[1]

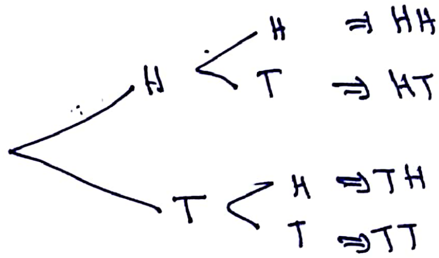
chapter 3 3.1 Random variables and thier distributions

المتغيرات العشوائية توزيعاتها

Ex: In case of tossing acoin twice. Find the Random (العشوائي) variable x (المتغير) which represent (تمثل) the number (عدد) of heads (الصور) appears (التي تظهر) and its distribution (وتوزيعه)

$S = \{HH, HT, TH, TT\}$

$X = \{2, 1, 0\}$



distribution

x	0	1	2	Σ
$P(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{4}{4} = 1$

Ex: in case of atossing acoin twice, If the coin is fair (متزنة), Find the distribution function (DF) of x that has the following form

$$F_X(x) = \begin{cases} P(\phi) = 0 & x < 0 \\ P(TT) = \frac{1}{4} & 0 \leq x < 1 \\ P(TH, HT, TT) = \frac{3}{4} & 1 \leq x < 2 \\ P(TT, HT, TH, HH) = \frac{4}{4} & x \geq 2 \end{cases}$$

$$F_X(x) = \begin{cases} \boxed{0} & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ \boxed{1} & x \geq 2 \end{cases}$$

3.2 Discrete random variables and their distributions

المتقطعة العشوائية المتغيرات

توزيعاتها

Ex: In case of ~~#~~ tossing a fair coin 3 time. find X variable which express the number of heads and the probability distribution and Mass function

دالة الكتلة

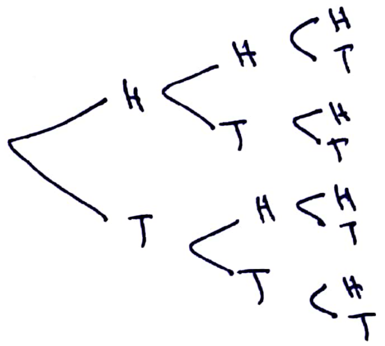
① $X = \{0, 1, 2, 3\}$

②

x	0	1	2	3	Σ
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

③ M-F

$$\begin{cases} \boxed{0} & \boxed{x < 0} \\ \frac{1}{8} & 0 \leq x < 1 \\ \frac{1}{8} + \frac{3}{8} = \frac{4}{8} & 1 \leq x < 2 \\ \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8} & 2 \leq x < 3 \\ 1 & \boxed{x \geq 3} \end{cases}$$



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Ex: 3.2.3 a fair die cube with each of its faces [3].

P. 101

shown different number of dots from 1 to 6

is thrown repeatedly until a 6 at the top.

(a) Determine the p.m.f for a number

times one throws the die

k	1	2	3	4	.....	i
$p(k)$	$\frac{1}{6}$	$\frac{5}{6^2}$	$\frac{5^2}{6^3}$	$\frac{5^3}{6^4}$		$\frac{5^{i-1}}{6^i}$

(b) p.m.f لإيجاد دالة الكتلة الاحتمالية

لابد توافر شرطين

① $\forall\ p(k_i) \geq 0$ ✓

② $\sum_i p(k_i) = 1$

$$\sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^{k-1} \cdot 6^1}$$

$$= \sum \underbrace{\frac{1}{6}}_{a_1} \Big(\underbrace{\frac{5}{6}}_{r}\Big)^{k-1}$$ متسلسلة هندسية لانهائية

$$S_\infty = \frac{a_1}{1-r}$$

$$= \frac{\frac{1}{6}}{1 - \frac{5}{6}} = 1$$

$$\sum a_1 r^{k-1}$$

Find the probability of getting 6 in not more than two trials [4].

$p(X \le 2) = p(1) + p(2)$

$= \frac{1}{6} + \frac{5}{6^2} = \frac{11}{36}$

ⓐ the prob. of getting 6 in more than 2 trials.

$p(X \geq 2) = 1 - p(X \leq 2)$

$p(X \geq a) = 1 - p(X < a)$

$p(X > a) = 1 - p(X \leq a)$

$= 1 - [p(x=1) + p(x=2)]$

$1 - [\frac{1}{6} + \frac{5}{36}] = \frac{25}{36}$

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Ex: 3.2.4
p. 113

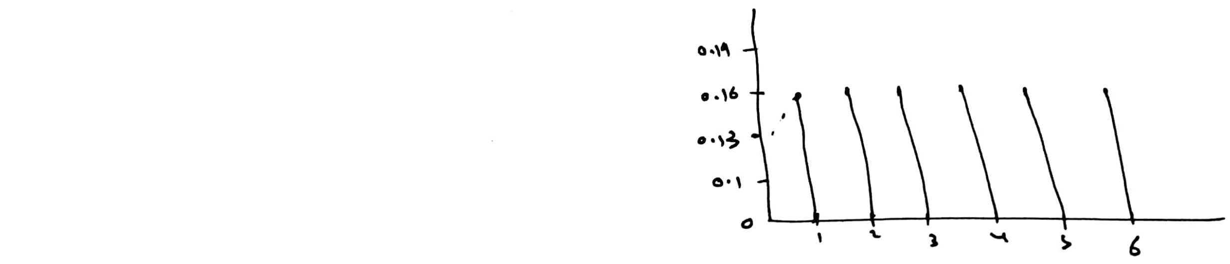
$X: x_1, x_2, x_3, x_4, x_5, x_6$ متغير [illegible]
r.v

(3).

tabular, and graphical and then find

the probability mass function, distribution function.

x	1	2	3	4	5	6	Σ
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1



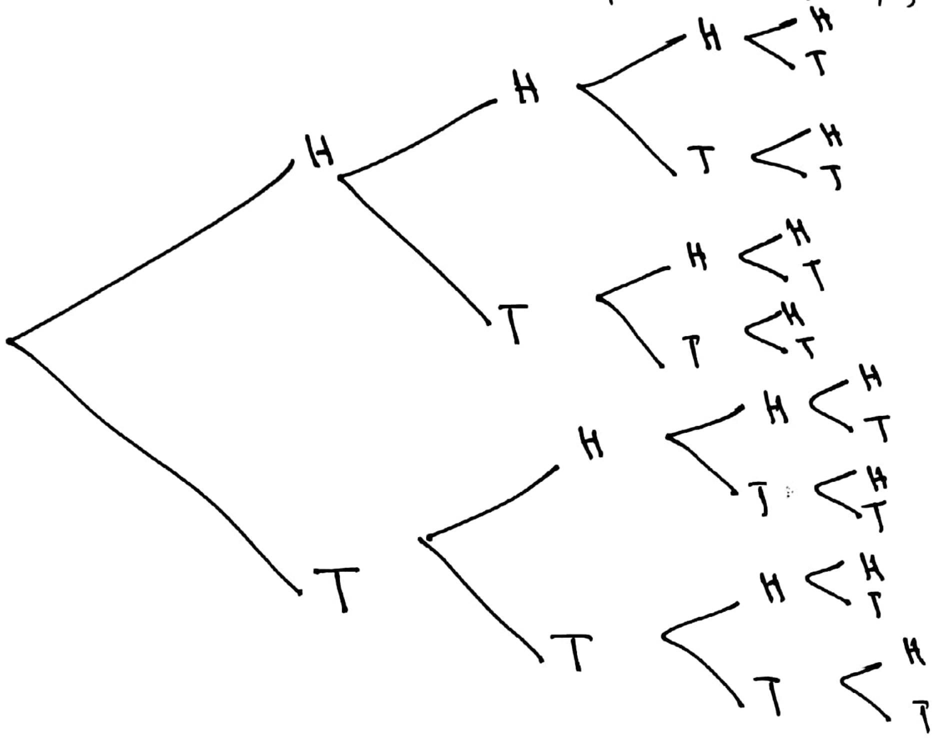
distribution function

$$F_x(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{6} & 1 \le x < 2 \\ \frac{2}{6} & 2 \le x < 3 \\ \frac{3}{6} & 3 \le x < 4 \\ \frac{4}{6} & 4 \le x < 5 \\ \frac{5}{6} & 5 \le x < 6 \\ 1 & x \geq 6 \end{cases}$$

Ex: Consider tossing <u>four</u> fair coins. let X [6]

arondom variable observes the number of heads on all coins. Now

ⓐ Determine the prob- space $\{\Omega, \mathcal{A}, P\}$ for exprement



HHHH
HHHT
HHTH
HHTT
HTHH
HTHT
HTTH
HTTT

THHH
THHT
THTH
THTT
TTHH
TTHT
TTTH
TTTT

$\Omega \Rightarrow$

عناصر فضاء العينة

$$\mathcal{A} = 2^{\Omega} =$$

عدد الأزواج

$$P(A) = \sum_{w \in A} p(w)$$

ⓑ what are the possible values of x

$X = \{0, 1, 2, 3, 4\}$

ⓒ Is random varsable of X is descr- or contin-

descrete

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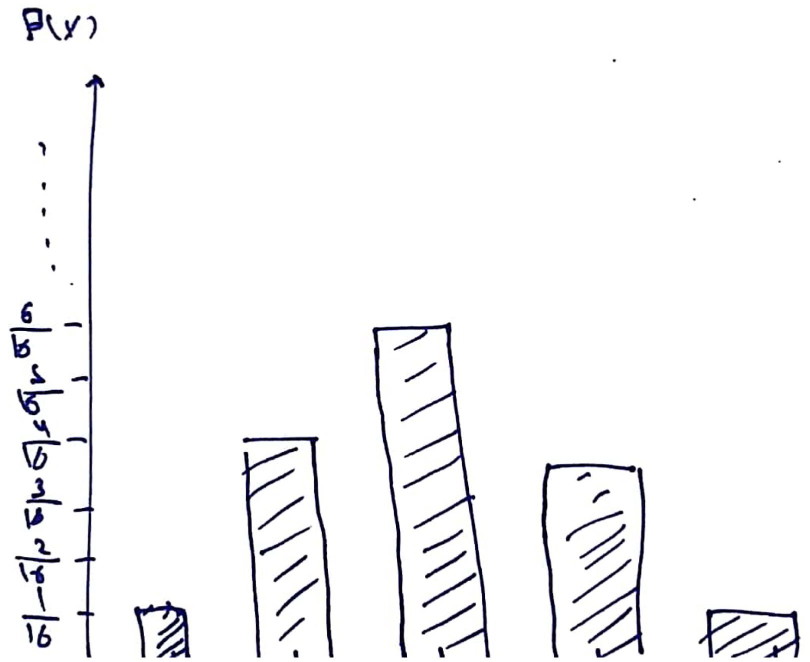
k	0	1	2	3	4
p	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F_x(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	$\frac{16}{16} = 1$

جدولى   البيانى

(e) Give the tabular and graphical representation of x

graphical



X	$p(x)$	$xp(x)$	$x^2p(x)$
0	$\frac{1}{16}$	0	0
1	$\frac{4}{16}$	0.25	0.25
2	$\frac{6}{16}$	~~0.375~~ 0.75	1.5
3	$\frac{4}{16}$	~~0.25~~ 0.75	2.25
4	$\frac{1}{16}$	0.25	1
Σ	1	2	5

$E(x)\ \Sigma\, xp(x) = \boxed{2}$

$V(x) = \Sigma x^2 p(x) - \left(\Sigma x\, p(x)\right)^2$

$5 - (2)^2 = \boxed{1}$

$\delta = \sqrt{V(x)} = \sqrt{1} = \boxed{1}$

properties of expected value and variance

1. $E(a) = a$

ثابت

2. $E(ax) = aE(x)$

3. $E(ax+b) = E(ax) + E(b)$

$= aE(x) + b$

4. $V(a) = \boxed{0}$

5. $V(ax) = a^2 V(x)$

6. $V(ax+b) = V(ax) + V(b)$

$= a^2 V(x) + 0$

Ex: 3.2.9
P:109

$$f(k) = \begin{cases} 2c & k=10 \\ c & k=20 \\ c-0.2 & k=30 \\ 0 & \text{otherwise} \end{cases}$$

where c is areal constant.

(a) determine the value of c

$\sum f(k) = 1$

$2c + c + c - 0.2 + 0 = 1$

$4c - 0.2 = 1$

$4c = 1 + 0.2$

$\frac{4c}{4} = \frac{1.2}{4}$

$\boxed{c = 0.3}$

(b) Determine the corresponding distribution function

$$f(x) = \begin{cases} \underset{}{\overset{0.6}{2(0.3)}} & k=10 \\ 0.3 & k=20 \\ \overset{0.1}{0.3-0.2} & k=30 \\ 0 & \text{o.w} \end{cases} = \begin{cases} 0.6 & k=10 \\ 0.3 & k=20 \\ 0.1 & k=30 \\ 0 & \text{o.w} \end{cases}$$

x	$p(x)$	$xp(x)$	$x^2p(x)$
10	0.6	6	60
20	0.3	6	120
30	0.1	3	90
		15	270

$E(x) = \sum_i x\,p(x) = 15$

$V(x) = \sum_i x^2 p(x) - \left(\sum x\,p(x)\right)^2$

$= 270 - (15)^2 = 45$

$\delta(x) = \sqrt{45} = 3\sqrt{5} \simeq 6.7$

calculate $E(5X+9)$

$E(5X) + E(9)$

$5E(X)+9 = 5(15)+9 =$

$V(3X+14)$

$V(3X) + V(14)$

$9V(X) + 0 = 9(45) =$

ذات الحدين

The bionomial distribution

prob- mass function

$$p(k) = \binom{n}{k} p^{k} (1-p)^{n-k}$$

n العدد الكلي للاختبار

k العدد المطلوب في السؤال

p الاحتمال المعطى في السؤال دو احتمال تحقق الحدث

mean: np

variance $= np(1-p)$

S.D $= \sqrt{np(1-p)}$

Ex: If amean and variance of abionomial distribution are $\underset{=}{\overset{\mu}{16}}$, and $\underset{=}{\overset{V(x)}{8}}$ then: [14]

@ determine the prob mass function

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np = 16 \Rightarrow n = \frac{16}{p}$$

$$\sigma^2 = np(1-p) = 8$$

$$\Rightarrow \frac{16}{\cancel{p}} \cancel{p}(1-p) = 8$$

$$16(1-p) = 8$$

$$16 - 16p = 8$$

$$-16p = 8 - 16$$

$$\frac{-16p}{-16} = \frac{-8}{-16} \qquad \boxed{p = 0.5}$$

$$n = \frac{16}{0.5}$$

$$\boxed{n = 32}$$

$$p(k) = \binom{32}{k} (0.5)^k (0.5)^{32-k}$$

Calculate $p(X=0)$ (15)

$$p(0) = \binom{32}{0} (0.5)^0 (0.5)^{32-0}$$

$\boxed{32}$ $\boxed{shift}$ $\boxed{\div}$ $\boxed{0}$

$$p(0) = (1)(1)\left(\tfrac{1}{2}\right)^{32} = \left(\tfrac{1}{2}\right)^{32}$$

@ Calculate $p(X \geq 2)$ $p(X \geq 2) = 1 - p(X < 2)$

$$= 1 - [p(X<2)]$$

$$= 1 - [p(X=1) + p(X=0)]$$

$$= 1 - \left[\binom{32}{1} (0.5)^1 (0.5)^{31} + \binom{32}{0} (0.5)^0 (0.5)^{32} \right]$$

$$1 - (\quad + \quad) = 1 - 33\left(\tfrac{1}{2}\right)^{32}$$

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3.3 Continuous Random variables

EX: let X be abionomial distribution [16]

r.v with parameters:

ⓐ $n = 12$, $p = \frac{1}{2}$

ⓑ $n = 8$, $p = \frac{1}{3}$

[1] draw the graph of X in the two case

ⓐ

-solu-

$X = \{0, 1, 2, 3, \ldots, 12\}$

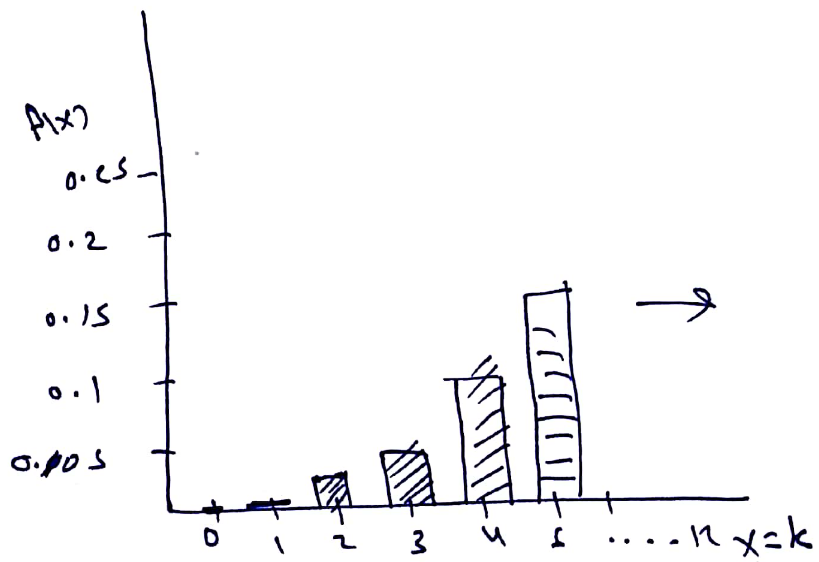
$p(X=0) = \binom{12}{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{12} = 0.00024$

$p(X=1) = \binom{12}{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{11} = 0.0029$

$p(X=2) = \binom{12}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{10} = 0.016$

⋮

$p(X=12)$



poisson distr

$$p(x=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\lambda = np$$

mean $= np = \lambda$

variance $= np = \lambda$

Ex: In alarge oil exploration firm company engineering accidents occur independently at the mean per month $\lambda = 3$. the prob - mass function is

ⓐ $p(x=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ ⓑ $p(x=4)$

$= \dfrac{3^k \cdot e^{-3}}{k!}$ $\qquad \dfrac{3^4 \cdot e^{-3}}{4!} = 0.168$

Ex: 3.2.17 H.W

118

3.3 Continuous Random variables and their distributions

المتغير العشوائي المتصل

Def: Continuous Random variable:

is a random variable whose set of its possible values is un Countable set. « يعبر بـ كسور »

(غير قابلة للعد)

Ex: Life length, temperature, blood pressure, electric voltage,

p.d.f function: (دالة كثافة الاحتمال)

$$* \; p(a \le x \le b) = \int_a^b f(x)\,dx$$

distribution function $\Leftarrow$

$$* \; F_x(x) = p(X \le x) = \int_{-\infty}^{x} f_x \, dx$$

probability density function (دالة كثافة الاحتمال)

$$* \; f_x(x) = \frac{dF_x}{dx} = F'(x)$$

[2]

Ex: 3.3.2 let X be a continuous random variable
p: 121

with distribution function F_X.

then we will find the pdf of X

ⓐ $F_X(x) = \frac{1x^4}{16}$ $0 \le x \le 2$

$$\text{pdf} = \frac{d}{dx}(F_X)$$

$$\text{pdf} = \frac{4x^3}{16} = \frac{x^3}{4}$$

$f(x) = 3 \Rightarrow f' = 0$

$f(x) = 3x \Rightarrow f' = 3$

$f(x) = 3x^2 \Rightarrow f'(x)$
$= 3 \cdot 2\, x^{2-1}$
$= 6x$

ⓑ $F_X(x) = 1 - e^{-5x}$, $x \ge 0$

$$\text{p.d.f} = \frac{d}{dx}(F_X)$$

$$\text{p.d.f} = \frac{d}{dx}\left(1 - e^{-5x}\right)$$

$$= 0 - (-5)\, e^{-5x}$$

$$= 5e^{-5x}$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)\, e^{f(x)}$$

مشتقة الأس

الدالة نفسها

3

Expectation $E(x) = \mu = \int_{-\infty}^{\infty} x f_x(x)\,dx$

variance $\delta^2 = v(x) = \cancel{\int_{-\infty}^{\infty} (x-\mu)^2 f_x(x)\,dx} = E(x^2) - (E(x))^2$

قواعد التكامل

ثابت

[1] $\int k\,dx = kx$ ex: $\int 3\,dx = 3x$

[2] $\int kx^n\,dx = \frac{kx^{n+1}}{n+1}$ ex: $\int 3x^4\,dx = \frac{3x^5}{5}$

Ex.: if $f_x(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{other wise} \end{cases}$

Calculate the mean and variance

$\mu = E(x) = \int_{-\infty}^{\infty} x f_x(x)\,dx$

$= \int_a^b x \left(\frac{1}{b-a}\right) dx = \frac{1}{b-a} \int_a^b x^1\,dx$

$= \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b = \frac{x^2}{2(b-a)} \Big|_a^b$

$= \left(\frac{b^2}{2(b-a)}\right) - \left(\frac{a^2}{2(b-a)}\right)$

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$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{b+a}{2}$$

$x^2-y^2=(x-y)(x+y)$

$$E(x) = \mu = \frac{b+a}{2}$$

② variance $\delta(x) = V(x)$

$$V(x) = E(x^2) - (E(x))^2$$

$$= ? - \left(\frac{b+a}{2}\right)^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$$

$$= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\Big|_a^b\right]$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right] = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$= \frac{1}{\cancel{b-a}}\left(\frac{\cancel{(b-a)}(b^2+ba+a^2)}{3}\right)$$

$b^3-a^3 = (b-a)(b^2+ba+a^2)$

$$E(x^2) = \frac{b^2 + ba + a^2}{3}$$

$$V(x) = \frac{b^2 + ba + a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{4(b^2 + ba + a^2)}{4 \cdot 3} - \frac{3(b^2 + 2ba + a^2)}{3 \cdot 4}$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$= \frac{4b^2 + 4ba + 4a^2 - 3b^2 - 6ba - 3a^2}{12}$$

$$= \frac{b^2 - 2ba + a^2}{12}$$

$$= \frac{(b-a)(b-a)}{12} = \frac{(b-a)^2}{12}$$

~~variance~~ = standard devition = $\sqrt{\text{variance}}$

$$= \sqrt{\frac{(b-a)^2}{12}}$$

6

* The exponential distribution:

① $f_x(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

② $F_x(x) = 1 - e^{-\lambda x}$

③ $\mu = E(x) = \text{mean} = \frac{1}{\lambda}$

④ $\text{variance} = \frac{1}{\lambda^2}$

⑤ $\text{standard devition} = \frac{1}{\lambda}$

Ex: 3.3.7
p. 126

متوسط
$\text{avrage} = \cancel{\lambda} = 15 = \mu \Rightarrow \lambda = ?? \quad \lambda = \frac{1}{15}$

~~ⓐ $p(x \leq 6) = \frac{1}{15} e^{-\frac{1}{15}(x)}$~~

$p(x) = \frac{1}{15} e^{-\frac{1}{15}(x)}$

ⓐ أقل
~~$p(x \leq 6) = p(x=6) + p(x=5) + p(x=4) + p(x=3)$~~
~~$+ p(x=2) + p(x=1) + p(x=0)$~~
~~$= \frac{1}{15} e^{-\frac{1}{15}(6)} + \frac{1}{15} e^{-\frac{1}{15}(5)} + \dots$~~
~~$+ \dots \frac{1}{15} e^{-\frac{1}{15}(0)} =$~~

صفر $= 1 - p(x=6) = 1 - \left(\frac{1}{15} e^{-\frac{1}{15}(6)}\right) = 0.3297$

$$p(x \leq a) = 1 - e^{-\frac{a}{\lambda}}$$

$$p(x \geq a) = e^{-\frac{a}{\lambda}}$$

ⓐ $p(x \leq 6) = 1 - e^{-\frac{6}{15}} = 0.3297$

ⓑ $p(x \geq 18) = e^{-\frac{18}{15}} = 0.3012$

ⓒ The variance $= \frac{1}{\lambda^2} = \frac{1}{(\frac{1}{15})^2} = 0.0044$

standard devition: $= \sqrt{0.0044} = 0.067$

The Normal distribution

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

standard Normal distrib—

$X \sim N(\mu, \sigma)$ Normal

$Z \sim N(0, 1)$

$$Z = \frac{x - \mu}{\sigma}$$

Ex: 3.38 follow normal distribution
P. 130

mean = 80 , S.D = 5

ⓐ $p(X < 75) = p\left(\frac{X-\mu}{\sigma} < \frac{75-\mu}{\sigma}\right)$

$p\left(Z < \frac{75-80}{5}\right)$

$p(Z < -1) = p(\phi) = \boxed{0}$

ⓑ $p(7\cancel{5} < X < \cancel{8}2)$

$p\left(\frac{76-80}{5} < \frac{X-\mu}{\sigma} < \frac{82-80}{5}\right)$

$p(-0.8 < Z < 0.4) = p(Z < 0.4) - p(Z < -0.8)$

$= 0.6554 - 0.2119 =$

Ex: 3.3.9 use the standard normal distr. to find
P. 131

Z - value

ⓐ $p(\cancel{Z} \leq z) = 0.4090$

$z = -0.23$

ⓑ $p(Z \leq z) = 0.80$

ⓒ $p(Z > z) = 0.4090$

$p(Z > z) = 1 - p(Z < z)$

$0.4090 = 1 - p(Z < z)$

$\boxed{0.23}$

$p(Z < z) = 1 - 0.4090 = \cancel{1 - 0.23 = 1 - 23} = 0.5910$ ابحث

Ex: 3.3.8 follow normal distribution
p: 130

mean = 80 , S.D = 5

ⓐ $p(X < 75) = p\left(\frac{X-\mu}{\sigma} < \frac{75-\mu}{\sigma}\right)$

$p\left(Z < \frac{75-80}{5}\right)$

$p(Z < -1) = p(\phi) = \boxed{0}$

ⓑ $p(75 < X < 82)$

$p\left(\frac{76-80}{5} < \frac{X-\mu}{\sigma} < \frac{82-80}{5}\right)$

$p(-0.8 < Z < 0.4) = p(Z < 0.4) - p(Z < -0.8)$

$= 0.6554 - 0.2119 =$

Ex: 3.3.9 use the standard normal distr. to find
P: 131

Z-value

ⓐ $p(Z \leq z) = \underline{\underline{0.4090}}$

$z = \underline{\underline{-0.23}}$

ⓑ $p(Z \leq z) = 0.80$

$\boxed{0.23}$

ⓒ $p(Z > z) = 0.4090$ $p(Z > z) = 1 - p(Z < z)$

$0.4090 = 1 - p(Z < z)$

$p(Z < z) = 1 - 0.4090 = \cancel{1 - 0.23} = 0.\underline{5910}$