u}{2} + C \\
&= \frac{1}{2}\ln[(x-2)^2+4] + \tan^{-1}\left(\frac{x-2}{2}\right) + C \quad ◀
\end{aligned}$$

✔ QUICK CHECK EXERCISES 7.4

(See page 514 for answers.)

1. For each expression, give a trigonometric substitution that will eliminate the radical.
   (a) $\sqrt{a^2 - x^2}$ ________ (b) $\sqrt{a^2 + x^2}$ ________
   (c) $\sqrt{x^2 - a^2}$ ________
2. If $x = 2\sec\theta$ and $0 < \theta < \pi/2$, then
   (a) $\sin\theta =$ ________ (b) $\cos\theta =$ ________
   (c) $\tan\theta =$ ________.
3. In each part, state the trigonometric substitution that you would try first to evaluate the integral. Do not evaluate the integral.
   (a) $\int \sqrt{9 + x^2}\, dx$ ________
   (b) $\int \sqrt{9 - x^2}\, dx$ ________
   (c) $\int \sqrt{1 - 9x^2}\, dx$ ________
   (d) $\int \sqrt{x^2 - 9}\, dx$ ________
   (e) $\int \sqrt{9 + 3x^2}\, dx$ ________
   (f) $\int \sqrt{1 + (9x)^2}\, dx$ ________
4. In each part, determine the substitution u.
   (a) $\int \frac{1}{x^2 - 2x + 10}\, dx = \int \frac{1}{u^2 + 3^2}\, du$;
   $u =$ ________
   (b) $\int \sqrt{x^2 - 6x + 8}\, dx = \int \sqrt{u^2 - 1}\, du$;
   $u =$ ________
   (c) $\int \sqrt{12 - 4x - x^2}\, dx = \int \sqrt{4^2 - u^2}\, du$;
   $u =$ ________

EXERCISE SET 7.4 [C] CAS

1–26 Evaluate the integral. ■

1. $\int \sqrt{4 - x^2}\, dx$

2. $\int \sqrt{1 - 4x^2}\, dx$

3. $\int \frac{x^2}{\sqrt{16 - x^2}}\, dx$

4. $\int \frac{dx}{x^2\sqrt{9 - x^2}}$

5. $\int \frac{dx}{(4 + x^2)^2}$

6. $\int \frac{x^2}{\sqrt{5 + x^2}}\, dx$

7. $\int \frac{\sqrt{x^2 - 9}}{x}\, dx$

8. $\int \frac{dx}{x^2\sqrt{x^2 - 16}}$

9. $\int \frac{3x^3}{\sqrt{1 - x^2}}\, dx$

10. $\int x^3\sqrt{5 - x^2}\, dx$

11. $\int \frac{dx}{x^2\sqrt{9x^2 - 4}}$

12. $\int \frac{\sqrt{1 + t^2}}{t}\, dt$

13. $\int \frac{dx}{(1 - x^2)^{3/2}}$

14. $\int \frac{dx}{x^2\sqrt{x^2 + 25}}$

15. $\int \frac{dx}{\sqrt{x^2 - 9}}$

16. $\int \frac{dx}{1 + 2x^2 + x^4}$

17. $\int \frac{dx}{(4x^2 - 9)^{3/2}}$

18. $\int \frac{3x^3}{\sqrt{x^2 - 25}}\, dx$

19. $\int e^x\sqrt{1 - e^{2x}}\, dx$

20. $\int \frac{\cos\theta}{\sqrt{2 - \sin^2\theta}}\, d\theta$

21. $\int_0^1 5x^3\sqrt{1 - x^2}\, dx$

22. $\int_0^{1/2} \frac{dx}{(1 - x^2)^2}$

23. $\int_{\sqrt{2}}^2 \frac{dx}{x^2\sqrt{x^2 - 1}}$

24. $\int_{\sqrt{2}}^2 \frac{\sqrt{2x^2 - 4}}{x}\, dx$

25. $\int_1^3 \frac{dx}{x^4\sqrt{x^2 + 3}}$

26. $\int_0^3 \frac{x^3}{(3 + x^2)^{5/2}}\, dx$

27–30 True–False Determine whether the statement is true or false. Explain your answer. ■

27. An integrand involving a radical of the form $\sqrt{a^2 - x^2}$ suggests the substitution $x = a\sin\theta$.

28. The trigonometric substitution $x = a\sin\theta$ is made with the restriction $0 \le \theta \le \pi$.

29. An integrand involving a radical of the form $\sqrt{x^2 - a^2}$ suggests the substitution $x = a\cos\theta$.

30. The area enclosed by the ellipse $x^2 + 4y^2 = 1$ is $\pi/2$.

FOCUS ON CONCEPTS

31. The integral
$$\int \frac{x}{x^2 + 4}\, dx$$
can be evaluated either by a trigonometric substitution or by the substitution $u = x^2 + 4$. Do it both ways and show that the results are equivalent.

32. The integral
$$\int \frac{x^2}{x^2 + 4}\, dx$$
can be evaluated either by a trigonometric substitution or by algebraically rewriting the numerator of the integrand as $(x^2 + 4) - 4$. Do it both ways and show that the results are equivalent.

33. Find the arc length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

34. Find the arc length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

35. Find the area of the surface generated when the curve in Exercise 34 is revolved about the x-axis.

36. Find the volume of the solid generated when the region enclosed by $x = y(1-y^2)^{1/4}$, $y = 0$, $y = 1$, and $x = 0$ is revolved about the y-axis.

37–48 Evaluate the integral. ■

37. $\displaystyle\int \frac{dx}{x^2-4x+5}$

38. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

39. $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$

40. $\displaystyle\int \frac{dx}{16x^2+16x+5}$

41. $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+10}}$

42. $\displaystyle\int \frac{x}{x^2+2x+2}\,dx$

43. $\displaystyle\int \sqrt{3-2x-x^2}\,dx$

44. $\displaystyle\int \frac{e^x}{\sqrt{1+e^x+e^{2x}}}\,dx$

45. $\displaystyle\int \frac{dx}{2x^2+4x+7}$

46. $\displaystyle\int \frac{2x+3}{4x^2+4x+5}\,dx$

47. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4x-x^2}}$

48. $\displaystyle\int_0^4 \sqrt{x(4-x)}\,dx$

C **49–50** There is a good chance that your CAS will not be able to evaluate these integrals as stated. If this is so, make a substitution that converts the integral into one that your CAS can evaluate. ■

49. $\displaystyle\int \cos x \sin x \sqrt{1-\sin^4 x}\,dx$

50. $\displaystyle\int (x\cos x + \sin x)\sqrt{1+x^2\sin^2 x}\,dx$

51. (a) Use the ***hyperbolic substitution*** $x = 3\sinh u$, the identity $\cosh^2 u - \sinh^2 u = 1$, and Theorem 6.9.4 to evaluate
$$\int \frac{dx}{\sqrt{x^2+9}}$$
(b) Evaluate the integral in part (a) using a trigonometric substitution and show that the result agrees with that obtained in part (a).

52. Use the hyperbolic substitutionn $x = \cosh u$, the identity $\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$, and the results referenced in Exercise 51 to evaluate
$$\int \sqrt{x^2-1}\,dx, \quad x \geq 1$$

53. Writing The trigonometric substitution $x = a\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$, is suggested for an integral whose integrand involves $\sqrt{a^2-x^2}$. Discuss the implications of restricting θ to $\pi/2 \leq \theta \leq 3\pi/2$, and explain why the restriction $-\pi/2 \leq \theta \leq \pi/2$ should be preferred.

54. Writing The trigonometric substitution $x = a\cos\theta$ could also be used for an integral whose integrand involves $\sqrt{a^2-x^2}$. Determine an appropriate restriction for θ with the substitution $x = a\cos\theta$, and discuss how to apply this substitution in appropriate integrals. Illustrate your discussion by evaluating the integral in Example 1 using a substitution of this type.

✔ QUICK CHECK ANSWERS 7.4

1. (a) $x = a\sin\theta$ (b) $x = a\tan\theta$ (c) $x = a\sec\theta$ **2.** (a) $\dfrac{\sqrt{x^2-4}}{x}$ (b) $\dfrac{2}{x}$ (c) $\dfrac{\sqrt{x^2-4}}{2}$ **3.** (a) $x = 3\tan\theta$ (b) $x = 3\sin\theta$ (c) $x = \frac{1}{3}\sin\theta$ (d) $x = 3\sec\theta$ (e) $x = \sqrt{3}\tan\theta$ (f) $x = \frac{1}{9}\tan\theta$ **4.** (a) $x-1$ (b) $x-3$ (c) $x+2$

7.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

Recall that a rational function is a ratio of two polynomials. In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions that can be integrated by the methods studied in earlier sections.

■ PARTIAL FRACTIONS

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2(x+1)+3(x-4)}{(x-4)(x+1)} = \frac{5x-10}{x^2-3x-4} \tag{1}$$

However, for purposes of integration, the left side of (1) is preferable to the right side since each of the terms is easy to integrate:

$$\int \frac{5x-10}{x^2-3x-4}\,dx = \int \frac{2}{x-4}\,dx + \int \frac{3}{x+1}\,dx = 2\ln|x-4| + 3\ln|x+1| + C$$

Thus, it is desirable to have some method that will enable us to obtain the left side of (1), starting with the right side. To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side. Thus, to find the left side of (1), starting from the right side, we could factor the denominator of the right side and look for constants A and B such that

$$\frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \tag{2}$$

One way to find the constants A and B is to multiply (2) through by $(x-4)(x+1)$ to clear fractions. This yields

$$5x-10 = A(x+1) + B(x-4) \tag{3}$$

This relationship holds for all x, so it holds in particular if $x=4$ or $x=-1$. Substituting $x=4$ in (3) makes the second term on the right drop out and yields the equation $10=5A$ or $A=2$; and substituting $x=-1$ in (3) makes the first term on the right drop out and yields the equation $-15=-5B$ or $B=3$. Substituting these values in (2) we obtain

$$\frac{5x-10}{(x-4)(x+1)} = \frac{2}{x-4} + \frac{3}{x+1} \tag{4}$$

which agrees with (1).

A second method for finding the constants A and B is to multiply out the right side of (3) and collect like powers of x to obtain

$$5x-10 = (A+B)x + (A-4B)$$

Since the polynomials on the two sides are identical, their corresponding coefficients must be the same. Equating the corresponding coefficients on the two sides yields the following system of equations in the unknowns A and B:

$$\begin{aligned} A + B &= 5 \\ A - 4B &= -10 \end{aligned}$$

Solving this system yields $A=2$ and $B=3$ as before (verify).

The terms on the right side of (4) are called ***partial fractions*** of the expression on the left side because they each constitute *part* of that expression. To find those partial fractions we first had to make a guess about their form, and then we had to find the unknown constants. Our next objective is to extend this idea to general rational functions. For this purpose, suppose that $P(x)/Q(x)$ is a ***proper rational function***, by which we mean that the degree of the numerator is less than the degree of the denominator. There is a theorem in advanced algebra which states that every proper rational function can be expressed as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where $F_1(x), F_2(x), \ldots, F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

in which the denominators are factors of $Q(x)$. The sum is called the ***partial fraction decomposition*** of $P(x)/Q(x)$, and the terms are called ***partial fractions***. As in our opening example, there are two parts to finding a partial fraction decomposition: determining the exact form of the decomposition and finding the unknown constants.

FINDING THE FORM OF A PARTIAL FRACTION DECOMPOSITION

The first step in finding the form of the partial fraction decomposition of a proper rational function $P(x)/Q(x)$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of *distinct* factors of the form

$$(ax+b)^m \quad \text{and} \quad (ax^2+bx+c)^m$$

From these factors we can determine the form of the partial fraction decomposition using two rules that we will now discuss.

LINEAR FACTORS

If all of the factors of $Q(x)$ are linear, then the partial fraction decomposition of $P(x)/Q(x)$ can be determined by using the following rule:

LINEAR FACTOR RULE For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$.

Solution. The integrand is a proper rational function that can be written as

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

The factors $x-1$ and $x+2$ are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \tag{5}$$

where A and B are constants to be determined. Multiplying this expression through by $(x-1)(x+2)$ yields

$$1 = A(x+2) + B(x-1) \tag{6}$$

As discussed earlier, there are two methods for finding A and B: we can substitute values of x that are chosen to make terms on the right drop out, or we can multiply out on the right and equate corresponding coefficients on the two sides to obtain a system of equations that can be solved for A and B. We will use the first approach.

Setting $x = 1$ makes the second term in (6) drop out and yields $1 = 3A$ or $A = \frac{1}{3}$; and setting $x = -2$ makes the first term in (6) drop out and yields $1 = -3B$ or $B = -\frac{1}{3}$. Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

The integration can now be completed as follows:

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C \blacktriangleleft$$

If the factors of $Q(x)$ are linear and none are repeated, as in the last example, then the recommended method for finding the constants in the partial fraction decomposition is to substitute appropriate values of x to make terms drop out. However, if some of the linear factors are repeated, then it will not be possible to find all of the constants in this way. In this case the recommended procedure is to find as many constants as possible by substitution and then find the rest by equating coefficients. This is illustrated in the next example.

▶ Example 2 Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$.

Solution. The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Although x^2 is a quadratic factor, it is *not* irreducible since $x^2 = xx$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

and the factor $x - 2$ introduces one term (since $m = 1$) of the form

$$\frac{C}{x-2}$$

so the partial fraction decomposition is

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \tag{7}$$

Multiplying by $x^2(x-2)$ yields

$$2x+4 = Ax(x-2) + B(x-2) + Cx^2 \tag{8}$$

which, after multiplying out and collecting like powers of x, becomes

$$2x+4 = (A+C)x^2 + (-2A+B)x - 2B \tag{9}$$

Setting $x = 0$ in (8) makes the first and third terms drop out and yields $B = -2$, and setting $x = 2$ in (8) makes the first and second terms drop out and yields $C = 2$ (verify). However, there is no substitution in (8) that produces A directly, so we look to Equation (9) to find this value. This can be done by equating the coefficients of x^2 on the two sides to obtain

$$A + C = 0 \quad \text{or} \quad A = -C = -2$$

Substituting the values $A = -2$, $B = -2$, and $C = 2$ in (7) yields the partial fraction decomposition

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

Thus,

$$\int \frac{2x+4}{x^2(x-2)}\,dx = -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$
$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C \blacktriangleleft$$

QUADRATIC FACTORS

If some of the factors of $Q(x)$ are irreducible quadratics, then the contribution of those factors to the partial fraction decomposition of $P(x)/Q(x)$ can be determined from the following rule:

QUADRATIC FACTOR RULE For each factor of the form $(ax^2+bx+c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m=1$, only the first term in the sum appears.

Example 3 Evaluate $\displaystyle\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$.

Solution. The denominator in the integrand can be factored by grouping:

$$3x^3-x^2+3x-1=x^2(3x-1)+(3x-1)=(3x-1)(x^2+1)$$

By the linear factor rule, the factor $3x-1$ introduces one term, namely,

$$\frac{A}{3x-1}$$

and by the quadratic factor rule, the factor x^2+1 introduces one term, namely,

$$\frac{Bx+C}{x^2+1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2+x-2}{(3x-1)(x^2+1)}=\frac{A}{3x-1}+\frac{Bx+C}{x^2+1} \tag{10}$$

Multiplying by $(3x-1)(x^2+1)$ yields

$$x^2+x-2=A(x^2+1)+(Bx+C)(3x-1) \tag{11}$$

We could find A by substituting $x=\frac{1}{3}$ to make the last term drop out, and then find the rest of the constants by equating corresponding coefficients. However, in this case it is just as easy to find *all* of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the right side of (11) and collect like terms:

$$x^2+x-2=(A+3B)x^2+(-B+3C)x+(A-C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A+3B \qquad\quad &= 1\\ -\ B+3C &= 1\\ A \qquad\ -\ C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A=-\frac{7}{5},\quad B=\frac{4}{5},\quad C=\frac{3}{5}$$

Thus, (10) becomes

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}$$

and

$$\begin{aligned}\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx &= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1} \\ &= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C \end{aligned}$$ ◀

TECHNOLOGY MASTERY

Computer algebra systems have built-in capabilities for finding partial fraction decompositions. If you have a CAS, use it to find the decompositions in Examples 1, 2, and 3.

▶ **Example 4** Evaluate $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$.

Solution. Observe that the integrand is a proper rational function since the numerator has degree 4 and the denominator has degree 5. Thus, the method of partial fractions is applicable. By the linear factor rule, the factor $x+2$ introduces the single term

$$\frac{A}{x+2}$$

and by the quadratic factor rule, the factor $(x^2+3)^2$ introduces two terms (since $m=2$):

$$\frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

Thus, the partial fraction decomposition of the integrand is

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2} \tag{12}$$

Multiplying by $(x+2)(x^2+3)^2$ yields

$$\begin{aligned}&3x^4+4x^3+16x^2+20x+9 \\ &\qquad = A(x^2+3)^2 + (Bx+C)(x^2+3)(x+2) + (Dx+E)(x+2)\end{aligned} \tag{13}$$

which, after multiplying out and collecting like powers of x, becomes

$$\begin{aligned}&3x^4+4x^3+16x^2+20x+9 \\ &\qquad = (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2 \\ &\qquad\qquad + (6B+3C+2D+E)x + (9A+6C+2E)\end{aligned} \tag{14}$$

Equating corresponding coefficients in (14) yields the following system of five linear equations in five unknowns:

$$\begin{aligned} A+B &= 3 \\ 2B+C &= 4 \\ 6A+3B+2C+D &= 16 \\ 6B+3C+2D+E &= 20 \\ 9A+6C+2E &= 9 \end{aligned} \tag{15}$$

Efficient methods for solving systems of linear equations such as this are studied in a branch of mathematics called ***linear algebra***; those methods are outside the scope of this text. However, as a practical matter most linear systems of any size are solved by computer, and most computer algebra systems have commands that in many cases can solve linear systems exactly. In this particular case we can simplify the work by first substituting $x=-2$

in (13), which yields $A = 1$. Substituting this known value of A in (15) yields the simpler system

$$\begin{aligned} B &= 2 \\ 2B + C &= 4 \\ 3B + 2C + D &= 10 \\ 6B + 3C + 2D + E &= 20 \\ 6C + 2E &= 0 \end{aligned} \tag{16}$$

This system can be solved by starting at the top and working down, first substituting $B = 2$ in the second equation to get $C = 0$, then substituting the known values of B and C in the third equation to get $D = 4$, and so forth. This yields

$$A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0$$

Thus, (12) becomes

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

and so

$$\begin{aligned} &\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\,dx \\ &\qquad = \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\,dx + 4\int \frac{x}{(x^2+3)^2}\,dx \\ &\qquad = \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C \end{aligned}$$

INTEGRATING IMPROPER RATIONAL FUNCTIONS

Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into partial fractions. This idea is illustrated in the following example.

Example 5 Evaluate $\displaystyle\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx.$

Solution. The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2. Thus, we first perform the long division

$$\begin{array}{r|l} & \quad\;\; 3x^2 \qquad\quad + 1 \\ \hline x^2 + x - 2 & 3x^4 + 3x^3 - 5x^2 + x - 1 \\ & 3x^4 + 3x^3 - 6x^2 \\ \hline & \qquad\qquad\quad x^2 + x - 1 \\ & \qquad\qquad\quad x^2 + x - 2 \\ \hline & \qquad\qquad\qquad\qquad\;\; 1 \end{array}$$

It follows that the integrand can be expressed as

$$\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}$$

and hence

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx = \int (3x^2 + 1)\,dx + \int \frac{dx}{x^2 + x - 2}$$

The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C \blacktriangleleft$$

CONCLUDING REMARKS

There are some cases in which the method of partial fractions is inappropriate. For example, it would be inefficient to use partial fractions to perform the integration

$$\int \frac{3x^2+2}{x^3+2x-8}\,dx = \ln|x^3+2x-8|+C$$

since the substitution $u = x^3+2x-8$ is more direct. Similarly, the integration

$$\int \frac{2x-1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} = \ln(x^2+1) - \tan^{-1}x + C$$

requires only a little algebra since the integrand is already in partial fraction form.

✔QUICK CHECK EXERCISES 7.5 *(See page 523 for answers.)*

1. A partial fraction is a rational function of the form ________ or of the form ________.

2. (a) What is a proper rational function?
(b) What condition must the degree of the numerator and the degree of the denominator of a rational function satisfy for the method of partial fractions to be applicable directly?
(c) If the condition in part (b) is not satisfied, what must you do if you want to use partial fractions?

3. Suppose that the function $f(x) = P(x)/Q(x)$ is a proper rational function.
(a) For each factor of $Q(x)$ of the form $(ax+b)^m$, the partial fraction decomposition of f contains the following sum of m partial fractions: ________
(b) For each factor of $Q(x)$ of the form $(ax^2+bx+c)^m$, where ax^2+bx+c is an irreducible quadratic, the partial fraction decomposition of f contains the following sum of m partial fractions: ________

4. Complete the partial fraction decomposition.
(a) $\dfrac{-3}{(x+1)(2x-1)} = \dfrac{A}{x+1} - \dfrac{2}{2x-1}$
(b) $\dfrac{2x^2-3x}{(x^2+1)(3x+2)} = \dfrac{B}{3x+2} - \dfrac{1}{x^2+1}$

5. Evaluate the integral.
(a) $\displaystyle\int \frac{3}{(x+1)(1-2x)}\,dx$ (b) $\displaystyle\int \frac{2x^2-3x}{(x^2+1)(3x+2)}\,dx$

EXERCISE SET 7.5 C CAS

1–8 Write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.)

1. $\dfrac{3x-1}{(x-3)(x+4)}$ **2.** $\dfrac{5}{x(x^2-4)}$

3. $\dfrac{2x-3}{x^3-x^2}$ **4.** $\dfrac{x^2}{(x+2)^3}$

5. $\dfrac{1-x^2}{x^3(x^2+2)}$ **6.** $\dfrac{3x}{(x-1)(x^2+6)}$

7. $\dfrac{4x^3-x}{(x^2+5)^2}$ **8.** $\dfrac{1-3x^4}{(x-2)(x^2+1)^2}$

9–34 Evaluate the integral.

9. $\displaystyle\int \frac{dx}{x^2-3x-4}$ **10.** $\displaystyle\int \frac{dx}{x^2-6x-7}$

11. $\displaystyle\int \frac{11x+17}{2x^2+7x-4}\,dx$ **12.** $\displaystyle\int \frac{5x-5}{3x^2-8x-3}\,dx$

13. $\displaystyle\int \frac{2x^2-9x-9}{x^3-9x}\,dx$ **14.** $\displaystyle\int \frac{dx}{x(x^2-1)}$

15. $\displaystyle\int \frac{x^2-8}{x+3}\,dx$ **16.** $\displaystyle\int \frac{x^2+1}{x-1}\,dx$

17. $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$ **18.** $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$

19. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$ **20.** $\displaystyle\int \frac{3x+1}{3x^2+2x-1}\,dx$

21. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$ **22.** $\displaystyle\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$

23. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$ **24.** $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

25. $\int \frac{2x^2 - 10x + 4}{(x+1)(x-3)^2}\,dx$ **26.** $\int \frac{2x^2 - 2x - 1}{x^3 - x^2}\,dx$

27. $\int \frac{x^2}{(x+1)^3}\,dx$ **28.** $\int \frac{2x^2 + 3x + 3}{(x+1)^3}\,dx$

29. $\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}\,dx$ **30.** $\int \frac{dx}{x^3 + 2x}$

31. $\int \frac{x^3 + 3x^2 + x + 9}{(x^2+1)(x^2+3)}\,dx$ **32.** $\int \frac{x^3 + x^2 + x + 2}{(x^2+1)(x^2+2)}\,dx$

33. $\int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1}\,dx$

34. $\int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10}\,dx$

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. The technique of partial fractions is used for integrals whose integrands are ratios of polynomials.

36. The integrand in

$$\int \frac{3x^4 + 5}{(x^2+1)^2}\,dx$$

is a proper rational function.

37. The partial fraction decomposition of

$$\frac{2x+3}{x^2} \quad \text{is} \quad \frac{2}{x} + \frac{3}{x^2}$$

38. If $f(x) = P(x)/(x+5)^3$ is a proper rational function, then the partial fraction decomposition of $f(x)$ has terms with constant numerators and denominators $(x+5)$, $(x+5)^2$, and $(x+5)^3$.

39–42 Evaluate the integral by making a substitution that converts the integrand to a rational function.

39. $\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5}\,d\theta$ **40.** $\int \frac{e^t}{e^{2t} - 4}\,dt$

41. $\int \frac{e^{3x}}{e^{2x} + 4}\,dx$ **42.** $\int \frac{5 + 2\ln x}{x(1 + \ln x)^2}\,dx$

43. Find the volume of the solid generated when the region enclosed by $y = x^2/(9 - x^2)$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the x-axis.

44. Find the area of the region under the curve $y = 1/(1 + e^x)$, over the interval $[-\ln 5, \ln 5]$. [*Hint:* Make a substitution that converts the integrand to a rational function.]

C **45–46** Use a CAS to evaluate the integral in two ways: (i) integrate directly; (ii) use the CAS to find the partial fraction decomposition and integrate the decomposition. Integrate by hand to check the results.

45. $\int \frac{x^2 + 1}{(x^2 + 2x + 3)^2}\,dx$

46. $\int \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4}{(x^2+2)^3}\,dx$

C **47–48** Integrate by hand and check your answers using a CAS.

47. $\int \frac{dx}{x^4 - 3x^3 - 7x^2 + 27x - 18}$

48. $\int \frac{dx}{16x^3 - 4x^2 + 4x - 1}$

FOCUS ON CONCEPTS

49. Show that

$$\int_0^1 \frac{x}{x^4 + 1}\,dx = \frac{\pi}{8}$$

50. Use partial fractions to derive the integration formula

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

51. Suppose that $ax^2 + bx + c$ is a quadratic polynomial and that the integration

$$\int \frac{1}{ax^2 + bx + c}\,dx$$

produces a function with no inverse tangent terms. What does this tell you about the roots of the polynomial?

52. Suppose that $ax^2 + bx + c$ is a quadratic polynomial and that the integration

$$\int \frac{1}{ax^2 + bx + c}\,dx$$

produces a function with neither logarithmic nor inverse tangent terms. What does this tell you about the roots of the polynomial?

53. Does there exist a quadratic polynomial $ax^2 + bx + c$ such that the integration

$$\int \frac{x}{ax^2 + bx + c}\,dx$$

produces a function with no logarithmic terms? If so, give an example; if not, explain why no such polynomial can exist.

54. Writing Suppose that $P(x)$ is a cubic polynomial. State the general form of the partial fraction decomposition for

$$f(x) = \frac{P(x)}{(x+5)^4}$$

and state the implications of this decomposition for evaluating the integral $\int f(x)\,dx$.

55. Writing Consider the functions

$$f(x) = \frac{1}{x^2 - 4} \quad \text{and} \quad g(x) = \frac{x}{x^2 - 4}$$

Each of the integrals $\int f(x)\,dx$ and $\int g(x)\,dx$ can be evaluated using partial fractions and using at least one other integration technique. Demonstrate two different techniques for evaluating each of these integrals, and then discuss the considerations that would determine which technique you would use.

✔ QUICK CHECK ANSWERS 7.5

1. $\dfrac{A}{(ax+b)^k}$; $\dfrac{Ax+B}{(ax^2+bx+c)^k}$ **2.** (a) A proper rational function is a rational function in which the degree of the numerator is less than the degree of the denominator. (b) The degree of the numerator must be less than the degree of the denominator. (c) Divide the denominator into the numerator, which results in the sum of a polynomial and a proper rational function.
3. (a) $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_m}{(ax+b)^m}$ (b) $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_mx+B_m}{(ax^2+bx+c)^m}$
4. (a) $A=1$ (b) $B=2$
5. (a) $\displaystyle\int\frac{3}{(x+1)(1-2x)}\,dx=\ln\left|\frac{x+1}{1-2x}\right|+C$ (b) $\displaystyle\int\frac{2x^2-3x}{(x^2+1)(3x+2)}\,dx=\frac{2}{3}\ln|3x+2|-\tan^{-1}x+C$

7.6 USING COMPUTER ALGEBRA SYSTEMS AND TABLES OF INTEGRALS

In this section we will discuss how to integrate using tables, and we will see some special substitutions to try when an integral doesn't match any of the forms in an integral table. In particular, we will discuss a method for integrating rational functions of $\sin x$ *and* $\cos x$*. We will also address some of the issues that relate to using computer algebra systems for integration. Readers who are not using computer algebra systems can skip that material.*

INTEGRAL TABLES

Tables of integrals are useful for eliminating tedious hand computation. The endpapers of this text contain a relatively brief table of integrals that we will refer to as the ***Endpaper Integral Table***; more comprehensive tables are published in standard reference books such as the *CRC Standard Mathematical Tables and Formulae*, CRC Press, Inc., 2002.

All integral tables have their own scheme for classifying integrals according to the form of the integrand. For example, the Endpaper Integral Table classifies the integrals into 15 categories; *Basic Functions*, *Reciprocals of Basic Functions*, *Powers of Trigonometric Functions*, *Products of Trigonometric Functions*, and so forth. The first step in working with tables is to read through the classifications so that you understand the classification scheme and know where to look in the table for integrals of different types.

PERFECT MATCHES

If you are lucky, the integral you are attempting to evaluate will match up perfectly with one of the forms in the table. However, when looking for matches you may have to make an adjustment for the variable of integration. For example, the integral

$$\int x^2 \sin x\,dx$$

is a perfect match with Formula (46) in the Endpaper Integral Table, except for the letter used for the variable of integration. Thus, to apply Formula (46) to the given integral we need to change the variable of integration in the formula from u to x. With that minor modification we obtain

$$\int x^2 \sin x\,dx = 2x\sin x + (2-x^2)\cos x + C$$

Here are some more examples of perfect matches.

Example 1 Use the Endpaper Integral Table to evaluate

(a) $\int \sin 7x \cos 2x\, dx$ (b) $\int x^2\sqrt{7+3x}\, dx$

(c) $\int \frac{\sqrt{2-x^2}}{x}\, dx$ (d) $\int (x^3+7x+1)\sin \pi x\, dx$

Solution (a). The integrand can be classified as a product of trigonometric functions. Thus, from Formula (40) with $m = 7$ and $n = 2$ we obtain

$$\int \sin 7x \cos 2x\, dx = -\frac{\cos 9x}{18} - \frac{\cos 5x}{10} + C$$

Solution (b). The integrand can be classified as a power of x multiplying $\sqrt{a+bx}$. Thus, from Formula (103) with $a = 7$ and $b = 3$ we obtain

$$\int x^2\sqrt{7+3x}\, dx = \frac{2}{2835}(135x^2 - 252x + 392)(7+3x)^{3/2} + C$$

Solution (c). The integrand can be classified as a power of x dividing $\sqrt{a^2-x^2}$. Thus, from Formula (79) with $a = \sqrt{2}$ we obtain

$$\int \frac{\sqrt{2-x^2}}{x}\, dx = \sqrt{2-x^2} - \sqrt{2}\ln\left|\frac{\sqrt{2}+\sqrt{2-x^2}}{x}\right| + C$$

Solution (d). The integrand can be classified as a polynomial multiplying a trigonometric function. Thus, we apply Formula (58) with $p(x) = x^3 + 7x + 1$ and $a = \pi$. The successive nonzero derivatives of $p(x)$ are

$$p'(x) = 3x^2 + 7, \quad p''(x) = 6x, \quad p'''(x) = 6$$

and so

$$\int (x^3+7x+1)\sin \pi x\, dx$$
$$= -\frac{x^3+7x+1}{\pi}\cos \pi x + \frac{3x^2+7}{\pi^2}\sin \pi x + \frac{6x}{\pi^3}\cos \pi x - \frac{6}{\pi^4}\sin \pi x + C$$ ◀

MATCHES REQUIRING SUBSTITUTIONS

Sometimes an integral that does not match any table entry can be made to match by making an appropriate substitution.

Example 2 Use the Endpaper Integral Table to evaluate

(a) $\int e^{\pi x}\sin^{-1}(e^{\pi x})\, dx$ (b) $\int x\sqrt{x^2-4x+5}\, dx$

Solution (a). The integrand does not even come close to matching any of the forms in the table. However, a little thought suggests the substitution

$$u = e^{\pi x}, \quad du = \pi e^{\pi x}\, dx$$

from which we obtain

$$\int e^{\pi x} \sin^{-1}(e^{\pi x})\, dx = \frac{1}{\pi} \int \sin^{-1} u\, du$$

The integrand is now a basic function, and Formula (7) yields

$$\begin{aligned}\int e^{\pi x} \sin^{-1}(e^{\pi x})\, dx &= \frac{1}{\pi}\left[u \sin^{-1} u + \sqrt{1-u^2}\right] + C \\ &= \frac{1}{\pi}\left[e^{\pi x} \sin^{-1}(e^{\pi x}) + \sqrt{1-e^{2\pi x}}\right] + C\end{aligned}$$

Solution (b). Again, the integrand does not closely match any of the forms in the table. However, a little thought suggests that it may be possible to bring the integrand closer to the form $x\sqrt{x^2+a^2}$ by completing the square to eliminate the term involving x inside the radical. Doing this yields

$$\int x\sqrt{x^2-4x+5}\, dx = \int x\sqrt{(x^2-4x+4)+1}\, dx = \int x\sqrt{(x-2)^2+1}\, dx \qquad (1)$$

At this point we are closer to the form $x\sqrt{x^2+a^2}$, but we are not quite there because of the $(x-2)^2$ rather than x^2 inside the radical. However, we can resolve that problem with the substitution

$$u = x-2, \qquad du = dx$$

With this substitution we have $x = u+2$, so (1) can be expressed in terms of u as

$$\int x\sqrt{x^2-4x+5}\, dx = \int (u+2)\sqrt{u^2+1}\, du = \int u\sqrt{u^2+1}\, du + 2\int \sqrt{u^2+1}\, du$$

The first integral on the right is now a perfect match with Formula (84) with $a = 1$, and the second is a perfect match with Formula (72) with $a = 1$. Thus, applying these formulas we obtain

$$\int x\sqrt{x^2-4x+5}\, dx = \tfrac{1}{3}(u^2+1)^{3/2} + 2\left[\tfrac{1}{2}u\sqrt{u^2+1} + \tfrac{1}{2}\ln(u+\sqrt{u^2+1})\right] + C$$

If we now replace u by $x-2$ (in which case $u^2+1 = x^2-4x+5$), we obtain

$$\begin{aligned}\int x\sqrt{x^2-4x+5}\, dx = \tfrac{1}{3}(x^2-4x+5)^{3/2} &+ (x-2)\sqrt{x^2-4x+5} \\ &+ \ln\left(x-2+\sqrt{x^2-4x+5}\right) + C\end{aligned}$$

Although correct, this form of the answer has an unnecessary mixture of radicals and fractional exponents. If desired, we can "clean up" the answer by writing

$$(x^2-4x+5)^{3/2} = (x^2-4x+5)\sqrt{x^2-4x+5}$$

from which it follows that (verify)

$$\begin{aligned}\int x\sqrt{x^2-4x+5}\, dx = \tfrac{1}{3}(x^2-x-1)&\sqrt{x^2-4x+5} \\ &+ \ln\left(x-2+\sqrt{x^2-4x+5}\right) + C\end{aligned}$$ ◀

MATCHES REQUIRING REDUCTION FORMULAS

In cases where the entry in an integral table is a reduction formula, that formula will have to be applied first to reduce the given integral to a form in which it can be evaluated.

▶ **Example 3** Use the Endpaper Integral Table to evaluate $\int \frac{x^3}{\sqrt{1+x}}\,dx$.

Solution. The integrand can be classified as a power of x multiplying the reciprocal of $\sqrt{a+bx}$. Thus, from Formula (107) with $a = 1$, $b = 1$, and $n = 3$, followed by Formula (106), we obtain

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x}}\,dx &= \frac{2x^3\sqrt{1+x}}{7} - \frac{6}{7}\int \frac{x^2}{\sqrt{1+x}}\,dx \\ &= \frac{2x^3\sqrt{1+x}}{7} - \frac{6}{7}\left[\frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x}\right] + C \\ &= \left(\frac{2x^3}{7} - \frac{12x^2}{35} + \frac{16x}{35} - \frac{32}{35}\right)\sqrt{1+x} + C \end{aligned}$$ ◀

SPECIAL SUBSTITUTIONS

The Endpaper Integral Table has numerous entries involving an exponent of $3/2$ or involving square roots (exponent $1/2$), but it has no entries with other fractional exponents. However, integrals involving fractional powers of x can often be simplified by making the substitution $u = x^{1/n}$ in which n is the least common multiple of the denominators of the exponents. The resulting integral will then involve integer powers of u.

▶ **Example 4** Evaluate

$$\text{(a)} \int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx \qquad \text{(b)} \int \sqrt{1+e^x}\,dx$$

Solution (a). The integrand contains $x^{1/2}$ and $x^{1/3}$, so we make the substitution $u = x^{1/6}$, from which we obtain

$$x = u^6, \quad dx = 6u^5\,du$$

Thus,

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int \frac{(u^6)^{1/2}}{1+(u^6)^{1/3}}(6u^5)\,du = 6\int \frac{u^8}{1+u^2}\,du$$

By long division

$$\frac{u^8}{1+u^2} = u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}$$

from which it follows that

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx &= 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}\right) du \\ &= \tfrac{6}{7}u^7 - \tfrac{6}{5}u^5 + 2u^3 - 6u + 6\tan^{-1} u + C \\ &= \tfrac{6}{7}x^{7/6} - \tfrac{6}{5}x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1}(x^{1/6}) + C\end{aligned}$$

Solution (b). The integral does not match any of the forms in the Endpaper Integral Table. However, the table does include several integrals containing $\sqrt{a+bu}$. This suggests the substitution $u = e^x$, from which we obtain

$$x = \ln u, \quad dx = \frac{1}{u}\,du$$

Thus, from Formula (110) with $a = 1$ and $b = 1$, followed by Formula (108), we obtain

$$\begin{aligned}
\int \sqrt{1+e^x}\, dx &= \int \frac{\sqrt{1+u}}{u}\, du \\
&= 2\sqrt{1+u} + \int \frac{du}{u\sqrt{1+u}} \\
&= 2\sqrt{1+u} + \ln\left|\frac{\sqrt{1+u}-1}{\sqrt{1+u}+1}\right| + C \\
&= 2\sqrt{1+e^x} + \ln\left[\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right] + C
\end{aligned}$$

Absolute value not needed

Try finding the antiderivative in Example 4(b) using the substitution

$$u = \sqrt{1+e^x}$$

Functions that consist of finitely many sums, differences, quotients, and products of $\sin x$ and $\cos x$ are called ***rational functions of sin x and cos x***. Some examples are

$$\frac{\sin x + 3\cos^2 x}{\cos x + 4\sin x}, \quad \frac{\sin x}{1+\cos x - \cos^2 x}, \quad \frac{3\sin^5 x}{1+4\sin x}$$

The Endpaper Integral Table gives a few formulas for integrating rational functions of $\sin x$ and $\cos x$ under the heading *Reciprocals of Basic Functions*. For example, it follows from Formula (18) that

$$\int \frac{1}{1+\sin x}\, dx = \tan x - \sec x + C \tag{2}$$

However, since the integrand is a rational function of $\sin x$, it may be desirable in a particular application to express the value of the integral in terms of $\sin x$ and $\cos x$ and rewrite (2) as

$$\int \frac{1}{1+\sin x}\, dx = \frac{\sin x - 1}{\cos x} + C$$

Many rational functions of $\sin x$ and $\cos x$ can be evaluated by an ingenious method that was discovered by the mathematician Karl Weierstrass (see p. 102 for biography). The idea is to make the substitution

$$u = \tan(x/2), \quad -\pi/2 < x/2 < \pi/2$$

from which it follows that

$$x = 2\tan^{-1} u, \quad dx = \frac{2}{1+u^2}\, du$$

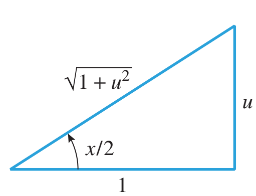
To implement this substitution we need to express $\sin x$ and $\cos x$ in terms of u. For this purpose we will use the identities

$$\sin x = 2\sin(x/2)\cos(x/2) \tag{3}$$

$$\cos x = \cos^2(x/2) - \sin^2(x/2) \tag{4}$$

and the following relationships suggested by Figure 7.6.1:

$$\sin(x/2) = \frac{u}{\sqrt{1+u^2}} \quad \text{and} \quad \cos(x/2) = \frac{1}{\sqrt{1+u^2}}$$



▲ **Figure 7.6.1**

Substituting these expressions in (3) and (4) yields

$$\sin x = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2}$$

$$\cos x = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$$

In summary, we have shown that the substitution $u = \tan(x/2)$ can be implemented in a rational function of $\sin x$ and $\cos x$ by letting

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\,du \tag{5}$$

▶ **Example 5** Evaluate $\displaystyle\int \frac{dx}{1 - \sin x + \cos x}$.

Solution. The integrand is a rational function of $\sin x$ and $\cos x$ that does not match any of the formulas in the Endpaper Integral Table, so we make the substitution $u = \tan(x/2)$. Thus, from (5) we obtain

$$\begin{aligned}
\int \frac{dx}{1-\sin x+\cos x} &= \int \frac{\dfrac{2\,du}{1+u^2}}{1 - \left(\dfrac{2u}{1+u^2}\right) + \left(\dfrac{1-u^2}{1+u^2}\right)} \\
&= \int \frac{2\,du}{(1+u^2) - 2u + (1-u^2)} \\
&= \int \frac{du}{1-u} = -\ln|1-u| + C = -\ln|1-\tan(x/2)| + C \quad ◀
\end{aligned}$$

The substitution $u = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ to an ordinary rational function of u. However, the method can lead to cumbersome partial fraction decompositions, so it may be worthwhile to consider other methods as well when hand computations are being used.

INTEGRATING WITH COMPUTER ALGEBRA SYSTEMS

Integration tables are rapidly giving way to computerized integration using computer algebra systems. However, as with many powerful tools, a knowledgeable operator is an important component of the system.

Sometimes computer algebra systems do not produce the most general form of the indefinite integral. For example, the integral formula

$$\int \frac{dx}{x-1} = \ln|x-1| + C$$

which can be obtained by inspection or by using the substitution $u = x - 1$, is valid for $x > 1$ or for $x < 1$. However, not all computer algebra systems produce this form of the answer. Some typical answers produced by various implementations of *Mathematica*, *Maple*, and the CAS on a handheld calculator are

$$\ln(-1+x), \quad \ln(x-1), \quad \ln(|x-1|)$$

Observe that none of the systems include the constant of integration—the answer produced is a particular antiderivative and not the most general antiderivative (indefinite integral). Observe also that only one of these answers includes the absolute value signs; the antiderivatives produced by the other systems are valid only for $x > 1$. All systems, however, are able to calculate the definite integral

$$\int_0^{1/2} \frac{dx}{x-1} = -\ln 2$$

correctly. Now let us examine how these systems handle the integral

$$\begin{aligned}
\int x\sqrt{x^2-4x+5}\,dx = {} & \tfrac{1}{3}(x^2-x-1)\sqrt{x^2-4x+5} \\
& + \ln\left(x-2+\sqrt{x^2-4x+5}\right)
\end{aligned} \tag{6}$$

which we obtained in Example 2(b) (with the constant of integration included). Some CAS implementations produce this result in slightly different algebraic forms, but a version of *Maple* produces the result

$$\int x\sqrt{x^2-4x+5}\,dx = \tfrac{1}{3}(x^2-4x+5)^{3/2} + \tfrac{1}{2}(2x-4)\sqrt{x^2-4x+5} + \sinh^{-1}(x-2)$$

This can be rewritten as (6) by expressing the fractional exponent in radical form and expressing $\sinh^{-1}(x-2)$ in logarithmic form using Theorem 6.9.4 (verify). A version of *Mathematica* produces the result

$$\int x\sqrt{x^2-4x+5}\,dx = \tfrac{1}{3}(x^2-x-1)\sqrt{x^2-4x+5} - \sinh^{-1}(2-x)$$

which can be rewritten in form (6) by using Theorem 6.9.4 together with the identity $\sinh^{-1}(-x) = -\sinh^{-1} x$ (verify).

Expanding the expression

$$\frac{(x+1)^8}{8}$$

produces a constant term of $\frac{1}{8}$, whereas the second expression in (7) has no constant term. What is the explanation?

Computer algebra systems can sometimes produce inconvenient or unnatural answers to integration problems. For example, various computer algebra systems produced the following results when asked to integrate $(x+1)^7$:

$$\frac{(x+1)^8}{8}, \quad \frac{1}{8}x^8 + x^7 + \frac{7}{2}x^6 + 7x^5 + \frac{35}{4}x^4 + 7x^3 + \frac{7}{2}x^2 + x \tag{7}$$

The first form is in keeping with the hand computation

$$\int (x+1)^7\,dx = \frac{(x+1)^8}{8} + C$$

that uses the substitution $u = x+1$, whereas the second form is based on expanding $(x+1)^7$ and integrating term by term.

In Example 2(a) of Section 7.3 we showed that

$$\int \sin^4 x \cos^5 x\,dx = \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C$$

However, a version of *Mathematica* integrates this as

$$\tfrac{3}{128}\sin x - \tfrac{1}{192}\sin 3x - \tfrac{1}{320}\sin 5x + \tfrac{1}{1792}\sin 7x + \tfrac{1}{2304}\sin 9x$$

whereas other computer algebra systems essentially integrate it as

$$-\tfrac{1}{9}\sin^3 x\cos^6 x - \tfrac{1}{21}\sin x\cos^6 x + \tfrac{1}{105}\cos^4 x\sin x + \tfrac{4}{315}\cos^2 x\sin x + \tfrac{8}{315}\sin x$$

Although these three results look quite different, they can be obtained from one another using appropriate trigonometric identities.

COMPUTER ALGEBRA SYSTEMS HAVE LIMITATIONS

A computer algebra system combines a set of integration rules (such as substitution) with a library of functions that it can use to construct antiderivatives. Such libraries contain elementary functions, such as polynomials, rational functions, trigonometric functions, as well as various nonelementary functions that arise in engineering, physics, and other applied fields. Just as our Endpaper Integral Table has only 121 indefinite integrals, these libraries are not exhaustive of all possible integrands. If the system cannot manipulate the integrand to a form matching one in its library, the program will give some indication that it cannot evaluate the integral. For example, when asked to evaluate the integral

$$\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx \tag{8}$$

all of the systems mentioned above respond by displaying some form of the unevaluated integral as an answer, indicating that they could not perform the integration.

TECHNOLOGY MASTERY

Sometimes integrals that cannot be evaluated by a CAS in their given form can be evaluated by first rewriting them in a different form or by making a substitution. If you have a CAS, make a u-substitution in (8) that will enable you to evaluate the integral with your CAS. Then evaluate the integral.

Sometimes computer algebra systems respond by expressing an integral in terms of another integral. For example, if you try to integrate e^{x^2} using *Mathematica*, *Maple*, or

Sage, you will obtain an expression involving erf (which stands for ***error function***). The function $\operatorname{erf}(x)$ is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

so all three programs essentially rewrite the given integral in terms of a closely related integral. From one point of view this is what we did in integrating $1/x$, since the natural logarithm function is (formally) defined as

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

(see Section 5.10).

Example 6 A particle moves along an x-axis in such a way that its velocity $v(t)$ at time t is

$$v(t) = 30 \cos^7 t \sin^4 t \quad (t \geq 0)$$

Graph the position versus time curve for the particle, given that the particle is at $x = 1$ when $t = 0$.

Solution. Since $dx/dt = v(t)$ and $x = 1$ when $t = 0$, the position function $x(t)$ is given by

$$x(t) = 1 + \int_0^t v(s)\, ds$$

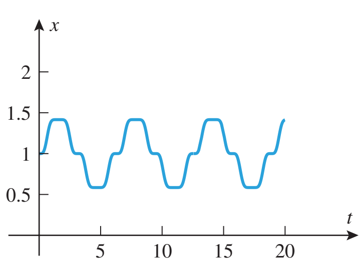
Some computer algebra systems will allow this expression to be entered directly into a command for plotting functions, but it is often more efficient to perform the integration first. The authors' integration utility yields

$$\begin{aligned} x &= \int 30 \cos^7 t \sin^4 t\, dt \\ &= -\tfrac{30}{11} \sin^{11} t + 10 \sin^9 t - \tfrac{90}{7} \sin^7 t + 6 \sin^5 t + C \end{aligned}$$

where we have added the required constant of integration. Using the initial condition $x(0) = 1$, we substitute the values $x = 1$ and $t = 0$ into this equation to find that $C = 1$, so

$$x(t) = -\tfrac{30}{11} \sin^{11} t + 10 \sin^9 t - \tfrac{90}{7} \sin^7 t + 6 \sin^5 t + 1 \quad (t \geq 0)$$

The graph of x versus t is shown in Figure 7.6.2. ◀



▲ **Figure 7.6.2**

✔ QUICK CHECK EXERCISES 7.6 *(See page 533 for answers.)*

1. Find an integral formula in the Endpaper Integral Table that can be used to evaluate the integral. Do not evaluate the integral.

(a) $\displaystyle\int \frac{2x}{3x+4}\, dx$ ________

(b) $\displaystyle\int \frac{1}{x\sqrt{5x-4}}\, dx$ ________

(c) $\displaystyle\int x\sqrt{3x+2}\, dx$ ________

(d) $\displaystyle\int x^2 \ln x\, dx$ ________

2. In each part, make the indicated u-substitution, and then find an integral formula in the Endpaper Integral Table that can be used to evaluate the integral. Do not evaluate the integral.

(a) $\displaystyle\int \frac{x}{1+e^{x^2}}\, dx;\ u = x^2$ ________

(b) $\displaystyle\int e^{\sqrt{x}}\, dx;\ u = \sqrt{x}$ ________

(c) $\displaystyle\int \frac{e^x}{1+\sin(e^x)}\, dx;\ u = e^x$ ________

(d) $\displaystyle\int \frac{1}{(1-4x^2)^{3/2}}\, dx;\ u = 2x$ ________

3. In each part, use the Endpaper Integral Table to evaluate the integral. (If necessary, first make an appropriate substitution or complete the square.) *(cont.)*

(a) $\displaystyle\int \frac{1}{4-x^2}\,dx = $ ________

(b) $\displaystyle\int \cos 2x \cos x\,dx = $ ________

(c) $\displaystyle\int \frac{e^{3x}}{\sqrt{1-e^{2x}}}\,dx = $ ________

(d) $\displaystyle\int \frac{x}{x^2-4x+8}\,dx = $ ________

EXERCISE SET 7.6 C CAS

C **1–24** (a) Use the Endpaper Integral Table to evaluate the given integral. (b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a). ■

1. $\displaystyle\int \frac{4x}{3x-1}\,dx$

2. $\displaystyle\int \frac{x}{(4-5x)^2}\,dx$

3. $\displaystyle\int \frac{1}{x(2x+5)}\,dx$

4. $\displaystyle\int \frac{1}{x^2(1-5x)}\,dx$

5. $\displaystyle\int x\sqrt{2x+3}\,dx$

6. $\displaystyle\int \frac{x}{\sqrt{2-x}}\,dx$

7. $\displaystyle\int \frac{1}{x\sqrt{4-3x}}\,dx$

8. $\displaystyle\int \frac{1}{x\sqrt{3x-4}}\,dx$

9. $\displaystyle\int \frac{1}{16-x^2}\,dx$

10. $\displaystyle\int \frac{1}{x^2-9}\,dx$

11. $\displaystyle\int \sqrt{x^2-3}\,dx$

12. $\displaystyle\int \frac{\sqrt{x^2-5}}{x^2}\,dx$

13. $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$

14. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-2}}\,dx$

15. $\displaystyle\int \sqrt{9-x^2}\,dx$

16. $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

17. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

18. $\displaystyle\int \frac{1}{x\sqrt{6x-x^2}}\,dx$

19. $\displaystyle\int \sin 3x \sin 4x\,dx$

20. $\displaystyle\int \sin 2x \cos 5x\,dx$

21. $\displaystyle\int x^3 \ln x\,dx$

22. $\displaystyle\int \frac{\ln x}{\sqrt{x^3}}\,dx$

23. $\displaystyle\int e^{-2x}\sin 3x\,dx$

24. $\displaystyle\int e^{x}\cos 2x\,dx$

C **25–36** (a) Make the indicated u-substitution, and then use the Endpaper Integral Table to evaluate the integral. (b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a). ■

25. $\displaystyle\int \frac{e^{4x}}{(4-3e^{2x})^2}\,dx,\ u = e^{2x}$

26. $\displaystyle\int \frac{\sin 2x}{(\cos 2x)(3-\cos 2x)}\,dx,\ u = \cos 2x$

27. $\displaystyle\int \frac{1}{\sqrt{x}(9x+4)}\,dx,\ u = 3\sqrt{x}$

28. $\displaystyle\int \frac{\cos 4x}{9+\sin^2 4x}\,dx,\ u = \sin 4x$

29. $\displaystyle\int \frac{1}{\sqrt{4x^2-9}}\,dx,\ u = 2x$

30. $\displaystyle\int x\sqrt{2x^4+3}\,dx,\ u = \sqrt{2}x^2$

31. $\displaystyle\int \frac{4x^5}{\sqrt{2-4x^4}}\,dx,\ u = 2x^2$

32. $\displaystyle\int \frac{1}{x^2\sqrt{3-4x^2}}\,dx,\ u = 2x$

33. $\displaystyle\int \frac{\sin^2(\ln x)}{x}\,dx,\ u = \ln x$

34. $\displaystyle\int e^{-2x}\cos^2(e^{-2x})\,dx,\ u = e^{-2x}$

35. $\displaystyle\int xe^{-2x}\,dx,\ u = -2x$

36. $\displaystyle\int \ln(3x+1)\,dx,\ u = 3x+1$

C **37–48** (a) Make an appropriate u-substitution, and then use the Endpaper Integral Table to evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a). ■

37. $\displaystyle\int \frac{\cos 3x}{(\sin 3x)(\sin 3x+1)^2}\,dx$

38. $\displaystyle\int \frac{\ln x}{x\sqrt{4\ln x-1}}\,dx$

39. $\displaystyle\int \frac{x}{16x^4-1}\,dx$

40. $\displaystyle\int \frac{e^x}{3-4e^{2x}}\,dx$

41. $\displaystyle\int e^x\sqrt{3-4e^{2x}}\,dx$

42. $\displaystyle\int \frac{\sqrt{4-9x^2}}{x^2}\,dx$

43. $\displaystyle\int \sqrt{5x-9x^2}\,dx$

44. $\displaystyle\int \frac{1}{x\sqrt{x-5x^2}}\,dx$

45. $\displaystyle\int 4x\sin 2x\,dx$

46. $\displaystyle\int \cos\sqrt{x}\,dx$

47. $\displaystyle\int e^{-\sqrt{x}}\,dx$

48. $\displaystyle\int x\ln(2+x^2)\,dx$

C **49–52** (a) Complete the square, make an appropriate u-substitution, and then use the Endpaper Integral Table to evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution or square completion), and then confirm that the result is equivalent to that in part (a). ■

49. $\displaystyle\int \frac{1}{x^2+6x-7}\,dx$

50. $\displaystyle\int \sqrt{3-2x-x^2}\,dx$

51. $\displaystyle\int \frac{x}{\sqrt{5+4x-x^2}}\,dx$ **52.** $\displaystyle\int \frac{x}{x^2+6x+13}\,dx$

C **53–64** (a) Make an appropriate u-substitution of the form $u = x^{1/n}$ or $u = (x+a)^{1/n}$, and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a). ■

53. $\displaystyle\int x\sqrt{x-2}\,dx$ **54.** $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

55. $\displaystyle\int x^5\sqrt{x^3+1}\,dx$ **56.** $\displaystyle\int \frac{1}{x\sqrt{x^3-1}}\,dx$

57. $\displaystyle\int \frac{dx}{x-\sqrt[3]{x}}$ **58.** $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

59. $\displaystyle\int \frac{dx}{x(1-x^{1/4})}$ **60.** $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx$

61. $\displaystyle\int \frac{dx}{x^{1/2}-x^{1/3}}$ **62.** $\displaystyle\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$

63. $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ **64.** $\displaystyle\int \frac{x}{(x+3)^{1/5}}\,dx$

C **65–70** (a) Make u-substitution (5) to convert the integrand to a rational function of u, and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a). ■

65. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$ **66.** $\displaystyle\int \frac{dx}{2+\sin x}$

67. $\displaystyle\int \frac{d\theta}{1-\cos\theta}$ **68.** $\displaystyle\int \frac{dx}{4\sin x-3\cos x}$

69. $\displaystyle\int \frac{dx}{\sin x+\tan x}$ **70.** $\displaystyle\int \frac{\sin x}{\sin x+\tan x}\,dx$

71–72 Use any method to solve for x. ■

71. $\displaystyle\int_2^x \frac{1}{t(4-t)}\,dt = 0.5,\ 2 < x < 4$

72. $\displaystyle\int_1^x \frac{1}{t\sqrt{2t-1}}\,dt = 1,\ x > \tfrac{1}{2}$

73–76 Use any method to find the area of the region enclosed by the curves. ■

73. $y = \sqrt{25-x^2},\ y = 0,\ x = 0,\ x = 4$

74. $y = \sqrt{9x^2-4},\ y = 0,\ x = 2$

75. $y = \dfrac{1}{25-16x^2},\ y = 0,\ x = 0,\ x = 1$

76. $y = \sqrt{x}\ln x,\ y = 0,\ x = 4$

77–80 Use any method to find the volume of the solid generated when the region enclosed by the curves is revolved about the y-axis. ■

77. $y = \cos x,\ y = 0,\ x = 0,\ x = \pi/2$

78. $y = \sqrt{x-4},\ y = 0,\ x = 8$

79. $y = e^{-x},\ y = 0,\ x = 0,\ x = 3$

80. $y = \ln x,\ y = 0,\ x = 5$

81–82 Use any method to find the arc length of the curve. ■

81. $y = 2x^2,\ 0 \le x \le 2$ **82.** $y = 3\ln x,\ 1 \le x \le 3$

83–84 Use any method to find the area of the surface generated by revolving the curve about the x-axis. ■

83. $y = \sin x,\ 0 \le x \le \pi$ **84.** $y = 1/x,\ 1 \le x \le 4$

C **85–86** Information is given about the motion of a particle moving along a coordinate line.

(a) Use a CAS to find the position function of the particle for $t \ge 0$.

(b) Graph the position versus time curve. ■

85. $v(t) = 20\cos^6 t\sin^3 t,\ s(0) = 2$

86. $a(t) = e^{-t}\sin 2t\sin 4t,\ v(0) = 0,\ s(0) = 10$

FOCUS ON CONCEPTS

87. (a) Use the substitution $u = \tan(x/2)$ to show that

$$\int \sec x\,dx = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C$$

and confirm that this is consistent with Formula (22) of Section 7.3.

(b) Use the result in part (a) to show that

$$\int \sec x\,dx = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

88. Use the substitution $u = \tan(x/2)$ to show that

$$\int \csc x\,dx = \frac{1}{2}\ln\left[\frac{1-\cos x}{1+\cos x}\right] + C$$

and confirm that this is consistent with the result in Exercise 65(a) of Section 7.3.

89. Find a substitution that can be used to integrate rational functions of $\sinh x$ and $\cosh x$ and use your substitution to evaluate

$$\int \frac{dx}{2\cosh x+\sinh x}$$

without expressing the integrand in terms of e^x and e^{-x}.

C **90–93** Some integrals that can be evaluated by hand cannot be evaluated by all computer algebra systems. Evaluate the integral by hand, and determine if it can be evaluated on your CAS. ■

90. $\displaystyle\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

91. $\displaystyle\int (\cos^{32} x\sin^{30} x - \cos^{30} x\sin^{32} x)\,dx$

92. $\displaystyle\int \sqrt{x-\sqrt{x^2-4}}\,dx$ [*Hint:* $\frac{1}{2}(\sqrt{x+2}-\sqrt{x-2})^2 = ?$]

93. $\displaystyle\int \frac{1}{x^{10}+x}\,dx$
[*Hint:* Rewrite the denominator as $x^{10}(1+x^{-9})$.]

C **94.** Let

$$f(x)=\frac{-2x^5+26x^4+15x^3+6x^2+20x+43}{x^6-x^5-18x^4-2x^3-39x^2-x-20}$$

(a) Use a CAS to factor the denominator, and then write down the form of the partial fraction decomposition. You need not find the values of the constants.

(b) Check your answer in part (a) by using the CAS to find the partial fraction decomposition of f.

(c) Integrate f by hand, and then check your answer by integrating with the CAS.

✔ QUICK CHECK ANSWERS 7.6

1. (a) Formula (60) (b) Formula (108) (c) Formula (102) (d) Formula (50) **2.** (a) Formula (25) (b) Formula (51) (c) Formula (18) (d) Formula (97) **3.** (a) $\frac{1}{4}\ln\left|\frac{x+2}{x-2}\right|+C$ (b) $\frac{1}{6}\sin 3x+\frac{1}{2}\sin x+C$ (c) $-\frac{e^x}{2}\sqrt{1-e^{2x}}+\frac{1}{2}\sin^{-1}e^x+C$ (d) $\frac{1}{2}\ln\left(x^2-4x+8\right)+\tan^{-1}\frac{x-2}{2}+C$

7.7 NUMERICAL INTEGRATION; SIMPSON'S RULE

If it is necessary to evaluate a definite integral of a function for which an antiderivative cannot be found, then one must settle for some kind of numerical approximation of the integral. In Section 5.4 we considered three such approximations in the context of areas—left endpoint approximation, right endpoint approximation, and midpoint approximation. In this section we will extend those methods to general definite integrals, and we will develop some new methods that often provide more accuracy with less computation. We will also discuss the errors that arise in integral approximations.

A REVIEW OF RIEMANN SUM APPROXIMATIONS

Recall from Section 5.5 that the definite integral of a continuous function f over an interval $[a, b]$ may be computed as

$$\int_a^b f(x)\,dx=\lim_{\max \Delta x_k\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

where the sum that appears on the right side is called a Riemann sum. In this formula, Δx_k is the width of the kth subinterval of a partition $a=x_0<x_1<x_2<\cdots<x_n=b$ of $[a,b]$ into n subintervals, and x_k^* denotes an arbitrary point in the kth subinterval. If we take all subintervals of the same width, so that $\Delta x_k=(b-a)/n$, then as n increases the Riemann sum will eventually be a good approximation to the definite integral. We denote this by writing

$$\int_a^b f(x)\,dx\approx\left(\frac{b-a}{n}\right)[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)] \tag{1}$$

If we denote the values of f at the endpoints of the subintervals by

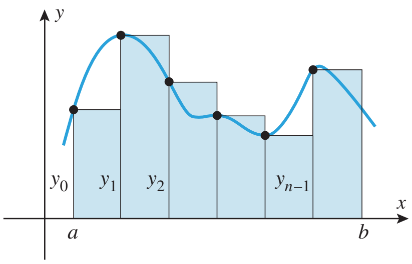
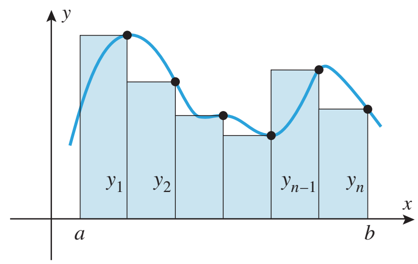
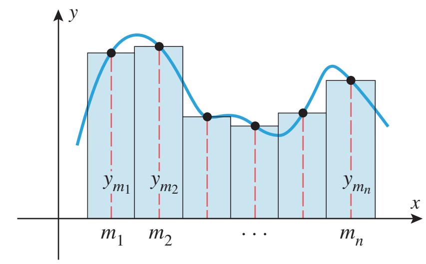
$$y_0=f(a),\quad y_1=f(x_1),\quad y_2=f(x_2),\ldots,y_{n-1}=f(x_{n-1}),\quad y_n=f(x_n)$$

and the values of f at the midpoints of the subintervals by

$$y_{m_1}, y_{m_2},\ldots,y_{m_n}$$

then it follows from (1) that the left endpoint, right endpoint, and midpoint approximations discussed in Section 5.4 can be expressed as shown in Table 7.7.1. Although we originally

Table 7.7.1

LEFT ENDPOINT APPROXIMATION	RIGHT ENDPOINT APPROXIMATION	MIDPOINT APPROXIMATION
$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[y_0 + y_1 + \cdots + y_{n-1}]$	$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[y_1 + y_2 + \cdots + y_n]$	$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[y_{m_1} + y_{m_2} + \cdots + y_{m_n}]$
		

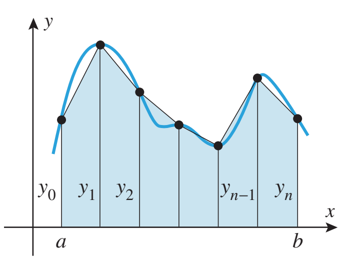
obtained these results for nonnegative functions in the context of approximating areas, they are applicable to any function that is continuous on $[a, b]$.

TRAPEZOIDAL APPROXIMATION

It will be convenient in this section to denote the left endpoint, right endpoint, and midpoint approximations with n subintervals by L_n, R_n, and M_n, respectively. Of the three approximations, the midpoint approximation is most widely used in applications. If we take the average of L_n and R_n, then we obtain another important approximation denoted by

$$T_n = \tfrac{1}{2}(L_n + R_n)$$

called the ***trapezoidal approximation***:



▲ **Figure 7.7.1**

Trapezoidal Approximation

$$\int_a^b f(x)\,dx \approx T_n = \left(\frac{b-a}{2n}\right)[y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n] \qquad (2)$$

The name "trapezoidal approximation" results from the fact that in the case where f is nonnegative on the interval of integration, the approximation T_n is the sum of the trapezoidal areas shown in Figure 7.7.1 (see Exercise 51).

Example 1 In Table 7.7.2 we have approximated

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

using the midpoint approximation and the trapezoidal approximation.* In each case we used $n = 10$ subdivisions of the interval $[1, 2]$, so that

$$\underbrace{\frac{b-a}{n} = \frac{2-1}{10} = 0.1}_{\text{Midpoint}} \quad \text{and} \quad \underbrace{\frac{b-a}{2n} = \frac{2-1}{20} = 0.05}_{\text{Trapezoidal}} \blacktriangleleft$$

*Throughout this section we will show numerical values to nine places to the right of the decimal point. If your calculating utility does not show this many places, then you will need to make the appropriate adjustments. What is important here is that you understand the principles being discussed.

Table 7.7.2

MIDPOINT AND TRAPEZOIDAL APPROXIMATIONS FOR $\int_1^2 \frac{1}{x}\,dx$

MIDPOINT APPROXIMATION			TRAPEZOIDAL APPROXIMATION				
i	MIDPOINT m_i	$y_{m_i} = f(m_i) = 1/m_i$	i	ENDPOINT x_i	$y_i = f(x_i) = 1/x_i$	MULTIPLIER w_i	$w_i y_i$
1	1.05	0.952380952	0	1.0	1.000000000	1	1.000000000
2	1.15	0.869565217	1	1.1	0.909090909	2	1.818181818
3	1.25	0.800000000	2	1.2	0.833333333	2	1.666666667
4	1.35	0.740740741	3	1.3	0.769230769	2	1.538461538
5	1.45	0.689655172	4	1.4	0.714285714	2	1.428571429
6	1.55	0.645161290	5	1.5	0.666666667	2	1.333333333
7	1.65	0.606060606	6	1.6	0.625000000	2	1.250000000
8	1.75	0.571428571	7	1.7	0.588235294	2	1.176470588
9	1.85	0.540540541	8	1.8	0.555555556	2	1.111111111
10	1.95	0.512820513	9	1.9	0.526315789	2	1.052631579
		6.928353603	10	2.0	0.500000000	1	0.500000000
							13.875428063
$\int_1^2 \frac{1}{x}\,dx \approx (0.1)(6.928353603) \approx 0.692835360$			$\int_1^2 \frac{1}{x}\,dx \approx (0.05)(13.875428063) \approx 0.693771403$				

COMPARISON OF THE MIDPOINT AND TRAPEZOIDAL APPROXIMATIONS

We define the ***errors*** in the midpoint and trapezoidal approximations to be

$$E_M = \int_a^b f(x)\,dx - M_n \quad \text{and} \quad E_T = \int_a^b f(x)\,dx - T_n \tag{3–4}$$

respectively, and we define $|E_M|$ and $|E_T|$ to be the ***absolute errors*** in these approximations. The absolute errors are nonnegative and do not distinguish between underestimates and overestimates.

By rewriting (3) and (4) in the form

$$\int_a^b f(x)\,dx = \text{approximation} + \text{error}$$

we see that positive values of E_M and E_T correspond to underestimates and negative values to overestimates.

Example 2 The value of ln 2 to nine decimal places is

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx \approx 0.693147181 \tag{5}$$

so we see from Tables 7.7.2 and 7.7.3 that the absolute errors in approximating ln 2 by M_{10} and T_{10} are

$$|E_M| = |\ln 2 - M_{10}| \approx 0.000311821$$

$$|E_T| = |\ln 2 - T_{10}| \approx 0.000624222$$

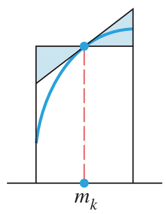
Thus, the midpoint approximation is more accurate than the trapezoidal approximation in this case. ◀

Table 7.7.3

ln 2 (NINE DECIMAL PLACES)	APPROXIMATION	ERROR
0.693147181	$M_{10} \approx 0.692835360$	$E_M = \ln 2 - M_{10} \approx 0.000311821$
0.693147181	$T_{10} \approx 0.693771403$	$E_T = \ln 2 - T_{10} \approx -0.000624222$

It is not accidental in Example 2 that the midpoint approximation of ln 2 was more accurate than the trapezoidal approximation. To see why this is so, we first need to look at the midpoint approximation from another point of view. To simplify our explanation, we will assume that f is nonnegative on $[a, b]$, though the conclusions we reach will be true without this assumption.

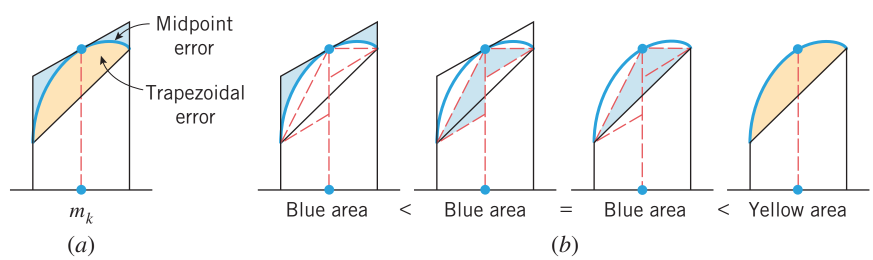
If f is a differentiable function, then the midpoint approximation is sometimes called the ***tangent line approximation*** because for each subinterval of $[a, b]$ the area of the rectangle used in the midpoint approximation is equal to the area of the trapezoid whose upper boundary is the tangent line to $y = f(x)$ at the midpoint of the subinterval (Figure 7.7.2). The equality of these areas follows from the fact that the shaded areas in the figure are congruent. We will now show how this point of view about midpoint approximations can be used to establish useful criteria for determining which of M_n or T_n produces the better approximation of a given integral.



▲ **Figure 7.7.2**

In Figure 7.7.3*a* we have isolated a subinterval of $[a, b]$ on which the graph of a function f is concave down, and we have shaded the areas that represent the errors in the midpoint and trapezoidal approximations over the subinterval. In Figure 7.7.3*b* we show a succession of four illustrations which make it evident that the error from the midpoint approximation is less than that from the trapezoidal approximation. If the graph of f were concave up, analogous figures would lead to the same conclusion. (This argument, due to Frank Buck, appeared in *The College Mathematics Journal*, Vol. 16, No. 1, 1985.)

Justify the conclusions in each step of Figure 7.7.3*b*.



▲ **Figure 7.7.3**

Figure 7.7.3*a* also suggests that on a subinterval where the graph is concave down, the midpoint approximation is larger than the value of the integral and the trapezoidal approximation is smaller. On an interval where the graph is concave up it is the other way around. In summary, we have the following result, which we state without formal proof:

7.7.1 THEOREM *Let f be continuous on $[a, b]$, and let $|E_M|$ and $|E_T|$ be the absolute errors that result from the midpoint and trapezoidal approximations of $\int_a^b f(x)\,dx$ using n subintervals.*

(a) *If the graph of f is either concave up or concave down on (a, b), then $|E_M| < |E_T|$, that is, the absolute error from the midpoint approximation is less than that from the trapezoidal approximation.*

(b) *If the graph of f is concave down on (a, b), then*

$$T_n < \int_a^b f(x)\,dx < M_n$$

(c) *If the graph of f is concave up on (a, b), then*

$$M_n < \int_a^b f(x)\,dx < T_n$$

▶ Example 3 Since the graph of $f(x) = 1/x$ is continuous on the interval $[1, 2]$ and concave up on the interval $(1, 2)$, it follows from part (a) of Theorem 7.7.1 that M_n will always provide a better approximation than T_n for

$$\int_1^2 \frac{1}{x}\, dx = \ln 2$$

Moreover, if follows from part (c) of Theorem 7.7.1 that $M_n < \ln 2 < T_n$ for every positive integer n. Note that this is consistent with our computations in Example 2. ◀

WARNING

Do not conclude that the midpoint approximation is always better than the trapezoidal approximation; the trapezoidal approximation may be better if the function changes concavity on the interval of integration.

▶ Example 4 The midpoint and trapezoidal approximations can be used to approximate sin 1 by using the integral

$$\sin 1 = \int_0^1 \cos x\, dx$$

Since $f(x) = \cos x$ is continuous on $[0, 1]$ and concave down on $(0, 1)$, it follows from parts (a) and (b) of Theorem 7.7.1 that the absolute error in M_n will be less than that in T_n, and that $T_n < \sin 1 < M_n$ for every positive integer n. This is consistent with the results in Table 7.7.4 for $n = 5$ (intermediate computations are omitted). ◀

Table 7.7.4

sin 1 (NINE DECIMAL PLACES)	APPROXIMATION	ERROR
0.841470985	$M_5 \approx 0.842875074$	$E_M = \sin 1 - M_5 \approx -0.001404089$
0.841470985	$T_5 \approx 0.838664210$	$E_T = \sin 1 - T_5 \approx 0.002806775$

▶ Example 5 Table 7.7.5 shows approximations for $\sin 3 = \int_0^3 \cos x\, dx$ using the midpoint and trapezoidal approximations with $n = 10$ subdivisions of the interval $[0, 3]$. Note that $|E_M| < |E_T|$ and $T_{10} < \sin 3 < M_{10}$, although these results are not guaranteed by Theorem 7.7.1 since $f(x) = \cos x$ changes concavity on the interval $[0, 3]$. ◀

Table 7.7.5

sin 3 (NINE DECIMAL PLACES)	APPROXIMATION	ERROR
0.141120008	$M_{10} \approx 0.141650601$	$E_M = \sin 3 - M_{10} \approx -0.000530592$
0.141120008	$T_{10} \approx 0.140060017$	$E_T = \sin 3 - T_{10} \approx 0.001059991$

SIMPSON'S RULE

When the left and right endpoint approximations are averaged to produce the trapezoidal approximation, a better approximation often results. We now see how a weighted average of the midpoint and trapezoidal approximations can yield an even better approximation.

The numerical evidence in Tables 7.7.3, 7.7.4, and 7.7.5 reveals that $E_T \approx -2E_M$, so that $2E_M + E_T \approx 0$ in these instances. This suggests that

$$\begin{aligned} 3\int_a^b f(x)\,dx &= 2\int_a^b f(x)\,dx + \int_a^b f(x)\,dx \\ &= 2(M_k + E_M) + (T_k + E_T) \\ &= (2M_k + T_k) + (2E_M + E_T) \\ &\approx 2M_k + T_k \end{aligned}$$

This gives

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}(2M_k + T_k) \tag{6}$$

The midpoint approximation M_k in (6) requires the evaluation of f at k points in the interval $[a, b]$, and the trapezoidal approximation T_k in (6) requires the evaluation of f at $k+1$ points in $[a, b]$. Thus, $\frac{1}{3}(2M_k + T_k)$ uses $2k+1$ values of f, taken at equally spaced points in the interval $[a, b]$. These points are obtained by partitioning $[a, b]$ into $2k$ equal subintervals indicated by the left endpoints, right endpoints, and midpoints used in T_k and M_k, respectively. Setting $n = 2k$, we use S_n to denote the weighted average of M_k and T_k in (6). That is,

WARNING

Note that in (7) the subscript n in S_n is always *even* since it is twice the value of the subscripts for the corresponding midpoint and trapezoidal approximations. For example,

$$S_{10} = \tfrac{1}{3}(2M_5 + T_5)$$

and

$$S_{20} = \tfrac{1}{3}(2M_{10} + T_{10})$$

$$S_n = S_{2k} = \tfrac{1}{3}(2M_k + T_k) \quad \text{or} \quad S_n = \tfrac{1}{3}(2M_{n/2} + T_{n/2}) \tag{7}$$

Table 7.7.6 displays the approximations S_n corresponding to the data in Tables 7.7.3 to 7.7.5.

Table 7.7.6

FUNCTION VALUE (NINE DECIMAL PLACES)	APPROXIMATION	ERROR
$\ln 2 \approx 0.693147181$	$\int_1^2 (1/x)\,dx \approx S_{20} = \frac{1}{3}(2M_{10} + T_{10}) \approx 0.693147375$	-0.000000194
$\sin 1 \approx 0.841470985$	$\int_0^1 \cos x\,dx \approx S_{10} = \frac{1}{3}(2M_5 + T_5) \approx 0.841471453$	-0.000000468
$\sin 3 \approx 0.141120008$	$\int_0^3 \cos x\,dx \approx S_{20} = \frac{1}{3}(2M_{10} + T_{10}) \approx 0.141120406$	-0.000000398

Using the midpoint approximation formula in Table 7.7.1 and Formula (2) for the trapezoidal approximation, we can derive a similar formula for S_n. We start by partitioning the interval $[a, b]$ into an *even* number of equal subintervals. If n is the number of subintervals, then each subinterval has length $(b-a)/n$. Label the endpoints of these subintervals successively by $a = x_0, x_1, x_2, \ldots, x_n = b$. Then $x_0, x_2, x_4, \ldots, x_n$ define a partition of $[a, b]$ into $n/2$ equal intervals, each of length $2(b-a)/n$, and the midpoints of these subintervals are $x_1, x_3, x_5, \ldots, x_{n-1}$, respectively, as illustrated in Figure 7.7.4. Using $y_i = f(x_i)$, we have

$$\begin{aligned} 2M_{n/2} &= 2\left(\frac{2(b-a)}{n}\right)[y_1 + y_3 + \cdots + y_{n-1}] \\ &= \left(\frac{b-a}{n}\right)[4y_1 + 4y_3 + \cdots + 4y_{n-1}] \end{aligned}$$

Noting that $(b-a)/[2(n/2)] = (b-a)/n$, we can express $T_{n/2}$ as

$$T_{n/2} = \left(\frac{b-a}{n}\right)[y_0 + 2y_2 + 2y_4 + \cdots + 2y_{n-2} + y_n]$$

Thus, $S_n = \frac{1}{3}(2M_{n/2} + T_{n/2})$ can be expressed as

$$S_n = \frac{1}{3}\left(\frac{b-a}{n}\right)[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n] \tag{8}$$

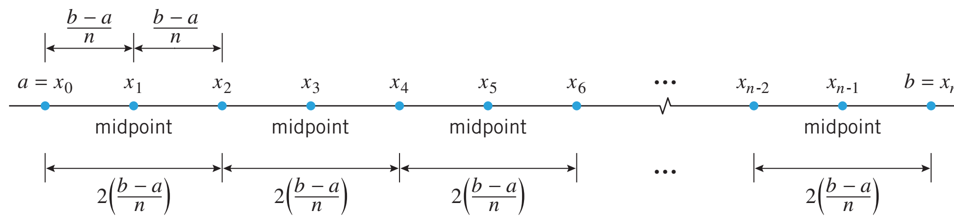
The approximation

$$\int_a^b f(x)\,dx \approx S_n \tag{9}$$

with S_n as given in (8) is known as ***Simpson's rule***. We denote the error in this approximation by

$$E_S = \int_a^b f(x)\,dx - S_n \tag{10}$$

As before, the absolute error in the approximation (9) is given by $|E_S|$.



▶ **Figure 7.7.4**

▶ **Example 6** In Table 7.7.7 we have used Simpson's rule with $n = 10$ subintervals to obtain the approximation

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx \approx S_{10} = 0.693150231$$

For this approximation,

$$\frac{1}{3}\left(\frac{b-a}{n}\right) = \frac{1}{3}\left(\frac{2-1}{10}\right) = \frac{1}{30}$$

Table 7.7.7

AN APPROXIMATION TO $\ln 2$ USING SIMPSON'S RULE

i	ENDPOINT x_i	$y_i = f(x_i) = 1/x_i$	MULTIPLIER w_i	$w_i y_i$
0	1.0	1.000000000	1	1.000000000
1	1.1	0.909090909	4	3.636363636
2	1.2	0.833333333	2	1.666666667
3	1.3	0.769230769	4	3.076923077
4	1.4	0.714285714	2	1.428571429
5	1.5	0.666666667	4	2.666666667
6	1.6	0.625000000	2	1.250000000
7	1.7	0.588235294	4	2.352941176
8	1.8	0.555555556	2	1.111111111
9	1.9	0.526315789	4	2.105263158
10	2.0	0.500000000	1	0.500000000
				20.794506921

$$\int_1^2 \frac{1}{x}\,dx \approx \left(\frac{1}{30}\right)(20.794506921) \approx 0.693150231$$

Thomas Simpson (1710–1761) English mathematician. Simpson was the son of a weaver. He was trained to follow in his father's footsteps and had little formal education in his early life. His interest in science and mathematics was aroused in 1724, when he witnessed an eclipse of the Sun and received two books from a peddler, one on astrology and the other on arithmetic. Simpson quickly absorbed their contents and soon became a successful local fortune teller. His improved financial situation enabled him to give up weaving and marry his landlady. Then in 1733 some mysterious "unfortunate incident" forced him to move. He settled in Derby, where he taught in an evening school and worked at weaving during the day. In 1736 he moved to London and published his first mathematical work in a periodical called the *Ladies' Diary* (of which he later became the editor). In 1737 he published a successful calculus textbook that enabled him to give up weaving completely and concentrate on textbook writing and teaching. His fortunes improved further in 1740 when one Robert Heath accused him of plagiarism. The publicity was marvelous, and Simpson proceeded to dash off a succession of best-selling textbooks: *Algebra* (ten editions plus translations), *Geometry* (twelve editions plus translations), *Trigonometry* (five editions plus translations), and numerous others. It is interesting to note that Simpson did not discover the rule that bears his name—it was a well-known result by Simpson's time.

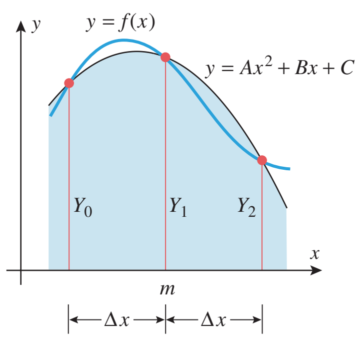
Although S_{10} is a weighted average of M_5 and T_5, it makes sense to compare S_{10} to M_{10} and T_{10}, since the sums for these three approximations involve the same number of terms. Using the values for M_{10} and T_{10} from Example 2 and the value for S_{10} in Table 7.7.7, we have

$$|E_M| = |\ln 2 - M_{10}| \approx |0.693147181 - 0.692835360| = 0.000311821$$
$$|E_T| = |\ln 2 - T_{10}| \approx |0.693147181 - 0.693771403| = 0.000624222$$
$$|E_S| = |\ln 2 - S_{10}| \approx |0.693147181 - 0.693150231| = 0.000003050$$

Comparing these absolute errors, it is clear that S_{10} is a much more accurate approximation of ln 2 than either M_{10} or T_{10}. ◄

GEOMETRIC INTERPRETATION OF SIMPSON'S RULE

The midpoint (or tangent line) approximation and the trapezoidal approximation for a definite integral are based on approximating a segment of the curve $y = f(x)$ by line segments. Intuition suggests that we might improve on these approximations using parabolic arcs rather than line segments, thereby accounting for concavity of the curve $y = f(x)$ more closely.



▲ **Figure 7.7.5**

At the heart of this idea is a formula, sometimes called the ***one-third rule***. The one-third rule expresses a definite integral of a quadratic function $g(x) = Ax^2 + Bx + C$ in terms of the values Y_0, Y_1, and Y_2 of g at the left endpoint, midpoint, and right endpoint, respectively, of the interval of integration $[m - \Delta x, m + \Delta x]$ (see Figure 7.7.5):

$$\int_{m-\Delta x}^{m+\Delta x} (Ax^2 + Bx + C)\, dx = \frac{\Delta x}{3}[Y_0 + 4Y_1 + Y_2] \tag{11}$$

Verification of the one-third rule is left for the reader (Exercise 53). By applying the one-third rule to subintervals $[x_{2k-2}, x_{2k}]$, $k = 1, \ldots, n/2$, one arrives at Formula (8) for Simpson's rule (Exercise 54). Thus, Simpson's rule corresponds to the integral of a piecewise-quadratic approximation to $f(x)$.

ERROR BOUNDS

With all the methods studied in this section, there are two sources of error: the *intrinsic* or *truncation error* due to the approximation formula, and the *roundoff error* introduced in the calculations. In general, increasing n reduces the truncation error but increases the roundoff error, since more computations are required for larger n. In practical applications, it is important to know how large n must be taken to ensure that a specified degree of accuracy is obtained. The analysis of roundoff error is complicated and will not be considered here. However, the following theorems, which are proved in books on numerical analysis, provide upper bounds on the truncation errors in the midpoint, trapezoidal, and Simpson's rule approximations.

7.7.2 THEOREM (***Midpoint and Trapezoidal Error Bounds***) *If f'' is continuous on $[a, b]$ and if K_2 is the maximum value of $|f''(x)|$ on $[a, b]$, then*

$$(a)\ |E_M| = \left| \int_a^b f(x)\, dx - M_n \right| \le \frac{(b-a)^3 K_2}{24n^2} \tag{12}$$

$$(b)\ |E_T| = \left| \int_a^b f(x)\, dx - T_n \right| \le \frac{(b-a)^3 K_2}{12n^2} \tag{13}$$

7.7.3 THEOREM (*Simpson Error Bound*) *If $f^{(4)}$ is continuous on $[a, b]$ and if K_4 is the maximum value of $|f^{(4)}(x)|$ on $[a, b]$, then*

$$|E_S| = \left|\int_a^b f(x)\,dx - S_n\right| \le \frac{(b-a)^5 K_4}{180n^4} \tag{14}$$

Example 7 Find an upper bound on the absolute error that results from approximating

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule S_{10}, each with $n = 10$ subintervals.

Solution. We will apply Formulas (12), (13), and (14) with

$$f(x) = \frac{1}{x}, \quad a = 1, \quad b = 2, \quad \text{and} \quad n = 10$$

We have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4}, \quad f^{(4)}(x) = \frac{24}{x^5}$$

Thus,

$$|f''(x)| = \left|\frac{2}{x^3}\right| = \frac{2}{x^3}, \quad |f^{(4)}(x)| = \left|\frac{24}{x^5}\right| = \frac{24}{x^5}$$

where we have dropped the absolute values because $f''(x)$ and $f^{(4)}(x)$ have positive values for $1 \le x \le 2$. Since $|f''(x)|$ and $|f^4(x)|$ are continuous and decreasing on $[1, 2]$, both functions have their maximum values at $x = 1$; for $|f''(x)|$ this maximum value is 2 and for $|f^4(x)|$ the maximum value is 24. Thus we can take $K_2 = 2$ in (12) and (13), and $K_4 = 24$ in (14). This yields

$$|E_M| \le \frac{(b-a)^3 K_2}{24n^2} = \frac{1^3 \cdot 2}{24 \cdot 10^2} \approx 0.000833333$$

$$|E_T| \le \frac{(b-a)^3 K_2}{12n^2} = \frac{1^3 \cdot 2}{12 \cdot 10^2} \approx 0.001666667$$

$$|E_S| \le \frac{(b-a)^5 K_4}{180n^4} = \frac{1^5 \cdot 24}{180 \cdot 10^4} \approx 0.000013333$$ ◄

Note that the upper bounds calculated in Example 7 are consistent with the values $|E_M|$, $|E_T|$, and $|E_S|$ calculated in Example 6 but are considerably greater than those values. It is quite common that the upper bounds on the absolute errors given in Theorems 7.7.2 and 7.7.3 substantially exceed the actual absolute errors. However, that does not diminish the utility of these bounds.

Example 8 How many subintervals should be used in approximating

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

by Simpson's rule for five decimal-place accuracy?

Solution. To obtain five decimal-place accuracy, we must choose the number of subintervals so that

$$|E_S| \le 0.000005 = 5 \times 10^{-6}$$

From (14), this can be achieved by taking n in Simpson's rule to satisfy

$$\frac{(b-a)^5 K_4}{180n^4} \le 5 \times 10^{-6}$$

Taking $a = 1$, $b = 2$, and $K_4 = 24$ (found in Example 7) in this inequality yields

$$\frac{1^5 \cdot 24}{180 \cdot n^4} \leq 5 \times 10^{-6}$$

which, on taking reciprocals, can be rewritten as

$$n^4 \geq \frac{2 \times 10^6}{75} = \frac{8 \times 10^4}{3}$$

Thus,

$$n \geq \frac{20}{\sqrt[4]{6}} \approx 12.779$$

Since n must be an even integer, the smallest value of n that satisfies this requirement is $n = 14$. Thus, the approximation S_{14} using 14 subintervals will produce five decimal-place accuracy. ◀

REMARK In cases where it is difficult to find the values of K_2 and K_4 in Formulas (12), (13), and (14), these constants may be replaced by any larger constants. For example, suppose that a constant K can be easily found with the certainty that $|f''(x)| < K$ on the interval. Then $K_2 \leq K$ and

$$|E_T| \leq \frac{(b-a)^3 K_2}{12n^2} \leq \frac{(b-a)^3 K}{12n^2} \tag{15}$$

so the right side of (15) is also an upper bound on the value of $|E_T|$. Using K, however, will likely increase the computed value of n needed for a given error tolerance. Many applications involve the resolution of competing practical issues, illustrated here through the trade-off between the convenience of finding a crude bound for $|f''(x)|$ versus the efficiency of using the smallest possible n for a desired accuracy.

▶ **Example 9** How many subintervals should be used in approximating

$$\int_0^1 \cos(x^2)\, dx$$

by the midpoint approximation for three decimal-place accuracy?

Solution. To obtain three decimal-place accuracy, we must choose n so that

$$|E_M| \leq 0.0005 = 5 \times 10^{-4} \tag{16}$$

From (12) with $f(x) = \cos(x^2)$, $a = 0$, and $b = 1$, an upper bound on $|E_M|$ is given by

$$|E_M| \leq \frac{K_2}{24n^2} \tag{17}$$

where $|K_2|$ is the maximum value of $|f''(x)|$ on the interval [0, 1]. However,

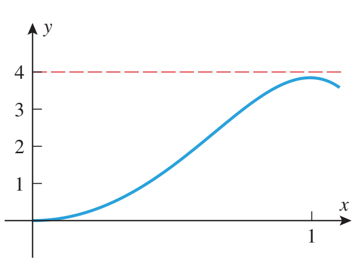
$$\begin{aligned} f'(x) &= -2x\sin(x^2) \\ f''(x) &= -4x^2\cos(x^2) - 2\sin(x^2) = -[4x^2\cos(x^2) + 2\sin(x^2)] \end{aligned}$$

so that

$$|f''(x)| = |4x^2\cos(x^2) + 2\sin(x^2)| \tag{18}$$

It would be tedious to look for the maximum value of this function on the interval [0, 1]. For x in [0, 1], it is easy to see that each of the expressions x^2, $\cos(x^2)$, and $\sin(x^2)$ is bounded in absolute value by 1, so $|4x^2\cos(x^2) + 2\sin(x^2)| \leq 4 + 2 = 6$ on [0, 1]. We can improve on this by using a graphing utility to sketch $|f''(x)|$, as shown in Figure 7.7.6. It is evident from the graph that

$$|f''(x)| < 4 \quad \text{for} \quad 0 \leq x \leq 1$$



$y = |f''(x)| = |4x^2\cos(x^2) + 2\sin(x^2)|$

▲ **Figure 7.7.6**

Thus, it follows from (17) that

$$|E_M| \leq \frac{K_2}{24n^2} < \frac{4}{24n^2} = \frac{1}{6n^2}$$

and hence we can satisfy (16) by choosing n so that

$$\frac{1}{6n^2} < 5 \times 10^{-4}$$

which, on taking reciprocals, can be written as

$$n^2 > \frac{10^4}{30} \quad \text{or} \quad n > \frac{10^2}{\sqrt{30}} \approx 18.257$$

The smallest integer value of n satisfying this inequality is $n = 19$. Thus, the midpoint approximation M_{19} using 19 subintervals will produce three decimal-place accuracy. ◀

A COMPARISON OF THE THREE METHODS

Of the three methods studied in this section, Simpson's rule generally produces more accurate results than the midpoint or trapezoidal approximations for the same amount of work. To make this plausible, let us express (12), (13), and (14) in terms of the subinterval width

$$\Delta x = \frac{b-a}{n}$$

We obtain

$$|E_M| \leq \frac{1}{24} K_2 (b-a)(\Delta x)^2 \tag{19}$$

$$|E_T| \leq \frac{1}{12} K_2 (b-a)(\Delta x)^2 \tag{20}$$

$$|E_S| \leq \frac{1}{180} K_4 (b-a)(\Delta x)^4 \tag{21}$$

(verify). For Simpson's rule, the upper bound on the absolute error is proportional to $(\Delta x)^4$, whereas the upper bound on the absolute error for the midpoint and trapezoidal approximations is proportional to $(\Delta x)^2$. Thus, reducing the interval width by a factor of 10, for example, reduces the error bound by a factor of 100 for the midpoint and trapezoidal approximations but reduces the error bound by a factor of 10,000 for Simpson's rule. This suggests that, as n increases, the accuracy of Simpson's rule improves much more rapidly than that of the other approximations.

As a final note, observe that if $f(x)$ is a polynomial of degree 3 or less, then we have $f^{(4)}(x) = 0$ for all x, so $K_4 = 0$ in (14) and consequently $|E_S| = 0$. Thus, Simpson's rule gives exact results for polynomials of degree 3 or less. Similarly, the midpoint and trapezoidal approximations give exact results for polynomials of degree 1 or less. (You should also be able to see that this is so geometrically.)

✔ QUICK CHECK EXERCISES 7.7

(See page 547 for answers.)

1. Let T_n be the trapezoidal approximation for the definite integral of $f(x)$ over an interval $[a, b]$ using n subintervals.
   (a) Expressed in terms of L_n and R_n (the left and right endpoint approximations), $T_n =$ ________.
   (b) Expressed in terms of the function values $y_0, y_1, \ldots, y_n$ at the endpoints of the subintervals, $T_n =$ ________.

2. Let I denote the definite integral of f over an interval $[a, b]$ with T_n and M_n the respective trapezoidal and midpoint approximations of I for a given n. Assume that the graph of f is concave up on the interval $[a, b]$ and order the quantities T_n, M_n, and I from smallest to largest: ________ $<$ ________ $<$ ________.

3. Let S_6 be the Simpson's rule approximation for $\int_a^b f(x)\,dx$ using $n = 6$ subintervals.
   (a) Expressed in terms of M_3 and T_3 (the midpoint and trapezoidal approximations), $S_6 =$ ________.
   (b) Using the function values $y_0, y_1, y_2, \ldots, y_6$ at the endpoints of the subintervals, $S_6 =$ ________.

4. Assume that $f^{(4)}$ is continuous on $[0, 1]$ and that $f^{(k)}(x)$ satisfies $|f^{(k)}(x)| \leq 1$ on $[0, 1]$, $k = 1, 2, 3, 4$. Find an upper bound on the absolute error that results from approximating the integral of f over $[0, 1]$ using (a) the midpoint approximation M_{10}; (b) the trapezoidal approximation T_{10}; and (c) Simpson's rule S_{10}.

5. Approximate $\displaystyle\int_1^3 \frac{1}{x^2}\,dx$ using the indicated method.
   (a) $M_1 =$ ________ (b) $T_1 =$ ________
   (c) $S_2 =$ ________

EXERCISE SET 7.7 C CAS

1–6 Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20} using Formula (7). In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places. ■

1. $\displaystyle\int_0^3 \sqrt{x+1}\,dx$
2. $\displaystyle\int_4^9 \frac{1}{\sqrt{x}}\,dx$
3. $\displaystyle\int_0^{\pi/2} \cos x\,dx$
4. $\displaystyle\int_0^2 \sin x\,dx$
5. $\displaystyle\int_1^3 e^{-2x}\,dx$
6. $\displaystyle\int_0^3 \frac{1}{3x+1}\,dx$

7–12 Use inequalities (12), (13), and (14) to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise. ■

7. Exercise 1
8. Exercise 2
9. Exercise 3
10. Exercise 4
11. Exercise 5
12. Exercise 6

13–18 Use inequalities (12), (13), and (14) to find a number n of subintervals for (a) the midpoint approximation M_n, (b) the trapezoidal approximation T_n, and (c) Simpson's rule approximation S_n to ensure that the absolute error will be less than the given value. ■

13. Exercise 1; 5×10^{-4}
14. Exercise 2; 5×10^{-4}
15. Exercise 3; 10^{-3}
16. Exercise 4; 10^{-3}
17. Exercise 5; 10^{-4}
18. Exercise 6; 10^{-4}

19–22 True–False Determine whether the statement is true or false. Explain your answer. ■

19. The midpoint approximation, M_n, is the average of the left and right endpoint approximations, L_n and R_n, respectively.

20. If $f(x)$ is concave down on the interval (a, b), then the trapezoidal approximation T_n underestimates $\int_a^b f(x)\,dx$.

21. The Simpson's rule approximation S_{50} for $\int_a^b f(x)\,dx$ is a weighted average of the approximations M_{50} and T_{50}, where M_{50} is given twice the weight of T_{50} in the average.

22. Simpson's rule approximation S_{50} for $\int_a^b f(x)\,dx$ corresponds to $\int_a^b q(x)\,dx$, where the graph of q is composed of 25 parabolic segments joined at points on the graph of f.

23–24 Find a function $g(x)$ of the form

$$g(x) = Ax^2 + Bx + C$$

whose graph contains the points $(m - \Delta x, f(m - \Delta x))$, $(m, f(m))$, and $(m + \Delta x, f(m + \Delta x))$, for the given function $f(x)$ and the given values of m and Δx. Then verify Formula (11):

$$\int_{m-\Delta x}^{m+\Delta x} g(x)\,dx = \frac{\Delta x}{3}[Y_0 + 4Y_1 + Y_2]$$

where $Y_0 = f(m - \Delta x)$, $Y_1 = f(m)$, and $Y_2 = f(m + \Delta x)$. ■

23. $f(x) = \dfrac{1}{x}$; $m = 3$, $\Delta x = 1$

24. $f(x) = \sin^2(\pi x)$; $m = \frac{1}{6}$, $\Delta x = \frac{1}{6}$

25–30 Approximate the integral using Simpson's rule S_{10} and compare your answer to that produced by a calculating utility with a numerical integration capability. Express your answers to at least four decimal places. ■

25. $\displaystyle\int_{-1}^1 e^{-x^2}\,dx$
26. $\displaystyle\int_0^3 \frac{x}{\sqrt{2x^3+1}}\,dx$
27. $\displaystyle\int_{-1}^2 x\sqrt{2+x^3}\,dx$
28. $\displaystyle\int_0^{\pi} \frac{x}{2+\sin x}\,dx$
29. $\displaystyle\int_0^1 \cos(x^2)\,dx$
30. $\displaystyle\int_1^2 (\ln x)^{3/2}\,dx$

31–32 The exact value of the given integral is π (verify). Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20} using Formula (7). Approximate the absolute error and express your answers to at least four decimal places. ■

31. $\displaystyle\int_0^2 \frac{8}{x^2+4}\,dx$
32. $\displaystyle\int_0^3 \frac{4}{9}\sqrt{9-x^2}\,dx$

33. In Example 8 we showed that taking $n = 14$ subdivisions ensures that the approximation of

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

by Simpson's rule is accurate to five decimal places. Confirm this by comparing the approximation of $\ln 2$ produced by Simpson's rule with $n = 14$ to the value produced directly by your calculating utility.

34. In each part, determine whether a trapezoidal approximation would be an underestimate or an overestimate for the definite integral.

(a) $\int_0^1 \cos(x^2)\,dx$ (b) $\int_{3/2}^2 \cos(x^2)\,dx$

35–36 Find a value of n to ensure that the absolute error in approximating the integral by the midpoint approximation will be less than 10^{-4}. Estimate the absolute error, and express your answers to at least four decimal places. ■

35. $\int_0^2 x \sin x\,dx$ **36.** $\int_0^1 e^{\cos x}\,dx$

37–38 Show that the inequalities (12) and (13) are of no value in finding an upper bound on the absolute error that results from approximating the integral using either the midpoint approximation or the trapezoidal approximation. ■

37. $\int_0^1 x\sqrt{x}\,dx$ **38.** $\int_0^1 \sin\sqrt{x}\,dx$

39–40 Use Simpson's rule approximation S_{10} to approximate the length of the curve over the stated interval. Express your answers to at least four decimal places. ■

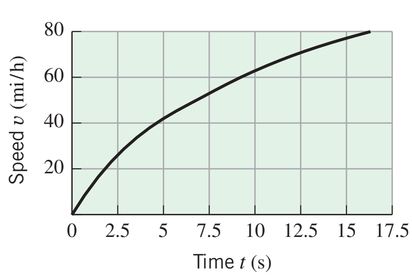
39. $y = \sin x$ from $x = 0$ to $x = \pi$

40. $y = x^{-2}$ from $x = 1$ to $x = 2$

FOCUS ON CONCEPTS

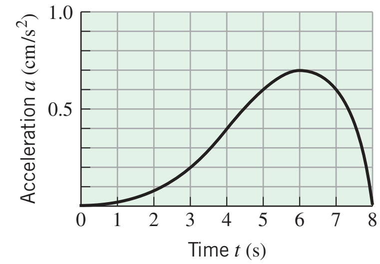
41. A graph of the speed v versus time t curve for a test run of a Mitsubishi Galant ES is shown in the accompanying figure. Estimate the speeds at times $t = 0$, 2.5, 5, 7.5, 10, 12.5, 15 s from the graph, convert to ft/s using 1 mi/h $= \frac{22}{15}$ ft/s, and use these speeds and Simpson's rule to approximate the number of feet traveled during the first 15 s. Round your answer to the nearest foot. [*Hint:* Distance traveled $= \int_0^{15} v(t)\,dt$.]

Source: Data from *Car and Driver*, November 2003.



◀ **Figure Ex-41**

42. A graph of the acceleration a versus time t for an object moving on a straight line is shown in the accompanying figure. Estimate the accelerations at $t = 0, 1, 2, \ldots, 8$ seconds (s) from the graph and use Simpson's rule to approximate the change in velocity from $t = 0$ to $t = 8$ s. Round your answer to the nearest tenth cm/s. [*Hint:* Change in velocity $= \int_0^8 a(t)\,dt$.]



◀ **Figure Ex-42**

43–46 Numerical integration methods can be used in problems where only measured or experimentally determined values of the integrand are available. Use Simpson's rule to estimate the value of the relevant integral in these exercises. ■

43. The accompanying table gives the speeds, in miles per second, at various times for a test rocket that was fired upward from the surface of the Earth. Use these values to approximate the number of miles traveled during the first 180 s. Round your answer to the nearest tenth of a mile. [*Hint:* Distance traveled $= \int_0^{180} v(t)\,dt$.]

TIME t (s)	SPEED v (mi/s)
0	0.00
30	0.03
60	0.08
90	0.16
120	0.27
150	0.42
180	0.65

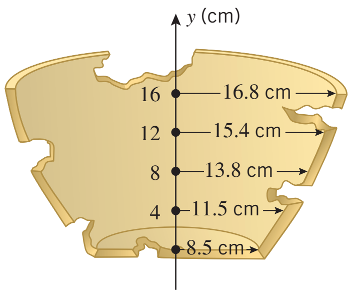
◀ **Table Ex-43**

44. The accompanying table gives the speeds of a bullet at various distances from the muzzle of a rifle. Use these values to approximate the number of seconds for the bullet to travel 1800 ft. Express your answer to the nearest hundredth of a second. [*Hint:* If v is the speed of the bullet and x is the distance traveled, then $v = dx/dt$ so that $dt/dx = 1/v$ and $t = \int_0^{1800}(1/v)\,dx$.]

DISTANCE x (ft)	SPEED v (ft/s)
0	3100
300	2908
600	2725
900	2549
1200	2379
1500	2216
1800	2059

◀ **Table Ex-44**

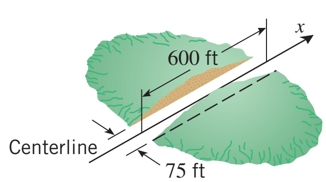
45. Measurements of a pottery shard recovered from an archaeological dig reveal that the shard came from a pot with a flat bottom and circular cross sections (see the accompanying figure below). The figure shows interior radius measurements of the shard made every 4 cm from the bottom of the pot to the top. Use those values to approximate the interior volume of the pot to the nearest tenth of a liter (1 L = 1000 cm^3). [*Hint:* Use 6.2.3 (volume by cross sections) to set up an appropriate integral for the volume.]



◀ **Figure Ex-45**

46. Engineers want to construct a straight and level road 600 ft long and 75 ft wide by making a vertical cut through an intervening hill (see the accompanying figure). Heights of the hill above the centerline of the proposed road, as obtained at various points from a contour map of the region, are shown in the accompanying figure. To estimate the construction costs, the engineers need to know the volume of earth that must be removed. Approximate this volume, rounded to the nearest cubic foot. [*Hint:* First set up an integral for the cross-sectional area of the cut along the centerline of the road, then assume that the height of the hill does not vary between the centerline and edges of the road.]

HORIZONTAL DISTANCE x (ft)	HEIGHT h (ft)
0	0
100	7
200	16
300	24
400	25
500	16
600	0



▲ **Figure Ex-46**

C **47.** Let $f(x) = \cos(x^2)$.
(a) Use a CAS to approximate the maximum value of $|f''(x)|$ on the interval $[0, 1]$.
(b) How large must n be in the midpoint approximation of $\int_0^1 f(x)\,dx$ to ensure that the absolute error is less than 5×10^{-4}? Compare your result with that obtained in Example 9.
(c) Estimate the integral using the midpoint approximation with the value of n obtained in part (b).

C **48.** Let $f(x) = \sqrt{1 + x^3}$.
(a) Use a CAS to approximate the maximum value of $|f''(x)|$ on the interval $[0, 1]$.
(b) How large must n be in the trapezoidal approximation of $\int_0^1 f(x)\,dx$ to ensure that the absolute error is less than 10^{-3}?
(c) Estimate the integral using the trapezoidal approximation with the value of n obtained in part (b).

C **49.** Let $f(x) = \cos(x - x^2)$.
(a) Use a CAS to approximate the maximum value of $|f^{(4)}(x)|$ on the interval $[0, 1]$.
(b) How large must the value of n be in the approximation S_n of $\int_0^1 f(x)\,dx$ by Simpson's rule to ensure that the absolute error is less than 10^{-4}?
(c) Estimate the integral using Simpson's rule approximation S_n with the value of n obtained in part (b).

C **50.** Let $f(x) = \sqrt{2 + x^3}$.
(a) Use a CAS to approximate the maximum value of $|f^{(4)}(x)|$ on the interval $[0, 1]$.
(b) How large must the value of n be in the approximation S_n of $\int_0^1 f(x)\,dx$ by Simpson's rule to ensure that the absolute error is less than 10^{-6}?
(c) Estimate the integral using Simpson's rule approximation S_n with the value of n obtained in part (b).

FOCUS ON CONCEPTS

51. (a) Verify that the average of the left and right endpoint approximations as given in Table 7.7.1 gives Formula (2) for the trapezoidal approximation.
(b) Suppose that f is a continuous nonnegative function on the interval $[a, b]$ and partition $[a, b]$ with equally spaced points, $a = x_0 < x_1 < \cdots < x_n = b$. Find the area of the trapezoid under the line segment joining points $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$ and above the interval $[x_k, x_{k+1}]$. Show that the right side of Formula (2) is the sum of these trapezoidal areas (Figure 7.7.1).

52. Let f be a function that is positive, continuous, decreasing, and concave down on the interval $[a, b]$. Assuming that $[a, b]$ is subdivided into n equal subintervals, arrange the following approximations of $\int_a^b f(x)\,dx$ in order of increasing value: left endpoint, right endpoint, midpoint, and trapezoidal.

53. Suppose that $\Delta x > 0$ and $g(x) = Ax^2 + Bx + C$. Let m be a number and set $Y_0 = g(m - \Delta x)$, $Y_1 = g(m)$, and $Y_2 = g(m + \Delta x)$. Verify Formula (11):

$$\int_{m-\Delta x}^{m+\Delta x} g(x)\,dx = \frac{\Delta x}{3}[Y_0 + 4Y_1 + Y_2]$$

54. Suppose that f is a continuous nonnegative function on the interval $[a, b]$, n is even, and $[a, b]$ is partitioned using $n + 1$ equally spaced points, $a = x_0 < x_1 < \cdots < x_n = b$. Set $y_0 = f(x_0)$, $y_1 = f(x_1)$, $\ldots$, $y_n = f(x_n)$. Let $g_1, g_2, \ldots, g_{n/2}$ be the quadratic functions of the form $g_i(x) = Ax^2 + Bx + C$ so that *(cont.)*

- the graph of g_1 passes through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2);
- the graph of g_2 passes through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4);
- ...
- the graph of $g_{n/2}$ passes through the points (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), and (x_n, y_n).

Verify that Formula (8) computes the area under a piecewise quadratic function by showing that

$$\sum_{j=1}^{n/2}\left(\int_{x_{2j-2}}^{x_{2j}} g_j(x)\,dx\right) = \frac{1}{3}\left(\frac{b-a}{n}\right)[y_0+4y_1+2y_2+4y_3+2y_4+\cdots + 2y_{n-2}+4y_{n-1}+y_n]$$

55. Writing Discuss two different circumstances under which numerical integration is necessary.

56. Writing For the numerical integration methods of this section, better accuracy of an approximation was obtained by increasing the number of subdivisions of the interval. Another strategy is to use the same number of subintervals, but to select subintervals of differing lengths. Discuss a scheme for doing this to approximate $\int_0^4 \sqrt{x}\,dx$ using a trapezoidal approximation with 4 subintervals. Comment on the advantages and disadvantages of your scheme.

✔QUICK CHECK ANSWERS 7.7

1. (a) $\frac{1}{2}(L_n+R_n)$ (b) $\left(\frac{b-a}{2n}\right)[y_0+2y_1+\cdots+2y_{n-1}+y_n]$ **2.** $M_n < I < T_n$ **3.** (a) $\frac{2}{3}M_3+\frac{1}{3}T_3$ (b) $\left(\frac{b-a}{18}\right)(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6)$ **4.** (a) $\frac{1}{2400}$ (b) $\frac{1}{1200}$ (c) $\frac{1}{1{,}800{,}000}$ **5.** (a) $M_1=\frac{1}{2}$ (b) $T_1=\frac{10}{9}$ (c) $S_2=\frac{19}{27}$

7.8 IMPROPER INTEGRALS

Up to now we have focused on definite integrals with continuous integrands and finite intervals of integration. In this section we will extend the concept of a definite integral to include infinite intervals of integration and integrands that become infinite within the interval of integration.

IMPROPER INTEGRALS

It is assumed in the definition of the definite integral

$$\int_a^b f(x)\,dx$$

that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable. We observed in Theorems 5.5.2 and 5.5.8 that continuous functions are integrable, as are bounded functions with finitely many points of discontinuity. We also observed in Theorem 5.5.8 that functions that are not bounded on the interval of integration are not integrable. Thus, for example, a function with a vertical asymptote within the interval of integration would not be integrable.

Our main objective in this section is to extend the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We will call the vertical asymptotes ***infinite discontinuities***, and we will call integrals with infinite intervals of integration or infinite discontinuities within the interval of integration ***improper integrals***. Here are some examples:

- Improper integrals with infinite intervals of integration:

$$\int_1^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_1^2 \frac{dx}{x-1}, \quad \int_0^{\pi} \tan x\,dx$$

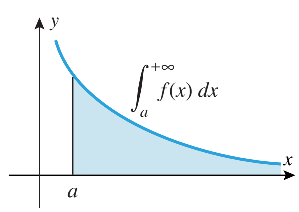
- Improper integrals with infinite discontinuities and infinite intervals of integration:

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_1^{+\infty} \sec x\,dx$$

INTEGRALS OVER INFINITE INTERVALS

To motivate a reasonable definition for improper integrals of the form

$$\int_a^{+\infty} f(x)\,dx$$

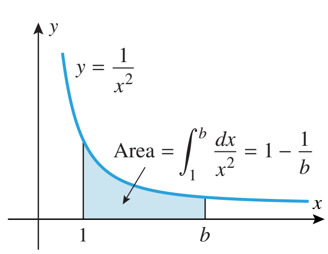


▲ **Figure 7.8.1**

let us begin with the case where f is continuous and nonnegative on $[a, +\infty)$, so we can think of the integral as the area under the curve $y = f(x)$ over the interval $[a, +\infty)$ (Figure 7.8.1). At first, you might be inclined to argue that this area is infinite because the region has infinite extent. However, such an argument would be based on vague intuition rather than precise mathematical logic, since the concept of area has only been defined over intervals of *finite extent*. Thus, before we can make any reasonable statements about the area of the region in Figure 7.8.1, we need to begin by defining what we mean by the area of this region. For that purpose, it will help to focus on a specific example.

Suppose we are interested in the area A of the region that lies below the curve $y = 1/x^2$ and above the interval $[1, +\infty)$ on the x-axis. Instead of trying to find the entire area at once, let us begin by calculating the portion of the area that lies above a finite interval $[1, b]$, where $b > 1$ is arbitrary. That area is

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg]_1^b = 1 - \frac{1}{b}$$

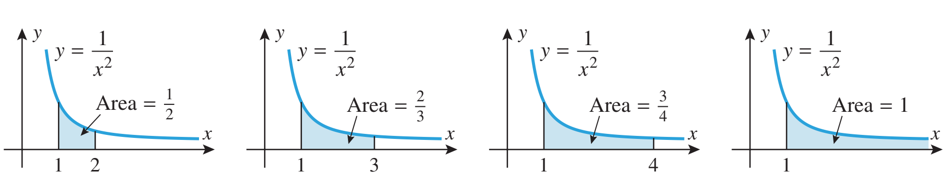


▲ **Figure 7.8.2**

(Figure 7.8.2). If we now allow b to increase so that $b \to +\infty$, then the portion of the area over the interval $[1, b]$ will begin to fill out the area over the entire interval $[1, +\infty)$ (Figure 7.8.3), and hence we can reasonably define the area A under $y = 1/x^2$ over the interval $[1, +\infty)$ to be

$$A = \int_1^{+\infty} \frac{dx}{x^2} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to+\infty} \left(1 - \frac{1}{b}\right) = 1 \tag{1}$$

Thus, the area has a finite value of 1 and is not infinite as we first conjectured.



▶ **Figure 7.8.3**

With the preceding discussion as our guide, we make the following definition (which is applicable to functions with both positive and negative values).

If f is nonnegative over the interval $[a, +\infty)$, then the improper integral in Definition 7.8.1 can be interpreted to be the area under the graph of f over the interval $[a, +\infty)$. If the integral converges, then the area is finite and equal to the value of the integral, and if the integral diverges, then the area is regarded to be infinite.

7.8.1 DEFINITION The ***improper integral of f over the interval*** $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Example 1 Evaluate

$$\text{(a)}\ \int_1^{+\infty}\frac{dx}{x^3} \qquad \text{(b)}\ \int_1^{+\infty}\frac{dx}{x}$$

Solution (a). Following the definition, we replace the infinite upper limit by a finite upper limit b, and then take the limit of the resulting integral. This yields

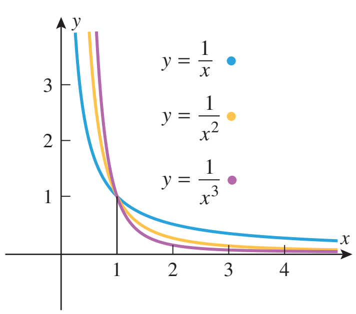
$$\int_1^{+\infty}\frac{dx}{x^3} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x^3} = \lim_{b\to+\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to+\infty}\left(\frac{1}{2}-\frac{1}{2b^2}\right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is $1/2$.

Solution (b).

$$\int_1^{+\infty}\frac{dx}{x} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x} = \lim_{b\to+\infty}\big[\ln x\big]_1^b = \lim_{b\to+\infty}\ln b = +\infty$$

In this case the integral diverges and hence has no value. ◀



▲ **Figure 7.8.4**

Because the functions $1/x^3$, $1/x^2$, and $1/x$ are nonnegative over the interval $[1, +\infty)$, it follows from (1) and the last example that over this interval the area under $y = 1/x^3$ is $\frac{1}{2}$, the area under $y = 1/x^2$ is 1, and the area under $y = 1/x$ is infinite. However, on the surface the graphs of the three functions seem very much alike (Figure 7.8.4), and there is nothing to suggest why one of the areas should be infinite and the other two finite. One explanation is that $1/x^3$ and $1/x^2$ approach zero more rapidly than $1/x$ as $x\to+\infty$, so that the area over the interval $[1, b]$ accumulates less rapidly under the curves $y = 1/x^3$ and $y = 1/x^2$ than under $y = 1/x$ as $b\to+\infty$, and the difference is just enough that the first two areas are finite and the third is infinite.

Example 2 For what values of p does the integral $\displaystyle\int_1^{+\infty}\frac{dx}{x^p}$ converge?

Solution. We know from the preceding example that the integral diverges if $p = 1$, so let us assume that $p \neq 1$. In this case we have

$$\int_1^{+\infty}\frac{dx}{x^p} = \lim_{b\to+\infty}\int_1^b x^{-p}\,dx = \lim_{b\to+\infty}\left.\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to+\infty}\left[\frac{b^{1-p}}{1-p}-\frac{1}{1-p}\right]$$

If $p > 1$, then the exponent $1-p$ is negative and $b^{1-p}\to 0$ as $b\to+\infty$; and if $p < 1$, then the exponent $1-p$ is positive and $b^{1-p}\to+\infty$ as $b\to+\infty$. Thus, the integral converges if $p > 1$ and diverges otherwise. In the convergent case the value of the integral is

$$\int_1^{+\infty}\frac{dx}{x^p} = \left[0-\frac{1}{1-p}\right] = \frac{1}{p-1} \quad (p > 1)$$ ◀

The following theorem summarizes this result.

7.8.2 THEOREM

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if} \quad p > 1 \\ \text{diverges} & \text{if} \quad p \le 1 \end{cases}$$

Example 3 Evaluate $\int_0^{+\infty} (1-x)e^{-x}\,dx$.

Solution. We begin by evaluating the indefinite integral using integration by parts. Setting $u = 1 - x$ and $dv = e^{-x}\,dx$ yields

$$\int (1-x)e^{-x}\,dx = -e^{-x}(1-x) - \int e^{-x}\,dx = -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C$$

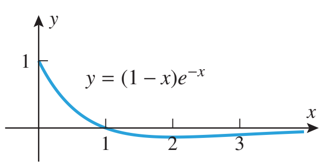
Thus,

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \Big[xe^{-x}\Big]_0^b = \lim_{b\to+\infty} \frac{b}{e^b}$$

The limit is an indeterminate form of type ∞/∞, so we will apply L'Hôpital's rule by differentiating the numerator and denominator with respect to b. This yields

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \frac{1}{e^b} = 0$$

We can interpret this to mean that the net signed area between the graph of $y = (1-x)e^{-x}$ and the interval $[0, +\infty)$ is 0 (Figure 7.8.5). ◀



The net signed area between the graph and the interval $[0, +\infty)$ is zero.

▲ **Figure 7.8.5**

If f is nonnegative over the interval $(-\infty, +\infty)$, then the improper integral

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

can be interpreted to be the area under the graph of f over the interval $(-\infty, +\infty)$. The area is finite and equal to the value of the integral if the integral converges and is infinite if it diverges.

7.8.3 DEFINITION The ***improper integral of f over the interval*** $(-\infty, b]$ is defined to be

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{2}$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not.

The ***improper integral of f over the interval*** $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx \tag{3}$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

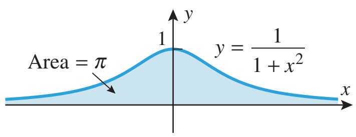
Example 4 Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution. We will evaluate the integral by choosing $c = 0$ in (3). With this value for c we obtain

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to+\infty} \Big[\tan^{-1} x\Big]_0^b = \lim_{b\to+\infty} (\tan^{-1} b) = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \Big[\tan^{-1} x\Big]_a^0 = \lim_{a\to-\infty} (-\tan^{-1} a) = \frac{\pi}{2}$$

Although we usually choose $c = 0$ in (3), the choice does not matter because it can be proved that neither the convergence nor the value of the integral is affected by the choice of c.



▲ **Figure 7.8.6**

Thus, the integral converges and its value is

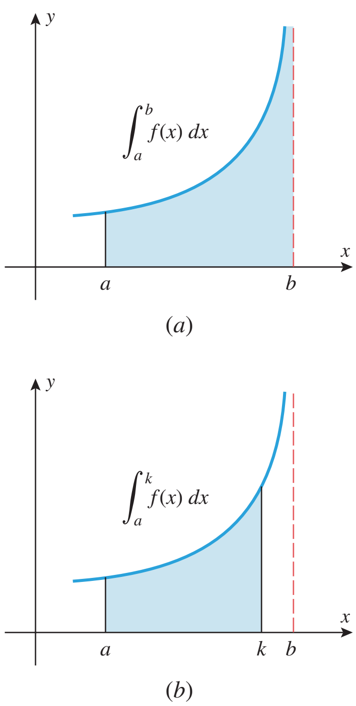
$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure 7.8.6. ◀

INTEGRALS WHOSE INTEGRANDS HAVE INFINITE DISCONTINUITIES

Next we will consider improper integrals whose integrands have infinite discontinuities. We will start with the case where the interval of integration is a finite interval $[a, b]$ and the infinite discontinuity occurs at the right-hand endpoint.

To motivate an appropriate definition for such an integral let us consider the case where f is nonnegative on $[a, b]$, so we can interpret the improper integral $\int_a^b f(x)\,dx$ as the area of the region in Figure 7.8.7*a*. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive y-direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $[a, k]$, where $a \le k < b$, and then letting k approach b to fill out the area of the entire region (Figure 7.8.7*b*). Motivated by this idea, we make the following definition.



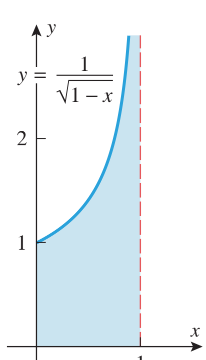
▲ **Figure 7.8.7**

7.8.4 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***improper integral of f over the interval* [*a*, *b*]** is defined as

$$\int_a^b f(x)\,dx = \lim_{k \to b^-} \int_a^k f(x)\,dx \tag{4}$$

In the case where the indicated limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

▶ **Example 5** Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.



▲ **Figure 7.8.8**

Solution. The integral is improper because the integrand approaches $+\infty$ as x approaches the upper limit 1 from the left (Figure 7.8.8). From (4),

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \Big[-2\sqrt{1-x}\Big]_0^k$$
$$= \lim_{k \to 1^-} \Big[-2\sqrt{1-k} + 2\Big] = 2 \;◀$$

Improper integrals with an infinite discontinuity at the left-hand endpoint or inside the interval of integration are defined as follows.

7.8.5 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval* [*a*, *b*]** is defined as

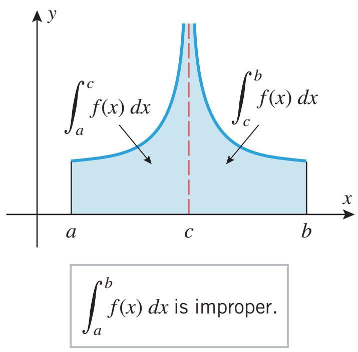
$$\int_a^b f(x)\,dx = \lim_{k\to a^+}\int_k^b f(x)\,dx \tag{5}$$

The integral is said to ***converge*** if the indicated limit exists and ***diverge*** if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval* [*a*, *b*]** is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{6}$$

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to ***converge*** if *both* terms on the right side converge and ***diverge*** if *either* term on the right side diverges (Figure 7.8.9).



▲ **Figure 7.8.9**

▶ **Example 6** Evaluate

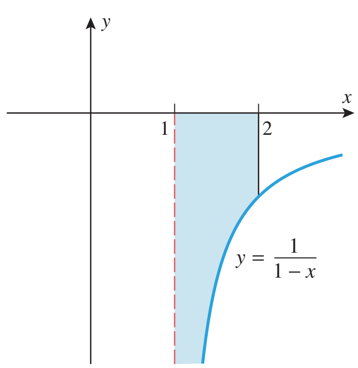
$$\text{(a)}\ \int_1^2 \frac{dx}{1-x} \qquad \text{(b)}\ \int_1^4 \frac{dx}{(x-2)^{2/3}}$$

Solution (a). The integral is improper because the integrand approaches $-\infty$ as x approaches the lower limit 1 from the right (Figure 7.8.10). From Definition 7.8.5 we obtain

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k\to 1^+}\Big[-\ln|-1| + \ln|1-k|\Big] = \lim_{k\to 1^+}\ln|1-k| = -\infty$$

so the integral diverges.



▲ **Figure 7.8.10**

Solution (b). The integral is improper because the integrand approaches $+\infty$ at $x = 2$, which is inside the interval of integration. From Definition 7.8.5 we obtain

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}} \tag{7}$$

and we must investigate the convergence of both improper integrals on the right. Since

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\Big[3(k-2)^{1/3} - 3(1-2)^{1/3}\Big] = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+}\int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+}\Big[3(4-2)^{1/3} - 3(k-2)^{1/3}\Big] = 3\sqrt[3]{2}$$

we have from (7) that

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2} \ ◀$$

WARNING It is sometimes tempting to apply the Fundamental Theorem of Calculus directly to an improper integral without taking the appropriate limits. To illustrate what can go wrong with this procedure, suppose we fail to recognize that the integral

$$\int_0^2 \frac{dx}{(x-1)^2} \tag{8}$$

is improper and mistakenly evaluate this integral as

$$-\frac{1}{x-1}\bigg]_0^2 = -1-(1) = -2$$

This result is clearly incorrect because the integrand is never negative and hence the integral cannot be negative! To evaluate (8) correctly we should first write

$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

and then treat each term as an improper integral. For the first term,

$$\int_0^1 \frac{dx}{(x-1)^2} = \lim_{k\to 1^-}\int_0^k \frac{dx}{(x-1)^2} = \lim_{k\to 1^-}\left[-\frac{1}{k-1}-1\right] = +\infty$$

so (8) diverges.

ARC LENGTH AND SURFACE AREA USING IMPROPER INTEGRALS

In Definitions 6.4.2 and 6.5.2 for arc length and surface area we required the function f to be smooth (continuous first derivative) to ensure the integrability in the resulting formula. However, smoothness is overly restrictive since some of the most basic formulas in geometry involve functions that are not smooth but lead to convergent improper integrals. Accordingly, let us agree to extend the definitions of arc length and surface area to allow functions that are not smooth, but for which the resulting integral in the formula converges.

Example 7 Derive the formula for the circumference of a circle of radius r.

Solution. For convenience, let us assume that the circle is centered at the origin, in which case its equation is $x^2+y^2=r^2$. We will find the arc length of the portion of the circle that lies in the first quadrant and then multiply by 4 to obtain the total circumference (Figure 7.8.11).

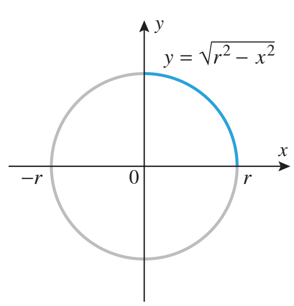


Figure 7.8.11

Since the equation of the upper semicircle is $y=\sqrt{r^2-x^2}$, it follows from Formula (4) of Section 6.4 that the circumference C is

$$C = 4\int_0^r \sqrt{1+(dy/dx)^2}\,dx = 4\int_0^r \sqrt{1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,dx$$
$$= 4r\int_0^r \frac{dx}{\sqrt{r^2-x^2}}$$

This integral is improper because of the infinite discontinuity at $x=r$, and hence we evaluate it by writing

$$C = 4r\lim_{k\to r^-}\int_0^k \frac{dx}{\sqrt{r^2-x^2}}$$
$$= 4r\lim_{k\to r^-}\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^k \qquad \text{Formula (77) in the Endpaper Integral Table}$$
$$= 4r\lim_{k\to r^-}\left[\sin^{-1}\left(\frac{k}{r}\right)-\sin^{-1}0\right]$$
$$= 4r[\sin^{-1}1-\sin^{-1}0] = 4r\left(\frac{\pi}{2}-0\right) = 2\pi r$$

✔ QUICK CHECK EXERCISES 7.8 (See page 557 for answers.)

1. In each part, determine whether the integral is improper, and if so, explain why. Do not evaluate the integrals.

(a) $\displaystyle\int_{\pi/4}^{3\pi/4} \cot x\, dx$ (b) $\displaystyle\int_{\pi/4}^{\pi} \cot x\, dx$

(c) $\displaystyle\int_{0}^{+\infty} \frac{1}{x^2+1}\, dx$ (d) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^2-1}\, dx$

2. Express each improper integral in Quick Check Exercise 1 in terms of one or more appropriate limits. Do not evaluate the limits.

3. The improper integral

$$\int_1^{+\infty} x^{-p}\, dx$$

converges to ________ provided ________.

4. Evaluate the integrals that converge.

(a) $\displaystyle\int_0^{+\infty} e^{-x}\, dx$ (b) $\displaystyle\int_0^{+\infty} e^{x}\, dx$

(c) $\displaystyle\int_0^{1} \frac{1}{x^3}\, dx$ (d) $\displaystyle\int_0^{1} \frac{1}{\sqrt[3]{x^2}}\, dx$

EXERCISE SET 7.8 Graphing Utility C CAS

1. In each part, determine whether the integral is improper, and if so, explain why.

(a) $\displaystyle\int_1^5 \frac{dx}{x-3}$ (b) $\displaystyle\int_1^5 \frac{dx}{x+3}$ (c) $\displaystyle\int_0^1 \ln x\, dx$

(d) $\displaystyle\int_1^{+\infty} e^{-x}\, dx$ (e) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{\sqrt[3]{x-1}}$ (f) $\displaystyle\int_0^{\pi/4} \tan x\, dx$

2. In each part, determine all values of p for which the integral is improper.

(a) $\displaystyle\int_0^1 \frac{dx}{x^p}$ (b) $\displaystyle\int_1^2 \frac{dx}{x-p}$ (c) $\displaystyle\int_0^1 e^{-px}\, dx$

3–32 Evaluate the integrals that converge. ■

3. $\displaystyle\int_0^{+\infty} e^{-2x}\, dx$

4. $\displaystyle\int_{-1}^{+\infty} \frac{x}{1+x^2}\, dx$

5. $\displaystyle\int_3^{+\infty} \frac{2}{x^2-1}\, dx$

6. $\displaystyle\int_0^{+\infty} xe^{-x^2}\, dx$

7. $\displaystyle\int_e^{+\infty} \frac{1}{x\ln^3 x}\, dx$

8. $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\, dx$

9. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

10. $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9}$

11. $\displaystyle\int_{-\infty}^{0} e^{3x}\, dx$

12. $\displaystyle\int_{-\infty}^{0} \frac{e^x\, dx}{3-2e^x}$

13. $\displaystyle\int_{-\infty}^{+\infty} x\, dx$

14. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\, dx$

15. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\, dx$

16. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\, dt$

17. $\displaystyle\int_0^4 \frac{dx}{(x-4)^2}$

18. $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{x}}$

19. $\displaystyle\int_0^{\pi/2} \tan x\, dx$

20. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

21. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

22. $\displaystyle\int_{-3}^{1} \frac{x\, dx}{\sqrt{9-x^2}}$

23. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}}\, dx$

24. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x}\, dx$

25. $\displaystyle\int_0^3 \frac{dx}{x-2}$

26. $\displaystyle\int_{-2}^{2} \frac{dx}{x^2}$

27. $\displaystyle\int_{-1}^{8} x^{-1/3}\, dx$

28. $\displaystyle\int_0^1 \frac{dx}{(x-1)^{2/3}}$

29. $\displaystyle\int_0^{+\infty} \frac{1}{x^2}\, dx$

30. $\displaystyle\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$

31. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}(x+1)}$

32. $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. $\displaystyle\int_1^{+\infty} x^{-4/3}\, dx$ converges to 3.

34. If f is continuous on $[a,+\infty]$ and $\lim_{x\to+\infty} f(x) = 1$, then $\int_a^{+\infty} f(x)\, dx$ converges.

35. $\displaystyle\int_1^2 \frac{1}{x(x-3)}\, dx$ is an improper integral.

36. $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\, dx = 0$

37–40 Make the u-substitution and evaluate the resulting definite integral. ■

37. $\displaystyle\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx;\ u=\sqrt{x}$ [*Note:* $u\to+\infty$ as $x\to+\infty$.]

38. $\displaystyle\int_{12}^{+\infty} \frac{dx}{\sqrt{x}(x+4)};\ u=\sqrt{x}$ [*Note:* $u\to+\infty$ as $x\to+\infty$.]

39. $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}\, dx;\ u=1-e^{-x}$
[*Note:* $u\to 1$ as $x\to+\infty$.]

40. $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx;\; u = e^{-x}$

C **41–42** Express the improper integral as a limit, and then evaluate that limit with a CAS. Confirm the answer by evaluating the integral directly with the CAS. ■

41. $\displaystyle\int_0^{+\infty} e^{-x}\cos x\,dx$ **42.** $\displaystyle\int_0^{+\infty} xe^{-3x}\,dx$

C **43.** In each part, try to evaluate the integral exactly with a CAS. If your result is not a simple numerical answer, then use the CAS to find a numerical approximation of the integral.

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^8+x+1}\,dx$ (b) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{1+x^3}}\,dx$

(c) $\displaystyle\int_1^{+\infty} \frac{\ln x}{e^x}\,dx$ (d) $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$

C **44.** In each part, confirm the result with a CAS.

(a) $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$ (b) $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$

(c) $\displaystyle\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$

45. Find the length of the curve $y = (4 - x^{2/3})^{3/2}$ over the interval $[0, 8]$.

46. Find the length of the curve $y = \sqrt{4-x^2}$ over the interval $[0, 2]$.

47–48 Use L'Hôpital's rule to help evaluate the improper integral. ■

47. $\displaystyle\int_0^1 \ln x\,dx$ **48.** $\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}\,dx$

49. Find the area of the region between the x-axis and the curve $y = e^{-3x}$ for $x \geq 0$.

50. Find the area of the region between the x-axis and the curve $y = 8/(x^2-4)$ for $x \geq 4$.

51. Suppose that the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the x-axis.
(a) Find the volume of the solid that is generated.
(b) Find the surface area of the solid.

FOCUS ON CONCEPTS

52. Suppose that f and g are continuous functions and that

$$0 \leq f(x) \leq g(x)$$

if $x \geq a$. Give a reasonable informal argument using areas to explain why the following results are true.
(a) If $\int_a^{+\infty} f(x)\,dx$ diverges, then $\int_a^{+\infty} g(x)\,dx$ diverges.
(b) If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges and $\int_a^{+\infty} f(x)\,dx \leq \int_a^{+\infty} g(x)\,dx$.
[*Note:* The results in this exercise are sometimes called ***comparison tests*** for improper integrals.]

53–56 Use the results in Exercise 52. ■

53. (a) Confirm graphically and algebraically that

$$e^{-x^2} \leq e^{-x} \quad (x \geq 1)$$

(b) Evaluate the integral

$$\int_1^{+\infty} e^{-x}\,dx$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_1^{+\infty} e^{-x^2}\,dx?$$

54. (a) Confirm graphically and algebraically that

$$\frac{1}{2x+1} \leq \frac{e^x}{2x+1} \quad (x \geq 0)$$

(b) Evaluate the integral

$$\int_0^{+\infty} \frac{dx}{2x+1}$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_0^{+\infty} \frac{e^x}{2x+1}\,dx?$$

55. Let R be the region to the right of $x = 1$ that is bounded by the x-axis and the curve $y = 1/x$. When this region is revolved about the x-axis it generates a solid whose surface is known as ***Gabriel's Horn*** (for reasons that should be clear from the accompanying figure). Show that the solid has a finite volume but its surface has an infinite area. [*Note:* It has been suggested that if one could saturate the interior of the solid with paint and allow it to seep through to the surface, then one could paint an infinite surface with a finite amount of paint! What do you think?]

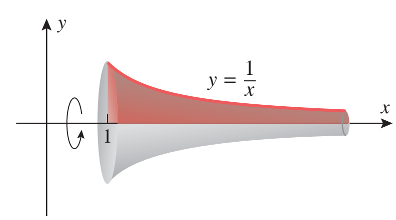


Figure Ex-55

56. In each part, use Exercise 52 to determine whether the integral converges or diverges. If it converges, then use part (b) of that exercise to find an upper bound on the value of the integral.

(a) $\displaystyle\int_2^{+\infty} \frac{\sqrt{x^3+1}}{x}\,dx$ (b) $\displaystyle\int_2^{+\infty} \frac{x}{x^5+1}\,dx$

(c) $\displaystyle\int_0^{+\infty} \frac{xe^x}{2x+1}\,dx$

FOCUS ON CONCEPTS

57. Sketch the region whose area is

$$\int_0^{+\infty} \frac{dx}{1+x^2}$$

and use your sketch to show that

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \int_0^1 \sqrt{\frac{1-y}{y}}\, dy$$

58. (a) Give a reasonable informal argument, based on areas, that explains why the integrals

$$\int_0^{+\infty} \sin x\, dx \quad \text{and} \quad \int_0^{+\infty} \cos x\, dx$$

diverge.

(b) Show that $\displaystyle\int_0^{+\infty} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$ diverges.

59. In electromagnetic theory, the magnetic potential at a point on the axis of a circular coil is given by

$$u = \frac{2\pi NIr}{k} \int_a^{+\infty} \frac{dx}{(r^2+x^2)^{3/2}}$$

where N, I, r, k, and a are constants. Find u.

C **60.** The *average speed*, $\bar{v}$, of the molecules of an ideal gas is given by

$$\bar{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^{+\infty} v^3 e^{-Mv^2/(2RT)}\, dv$$

and the *root-mean-square speed*, v_{rms}, by

$$v_{\text{rms}}^2 = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^{+\infty} v^4 e^{-Mv^2/(2RT)}\, dv$$

where v is the molecular speed, T is the gas temperature, M is the molecular weight of the gas, and R is the gas constant.

(a) Use a CAS to show that

$$\int_0^{+\infty} x^3 e^{-a^2x^2}\, dx = \frac{1}{2a^4}, \quad a > 0$$

and use this result to show that $\bar{v} = \sqrt{8RT/(\pi M)}$.

(b) Use a CAS to show that

$$\int_0^{+\infty} x^4 e^{-a^2x^2}\, dx = \frac{3\sqrt{\pi}}{8a^5}, \quad a > 0$$

and use this result to show that $v_{\text{rms}} = \sqrt{3RT/M}$.

61. In Exercise 25 of Section 6.6, we determined the work required to lift a 6000 lb satellite to an orbital position that is 1000 mi above the Earth's surface. The ideas discussed in that exercise will be needed here.

(a) Find a definite integral that represents the work required to lift a 6000 lb satellite to a position b miles above the Earth's surface.

(b) Find a definite integral that represents the work required to lift a 6000 lb satellite an "infinite distance" above the Earth's surface. Evaluate the integral. [*Note:* The result obtained here is sometimes called the work required to "escape" the Earth's gravity.]

62–63 A ***transform*** is a formula that converts or "transforms" one function into another. Transforms are used in applications to convert a difficult problem into an easier problem whose solution can then be used to solve the original difficult problem. The ***Laplace transform*** of a function $f(t)$, which plays an important role in the study of differential equations, is denoted by $\mathcal{L}\{f(t)\}$ and is defined by

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt$$

In this formula s is treated as a constant in the integration process; thus, the Laplace transform has the effect of transforming $f(t)$ into a function of s. Use this formula in these exercises.

62. Show that

(a) $\mathcal{L}\{1\} = \dfrac{1}{s}$, $s > 0$ (b) $\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$, $s > 2$

(c) $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, $s > 0$

(d) $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$, $s > 0$.

63. In each part, find the Laplace transform.

(a) $f(t) = t$, $s > 0$ (b) $f(t) = t^2$, $s > 0$

(c) $f(t) = \begin{cases} 0, & t < 3 \\ 1, & t \ge 3 \end{cases}$, $s > 0$

C **64.** Later in the text, we will show that

$$\int_0^{+\infty} e^{-x^2}\, dx = \tfrac{1}{2}\sqrt{\pi}$$

Confirm that this is reasonable by using a CAS or a calculator with a numerical integration capability.

65. Use the result in Exercise 64 to show that

(a) $\displaystyle\int_{-\infty}^{+\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$, $a > 0$

(b) $\displaystyle\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2}\, dx = 1$, $\sigma > 0$.

66–67 A convergent improper integral over an infinite interval can be approximated by first replacing the infinite limit(s) of integration by finite limit(s), then using a numerical integration technique, such as Simpson's rule, to approximate the integral with finite limit(s). This technique is illustrated in these exercises.

66. Suppose that the integral in Exercise 64 is approximated by first writing it as

$$\int_0^{+\infty} e^{-x^2}\, dx = \int_0^K e^{-x^2}\, dx + \int_K^{+\infty} e^{-x^2}\, dx$$

then dropping the second term, and then applying Simpson's rule to the integral

$$\int_0^K e^{-x^2}\, dx$$

The resulting approximation has two sources of error: the error from Simpson's rule and the error

$$E = \int_K^{+\infty} e^{-x^2}\, dx$$

(cont.)

that results from discarding the second term. We call E the ***truncation error***.

(a) Approximate the integral in Exercise 64 by applying Simpson's rule with $n = 10$ subdivisions to the integral
$$\int_0^3 e^{-x^2}\,dx$$
Round your answer to four decimal places and compare it to $\frac{1}{2}\sqrt{\pi}$ rounded to four decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $e^{-x^2} \le \frac{1}{3}xe^{-x^2}$ for $x \ge 3$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2.1 \times 10^{-5}$.

67. (a) It can be shown that
$$\int_0^{+\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$
Approximate this integral by applying Simpson's rule with $n = 20$ subdivisions to the integral
$$\int_0^4 \frac{1}{x^6+1}\,dx$$
Round your answer to three decimal places and compare it to $\pi/3$ rounded to three decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $1/(x^6+1) < 1/x^6$ for $x \ge 4$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2 \times 10^{-4}$.

68. For what values of p does $\displaystyle\int_0^{+\infty} e^{px}\,dx$ converge?

69. Show that $\displaystyle\int_0^1 dx/x^p$ converges if $p < 1$ and diverges if $p \ge 1$.

C **70.** It is sometimes possible to convert an improper integral into a "proper" integral having the same value by making an appropriate substitution. Evaluate the following integral by making the indicated substitution, and investigate what happens if you evaluate the integral directly using a CAS.
$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx;\ u = \sqrt{1-x}$$

71–72 Transform the given improper integral into a proper integral by making the stated u-substitution; then approximate the proper integral by Simpson's rule with $n = 10$ subdivisions. Round your answer to three decimal places. ■

71. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx;\ u = \sqrt{x}$

72. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{1-x}}\,dx;\ u = \sqrt{1-x}$

73. Writing What is "improper" about an integral over an infinite interval? Explain why Definition 5.5.1 for $\int_a^b f(x)\,dx$ fails for $\int_a^{+\infty} f(x)\,dx$. Discuss a strategy for assigning a value to $\int_a^{+\infty} f(x)\,dx$.

74. Writing What is "improper" about a definite integral over an interval on which the integrand has an infinite discontinuity? Explain why Definition 5.5.1 for $\int_a^b f(x)\,dx$ fails if the graph of f has a vertical asymptote at $x = a$. Discuss a strategy for assigning a value to $\int_a^b f(x)\,dx$ in this circumstance.

✔ QUICK CHECK ANSWERS 7.8

1. (a) proper (b) improper, since $\cot x$ has an infinite discontinuity at $x = \pi$ (c) improper, since there is an infinite interval of integration (d) improper, since there is an infinite interval of integration and the integrand has an infinite discontinuity at $x = 1$
2. (b) $\displaystyle\lim_{b\to\pi^-}\int_{\pi/4}^b \cot x\,dx$ (c) $\displaystyle\lim_{b\to+\infty}\int_0^b \frac{1}{x^2+1}\,dx$ (d) $\displaystyle\lim_{a\to 1^+}\int_a^2 \frac{1}{x^2-1}\,dx + \lim_{b\to+\infty}\int_2^b \frac{1}{x^2-1}\,dx$ **3.** $\dfrac{1}{p-1};\ p > 1$
4. (a) 1 (b) diverges (c) diverges (d) 3

CHAPTER 7 REVIEW EXERCISES

1–6 Evaluate the given integral with the aid of an appropriate u-substitution. ■

1. $\displaystyle\int \sqrt{4+9x}\,dx$

2. $\displaystyle\int \frac{1}{\sec \pi x}\,dx$

3. $\displaystyle\int \sqrt{\cos x}\sin x\,dx$

4. $\displaystyle\int \frac{dx}{x\ln x}$

5. $\displaystyle\int x\tan^2(x^2)\sec^2(x^2)\,dx$

6. $\displaystyle\int_0^9 \frac{\sqrt{x}}{x+9}\,dx$

7. (a) Evaluate the integral
$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$
three ways: using the substitution $u = \sqrt{x}$, using the substitution $u = \sqrt{2-x}$, and completing the square.

(b) Show that the answers in part (a) are equivalent.

8. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$
(a) using integration by parts
(b) using the substitution $u = \sqrt{x^2+1}$.

9–12 Use integration by parts to evaluate the integral. ■

9. $\int xe^{-x}\,dx$ **10.** $\int x\sin 2x\,dx$

11. $\int \ln(2x+3)\,dx$ **12.** $\int_0^{1/2} \tan^{-1}(2x)\,dx$

13. Evaluate $\int 8x^4\cos 2x\,dx$ using tabular integration by parts.

14. A particle moving along the x-axis has velocity function $v(t) = t^2e^{-t}$. How far does the particle travel from time $t = 0$ to $t = 5$?

15–20 Evaluate the integral. ■

15. $\int \sin^2 5\theta\,d\theta$ **16.** $\int \sin^3 2x\cos^2 2x\,dx$

17. $\int \sin x\cos 2x\,dx$ **18.** $\int_0^{\pi/6} \sin 2x\cos 4x\,dx$

19. $\int \sin^4 2x\,dx$ **20.** $\int x\cos^5(x^2)\,dx$

21–26 Evaluate the integral by making an appropriate trigonometric substitution. ■

21. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ **22.** $\int \frac{dx}{x^2\sqrt{16-x^2}}$

23. $\int \frac{dx}{\sqrt{x^2-1}}$ **24.** $\int \frac{x^2}{\sqrt{x^2-25}}\,dx$

25. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$ **26.** $\int \frac{\sqrt{1+4x^2}}{x}\,dx$

27–32 Evaluate the integral using the method of partial fractions. ■

27. $\int \frac{dx}{x^2+3x-4}$ **28.** $\int \frac{dx}{x^2+8x+7}$

29. $\int \frac{x^2+2}{x+2}\,dx$ **30.** $\int \frac{x^2+x-16}{(x-1)(x-3)^2}\,dx$

31. $\int \frac{x^2}{(x+2)^3}\,dx$ **32.** $\int \frac{dx}{x^3+x}$

33. Consider the integral $\int \frac{1}{x^3-x}\,dx$.
(a) Evaluate the integral using the substitution $x = \sec\theta$. For what values of x is your result valid?
(b) Evaluate the integral using the substitution $x = \sin\theta$. For what values of x is your result valid?
(c) Evaluate the integral using the method of partial fractions. For what values of x is your result valid?

34. Find the area of the region that is enclosed by the curves $y = (x-3)/(x^3+x^2)$, $y = 0$, $x = 1$, and $x = 2$.

35–40 Use the Endpaper Integral Table to evaluate the integral. ■

35. $\int \sin 7x\cos 9x\,dx$ **36.** $\int (x^3-x^2)e^{-x}\,dx$

37. $\int x\sqrt{x-x^2}\,dx$ **38.** $\int \frac{dx}{x\sqrt{4x+3}}$

39. $\int \tan^2 2x\,dx$ **40.** $\int \frac{3x-1}{2+x^2}\,dx$

41–42 Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20}. In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places. ■

41. $\int_1^3 \frac{1}{\sqrt{x+1}}\,dx$ **42.** $\int_{-1}^1 \frac{1}{1+x^2}\,dx$

43–44 Use inequalities (12), (13), and (14) of Section 7.7 to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise. ■

43. Exercise 41 **44.** Exercise 42

45–46 Use inequalities (12), (13), and (14) of Section 7.7 to find a number n of subintervals for (a) the midpoint approximation M_n, (b) the trapezoidal approximation T_n, and (c) Simpson's rule approximation S_n to ensure the absolute error will be less than 10^{-4}. ■

45. Exercise 41 **46.** Exercise 42

47–50 Evaluate the integral if it converges. ■

47. $\int_0^{+\infty} e^{-x}\,dx$ **48.** $\int_{-\infty}^2 \frac{dx}{x^2+4}$

49. $\int_0^9 \frac{dx}{\sqrt{9-x}}$ **50.** $\int_0^1 \frac{1}{2x-1}\,dx$

51. Find the area that is enclosed between the x-axis and the curve $y = (\ln x - 1)/x^2$ for $x \ge e$.

52. Find the volume of the solid that is generated when the region between the x-axis and the curve $y = e^{-x}$ for $x \ge 0$ is revolved about the y-axis.

53. Find a positive value of a that satisfies the equation

$$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = 1$$

54. Consider the following methods for evaluating integrals: u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitutions. In each part, state the approach that you would try first to evaluate the integral. If none of them seems appropriate, then say so. You need not evaluate the integral.
(a) $\int x\sin x\,dx$ (b) $\int \cos x\sin x\,dx$

(cont.)

(c) $\int \tan^7 x\,dx$
(d) $\int \tan^7 x \sec^2 x\,dx$
(e) $\int \frac{3x^2}{x^3+1}\,dx$
(f) $\int \frac{3x^2}{(x+1)^3}\,dx$
(g) $\int \tan^{-1} x\,dx$
(h) $\int \sqrt{4-x^2}\,dx$
(i) $\int x\sqrt{4-x^2}\,dx$

55–74 Evaluate the integral. ■

55. $\int \frac{dx}{(3+x^2)^{3/2}}$

56. $\int x\cos 3x\,dx$

57. $\int_0^{\pi/4} \tan^7 \theta\,d\theta$

58. $\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}\,d\theta$

59. $\int \sin^2 2x\cos^3 2x\,dx$

60. $\int_0^4 \frac{1}{(x-3)^2}\,dx$

61. $\int e^{2x}\cos 3x\,dx$

62. $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\,dx$

63. $\int \frac{dx}{(x-1)(x+2)(x-3)}$

64. $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

65. $\int_4^8 \frac{\sqrt{x-4}}{x}\,dx$

66. $\int_0^{\ln 2} \sqrt{e^x-1}\,dx$

67. $\int \frac{1}{\sqrt{e^x+1}}\,dx$

68. $\int \frac{dx}{x(x^2+x+1)}$

69. $\int_0^{1/2} \sin^{-1} x\,dx$

70. $\int \tan^5 4x\sec^4 4x\,dx$

71. $\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$

72. $\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\,d\theta$

73. $\int_a^{+\infty} \frac{x}{(x^2+1)^2}\,dx$

74. $\int_0^{+\infty} \frac{dx}{a^2+b^2x^2}, \quad a, b > 0$

CHAPTER 7 MAKING CONNECTIONS

C CAS

1. Recall from Theorem 3.3.1 and the discussion preceding it that if $f'(x) > 0$, then the function f is increasing and has an inverse function. Parts (a), (b), and (c) of this problem show that if this condition is satisfied and if f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.

(a) Use integration by parts to show that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx$$

(b) Use the result in part (a) to show that if $y = f(x)$, then

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

(c) Show that if we let $\alpha = f(a)$ and $\beta = f(b)$, then the result in part (b) can be written as

$$\int_\alpha^\beta f^{-1}(x)\,dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$$

2. In each part, use the result in Exercise 1 to obtain the equation, and then confirm that the equation is correct by performing the integrations.

(a) $\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x\,dx$

(b) $\int_e^{e^2} \ln x\,dx = (2e^2 - e) - \int_1^2 e^x\,dx$

3. The ***Gamma function***, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt$$

It can be shown that this improper integral converges if and only if $x > 0$.

(a) Find $\Gamma(1)$.
(b) Prove: $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. [*Hint:* Use integration by parts.]
(c) Use the results in parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$; and then make a conjecture about $\Gamma(n)$ for positive integer values of n.
(d) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [*Hint:* See Exercise 64 of Section 7.8.]
(e) Use the results obtained in parts (b) and (d) to show that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

4. Refer to the Gamma function defined in Exercise 3 to show that

(a) $\int_0^1 (\ln x)^n\,dx = (-1)^n\Gamma(n+1), \quad n > 0$

[*Hint:* Let $t = -\ln x$.]

(b) $\int_0^{+\infty} e^{-x^n}\,dx = \Gamma\left(\frac{n+1}{n}\right), \quad n > 0.$

[*Hint:* Let $t = x^n$. Use the result in Exercise 3(b).]

C **5.** A ***simple pendulum*** consists of a mass that swings in a vertical plane at the end of a massless rod of length L, as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle θ_0 and released from rest. It can be shown that in the absence of friction, the time T required for the pendulum to make one complete back-and-forth swing, called the ***period***, is given by

$$T = \sqrt{\frac{8L}{g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta \qquad (1)$$

where $\theta = \theta(t)$ is the angle the pendulum makes with the vertical at time t. The improper integral in (1) is difficult to evaluate numerically. By a substitution outlined below it can be shown that the period can be expressed as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\phi}}\, d\phi \qquad (2)$$

where $k = \sin(\theta_0/2)$. The integral in (2) is called a ***complete elliptic integral of the first kind*** and is more easily evaluated by numerical methods.

(a) Obtain (2) from (1) by substituting

$$\cos\theta = 1 - 2\sin^2(\theta/2)$$
$$\cos\theta_0 = 1 - 2\sin^2(\theta_0/2)$$
$$k = \sin(\theta_0/2)$$

and then making the change of variable

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} = \frac{\sin(\theta/2)}{k}$$

(b) Use (2) and the numerical integration capability of your CAS to estimate the period of a simple pendulum for which $L = 1.5$ ft, $\theta_0 = 20°$, and $g = 32$ ft/s^2.

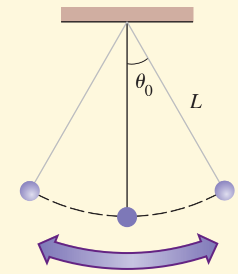
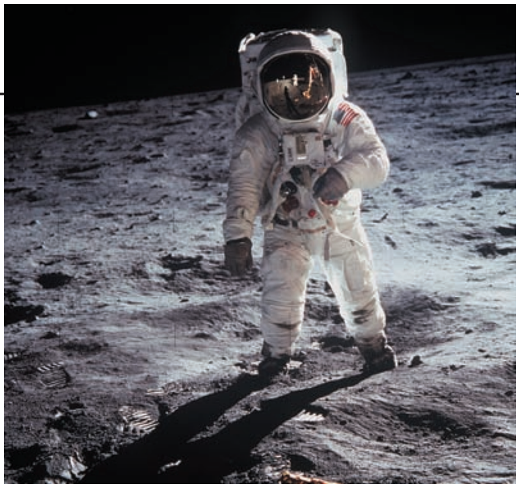


Figure Ex-5

EXPANDING THE CALCULUS HORIZON

To learn how numerical integration can be applied to the cost analysis of an engineering project, see the module entitled **Railroad Design** at:

www.wiley.com/college/anton



Courtesy NASA

6

APPLICATIONS OF THE DEFINITE INTEGRAL IN GEOMETRY, SCIENCE, AND ENGINEERING

Calculus is essential for the computations required to land an astronaut on the moon.

In the last chapter we introduced the definite integral as the limit of Riemann sums in the context of finding areas. However, Riemann sums and definite integrals have applications that extend far beyond the area problem. In this chapter we will show how Riemann sums and definite integrals arise in such problems as finding the volume and surface area of a solid, finding the length of a plane curve, calculating the work done by a force, finding the center of gravity of a planar region, finding the pressure and force exerted by a fluid on a submerged object, and finding properties of suspended cables.

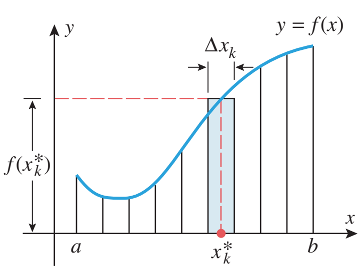
Although these problems are diverse, the required calculations can all be approached by the same procedure that we used to find areas—breaking the required calculation into "small parts," making an approximation for each part, adding the approximations from the parts to produce a Riemann sum that approximates the entire quantity to be calculated, and then taking the limit of the Riemann sums to produce an exact result.

6.1 AREA BETWEEN TWO CURVES

In the last chapter we showed how to find the area between a curve $y = f(x)$ and an interval on the x-axis. Here we will show how to find the area between two curves.

A REVIEW OF RIEMANN SUMS

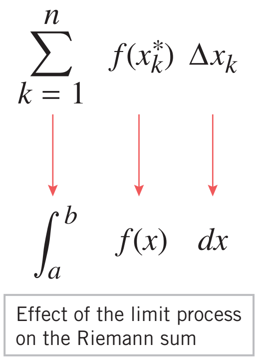
Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if f is continuous and nonnegative on $[a, b]$, then the definite integral for the area A under $y = f(x)$ over the interval $[a, b]$ is obtained in four steps (Figure 6.1.1):



▲ **Figure 6.1.1**

- Divide the interval $[a, b]$ into n subintervals, and use those subintervals to divide the region under the curve $y = f(x)$ into n strips.
- Assuming that the width of the kth strip is Δx_k, approximate the area of that strip by the area $f(x_k^*)\Delta x_k$ of a rectangle of width Δx_k and height $f(x_k^*)$, where x_k^* is a point in the kth subinterval.
- Add the approximate areas of the strips to approximate the entire area A by the Riemann sum:

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$



▲ **Figure 6.1.2**

- Take the limit of the Riemann sums as the number of subintervals increases and all their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area A:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

Figure 6.1.2 illustrates the effect that the limit process has on the various parts of the Riemann sum:

- The quantity x_k^* in the Riemann sum becomes the variable x in the definite integral.
- The interval width Δx_k in the Riemann sum becomes the dx in the definite integral.
- The interval $[a, b]$, which is the union of the subintervals with widths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, does not appear explicitly in the Riemann sum but is represented by the upper and lower limits of integration in the definite integral.

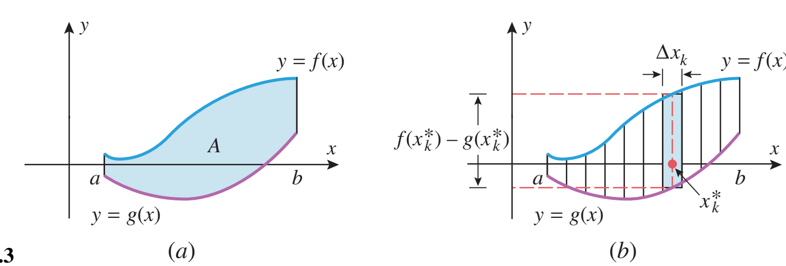
AREA BETWEEN $y = f(x)$ AND $y = g(x)$

We will now consider the following extension of the area problem.

6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.3a).



▶ **Figure 6.1.3**

To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of subdividing the region into n strips (Figure 6.1.3b). If we assume that the width of the kth strip is Δx_k, then the area of the strip can be approximated by the area of a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$, where x_k^* is a point in the kth subinterval. Adding these approximations yields the following Riemann sum that approximates the area A:

$$A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k$$

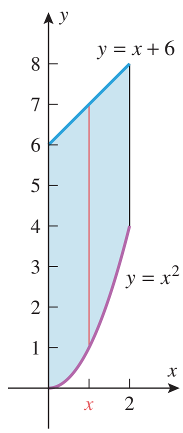
Taking the limit as n increases and the widths of all the subintervals approach zero yields the following definite integral for the area A between the curves:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k = \int_a^b [f(x) - g(x)]\,dx$$

In summary, we have the following result.

6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx \tag{1}$$



▲ **Figure 6.1.4**

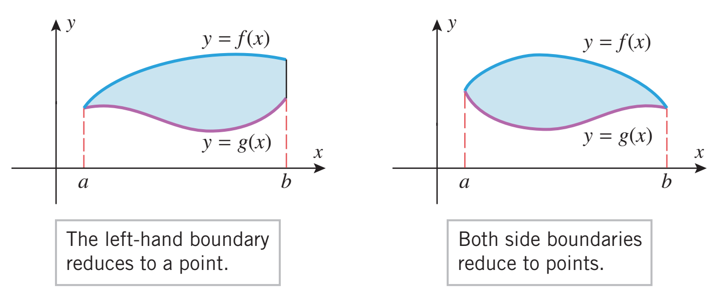
▶ Example 1 Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_0^2 [(x + 6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3} \quad ◀$$

What does the integral in (1) represent if the graphs of f and g cross each other over the interval $[a, b]$? How would you find the area between the curves in this case?

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.



▶ **Figure 6.1.5**

▶ Example 2 Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. A sketch of the region (Figure 6.1.6) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower boundaries have the same y-coordinates; thus, to find the endpoints we equate

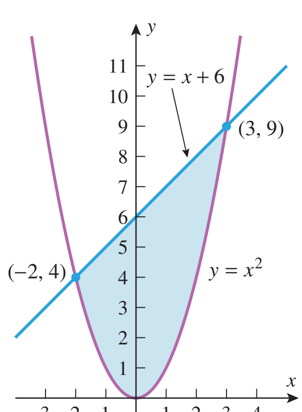
$$y = x^2 \quad \text{and} \quad y = x + 6 \tag{2}$$

This yields

$$x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x + 2)(x - 3) = 0$$

from which we obtain

$$x = -2 \quad \text{and} \quad x = 3$$



▲ **Figure 6.1.6**

Although the y-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $(-2, 4)$ and $(3, 9)$.

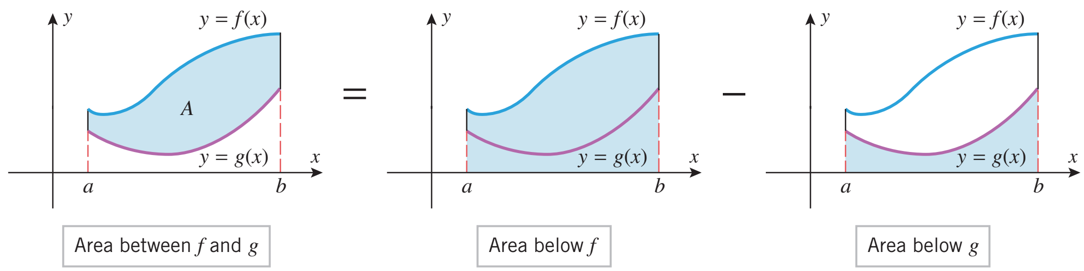
From (1) with $f(x) = x + 6$, $g(x) = x^2$, $a = -2$, and $b = 3$, we obtain the area

$$A = \int_{-2}^{3} [(x+6) - x^2]\, dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6} \blacktriangleleft$$

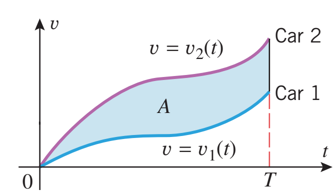
In the case where f and g are *nonnegative* on the interval $[a, b]$, the formula

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

states that the area A between the curves can be obtained by subtracting the area under $y = g(x)$ from the area under $y = f(x)$ (Figure 6.1.7).



▲ **Figure 6.1.7**



▲ **Figure 6.1.8**

▶ Example 3 Figure 6.1.8 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \le t \le T$.

Solution. From (1)

$$A = \int_0^T [v_2(t) - v_1(t)]\, dt = \int_0^T v_2(t)\, dt - \int_0^T v_1(t)\, dt$$

Since v_1 and v_2 are nonnegative functions on $[0, T]$, it follows from Formula (4) of Section 5.7 that the integral of v_1 over $[0, T]$ is the distance traveled by car 1 during the time interval $0 \le t \le T$, and the integral of v_2 over $[0, T]$ is the distance traveled by car 2 during the same time interval. Since $v_1(t) \le v_2(t)$ on $[0, T]$, car 2 travels farther than car 1 does over the time interval $0 \le t \le T$, and the area A represents the distance by which car 2 is ahead of car 1 at time T. ◀

Some regions may require careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

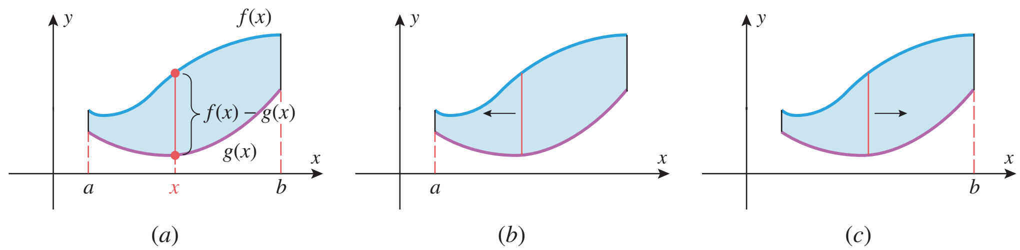
It is not necessary to make an extremely accurate sketch in Step 1; the only purpose of the sketch is to determine which curve is the upper boundary and which is the lower boundary.

Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x-axis, connecting the top and bottom boundaries (Figure 6.1.9*a*).

Step 2. The y-coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.9*b* and 6.1.9*c*).



▲ **Figure 6.1.9**

There is a useful way of thinking about this procedure:

> *If you view the vertical line segment as the "cross section" of the region at the point x, then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval $[a, b]$.*

It is possible for the upper or lower boundary of a region to consist of two or more different curves, in which case it will be convenient to subdivide the region into smaller pieces in order to apply Formula (1). This is illustrated in the next example.

► **Example 4** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Solution. To determine the appropriate boundaries of the region, we need to know where the curves $x = y^2$ and $y = x - 2$ intersect. In Example 2 we found intersections by equating the expressions for y. Here it is easier to rewrite the latter equation as $x = y + 2$ and equate the expressions for x, namely,

$$x = y^2 \quad \text{and} \quad x = y + 2 \tag{3}$$

This yields

$$y^2 = y + 2 \quad \text{or} \quad y^2 - y - 2 = 0 \quad \text{or} \quad (y + 1)(y - 2) = 0$$

from which we obtain $y = -1$, $y = 2$. Substituting these values in either equation in (3) we see that the corresponding x-values are $x = 1$ and $x = 4$, respectively, so the points of intersection are $(1, -1)$ and $(4, 2)$ (Figure 6.1.10*a*).

To apply Formula (1), the equations of the boundaries must be written so that y is expressed explicitly as a function of x. The upper boundary can be written as $y = \sqrt{x}$ (rewrite $x = y^2$ as $y = \pm\sqrt{x}$ and choose the $+$ for the upper portion of the curve). The lower boundary consists of two parts:

$$y = -\sqrt{x} \quad \text{for} \quad 0 \le x \le 1 \qquad \text{and} \qquad y = x - 2 \quad \text{for} \quad 1 \le x \le 4$$

(Figure 6.1.10*b*). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

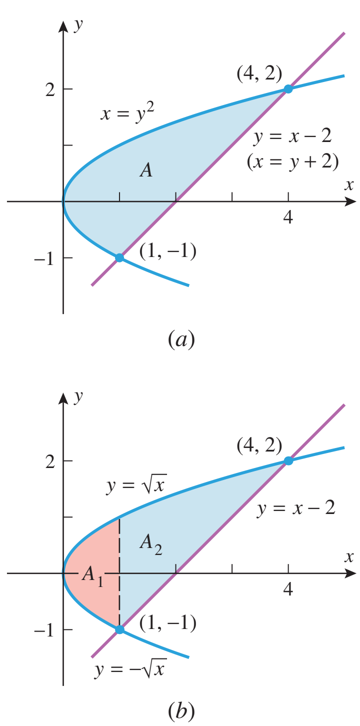
From (1) with $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$, and $b = 1$, we obtain

$$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\, dx = 2\int_0^1 \sqrt{x}\, dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}$$

From (1) with $f(x) = \sqrt{x}$, $g(x) = x - 2$, $a = 1$, and $b = 4$, we obtain

$$A_2 = \int_1^4 [\sqrt{x} - (x - 2)]\, dx = \int_1^4 (\sqrt{x} - x + 2)\, dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{19}{6}$$



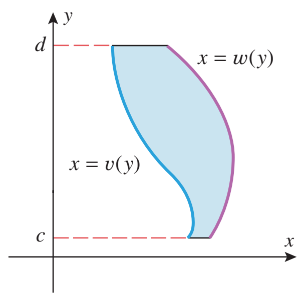
▲ **Figure 6.1.10**

Thus, the area of the entire region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$ ◀

REVERSING THE ROLES OF x AND y

Sometimes it is much easier to find the area of a region by integrating with respect to y rather than x. We will now show how this can be done.



▲ **Figure 6.1.11**

6.1.3 SECOND AREA PROBLEM Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

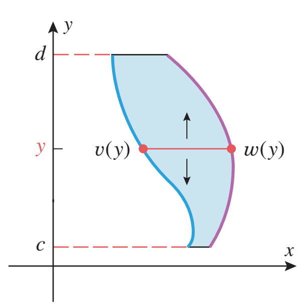
$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.11).

Proceeding as in the derivation of (1), but with the roles of x and y reversed, leads to the following analog of 6.1.2.

6.1.4 AREA FORMULA If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$



▲ **Figure 6.1.12**

The guiding principle in applying this formula is the same as with (1): The integrand in (4) can be viewed as the length of the horizontal cross section at an arbitrary point y on the y-axis, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval $[c, d]$ on the y-axis (Figure 6.1.12).

In Example 4, we split the region into two parts to facilitate integrating with respect to x. In the next example we will see that splitting this region can be avoided if we integrate with respect to y.

The choice between Formulas (1) and (4) is usually dictated by the shape of the region and which formula requires the least amount of splitting. However, sometimes one might choose the formula that requires more splitting because it is easier to evaluate the resulting integrals.

▶ **Example 5** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

Solution. As indicated in Figure 6.1.10 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \leq y \leq 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y. Thus, we rewrite $y = x - 2$ as $x = y + 2$. It now follows from (4) that

$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

which agrees with the result obtained in Example 4. ◀

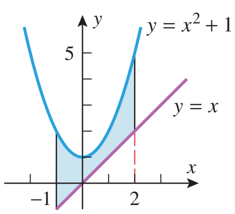
✔ QUICK CHECK EXERCISES 6.1

(See page 421 for answers.)

1. An integral expression for the area of the region between the curves $y = 20 - 3x^2$ and $y = e^x$ and bounded on the sides by $x = 0$ and $x = 2$ is ______.
2. An integral expression for the area of the parallelogram bounded by $y = 2x + 8$, $y = 2x - 3$, $x = -1$, and $x = 5$ is ______. The value of this integral is ______.
3. (a) The points of intersection for the circle $x^2 + y^2 = 4$ and the line $y = x + 2$ are ______ and ______.
   (b) Expressed as a definite integral with respect to x, ______ gives the area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = x + 2$.
   (c) Expressed as a definite integral with respect to y, ______ gives the area of the region described in part (b).
4. The area of the region enclosed by the curves $y = x^2$ and $y = \sqrt[3]{x}$ is ______.

EXERCISE SET 6.1 Graphing Utility C CAS

1–4 Find the area of the shaded region. ■

1.


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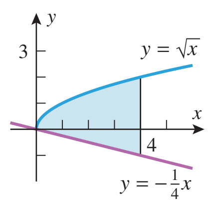

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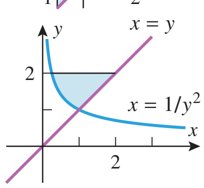

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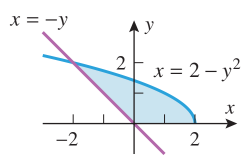
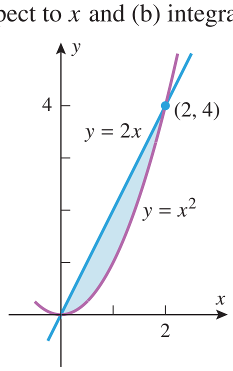
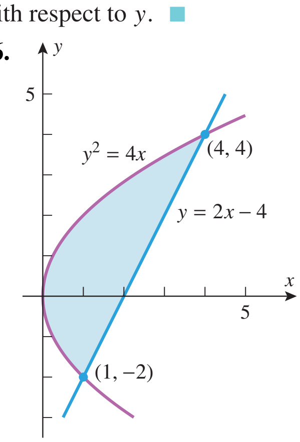


5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y. ■

5.


6.


7–18 Sketch the region enclosed by the curves and find its area. ■

7. $y = x^2,\ y = \sqrt{x},\ x = \frac{1}{4},\ x = 1$
8. $y = x^3 - 4x,\ y = 0,\ x = 0,\ x = 2$
9. $y = \cos 2x,\ y = 0,\ x = \pi/4,\ x = \pi/2$
10. $y = \sec^2 x,\ y = 2,\ x = -\pi/4,\ x = \pi/4$
11. $x = \sin y,\ x = 0,\ y = \pi/4,\ y = 3\pi/4$
12. $x^2 = y,\ x = y - 2$
13. $y = e^x,\ y = e^{2x},\ x = 0,\ x = \ln 2$
14. $x = 1/y,\ x = 0,\ y = 1,\ y = e$
15. $y = \dfrac{2}{1 + x^2},\ y = |x|$
16. $y = \dfrac{1}{\sqrt{1 - x^2}},\ y = 2$
17. $y = 2 + |x - 1|,\ y = -\frac{1}{5}x + 7$
18. $y = x,\ y = 4x,\ y = -x + 2$

19–26 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves. ■

19. $y = x^3 - 4x^2 + 3x,\ y = 0$
20. $y = x^3 - 2x^2,\ y = 2x^2 - 3x$
21. $y = \sin x,\ y = \cos x,\ x = 0,\ x = 2\pi$
22. $y = x^3 - 4x,\ y = 0$
23. $x = y^3 - y,\ x = 0$
24. $x = y^3 - 4y^2 + 3y,\ x = y^2 - y$
25. $y = xe^{x^2},\ y = 2|x|$
26. $y = \dfrac{1}{x\sqrt{1 - (\ln x)^2}},\ y = \dfrac{3}{x}$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that f and g are distinct continuous functions on $[a, b]$ and that A denotes the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.] ■

27. If f and g differ by a positive constant c, then $A = c(b - a)$.
28. If
$$\int_a^b [f(x) - g(x)]\,dx = -3$$
then $A = 3$.
29. If
$$\int_a^b [f(x) - g(x)]\,dx = 0$$
then the graphs of $y = f(x)$ and $y = g(x)$ cross at least once on $[a, b]$.
30. If
$$A = \left|\int_a^b [f(x) - g(x)]\,dx\right|$$

then the graphs of $y = f(x)$ and $y = g(x)$ don't cross on $[a, b]$.

31. Estimate the value of k $(0 < k < 1)$ so that the region enclosed by $y = 1/\sqrt{1 - x^2}$, $y = x$, $x = 0$, and $x = k$ has an area of 1 square unit.

32. Estimate the area of the region in the first quadrant enclosed by $y = \sin 2x$ and $y = \sin^{-1} x$.

33. Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.

34. Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.

35. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.

36. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{y}$, $x = 2$, and $y = 0$ into two equal parts.

37. (a) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x-axis.
(b) Find the value of m so that the line $y = mx$ divides the region in part (a) into two regions of equal area.

38. Find the area between the curve $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(5\pi/6, 1/2)$ on the curve.

39–43 Use Newton's Method (Section 4.7), where needed, to approximate the x-coordinates of the intersections of the curves to at least four decimal places, and then use those approximations to approximate the area of the region. ■

39. The region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.

40. The region enclosed by the graphs of $y = x^2$ and $y = \cos x$.

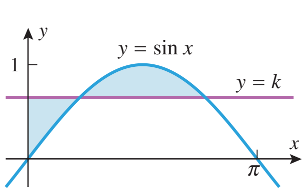
41. The region enclosed by the graphs of $y = (\ln x)/x$ and $y = x - 2$.

42. The region enclosed by the graphs of $y = 3 - 2\cos x$ and $y = 2/(1 + x^2)$.

43. The region enclosed by the graphs of $y = x^2 - 1$ and $y = 2\sin x$.

44. Referring to the accompanying figure, use a CAS to estimate the value of k so that the areas of the shaded regions are equal.

Source: This exercise is based on Problem A1 that was posed in the Fifty-Fourth Annual William Lowell Putnam Mathematical Competition.



◀ **Figure Ex-44**

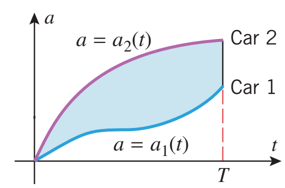
FOCUS ON CONCEPTS

45. Two racers in adjacent lanes move with velocity functions $v_1(t)$ m/s and $v_2(t)$ m/s, respectively. Suppose that the racers are even at time $t = 60$ s. Interpret the value of the integral

$$\int_0^{60} [v_2(t) - v_1(t)]\,dt$$

in this context.

46. The accompanying figure shows acceleration versus time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area A between the curves over the interval $0 \leq t \leq T$ represent? Justify your answer.



◀ **Figure Ex-46**

47. Suppose that f and g are integrable on $[a, b]$, but neither $f(x) \geq g(x)$ nor $g(x) \geq f(x)$ holds for all x in $[a, b]$ [i.e., the curves $y = f(x)$ and $y = g(x)$ are intertwined].
(a) What is the geometric significance of the integral

$$\int_a^b [f(x) - g(x)]\,dx?$$

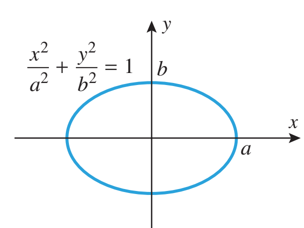
(b) What is the geometric significance of the integral

$$\int_a^b |f(x) - g(x)|\,dx?$$

48. Let $A(n)$ be the area in the first quadrant enclosed by the curves $y = \sqrt[n]{x}$ and $y = x$.
(a) By considering how the graph of $y = \sqrt[n]{x}$ changes as n increases, make a conjecture about the limit of $A(n)$ as $n \to +\infty$.
(b) Confirm your conjecture by calculating the limit.

49. Find the area of the region enclosed between the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.

50. Show that the area of the ellipse in the accompanying figure is πab. [*Hint:* Use a formula from geometry.]



◀ **Figure Ex-50**

51. Writing Suppose that f and g are continuous on $[a, b]$ but that the graphs of $y = f(x)$ and $y = g(x)$ cross several times. Describe a step-by-step procedure for determining the area bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.

52. **Writing** Suppose that R and S are two regions in the xy-plane that lie between a pair of lines L_1 and L_2 that are parallel to the y-axis. Assume that each line between L_1 and L_2 that is parallel to the y-axis intersects R and S in line segments of equal length. Give an informal argument that the area of R is equal to the area of S. (Make reasonable assumptions about the boundaries of R and S.)

✔ QUICK CHECK ANSWERS 6.1

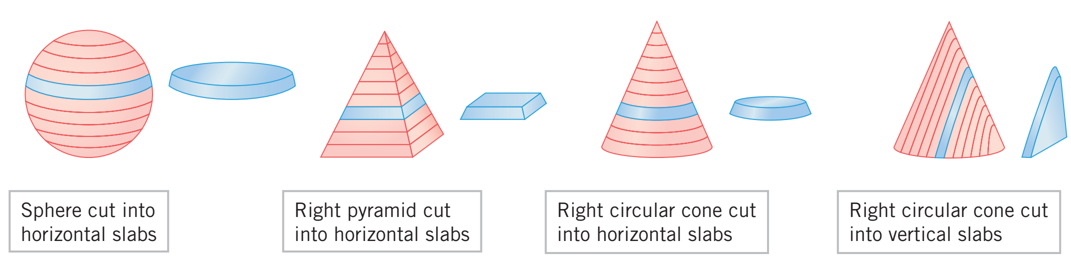
1. $\int_0^2 [(20-3x^2)-e^x]\,dx$ **2.** $\int_{-1}^5 [(2x+8)-(2x-3)]\,dx$; 66 **3.** (a) $(-2,0)$; $(0,2)$ (b) $\int_{-2}^0 [\sqrt{4-x^2}-(x+2)]\,dx$ (c) $\int_0^2 [(y-2)+\sqrt{4-y^2}]\,dy$ **4.** $\frac{5}{12}$

6.2 VOLUMES BY SLICING; DISKS AND WASHERS

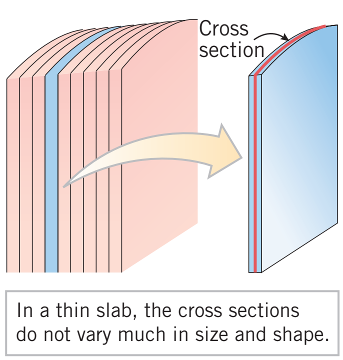
In the last section we showed that the area of a plane region bounded by two curves can be obtained by integrating the length of a general cross section over an appropriate interval. In this section we will see that the same basic principle can be used to find volumes of certain three-dimensional solids.

VOLUMES BY SLICING

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).



▲ **Figure 6.2.1**



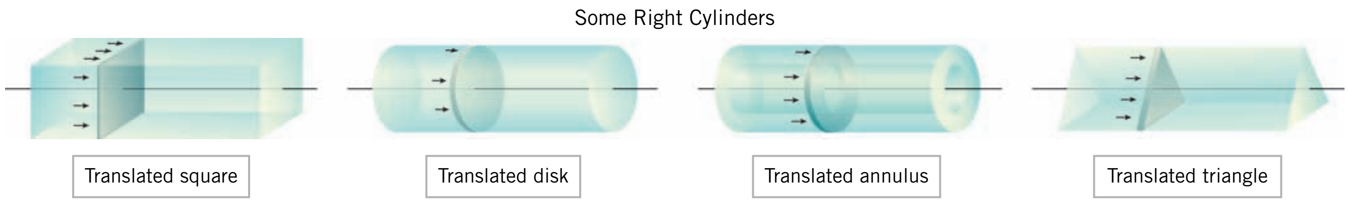
▲ **Figure 6.2.2**

What makes this method work is the fact that a *thin* slab has a cross section that does not vary much in size or shape, which, as we will see, makes its volume easy to approximate (Figure 6.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set up a Riemann sum whose limit is the volume of the entire solid. We will give the details shortly, but first we need to discuss how to find the volume of a solid whose cross sections do not vary in size and shape (i.e., are congruent).

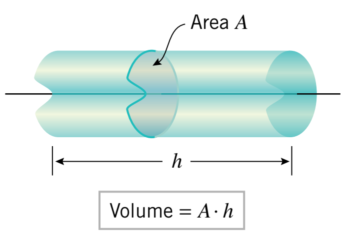
One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius r, since all cross sections taken perpendicular to the central axis are circular regions of radius r. The volume V of a right circular cylinder of radius r and height h can be expressed in terms of the height and the area of a cross section as

$$V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}] \tag{1}$$

This is a special case of a more general volume formula that applies to solids called right cylinders. A ***right cylinder*** is a solid that is generated when a plane region is translated along a line or ***axis*** that is perpendicular to the region (Figure 6.2.3).



▲ **Figure 6.2.3**



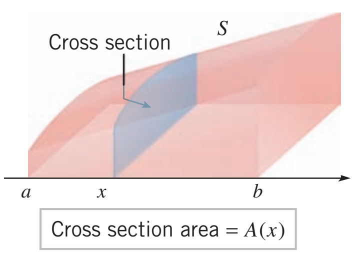
▲ **Figure 6.2.4**

If a right cylinder is generated by translating a region of area A through a distance h, then h is called the ***height*** (or sometimes the ***width***) of the cylinder, and the volume V of the cylinder is defined to be

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \tag{2}$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right *circular* cylinder.

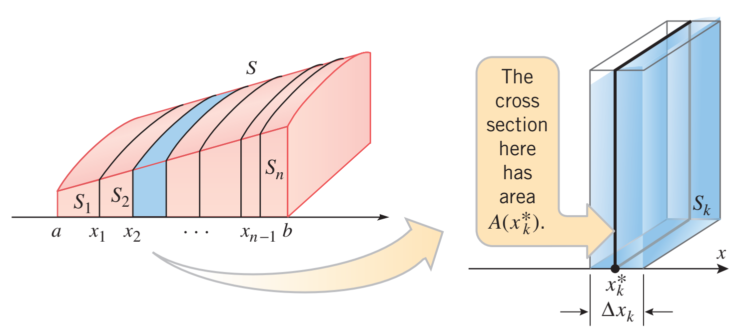
We now have all of the tools required to solve the following problem.



▲ **Figure 6.2.5**

6.2.1 PROBLEM Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at $x = a$ and $x = b$ (Figure 6.2.5). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.

To solve this problem we begin by dividing the interval $[a, b]$ into n subintervals, thereby dividing the solid into n slabs as shown in the left part of Figure 6.2.6. If we assume that the width of the kth subinterval is Δx_k, then the volume of the kth slab can be approximated by the volume $A(x_k^*)\Delta x_k$ of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is a point in the kth subinterval (see the right part of Figure 6.2.6).



▶ **Figure 6.2.6**

Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result.

It is understood in our calculations of volume that the units of volume are the cubed units of length [e.g., cubic inches (in^3) or cubic meters (m^3)].

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

There is a similar result for cross sections perpendicular to the y-axis.

6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

In words, these formulas state:

The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

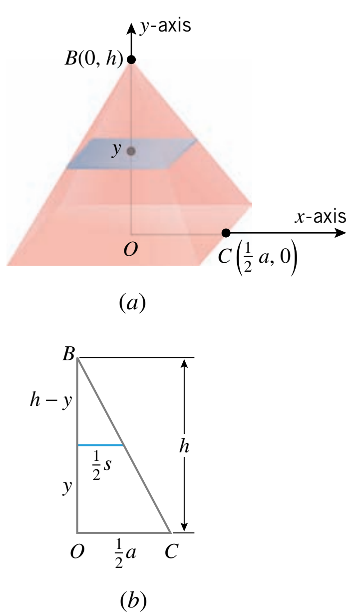


Figure 6.2.7

Example 1 Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution. As illustrated in Figure 6.2.7a, we introduce a rectangular coordinate system in which the y-axis passes through the apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.

At any y in the interval $[0, h]$ on the y-axis, the cross section perpendicular to the y-axis is a square. If s denotes the length of a side of this square, then by similar triangles (Figure 6.2.7b)

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h}(h-y)$$

Thus, the area $A(y)$ of the cross section at y is

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$

and by (4) the volume is

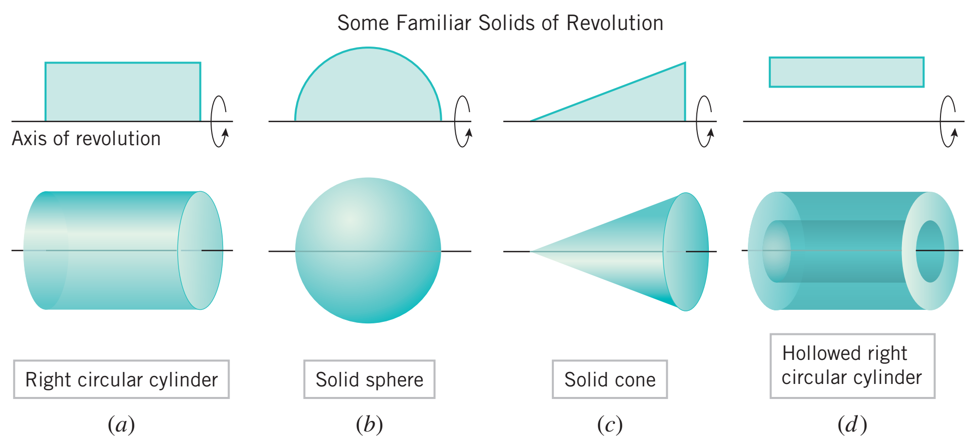
$$V = \int_0^h A(y)\,dy = \int_0^h \frac{a^2}{h^2}(h-y)^2\,dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy$$

$$= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_{y=0}^{h} = \frac{a^2}{h^2}\left[0+\frac{1}{3}h^3\right] = \frac{1}{3}a^2h$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude. ◀

SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type (Figure 6.2.8).

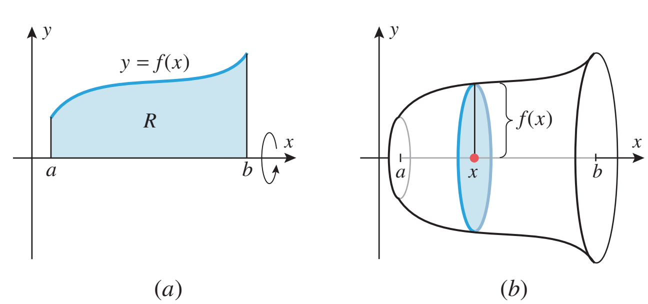


▶ **Figure 6.2.8**

VOLUMES BY DISKS PERPENDICULAR TO THE *x*-AXIS

We will be interested in the following general problem.

> **6.2.4 PROBLEM** Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.



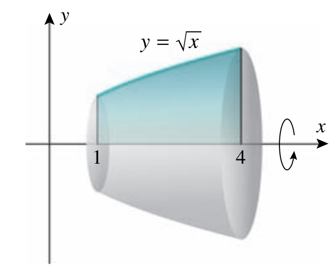
▶ **Figure 6.2.9**

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius $f(x)$ (Figure 6.2.9b). The area of this region is

$$A(x) = \pi[f(x)]^2$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2\,dx \tag{5}$$



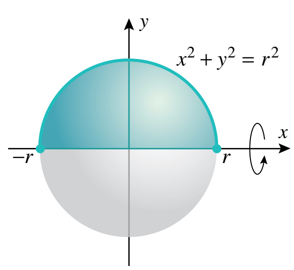
▲ **Figure 6.2.10**

Because the cross sections are disk shaped, the application of this formula is called the ***method of disks***.

▶ **Example 2** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_1^4 \pi x\,dx = \frac{\pi x^2}{2}\bigg]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2} \blacktriangleleft$$



▲ **Figure 6.2.11**

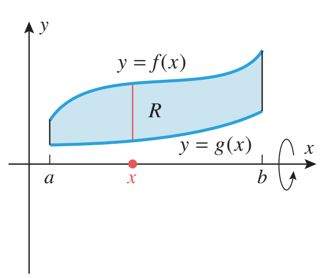
▶ **Example 3** Derive the formula for the volume of a sphere of radius r.

Solution. As indicated in Figure 6.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and

$$x^2 + y^2 = r^2$$

about the x-axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

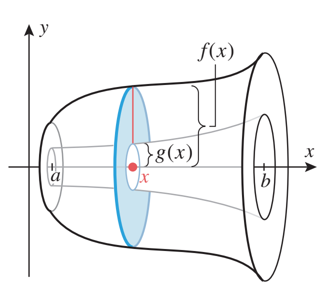
$$V = \int_a^b \pi[f(x)]^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3 \blacktriangleleft$$



(a)

VOLUMES BY WASHERS PERPENDICULAR TO THE *x*-AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in Figure 6.2.8d. So we will also be interested in problems of the following type.



(b)

▲ **Figure 6.2.12**

> **6.2.5 PROBLEM** Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis (Figure 6.2.12b).

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 6.2.12*b*); its area is

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \tag{6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers***.

Example 4 Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Solution. First sketch the region (Figure 6.2.13*a*); then imagine revolving it about the x-axis (Figure 6.2.13*b*). From (6) the volume is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right)dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right)dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \frac{69\pi}{10}$$

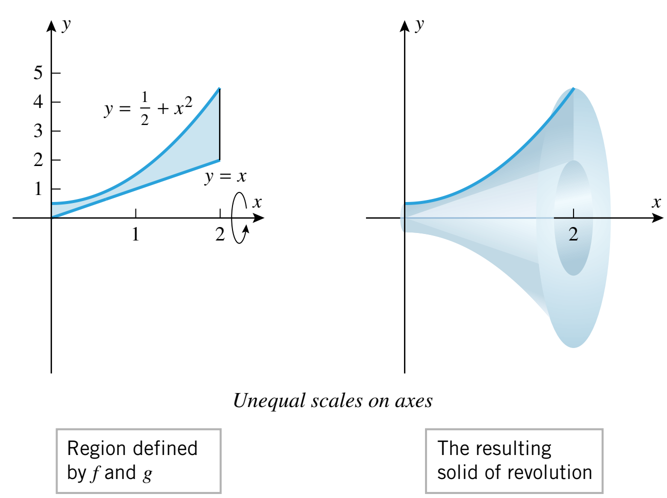
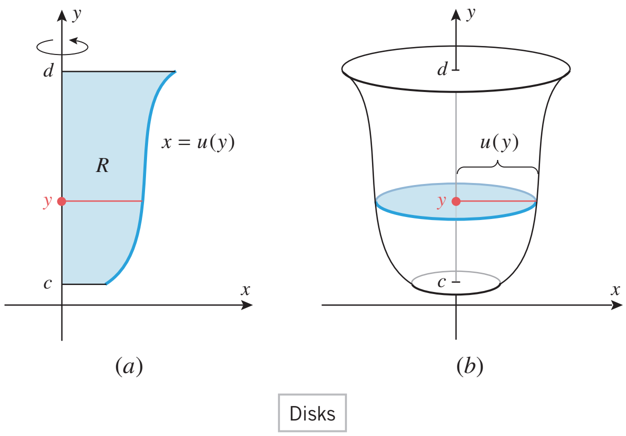


Figure 6.2.13 (*a*) (*b*)

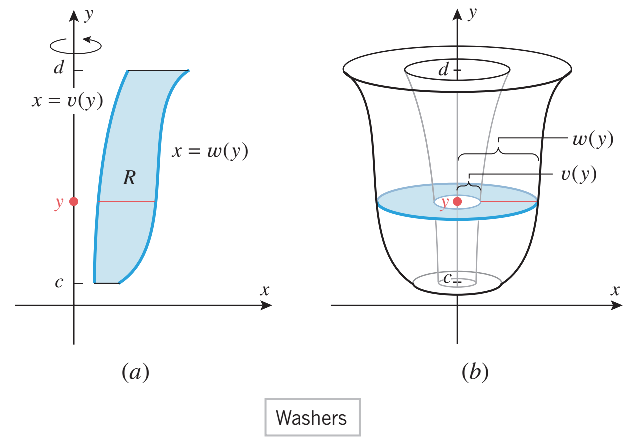
VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE *y*-AXIS

The methods of disks and washers have analogs for regions that are revolved about the y-axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4), you should be able to deduce the following formulas for the volumes of the solids in the figures.

$$V = \int_c^d \pi[u(y)]^2\,dy \quad \text{Disks} \qquad V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy \quad \text{Washers} \tag{7–8}$$



▲ Figure 6.2.14



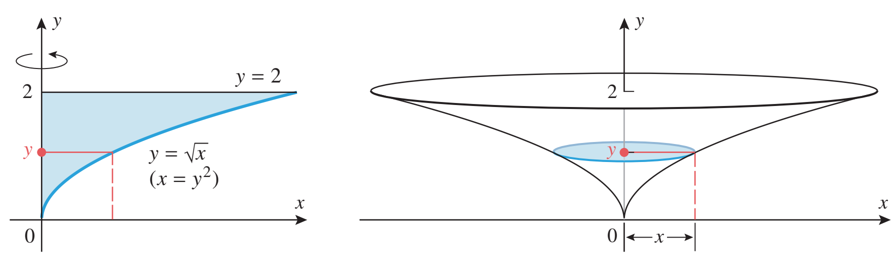
▲ Figure 6.2.15

▶ **Example 5** Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Solution. First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the y-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^2 \pi y^4\,dy = \frac{\pi y^5}{5}\bigg]_0^2 = \frac{32\pi}{5} \blacktriangleleft$$

▶ Figure 6.2.16



■ OTHER AXES OF REVOLUTION

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (3) and (4) and integrate an appropriate cross-sectional area to find the volume.

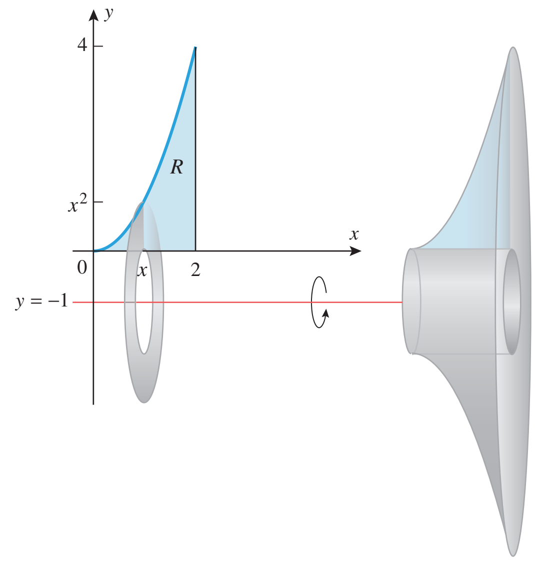
▶ **Example 6** Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval $[0, 2]$ is rotated about the line $y = -1$.

Solution. First sketch the region and the axis of revolution; then imagine revolving the region about the axis (Figure 6.2.17). At each x in the interval $0 \le x \le 2$, the cross section of the solid perpendicular to the axis $y = -1$ is a washer with outer radius $x^2 + 1$ and inner radius 1. Since the area of this washer is

$$A(x) = \pi([x^2 + 1]^2 - 1^2) = \pi(x^4 + 2x^2)$$

it follows by (3) that the volume of the solid is

$$V = \int_0^2 A(x)\,dx = \int_0^2 \pi\left(x^4 + 2x^2\right)dx = \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3\right]_0^2 = \frac{176\pi}{15}$$ ◄



► **Figure 6.2.17**

✔QUICK CHECK EXERCISES 6.2 *(See page 431 for answers.)*

1. A solid S extends along the x-axis from $x = 1$ to $x = 3$. For x between 1 and 3, the cross-sectional area of S perpendicular to the x-axis is $3x^2$. An integral expression for the volume of S is ________. The value of this integral is ________.

2. A solid S is generated by revolving the region between the x-axis and the curve $y = \sqrt{\sin x}$ $(0 \le x \le \pi)$ about the x-axis.
(a) For x between 0 and π, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ ________.
(b) An integral expression for the volume of S is ________.
(c) The value of the integral in part (b) is ________.

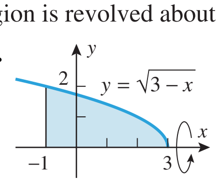
3. A solid S is generated by revolving the region enclosed by the line $y = 2x + 1$ and the curve $y = x^2 + 1$ about the x-axis.
(a) For x between ________ and ________, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ ________.
(b) An integral expression for the volume of S is ________.

4. A solid S is generated by revolving the region enclosed by the line $y = x + 1$ and the curve $y = x^2 + 1$ about the y-axis.
(a) For y between ________ and ________, the cross-sectional area of S perpendicular to the y-axis at y is $A(y) =$ ________.
(b) An integral expression for the volume of S is ________.

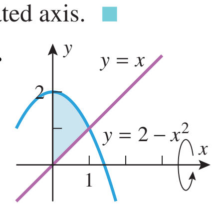
EXERCISE SET 6.2 [C] CAS

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis. ■

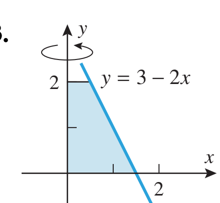
1.



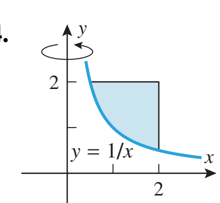
2.



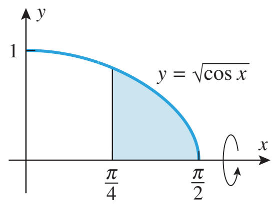
3.



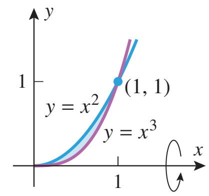
4.



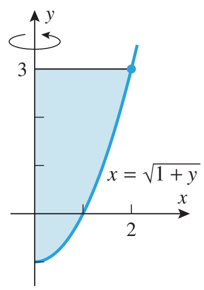
5.



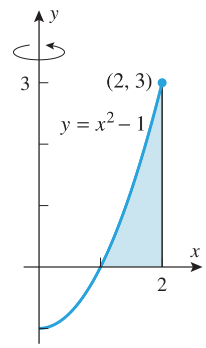
6.



7.



8.



9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are squares.

10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.

11–18 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis. ■

11. $y = \sqrt{25 - x^2}, \ y = 3$

12. $y = 9 - x^2, \ y = 0$

13. $x = \sqrt{y}, \ x = y/4$

14. $y = \sin x, \ y = \cos x, \ x = 0, \ x = \pi/4$
[*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. $y = e^x, \ y = 0, \ x = 0, \ x = \ln 3$

16. $y = e^{-2x}, \ y = 0, \ x = 0, \ x = 1$

17. $y = \dfrac{1}{\sqrt{4 + x^2}}, \ x = -2, \ x = 2, \ y = 0$

18. $y = \dfrac{e^{3x}}{\sqrt{1 + e^{6x}}}, \ x = 0, \ x = 1, \ y = 0$

19. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.

20. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

21–26 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis. ■

21. $x = \csc y, \ y = \pi/4, \ y = 3\pi/4, \ x = 0$

22. $y = x^2, \ x = y^2$

23. $x = y^2, \ x = y + 2$

24. $x = 1 - y^2, \ x = 2 + y^2, \ y = -1, \ y = 1$

25. $y = \ln x, \ x = 0, \ y = 0, \ y = 1$

26. $y = \sqrt{\dfrac{1 - x^2}{x^2}} \quad (x > 0), \ x = 0, \ y = 0, \ y = 2$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid S of volume V is bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$ and that for each x in $[a, b]$, $A(x)$ denotes the cross-sectional area of S perpendicular to the x-axis.] ■

27. If each cross section of S perpendicular to the x-axis is a square, then S is a rectangular parallelepiped (i.e., is box shaped).

28. If each cross section of S is a disk or a washer, then S is a solid of revolution.

29. If x is in centimeters (cm), then $A(x)$ must be a quadratic function of x, since units of $A(x)$ will be square centimeters (cm^2).

30. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b - a)$.

31. Find the volume of the solid that results when the region above the x-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x-axis.

32. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

33. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x + 1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

34. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

FOCUS ON CONCEPTS

35. Suppose that f is a continuous function on $[a, b]$, and let R be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving R about the line $y = k$. State and explain additional assumptions, if any, that you need about f for your formula.

36. Suppose that v and w are continuous functions on $[c, d]$, and let R be the region between the curves $x = v(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving R about the line

$x = k$. State and explain additional assumptions, if any, that you need about v and w for your formula.

37. Consider the solid generated by revolving the shaded region in Exercise 1 about the line $y = 2$.
(a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 1. Explain the basis of your conjecture.
(b) Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 1.

38. Consider the solid generated by revolving the shaded region in Exercise 4 about the line $x = 2.5$.
(a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 4. Explain the basis of your conjecture.
(b) Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 4.

39. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.

40. Find the volume of the solid that results when the region in Exercise 39 is revolved about the line $y = 3$.

41. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.

42. Find the volume of the solid that results when the region in Exercise 41 is revolved about the line $x = -1$.

43. Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved about the line $x = 1$.

44. Find the volume of the solid that results when the region in Exercise 43 is revolved about the line $y = -1$.

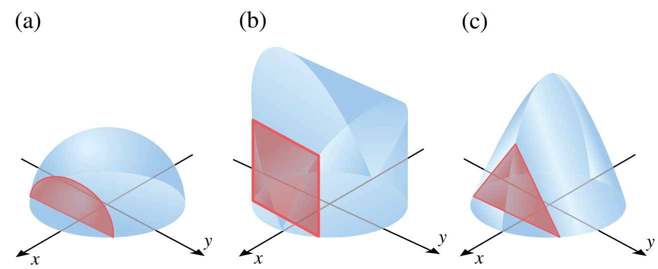
45. A nose cone for a space reentry vehicle is designed so that a cross section, taken x ft from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.

46. A certain solid is 1 ft high, and a horizontal cross section taken x ft above the bottom of the solid is an annulus of inner radius x^2 ft and outer radius $\sqrt{x}$ ft. Find the volume of the solid.

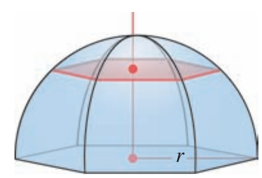
47. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are squares.

48. The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the x-axis is a semicircle with its diameter across the base. Find the volume of the solid.

49. In parts (a)–(c) find the volume of the solid whose base is enclosed by the circle $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the x-axis are
(a) semicircles (b) squares
(c) equilateral triangles.



50. As shown in the accompanying figure, a cathedral dome is designed with three semicircular supports of radius r so that each horizontal cross section is a regular hexagon. Show that the volume of the dome is $r^3\sqrt{3}$.



◀ **Figure Ex-50**

C **51–54** Use a CAS to estimate the volume of the solid that results when the region enclosed by the curves is revolved about the stated axis. ■

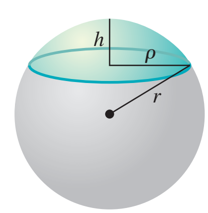
51. $y = \sin^8 x,\ \ y = 2x/\pi,\ \ x = 0,\ \ x = \pi/2;$ x-axis

52. $y = \pi^2 \sin x \cos^3 x,\ \ y = 4x^2,\ \ x = 0,\ \ x = \pi/4;$ x-axis

53. $y = e^x,\ \ x = 1,\ \ y = 1;$ y-axis

54. $y = x\sqrt{\tan^{-1} x},\ \ y = x;$ x-axis

55. The accompanying figure shows a ***spherical cap*** of radius ρ and height h cut from a sphere of radius r. Show that the volume V of the spherical cap can be expressed as
(a) $V = \frac{1}{3}\pi h^2(3r - h)$ (b) $V = \frac{1}{6}\pi h(3\rho^2 + h^2)$.



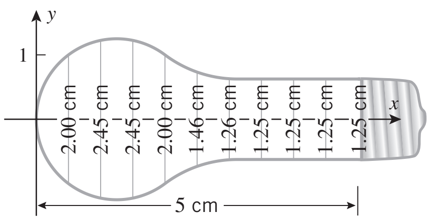
◀ **Figure Ex-55**

56. If fluid enters a hemispherical bowl with a radius of 10 ft at a rate of $\frac{1}{2}$ ft^3/min, how fast will the fluid be rising when the depth is 5 ft? [*Hint:* See Exercise 55.]

57. The accompanying figure (on the next page) shows the dimensions of a small lightbulb at 10 equally spaced points.
(a) Use formulas from geometry to make a rough estimate of the volume enclosed by the glass portion of the bulb.

(*cont.*)

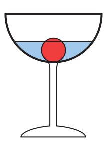
(b) Use the average of left and right endpoint approximations to approximate the volume.



▲ **Figure Ex-57**

58. Use the result in Exercise 55 to find the volume of the solid that remains when a hole of radius $r/2$ is drilled through the center of a sphere of radius r, and then check your answer by integrating.

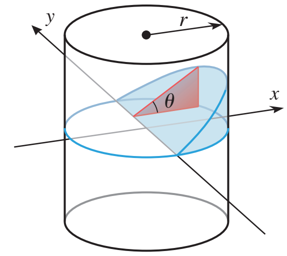
59. As shown in the accompanying figure, a cocktail glass with a bowl shaped like a hemisphere of diameter 8 cm contains a cherry with a diameter of 2 cm. If the glass is filled to a depth of h cm, what is the volume of liquid it contains? [*Hint:* First consider the case where the cherry is partially submerged, then the case where it is totally submerged.]



◀ **Figure Ex-59**

60. Find the volume of the torus that results when the region enclosed by the circle of radius r with center at $(h, 0)$, $h > r$, is revolved about the y-axis. [*Hint:* Use an appropriate formula from plane geometry to help evaluate the definite integral.]

61. A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder and the other making an angle θ with the first. Find the volume of the wedge by slicing perpendicular to the y-axis as shown in the accompanying figure.

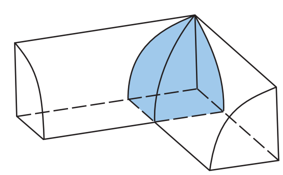


◀ **Figure Ex-61**

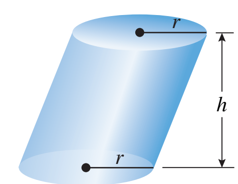
62. Find the volume of the wedge described in Exercise 61 by slicing perpendicular to the x-axis.

63. Two right circular cylinders of radius r have axes that intersect at right angles. Find the volume of the solid common to the two cylinders. [*Hint:* One-eighth of the solid is sketched in the accompanying figure.]

64. In 1635 Bonaventura Cavalieri, a student of Galileo, stated the following result, called ***Cavalieri's principle***: *If two solids have the same height, and if the areas of their cross sections taken parallel to and at equal distances from their bases are always equal, then the solids have the same volume.* Use this result to find the volume of the oblique cylinder in the accompanying figure. (See Exercise 52 of Section 6.1 for a planar version of Cavalieri's principle.)



▲ **Figure Ex-63**



▲ **Figure Ex-64**

65. **Writing** Use the results of this section to derive Cavalieri's principle (Exercise 64).

66. **Writing** Write a short paragraph that explains how Formulas (4)–(8) may all be viewed as consequences of Formula (3).

✔ QUICK CHECK ANSWERS 6.2

1. $\int_1^3 3x^2\,dx$; 26 **2.** (a) $\pi \sin x$ (b) $\int_0^{\pi} \pi \sin x\,dx$ (c) 2π **3.** (a) 0; 2; $\pi[(2x+1)^2 - (x^2+1)^2] = \pi[-x^4 + 2x^2 + 4x]$ (b) $\int_0^2 \pi[-x^4 + 2x^2 + 4x]\,dx$ **4.** (a) 1; 2; $\pi[(y-1) - (y-1)^2] = \pi[-y^2 + 3y - 2]$ (b) $\int_1^2 \pi[-y^2 + 3y - 2]\,dy$

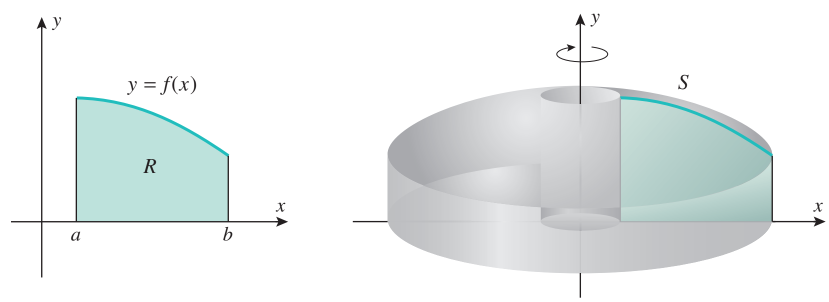
6.3 VOLUMES BY CYLINDRICAL SHELLS

The methods for computing volumes that have been discussed so far depend on our ability to compute the cross-sectional area of the solid and to integrate that area across the solid. In this section we will develop another method for finding volumes that may be applicable when the cross-sectional area cannot be found or the integration is too difficult.

CYLINDRICAL SHELLS

In this section we will be interested in the following problem.

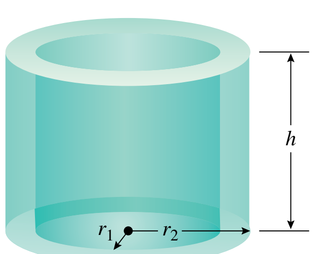
6.3.1 PROBLEM Let f be continuous and nonnegative on $[a, b]$ $(0 \leq a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume V of the solid of revolution S that is generated by revolving the region R about the y-axis (Figure 6.3.1).



▶ **Figure 6.3.1**

Sometimes problems of the above type can be solved by the method of disks or washers perpendicular to the y-axis, but when that method is not applicable or the resulting integral is difficult, the *method of cylindrical shells*, which we will discuss here, will often work.

A ***cylindrical shell*** is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h can be written as



▲ **Figure 6.3.2**

$$\begin{aligned} V &= [\text{area of cross section}] \cdot [\text{height}] \\ &= (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \cdot \left[\tfrac{1}{2}(r_1 + r_2)\right] \cdot h \cdot (r_2 - r_1) \end{aligned}$$

But $\frac{1}{2}(r_1 + r_2)$ is the average radius of the shell and $r_2 - r_1$ is its thickness, so

$$V = 2\pi \cdot [\text{average radius}] \cdot [\text{height}] \cdot [\text{thickness}] \tag{1}$$

We will now show how this formula can be used to solve Problem 6.3.1. The underlying idea is to divide the interval $[a, b]$ into n subintervals, thereby subdividing the region R into n strips, $R_1, R_2, \ldots, R_n$ (Figure 6.3.3*a*). When the region R is revolved about the y-axis, these strips generate "tube-like" solids $S_1, S_2, \ldots, S_n$ that are nested one inside the other and together comprise the entire solid S (Figure 6.3.3*b*). Thus, the volume V of the solid can be obtained by adding together the volumes of the tubes; that is,

$$V = V(S_1) + V(S_2) + \cdots + V(S_n)$$

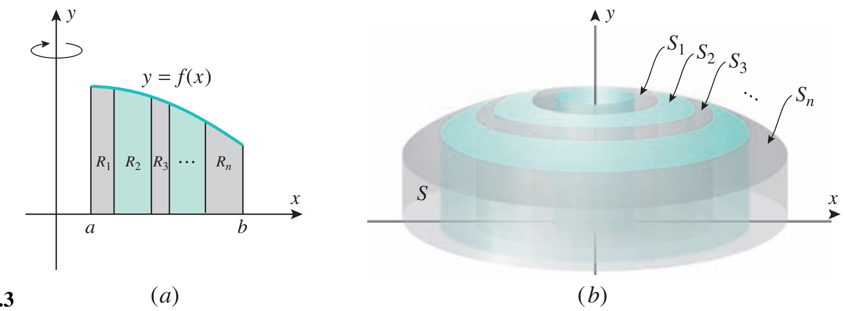


Figure 6.3.3

As a rule, the tubes will have curved upper surfaces, so there will be no simple formulas for their volumes. However, if the strips are thin, then we can approximate each strip by a rectangle (Figure 6.3.4*a*). These rectangles, when revolved about the y-axis, will produce cylindrical shells whose volumes closely approximate the volumes of the tubes generated by the original strips (Figure 6.3.4*b*). We will show that by adding the volumes of the cylindrical shells we can obtain a Riemann sum that approximates the volume V, and by taking the limit of the Riemann sums we can obtain an integral for the exact volume V.

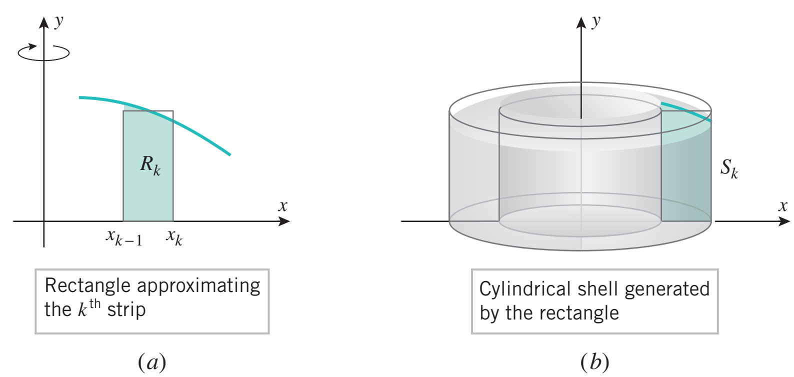


Figure 6.3.4

To implement this idea, suppose that the kth strip extends from x_{k-1} to x_k and that the width of this strip is

$$\Delta x_k = x_k - x_{k-1}$$

If we let x_k^* be the *midpoint* of the interval $[x_{k-1}, x_k]$, and if we construct a rectangle of height $f(x_k^*)$ over the interval, then revolving this rectangle about the y-axis produces a cylindrical shell of average radius x_k^*, height $f(x_k^*)$, and thickness Δx_k (Figure 6.3.5). From (1), the volume V_k of this cylindrical shell is

$$V_k = 2\pi x_k^* f(x_k^*)\Delta x_k$$

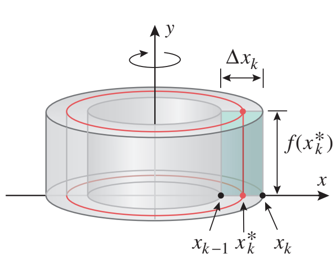


Figure 6.3.5

Adding the volumes of the n cylindrical shells yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k$$

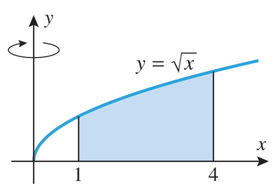
Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k = \int_a^b 2\pi x f(x)\, dx$$

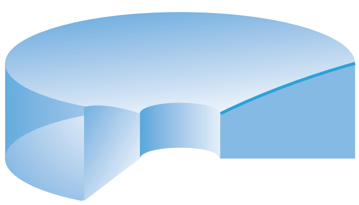
In summary, we have the following result.

6.3.2 VOLUME BY CYLINDRICAL SHELLS ABOUT THE y-AXIS Let f be continuous and nonnegative on $[a, b]$ $(0 \le a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the y-axis is given by

$$V = \int_a^b 2\pi x f(x)\, dx \qquad (2)$$



(a)


Cutaway view of the solid

(b)

Figure 6.3.6

Example 1 Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis.

Solution. First sketch the region (Figure 6.3.6a); then imagine revolving it about the y-axis (Figure 6.3.6b). Since $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$, Formula (2) yields

$$V = \int_1^4 2\pi x\sqrt{x}\, dx = 2\pi \int_1^4 x^{3/2}\, dx = \left[2\pi \cdot \frac{2}{5}x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32 - 1] = \frac{124\pi}{5}$$

VARIATIONS OF THE METHOD OF CYLINDRICAL SHELLS

The method of cylindrical shells is applicable in a variety of situations that do not fit the conditions required by Formula (2). For example, the region may be enclosed between two curves, or the axis of revolution may be some line other than the y-axis. However, rather than develop a separate formula for every possible situation, we will give a general way of thinking about the method of cylindrical shells that can be adapted to each new situation as it arises.

For this purpose, we will need to reexamine the integrand in Formula (2): At each x in the interval $[a, b]$, the vertical line segment from the x-axis to the curve $y = f(x)$ can be viewed as the cross section of the region R at x (Figure 6.3.7a). When the region R is revolved about the y-axis, the cross section at x sweeps out the *surface* of a right circular cylinder of height $f(x)$ and radius x (Figure 6.3.7b). The area of this surface is

$$2\pi x f(x)$$

(Figure 6.3.7c), which is the integrand in (2). Thus, Formula (2) can be viewed informally in the following way.

6.3.3 AN INFORMAL VIEWPOINT ABOUT CYLINDRICAL SHELLS The volume V of a solid of revolution that is generated by revolving a region R about an axis can be obtained by integrating the area of the surface generated by an arbitrary cross section of R taken parallel to the axis of revolution.

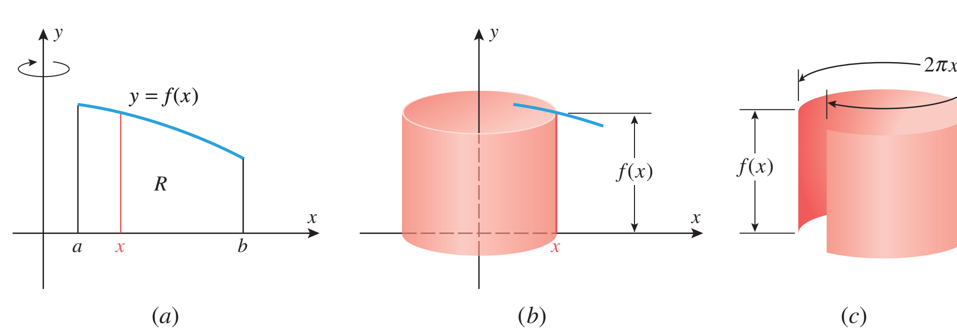


Figure 6.3.7

The following examples illustrate how to apply this result in situations where Formula (2) is not applicable.

▶ **Example 2** Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the y-axis (Figure 6.3.8a).

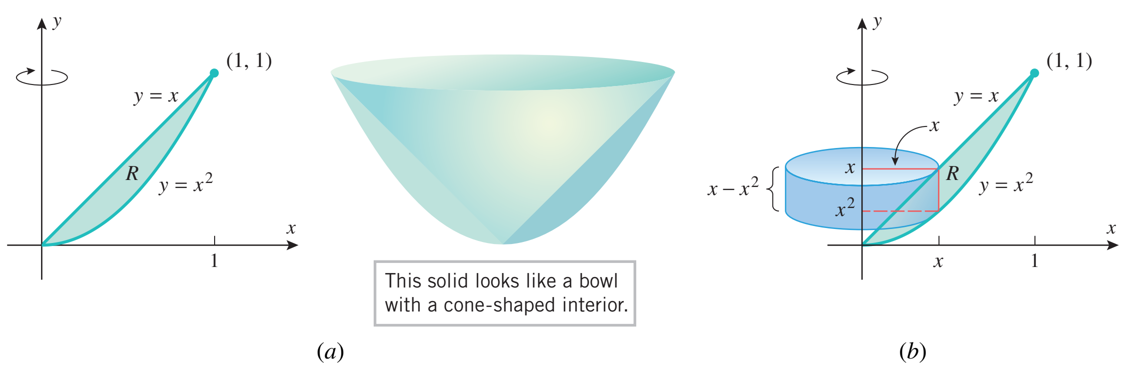
Solution. As illustrated in part (b) of Figure 6.3.8, at each x in [0, 1] the cross section of R parallel to the y-axis generates a cylindrical surface of height $x - x^2$ and radius x. Since the area of this surface is

$$2\pi x(x - x^2)$$

the volume of the solid is

$$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6} \blacktriangleleft$$



▲ **Figure 6.3.8**

▶ **Example 3** Use cylindrical shells to find the volume of the solid generated when the region R under $y = x^2$ over the interval [0, 2] is revolved about the line $y = -1$.

Solution. First draw the axis of revolution; then imagine revolving the region about the axis (Figure 6.3.9a). As illustrated in Figure 6.3.9b, at each y in the interval $0 \le y \le 4$, the cross section of R parallel to the x-axis generates a cylindrical surface of height $2 - \sqrt{y}$ and radius $y + 1$. Since the area of this surface is

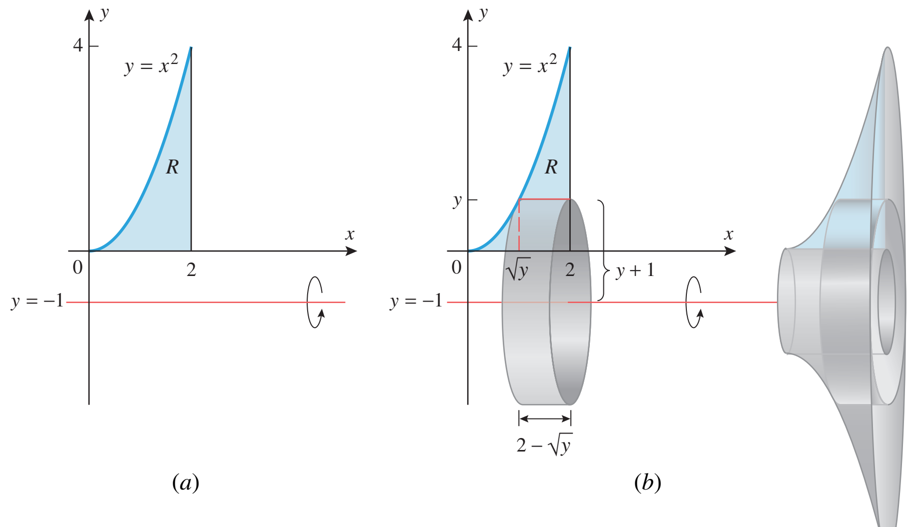
$$2\pi(y + 1)(2 - \sqrt{y})$$

it follows that the volume of the solid is

$$\int_0^4 2\pi(y + 1)(2 - \sqrt{y}\,)\,dy = 2\pi \int_0^4 (2y - y^{3/2} + 2 - y^{1/2})\,dy$$

$$= 2\pi\left[y^2 - \frac{2}{5}y^{5/2} + 2y - \frac{2}{3}y^{3/2}\right]_0^4 = \frac{176\pi}{15} \blacktriangleleft$$

Note that the volume found in Example 3 agrees with the volume of the same solid found by the method of washers in Example 6 of Section 6.2. Confirm that the volume in Example 2 found by the method of cylindrical shells can also be obtained by the method of washers.



▶ **Figure 6.3.9**

✔ QUICK CHECK EXERCISES 6.3 *(See page 438 for answers.)*

1. Let R be the region between the x-axis and the curve $y = 1 + \sqrt{x}$ for $1 \le x \le 4$.
(a) For x between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of R at x about the y-axis is ________.
(b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving R about the y-axis is ________.

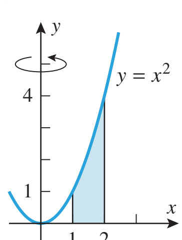
2. Let R be the region described in Quick Check Exercise 1.
(a) For x between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of R at x about the line $x = 5$ is ________.
(b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving R about the line $x = 5$ is ________.

3. A solid S is generated by revolving the region enclosed by the curves $x = (y - 2)^2$ and $x = 4$ about the x-axis. Using cylindrical shells, an integral expression for the volume of S is ________.

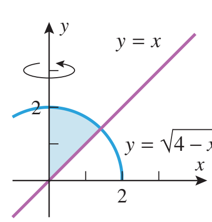
EXERCISE SET 6.3 $\boxed{C}$ CAS

1–4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis. ■

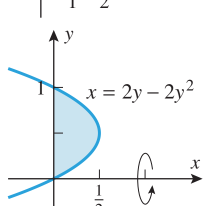
1.



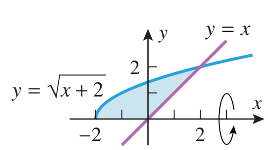
2.



3.



4.



5–12 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis. ■

5. $y = x^3, \ x = 1, \ y = 0$

6. $y = \sqrt{x}, \ x = 4, \ x = 9, \ y = 0$

7. $y = 1/x, \ y = 0, \ x = 1, \ x = 3$

8. $y = \cos(x^2), \ x = 0, \ x = \frac{1}{2}\sqrt{\pi}, \ y = 0$

9. $y = 2x - 1, \ y = -2x + 3, \ x = 2$

10. $y = 2x - x^2, \ y = 0$

11. $y = \dfrac{1}{x^2 + 1}, \ x = 0, \ x = 1, \ y = 0$

12. $y = e^{x^2}, \ x = 1, \ x = \sqrt{3}, \ y = 0$

13–16 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x-axis. ■

13. $y^2 = x,\ y = 1,\ x = 0$

14. $x = 2y,\ y = 2,\ y = 3,\ x = 0$

15. $y = x^2,\ x = 1,\ y = 0$ **16.** $xy = 4,\ x + y = 5$

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. The volume of a cylindrical shell is equal to the product of the thickness of the shell with the surface area of a cylinder whose height is that of the shell and whose radius is equal to the average of the inner and outer radii of the shell.

18. The method of cylindrical shells is a special case of the method of integration of cross-sectional area that was discussed in Section 6.2.

19. In the method of cylindrical shells, integration is over an interval on a coordinate axis that is *perpendicular* to the axis of revolution of the solid.

20. The Riemann sum approximation

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k \quad \left(\text{where } x_k^* = \frac{x_k + x_{k-1}}{2}\right)$$

for the volume of a solid of revolution is exact when f is a constant function.

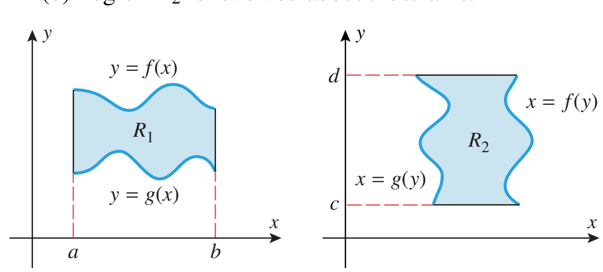
C **21.** Use a CAS to find the volume of the solid generated when the region enclosed by $y = e^x$ and $y = 0$ for $1 \le x \le 2$ is revolved about the y-axis.

C **22.** Use a CAS to find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = 0$, and $x = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

C **23.** Consider the region to the right of the y-axis, to the left of the vertical line $x = k$ $(0 < k < \pi)$, and between the curve $y = \sin x$ and the x-axis. Use a CAS to estimate the value of k so that the solid generated by revolving the region about the y-axis has a volume of 8 cubic units.

FOCUS ON CONCEPTS

24. Let R_1 and R_2 be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when
(a) region R_1 is revolved about the y-axis
(b) region R_2 is revolved about the x-axis.



▲ **Figure Ex-24**

25. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve

$$y = x^3 - 3x^2 + 2x$$

over $[0, 1]$ is revolved about the y-axis.
(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

26. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is enclosed by $y = f(x)$ and $y = 0$ for $a \le x \le b$. Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving R about the line $x = k$, where $k \le a$.

27–28 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region R is revolved about (a) the line $x = 1$ and (b) the line $y = -1$.

27. R is the region bounded by the graphs of $y = x$, $y = 0$, and $x = 1$.

28. R is the region in the first quadrant bounded by the graphs of $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$.

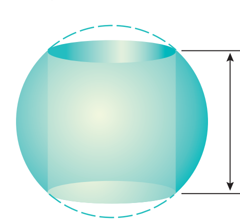
29. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

30. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

31. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0, 0)$, $(0, r)$, $(h, 0)$, where $r > 0$ and $h > 0$, is revolved about the x-axis.

32. The region enclosed between the curve $y^2 = kx$ and the line $x = \frac{1}{4}k$ is revolved about the line $x = \frac{1}{2}k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k > 0$.)

33. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.



◀ **Figure Ex-33**

34. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^2 + y^2 = a^2$ about the line

$x = b$, where $b > a > 0$. [*Hint:* It may help in the integration to think of an integral as an area.]

35. Let V_x and V_y be the volumes of the solids that result when the region enclosed by $y = 1/x$, $y = 0$, $x = \frac{1}{2}$, and $x = b$ $\left(b > \frac{1}{2}\right)$ is revolved about the x-axis and y-axis, respectively. Is there a value of b for which $V_x = V_y$?

36. (a) Find the volume V of the solid generated when the region bounded by $y = 1/(1 + x^4)$, $y = 0$, $x = 1$, and $x = b$ $(b > 1)$ is revolved about the y-axis.
(b) Find $\lim_{b \to +\infty} V$.

37. Writing Faced with the problem of computing the volume of a solid of revolution, how would you go about deciding whether to use the method of disks/washers or the method of cylindrical shells?

38. Writing With both the method of disks/washers and with the method of cylindrical shells, we integrate an "area" to get the volume of a solid of revolution. However, these two approaches differ in very significant ways. Write a brief paragraph that discusses these differences.

✔QUICK CHECK ANSWERS 6.3

1. (a) $2\pi x(1 + \sqrt{x})$ (b) $\int_1^4 2\pi x(1 + \sqrt{x})\,dx$ **2.** (a) $2\pi(5 - x)(1 + \sqrt{x})$ (b) $\int_1^4 2\pi(5 - x)(1 + \sqrt{x})\,dx$

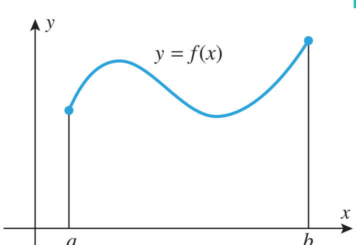
3. $\int_0^4 2\pi y[4 - (y - 2)^2]\,dy$

6.4 LENGTH OF A PLANE CURVE

In this section we will use the tools of calculus to study the problem of finding the length of a plane curve.

ARC LENGTH

Our first objective is to define what we mean by the ***length*** (also called the ***arc length***) of a plane curve $y = f(x)$ over an interval $[a, b]$ (Figure 6.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on $[a, b]$, in which case we will say that $y = f(x)$ is a ***smooth curve*** on $[a, b]$ or that f is a ***smooth function*** on $[a, b]$. Thus, we will be concerned with the following problem.



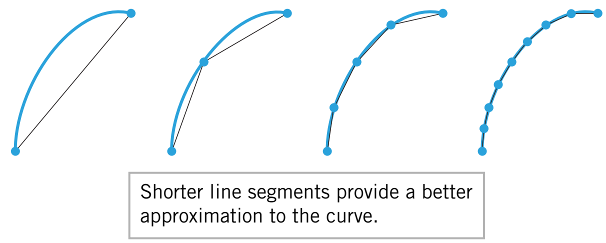
▲ **Figure 6.4.1**

6.4.1 ARC LENGTH PROBLEM Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$. Define and find a formula for the arc length L of the curve $y = f(x)$ over the interval $[a, b]$.

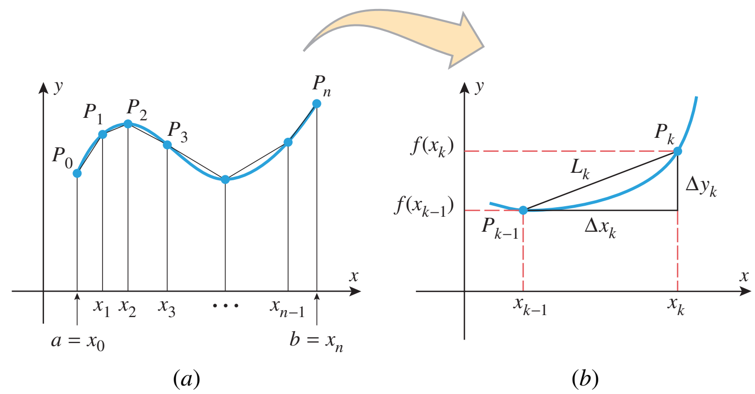
Intuitively, you might think of the arc length of a curve as the number obtained by aligning a piece of string with the curve and then measuring the length of the string after it is straightened out.

To define the arc length of a curve we start by breaking the curve into small segments. Then we approximate the curve segments by line segments and add the lengths of the line segments to form a Riemann sum. Figure 6.4.2 illustrates how such line segments tend to become better and better approximations to a curve as the number of segments increases. As the number of segments increases, the corresponding Riemann sums approach a definite integral whose value we will take to be the arc length L of the curve.

To implement our idea for solving Problem 6.4.1, divide the interval $[a, b]$ into n subintervals by inserting points $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As shown in Figure 6.4.3*a*, let $P_0, P_1, \ldots, P_n$ be the points on the curve with x-coordinates $a = x_0$,



▶ **Figure 6.4.2**



▶ **Figure 6.4.3**

$x_1, x_2, \ldots, x_{n-1}, b = x_n$ and join these points with straight line segments. These line segments form a ***polygonal path*** that we can regard as an approximation to the curve $y = f(x)$. As indicated in Figure 6.4.3*b*, the length L_k of the kth line segment in the polygonal path is

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{1}$$

If we now add the lengths of these line segments, we obtain the following approximation to the length L of the curve

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_k^*)]^2(\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$

Explain why the approximation in (2) cannot be greater than L.

Thus, taking the limit as n increases and the widths of all the subintervals approach zero yields the following integral that defines the arc length L:

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In summary, we have the following definition.

6.4.2 DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

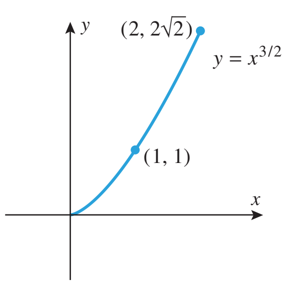
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{3}$$

This result provides both a definition and a formula for computing arc lengths. Where convenient, (3) can also be expressed as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4}$$

Moreover, for a curve expressed in the form $x = g(y)$, where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{5}$$



▲ **Figure 6.4.4**

▶ **Example 1** Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$ (Figure 6.4.4) in two ways: (a) using Formula (4) and (b) using Formula (5).

Solution (a).

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

and since the curve extends from $x = 1$ to $x = 2$, it follows from (4) that

$$L = \int_1^2 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx = \int_1^2 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

To evaluate this integral we make the u-substitution

$$u = 1 + \tfrac{9}{4}x, \quad du = \tfrac{9}{4}\, dx$$

and then change the x-limits of integration $(x = 1, x = 2)$ to the corresponding u-limits $\left(u = \frac{13}{4}, u = \frac{22}{4}\right)$:

$$L = \frac{4}{9}\int_{13/4}^{22/4} u^{1/2}\, du = \frac{8}{27}u^{3/2}\bigg]_{13/4}^{22/4} = \frac{8}{27}\left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09$$

Solution (b). To apply Formula (5) we must first rewrite the equation $y = x^{3/2}$ so that x is expressed as a function of y. This yields $x = y^{2/3}$ and

$$\frac{dx}{dy} = \frac{2}{3}y^{-1/3}$$

Since the curve extends from $y = 1$ to $y = 2\sqrt{2}$, it follows from (5) that

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \tfrac{4}{9}y^{-2/3}}\, dy = \frac{1}{3}\int_1^{2\sqrt{2}} y^{-1/3}\sqrt{9y^{2/3} + 4}\, dy$$

To evaluate this integral we make the u-substitution

$$u = 9y^{2/3} + 4, \quad du = 6y^{-1/3}\,dy$$

and change the y-limits of integration ($y = 1$, $y = 2\sqrt{2}$) to the corresponding u-limits ($u = 13$, $u = 22$). This gives

$$L = \frac{1}{18}\int_{13}^{22} u^{1/2}\,du = \frac{1}{27}u^{3/2}\Big]_{13}^{22} = \frac{1}{27}[(22)^{3/2} - (13)^{3/2}] = \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

The arc from the point $(1, 1)$ to the point $(2, 2\sqrt{2})$ in Figure 6.4.4 is nearly a straight line, so the arc length should be only slightly larger than the straight-line distance between these points. Show that this is so.

The answer in part (b) agrees with that in part (a); however, the integration in part (b) is more tedious. In problems where there is a choice between using (4) or (5), it is often the case that one of the formulas leads to a simpler integral than the other. ◀

FINDING ARC LENGTH BY NUMERICAL METHODS

In the next chapter we will develop some techniques of integration that will enable us to find exact values of more integrals encountered in arc length calculations; however, generally speaking, most such integrals are impossible to evaluate in terms of elementary functions. In these cases one usually approximates the integral using a numerical method such as the midpoint rule discussed in Section 5.4.

TECHNOLOGY MASTERY

If your calculating utility has a numerical integration capability, use it to confirm that the arc length L in Example 2 is approximately $L \approx 3.8202$.

▶ **Example 2** From (4), the arc length of $y = \sin x$ from $x = 0$ to $x = \pi$ is given by the integral

$$L = \int_0^{\pi} \sqrt{1 + (\cos x)^2}\,dx$$

This integral cannot be evaluated in terms of elementary functions; however, using a calculating utility with a numerical integration capability yields the approximation $L \approx 3.8202$. ◀

✔ QUICK CHECK EXERCISES 6.4 *(See page 443 for answers.)*

1. A function f is smooth on $[a, b]$ if f' is ________ on $[a, b]$.
2. If a function f is smooth on $[a, b]$, then the length of the curve $y = f(x)$ over $[a, b]$ is ________.
3. The distance between points $(1, 0)$ and $(e, 1)$ is ________.
4. Let L be the length of the curve $y = \ln x$ from $(1, 0)$ to $(e, 1)$.
   (a) Integrating with respect to x, an integral expression for L is ________.
   (b) Integrating with respect to y, an integral expression for L is ________.

EXERCISE SET 6.4

C CAS

1. Use the Theorem of Pythagoras to find the length of the line segment $y = 2x$ from $(1, 2)$ to $(2, 4)$, and confirm that the value is consistent with the length computed using
   (a) Formula (4) (b) Formula (5).
2. Use the Theorem of Pythagoras to find the length of the line segment $y = 5x$ from $(0, 0)$ and $(1, 5)$, and confirm that the value is consistent with the length computed using
   (a) Formula (4) (b) Formula (5).

3–8 Find the exact arc length of the curve over the interval.

3. $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$
4. $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y = 0$ to $y = 1$
5. $y = x^{2/3}$ from $x = 1$ to $x = 8$
6. $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$
7. $24xy = y^4 + 48$ from $y = 2$ to $y = 4$
8. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$

9–12 True–False Determine whether the statement is true or false. Explain your answer.

9. The graph of $y = \sqrt{1 - x^2}$ is a smooth curve on $[-1, 1]$.

10. The approximation
$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$
for arc length is not expressed in the form of a Riemann sum.

11. The approximation
$$L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$
for arc length is exact when f is a linear function of x.

12. In our definition of the arc length for the graph of $y = f(x)$, we need $f'(x)$ to be a continuous function in order for f to satisfy the hypotheses of the Mean-Value Theorem (4.8.2).

C **13–14** Express the exact arc length of the curve over the given interval as an integral that has been simplified to eliminate the radical, and then evaluate the integral using a CAS. ■

13. $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$

14. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$

FOCUS ON CONCEPTS

15. Consider the curve $y = x^{2/3}$.
(a) Sketch the portion of the curve between $x = -1$ and $x = 8$.
(b) Explain why Formula (4) cannot be used to find the arc length of the curve sketched in part (a).
(c) Find the arc length of the curve sketched in part (a).

16. The curve segment $y = x^2$ from $x = 1$ to $x = 2$ may also be expressed as the graph of $x = \sqrt{y}$ from $y = 1$ to $y = 4$. Set up two integrals that give the arc length of this curve segment, one by integrating with respect to x, and the other by integrating with respect to y. Demonstrate a substitution that verifies that these two integrals are equal.

17. Consider the curve segments $y = x^2$ from $x = \frac{1}{2}$ to $x = 2$ and $y = \sqrt{x}$ from $x = \frac{1}{4}$ to $x = 4$.
(a) Graph the two curve segments and use your graphs to explain why the lengths of these two curve segments should be equal.
(b) Set up integrals that give the arc lengths of the curve segments by integrating with respect to x. Demonstrate a substitution that verifies that these two integrals are equal.
(c) Set up integrals that give the arc lengths of the curve segments by integrating with respect to y.
(d) Approximate the arc length of each curve segment using Formula (2) with $n = 10$ equal subintervals.
(e) Which of the two approximations in part (d) is more accurate? Explain.
(f) Use the midpoint approximation with $n = 10$ subintervals to approximate each arc length integral in part (b).
(g) Use a calculating utility with numerical integration capabilities to approximate the arc length integrals in part (b) to four decimal places.

18. Follow the directions of Exercise 17 for the curve segments $y = x^{8/3}$ from $x = 10^{-3}$ to $x = 1$ and $y = x^{3/8}$ from $x = 10^{-8}$ to $x = 1$.

19. Follow the directions of Exercise 17 for the curve segment $y = \tan x$ from $x = 0$ to $x = \pi/3$ and for the curve segment $y = \tan^{-1} x$ from $x = 0$ to $x = \sqrt{3}$.

20. Let $y = f(x)$ be a smooth curve on the closed interval $[a, b]$. Prove that if m and M are nonnegative numbers such that $m \le |f'(x)| \le M$ for all x in $[a, b]$, then the arc length L of $y = f(x)$ over the interval $[a, b]$ satisfies the inequalities
$$(b-a)\sqrt{1+m^2} \le L \le (b-a)\sqrt{1+M^2}$$

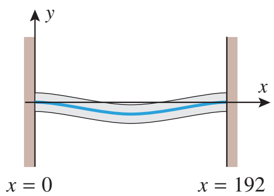
21. Use the result of Exercise 20 to show that the arc length L of $y = \sec x$ over the interval $0 \le x \le \pi/3$ satisfies
$$\frac{\pi}{3} \le L \le \frac{\pi}{3}\sqrt{13}$$

C **22.** A basketball player makes a successful shot from the free throw line. Suppose that the path of the ball from the moment of release to the moment it enters the hoop is described by
$$y = 2.15 + 2.09x - 0.41x^2, \quad 0 \le x \le 4.6$$
where x is the horizontal distance (in meters) from the point of release, and y is the vertical distance (in meters) above the floor. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance the ball travels from the moment it is released to the moment it enters the hoop. Round your answer to two decimal places.

C **23.** Find a positive value of k (to two decimal places) such that the curve $y = k \sin x$ has an arc length of $L = 5$ units over the interval from $x = 0$ to $x = \pi$. [*Hint:* Find an integral for the arc length L in terms of k, and then use a CAS or a scientific calculator with a numerical integration capability to find integer values of k at which the values of $L - 5$ have opposite signs. Complete the solution by using the Intermediate-Value Theorem (1.5.7) to approximate the value of k to two decimal places.]

C **24.** As shown in the accompanying figure on the next page, a horizontal beam with dimensions 2 in × 6 in × 16 ft is fixed at both ends and is subjected to a uniformly distributed load of 120 lb/ft. As a result of the load, the centerline of the beam undergoes a deflection that is described by
$$y = -1.67 \times 10^{-8}(x^4 - 2Lx^3 + L^2x^2)$$
($0 \le x \le 192$), where $L = 192$ in is the length of the unloaded beam, x is the horizontal distance along the beam measured in inches from the left end, and y is the deflection of the centerline in inches.
(a) Graph y versus x for $0 \le x \le 192$.
(b) Find the maximum deflection of the centerline. *(cont.)*

(c) Use a CAS or a calculator with a numerical integration capability to find the length of the centerline of the loaded beam. Round your answer to two decimal places.



◀ **Figure Ex-24**

C **25.** A golfer makes a successful chip shot to the green. Suppose that the path of the ball from the moment it is struck to the moment it hits the green is described by

$$y = 12.54x - 0.41x^2$$

where x is the horizontal distance (in yards) from the point where the ball is struck, and y is the vertical distance (in yards) above the fairway. Use a CAS or a calculating utility with a numerical integration capability to find the distance the ball travels from the moment it is struck to the moment it hits the green. Assume that the fairway and green are at the same level and round your answer to two decimal places.

26–34 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I. ■

C **26.** Assume that no segment of the curve

$$x = x(t), \quad y = y(t), \qquad (a \le t \le b)$$

is traced more than once as t increases from a to b. Divide the interval $[a, b]$ into n subintervals by inserting points $t_1, t_2, \ldots, t_{n-1}$ between $a = t_0$ and $b = t_n$. Let L denote the arc length of the curve. Give an informal argument for the approximation

$$L \approx \sum_{k=1}^{n} \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}$$

If dx/dt and dy/dt are continuous functions for $a \le t \le b$, then it can be shown that as $\max \Delta t_k \to 0$, this sum converges to

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

27–32 Use the arc length formula from Exercise 26 to find the arc length of the curve. ■

27. $x = \frac{1}{3}t^3,\ \ y = \frac{1}{2}t^2 \quad (0 \le t \le 1)$

28. $x = (1+t)^2,\ \ y = (1+t)^3 \quad (0 \le t \le 1)$

29. $x = \cos 2t,\ \ y = \sin 2t \quad (0 \le t \le \pi/2)$

30. $x = \cos t + t \sin t,\ \ y = \sin t - t \cos t \quad (0 \le t \le \pi)$

31. $x = e^t \cos t,\ \ y = e^t \sin t \quad (0 \le t \le \pi/2)$

32. $x = e^t(\sin t + \cos t),\ \ y = e^t(\cos t - \sin t) \quad (1 \le t \le 4)$

C **33.** (a) Show that the total arc length of the ellipse

$$x = 2\cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

is given by

$$4\int_0^{\pi/2} \sqrt{1 + 3\sin^2 t}\, dt$$

(b) Use a CAS or a scientific calculator with a numerical integration capability to approximate the arc length in part (a). Round your answer to two decimal places.

(c) Suppose that the parametric equations in part (a) describe the path of a particle moving in the xy-plane, where t is time in seconds and x and y are in centimeters. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance traveled by the particle from $t = 1.5$ s to $t = 4.8$ s. Round your answer to two decimal places.

34. Show that the total arc length of the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ for $a > b > 0$ is given by

$$4a\int_0^{\pi/2} \sqrt{1 - k^2\cos^2 t}\, dt$$

where $k = \sqrt{a^2 - b^2}/a$.

35. Writing In our discussion of Arc Length Problem 6.4.1, we derived the approximation

$$L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$

Discuss the geometric meaning of this approximation. (Be sure to address the appearance of the derivative f'.)

36. Writing Give examples in which Formula (4) for arc length cannot be applied directly, and describe how you would go about finding the arc length of the curve in each case. (Discuss both the use of alternative formulas and the use of numerical methods.)

✔QUICK CHECK ANSWERS 6.4

1. continuous **2.** $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ **3.** $\sqrt{(e-1)^2 + 1}$ **4.** (a) $\int_1^e \sqrt{1 + (1/x)^2}\, dx$ (b) $\int_0^1 \sqrt{1 + e^{2y}}\, dy$

6.5 AREA OF A SURFACE OF REVOLUTION

In this section we will consider the problem of finding the area of a surface that is generated by revolving a plane curve about a line.

SURFACE AREA

A ***surface of revolution*** is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).

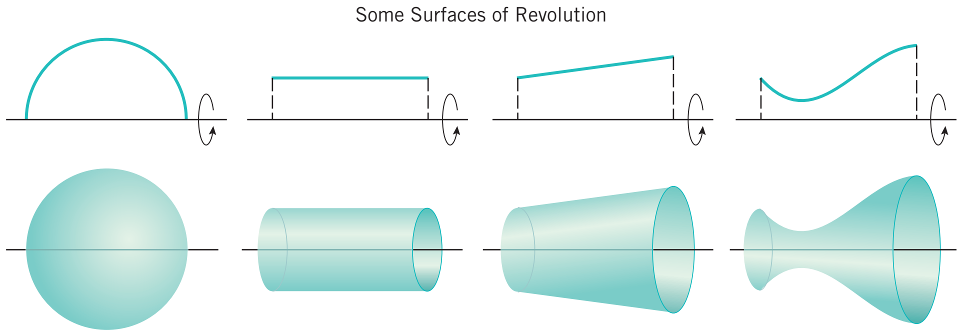


Figure 6.5.1

In this section we will be concerned with the following problem.

6.5.1 SURFACE AREA PROBLEM Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.

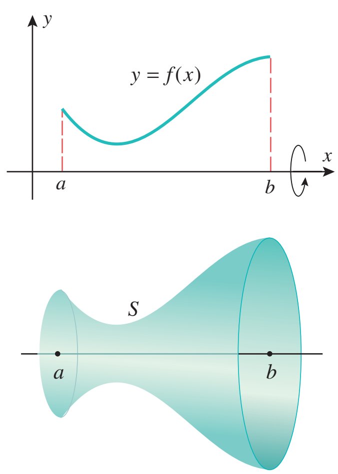


Figure 6.5.2

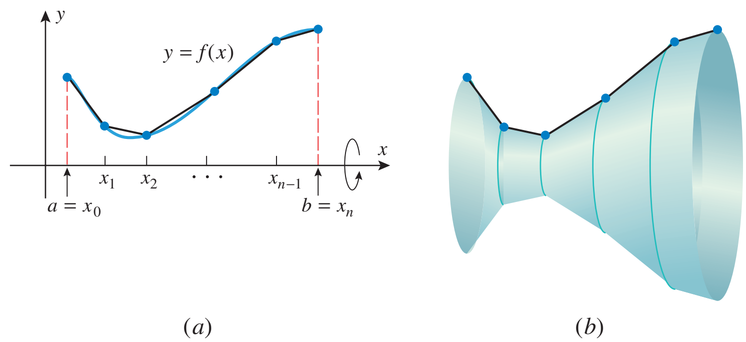
To motivate an appropriate definition for the area S of a surface of revolution, we will decompose the surface into small sections whose areas can be approximated by elementary formulas, add the approximations of the areas of the sections to form a Riemann sum that approximates S, and then take the limit of the Riemann sums to obtain an integral for the exact value of S.

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting points $x_1, x_2, \dots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As illustrated in Figure 6.5.3*a*, the corresponding points on the graph of f define a polygonal path that approximates the curve $y = f(x)$ over the interval $[a, b]$. As illustrated in Figure 6.5.3*b*, when this polygonal path is revolved about the x-axis, it generates a surface consisting of n parts, each of which is a portion of a right circular cone called a ***frustum*** (from the Latin meaning "bit" or "piece"). Thus, the area of each part of the approximating surface can be obtained from the formula

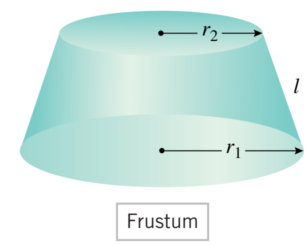
$$S = \pi(r_1 + r_2)l \tag{1}$$

for the lateral area S of a frustum of slant height l and base radii r_1 and r_2 (Figure 6.5.4). As suggested by Figure 6.5.5, the kth frustum has radii $f(x_{k-1})$ and $f(x_k)$ and height Δx_k. Its slant height is the length L_k of the kth line segment in the polygonal path, which from Formula (1) of Section 6.4 is

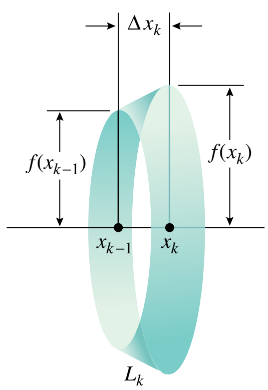
$$L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$



▲ Figure 6.5.3



▲ Figure 6.5.4



▲ Figure 6.5.5

This makes the lateral area S_k of the kth frustum

$$S_k = \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

If we add these areas, we obtain the following approximation to the area S of the entire surface:

$$S \approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$\begin{aligned} S &\approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f'(x_k^*)]^2(\Delta x_k)^2} \\ &= \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k \end{aligned} \tag{3}$$

However, this is not yet a Riemann sum because it involves the variables x_{k-1} and x_k. To eliminate these variables from the expression, observe that the average value of the numbers $f(x_{k-1})$ and $f(x_k)$ lies between these numbers, so the continuity of f and the Intermediate-Value Theorem (1.5.7) imply that there is a point x_k^{**} between x_{k-1} and x_k such that

$$\tfrac{1}{2}[f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

Thus, (2) can be expressed as

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$

Although this expression is close to a Riemann sum in form, it is not a true Riemann sum because it involves two variables x_k^* and x_k^{**}, rather than x_k^* alone. However, it is proved in advanced calculus courses that this has no effect on the limit because of the continuity of f. Thus, we can assume that $x_k^{**} = x_k^*$ when taking the limit, and this suggests that S can be defined as

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi f(x_k^*)\sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

In summary, we have the following definition.

6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

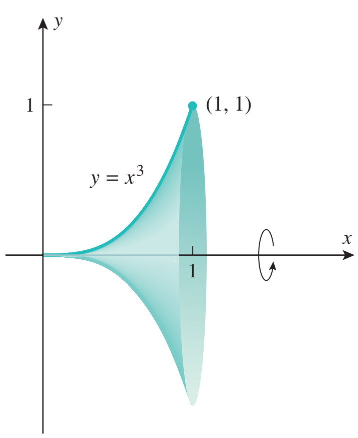
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

This result provides both a definition and a formula for computing surface areas. Where convenient, this formula can also be expressed as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4}$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis can be expressed as

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{5}$$

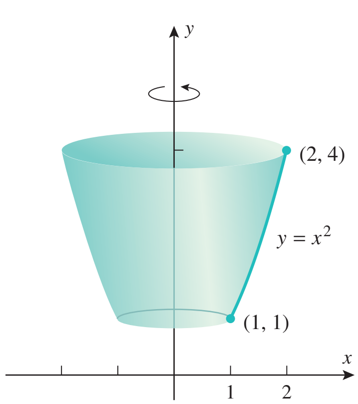


▲ Figure 6.5.6

▶ **Example 1** Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

Solution. First sketch the curve; then imagine revolving it about the x-axis (Figure 6.5.6). Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

$$\begin{aligned}
S &= \int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}\, dx \\
&= 2\pi \int_0^1 x^3(1 + 9x^4)^{1/2}\, dx \\
&= \frac{2\pi}{36}\int_1^{10} u^{1/2}\, du \qquad \boxed{\begin{array}{c} u = 1 + 9x^4 \\ du = 36x^3\, dx \end{array}} \\
&= \frac{2\pi}{36}\cdot \frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2} - 1) \approx 3.56 \blacktriangleleft
\end{aligned}$$



▲ Figure 6.5.7

▶ **Example 2** Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.

Solution. First sketch the curve; then imagine revolving it about the y-axis (Figure 6.5.7). Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and

$x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$\begin{aligned} S &= \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= \int_1^4 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy \\ &= \pi \int_1^4 \sqrt{4y+1}\, dy \\ &= \frac{\pi}{4}\int_5^{17} u^{1/2}\, du \qquad \boxed{\begin{aligned} u &= 4y+1 \\ du &= 4\,dy \end{aligned}} \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\bigg]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \end{aligned}$$

✔ QUICK CHECK EXERCISES 6.5

(See page 449 for answers.)

1. If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is ________.
2. The lateral area of the frustum with slant height $\sqrt{10}$ and base radii $r_1 = 1$ and $r_2 = 2$ is ________.
3. An integral expression for the area of the surface generated by rotating the line segment joining (3, 1) and (6, 2) about the x-axis is ________.
4. An integral expression for the area of the surface generated by rotating the line segment joining (3, 1) and (6, 2) about the y-axis is ________.

EXERCISE SET 6.5 C CAS

1–4 Find the area of the surface generated by revolving the given curve about the x-axis.

1. $y = 7x,\ 0 \le x \le 1$
2. $y = \sqrt{x},\ 1 \le x \le 4$
3. $y = \sqrt{4 - x^2},\ -1 \le x \le 1$
4. $x = \sqrt[3]{y},\ 1 \le y \le 8$

5–8 Find the area of the surface generated by revolving the given curve about the y-axis.

5. $x = 9y + 1,\ 0 \le y \le 2$
6. $x = y^3,\ 0 \le y \le 1$
7. $x = \sqrt{9 - y^2},\ -2 \le y \le 2$
8. $x = 2\sqrt{1 - y},\ -1 \le y \le 0$

C **9–12** Use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis.

9. $y = \sqrt{x} - \frac{1}{3}x^{3/2},\ 1 \le x \le 3;$ x-axis
10. $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1},\ 1 \le x \le 2;$ x-axis
11. $8xy^2 = 2y^6 + 1,\ 1 \le y \le 2;$ y-axis
12. $x = \sqrt{16 - y},\ 0 \le y \le 15;$ y-axis

C **13–16** Use a CAS or a calculating utility with a numerical integration capability to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

13. $y = \sin x,\ 0 \le x \le \pi;$ x-axis
14. $x = \tan y,\ 0 \le y \le \pi/4;$ y-axis
15. $y = e^x,\ 0 \le x \le 1;$ x-axis
16. $y = e^x,\ 1 \le y \le e;$ y-axis

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. The lateral surface area S of a right circular cone with height h and base radius r is $S = \pi r\sqrt{r^2 + h^2}$.
18. The lateral surface area of a frustum of slant height l and base radii r_1 and r_2 is equal to the lateral surface area of a right circular cylinder of height l and radius equal to the average of r_1 and r_2.

19. The approximation

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1+[f'(x_k^*)]^2}\,\Delta x_k$$

for surface area is exact if f is a positive-valued constant function.

20. The expression

$$\sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1+[f'(x_k^*)]^2}\,\Delta x_k$$

is not a true Riemann sum for

$$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

21–22 Approximate the area of the surface using Formula (2) with $n = 20$ subintervals of equal width. Round your answer to two decimal places. ■

21. The surface of Exercise 13.

22. The surface of Exercise 16.

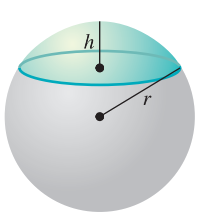
FOCUS ON CONCEPTS

23. Assume that $y = f(x)$ is a smooth curve on the interval $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. Derive a formula for the surface area generated when the curve $y = f(x)$, $a \leq x \leq b$, is revolved about the line $y = -k$ $(k > 0)$.

24. Would it be circular reasoning to use Definition 6.5.2 to find the surface area of a frustum of a right circular cone? Explain your answer.

25. Show that the area of the surface of a sphere of radius r is $4\pi r^2$. [*Hint:* Revolve the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis.]

26. The accompanying figure shows a spherical cap of height h cut from a sphere of radius r. Show that the surface area S of the cap is $S = 2\pi rh$. [*Hint:* Revolve an appropriate portion of the circle $x^2 + y^2 = r^2$ about the y-axis.]



◀ **Figure Ex-26**

27. The portion of a sphere that is cut by two parallel planes is called a ***zone***. Use the result of Exercise 26 to show that the surface area of a zone depends on the radius of the sphere and the distance between the planes, but not on the location of the zone.

28. Let $y = f(x)$ be a smooth curve on the interval $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. By the Extreme-Value Theorem (4.4.2), the function f has a maximum value K and a minimum value k on $[a, b]$. Prove: If L is the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$, and if S is the area of the surface that is generated by revolving this curve about the x-axis, then

$$2\pi kL \leq S \leq 2\pi KL$$

29. Use the results of Exercise 28 above and Exercise 21 in Section 6.4 to show that the area S of the surface generated by revolving the curve $y = \sec x$, $0 \leq x \leq \pi/3$, about the x-axis satisfies

$$\frac{2\pi^2}{3} \leq S \leq \frac{4\pi^2}{3}\sqrt{13}$$

30. Let $y = f(x)$ be a smooth curve on $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. Let A be the area under the curve $y = f(x)$ between $x = a$ and $x = b$, and let S be the area of the surface obtained when this section of curve is revolved about the x-axis.

(a) Prove that $2\pi A \leq S$.

(b) For what functions f is $2\pi A = S$?

31–37 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I. ■

31–32 For these exercises, divide the interval $[a, b]$ into n subintervals by inserting points $t_1, t_2, \ldots, t_{n-1}$ between $a = t_0$ and $b = t_n$, and assume that $x'(t)$ and $y'(t)$ are continuous functions and that no segment of the curve

$$x = x(t), \quad y = y(t) \qquad (a \leq t \leq b)$$

is traced more than once. ■

31. Let S be the area of the surface generated by revolving the curve $x = x(t)$, $y = y(t)$ $(a \leq t \leq b)$ about the x-axis. Explain how S can be approximated by

$$S \approx \sum_{k=1}^{n} (\pi[y(t_{k-1}) + y(t_k)] \times \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}\,)$$

Using results from advanced calculus, it can be shown that as $\max \Delta t_k \to 0$, this sum converges to

$$S = \int_a^b 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt \qquad \text{(A)}$$

32. Let S be the area of the surface generated by revolving the curve $x = x(t)$, $y = y(t)$ $(a \leq t \leq b)$ about the y-axis. Explain how S can be approximated by

$$S \approx \sum_{k=1}^{n} (\pi[x(t_{k-1}) + x(t_k)] \times \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}\,)$$

Using results from advanced calculus, it can be shown that as $\max \Delta t_k \to 0$, this sum converges to

$$S = \int_a^b 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt \qquad \text{(B)}$$

33–37 Use Formulas (A) and (B) from Exercises 31 and 32.

33. Find the area of the surface generated by revolving the parametric curve $x = t^2$, $y = 2t$ $(0 \le t \le 4)$ about the x-axis.

C **34.** Use a CAS to find the area of the surface generated by revolving the parametric curve

$$x = \cos^2 t, \quad y = 5 \sin t \qquad (0 \le t \le \pi/2)$$

about the x-axis.

35. Find the area of the surface generated by revolving the parametric curve $x = t$, $y = 2t^2$ $(0 \le t \le 1)$ about the y-axis.

36. Find the area of the surface generated by revolving the parametric curve $x = \cos^2 t$, $y = \sin^2 t$ $(0 \le t \le \pi/2)$ about the y-axis.

37. By revolving the semicircle

$$x = r \cos t, \quad y = r \sin t \qquad (0 \le t \le \pi)$$

about the x-axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

38. **Writing** Compare the derivation of Definition 6.5.2 with that of Definition 6.4.2. Discuss the geometric features that result in similarities in the two definitions.

39. **Writing** Discuss what goes wrong if we replace the frustums of right circular cones by right circular cylinders in the derivation of Definition 6.5.2.

✔ QUICK CHECK ANSWERS 6.5

1. $\displaystyle\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$ **2.** $3\sqrt{10}\,\pi$ **3.** $\displaystyle\int_3^6 (2\pi)\left(\frac{x}{3}\right)\sqrt{\frac{10}{9}}\,dx = \int_3^6 \frac{2\sqrt{10}\,\pi}{9}x\,dx$ **4.** $\displaystyle\int_1^2 (2\pi)(3y)\sqrt{10}\,dy$

6.6 WORK

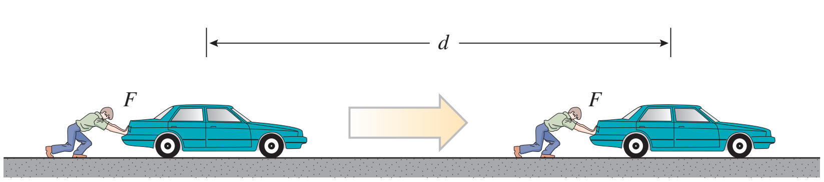
In this section we will use the integration tools developed in the preceding chapter to study some of the basic principles of "work," which is one of the fundamental concepts in physics and engineering.

THE ROLE OF WORK IN PHYSICS AND ENGINEERING

In this section we will be concerned with two related concepts, *work* and *energy*. To put these ideas in a familiar setting, when you push a stalled car for a certain distance you are performing work, and the effect of your work is to make the car move. The energy of motion caused by the work is called the *kinetic energy* of the car. The exact connection between work and kinetic energy is governed by a principle of physics called the *work–energy relationship*. Although we will touch on this idea in this section, a detailed study of the relationship between work and energy will be left for courses in physics and engineering. Our primary goal here will be to explain the role of integration in the study of work.

WORK DONE BY A CONSTANT FORCE APPLIED IN THE DIRECTION OF MOTION

When a stalled car is pushed, the speed that the car attains depends on the force F with which it is pushed and the distance d over which that force is applied (Figure 6.6.1). Force and distance appear in the following definition of work.



▶ **Figure 6.6.1**

6.6.1 DEFINITION If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d, then we define the ***work*** W performed by the force on the object to be

$$W = F \cdot d \tag{1}$$

If you push against an immovable object, such as a brick wall, you may tire yourself out, but you will not perform any work. Why?

Common units for measuring force are newtons (N) in the International System of Units (SI), dynes (dyn) in the centimeter-gram-second (CGS) system, and pounds (lb) in the British Engineering (BE) system. One newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2, one dyne is the force required to give a mass of 1 g an acceleration of 1 cm/s^2, and one pound of force is the force required to give a mass of 1 slug an acceleration of 1 ft/s^2.

It follows from Definition 6.6.1 that work has units of force times distance. The most common units of work are newton-meters (N·m), dyne-centimeters (dyn·cm), and foot-pounds (ft·lb). As indicated in Table 6.6.1, one newton-meter is also called a ***joule*** (J), and one dyne-centimeter is also called an ***erg***. One foot-pound is approximately 1.36 J.

Table 6.6.1

SYSTEM	FORCE ×	DISTANCE =	WORK
SI	newton (N)	meter (m)	joule (J)
CGS	dyne (dyn)	centimeter (cm)	erg
BE	pound (lb)	foot (ft)	foot-pound (ft·lb)

CONVERSION FACTORS:

$1 \text{ N} = 10^5 \text{ dyn} \approx 0.225 \text{ lb}$ $\quad$ $1 \text{ lb} \approx 4.45 \text{ N}$

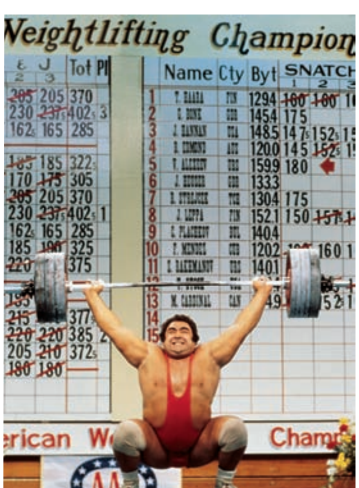
$1 \text{ J} = 10^7 \text{ erg} \approx 0.738 \text{ ft·lb}$ $\quad$ $1 \text{ ft·lb} \approx 1.36 \text{ J} = 1.36 \times 10^7 \text{ erg}$

▶ **Example 1** An object moves 5 ft along a line while subjected to a constant force of 100 lb in its direction of motion. The work done is

$$W = F \cdot d = 100 \cdot 5 = 500 \text{ ft·lb}$$

An object moves 25 m along a line while subjected to a constant force of 4 N in its direction of motion. The work done is

$$W = F \cdot d = 4 \cdot 25 = 100 \text{ N·m} = 100 \text{ J} \blacktriangleleft$$



Vasili Alexeev shown lifting a record-breaking 562 lb in the 1976 Olympics. In eight successive years he won Olympic gold medals, captured six world championships, and broke 80 world records. In 1999 he was honored in Greece as the best sportsman of the 20th Century.

▶ **Example 2** In the 1976 Olympics, Vasili Alexeev astounded the world by lifting a record-breaking 562 lb from the floor to above his head (about 2 m). Equally astounding was the feat of strongman Paul Anderson, who in 1957 braced himself on the floor and used his back to lift 6270 lb of lead and automobile parts a distance of 1 cm. Who did more work?

Solution. To lift an object one must apply sufficient force to overcome the gravitational force that the Earth exerts on that object. The force that the Earth exerts on an object is that object's weight; thus, in performing their feats, Alexeev applied a force of 562 lb over a distance of 2 m and Anderson applied a force of 6270 lb over a distance of 1 cm. Pounds are units in the BE system, meters are units in SI, and centimeters are units in the CGS system. We will need to decide on the measurement system we want to use and be consistent. Let us agree to use SI and express the work of the two men in joules. Using the conversion factor in Table 6.6.1 we obtain

$$562 \text{ lb} \approx 562 \text{ lb} \times 4.45 \text{ N/lb} \approx 2500 \text{ N}$$
$$6270 \text{ lb} \approx 6270 \text{ lb} \times 4.45 \text{ N/lb} \approx 27{,}900 \text{ N}$$

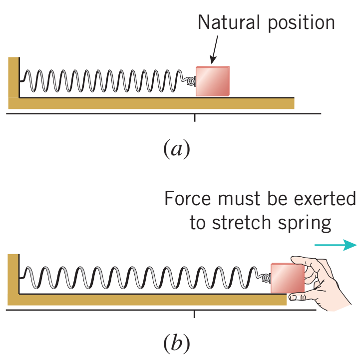
Using these values and the fact that 1 cm = 0.01 m we obtain

$$\text{Alexeev's work} = (2500 \text{ N}) \times (2 \text{ m}) = 5000 \text{ J}$$
$$\text{Anderson's work} = (27{,}900 \text{ N}) \times (0.01 \text{ m}) = 279 \text{ J}$$

Therefore, even though Anderson's lift required a tremendous upward force, it was applied over such a short distance that Alexeev did more work. ◀

WORK DONE BY A VARIABLE FORCE APPLIED IN THE DIRECTION OF MOTION

Many important problems are concerned with finding the work done by a *variable* force that is applied in the direction of motion. For example, Figure 6.6.2a shows a spring in its natural state (neither compressed nor stretched). If we want to pull the block horizontally (Figure 6.6.2b), then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring. Thus, our next objective is to define what is meant by the work performed by a variable force and to find a formula for computing it. This will require calculus.



▲ **Figure 6.6.2**

6.6.2 PROBLEM Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the *work* W performed by the force on the object as the object moves from $x = a$ to $x = b$, and find a formula for computing the work.

The basic idea for solving this problem is to break up the interval $[a, b]$ into subintervals that are sufficiently small that the force does not vary much on each subinterval. This will allow us to treat the force as constant on each subinterval and to approximate the work on each subinterval using Formula (1). By adding the approximations to the work on the subintervals, we will obtain a Riemann sum that approximates the work W over the entire interval, and by taking the limit of the Riemann sums we will obtain an integral for W.

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting points $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. We can use Formula (1) to approximate the work W_k done in the kth subinterval by choosing any point x_k^* in this interval and regarding the force to have a constant value $F(x_k^*)$ throughout the interval. Since the width of the kth subinterval is $x_k - x_{k-1} = \Delta x_k$, this yields the approximation

$$W_k \approx F(x_k^*)\Delta x_k$$

Adding these approximations yields the following Riemann sum that approximates the work W done over the entire interval:

$$W \approx \sum_{k=1}^{n} F(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} F(x_k^*)\Delta x_k = \int_a^b F(x)\, dx$$

In summary, we have the following result.

6.6.3 DEFINITION Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the ***work*** W performed by the force on the object to be

$$W = \int_a^b F(x)\,dx \tag{2}$$

Hooke's law [Robert Hooke (1635–1703), English physicist] states that under appropriate conditions a spring that is stretched x units beyond its natural length pulls back with a force

$$F(x) = kx$$

where k is a constant (called the ***spring constant*** or ***spring stiffness***). The value of k depends on such factors as the thickness of the spring and the material used in its composition. Since $k = F(x)/x$, the constant k has units of force per unit length.

▶ Example 3 A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

(a) Find the spring constant k.

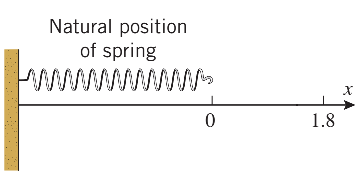
(b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a). From Hooke's law,

$$F(x) = kx$$

From the data, $F(x) = 5$ N when $x = 1$ m, so $5 = k \cdot 1$. Thus, the spring constant is $k = 5$ newtons per meter (N/m). This means that the force $F(x)$ required to stretch the spring x meters is

$$F(x) = 5x \tag{3}$$



▲ **Figure 6.6.3**

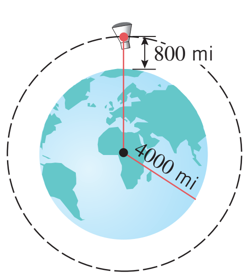
Solution (b). Place the spring along a coordinate line as shown in Figure 6.6.3. We want to find the work W required to stretch the spring over the interval from $x = 0$ to $x = 1.8$. From (2) and (3) the work W required is

$$W = \int_a^b F(x)\,dx = \int_0^{1.8} 5x\,dx = \frac{5x^2}{2}\bigg]_0^{1.8} = 8.1 \text{ J} \blacktriangleleft$$

▶ Example 4 An astronaut's *weight* (or more precisely, *Earth weight*) is the force exerted on the astronaut by the Earth's gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. If the Earth is assumed to be a sphere of radius 4000 mi, then it can be shown using physics that an astronaut who weighs 150 lb on Earth will have a weight of

$$w(x) = \frac{2{,}400{,}000{,}000}{x^2} \text{ lb}, \quad x \geq 4000$$

at a distance of x mi from the Earth's center (Exercise 25). Use this formula to determine the work in foot-pounds required to lift the astronaut to a point that is 800 mi above the surface of the Earth (Figure 6.6.4).



▲ **Figure 6.6.4**

Solution. Since the Earth has a radius of 4000 mi, the astronaut is lifted from a point that is 4000 mi from the Earth's center to a point that is 4800 mi from the Earth's center. Thus, from (2), the work W required to lift the astronaut is

$$\begin{aligned} W &= \int_{4000}^{4800} \frac{2{,}400{,}000{,}000}{x^2}\,dx \\ &= -\frac{2{,}400{,}000{,}000}{x}\bigg]_{4000}^{4800} \\ &= -500{,}000 + 600{,}000 \\ &= 100{,}000 \text{ mile-pounds} \\ &= (100{,}000 \text{ mi}\cdot\text{lb}) \times (5280 \text{ ft/mi}) \\ &= 5.28 \times 10^8 \text{ ft}\cdot\text{lb} \end{aligned}$$ ◀

CALCULATING WORK FROM BASIC PRINCIPLES

Some problems cannot be solved by mechanically substituting into formulas, and one must return to basic principles to obtain solutions. This is illustrated in the next example.

▶ **Example 5** Figure 6.6.5a shows a conical container of radius 10 ft and height 30 ft. Suppose that this container is filled with water to a depth of 15 ft. How much work is required to pump all of the water out through a hole in the top of the container?

Solution. Our strategy will be to divide the water into thin layers, approximate the work required to move each layer to the top of the container, add the approximations for the layers to obtain a Riemann sum that approximates the total work, and then take the limit of the Riemann sums to produce an integral for the total work.

To implement this idea, introduce an x-axis as shown in Figure 6.6.5a, and divide the water into n layers with Δx_k denoting the thickness of the kth layer. This division induces a partition of the interval [15, 30] into n subintervals. Although the upper and lower surfaces of the kth layer are at different distances from the top, the difference will be small if the layer is thin, and we can reasonably assume that the entire layer is concentrated at a single point x_k^* (Figure 6.6.5a). Thus, the work W_k required to move the kth layer to the top of the container is approximately

$$W_k \approx F_k x_k^* \tag{4}$$

where F_k is the force required to lift the kth layer. But the force required to lift the kth layer is the force needed to overcome gravity, and this is the same as the weight of the layer. If the layer is very thin, we can approximate the volume of the kth layer with the volume of a cylinder of height Δx_k and radius r_k, where (by similar triangles)

$$\frac{r_k}{x_k^*} = \frac{10}{30} = \frac{1}{3}$$

or, equivalently, $r_k = x_k^*/3$ (Figure 6.6.5b). Therefore, the volume of the kth layer of water is approximately

$$\pi r_k^2 \Delta x_k = \pi(x_k^*/3)^2 \Delta x_k = \frac{\pi}{9}(x_k^*)^2 \Delta x_k$$

Since the weight density of water is 62.4 lb/ft^3, it follows that

$$F_k \approx \frac{62.4\pi}{9}(x_k^*)^2 \Delta x_k$$

Thus, from (4)

$$W_k \approx \left(\frac{62.4\pi}{9}(x_k^*)^2 \Delta x_k\right) x_k^* = \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k$$

and hence the work W required to move all n layers has the approximation

$$W = \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k$$

To find the *exact* value of the work we take the limit as $\max \Delta x_k \to 0$. This yields

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k = \int_{15}^{30} \frac{62.4\pi}{9} x^3\, dx$$

$$= \frac{62.4\pi}{9}\left(\frac{x^4}{4}\right)\bigg]_{15}^{30} = 1{,}316{,}250\pi \approx 4{,}135{,}000 \text{ ft·lb} \blacktriangleleft$$

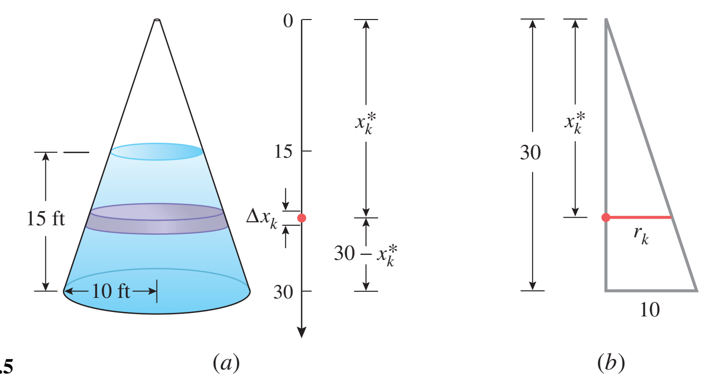


Figure 6.6.5

THE WORK–ENERGY RELATIONSHIP

Mike Brinson/Getty Images

The work performed by the skater's stick in a brief interval of time produces the blinding speed of the hockey puck.

When you see an object in motion, you can be certain that somehow work has been expended to create that motion. For example, when you drop a stone from a building, the stone gathers speed because the force of the Earth's gravity is performing work on it, and when a hockey player strikes a puck with a hockey stick, the work performed on the puck during the brief period of contact with the stick creates the enormous speed of the puck across the ice. However, experience shows that the speed obtained by an object depends not only on the amount of work done, but also on the mass of the object. For example, the work required to throw a 5 oz baseball 50 mi/h would accelerate a 10 lb bowling ball to less than 9 mi/h.

Using the method of substitution for definite integrals, we will derive a simple equation that relates the work done on an object to the object's mass and velocity. Furthermore, this equation will allow us to motivate an appropriate definition for the "energy of motion" of an object. As in Definition 6.6.3, we will assume that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a force $F(x)$ that is applied in the direction of motion. We let m denote the mass of the object, and we let $x = x(t)$, $v = v(t) = x'(t)$, and $a = a(t) = v'(t)$ denote the respective position, velocity, and acceleration of the object at time t. We will need the following important result from physics that relates the force acting on an object with the mass and acceleration of the object.

6.6.4 NEWTON'S SECOND LAW OF MOTION If an object with mass m is subjected to a force F, then the object undergoes an acceleration a that satisfies the equation

$$F = ma \tag{5}$$

It follows from Newton's Second Law of Motion that

$$F(x(t)) = ma(t) = mv'(t)$$

Assume that

$$x(t_0) = a \quad \text{and} \quad x(t_1) = b$$

with

$$v(t_0) = v_i \quad \text{and} \quad v(t_1) = v_f$$

the initial and final velocities of the object, respectively. Then

$$\begin{aligned}
W &= \int_a^b F(x)\,dx = \int_{x(t_0)}^{x(t_1)} F(x)\,dx \\
&= \int_{t_0}^{t_1} F(x(t))x'(t)\,dt \qquad \boxed{\text{By Theorem 5.9.1 with } x = x(t),\, dx = x'(t)\,dt} \\
&= \int_{t_0}^{t_1} mv'(t)v(t)\,dt = \int_{t_0}^{t_1} mv(t)v'(t)\,dt \\
&= \int_{v(t_0)}^{v(t_1)} mv\,dv \qquad \boxed{\text{By Theorem 5.9.1 with } v = v(t),\, dv = v'(t)\,dt} \\
&= \int_{v_i}^{v_f} mv\,dv = \tfrac{1}{2}mv^2\Big|_{v_i}^{v_f} = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2
\end{aligned}$$

We see from the equation

$$W = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 \tag{6}$$

that the work done on the object is equal to the change in the quantity $\frac{1}{2}mv^2$ from its initial value to its final value. We will refer to Equation (6) as the ***work–energy relationship***. If we define the "energy of motion" or ***kinetic energy*** of our object to be given by

$$K = \tfrac{1}{2}mv^2 \tag{7}$$

then Equation (6) tells us that the work done on an object is equal to the *change* in the object's kinetic energy. Loosely speaking, we may think of work done on an object as being "transformed" into kinetic energy of the object. The units of kinetic energy are the same as the units of work. For example, in SI kinetic energy is measured in joules (J).

▶ Example 6 A space probe of mass $m = 5.00 \times 10^4$ kg travels in deep space subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.10 \times 10^4$ m/s, the engine is fired continuously over a distance of 2.50×10^6 m with a constant force of 4.00×10^5 N in the direction of motion. What is the final speed of the probe?

Solution. Since the force applied by the engine is constant and in the direction of motion, the work W expended by the engine on the probe is

$$W = \text{force} \times \text{distance} = (4.00 \times 10^5 \text{ N}) \times (2.50 \times 10^6 \text{ m}) = 1.00 \times 10^{12} \text{ J}$$

From (6), the final kinetic energy $K_f = \frac{1}{2}mv_f^2$ of the probe can be expressed in terms of the work W and the initial kinetic energy $K_i = \frac{1}{2}mv_i^2$ as

$$K_f = W + K_i$$

Thus, from the known mass and initial speed we have

$$K_f = (1.00 \times 10^{12} \text{ J}) + \tfrac{1}{2}(5.00 \times 10^4 \text{ kg})(1.10 \times 10^4 \text{ m/s})^2 = 4.025 \times 10^{12} \text{ J}$$

The final kinetic energy is $K_f = \frac{1}{2}mv_f^2$, so the final speed of the probe is

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(4.025 \times 10^{12})}{5.00 \times 10^4}} \approx 1.27 \times 10^4 \text{ m/s} \quad ◀$$

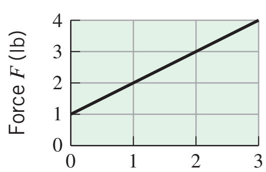
✔QUICK CHECK EXERCISES 6.6 *(See page 458 for answers.)*

1. If a constant force of 5 lb moves an object 10 ft, then the work done by the force on the object is ________.
2. A newton-meter is also called a ________. A dyne-centimeter is also called an ________.
3. Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$. The work performed on the object by a variable force $F(x)$ applied in the direction of motion is $W =$ ________.
4. A force $F(x) = 10 - 2x$ N applied in the positive x-direction moves an object 3 m from $x = 2$ to $x = 5$. The work done by the force on the object is ________.

EXERCISE SET 6.6

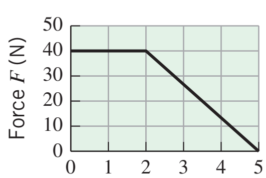
FOCUS ON CONCEPTS

1. A variable force $F(x)$ in the positive x-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x = 0$ to $x = 3$.



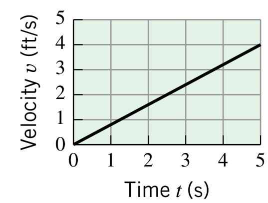
◀ **Figure Ex-1**

2. A variable force $F(x)$ in the positive x-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x = 0$ to $x = 5$.



◀ **Figure Ex-2**

3. For the variable force $F(x)$ in Exercise 2, consider the distance d for which the work done by the force on the particle when the particle moves from $x = 0$ to $x = d$ is half of the work done when the particle moves from $x = 0$ to $x = 5$. By inspecting the graph of F, is d more or less than 2.5? Explain, and then find the exact value of d.
4. Suppose that a variable force $F(x)$ is applied in the positive x-direction so that an object moves from $x = a$ to $x = b$. Relate the work done by the force on the object and the average value of F over $[a, b]$, and illustrate this relationship graphically.
5. A constant force of 10 lb in the positive x-direction is applied to a particle whose velocity versus time curve is shown in the accompanying figure. Find the work done by the force on the particle from time $t = 0$ to $t = 5$.



◀ **Figure Ex-5**

6. A spring exerts a force of 6 N when it is stretched from its natural length of 4 m to a length of $4\frac{1}{2}$ m. Find the work required to stretch the spring from its natural length to a length of 6 m.
7. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?
8. A spring whose natural length is 15 cm exerts a force of 45 N when stretched to a length of 20 cm.
   (a) Find the spring constant (in newtons/meter).
   (b) Find the work that is done in stretching the spring 3 cm beyond its natural length.
   (c) Find the work done in stretching the spring from a length of 20 cm to a length of 25 cm.
9. Assume that 10 ft·lb of work is required to stretch a spring 1 ft beyond its natural length. What is the spring constant?

10–13 True–False Determine whether the statement is true or false. Explain your answer. ■

10. In order to support the weight of a parked automobile, the surface of a driveway must do work against the force of gravity on the vehicle.
11. A force of 10 lb in the direction of motion of an object that moves 5 ft in 2 s does six times the work of a force of 10 lb in the direction of motion of an object that moves 5 ft in 12 s.
12. It follows from Hooke's law that in order to double the distance a spring is stretched beyond its natural length, four times as much work is required.
13. In the International System of Units, work and kinetic energy have the same units.

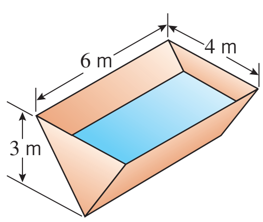
14. A cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with water. Find the work required to pump all the water over the upper rim.

15. Solve Exercise 14 assuming that the tank is half-filled with water.

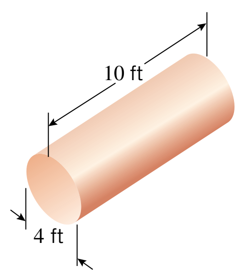
16. A cone-shaped water reservoir is 20 ft in diameter across the top and 15 ft deep. If the reservoir is filled to a depth of 10 ft, how much work is required to pump all the water to the top of the reservoir?

17. The vat shown in the accompanying figure contains water to a depth of 2 m. Find the work required to pump all the water to the top of the vat. [Use 9810 N/m^3 as the weight density of water.]

18. The cylindrical tank shown in the accompanying figure is filled with a liquid weighing 50 lb/ft^3. Find the work required to pump all the liquid to a level 1 ft above the top of the tank.



▲ **Figure Ex-17**



▲ **Figure Ex-18**

19. A swimming pool is built in the shape of a rectangular parallelepiped 10 ft deep, 15 ft wide, and 20 ft long.
(a) If the pool is filled to 1 ft below the top, how much work is required to pump all the water into a drain at the top edge of the pool?
(b) A one-horsepower motor can do 550 ft·lb of work per second. What size motor is required to empty the pool in 1 hour?

20. How much work is required to fill the swimming pool in Exercise 19 to 1 ft below the top if the water is pumped in through an opening located at the bottom of the pool?

21. A 100 ft length of steel chain weighing 15 lb/ft is dangling from a pulley. How much work is required to wind the chain onto the pulley?

22. A 3 lb bucket containing 20 lb of water is hanging at the end of a 20 ft rope that weighs 4 oz/ft. The other end of the rope is attached to a pulley. How much work is required to wind the length of rope onto the pulley, assuming that the rope is wound onto the pulley at a rate of 2 ft/s and that as the bucket is being lifted, water leaks from the bucket at a rate of 0.5 lb/s?

23. A rocket weighing 3 tons is filled with 40 tons of liquid fuel. In the initial part of the flight, fuel is burned off at a constant rate of 2 tons per 1000 ft of vertical height. How much work in foot-tons (ft·ton) is done lifting the rocket 3000 ft?

24. It follows from Coulomb's law in physics that two like electrostatic charges repel each other with a force inversely proportional to the square of the distance between them. Suppose that two charges A and B repel with a force of k newtons when they are positioned at points $A(-a, 0)$ and $B(a, 0)$, where a is measured in meters. Find the work W required to move charge A along the x-axis to the origin if charge B remains stationary.

25. It is a law of physics that the gravitational force exerted by the Earth on an object above the Earth's surface varies inversely as the square of its distance from the Earth's center. Thus, an object's weight $w(x)$ is related to its distance x from the Earth's center by a formula of the form

$$w(x) = \frac{k}{x^2}$$

where k is a constant of proportionality that depends on the mass of the object.
(a) Use this fact and the assumption that the Earth is a sphere of radius 4000 mi to obtain the formula for $w(x)$ in Example 4.
(b) Find a formula for the weight $w(x)$ of a satellite that is x mi from the Earth's surface if its weight on Earth is 6000 lb.
(c) How much work is required to lift the satellite from the surface of the Earth to an orbital position that is 1000 mi high?

26. (a) The formula $w(x) = k/x^2$ in Exercise 25 is applicable to all celestial bodies. Assuming that the Moon is a sphere of radius 1080 mi, find the force that the Moon exerts on an astronaut who is x mi from the surface of the Moon if her weight on the Moon's surface is 20 lb.
(b) How much work is required to lift the astronaut to a point that is 10.8 mi above the Moon's surface?

27. The world's first commercial high-speed magnetic levitation (MAGLEV) train, a 30 km double-track project connecting Shanghai, China, to Pudong International Airport, began full revenue service in 2003. Suppose that a MAGLEV train has a mass $m = 4.00 \times 10^5$ kg and that starting at a time when the train has a speed of 20 m/s the engine applies a force of 6.40×10^5 N in the direction of motion over a distance of 3.00×10^3 m. Use the work–energy relationship (6) to find the final speed of the train.

28. Assume that a Mars probe of mass $m = 2.00 \times 10^3$ kg is subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.00 \times 10^4$ m/s, the engine is fired continuously over a distance of 1.50×10^5 m with a constant force of 2.00×10^5 N in the direction of motion. Use the work–energy relationship (6) to find the final speed of the probe.

29. On August 10, 1972 a meteorite with an estimated mass of 4×10^6 kg and an estimated speed of 15 km/s skipped across the atmosphere above the western United States and Canada but fortunately did not hit the Earth. *(cont.)*

(a) Assuming that the meteorite had hit the Earth with a speed of 15 km/s, what would have been its change in kinetic energy in joules (J)?

(b) Express the energy as a multiple of the explosive energy of 1 megaton of TNT, which is 4.2×10^{15} J.

(c) The energy associated with the Hiroshima atomic bomb was 13 kilotons of TNT. To how many such bombs would the meteorite impact have been equivalent?

30. Writing After reading Examples 3–5, a student classifies work problems as either "pushing/pulling" or "pumping." Describe these categories in your own words and discuss the methods used to solve each type. Give examples to illustrate that these categories are not mutually exclusive.

31. Writing How might you recognize that a problem can be solved by means of the work–energy relationship? That is, what sort of "givens" and "unknowns" would suggest such a solution? Discuss two or three examples.

✔QUICK CHECK ANSWERS 6.6

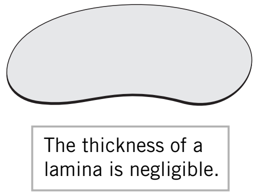
1. 50 ft·lb **2.** joule; erg **3.** $\int_a^b F(x)\,dx$ **4.** 9 J

6.7 MOMENTS, CENTERS OF GRAVITY, AND CENTROIDS

Suppose that a rigid physical body is acted on by a constant gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of the gravitational field on the body consists of a large number of forces distributed over the entire body. However, it is a fact of physics that these individual forces can be replaced by a single force acting at a point called the ***center of gravity*** *of the body. In this section we will show how integrals can be used to locate centers of gravity.*

DENSITY AND MASS OF A LAMINA

Let us consider an idealized flat object that is thin enough to be viewed as a two-dimensional plane region (Figure 6.7.1). Such an object is called a ***lamina***. A lamina is called ***homogeneous*** if its composition is uniform throughout and ***inhomogeneous*** otherwise. We will consider homogeneous laminas in this section. Inhomogeneous laminas will be discussed in Chapter 14. The ***density*** of a *homogeneous* lamina is defined to be its mass per unit area. Thus, the density δ of a homogeneous lamina of mass M and area A is given by $\delta = M/A$. Notice that the mass M of a homogeneous lamina can be expressed as



▲ **Figure 6.7.1**

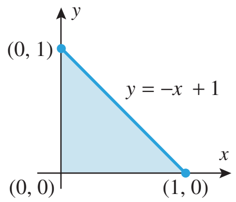
$$M = \delta A \tag{1}$$

The units in Equation (1) are consistent since mass = (mass/area) × area.

▶ **Example 1** A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density $\delta = 3$. Find its total mass.

Solution. Referring to (1) and Figure 6.7.2, the mass M of the lamina is

$$M = \delta A = 3 \cdot \frac{1}{2} = \frac{3}{2} \text{ (unit of mass)} ◀$$



▲ **Figure 6.7.2**

CENTER OF GRAVITY OF A LAMINA

Assume that the acceleration due to the force of gravity is constant and acts downward, and suppose that a lamina occupies a region R in a horizontal xy-plane. It can be shown that there exists a unique point $(\bar{x}, \bar{y})$ (which may or may not belong to R) such that the effect of gravity on the lamina is "equivalent" to that of a single force acting at the point $(\bar{x}, \bar{y})$. This point is called the ***center of gravity*** of the lamina, and if it is in R, then the lamina will balance horizontally on the point of a support placed at $(\bar{x}, \bar{y})$. For example, the center of gravity of a homogeneous disk is at the center of the disk, and the center of gravity of a homogeneous rectangular region is at the center of the rectangle. For an irregularly shaped homogeneous lamina, locating the center of gravity requires calculus.

6.7.1 PROBLEM Let f be a positive continuous function on the interval $[a, b]$. Suppose that a homogeneous lamina with constant density δ occupies a region R in a horizontal xy-plane bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity of the lamina.

To motivate the solution, consider what happens if we try to balance the lamina on a knife-edge parallel to the x-axis. Suppose the lamina in Figure 6.7.3 is placed on a knife-edge along a line $y = c$ that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity $(\bar{x}, \bar{y})$, the lamina will be rotationally unstable and the force of gravity will cause a rotation about $y = c$. Similarly, the lamina will undergo a rotation if placed on a knife-edge along $y = d$. However, if the knife-edge runs along the line $y = \bar{y}$ through the center of gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife-edge along the line $x = \bar{x}$ through the center of gravity. This suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the x-axis and the other parallel to the y-axis. In order to find these lines of balance, we will need some preliminary results about rotations.

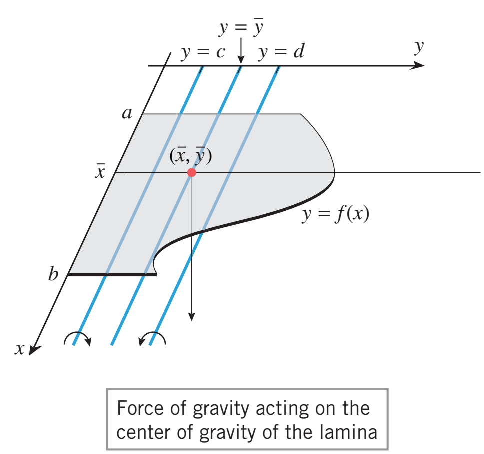
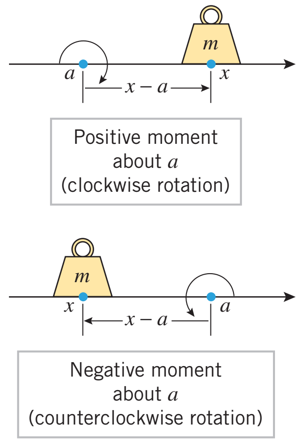


Figure 6.7.3

Children on a seesaw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for an object to produce rotation is proportional not only to its mass but also to the distance between the object and the fulcrum. To make this more precise, consider an x-axis, which we view as a weightless beam. If a mass m is located on the axis at x, then the tendency for that mass to produce a rotation of the beam about a point a on the axis is measured by the following quantity, called the ***moment of m about $x = a$***:

$$\begin{bmatrix} \text{moment of } m \\ \text{about } a \end{bmatrix} = m(x - a)$$



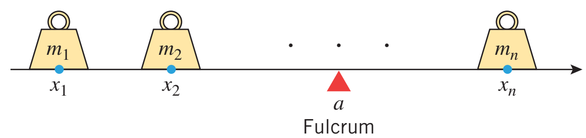
▲ **Figure 6.7.4**

The number $x - a$ is called the ***lever arm***. Depending on whether the mass is to the right or left of a, the lever arm is either the distance between x and a or the negative of this distance (Figure 6.7.4). Positive lever arms result in positive moments and clockwise rotations, and negative lever arms result in negative moments and counterclockwise rotations.

Suppose that masses $m_1, m_2, \dots, m_n$ are located at $x_1, x_2, \dots, x_n$ on a coordinate axis and a fulcrum is positioned at the point a (Figure 6.7.5). Depending on whether the sum of the moments about a,

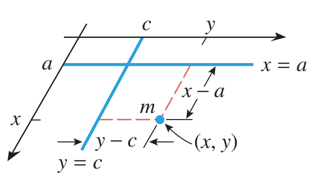
$$\sum_{k=1}^{n} m_k(x_k - a) = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a)$$

is positive, negative, or zero, a weightless beam along the axis will rotate clockwise about a, rotate counterclockwise about a, or balance perfectly. In the last case, the system of masses is said to be in ***equilibrium***.



▶ **Figure 6.7.5**

The preceding ideas can be extended to masses distributed in two-dimensional space. If we imagine the xy-plane to be a weightless sheet supporting a mass m located at a point (x, y), then the tendency for the mass to produce a rotation of the sheet about the line $x = a$ is $m(x - a)$, called the ***moment of m about x = a***, and the tendency for the mass to produce a rotation about the line $y = c$ is $m(y - c)$, called the ***moment of m about y = c*** (Figure 6.7.6). In summary,

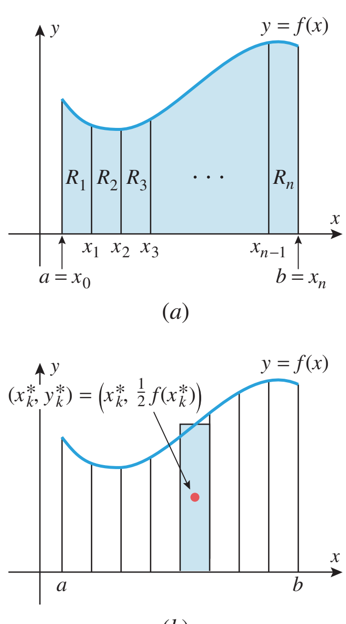


▲ **Figure 6.7.6**

$$\begin{bmatrix}\text{moment of } m\\ \text{about the}\\ \text{line } x = a\end{bmatrix} = m(x - a) \quad \text{and} \quad \begin{bmatrix}\text{moment of } m\\ \text{about the}\\ \text{line } y = c\end{bmatrix} = m(y - c) \qquad (2\text{–}3)$$

If a number of masses are distributed throughout the xy-plane, then the plane (viewed as a weightless sheet) will balance on a knife-edge along the line $x = a$ if the sum of the moments about the line is zero. Similarly, the plane will balance on a knife-edge along the line $y = c$ if the sum of the moments about that line is zero.

We are now ready to solve Problem 6.7.1. The basic idea for solving this problem is to divide the lamina into strips whose areas may be approximated by the areas of rectangles. These area approximations, along with Formulas (2) and (3), will allow us to create a Riemann sum that approximates the moment of the lamina about a horizontal or vertical line. By taking the limit of Riemann sums we will then obtain an integral for the moment of a lamina about a horizontal or vertical line. We observe that since the lamina balances on the lines $x = \bar{x}$ and $y = \bar{y}$, the moment of the lamina about those lines should be zero. This observation will enable us to calculate $\bar{x}$ and $\bar{y}$.



▲ **Figure 6.7.7**

To implement this idea, we divide the interval $[a, b]$ into n subintervals by inserting the points $x_1, x_2, \dots, x_{n-1}$ between $a = x_0$ and $b = x_n$. This has the effect of dividing the lamina R into n strips $R_1, R_2, \dots, R_n$ (Figure 6.7.7a). Suppose that the kth strip extends from x_{k-1} to x_k and that the width of this strip is

$$\Delta x_k = x_k - x_{k-1}$$

We will let x_k^* be the midpoint of the kth subinterval and we will approximate R_k by a rectangle of width Δx_k and height $f(x_k^*)$. From (1), the mass ΔM_k of this rectangle is $\Delta M_k = \delta f(x_k^*)\Delta x_k$, and we will assume that the rectangle behaves as if its entire mass is concentrated at its center $(x_k^*, y_k^*) = (x_k^*, \frac{1}{2}f(x_k^*))$ (Figure 6.7.7b). It then follows from (2) and (3) that the moments of R_k about the lines $x = \bar{x}$ and $y = \bar{y}$ may be approximated by $(x_k^* - \bar{x})\Delta M_k$ and $(y_k^* - \bar{y})\Delta M_k$, respectively. Adding these approximations yields the following Riemann sums that approximate the moment of the entire lamina about the lines $x = \bar{x}$ and $y = \bar{y}$:

$$\sum_{k=1}^{n}(x_k^* - \bar{x})\Delta M_k = \sum_{k=1}^{n}(x_k^* - \bar{x})\delta f(x_k^*)\Delta x_k$$

$$\sum_{k=1}^{n}(y_k^* - \bar{y})\Delta M_k = \sum_{k=1}^{n}\left(\frac{f(x_k^*)}{2} - \bar{y}\right)\delta f(x_k^*)\Delta x_k$$

Taking the limits as n increases and the widths of all the rectangles approach zero yields the definite integrals

$$\int_a^b (x - \bar{x})\delta f(x)\,dx \quad \text{and} \quad \int_a^b \left(\frac{f(x)}{2} - \bar{y}\right)\delta f(x)\,dx$$

that represent the moments of the lamina about the lines $x = \bar{x}$ and $y = \bar{y}$. Since the lamina balances on those lines, the moments of the lamina about those lines should be zero:

$$\int_a^b (x - \bar{x})\delta f(x)\,dx = \int_a^b \left(\frac{f(x)}{2} - \bar{y}\right)\delta f(x)\,dx = 0$$

Since $\bar{x}$ and $\bar{y}$ are constant, these equations can be rewritten as

$$\int_a^b \delta x f(x)\,dx = \bar{x}\int_a^b \delta f(x)\,dx$$

$$\int_a^b \frac{1}{2}\delta (f(x))^2\,dx = \bar{y}\int_a^b \delta f(x)\,dx$$

from which we obtain the following formulas for the center of gravity of the lamina:

Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{\displaystyle\int_a^b \delta x f(x)\,dx}{\displaystyle\int_a^b \delta f(x)\,dx}, \qquad \bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}\delta\,(f(x))^2\,dx}{\displaystyle\int_a^b \delta f(x)\,dx} \tag{4–5}$$

Observe that in both formulas the denominator is the mass M of the lamina. The numerator in the formula for $\bar{x}$ is denoted by M_y and is called the ***first moment of the lamina about the y-axis***; the numerator of the formula for $\bar{y}$ is denoted by M_x and is called the ***first moment of the lamina about the x-axis***. Thus, we can write (4) and (5) as

Alternative Formulas for Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R}\int_a^b \delta x f(x)\,dx \tag{6}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R}\int_a^b \frac{1}{2}\delta\,(f(x))^2\,dx \tag{7}$$

▶ **Example 2** Find the center of gravity of the triangular lamina with vertices (0, 0), (0, 1), and (1, 0) and density $\delta = 3$.

Solution. The lamina is shown in Figure 6.7.2. In Example 1 we found the mass of the lamina to be

$$M = \frac{3}{2}$$

The moment of the lamina about the y-axis is

$$M_y = \int_0^1 \delta x f(x)\,dx = \int_0^1 3x(-x+1)\,dx$$
$$= \int_0^1 (-3x^2+3x)\,dx = \left(-x^3 + \frac{3}{2}x^2\right)\Bigg]_0^1 = -1 + \frac{3}{2} = \frac{1}{2}$$

and the moment about the x-axis is

$$M_x = \int_0^1 \frac{1}{2}\delta(f(x))^2\,dx = \int_0^1 \frac{3}{2}(-x+1)^2\,dx$$
$$= \int_0^1 \frac{3}{2}(x^2-2x+1)\,dx = \frac{3}{2}\left(\frac{1}{3}x^3 - x^2 + x\right)\Bigg]_0^1 = \frac{3}{2}\left(\frac{1}{3}\right) = \frac{1}{2}$$

From (6) and (7),

$$\bar{x} = \frac{M_y}{M} = \frac{1/2}{3/2} = \frac{1}{3}, \qquad \bar{y} = \frac{M_x}{M} = \frac{1/2}{3/2} = \frac{1}{3}$$

so the center of gravity is $(\frac{1}{3}, \frac{1}{3})$. ◀

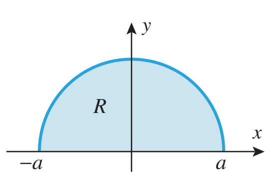
In the case of a *homogeneous* lamina, the center of gravity of a lamina occupying the region R is called the ***centroid of the region R***. Since the lamina is homogeneous, δ is constant. The factor δ in (4) and (5) may thus be moved through the integral signs and canceled, and (4) and (5) can be expressed as

Since the density factor has canceled, we may interpret the centroid as a ***geometric property*** of the region, and distinguish it from the center of gravity, which is a ***physical property*** of an idealized object that occupies the region.

Centroid of a Region R

$$\bar{x} = \frac{\displaystyle\int_a^b x f(x)\,dx}{\displaystyle\int_a^b f(x)\,dx} = \frac{1}{\text{area of } R}\int_a^b x f(x)\,dx \tag{8}$$

$$\bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}(f(x))^2\,dx}{\displaystyle\int_a^b f(x)\,dx} = \frac{1}{\text{area of } R}\int_a^b \frac{1}{2}(f(x))^2\,dx \tag{9}$$



▲ **Figure 6.7.8**

▶ **Example 3** Find the centroid of the semicircular region in Figure 6.7.8.

Solution. By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance. To find $\bar{y}$, first note that the equation of the semicircle is $y = f(x) = \sqrt{a^2 - x^2}$. From (9),

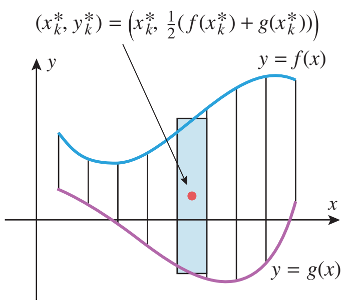
$$\bar{y} = \frac{1}{\text{area of } R}\int_{-a}^{a} \frac{1}{2}(f(x))^2\,dx = \frac{1}{\frac{1}{2}\pi a^2}\int_{-a}^{a} \frac{1}{2}(a^2 - x^2)\,dx$$
$$= \frac{1}{\pi a^2}\left(a^2 x - \frac{1}{3}x^3\right)\Bigg]_{-a}^{a}$$
$$= \frac{1}{\pi a^2}\left[\left(a^3 - \frac{1}{3}a^3\right) - \left(-a^3 + \frac{1}{3}a^3\right)\right]$$
$$= \frac{1}{\pi a^2}\left(\frac{4a^3}{3}\right) = \frac{4a}{3\pi}$$

so the centroid is $(0, 4a/3\pi)$. ◀

OTHER TYPES OF REGIONS

The strategy used to find the center of gravity of the region in Problem 6.7.1 can be used to find the center of gravity of regions that are not of that form.

Consider a homogeneous lamina that occupies the region R between two continuous functions $f(x)$ and $g(x)$ over the interval $[a, b]$, where $f(x) \geq g(x)$ for $a \leq x \leq b$. To find the center of gravity of this lamina we can subdivide it into n strips using lines parallel to the y-axis. If x_k^* is the midpoint of the kth strip, the strip can be approximated by a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$. We assume that the entire mass of the kth rectangle is concentrated at its center $(x_k^*, y_k^*) = (x_k^*, \frac{1}{2}(f(x_k^*) + g(x_k^*)))$ (Figure 6.7.9). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is



▲ **Figure 6.7.9**

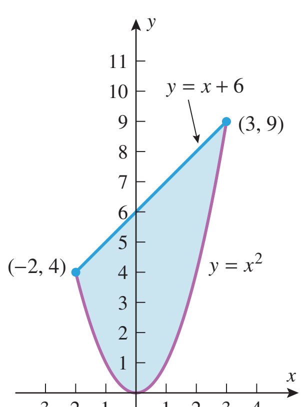
$$\bar{x} = \frac{\displaystyle\int_a^b x(f(x) - g(x))\,dx}{\displaystyle\int_a^b (f(x) - g(x))\,dx} = \frac{1}{\text{area of } R} \int_a^b x(f(x) - g(x))\,dx \tag{10}$$

$$\bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\,dx}{\displaystyle\int_a^b (f(x) - g(x))\,dx} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\,dx \tag{11}$$

Note that the density of the lamina does not appear in Equations (10) and (11). This reflects the fact that the centroid is a geometric property of R.

▶ **Example 4** Find the centroid of the region R enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. To begin, we note that the two curves intersect when $x = -2$ and $x = 3$ and that $x + 6 \geq x^2$ over that interval (Figure 6.7.10). The area of R is



▲ **Figure 6.7.10**

$$\int_{-2}^{3} [(x + 6) - x^2]\,dx = \frac{125}{6}$$

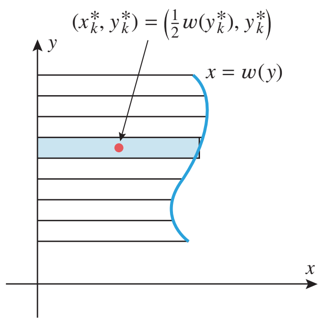
From (10) and (11),

$$\begin{aligned}
\bar{x} &= \frac{1}{\text{area of } R} \int_{-2}^{3} x[(x + 6) - x^2]\,dx \\
&= \frac{6}{125}\left(\frac{1}{3}x^3 + 3x^2 - \frac{1}{4}x^4\right)\Bigg]_{-2}^{3} \\
&= \frac{6}{125} \cdot \frac{125}{12} = \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\bar{y} &= \frac{1}{\text{area of } R} \int_{-2}^{3} \frac{1}{2}((x + 6)^2 - (x^2)^2)\,dx \\
&= \frac{6}{125} \int_{-2}^{3} \frac{1}{2}(x^2 + 12x + 36 - x^4)\,dx \\
&= \frac{6}{125} \cdot \frac{1}{2}\left(\frac{1}{3}x^3 + 6x^2 + 36x - \frac{1}{5}x^5\right)\Bigg]_{-2}^{3} \\
&= \frac{6}{125} \cdot \frac{250}{3} = 4
\end{aligned}$$

so the centroid of R is $(\frac{1}{2}, 4)$. ◀



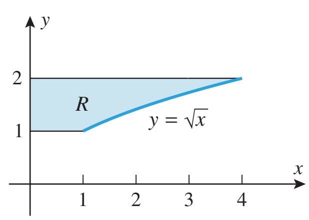
▲ **Figure 6.7.11**

Suppose that w is a continuous function of y on an interval $[c, d]$ with $w(y) \geq 0$ for $c \leq y \leq d$. Consider a lamina that occupies a region R bounded above by $y = d$, below by $y = c$, on the left by the y-axis, and on the right by $x = w(y)$ (Figure 6.7.11). To find the center of gravity of this lamina, we note that the roles of x and y in Problem 6.7.1 have been reversed. We now imagine the lamina to be subdivided into n strips using lines parallel to the x-axis. We let y_k^* be the midpoint of the kth subinterval and approximate the strip by a rectangle of width Δy_k and height $w(y_k^*)$. We assume that the entire mass of the kth rectangle is concentrated at its center $(x_k^*, y_k^*) = (\frac{1}{2}w(y_k^*), y_k^*)$ (Figure 6.7.11). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is

$$\bar{x} = \frac{\displaystyle\int_c^d \frac{1}{2}(w(y))^2\,dy}{\displaystyle\int_c^d w(y)\,dy} = \frac{1}{\text{area of } R}\int_c^d \frac{1}{2}(w(y))^2\,dy \tag{12}$$

$$\bar{y} = \frac{\displaystyle\int_c^d yw(y)\,dy}{\displaystyle\int_c^d w(y)\,dy} = \frac{1}{\text{area of } R}\int_c^d yw(y)\,dy \tag{13}$$

Once again, the absence of the density in Equations (12) and (13) reflects the geometric nature of the centroid.



▲ **Figure 6.7.12**

▶ **Example 5** Find the centroid of the region R enclosed between the curves $y = \sqrt{x}$, $y = 1$, $y = 2$, and the y-axis (Figure 6.7.12).

Solution. Note that $x = w(y) = y^2$ and that the area of R is

$$\int_1^2 y^2\,dy = \frac{7}{3}$$

From (12) and (13),

$$\bar{x} = \frac{1}{\text{area of } R}\int_1^2 \frac{1}{2}(y^2)^2\,dy = \frac{3}{7}\cdot\frac{1}{10}y^5\bigg]_1^2 = \frac{3}{7}\cdot\frac{31}{10} = \frac{93}{70}$$

$$\bar{y} = \frac{1}{\text{area of } R}\int_1^2 y(y^2)\,dy = \frac{3}{7}\cdot\frac{1}{4}y^4\bigg]_1^2 = \frac{3}{7}\cdot\frac{15}{4} = \frac{45}{28}$$

so the centroid of R is $(93/70, 45/28) \approx (1.329, 1.607)$. ◀

THEOREM OF PAPPUS

The following theorem, due to the Greek mathematician Pappus, gives an important relationship between the centroid of a plane region R and the volume of the solid generated when the region is revolved about a line.

6.7.2 THEOREM (*Theorem of Pappus*) *If R is a bounded plane region and L is a line that lies in the plane of R such that R is entirely on one side of L, then the volume of the solid formed by revolving R about L is given by*

$$\text{volume} = (\text{area of } R)\cdot\begin{pmatrix}\text{distance traveled}\\ \text{by the centroid}\end{pmatrix}$$

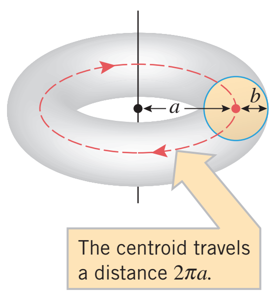
PROOF We prove this theorem in the special case where L is the y-axis, the region R is in the first quadrant, and the region R is of the form given in Problem 6.7.1. (A more general proof will be outlined in the Exercises of Section 14.8.) In this case, the volume V of the solid formed by revolving R about L can be found by the method of cylindrical shells (Section 6.3) to be

$$V = 2\pi \int_a^b x f(x)\, dx$$

Thus, it follows from (8) that

$$V = 2\pi \bar{x}[\text{area of } R]$$

This completes the proof since $2\pi\bar{x}$ is the distance traveled by the centroid when R is revolved about the y-axis. ■



▲ **Figure 6.7.13**

▶ **Example 6** Use Pappus' Theorem to find the volume V of the torus generated by revolving a circular region of radius b about a line at a distance a (greater than b) from the center of the circle (Figure 6.7.13).

Solution. By symmetry, the centroid of a circular region is its center. Thus, the distance traveled by the centroid is $2\pi a$. Since the area of a circle of radius b is πb^2, it follows from Pappus' Theorem that the volume of the torus is

$$V = (2\pi a)(\pi b^2) = 2\pi^2 ab^2 \quad ◀$$

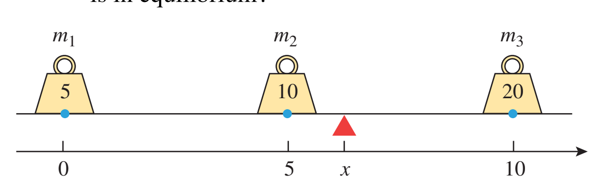
✔ QUICK CHECK EXERCISES 6.7 *(See page 467 for answers.)*

1. The total mass of a homogeneous lamina of area A and density δ is ________.
2. A homogeneous lamina of mass M and density δ occupies a region in the xy-plane bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$, where f is a nonnegative continuous function defined on an interval $[a, b]$. The x-coordinate of the center of gravity of the lamina is M_y/M, where M_y is called the ________ and is given by the integral ________.
3. Let R be the region between the graphs of $y = x^2$ and $y = 2 - x$ for $0 \le x \le 1$. The area of R is $\frac{7}{6}$ and the centroid of R is ________.
4. If the region R in Quick Check Exercise 3 is used to generate a solid G by rotating R about a horizontal line 6 units above its centroid, then the volume of G is ________.

EXERCISE SET 6.7 [C] CAS

FOCUS ON CONCEPTS

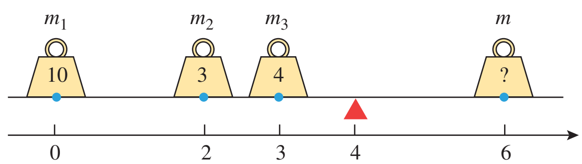
1. Masses $m_1 = 5$, $m_2 = 10$, and $m_3 = 20$ are positioned on a weightless beam as shown in the accompanying figure.
   (a) Suppose that the fulcrum is positioned at $x = 5$. Without computing the sum of moments about 5, determine whether the sum is positive, zero, or negative. Explain.
   (b) Where should the fulcrum be placed so that the beam is in equilibrium?



▲ **Figure Ex-1**

Pappus of Alexandria (**4th century** A.D.) Greek mathematician. Pappus lived during the early Christian era when mathematical activity was in a period of decline. His main contributions to mathematics appeared in a series of eight books called *The Collection* (written about 340 A.D.). This work, which survives only partially, contained some original results but was devoted mostly to statements, refinements, and proofs of results by earlier mathematicians. Pappus' Theorem, stated without proof in Book VII of *The Collection*, was probably known and proved in earlier times. This result is sometimes called Guldin's Theorem in recognition of the Swiss mathematician, Paul Guldin (1577–1643), who rediscovered it independently.

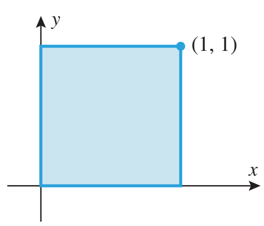
2. Masses $m_1 = 10, m_2 = 3, m_3 = 4$, and m are positioned on a weightless beam, with the fulcrum positioned at point 4, as shown in the accompanying figure.
   (a) Suppose that $m = 14$. Without computing the sum of the moments about 4, determine whether the sum is positive, zero, or negative. Explain.
   (b) For what value of m is the beam in equilibrium?



▲ **Figure Ex-2**

3–6 Find the centroid of the region by inspection and confirm your answer by integrating. ■

3.



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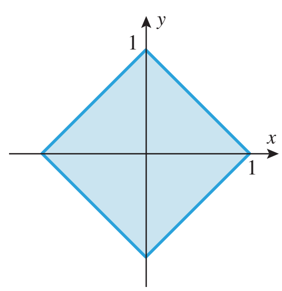
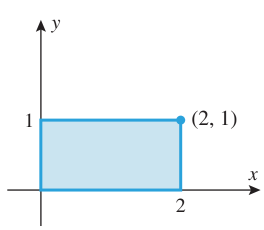
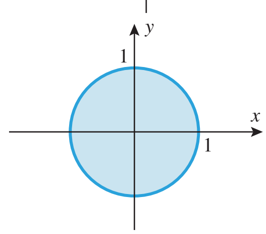


5.

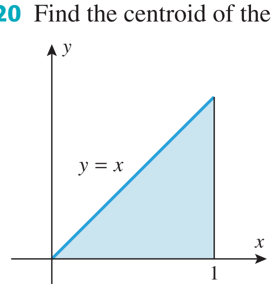


6.



7–20 Find the centroid of the region. ■

7.



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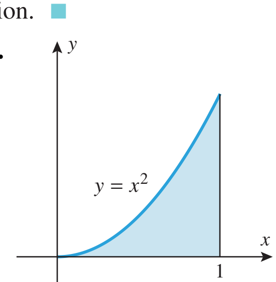

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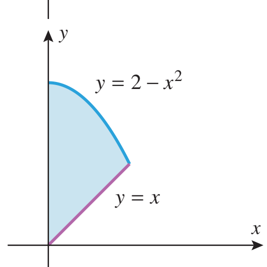
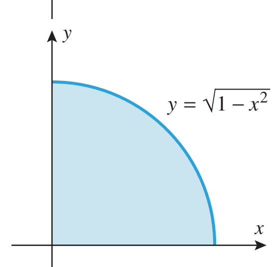


10.



11. The triangle with vertices (0, 0), (2, 0), and (0, 1).

12. The triangle with vertices (0, 0), (1, 1), and (2, 0).

13. The region bounded by the graphs of $y = x^2$ and $x + y = 6$.

14. The region bounded on the left by the y-axis, on the right by the line $x = 2$, below by the parabola $y = x^2$, and above by the line $y = x + 6$.

15. The region bounded by the graphs of $y = x^2$ and $y = x + 2$.

16. The region bounded by the graphs of $y = x^2$ and $y = 1$.

17. The region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$.

18. The region bounded by the graphs of $x = 1/y$, $x = 0$, $y = 1$, and $y = 2$.

19. The region bounded by the graphs of $y = x$, $x = 1/y^2$, and $y = 2$.

20. The region bounded by the graphs of $xy = 4$ and $x + y = 5$.

FOCUS ON CONCEPTS

21. Use symmetry considerations to argue that the centroid of an isosceles triangle lies on the median to the base of the triangle.

22. Use symmetry considerations to argue that the centroid of an ellipse lies at the intersection of the major and minor axes of the ellipse.

23–26 Find the mass and center of gravity of the lamina with density δ. ■

23. A lamina bounded by the x-axis, the line $x = 1$, and the curve $y = \sqrt{x}$; $\delta = 2$.

24. A lamina bounded by the graph of $x = y^4$ and the line $x = 1$; $\delta = 15$.

25. A lamina bounded by the graph of $y = |x|$ and the line $y = 1$; $\delta = 3$.

26. A lamina bounded by the x-axis and the graph of the equation $y = 1 - x^2$; $\delta = 3$.

C **27–30** Use a CAS to find the mass and center of gravity of the lamina with density δ. ■

27. A lamina bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$; $\delta = 4$.

28. A lamina bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$; $\delta = 1/(e - 1)$.

29. A lamina bounded by the graph of $y = \ln x$, the x-axis, and the line $x = 2$; $\delta = 1$.

30. A lamina bounded by the graphs of $y = \cos x$, $y = \sin x$, $x = 0$, and $x = \pi/4$; $\delta = 1 + \sqrt{2}$.

31–34 True–False Determine whether the statement is true or false. Explain your answer. [In Exercise 34, assume that the (rotated) square lies in the xy-plane to the right of the y-axis.] ■

31. The centroid of a rectangle is the intersection of the diagonals of the rectangle.

32. The centroid of a rhombus is the intersection of the diagonals of the rhombus.

33. The centroid of an equilateral triangle is the intersection of the medians of the triangle.

34. By rotating a square about its center, it is possible to change the volume of the solid of revolution generated by revolving the square about the y-axis.

35. Find the centroid of the triangle with vertices $(0, 0)$, (a, b), and $(a, -b)$.

36. Prove that the centroid of a triangle is the point of intersection of the three medians of the triangle. [*Hint:* Choose coordinates so that the vertices of the triangle are located at $(0, -a)$, $(0, a)$, and (b, c).]

37. Find the centroid of the isosceles trapezoid with vertices $(-a, 0)$, $(a, 0)$, $(-b, c)$, and (b, c).

38. Prove that the centroid of a parallelogram is the point of intersection of the diagonals of the parallelogram. [*Hint:* Choose coordinates so that the vertices of the parallelogram are located at $(0, 0)$, $(0, a)$, (b, c), and $(b, a + c)$.]

39. Use the Theorem of Pappus and the fact that the volume of a sphere of radius a is $V = \frac{4}{3}\pi a^3$ to show that the centroid of the lamina that is bounded by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ is $(0, 4a/(3\pi))$. (This problem was solved directly in Example 3.)

40. Use the Theorem of Pappus and the result of Exercise 39 to find the volume of the solid generated when the region bounded by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about

(a) the line $y = -a$ (b) the line $y = x - a$.

41. Use the Theorem of Pappus and the fact that the area of an ellipse with semiaxes a and b is πab to find the volume of the elliptical torus generated by revolving the ellipse

$$\frac{(x - k)^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the y-axis. Assume that $k > a$.

42. Use the Theorem of Pappus to find the volume of the solid that is generated when the region enclosed by $y = x^2$ and $y = 8 - x^2$ is revolved about the x-axis.

43. Use the Theorem of Pappus to find the centroid of the triangular region with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a > 0$ and $b > 0$. [*Hint:* Revolve the region about the x-axis to obtain $\bar{y}$ and about the y-axis to obtain $\bar{x}$.]

44. **Writing** Suppose that a region R in the plane is decomposed into two regions R_1 and R_2 whose areas are A_1 and A_2, respectively, and whose centroids are $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$, respectively. Investigate the problem of expressing the centroid of R in terms of A_1, A_2, $(\bar{x}_1, \bar{y}_1)$, and $(\bar{x}_2, \bar{y}_2)$. Write a short report on your investigations, supporting your reasoning with plausible arguments. Can you extend your results to decompositions of R into more than two regions?

45. **Writing** How might you recognize that a problem can be solved by means of the Theorem of Pappus? That is, what sort of "givens" and "unknowns" would suggest such a solution? Discuss two or three examples.

✔ QUICK CHECK ANSWERS 6.7

1. δA **2.** first moment about the y-axis; $\int_a^b \delta x f(x)\,dx$ **3.** $\left(\frac{5}{14}, \frac{32}{35}\right)$ **4.** 14π

6.8 FLUID PRESSURE AND FORCE

In this section we will use the integration tools developed in the preceding chapter to study the pressures and forces exerted by fluids on submerged objects.

WHAT IS A FLUID?

A ***fluid*** is a substance that flows to conform to the boundaries of any container in which it is placed. Fluids include *liquids*, such as water, oil, and mercury, as well as *gases*, such as helium, oxygen, and air. The study of fluids falls into two categories: *fluid statics* (the study of fluids at rest) and *fluid dynamics* (the study of fluids in motion). In this section we will be concerned only with fluid statics; toward the end of this text we will investigate problems in fluid dynamics.

THE CONCEPT OF PRESSURE

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Snowshoes prevent the woman from sinking by spreading her weight over a large area to reduce her pressure on the snow.

The effect that a force has on an object depends on how that force is spread over the surface of the object. For example, when you walk on soft snow with boots, the weight of your body crushes the snow and you sink into it. However, if you put on a pair of snowshoes to spread the weight of your body over a greater surface area, then the weight of your body has less of a crushing effect on the snow. The concept that accounts for both the magnitude of a force and the area over which it is applied is called *pressure*.

6.8.1 DEFINITION If a force of magnitude F is applied to a surface of area A, then we define the ***pressure*** P exerted by the force on the surface to be

$$P = \frac{F}{A} \tag{1}$$

It follows from this definition that pressure has units of force per unit area. The most common units of pressure are newtons per square meter ($\mathrm{N/m^2}$) in SI and pounds per square inch ($\mathrm{lb/in^2}$) or pounds per square foot ($\mathrm{lb/ft^2}$) in the BE system. As indicated in Table 6.8.1, one newton per square meter is called a *pascal* (Pa). A pressure of 1 Pa is quite small ($1\ \mathrm{Pa} = 1.45 \times 10^{-4}\ \mathrm{lb/in^2}$), so in countries using SI, tire pressure gauges are usually calibrated in kilopascals (kPa), which is 1000 pascals.

Table 6.8.1

UNITS OF FORCE AND PRESSURE

SYSTEM	FORCE	÷	AREA	=	PRESSURE
SI	newton (N)		square meter ($\mathrm{m^2}$)		pascal (Pa)
BE	pound (lb)		square foot ($\mathrm{ft^2}$)		$\mathrm{lb/ft^2}$
BE	pound (lb)		square inch ($\mathrm{in^2}$)		$\mathrm{lb/in^2}$ (psi)

CONVERSION FACTORS:

$1\ \mathrm{Pa} \approx 1.45 \times 10^{-4}\ \mathrm{lb/in^2} \approx 2.09 \times 10^{-2}\ \mathrm{lb/ft^2}$

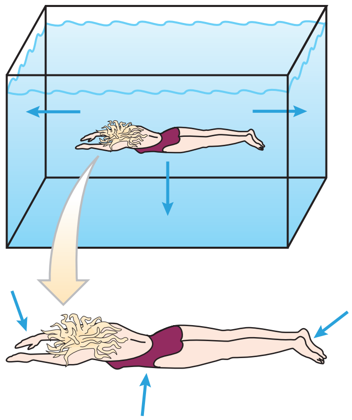
$1\ \mathrm{lb/in^2} \approx 6.89 \times 10^{3}\ \mathrm{Pa}$ $\quad$ $1\ \mathrm{lb/ft^2} \approx 47.9\ \mathrm{Pa}$

Blaise Pascal (1623–1662) French mathematician and scientist. Pascal's mother died when he was three years old and his father, a highly educated magistrate, personally provided the boy's early education. Although Pascal showed an inclination for science and mathematics, his father refused to tutor him in those subjects until he mastered Latin and Greek. Pascal's sister and primary biographer claimed that he independently discovered the first thirty-two propositions of Euclid without ever reading a book on geometry. (However, it is generally agreed that the story is apocryphal.) Nevertheless, the precocious Pascal published a highly respected essay on conic sections by the time he was sixteen years old. Descartes, who read the essay, thought it so brilliant that he could not believe that it was written by such a young man. By age 18 his health began to fail and until his death he was in frequent pain. However, his creativity was unimpaired.

Pascal's contributions to physics include the discovery that air pressure decreases with altitude and the principle of fluid pressure that bears his name. However, the originality of his work is questioned by some historians. Pascal made major contributions to a branch of mathematics called "projective geometry," and he helped to develop probability theory through a series of letters with Fermat.

In 1646, Pascal's health problems resulted in a deep emotional crisis that led him to become increasingly concerned with religious matters. Although born a Catholic, he converted to a religious doctrine called Jansenism and spent most of his final years writing on religion and philosophy.

In this section we will be interested in pressures and forces on objects submerged in fluids. Pressures themselves have no directional characteristics, but the forces that they create always act perpendicular to the face of the submerged object. Thus, in Figure 6.8.1 the water pressure creates horizontal forces on the sides of the tank, vertical forces on the bottom of the tank, and forces that vary in direction, so as to be perpendicular to the different parts of the swimmer's body.



Fluid forces always act perpendicular to the surface of a submerged object.

▲ **Figure 6.8.1**

▶ **Example 1** Referring to Figure 6.8.1, suppose that the back of the swimmer's hand has a surface area of 8.4×10^{-3} m^2 and that the pressure acting on it is 5.1×10^4 Pa (a realistic value near the bottom of a deep diving pool). Find the force that acts on the swimmer's hand.

Solution. From (1), the force F is

$$F = PA = (5.1 \times 10^4 \text{ N/m}^2)(8.4 \times 10^{-3} \text{ m}^2) \approx 4.3 \times 10^2 \text{ N}$$

This is quite a large force (nearly 100 lb in the BE system). ◀

FLUID DENSITY

Scuba divers know that the pressure and forces on their bodies increase with the depth they dive. This is caused by the weight of the water and air above—the deeper the diver goes, the greater the weight above and so the greater the pressure and force exerted on the diver.

To calculate pressures and forces on submerged objects, we need to know something about the characteristics of the fluids in which they are submerged. For simplicity, we will assume that the fluids under consideration are *homogeneous*, by which we mean that any two samples of the fluid with the same volume have the same mass. It follows from this assumption that the mass per unit volume is a constant δ that depends on the physical characteristics of the fluid but not on the size or location of the sample; we call

$$\delta = \frac{m}{V} \tag{2}$$

the ***mass density*** of the fluid. Sometimes it is more convenient to work with weight per unit volume than with mass per unit volume. Thus, we define the ***weight density*** ρ of a fluid to be

$$\rho = \frac{w}{V} \tag{3}$$

where w is the weight of a fluid sample of volume V. Thus, if the weight density of a fluid is known, then the weight w of a fluid sample of volume V can be computed from the formula $w = \rho V$. Table 6.8.2 shows some typical weight densities.

Table 6.8.2

WEIGHT DENSITIES

SI	N/m^3
Machine oil	4708
Gasoline	6602
Fresh water	9810
Seawater	10,045
Mercury	133,416
BE SYSTEM	**lb/ft^3**
Machine oil	30.0
Gasoline	42.0
Fresh water	62.4
Seawater	64.0
Mercury	849.0

All densities are affected by variations in temperature and pressure. Weight densities are also affected by variations in g.

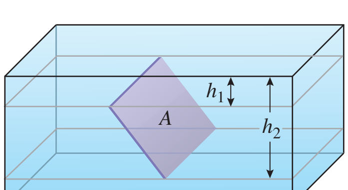
FLUID PRESSURE

To calculate fluid pressures and forces we will need to make use of an experimental observation. Suppose that a flat surface of area A is submerged in a homogeneous fluid of weight density ρ such that the entire surface lies between depths h_1 and h_2, where $h_1 \le h_2$ (Figure 6.8.2). Experiments show that on both sides of the surface, the fluid exerts a force that is perpendicular to the surface and whose magnitude F satisfies the inequalities

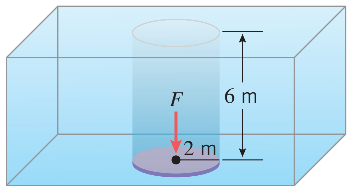
$$\rho h_1 A \le F \le \rho h_2 A \tag{4}$$

Thus, it follows from (1) that the pressure $P = F/A$ on a given side of the surface satisfies the inequalities

$$\rho h_1 \le P \le \rho h_2 \tag{5}$$



▲ **Figure 6.8.2**



The fluid force is the fluid pressure times the area.

▲ Figure 6.8.3

Note that it is now a straightforward matter to calculate fluid force and pressure on a flat surface that is submerged *horizontally* at depth h, for then $h = h_1 = h_2$ and inequalities (4) and (5) become the *equalities*

$$F = \rho h A \tag{6}$$

and

$$P = \rho h \tag{7}$$

▶ **Example 2** Find the fluid pressure and force on the top of a flat circular plate of radius 2 m that is submerged horizontally in water at a depth of 6 m (Figure 6.8.3).

Solution. Since the weight density of water is $\rho = 9810\ \text{N/m}^3$, it follows from (7) that the fluid pressure is

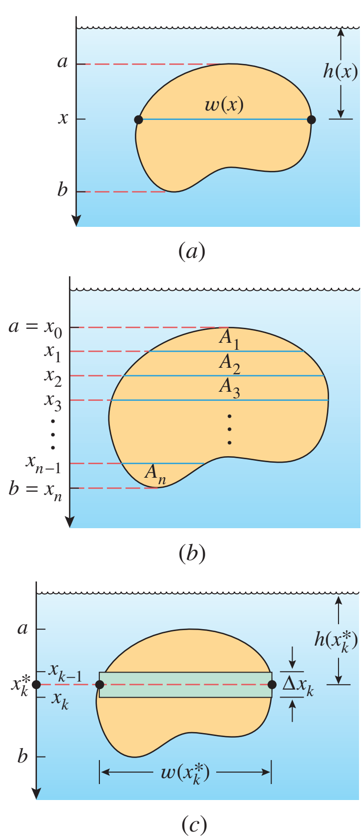
$$P = \rho h = (9810)(6) = 58{,}860\ \text{Pa}$$

and it follows from (6) that the fluid force is

$$F = \rho h A = \rho h(\pi r^2) = (9810)(6)(4\pi) = 235{,}440\pi \approx 739{,}700\ \text{N} \blacktriangleleft$$

FLUID FORCE ON A VERTICAL SURFACE

It was easy to calculate the fluid force on the horizontal plate in Example 2 because each point on the plate was at the same depth. The problem of finding the fluid force on a vertical surface is more complicated because the depth, and hence the pressure, is not constant over the surface. To find the fluid force on a vertical surface we will need calculus.



▲ Figure 6.8.4

6.8.2 PROBLEM Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x-axis whose positive direction is down (Figure 6.8.4a). For $a \le x \le b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x. Define what is meant by the *fluid force* F on the surface, and find a formula for computing it.

The basic idea for solving this problem is to divide the surface into horizontal strips whose areas may be approximated by areas of rectangles. These area approximations, along with inequalities (4), will allow us to create a Riemann sum that approximates the total force on the surface. By taking a limit of Riemann sums we will then obtain an integral for F.

To implement this idea, we divide the interval $[a, b]$ into n subintervals by inserting the points $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. This has the effect of dividing the surface into n strips of area A_k, $k = 1, 2, \ldots, n$ (Figure 6.8.4b). It follows from (4) that the force F_k on the kth strip satisfies the inequalities

$$\rho h(x_{k-1})A_k \le F_k \le \rho h(x_k)A_k$$

or, equivalently,

$$h(x_{k-1}) \le \frac{F_k}{\rho A_k} \le h(x_k)$$

Since the depth function $h(x)$ increases linearly, there must exist a point x_k^* between x_{k-1} and x_k such that

$$h(x_k^*) = \frac{F_k}{\rho A_k}$$

or, equivalently,

$$F_k = \rho h(x_k^*)A_k$$

We now approximate the area A_k of the kth strip of the surface by the area of a rectangle of width $w(x_k^*)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.8.4c). It follows that F_k may be approximated as

$$F_k = \rho h(x_k^*)A_k \approx \rho h(x_k^*) \cdot \underbrace{w(x_k^*)\Delta x_k}_{\text{Area of rectangle}}$$

Adding these approximations yields the following Riemann sum that approximates the total force F on the surface:

$$F = \sum_{k=1}^{n} F_k \approx \sum_{k=1}^{n} \rho h(x_k^*)w(x_k^*)\Delta x_k$$

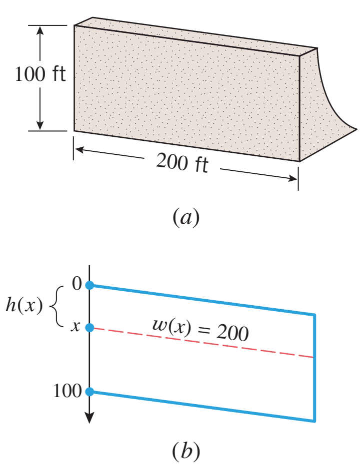
Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$F = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \rho h(x_k^*)w(x_k^*)\Delta x_k = \int_a^b \rho h(x)w(x)\,dx$$

In summary, we have the following result.

6.8.3 DEFINITION Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x-axis whose positive direction is down (Figure 6.8.4a). For $a \le x \le b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x. Then we define the ***fluid force*** F on the surface to be

$$F = \int_a^b \rho h(x)w(x)\,dx \tag{8}$$



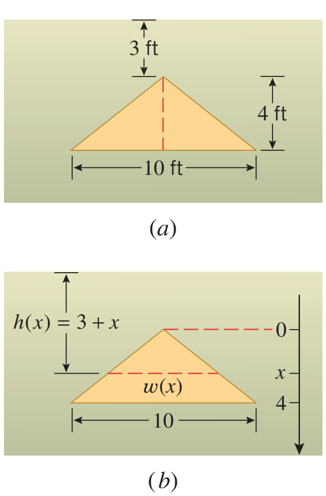
▲ **Figure 6.8.5**

▶ Example 3 The face of a dam is a vertical rectangle of height 100 ft and width 200 ft (Figure 6.8.5a). Find the total fluid force exerted on the face when the water surface is level with the top of the dam.

Solution. Introduce an x-axis with its origin at the water surface as shown in Figure 6.8.5b. At a point x on this axis, the width of the dam in feet is $w(x) = 200$ and the depth in feet is $h(x) = x$. Thus, from (8) with $\rho = 62.4$ lb/ft^3 (the weight density of water) we obtain as the total force on the face

$$F = \int_0^{100} (62.4)(x)(200)\,dx = 12{,}480 \int_0^{100} x\,dx$$

$$= 12{,}480 \left.\frac{x^2}{2}\right]_0^{100} = 62{,}400{,}000 \text{ lb} \blacktriangleleft$$



▲ **Figure 6.8.6**

▶ Example 4 A plate in the form of an isosceles triangle with base 10 ft and altitude 4 ft is submerged vertically in machine oil as shown in Figure 6.8.6a. Find the fluid force F against the plate surface if the oil has weight density $\rho = 30$ lb/ft^3.

Solution. Introduce an x-axis as shown in Figure 6.8.6b. By similar triangles, the width of the plate, in feet, at a depth of $h(x) = (3 + x)$ ft satisfies

$$\frac{w(x)}{10} = \frac{x}{4}, \quad \text{so} \quad w(x) = \frac{5}{2}x$$

Thus, it follows from (8) that the force on the plate is

$$F = \int_a^b \rho h(x) w(x)\, dx = \int_0^4 (30)(3+x)\left(\frac{5}{2}x\right) dx$$

$$= 75 \int_0^4 (3x + x^2)\, dx = 75 \left[\frac{3x^2}{2} + \frac{x^3}{3}\right]_0^4 = 3400 \text{ lb} \blacktriangleleft$$

✔ QUICK CHECK EXERCISES 6.8

(See page 473 for answers.)

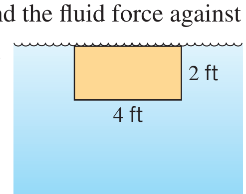
1. The pressure unit equivalent to a newton per square meter (N/m^2) is called a ________. The pressure unit psi stands for ________.
2. Given that the weight density of water is 9810 N/m^3, the fluid pressure on a rectangular 2 m × 3 m flat plate submerged horizontally in water at a depth of 10 m is ________. The fluid force on the plate is ________.
3. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x-axis whose positive direction is down. If, for $a \le x \le b$, the surface has width $w(x)$ and depth $h(x)$, then the fluid force on the surface is $F =$ ________.
4. A rectangular plate 2 m wide and 3 m high is submerged vertically in water so that the top of the plate is 5 m below the water surface. An integral expression for the force of the water on the plate surface is $F =$ ________.

EXERCISE SET 6.8

In this exercise set, refer to Table 6.8.2 for weight densities of fluids, where needed. ■

1. A flat rectangular plate is submerged horizontally in water.
   (a) Find the force (in lb) and the pressure (in lb/ft^2) on the top surface of the plate if its area is 100 ft^2 and the surface is at a depth of 5 ft.
   (b) Find the force (in N) and the pressure (in Pa) on the top surface of the plate if its area is 25 m^2 and the surface is at a depth of 10 m.
2. (a) Find the force (in N) on the deck of a sunken ship if its area is 160 m^2 and the pressure acting on it is 6.0×10^5 Pa.
   (b) Find the force (in lb) on a diver's face mask if its area is 60 in^2 and the pressure acting on it is 100 lb/in^2.

3–8 The flat surfaces shown are submerged vertically in water. Find the fluid force against each surface. ■

3.


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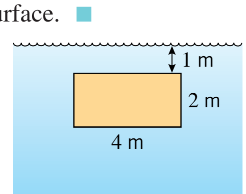

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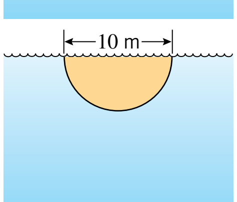

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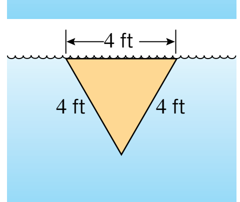

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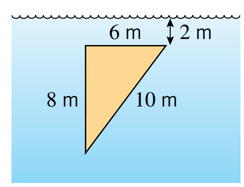

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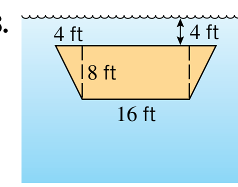


9. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ. If ρ is doubled, is the force on the plate also doubled? Explain your reasoning.
10. An oil tank is shaped like a right circular cylinder of diameter 4 ft. Find the total fluid force against one end when the axis is horizontal and the tank is half filled with oil of weight density 50 lb/ft^3.
11. A square plate of side a feet is dipped in a liquid of weight density ρ lb/ft^3. Find the fluid force on the plate if a vertex is at the surface and a diagonal is perpendicular to the surface.

12–15 True–False Determine whether the statement is true or false. Explain your answer. ■

12. In the International System of Units, pressure and force have the same units.
13. In a cylindrical water tank (with vertical axis), the fluid force on the base of the tank is equal to the weight of water in the tank.
14. In a rectangular water tank, the fluid force on any side of the tank must be less than the fluid force on the base of the tank.

15. In any water tank with a flat base, no matter what the shape of the tank, the fluid force on the base is at most equal to the weight of water in the tank.

16–19 Formula (8) gives the fluid force on a flat surface immersed vertically in a fluid. More generally, if a flat surface is immersed so that it makes an angle of $0 \leq \theta < \pi/2$ with the vertical, then the fluid force on the surface is given by

$$F = \int_a^b \rho h(x)w(x)\sec\theta\, dx$$

Use this formula in these exercises.

16. Derive the formula given above for the fluid force on a flat surface immersed at an angle in a fluid.

17. The accompanying figure shows a rectangular swimming pool whose bottom is an inclined plane. Find the fluid force on the bottom when the pool is filled to the top.

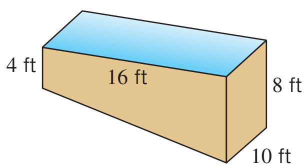


Figure Ex-17

18. By how many feet should the water in the pool of Exercise 17 be lowered in order for the force on the bottom to be reduced by a factor of $\frac{1}{2}$?

19. The accompanying figure shows a dam whose face is an inclined rectangle. Find the fluid force on the face when the water is level with the top of this dam.

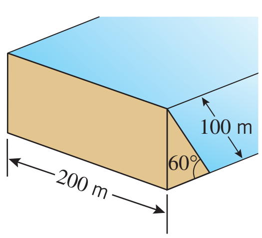


Figure Ex-19

20. An observation window on a submarine is a square with 2 ft sides. Using ρ_0 for the weight density of seawater, find the fluid force on the window when the submarine has descended so that the window is vertical and its top is at a depth of h feet.

FOCUS ON CONCEPTS

21. (a) Show: If the submarine in Exercise 20 descends vertically at a constant rate, then the fluid force on the window increases at a constant rate.
(b) At what rate is the force on the window increasing if the submarine is descending vertically at 20 ft/min?

22. (a) Let $D = D_a$ denote a disk of radius a submerged in a fluid of weight density ρ such that the center of D is h units below the surface of the fluid. For each value of r in the interval $(0, a]$, let D_r denote the disk of radius r that is concentric with D. Select a side of the disk D and define $P(r)$ to be the fluid pressure on the chosen side of D_r. Use (5) to prove that

$$\lim_{r\to 0^+} P(r) = \rho h$$

(b) Explain why the result in part (a) may be interpreted to mean that *fluid pressure at a given depth is the same in all directions*. (This statement is one version of a result known as ***Pascal's Principle***.)

23. **Writing** Suppose that we model the Earth's atmosphere as a "fluid." Atmospheric pressure at sea level is $P = 14.7$ lb/in^2 and the weight density of air at sea level is about $\rho = 4.66 \times 10^{-5}$ lb/in^3. With these numbers, what would Formula (7) yield as the height of the atmosphere above the Earth? Do you think this answer is reasonable? If not, explain how we might modify our assumptions to yield a more plausible answer.

24. **Writing** Suppose that the weight density ρ of a fluid is a function $\rho = \rho(x)$ of the depth x within the fluid. How do you think that Formula (7) for fluid pressure will need to be modified? Support your answer with plausible arguments.

✔ QUICK CHECK ANSWERS 6.8

1. pascal; pounds per square inch **2.** 98,100 Pa; 588,600 N **3.** $\int_a^b \rho h(x)w(x)\,dx$ **4.** $\int_0^3 9810\,[(5+x)2]\,dx$

6.9 HYPERBOLIC FUNCTIONS AND HANGING CABLES

In this section we will study certain combinations of e^x and e^{-x}, called "hyperbolic functions." These functions, which arise in various engineering applications, have many properties in common with the trigonometric functions. This similarity is somewhat surprising, since there is little on the surface to suggest that there should be any relationship between exponential and trigonometric functions. This is because the relationship occurs within the context of complex numbers, a topic which we will leave for more advanced courses.

DEFINITIONS OF HYPERBOLIC FUNCTIONS

To introduce the hyperbolic functions, observe from Exercise 61 in Section 0.2 that the function e^x can be expressed in the following way as the sum of an even function and an odd function:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd}}$$

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the *hyperbolic sine* of x and the even function is called the *hyperbolic cosine* of x. They are denoted by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

where sinh is pronounced "cinch" and cosh rhymes with "gosh." From these two building blocks we can create four more functions to produce the following set of six ***hyperbolic functions***.

The terms "tanh," "sech," and "csch" are pronounced "tanch," "seech," and "coseech," respectively.

6.9.1 DEFINITION

Hyperbolic sine $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine $\quad \cosh x = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent $\quad \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent $\quad \coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant $\quad \operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

Hyperbolic cosecant $\quad \operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

TECHNOLOGY MASTERY

Computer algebra systems have built-in capabilities for evaluating hyperbolic functions directly, but some calculators do not. However, if you need to evaluate a hyperbolic function on a calculator, you can do so by expressing it in terms of exponential functions, as in Example 1.

Example 1

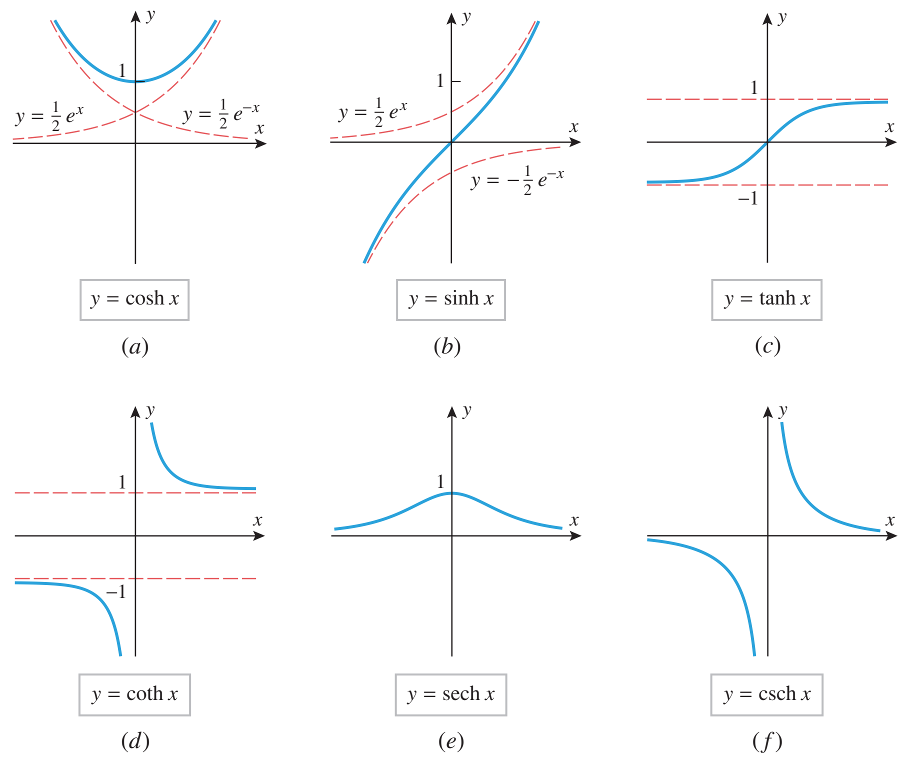
$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\sinh 2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269$$

GRAPHS OF THE HYPERBOLIC FUNCTIONS

The graphs of the hyperbolic functions, which are shown in Figure 6.9.1, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of $y = \cosh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ separately and adding the corresponding y-coordinates [see part (a) of the figure]. Similarly, the general shape of the graph of $y = \sinh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$ separately and adding corresponding y-coordinates [see part (b) of the figure].



▲ **Figure 6.9.1**

Glen Allison/Stone/Getty Images

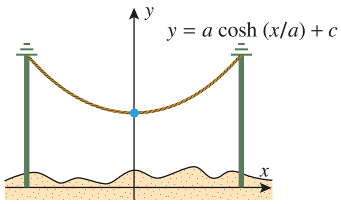
The design of the Gateway Arch near St. Louis is based on an inverted hyperbolic cosine curve (Exercise 73).

Observe that $\sinh x$ has a domain of $(-\infty, +\infty)$ and a range of $(-\infty, +\infty)$, whereas $\cosh x$ has a domain of $(-\infty, +\infty)$ and a range of $[1, +\infty)$. Observe also that $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ are ***curvilinear asymptotes*** for $y = \cosh x$ in the sense that the graph of $y = \cosh x$ gets closer and closer to the graph of $y = \frac{1}{2}e^x$ as $x \to +\infty$ and gets closer and closer to the graph of $y = \frac{1}{2}e^{-x}$ as $x \to -\infty$. (See Section 4.3.) Similarly, $y = \frac{1}{2}e^x$ is a curvilinear asymptote for $y = \sinh x$ as $x \to +\infty$ and $y = -\frac{1}{2}e^{-x}$ is a curvilinear asymptote as $x \to -\infty$. Other properties of the hyperbolic functions are explored in the exercises.

HANGING CABLES AND OTHER APPLICATIONS

Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a ***catenary*** (from the Latin *catena*, meaning "chain"). If, as in Figure 6.9.2, a coordinate system is introduced so that the low point of the cable lies on the y-axis, then it can be shown using principles of physics that the cable has an equation of the form

$$y = a\cosh\left(\frac{x}{a}\right) + c$$



▲ Figure 6.9.2

where the parameters a and c are determined by the distance between the poles and the composition of the cable.

HYPERBOLIC IDENTITIES

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$$\cosh^2 x - \sinh^2 x = 1 \tag{1}$$

which can be proved by writing

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= (\cosh x + \sinh x)(\cosh x - \sinh x)\\ &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)\\ &= e^x \cdot e^{-x} = 1\end{aligned}$$

Larry Auippy/Mira.com/Digital Railroad, Inc.

A flexible cable suspended between two poles forms a catenary.

Other hyperbolic identities can be derived in a similar manner or, alternatively, by performing algebraic operations on known identities. For example, if we divide (1) by $\cosh^2 x$, we obtain

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

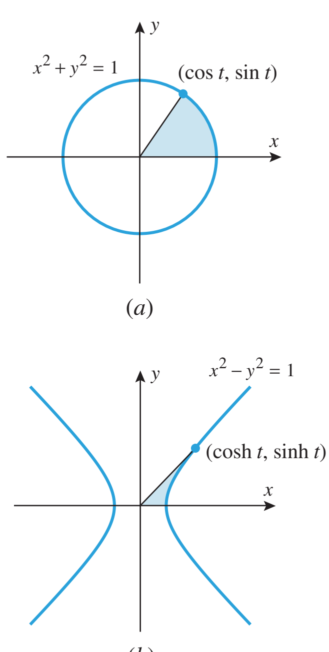
and if we divide (1) by $\sinh^2 x$, we obtain

$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

The following theorem summarizes some of the more useful hyperbolic identities. The proofs of those not already obtained are left as exercises.

6.9.2 THEOREM

$\cosh x + \sinh x = e^x$	$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
$\cosh x - \sinh x = e^{-x}$	$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh(-x) = \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\sinh(-x) = -\sinh x$	$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$



▲ Figure 6.9.3

WHY THEY ARE CALLED HYPERBOLIC FUNCTIONS

Recall that the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

represent the unit circle $x^2 + y^2 = 1$ (Figure 6.9.3a), as may be seen by writing

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

If $0 \le t \le 2\pi$, then the parameter t can be interpreted as the angle in radians from the positive x-axis to the point $(\cos t, \sin t)$ or, alternatively, as twice the shaded area of the sector in Figure 6.9.3a (verify). Analogously, the parametric equations

$$x = \cosh t, \quad y = \sinh t \qquad (-\infty < t < +\infty)$$

represent a portion of the curve $x^2 - y^2 = 1$, as may be seen by writing

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$$

and observing that $x = \cosh t > 0$. This curve, which is shown in Figure 6.9.3*b*, is the right half of a larger curve called the ***unit hyperbola***; this is the reason why the functions in this section are called *hyperbolic* functions. It can be shown that if $t \geq 0$, then the parameter t can be interpreted as twice the shaded area in Figure 6.9.3*b*. (We omit the details.)

DERIVATIVE AND INTEGRAL FORMULAS

Derivative formulas for $\sinh x$ and $\cosh x$ can be obtained by expressing these functions in terms of e^x and e^{-x}:

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

Derivatives of the remaining hyperbolic functions can be obtained by expressing them in terms of sinh and cosh and applying appropriate identities. For example,

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \frac{d}{dx}[\sinh x] - \sinh x \frac{d}{dx}[\cosh x]}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

6.9.3 THEOREM

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

▶ Example 2

$$\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx}[x^3] = 3x^2 \sinh(x^3)$$

$$\frac{d}{dx}[\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx}[\tanh x] = \frac{\operatorname{sech}^2 x}{\tanh x} \blacktriangleleft$$

▶ **Example 3**

$$\int \sinh^5 x \cosh x \, dx = \tfrac{1}{6} \sinh^6 x + C$$

$u = \sinh x$
$du = \cosh x \, dx$

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$
$$= \ln|\cosh x| + C$$

$u = \cosh x$
$du = \sinh x \, dx$

$$= \ln(\cosh x) + C$$

We were justified in dropping the absolute value signs since $\cosh x > 0$ for all x. ◀

▶ **Example 4** A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution. From above, the wire forms a catenary curve with equation

$$y = a \cosh\left(\frac{x}{a}\right) + c$$

where the origin is on the ground midway between the poles. Using Formula (4) of Section 6.4 for the length of the catenary, we have

$$100 = \int_{-45}^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$
$$= 2\int_{0}^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

By symmetry about the y-axis

$$= 2\int_{0}^{45} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)} \, dx$$
$$= 2\int_{0}^{45} \cosh\left(\frac{x}{a}\right) dx$$

By (1) and the fact that $\cosh x > 0$

$$= 2a \sinh\left(\frac{x}{a}\right)\Bigg]_0^{45} = 2a \sinh\left(\frac{45}{a}\right)$$

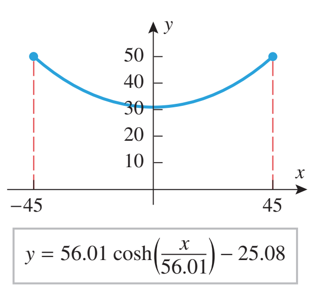
Using a calculating utility's numeric solver to solve

$$100 = 2a \sinh\left(\frac{45}{a}\right)$$

for a gives $a \approx 56.01$. Then

$$50 = y(45) = 56.01 \cosh\left(\frac{45}{56.01}\right) + c \approx 75.08 + c$$

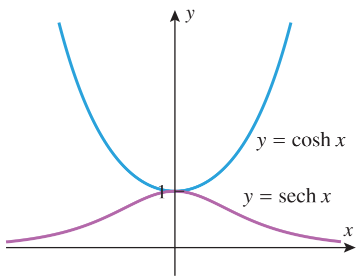
so $c \approx -25.08$. Thus, the middle of the wire is $y(0) \approx 56.01 - 25.08 = 30.93$ ft above the ground (Figure 6.9.4). ◀



▲ **Figure 6.9.4**

INVERSES OF HYPERBOLIC FUNCTIONS

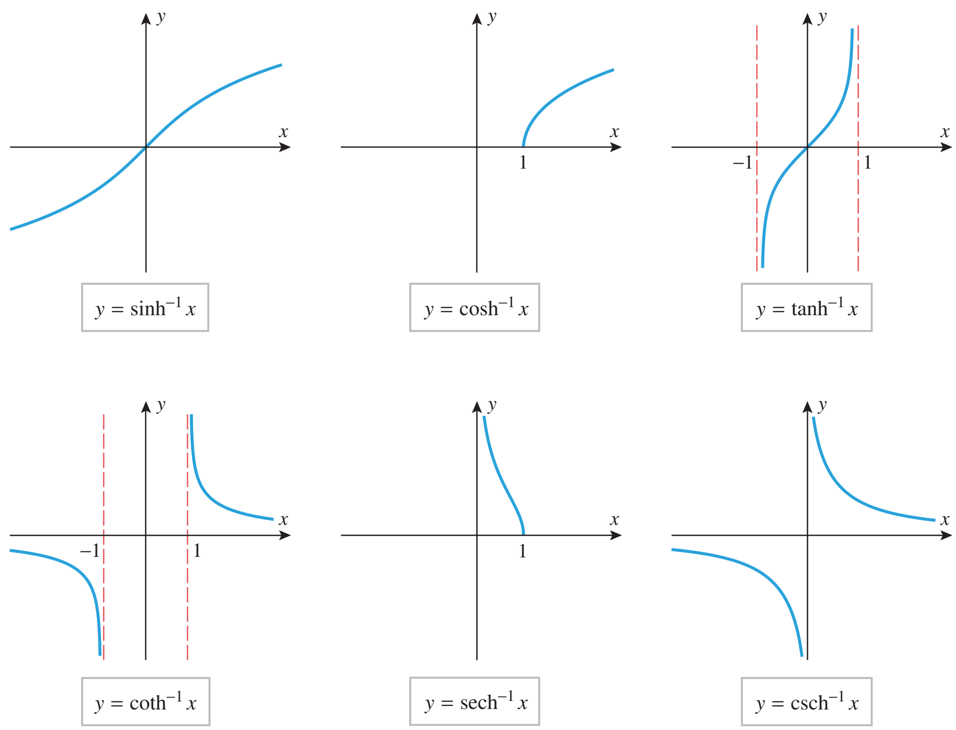
Referring to Figure 6.9.1, it is evident that the graphs of $\sinh x$, $\tanh x$, $\coth x$, and $\operatorname{csch} x$ pass the horizontal line test, but the graphs of $\cosh x$ and $\operatorname{sech} x$ do not. In the latter case, restricting x to be nonnegative makes the functions invertible (Figure 6.9.5). The graphs of the six inverse hyperbolic functions in Figure 6.9.6 were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line $y = x$.



With the restriction that $x \geq 0$, the curves $y = \cosh x$ and $y = \operatorname{sech} x$ pass the horizontal line test.

▲ **Figure 6.9.5**

Table 6.9.1 summarizes the basic properties of the inverse hyperbolic functions. You should confirm that the domains and ranges listed in this table agree with the graphs in Figure 6.9.6.



▶ **Figure 6.9.6**

Table 6.9.1

PROPERTIES OF INVERSE HYPERBOLIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \geq 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \leq 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Because the hyperbolic functions are expressible in terms of e^x, it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.

6.9.4 THEOREM *The following relationships hold for all x in the domains of the stated inverse hyperbolic functions:*

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

We will show how to derive the first formula in this theorem and leave the rest as exercises. The basic idea is to write the equation $x = \sinh y$ in terms of exponential functions and solve this equation for y as a function of x. This will produce the equation $y = \sinh^{-1} x$ with $\sinh^{-1} x$ expressed in terms of natural logarithms. Expressing $x = \sinh y$ in terms of exponentials yields

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

which can be rewritten as

$$e^y - 2x - e^{-y} = 0$$

Multiplying this equation through by e^y we obtain

$$e^{2y} - 2xe^y - 1 = 0$$

and applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $e^y > 0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$e^y = x + \sqrt{x^2+1}$$

Taking natural logarithms yields

$$y = \ln(x + \sqrt{x^2+1}) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

▶ Example 5

$$\sinh^{-1} 1 = \ln(1 + \sqrt{1^2+1}) = \ln(1+\sqrt{2}) \approx 0.8814$$

$$\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln 3 \approx 0.5493 \ ◀$$

DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Show that the derivative of the function $\sinh^{-1} x$ can also be obtained by letting $y = \sinh^{-1} x$ and then differentiating $x = \sinh y$ implicitly.

Formulas for the derivatives of the inverse hyperbolic functions can be obtained from Theorem 6.9.4. For example,

$$\begin{aligned}\frac{d}{dx}[\sinh^{-1} x] &= \frac{d}{dx}[\ln(x + \sqrt{x^2+1}\,)] = \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1}\,)(\sqrt{x^2+1}\,)} = \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

This computation leads to two integral formulas, a formula that involves $\sinh^{-1} x$ and an equivalent formula that involves logarithms:

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C = \ln(x + \sqrt{x^2+1}\,) + C$$

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

6.9.5 THEOREM

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1 \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1 \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

6.9.6 THEOREM *If $a > 0$, then*

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad \text{or} \quad \ln(u + \sqrt{u^2+a^2}\,) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad \text{or} \quad \ln(u + \sqrt{u^2-a^2}\,) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & |u| > a \end{cases} \quad \text{or} \quad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \quad \text{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \quad \text{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

Example 6 Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}$, $x > \frac{3}{2}$.

Solution. Let $u = 2x$. Thus, $du = 2\,dx$ and

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-3^2}}$$
$$= \frac{1}{2}\cosh^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

Alternatively, we can use the logarithmic equivalent of $\cosh^{-1}(2x/3)$,

$$\cosh^{-1}\left(\frac{2x}{3}\right) = \ln(2x + \sqrt{4x^2-9}) - \ln 3$$

(verify), and express the answer as

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\ln(2x + \sqrt{4x^2-9}) + C$$

✔ QUICK CHECK EXERCISES 6.9 (See page 485 for answers.)

1. $\cosh x =$ ________ $\sinh x =$ ________
$\tanh x =$ ________

2. Complete the table.

	$\cosh x$	$\sinh x$	$\tanh x$	$\coth x$	$\operatorname{sech} x$	$\operatorname{csch} x$
DOMAIN						
RANGE						

3. The parametric equations

$$x = \cosh t, \quad y = \sinh t \quad (-\infty < t < +\infty)$$

represent the right half of the curve called a ________. Eliminating the parameter, the equation of this curve is ________.

4. $\frac{d}{dx}[\cosh x] =$ ________ $\frac{d}{dx}[\sinh x] =$ ________
$\frac{d}{dx}[\tanh x] =$ ________

5. $\int \cosh x\,dx =$ ________ $\int \sinh x\,dx =$ ________
$\int \tanh x\,dx =$ ________

6. $\frac{d}{dx}[\cosh^{-1} x] =$ ________ $\frac{d}{dx}[\sinh^{-1} x] =$ ________
$\frac{d}{dx}[\tanh^{-1} x] =$ ________

EXERCISE SET 6.9 Graphing Utility

1–2 Approximate the expression to four decimal places. ■

1. (a) $\sinh 3$ (b) $\cosh(-2)$ (c) $\tanh(\ln 4)$
(d) $\sinh^{-1}(-2)$ (e) $\cosh^{-1} 3$ (f) $\tanh^{-1}\frac{3}{4}$

2. (a) $\operatorname{csch}(-1)$ (b) $\operatorname{sech}(\ln 2)$ (c) $\coth 1$
(d) $\operatorname{sech}^{-1}\frac{1}{2}$ (e) $\coth^{-1} 3$ (f) $\operatorname{csch}^{-1}(-\sqrt{3})$

3. Find the exact numerical value of each expression.
(a) $\sinh(\ln 3)$ (b) $\cosh(-\ln 2)$
(c) $\tanh(2\ln 5)$ (d) $\sinh(-3\ln 2)$

4. In each part, rewrite the expression as a ratio of polynomials.
(a) $\cosh(\ln x)$ (b) $\sinh(\ln x)$
(c) $\tanh(2\ln x)$ (d) $\cosh(-\ln x)$

5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number x_0. Use appropriate identities to find the exact values of the remaining five hyperbolic functions at x_0.
(a) $\sinh x_0 = 2$ (b) $\cosh x_0 = \frac{5}{4}$ (c) $\tanh x_0 = \frac{4}{5}$

6. Obtain the derivative formulas for $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$ from the derivative formulas for $\sinh x$, $\cosh x$, and $\tanh x$.

7. Find the derivatives of $\cosh^{-1} x$ and $\tanh^{-1} x$ by differentiating the formulas in Theorem 6.9.4.

8. Find the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$ by differentiating the equations $x = \sinh y$, $x = \cosh y$, and $x = \tanh y$ implicitly.

9–28 Find dy/dx. ■

9. $y = \sinh(4x - 8)$ **10.** $y = \cosh(x^4)$

11. $y = \coth(\ln x)$

12. $y = \ln(\tanh 2x)$

13. $y = \operatorname{csch}(1/x)$

14. $y = \operatorname{sech}(e^{2x})$

15. $y = \sqrt{4x + \cosh^2(5x)}$

16. $y = \sinh^3(2x)$

17. $y = x^3 \tanh^2(\sqrt{x})$

18. $y = \sinh(\cos 3x)$

19. $y = \sinh^{-1}\left(\frac{1}{3}x\right)$

20. $y = \sinh^{-1}(1/x)$

21. $y = \ln(\cosh^{-1} x)$

22. $y = \cosh^{-1}(\sinh^{-1} x)$

23. $y = \dfrac{1}{\tanh^{-1} x}$

24. $y = (\coth^{-1} x)^2$

25. $y = \cosh^{-1}(\cosh x)$

26. $y = \sinh^{-1}(\tanh x)$

27. $y = e^x \operatorname{sech}^{-1}\sqrt{x}$

28. $y = (1 + x \operatorname{csch}^{-1} x)^{10}$

29–44 Evaluate the integrals. ■

29. $\displaystyle\int \sinh^6 x \cosh x \, dx$

30. $\displaystyle\int \cosh(2x - 3) \, dx$

31. $\displaystyle\int \sqrt{\tanh x} \operatorname{sech}^2 x \, dx$

32. $\displaystyle\int \operatorname{csch}^2(3x) \, dx$

33. $\displaystyle\int \tanh 2x \, dx$

34. $\displaystyle\int \coth^2 x \operatorname{csch}^2 x \, dx$

35. $\displaystyle\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^3 x \, dx$

36. $\displaystyle\int_0^{\ln 3} \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

37. $\displaystyle\int \frac{dx}{\sqrt{1 + 9x^2}}$

38. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 2}} \quad (x > \sqrt{2})$

39. $\displaystyle\int \frac{dx}{\sqrt{1 - e^{2x}}} \quad (x < 0)$

40. $\displaystyle\int \frac{\sin\theta \, d\theta}{\sqrt{1 + \cos^2\theta}}$

41. $\displaystyle\int \frac{dx}{x\sqrt{1 + 4x^2}}$

42. $\displaystyle\int \frac{dx}{\sqrt{9x^2 - 25}} \quad (x > 5/3)$

43. $\displaystyle\int_0^{1/2} \frac{dx}{1 - x^2}$

44. $\displaystyle\int_0^{\sqrt{3}} \frac{dt}{\sqrt{t^2 + 1}}$

45–48 True–False Determine whether the statement is true or false. Explain your answer. ■

45. The equation $\cosh x = \sinh x$ has no solutions.

46. Exactly two of the hyperbolic functions are bounded.

47. There is exactly one hyperbolic function $f(x)$ such that for all real numbers a, the equation $f(x) = a$ has a unique solution x.

48. The identities in Theorem 6.9.2 may be obtained from the corresponding trigonometric identities by replacing each trigonometric function with its hyperbolic analogue.

49. Find the area enclosed by $y = \sinh 2x$, $y = 0$, and $x = \ln 3$.

50. Find the volume of the solid that is generated when the region enclosed by $y = \operatorname{sech} x$, $y = 0$, $x = 0$, and $x = \ln 2$ is revolved about the x-axis.

51. Find the volume of the solid that is generated when the region enclosed by $y = \cosh 2x$, $y = \sinh 2x$, $x = 0$, and $x = 5$ is revolved about the x-axis.

52. Approximate the positive value of the constant a such that the area enclosed by $y = \cosh ax$, $y = 0$, $x = 0$, and $x = 1$ is 2 square units. Express your answer to at least five decimal places.

53. Find the arc length of the catenary $y = \cosh x$ between $x = 0$ and $x = \ln 2$.

54. Find the arc length of the catenary $y = a\cosh(x/a)$ between $x = 0$ and $x = x_1$ $(x_1 > 0)$.

55. In parts (a)–(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.9.1 and 6.9.6.

(a) $\displaystyle\lim_{x\to+\infty} \sinh x$ (b) $\displaystyle\lim_{x\to-\infty} \sinh x$

(c) $\displaystyle\lim_{x\to+\infty} \tanh x$ (d) $\displaystyle\lim_{x\to-\infty} \tanh x$

(e) $\displaystyle\lim_{x\to+\infty} \sinh^{-1} x$ (f) $\displaystyle\lim_{x\to1^-} \tanh^{-1} x$

FOCUS ON CONCEPTS

56. Explain how to obtain the asymptotes for $y = \tanh x$ from the curvilinear asymptotes for $y = \cosh x$ and $y = \sinh x$.

57. Prove that $\sinh x$ is an odd function of x and that $\cosh x$ is an even function of x, and check that this is consistent with the graphs in Figure 6.9.1.

58–59 Prove the identities. ■

58. (a) $\cosh x + \sinh x = e^x$

(b) $\cosh x - \sinh x = e^{-x}$

(c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

(d) $\sinh 2x = 2\sinh x \cosh x$

(e) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(f) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(g) $\cosh 2x = 2\sinh^2 x + 1$

(h) $\cosh 2x = 2\cosh^2 x - 1$

59. (a) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(b) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

(c) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

60. Prove:

(a) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$

(b) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right), \quad -1 < x < 1.$

61. Use Exercise 60 to obtain the derivative formulas for $\cosh^{-1} x$ and $\tanh^{-1} x$.

62. Prove:

$$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x), \quad 0 < x \leq 1$$
$$\coth^{-1} x = \tanh^{-1}(1/x), \quad |x| > 1$$
$$\operatorname{csch}^{-1} x = \sinh^{-1}(1/x), \quad x \neq 0$$

63. Use Exercise 62 to express the integral

$$\int \frac{du}{1 - u^2}$$

entirely in terms of $\tanh^{-1}$.

64. Show that

(a) $\dfrac{d}{dx}[\operatorname{sech}^{-1}|x|] = -\dfrac{1}{x\sqrt{1-x^2}}$

(b) $\dfrac{d}{dx}[\operatorname{csch}^{-1}|x|] = -\dfrac{1}{x\sqrt{1+x^2}}$.

65. In each part, find the limit.

(a) $\lim\limits_{x\to+\infty}(\cosh^{-1} x - \ln x)$ (b) $\lim\limits_{x\to+\infty}\dfrac{\cosh x}{e^x}$

66. Use the first and second derivatives to show that the graph of $y = \tanh^{-1} x$ is always increasing and has an inflection point at the origin.

67. The integration formulas for $1/\sqrt{u^2-a^2}$ in Theorem 6.9.6 are valid for $u > a$. Show that the following formula is valid for $u < -a$:

$$\int \frac{du}{\sqrt{u^2-a^2}} = -\cosh^{-1}\left(-\frac{u}{a}\right) + C \quad \text{or} \quad \ln\left|u + \sqrt{u^2-a^2}\right| + C$$

68. Show that $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$.

69. Show that

$$\int_{-a}^{a} e^{tx}\,dx = \frac{2\sinh at}{t}$$

70. A cable is suspended between two poles as shown in Figure 6.9.2. Assume that the equation of the curve formed by the cable is $y = a\cosh(x/a)$, where a is a positive constant. Suppose that the x-coordinates of the points of support are $x = -b$ and $x = b$, where $b > 0$.

(a) Show that the length L of the cable is given by

$$L = 2a\sinh\frac{b}{a}$$

(b) Show that the sag S (the vertical distance between the highest and lowest points on the cable) is given by

$$S = a\cosh\frac{b}{a} - a$$

71–72 These exercises refer to the hanging cable described in Exercise 70. ■

71. Assuming that the poles are 400 ft apart and the sag in the cable is 30 ft, approximate the length of the cable by approximating a. Express your final answer to the nearest tenth of a foot. [*Hint:* First let $u = 200/a$.]

72. Assuming that the cable is 120 ft long and the poles are 100 ft apart, approximate the sag in the cable by approximating a. Express your final answer to the nearest tenth of a foot. [*Hint:* First let $u = 50/a$.]

73. The design of the Gateway Arch in St. Louis, Missouri, by architect Eero Saarinan was implemented using equations provided by Dr. Hannskarl Badel. The equation used for the centerline of the arch was

$$y = 693.8597 - 68.7672\cosh(0.0100333x) \text{ ft}$$

for x between -299.2239 and 299.2239.

(a) Use a graphing utility to graph the centerline of the arch.

(b) Find the length of the centerline to four decimal places.

(c) For what values of x is the height of the arch 100 ft? Round your answers to four decimal places.

(d) Approximate, to the nearest degree, the acute angle that the tangent line to the centerline makes with the ground at the ends of the arch.

74. Suppose that a hollow tube rotates with a constant angular velocity of ω rad/s about a horizontal axis at one end of the tube, as shown in the accompanying figure. Assume that an object is free to slide without friction in the tube while the tube is rotating. Let r be the distance from the object to the pivot point at time $t \geq 0$, and assume that the object is at rest and $r = 0$ when $t = 0$. It can be shown that if the tube is horizontal at time $t = 0$ and rotating as shown in the figure, then

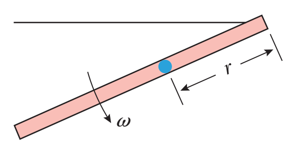
$$r = \frac{g}{2\omega^2}[\sinh(\omega t) - \sin(\omega t)]$$

during the period that the object is in the tube. Assume that t is in seconds and r is in meters, and use $g = 9.8$ m/s^2 and $\omega = 2$ rad/s.

(a) Graph r versus t for $0 \leq t \leq 1$.

(b) Assuming that the tube has a length of 1 m, approximately how long does it take for the object to reach the end of the tube?

(c) Use the result of part (b) to approximate dr/dt at the instant that the object reaches the end of the tube.



◀ **Figure Ex-74**

75. The accompanying figure (on the next page) shows a person pulling a boat by holding a rope of length a attached to the bow and walking along the edge of a dock. If we assume that the rope is always tangent to the curve traced by the bow of the boat, then this curve, which is called a ***tractrix***, has the property that the segment of the tangent line between the curve and the y-axis has a constant length a. It can be proved that the equation of this tractrix is

$$y = a\operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$$

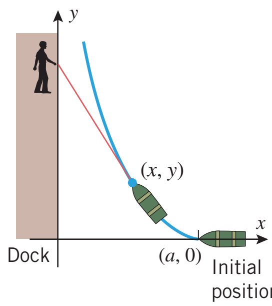
(a) Show that to move the bow of the boat to a point (x, y), the person must walk a distance

$$D = a\operatorname{sech}^{-1}\frac{x}{a}$$

from the origin.

(b) If the rope has a length of 15 m, how far must the person walk from the origin to bring the boat 10 m from the dock? Round your answer to two decimal places.

(c) Find the distance traveled by the bow along the tractrix as it moves from its initial position to the point where it is 5 m from the dock.



◀ **Figure Ex-75**

76. Writing Suppose that, by analogy with the trigonometric functions, we *define* $\cosh t$ and $\sinh t$ geometrically using Figure 6.9.3*b*:
"For any real number t, define $x = \cosh t$ and $y = \sinh t$ to be the unique values of x and y such that

(i) $P(x, y)$ is on the right branch of the unit hyperbola $x^2 - y^2 = 1$;
(ii) t and y have the same sign (or are both 0);
(iii) the area of the region bounded by the x-axis, the right branch of the unit hyperbola, and the segment from the origin to P is $|t|/2$."

Discuss what properties would first need to be verified in order for this to be a legitimate definition.

77. Writing Investigate what properties of $\cosh t$ and $\sinh t$ can be proved directly from the geometric definition in Exercise 76. Write a short description of the results of your investigation.

✔ QUICK CHECK ANSWERS 6.9

1. $\dfrac{e^x + e^{-x}}{2}$; $\dfrac{e^x - e^{-x}}{2}$; $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

2.

	$\cosh x$	$\sinh x$	$\tanh x$	$\coth x$	$\operatorname{sech} x$	$\operatorname{csch} x$
DOMAIN	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$
RANGE	$[1, +\infty)$	$(-\infty, +\infty)$	$(-1, 1)$	$(-\infty, -1) \cup (1, +\infty)$	$(0, 1]$	$(-\infty, 0) \cup (0, +\infty)$

3. unit hyperbola; $x^2 - y^2 = 1$ **4.** $\sinh x$; $\cosh x$; $\operatorname{sech}^2 x$ **5.** $\sinh x + C$; $\cosh x + C$; $\ln(\cosh x) + C$

6. $\dfrac{1}{\sqrt{x^2 - 1}}$; $\dfrac{1}{\sqrt{1 + x^2}}$; $\dfrac{1}{1 - x^2}$

CHAPTER 6 REVIEW EXERCISES

1. Describe the method of slicing for finding volumes, and use that method to derive an integral formula for finding volumes by the method of disks.

2. State an integral formula for finding a volume by the method of cylindrical shells, and use Riemann sums to derive the formula.

3. State an integral formula for finding the arc length of a smooth curve $y = f(x)$ over an interval $[a, b]$, and use Riemann sums to derive the formula.

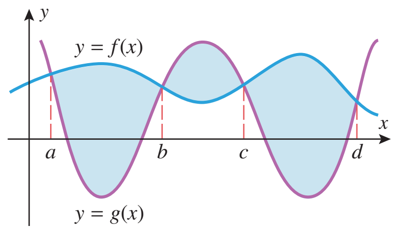
4. State an integral formula for the work W done by a variable force $F(x)$ applied in the direction of motion to an object moving from $x = a$ to $x = b$, and use Riemann sums to derive the formula.

5. State an integral formula for the fluid force F exerted on a vertical flat surface immersed in a fluid of weight density ρ, and use Riemann sums to derive the formula.

6. Let R be the region in the first quadrant enclosed by $y = x^2$, $y = 2 + x$, and $x = 0$. In each part, set up, but *do not evaluate*, an integral or a sum of integrals that will solve the problem.
(a) Find the area of R by integrating with respect to x.
(b) Find the area of R by integrating with respect to y.
(c) Find the volume of the solid generated by revolving R about the x-axis by integrating with respect to x.
(d) Find the volume of the solid generated by revolving R about the x-axis by integrating with respect to y.
(e) Find the volume of the solid generated by revolving R about the y-axis by integrating with respect to x.
(f) Find the volume of the solid generated by revolving R about the y-axis by integrating with respect to y.
(g) Find the volume of the solid generated by revolving R about the line $y = -3$ by integrating with respect to x.
(h) Find the volume of the solid generated by revolving R about the line $x = 5$ by integrating with respect to x.

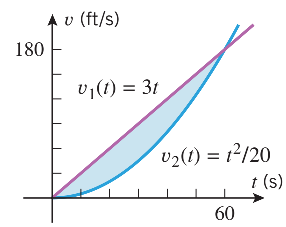
7. (a) Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$ in the accompanying figure on the next page. *(cont.)*

(b) Find the total area enclosed between $y = x^3$ and $y = x$ over the interval $[-1, 2]$.



◀ **Figure Ex-7**

8. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.
(a) How far apart are the cars after 60 seconds?
(b) How far apart are the cars after T seconds, where $0 \leq T \leq 60$?



◀ **Figure Ex-8**

9. Let R be the region enclosed by the curves $y = x^2 + 4$, $y = x^3$, and the y-axis. Find and evaluate a definite integral that represents the volume of the solid generated by revolving R about the x-axis.

10. A football has the shape of the solid generated by revolving the region bounded between the x-axis and the parabola $y = 4R(x^2 - \frac{1}{4}L^2)/L^2$ about the x-axis. Find its volume.

11. Find the volume of the solid whose base is the region bounded between the curves $y = \sqrt{x}$ and $y = 1/\sqrt{x}$ for $1 \leq x \leq 4$ and whose cross sections perpendicular to the x-axis are squares.

12. Consider the region enclosed by $y = \sin^{-1} x$, $y = 0$, and $x = 1$. Set up, but *do not evaluate*, an integral that represents the volume of the solid generated by revolving the region about the x-axis using
(a) disks (b) cylindrical shells.

13. Find the arc length in the second quadrant of the curve $x^{2/3} + y^{2/3} = 4$ from $x = -8$ to $x = -1$.

14. Let C be the curve $y = e^x$ between $x = 0$ and $x = \ln 10$. In each part, set up, but *do not evaluate*, an integral that solves the problem.
(a) Find the arc length of C by integrating with respect to x.
(b) Find the arc length of C by integrating with respect to y.

15. Find the area of the surface generated by revolving the curve $y = \sqrt{25 - x}$, $9 \leq x \leq 16$, about the x-axis.

16. Let C be the curve $27x - y^3 = 0$ between $y = 0$ and $y = 2$. In each part, set up, but *do not evaluate*, an integral or a sum of integrals that solves the problem.
(a) Find the area of the surface generated by revolving C about the x-axis by integrating with respect to x.
(b) Find the area of the surface generated by revolving C about the y-axis by integrating with respect to y.
(c) Find the area of the surface generated by revolving C about the line $y = -2$ by integrating with respect to y.

17. (a) A spring exerts a force of 0.5 N when stretched 0.25 m beyond its natural length. Assuming that Hooke's law applies, how much work was performed in stretching the spring to this length?
(b) How far beyond its natural length can the spring be stretched with 25 J of work?

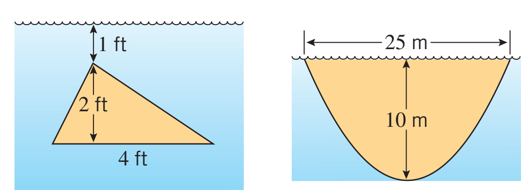
18. A boat is anchored so that the anchor is 150 ft below the surface of the water. In the water, the anchor weighs 2000 lb and the chain weighs 30 lb/ft. How much work is required to raise the anchor to the surface?

19–20 Find the centroid of the region. ■

19. The region bounded by $y^2 = 4x$ and $y^2 = 8(x - 2)$.

20. The upper half of the ellipse $(x/a)^2 + (y/b)^2 = 1$.

21. In each part, set up, but *do not evaluate*, an integral that solves the problem.
(a) Find the fluid force exerted on a side of a box that has a 3 m square base and is filled to a depth of 1 m with a liquid of weight density ρ N/m^3.
(b) Find the fluid force exerted by a liquid of weight density ρ lb/ft^3 on a face of the vertical plate shown in part (*a*) of the accompanying figure.
(c) Find the fluid force exerted on the parabolic dam in part (*b*) of the accompanying figure by water that extends to the top of the dam.



(*a*) (*b*)

▲ **Figure Ex-21**

22. Show that for any constant a, the function $y = \sinh(ax)$ satisfies the equation $y'' = a^2 y$.

23. In each part, prove the identity.
(a) $\cosh 3x = 4\cosh^3 x - 3\cosh x$
(b) $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$
(c) $\sinh \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(\cosh x - 1)}$

CHAPTER 6 MAKING CONNECTIONS

1. Suppose that f is a nonnegative function defined on [0, 1] such that the area between the graph of f and the interval [0, 1] is A_1 and such that the area of the region R between the graph of $g(x) = f(x^2)$ and the interval [0, 1] is A_2. In each part, express your answer in terms of A_1 and A_2.
(a) What is the volume of the solid of revolution generated by revolving R about the y-axis?
(b) Find a value of a such that if the xy-plane were horizontal, the region R would balance on the line $x = a$.

2. A water tank has the shape of a conical frustum with radius of the base 5 ft, radius of the top 10 ft and (vertical) height 15 ft. Suppose the tank is filled with water and consider the problem of finding the work required to pump all the water out through a hole in the top of the tank.
(a) Solve this problem using the method of Example 5 in Section 6.6.
(b) Solve this problem using Definition 6.6.3. [*Hint:* Think of the base as the head of a piston that expands to a water-tight fit against the sides of the tank as the piston is pushed upward. What important result about water pressure do you need to use?]

3. A disk of radius a is an inhomogeneous lamina whose density is a function $f(r)$ of the distance r to the center of the lamina. Modify the argument used to derive the method of cylindrical shells to find a formula for the mass of the lamina.

4. Compare Formula (10) in Section 6.7 with Formula (8) in Section 6.8. Then give a plausible argument that the force on a flat surface immersed vertically in a fluid of constant weight density is equal to the product of the area of the surface and the pressure at the centroid of the surface. Conclude that the force on the surface is the same as if the surface were immersed horizontally at the depth of the centroid.

5. ***Archimedes' Principle*** states that a solid immersed in a fluid experiences a buoyant force equal to the weight of the fluid displaced by the solid.
(a) Use the results of Section 6.8 to verify Archimedes' Principle in the case of (i) a box-shaped solid with a pair of faces parallel to the surface of the fluid, (ii) a solid cylinder with vertical axis, and (iii) a cylindrical shell with vertical axis.
(b) Give a plausible argument for Archimedes' Principle in the case of a solid of revolution immersed in fluid such that the axis of revolution of the solid is vertical. [*Hint:* Approximate the solid by a union of cylindrical shells and use the result from part (a).]

Dwight R. Kuhn

10 PARAMETRIC AND POLAR CURVES; CONIC SECTIONS

Mathematical curves, such as the spirals in the center of a sunflower, can be described conveniently using ideas developed in this chapter.

In this chapter we will study alternative ways of expressing curves in the plane. We will begin by studying parametric curves: curves described in terms of component functions. This study will include methods for finding tangent lines to parametric curves. We will then introduce polar coordinate systems and discuss methods for finding tangent lines to polar curves, arc length of polar curves, and areas enclosed by polar curves. Our attention will then turn to a review of the basic properties of conic sections: parabolas, ellipses, and hyperbolas. Finally, we will consider conic sections in the context of polar coordinates and discuss some applications in astronomy.

10.1 PARAMETRIC EQUATIONS; TANGENT LINES AND ARC LENGTH FOR PARAMETRIC CURVES

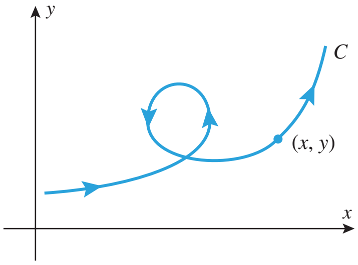
Graphs of functions must pass the vertical line test, a limitation that excludes curves with self-intersections or even such basic curves as circles. In this section we will study an alternative method for describing curves algebraically that is not subject to the severe restriction of the vertical line test. We will then derive formulas required to find slopes, tangent lines, and arc lengths of these parametric curves. We will conclude with an investigation of a classic parametric curve known as the cycloid.

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve C in the xy-plane in such a way that its x- and y-coordinates, as functions of time, are

$$x = f(t), \quad y = g(t)$$

We call these the ***parametric equations*** of motion for the particle and refer to C as the ***trajectory*** of the particle or the ***graph*** of the equations (Figure 10.1.1). The variable t is called the ***parameter*** for the equations.



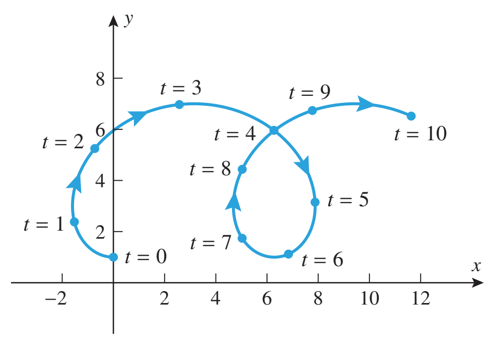
A moving particle with trajectory C

▲ **Figure 10.1.1**

▶ **Example 1** Sketch the trajectory over the time interval $0 \le t \le 10$ of the particle whose parametric equations of motion are

$$x = t - 3\sin t, \quad y = 4 - 3\cos t \tag{1}$$

Solution. One way to sketch the trajectory is to choose a representative succession of times, plot the (x, y) coordinates of points on the trajectory at those times, and connect the points with a smooth curve. The trajectory in Figure 10.1.2 was obtained in this way from the data in Table 10.1.1 in which the approximate coordinates of the particle are given at time increments of 1 unit. Observe that there is no t-axis in the picture; the values of t appear only as labels on the plotted points, and even these are usually omitted unless it is important to emphasize the locations of the particle at specific times. ◀



▲ **Figure 10.1.2**

Table 10.1.1

t	x	y
0	0.0	1.0
1	−1.5	2.4
2	−0.7	5.2
3	2.6	7.0
4	6.3	6.0
5	7.9	3.1
6	6.8	1.1
7	5.0	1.7
8	5.0	4.4
9	7.8	6.7
10	11.6	6.5

TECHNOLOGY MASTERY

Read the documentation for your graphing utility to learn how to graph parametric equations, and then generate the trajectory in Example 1. Explore the behavior of the particle beyond time $t = 10$.

Although parametric equations commonly arise in problems of motion with time as the parameter, they arise in other contexts as well. Thus, unless the problem dictates that the parameter t in the equations

$$x = f(t), \quad y = g(t)$$

represents time, it should be viewed simply as an independent variable that varies over some interval of real numbers. (In fact, there is no need to use the letter t for the parameter; any letter not reserved for another purpose can be used.) If no restrictions on the parameter are stated explicitly or implied by the equations, then it is understood that it varies from $-\infty$ to $+\infty$. To indicate that a parameter t is restricted to an interval $[a, b]$, we will write

$$x = f(t), \quad y = g(t) \qquad (a \le t \le b)$$

▶ **Example 2** Find the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi) \tag{2}$$

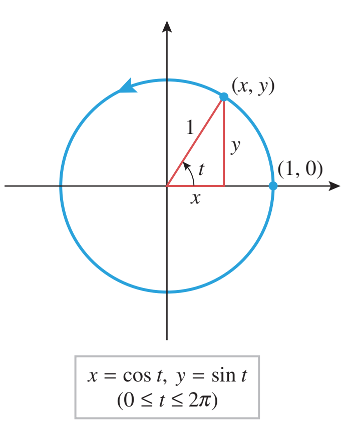
Solution. One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

Thus, the graph is contained in the unit circle $x^2 + y^2 = 1$. Geometrically, the parameter t can be interpreted as the angle swept out by the radial line from the origin to the point $(x, y) = (\cos t, \sin t)$ on the unit circle (Figure 10.1.3). As t increases from 0 to 2π, the point traces the circle counterclockwise, starting at $(1, 0)$ when $t = 0$ and completing one full revolution when $t = 2\pi$. One can obtain different portions of the circle by varying the interval over which the parameter varies. For example,

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le \pi) \tag{3}$$

represents just the upper semicircle in Figure 10.1.3. ◀



▲ **Figure 10.1.3**

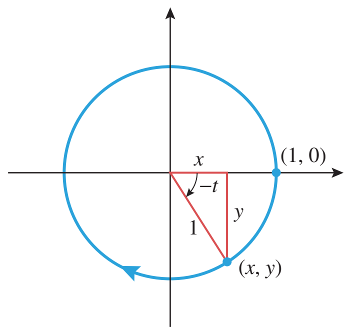
ORIENTATION

The direction in which the graph of a pair of parametric equations is traced as the parameter increases is called the ***direction of increasing parameter*** or sometimes the ***orientation*** imposed on the curve by the equations. Thus, we make a distinction between a ***curve***, which is a set of points, and a ***parametric curve***, which is a curve with an orientation imposed on it by a set of parametric equations. For example, we saw in Example 2 that the circle represented parametrically by (2) is traced counterclockwise as t increases and hence has *counterclockwise orientation*. As shown in Figures 10.1.2 and 10.1.3, the orientation of a parametric curve can be indicated by arrowheads.

To obtain parametric equations for the unit circle with *clockwise orientation*, we can replace t by $-t$ in (2) and use the identities $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. This yields

$$x = \cos t, \quad y = -\sin t \qquad (0 \le t \le 2\pi)$$

Here, the circle is traced clockwise by a point that starts at $(1, 0)$ when $t = 0$ and completes one full revolution when $t = 2\pi$ (Figure 10.1.4).



$x = \cos(-t),\ y = \sin(-t)$
$(0 \le t \le 2\pi)$

▲ **Figure 10.1.4**

TECHNOLOGY MASTERY

When parametric equations are graphed using a calculator, the orientation can often be determined by watching the direction in which the graph is traced on the screen. However, many computers graph so fast that it is often hard to discern the orientation. See if you can use your graphing utility to confirm that (3) has a counterclockwise orientation.

▶ **Example 3** Graph the parametric curve

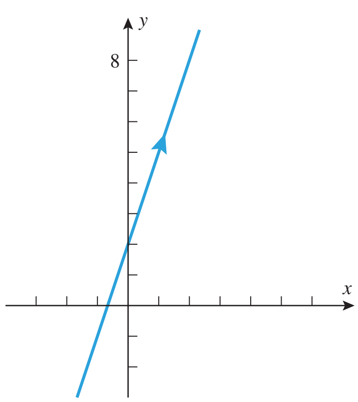
$$x = 2t - 3, \quad y = 6t - 7$$

by eliminating the parameter, and indicate the orientation on the graph.

Solution. To eliminate the parameter we will solve the first equation for t as a function of x, and then substitute this expression for t into the second equation:

$$t = \left(\tfrac{1}{2}\right)(x + 3)$$
$$y = 6\left(\tfrac{1}{2}\right)(x + 3) - 7$$
$$y = 3x + 2$$

Thus, the graph is a line of slope 3 and y-intercept 2. To find the orientation we must look to the original equations; the direction of increasing t can be deduced by observing that x increases as t increases *or* by observing that y increases as t increases. Either piece of information tells us that the line is traced left to right as shown in Figure 10.1.5. ◀



$x = 2t - 3,\ y = 6t - 7$

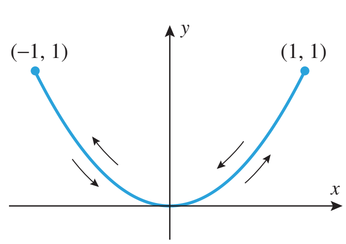
▲ **Figure 10.1.5**

REMARK

Not all parametric equations produce curves with definite orientations; if the equations are badly behaved, then the point tracing the curve may leap around sporadically or move back and forth, failing to determine a definite direction. For example, if

$$x = \sin t, \quad y = \sin^2 t$$

then the point (x, y) moves along the parabola $y = x^2$. However, the value of x varies periodically between -1 and 1, so the point (x, y) moves periodically back and forth along the parabola between the points $(-1, 1)$ and $(1, 1)$ (as shown in Figure 10.1.6). Later in the text we will discuss restrictions that eliminate such erratic behavior, but for now we will just avoid such complications.



▲ **Figure 10.1.6**

EXPRESSING ORDINARY FUNCTIONS PARAMETRICALLY

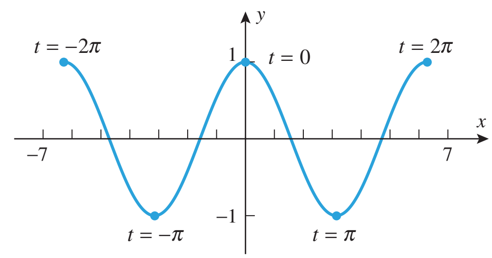
An equation $y = f(x)$ can be expressed in parametric form by introducing the parameter $t = x$; this yields the parametric equations

$$x = t, \quad y = f(t)$$

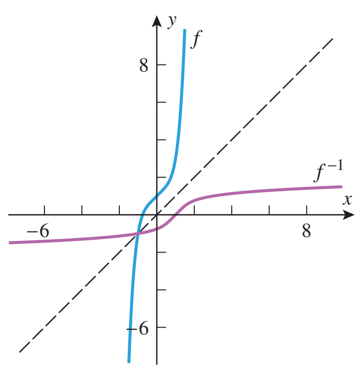
For example, the portion of the curve $y = \cos x$ over the interval $[-2\pi, 2\pi]$ can be expressed parametrically as

$$x = t, \quad y = \cos t \qquad (-2\pi \le t \le 2\pi)$$

(Figure 10.1.7).



▶ **Figure 10.1.7**



▲ **Figure 10.1.8**

If a function f is one-to-one, then it has an inverse function f^{-1}. In this case the equation $y = f^{-1}(x)$ is equivalent to $x = f(y)$. We can express the graph of f^{-1} in parametric form by introducing the parameter $y = t$; this yields the parametric equations

$$x = f(t), \quad y = t$$

For example, Figure 10.1.8 shows the graph of $f(x) = x^5 + x + 1$ and its inverse. The graph of f can be represented parametrically as

$$x = t, \quad y = t^5 + t + 1$$

and the graph of f^{-1} can be represented parametrically as

$$x = t^5 + t + 1, \quad y = t$$

TANGENT LINES TO PARAMETRIC CURVES

We will be concerned with curves that are given by parametric equations

$$x = f(t), \quad y = g(t)$$

in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $dx/dt \ne 0$, then y is a differentiable function of x, in which case the chain rule implies that

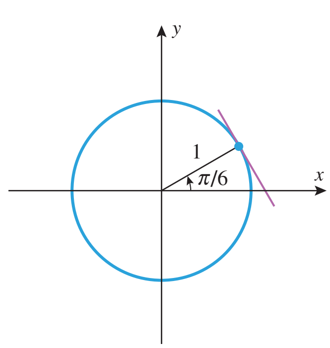
$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}} \tag{4}$$

This formula makes it possible to find dy/dx directly from the parametric equations without eliminating the parameter.

▶ **Example 4** Find the slope of the tangent line to the unit circle

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

at the point where $t = \pi/6$ (Figure 10.1.9).



▲ **Figure 10.1.9**

Solution. From (4), the slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t \tag{5}$$

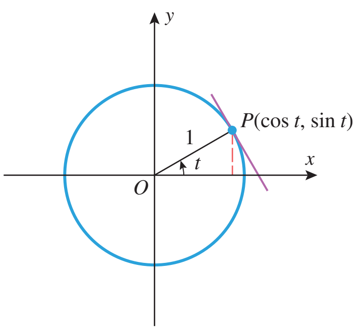
Thus, the slope at $t = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\cot\frac{\pi}{6} = -\sqrt{3}$$ ◀

Note that Formula (5) makes sense geometrically because the radius from the origin to the point $P(\cos t, \sin t)$ has slope $m = \tan t$. Thus the tangent line at P, being perpendicular to the radius, has slope

$$-\frac{1}{m} = -\frac{1}{\tan t} = -\cot t$$

(Figure 10.1.10).



Radius OP has slope $m = \tan t$.

Figure 10.1.10

Stanislovas Kairys/iStockphoto

The complicated motion of a paper airplane is best described mathematically using parametric equations.

It follows from Formula (4) that the tangent line to a parametric curve will be horizontal at those points where $dy/dt = 0$ and $dx/dt \neq 0$, since $dy/dx = 0$ at such points. Two different situations occur when $dx/dt = 0$. At points where $dx/dt = 0$ and $dy/dt \neq 0$, the right side of (4) has a nonzero numerator and a zero denominator; we will agree that the curve has ***infinite slope*** and a ***vertical tangent line*** at such points. At points where dx/dt and dy/dt are both zero, the right side of (4) becomes an indeterminate form; we call such points ***singular points***. No general statement can be made about the behavior of parametric curves at singular points; they must be analyzed case by case.

Example 5 In a disastrous first flight, an experimental paper airplane follows the trajectory of the particle in Example 1:

$$x = t - 3\sin t, \quad y = 4 - 3\cos t \qquad (t \geq 0)$$

but crashes into a wall at time $t = 10$ (Figure 10.1.11).

(a) At what times was the airplane flying horizontally?

(b) At what times was it flying vertically?

Solution (a). The airplane was flying horizontally at those times when $dy/dt = 0$ and $dx/dt \neq 0$. From the given trajectory we have

$$\frac{dy}{dt} = 3\sin t \quad \text{and} \quad \frac{dx}{dt} = 1 - 3\cos t \tag{6}$$

Setting $dy/dt = 0$ yields the equation $3\sin t = 0$, or, more simply, $\sin t = 0$. This equation has four solutions in the time interval $0 \leq t \leq 10$:

$$t = 0, \quad t = \pi, \quad t = 2\pi, \quad t = 3\pi$$

Since $dx/dt = 1 - 3\cos t \neq 0$ for these values of t (verify), the airplane was flying horizontally at times

$$t = 0, \quad t = \pi \approx 3.14, \quad t = 2\pi \approx 6.28, \quad \text{and} \quad t = 3\pi \approx 9.42$$

which is consistent with Figure 10.1.11.

Solution (b). The airplane was flying vertically at those times when $dx/dt = 0$ and $dy/dt \neq 0$. Setting $dx/dt = 0$ in (6) yields the equation

$$1 - 3\cos t = 0 \quad \text{or} \quad \cos t = \tfrac{1}{3}$$

This equation has three solutions in the time interval $0 \leq t \leq 10$ (Figure 10.1.12):

$$t = \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi - \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi + \cos^{-1}\tfrac{1}{3}$$

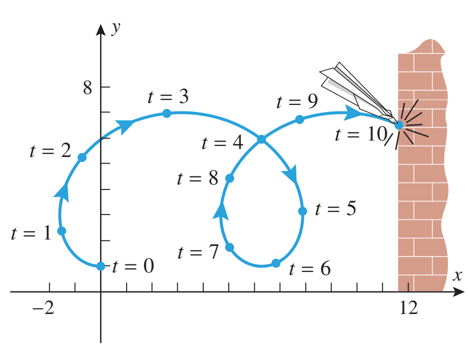


Figure 10.1.11

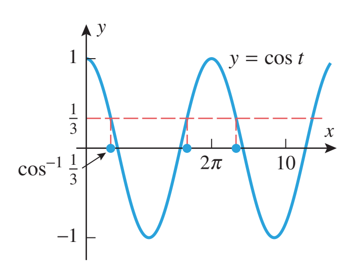
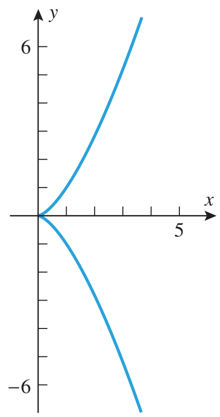


Figure 10.1.12

Since $dy/dt = 3 \sin t$ is not zero at these points (why?), it follows that the airplane was flying vertically at times

$$t = \cos^{-1} \tfrac{1}{3} \approx 1.23, \quad t \approx 2\pi - 1.23 \approx 5.05, \quad t \approx 2\pi + 1.23 \approx 7.51$$

which again is consistent with Figure 10.1.11. ◀



$x = t^2$, $y = t^3$
$(-\infty < t < +\infty)$

▲ **Figure 10.1.13**

WARNING

Although it is true that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

you cannot conclude that d^2y/dx^2 is the quotient of d^2y/dt^2 and d^2x/dt^2. To illustrate that this conclusion is erroneous, show that for the parametric curve in Example 7,

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} \neq \left.\frac{d^2y/dt^2}{d^2x/dt^2}\right|_{t=1}$$

▶ **Example 6** The curve represented by the parametric equations

$$x = t^2, \quad y = t^3 \qquad (-\infty < t < +\infty)$$

is called a ***semicubical parabola***. The parameter t can be eliminated by cubing x and squaring y, from which it follows that $y^2 = x^3$. The graph of this equation, shown in Figure 10.1.13, consists of two branches: an upper branch obtained by graphing $y = x^{3/2}$ and a lower branch obtained by graphing $y = -x^{3/2}$. The two branches meet at the origin, which corresponds to $t = 0$ in the parametric equations. This is a singular point because the derivatives $dx/dt = 2t$ and $dy/dt = 3t^2$ are both zero there. ◀

▶ **Example 7** Without eliminating the parameter, find dy/dx and d^2y/dx^2 at (1, 1) and (1, −1) on the semicubical parabola given by the parametric equations in Example 6.

Solution. From (4) we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \qquad (t \neq 0) \tag{7}$$

and from (4) applied to $y' = dy/dx$ we have

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t} \tag{8}$$

Since the point (1, 1) on the curve corresponds to $t = 1$ in the parametric equations, it follows from (7) and (8) that

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{3}{4}$$

Similarly, the point (1, −1) corresponds to $t = -1$ in the parametric equations, so applying (7) and (8) again yields

$$\left.\frac{dy}{dx}\right|_{t=-1} = -\frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=-1} = -\frac{3}{4}$$

Note that the values we obtained for the first and second derivatives are consistent with the graph in Figure 10.1.13, since at (1, 1) on the upper branch the tangent line has positive slope and the curve is concave up, and at (1, −1) on the lower branch the tangent line has negative slope and the curve is concave down.

Finally, observe that we were able to apply Formulas (7) and (8) for both $t = 1$ and $t = -1$, even though the points (1, 1) and (1, −1) lie on different branches. In contrast, had we chosen to perform the same computations by eliminating the parameter, we would have had to obtain separate derivative formulas for $y = x^{3/2}$ and $y = -x^{3/2}$. ◀

ARC LENGTH OF PARAMETRIC CURVES

The following result provides a formula for finding the arc length of a curve from parametric equations for the curve. Its derivation is similar to that of Formula (3) in Section 6.4 and will be omitted.

Formulas (4) and (5) in Section 6.4 can be viewed as special cases of (9). For example, Formula (4) in Section 6.4 can be obtained from (9) by writing $y = f(x)$ parametrically as

$$x = t, \quad y = f(t)$$

and Formula (5) in Section 6.4 can be obtained by writing $x = g(y)$ parametrically as

$$x = g(t), \quad y = t$$

10.1.1 ARC LENGTH FORMULA FOR PARAMETRIC CURVES If no segment of the curve represented by the parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

is traced more than once as t increases from a to b, and if dx/dt and dy/dt are continuous functions for $a \le t \le b$, then the arc length L of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{9}$$

Example 8 Use (9) to find the circumference of a circle of radius a from the parametric equations

$$x = a\cos t, \quad y = a\sin t \qquad (0 \le t \le 2\pi)$$

Solution.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$= \int_0^{2\pi} a\, dt = at\Big]_0^{2\pi} = 2\pi a$$

THE CYCLOID (THE APPLE OF DISCORD)

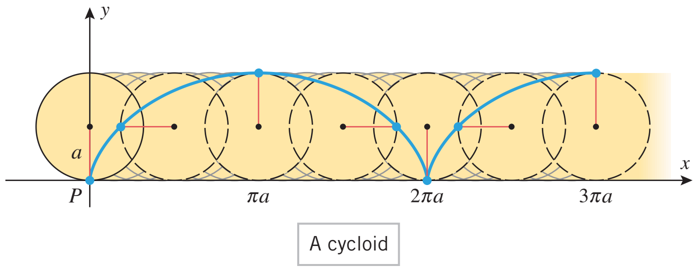
The results of this section can be used to investigate a curve known as a ***cycloid***. This curve, which is one of the most significant in the history of mathematics, can be generated by a point on a circle that rolls along a straight line (Figure 10.1.14). This curve has a fascinating history, which we will discuss shortly; but first we will show how to obtain parametric equations for it. For this purpose, let us assume that the circle has radius a and rolls along the positive x-axis of a rectangular coordinate system. Let $P(x, y)$ be the point on the circle that traces the cycloid, and assume that P is initially at the origin. We will take as our parameter the angle θ that is swept out by the radial line to P as the circle rolls (Figure 10.1.14). It is standard here to regard θ as positive, even though it is generated by a clockwise rotation.

The motion of P is a combination of the movement of the circle's center parallel to the x-axis and the rotation of P about the center. As the radial line sweeps out an angle θ, the point P traverses an arc of length $a\theta$, and the circle moves a distance $a\theta$ along the x-axis. Thus, as suggested by Figure 10.1.15, the center moves to the point $(a\theta, a)$, and the coordinates of P are

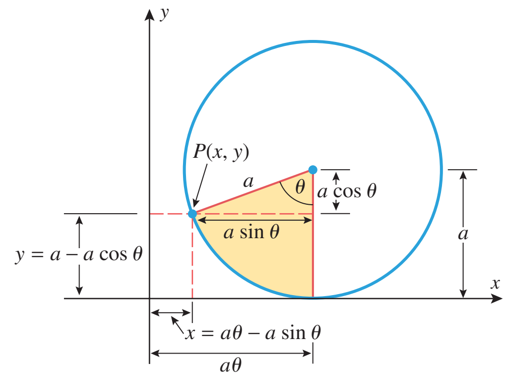
$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta \tag{10}$$

These are the equations of the cycloid in terms of the parameter θ.

One of the reasons the cycloid is important in the history of mathematics is that the study of its properties helped to spur the development of early versions of differentiation and integration. Work on the cycloid was carried out by some of the most famous names in seventeenth century mathematics, including Johann and Jakob Bernoulli, Descartes, L'Hôpital, Newton, and Leibniz. The curve was named the "cycloid" by the Italian mathematician and astronomer, Galileo, who spent over 40 years investigating its properties. An early problem of interest was that of constructing tangent lines to the cycloid. This problem was first solved by Descartes, and then by Fermat, whom Descartes had challenged with the question. A modern solution to this problem follows directly from the parametric equations (10) and Formula (4). For example, using Formula (4), it is straightforward to show that the x-intercepts of the cycloid are cusps and that there is a horizontal tangent line to the cycloid halfway between adjacent x-intercepts (Exercise 60).



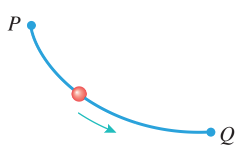
▲ Figure 10.1.14



▲ Figure 10.1.15

Another early problem was determining the arc length of an arch of the cycloid. This was solved in 1658 by the famous British architect and mathematician, Sir Christopher Wren. He showed that the arc length of one arch of the cycloid is exactly eight times the radius of the generating circle. [For a solution to this problem using Formula (9), see Exercise 71.]

The cycloid is also important historically because it provides the solution to two famous mathematical problems—the ***brachistochrone problem*** (from Greek words meaning "shortest time") and the ***tautochrone problem*** (from Greek words meaning "equal time"). The brachistochrone problem is to determine the shape of a wire along which a bead might slide from a point P to another point Q, not directly below, in the *shortest time*. The tautochrone problem is to find the shape of a wire from P to Q such that two beads started at any points on the wire between P and Q reach Q in the *same amount of time*. The solution to both problems turns out to be an inverted cycloid (Figure 10.1.16).

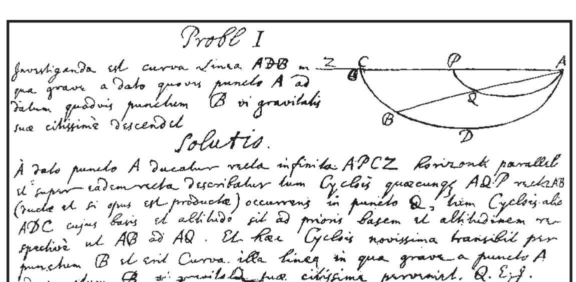


▲ Figure 10.1.16

In June of 1696, Johann Bernoulli posed the brachistochrone problem in the form of a challenge to other mathematicians. At first, one might conjecture that the wire should form a straight line, since that shape results in the shortest distance from P to Q. However, the inverted cycloid allows the bead to fall more rapidly at first, building up sufficient speed to reach Q in the shortest time, even though it travels a longer distance. The problem was solved by Newton, Leibniz, and L'Hôpital, as well as by Johann Bernoulli and his older brother Jakob; it was formulated and solved *incorrectly* years earlier by Galileo, who thought the answer was a circular arc. In fact, Johann was so impressed with his brother Jakob's solution that he claimed it to be his own. (This was just one of many disputes about the cycloid that eventually led to the curve being known as the "apple of discord.") One solution of the brachistochrone problem leads to the differential equation

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right) y = 2a \tag{11}$$

where a is a positive constant. We leave it as an exercise (Exercise 72) to show that the cycloid provides a solution to this differential equation.



Probl I

Investiganda est curva Linea ADB in qua grave a dato quovis puncto A ad datum quodvis punctum B vi gravitatis suae citissime descendet.

Solutio.

A dato puncto A ducatur recta infinita APCZ Horizonti parallela et super eadem recta describatur tum Cyclois quaecunque AQP rectae AB (ductae et si opus est productae) occurrens in puncto Q, tum Cyclois alia ADC cujus basis et altitudo sit ad prioris basem et altitudinem respective ut AB ad AQ. Et haec Cyclois novissima transibit per punctum B et erit Curva illa linea in qua grave a puncto A ad punctum B vi gravitatis suae citissime perveniet. Q.E.I.

Newton's solution of the brachistochrone problem in his own handwriting

✔ QUICK CHECK EXERCISES 10.1

(See page 705 for answers.)

1. Find parametric equations for a circle of radius 2, centered at (3, 5).
2. The graph of the curve described by the parametric equations $x = 4t - 1$, $y = 3t + 2$ is a straight line with slope ________ and y-intercept ________.
3. Suppose that a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$ for $0 \le t \le 1$. Find parametric equations for C that reverse the direction the curve is traced as the parameter increases from 0 to 1.
4. To find dy/dx directly from the parametric equations
$$x = f(t), \quad y = g(t)$$
we can use the formula $dy/dx =$ ________.
5. Let L be the length of the curve
$$x = \ln t, \quad y = \sin t \qquad (1 \le t \le \pi)$$
An integral expression for L is ________.

EXERCISE SET 10.1 Graphing Utility C CAS

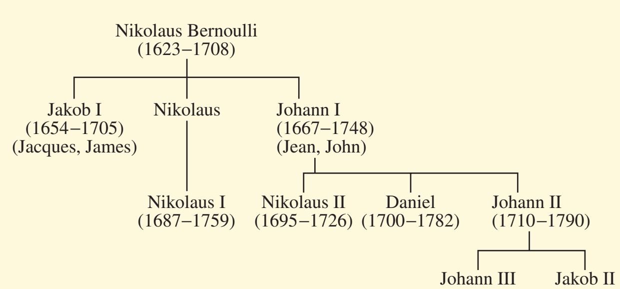
1. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 5$ of the particle whose parametric equations of motion are
$$x = t - 1, \quad y = t + 1$$
(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.

Johann (left) **and Jakob** (right) **Bernoulli** Members of an amazing Swiss family that included several generations of outstanding mathematicians and scientists. Nikolaus Bernoulli (1623–1708), a druggist, fled from Antwerp to escape religious persecution and ultimately settled in Basel, Switzerland. There he had three sons, Jakob I (also called Jacques or James), Nikolaus, and Johann I (also called Jean or John). The Roman numerals are used to distinguish family members with identical names (see the family tree below). Following Newton and Leibniz, the Bernoulli brothers, Jakob I and Johann I, are considered by some to be the two most important founders of calculus. Jakob I was self-taught in mathematics. His father wanted him to study for the ministry, but he turned to mathematics and in 1686 became a professor at the University of Basel. When he started working in mathematics, he knew nothing of Newton's and Leibniz' work. He eventually became familiar with Newton's results, but because so little of Leibniz' work was published, Jakob duplicated many of Leibniz' results.

Jakob's younger brother Johann I was urged to enter into business by his father. Instead, he turned to medicine and studied mathematics under the guidance of his older brother. He eventually became a mathematics professor at Gröningen in Holland, and then, when Jakob died in 1705, Johann succeeded him as mathematics professor at Basel. Throughout their lives, Jakob I and Johann I had a mutual passion for criticizing each other's work, which frequently erupted into ugly confrontations. Leibniz tried to mediate the disputes, but Jakob, who resented Leibniz' superior intellect, accused him of siding with Johann, and thus Leibniz became entangled in the arguments. The brothers often worked on common problems that they posed as challenges to one another. Johann, interested in gaining fame, often used unscrupulous means to make himself appear the originator of his brother's results; Jakob occasionally retaliated. Thus, it is often difficult to determine who deserves credit for many results. However, both men made major contributions to the development of calculus. In addition to his work on calculus, Jakob helped establish fundamental principles in probability, including the Law of Large Numbers, which is a cornerstone of modern probability theory.

Among the other members of the Bernoulli family, Daniel, son of Johann I, is the most famous. He was a professor of mathematics at St. Petersburg Academy in Russia and subsequently a professor of anatomy and then physics at Basel. He did work in calculus and probability, but is best known for his work in physics. A basic law of fluid flow, called Bernoulli's principle, is named in his honor. He won the annual prize of the French Academy 10 times for work on vibrating strings, tides of the sea, and kinetic theory of gases.

Johann II succeeded his father as professor of mathematics at Basel. His research was on the theory of heat and sound. Nikolaus I was a mathematician and law scholar who worked on probability and series. On the recommendation of Leibniz, he was appointed professor of mathematics at Padua and then went to Basel as a professor of logic and then law. Nikolaus II was professor of jurisprudence in Switzerland and then professor of mathematics at St. Petersburg Academy. Johann III was a professor of mathematics and astronomy in Berlin and Jakob II succeeded his uncle Daniel as professor of mathematics at St. Petersburg Academy in Russia. Truly an incredible family!



2. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 1$ of the particle whose parametric equations of motion are

$$x = \cos(\pi t), \quad y = \sin(\pi t)$$

(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the particle at times $t = 0, 0.25, 0.5, 0.75, 1$.
(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.

3–12 Sketch the curve by eliminating the parameter, and indicate the direction of increasing t. ■

3. $x = 3t - 4,\ y = 6t + 2$
4. $x = t - 3,\ y = 3t - 7 \quad (0 \le t \le 3)$
5. $x = 2\cos t,\ y = 5\sin t \quad (0 \le t \le 2\pi)$
6. $x = \sqrt{t},\ y = 2t + 4$
7. $x = 3 + 2\cos t,\ y = 2 + 4\sin t \quad (0 \le t \le 2\pi)$
8. $x = \sec t,\ y = \tan t \quad (\pi \le t < 3\pi/2)$
9. $x = \cos 2t,\ y = \sin t \quad (-\pi/2 \le t \le \pi/2)$
10. $x = 4t + 3,\ y = 16t^2 - 9$
11. $x = 2\sin^2 t,\ y = 3\cos^2 t \quad (0 \le t \le \pi/2)$
12. $x = \sec^2 t,\ y = \tan^2 t \quad (0 \le t < \pi/2)$

13–18 Find parametric equations for the curve, and check your work by generating the curve with a graphing utility. ■

13. A circle of radius 5, centered at the origin, oriented clockwise.
14. The portion of the circle $x^2 + y^2 = 1$ that lies in the third quadrant, oriented counterclockwise.
15. A vertical line intersecting the x-axis at $x = 2$, oriented upward.
16. The ellipse $x^2/4 + y^2/9 = 1$, oriented counterclockwise.
17. The portion of the parabola $x = y^2$ joining $(1, -1)$ and $(1, 1)$, oriented down to up.
18. The circle of radius 4, centered at $(1, -3)$, oriented counterclockwise.

19. (a) Use a graphing utility to generate the trajectory of a particle whose equations of motion over the time interval $0 \le t \le 5$ are

$$x = 6t - \tfrac{1}{2}t^3, \quad y = 1 + \tfrac{1}{2}t^2$$

(b) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(c) At what times is the particle on the y-axis?
(d) During what time interval is $y < 5$?
(e) At what time does the x-coordinate of the particle reach a maximum?

20. (a) Use a graphing utility to generate the trajectory of a paper airplane whose equations of motion for $t \ge 0$ are

$$x = t - 2\sin t, \quad y = 3 - 2\cos t$$

(b) Assuming that the plane flies in a room in which the floor is at $y = 0$, explain why the plane will not crash into the floor. [For simplicity, ignore the physical size of the plane by treating it as a particle.]
(c) How high must the ceiling be to ensure that the plane does not touch or crash into it?

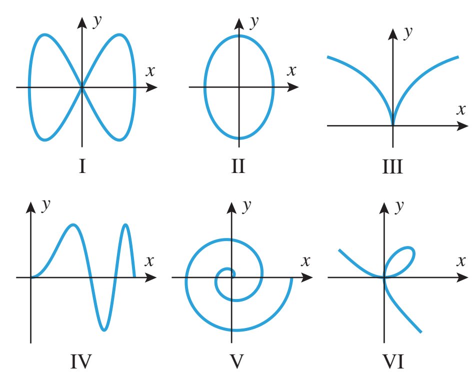
21–22 Graph the equation using a graphing utility. ■

21. (a) $x = y^2 + 2y + 1$
(b) $x = \sin y,\ -2\pi \le y \le 2\pi$

22. (a) $x = y + 2y^3 - y^5$
(b) $x = \tan y,\ -\pi/2 < y < \pi/2$

FOCUS ON CONCEPTS

23. In each part, match the parametric equation with one of the curves labeled (I)–(VI), and explain your reasoning.
(a) $x = \sqrt{t},\ y = \sin 3t$ (b) $x = 2\cos t,\ y = 3\sin t$
(c) $x = t\cos t,\ y = t\sin t$
(d) $x = \dfrac{3t}{1+t^3},\ y = \dfrac{3t^2}{1+t^3}$
(e) $x = \dfrac{t^3}{1+t^2},\ y = \dfrac{2t^2}{1+t^2}$
(f) $x = \tfrac{1}{2}\cos t,\ y = \sin 2t$



▲ **Figure Ex-23**

24. (a) Identify the orientation of the curves in Exercise 23.
(b) Explain why the parametric curve

$$x = t^2, \quad y = t^4 \quad (-1 \le t \le 1)$$

does not have a definite orientation.

25. (a) Suppose that the line segment from the point $P(x_0, y_0)$ to $Q(x_1, y_1)$ is represented parametrically by

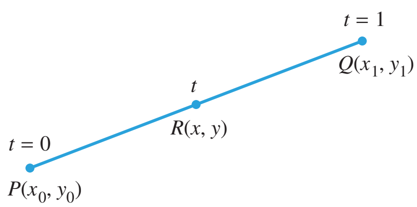
$$\begin{aligned} x &= x_0 + (x_1 - x_0)t, \\ y &= y_0 + (y_1 - y_0)t \end{aligned} \quad (0 \le t \le 1)$$

and that $R(x, y)$ is the point on the line segment corresponding to a specified value of t (see the accompanying figure on the next page). Show that $t = r/q$, where r is the distance from P to R and q is the distance from P to Q.

(cont.)

(b) What value of t produces the midpoint between points P and Q?

(c) What value of t produces the point that is three-fourths of the way from P to Q?



◀ **Figure Ex-25**

26. Find parametric equations for the line segment joining $P(2, -1)$ and $Q(3, 1)$, and use the result in Exercise 25 to find
(a) the midpoint between P and Q
(b) the point that is one-fourth of the way from P to Q
(c) the point that is three-fourths of the way from P to Q.

27. (a) Show that the line segment joining the points (x_0, y_0) and (x_1, y_1) can be represented parametrically as

$$x = x_0 + (x_1 - x_0)\frac{t - t_0}{t_1 - t_0}, \qquad (t_0 \le t \le t_1)$$
$$y = y_0 + (y_1 - y_0)\frac{t - t_0}{t_1 - t_0}$$

(b) Which way is the line segment oriented?

(c) Find parametric equations for the line segment traced from $(3, -1)$ to $(1, 4)$ as t varies from 1 to 2, and check your result with a graphing utility.

28. (a) By eliminating the parameter, show that if a and c are not both zero, then the graph of the parametric equations

$$x = at + b, \quad y = ct + d \qquad (t_0 \le t \le t_1)$$

is a line segment.

(b) Sketch the parametric curve

$$x = 2t - 1, \quad y = t + 1 \qquad (1 \le t \le 2)$$

and indicate its orientation.

(c) What can you say about the line in part (a) if a or c (but not both) is zero?

(d) What do the equations represent if a and c are both zero?

29–32 Use a graphing utility and parametric equations to display the graphs of f and f^{-1} on the same screen. ■

29. $f(x) = x^3 + 0.2x - 1, \quad -1 \le x \le 2$

30. $f(x) = \sqrt{x^2 + 2} + x, \quad -5 \le x \le 5$

31. $f(x) = \cos(\cos 0.5x), \quad 0 \le x \le 3$

32. $f(x) = x + \sin x, \quad 0 \le x \le 6$

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. The equation $y = 1 - x^2$ can be described parametrically by $x = \sin t$, $y = \cos^2 t$.

34. The graph of the parametric equations $x = f(t)$, $y = t$ is the reflection of the graph of $y = f(x)$ about the x-axis.

35. For the parametric curve $x = x(t)$, $y = 3t^4 - 2t^3$, the derivative of y with respect to x is computed by

$$\frac{dy}{dx} = \frac{12t^3 - 6t^2}{x'(t)}$$

36. The curve represented by the parametric equations

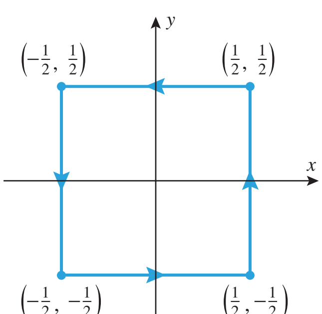
$$x = t^3, \quad y = t + t^6 \qquad (-\infty < t < +\infty)$$

is concave down for $t < 0$.

37. Parametric curves can be defined piecewise by using different formulas for different values of the parameter. Sketch the curve that is represented piecewise by the parametric equations

$$\begin{cases} x = 2t, & y = 4t^2 \qquad \left(0 \le t \le \frac{1}{2}\right) \\ x = 2 - 2t, & y = 2t \qquad \left(\frac{1}{2} \le t \le 1\right) \end{cases}$$

38. Find parametric equations for the rectangle in the accompanying figure, assuming that the rectangle is traced counterclockwise as t varies from 0 to 1, starting at $\left(\frac{1}{2}, \frac{1}{2}\right)$ when $t = 0$. [*Hint:* Represent the rectangle piecewise, letting t vary from 0 to $\frac{1}{4}$ for the first edge, from $\frac{1}{4}$ to $\frac{1}{2}$ for the second edge, and so forth.]



◀ **Figure Ex-38**

39. (a) Find parametric equations for the ellipse that is centered at the origin and has intercepts $(4, 0)$, $(-4, 0)$, $(0, 3)$, and $(0, -3)$.

(b) Find parametric equations for the ellipse that results by translating the ellipse in part (a) so that its center is at $(-1, 2)$.

(c) Confirm your results in parts (a) and (b) using a graphing utility.

40. We will show later in the text that if a projectile is fired from ground level with an initial speed of v_0 meters per second at an angle α with the horizontal, and if air resistance is neglected, then its position after t seconds, relative to the coordinate system in the accompanying figure on the next page is

$$x = (v_0 \cos\alpha)t, \quad y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

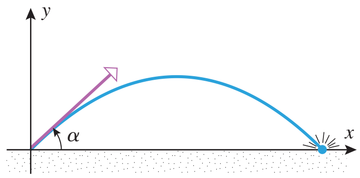
where $g \approx 9.8\ \text{m/s}^2$.

(a) By eliminating the parameter, show that the trajectory lies on the graph of a quadratic polynomial.

(b) Use a graphing utility to sketch the trajectory if $\alpha = 30°$ and $v_0 = 1000\ \text{m/s}$.

(c) Using the trajectory in part (b), how high does the shell rise? (cont.)

(d) Using the trajectory in part (b), how far does the shell travel horizontally?



◀ **Figure Ex-40**

FOCUS ON CONCEPTS

41. (a) Find the slope of the tangent line to the parametric curve $x = t/2$, $y = t^2 + 1$ at $t = -1$ and at $t = 1$ without eliminating the parameter.
(b) Check your answers in part (a) by eliminating the parameter and differentiating an appropriate function of x.

42. (a) Find the slope of the tangent line to the parametric curve $x = 3\cos t$, $y = 4\sin t$ at $t = \pi/4$ and at $t = 7\pi/4$ without eliminating the parameter.
(b) Check your answers in part (a) by eliminating the parameter and differentiating an appropriate function of x.

43. For the parametric curve in Exercise 41, make a conjecture about the sign of d^2y/dx^2 at $t = -1$ and at $t = 1$, and confirm your conjecture without eliminating the parameter.

44. For the parametric curve in Exercise 42, make a conjecture about the sign of d^2y/dx^2 at $t = \pi/4$ and at $t = 7\pi/4$, and confirm your conjecture without eliminating the parameter.

45–50 Find dy/dx and d^2y/dx^2 at the given point without eliminating the parameter. ■

45. $x = \sqrt{t}$, $y = 2t + 4$; $t = 1$

46. $x = \frac{1}{2}t^2 + 1$, $y = \frac{1}{3}t^3 - t$; $t = 2$

47. $x = \sec t$, $y = \tan t$; $t = \pi/3$

48. $x = \sinh t$, $y = \cosh t$; $t = 0$

49. $x = \theta + \cos\theta$, $y = 1 + \sin\theta$; $\theta = \pi/6$

50. $x = \cos\phi$, $y = 3\sin\phi$; $\phi = 5\pi/6$

51. (a) Find the equation of the tangent line to the curve
$$x = e^t, \quad y = e^{-t}$$
at $t = 1$ without eliminating the parameter.
(b) Find the equation of the tangent line in part (a) by eliminating the parameter.

52. (a) Find the equation of the tangent line to the curve
$$x = 2t + 4, \quad y = 8t^2 - 2t + 4$$
at $t = 1$ without eliminating the parameter.
(b) Find the equation of the tangent line in part (a) by eliminating the parameter.

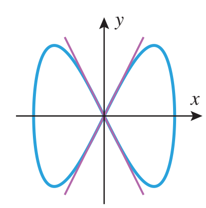
53–54 Find all values of t at which the parametric curve has (a) a horizontal tangent line and (b) a vertical tangent line. ■

53. $x = 2\sin t$, $y = 4\cos t$ $(0 \le t \le 2\pi)$

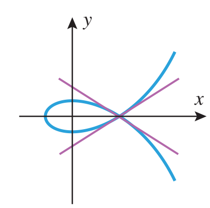
54. $x = 2t^3 - 15t^2 + 24t + 7$, $y = t^2 + t + 1$

55. In the mid-1850s the French physicist Jules Antoine Lissajous (1822–1880) became interested in parametric equations of the form
$$x = \sin at, \quad y = \sin bt$$
in the course of studying vibrations that combine two perpendicular sinusoidal motions. If a/b is a rational number, then the combined effect of the oscillations is a periodic motion along a path called a ***Lissajous curve***.
(a) Use a graphing utility to generate the complete graph of the Lissajous curves corresponding to $a = 1, b = 2$; $a = 2, b = 3$; $a = 3, b = 4$; and $a = 4, b = 5$.
(b) The Lissajous curve
$$x = \sin t, \quad y = \sin 2t \qquad (0 \le t \le 2\pi)$$
crosses itself at the origin (see Figure Ex-55). Find equations for the two tangent lines at the origin.

56. The ***prolate cycloid***
$$x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t \qquad (-\pi \le t \le \pi)$$
crosses itself at a point on the x-axis (see the accompanying figure). Find equations for the two tangent lines at that point.



▲ **Figure Ex-55**



▲ **Figure Ex-56**

57. Show that the curve $x = t^2$, $y = t^3 - 4t$ intersects itself at the point $(4, 0)$, and find equations for the two tangent lines to the curve at the point of intersection.

58. Show that the curve with parametric equations
$$x = t^2 - 3t + 5, \quad y = t^3 + t^2 - 10t + 9$$
intersects itself at the point $(3, 1)$, and find equations for the two tangent lines to the curve at the point of intersection.

59. (a) Use a graphing utility to generate the graph of the parametric curve
$$x = \cos^3 t, \quad y = \sin^3 t \qquad (0 \le t \le 2\pi)$$
and make a conjecture about the values of t at which singular points occur.
(b) Confirm your conjecture in part (a) by calculating appropriate derivatives.

60. Verify that the cycloid described by Formula (10) has cusps at its x-intercepts and horizontal tangent lines at midpoints between adjacent x-intercepts (see Figure 10.1.14).

61. (a) What is the slope of the tangent line at time t to the trajectory of the paper airplane in Example 5?
(b) What was the airplane's approximate angle of inclination when it crashed into the wall?

62. Suppose that a bee follows the trajectory

$$x = t - 2\cos t, \quad y = 2 - 2\sin t \qquad (0 \le t \le 10)$$

(a) At what times was the bee flying horizontally?
(b) At what times was the bee flying vertically?

63. Consider the family of curves described by the parametric equations

$$x = a\cos t + h, \quad y = b\sin t + k \qquad (0 \le t < 2\pi)$$

where $a \neq 0$ and $b \neq 0$. Describe the curves in this family if
(a) h and k are fixed but a and b can vary
(b) a and b are fixed but h and k can vary
(c) $a = 1$ and $b = 1$, but h and k vary so that $h = k + 1$.

64. (a) Use a graphing utility to study how the curves in the family

$$x = 2a\cos^2 t, \quad y = 2a\cos t\sin t \qquad (-2\pi < t < 2\pi)$$

change as a varies from 0 to 5.
(b) Confirm your conclusion algebraically.
(c) Write a brief paragraph that describes your findings.

65–70 Find the exact arc length of the curve over the stated interval. ■

65. $x = t^2,\ y = \frac{1}{3}t^3 \quad (0 \le t \le 1)$

66. $x = \sqrt{t} - 2,\ y = 2t^{3/4} \quad (1 \le t \le 16)$

67. $x = \cos 3t,\ y = \sin 3t \quad (0 \le t \le \pi)$

68. $x = \sin t + \cos t,\ y = \sin t - \cos t \quad (0 \le t \le \pi)$

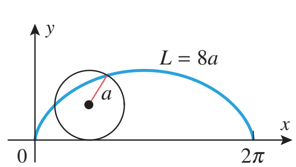
69. $x = e^{2t}(\sin t + \cos t),\ y = e^{2t}(\sin t - \cos t)\ (-1 \le t \le 1)$

70. $x = 2\sin^{-1} t,\ y = \ln(1 - t^2) \quad \left(0 \le t \le \frac{1}{2}\right)$

C **71.** (a) Use Formula (9) to show that the length L of one arch of a cycloid is given by

$$L = a\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

(b) Use a CAS to show that L is eight times the radius of the wheel that generates the cycloid (see the accompanying figure).



◄ **Figure Ex-71**

72. Use the parametric equations in Formula (10) to verify that the cycloid provides one solution to the differential equation

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right) y = 2a$$

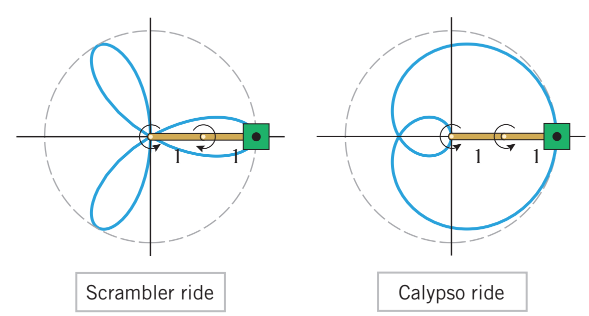
where a is a positive constant.

FOCUS ON CONCEPTS

73. The amusement park rides illustrated in the accompanying figure consist of two connected rotating arms of length 1—an inner arm that rotates counterclockwise at 1 radian per second and an outer arm that can be programmed to rotate either clockwise at 2 radians per second (the Scrambler ride) or counterclockwise at 2 radians per second (the Calypso ride). The center of the rider cage is at the end of the outer arm.
(a) Show that in the Scrambler ride the center of the cage has parametric equations

$$x = \cos t + \cos 2t, \quad y = \sin t - \sin 2t$$

(b) Find parametric equations for the center of the cage in the Calypso ride, and use a graphing utility to confirm that the center traces the curve shown in the accompanying figure.
(c) Do you think that a rider travels the same distance in one revolution of the Scrambler ride as in one revolution of the Calypso ride? Justify your conclusion.

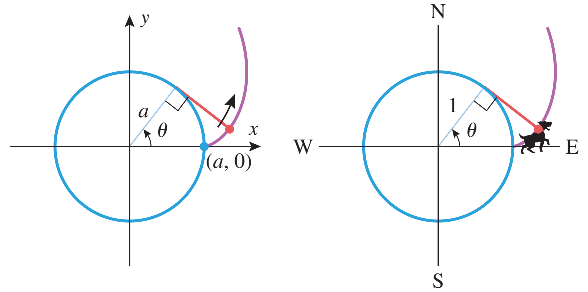


▲ **Figure Ex-73**

74. (a) If a thread is unwound from a fixed circle while being held taut (i.e., tangent to the circle), then the end of the thread traces a curve called an ***involute of a circle***. Show that if the circle is centered at the origin, has radius a, and the end of the thread is initially at the point $(a, 0)$, then the involute can be expressed parametrically as

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta)$$

where θ is the angle shown in part (*a*) of the accompanying figure on the next page.
(b) Assuming that the dog in part (*b*) of the accompanying figure on the next page unwinds its leash while keeping it taut, for what values of θ in the interval $0 \le \theta \le 2\pi$ will the dog be walking North? South? East? West?
(c) Use a graphing utility to generate the curve traced by the dog, and show that it is consistent with your answer in part (*b*).



(a) (b)

▲ **Figure Ex-74**

75–80 If $f'(t)$ and $g'(t)$ are continuous functions, and if no segment of the curve

$$x = f(t), \quad y = g(t) \qquad (a \le t \le b)$$

is traced more than once, then it can be shown that the area of the surface generated by revolving this curve about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and the area of the surface generated by revolving the curve about the y-axis is

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

[The derivations are similar to those used to obtain Formulas (4) and (5) in Section 6.5.] Use the formulas above in these exercises. ■

75. Find the area of the surface generated by revolving $x = t^2$, $y = 3t$ $(0 \le t \le 2)$ about the x-axis.

76. Find the area of the surface generated by revolving the curve $x = e^t \cos t$, $y = e^t \sin t$ $(0 \le t \le \pi/2)$ about the x-axis.

77. Find the area of the surface generated by revolving the curve $x = \cos^2 t$, $y = \sin^2 t$ $(0 \le t \le \pi/2)$ about the y-axis.

78. Find the area of the surface generated by revolving $x = 6t$, $y = 4t^2$ $(0 \le t \le 1)$ about the y-axis.

79. By revolving the semicircle

$$x = r\cos t, \quad y = r\sin t \qquad (0 \le t \le \pi)$$

about the x-axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

80. The equations

$$x = a\phi - a\sin\phi, \quad y = a - a\cos\phi \qquad (0 \le \phi \le 2\pi)$$

represent one arch of a cycloid. Show that the surface area generated by revolving this curve about the x-axis is given by $S = 64\pi a^2/3$.

81. Writing Consult appropriate reference works and write an essay on American mathematician Nathaniel Bowditch (1773–1838) and his investigation of ***Bowditch curves*** (better known as Lissajous curves; see Exercise 55).

82. Writing What are some of the advantages of expressing a curve parametrically rather than the form $y = f(x)$?

✔ QUICK CHECK ANSWERS 10.1

1. $x = 3 + 2\cos t,\ y = 5 + 2\sin t \quad (0 \le t \le 2\pi)$ **2.** $\frac{3}{4}$; 2.75 **3.** $x = f(1-t),\ y = g(1-t)$ **4.** $\dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$

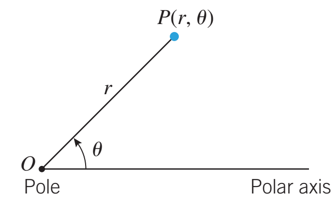
5. $\displaystyle\int_1^\pi \sqrt{(1/t)^2 + \cos^2 t}\, dt$

10.2 POLAR COORDINATES

Up to now we have specified the location of a point in the plane by means of coordinates relative to two perpendicular coordinate axes. However, sometimes a moving point has a special affinity for some fixed point, such as a planet moving in an orbit under the central attraction of the Sun. In such cases, the path of the particle is best described by its angular direction and its distance from the fixed point. In this section we will discuss a new kind of coordinate system that is based on this idea.

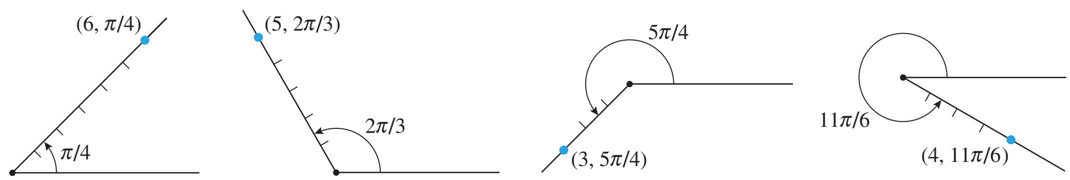
■ POLAR COORDINATE SYSTEMS

A ***polar coordinate system*** in a plane consists of a fixed point O, called the ***pole*** (or ***origin***), and a ray emanating from the pole, called the ***polar axis***. In such a coordinate system



▲ **Figure 10.2.1**

we can associate with each point P in the plane a pair of ***polar coordinates*** (r, θ), where r is the distance from P to the pole and θ is an angle from the polar axis to the ray OP (Figure 10.2.1). The number r is called the ***radial coordinate*** of P and the number θ the ***angular coordinate*** (or ***polar angle***) of P. In Figure 10.2.2, the points $(6, \pi/4)$, $(5, 2\pi/3)$, $(3, 5\pi/4)$, and $(4, 11\pi/6)$ are plotted in polar coordinate systems. If P is the pole, then $r = 0$, but there is no clearly defined polar angle. We will agree that an arbitrary angle can be used in this case; that is, $(0, \theta)$ are polar coordinates of the pole for all choices of θ.

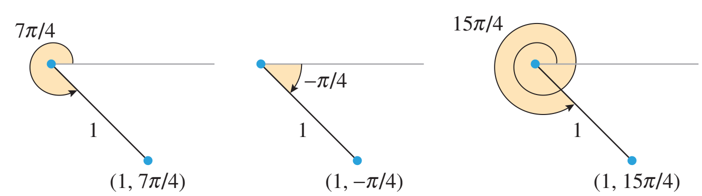


▲ **Figure 10.2.2**

The polar coordinates of a point are not unique. For example, the polar coordinates

$$(1, 7\pi/4), \quad (1, -\pi/4), \quad \text{and} \quad (1, 15\pi/4)$$

all represent the same point (Figure 10.2.3).



▶ **Figure 10.2.3**

In general, if a point P has polar coordinates (r, θ), then

$$(r, \theta + 2n\pi) \quad \text{and} \quad (r, \theta - 2n\pi)$$

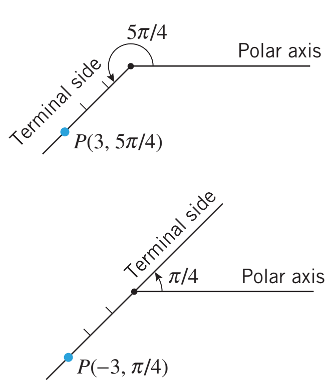
are also polar coordinates of P for any nonnegative integer n. Thus, every point has infinitely many pairs of polar coordinates.

As defined above, the radial coordinate r of a point P is nonnegative, since it represents the distance from P to the pole. However, it will be convenient to allow for negative values of r as well. To motivate an appropriate definition, consider the point P with polar coordinates $(3, 5\pi/4)$. As shown in Figure 10.2.4, we can reach this point by rotating the polar axis through an angle of $5\pi/4$ and then moving 3 units from the pole along the terminal side of the angle, or we can reach the point P by rotating the polar axis through an angle of $\pi/4$ and then moving 3 units from the pole along the extension of the terminal side. This suggests that the point $(3, 5\pi/4)$ might also be denoted by $(-3, \pi/4)$, with the minus sign serving to indicate that the point is on the *extension* of the angle's terminal side rather than on the terminal side itself.

In general, the terminal side of the angle $\theta + \pi$ is the extension of the terminal side of θ, so we define negative radial coordinates by agreeing that

$$(-r, \theta) \quad \text{and} \quad (r, \theta + \pi)$$

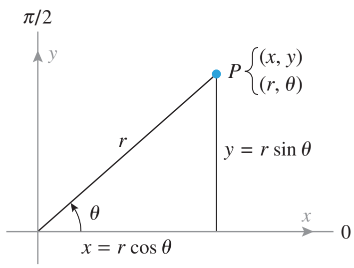
are polar coordinates of the same point.



▲ **Figure 10.2.4**

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

Frequently, it will be useful to superimpose a rectangular xy-coordinate system on top of a polar coordinate system, making the positive x-axis coincide with the polar axis. If this is done, then every point P will have both rectangular coordinates (x, y) and polar coordinates



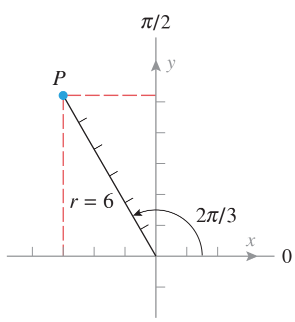
▲ **Figure 10.2.5**

(r, θ). As suggested by Figure 10.2.5, these coordinates are related by the equations

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

These equations are well suited for finding x and y when r and θ are known. However, to find r and θ when x and y are known, it is preferable to use the identities $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta/\cos\theta$ to rewrite (1) as

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \tag{2}$$



▲ **Figure 10.2.6**

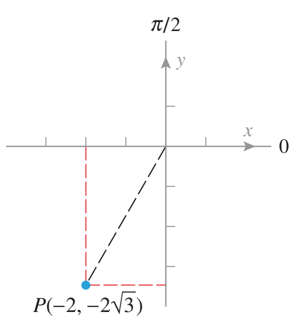
▶ **Example 1** Find the rectangular coordinates of the point P whose polar coordinates are $(r, \theta) = (6, 2\pi/3)$ (Figure 10.2.6).

Solution. Substituting the polar coordinates $r = 6$ and $\theta = 2\pi/3$ in (1) yields

$$x = 6\cos\frac{2\pi}{3} = 6\left(-\frac{1}{2}\right) = -3$$

$$y = 6\sin\frac{2\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

Thus, the rectangular coordinates of P are $(x, y) = (-3, 3\sqrt{3})$. ◀



▲ **Figure 10.2.7**

▶ **Example 2** Find polar coordinates of the point P whose rectangular coordinates are $(-2, -2\sqrt{3})$ (Figure 10.2.7).

Solution. We will find the polar coordinates (r, θ) of P that satisfy the conditions $r > 0$ and $0 \le \theta < 2\pi$. From the first equation in (2),

$$r^2 = x^2 + y^2 = (-2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16$$

so $r = 4$. From the second equation in (2),

$$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$$

From this and the fact that $(-2, -2\sqrt{3})$ lies in the third quadrant, it follows that the angle satisfying the requirement $0 \le \theta < 2\pi$ is $\theta = 4\pi/3$. Thus, $(r, \theta) = (4, 4\pi/3)$ are polar coordinates of P. All other polar coordinates of P are expressible in the form

$$\left(4, \frac{4\pi}{3} + 2n\pi\right) \quad \text{or} \quad \left(-4, \frac{\pi}{3} + 2n\pi\right)$$

where n is an integer. ◀

GRAPHS IN POLAR COORDINATES

We will now consider the problem of graphing equations in r and θ, where θ is assumed to be measured in radians. Some examples of such equations are

$$r = 1, \quad \theta = \pi/4, \quad r = \theta, \quad r = \sin\theta, \quad r = \cos 2\theta$$

In a rectangular coordinate system the graph of an equation in x and y consists of all points whose coordinates (x, y) satisfy the equation. However, in a polar coordinate system, points have infinitely many different pairs of polar coordinates, so that a given point may have some polar coordinates that satisfy an equation and others that do not. Given an equation in r and θ, we define its ***graph in polar coordinates*** to consist of all points with *at least one* pair of coordinates (r, θ) that satisfy the equation.

Example 3 Sketch the graphs of

$$\text{(a)}\ r = 1 \qquad \text{(b)}\ \theta = \frac{\pi}{4}$$

in polar coordinates.

Solution (a). For all values of θ, the point $(1, \theta)$ is 1 unit away from the pole. Since θ is arbitrary, the graph is the circle of radius 1 centered at the pole (Figure 10.2.8*a*).

Solution (b). For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure 10.2.8*b*). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \geq 0$, the graph would have been just the ray in the first quadrant. ◀

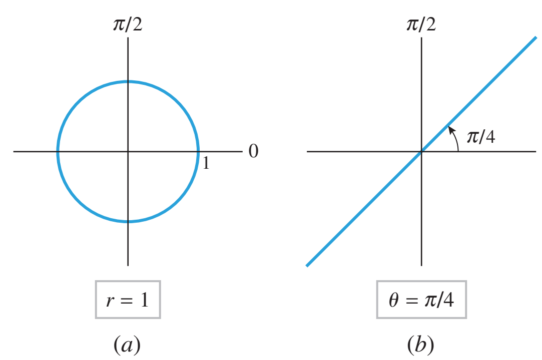


Figure 10.2.8

Equations $r = f(\theta)$ that express r as a function of θ are especially important. One way to graph such an equation is to choose some typical values of θ, calculate the corresponding values of r, and then plot the resulting pairs (r, θ) in a polar coordinate system. The next two examples illustrate this process.

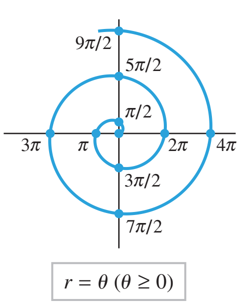


Figure 10.2.9

Example 4 Sketch the graph of $r = \theta$ $(\theta \geq 0)$ in polar coordinates by plotting points.

Solution. Observe that as θ increases, so does r; thus, the graph is a curve that spirals out from the pole as θ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the points that correspond to values of θ that are integer multiples of $\pi/2$, keeping in mind that the value of r is always equal to the value of θ (Figure 10.2.9). ◀

Graph the spiral $r = \theta$ $(\theta \leq 0)$. Compare your graph to that in Figure 10.2.9.

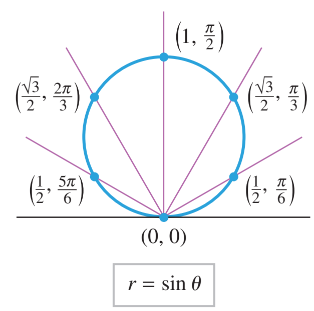
Example 5 Sketch the graph of the equation $r = \sin\theta$ in polar coordinates by plotting points.

Solution. Table 10.2.1 shows the coordinates of points on the graph at increments of $\pi/6$.

These points are plotted in Figure 10.2.10. Note, however, that there are 13 points listed in the table but only 6 distinct plotted points. This is because the pairs from $\theta = \pi$ on yield duplicates of the preceding points. For example, $(-1/2, 7\pi/6)$ and $(1/2, \pi/6)$ represent the same point. ◂

Table 10.2.1

θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
(r, θ)	$(0, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(0, \pi)$	$\left(-\frac{1}{2}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$	$\left(-1, \frac{3\pi}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$



▲ **Figure 10.2.10**

Observe that the points in Figure 10.2.10 appear to lie on a circle. We can confirm that this is so by expressing the polar equation $r = \sin\theta$ in terms of x and y. To do this, we multiply the equation through by r to obtain

$$r^2 = r\sin\theta$$

which now allows us to apply Formulas (1) and (2) to rewrite the equation as

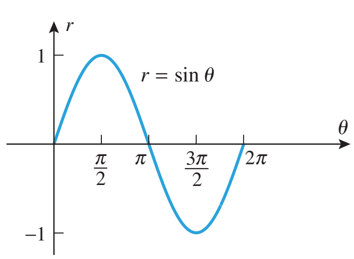
$$x^2 + y^2 = y$$

Rewriting this equation as $x^2 + y^2 - y = 0$ and then completing the square yields

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

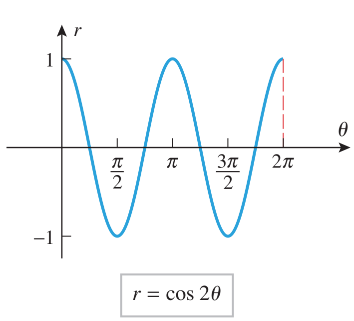
which is a circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$ in the xy-plane.

It is often useful to view the equation $r = f(\theta)$ as an equation in rectangular coordinates (rather than polar coordinates) and graphed in a rectangular θr-coordinate system. For example, Figure 10.2.11 shows the graph of $r = \sin\theta$ displayed using rectangular θr-coordinates. This graph can actually help to visualize how the polar graph in Figure 10.2.10 is generated:



▲ **Figure 10.2.11**

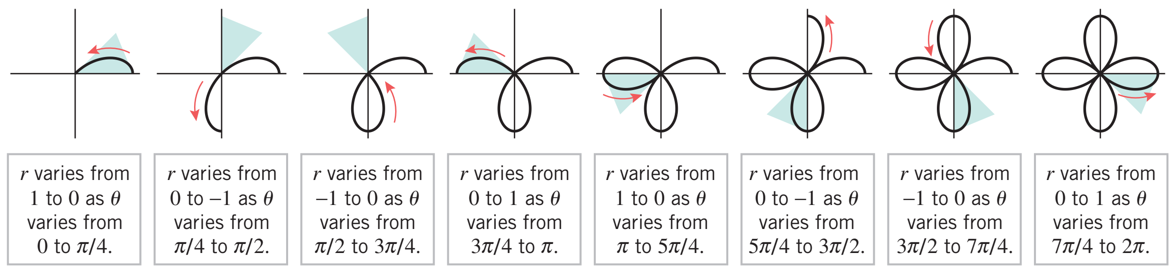
- At $\theta = 0$ we have $r = 0$, which corresponds to the pole $(0, 0)$ on the polar graph.
- As θ varies from 0 to $\pi/2$, the value of r increases from 0 to 1, so the point (r, θ) moves along the circle from the pole to the high point at $(1, \pi/2)$.
- As θ varies from $\pi/2$ to π, the value of r decreases from 1 back to 0, so the point (r, θ) moves along the circle from the high point back to the pole.
- As θ varies from π to $3\pi/2$, the values of r are negative, varying from 0 to -1. Thus, the point (r, θ) moves along the circle from the pole to the high point at $(1, \pi/2)$, which is the same as the point $(-1, 3\pi/2)$. This duplicates the motion that occurred for $0 \le \theta \le \pi/2$.
- As θ varies from $3\pi/2$ to 2π, the value of r varies from -1 to 0. Thus, the point (r, θ) moves along the circle from the high point back to the pole, duplicating the motion that occurred for $\pi/2 \le \theta \le \pi$.



▲ **Figure 10.2.12**

▸ **Example 6** Sketch the graph of $r = \cos 2\theta$ in polar coordinates.

Solution. Instead of plotting points, we will use the graph of $r = \cos 2\theta$ in rectangular coordinates (Figure 10.2.12) to visualize how the polar graph of this equation is generated. The analysis and the resulting polar graph are shown in Figure 10.2.13. This curve is called a *four-petal rose*. ◂



▲ **Figure 10.2.13**

SYMMETRY TESTS

Observe that the polar graph of $r = \cos 2\theta$ in Figure 10.2.13 is symmetric about the x-axis and the y-axis. This symmetry could have been predicted from the following theorem, which is suggested by Figure 10.2.14 (we omit the proof).

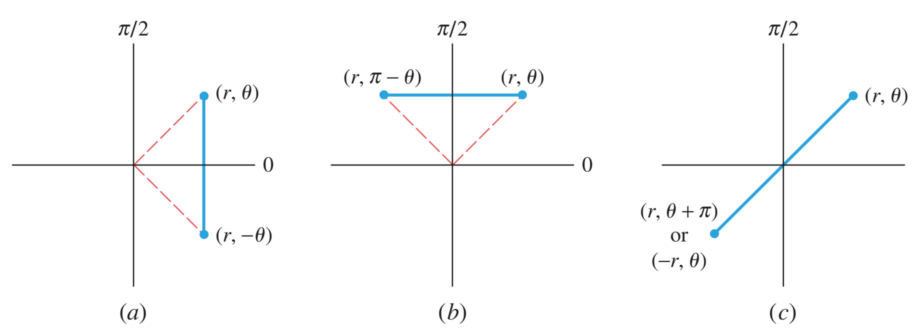
The converse of each part of Theorem 10.2.1 is false. See Exercise 79.

10.2.1 THEOREM (*Symmetry Tests*)

(*a*) *A curve in polar coordinates is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation (Figure 10.2.14a).*

(*b*) *A curve in polar coordinates is symmetric about the y-axis if replacing θ by $\pi - \theta$ in its equation produces an equivalent equation (Figure 10.2.14b).*

(*c*) *A curve in polar coordinates is symmetric about the origin if replacing θ by $\theta + \pi$, or replacing r by $-r$ in its equation produces an equivalent equation (Figure 10.2.14c).*



► **Figure 10.2.14**

► **Example 7** Use Theorem 10.2.1 to confirm that the graph of $r = \cos 2\theta$ in Figure 10.2.13 is symmetric about the x-axis and y-axis.

A graph that is symmetric about both the x-axis and the y-axis is also symmetric about the origin. Use Theorem 10.2.1(*c*) to verify that the curve in Example 7 is symmetric about the origin.

Solution. To test for symmetry about the x-axis, we replace θ by $-\theta$. This yields

$$r = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $-\theta$ does not alter the equation.

To test for symmetry about the y-axis, we replace θ by $\pi - \theta$. This yields

$$r = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $\pi - \theta$ does not alter the equation. ◄

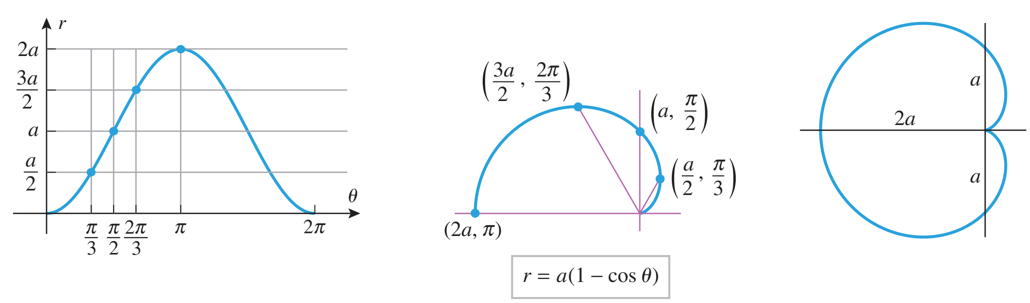
▶ Example 8 Sketch the graph of $r = a(1 - \cos\theta)$ in polar coordinates, assuming a to be a positive constant.

Solution. Observe first that replacing θ by $-\theta$ does not alter the equation, so we know in advance that the graph is symmetric about the polar axis. Thus, if we graph the upper half of the curve, then we can obtain the lower half by reflection about the polar axis.

As in our previous examples, we will first graph the equation in rectangular θr-coordinates. This graph, which is shown in Figure 10.2.15*a*, can be obtained by rewriting the given equation as $r = a - a\cos\theta$, from which we see that the graph in rectangular θr-coordinates can be obtained by first reflecting the graph of $r = a\cos\theta$ about the x-axis to obtain the graph of $r = -a\cos\theta$, and then translating that graph up a units to obtain the graph of $r = a - a\cos\theta$. Now we can see the following:

- As θ varies from 0 to $\pi/3$, r increases from 0 to $a/2$.
- As θ varies from $\pi/3$ to $\pi/2$, r increases from $a/2$ to a.
- As θ varies from $\pi/2$ to $2\pi/3$, r increases from a to $3a/2$.
- As θ varies from $2\pi/3$ to π, r increases from $3a/2$ to $2a$.

This produces the polar curve shown in Figure 10.2.15*b*. The rest of the curve can be obtained by continuing the preceding analysis from π to 2π or, as noted above, by reflecting the portion already graphed about the x-axis (Figure 10.2.15*c*). This heart-shaped curve is called a ***cardioid*** (from the Greek word *kardia* meaning "heart"). ◀



(*a*) (*b*) (*c*)

▲ **Figure 10.2.15**

▶ Example 9 Sketch the graph of $r^2 = 4\cos 2\theta$ in polar coordinates.

Solution. This equation does not express r as a function of θ, since solving for r in terms of θ yields two functions:

$$r = 2\sqrt{\cos 2\theta} \quad \text{and} \quad r = -2\sqrt{\cos 2\theta}$$

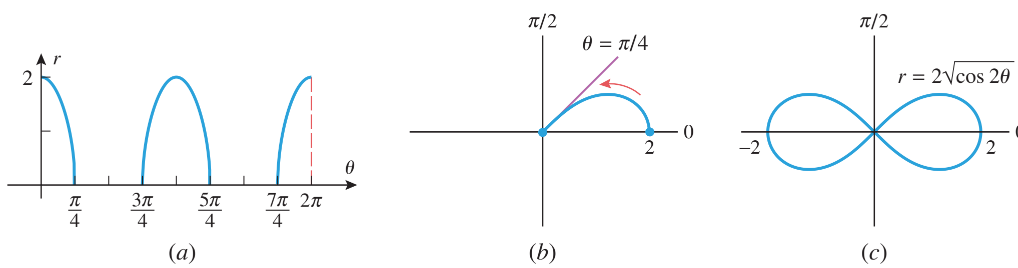
Thus, to graph the equation $r^2 = 4\cos 2\theta$ we will have to graph the two functions separately and then combine those graphs.

We will start with the graph of $r = 2\sqrt{\cos 2\theta}$. Observe first that this equation is not changed if we replace θ by $-\theta$ or if we replace θ by $\pi - \theta$. Thus, the graph is symmetric about the x-axis and the y-axis. This means that the entire graph can be obtained by graphing the portion in the first quadrant, reflecting that portion about the y-axis to obtain the portion in the second quadrant, and then reflecting those two portions about the x-axis to obtain the portions in the third and fourth quadrants.

To begin the analysis, we will graph the equation $r = 2\sqrt{\cos 2\theta}$ in rectangular θr-coordinates (see Figure 10.2.16*a*). Note that there are gaps in that graph over the intervals $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ because $\cos 2\theta$ is negative for those values of θ. From this graph we can see the following:

- As θ varies from 0 to $\pi/4$, r decreases from 2 to 0.
- As θ varies from $\pi/4$ to $\pi/2$, no points are generated on the polar graph.

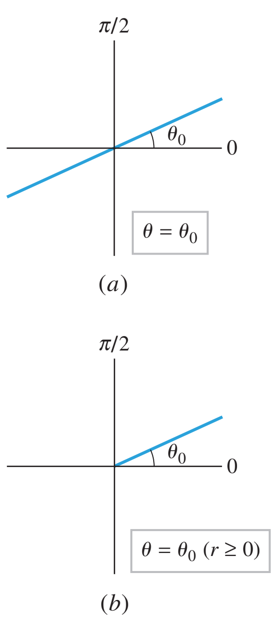
This produces the portion of the graph shown in Figure 10.2.16*b*. As noted above, we can complete the graph by a reflection about the y-axis followed by a reflection about the x-axis (Figure 10.2.16*c*). The resulting propeller-shaped graph is called a ***lemniscate*** (from the Greek word *lemniscos* for a looped ribbon resembling the number 8). We leave it for you to verify that the equation $r = 2\sqrt{\cos 2\theta}$ has the same graph as $r = -2\sqrt{\cos 2\theta}$, but traced in a diagonally opposite manner. Thus, the graph of the equation $r^2 = 4\cos 2\theta$ consists of two identical superimposed lemniscates. ◄



▲ **Figure 10.2.16**

FAMILIES OF LINES AND RAYS THROUGH THE POLE

If θ_0 is a fixed angle, then for all values of r the point (r, θ_0) lies on the line that makes an angle of $\theta = \theta_0$ with the polar axis; and, conversely, every point on this line has a pair of polar coordinates of the form (r, θ_0). Thus, the equation $\theta = \theta_0$ represents the line that passes through the pole and makes an angle of θ_0 with the polar axis (Figure 10.2.17*a*). If r is restricted to be nonnegative, then the graph of the equation $\theta = \theta_0$ is the ray that emanates from the pole and makes an angle of θ_0 with the polar axis (Figure 10.2.17*b*). Thus, as θ_0 varies, the equation $\theta = \theta_0$ produces either a family of lines through the pole or a family of rays through the pole, depending on the restrictions on r.



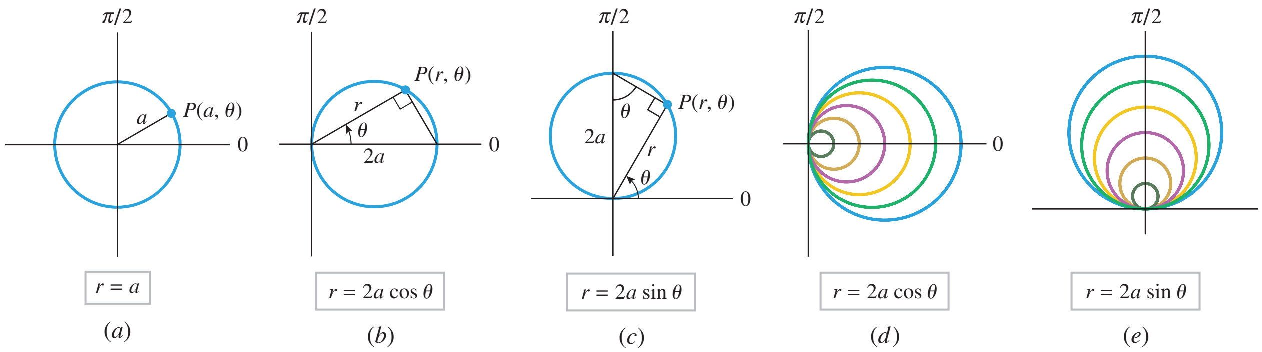
▲ **Figure 10.2.17**

FAMILIES OF CIRCLES

We will consider three families of circles in which a is assumed to be a positive constant:

$$r = a \qquad r = 2a\cos\theta \qquad r = 2a\sin\theta \tag{3–5}$$

The equation $r = a$ represents a circle of radius a centered at the pole (Figure 10.2.18*a*). Thus, as a varies, this equation produces a family of circles centered at the pole. For families (4) and (5), recall from plane geometry that a triangle that is inscribed in a circle with a diameter of the circle for a side must be a right triangle. Thus, as indicated in Figures 10.2.18*b* and 10.2.18*c*, the equation $r = 2a\cos\theta$ represents a circle of radius a, centered on the x-axis and tangent to the y-axis at the origin; similarly, the equation $r = 2a\sin\theta$ represents a circle of radius a, centered on the y-axis and tangent to the x-axis at the origin. Thus, as a varies, Equations (4) and (5) produce the families illustrated in Figures 10.2.18*d* and 10.2.18*e*.



▲ **Figure 10.2.18**

Observe that replacing θ by $-\theta$ does not change the equation $r = 2a\cos\theta$ and that replacing θ by $\pi - \theta$ does not change the equation $r = 2a\sin\theta$. This explains why the circles in Figure 10.2.18*d* are symmetric about the x-axis and those in Figure 10.2.18e are symmetric about the y-axis.

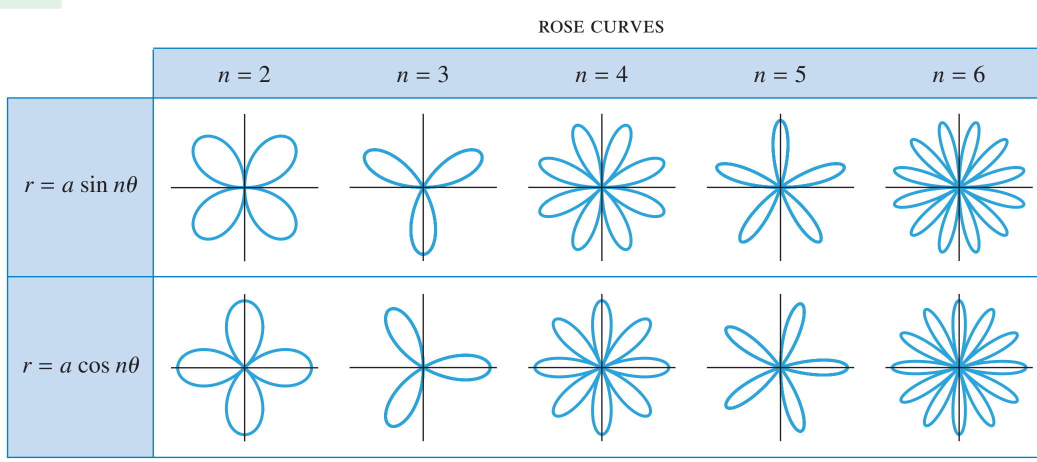
What do the graphs of one-petal roses look like?

FAMILIES OF ROSE CURVES

In polar coordinates, equations of the form

$$r = a\sin n\theta \qquad r = a\cos n\theta \tag{6–7}$$

in which $a > 0$ and n is a positive integer represent families of flower-shaped curves called ***roses*** (Figure 10.2.19). The rose consists of n equally spaced petals of radius a if n is odd and $2n$ equally spaced petals of radius a if n is even. It can be shown that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$ (Exercise 78). A four-petal rose of radius 1 was graphed in Example 6.



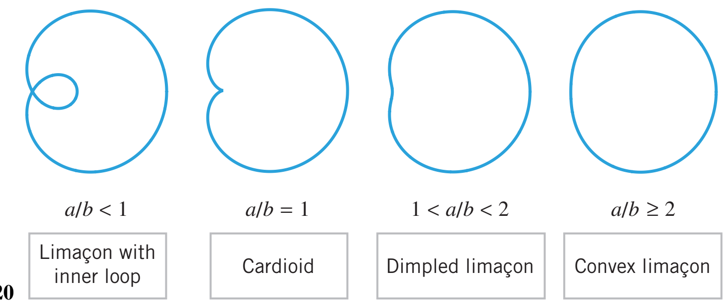
▲ **Figure 10.2.19**

FAMILIES OF CARDIOIDS AND LIMAÇONS

Equations with any of the four forms

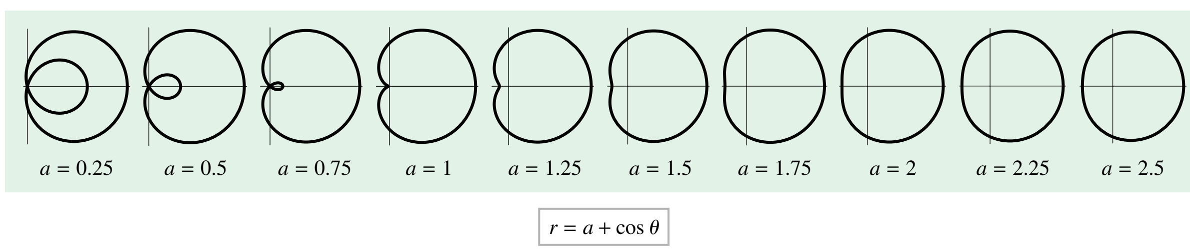
$$r = a \pm b\sin\theta \qquad r = a \pm b\cos\theta \tag{8–9}$$

in which $a > 0$ and $b > 0$ represent polar curves called ***limaçons*** (from the Latin word *limax* for a snail-like creature that is commonly called a "slug"). There are four possible shapes for a limaçon that are determined by the ratio a/b (Figure 10.2.20). If $a = b$ (the case $a/b = 1$), then the limaçon is called a ***cardioid*** because of its heart-shaped appearance, as noted in Example 8.



▶ **Figure 10.2.20**

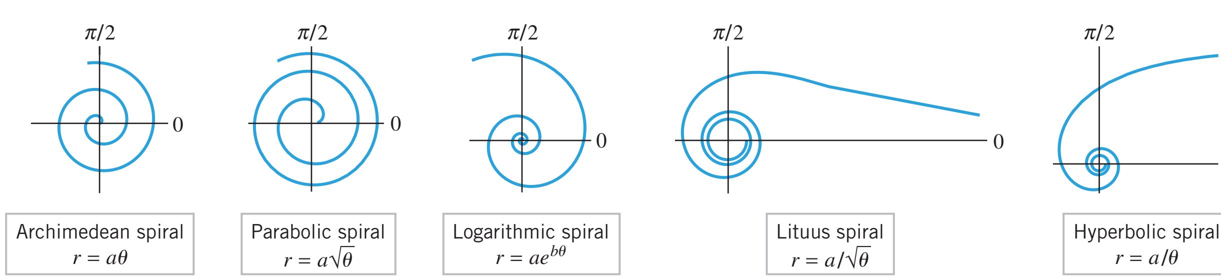
▶ **Example 10** Figure 10.2.21 shows the family of limaçons $r = a + \cos\theta$ with the constant a varying from 0.25 to 2.50 in steps of 0.25. In keeping with Figure 10.2.20, the limaçons evolve from the loop type to the convex type. As a increases from the starting value of 0.25, the loops get smaller and smaller until the cardioid is reached at $a = 1$. As a increases further, the limaçons evolve through the dimpled type into the convex type. ◀



▲ **Figure 10.2.21**

FAMILIES OF SPIRALS

A ***spiral*** is a curve that coils around a central point. Spirals generally have "left-hand" and "right-hand" versions that coil in opposite directions, depending on the restrictions on the polar angle and the signs of constants that appear in their equations. Some of the more common types of spirals are shown in Figure 10.2.22 for nonnegative values of θ, a, and b.



▲ **Figure 10.2.22**

SPIRALS IN NATURE

Spirals of many kinds occur in nature. For example, the shell of the chambered nautilus (below) forms a logarithmic spiral, and a coiled sailor's rope forms an Archimedean spiral. Spirals also occur in flowers, the tusks of certain animals, and in the shapes of galaxies.

Thomas Taylor/Photo Researchers

The shell of the chambered nautilus reveals a logarithmic spiral. The animal lives in the outermost chamber.

Rex Ziak/Stone/Getty Images

A sailor's coiled rope forms an Archimedean spiral.

Courtesy NASA & The Hubble Heritage Team

A spiral galaxy.

GENERATING POLAR CURVES WITH GRAPHING UTILITIES

For polar curves that are too complicated for hand computation, graphing utilities can be used. Although many graphing utilities are capable of graphing polar curves directly, some are not. However, if a graphing utility is capable of graphing parametric equations, then it can be used to graph a polar curve $r = f(\theta)$ by converting this equation to parametric form. This can be done by substituting $f(\theta)$ for r in (1). This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta \tag{10}$$

which is a pair of parametric equations for the polar curve in terms of the parameter θ.

Example 11 Express the polar equation

$$r = 2 + \cos\frac{5\theta}{2}$$

parametrically, and generate the polar graph from the parametric equations using a graphing utility.

Solution. Substituting the given expression for r in $x = r\cos\theta$ and $y = r\sin\theta$ yields the parametric equations

$$x = \left[2 + \cos\frac{5\theta}{2}\right]\cos\theta, \quad y = \left[2 + \cos\frac{5\theta}{2}\right]\sin\theta$$

Next, we need to find an interval over which to vary θ to produce the entire graph. To find such an interval, we will look for the smallest number of complete revolutions that must occur until the value of r begins to repeat. Algebraically, this amounts to finding the smallest positive integer n such that

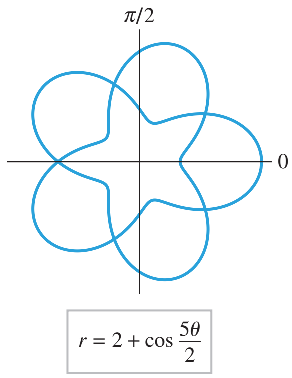
$$2 + \cos\left(\frac{5(\theta + 2n\pi)}{2}\right) = 2 + \cos\frac{5\theta}{2}$$

or

$$\cos\left(\frac{5\theta}{2} + 5n\pi\right) = \cos\frac{5\theta}{2}$$

TECHNOLOGY MASTERY

Use a graphing utility to duplicate the curve in Figure 10.2.23. If your graphing utility requires that t be used as the parameter, then you will have to replace θ by t in (10) to generate the graph.



▲ **Figure 10.2.23**

For this equality to hold, the quantity $5n\pi$ must be an even multiple of π; the smallest n for which this occurs is $n = 2$. Thus, the entire graph will be traced in two revolutions, which means it can be generated from the parametric equations

$$x = \left[2 + \cos\frac{5\theta}{2}\right]\cos\theta, \quad y = \left[2 + \cos\frac{5\theta}{2}\right]\sin\theta \qquad (0 \le \theta \le 4\pi)$$

This yields the graph in Figure 10.2.23. ◀

✔ QUICK CHECK EXERCISES 10.2

(See page 719 for answers.)

1. (a) Rectangular coordinates of a point (x, y) may be recovered from its polar coordinates (r, θ) by means of the equations $x =$ ________ and $y =$ ________.

(b) Polar coordinates (r, θ) may be recovered from rectangular coordinates (x, y) by means of the equations $r^2 =$ ________ and $\tan\theta =$ ________.

2. Find the rectangular coordinates of the points whose polar coordinates are given.
(a) $(4, \pi/3)$ (b) $(2, -\pi/6)$
(c) $(6, -2\pi/3)$ (d) $(4, 5\pi/4)$

3. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(1, \sqrt{3})$.
(a) $r \ge 0$ and $0 \le \theta < 2\pi$
(b) $r \le 0$ and $0 \le \theta < 2\pi$

4. In each part, state the name that describes the polar curve most precisely: a rose, a line, a circle, a limaçon, a cardioid, a spiral, a lemniscate, or none of these.
(a) $r = 1 - \theta$ (b) $r = 1 + 2\sin\theta$
(c) $r = \sin 2\theta$ (d) $r = \cos^2\theta$
(e) $r = \csc\theta$ (f) $r = 2 + 2\cos\theta$
(g) $r = -2\sin\theta$

EXERCISE SET 10.2

Graphing Utility

1–2 Plot the points in polar coordinates.

1. (a) $(3, \pi/4)$ (b) $(5, 2\pi/3)$ (c) $(1, \pi/2)$
(d) $(4, 7\pi/6)$ (e) $(-6, -\pi)$ (f) $(-1, 9\pi/4)$

2. (a) $(2, -\pi/3)$ (b) $(3/2, -7\pi/4)$ (c) $(-3, 3\pi/2)$
(d) $(-5, -\pi/6)$ (e) $(2, 4\pi/3)$ (f) $(0, \pi)$

3–4 Find the rectangular coordinates of the points whose polar coordinates are given.

3. (a) $(6, \pi/6)$ (b) $(7, 2\pi/3)$ (c) $(-6, -5\pi/6)$
(d) $(0, -\pi)$ (e) $(7, 17\pi/6)$ (f) $(-5, 0)$

4. (a) $(-2, \pi/4)$ (b) $(6, -\pi/4)$ (c) $(4, 9\pi/4)$
(d) $(3, 0)$ (e) $(-4, -3\pi/2)$ (f) $(0, 3\pi)$

5. In each part, a point is given in rectangular coordinates. Find two pairs of polar coordinates for the point, one pair satisfying $r \ge 0$ and $0 \le \theta < 2\pi$, and the second pair satisfying $r \ge 0$ and $-2\pi < \theta \le 0$.
(a) $(-5, 0)$ (b) $(2\sqrt{3}, -2)$ (c) $(0, -2)$
(d) $(-8, -8)$ (e) $(-3, 3\sqrt{3})$ (f) $(1, 1)$

6. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(-\sqrt{3}, 1)$.
(a) $r \ge 0$ and $0 \le \theta < 2\pi$
(b) $r \le 0$ and $0 \le \theta < 2\pi$
(c) $r \ge 0$ and $-2\pi < \theta \le 0$
(d) $r \le 0$ and $-\pi < \theta \le \pi$

7–8 Use a calculating utility, where needed, to approximate the polar coordinates of the points whose rectangular coordinates are given.

7. (a) $(3, 4)$ (b) $(6, -8)$ (c) $(-1, \tan^{-1} 1)$

8. (a) $(-3, 4)$ (b) $(-3, 1.7)$ (c) $\left(2, \sin^{-1}\frac{1}{2}\right)$

9–10 Identify the curve by transforming the given polar equation to rectangular coordinates.

9. (a) $r = 2$ (b) $r\sin\theta = 4$
(c) $r = 3\cos\theta$ (d) $r = \dfrac{6}{3\cos\theta + 2\sin\theta}$

10. (a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$
(c) $r = 4\cos\theta + 4\sin\theta$ (d) $r = \sec\theta\tan\theta$

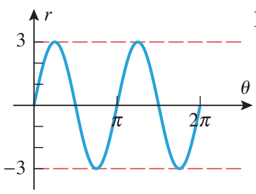
11–12 Express the given equations in polar coordinates.

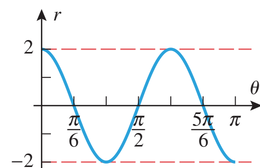
11. (a) $x = 3$ (b) $x^2 + y^2 = 7$
(c) $x^2 + y^2 + 6y = 0$ (d) $9xy = 4$

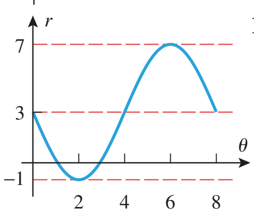
12. (a) $y = -3$ (b) $x^2 + y^2 = 5$
(c) $x^2 + y^2 + 4x = 0$ (d) $x^2(x^2 + y^2) = y^2$

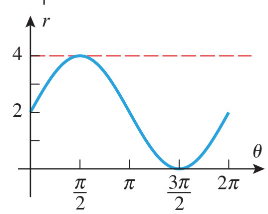
FOCUS ON CONCEPTS

13–16 A graph is given in a rectangular θr-coordinate system. Sketch the corresponding graph in polar coordinates. ■

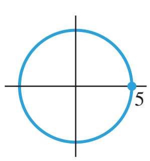
13.


14.


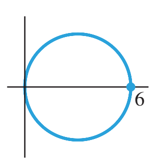
15.


16.


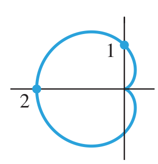
17–20 Find an equation for the given polar graph. [*Note:* Numeric labels on these graphs represent distances to the origin.] ■

17. (a)


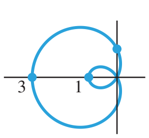
Circle

(b)


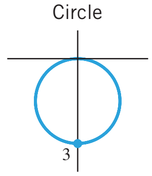
Circle

(c)


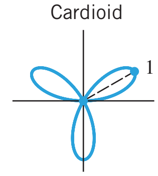
Cardioid

18. (a)


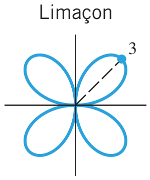
Limaçon

(b)


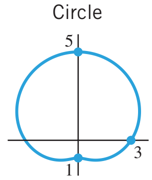
Circle

(c)


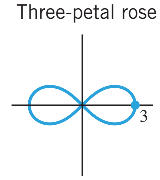
Three-petal rose

19. (a)


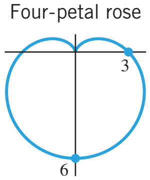
Four-petal rose

(b)


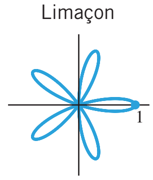
Limaçon

(c)


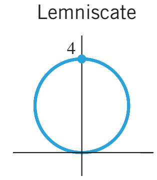
Lemniscate

20. (a)


Cardioid

(b)


Five-petal rose

(c)


Circle

21–46 Sketch the curve in polar coordinates. ■

21. $\theta = \dfrac{\pi}{3}$
22. $\theta = -\dfrac{3\pi}{4}$
23. $r = 3$
24. $r = 4\cos\theta$
25. $r = 6\sin\theta$
26. $r - 2 = 2\cos\theta$
27. $r = 3(1 + \sin\theta)$
28. $r = 5 - 5\sin\theta$
29. $r = 4 - 4\cos\theta$
30. $r = 1 + 2\sin\theta$
31. $r = -1 - \cos\theta$
32. $r = 4 + 3\cos\theta$
33. $r = 3 - \sin\theta$
34. $r = 3 + 4\cos\theta$
35. $r - 5 = 3\sin\theta$
36. $r = 5 - 2\cos\theta$
37. $r = -3 - 4\sin\theta$
38. $r^2 = \cos 2\theta$
39. $r^2 = 16\sin 2\theta$
40. $r = 4\theta \quad (\theta \geq 0)$
41. $r = 4\theta \quad (\theta \leq 0)$
42. $r = 4\theta$
43. $r = -2\cos 2\theta$
44. $r = 3\sin 2\theta$
45. $r = 9\sin 4\theta$
46. $r = 2\cos 3\theta$

47–50 True–False Determine whether the statement is true or false. Explain your answer. ■

47. The polar coordinate pairs $(-1, \pi/3)$ and $(1, -2\pi/3)$ describe the same point.

48. If the graph of $r = f(\theta)$ drawn in rectangular θr-coordinates is symmetric about the r-axis, then the graph of $r = f(\theta)$ drawn in polar coordinates is symmetric about the x-axis.

49. The portion of the polar graph of $r = \sin 2\theta$ for values of θ between $\pi/2$ and π is contained in the second quadrant.

50. The graph of a dimpled limaçon passes through the polar origin.

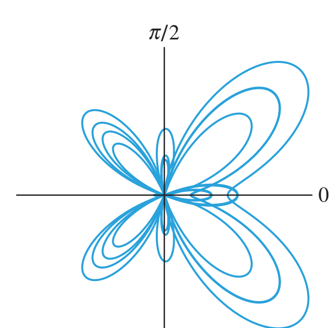
51–55 Determine a shortest parameter interval on which a complete graph of the polar equation can be generated, and then use a graphing utility to generate the polar graph. ■

51. $r = \cos\dfrac{\theta}{2}$
52. $r = \sin\dfrac{\theta}{2}$
53. $r = 1 - 2\sin\dfrac{\theta}{4}$
54. $r = 0.5 + \cos\dfrac{\theta}{3}$
55. $r = \cos\dfrac{\theta}{5}$

56. The accompanying figure shows the graph of the "butterfly curve"

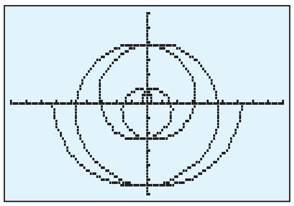
$$r = e^{\cos\theta} - 2\cos 4\theta + \sin^3\frac{\theta}{4}$$

Determine a shortest parameter interval on which the complete butterfly can be generated, and then check your answer using a graphing utility.



◀ **Figure Ex-56**

57. The accompanying figure shows the Archimedean spiral $r = \theta/2$ produced with a graphing calculator.
(a) What interval of values for θ do you think was used to generate the graph?
(b) Duplicate the graph with your own graphing utility.



$[-9, 9] \times [-6, 6]$
$x\text{Scl} = 1$, $y\text{Scl} = 1$

Figure Ex-57

58. Find equations for the two families of circles in the accompanying figure.

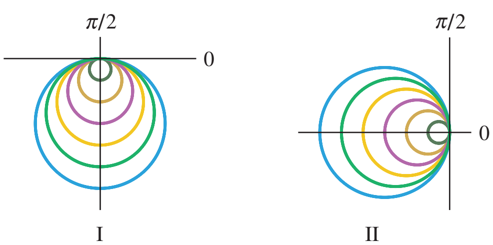


Figure Ex-58

59. (a) Show that if a varies, then the polar equation

$$r = a \sec\theta \quad (-\pi/2 < \theta < \pi/2)$$

describes a family of lines perpendicular to the polar axis.

(b) Show that if b varies, then the polar equation

$$r = b \csc\theta \quad (0 < \theta < \pi)$$

describes a family of lines parallel to the polar axis.

FOCUS ON CONCEPTS

60. The accompanying figure shows graphs of the Archimedean spiral $r = \theta$ and the parabolic spiral $r = \sqrt{\theta}$. Which is which? Explain your reasoning.

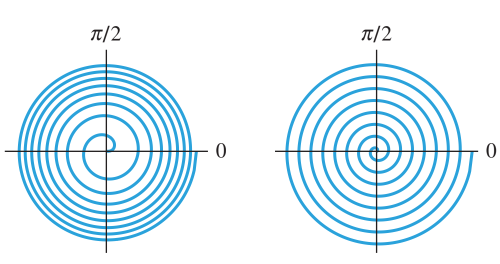


Figure Ex-60

61–62 A polar graph of $r = f(\theta)$ is given over the stated interval. Sketch the graph of

(a) $r = f(-\theta)$ (b) $r = f\left(\theta - \dfrac{\pi}{2}\right)$
(c) $r = f\left(\theta + \dfrac{\pi}{2}\right)$ (d) $r = -f(\theta)$.

61. $0 \le \theta \le \pi/2$ **62.** $\pi/2 \le \theta \le \pi$

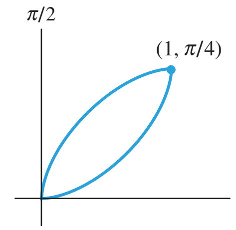


Figure Ex-61

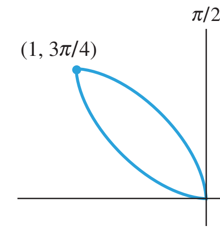


Figure Ex-62

63–64 Use the polar graph from the indicated exercise to sketch the graph of

(a) $r = f(\theta) + 1$ (b) $r = 2f(\theta) - 1$.

63. Exercise 61 **64.** Exercise 62

65. Show that if the polar graph of $r = f(\theta)$ is rotated counterclockwise around the origin through an angle α, then $r = f(\theta - \alpha)$ is an equation for the rotated curve. [*Hint:* If (r_0, θ_0) is any point on the original graph, then $(r_0, \theta_0 + \alpha)$ is a point on the rotated graph.]

66. Use the result in Exercise 65 to find an equation for the lemniscate that results when the lemniscate in Example 9 is rotated counterclockwise through an angle of $\pi/2$.

67. Use the result in Exercise 65 to find an equation for the cardioid $r = 1 + \cos\theta$ after it has been rotated through the given angle, and check your answer with a graphing utility.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) $\dfrac{5\pi}{4}$

68. (a) Show that if A and B are not both zero, then the graph of the polar equation

$$r = A\sin\theta + B\cos\theta$$

is a circle. Find its radius.

(b) Derive Formulas (4) and (5) from the formula given in part (a).

69. Find the highest point on the cardioid $r = 1 + \cos\theta$.

70. Find the leftmost point on the upper half of the cardioid $r = 1 + \cos\theta$.

71. Show that in a polar coordinate system the distance d between the points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

72–74 Use the formula obtained in Exercise 71 to find the distance between the two points indicated in polar coordinates.

72. $(3, \pi/6)$ and $(2, \pi/3)$

73. Successive tips of the four-petal rose $r = \cos 2\theta$. Check your answer using geometry.

74. Successive tips of the three-petal rose $r = \sin 3\theta$. Check your answer using trigonometry.

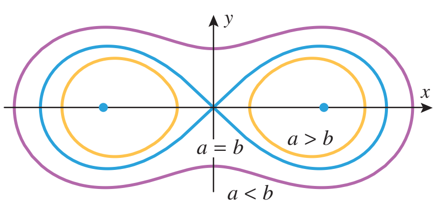
75. In the late seventeenth century the Italian astronomer Giovanni Domenico Cassini (1625–1712) introduced the family of curves

$$(x^2 + y^2 + a^2)^2 - b^4 - 4a^2x^2 = 0 \quad (a > 0, b > 0)$$

in his studies of the relative motions of the Earth and the Sun. These curves, which are called ***Cassini ovals***, have one of the three basic shapes shown in the accompanying figure.

(a) Show that if $a = b$, then the polar equation of the Cassini oval is $r^2 = 2a^2 \cos 2\theta$, which is a lemniscate.

(b) Use the formula in Exercise 71 to show that the lemniscate in part (a) is the curve traced by a point that moves in such a way that the product of its distances from the polar points $(a, 0)$ and (a, π) is a^2.



▲ **Figure Ex-75**

76–77 Vertical and horizontal asymptotes of polar curves can sometimes be detected by investigating the behavior of $x = r\cos\theta$ and $y = r\sin\theta$ as θ varies. This idea is used in these exercises. ■

76. Show that the ***hyperbolic spiral*** $r = 1/\theta$ $(\theta > 0)$ has a horizontal asymptote at $y = 1$ by showing that $y \to 1$ and $x \to +\infty$ as $\theta \to 0^+$. Confirm this result by generating the spiral with a graphing utility.

77. Show that the spiral $r = 1/\theta^2$ does not have any horizontal asymptotes.

78. Prove that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$.

79. (a) Use a graphing utility to confirm that the graph of $r = 2 - \sin(\theta/2)$ $(0 \le \theta \le 4\pi)$ is symmetric about the x-axis.

(b) Show that replacing θ by $-\theta$ in the polar equation $r = 2 - \sin(\theta/2)$ does not produce an equivalent equation. Why does this not contradict the symmetry demonstrated in part (a)?

80. **Writing** Use a graphing utility to investigate how the family of polar curves $r = 1 + a\cos n\theta$ is affected by changing the values of a and n, where a is a positive real number and n is a positive integer. Write a brief paragraph to explain your conclusions.

81. **Writing** Why do you think the adjective "polar" was chosen in the name "polar coordinates"?

✔ QUICK CHECK ANSWERS 10.2

1. (a) $r\cos\theta$; $r\sin\theta$ (b) $x^2 + y^2$; y/x **2.** (a) $(2, 2\sqrt{3})$ (b) $(\sqrt{3}, -1)$ (c) $(-3, -3\sqrt{3})$ (d) $(-2\sqrt{2}, -2\sqrt{2})$ **3.** (a) $(2, \pi/3)$ (b) $(-2, 4\pi/3)$ **4.** (a) spiral (b) limaçon (c) rose (d) none of these (e) line (f) cardioid (g) circle

10.3 TANGENT LINES, ARC LENGTH, AND AREA FOR POLAR CURVES

In this section we will derive the formulas required to find slopes, tangent lines, and arc lengths of polar curves. We will then show how to find areas of regions that are bounded by polar curves.

■ TANGENT LINES TO POLAR CURVES

Our first objective in this section is to find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$ in which r is a differentiable function of θ. We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

from which we obtain

$$\begin{aligned} \frac{dx}{d\theta} &= -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta \\ \frac{dy}{d\theta} &= f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta \end{aligned} \tag{1}$$

Thus, if $dx/d\theta$ and $dy/d\theta$ are continuous and if $dx/d\theta \neq 0$, then y is a differentiable function of x, and Formula (4) in Section 10.1 with θ in place of t yields

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}} \tag{2}$$

Example 1 Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.

Solution. From (2) with $r = 4\cos\theta$, so that $dr/d\theta = -4\sin\theta$, we obtain

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = -\frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

Using the double-angle formulas for sine and cosine,

$$\frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$$

Thus, at the point where $\theta = \pi/4$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\cot\frac{\pi}{2} = 0$$

which implies that the circle has a horizontal tangent line at the point where $\theta = \pi/4$ (Figure 10.3.1). ◄

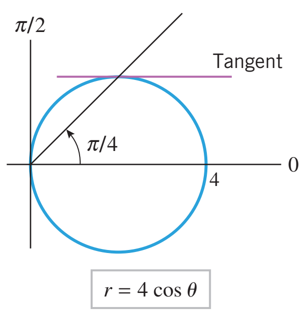


Figure 10.3.1

Example 2 Find the points on the cardioid $r = 1 - \cos\theta$ at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

Solution. A horizontal tangent line will occur where $dy/d\theta = 0$ and $dx/d\theta \neq 0$, a vertical tangent line where $dy/d\theta \neq 0$ and $dx/d\theta = 0$, and a singular point where $dy/d\theta = 0$ and $dx/d\theta = 0$. We could find these derivatives from the formulas in (1). However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting $r = 1 - \cos\theta$ in the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. This yields

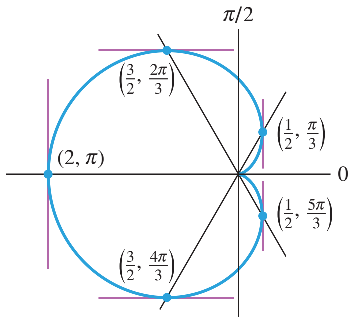
$$x = (1 - \cos\theta)\cos\theta, \quad y = (1 - \cos\theta)\sin\theta \qquad (0 \le \theta \le 2\pi)$$

Differentiating these equations with respect to θ and then simplifying yields (verify)

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1), \quad \frac{dy}{d\theta} = (1 - \cos\theta)(1 + 2\cos\theta)$$

Thus, $dx/d\theta = 0$ if $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and $dy/d\theta = 0$ if $\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$. We leave it for you to solve these equations and show that the solutions of $dx/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dx}{d\theta} = 0: \quad \theta = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$$



$r = 1 - \cos\theta$

▲ **Figure 10.3.2**

and the solutions of $dy/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

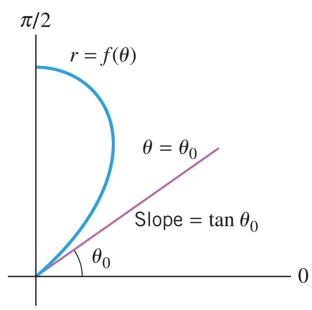
$$\frac{dy}{d\theta} = 0: \quad \theta = 0, \ \frac{2\pi}{3}, \ \frac{4\pi}{3}, \ 2\pi$$

Thus, horizontal tangent lines occur at $\theta = 2\pi/3$ and $\theta = 4\pi/3$; vertical tangent lines occur at $\theta = \pi/3$, π, and $5\pi/3$; and singular points occur at $\theta = 0$ and $\theta = 2\pi$ (Figure 10.3.2). Note, however, that $r = 0$ at both singular points, so there is really only one singular point on the cardioid—the pole. ◀

TANGENT LINES TO POLAR CURVES AT THE ORIGIN

Formula (2) reveals some useful information about the behavior of a polar curve $r = f(\theta)$ that passes through the origin. If we assume that $r = 0$ and $dr/d\theta \ne 0$ when $\theta = \theta_0$, then it follows from Formula (2) that the slope of the tangent line to the curve at $\theta = \theta_0$ is

$$\frac{dy}{dx} = \frac{0 + \sin\theta_0 \dfrac{dr}{d\theta}}{0 + \cos\theta_0 \dfrac{dr}{d\theta}} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$$

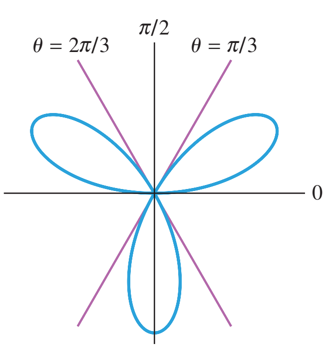


▲ **Figure 10.3.3**

(Figure 10.3.3). However, $\tan\theta_0$ is also the slope of the line $\theta = \theta_0$, so we can conclude that this line is tangent to the curve at the origin. Thus, we have established the following result.

> **10.3.1 THEOREM** *If the polar curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, and if $dr/d\theta \ne 0$ at $\theta = \theta_0$, then the line $\theta = \theta_0$ is tangent to the curve at the origin.*

This theorem tells us that equations of the tangent lines at the origin to the curve $r = f(\theta)$ can be obtained by solving the equation $f(\theta) = 0$. It is important to keep in mind, however, that $r = f(\theta)$ may be zero for more than one value of θ, so there may be more than one tangent line at the origin. This is illustrated in the next example.



$r = \sin 3\theta$

▲ **Figure 10.3.4**

▶ **Example 3** The three-petal rose $r = \sin 3\theta$ in Figure 10.3.4 has three tangent lines at the origin, which can be found by solving the equation

$$\sin 3\theta = 0$$

It was shown in Exercise 78 of Section 10.2 that the complete rose is traced once as θ varies over the interval $0 \le \theta < \pi$, so we need only look for solutions in this interval. We leave it for you to confirm that these solutions are

$$\theta = 0, \quad \theta = \frac{\pi}{3}, \quad \text{and} \quad \theta = \frac{2\pi}{3}$$

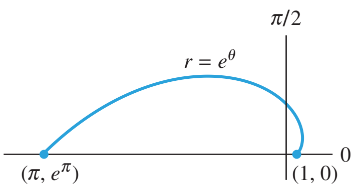
Since $dr/d\theta = 3\cos 3\theta \ne 0$ for these values of θ, these three lines are tangent to the rose at the origin, which is consistent with the figure. ◀

ARC LENGTH OF A POLAR CURVE

A formula for the arc length of a polar curve $r = f(\theta)$ can be derived by expressing the curve in parametric form and applying Formula (9) of Section 10.1 for the arc length of a parametric curve. We leave it as an exercise to show the following.

10.3.2 ARC LENGTH FORMULA FOR POLAR CURVES If no segment of the polar curve $r = f(\theta)$ is traced more than once as θ increases from α to β, and if $dr/d\theta$ is continuous for $\alpha \le \theta \le \beta$, then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{3}$$



▲ **Figure 10.3.5**

▶ Example 4 Find the arc length of the spiral $r = e^{\theta}$ in Figure 10.3.5 between $\theta = 0$ and $\theta = \pi$.

Solution.

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta$$

$$= \int_{0}^{\pi} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2}\, e^{\theta}\Big]_0^{\pi} = \sqrt{2}(e^{\pi} - 1) \approx 31.3 \blacktriangleleft$$

▶ Example 5 Find the total arc length of the cardioid $r = 1 + \cos\theta$.

Solution. The cardioid is traced out once as θ varies from $\theta = 0$ to $\theta = 2\pi$. Thus,

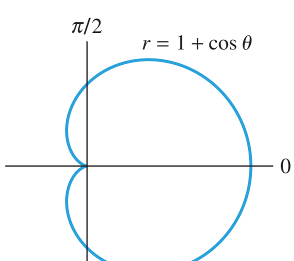
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \sqrt{2}\int_{0}^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

$$= 2\int_{0}^{2\pi} \sqrt{\cos^2 \tfrac{1}{2}\theta}\, d\theta$$

Identity (45) of Appendix B

$$= 2\int_{0}^{2\pi} \left|\cos \tfrac{1}{2}\theta\right| d\theta$$



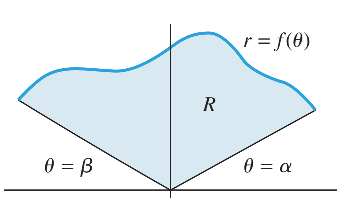
▲ **Figure 10.3.6**

Since $\cos \frac{1}{2}\theta$ changes sign at π, we must split the last integral into the sum of two integrals: the integral from 0 to π plus the integral from π to 2π. However, the integral from π to 2π is equal to the integral from 0 to π, since the cardioid is symmetric about the polar axis (Figure 10.3.6). Thus,

$$L = 2\int_{0}^{2\pi} \left|\cos \tfrac{1}{2}\theta\right| d\theta = 4\int_{0}^{\pi} \cos \tfrac{1}{2}\theta\, d\theta = 8 \sin \tfrac{1}{2}\theta\Big]_0^{\pi} = 8 \blacktriangleleft$$

AREA IN POLAR COORDINATES

We begin our investigation of area in polar coordinates with a simple case.



▲ **Figure 10.3.7**

10.3.3 AREA PROBLEM IN POLAR COORDINATES Suppose that α and β are angles that satisfy the condition

$$\alpha < \beta \le \alpha + 2\pi$$

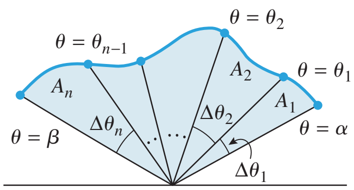
and suppose that $f(\theta)$ is continuous and nonnegative for $\alpha \le \theta \le \beta$. Find the area of the region R enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ (Figure 10.3.7).

In rectangular coordinates we obtained areas under curves by dividing the region into an increasing number of vertical strips, approximating the strips by rectangles, and taking a limit. In polar coordinates rectangles are clumsy to work with, and it is better to partition the region into ***wedges*** by using rays

$$\theta = \theta_1,\ \theta = \theta_2, \ldots,\ \theta = \theta_{n-1}$$

such that

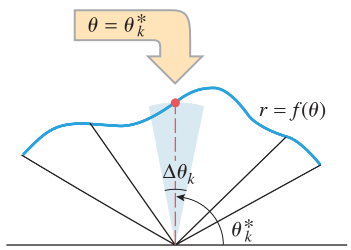
$$\alpha < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \beta$$



▲ **Figure 10.3.8**

(Figure 10.3.8). As shown in that figure, the rays divide the region R into n wedges with areas $A_1, A_2, \ldots, A_n$ and central angles $\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n$. The area of the entire region can be written as

$$A = A_1 + A_2 + \cdots + A_n = \sum_{k=1}^{n} A_k \tag{4}$$



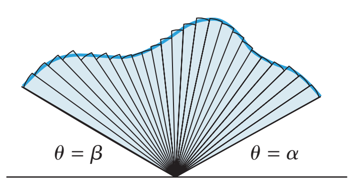
▲ **Figure 10.3.9**

If $\Delta\theta_k$ is small, then we can approximate the area A_k of the kth wedge by the area of a sector with central angle $\Delta\theta_k$ and radius $f(\theta_k^*)$, where $\theta = \theta_k^*$ is any ray that lies in the kth wedge (Figure 10.3.9). Thus, from (4) and Formula (5) of Appendix B for the area of a sector, we obtain

$$A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k \tag{5}$$

If we now increase n in such a way that $\max \Delta\theta_k \to 0$, then the sectors will become better and better approximations of the wedges and it is reasonable to expect that (5) will approach the exact value of the area A (Figure 10.3.10); that is,

$$A = \lim_{\max \Delta\theta_k \to 0} \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k = \int_{\alpha}^{\beta} \tfrac{1}{2}[f(\theta)]^2\, d\theta$$



▲ **Figure 10.3.10**

Note that the discussion above can easily be adapted to the case where $f(\theta)$ is nonpositive for $\alpha \le \theta \le \beta$. We summarize this result below.

10.3.4 AREA IN POLAR COORDINATES If α and β are angles that satisfy the condition

$$\alpha < \beta \le \alpha + 2\pi$$

and if $f(\theta)$ is continuous and either nonnegative or nonpositive for $\alpha \le \theta \le \beta$, then the area A of the region R enclosed by the polar curve $r = f(\theta)$ $(\alpha \le \theta \le \beta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \tfrac{1}{2}[f(\theta)]^2\, d\theta = \int_{\alpha}^{\beta} \tfrac{1}{2} r^2\, d\theta \tag{6}$$

The hardest part of applying (6) is determining the limits of integration. This can be done as follows:

Area in Polar Coordinates: Limits of Integration

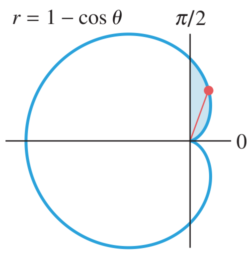
Step 1. Sketch the region R whose area is to be determined.

Step 2. Draw an arbitrary "radial line" from the pole to the boundary curve $r = f(\theta)$.

Step 3. Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"

Step 4. Your answer in Step 3 will determine the lower and upper limits of integration.

▶ **Example 6** Find the area of the region in the first quadrant that is within the cardioid $r = 1 - \cos\theta$.



The shaded region is swept out by the radial line as θ varies from 0 to $\pi/2$.

▲ **Figure 10.3.11**

Solution. The region and a typical radial line are shown in Figure 10.3.11. For the radial line to sweep out the region, θ must vary from 0 to $\pi/2$. Thus, from (6) with $\alpha = 0$ and $\beta = \pi/2$, we obtain

$$A = \int_0^{\pi/2} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

With the help of the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, this can be rewritten as

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3}{8}\pi - 1 \quad ◀$$

▶ **Example 7** Find the entire area within the cardioid of Example 6.

Solution. For the radial line to sweep out the entire cardioid, θ must vary from 0 to 2π. Thus, from (6) with $\alpha = 0$ and $\beta = 2\pi$,

$$A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

If we proceed as in Example 6, this reduces to

$$A = \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{3\pi}{2}$$

Alternative Solution. Since the cardioid is symmetric about the x-axis, we can calculate the portion of the area above the x-axis and double the result. In the portion of the cardioid above the x-axis, θ ranges from 0 to π, so that

$$A = 2\int_0^{\pi}\frac{1}{2}r^2\,d\theta = \int_0^{\pi}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2} \quad ◀$$

USING SYMMETRY

Although Formula (6) is applicable if $r = f(\theta)$ is negative, area computations can sometimes be simplified by using symmetry to restrict the limits of integration to intervals where $r \geq 0$. This is illustrated in the next example.

▶ **Example 8** Find the area of the region enclosed by the rose curve $r = \cos 2\theta$.

Solution. Referring to Figure 10.2.13 and using symmetry, the area in the first quadrant that is swept out for $0 \leq \theta \leq \pi/4$ is one-eighth of the total area inside the rose. Thus, from Formula (6)

$$\begin{aligned} A &= 8\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta = 4\int_0^{\pi/4}\cos^2 2\theta\,d\theta \\ &= 4\int_0^{\pi/4}\frac{1}{2}(1+\cos 4\theta)\,d\theta = 2\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta \\ &= 2\theta + \frac{1}{2}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{2} \quad ◀ \end{aligned}$$

Sometimes the most natural way to satisfy the restriction $\alpha < \beta \le \alpha + 2\pi$ required by Formula (6) is to use a negative value for α. For example, suppose that we are interested in finding the area of the shaded region in Figure 10.3.12*a*. The first step would be to determine the intersections of the cardioid $r = 4 + 4\cos\theta$ and the circle $r = 6$, since this information is needed for the limits of integration. To find the points of intersection, we can equate the two expressions for r. This yields

$$4 + 4\cos\theta = 6 \quad \text{or} \quad \cos\theta = \frac{1}{2}$$

which is satisfied by the positive angles

$$\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}$$

However, there is a problem here because the radial lines to the circle and cardioid do not sweep through the shaded region shown in Figure 10.3.12*b* as θ varies over the interval $\pi/3 \le \theta \le 5\pi/3$. There are two ways to circumvent this problem—one is to take advantage of the symmetry by integrating over the interval $0 \le \theta \le \pi/3$ and doubling the result, and the second is to use a negative lower limit of integration and integrate over the interval $-\pi/3 \le \theta \le \pi/3$ (Figure 10.3.12*c*). The two methods are illustrated in the next example.

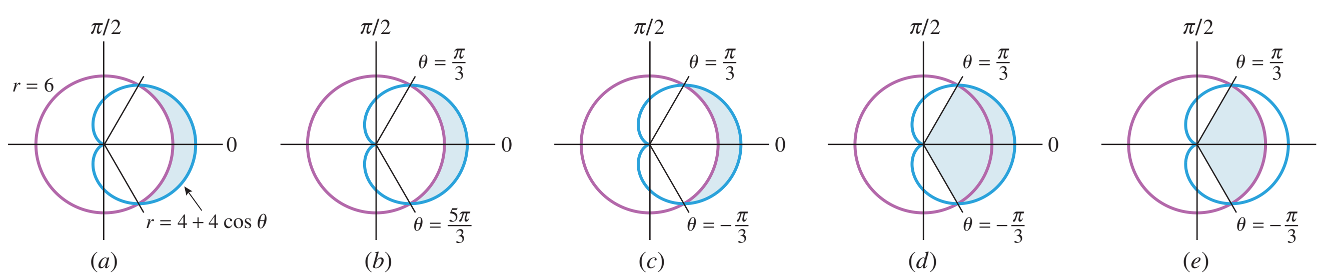


Figure 10.3.12

Example 9 Find the area of the region that is inside of the cardioid $r = 4 + 4\cos\theta$ and outside of the circle $r = 6$.

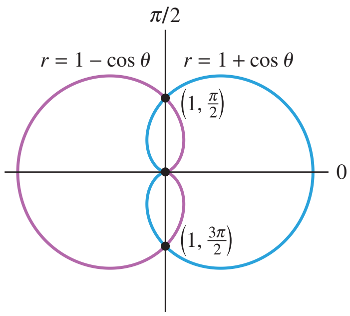
Solution Using a Negative Angle. The area of the region can be obtained by subtracting the areas in Figures 10.3.12*d* and 10.3.12*e*:

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4 + 4\cos\theta)^2\,d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(6)^2\,d\theta \qquad \boxed{\text{Area inside cardioid minus area inside circle.}} \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}[(4 + 4\cos\theta)^2 - 36]\,d\theta = \int_{-\pi/3}^{\pi/3} (16\cos\theta + 8\cos^2\theta - 10)\,d\theta \\ &= \Big[16\sin\theta + (4\theta + 2\sin 2\theta) - 10\,\theta\Big]_{-\pi/3}^{\pi/3} = 18\sqrt{3} - 4\pi \end{aligned}$$

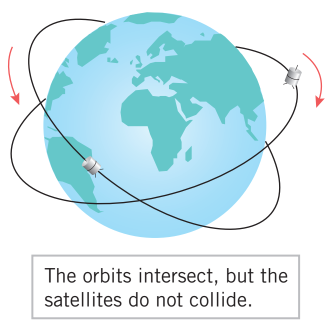
Solution Using Symmetry. Using symmetry, we can calculate the area above the polar axis and double it. This yields (verify)

$$A = 2\int_0^{\pi/3} \frac{1}{2}[(4 + 4\cos\theta)^2 - 36]\,d\theta = 2(9\sqrt{3} - 2\pi) = 18\sqrt{3} - 4\pi$$

which agrees with the preceding result. ◀



▲ **Figure 10.3.13**



▲ **Figure 10.3.14**

INTERSECTIONS OF POLAR GRAPHS

In the last example we found the intersections of the cardioid and circle by equating their expressions for r and solving for θ. However, because a point can be represented in different ways in polar coordinates, this procedure will not always produce all of the intersections. For example, the cardioids

$$r = 1 - \cos\theta \quad \text{and} \quad r = 1 + \cos\theta \tag{7}$$

intersect at three points: the pole, the point $(1, \pi/2)$, and the point $(1, 3\pi/2)$ (Figure 10.3.13). Equating the right-hand sides of the equations in (7) yields $1 - \cos\theta = 1 + \cos\theta$ or $\cos\theta = 0$, so

$$\theta = \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

Substituting any of these values in (7) yields $r = 1$, so that we have found only two distinct points of intersection, $(1, \pi/2)$ and $(1, 3\pi/2)$; the pole has been missed. This problem occurs because the two cardioids pass through the pole at different values of θ—the cardioid $r = 1 - \cos\theta$ passes through the pole at $\theta = 0$, and the cardioid $r = 1 + \cos\theta$ passes through the pole at $\theta = \pi$.

The situation with the cardioids is analogous to two satellites circling the Earth in intersecting orbits (Figure 10.3.14). The satellites will not collide unless they reach the same point at the same time. In general, when looking for intersections of polar curves, it is a good idea to graph the curves to determine how many intersections there should be.

✔ QUICK CHECK EXERCISES 10.3 *(See page 729 for answers.)*

1. (a) To obtain dy/dx directly from the polar equation $r = f(\theta)$, we can use the formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \_\_\_\_\_\_\_\_$$

(b) Use the formula in part (a) to find dy/dx directly from the polar equation $r = \csc\theta$.

2. (a) What conditions on $f(\theta_0)$ and $f'(\theta_0)$ guarantee that the line $\theta = \theta_0$ is tangent to the polar curve $r = f(\theta)$ at the origin?

(b) What are the values of θ_0 in $[0, 2\pi]$ at which the lines $\theta = \theta_0$ are tangent at the origin to the four-petal rose $r = \cos 2\theta$?

3. (a) To find the arc length L of the polar curve $r = f(\theta)$ $(\alpha \le \theta \le \beta)$, we can use the formula $L =$ ________.

(b) The polar curve $r = \sec\theta$ $(0 \le \theta \le \pi/4)$ has arc length $L =$ ________.

4. The area of the region enclosed by a nonnegative polar curve $r = f(\theta)$ $(\alpha \le \theta \le \beta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is given by the definite integral ________.

5. Find the area of the circle $r = a$ by integration.

EXERCISE SET 10.3 Graphing Utility C CAS

1–6 Find the slope of the tangent line to the polar curve for the given value of θ.

1. $r = 2\sin\theta$; $\theta = \pi/6$

2. $r = 1 + \cos\theta$; $\theta = \pi/2$

3. $r = 1/\theta$; $\theta = 2$

4. $r = a\sec 2\theta$; $\theta = \pi/6$

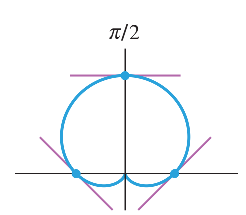
5. $r = \sin 3\theta$; $\theta = \pi/4$

6. $r = 4 - 3\sin\theta$; $\theta = \pi$

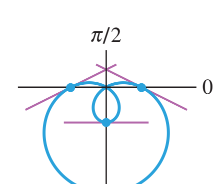
7–8 Calculate the slopes of the tangent lines indicated in the accompanying figures.

7. $r = 2 + 2\sin\theta$

8. $r = 1 - 2\sin\theta$



▲ **Figure Ex-7**



▲ **Figure Ex-8**

9–10 Find polar coordinates of all points at which the polar curve has a horizontal or a vertical tangent line.

9. $r = a(1 + \cos\theta)$

10. $r = a\sin\theta$

11–12 Use a graphing utility to make a conjecture about the number of points on the polar curve at which there is a horizontal tangent line, and confirm your conjecture by finding appropriate derivatives. ■

11. $r = \sin\theta\cos^2\theta$ **12.** $r = 1 - 2\sin\theta$

13–18 Sketch the polar curve and find polar equations of the tangent lines to the curve at the pole. ■

13. $r = 2\cos 3\theta$ **14.** $r = 4\sin\theta$ **15.** $r = 4\sqrt{\cos 2\theta}$

16. $r = \sin 2\theta$ **17.** $r = 1 - 2\cos\theta$ **18.** $r = 2\theta$

19–22 Use Formula (3) to calculate the arc length of the polar curve. ■

19. The entire circle $r = a$

20. The entire circle $r = 2a\cos\theta$

21. The entire cardioid $r = a(1 - \cos\theta)$

22. $r = e^{3\theta}$ from $\theta = 0$ to $\theta = 2$

23. (a) Show that the arc length of one petal of the rose $r = \cos n\theta$ is given by

$$2\int_0^{\pi/(2n)} \sqrt{1 + (n^2 - 1)\sin^2 n\theta}\, d\theta$$

(b) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the four-petal rose $r = \cos 2\theta$.

(c) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the n-petal rose $r = \cos n\theta$ for $n = 2, 3, 4, \ldots, 20$; then make a conjecture about the limit of these arc lengths as $n \to +\infty$.

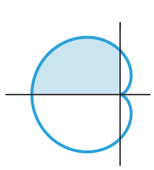
24. (a) Sketch the spiral $r = e^{-\theta/8}$ $(0 \le \theta < +\infty)$.

(b) Find an improper integral for the total arc length of the spiral.

(c) Show that the integral converges and find the total arc length of the spiral.

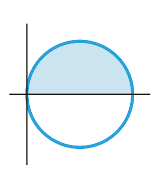
25. Write down, but do not evaluate, an integral for the area of each shaded region.

(a)



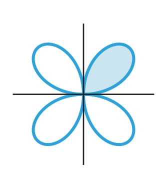
$r = 1 - \cos\theta$

(b)



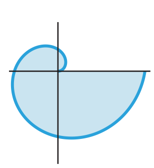
$r = 2\cos\theta$

(c)



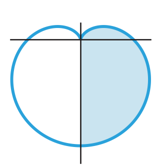
$r = \sin 2\theta$

(d)



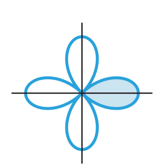
$r = \theta$

(e)



$r = 1 - \sin\theta$

(f)



$r = \cos 2\theta$

26. Find the area of the shaded region in Exercise 25(d).

27. In each part, find the area of the circle by integration.
(a) $r = 2a\sin\theta$ (b) $r = 2a\cos\theta$

28. (a) Show that $r = 2\sin\theta + 2\cos\theta$ is a circle.
(b) Find the area of the circle using a geometric formula and then by integration.

29–34 Find the area of the region described. ■

29. The region that is enclosed by the cardioid $r = 2 + 2\sin\theta$.

30. The region in the first quadrant within the cardioid $r = 1 + \cos\theta$.

31. The region enclosed by the rose $r = 4\cos 3\theta$.

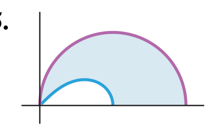
32. The region enclosed by the rose $r = 2\sin 2\theta$.

33. The region enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$. [*Hint:* $r \le 0$ over the interval of integration.]

34. The region swept out by a radial line from the pole to the curve $r = 2/\theta$ as θ varies over the interval $1 \le \theta \le 3$.

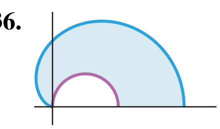
35–38 Find the area of the shaded region. ■

35.



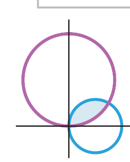
$r = \sqrt{\cos 2\theta}$
$r = 2\cos\theta$

36.



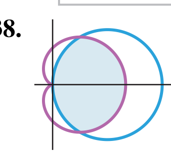
$r = 1 + \cos\theta$
$r = \cos\theta$

37.



$r = 4\cos\theta$
$r = 4\sqrt{3}\sin\theta$

38.



$r = 1 + \cos\theta$
$r = 3\cos\theta$

39–46 Find the area of the region described. ■

39. The region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

40. The region outside the cardioid $r = 2 - 2\cos\theta$ and inside the circle $r = 4$.

41. The region inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

42. The region that is common to the circles $r = 2\cos\theta$ and $r = 2\sin\theta$.

43. The region between the loops of the limaçon $r = \frac{1}{2} + \cos\theta$.

44. The region inside the cardioid $r = 2 + 2\cos\theta$ and to the right of the line $r\cos\theta = \frac{3}{2}$.

45. The region inside the circle $r = 2$ and to the right of the line $r = \sqrt{2}\sec\theta$.

46. The region inside the rose $r = 2a\cos 2\theta$ and outside the circle $r = a\sqrt{2}$.

47–50 True–False Determine whether the statement is true or false. Explain your answer.

47. The x-axis is tangent to the polar curve $r = \cos(\theta/2)$ at $\theta = 3\pi$.

48. The arc length of the polar curve $r = \sqrt{\theta}$ for $0 \le \theta \le \pi/2$ is given by

$$L = \int_0^{\pi/2} \sqrt{1 + \frac{1}{4\theta}}\, d\theta$$

49. The area of a sector with central angle θ taken from a circle of radius r is θr^2.

50. The expression

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (1 - \sqrt{2}\cos\theta)^2\, d\theta$$

computes the area enclosed by the inner loop of the limaçon $r = 1 - \sqrt{2}\cos\theta$.

FOCUS ON CONCEPTS

51. (a) Find the error: The area that is inside the lemniscate $r^2 = a^2\cos 2\theta$ is

$$A = \int_0^{2\pi} \tfrac{1}{2}r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}a^2\cos 2\theta\, d\theta$$
$$= \tfrac{1}{4}a^2 \sin 2\theta\Big]_0^{2\pi} = 0$$

(b) Find the correct area.

(c) Find the area inside the lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \sqrt{2}$.

52. Find the area inside the curve $r^2 = \sin 2\theta$.

53. A radial line is drawn from the origin to the spiral $r = a\theta$ ($a > 0$ and $\theta \ge 0$). Find the area swept out during the second revolution of the radial line that was not swept out during the first revolution.

54. As illustrated in the accompanying figure, suppose that a rod with one end fixed at the pole of a polar coordinate system rotates counterclockwise at the constant rate of 1 rad/s. At time $t = 0$ a bug on the rod is 10 mm from the pole and is moving outward along the rod at the constant speed of 2 mm/s.

(a) Find an equation of the form $r = f(\theta)$ for the path of motion of the bug, assuming that $\theta = 0$ when $t = 0$.

(b) Find the distance the bug travels along the path in part (a) during the first 5 s. Round your answer to the nearest tenth of a millimeter.

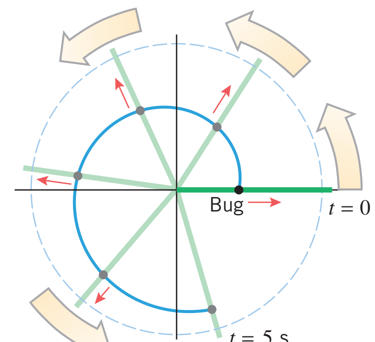


Figure Ex-54

C **55.** (a) Show that the Folium of Descartes $x^3 - 3xy + y^3 = 0$ can be expressed in polar coordinates as

$$r = \frac{3\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

(b) Use a CAS to show that the area inside of the loop is $\frac{3}{2}$ (Figure 3.1.3a).

C **56.** (a) What is the area that is enclosed by one petal of the rose $r = a\cos n\theta$ if n is an even integer?

(b) What is the area that is enclosed by one petal of the rose $r = a\cos n\theta$ if n is an odd integer?

(c) Use a CAS to show that the total area enclosed by the rose $r = a\cos n\theta$ is $\pi a^2/2$ if the number of petals is even. [*Hint:* See Exercise 78 of Section 10.2.]

(d) Use a CAS to show that the total area enclosed by the rose $r = a\cos n\theta$ is $\pi a^2/4$ if the number of petals is odd.

57. One of the most famous problems in Greek antiquity was "squaring the circle," that is, using a straightedge and compass to construct a square whose area is equal to that of a given circle. It was proved in the nineteenth century that no such construction is possible. However, show that the shaded areas in the accompanying figure are equal, thereby "squaring the crescent."

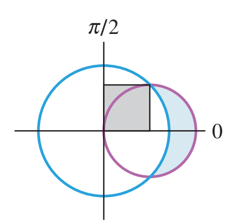


Figure Ex-57

58. Use a graphing utility to generate the polar graph of the equation $r = \cos 3\theta + 2$, and find the area that it encloses.

59. Use a graphing utility to generate the graph of the ***bifolium*** $r = 2\cos\theta\sin^2\theta$, and find the area of the upper loop.

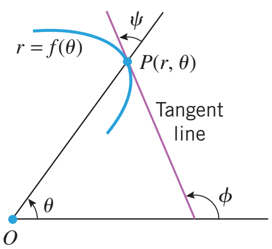
60. Use Formula (9) of Section 10.1 to derive the arc length formula for polar curves, Formula (3).

61. As illustrated in the accompanying figure, let $P(r, \theta)$ be a point on the polar curve $r = f(\theta)$, let ψ be the smallest counterclockwise angle from the extended radius OP to the tangent line at P, and let ϕ be the angle of inclination of the tangent line. Derive the formula

$$\tan\psi = \frac{r}{dr/d\theta}$$

by substituting $\tan\phi$ for dy/dx in Formula (2) and applying the trigonometric identity

$$\tan(\phi-\theta) = \frac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta}$$



◀ **Figure Ex-61**

62–63 Use the formula for ψ obtained in Exercise 61. ■

62. (a) Use the trigonometric identity

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

to show that if (r,θ) is a point on the cardioid

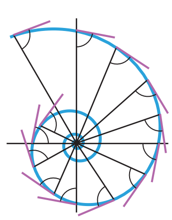
$$r = 1-\cos\theta \qquad (0 \le \theta < 2\pi)$$

then $\psi = \theta/2$.

(b) Sketch the cardioid and show the angle ψ at the points where the cardioid crosses the y-axis.

(c) Find the angle ψ at the points where the cardioid crosses the y-axis.

63. Show that for a logarithmic spiral $r = ae^{b\theta}$, the angle from the radial line to the tangent line is constant along the spiral (see the accompanying figure). [*Note:* For this reason, logarithmic spirals are sometimes called ***equiangular spirals***.]



◀ **Figure Ex-63**

64. (a) In the discussion associated with Exercises 75–80 of Section 10.1, formulas were given for the area of the surface of revolution that is generated by revolving a parametric curve about the x-axis or y-axis. Use those formulas to derive the following formulas for the areas of the surfaces of revolution that are generated by revolving the portion of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the polar axis and about the line $\theta = \pi/2$:

$$S = \int_\alpha^\beta 2\pi r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \boxed{\text{About } \theta = 0}$$

$$S = \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \boxed{\text{About } \theta = \pi/2}$$

(b) State conditions under which these formulas hold.

65–68 Sketch the surface, and use the formulas in Exercise 64 to find the surface area. ■

65. The surface generated by revolving the circle $r = \cos\theta$ about the line $\theta = \pi/2$.

66. The surface generated by revolving the spiral $r = e^\theta$ $(0 \le \theta \le \pi/2)$ about the line $\theta = \pi/2$.

67. The "apple" generated by revolving the upper half of the cardioid $r = 1-\cos\theta$ $(0 \le \theta \le \pi)$ about the polar axis.

68. The sphere of radius a generated by revolving the semicircle $r = a$ in the upper half-plane about the polar axis.

69. Writing

(a) Show that if $0 \le \theta_1 < \theta_2 \le \pi$ and if r_1 and r_2 are positive, then the area A of a triangle with vertices $(0,0)$, (r_1,θ_1), and (r_2,θ_2) is

$$A = \tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1)$$

(b) Use the formula obtained in part (a) to describe an approach to answer Area Problem 10.3.3 that uses an approximation of the region R by triangles instead of circular wedges. Reconcile your approach with Formula (6).

70. Writing In order to find the area of a region bounded by two polar curves it is often necessary to determine their points of intersection. Give an example to illustrate that the points of intersection of curves $r = f(\theta)$ and $r = g(\theta)$ may not coincide with solutions to $f(\theta) = g(\theta)$. Discuss some strategies for determining intersection points of polar curves and provide examples to illustrate your strategies.

✔ QUICK CHECK ANSWERS 10.3

1. (a) $\dfrac{r\cos\theta+\sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta+\cos\theta\dfrac{dr}{d\theta}}$ (b) $\dfrac{dy}{dx}=0$ **2.** (a) $f(\theta_0)=0,\ f'(\theta_0)\ne 0$ (b) $\theta_0 = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

3. (a) $\displaystyle\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ (b) 1 **4.** $\displaystyle\int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta$ **5.** $\displaystyle\int_0^{2\pi}\tfrac{1}{2}a^2\,d\theta = \pi a^2$

10.4 CONIC SECTIONS

In this section[*] we will discuss some of the basic geometric properties of parabolas, ellipses, and hyperbolas. These curves play an important role in calculus and also arise naturally in a broad range of applications in such fields as planetary motion, design of telescopes and antennas, geodetic positioning, and medicine, to name a few.

CONIC SECTIONS

Circles, ellipses, parabolas, and hyperbolas are called ***conic sections*** or ***conics*** because they can be obtained as intersections of a plane with a double-napped circular cone (Figure 10.4.1). If the plane passes through the vertex of the double-napped cone, then the intersection is a point, a pair of intersecting lines, or a single line. These are called ***degenerate conic sections***.

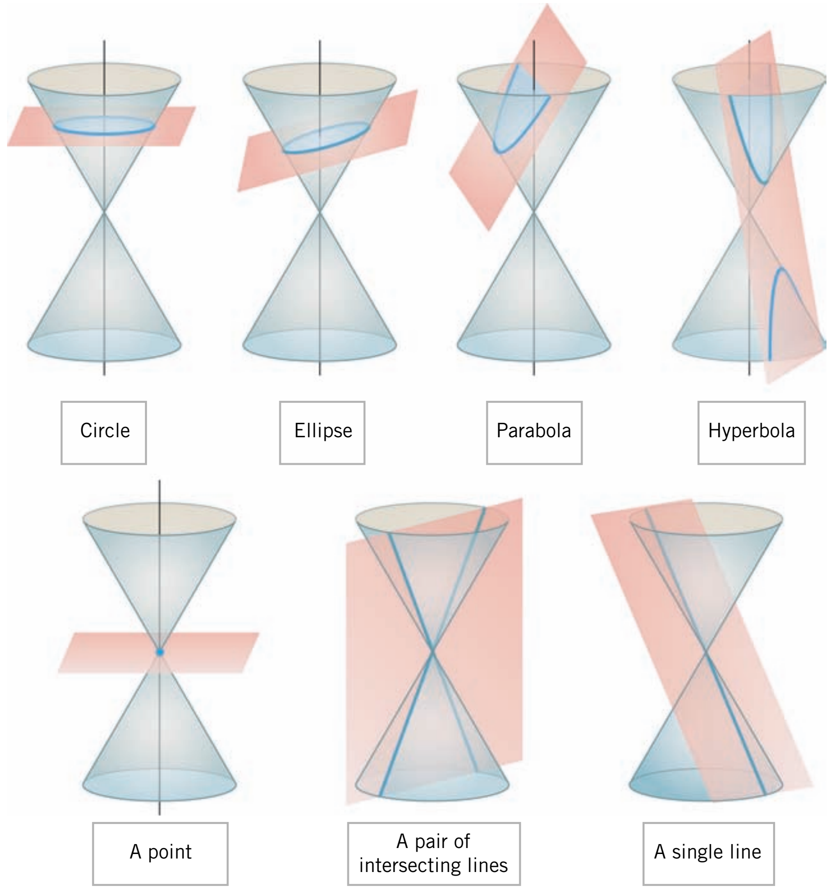


Figure 10.4.1

[*]Some students may already be familiar with the material in this section, in which case it can be treated as a review. Instructors who want to spend some additional time on precalculus review may want to allocate more than one lecture on this material.

DEFINITIONS OF THE CONIC SECTIONS

Although we could derive properties of parabolas, ellipses, and hyperbolas by defining them as intersections with a double-napped cone, it will be better suited to calculus if we begin with equivalent definitions that are based on their geometric properties.

10.4.1 DEFINITION A ***parabola*** is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

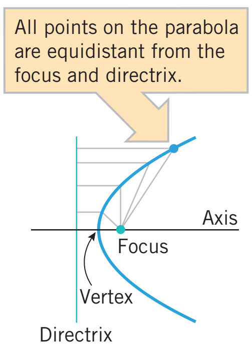


Figure 10.4.2

The line is called the ***directrix*** of the parabola, and the point is called the ***focus*** (Figure 10.4.2). A parabola is symmetric about the line that passes through the focus at right angles to the directrix. This line, called the ***axis*** or the ***axis of symmetry*** of the parabola, intersects the parabola at a point called the ***vertex***.

10.4.2 DEFINITION An ***ellipse*** is the set of all points in the plane, the sum of whose distances from two fixed points is a given positive constant that is greater than the distance between the fixed points.

The two fixed points are called the ***foci*** (plural of "focus") of the ellipse, and the midpoint of the line segment joining the foci is called the ***center*** (Figure 10.4.3*a*). To help visualize Definition 10.4.2, imagine that two ends of a string are tacked to the foci and a pencil traces a curve as it is held tight against the string (Figure 10.4.3*b*). The resulting curve will be an ellipse since the sum of the distances to the foci is a constant, namely, the total length of the string. Note that if the foci coincide, the ellipse reduces to a circle. For ellipses other than circles, the line segment through the foci and across the ellipse is called the ***major axis*** (Figure 10.4.3*c*), and the line segment across the ellipse, through the center, and perpendicular to the major axis is called the ***minor axis***. The endpoints of the major axis are called ***vertices***.

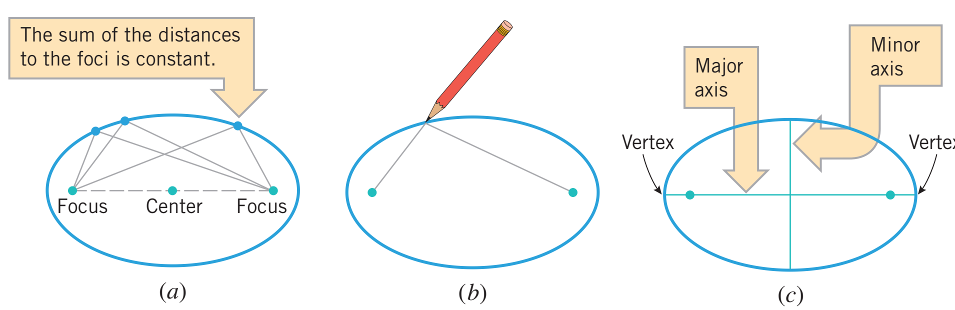
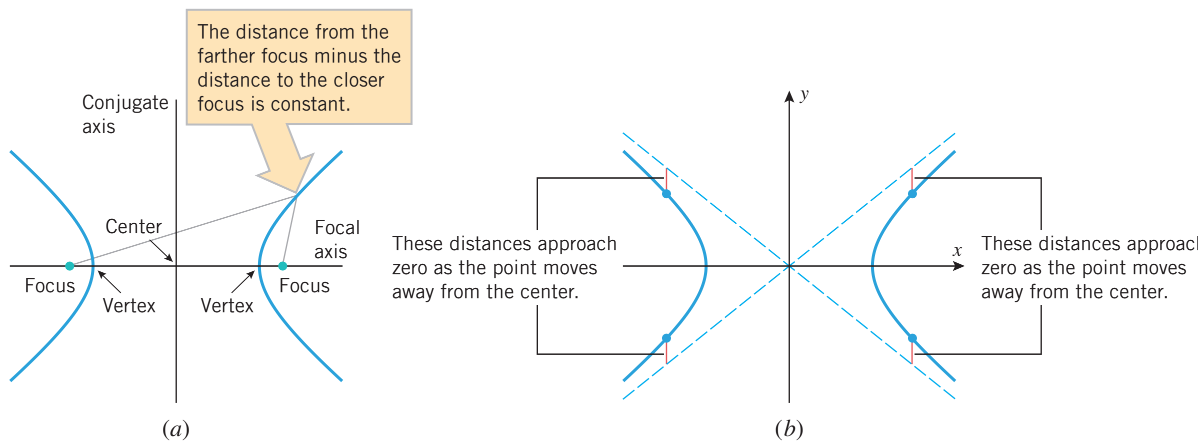


Figure 10.4.3

10.4.3 DEFINITION A ***hyperbola*** is the set of all points in the plane, the difference of whose distances from two fixed distinct points is a given positive constant that is less than the distance between the fixed points.

The two fixed points are called the ***foci*** of the hyperbola, and the term "difference" that is used in the definition is understood to mean the distance to the farther focus minus the distance to the closer focus. As a result, the points on the hyperbola form two ***branches***, each "wrapping around" the closer focus (Figure 10.4.4*a*). The midpoint of the line segment joining the foci is called the ***center*** of the hyperbola, the line through the foci is called the ***focal axis***, and the line through the center that is perpendicular to the focal axis is called the ***conjugate axis***. The hyperbola intersects the focal axis at two points called the ***vertices***.

Associated with every hyperbola is a pair of lines, called the ***asymptotes*** of the hyperbola. These lines intersect at the center of the hyperbola and have the property that as a point P moves along the hyperbola away from the center, the vertical distance between P and one of the asymptotes approaches zero (Figure 10.4.4*b*).

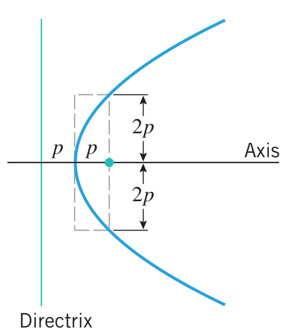


▲ **Figure 10.4.4**

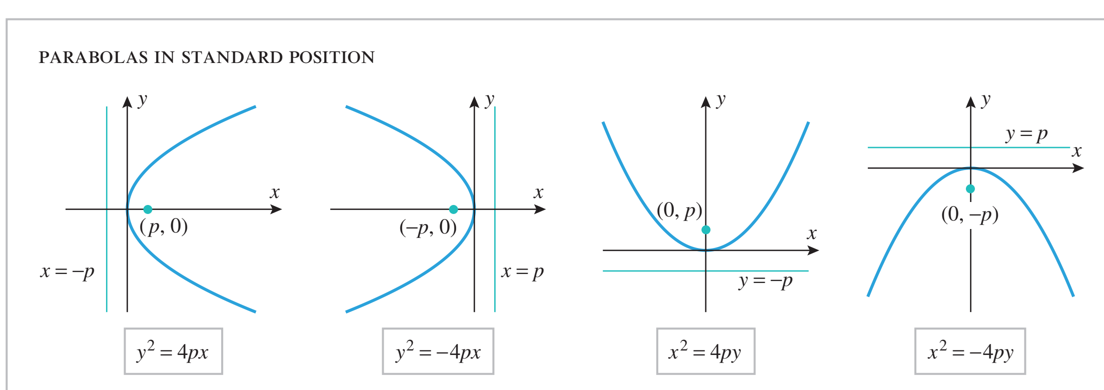
EQUATIONS OF PARABOLAS IN STANDARD POSITION

It is traditional in the study of parabolas to denote the distance between the focus and the vertex by p. The vertex is equidistant from the focus and the directrix, so the distance between the vertex and the directrix is also p; consequently, the distance between the focus and the directrix is $2p$ (Figure 10.4.5). As illustrated in that figure, the parabola passes through two of the corners of a box that extends from the vertex to the focus along the axis of symmetry and extends $2p$ units above and $2p$ units below the axis of symmetry.

The equation of a parabola is simplest if the vertex is the origin and the axis of symmetry is along the x-axis or y-axis. The four possible such orientations are shown in Figure 10.4.6. These are called the ***standard positions*** of a parabola, and the resulting equations are called the ***standard equations*** of a parabola.



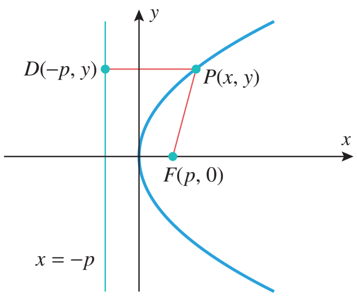
▲ **Figure 10.4.5**



▲ **Figure 10.4.6**

To illustrate how the equations in Figure 10.4.6 are obtained, we will derive the equation for the parabola with focus $(p, 0)$ and directrix $x = -p$. Let $P(x, y)$ be any point on the parabola. Since P is equidistant from the focus and directrix, the distances PF and PD in Figure 10.4.7 are equal; that is,

$$PF = PD \tag{1}$$



▲ **Figure 10.4.7**

where $D(-p, y)$ is the foot of the perpendicular from P to the directrix. From the distance formula, the distances PF and PD are

$$PF = \sqrt{(x-p)^2 + y^2} \quad \text{and} \quad PD = \sqrt{(x+p)^2} \tag{2}$$

Substituting in (1) and squaring yields

$$(x-p)^2 + y^2 = (x+p)^2 \tag{3}$$

and after simplifying

$$y^2 = 4px \tag{4}$$

The derivations of the other equations in Figure 10.4.6 are similar.

A TECHNIQUE FOR SKETCHING PARABOLAS

Parabolas can be sketched from their *standard equations* using four basic steps:

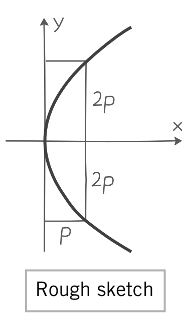
Sketching a Parabola from Its Standard Equation

Step 1. Determine whether the axis of symmetry is along the x-axis or the y-axis. Referring to Figure 10.4.6, the axis of symmetry is along the x-axis if the equation has a y^2-term, and it is along the y-axis if it has an x^2-term.

Step 2. Determine which way the parabola opens. If the axis of symmetry is along the x-axis, then the parabola opens to the right if the coefficient of x is positive, and it opens to the left if the coefficient is negative. If the axis of symmetry is along the y-axis, then the parabola opens up if the coefficient of y is positive, and it opens down if the coefficient is negative.

Step 3. Determine the value of p and draw a box extending p units from the origin along the axis of symmetry in the direction in which the parabola opens and extending $2p$ units on each side of the axis of symmetry.

Step 4. Using the box as a guide, sketch the parabola so that its vertex is at the origin and it passes through the corners of the box (Figure 10.4.8).



▲ **Figure 10.4.8**

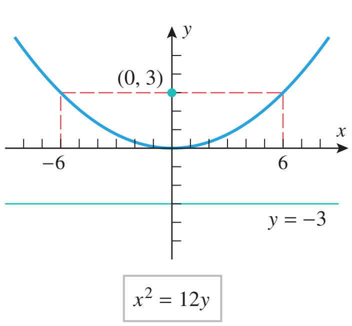
▶ **Example 1** Sketch the graphs of the parabolas

$$\text{(a)}\ x^2 = 12y \qquad \text{(b)}\ y^2 + 8x = 0$$

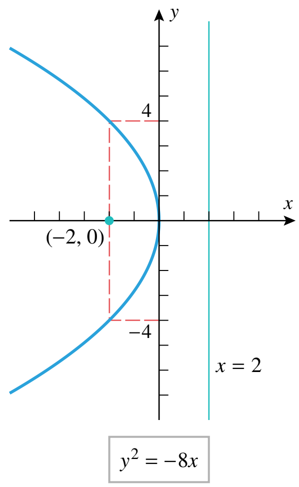
and show the focus and directrix of each.

Solution (a). This equation involves x^2, so the axis of symmetry is along the y-axis, and the coefficient of y is positive, so the parabola opens upward. From the coefficient of y, we obtain $4p = 12$ or $p = 3$. Drawing a box extending $p = 3$ units up from the origin and $2p = 6$ units to the left and $2p = 6$ units to the right of the y-axis, then using corners of the box as a guide, yields the graph in Figure 10.4.9.

The focus is $p = 3$ units from the vertex along the axis of symmetry in the direction in which the parabola opens, so its coordinates are (0, 3). The directrix is perpendicular to the axis of symmetry at a distance of $p = 3$ units from the vertex on the opposite side from the focus, so its equation is $y = -3$.



▲ **Figure 10.4.9**



▲ Figure 10.4.10

Solution (b). We first rewrite the equation in the standard form

$$y^2 = -8x$$

This equation involves y^2, so the axis of symmetry is along the x-axis, and the coefficient of x is negative, so the parabola opens to the left. From the coefficient of x we obtain $4p = 8$, so $p = 2$. Drawing a box extending $p = 2$ units left from the origin and $2p = 4$ units above and $2p = 4$ units below the x-axis, then using corners of the box as a guide, yields the graph in Figure 10.4.10. ◀

▶ **Example 2** Find an equation of the parabola that is symmetric about the y-axis, has its vertex at the origin, and passes through the point $(5, 2)$.

Solution. Since the parabola is symmetric about the y-axis and has its vertex at the origin, the equation is of the form

$$x^2 = 4py \quad \text{or} \quad x^2 = -4py$$

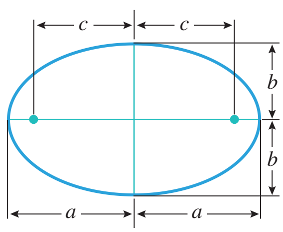
where the sign depends on whether the parabola opens up or down. But the parabola must open up since it passes through the point $(5, 2)$, which lies in the first quadrant. Thus, the equation is of the form

$$x^2 = 4py \tag{5}$$

Since the parabola passes through $(5, 2)$, we must have $5^2 = 4p \cdot 2$ or $4p = \frac{25}{2}$. Therefore, (5) becomes

$$x^2 = \tfrac{25}{2}y \quad ◀$$

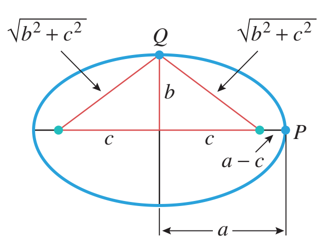
EQUATIONS OF ELLIPSES IN STANDARD POSITION



▲ Figure 10.4.11

It is traditional in the study of ellipses to denote the length of the major axis by $2a$, the length of the minor axis by $2b$, and the distance between the foci by $2c$ (Figure 10.4.11). The number a is called the ***semimajor axis*** and the number b the ***semiminor axis*** (standard but odd terminology, since a and b are numbers, not geometric axes).

There is a basic relationship between the numbers a, b, and c that can be obtained by examining the sum of the distances to the foci from a point P at the end of the major axis and from a point Q at the end of the minor axis (Figure 10.4.12). From Definition 10.4.2, these sums must be equal, so we obtain



▲ Figure 10.4.12

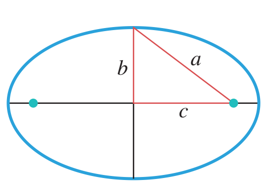
$$2\sqrt{b^2 + c^2} = (a - c) + (a + c)$$

from which it follows that

$$\boxed{a = \sqrt{b^2 + c^2}} \tag{6}$$

or, equivalently,

$$\boxed{c = \sqrt{a^2 - b^2}} \tag{7}$$



▲ Figure 10.4.13

From (6), the distance from a focus to an end of the minor axis is a (Figure 10.4.13), which implies that for *all* points on the ellipse the sum of the distances to the foci is $2a$.

It also follows from (6) that $a \geq b$ with the equality holding only when $c = 0$. Geometrically, this means that the major axis of an ellipse is at least as large as the minor axis and that the two axes have equal length only when the foci coincide, in which case the ellipse is a circle.

The equation of an ellipse is simplest if the center of the ellipse is at the origin and the foci are on the x-axis or y-axis. The two possible such orientations are shown in Figure 10.4.14.

These are called the ***standard positions*** of an ellipse, and the resulting equations are called the ***standard equations*** of an ellipse.

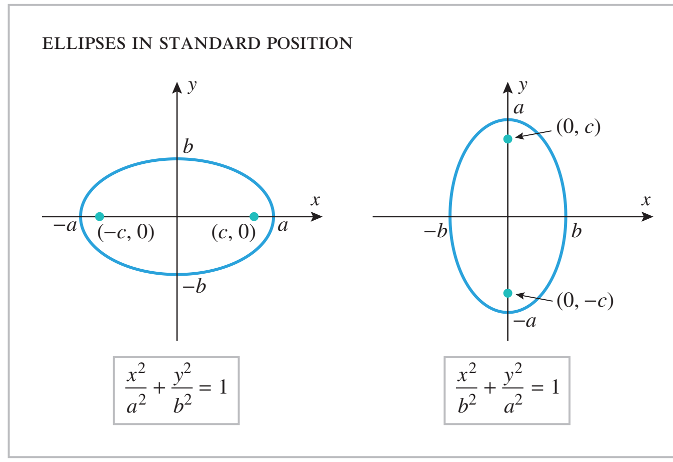


Figure 10.4.14

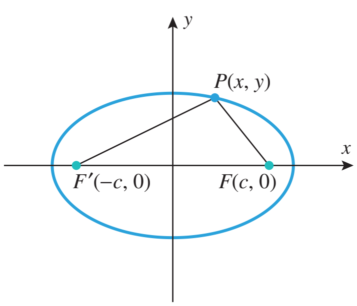


Figure 10.4.15

To illustrate how the equations in Figure 10.4.14 are obtained, we will derive the equation for the ellipse with foci on the x-axis. Let $P(x, y)$ be any point on that ellipse. Since the sum of the distances from P to the foci is $2a$, it follows (Figure 10.4.15) that

$$PF' + PF = 2a$$

so

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Transposing the second radical to the right side of the equation and squaring yields

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

and, on simplifying,

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x \tag{8}$$

Squaring again and simplifying yields

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

which, by virtue of (6), can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

Conversely, it can be shown that any point whose coordinates satisfy (9) has $2a$ as the sum of its distances from the foci, so that such a point is on the ellipse.

A TECHNIQUE FOR SKETCHING ELLIPSES

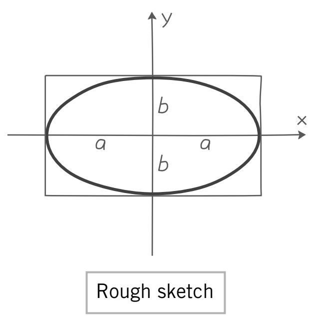
Ellipses can be sketched from their *standard equations* using three basic steps:

Sketching an Ellipse from Its Standard Equation

Step 1. Determine whether the major axis is on the x-axis or the y-axis. This can be ascertained from the sizes of the denominators in the equation. Referring to Figure 10.4.14, and keeping in mind that $a^2 > b^2$ (since $a > b$), the major axis is along the x-axis if x^2 has the larger denominator, and it is along the y-axis if y^2 has the larger denominator. If the denominators are equal, the ellipse is a circle.

Step 2. Determine the values of a and b and draw a box extending a units on each side of the center along the major axis and b units on each side of the center along the minor axis.

Step 3. Using the box as a guide, sketch the ellipse so that its center is at the origin and it touches the sides of the box where the sides intersect the coordinate axes (Figure 10.4.16).



▲ **Figure 10.4.16**

▶ **Example 3** Sketch the graphs of the ellipses

$$\text{(a)}\ \frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{(b)}\ x^2 + 2y^2 = 4$$

showing the foci of each.

Solution (a). Since y^2 has the larger denominator, the major axis is along the y-axis. Moreover, since $a^2 > b^2$, we must have $a^2 = 16$ and $b^2 = 9$, so

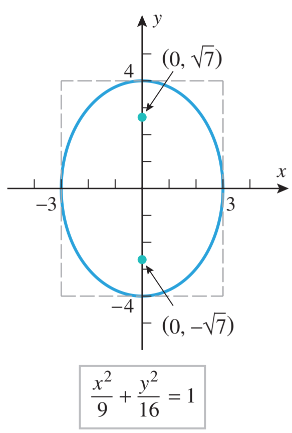
$$a = 4 \quad \text{and} \quad b = 3$$

Drawing a box extending 4 units on each side of the origin along the y-axis and 3 units on each side of the origin along the x-axis as a guide yields the graph in Figure 10.4.17.

The foci lie c units on each side of the center along the major axis, where c is given by (7). From the values of a^2 and b^2 above, we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7} \approx 2.6$$

Thus, the coordinates of the foci are $(0, \sqrt{7})$ and $(0, -\sqrt{7})$, since they lie on the y-axis.



▲ **Figure 10.4.17**

Solution (b). We first rewrite the equation in the standard form

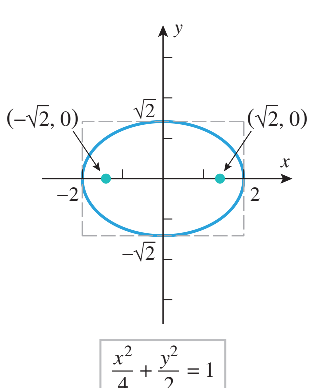
$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$

Since x^2 has the larger denominator, the major axis lies along the x-axis, and we have $a^2 = 4$ and $b^2 = 2$. Drawing a box extending $a = 2$ units on each side of the origin along the x-axis and extending $b = \sqrt{2} \approx 1.4$ units on each side of the origin along the y-axis as a guide yields the graph in Figure 10.4.18.

From (7), we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{2} \approx 1.4$$

Thus, the coordinates of the foci are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$, since they lie on the x-axis. ◀



▲ **Figure 10.4.18**

► Example 4 Find an equation for the ellipse with foci $(0, \pm 2)$ and major axis with endpoints $(0, \pm 4)$.

Solution. From Figure 10.4.14, the equation has the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

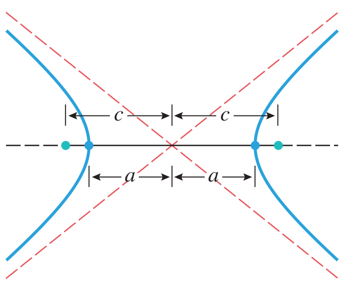
and from the given information, $a = 4$ and $c = 2$. It follows from (6) that

$$b^2 = a^2 - c^2 = 16 - 4 = 12$$

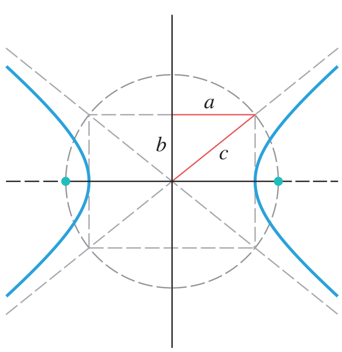
so the equation of the ellipse is

$$\frac{x^2}{12} + \frac{y^2}{16} = 1 \quad ◄$$

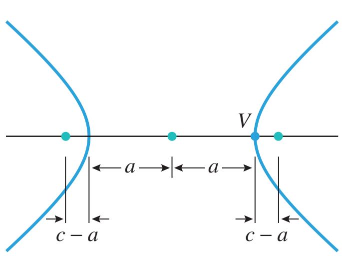
EQUATIONS OF HYPERBOLAS IN STANDARD POSITION



▲ **Figure 10.4.19**



▲ **Figure 10.4.20**



▲ **Figure 10.4.21**

It is traditional in the study of hyperbolas to denote the distance between the vertices by $2a$, the distance between the foci by $2c$ (Figure 10.4.19), and to define the quantity b as

$$\boxed{b = \sqrt{c^2 - a^2}} \qquad (10)$$

This relationship, which can also be expressed as

$$\boxed{c = \sqrt{a^2 + b^2}} \qquad (11)$$

is pictured geometrically in Figure 10.4.20. As illustrated in that figure, and as we will show later in this section, the asymptotes pass through the corners of a box extending b units on each side of the center along the conjugate axis and a units on each side of the center along the focal axis. The number a is called the ***semifocal axis*** of the hyperbola and the number b the ***semiconjugate axis***. (As with the semimajor and semiminor axes of an ellipse, these are numbers, not geometric axes.)

If V is one vertex of a hyperbola, then, as illustrated in Figure 10.4.21, the distance from V to the farther focus minus the distance from V to the closer focus is

$$[(c - a) + 2a] - (c - a) = 2a$$

Thus, for *all* points on a hyperbola, the distance to the farther focus minus the distance to the closer focus is $2a$.

The equation of a hyperbola has an especially convenient form if the center of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two possible such orientations are shown in Figure 10.4.22. These are called the ***standard positions*** of a hyperbola, and the resulting equations are called the ***standard equations*** of a hyperbola.

The derivations of these equations are similar to those already given for parabolas and ellipses, so we will leave them as exercises. However, to illustrate how the equations of the asymptotes are derived, we will derive those equations for the hyperbola

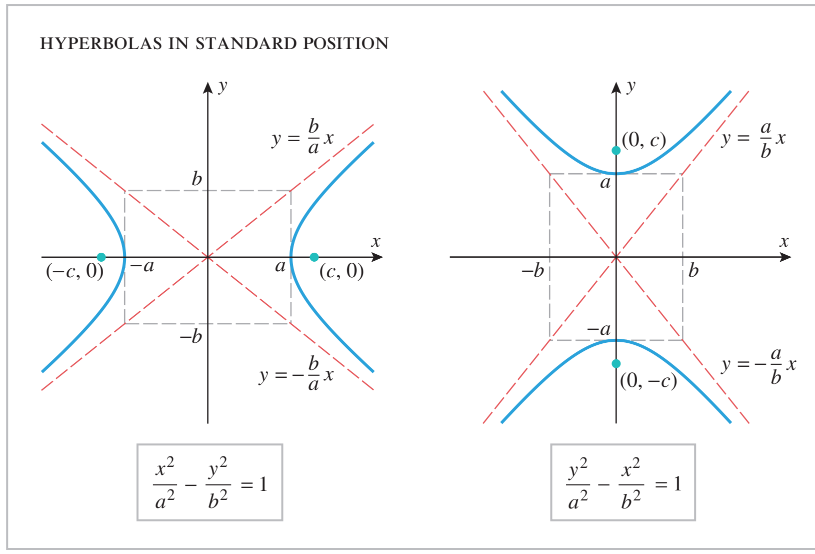
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We can rewrite this equation as

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

which is equivalent to the pair of equations

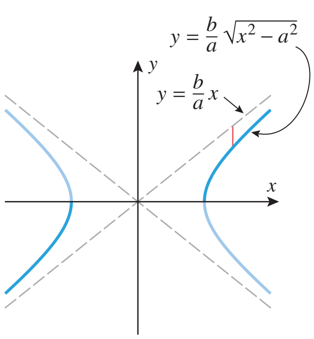
$$y = \frac{b}{a}\sqrt{x^2 - a^2} \quad \text{and} \quad y = -\frac{b}{a}\sqrt{x^2 - a^2}$$



▶ **Figure 10.4.22**

Thus, in the first quadrant, the vertical distance between the line $y = (b/a)x$ and the hyperbola can be written as

$$\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}$$



▲ **Figure 10.4.23**

(Figure 10.4.23). But this distance tends to zero as $x \to +\infty$ since

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) &= \lim_{x\to+\infty}\frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right) \\
&= \lim_{x\to+\infty}\frac{b}{a}\frac{\left(x - \sqrt{x^2 - a^2}\right)\left(x + \sqrt{x^2 - a^2}\right)}{x + \sqrt{x^2 - a^2}} \\
&= \lim_{x\to+\infty}\frac{ab}{x + \sqrt{x^2 - a^2}} = 0
\end{aligned}$$

The analysis in the remaining quadrants is similar.

A QUICK WAY TO FIND ASYMPTOTES

There is a trick that can be used to avoid memorizing the equations of the asymptotes of a hyperbola. They can be obtained, when needed, by replacing 1 by 0 on the right side of the hyperbola equation, and then solving for y in terms of x. For example, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we would write

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y^2 = \frac{b^2}{a^2}x^2 \quad \text{or} \quad y = \pm\frac{b}{a}x$$

which are the equations for the asymptotes.

A TECHNIQUE FOR SKETCHING HYPERBOLAS

Hyperbolas can be sketched from their *standard equations* using four basic steps:

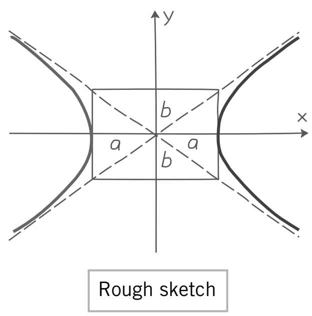
Sketching a Hyperbola from Its Standard Equation

Step 1. Determine whether the focal axis is on the x-axis or the y-axis. This can be ascertained from the location of the minus sign in the equation. Referring to Figure 10.4.22, the focal axis is along the x-axis when the minus sign precedes the y^2-term, and it is along the y-axis when the minus sign precedes the x^2-term.

Step 2. Determine the values of a and b and draw a box extending a units on either side of the center along the focal axis and b units on either side of the center along the conjugate axis. (The squares of a and b can be read directly from the equation.)

Step 3. Draw the asymptotes along the diagonals of the box.

Step 4. Using the box and the asymptotes as a guide, sketch the graph of the hyperbola (Figure 10.4.24).



▲ Figure 10.4.24

Example 5 Sketch the graphs of the hyperbolas

$$\text{(a)}\ \frac{x^2}{4} - \frac{y^2}{9} = 1 \qquad \text{(b)}\ y^2 - x^2 = 1$$

showing their vertices, foci, and asymptotes.

Solution (a). The minus sign precedes the y^2-term, so the focal axis is along the x-axis. From the denominators in the equation we obtain

$$a^2 = 4 \quad \text{and} \quad b^2 = 9$$

Since a and b are positive, we must have $a = 2$ and $b = 3$. Recalling that the vertices lie a units on each side of the center on the focal axis, it follows that their coordinates in this case are $(2, 0)$ and $(-2, 0)$. Drawing a box extending $a = 2$ units along the x-axis on each side of the origin and $b = 3$ units on each side of the origin along the y-axis, then drawing the asymptotes along the diagonals of the box as a guide, yields the graph in Figure 10.4.25.

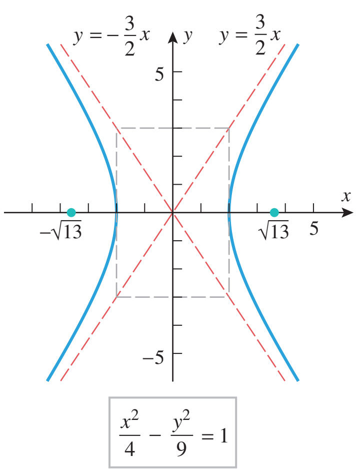
To obtain equations for the asymptotes, we replace 1 by 0 in the given equation; this yields

$$\frac{x^2}{4} - \frac{y^2}{9} = 0 \quad \text{or} \quad y = \pm\frac{3}{2}x$$

The foci lie c units on each side of the center along the focal axis, where c is given by (11). From the values of a^2 and b^2 above we obtain

$$c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13} \approx 3.6$$

Since the foci lie on the x-axis in this case, their coordinates are $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$.

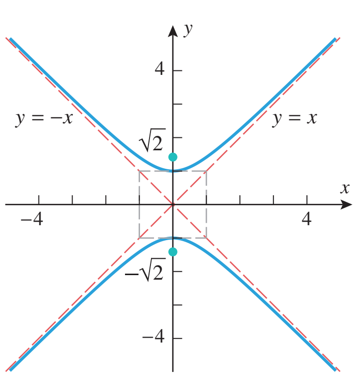


▲ Figure 10.4.25

Solution (b). The minus sign precedes the x^2-term, so the focal axis is along the y-axis. From the denominators in the equation we obtain $a^2 = 1$ and $b^2 = 1$, from which it follows that

$$a = 1 \quad \text{and} \quad b = 1$$

Thus, the vertices are at $(0, -1)$ and $(0, 1)$. Drawing a box extending $a = 1$ unit on either side of the origin along the y-axis and $b = 1$ unit on either side of the origin along the x-axis, then drawing the asymptotes, yields the graph in Figure 10.4.26. Since the box is actually



▲ Figure 10.4.26

A hyperbola in which $a = b$, as in part (b) of Example 5, is called an ***equilateral hyperbola***. Such hyperbolas always have perpendicular asymptotes.

a square, the asymptotes are perpendicular and have equations $y = \pm x$. This can also be seen by replacing 1 by 0 in the given equation, which yields $y^2 - x^2 = 0$ or $y = \pm x$. Also,

$$c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2}$$

so the foci, which lie on the y-axis, are $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. ◀

▶ Example 6 Find the equation of the hyperbola with vertices $(0, \pm 8)$ and asymptotes $y = \pm\frac{4}{3}x$.

Solution. Since the vertices are on the y-axis, the equation of the hyperbola has the form $(y^2/a^2) - (x^2/b^2) = 1$ and the asymptotes are

$$y = \pm\frac{a}{b}x$$

From the locations of the vertices we have $a = 8$, so the given equations of the asymptotes yield

$$y = \pm\frac{a}{b}x = \pm\frac{8}{b}x = \pm\frac{4}{3}x$$

from which it follows that $b = 6$. Thus, the hyperbola has the equation

$$\frac{y^2}{64} - \frac{x^2}{36} = 1 \quad ◀$$

TRANSLATED CONICS

Equations of conics that are translated from their standard positions can be obtained by replacing x by $x - h$ and y by $y - k$ in their standard equations. For a parabola, this translates the vertex from the origin to the point (h, k); and for ellipses and hyperbolas, this translates the center from the origin to the point (h, k).

Parabolas with vertex (h, k) and axis parallel to x-axis

$$(y - k)^2 = 4p(x - h) \quad \text{[Opens right]} \tag{12}$$

$$(y - k)^2 = -4p(x - h) \quad \text{[Opens left]} \tag{13}$$

Parabolas with vertex (h, k) and axis parallel to y-axis

$$(x - h)^2 = 4p(y - k) \quad \text{[Opens up]} \tag{14}$$

$$(x - h)^2 = -4p(y - k) \quad \text{[Opens down]} \tag{15}$$

Ellipse with center (h, k) and major axis parallel to x-axis

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad [b < a] \tag{16}$$

Ellipse with center (h, k) and major axis parallel to y-axis

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \quad [b < a] \tag{17}$$

Hyperbola with center (h, k) and focal axis parallel to x-axis

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \tag{18}$$

Hyperbola with center (h, k) and focal axis parallel to y-axis

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \tag{19}$$

Example 7 Find an equation for the parabola that has its vertex at $(1, 2)$ and its focus at $(4, 2)$.

Solution. Since the focus and vertex are on a horizontal line, and since the focus is to the right of the vertex, the parabola opens to the right and its equation has the form

$$(y-k)^2 = 4p(x-h)$$

Since the vertex and focus are 3 units apart, we have $p = 3$, and since the vertex is at $(h, k) = (1, 2)$, we obtain

$$(y-2)^2 = 12(x-1)$$

Sometimes the equations of translated conics occur in expanded form, in which case we are faced with the problem of identifying the graph of a quadratic equation in x and y:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{20}$$

The basic procedure for determining the nature of such a graph is to complete the squares of the quadratic terms and then try to match up the resulting equation with one of the forms of a translated conic.

Example 8 Describe the graph of the equation

$$y^2 - 8x - 6y - 23 = 0$$

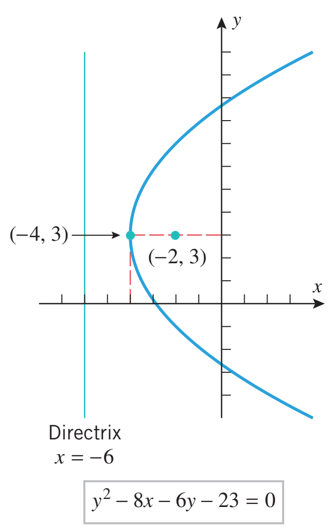


Figure 10.4.27

Solution. The equation involves quadratic terms in y but none in x, so we first take all of the y-terms to one side:

$$y^2 - 6y = 8x + 23$$

Next, we complete the square on the y-terms by adding 9 to both sides:

$$(y-3)^2 = 8x + 32$$

Finally, we factor out the coefficient of the x-term to obtain

$$(y-3)^2 = 8(x+4)$$

This equation is of form (12) with $h = -4$, $k = 3$, and $p = 2$, so the graph is a parabola with vertex $(-4, 3)$ opening to the right. Since $p = 2$, the focus is 2 units to the right of the vertex, which places it at the point $(-2, 3)$; and the directrix is 2 units to the left of the vertex, which means that its equation is $x = -6$. The parabola is shown in Figure 10.4.27.

Example 9 Describe the graph of the equation

$$16x^2 + 9y^2 - 64x - 54y + 1 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

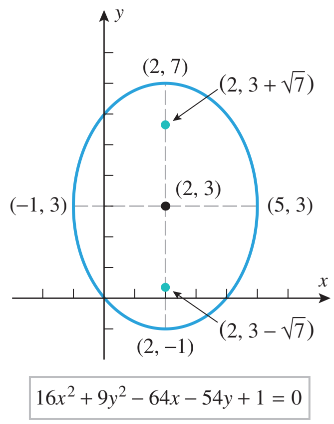
$$(16x^2 - 64x) + (9y^2 - 54y) = -1$$

Next, factor out the coefficients of x^2 and y^2 and complete the squares:

$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$

or

$$16(x-2)^2 + 9(y-3)^2 = 144$$



▲ **Figure 10.4.28**

Finally, divide through by 144 to introduce a 1 on the right side:

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

This is an equation of form (17), with $h = 2$, $k = 3$, $a^2 = 16$, and $b^2 = 9$. Thus, the graph of the equation is an ellipse with center $(2, 3)$ and major axis parallel to the y-axis. Since $a = 4$, the major axis extends 4 units above and 4 units below the center, so its endpoints are $(2, 7)$ and $(2, -1)$ (Figure 10.4.28). Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are $(-1, 3)$ and $(5, 3)$. Since

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

the foci lie $\sqrt{7}$ units above and below the center, placing them at the points $(2, 3 + \sqrt{7})$ and $(2, 3 - \sqrt{7})$. ◀

▶ **Example 10** Describe the graph of the equation

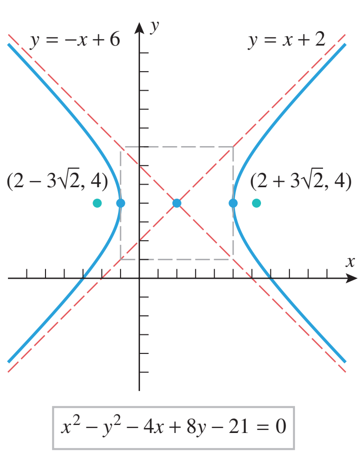
$$x^2 - y^2 - 4x + 8y - 21 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(x^2 - 4x) - (y^2 - 8y) = 21$$

We leave it for you to verify by completing the squares that this equation can be written as

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 1 \tag{21}$$



▲ **Figure 10.4.29**

This is an equation of form (18) with $h = 2$, $k = 4$, $a^2 = 9$, and $b^2 = 9$. Thus, the equation represents a hyperbola with center $(2, 4)$ and focal axis parallel to the x-axis. Since $a = 3$, the vertices are located 3 units to the left and 3 units to the right of the center, or at the points $(-1, 4)$ and $(5, 4)$. From (11), $c = \sqrt{a^2 + b^2} = \sqrt{9 + 9} = 3\sqrt{2}$, so the foci are located $3\sqrt{2}$ units to the left and right of the center, or at the points $(2 - 3\sqrt{2}, 4)$ and $(2 + 3\sqrt{2}, 4)$.

The equations of the asymptotes may be found using the trick of replacing 1 by 0 in (21) to obtain

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 0$$

This can be written as $y - 4 = \pm(x - 2)$, which yields the asymptotes

$$y = x + 2 \quad \text{and} \quad y = -x + 6$$

With the aid of a box extending $a = 3$ units left and right of the center and $b = 3$ units above and below the center, we obtain the sketch in Figure 10.4.29. ◀

REFLECTION PROPERTIES OF THE CONIC SECTIONS

Parabolas, ellipses, and hyperbolas have certain reflection properties that make them extremely valuable in various applications. In the exercises we will ask you to prove the following results.

> **10.4.4 THEOREM (*Reflection Property of Parabolas*)** *The tangent line at a point P on a parabola makes equal angles with the line through P parallel to the axis of symmetry and the line through P and the focus (Figure 10.4.30a).*

10.4.5 THEOREM (*Reflection Property of Ellipses*) *A line tangent to an ellipse at a point P makes equal angles with the lines joining P to the foci (Figure 10.4.30b).*

10.4.6 THEOREM (*Reflection Property of Hyperbolas*) *A line tangent to a hyperbola at a point P makes equal angles with the lines joining P to the foci (Figure 10.4.30c).*

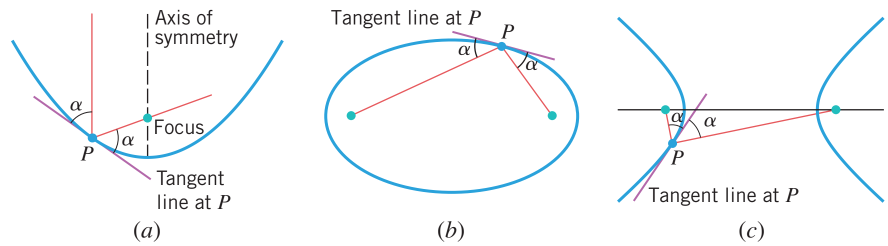


Figure 10.4.30

APPLICATIONS OF THE CONIC SECTIONS

John Mead/Science Photo Library/Photo Researchers

Incoming signals are reflected by the parabolic antenna to the receiver at the focus.

Fermat's principle in optics implies that light reflects off of a surface at an angle equal to its angle of incidence. (See Exercise 62 in Section 4.5.) In particular, if a reflecting surface is generated by revolving a parabola about its axis of symmetry, it follows from Theorem 10.4.4 that all light rays entering parallel to the axis will be reflected to the focus (Figure 10.4.31*a*); conversely, if a light source is located at the focus, then the reflected rays will all be parallel to the axis (Figure 10.4.31*b*). This principle is used in certain telescopes to reflect the approximately parallel rays of light from the stars and planets off of a parabolic mirror to an eyepiece at the focus; and the parabolic reflectors in flashlights and automobile headlights utilize this principle to form a parallel beam of light rays from a bulb placed at the focus. The same optical principles apply to radar signals and sound waves, which explains the parabolic shape of many antennas.

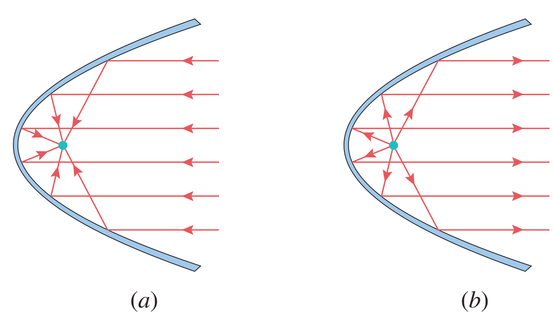
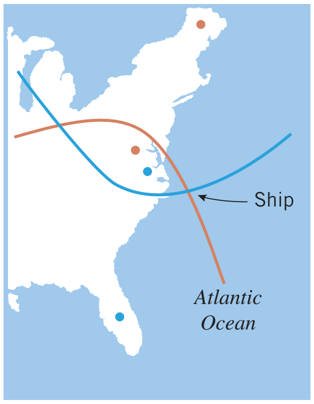


Figure 10.4.31

Visitors to various rooms in the United States Capitol Building and in St. Paul's Cathedral in London are often astonished by the "whispering gallery" effect in which two people at opposite ends of the room can hear one another's whispers very clearly. Such rooms have ceilings with elliptical cross sections and common foci. Thus, when the two people stand at the foci, their whispers are reflected directly to one another off of the elliptical ceiling.

Hyperbolic navigation systems, which were developed in World War II as navigational aids to ships, are based on the definition of a hyperbola. With these systems the ship receives



▲ **Figure 10.4.32**

synchronized radio signals from two widely spaced transmitters with known positions. The ship's electronic receiver measures the difference in reception times between the signals and then uses that difference to compute the difference $2a$ between its distances from the two transmitters. This information places the ship somewhere on the hyperbola whose foci are at the transmitters and whose points have $2a$ as the difference in their distances from the foci. By repeating the process with a second set of transmitters, the position of the ship can be approximated as the intersection of two hyperbolas (Figure 10.4.32). (The modern global positioning system (GPS) is based on the same principle.)

✔ QUICK CHECK EXERCISES 10.4 (See page 748 for answers.)

1. Identify the conic.
(a) The set of points in the plane, the sum of whose distances to two fixed points is a positive constant greater than the distance between the fixed points is ________.
(b) The set of points in the plane, the difference of whose distances to two fixed points is a positive constant less than the distance between the fixed points is ________.
(c) The set of points in the plane that are equidistant from a fixed line and a fixed point not on the line is ________.

2. (a) The equation of the parabola with focus $(p, 0)$ and directrix $x = -p$ is ________.
(b) The equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is ________.

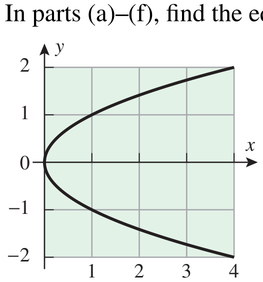
3. (a) Suppose that an ellipse has semimajor axis a and semiminor axis b. Then for all points on the ellipse, the sum of the distances to the foci is equal to ________.
(b) The two standard equations of an ellipse with semimajor axis a and semiminor axis b are ________ and ________.
(c) Suppose that an ellipse has semimajor axis a, semiminor axis b, and foci $(\pm c, 0)$. Then c may be obtained from a and b by the equation $c =$ ________.

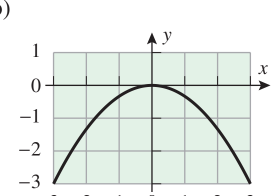
4. (a) Suppose that a hyperbola has semifocal axis a and semiconjugate axis b. Then for all points on the hyperbola, the difference of the distance to the farther focus minus the distance to the closer focus is equal to ________.
(b) The two standard equations of a hyperbola with semifocal axis a and semiconjugate axis b are ________ and ________.
(c) Suppose that a hyperbola in standard position has semifocal axis a, semiconjugate axis b, and foci $(\pm c, 0)$. Then c may be obtained from a and b by the equation $c =$ ________. The equations of the asymptotes of this hyperbola are $y = \pm$ ________.

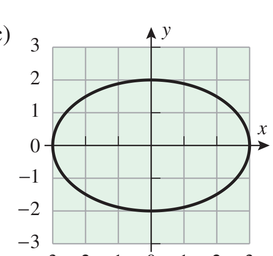
EXERCISE SET 10.4 Graphing Utility

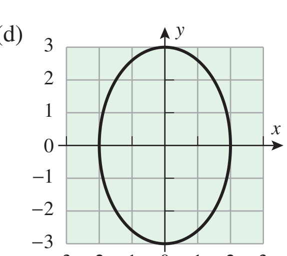
FOCUS ON CONCEPTS

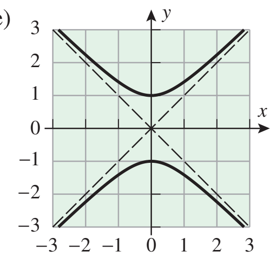
1. In parts (a)–(f), find the equation of the conic.

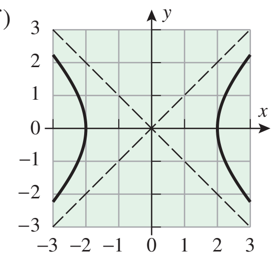
(a)


(b)


(c)


(d)


(e)


(f)


2. (a) Find the focus and directrix for each parabola in Exercise 1.
(b) Find the foci of the ellipses in Exercise 1.
(c) Find the foci and the equations of the asymptotes of the hyperbolas in Exercise 1.

3–6 Sketch the parabola, and label the focus, vertex, and directrix. ■

3. (a) $y^2 = 4x$ (b) $x^2 = -8y$

4. (a) $y^2 = -10x$ (b) $x^2 = 4y$

5. (a) $(y-1)^2 = -12(x+4)$ (b) $(x-1)^2 = 2\left(y - \frac{1}{2}\right)$

6. (a) $y^2 - 6y - 2x + 1 = 0$ (b) $y = 4x^2 + 8x + 5$

7–10 Sketch the ellipse, and label the foci, vertices, and ends of the minor axis. ■

7. (a) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ (b) $9x^2 + y^2 = 9$

8. (a) $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ (b) $4x^2 + y^2 = 36$

9. (a) $(x+3)^2 + 4(y-5)^2 = 16$
(b) $\frac{1}{4}x^2 + \frac{1}{9}(y+2)^2 - 1 = 0$

10. (a) $9x^2 + 4y^2 - 18x + 24y + 9 = 0$
(b) $5x^2 + 9y^2 + 20x - 54y = -56$

11–14 Sketch the hyperbola, and label the vertices, foci, and asymptotes. ■

11. (a) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ (b) $9y^2 - x^2 = 36$

12. (a) $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$ (b) $16x^2 - 25y^2 = 400$

13. (a) $\dfrac{(y+4)^2}{3} - \dfrac{(x-2)^2}{5} = 1$
(b) $16(x+1)^2 - 8(y-3)^2 = 16$

14. (a) $x^2 - 4y^2 + 2x + 8y - 7 = 0$
(b) $16x^2 - y^2 - 32x - 6y = 57$

15–18 Find an equation for the parabola that satisfies the given conditions. ■

15. (a) Vertex $(0, 0)$; focus $(3, 0)$.
(b) Vertex $(0, 0)$; directrix $y = \frac{1}{4}$.

16. (a) Focus $(6, 0)$; directrix $x = -6$.
(b) Focus $(1, 1)$; directrix $y = -2$.

17. Axis $y = 0$; passes through $(3, 2)$ and $(2, -\sqrt{2})$.

18. Vertex $(5, -3)$; axis parallel to the y-axis; passes through $(9, 5)$.

19–22 Find an equation for the ellipse that satisfies the given conditions. ■

19. (a) Ends of major axis $(\pm 3, 0)$; ends of minor axis $(0, \pm 2)$.
(b) Length of minor axis 8; foci $(0, \pm 3)$.

20. (a) Foci $(\pm 1, 0)$; $b = \sqrt{2}$.
(b) $c = 2\sqrt{3}$; $a = 4$; center at the origin; foci on a coordinate axis (two answers).

21. (a) Ends of major axis $(0, \pm 6)$; passes through $(-3, 2)$.
(b) Foci $(-1, 1)$ and $(-1, 3)$; minor axis of length 4.

22. (a) Center at $(0, 0)$; major and minor axes along the coordinate axes; passes through $(3, 2)$ and $(1, 6)$.
(b) Foci $(2, 1)$ and $(2, -3)$; major axis of length 6.

23–26 Find an equation for a hyperbola that satisfies the given conditions. [*Note:* In some cases there may be more than one hyperbola.] ■

23. (a) Vertices $(\pm 2, 0)$; foci $(\pm 3, 0)$.
(b) Vertices $(0, \pm 2)$; asymptotes $y = \pm\frac{2}{3}x$.

24. (a) Asymptotes $y = \pm\frac{3}{2}x$; $b = 4$.
(b) Foci $(0, \pm 5)$; asymptotes $y = \pm 2x$.

25. (a) Asymptotes $y = \pm\frac{3}{4}x$; $c = 5$.
(b) Foci $(\pm 3, 0)$; asymptotes $y = \pm 2x$.

26. (a) Vertices $(0, 6)$ and $(6, 6)$; foci 10 units apart.
(b) Asymptotes $y = x - 2$ and $y = -x + 4$; passes through the origin.

27–30 True–False Determine whether the statement is true or false. Explain your answer. ■

27. A hyperbola is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

28. If an ellipse is not a circle, then the foci of an ellipse lie on the major axis of the ellipse.

29. If a parabola has equation $y^2 = 4px$, where p is a positive constant, then the perpendicular distance from the parabola's focus to its directrix is p.

30. The hyperbola $(y^2/a^2) - x^2 = 1$ has asymptotes the lines $y = \pm x/a$.

31. (a) As illustrated in the accompanying figure, a parabolic arch spans a road 40 ft wide. How high is the arch if a center section of the road 20 ft wide has a minimum clearance of 12 ft?
(b) How high would the center be if the arch were the upper half of an ellipse?

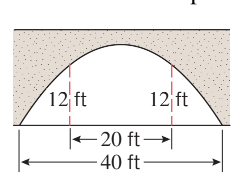


Figure Ex-31

32. (a) Find an equation for the parabolic arch with base b and height h, shown in the accompanying figure.
(b) Find the area under the arch.

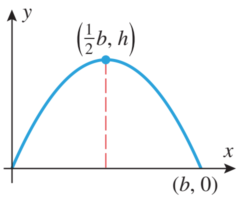


Figure Ex-32

33. Show that the vertex is the closest point on a parabola to the focus. [*Suggestion:* Introduce a convenient coordinate system and use Definition 10.4.1.]

34. As illustrated in the accompanying figure, suppose that a comet moves in a parabolic orbit with the Sun at its focus and that the line from the Sun to the comet makes an angle of 60° with the axis of the parabola when the comet is 40 million miles from the center of the Sun. Use the result in Exercise 33 to determine how close the comet will come to the center of the Sun.

35. For the parabolic reflector in the accompanying figure, how far from the vertex should the light source be placed to produce a beam of parallel rays?

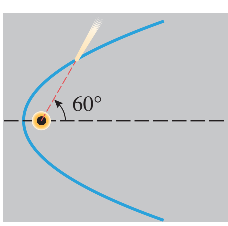


Figure Ex-34

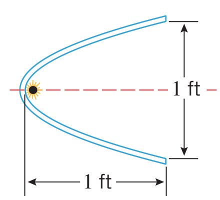


Figure Ex-35

36. (a) Show that the right and left branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be represented parametrically as

$$x = a\cosh t, \quad y = b\sinh t \qquad (-\infty < t < +\infty)$$
$$x = -a\cosh t, \quad y = b\sinh t \qquad (-\infty < t < +\infty)$$

(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

37. (a) Show that the right and left branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be represented parametrically as

$$x = a\sec t, \quad y = b\tan t \qquad (-\pi/2 < t < \pi/2)$$
$$x = -a\sec t, \quad y = b\tan t \qquad (-\pi/2 < t < \pi/2)$$

(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

38. Find an equation of the parabola traced by a point that moves so that its distance from (2, 4) is the same as its distance to the x-axis.

39. Find an equation of the ellipse traced by a point that moves so that the sum of its distances to (4, 1) and (4, 5) is 12.

40. Find the equation of the hyperbola traced by a point that moves so that the difference between its distances to (0, 0) and (1, 1) is 1.

41. Show that an ellipse with semimajor axis a and semiminor axis b has area $A = \pi ab$.

FOCUS ON CONCEPTS

42. Show that if a plane is not parallel to the axis of a right circular cylinder, then the intersection of the plane and cylinder is an ellipse (possibly a circle). [*Hint:* Let θ be the angle shown in the accompanying figure, introduce coordinate axes as shown, and express x' and y' in terms of x and y.]

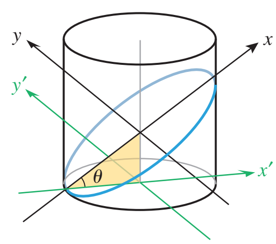


Figure Ex-42

43. As illustrated in the accompanying figure, a carpenter needs to cut an elliptical hole in a sloped roof through which a circular vent pipe of diameter D is to be inserted vertically. The carpenter wants to draw the outline of the hole on the roof using a pencil, two tacks, and a piece of string (as in Figure 10.4.3b). The center point of the ellipse is known, and common sense suggests that its major axis must be perpendicular to the drip line of the roof. The carpenter needs to determine the length L of the string and the distance T between a tack and the center point. The architect's plans show that the pitch of the roof is p (pitch = rise over run; see the accompanying figure). Find T and L in terms of D and p.

Source: This exercise is based on an article by William H. Enos, which appeared in the *Mathematics Teacher*, Feb. 1991, p. 148.

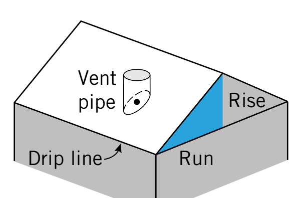
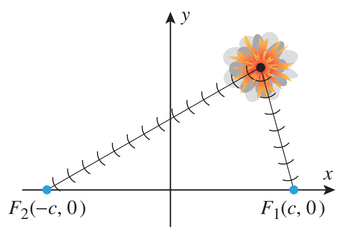


Figure Ex-43

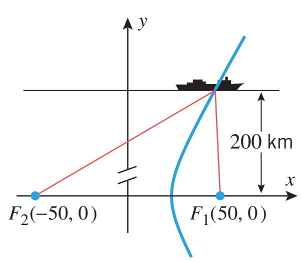
44. As illustrated in the accompanying figure on the next page, suppose that two observers are stationed at the points $F_1(c, 0)$ and $F_2(-c, 0)$ in an xy-coordinate system. Suppose also that the sound of an explosion in the xy-plane is heard by the F_1 observer t seconds before it is heard by the F_2 observer. Assuming that the speed of sound is a constant v, show that the explosion occurred somewhere on the hyperbola

$$\frac{x^2}{v^2t^2/4} - \frac{y^2}{c^2 - (v^2t^2/4)} = 1$$



◀ **Figure Ex-44**

45. As illustrated in the accompanying figure, suppose that two transmitting stations are positioned 100 km apart at points $F_1(50, 0)$ and $F_2(-50, 0)$ on a straight shoreline in an xy-coordinate system. Suppose also that a ship is traveling parallel to the shoreline but 200 km at sea. Find the coordinates of the ship if the stations transmit a pulse simultaneously, but the pulse from station F_1 is received by the ship 100 microseconds sooner than the pulse from station F_2. [*Hint:* Use the formula obtained in Exercise 44, assuming that the pulses travel at the speed of light (299,792,458 m/s).]



◀ **Figure Ex-45**

46. A nuclear cooling tower is to have a height of h feet and the shape of the solid that is generated by revolving the region R enclosed by the right branch of the hyperbola $1521x^2 - 225y^2 = 342{,}225$ and the lines $x = 0$, $y = -h/2$, and $y = h/2$ about the y-axis.
(a) Find the volume of the tower.
(b) Find the lateral surface area of the tower.

47. Let R be the region that is above the x-axis and enclosed between the curve $b^2x^2 - a^2y^2 = a^2b^2$ and the line $x = \sqrt{a^2 + b^2}$.
(a) Sketch the solid generated by revolving R about the x-axis, and find its volume.
(b) Sketch the solid generated by revolving R about the y-axis, and find its volume.

48. Prove: The line tangent to the parabola $x^2 = 4py$ at the point (x_0, y_0) is $x_0x = 2p(y + y_0)$.

49. Prove: The line tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has the equation

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

50. Prove: The line tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has the equation

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

51. Use the results in Exercises 49 and 50 to show that if an ellipse and a hyperbola have the same foci, then at each point of intersection their tangent lines are perpendicular.

52. Consider the second-degree equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0. Show by completing the square:
(a) If $AC > 0$, then the equation represents an ellipse, a circle, a point, or has no graph.
(b) If $AC < 0$, then the equation represents a hyperbola or a pair of intersecting lines.
(c) If $AC = 0$, then the equation represents a parabola, a pair of parallel lines, or has no graph.

53. In each part, use the result in Exercise 52 to make a statement about the graph of the equation, and then check your conclusion by completing the square and identifying the graph.
(a) $x^2 - 5y^2 - 2x - 10y - 9 = 0$
(b) $x^2 - 3y^2 - 6y - 3 = 0$
(c) $4x^2 + 8y^2 + 16x + 16y + 20 = 0$
(d) $3x^2 + y^2 + 12x + 2y + 13 = 0$
(e) $x^2 + 8x + 2y + 14 = 0$
(f) $5x^2 + 40x + 2y + 94 = 0$

54. Derive the equation $x^2 = 4py$ in Figure 10.4.6.

55. Derive the equation $(x^2/b^2) + (y^2/a^2) = 1$ given in Figure 10.4.14.

56. Derive the equation $(x^2/a^2) - (y^2/b^2) = 1$ given in Figure 10.4.22.

57. Prove Theorem 10.4.4. [*Hint:* Choose coordinate axes so that the parabola has the equation $x^2 = 4py$. Show that the tangent line at $P(x_0, y_0)$ intersects the y-axis at $Q(0, -y_0)$ and that the triangle whose three vertices are at P, Q, and the focus is isosceles.]

58. Given two intersecting lines, let L_2 be the line with the larger angle of inclination ϕ_2, and let L_1 be the line with the smaller angle of inclination ϕ_1. We define the ***angle θ between L_1 and L_2*** by $\theta = \phi_2 - \phi_1$. (See the accompanying figure on the next page.)
(a) Prove: If L_1 and L_2 are not perpendicular, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$$

where L_1 and L_2 have slopes m_1 and m_2.
(b) Prove Theorem 10.4.5. [*Hint:* Introduce coordinates so that the equation $(x^2/a^2) + (y^2/b^2) = 1$ describes the ellipse, and use part (a).]

(cont.)

(c) Prove Theorem 10.4.6. [*Hint:* Introduce coordinates so that the equation $(x^2/a^2) - (y^2/b^2) = 1$ describes the hyperbola, and use part (a).]

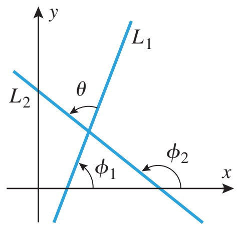


Figure Ex-58

59. Writing Suppose that you want to draw an ellipse that has given values for the lengths of the major and minor axes by using the method shown in Figure 10.4.3*b*. Assuming that the axes are drawn, explain how a compass can be used to locate the positions for the tacks.

60. Writing List the forms for standard equations of parabolas, ellipses, and hyperbolas, and write a summary of techniques for sketching conic sections from their standard equations.

✔QUICK CHECK ANSWERS 10.4

1. (a) an ellipse (b) a hyperbola (c) a parabola **2.** (a) $y^2 = 4px$ (b) $x^2 = 4py$
3. (a) $2a$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ (c) $\sqrt{a^2 - b^2}$ **4.** (a) $2a$ (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ (c) $\sqrt{a^2 + b^2}$; $\dfrac{b}{a}x$

10.5 ROTATION OF AXES; SECOND-DEGREE EQUATIONS

In the preceding section we obtained equations of conic sections with axes parallel to the coordinate axes. In this section we will study the equations of conics that are "tilted" relative to the coordinate axes. This will lead us to investigate rotations of coordinate axes.

QUADRATIC EQUATIONS IN *x* AND *y*

We saw in Examples 8 to 10 of the preceding section that equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

can represent conic sections. Equation (1) is a special case of the more general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

which, if A, B, and C are not all zero, is called a ***quadratic equation*** in x and y. It is usually the case that the graph of any second-degree equation is a conic section. If $B = 0$, then (2) reduces to (1) and the conic section has its axis or axes parallel to the coordinate axes. However, if $B \neq 0$, then (2) contains a ***cross-product term*** Bxy, and the graph of the conic section represented by the equation has its axis or axes "tilted" relative to the coordinate axes. As an illustration, consider the ellipse with foci $F_1(1, 2)$ and $F_2(-1, -2)$ and such that the sum of the distances from each point $P(x, y)$ on the ellipse to the foci is 6 units. Expressing this condition as an equation, we obtain (Figure 10.5.1)

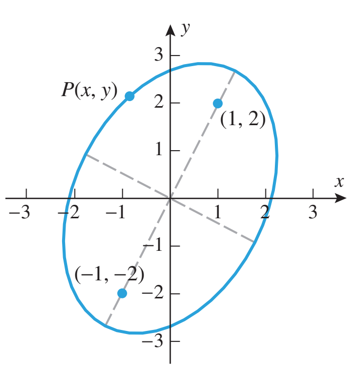


Figure 10.5.1

$$\sqrt{(x-1)^2 + (y-2)^2} + \sqrt{(x+1)^2 + (y+2)^2} = 6$$

Squaring both sides, then isolating the remaining radical, then squaring again ultimately yields

$$8x^2 - 4xy + 5y^2 = 36$$

as the equation of the ellipse. This is of form (2) with $A = 8$, $B = -4$, $C = 5$, $D = 0$, $E = 0$, and $F = -36$.

ROTATION OF AXES

To study conics that are tilted relative to the coordinate axes it is frequently helpful to rotate the coordinate axes, so that the rotated coordinate axes are parallel to the axes of the conic. Before we can discuss the details, we need to develop some ideas about rotation of coordinate axes.

In Figure 10.5.2*a* the axes of an xy-coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'$-coordinate system. As shown in the figure, each point P in the plane has coordinates (x', y') as well as coordinates (x, y). To see how the two are related, let r be the distance from the common origin to the point P, and let α be the angle shown in Figure 10.5.2*b*. It follows that

$$x = r\cos(\theta + \alpha), \quad y = r\sin(\theta + \alpha) \tag{3}$$

and

$$x' = r\cos\alpha, \quad y' = r\sin\alpha \tag{4}$$

Using familiar trigonometric identities, the relationships in (3) can be written as

$$\begin{aligned} x &= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ y &= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{aligned}$$

and on substituting (4) in these equations we obtain the following relationships called the ***rotation equations***:

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \tag{5}$$

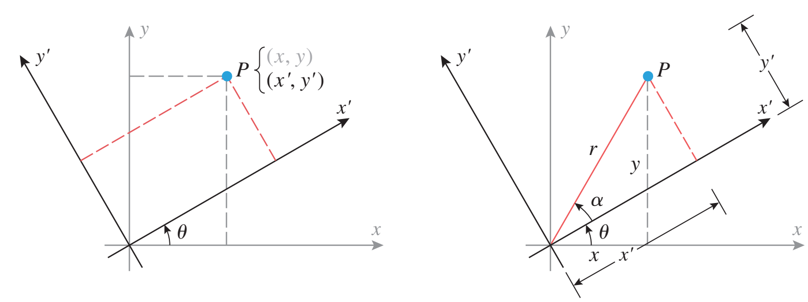


Figure 10.5.2 (*a*) (*b*)

Example 1 Suppose that the axes of an xy-coordinate system are rotated through an angle of $\theta = 45°$ to obtain an $x'y'$-coordinate system. Find the equation of the curve

$$x^2 - xy + y^2 - 6 = 0$$

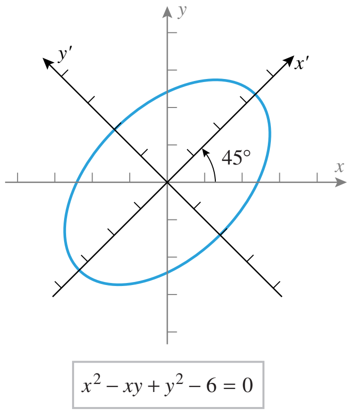
in $x'y'$-coordinates.

Solution. Substituting $\sin\theta = \sin 45° = 1/\sqrt{2}$ and $\cos\theta = \cos 45° = 1/\sqrt{2}$ in (5) yields the rotation equations

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

Substituting these into the given equation yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - 6 = 0$$



▲ **Figure 10.5.3**

or

$$\frac{x'^2 - 2x'y' + y'^2 - x'^2 + y'^2 + x'^2 + 2x'y' + y'^2}{2} = 6$$

or

$$\frac{x'^2}{12} + \frac{y'^2}{4} = 1$$

which is the equation of an ellipse (Figure 10.5.3). ◀

If the rotation equations (5) are solved for x' and y' in terms of x and y, one obtains (Exercise 16):

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \tag{6}$$

▶ **Example 2** Find the new coordinates of the point (2, 4) if the coordinate axes are rotated through an angle of $\theta = 30^\circ$.

Solution. Using the rotation equations in (6) with $x = 2$, $y = 4$, $\cos\theta = \cos 30^\circ = \sqrt{3}/2$, and $\sin\theta = \sin 30^\circ = 1/2$, we obtain

$$\begin{aligned} x' &= 2(\sqrt{3}/2) + 4(1/2) = \sqrt{3} + 2 \\ y' &= -2(1/2) + 4(\sqrt{3}/2) = -1 + 2\sqrt{3} \end{aligned}$$

Thus, the new coordinates are $(\sqrt{3} + 2, -1 + 2\sqrt{3})$. ◀

ELIMINATING THE CROSS-PRODUCT TERM

In Example 1 we were able to identify the curve $x^2 - xy + y^2 - 6 = 0$ as an ellipse because the rotation of axes eliminated the xy-term, thereby reducing the equation to a familiar form. This occurred because the new $x'y'$-axes were aligned with the axes of the ellipse. The following theorem tells how to determine an appropriate rotation of axes to eliminate the cross-product term of a second-degree equation in x and y.

10.5.1 THEOREM *If the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{7}$$

is such that $B \neq 0$, and if an $x'y'$-coordinate system is obtained by rotating the xy-axes through an angle θ satisfying

$$\cot 2\theta = \frac{A - C}{B} \tag{8}$$

then, in $x'y'$-coordinates, Equation (7) will have the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

It is always possible to satisfy (8) with an angle θ in the interval

$$0 < \theta < \pi/2$$

We will always choose θ in this way.

PROOF Substituting (5) into (7) and simplifying yields

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

where

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
B' &= B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
D' &= D\cos\theta + E\sin\theta \\
E' &= -D\sin\theta + E\cos\theta \\
F' &= F
\end{aligned} \tag{9}$$

(Verify.) To complete the proof we must show that $B' = 0$ if

$$\cot 2\theta = \frac{A - C}{B}$$

or, equivalently,

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B} \tag{10}$$

However, by using the trigonometric double-angle formulas, we can rewrite B' in the form

$$B' = B\cos 2\theta - (A - C)\sin 2\theta$$

Thus, $B' = 0$ if θ satisfies (10). ■

▶ **Example 3** Identify and sketch the curve $xy = 1$.

Solution. As a first step, we will rotate the coordinate axes to eliminate the cross-product term. Comparing the given equation to (7), we have

$$A = 0, \quad B = 1, \quad C = 0$$

Thus, the desired angle of rotation must satisfy

$$\cot 2\theta = \frac{A - C}{B} = \frac{0 - 0}{1} = 0$$

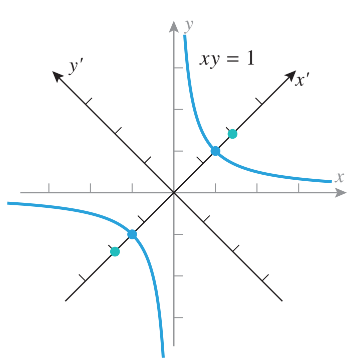
This condition can be met by taking $2\theta = \pi/2$ or $\theta = \pi/4 = 45°$. Making the substitutions $\cos\theta = \cos 45° = 1/\sqrt{2}$ and $\sin\theta = \sin 45° = 1/\sqrt{2}$ in (5) yields

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

Substituting these in the equation $xy = 1$ yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 1 \quad \text{or} \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

which is the equation in the $x'y'$-coordinate system of an equilateral hyperbola with vertices at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ in that coordinate system (Figure 10.5.4). ◀



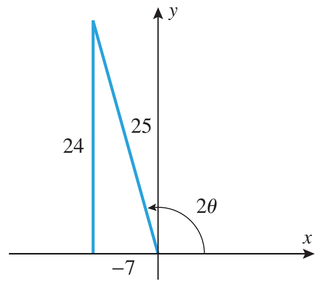
▲ Figure 10.5.4

In problems where it is inconvenient to solve

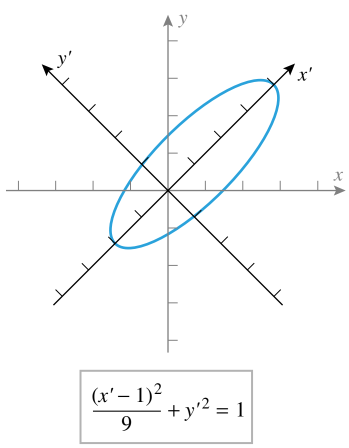
$$\cot 2\theta = \frac{A - C}{B}$$

for θ, the values of $\sin\theta$ and $\cos\theta$ needed for the rotation equations can be obtained by first calculating $\cos 2\theta$ and then computing $\sin\theta$ and $\cos\theta$ from the identities

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$$



▲ Figure 10.5.5



▲ Figure 10.5.6

There is a method for deducing the kind of curve represented by a second-degree equation directly from the equation itself without rotating coordinate axes. For a discussion of this topic, see the section on the ***discriminant*** that appears in Web Appendix K.

▶ **Example 4** Identify and sketch the curve

$$153x^2 - 192xy + 97y^2 - 30x - 40y - 200 = 0$$

Solution. We have $A = 153$, $B = -192$, and $C = 97$, so

$$\cot 2\theta = \frac{A-C}{B} = -\frac{56}{192} = -\frac{7}{24}$$

Since θ is to be chosen in the range $0 < \theta < \pi/2$, this relationship is represented by the triangle in Figure 10.5.5. From that triangle we obtain $\cos 2\theta = -\frac{7}{25}$, which implies that

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \frac{3}{5}$$

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{7}{25}}{2}} = \frac{4}{5}$$

Substituting these values in (5) yields the rotation equations

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y' \quad \text{and} \quad y = \tfrac{4}{5}x' + \tfrac{3}{5}y'$$

and substituting these in turn in the given equation yields

$$\tfrac{153}{25}(3x' - 4y')^2 - \tfrac{192}{25}(3x' - 4y')(4x' + 3y') + \tfrac{97}{25}(4x' + 3y')^2$$
$$- \tfrac{30}{5}(3x' - 4y') - \tfrac{40}{5}(4x' + 3y') - 200 = 0$$

which simplifies to

$$25x'^2 + 225y'^2 - 50x' - 200 = 0$$

or

$$x'^2 + 9y'^2 - 2x' - 8 = 0$$

Completing the square yields

$$\frac{(x'-1)^2}{9} + y'^2 = 1$$

which is the equation in the $x'y'$-coordinate system of an ellipse with center $(1, 0)$ in that coordinate system and semiaxes $a = 3$ and $b = 1$ (Figure 10.5.6). ◀

✔ QUICK CHECK EXERCISES 10.5

(See page 754 for answers.)

1. Suppose that an xy-coordinate system is rotated θ radians to produce a new $x'y'$-coordinate system.
(a) x and y may be obtained from x', y', and θ using the rotation equations $x =$ ________ and $y =$ ________.
(b) x' and y' may be obtained from x, y, and θ using the equations $x' =$ ________ and $y' =$ ________.

2. If the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is such that $B \neq 0$, then the xy-term in this equation can be eliminated by a rotation of axes through an angle θ satisfying $\cot 2\theta =$ ________.

3. In each part, determine a rotation angle θ that will eliminate the xy-term.
(a) $2x^2 + xy + 2y^2 + x - y = 0$
(b) $x^2 + 2\sqrt{3}xy + 3y^2 - 2x + y = 1$
(c) $3x^2 + \sqrt{3}xy + 2y^2 + y = 0$

4. Express $2x^2 + xy + 2y^2 = 1$ in the $x'y'$-coordinate system obtained by rotating the xy-coordinate system through the angle $\theta = \pi/4$.

EXERCISE SET 10.5

1. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of $\theta = 60°$.
   (a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(-2, 6)$.
   (b) Find an equation of the curve $\sqrt{3}xy + y^2 = 6$ in $x'y'$-coordinates.
   (c) Sketch the curve in part (b), showing both xy-axes and $x'y'$-axes.

2. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of $\theta = 30°$.
   (a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(1, -\sqrt{3})$.
   (b) Find an equation of the curve $2x^2 + 2\sqrt{3}xy = 3$ in $x'y'$-coordinates.
   (c) Sketch the curve in part (b), showing both xy-axes and $x'y'$-axes.

3–12 Rotate the coordinate axes to remove the xy-term. Then identify the type of conic and sketch its graph. ■

3. $xy = -9$
4. $x^2 - xy + y^2 - 2 = 0$
5. $x^2 + 4xy - 2y^2 - 6 = 0$
6. $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$
7. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$
8. $34x^2 - 24xy + 41y^2 - 25 = 0$
9. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$
10. $5x^2 - 6xy + 5y^2 - 8\sqrt{2}x + 8\sqrt{2}y = 8$
11. $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$
12. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$
13. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of 45°. Use (6) to find an equation of the curve $3x'^2 + y'^2 = 6$ in xy-coordinates.
14. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of 30°. Use (5) to find an equation in $x'y'$-coordinates of the curve $y = x^2$.

FOCUS ON CONCEPTS

15. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle θ. Prove: For every value of θ, the equation $x^2 + y^2 = r^2$ becomes the equation $x'^2 + y'^2 = r^2$. Give a geometric explanation.
16. Derive (6) by solving the rotation equations in (5) for x' and y' in terms of x and y.
17. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle θ. Explain how to find the xy-coordinates of a point whose $x'y'$-coordinates are known.
18. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle θ. Explain how to find the xy-equation of a line whose $x'y'$-equation is known.

19–22 Show that the graph of the given equation is a parabola. Find its vertex, focus, and directrix. ■

19. $x^2 + 2xy + y^2 + 4\sqrt{2}x - 4\sqrt{2}y = 0$
20. $x^2 - 2\sqrt{3}xy + 3y^2 - 8\sqrt{3}x - 8y = 0$
21. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$
22. $x^2 + 2\sqrt{3}xy + 3y^2 + 16\sqrt{3}x - 16y - 96 = 0$

23–26 Show that the graph of the given equation is an ellipse. Find its foci, vertices, and the ends of its minor axis. ■

23. $288x^2 - 168xy + 337y^2 - 3600 = 0$
24. $25x^2 - 14xy + 25y^2 - 288 = 0$
25. $31x^2 + 10\sqrt{3}xy + 21y^2 - 32x + 32\sqrt{3}y - 80 = 0$
26. $43x^2 - 14\sqrt{3}xy + 57y^2 - 36\sqrt{3}x - 36y - 540 = 0$

27–30 Show that the graph of the given equation is a hyperbola. Find its foci, vertices, and asymptotes. ■

27. $x^2 - 10\sqrt{3}xy + 11y^2 + 64 = 0$
28. $17x^2 - 312xy + 108y^2 - 900 = 0$
29. $32y^2 - 52xy - 7x^2 + 72\sqrt{5}x - 144\sqrt{5}y + 900 = 0$
30. $2\sqrt{2}y^2 + 5\sqrt{2}xy + 2\sqrt{2}x^2 + 18x + 18y + 36\sqrt{2} = 0$
31. Show that the graph of the equation
$$\sqrt{x} + \sqrt{y} = 1$$
is a portion of a parabola. [*Hint:* First rationalize the equation and then perform a rotation of axes.]

FOCUS ON CONCEPTS

32. Derive the expression for B' in (9).
33. Use (9) to prove that $B^2 - 4AC = B'^2 - 4A'C'$ for all values of θ.
34. Use (9) to prove that $A + C = A' + C'$ for all values of θ.
35. Prove: If $A = C$ in (7), then the cross-product term can be eliminated by rotating through 45°.
36. Prove: If $B \neq 0$, then the graph of $x^2 + Bxy + F = 0$ is a hyperbola if $F \neq 0$ and two intersecting lines if $F = 0$.

✔QUICK CHECK ANSWERS 10.5

1. (a) $x'\cos\theta - y'\sin\theta;\ x'\sin\theta + y'\cos\theta$ (b) $x\cos\theta + y\sin\theta;\ -x\sin\theta + y\cos\theta$ **2.** $\dfrac{A-C}{B}$ **3.** (a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{6}$
4. $5x'^2 + 3y'^2 = 2$

10.6 CONIC SECTIONS IN POLAR COORDINATES

It will be shown later in the text that if an object moves in a gravitational field that is directed toward a fixed point (such as the center of the Sun), then the path of that object must be a conic section with the fixed point at a focus. For example, planets in our solar system move along elliptical paths with the Sun at a focus, and the comets move along parabolic, elliptical, or hyperbolic paths with the Sun at a focus, depending on the conditions under which they were born. For applications of this type it is usually desirable to express the equations of the conic sections in polar coordinates with the pole at a focus. In this section we will show how to do this.

THE FOCUS–DIRECTRIX CHARACTERIZATION OF CONICS

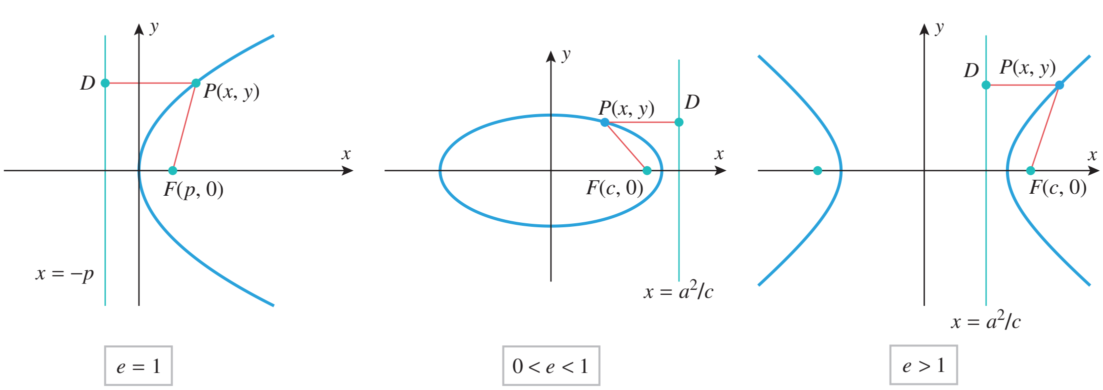
To obtain polar equations for the conic sections we will need the following theorem.

It is an unfortunate historical accident that the letter e is used for the base of the natural logarithm as well as for the eccentricity of conic sections. However, as a practical matter the appropriate interpretation will usually be clear from the context in which the letter is used.

10.6.1 THEOREM (*Focus–Directrix Property of Conics*) *Suppose that a point P moves in the plane determined by a fixed point (called the **focus**) and a fixed line (called the **directrix**), where the focus does not lie on the directrix. If the point moves in such a way that its distance to the focus divided by its distance to the directrix is some constant e (called the **eccentricity**), then the curve traced by the point is a conic section. Moreover, the conic is*

(a) a parabola if $e = 1$ (b) an ellipse if $0 < e < 1$ (c) a hyperbola if $e > 1$.

We will not give a formal proof of this theorem; rather, we will use the specific cases in Figure 10.6.1 to illustrate the basic ideas. For the parabola, we will take the directrix to be $x = -p$, as usual; and for the ellipse and the hyperbola we will take the directrix to be $x = a^2/c$. We want to show in all three cases that if P is a point on the graph, F is the focus, and D is the directrix, then the ratio PF/PD is some constant e, where $e = 1$ for the parabola, $0 < e < 1$ for the ellipse, and $e > 1$ for the hyperbola. We will give the arguments for the parabola and ellipse and leave the argument for the hyperbola as an exercise.



▲ **Figure 10.6.1**

For the parabola, the distance PF to the focus is equal to the distance PD to the directrix, so that $PF/PD = 1$, which is what we wanted to show. For the ellipse, we rewrite Equation (8) of Section 10.4 as

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x = \frac{c}{a}\left(\frac{a^2}{c} - x\right)$$

But the expression on the left side is the distance PF, and the expression in the parentheses on the right side is the distance PD, so we have shown that

$$PF = \frac{c}{a}PD$$

Thus, PF/PD is constant, and the eccentricity is

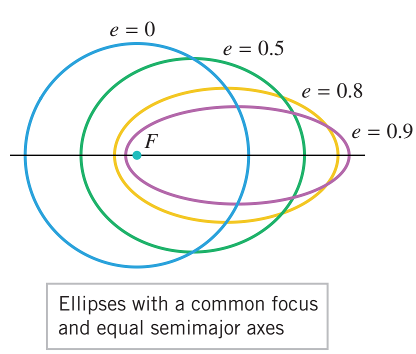
$$e = \frac{c}{a} \tag{1}$$

If we rule out the degenerate case where $a = 0$ or $c = 0$, then it follows from Formula (7) of Section 10.4 that $0 < c < a$, so $0 < e < 1$, which is what we wanted to show.

We will leave it as an exercise to show that the eccentricity of the hyperbola in Figure 10.6.1 is also given by Formula (1), but in this case it follows from Formula (11) of Section 10.4 that $c > a$, so $e > 1$.

ECCENTRICITY OF AN ELLIPSE AS A MEASURE OF FLATNESS

The eccentricity of an ellipse can be viewed as a measure of its flatness—as e approaches 0 the ellipses become more and more circular, and as e approaches 1 they become more and more flat (Figure 10.6.2). Table 10.6.1 shows the orbital eccentricities of various celestial objects. Note that most of the planets actually have fairly circular orbits.



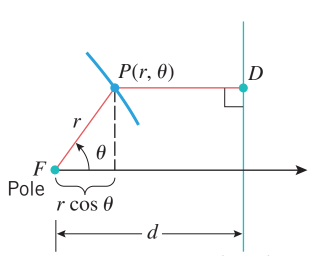
▲ **Figure 10.6.2**

Table 10.6.1

CELESTIAL BODY	ECCENTRICITY
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.046
Neptune	0.010
Pluto	0.249
Halley's comet	0.970

POLAR EQUATIONS OF CONICS

Our next objective is to derive polar equations for the conic sections from their focus–directrix characterizations. We will assume that the focus is at the pole and the directrix is either parallel or perpendicular to the polar axis. If the directrix is parallel to the polar axis, then it can be above or below the pole; and if the directrix is perpendicular to the polar axis, then it can be to the left or right of the pole. Thus, there are four cases to consider. We will derive the formulas for the case in which the directrix is perpendicular to the polar axis and to the right of the pole.



▲ **Figure 10.6.3**

As illustrated in Figure 10.6.3, let us assume that the directrix is perpendicular to the polar axis and d units to the right of the pole, where the constant d is known. If P is a point on the conic and if the eccentricity of the conic is e, then it follows from Theorem 10.6.1 that $PF/PD = e$ or, equivalently, that

$$PF = ePD \tag{2}$$

However, it is evident from Figure 10.6.3 that $PF = r$ and $PD = d - r\cos\theta$. Thus, (2) can be written as

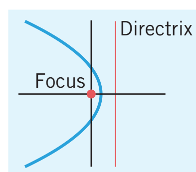
$$r = e(d - r\cos\theta)$$

which can be solved for r and expressed as

$$r = \frac{ed}{1 + e\cos\theta}$$

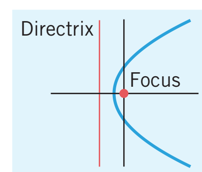
(verify). Observe that this single polar equation can represent a parabola, an ellipse, or a hyperbola, depending on the value of e. In contrast, the rectangular equations for these conics all have different forms. The derivations in the other three cases are similar.

10.6.2 THEOREM *If a conic section with eccentricity e is positioned in a polar coordinate system so that its focus is at the pole and the corresponding directrix is d units from the pole and is either parallel or perpendicular to the polar axis, then the equation of the conic has one of four possible forms, depending on its orientation:*



$$r = \frac{ed}{1 + e\cos\theta}$$

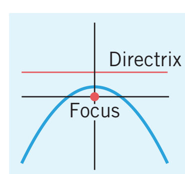
Directrix right of pole



$$r = \frac{ed}{1 - e\cos\theta}$$

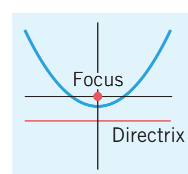
Directrix left of pole

(3–4)



$$r = \frac{ed}{1 + e\sin\theta}$$

Directrix above pole



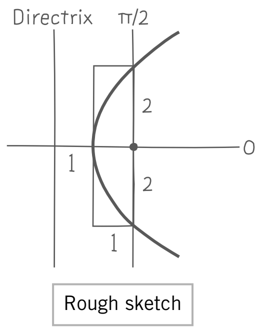
$$r = \frac{ed}{1 - e\sin\theta}$$

Directrix below pole

(5–6)

SKETCHING CONICS IN POLAR COORDINATES

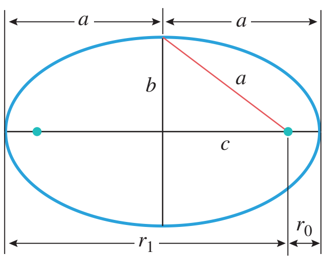
Precise graphs of conic sections in polar coordinates can be generated with graphing utilities. However, it is often useful to be able to make quick sketches of these graphs that show their orientations and give some sense of their dimensions. The orientation of a conic relative to the polar axis can be deduced by matching its equation with one of the four forms in Theorem 10.6.2. The key dimensions of a parabola are determined by the constant p (Figure 10.4.5) and those of ellipses and hyperbolas by the constants a, b, and c (Figures 10.4.11 and 10.4.20). Thus, we need to show how these constants can be obtained from the polar equations.



▲ **Figure 10.6.4**

▶ Example 1 Sketch the graph of $r = \dfrac{2}{1 - \cos\theta}$ in polar coordinates.

Solution. The equation is an exact match to (4) with $d = 2$ and $e = 1$. Thus, the graph is a parabola with the focus at the pole and the directrix 2 units to the left of the pole. This tells us that the parabola opens to the right along the polar axis and $p = 1$. Thus, the parabola looks roughly like that sketched in Figure 10.6.4. ◀



▲ **Figure 10.6.5**

All of the important geometric information about an ellipse can be obtained from the values of a, b, and c in Figure 10.6.5. One way to find these values from the polar equation of an ellipse is based on finding the distances from the focus to the vertices. As shown in the figure, let r_0 be the distance from the focus to the closest vertex and r_1 the distance to the farthest vertex. Thus,

$$r_0 = a - c \quad \text{and} \quad r_1 = a + c \tag{7}$$

from which it follows that

$$\boxed{a = \tfrac{1}{2}(r_1 + r_0)} \qquad \boxed{c = \tfrac{1}{2}(r_1 - r_0)} \tag{8–9}$$

Moreover, it also follows from (7) that

$$r_0 r_1 = a^2 - c^2 = b^2$$

Thus,

$$\boxed{b = \sqrt{r_0 r_1}} \tag{10}$$

In words, Formula (8) states that a is the ***arithmetic average*** (also called the ***arithmetic mean***) of r_0 and r_1, and Formula (10) states that b is the ***geometric mean*** of r_0 and r_1.

▶ Example 2 Find the constants a, b, and c for the ellipse $r = \dfrac{6}{2 + \cos\theta}$.

Solution. This equation does not match any of the forms in Theorem 10.6.2 because they all require a constant term of 1 in the denominator. However, we can put the equation into one of these forms by dividing the numerator and denominator by 2 to obtain

$$r = \frac{3}{1 + \frac{1}{2}\cos\theta}$$

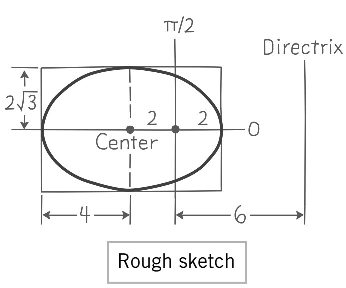
This is an exact match to (3) with $d = 6$ and $e = \frac{1}{2}$, so the graph is an ellipse with the directrix 6 units to the right of the pole. The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = 0$ in this equation, and the distance r_1 to the farthest vertex can be obtained by setting $\theta = \pi$. This yields

$$r_0 = \frac{3}{1 + \frac{1}{2}\cos 0} = \frac{3}{\frac{3}{2}} = 2, \quad r_1 = \frac{3}{1 + \frac{1}{2}\cos\pi} = \frac{3}{\frac{1}{2}} = 6$$

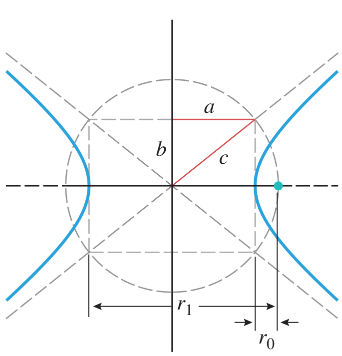
Thus, from Formulas (8), (10), and (9), respectively, we obtain

$$a = \tfrac{1}{2}(r_1 + r_0) = 4, \quad b = \sqrt{r_0 r_1} = 2\sqrt{3}, \quad c = \tfrac{1}{2}(r_1 - r_0) = 2$$

The ellipse looks roughly like that sketched in Figure 10.6.6. ◀



▲ **Figure 10.6.6**



▲ **Figure 10.6.7**

All of the important information about a hyperbola can be obtained from the values of a, b, and c in Figure 10.6.7. As with the ellipse, one way to find these values from the polar equation of a hyperbola is based on finding the distances from the focus to the vertices. As shown in the figure, let r_0 be the distance from the focus to the closest vertex and r_1 the distance to the farthest vertex. Thus,

$$r_0 = c - a \quad \text{and} \quad r_1 = c + a \tag{11}$$

from which it follows that

$$\boxed{a = \tfrac{1}{2}(r_1 - r_0)} \qquad \boxed{c = \tfrac{1}{2}(r_1 + r_0)} \tag{12–13}$$

In words, Formula (13) states that c is the ***arithmetic mean*** of r_0 and r_1, and Formula (14) states that b is the ***geometric mean*** of r_0 and r_1.

Moreover, it also follows from (11) that

$$r_0 r_1 = c^2 - a^2 = b^2$$

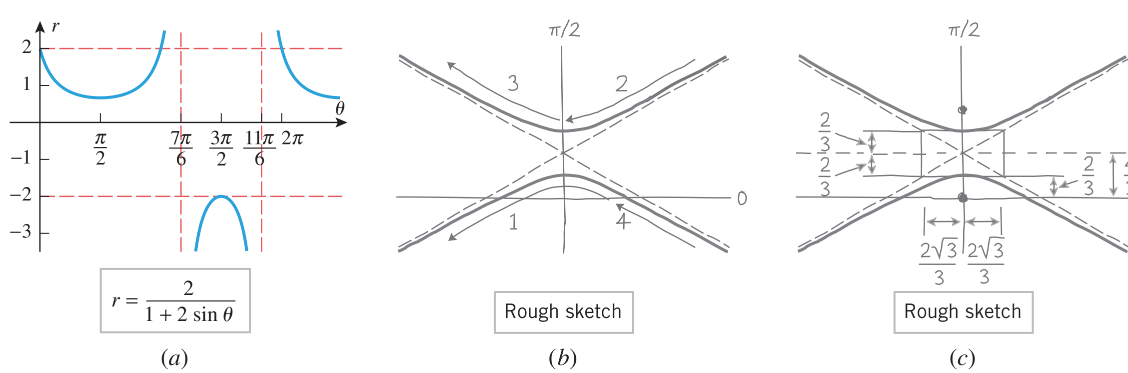
from which it follows that

$$\boxed{b = \sqrt{r_0 r_1}} \tag{14}$$

▶ Example 3 Sketch the graph of $r = \dfrac{2}{1 + 2\sin\theta}$ in polar coordinates.

Solution. This equation is an exact match to (5) with $d = 1$ and $e = 2$. Thus, the graph is a hyperbola with its directrix 1 unit above the pole. However, it is not so straightforward to compute the values of r_0 and r_1, since hyperbolas in polar coordinates are generated in a strange way as θ varies from 0 to 2π. This can be seen from Figure 10.6.8a, which is the graph of the given equation in rectangular θr-coordinates. It follows from this graph that the corresponding polar graph is generated in pieces (see Figure 10.6.8b):

- As θ varies over the interval $0 \le \theta < 7\pi/6$, the value of r is positive and varies from 2 down to 2/3 and then to $+\infty$, which generates part of the lower branch.
- As θ varies over the interval $7\pi/6 < \theta \le 3\pi/2$, the value of r is negative and varies from $-\infty$ to -2, which generates the right part of the upper branch.
- As θ varies over the interval $3\pi/2 \le \theta < 11\pi/6$, the value of r is negative and varies from -2 to $-\infty$, which generates the left part of the upper branch.
- As θ varies over the interval $11\pi/6 < \theta \le 2\pi$, the value of r is positive and varies from $+\infty$ to 2, which fills in the missing piece of the lower right branch.



▲ **Figure 10.6.8**

To obtain a rough sketch of a hyperbola, it is generally sufficient to locate the center, the asymptotes, and the points where $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$.

It is now clear that we can obtain r_0 by setting $\theta = \pi/2$ and r_1 by setting $\theta = 3\pi/2$. Keeping in mind that r_0 and r_1 are positive, this yields

$$r_0 = \frac{2}{1 + 2\sin(\pi/2)} = \frac{2}{3}, \qquad r_1 = \left|\frac{2}{1 + 2\sin(3\pi/2)}\right| = \left|\frac{2}{-1}\right| = 2$$

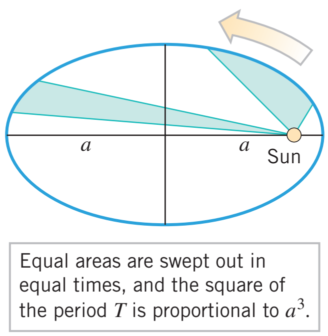
Thus, from Formulas (12), (14), and (13), respectively, we obtain

$$a = \frac{1}{2}(r_1 - r_0) = \frac{2}{3}, \quad b = \sqrt{r_0 r_1} = \frac{2\sqrt{3}}{3}, \quad c = \frac{1}{2}(r_1 + r_0) = \frac{4}{3}$$

Thus, the hyperbola looks roughly like that sketched in Figure 10.6.8*c*. ◀

APPLICATIONS IN ASTRONOMY

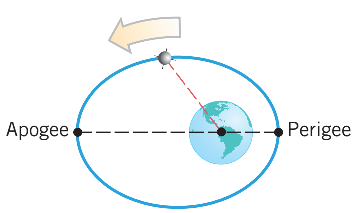
In 1609 Johannes Kepler published a book known as *Astronomia Nova* (or sometimes *Commentaries on the Motions of Mars*) in which he succeeded in distilling thousands of years of observational astronomy into three beautiful laws of planetary motion (Figure 10.6.9).



▲ **Figure 10.6.9**

10.6.3 KEPLER'S LAWS

- First law (***Law of Orbits***). Each planet moves in an elliptical orbit with the Sun at a focus.
- Second law (***Law of Areas***). The radial line from the center of the Sun to the center of a planet sweeps out equal areas in equal times.
- Third law (***Law of Periods***). The square of a planet's period (the time it takes the planet to complete one orbit about the Sun) is proportional to the cube of the semimajor axis of its orbit.



▲ **Figure 10.6.10**

Kepler's laws, although stated for planetary motion around the Sun, apply to all orbiting celestial bodies that are subjected to a *single* central gravitational force—artificial satellites subjected only to the central force of Earth's gravity and moons subjected only to the central gravitational force of a planet, for example. Later in the text we will derive Kepler's laws from basic principles, but for now we will show how they can be used in basic astronomical computations.

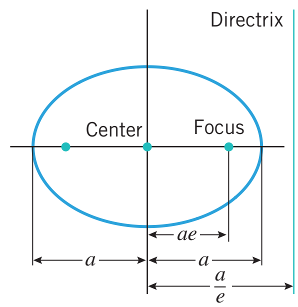
In an elliptical orbit, the closest point to the focus is called the ***perigee*** and the farthest point the ***apogee*** (Figure 10.6.10). The distances from the focus to the perigee and apogee

Johannes Kepler (1571–1630) German astronomer and physicist. Kepler, whose work provided our contemporary view of planetary motion, led a fascinating but ill-starred life. His alcoholic father made him work in a family-owned tavern as a child, later withdrawing him from elementary school and hiring him out as a field laborer, where the boy contracted smallpox, permanently crippling his hands and impairing his eyesight. In later years, Kepler's first wife and several children died, his mother was accused of witchcraft, and being a Protestant he was often subjected to persecution by Catholic authorities. He was often impoverished, eking out a living as an astrologer and prognosticator. Looking back on his unhappy childhood, Kepler described his father as "criminally inclined" and "quarrelsome" and his mother as "garrulous" and "bad-tempered." However, it was his mother who left an indelible mark on the six-year-old Kepler by showing him the comet of 1577; and in later life he personally prepared her defense against the witchcraft charges. Kepler became acquainted with the work of Copernicus as a student at the University of Tübingen, where he received his master's degree in 1591. He continued on as a theological student, but at the urging of the university officials he abandoned his clerical studies and accepted a position as a mathematician and teacher in Graz, Austria. However, he was expelled from the city when it came under Catholic control, and in 1600 he finally moved on to Prague, where he became an assistant at the observatory of the famous Danish astronomer Tycho Brahe. Brahe was a brilliant and meticulous astronomical observer who amassed the most accurate astronomical data known at that time; and when Brahe died in 1601 Kepler inherited the treasure-trove of data. After eight years of intense labor, Kepler deciphered the underlying principles buried in the data and in 1609 published his monumental work, *Astronomia Nova*, in which he stated his first two laws of planetary motion. Commenting on his discovery of elliptical orbits, Kepler wrote, "I was almost driven to madness in considering and calculating this matter. I could not find out why the planet would rather go on an elliptical orbit (rather than a circle). Oh ridiculous me!" It ultimately remained for Isaac Newton to discover the laws of gravitation that explained the reason for elliptical orbits.

are called the ***perigee distance*** and ***apogee distance***, respectively. For orbits around the Sun, it is more common to use the terms ***perihelion*** and ***aphelion***, rather than perigee and apogee, and to measure time in Earth years and distances in astronomical units (AU), where 1 AU is the semimajor axis a of the Earth's orbit (approximately 150×10^6 km or 92.9×10^6 mi). With this choice of units, the constant of proportionality in Kepler's third law is 1, since $a = 1$ AU produces a period of $T = 1$ Earth year. In this case Kepler's third law can be expressed as

$$T = a^{3/2} \tag{15}$$

Shapes of elliptical orbits are often specified by giving the eccentricity e and the semimajor axis a, so it is useful to express the polar equations of an ellipse in terms of these constants. Figure 10.6.11, which can be obtained from the ellipse in Figure 10.6.1 and the relationship $c = ea$, implies that the distance d between the focus and the directrix is



▲ **Figure 10.6.11**

$$d = \frac{a}{e} - c = \frac{a}{e} - ea = \frac{a(1-e^2)}{e} \tag{16}$$

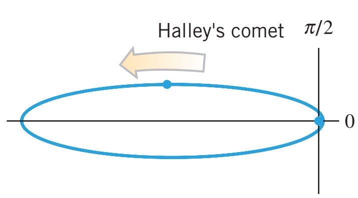
from which it follows that $ed = a(1 - e^2)$. Thus, depending on the orientation of the ellipse, the formulas in Theorem 10.6.2 can be expressed in terms of a and e as

$$r = \frac{a(1-e^2)}{1 \pm e\cos\theta} \qquad r = \frac{a(1-e^2)}{1 \pm e\sin\theta} \tag{17–18}$$

+: Directrix right of pole | +: Directrix above pole
−: Directrix left of pole | −: Directrix below pole

Moreover, it is evident from Figure 10.6.11 that the distances from the focus to the closest and farthest vertices can be expressed in terms of a and e as

$$r_0 = a - ea = a(1 - e) \quad \text{and} \quad r_1 = a + ea = a(1 + e) \tag{19–20}$$



▲ **Figure 10.6.12**

Science Photo Library/Photo Researchers
Halley's comet photographed April 21, 1910 in Peru.

▶ Example 4 Halley's comet (last seen in 1986) has an eccentricity of 0.97 and a semimajor axis of $a = 18.1$ AU.

(a) Find the equation of its orbit in the polar coordinate system shown in Figure 10.6.12.
(b) Find the period of its orbit.
(c) Find its perihelion and aphelion distances.

Solution (a). From (17), the polar equation of the orbit has the form

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

But $a(1 - e^2) = 18.1[1 - (0.97)^2] \approx 1.07$. Thus, the equation of the orbit is

$$r = \frac{1.07}{1 + 0.97\cos\theta}$$

Solution (b). From (15), with $a = 18.1$, the period of the orbit is

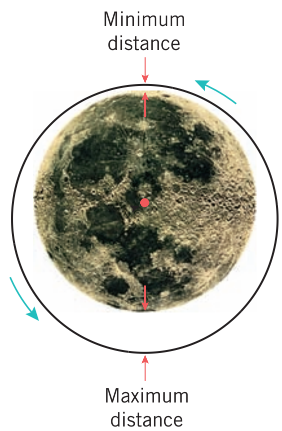
$$T = (18.1)^{3/2} \approx 77 \text{ years}$$

Solution (c). Since the perihelion and aphelion distances are the distances to the closest and farthest vertices, respectively, it follows from (19) and (20) that

$$r_0 = a - ea = a(1 - e) = 18.1(1 - 0.97) \approx 0.543 \text{ AU}$$
$$r_1 = a + ea = a(1 + e) = 18.1(1 + 0.97) \approx 35.7 \text{ AU}$$

or since 1 AU $\approx 150 \times 10^6$ km, the perihelion and aphelion distances in kilometers are

$$r_0 = 18.1(1 - 0.97)(150 \times 10^6) \approx 81{,}500{,}000 \text{ km}$$
$$r_1 = 18.1(1 + 0.97)(150 \times 10^6) \approx 5{,}350{,}000{,}000 \text{ km} \blacktriangleleft$$



▲ **Figure 10.6.13**

► **Example 5** An Apollo lunar lander orbits the Moon in an elliptic orbit with eccentricity $e = 0.12$ and semimajor axis $a = 2015$ km. Assuming the Moon to be a sphere of radius 1740 km, find the minimum and maximum heights of the lander above the lunar surface (Figure 10.6.13).

Solution. If we let r_0 and r_1 denote the minimum and maximum distances from the center of the Moon, then the minimum and maximum distances from the surface of the Moon will be

$$d_{\min} = r_0 - 1740$$
$$d_{\max} = r_1 - 1740$$

or from Formulas (19) and (20)

$$d_{\min} = r_0 - 1740 = a(1 - e) - 1740 = 2015(0.88) - 1740 = 33.2 \text{ km}$$
$$d_{\max} = r_1 - 1740 = a(1 + e) - 1740 = 2015(1.12) - 1740 = 516.8 \text{ km} \blacktriangleleft$$

✔ QUICK CHECK EXERCISES 10.6 (See page 763 for answers.)

1. In each part, name the conic section described.
(a) The set of points whose distance to the point (2, 3) is half the distance to the line $x + y = 1$ is ______.
(b) The set of points whose distance to the point (2, 3) is equal to the distance to the line $x + y = 1$ is ______.
(c) The set of points whose distance to the point (2, 3) is twice the distance to the line $x + y = 1$ is ______.

2. In each part: (i) Identify the polar graph as a parabola, an ellipse, or a hyperbola; (ii) state whether the directrix is above, below, to the left, or to the right of the pole; and (iii) find the distance from the pole to the directrix.
(a) $r = \dfrac{1}{4 + \cos\theta}$ (b) $r = \dfrac{1}{1 - 4\cos\theta}$
(c) $r = \dfrac{1}{4 + 4\sin\theta}$ (d) $r = \dfrac{4}{1 - \sin\theta}$

3. If the distance from a vertex of an ellipse to the nearest focus is r_0, and if the distance from that vertex to the farthest focus is r_1, then the semimajor axis is $a =$ ______ and the semiminor axis is $b =$ ______.

4. If the distance from a vertex of a hyperbola to the nearest focus is r_0, and if the distance from that vertex to the farthest focus is r_1, then the semifocal axis is $a =$ ______ and the semiconjugate axis is $b =$ ______.

EXERCISE SET 10.6 Graphing Utility

1–2 Find the eccentricity and the distance from the pole to the directrix, and sketch the graph in polar coordinates. ■

1. (a) $r = \dfrac{3}{2 - 2\cos\theta}$ (b) $r = \dfrac{3}{2 + \sin\theta}$

2. (a) $r = \dfrac{4}{2 + 3\cos\theta}$ (b) $r = \dfrac{5}{3 + 3\sin\theta}$

3–4 Use Formulas (3)–(6) to identify the type of conic and its orientation. Check your answer by generating the graph with a graphing utility. ■

3. (a) $r = \dfrac{8}{1 - \sin\theta}$ (b) $r = \dfrac{16}{4 + 3\sin\theta}$

4. (a) $r = \dfrac{4}{2 - 3\sin\theta}$ (b) $r = \dfrac{12}{4 + \cos\theta}$

5–6 Find a polar equation for the conic that has its focus at the pole and satisfies the stated conditions. Points are in polar coordinates and directrices in rectangular coordinates for simplicity. (In some cases there may be more than one conic that satisfies the conditions.) ■

5. (a) Ellipse; $e = \frac{3}{4}$; directrix $x = 2$.
(b) Parabola; directrix $x = 1$.
(c) Hyperbola; $e = \frac{4}{3}$; directrix $y = 3$.

6. (a) Ellipse; ends of major axis $(2, \pi/2)$ and $(6, 3\pi/2)$.
   (b) Parabola; vertex $(2, \pi)$.
   (c) Hyperbola; $e = \sqrt{2}$; vertex $(2, 0)$.

7–8 Find the distances from the pole to the vertices, and then apply Formulas (8)–(10) to find the equation of the ellipse in rectangular coordinates. ■

7. (a) $r = \dfrac{6}{2 + \sin\theta}$ (b) $r = \dfrac{1}{2 - \cos\theta}$

8. (a) $r = \dfrac{6}{5 + 2\cos\theta}$ (b) $r = \dfrac{8}{4 - 3\sin\theta}$

9–10 Find the distances from the pole to the vertices, and then apply Formulas (12)–(14) to find the equation of the hyperbola in rectangular coordinates. ■

9. (a) $r = \dfrac{3}{1 + 2\sin\theta}$ (b) $r = \dfrac{5}{2 - 3\cos\theta}$

10. (a) $r = \dfrac{4}{1 - 2\sin\theta}$ (b) $r = \dfrac{15}{2 + 8\cos\theta}$

11–12 Find a polar equation for the ellipse that has its focus at the pole and satisfies the stated conditions. ■

11. (a) Directrix to the right of the pole; $a = 8$; $e = \frac{1}{2}$.
    (b) Directrix below the pole; $a = 4$; $e = \frac{3}{5}$.

12. (a) Directrix to the left of the pole; $b = 4$; $e = \frac{3}{5}$.
    (b) Directrix above the pole; $c = 5$; $e = \frac{1}{5}$.

13. Find the polar equation of an equilateral hyperbola with a focus at the pole and vertex $(5, 0)$.

FOCUS ON CONCEPTS

14. Prove that a hyperbola is an equilateral hyperbola if and only if $e = \sqrt{2}$.

15. (a) Show that the coordinates of the point P on the hyperbola in Figure 10.6.1 satisfy the equation
$$\sqrt{(x - c)^2 + y^2} = \frac{c}{a}x - a$$
    (b) Use the result obtained in part (a) to show that $PF/PD = c/a$.

16. (a) Show that the eccentricity of an ellipse can be expressed in terms of r_0 and r_1 as
$$e = \frac{r_1 - r_0}{r_1 + r_0}$$
    (b) Show that
$$\frac{r_1}{r_0} = \frac{1 + e}{1 - e}$$

17. (a) Show that the eccentricity of a hyperbola can be expressed in terms of r_0 and r_1 as
$$e = \frac{r_1 + r_0}{r_1 - r_0}$$
    (b) Show that
$$\frac{r_1}{r_0} = \frac{e + 1}{e - 1}$$

18. (a) Sketch the curves
$$r = \frac{1}{1 + \cos\theta} \quad \text{and} \quad r = \frac{1}{1 - \cos\theta}$$
    (b) Find polar coordinates of the intersections of the curves in part (a).
    (c) Show that the curves are *orthogonal*, that is, their tangent lines are perpendicular at the points of intersection.

19–22 True–False Determine whether the statement is true or false. Explain your answer. ■

19. If an ellipse is not a circle, then the eccentricity of the ellipse is less than one.

20. A parabola has eccentricity greater than one.

21. If one ellipse has foci that are farther apart than those of a second ellipse, then the eccentricity of the first is greater than that of the second.

22. If d is a positive constant, then the conic section with polar equation
$$r = \frac{d}{1 + \cos\theta}$$
is a parabola.

23–28 Use the following values, where needed:

radius of the Earth = 4000 mi = 6440 km
1 year (Earth year) = 365 days (Earth days)
1 AU = 92.9×10^6 mi = 150×10^6 km ■

23. The dwarf planet Pluto has eccentricity $e = 0.249$ and semimajor axis $a = 39.5$ AU.
    (a) Find the period T in years.
    (b) Find the perihelion and aphelion distances.
    (c) Choose a polar coordinate system with the center of the Sun at the pole, and find a polar equation of Pluto's orbit in that coordinate system.
    (d) Make a sketch of the orbit with reasonably accurate proportions.

24. (a) Let a be the semimajor axis of a planet's orbit around the Sun, and let T be its period. Show that if T is measured in days and a is measured in kilometers, then $T = (365 \times 10^{-9})(a/150)^{3/2}$.
    (b) Use the result in part (a) to find the period of the planet Mercury in days, given that its semimajor axis is $a = 57.95 \times 10^6$ km.
    (c) Choose a polar coordinate system with the Sun at the pole, and find an equation for the orbit of Mercury in that coordinate system given that the eccentricity of the orbit is $e = 0.206$.
    (d) Use a graphing utility to generate the orbit of Mercury from the equation obtained in part (c).

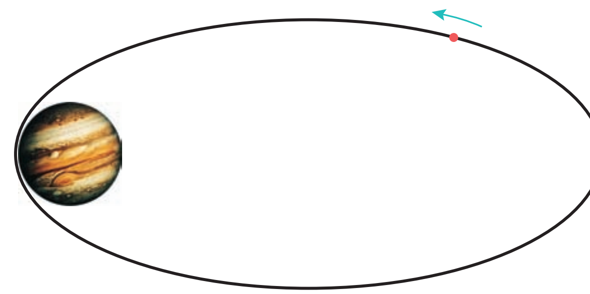
25. The Hale–Bopp comet, discovered independently on July 23, 1995 by Alan Hale and Thomas Bopp, has an orbital eccentricity of $e = 0.9951$ and a period of 2380 years.
    (a) Find its semimajor axis in astronomical units (AU).
    (b) Find its perihelion and aphelion distances. *(cont.)*

(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the Hale–Bopp orbit in that coordinate system.
(d) Make a sketch of the Hale–Bopp orbit with reasonably accurate proportions.

26. Mars has a perihelion distance of 204,520,000 km and an aphelion distance of 246,280,000 km.
(a) Use these data to calculate the eccentricity, and compare your answer to the value given in Table 10.6.1.
(b) Find the period of Mars.
(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the orbit of Mars in that coordinate system.
(d) Use a graphing utility to generate the orbit of Mars from the equation obtained in part (c).

27. *Vanguard 1* was launched in March 1958 into an orbit around the Earth with eccentricity $e = 0.21$ and semimajor axis 8864.5 km. Find the minimum and maximum heights of *Vanguard 1* above the surface of the Earth.

28. The planet Jupiter is believed to have a rocky core of radius 10,000 km surrounded by two layers of hydrogen—a 40,000 km thick layer of compressed metallic-like hydrogen and a 20,000 km thick layer of ordinary molecular hydrogen. The visible features, such as the Great Red Spot, are at the outer surface of the molecular hydrogen layer. On November 6, 1997 the spacecraft *Galileo* was placed in a Jovian orbit to study the moon Europa. The orbit had eccentricity 0.814580 and semimajor axis 3,514,918.9 km. Find *Galileo*'s minimum and maximum heights above the molecular hydrogen layer (see the accompanying figure).



▲ **Figure Ex-28**

29. Writing Discuss how a hyperbola's eccentricity e affects the shape of the hyperbola. How is the shape affected as e approaches 1? As e approaches $+\infty$? Draw some pictures to illustrate your conclusions.

30. Writing Discuss the relationship between the eccentricity e of an ellipse and the distance z between the directrix and center of the ellipse. For example, if the foci remain fixed, what happens to z as e approaches 0?

✔ QUICK CHECK ANSWERS 10.6

1. (a) an ellipse (b) a parabola (c) a hyperbola **2.** (a) (i) ellipse (ii) to the right of the pole (iii) distance $= 1$ (b) (i) hyperbola (ii) to the left of the pole (iii) distance $= \frac{1}{4}$ (c) (i) parabola (ii) above the pole (iii) distance $= \frac{1}{4}$ (d) (i) parabola (ii) below the pole (iii) distance $= 4$ **3.** $\frac{1}{2}(r_1 + r_0)$; $\sqrt{r_0 r_1}$ **4.** $\frac{1}{2}(r_1 - r_0)$; $\sqrt{r_0 r_1}$

CHAPTER 10 REVIEW EXERCISES

Graphing Utility

1. Find parametric equations for the portion of the circle $x^2 + y^2 = 2$ that lies outside the first quadrant, oriented clockwise. Check your work by generating the curve with a graphing utility.

2. (a) Suppose that the equations $x = f(t)$, $y = g(t)$ describe a curve C as t increases from 0 to 1. Find parametric equations that describe the same curve C but traced in the opposite direction as t increases from 0 to 1.
(b) Check your work using the parametric graphing feature of a graphing utility by generating the line segment between $(1, 2)$ and $(4, 0)$ in both possible directions as t increases from 0 to 1.

3. (a) Find the slope of the tangent line to the parametric curve $x = t^2 + 1$, $y = t/2$ at $t = -1$ and $t = 1$ without eliminating the parameter.
(b) Check your answers in part (a) by eliminating the parameter and differentiating a function of x.

4. Find dy/dx and d^2y/dx^2 at $t = 2$ for the parametric curve $x = \frac{1}{2}t^2$, $y = \frac{1}{3}t^3$.

5. Find all values of t at which a tangent line to the parametric curve $x = 2\cos t$, $y = 4\sin t$ is
(a) horizontal (b) vertical.

6. Find the exact arc length of the curve

$$x = 1 - 5t^4, \quad y = 4t^5 - 1 \qquad (0 \le t \le 1)$$

7. In each part, find the rectangular coordinates of the point whose polar coordinates are given.
(a) $(-8, \pi/4)$ (b) $(7, -\pi/4)$ (c) $(8, 9\pi/4)$
(d) $(5, 0)$ (e) $(-2, -3\pi/2)$ (f) $(0, \pi)$

8. Express the point whose xy-coordinates are $(-1, 1)$ in polar coordinates with
(a) $r > 0,\ 0 \le \theta < 2\pi$ (b) $r < 0,\ 0 \le \theta < 2\pi$
(c) $r > 0,\ -\pi < \theta \le \pi$ (d) $r < 0,\ -\pi < \theta \le \pi$.

9. In each part, use a calculating utility to approximate the polar coordinates of the point whose rectangular coordinates are given.
(a) $(4, 3)$ (b) $(2, -5)$ (c) $(1, \tan^{-1} 1)$

10. In each part, state the name that describes the polar curve most precisely: a rose, a line, a circle, a limaçon, a cardioid, a spiral, a lemniscate, or none of these.
(a) $r = 3\cos\theta$ (b) $r = \cos 3\theta$
(c) $r = \dfrac{3}{\cos\theta}$ (d) $r = 3 - \cos\theta$
(e) $r = 1 - 3\cos\theta$ (f) $r^2 = 3\cos\theta$
(g) $r = (3\cos\theta)^2$ (h) $r = 1 + 3\theta$

11. In each part, identify the curve by converting the polar equation to rectangular coordinates. Assume that $a > 0$.
(a) $r = a\sec^2\dfrac{\theta}{2}$ (b) $r^2\cos 2\theta = a^2$
(c) $r = 4\csc\left(\theta - \dfrac{\pi}{4}\right)$ (d) $r = 4\cos\theta + 8\sin\theta$

12. In each part, express the given equation in polar coordinates.
(a) $x = 7$ (b) $x^2 + y^2 = 9$
(c) $x^2 + y^2 - 6y = 0$ (d) $4xy = 9$

13–17 Sketch the curve in polar coordinates. ■

13. $\theta = \dfrac{\pi}{6}$ **14.** $r = 6\cos\theta$

15. $r = 3(1 - \sin\theta)$ **16.** $r^2 = \sin 2\theta$

17. $r = 3 - \cos\theta$

18. (a) Show that the maximum value of the y-coordinate of points on the curve $r = 1/\sqrt{\theta}$ for θ in the interval $(0, \pi]$ occurs when $\tan\theta = 2\theta$.
(b) Use a calculating utility to solve the equation in part (a) to at least four decimal-place accuracy.
(c) Use the result of part (b) to approximate the maximum value of y for $0 < \theta \le \pi$.

19. (a) Find the minimum and maximum x-coordinates of points on the cardioid $r = 1 - \cos\theta$.
(b) Find the minimum and maximum y-coordinates of points on the cardioid in part (a).

20. Determine the slope of the tangent line to the polar curve $r = 1 + \sin\theta$ at $\theta = \pi/4$.

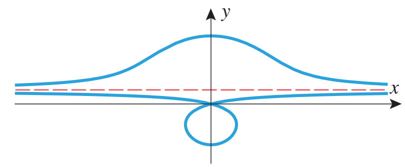
21. A parametric curve of the form

$$x = a\cot t + b\cos t, \quad y = a + b\sin t \qquad (0 < t < 2\pi)$$

is called a ***conchoid of Nicomedes*** (see the accompanying figure for the case $0 < a < b$).
(a) Describe how the conchoid

$$x = \cot t + 4\cos t, \quad y = 1 + 4\sin t$$

is generated as t varies over the interval $0 < t < 2\pi$.
(b) Find the horizontal asymptote of the conchoid given in part (a).
(c) For what values of t does the conchoid in part (a) have a horizontal tangent line? A vertical tangent line?
(d) Find a polar equation $r = f(\theta)$ for the conchoid in part (a), and then find polar equations for the tangent lines to the conchoid at the pole.



◄ **Figure Ex-21**

22. (a) Find the arc length of the polar curve $r = 1/\theta$ for $\pi/4 \le \theta \le \pi/2$.
(b) What can you say about the arc length of the portion of the curve that lies inside the circle $r = 1$?

23. Find the area of the region that is enclosed by the cardioid $r = 2 + 2\cos\theta$.

24. Find the area of the region in the first quadrant within the cardioid $r = 1 + \sin\theta$.

25. Find the area of the region that is common to the circles $r = 1$, $r = 2\cos\theta$, and $r = 2\sin\theta$.

26. Find the area of the region that is inside the cardioid $r = a(1 + \sin\theta)$ and outside the circle $r = a\sin\theta$.

27–30 Sketch the parabola, and label the focus, vertex, and directrix. ■

27. $y^2 = 6x$ **28.** $x^2 = -9y$

29. $(y + 1)^2 = -7(x - 4)$ **30.** $\left(x - \frac{1}{2}\right)^2 = 2(y - 1)$

31–34 Sketch the ellipse, and label the foci, the vertices, and the ends of the minor axis. ■

31. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ **32.** $4x^2 + 9y^2 = 36$

33. $9(x - 1)^2 + 16(y - 3)^2 = 144$

34. $3(x + 2)^2 + 4(y + 1)^2 = 12$

35–37 Sketch the hyperbola, and label the vertices, foci, and asymptotes. ■

35. $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ **36.** $9y^2 - 4x^2 = 36$

37. $\dfrac{(x - 2)^2}{9} - \dfrac{(y - 4)^2}{4} = 1$

38. In each part, sketch the graph of the conic section with reasonably accurate proportions.
(a) $x^2 - 4x + 8y + 36 = 0$
(b) $3x^2 + 4y^2 - 30x - 8y + 67 = 0$
(c) $4x^2 - 5y^2 - 8x - 30y - 21 = 0$

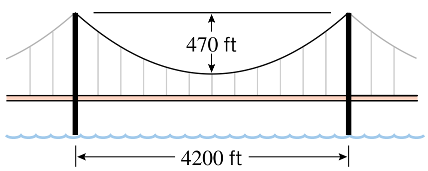
39–41 Find an equation for the conic described. ■

39. A parabola with vertex $(0, 0)$ and focus $(0, -4)$.

40. An ellipse with the ends of the major axis $(0, \pm\sqrt{5})$ and the ends of the minor axis $(\pm 1, 0)$.

41. A hyperbola with vertices $(0, \pm 3)$ and asymptotes $y = \pm x$.

42. It can be shown in the accompanying figure that hanging cables form parabolic arcs rather than catenaries if they are subjected to uniformly distributed downward forces along their length. For example, if the weight of the roadway in a suspension bridge is assumed to be uniformly distributed along the supporting cables, then the cables can be modeled by parabolas.
(a) Assuming a parabolic model, find an equation for the cable in the accompanying figure, taking the y-axis to be vertical and the origin at the low point of the cable.
(b) Find the length of the cable between the supports.

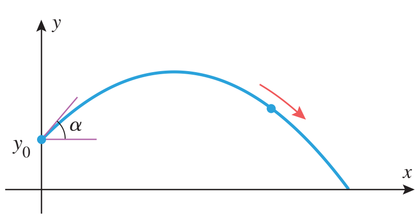


▲ **Figure Ex-42**

43. It will be shown later in this text that if a projectile is launched with speed v_0 at an angle α with the horizontal and at a height y_0 above ground level, then the resulting trajectory relative to the coordinate system in the accompanying figure will have parametric equations

$$x = (v_0 \cos\alpha)t, \quad y = y_0 + (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity.
(a) Show that the trajectory is a parabola.
(b) Find the coordinates of the vertex.



◀ **Figure Ex-43**

44. Mickey Mantle is recognized as baseball's unofficial king of long home runs. On April 17, 1953 Mantle blasted a pitch by Chuck Stobbs of the hapless Washington Senators out of Griffith Stadium, just clearing the 50 ft wall at the 391 ft marker in left center. Assuming that the ball left the bat at a height of 3 ft above the ground and at an angle of 45°, use the parametric equations in Exercise 43 with $g = 32\ \text{ft/s}^2$ to find
(a) the speed of the ball as it left the bat
(b) the maximum height of the ball
(c) the distance along the ground from home plate to where the ball struck the ground.

45–47 Rotate the coordinate axes to remove the xy-term, and then name the conic. ■

45. $x^2 + y^2 - 3xy - 3 = 0$

46. $7x^2 + 2\sqrt{3}xy + 5y^2 - 4 = 0$

47. $4\sqrt{5}x^2 + 4\sqrt{5}xy + \sqrt{5}y^2 + 5x - 10y = 0$

48. Rotate the coordinate axes to show that the graph of

$$17x^2 - 312xy + 108y^2 + 1080x - 1440y + 4500 = 0$$

is a hyperbola. Then find its vertices, foci, and asymptotes.

49. In each part: (i) Identify the polar graph as a parabola, an ellipse, or a hyperbola; (ii) state whether the directrix is above, below, to the left, or to the right of the pole; and (iii) find the distance from the pole to the directrix.
(a) $r = \dfrac{1}{3 + \cos\theta}$ (b) $r = \dfrac{1}{1 - 3\cos\theta}$
(c) $r = \dfrac{1}{3(1 + \sin\theta)}$ (d) $r = \dfrac{3}{1 - \sin\theta}$

50–51 Find an equation in xy-coordinates for the conic section that satisfies the given conditions. ■

50. (a) Ellipse with eccentricity $e = \frac{2}{7}$ and ends of the minor axis at the points $(0, \pm 3)$.
(b) Parabola with vertex at the origin, focus on the y-axis, and directrix passing through the point $(7, 4)$.
(c) Hyperbola that has the same foci as the ellipse $3x^2 + 16y^2 = 48$ and asymptotes $y = \pm 2x/3$.

51. (a) Ellipse with center $(-3, 2)$, vertex $(2, 2)$, and eccentricity $e = \frac{4}{5}$.
(b) Parabola with focus $(-2, -2)$ and vertex $(-2, 0)$.
(c) Hyperbola with vertex $(-1, 7)$ and asymptotes $y - 5 = \pm 8(x + 1)$.

52. Use the parametric equations $x = a\cos t$, $y = b\sin t$ to show that the circumference C of an ellipse with semimajor axis a and eccentricity e is

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 u}\, du$$

53. Use Simpson's rule or the numerical integration capability of a graphing utility to approximate the circumference of the ellipse $4x^2 + 9y^2 = 36$ from the integral obtained in Exercise 52.

54. (a) Calculate the eccentricity of the Earth's orbit, given that the ratio of the distance between the center of the Earth and the center of the Sun at perihelion to the distance between the centers at aphelion is $\frac{59}{61}$.
(b) Find the distance between the center of the Earth and the center of the Sun at perihelion, given that the average value of the perihelion and aphelion distances between the centers is 93 million miles.
(c) Use the result in Exercise 52 and Simpson's rule or the numerical integration capability of a graphing utility to approximate the distance that the Earth travels in 1 year (one revolution around the Sun).

CHAPTER 10 MAKING CONNECTIONS

C CAS

C **1.** Recall from Section 5.10 that the Fresnel sine and cosine functions are defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$$

The following parametric curve, which is used to study amplitudes of light waves in optics, is called a ***clothoid*** or ***Cornu spiral*** in honor of the French scientist Marie Alfred Cornu (1841–1902):

$$\begin{aligned} x &= C(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du \\ y &= S(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du \end{aligned} \qquad (-\infty < t < +\infty)$$

(a) Use a CAS to graph the Cornu spiral.

(b) Describe the behavior of the spiral as $t \to +\infty$ and as $t \to -\infty$.

(c) Find the arc length of the spiral for $-1 \le t \le 1$.

2. (a) The accompanying figure shows an ellipse with semimajor axis a and semiminor axis b. Express the coordinates of the points P, Q, and R in terms of t.

(b) How does the geometric interpretation of the parameter t differ between a circle

$$x = a\cos t, \quad y = a\sin t$$

and an ellipse

$$x = a\cos t, \quad y = b\sin t?$$

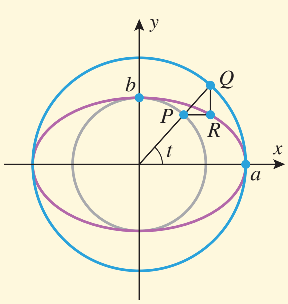


Figure Ex-2

3. The accompanying figure shows Kepler's method for constructing a parabola. A piece of string the length of the left edge of the drafting triangle is tacked to the vertex Q of the triangle and the other end to a fixed point F. A pencil holds the string taut against the base of the triangle as the edge opposite Q slides along a horizontal line L below F. Show that the pencil traces an arc of a parabola with focus F and directrix L.

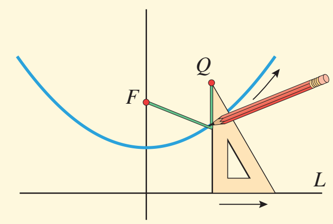


Figure Ex-3

4. The accompanying figure shows a method for constructing a hyperbola. A corner of a ruler is pinned to a fixed point F_1 and the ruler is free to rotate about that point. A piece of string whose length is less than that of the ruler is tacked to a point F_2 and to the free corner Q of the ruler on the same edge as F_1. A pencil holds the string taut against the top edge of the ruler as the ruler rotates about the point F_1. Show that the pencil traces an arc of a hyperbola with foci F_1 and F_2.

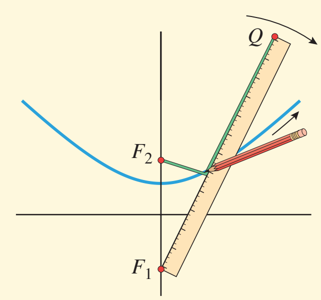


Figure Ex-4

5. Consider an ellipse E with semimajor axis a and semiminor axis b, and set $c = \sqrt{a^2 - b^2}$.

(a) Show that the ellipsoid that results when E is revolved about its major axis has volume $V = \frac{4}{3}\pi ab^2$ and surface area

$$S = 2\pi ab\left(\frac{b}{a} + \frac{a}{c}\sin^{-1}\frac{c}{a}\right)$$

(b) Show that the ellipsoid that results when E is revolved about its minor axis has volume $V = \frac{4}{3}\pi a^2 b$ and surface area

$$S = 2\pi ab\left(\frac{a}{b} + \frac{b}{c}\ln\frac{a+c}{b}\right)$$

EXPANDING THE CALCULUS HORIZON

To learn how polar coordinates and conic sections can be used to analyze the possibility of a collision between a comet and Earth, see the module entitled **Comet Collision** at:

www.wiley.com/college/anton

Craig Lovell/Corbis Images

3 TOPICS IN DIFFERENTIATION

The growth and decline of animal populations and natural resources can be modeled using basic functions studied in calculus.

We begin this chapter by extending the process of differentiation to functions that are either difficult or impossible to differentiate directly. We will discuss a combination of direct and indirect methods of differentiation that will allow us to develop a number of new derivative formulas that include the derivatives of logarithmic, exponential, and inverse trigonometric functions. Later in the chapter, we will consider some applications of the derivative. These will include ways in which different rates of change can be related as well as the use of linear functions to approximate nonlinear functions. Finally, we will discuss L'Hôpital's rule, a powerful tool for evaluating limits.

3.1 IMPLICIT DIFFERENTIATION

Up to now we have been concerned with differentiating functions that are given by equations of the form $y = f(x)$. In this section we will consider methods for differentiating functions for which it is inconvenient or impossible to express them in this form.

FUNCTIONS DEFINED EXPLICITLY AND IMPLICITLY

An equation of the form $y = f(x)$ is said to define y ***explicitly*** as a function of x because the variable y appears alone on one side of the equation and does not appear at all on the other side. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that (1) defines y ***implicitly*** as a function of x, the function being

$$f(x) = \frac{x-1}{x+1}$$

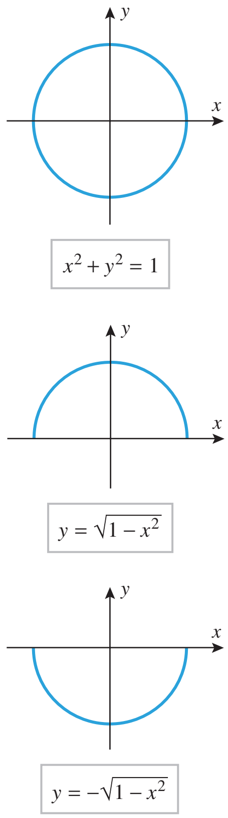
An equation in x and y can implicitly define more than one function of x. This can occur when the graph of the equation fails the vertical line test, so it is not the graph of a function of x. For example, if we solve the equation of the circle

$$x^2 + y^2 = 1 \tag{2}$$

for y in terms of x, we obtain $y = \pm\sqrt{1 - x^2}$, so we have found two functions that are defined implicitly by (2), namely,

$$f_1(x) = \sqrt{1 - x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1 - x^2} \tag{3}$$

The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 3.1.1). This leads us to the following definition.

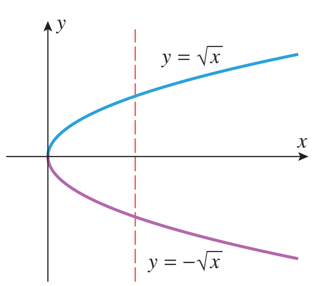


▲ **Figure 3.1.1**

3.1.1 DEFINITION We will say that a given equation in x and y defines the function f ***implicitly*** if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

▶ **Example 1** The graph of $x = y^2$ is not the graph of a function of x, since it does not pass the vertical line test (Figure 3.1.2). However, if we solve this equation for y in terms of x, we obtain the equations $y = \sqrt{x}$ and $y = -\sqrt{x}$, whose graphs pass the vertical line test and are portions of the graph of $x = y^2$ (Figure 3.1.2). Thus, the equation $x = y^2$ implicitly defines the functions

$$f_1(x) = \sqrt{x} \quad \text{and} \quad f_2(x) = -\sqrt{x} \; ◀$$



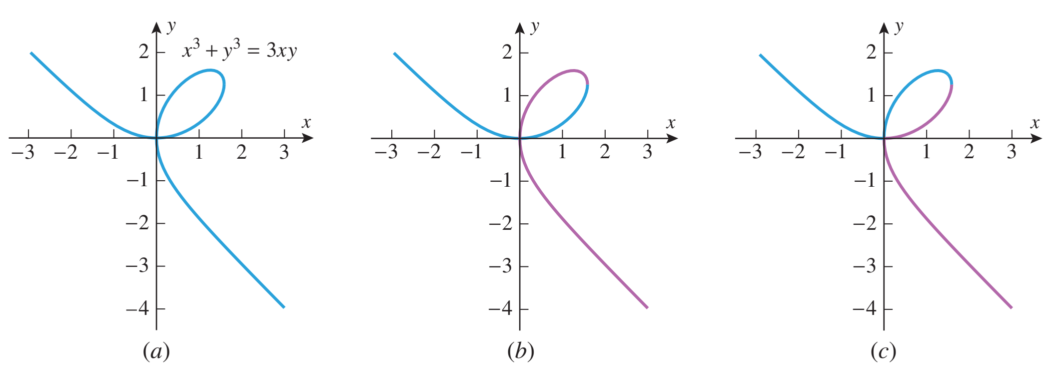
▲ **Figure 3.1.2** The graph of $x = y^2$ does not pass the vertical line test, but the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ do.

Although it was a trivial matter in the last example to solve the equation $x = y^2$ for y in terms of x, it is difficult or impossible to do this for some equations. For example, the equation

$$x^3 + y^3 = 3xy \tag{4}$$

can be solved for y in terms of x, but the resulting formulas are too complicated to be practical. Other equations, such as $\sin(xy) = y$, cannot be solved for y by any elementary method. Thus, even though an equation may define one or more functions of x, it may not be possible or practical to find explicit formulas for those functions.

Fortunately, CAS programs, such as *Mathematica* and *Maple*, have "implicit plotting" capabilities that can graph equations such as (4). The graph of this equation, which is called the ***Folium of Descartes***, is shown in Figure 3.1.3*a*. Parts (*b*) and (*c*) of the figure show the graphs (in blue) of two functions that are defined implicitly by (4).



▲ **Figure 3.1.3**

IMPLICIT DIFFERENTIATION

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this, let us consider the simple equation

$$xy = 1 \tag{5}$$

One way to find dy/dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{7}$$

Another way to obtain this derivative is to differentiate both sides of (5) *before* solving for y in terms of x, treating y as a (temporarily unspecified) differentiable function of x. With this approach we obtain

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

which agrees with Equation (7). This method of obtaining derivatives is called ***implicit differentiation***.

Example 2 Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x.

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

René Descartes (1596–1650) Descartes, a French aristocrat, was the son of a government official. He graduated from the University of Poitiers with a law degree at age 20. After a brief probe into the pleasures of Paris he became a military engineer, first for the Dutch Prince of Nassau and then for the German Duke of Bavaria. It was during his service as a soldier that Descartes began to pursue mathematics seriously and develop his analytic geometry. After the wars, he returned to Paris where he stalked the city as an eccentric, wearing a sword in his belt and a plumed hat. He lived in leisure, seldom arose before 11 A.M., and dabbled in the study of human physiology, philosophy, glaciers, meteors, and rainbows. He eventually moved to Holland, where he published his *Discourse on the Method*, and finally to Sweden where he died while serving as tutor to Queen Christina. Descartes is regarded as a genius of the first magnitude. In addition to major contributions in mathematics and philosophy he is considered, along with William Harvey, to be a founder of modern physiology.

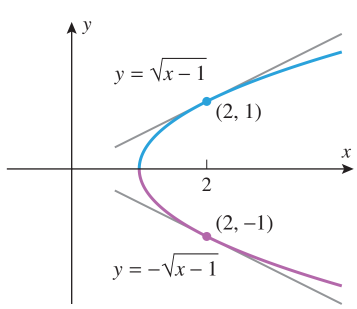
Solving for dy/dx we obtain
$$\frac{dy}{dx} = \frac{2x}{10y + \cos y} \tag{8}$$
Note that this formula involves both x and y. In order to obtain a formula for dy/dx that involves x alone, we would have to solve the original equation for y in terms of x and then substitute in (8). However, it is impossible to do this, so we are forced to leave the formula for dy/dx in terms of x and y. ◀

▶ Example 3 Use implicit differentiation to find d^2y/dx^2 if $4x^2 - 2y^2 = 9$.

Solution. Differentiating both sides of $4x^2 - 2y^2 = 9$ with respect to x yields
$$8x - 4y\frac{dy}{dx} = 0$$
from which we obtain
$$\frac{dy}{dx} = \frac{2x}{y} \tag{9}$$
Differentiating both sides of (9) yields
$$\frac{d^2y}{dx^2} = \frac{(y)(2) - (2x)(dy/dx)}{y^2} \tag{10}$$
Substituting (9) into (10) and simplifying using the original equation, we obtain
$$\frac{d^2y}{dx^2} = \frac{2y - 2x(2x/y)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -\frac{9}{y^3}$$ ◀

In Examples 2 and 3, the resulting formulas for dy/dx involved both x and y. Although it is usually more desirable to have the formula for dy/dx expressed in terms of x alone, having the formula in terms of x and y is not an impediment to finding slopes and equations of tangent lines provided the x- and y-coordinates of the point of tangency are known. This is illustrated in the following example.

▶ Example 4 Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.



▲ **Figure 3.1.4**

Solution. We could proceed by solving the equation for y in terms of x, and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2, 1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$ (Figure 3.1.4). However, implicit differentiation is more efficient since it can be used for the slopes of *both* tangent lines. Differentiating implicitly yields
$$\begin{aligned}
\frac{d}{dx}[y^2 - x + 1] &= \frac{d}{dx}[0] \\
\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] &= \frac{d}{dx}[0] \\
2y\frac{dy}{dx} - 1 &= 0 \\
\frac{dy}{dx} &= \frac{1}{2y}
\end{aligned}$$
At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the tangent lines to the curve at those points are
$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2}$$ ◀

▶ Example 5

(a) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.

(b) Find an equation for the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(c) At what point(s) in the first quadrant is the tangent line to the Folium of Descartes horizontal?

Solution (a). Differentiating implicitly yields

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

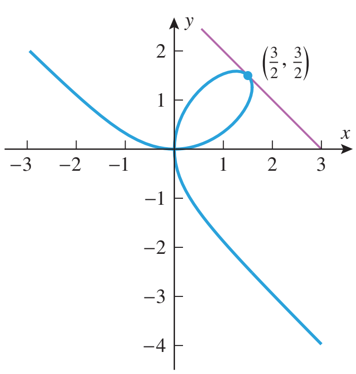
$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \tag{11}$$

Formula (11) cannot be evaluated at (0, 0) and hence provides no information about the nature of the Folium of Descartes at the origin. Based on the graphs in Figure 3.1.3, what can you say about the differentiability of the implicitly defined functions graphed in blue in parts (*b*) and (*c*) of the figure?

Solution (b). At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have $x = \frac{3}{2}$ and $y = \frac{3}{2}$, so from (11) the slope $m_{\tan}$ of the tangent line at this point is

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=3/2\\y=3/2}} = \frac{(3/2) - (3/2)^2}{(3/2)^2 - (3/2)} = -1$$



▲ **Figure 3.1.5**

Thus, the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \tfrac{3}{2} = -1\left(x - \tfrac{3}{2}\right) \quad \text{or} \quad x + y = 3$$

which is consistent with Figure 3.1.5.

Solution (c). The tangent line is horizontal at the points where $dy/dx = 0$, and from (11) this occurs only where $y - x^2 = 0$ or

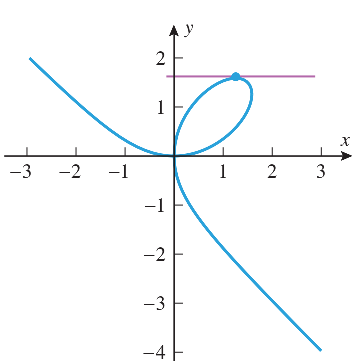
$$y = x^2 \tag{12}$$

Substituting this expression for y in the equation $x^3 + y^3 = 3xy$ for the curve yields

$$x^3 + (x^2)^3 = 3x^3$$

$$x^6 - 2x^3 = 0$$

$$x^3(x^3 - 2) = 0$$



▲ **Figure 3.1.6**

whose solutions are $x = 0$ and $x = 2^{1/3}$. From (12), the solutions $x = 0$ and $x = 2^{1/3}$ yield the points (0, 0) and $(2^{1/3}, 2^{2/3})$, respectively. Of these two, only $(2^{1/3}, 2^{2/3})$ is in the first quadrant. Substituting $x = 2^{1/3}$, $y = 2^{2/3}$ into (11) yields

$$\left.\frac{dy}{dx}\right|_{\substack{x=2^{1/3}\\y=2^{2/3}}} = \frac{0}{2^{4/3} - 2^{2/3}} = 0$$

We conclude that $(2^{1/3}, 2^{2/3}) \approx (1.26, 1.59)$ is the only point on the Folium of Descartes in the first quadrant at which the tangent line is horizontal (Figure 3.1.6). ◀

DIFFERENTIABILITY OF FUNCTIONS DEFINED IMPLICITLY

When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense. For example, if we differentiate the equation

$$x^2 + y^2 + 1 = 0 \tag{13}$$

we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

However, this derivative is meaningless because there are no real values of x and y that satisfy (13) (why?); and hence (13) does not define any real functions implicitly.

The nonsensical conclusion of these computations conveys the importance of knowing whether an equation in x and y that is to be differentiated implicitly actually defines some differentiable function of x implicitly. Unfortunately, this can be a difficult problem, so we will leave the discussion of such matters for more advanced courses in analysis.

✔QUICK CHECK EXERCISES 3.1 (See page 192 for answers.)

1. The equation $xy + 2y = 1$ defines implicitly the function $y =$ ______.
2. Use implicit differentiation to find dy/dx for $x^2 - y^3 = xy$.
3. The slope of the tangent line to the graph of $x + y + xy = 3$ at $(1, 1)$ is ______.
4. Use implicit differentiation to find d^2y/dx^2 for $\sin y = x$.

EXERCISE SET 3.1 C CAS

1–2
(a) Find dy/dx by differentiating implicitly.
(b) Solve the equation for y as a function of x, and find dy/dx from that equation.
(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

1. $x + xy - 2x^3 = 2$
2. $\sqrt{y} - \sin x = 2$

3–12 Find dy/dx by implicit differentiation.

3. $x^2 + y^2 = 100$
4. $x^3 + y^3 = 3xy^2$
5. $x^2y + 3xy^3 - x = 3$
6. $x^3y^2 - 5x^2y + x = 1$
7. $\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} = 1$
8. $x^2 = \dfrac{x+y}{x-y}$
9. $\sin(x^2y^2) = x$
10. $\cos(xy^2) = y$
11. $\tan^3(xy^2 + y) = x$
12. $\dfrac{xy^3}{1 + \sec y} = 1 + y^4$

13–18 Find d^2y/dx^2 by implicit differentiation.

13. $2x^2 - 3y^2 = 4$
14. $x^3 + y^3 = 1$
15. $x^3y^3 - 4 = 0$
16. $xy + y^2 = 2$
17. $y + \sin y = x$
18. $x \cos y = y$

19–20 Find the slope of the tangent line to the curve at the given points in two ways: first by solving for y in terms of x and differentiating and then by implicit differentiation.

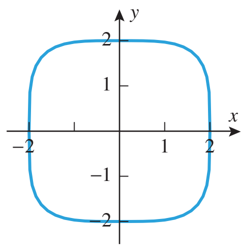
19. $x^2 + y^2 = 1$; $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$
20. $y^2 - x + 1 = 0$; $(10, 3)$, $(10, -3)$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

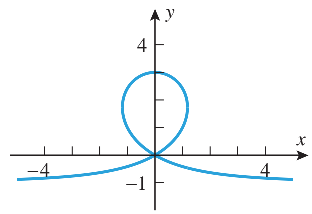
21. If an equation in x and y defines a function $y = f(x)$ implicitly, then the graph of the equation and the graph of f are identical.
22. The function
$$f(x) = \begin{cases} \sqrt{1-x^2}, & 0 < x \le 1 \\ -\sqrt{1-x^2}, & -1 \le x \le 0 \end{cases}$$
is defined implicitly by the equation $x^2 + y^2 = 1$.
23. The function $|x|$ is not defined implicitly by the equation $(x + y)(x - y) = 0$.
24. If y is defined implicitly as a function of x by the equation $x^2 + y^2 = 1$, then $dy/dx = -x/y$.

25–28 Use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph on the next page.

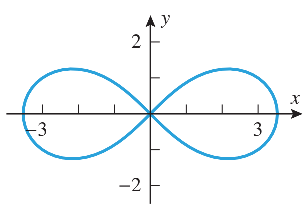
25. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [***Lamé's special quartic***]
26. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [***trisectrix***]
27. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$; $(3, 1)$ [***lemniscate***]
28. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [***four-cusped hypocycloid***]



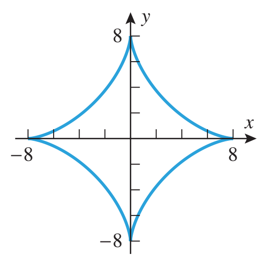
▲ Figure Ex-25



▲ Figure Ex-26



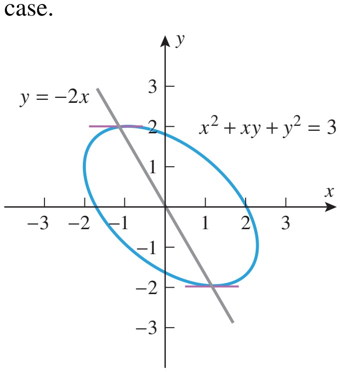
▲ Figure Ex-27



▲ Figure Ex-28

FOCUS ON CONCEPTS

29. In the accompanying figure, it appears that the ellipse $x^2 + xy + y^2 = 3$ has horizontal tangent lines at the points of intersection of the ellipse and the line $y = -2x$. Use implicit differentiation to explain why this is the case.



◀ Figure Ex-29

30. (a) A student claims that the ellipse $x^2 - xy + y^2 = 1$ has a horizontal tangent line at the point (1, 1). Without doing any computations, explain why the student's claim must be incorrect.
(b) Find all points on the ellipse $x^2 - xy + y^2 = 1$ at which the tangent line is horizontal.

C **31.** (a) Use the implicit plotting capability of a CAS to graph the equation $y^4 + y^2 = x(x - 1)$.
(b) Use implicit differentiation to help explain why the graph in part (a) has no horizontal tangent lines.
(c) Solve the equation $y^4 + y^2 = x(x - 1)$ for x in terms of y and explain why the graph in part (a) consists of two parabolas.

32. Use implicit differentiation to find all points on the graph of $y^4 + y^2 = x(x - 1)$ at which the tangent line is vertical.

33–34 These exercises deal with the rotated ellipse C whose equation is $x^2 - xy + y^2 = 4$. ■

33. Show that the line $y = x$ intersects C at two points P and Q and that the tangent lines to C at P and Q are parallel.

34. Prove that if $P(a, b)$ is a point on C, then so is $Q(-a, -b)$ and that the tangent lines to C through P and through Q are parallel.

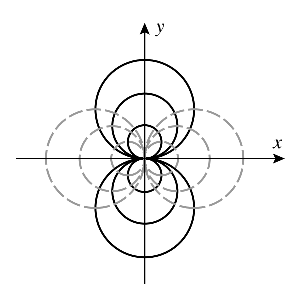
35. Find the values of a and b for the curve $x^2y + ay^2 = b$ if the point (1, 1) is on its graph and the tangent line at (1, 1) has the equation $4x + 3y = 7$.

36. At what point(s) is the tangent line to the curve $y^3 = 2x^2$ perpendicular to the line $x + 2y - 2 = 0$?

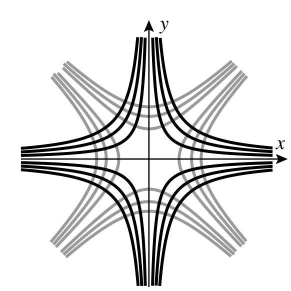
37–38 Two curves are said to be ***orthogonal*** if their tangent lines are perpendicular at each point of intersection, and two families of curves are said to be ***orthogonal trajectories*** of one another if each member of one family is orthogonal to each member of the other family. This terminology is used in these exercises. ■

37. The accompanying figure shows some typical members of the families of circles $x^2 + (y - c)^2 = c^2$ (black curves) and $(x - k)^2 + y^2 = k^2$ (gray curves). Show that these families are orthogonal trajectories of one another. [*Hint:* For the tangent lines to be perpendicular at a point of intersection, the slopes of those tangent lines must be negative reciprocals of one another.]

38. The accompanying figure shows some typical members of the families of hyperbolas $xy = c$ (black curves) and $x^2 - y^2 = k$ (gray curves), where $c \neq 0$ and $k \neq 0$. Use the hint in Exercise 37 to show that these families are orthogonal trajectories of one another.



▲ Figure Ex-37



▲ Figure Ex-38

C **39.** (a) Use the implicit plotting capability of a CAS to graph the curve C whose equation is $x^3 - 2xy + y^3 = 0$.
(b) Use the graph in part (a) to estimate the x-coordinates of a point in the first quadrant that is on C and at which the tangent line to C is parallel to the x-axis.
(c) Find the exact value of the x-coordinate in part (b).

C **40.** (a) Use the implicit plotting capability of a CAS to graph the curve C whose equation is $x^3 - 2xy + y^3 = 0$.
(b) Use the graph to guess the coordinates of a point in the first quadrant that is on C and at which the tangent line to C is parallel to the line $y = -x$. *(cont.)*

(c) Use implicit differentiation to verify your conjecture in part (b).

41. Prove that for every nonzero rational number r, the tangent line to the graph of $x^r + y^r = 2$ at the point $(1, 1)$ has slope -1.

42. Find equations for two lines through the origin that are tangent to the ellipse $2x^2 - 4x + y^2 + 1 = 0$.

43. Writing Write a paragraph that compares the concept of an *explicit* definition of a function with that of an *implicit* definition of a function.

44. Writing A student asks: "Suppose implicit differentiation yields an undefined expression at a point. Does this mean that dy/dx is undefined at that point?" Using the equation $x^2 - 2xy + y^2 = 0$ as a basis for your discussion, write a paragraph that answers the student's question.

✔QUICK CHECK ANSWERS 3.1

1. $\dfrac{1}{x+2}$ **2.** $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 3y^2}$ **3.** -1 **4.** $\dfrac{d^2y}{dx^2} = \sec^2 y \tan y$

3.2 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In this section we will obtain derivative formulas for logarithmic functions, and we will explain why the natural logarithm function is preferred over logarithms with other bases in calculus.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

We will establish that $f(x) = \ln x$ is differentiable for $x > 0$ by applying the derivative definition to $f(x)$. To evaluate the resulting limit, we will need the fact that $\ln x$ is continuous for $x > 0$ (Theorem 1.6.3), and we will need the limit

$$\lim_{v \to 0} (1 + v)^{1/v} = e \tag{1}$$

This limit can be obtained from limits (7) and (8) of Section 1.3 by making the substitution $v = 1/x$ and using the fact that $v \to 0^+$ as $x \to +\infty$ and $v \to 0^-$ as $x \to -\infty$. This produces two equal one-sided limits that together imply (1) (see Exercise 64 of Section 1.3).

$$\begin{aligned}
\frac{d}{dx}[\ln x] &= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) && \boxed{\text{The quotient property of logarithms in Theorem 0.5.2}} \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) \\
&= \lim_{v \to 0} \frac{1}{vx} \ln(1+v) && \boxed{\text{Let } v = h/x \text{ and note that } v \to 0 \text{ if and only if } h \to 0.} \\
&= \frac{1}{x} \lim_{v \to 0} \frac{1}{v} \ln(1+v) && \boxed{x \text{ is fixed in this limit computation, so } 1/x \text{ can be moved through the limit sign.}} \\
&= \frac{1}{x} \lim_{v \to 0} \ln(1+v)^{1/v} && \boxed{\text{The power property of logarithms in Theorem 0.5.2}} \\
&= \frac{1}{x} \ln\left[\lim_{v \to 0} (1+v)^{1/v}\right] && \boxed{\ln x \text{ is continuous on } (0, +\infty) \text{ so we can move the limit through the function symbol.}} \\
&= \frac{1}{x} \ln e \\
&= \frac{1}{x} && \boxed{\text{Since } \ln e = 1}
\end{aligned}$$

Thus,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0 \tag{2}$$

A derivative formula for the general logarithmic function $\log_b x$ can be obtained from (2) by using Formula (6) of Section 0.5 to write

$$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$$

It follows from this that

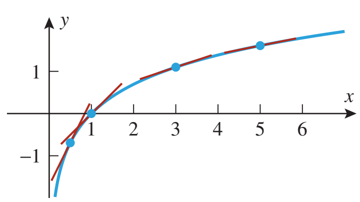
$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0 \tag{3}$$

Note that, among all possible bases, the base $b = e$ produces the simplest formula for the derivative of $\log_b x$. This is one of the reasons why the natural logarithm function is preferred over other logarithms in calculus.

▶ **Example 1**

(a) Figure 3.2.1 shows the graph of $y = \ln x$ and its tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5. Find the slopes of those tangent lines.

(b) Does the graph of $y = \ln x$ have any horizontal tangent lines? Use the derivative of $\ln x$ to justify your answer.



$y = \ln x$ with tangent lines

▲ **Figure 3.2.1**

Solution (a). From (2), the slopes of the tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5 are $1/x = 2, 1, \frac{1}{3}$, and $\frac{1}{5}$, respectively, which is consistent with Figure 3.2.1.

Solution (b). It does not appear from the graph of $y = \ln x$ that there are any horizontal tangent lines. This is confirmed by the fact that $dy/dx = 1/x$ is not equal to zero for any real value of x. ◀

If u is a differentiable function of x, and if $u(x) > 0$, then applying the chain rule to (2) and (3) produces the following generalized derivative formulas:

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\log_b u] = \frac{1}{u \ln b}\cdot\frac{du}{dx} \tag{4–5}$$

▶ **Example 2** Find $\dfrac{d}{dx}[\ln(x^2+1)]$.

Solution. Using (4) with $u = x^2 + 1$ we obtain

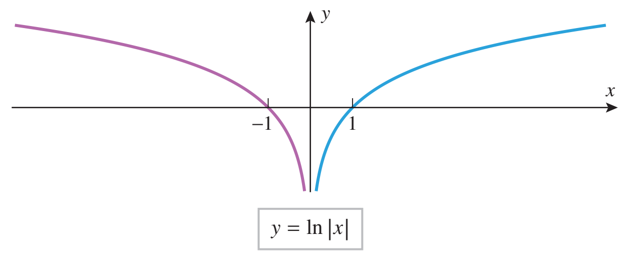
$$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1}\cdot\frac{d}{dx}[x^2+1] = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1} \quad ◀$$

When possible, the properties of logarithms in Theorem 0.5.2 should be used to convert products, quotients, and exponents into sums, differences, and constant multiples *before* differentiating a function involving logarithms.

▶ **Example 3**

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right] &= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right] \\ &= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)} \\ &= \frac{2}{x} + \cot x - \frac{1}{2+2x} \quad ◀\end{aligned}$$

Figure 3.2.2 shows the graph of $f(x) = \ln|x|$. This function is important because it "extends" the domain of the natural logarithm function in the sense that the values of $\ln|x|$ and $\ln x$ are the same for $x > 0$, but $\ln|x|$ is defined for all nonzero values of x, and $\ln x$ is only defined for positive values of x.



▶ **Figure 3.2.2**

The derivative of $\ln|x|$ for $x \neq 0$ can be obtained by considering the cases $x > 0$ and $x < 0$ separately:

Case $x > 0$. In this case $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Case $x < 0$. In this case $|x| = -x$, so it follows from (4) that

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{x}$$

Since the same formula results in both cases, we have shown that

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{if } x \neq 0 \tag{6}$$

▶ **Example 4** From (6) and the chain rule,

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \cot x \ ◀$$

LOGARITHMIC DIFFERENTIATION

We now consider a technique called ***logarithmic differentiation*** that is useful for differentiating functions that are composed of products, quotients, and powers.

▶ **Example 5** The derivative of

$$y = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4} \tag{7}$$

is messy to calculate directly. However, if we first take the natural logarithm of both sides and then use its properties, we can write

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x - 14) - 4\ln(1 + x^2)$$

Differentiating both sides with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x - 14} - \frac{8x}{1 + x^2}$$

Thus, on solving for dy/dx and using (7) we obtain

$$\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right] \blacktriangleleft$$

REMARK Since $\ln y$ is only defined for $y > 0$, the computations in Example 5 are only valid for $x > 2$ (verify). However, because the derivative of $\ln y$ is the same as the derivative of $\ln|y|$, and because $\ln|y|$ is defined for $y < 0$ as well as $y > 0$, it follows that the formula obtained for dy/dx is valid for $x < 2$ as well as $x > 2$. In general, whenever a derivative dy/dx is obtained by logarithmic differentiation, the resulting derivative formula will be valid for all values of x for which $y \neq 0$. It may be valid at those points as well, but it is not guaranteed.

DERIVATIVES OF REAL POWERS OF x

We know from Theorem 2.3.2 and Exercise 82 in Section 2.3 that the differentiation formula

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{8}$$

holds for constant integer values of r. We will now use logarithmic differentiation to show that this formula holds if r is *any* real number (rational or irrational). In our computations we will assume that x^r is a differentiable function and that the familiar laws of exponents hold for real exponents.

In the next section we will discuss differentiating functions that have exponents which are not constant.

Let $y = x^r$, where r is a real number. The derivative dy/dx can be obtained by logarithmic differentiation as follows:

$$\ln|y| = \ln|x^r| = r\ln|x|$$
$$\frac{d}{dx}[\ln|y|] = \frac{d}{dx}[r\ln|x|]$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$
$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

✔QUICK CHECK EXERCISES 3.2

(See page 196 for answers.)

1. The equation of the tangent line to the graph of $y = \ln x$ at $x = e^2$ is ________.

2. Find dy/dx.
(a) $y = \ln 3x$ (b) $y = \ln\sqrt{x}$
(c) $y = \log(1/|x|)$

3. Use logarithmic differentiation to find the derivative of

$$f(x) = \frac{\sqrt{x+1}}{\sqrt[3]{x-1}}$$

4. $\lim_{h\to 0}\frac{\ln(1+h)}{h} =$ ________

EXERCISE SET 3.2

1–26 Find dy/dx.

1. $y = \ln 5x$
2. $y = \ln\frac{x}{3}$
3. $y = \ln|1+x|$
4. $y = \ln(2+\sqrt{x})$
5. $y = \ln|x^2-1|$
6. $y = \ln|x^3-7x^2-3|$
7. $y = \ln\left(\frac{x}{1+x^2}\right)$
8. $y = \ln\left|\frac{1+x}{1-x}\right|$
9. $y = \ln x^2$
10. $y = (\ln x)^3$
11. $y = \sqrt{\ln x}$
12. $y = \ln\sqrt{x}$
13. $y = x\ln x$
14. $y = x^3\ln x$
15. $y = x^2\log_2(3-2x)$
16. $y = x[\log_2(x^2-2x)]^3$
17. $y = \frac{x^2}{1+\log x}$
18. $y = \frac{\log x}{1+\log x}$

19. $y = \ln(\ln x)$

20. $y = \ln(\ln(\ln x))$

21. $y = \ln(\tan x)$

22. $y = \ln(\cos x)$

23. $y = \cos(\ln x)$

24. $y = \sin^2(\ln x)$

25. $y = \log(\sin^2 x)$

26. $y = \log(1 - \sin^2 x)$

27–30 Use the method of Example 3 to help perform the indicated differentiation. ■

27. $\dfrac{d}{dx}[\ln((x-1)^3(x^2+1)^4)]$

28. $\dfrac{d}{dx}[\ln((\cos^2 x)\sqrt{1+x^4})]$

29. $\dfrac{d}{dx}\left[\ln \dfrac{\cos x}{\sqrt{4-3x^2}}\right]$

30. $\dfrac{d}{dx}\left[\ln \sqrt{\dfrac{x-1}{x+1}}\right]$

31–34 True–False Determine whether the statement is true or false. Explain your answer. ■

31. The slope of the tangent line to the graph of $y = \ln x$ at $x = a$ approaches infinity as $a \to 0^+$.

32. If $\lim_{x\to+\infty} f'(x) = 0$, then the graph of $y = f(x)$ has a horizontal asymptote.

33. The derivative of $\ln|x|$ is an odd function.

34. We have

$$\frac{d}{dx}((\ln x)^2) = \frac{d}{dx}(2(\ln x)) = \frac{2}{x}$$

35–38 Find dy/dx using logarithmic differentiation. ■

35. $y = x\sqrt[3]{1+x^2}$

36. $y = \sqrt[5]{\dfrac{x-1}{x+1}}$

37. $y = \dfrac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}$

38. $y = \dfrac{\sin x \cos x \tan^3 x}{\sqrt{x}}$

39. Find
(a) $\dfrac{d}{dx}[\log_x e]$ (b) $\dfrac{d}{dx}[\log_x 2]$.

40. Find
(a) $\dfrac{d}{dx}[\log_{(1/x)} e]$ (b) $\dfrac{d}{dx}[\log_{(\ln x)} e]$.

41–44 Find the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$. ■

41. $f(x) = \ln x;\ x_0 = e^{-1}$

42. $f(x) = \log x;\ x_0 = 10$

43. $f(x) = \ln(-x);\ x_0 = -e$

44. $f(x) = \ln|x|;\ x_0 = -2$

FOCUS ON CONCEPTS

45. (a) Find the equation of a line through the origin that is tangent to the graph of $y = \ln x$.
(b) Explain why the y-intercept of a tangent line to the curve $y = \ln x$ must be 1 unit less than the y-coordinate of the point of tangency.

46. Use logarithmic differentiation to verify the product and quotient rules. Explain what properties of $\ln x$ are important for this verification.

47. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y = \ln x$ at $P(w, \ln w)$, the horizontal line through P, and the y-axis.

48. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y = \ln x^2$ at $P(w, \ln w^2)$, the horizontal line through P, and the y-axis.

49. Verify that $y = \ln(x+e)$ satisfies $dy/dx = e^{-y}$, with $y = 1$ when $x = 0$.

50. Verify that $y = -\ln(e^2 - x)$ satisfies $dy/dx = e^y$, with $y = -2$ when $x = 0$.

51. Find a function f such that $y = f(x)$ satisfies $dy/dx = e^{-y}$, with $y = 0$ when $x = 0$.

52. Find a function f such that $y = f(x)$ satisfies $dy/dx = e^y$, with $y = -\ln 2$ when $x = 0$.

53–55 Find the limit by interpreting the expression as an appropriate derivative. ■

53. (a) $\displaystyle\lim_{x\to 0} \frac{\ln(1+3x)}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\ln(1-5x)}{x}$

54. (a) $\displaystyle\lim_{\Delta x\to 0} \frac{\ln(e^2+\Delta x)-2}{\Delta x}$ (b) $\displaystyle\lim_{w\to 1} \frac{\ln w}{w-1}$

55. (a) $\displaystyle\lim_{x\to 0} \frac{\ln(\cos x)}{x}$ (b) $\displaystyle\lim_{h\to 0} \frac{(1+h)^{\sqrt{2}}-1}{h}$

56. Modify the derivation of Equation (2) to give another proof of Equation (3).

57. Writing Review the derivation of the formula

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

and then write a paragraph that discusses all the ingredients (theorems, limit properties, etc.) that are needed for this derivation.

58. Writing Write a paragraph that explains how logarithmic differentiation can replace a difficult differentiation computation with a simpler computation.

✔ QUICK CHECK ANSWERS 3.2

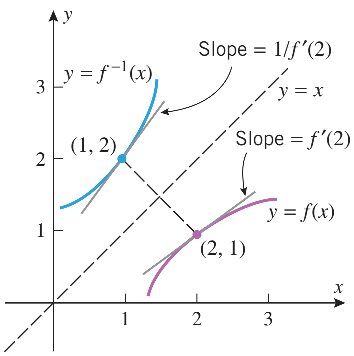
1. $y = \dfrac{x}{e^2} + 1$ **2.** (a) $\dfrac{dy}{dx} = \dfrac{1}{x}$ (b) $\dfrac{dy}{dx} = \dfrac{1}{2x}$ (c) $\dfrac{dy}{dx} = -\dfrac{1}{x\ln 10}$ **3.** $\dfrac{\sqrt{x+1}}{\sqrt[3]{x-1}}\left[\dfrac{1}{2(x+1)} - \dfrac{1}{3(x-1)}\right]$ **4.** 1

3.3 DERIVATIVES OF EXPONENTIAL AND INVERSE TRIGONOMETRIC FUNCTIONS

See Section 0.4 for a review of one-to-one functions and inverse functions.

In this section we will show how the derivative of a one-to-one function can be used to obtain the derivative of its inverse function. This will provide the tools we need to obtain derivative formulas for exponential functions from the derivative formulas for logarithmic functions and to obtain derivative formulas for inverse trigonometric functions from the derivative formulas for trigonometric functions.

Our first goal in this section is to obtain a formula relating the derivative of the inverse function f^{-1} to the derivative of the function f.



▲ **Figure 3.3.1**

▶ **Example 1** Suppose that f is a one-to-one differentiable function such that $f(2) = 1$ and $f'(2) = \frac{3}{4}$. Then the tangent line to $y = f(x)$ at the point (2, 1) has equation

$$y - 1 = \tfrac{3}{4}(x - 2)$$

The tangent line to $y = f^{-1}(x)$ at the point (1, 2) is the reflection about the line $y = x$ of the tangent line to $y = f(x)$ at the point (2, 1) (Figure 3.3.1), and its equation can be obtained by interchanging x and y:

$$x - 1 = \tfrac{3}{4}(y - 2) \quad \text{or} \quad y - 2 = \tfrac{4}{3}(x - 1)$$

Notice that the slope of the tangent line to $y = f^{-1}(x)$ at $x = 1$ is the reciprocal of the slope of the tangent line to $y = f(x)$ at $x = 2$. That is,

$$(f^{-1})'(1) = \frac{1}{f'(2)} = \frac{4}{3} \quad \blacktriangleleft \tag{1}$$

Since $2 = f^{-1}(1)$ for the function f in Example 1, it follows that $f'(2) = f'(f^{-1}(1))$. Thus, Formula (1) can also be expressed as

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

In general, if f is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \tag{2}$$

provided $f'(f^{-1}(x)) \neq 0$.

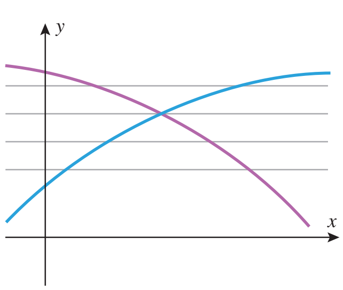
Formula (2) can be confirmed using implicit differentiation. The equation $y = f^{-1}(x)$ is equivalent to $x = f(y)$. Differentiating with respect to x we obtain

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$$

so that
$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Also from $x = f(y)$ we have $dx/dy = f'(y)$, which gives the following alternative version of Formula (2):

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{3}$$



▲ **Figure 3.3.2** The graph of an increasing function (blue) or a decreasing function (purple) is cut at most once by any horizontal line.

INCREASING OR DECREASING FUNCTIONS ARE ONE-TO-ONE

If the graph of a function f is always increasing or always decreasing over the domain of f, then a horizontal line will cut the graph of f in at most one point (Figure 3.3.2), so f

must have an inverse function (see Section 0.4). We will prove in the next chapter that f is increasing on any interval on which $f'(x) > 0$ (since the graph has positive slope) and that f is decreasing on any interval on which $f'(x) < 0$ (since the graph has negative slope). These intuitive observations, together with Formula (2), suggest the following theorem, which we state without formal proof.

3.3.1 THEOREM *Suppose that the domain of a function f is an open interval on which $f'(x) > 0$ or on which $f'(x) < 0$. Then f is one-to-one, $f^{-1}(x)$ is differentiable at all values of x in the range of f, and the derivative of $f^{-1}(x)$ is given by Formula (2).*

► **Example 2** Consider the function $f(x) = x^5 + x + 1$.

(a) Show that f is one-to-one on the interval $(-\infty, +\infty)$.
(b) Find a formula for the derivative of f^{-1}.
(c) Compute $(f^{-1})'(1)$.

In general, once it is established that f^{-1} is differentiable, one has the option of calculating the derivative of f^{-1} using Formula (2) or (3), or by differentiating implicitly, as in Example 2.

Solution (a). Since

$$f'(x) = 5x^4 + 1 > 0$$

for all real values of x, it follows from Theorem 3.3.1 that f is one-to-one on the interval $(-\infty, +\infty)$.

Solution (b). Let $y = f^{-1}(x)$. Differentiating $x = f(y) = y^5 + y + 1$ implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1} \tag{4}$$

We cannot solve $x = y^5 + y + 1$ for y in terms of x, so we leave the expression for dy/dx in Equation (4) in terms of y.

Solution (c). From Equation (4),

$$(f^{-1})'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{5y^4 + 1}\right|_{x=1}$$

Thus, we need to know the value of $y = f^{-1}(x)$ at $x = 1$, which we can obtain by solving the equation $f(y) = 1$ for y. This equation is $y^5 + y + 1 = 1$, which, by inspection, is satisfied by $y = 0$. Thus,

$$(f^{-1})'(1) = \left.\frac{1}{5y^4 + 1}\right|_{y=0} = 1 \quad ◄$$

DERIVATIVES OF EXPONENTIAL FUNCTIONS

Our next objective is to show that the general exponential function b^x $(b > 0, b \neq 1)$ is differentiable everywhere and to find its derivative. To do this, we will use the fact that

b^x is the inverse of the function $f(x) = \log_b x$. We will assume that $b > 1$. With this assumption we have $\ln b > 0$, so

$$f'(x) = \frac{d}{dx}[\log_b x] = \frac{1}{x \ln b} > 0 \quad \text{for all } x \text{ in the interval } (0, +\infty)$$

It now follows from Theorem 3.3.1 that $f^{-1}(x) = b^x$ is differentiable for all x in the range of $f(x) = \log_b x$. But we know from Table 0.5.3 that the range of $\log_b x$ is $(-\infty, +\infty)$, so we have established that b^x is differentiable everywhere.

To obtain a derivative formula for b^x we rewrite $y = b^x$ as

$$x = \log_b y$$

and differentiate implicitly using Formula (5) of Section 3.2 to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for dy/dx and replacing y by b^x we have

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

> How does the derivation of Formula (5) change if $0 < b < 1$?

Thus, we have shown that

$$\frac{d}{dx}[b^x] = b^x \ln b \tag{5}$$

In the special case where $b = e$ we have $\ln e = 1$, so that (5) becomes

$$\frac{d}{dx}[e^x] = e^x \tag{6}$$

> In Section 0.5 we stated that $b = e$ is the only base for which the slope of the tangent line to the curve $y = b^x$ at any point P on the curve is the y-coordinate at P (see page 54). Verify this statement.

Moreover, if u is a differentiable function of x, then it follows from (5) and (6) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \tag{7–8}$$

> It is important to distinguish between differentiating an exponential function b^x (variable exponent and constant base) and a power function x^b (variable base and constant exponent). For example, compare the derivative
>
> $$\frac{d}{dx}[x^2] = 2x$$
>
> to the derivative of 2^x in Example 3.

Example 3 The following computations use Formulas (7) and (8).

$$\frac{d}{dx}[2^x] = 2^x \ln 2$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x}$$ ◀

Functions of the form $f(x) = u^v$ in which u and v are *nonconstant* functions of x are neither exponential functions nor power functions. Functions of this form can be differentiated using logarithmic differentiation.

Example 4 Use logarithmic differentiation to find $\frac{d}{dx}[(x^2+1)^{\sin x}]$.

Solution. Setting $y = (x^2+1)^{\sin x}$ we have

$$\ln y = \ln[(x^2+1)^{\sin x}] = (\sin x)\ln(x^2+1)$$

Differentiating both sides with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[(\sin x)\ln(x^2+1)]$$

$$= (\sin x)\frac{1}{x^2+1}(2x) + (\cos x)\ln(x^2+1)$$

Thus,

$$\frac{dy}{dx} = y\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]$$

$$= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right] \blacktriangleleft$$

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

To obtain formulas for the derivatives of the inverse trigonometric functions, we will need to use some of the identities given in Formulas (11) to (17) of Section 0.4. Rather than memorize those identities, we recommend that you review the "triangle technique" that we used to obtain them.

To begin, consider the function $\sin^{-1} x$. If we let $f(x) = \sin x$ $(-\pi/2 \le x \le \pi/2)$, then it follows from Formula (2) that $f^{-1}(x) = \sin^{-1} x$ will be differentiable at any point x where $\cos(\sin^{-1} x) \ne 0$. This is equivalent to the condition

$$\sin^{-1} x \ne -\frac{\pi}{2} \quad \text{and} \quad \sin^{-1} x \ne \frac{\pi}{2}$$

so it follows that $\sin^{-1} x$ is differentiable on the interval $(-1, 1)$.

A derivative formula for $\sin^{-1} x$ on $(-1, 1)$ can be obtained by using Formula (2) or (3) or by differentiating implicitly. We will use the latter method. Rewriting the equation $y = \sin^{-1} x$ as $x = \sin y$ and differentiating implicitly with respect to x, we obtain

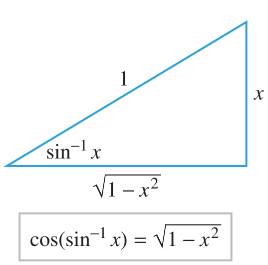
Observe that $\sin^{-1} x$ is only differentiable on the interval $(-1, 1)$, even though its domain is $[-1, 1]$. This is because the graph of $y = \sin x$ has horizontal tangent lines at the points $(\pi/2, 1)$ and $(-\pi/2, -1)$, so the graph of $y = \sin^{-1} x$ has vertical tangent lines at $x = \pm 1$.

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

At this point we have succeeded in obtaining the derivative; however, this derivative formula can be simplified using the identity indicated in Figure 3.3.3. This yields



▲ **Figure 3.3.3**

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Thus, we have shown that

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

More generally, if u is a differentiable function of x, then the chain rule produces the following generalized version of this formula:

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (-1 < u < 1)$$

The method used to derive this formula can be used to obtain generalized derivative formulas for the remaining inverse trigonometric functions. The following is a complete list of these formulas, each of which is valid on the natural domain of the function that multiplies du/dx.

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{9–10}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx} \tag{11–12}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \tag{13–14}$$

The appearance of $|u|$ in (13) and (14) will be explained in Exercise 58.

Example 5 Find dy/dx if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution (a). From (9)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

Solution (b). From (13)

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}}$$ ◀

✔QUICK CHECK EXERCISES 3.3 *(See page 203 for answers.)*

1. Suppose that a one-to-one function f has tangent line $y = 5x + 3$ at the point $(1, 8)$. Evaluate $(f^{-1})'(8)$.
2. In each case, from the given derivative, determine whether the function f is invertible.
   (a) $f'(x) = x^2 + 1$ (b) $f'(x) = x^2 - 1$
   (c) $f'(x) = \sin x$ (d) $f'(x) = \dfrac{\pi}{2} + \tan^{-1} x$
3. Evaluate the derivative.
   (a) $\dfrac{d}{dx}[e^x]$ (b) $\dfrac{d}{dx}[7^x]$
   (c) $\dfrac{d}{dx}[\cos(e^x + 1)]$ (d) $\dfrac{d}{dx}[e^{3x-2}]$
4. Let $f(x) = e^{x^3+x}$. Use $f'(x)$ to verify that f is one-to-one.

EXERCISE SET 3.3 Graphing Utility

FOCUS ON CONCEPTS

1. Let $f(x) = x^5 + x^3 + x$.
   (a) Show that f is one-to-one and confirm that $f(1) = 3$.
   (b) Find $(f^{-1})'(3)$.
2. Let $f(x) = x^3 + 2e^x$.
   (a) Show that f is one-to-one and confirm that $f(0) = 2$.
   (b) Find $(f^{-1})'(2)$.

3–4 Find $(f^{-1})'(x)$ using Formula (2), and check your answer by differentiating f^{-1} directly. ■

3. $f(x) = 2/(x + 3)$
4. $f(x) = \ln(2x + 1)$

5–6 Determine whether the function f is one-to-one by examining the sign of $f'(x)$. ■

5. (a) $f(x) = x^2 + 8x + 1$
   (b) $f(x) = 2x^5 + x^3 + 3x + 2$
   (c) $f(x) = 2x + \sin x$
   (d) $f(x) = \left(\frac{1}{2}\right)^x$
6. (a) $f(x) = x^3 + 3x^2 - 8$
   (b) $f(x) = x^5 + 8x^3 + 2x - 1$
   (c) $f(x) = \dfrac{x}{x+1}$
   (d) $f(x) = \log_b x, \quad 0 < b < 1$

7–10 Find the derivative of f^{-1} by using Formula (3), and check your result by differentiating implicitly.

7. $f(x) = 5x^3 + x - 7$
8. $f(x) = 1/x^2, \quad x > 0$
9. $f(x) = 2x^5 + x^3 + 1$
10. $f(x) = 5x - \sin 2x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$

FOCUS ON CONCEPTS

11. Figure 0.4.8 is a "proof by picture" that the reflection of a point $P(a, b)$ about the line $y = x$ is the point $Q(b, a)$. Establish this result rigorously by completing each part.
    (a) Prove that if P is not on the line $y = x$, then P and Q are distinct, and the line $\overleftrightarrow{PQ}$ is perpendicular to the line $y = x$.
    (b) Prove that if P is not on the line $y = x$, the midpoint of segment PQ is on the line $y = x$.
    (c) Carefully explain what it means geometrically to reflect P about the line $y = x$.
    (d) Use the results of parts (a)–(c) to prove that Q is the reflection of P about the line $y = x$.
12. Prove that the reflection about the line $y = x$ of a line with slope m, $m \neq 0$, is a line with slope $1/m$. [*Hint:* Apply the result of the previous exercise to a pair of points on the line of slope m and to a corresponding pair of points on the reflection of this line about the line $y = x$.]
13. Suppose that f and g are increasing functions. Determine which of the functions $f(x) + g(x)$, $f(x)g(x)$, and $f(g(x))$ must also be increasing.
14. Suppose that f and g are one-to-one functions. Determine which of the functions $f(x) + g(x)$, $f(x)g(x)$, and $f(g(x))$ must also be one-to-one.

15–26 Find dy/dx.

15. $y = e^{7x}$
16. $y = e^{-5x^2}$
17. $y = x^3 e^x$
18. $y = e^{1/x}$
19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
20. $y = \sin(e^x)$
21. $y = e^{x \tan x}$
22. $y = \dfrac{e^x}{\ln x}$
23. $y = e^{(x - e^{3x})}$
24. $y = \exp(\sqrt{1 + 5x^3})$
25. $y = \ln(1 - xe^{-x})$
26. $y = \ln(\cos e^x)$

27–30 Find $f'(x)$ by Formula (7) and then by logarithmic differentiation.

27. $f(x) = 2^x$
28. $f(x) = 3^{-x}$
29. $f(x) = \pi^{\sin x}$
30. $f(x) = \pi^{x \tan x}$

31–35 Find dy/dx using the method of logarithmic differentiation.

31. $y = (x^3 - 2x)^{\ln x}$
32. $y = x^{\sin x}$
33. $y = (\ln x)^{\tan x}$
34. $y = (x^2 + 3)^{\ln x}$
35. $y = (\ln x)^{\ln x}$
36. (a) Explain why Formula (5) cannot be used to find $(d/dx)[x^x]$.
    (b) Find this derivative by logarithmic differentiation.

37–52 Find dy/dx.

37. $y = \sin^{-1}(3x)$
38. $y = \cos^{-1}\left(\dfrac{x+1}{2}\right)$
39. $y = \sin^{-1}(1/x)$
40. $y = \cos^{-1}(\cos x)$
41. $y = \tan^{-1}(x^3)$
42. $y = \sec^{-1}(x^5)$
43. $y = (\tan x)^{-1}$
44. $y = \dfrac{1}{\tan^{-1} x}$
45. $y = e^x \sec^{-1} x$
46. $y = \ln(\cos^{-1} x)$
47. $y = \sin^{-1} x + \cos^{-1} x$
48. $y = x^2(\sin^{-1} x)^3$
49. $y = \sec^{-1} x + \csc^{-1} x$
50. $y = \csc^{-1}(e^x)$
51. $y = \cot^{-1}(\sqrt{x})$
52. $y = \sqrt{\cot^{-1} x}$

53–56 True–False Determine whether the statement is true or false. Explain your answer.

53. If a function $y = f(x)$ satisfies $dy/dx = y$, then $y = e^x$.
54. If $y = f(x)$ is a function such that dy/dx is a rational function, then $f(x)$ is also a rational function.
55. $\dfrac{d}{dx}(\log_b |x|) = \dfrac{1}{x \ln b}$
56. We can conclude from the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ that $\sin^{-1} x + \cos^{-1} x$ is constant.
57. (a) Use Formula (2) to prove that
$$\frac{d}{dx}[\cot^{-1} x]\bigg|_{x=0} = -1$$
    (b) Use part (a) above, part (a) of Exercise 48 in Section 0.4, and the chain rule to show that
$$\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1+x^2}$$
    for $-\infty < x < +\infty$.
    (c) Conclude from part (b) that
$$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$
    for $-\infty < u < +\infty$.
58. (a) Use part (c) of Exercise 48 in Section 0.4 and the chain rule to show that
$$\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$
    for $1 < |x|$.
    (b) Conclude from part (a) that
$$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$
    for $1 < |u|$.

(cont.)

(c) Use Equation (11) in Section 0.4 and parts (b) and (c) of Exercise 48 in that section to show that if $|x| \geq 1$ then, $\sec^{-1} x + \csc^{-1} x = \pi/2$. Conclude from part (a) that

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

(d) Conclude from part (c) that

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

59–60 Find dy/dx by implicit differentiation. ■

59. $x^3 + x\tan^{-1} y = e^y$

60. $\sin^{-1}(xy) = \cos^{-1}(x-y)$

61. (a) Show that $f(x) = x^3 - 3x^2 + 2x$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the largest value of k such that f is one-to-one on the interval $(-k, k)$.

62. (a) Show that the function $f(x) = x^4 - 2x^3$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the smallest value of k such that f is one-to-one on the interval $[k, +\infty)$.

63. Let $f(x) = x^4 + x^3 + 1$, $0 \leq x \leq 2$.
(a) Show that f is one-to-one.
(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f(2g(x))$. Find an equation for the tangent line to $y = F(x)$ at $x = 3$.

64. Let $f(x) = \dfrac{\exp(4-x^2)}{x}$, $x > 0$.
(a) Show that f is one-to-one.
(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f([g(x)]^2)$. Find $F'\left(\frac{1}{2}\right)$.

65. Show that for any constants A and k, the function $y = Ae^{kt}$ satisfies the equation $dy/dt = ky$.

66. Show that for any constants A and B, the function

$$y = Ae^{2x} + Be^{-4x}$$

satisfies the equation

$$y'' + 2y' - 8y = 0$$

67. Show that
(a) $y = xe^{-x}$ satisfies the equation $xy' = (1-x)y$
(b) $y = xe^{-x^2/2}$ satisfies the equation $xy' = (1-x^2)y$.

68. Show that the rate of change of $y = 100e^{-0.2x}$ with respect to x is proportional to y.

69. Show that

$$y = \frac{60}{5+7e^{-t}} \quad \text{satisfies} \quad \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

for some constants r and K, and determine the values of these constants.

70. Suppose that the population of oxygen-dependent bacteria in a pond is modeled by the equation

$$P(t) = \frac{60}{5+7e^{-t}}$$

where $P(t)$ is the population (in billions) t days after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

71–76 Find the limit by interpreting the expression as an appropriate derivative. ■

71. $\displaystyle\lim_{x\to 0}\frac{e^{3x}-1}{x}$

72. $\displaystyle\lim_{x\to 0}\frac{\exp(x^2)-1}{x}$

73. $\displaystyle\lim_{h\to 0}\frac{10^h-1}{h}$

74. $\displaystyle\lim_{h\to 0}\frac{\tan^{-1}(1+h)-\pi/4}{h}$

75. $\displaystyle\lim_{\Delta x\to 0}\frac{9\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}+\Delta x\right)\right]^2-\pi^2}{\Delta x}$

76. $\displaystyle\lim_{w\to 2}\frac{3\sec^{-1} w-\pi}{w-2}$

77. **Writing** Let G denote the graph of an invertible function f and consider G as a fixed set of points in the plane. Suppose we relabel the coordinate axes so that the x-axis becomes the y-axis and vice versa. Carefully explain why now the same set of points G becomes the graph of f^{-1} (with the coordinate axes in a nonstandard position). Use this result to explain Formula (2).

78. **Writing** Suppose that f has an inverse function. Carefully explain the connection between Formula (2) and implicit differentiation of the equation $x = f(y)$.

✔ QUICK CHECK ANSWERS 3.3

1. $\frac{1}{5}$ **2.** (a) yes (b) no (c) no (d) yes **3.** (a) e^x (b) $7^x \ln 7$ (c) $-e^x\sin(e^x+1)$ (d) $3e^{3x-2}$
4. $f'(x) = e^{x^3+x}\cdot(3x^2+1) > 0$ for all x

3.4 RELATED RATES

In this section we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known.

DIFFERENTIATING EQUATIONS TO RELATE RATES

Figure 3.4.1 shows a liquid draining through a conical filter. As the liquid drains, its volume V, height h, and radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3}r^2h$$

If we were interested in finding the rate of change of the volume V with respect to the time t, we could begin by differentiating both sides of this equation with respect to t to obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Thus, to find dV/dt at a specific time t from this equation we would need to have values for r, h, dh/dt, and dr/dt at that time. This is called a ***related rates problem*** because the goal is to find an unknown rate of change by *relating* it to other variables whose values and whose rates of change at time t are known or can be found in some way. Let us begin with a simple example.

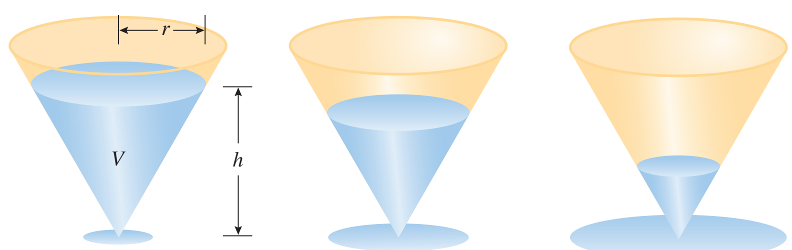


Figure 3.4.1

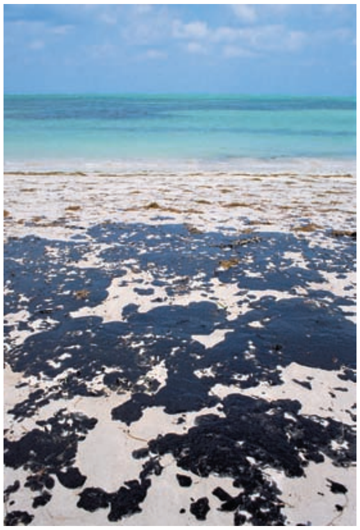
Example 1 Suppose that x and y are differentiable functions of t and are related by the equation $y = x^3$. Find dy/dt at time $t = 1$ if $x = 2$ and $dx/dt = 4$ at time $t = 1$.

Solution. Using the chain rule to differentiate both sides of the equation $y = x^3$ with respect to t yields

$$\frac{dy}{dt} = \frac{d}{dt}[x^3] = 3x^2\frac{dx}{dt}$$

Thus, the value of dy/dt at time $t = 1$ is

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(2)^2 \left.\frac{dx}{dt}\right|_{t=1} = 12 \cdot 4 = 48 \blacktriangleleft$$



Arni Katz/Phototake

Oil spill from a ruptured tanker.

▶ **Example 2** Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

Solution. Let

t = number of seconds elapsed from the time of the spill
r = radius of the spill in feet after t seconds
A = area of the spill in square feet after t seconds

(Figure 3.4.2). We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $r = 60$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

This suggests that we look for an equation relating A and r that we can differentiate with respect to t to produce a relationship between dA/dt and dr/dt. But A is the area of a circle of radius r, so

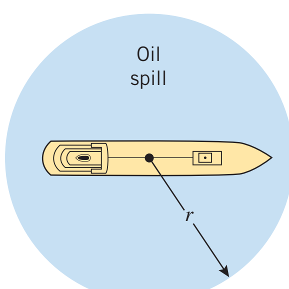
$$A = \pi r^2 \tag{1}$$

Differentiating both sides of (1) with respect to t yields

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \tag{2}$$

Thus, when $r = 60$ the area of the spill is increasing at the rate of

$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s} \approx 754 \text{ ft}^2/\text{s} \blacktriangleleft$$



▲ **Figure 3.4.2**

With some minor variations, the method used in Example 2 can be used to solve a variety of related rates problems. We can break the method down into five steps.

A Strategy for Solving Related Rates Problems

Step 1. Assign letters to all quantities that vary with time and any others that seem relevant to the problem. Give a definition for each letter.

Step 2. Identify the rates of change that are known and the rate of change that is to be found. Interpret each rate as a derivative.

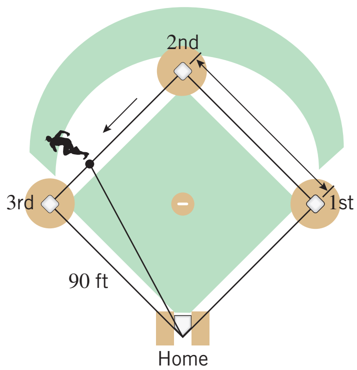
Step 3. Find an equation that relates the variables whose rates of change were identified in Step 2. To do this, it will often be helpful to draw an appropriately labeled figure that illustrates the relationship.

Step 4. Differentiate both sides of the equation obtained in Step 3 with respect to time to produce a relationship between the known rates of change and the unknown rate of change.

Step 5. *After* completing Step 4, substitute all known values for the rates of change and the variables, and then solve for the unknown rate of change.

WARNING

We have italicized the word "After" in Step 5 because it is a common error to substitute numerical values before performing the differentiation. For instance, in Example 2 had we substituted the known value of $r = 60$ in (1) before differentiating, we would have obtained $dA/dt = 0$, which is obviously incorrect.

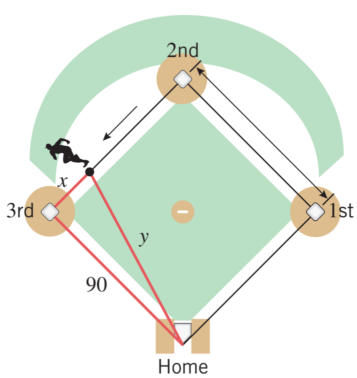


▲ **Figure 3.4.3**

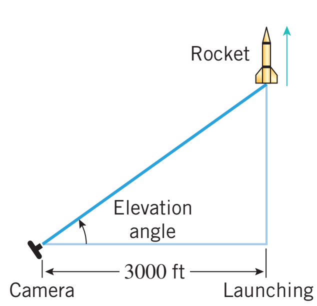
The quantity

$$\left.\frac{dx}{dt}\right|_{x=20}$$

is negative because x is decreasing with respect to t.



▲ **Figure 3.4.4**



▲ **Figure 3.4.5**

▶ **Example 3** A baseball diamond is a square whose sides are 90 ft long (Figure 3.4.3). Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?

Solution. We are given a constant speed with which the player is approaching third base, and we want to find the rate of change of the distance between the player and home plate at a particular instant. Thus, let

t = number of seconds since the player left second base
x = distance in feet from the player to third base
y = distance in feet from the player to home plate

(Figure 3.4.4). Thus, we want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

As suggested by Figure 3.4.4, an equation relating the variables x and y can be obtained using the Theorem of Pythagoras:

$$x^2 + 90^2 = y^2 \tag{3}$$

Differentiating both sides of this equation with respect to t yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

from which we obtain

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \tag{4}$$

When $x = 20$, it follows from (3) that

$$y = \sqrt{20^2 + 90^2} = \sqrt{8500} = 10\sqrt{85}$$

so that (4) yields

$$\left.\frac{dy}{dt}\right|_{x=20} = \frac{20}{10\sqrt{85}}(-30) = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

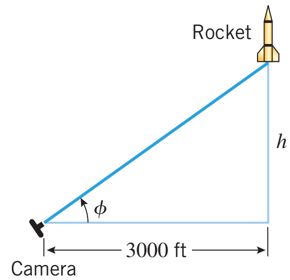
The negative sign in the answer tells us that y is decreasing, which makes sense physically from Figure 3.4.4. ◀

▶ **Example 4** In Figure 3.4.5 we have shown a camera mounted at a point 3000 ft from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 ft above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

Solution. Let

t = number of seconds elapsed from the time of launch
ϕ = camera elevation angle in radians after t seconds
h = height of the rocket in feet after t seconds

(Figure 3.4.6). At each instant the rate at which the camera elevation angle must change



▲ **Figure 3.4.6**

is $d\phi/dt$, and the rate at which the rocket is rising is dh/dt. We want to find

$$\left.\frac{d\phi}{dt}\right|_{h=4000} \quad \text{given that} \quad \left.\frac{dh}{dt}\right|_{h=4000} = 880 \text{ ft/s}$$

From Figure 3.4.6 we see that

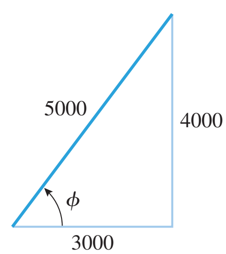
$$\tan\phi = \frac{h}{3000} \tag{5}$$

Differentiating both sides of (5) with respect to t yields

$$(\sec^2\phi)\frac{d\phi}{dt} = \frac{1}{3000}\frac{dh}{dt} \tag{6}$$

When $h = 4000$, it follows that

$$(\sec\phi)\big|_{h=4000} = \frac{5000}{3000} = \frac{5}{3}$$



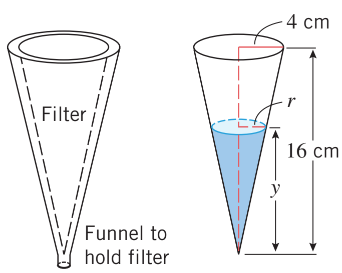
▲ **Figure 3.4.7**

(see Figure 3.4.7), so that from (6)

$$\left(\frac{5}{3}\right)^2 \left.\frac{d\phi}{dt}\right|_{h=4000} = \frac{1}{3000}\cdot 880 = \frac{22}{75}$$

$$\left.\frac{d\phi}{dt}\right|_{h=4000} = \frac{22}{75}\cdot\frac{9}{25} = \frac{66}{625} \approx 0.11 \text{ rad/s} \approx 6.05 \text{ deg/s} \blacktriangleleft$$

▶ **Example 5** Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top (Figure 3.4.8). Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3/min.



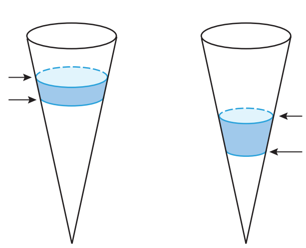
▲ **Figure 3.4.8**

(a) Do you think that the depth of the liquid will decrease at a constant rate? Give a verbal argument that justifies your conclusion.

(b) Find a formula that expresses the rate at which the depth of the liquid is changing in terms of the depth, and use that formula to determine whether your conclusion in part (a) is correct.

(c) At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?

Solution (a). For the volume of liquid to decrease by a *fixed amount*, it requires a greater decrease in depth when the cone is close to empty than when it is almost full (Figure 3.4.9). This suggests that for the volume to decrease at a constant rate, the depth must decrease at an increasing rate.


The same volume has drained, but the change in height is greater near the bottom than near the top.

▲ **Figure 3.4.9**

Solution (b). Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
r = radius of the liquid surface at time t (cm)

(Figure 3.4.8). At each instant the rate at which the volume of liquid is changing is dV/dt, and the rate at which the depth is changing is dy/dt. We want to express dy/dt in terms of y given that dV/dt has a constant value of $dV/dt = -2$. (We must use a minus sign here because V *decreases* as t increases.)

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \tfrac{1}{3}\pi r^2 y \tag{7}$$

If we differentiate both sides of (7) with respect to t, the right side will involve the quantity dr/dt. Since we have no direct information about dr/dt, it is desirable to eliminate r from (7) before differentiating. This can be done using similar triangles. From Figure 3.4.8 we see that

$$\frac{r}{y} = \frac{4}{16} \quad \text{or} \quad r = \frac{1}{4}y$$

Substituting this expression in (7) gives

$$V = \frac{\pi}{48}y^3 \tag{8}$$

Differentiating both sides of (8) with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{9}$$

which expresses dy/dt in terms of y. The minus sign tells us that y is decreasing with time, and

$$\left|\frac{dy}{dt}\right| = \frac{32}{\pi y^2}$$

tells us how fast y is decreasing. From this formula we see that $|dy/dt|$ increases as y decreases, which confirms our conjecture in part (a) that the depth of the liquid decreases more quickly as the liquid drains through the filter.

Solution (c). The rate at which the depth is changing when the depth is 8 cm can be obtained from (9) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min} \blacktriangleleft$$

✔ QUICK CHECK EXERCISES 3.4 (See page 211 for answers.)

1. If $A = x^2$ and $\dfrac{dx}{dt} = 3$, find $\left.\dfrac{dA}{dt}\right|_{x=10}$.

2. If $A = x^2$ and $\dfrac{dA}{dt} = 3$, find $\left.\dfrac{dx}{dt}\right|_{x=10}$.

3. A 10-foot ladder stands on a horizontal floor and leans against a vertical wall. Use x to denote the distance along the floor from the wall to the foot of the ladder, and use y to denote the distance along the wall from the floor to the top of the ladder. If the foot of the ladder is dragged away from the wall, find an equation that relates rates of change of x and y with respect to time.

4. Suppose that a block of ice in the shape of a right circular cylinder melts so that it retains its cylindrical shape. Find an equation that relates the rates of change of the volume (V), height (h), and radius (r) of the block of ice.

EXERCISE SET 3.4

1–4 Both x and y denote functions of t that are related by the given equation. Use this equation and the given derivative information to find the specified derivative. ■

1. Equation: $y = 3x + 5$.
   (a) Given that $dx/dt = 2$, find dy/dt when $x = 1$.
   (b) Given that $dy/dt = -1$, find dx/dt when $x = 0$.

2. Equation: $x + 4y = 3$.
   (a) Given that $dx/dt = 1$, find dy/dt when $x = 2$.
   (b) Given that $dy/dt = 4$, find dx/dt when $x = 3$.

3. Equation: $4x^2 + 9y^2 = 1$.
   (a) Given that $dx/dt = 3$, find dy/dt when $(x, y) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$.

(cont.)

(b) Given that $dy/dt = 8$, find dx/dt when $(x, y) = \left(\frac{1}{3}, -\frac{\sqrt{5}}{9}\right)$.

4. Equation: $x^2 + y^2 = 2x + 4y$.

(a) Given that $dx/dt = -5$, find dy/dt when $(x, y) = (3, 1)$.

(b) Given that $dy/dt = 6$, find dx/dt when $(x, y) = (1 + \sqrt{2}, 2 + \sqrt{3})$.

FOCUS ON CONCEPTS

5. Let A be the area of a square whose sides have length x, and assume that x varies with the time t.

(a) Draw a picture of the square with the labels A and x placed appropriately.

(b) Write an equation that relates A and x.

(c) Use the equation in part (b) to find an equation that relates dA/dt and dx/dt.

(d) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

6. In parts (a)–(d), let A be the area of a circle of radius r, and assume that r increases with the time t.

(a) Draw a picture of the circle with the labels A and r placed appropriately.

(b) Write an equation that relates A and r.

(c) Use the equation in part (b) to find an equation that relates dA/dt and dr/dt.

(d) At a certain instant the radius is 5 cm and increasing at the rate of 2 cm/s. How fast is the area increasing at that instant?

7. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time.

(a) How are dV/dt, dh/dt, and dr/dt related?

(b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

8. Let l be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time.

(a) How are dl/dt, dx/dt, and dy/dt related?

(b) If x increases at a constant rate of $\frac{1}{2}$ ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft/s, how fast is the size of the diagonal changing when $x = 3$ ft and $y = 4$ ft? Is the diagonal increasing or decreasing at that instant?

9. Let θ (in radians) be an acute angle in a right triangle, and let x and y, respectively, be the lengths of the sides adjacent to and opposite θ. Suppose also that x and y vary with time.

(a) How are $d\theta/dt$, dx/dt, and dy/dt related?

(b) At a certain instant, $x = 2$ units and is increasing at 1 unit/s, while $y = 2$ units and is decreasing at $\frac{1}{4}$ unit/s. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

10. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

11. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

12. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

13. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

14. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

15. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

16. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

17. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

18. A 10 ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

19. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

20. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

21. For the camera and rocket shown in Figure 3.4.5, at what rate is the camera-to-rocket distance changing when the rocket is 4000 ft up and rising vertically at 880 ft/s?

22. For the camera and rocket shown in Figure 3.4.5, at what rate is the rocket rising when the elevation angle is $\pi/4$ radians and increasing at a rate of 0.2 rad/s?

23. A satellite is in an elliptical orbit around the Earth. Its distance r (in miles) from the center of the Earth is given by

$$r = \frac{4995}{1 + 0.12\cos\theta}$$

where θ is the angle measured from the point on the orbit nearest the Earth's surface (see the accompanying figure).

(a) Find the altitude of the satellite at ***perigee*** (the point nearest the surface of the Earth) and at ***apogee*** (the point farthest from the surface of the Earth). Use 3960 mi as the radius of the Earth.

(b) At the instant when θ is 120°, the angle θ is increasing at the rate of 2.7°/min. Find the altitude of the satellite and the rate at which the altitude is changing at this instant. Express the rate in units of mi/min.

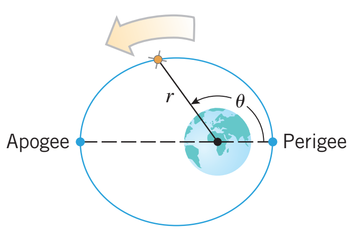


Figure Ex-23

24. An aircraft is flying horizontally at a constant height of 4000 ft above a fixed observation point (see the accompanying figure). At a certain instant the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 300 mi/h.

(a) How fast is θ decreasing at this instant? Express the result in units of deg/s.

(b) How fast is the distance between the aircraft and the observation point changing at this instant? Express the result in units of ft/s. Use 1 mi = 5280 ft.

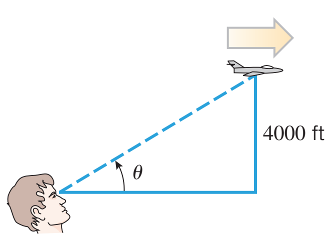


Figure Ex-24

25. A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at a rate of 20 ft^3/min, how fast is the depth of the water increasing when the water is 16 ft deep?

26. Grain pouring from a chute at the rate of 8 ft^3/min forms a conical pile whose height is always twice its radius. How fast is the height of the pile increasing at the instant when the pile is 6 ft high?

27. Sand pouring from a chute forms a conical pile whose height is always equal to the diameter. If the height increases at a constant rate of 5 ft/min, at what rate is sand pouring from the chute when the pile is 10 ft high?

28. Wheat is poured through a chute at the rate of 10 ft^3/min and falls in a conical pile whose bottom radius is always half the altitude. How fast will the circumference of the base be increasing when the pile is 8 ft high?

29. An aircraft is climbing at a 30° angle to the horizontal. How fast is the aircraft gaining altitude if its speed is 500 mi/h?

30. A boat is pulled into a dock by means of a rope attached to a pulley on the dock (see the accompanying figure). The rope is attached to the bow of the boat at a point 10 ft below the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 125 ft of rope is out?

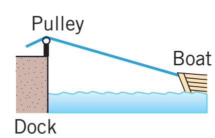


Figure Ex-30

31. For the boat in Exercise 30, how fast must the rope be pulled if we want the boat to approach the dock at a rate of 12 ft/min at the instant when 125 ft of rope is out?

32. A man 6 ft tall is walking at the rate of 3 ft/s toward a streetlight 18 ft high (see the accompanying figure).

(a) At what rate is his shadow length changing?

(b) How fast is the tip of his shadow moving?

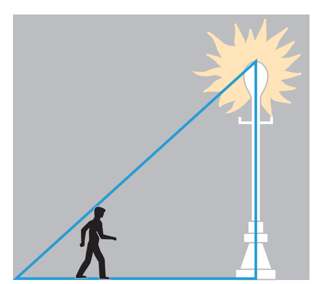


Figure Ex-32

33. A beacon that makes one revolution every 10 s is located on a ship anchored 4 kilometers from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

34. An aircraft is flying at a constant altitude with a constant speed of 600 mi/h. An antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point P (see the accompanying figure). At the instant the aircraft is 2 mi from the impact point P the missile is 4 mi from P and flying at 1200 mi/h. At that instant, how rapidly is the distance between missile and aircraft decreasing?

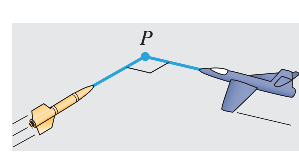


Figure Ex-34

35. Solve Exercise 34 under the assumption that the angle between the flight paths is 120° instead of the assumption that the paths are perpendicular. [*Hint:* Use the law of cosines.]

36. A police helicopter is flying due north at 100 mi/h and at a constant altitude of $\frac{1}{2}$ mi. Below, a car is traveling west on a highway at 75 mi/h. At the moment the helicopter crosses over the highway the car is 2 mi east of the helicopter.
(a) How fast is the distance between the car and helicopter changing at the moment the helicopter crosses the highway?
(b) Is the distance between the car and helicopter increasing or decreasing at that moment?

37. A particle is moving along the curve whose equation is

$$\frac{xy^3}{1+y^2} = \frac{8}{5}$$

Assume that the x-coordinate is increasing at the rate of 6 units/s when the particle is at the point $(1, 2)$.
(a) At what rate is the y-coordinate of the point changing at that instant?
(b) Is the particle rising or falling at that instant?

38. A point P is moving along the curve whose equation is $y = \sqrt{x^3 + 17}$. When P is at $(2, 5)$, y is increasing at the rate of 2 units/s. How fast is x changing?

39. A point P is moving along the line whose equation is $y = 2x$. How fast is the distance between P and the point $(3, 0)$ changing at the instant when P is at $(3, 6)$ if x is decreasing at the rate of 2 units/s at that instant?

40. A point P is moving along the curve whose equation is $y = \sqrt{x}$. Suppose that x is increasing at the rate of 4 units/s when $x = 3$.
(a) How fast is the distance between P and the point $(2, 0)$ changing at this instant?
(b) How fast is the angle of inclination of the line segment from P to $(2, 0)$ changing at this instant?

41. A particle is moving along the curve $y = x/(x^2 + 1)$. Find all values of x at which the rate of change of x with respect to time is three times that of y. [Assume that dx/dt is never zero.]

42. A particle is moving along the curve $16x^2 + 9y^2 = 144$. Find all points (x, y) at which the rates of change of x and y with respect to time are equal. [Assume that dx/dt and dy/dt are never both zero at the same point.]

43. The ***thin lens equation*** in physics is

$$\frac{1}{s} + \frac{1}{S} = \frac{1}{f}$$

where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 cm and that an object is moving toward the lens at the rate of 2 cm/s. How fast is the image distance changing at the instant when the object is 10 cm from the lens? Is the image moving away from the lens or toward the lens?

44. Water is stored in a cone-shaped reservoir (vertex down). Assuming the water evaporates at a rate proportional to the surface area exposed to the air, show that the depth of the water will decrease at a constant rate that does not depend on the dimensions of the reservoir.

45. A meteor enters the Earth's atmosphere and burns up at a rate that, at each instant, is proportional to its surface area. Assuming that the meteor is always spherical, show that the radius decreases at a constant rate.

46. On a certain clock the minute hand is 4 in long and the hour hand is 3 in long. How fast is the distance between the tips of the hands changing at 9 o'clock?

47. Coffee is poured at a uniform rate of 20 cm^3/s into a cup whose inside is shaped like a truncated cone (see the accompanying figure). If the upper and lower radii of the cup are 4 cm and 2 cm and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up? [*Hint:* Extend the cup downward to form a cone.]

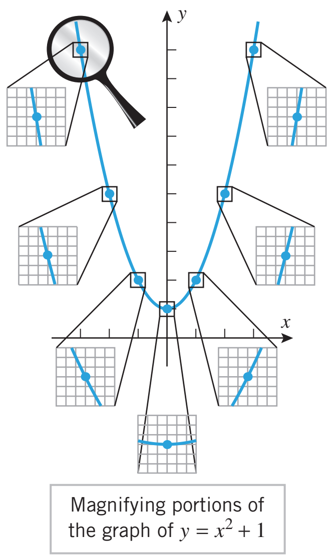
◀ **Figure Ex-47**

✔ QUICK CHECK ANSWERS 3.4

1. 60 **2.** $\frac{3}{20}$ **3.** $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ **4.** $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

3.5 LOCAL LINEAR APPROXIMATION; DIFFERENTIALS

In this section we will show how derivatives can be used to approximate nonlinear functions by linear functions. Also, up to now we have been interpreting dy/dx as a single entity representing the derivative. In this section we will define the quantities dx and dy themselves, thereby allowing us to interpret dy/dx as an actual ratio.



▲ **Figure 3.5.1**

Recall from Section 2.2 that if a function f is differentiable at x_0, then a sufficiently magnified portion of the graph of f centered at the point $P(x_0, f(x_0))$ takes on the appearance of a straight line segment. Figure 3.5.1 illustrates this at several points on the graph of $y = x^2 + 1$. For this reason, a function that is differentiable at x_0 is sometimes said to be ***locally linear*** at x_0.

The line that best approximates the graph of f in the vicinity of $P(x_0, f(x_0))$ is the tangent line to the graph of f at x_0, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

[see Formula (3) of Section 2.2]. Thus, for values of x near x_0 we can approximate values of $f(x)$ by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \qquad (1)$$

This is called the ***local linear approximation*** of f at x_0. This formula can also be expressed in terms of the increment $\Delta x = x - x_0$ as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \qquad (2)$$

▶ Example 1

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by a calculating utility.

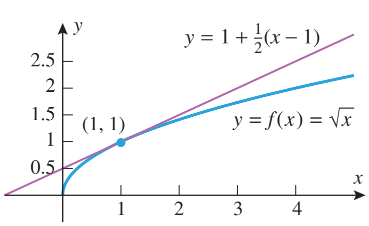
Solution (a). Since $f'(x) = 1/(2\sqrt{x})$, it follows from (1) that the local linear approximation of $\sqrt{x}$ at a point x_0 is

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0)$$

Thus, the local linear approximation at $x_0 = 1$ is

$$\sqrt{x} \approx 1 + \tfrac{1}{2}(x - 1) \qquad (3)$$

The graphs of $y = \sqrt{x}$ and the local linear approximation $y = 1 + \frac{1}{2}(x - 1)$ are shown in Figure 3.5.2.



▲ **Figure 3.5.2**

Solution (b). Applying (3) with $x = 1.1$ yields

$$\sqrt{1.1} \approx 1 + \tfrac{1}{2}(1.1 - 1) = 1.05$$

Since the tangent line $y = 1 + \frac{1}{2}(x - 1)$ in Figure 3.5.2 lies above the graph of $f(x) = \sqrt{x}$, we would expect this approximation to be slightly too large. This expectation is confirmed by the calculator approximation $\sqrt{1.1} \approx 1.04881$. ◀

Examples 1 and 2 illustrate important ideas and are not meant to suggest that you should use local linear approximations for computations that your calculating utility can perform. The main application of local linear approximation is in modeling problems where it is useful to replace complicated functions by simpler ones.

▶ Example 2

(a) Find the local linear approximation of $f(x) = \sin x$ at $x_0 = 0$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sin 2°$, and compare your approximation to the result produced directly by your calculating device.

Solution (a). Since $f'(x) = \cos x$, it follows from (1) that the local linear approximation of $\sin x$ at a point x_0 is

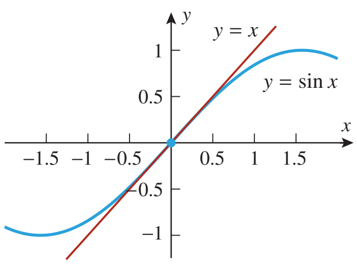
$$\sin x \approx \sin x_0 + (\cos x_0)(x - x_0)$$

Thus, the local linear approximation at $x_0 = 0$ is

$$\sin x \approx \sin 0 + (\cos 0)(x - 0)$$

which simplifies to

$$\sin x \approx x \tag{4}$$



▲ **Figure 3.5.3**

Solution (b). The variable x in (4) is in radian measure, so we must first convert 2° to radians before we can apply this approximation. Since

$$2° = 2\left(\frac{\pi}{180}\right) = \frac{\pi}{90} \approx 0.0349066 \text{ radian}$$

it follows from (4) that $\sin 2° \approx 0.0349066$. Comparing the two graphs in Figure 3.5.3, we would expect this approximation to be slightly larger than the exact value. The calculator approximation $\sin 2° \approx 0.0348995$ shows that this is indeed the case. ◀

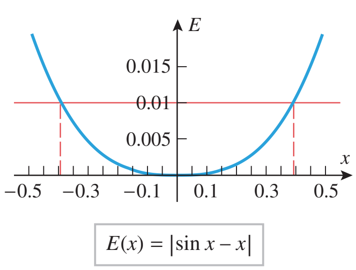
ERROR IN LOCAL LINEAR APPROXIMATIONS

As a general rule, the accuracy of the local linear approximation to $f(x)$ at x_0 will deteriorate as x gets progressively farther from x_0. To illustrate this for the approximation $\sin x \approx x$ in Example 2, let us graph the function

$$E(x) = |\sin x - x|$$

which is the absolute value of the error in the approximation (Figure 3.5.4).

In Figure 3.5.4, the graph shows how the absolute error in the local linear approximation of $\sin x$ increases as x moves progressively farther from 0 in either the positive or negative direction. The graph also tells us that for values of x between the two vertical lines, the absolute error does not exceed 0.01. Thus, for example, we could use the local linear approximation $\sin x \approx x$ for all values of x in the interval $-0.35 < x < 0.35$ (radians) with confidence that the approximation is within ± 0.01 of the exact value.



▲ **Figure 3.5.4**

DIFFERENTIALS

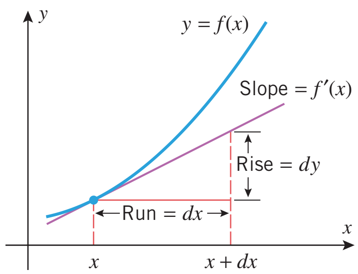
Newton and Leibniz each used a different notation when they published their discoveries of calculus, thereby creating a notational divide between Britain and the European continent that lasted for more than 50 years. The ***Leibniz notation*** dy/dx eventually prevailed because it suggests correct formulas in a natural way, the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

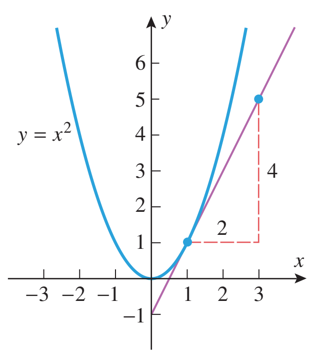
being a good example.

Up to now we have interpreted dy/dx as a single entity representing the derivative of y with respect to x; the symbols "dy" and "dx," which are called ***differentials***, have had no meanings attached to them. Our next goal is to define these symbols in such a way that dy/dx can be treated as an actual ratio. To do this, assume that f is differentiable at a point x, *define* dx to be an independent variable that can have any real value, and *define* dy by the formula

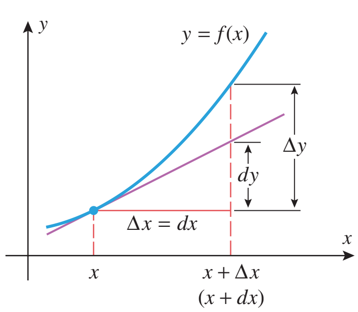
$$dy = f'(x)\,dx \tag{5}$$



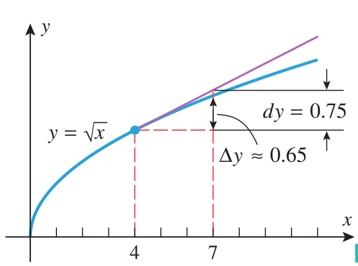
▲ **Figure 3.5.5**



▲ **Figure 3.5.6**



▲ **Figure 3.5.7**



▲ **Figure 3.5.8**

If $dx \neq 0$, then we can divide both sides of (5) by dx to obtain

$$\frac{dy}{dx} = f'(x) \tag{6}$$

Thus, we have achieved our goal of defining dy and dx so their ratio is $f'(x)$. Formula (5) is said to express (6) in ***differential form***.

To interpret (5) geometrically, note that $f'(x)$ is the slope of the tangent line to the graph of f at x. The differentials dy and dx can be viewed as a corresponding rise and run of this tangent line (Figure 3.5.5).

▶ **Example 3** Express the derivative with respect to x of $y = x^2$ in differential form, and discuss the relationship between dy and dx at $x = 1$.

Solution. The derivative of y with respect to x is $dy/dx = 2x$, which can be expressed in differential form as

$$dy = 2x\,dx$$

When $x = 1$ this becomes

$$dy = 2\,dx$$

This tells us that if we travel along the tangent line to the curve $y = x^2$ at $x = 1$, then a change of dx units in x produces a change of $2\,dx$ units in y. Thus, for example, a run of $dx = 2$ units produces a rise of $dy = 4$ units along the tangent line (Figure 3.5.6). ◀

It is important to understand the distinction between the increment Δy and the differential dy. To see the difference, let us assign the independent variables dx and Δx the same value, so $dx = \Delta x$. Then Δy represents the change in y that occurs when we start at x and travel *along the curve* $y = f(x)$ until we have moved $\Delta x\ (= dx)$ units in the x-direction, while dy represents the change in y that occurs if we start at x and travel *along the tangent line* until we have moved $dx\ (= \Delta x)$ units in the x-direction (Figure 3.5.7).

▶ **Example 4** Let $y = \sqrt{x}$. Find dy and Δy at $x = 4$ with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$, showing dy and Δy in the picture.

Solution. With $f(x) = \sqrt{x}$ we obtain

$$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{7} - \sqrt{4} \approx 0.65$$

If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$$

Figure 3.5.8 shows the curve $y = \sqrt{x}$ together with dy and Δy. ◀

LOCAL LINEAR APPROXIMATION FROM THE DIFFERENTIAL POINT OF VIEW

Although Δy and dy are generally different, the differential dy will nonetheless be a good approximation of Δy provided $dx = \Delta x$ is close to 0. To see this, recall from Section 2.2 that

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

It follows that if Δx is close to 0, then we will have $f'(x) \approx \Delta y/\Delta x$ or, equivalently,

$$\Delta y \approx f'(x)\Delta x$$

If we agree to let $dx = \Delta x$, then we can rewrite this as

$$\Delta y \approx f'(x)\,dx = dy \tag{7}$$

In words, this states that for values of dx near zero the differential dy closely approximates the increment Δy (Figure 3.5.7). But this is to be expected since the graph of the tangent line at x is the local linear approximation of the graph of f.

ERROR PROPAGATION

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Real-world measurements inevitably have small errors.

In real-world applications, small errors in measured quantities will invariably occur. These measurement errors are of importance in scientific research—all scientific measurements come with measurement errors included. For example, your height might be measured as 170 ± 0.5 cm, meaning that your exact height lies somewhere between 169.5 and 170.5 cm. Researchers often must use these inexactly measured quantities to compute other quantities, thereby *propagating* the errors from the measured quantities to the computed quantities. This phenomenon is called ***error propagation***. Researchers must be able to estimate errors in the computed quantities. Our goal is to show how to estimate these errors using local linear approximation and differentials. For this purpose, suppose

x_0 is the exact value of the quantity being measured
$y_0 = f(x_0)$ is the exact value of the quantity being computed
x is the measured value of x_0
$y = f(x)$ is the computed value of y

We define

$dx\ (= \Delta x) = x - x_0$ to be the ***measurement error*** of x
$\Delta y = f(x) - f(x_0)$ to be the ***propagated error*** of y

It follows from (7) with x_0 replacing x that the propagated error Δy can be approximated by

$$\Delta y \approx dy = f'(x_0)\,dx \tag{8}$$

Unfortunately, there is a practical difficulty in applying this formula since the value of x_0 is unknown. (Keep in mind that only the measured value x is known to the researcher.) This being the case, it is standard practice in research to use the measured value x in place of x_0 in (8) and use the approximation

$$\Delta y \approx dy = f'(x)\,dx \tag{9}$$

for the propagated error.

Note that measurement error is positive if the measured value is greater than the exact value and is negative if it is less than the exact value. The sign of the propagated error conveys similar information.

Example 5 Suppose that the side of a square is measured with a ruler to be 10 inches with a measurement error of at most $\pm\frac{1}{32}$ in. Estimate the error in the computed area of the square.

Explain why an error estimate of at most $\pm\frac{1}{32}$ inch is reasonable for a ruler that is calibrated in sixteenths of an inch.

Solution. Let x denote the exact length of a side and y the exact area so that $y = x^2$. It follows from (9) with $f(x) = x^2$ that if dx is the measurement error, then the propagated error Δy can be approximated as

$$\Delta y \approx dy = 2x\,dx$$

Substituting the measured value $x = 10$ into this equation yields

$$dy = 20\,dx \tag{10}$$

But to say that the measurement error is at most $\pm\frac{1}{32}$ means that

$$-\frac{1}{32} \le dx \le \frac{1}{32}$$

Multiplying these inequalities through by 20 and applying (10) yields

$$20\left(-\tfrac{1}{32}\right) \le dy \le 20\left(\tfrac{1}{32}\right) \quad \text{or equivalently} \quad -\tfrac{5}{8} \le dy \le \tfrac{5}{8}$$

Thus, the propagated error in the area is estimated to be within $\pm\frac{5}{8}$ in^2. ◀

If the true value of a quantity is q and a measurement or calculation produces an error Δq, then $\Delta q/q$ is called the ***relative error*** in the measurement or calculation; when expressed as a percentage, $\Delta q/q$ is called the ***percentage error***. As a practical matter, the true value q is usually unknown, so that the measured or calculated value of q is used instead; and the relative error is approximated by dq/q.

Example 6 The radius of a sphere is measured with a percentage error within $\pm 0.04\%$. Estimate the percentage error in the calculated volume of the sphere.

Formula (11) tells us that, as a rule of thumb, the percentage error in the computed volume of a sphere is approximately 3 times the percentage error in the measured value of its radius. As a rule of thumb, how is the percentage error in the computed area of a square related to the percentage error in the measured value of a side?

Solution. The volume V of a sphere is $V = \frac{4}{3}\pi r^3$, so

$$\frac{dV}{dr} = 4\pi r^2$$

from which it follows that $dV = 4\pi r^2\, dr$. Thus, the relative error in V is approximately

$$\frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r} \tag{11}$$

We are given that the relative error in the measured value of r is $\pm 0.04\%$, which means that

$$-0.0004 \le \frac{dr}{r} \le 0.0004$$

Multiplying these inequalities through by 3 and applying (11) yields

$$3(-0.0004) \le \frac{dV}{V} \le 3(0.0004) \quad \text{or equivalently} \quad -0.0012 \le \frac{dV}{V} \le 0.0012$$

Thus, we estimate the percentage error in the calculated value of V to be within $\pm 0.12\%$. ◀

MORE NOTATION; DIFFERENTIAL FORMULAS

The symbol df is another common notation for the differential of a function $y = f(x)$. For example, if $f(x) = \sin x$, then we can write $df = \cos x\, dx$. We can also view the symbol "d" as an *operator* that acts on a function to produce the corresponding differential. For example, $d[x^2] = 2x\, dx$, $d[\sin x] = \cos x\, dx$, and so on. All of the general rules of differentiation then have corresponding differential versions:

DERIVATIVE FORMULA	DIFFERENTIAL FORMULA
$\frac{d}{dx}[c] = 0$	$d[c] = 0$
$\frac{d}{dx}[cf] = c\frac{df}{dx}$	$d[cf] = c\, df$
$\frac{d}{dx}[f+g] = \frac{df}{dx} + \frac{dg}{dx}$	$d[f+g] = df + dg$
$\frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$	$d[fg] = f\, dg + g\, df$
$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$	$d\left[\frac{f}{g}\right] = \frac{g\, df - f\, dg}{g^2}$

For example,

$$\begin{aligned} d[x^2 \sin x] &= (x^2 \cos x + 2x \sin x)\, dx \\ &= x^2(\cos x\, dx) + (2x\, dx) \sin x \\ &= x^2 d[\sin x] + (\sin x)\, d[x^2] \end{aligned}$$

illustrates the differential version of the product rule.

✔QUICK CHECK EXERCISES 3.5 (See page 219 for answers.)

1. The local linear approximation of f at x_0 uses the ________ line to the graph of $y = f(x)$ at $x = x_0$ to approximate values of ________ for values of x near ________.
2. Find an equation for the local linear approximation to $y = 5 - x^2$ at $x_0 = 2$.
3. Let $y = 5 - x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 0.1$.
4. The intensity of light from a light source is a function $I = f(x)$ of the distance x from the light source. Suppose that a small gemstone is measured to be 10 m from a light source, $f(10) = 0.2\ \text{W/m}^2$, and $f'(10) = -0.04\ \text{W/m}^3$. If the distance $x = 10$ m was obtained with a measurement error within ± 0.05 m, estimate the percentage error in the calculated intensity of the light on the gemstone.

EXERCISE SET 3.5 Graphing Utility

1. (a) Use Formula (1) to obtain the local linear approximation of x^3 at $x_0 = 1$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $(1.02)^3$, and confirm that the formula obtained in part (b) produces the same result.
2. (a) Use Formula (1) to obtain the local linear approximation of $1/x$ at $x_0 = 2$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $1/2.05$, and confirm that the formula obtained in part (b) produces the same result.

FOCUS ON CONCEPTS

3. (a) Find the local linear approximation of the function $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.
   (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $\sqrt{0.9}$ and $\sqrt{1.1}$.
4. A student claims that whenever a local linear approximation is used to approximate the square root of a number, the approximation is too large.
   (a) Write a few sentences that make the student's claim precise, and justify this claim geometrically.
   (b) Verify the student's claim algebraically using approximation (1).

5–10 Confirm that the stated formula is the local linear approximation at $x_0 = 0$. ■

5. $(1+x)^{15} \approx 1 + 15x$
6. $\dfrac{1}{\sqrt{1-x}} \approx 1 + \dfrac{1}{2}x$
7. $\tan x \approx x$
8. $\dfrac{1}{1+x} \approx 1 - x$
9. $e^x \approx 1 + x$
10. $\ln(1+x) \approx x$

11–16 Confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\Delta x = x - 1$. ■

11. $f(x) = x^4$; $(1+\Delta x)^4 \approx 1 + 4\Delta x$
12. $f(x) = \sqrt{x}$; $\sqrt{1+\Delta x} \approx 1 + \frac{1}{2}\Delta x$
13. $f(x) = \dfrac{1}{2+x}$; $\dfrac{1}{3+\Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$
14. $f(x) = (4+x)^3$; $(5+\Delta x)^3 \approx 125 + 75\Delta x$
15. $\tan^{-1} x$; $\tan^{-1}(1+\Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x$
16. $\sin^{-1}\left(\dfrac{x}{2}\right)$; $\sin^{-1}\left(\dfrac{1}{2} + \dfrac{1}{2}\Delta x\right) \approx \dfrac{\pi}{6} + \dfrac{1}{\sqrt{3}}\Delta x$

17–20 Confirm that the formula is the local linear approximation at $x_0 = 0$, and use a graphing utility to estimate an interval of x-values on which the error is at most ± 0.1. ■

17. $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$
18. $\dfrac{1}{\sqrt{9-x}} \approx \dfrac{1}{3} + \dfrac{1}{54}x$
19. $\tan 2x \approx 2x$
20. $\dfrac{1}{(1+2x)^5} \approx 1 - 10x$
21. (a) Use the local linear approximation of $\sin x$ at $x_0 = 0$ obtained in Example 2 to approximate $\sin 1°$, and compare the approximation to the result produced directly by your calculating device.
    (b) How would you choose x_0 to approximate $\sin 44°$?
    (c) Approximate $\sin 44°$; compare the approximation to the result produced directly by your calculating device.
22. (a) Use the local linear approximation of $\tan x$ at $x_0 = 0$ to approximate $\tan 2°$, and compare the approximation to the result produced directly by your calculating device.
    (b) How would you choose x_0 to approximate $\tan 61°$?
    (c) Approximate $\tan 61°$; compare the approximation to the result produced directly by your calculating device.

23–31 Use an appropriate local linear approximation to estimate the value of the given quantity. ■

23. $(3.02)^4$
24. $(1.97)^3$
25. $\sqrt{65}$

26. $\sqrt{24}$ **27.** $\sqrt{80.9}$ **28.** $\sqrt{36.03}$

29. $\sin 0.1$ **30.** $\tan 0.2$ **31.** $\cos 31°$

32. $\ln(1.01)$ **33.** $\tan^{-1}(0.99)$

FOCUS ON CONCEPTS

34. The approximation $(1+x)^k \approx 1+kx$ is commonly used by engineers for quick calculations.
(a) Derive this result, and use it to make a rough estimate of $(1.001)^{37}$.
(b) Compare your estimate to that produced directly by your calculating device.
(c) If k is a positive integer, how is the approximation $(1+x)^k \approx 1+kx$ related to the expansion of $(1+x)^k$ using the binomial theorem?

35. Use the approximation $(1+x)^k \approx 1+kx$, along with some mental arithmetic to show that $\sqrt[3]{8.24} \approx 2.02$ and $4.08^{3/2} \approx 8.24$.

36. Referring to the accompanying figure, suppose that the angle of elevation of the top of the building, as measured from a point 500 ft from its base, is found to be $\theta = 6°$. Use an appropriate local linear approximation, along with some mental arithmetic to show that the building is about 52 ft high.

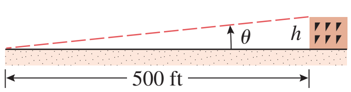


Figure Ex-36

37. (a) Let $y = x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 1$.
(b) Sketch the graph of $y = x^2$, showing dy and Δy in the picture.

38. (a) Let $y = x^3$. Find dy and Δy at $x = 1$ with $dx = \Delta x = 1$.
(b) Sketch the graph of $y = x^3$, showing dy and Δy in the picture.

39–42 Find formulas for dy and Δy. ■

39. $y = x^3$ **40.** $y = 8x - 4$

41. $y = x^2 - 2x + 1$ **42.** $y = \sin x$

43–46 Find the differential dy. ■

43. (a) $y = 4x^3 - 7x^2$ (b) $y = x\cos x$

44. (a) $y = 1/x$ (b) $y = 5\tan x$

45. (a) $y = x\sqrt{1-x}$ (b) $y = (1+x)^{-17}$

46. (a) $y = \dfrac{1}{x^3 - 1}$ (b) $y = \dfrac{1-x^3}{2-x}$

47–50 True–False Determine whether the statement is true or false. Explain your answer. ■

47. A differential dy is defined to be a very small change in y.

48. The error in approximation (2) is the same as the error in approximation (7).

49. A local linear approximation to a function can never be identically equal to the function.

50. A local linear approximation to a nonconstant function can never be constant.

51–54 Use the differential dy to approximate Δy when x changes as indicated. ■

51. $y = \sqrt{3x-2}$; from $x = 2$ to $x = 2.03$

52. $y = \sqrt{x^2+8}$; from $x = 1$ to $x = 0.97$

53. $y = \dfrac{x}{x^2+1}$; from $x = 2$ to $x = 1.96$

54. $y = x\sqrt{8x+1}$; from $x = 3$ to $x = 3.05$

55. The side of a square is measured to be 10 ft, with a possible error of ± 0.1 ft.
(a) Use differentials to estimate the error in the calculated area.
(b) Estimate the percentage errors in the side and the area.

56. The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm.
(a) Use differentials to estimate the error in the calculated volume.
(b) Estimate the percentage errors in the side and volume.

57. The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be $30°$, with a possible error of $\pm 1°$.
(a) Use differentials to estimate the errors in the sides opposite and adjacent to the measured angle.
(b) Estimate the percentage errors in the sides.

58. One side of a right triangle is known to be 25 cm exactly. The angle opposite to this side is measured to be $60°$, with a possible error of $\pm 0.5°$.
(a) Use differentials to estimate the errors in the adjacent side and the hypotenuse.
(b) Estimate the percentage errors in the adjacent side and hypotenuse.

59. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of $\pm 5\%$, use differentials to estimate the percentage error in R. (Assume k is exact.)

60. A 12-foot ladder leaning against a wall makes an angle θ with the floor. If the top of the ladder is h feet up the wall, express h in terms of θ and then use dh to estimate the change in h if θ changes from $60°$ to $59°$.

61. The area of a right triangle with a hypotenuse of H is calculated using the formula $A = \frac{1}{4}H^2 \sin 2\theta$, where θ is one of the acute angles. Use differentials to approximate the error in calculating A if $H = 4$ cm (exactly) and θ is measured to be $30°$, with a possible error of $\pm 15'$.

62. The side of a square is measured with a possible percentage error of $\pm 1\%$. Use differentials to estimate the percentage error in the area.

63. The side of a cube is measured with a possible percentage error of $\pm 2\%$. Use differentials to estimate the percentage error in the volume.

64. The volume of a sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within $\pm 3\%$. ($V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.)

65. The area of a circle is to be computed from a measured value of its diameter. Estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept within $\pm 1\%$.

66. A steel cube with 1-inch sides is coated with 0.01 inch of copper. Use differentials to estimate the volume of copper in the coating. [*Hint:* Let ΔV be the change in the volume of the cube.]

67. A metal rod 15 cm long and 5 cm in diameter is to be covered (except for the ends) with insulation that is 0.1 cm thick. Use differentials to estimate the volume of insulation. [*Hint:* Let ΔV be the change in volume of the rod.]

68. The time required for one complete oscillation of a pendulum is called its ***period***. If L is the length of the pendulum and the oscillation is small, then the period is given by $P = 2\pi\sqrt{L/g}$, where g is the constant acceleration due to gravity. Use differentials to show that the percentage error in P is approximately half the percentage error in L.

69. If the temperature T of a metal rod of length L is changed by an amount ΔT, then the length will change by the amount $\Delta L = \alpha L \Delta T$, where α is called the ***coefficient of linear expansion***. For moderate changes in temperature α is taken as constant.
(a) Suppose that a rod 40 cm long at 20°C is found to be 40.006 cm long when the temperature is raised to 30°C. Find α.
(b) If an aluminum pole is 180 cm long at 15°C, how long is the pole if the temperature is raised to 40°C? [Take $\alpha = 2.3 \times 10^{-5}/°\text{C}$.]

70. If the temperature T of a solid or liquid of volume V is changed by an amount ΔT, then the volume will change by the amount $\Delta V = \beta V \Delta T$, where β is called the ***coefficient of volume expansion***. For moderate changes in temperature β is taken as constant. Suppose that a tank truck loads 4000 gallons of ethyl alcohol at a temperature of 35°C and delivers its load sometime later at a temperature of 15°C. Using $\beta = 7.5 \times 10^{-4}/°\text{C}$ for ethyl alcohol, find the number of gallons delivered.

71. Writing Explain why the local linear approximation of a function value is equivalent to the use of a differential to approximate a change in the function.

72. Writing The local linear approximation

$$\sin x \approx x$$

is known as the *small angle approximation* and has both practical and theoretical applications. Do some research on some of these applications, and write a short report on the results of your investigations.

✔ QUICK CHECK ANSWERS 3.5

1. tangent; $f(x)$; x_0 **2.** $y = 1 + (-4)(x-2)$ or $y = -4x + 9$ **3.** $dy = -0.4$, $\Delta y = -0.41$ **4.** within $\pm 1\%$

3.6 L'HÔPITAL'S RULE; INDETERMINATE FORMS

In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

INDETERMINATE FORMS OF TYPE 0/0

Recall that a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)} \tag{1}$$

in which $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ is called an ***indeterminate form of type 0/0***. Some examples encountered earlier in the text are

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2, \quad \lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

The first limit was obtained algebraically by factoring the numerator and canceling the common factor of $x - 1$, and the second two limits were obtained using geometric methods. However, there are many indeterminate forms for which neither algebraic nor geometric methods will produce the limit, so we need to develop a more general method.

To motivate such a method, suppose that (1) is an indeterminate form of type $0/0$ in which f' and g' are continuous at $x = a$ and $g'(a) \neq 0$. Since f and g can be closely approximated by their local linear approximations near a, it is reasonable to expect that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} \tag{2}$$

Since we are assuming that f' and g' are continuous at $x = a$, we have

$$\lim_{x \to a} f'(x) = f'(a) \quad \text{and} \quad \lim_{x \to a} g'(x) = g'(a)$$

and since the differentiability of f and g at $x = a$ implies the continuity of f and g at $x = a$, we have

$$f(a) = \lim_{x \to a} f(x) = 0 \quad \text{and} \quad g(a) = \lim_{x \to a} g(x) = 0$$

Thus, we can rewrite (2) as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x - a)}{g'(a)(x - a)} = \lim_{x \to a} \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{3}$$

This result, called ***L'Hôpital's rule***, converts the given indeterminate form into a limit involving derivatives that is often easier to evaluate.

Although we motivated (3) by assuming that f and g have continuous derivatives at $x = a$ and that $g'(a) \neq 0$, the result is true under less stringent conditions and is also valid for one-sided limits and limits at $+\infty$ and $-\infty$. The proof of the following precise statement of L'Hôpital's rule is omitted.

3.6.1 THEOREM (*L'Hôpital's Rule for Form* 0/0) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

WARNING

Note that in L'Hôpital's rule the numerator and denominator are differentiated individually. This is *not* the same as differentiating $f(x)/g(x)$.

In the examples that follow we will apply L'Hôpital's rule using the following three-step process:

Applying L'Hôpital's Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

▶ Example 1 Find the limit

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\dfrac{d}{dx}[x^2-4]}{\dfrac{d}{dx}[x-2]} = \lim_{x\to 2} \frac{2x}{1} = 4$$

This agrees with the computation

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4 \ ◀$$

The limit in Example 1 can be interpreted as the limit form of a certain derivative. Use that derivative to evaluate the limit.

▶ Example 2 In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

$$\text{(a)}\ \lim_{x\to 0} \frac{\sin 2x}{x} \qquad \text{(b)}\ \lim_{x\to \pi/2} \frac{1-\sin x}{\cos x} \qquad \text{(c)}\ \lim_{x\to 0} \frac{e^x-1}{x^3}$$

$$\text{(d)}\ \lim_{x\to 0^-} \frac{\tan x}{x^2} \qquad \text{(e)}\ \lim_{x\to 0} \frac{1-\cos x}{x^2} \qquad \text{(f)}\ \lim_{x\to +\infty} \frac{x^{-4/3}}{\sin(1/x)}$$

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{\dfrac{d}{dx}[\sin 2x]}{\dfrac{d}{dx}[x]} = \lim_{x\to 0} \frac{2\cos 2x}{1} = 2$$

Observe that this result agrees with that obtained by substitution in Example 4(b) of Section 1.6.

WARNING

Applying L'Hôpital's rule to limits that are not indeterminate forms can produce incorrect results. For example, the computation

$$\lim_{x\to 0} \frac{x+6}{x+2} = \lim_{x\to 0} \frac{\dfrac{d}{dx}[x+6]}{\dfrac{d}{dx}[x+2]} = \lim_{x\to 0} \frac{1}{1} = 1$$

is ***not valid***, since the limit is not an indeterminate form. The correct result is

$$\lim_{x\to 0} \frac{x+6}{x+2} = \frac{0+6}{0+2} = 3$$

Solution (b). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to \pi/2} \frac{1-\sin x}{\cos x} = \lim_{x\to \pi/2} \frac{\dfrac{d}{dx}[1-\sin x]}{\dfrac{d}{dx}[\cos x]} = \lim_{x\to \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

Guillaume François Antoine de L'Hôpital (1661–1704) French mathematician. L'Hôpital, born to parents of the French high nobility, held the title of Marquis de Sainte-Mesme Comte d'Autrement. He showed mathematical talent quite early and at age 15 solved a difficult problem about cycloids posed by Pascal. As a young man he served briefly as a cavalry officer, but resigned because of nearsightedness. In his own time he gained fame as the author of the first textbook ever published on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). L'Hôpital's rule appeared for the first time in that book. Actually, L'Hôpital's rule and most of the material in the calculus text were due to John Bernoulli, who was L'Hôpital's teacher. L'Hôpital dropped his plans for a book on integral calculus when Leibniz informed him that he intended to write such a text. L'Hôpital was apparently generous and personable, and his many contacts with major mathematicians provided the vehicle for disseminating major discoveries in calculus throughout Europe.

Solution (c). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[e^x-1]}{\dfrac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to+\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to+\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=+\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to +\infty} f(x)=\infty \quad \text{means} \quad \lim_{x\to +\infty} f(x)=+\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x)=-\infty$$

$$\lim_{x\to a} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type*** ∞/∞. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

3.6.2 THEOREM (*L'Hôpital's Rule for Form* ∞/∞) *Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that*

$$\lim_{x\to a} f(x)=\infty \quad \textit{and} \quad \lim_{x\to a} g(x)=\infty$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ *exists, or if this limit is $+\infty$ or $-\infty$, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, or as $x\to +\infty$.

▶ **Example 3** In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)} \quad \lim_{x\to+\infty} \frac{x}{e^x} \qquad \text{(b)} \quad \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} \tag{4}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

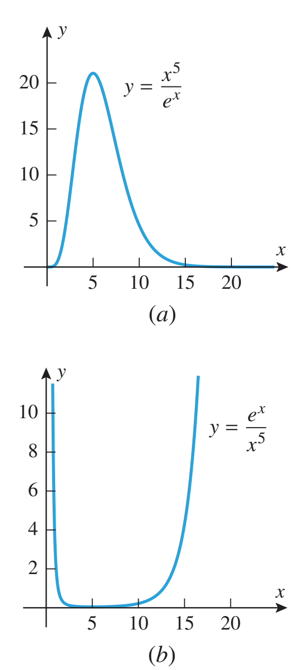
$$\lim_{x\to 0^+} \left(-\frac{\sin x}{x} \tan x\right) = -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \tan x = -(1)(0) = 0$$

Thus,

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = 0 \;◀$$

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n \to +\infty$ as $x \to +\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x \to +\infty$ as $x \to +\infty$ and that the growth of e^x is very rapid (Table 0.5.5); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x \to +\infty$. For example, Figure 3.6.1a shows the graph of $y = x^5/e^x$. This graph suggests that $x^5/e^x \to 0$ as $x \to +\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x \to +\infty$ (Figure 3.6.1b). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,



▲ **Figure 3.6.1**

$$\boxed{\lim_{x\to+\infty} \frac{x^n}{e^x} = 0} \quad \text{and} \quad \boxed{\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty} \tag{5–6}$$

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (5), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is $n(n-1)(n-2)\cdots 1 = n!$ (verify).* Thus, applying L'Hôpital's rule n times to (5) yields

$$\lim_{x\to+\infty}\frac{x^n}{e^x} = \lim_{x\to+\infty}\frac{n!}{e^x} = 0$$

Limit (6) can be established similarly.

INDETERMINATE FORMS OF TYPE $0\cdot\infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x)\cdot g(x), \quad f(x)^{g(x)}, \quad f(x)-g(x), \quad f(x)+g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x\to 0^+} x\ln x$$

is an ***indeterminate form of type*** $\mathbf{0\cdot\infty}$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x\to+\infty}[\sqrt{x}(1-x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Indeterminate forms of type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

WARNING

It is tempting to argue that an indeterminate form of type $0\cdot\infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0\cdot\infty$ is not a product of numbers, but rather a statement about limits. For example, here are two indeterminate forms of type $0\cdot\infty$ whose limits are *not* zero:

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0^+}\left(\sqrt{x}\cdot\frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}\right) = +\infty$$

▶ Example 4 Evaluate

(a) $\lim_{x\to 0^+} x\ln x$ (b) $\lim_{x\to\pi/4}(1-\tan x)\sec 2x$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+}\frac{\ln x}{1/x} \quad\text{or}\quad \lim_{x\to 0^+}\frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4}(1-\tan x)\sec 2x = \lim_{x\to\pi/4}\frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4}\frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1 \ ◀$$

*Recall that for $n\geq 1$ the expression $n!$, read ***n-factorial***, denotes the product of the first n positive integers.

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty) - (+\infty), \quad (-\infty) - (-\infty),$$
$$(+\infty) + (-\infty), \quad (-\infty) + (+\infty)$$

is called an ***indeterminate form of type*** $\infty - \infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty) + (+\infty), \quad (+\infty) - (-\infty),$$
$$(-\infty) + (-\infty), \quad (-\infty) - (+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

Example 5 Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+}\frac{\sin x - x}{x \sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+}\frac{\sin x - x}{x \sin x} = \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to ***indeterminate forms of the types*** $\mathbf{0^0}$, $\boldsymbol{\infty^0}$, ***and*** $\mathbf{1^\infty}$. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x\to 0^+}(1 + x)^{1/x}$$

whose value we know to be e [see Formula (1) of Section 3.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1 + x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x) \cdot \ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0 \cdot \infty$ (verify), which can be evaluated by methods we have already studied. Once the limit of $\ln y$ is known, it is a straightforward matter to determine the limit of $y = f(x)^{g(x)}$, as we will illustrate in the next example.

▶ **Example 6** Find $\lim_{x\to 0}(1+\sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1+\sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{1/x} = \frac{1}{x}\ln(1+\sin x) = \frac{\ln(1+\sin x)}{x}$$

Thus,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type 0/0, so by L'Hôpital's rule

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x} = \lim_{x\to 0}\frac{(\cos x)/(1+\sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x\to 0$, the continuity of the exponential function implies that $e^{\ln y}\to e^1$ as $x\to 0$, and this implies that $y\to e$ as $x\to 0$. Thus,

$$\lim_{x\to 0}(1+\sin x)^{1/x} = e \blacktriangleleft$$

✔QUICK CHECK EXERCISES 3.6

(See page 228 for answers.)

1. In each part, does L'Hôpital's rule apply to the given limit?
   (a) $\lim_{x\to 1}\frac{2x-2}{x^3+x-2}$ (b) $\lim_{x\to 0}\frac{\cos x}{x}$
   (c) $\lim_{x\to 0}\frac{e^{2x}-1}{\tan x}$
2. Evaluate each of the limits in Quick Check Exercise 1.
3. Using L'Hôpital's rule, $\lim_{x\to+\infty}\frac{e^x}{500x^2}=$ ______.

EXERCISE SET 3.6 Graphing Utility C CAS

1–2 Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule. ■

1. (a) $\lim_{x\to 2}\frac{x^2-4}{x^2+2x-8}$ (b) $\lim_{x\to+\infty}\frac{2x-5}{3x+7}$
2. (a) $\lim_{x\to 0}\frac{\sin x}{\tan x}$ (b) $\lim_{x\to 1}\frac{x^2-1}{x^3-1}$

3–6 True–False Determine whether the statement is true or false. Explain your answer. ■

3. L'Hôpital's rule does not apply to $\lim_{x\to-\infty}\frac{\ln x}{x}$.
4. For any polynomial $p(x)$, $\lim_{x\to+\infty}\frac{p(x)}{e^x}=0$.
5. If n is chosen sufficiently large, then $\lim_{x\to+\infty}\frac{(\ln x)^n}{x}=+\infty$.
6. $\lim_{x\to 0^+}(\sin x)^{1/x}=0$

7–45 Find the limits. ■

7. $\lim_{x\to 0}\frac{e^x-1}{\sin x}$
8. $\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}$
9. $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$
10. $\lim_{t\to 0}\frac{te^t}{1-e^t}$
11. $\lim_{x\to\pi^+}\frac{\sin x}{x-\pi}$
12. $\lim_{x\to 0^+}\frac{\sin x}{x^2}$
13. $\lim_{x\to+\infty}\frac{\ln x}{x}$
14. $\lim_{x\to+\infty}\frac{e^{3x}}{x^2}$
15. $\lim_{x\to 0^+}\frac{\cot x}{\ln x}$
16. $\lim_{x\to 0^+}\frac{1-\ln x}{e^{1/x}}$
17. $\lim_{x\to+\infty}\frac{x^{100}}{e^x}$
18. $\lim_{x\to 0^+}\frac{\ln(\sin x)}{\ln(\tan x)}$
19. $\lim_{x\to 0}\frac{\sin^{-1}2x}{x}$
20. $\lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3}$
21. $\lim_{x\to+\infty}xe^{-x}$
22. $\lim_{x\to\pi^-}(x-\pi)\tan\frac{1}{2}x$
23. $\lim_{x\to+\infty}x\sin\frac{\pi}{x}$
24. $\lim_{x\to 0^+}\tan x\ln x$
25. $\lim_{x\to\pi/2^-}\sec 3x\cos 5x$
26. $\lim_{x\to\pi}(x-\pi)\cot x$
27. $\lim_{x\to+\infty}(1-3/x)^x$
28. $\lim_{x\to 0}(1+2x)^{-3/x}$

29. $\lim_{x\to 0}(e^x + x)^{1/x}$

30. $\lim_{x\to +\infty}(1 + a/x)^{bx}$

31. $\lim_{x\to 1}(2 - x)^{\tan[(\pi/2)x]}$

32. $\lim_{x\to +\infty}[\cos(2/x)]^{x^2}$

33. $\lim_{x\to 0}(\csc x - 1/x)$

34. $\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{\cos 3x}{x^2}\right)$

35. $\lim_{x\to +\infty}(\sqrt{x^2 + x} - x)$

36. $\lim_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

37. $\lim_{x\to +\infty}[x - \ln(x^2 + 1)]$

38. $\lim_{x\to +\infty}[\ln x - \ln(1 + x)]$

39. $\lim_{x\to 0^+} x^{\sin x}$

40. $\lim_{x\to 0^+}(e^{2x} - 1)^x$

41. $\lim_{x\to 0^+}\left[-\dfrac{1}{\ln x}\right]^x$

42. $\lim_{x\to +\infty} x^{1/x}$

43. $\lim_{x\to +\infty}(\ln x)^{1/x}$

44. $\lim_{x\to 0^+}(-\ln x)^x$

45. $\lim_{x\to \pi/2^-}(\tan x)^{(\pi/2)-x}$

46. Show that for any positive integer n

(a) $\lim_{x\to +\infty}\dfrac{\ln x}{x^n} = 0$ (b) $\lim_{x\to +\infty}\dfrac{x^n}{\ln x} = +\infty.$

FOCUS ON CONCEPTS

47. (a) Find the error in the following calculation:

$$\lim_{x\to 1}\frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x\to 1}\frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x\to 1}\frac{6x - 2}{6x - 2} = 1$$

(b) Find the correct limit.

48. (a) Find the error in the following calculation:

$$\lim_{x\to 2}\frac{e^{3x^2-12x+12}}{x^4 - 16} = \lim_{x\to 2}\frac{(6x - 12)e^{3x^2-12x+12}}{4x^3} = 0$$

(b) Find the correct limit.

49–52 Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule. ■

49. $\lim_{x\to +\infty}\dfrac{\ln(\ln x)}{\sqrt{x}}$

50. $\lim_{x\to 0^+} x^x$

51. $\lim_{x\to 0^+}(\sin x)^{3/\ln x}$

52. $\lim_{x\to(\pi/2)^-}\dfrac{4\tan x}{1 + \sec x}$

53–56 Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule. ■

53. $y = \ln x - e^x$

54. $y = x - \ln(1 + 2e^x)$

55. $y = (\ln x)^{1/x}$

56. $y = \left(\dfrac{x+1}{x+2}\right)^x$

57. Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty + (+\infty),$$
$$+\infty - (-\infty),\quad -\infty + (-\infty),\quad -\infty - (+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+}\dfrac{x}{\ln x}$ (b) $\lim_{x\to +\infty}\dfrac{x^3}{e^{-x}}$

(c) $\lim_{x\to(\pi/2)^-}(\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+}(\ln x)\cot x$

(e) $\lim_{x\to 0^+}\left(\dfrac{1}{x} - \ln x\right)$ (f) $\lim_{x\to -\infty}(x + x^3)$

58. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim_{x\to 0^+}[x^{(\ln a)/(1+\ln x)}] = a$ (form 0^0)

(b) $\lim_{x\to +\infty}[x^{(\ln a)/(1+\ln x)}] = a$ (form ∞^0)

(c) $\lim_{x\to 0}[(x + 1)^{(\ln a)/x}] = a$ (form 1^∞).

Verify these results.

59–62 Verify that L'Hôpital's rule is of no help in finding the limit; then find the limit, if it exists, by some other method. ■

59. $\lim_{x\to +\infty}\dfrac{x + \sin 2x}{x}$

60. $\lim_{x\to +\infty}\dfrac{2x - \sin x}{3x + \sin x}$

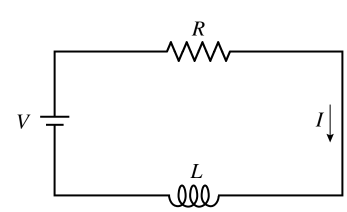
61. $\lim_{x\to +\infty}\dfrac{x(2 + \sin 2x)}{x + 1}$

62. $\lim_{x\to +\infty}\dfrac{x(2 + \sin x)}{x^2 + 1}$

63. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R\to 0^+$)?



◀ **Figure Ex-63**

64. (a) Show that $\lim_{x\to \pi/2}(\pi/2 - x)\tan x = 1$.

(b) Show that

$$\lim_{x\to \pi/2}\left(\frac{1}{\pi/2 - x} - \tan x\right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$

should be good for values of x near $\pi/2$. Use a calculator to find $\tan x$ and $1/(\pi/2 - x)$ for $x = 1.57$; compare the results.

C **65.** (a) Use a CAS to show that if k is a positive constant, then

$$\lim_{x\to+\infty} x(k^{1/x} - 1) = \ln k$$

(b) Confirm this result using L'Hôpital's rule. [*Hint:* Express the limit in terms of $t = 1/x$.]

(c) If n is a positive integer, then it follows from part (a) with $x = n$ that the approximation

$$n(\sqrt[n]{k} - 1) \approx \ln k$$

should be good when n is large. Use this result and the square root key on a calculator to approximate the values of $\ln 0.3$ and $\ln 2$ with $n = 1024$, then compare the values obtained with values of the logarithms generated directly from the calculator. [*Hint:* The nth roots for which n is a power of 2 can be obtained as successive square roots.]

66. Find all values of k and l such that

$$\lim_{x\to 0} \frac{k + \cos lx}{x^2} = -4$$

FOCUS ON CONCEPTS

67. Let $f(x) = x^2 \sin(1/x)$.

(a) Are the limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ indeterminate forms?

(b) Use a graphing utility to generate the graph of f, and use the graph to make conjectures about the limits in part (a).

(c) Use the Squeezing Theorem (1.6.4) to confirm that your conjectures in part (b) are correct.

68. (a) Explain why L'Hôpital's rule does not apply to the problem

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$$

(b) Find the limit.

69. Find $\lim_{x\to 0^+} \dfrac{x \sin(1/x)}{\sin x}$ if it exists.

70. Suppose that functions f and g are differentiable at $x = a$ and that $f(a) = g(a) = 0$. If $g'(a) \neq 0$, show that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

without using L'Hôpital's rule. [*Hint:* Divide the numerator and denominator of $f(x)/g(x)$ by $x - a$ and use the definitions for $f'(a)$ and $g'(a)$.]

71. **Writing** Were we to use L'Hôpital's rule to evaluate either

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{or} \quad \lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x$$

we could be accused of circular reasoning. Explain why.

72. **Writing** Exercise 58 shows that the indeterminate forms 0^0 and ∞^0 can assume any positive real value. However, it is often the case that these indeterminate forms have value 1. Read the article "Indeterminate Forms of Exponential Type" by John Baxley and Elmer Hayashi in the June–July 1978 issue of *The American Mathematical Monthly*, and write a short report on why this is the case.

✔ QUICK CHECK ANSWERS 3.6

1. (a) yes (b) no (c) yes **2.** (a) $\frac{1}{2}$ (b) does not exist (c) 2 **3.** $+\infty$

CHAPTER 3 REVIEW EXERCISES

Graphing Utility

1–2 (a) Find dy/dx by differentiating implicitly. (b) Solve the equation for y as a function of x, and find dy/dx from that equation. (c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

1. $x^3 + xy - 2x = 1$

2. $xy = x - y$

3–6 Find dy/dx by implicit differentiation.

3. $\dfrac{1}{y} + \dfrac{1}{x} = 1$

4. $x^3 - y^3 = 6xy$

5. $\sec(xy) = y$

6. $x^2 = \dfrac{\cot y}{1 + \csc y}$

7–8 Find d^2y/dx^2 by implicit differentiation.

7. $3x^2 - 4y^2 = 7$

8. $2xy - y^2 = 3$

9. Use implicit differentiation to find the slope of the tangent line to the curve $y = x \tan(\pi y/2)$, $x > 0$, $y > 0$ (***the quadratrix of Hippias***) at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

10. At what point(s) is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

11. Prove that if P and Q are two distinct points on the rotated ellipse $x^2 + xy + y^2 = 4$ such that P, Q, and the origin are collinear, then the tangent lines to the ellipse at P and Q are parallel.

12. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the x-axis.

13. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the y-axis.

14. Use implicit differentiation to show that the equation of the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is

$$y_0 y = \tfrac{1}{2}k(x + x_0)$$

15–16 Find dy/dx by first using algebraic properties of the natural logarithm function. ■

15. $y = \ln\left(\dfrac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$ **16.** $y = \ln\left(\dfrac{\sqrt{x}\,\sqrt[3]{x+1}}{\sin x \sec x}\right)$

17–34 Find dy/dx. ■

17. $y = \ln 2x$ **18.** $y = (\ln x)^2$

19. $y = \sqrt[3]{\ln x + 1}$ **20.** $y = \ln(\sqrt[3]{x+1})$

21. $y = \log(\ln x)$ **22.** $y = \dfrac{1 + \log x}{1 - \log x}$

23. $y = \ln(x^{3/2}\sqrt{1 + x^4})$ **24.** $y = \ln\left(\dfrac{\sqrt{x}\cos x}{1 + x^2}\right)$

25. $y = e^{\ln(x^2+1)}$ **26.** $y = \ln\left(\dfrac{1 + e^x + e^{2x}}{1 - e^{3x}}\right)$

27. $y = 2xe^{\sqrt{x}}$ **28.** $y = \dfrac{a}{1 + be^{-x}}$

29. $y = \dfrac{1}{\pi}\tan^{-1} 2x$ **30.** $y = 2^{\sin^{-1} x}$

31. $y = x^{(e^x)}$ **32.** $y = (1 + x)^{1/x}$

33. $y = \sec^{-1}(2x + 1)$ **34.** $y = \sqrt{\cos^{-1} x^2}$

35–36 Find dy/dx using logarithmic differentiation. ■

35. $y = \dfrac{x^3}{\sqrt{x^2 + 1}}$ **36.** $y = \sqrt[3]{\dfrac{x^2 - 1}{x^2 + 1}}$

37. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

38. Recall from Section 0.5 that the loudness β of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$.

39. A particle is moving along the curve $y = x\ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

40. Find the equation of the tangent line to the graph of $y = \ln(5 - x^2)$ at $x = 2$.

41. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

42. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

43. If f and g are inverse functions and f is differentiable on its domain, must g be differentiable on its domain? Give a reasonable informal argument to support your answer.

44. In each part, find $(f^{-1})'(x)$ using Formula (2) of Section 3.3, and check your answer by differentiating f^{-1} directly.
(a) $f(x) = 3/(x + 1)$ (b) $f(x) = \sqrt{e^x}$

45. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

46. Show that the rate of change of $y = 5000e^{1.07x}$ is proportional to y.

47. Show that the rate of change of $y = 3^{2x}5^{7x}$ is proportional to y.

48. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law

$$k = k_0 \exp\left(-\frac{q(T - T_0)}{2T_0 T}\right)$$

where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

49. Show that the function $y = e^{ax}\sin bx$ satisfies

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

for any real constants a and b.

50. Show that the function $y = \tan^{-1} x$ satisfies

$$y'' = -2\sin y\cos^3 y$$

51. Suppose that the population of deer on an island is modeled by the equation

$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$

where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

52. In each part, find each limit by interpreting the expression as an appropriate derivative.
(a) $\displaystyle\lim_{h\to 0}\frac{(1 + h)^\pi - 1}{h}$ (b) $\displaystyle\lim_{x\to e}\frac{1 - \ln x}{(x - e)\ln x}$

53. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

54. (a) Under what conditions will a limit of the form

$$\lim_{x \to a} [f(x)/g(x)]$$

be an indeterminate form?

(b) If $\lim_{x \to a} g(x) = 0$, must $\lim_{x \to a}[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

55–58 Evaluate the given limit. ■

55. $\lim\limits_{x \to +\infty} (e^x - x^2)$

56. $\lim\limits_{x \to 1} \sqrt{\dfrac{\ln x}{x^4 - 1}}$

57. $\lim\limits_{x \to 0} \dfrac{x^2 e^x}{\sin^2 3x}$

58. $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x}, \quad a > 0$

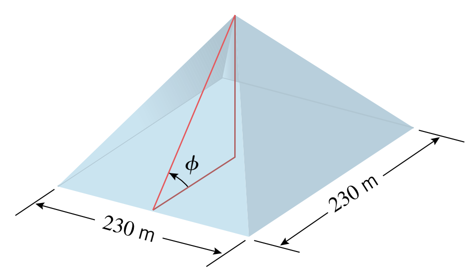
59. An oil slick on a lake is surrounded by a floating circular containment boom. As the boom is pulled in, the circular containment area shrinks. If the boom is pulled in at the rate of 5 m/min, at what rate is the containment area shrinking when the containment area has a diameter of 100 m?

60. The hypotenuse of a right triangle is growing at a constant rate of a centimeters per second and one leg is decreasing at a constant rate of b centimeters per second. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are 1 cm?

61. In each part, use the given information to find Δx, Δy, and dy.
(a) $y = 1/(x - 1)$; x decreases from 2 to 1.5.
(b) $y = \tan x$; x increases from $-\pi/4$ to 0.
(c) $y = \sqrt{25 - x^2}$; x increases from 0 to 3.

62. Use an appropriate local linear approximation to estimate the value of $\cot 46°$, and compare your answer to the value obtained with a calculating device.

63. The base of the Great Pyramid at Giza is a square that is 230 m on each side.
(a) As illustrated in the accompanying figure, suppose that an archaeologist standing at the center of a side measures the angle of elevation of the apex to be $\phi = 51°$ with an error of $\pm 0.5°$. What can the archaeologist reasonably say about the height of the pyramid?
(b) Use differentials to estimate the allowable error in the elevation angle that will ensure that the error in calculating the height is at most ± 5 m.



▲ **Figure Ex-63**

CHAPTER 3 MAKING CONNECTIONS

In these exercises we explore an application of exponential functions to radioactive decay, and we consider another approach to computing the derivative of the natural exponential function.

1. Consider a simple model of radioactive decay. We assume that given any quantity of a radioactive element, the fraction of the quantity that decays over a period of time will be a constant that depends on only the particular element and the length of the time period. We choose a time parameter $-\infty < t < +\infty$ and let $A = A(t)$ denote the amount of the element remaining at time t. We also choose units of measure such that the initial amount of the element is $A(0) = 1$, and we let $b = A(1)$ denote the amount at time $t = 1$. Prove that the function $A(t)$ has the following properties.

(a) $A(-t) = \dfrac{1}{A(t)}$ [*Hint:* For $t > 0$, you can interpret $A(t)$ as the fraction of any given amount that remains after a time period of length t.]

(b) $A(s + t) = A(s) \cdot A(t)$ [*Hint:* First consider positive s and t. For the other cases use the property in part (a).]

(c) If n is any nonzero integer, then

$$A\left(\frac{1}{n}\right) = (A(1))^{1/n} = b^{1/n}$$

(d) If m and n are integers with $n \neq 0$, then

$$A\left(\frac{m}{n}\right) = (A(1))^{m/n} = b^{m/n}$$

(e) Assuming that $A(t)$ is a continuous function of t, then $A(t) = b^t$. [*Hint:* Prove that if two continuous functions agree on the set of rational numbers, then they are equal.]

(f) If we replace the assumption that $A(0) = 1$ by the condition $A(0) = A_0$, prove that $A = A_0 b^t$.

2. Refer to Figure 1.3.4.

(a) Make the substitution $h = 1/x$ and conclude that

$$(1+h)^{1/h} < e < (1-h)^{-1/h} \quad \text{for } h > 0$$

and

$$(1-h)^{-1/h} < e < (1+h)^{1/h} \quad \text{for } h < 0$$

(b) Use the inequalities in part (a) and the Squeezing Theorem to prove that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(c) Explain why the limit in part (b) confirms Figure 0.5.4.

(d) Use the limit in part (b) to prove that

$$\frac{d}{dx}(e^x) = e^x$$