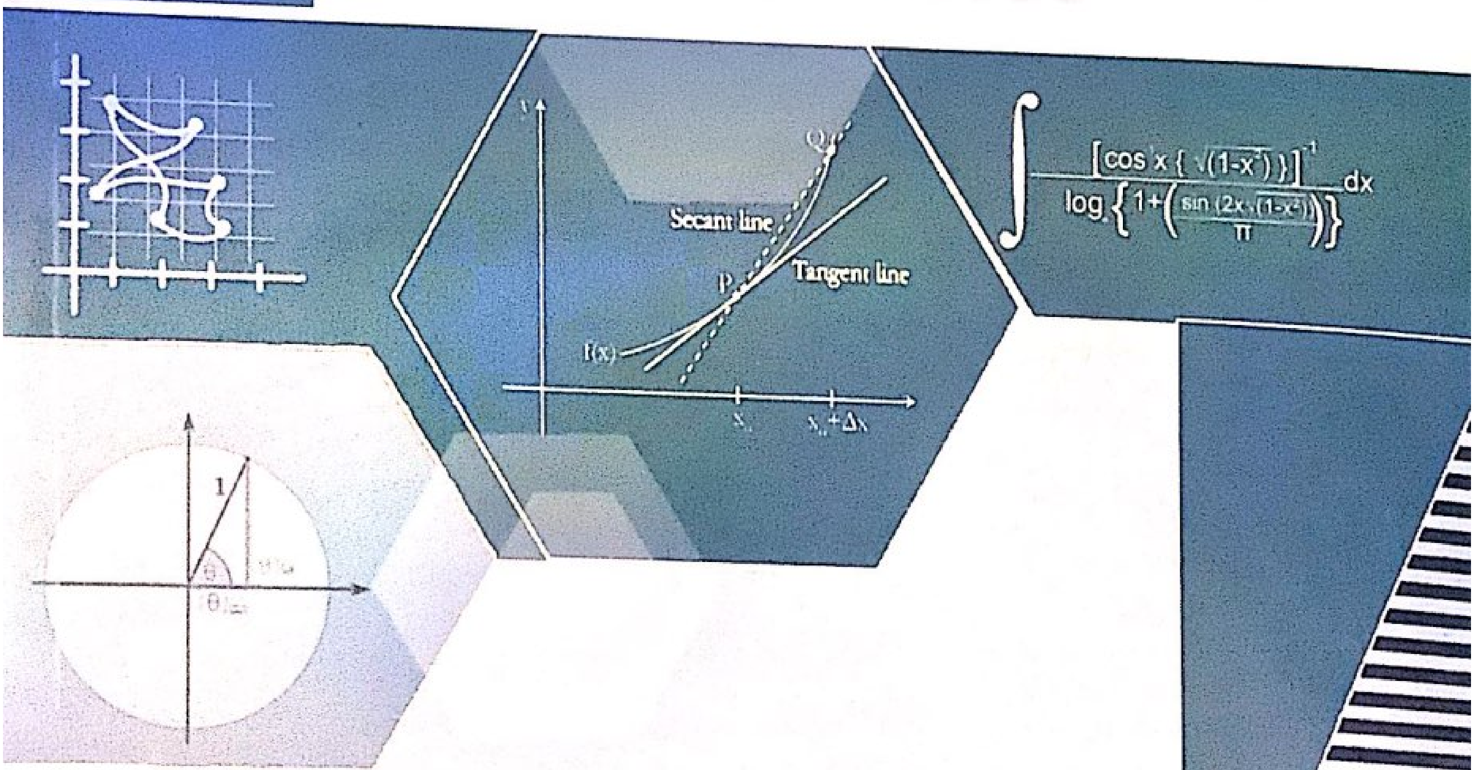
GENERAL MATHEMATICS

GENERAL MATHEMATICS



Written by
Agroup of professors of mathematics
University of Bisha

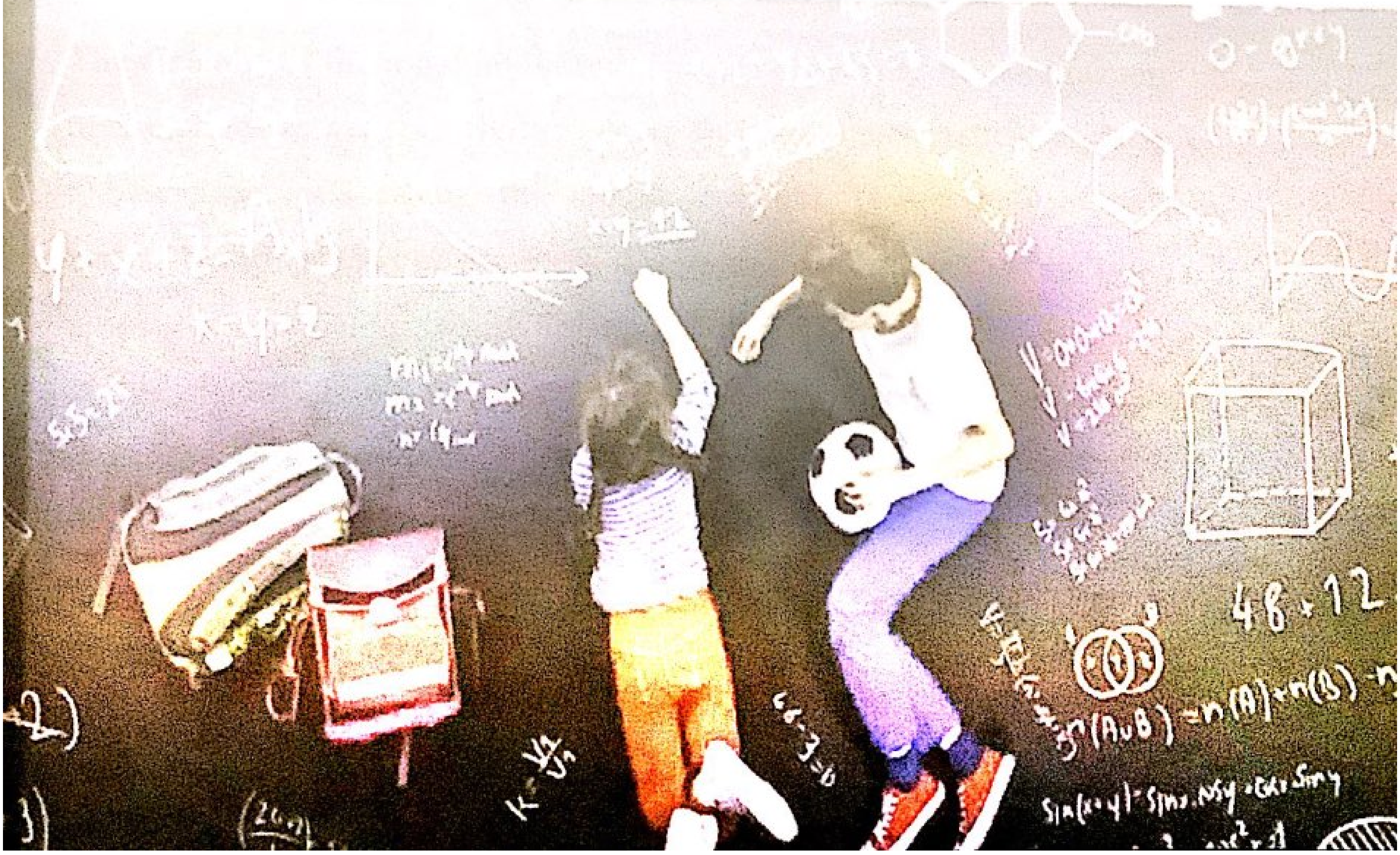
First edition
2021 AD

يأتي هذا الكتاب ضمن سلسلة:

Chapter 1

Functions

1



Chapter 1 : Functions

Learning Outcomes:

By completing the study of this chapter, it is expected that the student will be able to:

- Define the function and discuss its basic properties.
- Do arithmetic operations on functions.
- Draw and discuss curves of some fundamental functions.
- Define polynomials and algebraic functions and determine their properties.
- Define trigonometric, inverse trigonometric, power, and logarithmic functions and discuss their basic properties.
- Apply the properties of the trigonometric, inverse trigonometric, power, and logarithmic functions.

The function is the most important concept especially in calculus and in mathematics in general. Before we discuss the mathematical concept of the function, let's discuss this simple example. That is how to make a loaf of bread (output or product). The ingredients (inputs) are a glass of water, a glass of flour, with a little salt and a spoon of yeast. Every input of these inputs is a value that can be changed (weight, type, etc.) and according to its variation the specifications of the output or the final product will change, i.e., the loaf of bread (weight, taste, etc.). From this point of view, we can call these inputs "the independent variables" and often denoted by $x_1, x_2, ...$ and according to their variation, the value of the dependent variable "the output" change. The dependent variable is often denoted by y. We can say that y is a function of the variables $x_1, x_2, ...$

Thus, any natural system can be expressed as a function and therefore, it is converted into an abstract mathematical form that can be handled, and its properties cab be mathematically studied at least from the theoretical point of view.

In the following section, we will define the set to get the definition of the function. We will concern only with the basic concepts and simple examples for clarity and let details to more dedicated courses.

1.1 Sets and Subsets :

Sets are used in many scientific fields and they are one of the most important basic concepts in mathematics. We will use the concept of the set in this book according to the following definition:

Definition 1.1.1 "Set":

A set is a collection of well-defined objects. Capital letters are commonly used to denote sets $(A, B, C, etc.)$. The objects of the set are called elements and will be denoted by lowercase letters $(a, b, c, etc.)$.

The set can be written in one of two ways, as in the following definition.

Definition 1.1.2:

(1) Roster or List Method: in this method, the elements of the set are placed between braces "{}", called the set braces, and the elements are separated from each other by a comma ",".

(2) Characteristic or Rule Method: in this method the set is expressed bya characteristic or rule that combine its elements. It will be written in the form $\{x: p(x)\}$ and is read as "all elements $\{x\}$ such that the characteristic or the rule$\{p(x)\}$ is true."

Example 1.1.1:

Let A be the set of the squares of all natural numbers, we may write $A = \{x: x = n^2, n \in N\}$ (characteristic or rule method) or $A = \{1,4,9, ...\}$ (roster or list method).

The relationship between the element and the set is called the belonging relationship and is given in the following definition.

Definition 1.1.3 Belonging:

It is said that an element a belongs to a set A and is denoted by $a \in A$ if and only if it is one of the elements of the set A. It is said that it does not belong to A and is denoted by $a \notin A$ if and only if it is not an element of A.

Sets, according to the number of their elements, are divided into two types, finite and infinite as in the following definition:

Definition 1.1.4:

(1) A set is said to be finite if it contains a finite number of elements.

(2) A set is said to be infinite if it contains an infinite number of elements. The number of elements of the set A is denoted by$|A|$.

Numbers Sets:

One of the most important infinite sets is the sets of numbers. The set of natural numbers that is denoted by $\mathbb{N}$ where $\mathbb{N} = \{1,2,3, ...\}$, the set of integers that is denoted by $\mathbb{Z}$ where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, the set of rational numbers $\mathbb{Q}$ where $\mathbb{Q} = \{\frac{a}{b}, : a, b \in \mathbb{Z}, b \neq 0\}$, the set of irrational numbers that is denoted by $\dot{\mathbb{Q}}$ includes all the numbers that cannot be written in a rational form such as $\sqrt{2}, \sqrt{3}, \pi, e, \dots$, the set of real numbers that is denoted by $\mathbb{R}$ is the set of all numbers on the real line $\mathbb{R} = \mathbb{Q} \cup \dot{\mathbb{Q}} = (-\infty, \infty)$.

Definition 1.1.5 Empty Set:

The set that does not have any element is called empty set and is denoted by φ (phi) or $\{\}$.

Definition 1.1.6 Subset:

Let A and B be two sets. A is said to be a subset of B and is denoted by $A \subseteq B$ if and only if every element that belongs to A also belongs to B. This is expressed mathematically as:

$$A \subseteq B \Leftrightarrow x \in B \ \forall x \in A,$$

or in the contrapositive form as:

$$A \subseteq B \Leftrightarrow x \notin A \ \forall x \notin B.$$

The set itself and the empty set are called improper subsets of a given set whereas any other subset is called a proper subset.

Definition 1.1.7 Equal Sets:

Two sets A and B are said to be equal and denoted by $A = B$ if and only if they have exactly the same elements, i.e., every element that belongs to A also belongs to B and every element that belongs to B also belongs A.

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

1.2 Operations on Sets:

It is customary in mathematics to define a mathematical concept and then using it to form other forms of it, by defining operations on this concept and studying the properties of these operations and this is what we will do, Insha Allah in this section for sets.

Definition 1.2.1 Union:

The union of two sets A and B which is denoted by $A \cup B$ is the set which consists of all the elements that belong to A or B or both; i.e., the set that consists of all the elements of A in addition to all the elements of B, without repetition and can be written as:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Definition 1.2.2 Intersection:

The intersection of two sets A and B which is denoted by $A \cap B$ is the set which consists of all elements that belong both to A and B; i.e., the set that contains the common elements of A and B, and can be written as:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Definition 1.2.3 Difference:

If A and B are two sets, the difference of B from A which is denoted by $A - B$ is the set of all elements of A that do not belong to B; i.e.,

$$A - B = \{x : x \in A \text{ and } x \notin B\},$$

and thus:

$$B - A = \{x : x \in B \text{ and } x \notin A\}.$$

Definition 1.2.4 Complement:

Let U be a universal set (i.e., contains all sets) and $A \subset U$, the complement of A, which is denoted by A^c or À is the set of all elements are not belonging to A; i.e.,

$$A^c = \{x : x \in U \ and \ x \notin A\}.$$

The properties of operations on sets is given in the following theorem.

Theorem 1.2.1 Properties of Operations on Sets:

If A, B, and C are subsets of the universal set U, then

$(i) A \cup B = B \cup A,\ A \cap B = B \cap A,$

this property is known as the *Commutative Property.*

$(ii) (A \cup B) \cup C = A \cup (B \cup A), (A \cap B) \cap C = A \cap (B \cap C),$

this property is known as the *Associative Property.*

$(iii)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C),\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$

this property is known as the *Distribution Property.*

$(iv) (A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c,$

this property is known as *De Morgan's Laws.*

1.3 Real Numbers and Intervals

Calculus and its theories are built on the properties of the set of real numbers the most important of which is ordering and completeness. We will mention here some of these properties and they will be studied in detail, Insha Allah, in other advanced courses.

Property 1.3.1 Properties of Real Numbers:

If a and b are two real numbers, they define the real number $a + b$ that is known as their summation (addition operation) and the real number ab that is known as their product (multiplication operation) and satisfy the following properties:

$(i) a + b = b + a, \quad ab = ba.$

The *Commutative Property* of addition and multiplication.

$(ii) (a + b) + c = a + (b + c), (ab)c = a(bc).$

The *Associative Property* of addition and multiplication.

$(iii)\ a(b + c) = ab + bc.$

The *Distribution Property*.

There are $0, 1 \in \mathbb{R}$ where:

$(iv)\ 0 + a = a + 0 = a, 1a = a1 = a, 0 \neq 1$

The *Addition and Multiplication identities*.

Property 1.3.2 Properties of Real Numbers:

If a and b are two real numbers, then only one of the following alternatives is true:

$(v)\ a > b \text{ or } b > a \text{ or } a = b,$

this property is known as the order property and the order relation is (< less than).

Only one of the following alternatives is true:

a is a positive number or $-a$ is a positive number or $a = 0$.

Definition 1.3.1 Intervals:

If a and b are real numbers such that $a < b$, then

(1) The set of all real numbers between a and $b$is denoted by (a,b) and is called the open interval of$a$and b, i.e.

$(a,b) = \{x: a < x < b, x \in \mathbb{R}\}$.

(2) The set of all real numbers between a and b including a is denoted by $[a,b)$ and is called the right half-open interval of a and b, i.e.

$[a,b) = \{x: a \leq x < b, x \in \mathbb{R}\}$.

(3) The set of all real numbers between a and b including b is denoted by $(a,b]$ and is called the left half-open interval of a and b, i.e.

$(a,b] = \{x: a < x \leq b, x \in \mathbb{R}\}$.

(4) The set of all real numbers between a and b including a and b is denoted by $[a,b]$ and is called the closed interval of $a$and b, i.e.

$[a,b] = \{x: a \leq x \leq b, x \in \mathbb{R}\}$.

(5) The set of all real numbers greater than a is denoted by (a,∞), i.e.

$(a,\infty) = \{x: x > a, x \in \mathbb{R}\}$.

(6) The set of all real numbers greater than or equal a is denoted by $[a,\infty)$, i.e.,

$[a,\infty) = \{x: x \geq a, x \in \mathbb{R}\}$.

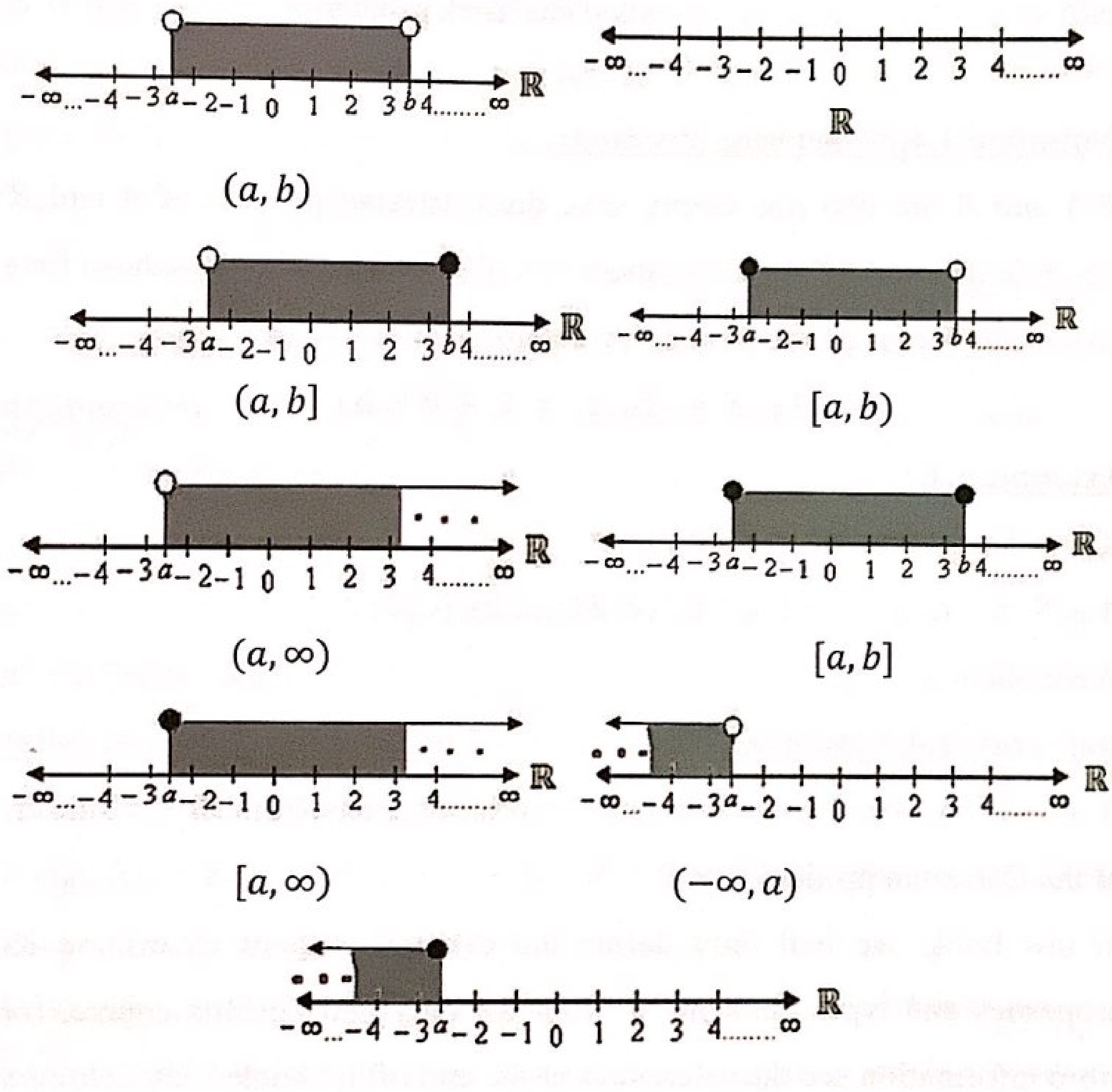
(7) The set of all real numbers less than a is denoted by $(-\infty,a)$, i.e.

$(-\infty,a) = \{x: x < a, x \in \mathbb{R}\}$.

(8) The set of all real numbers less than or equal a is denoted by $(-\infty,a]$, i.e.,

$$(-\infty,a] = \{x: x \leq a, x \in \mathbb{R}\}.$$

The set of real numbers and intervals are represented graphically as in Figure (1-1).



(a,b) $(a,b]$ $[a,b)$ (a,∞) $[a,b]$ $[a,\infty)$ $(-\infty,a)$ $(-\infty,a]$

Figure (1-1)

1.4 Functions in One Variable:

We can define a function in one variable as a dependent variable that changes with only one independent variable. We will use the concept "function" in this book to express the function in one variable.

Definition 1.4.1 Ordered Pairs:

The ordered pair (2-tuple) is an ordered array of two elements and is written as (a, b) where a is called the first projection (entry) and b is called the second projection (entry).

Definition 1.4.2 Cartesian Product:

If A and B are two non-empty sets, the Cartesian product of A and B which is denoted by $A \times B$, is the set of all the ordered pairs whose first projection is in A and its second projection is in B, that is

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

Example 1.4.1:

If $A = \{a, b, c\}$and $B = \{1, 2\}$, then

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

What about $B \times A$?

Definition 1.4.3 Relation:

If A and B are two non-empty sets, the relation from A into B is a subset of the Cartesian product $A \times B$.

In this book, we will only define the relation without examining its properties and types, and this is what we only need in this course, for more information see the references at the end of the book.

Now, we can define the function.

Definition 1.4.4 Function:

If X and Y are two non-empty sets, then the function f from X into Y is a relation from X into Y such that every element of X is associated with only one element of Y. We can formulate it in another way, that is, every element of X appears as a first projection of the elements of the function f only once. That is denoted by $f: X \to Y$ or $y = f(x)$ and y is called the image of the element (origin) x. X is called the domain of the function, and Y is called the co-domain. See Figure (1-2).

Based on this, the relation $y = f(x)$ is a function if and only if:

$f(x_1) = f(x_2)\ \forall\, x_1 = x_2,\ x_1, x_2 \in X.$

It is clear from the definition that the function from X into Y is a subset of the Cartesian product $X \times Y$, but every element of X must appear as a first projection only once. If y is the image of the element x under the effect of the function f expressed as $y = f(x)$ and assuming that x is a general element in the domain of the function, x is known as the independent variable and y is known as the dependent variable. We can define this as follows.

Definition 1.4.5 Independent Variable and Dependent Variable:

Generally, the independent variable could be defined as a free changeable value, according to its change another value is changed which is called dependent variable.

To clarify the latter definition, let us discuss the following simple example, we can say that the height of a student is an independent variable. The weight of the student will be changed according to the change of the student's height, thus the weight is a dependent variable.

If the domainof the function is not stated explicitly, there is what is known as the natural or possible domain, which is given in the following definition.

Definition 1.4.6 Natural or Possible Domain of a Function:

The natural or possible domain of the function f is all the possible values of x for which $f(x)$ is defined.

The division by zero is undefined. The definition will be sufficient at this stage; examples will be discussed in details after defining the algebraic functions.

Definition 1.4.7 Range of a Function:

The range of the function $f: X \to Y$ is the set of all values of Y that appear as images of the elements of X (sometimes called the set of the images of the function and X may be called the origin set) and is denoted by R_f. i.e.,$R_f = \{y \in Y, y = f(x) \; \forall \, x \in X\}$.

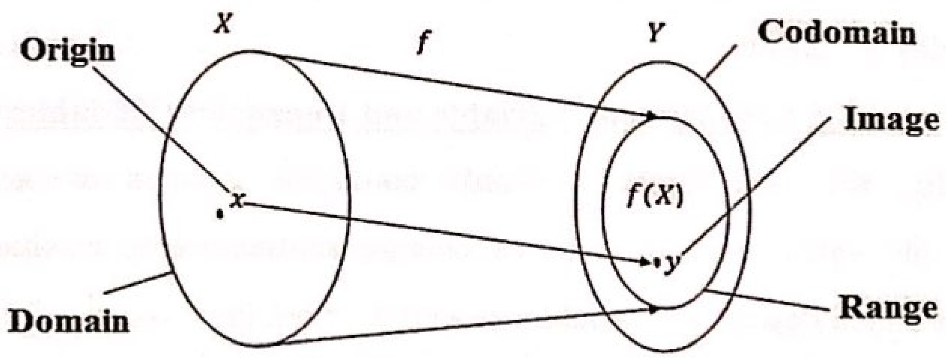


Figure (1-2)

Example 1.4.1:

If $X = \{1,2,3\}$ and $Y = \{a, b, c, d, e\}$, determine which of the following sets is a function from X into Y. Explain.

(i) $f_1 = \{(1, a), (2, c), (3, e), (3, c)\}$.

(ii) $f_2 = \{(1, a), (2, a), (3, c)\}$.

(iii)$f_3 = \{(2, a), (3, c)\}$.

Solution:

f_1 is not a function because the element 3 appeared as a first projection twice. f_2 is a function (note that the element a appeared twice, but this does not affect it being a function). f_3 is not a function because the element 1 did not appear as a first projection.

Definition 1.4.8 Real-Valued Functions:

A real-valued function is a function whose codomain is the set of real numbers. A real-valued function of a real variable is a function whose domain is the set of real numbers or a subset of it and its codomain is the set of real numbers.

In this course we will be concerned with studying the real-valued functions in real variable, the term function will be used to refer to it unless mentioned otherwise. Examples of such functions are, polynomials, rational functions, absolute value function, square root function, trigonometric functions, etc. We will study these functions in details.

1.5 Graph of Function:

Graph of a function gives a pictorial form of the function which is called the curve of the function. Let $y = f(x)$, i.e., every x (origin) in the domain of the function is connected to a single value y (image) in the range of the function f. In other words, if we assume that the value of x

is x_1 in the domain of f, then there is only one value, say, y_1 is associated with it. Thus, the ordered pairs (x, y) are different for all the values of x in the domain of f. In the Cartesian plan oxy, every ordered pair (x, y) is represented by only one point in the plane; see Figure (1-3). We can define the curve of a function as follows:

1.5.1 Curve of Function:

The curve of the function $y = f(x)$ is all the points in the Cartesian plane oxy corresponding to the ordered pairs (x, y).

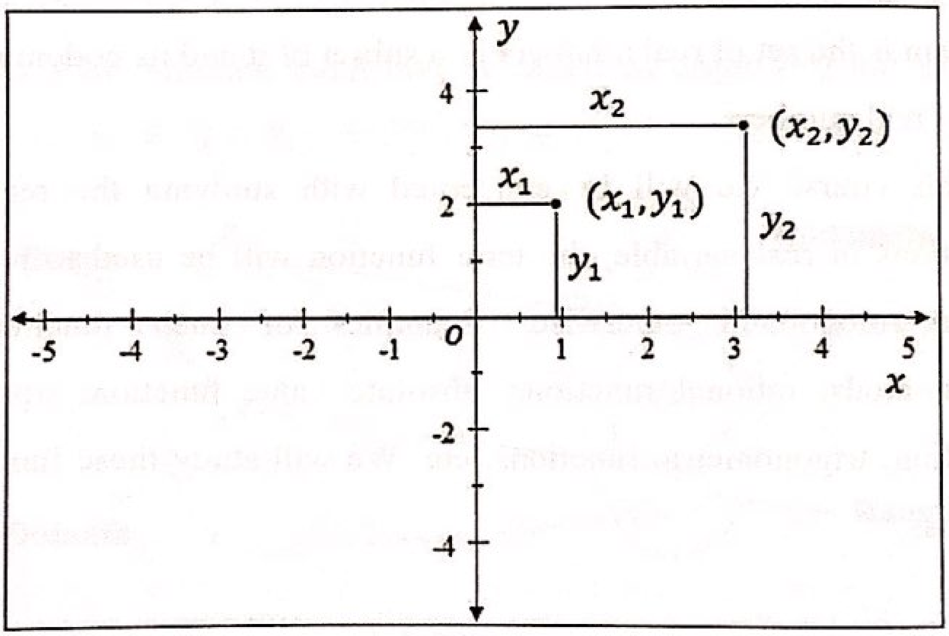


Figure (1-3)

From the previous discussion, any vertical line cannot intersect the curve of a function in more than one point. Assume a curve which is intersected by a vertical line in two points, i.e. they have the same value of x, say, x_3 but there are two values for y, say the values y_3, y_4, thus there are two ordered pairs (x_3, y_3), (x_3, y_4) which indicates that x_3

appeared as a first projection twice and so this curve does not represent a function, and this test is known as the vertical line test.

<u>Definition 1.5.2 Vertical Line Test:</u>

A curve in the Cartesian plane oxy represents a function if and only if any vertical line does not intersect the curve in more than one point.

Using vertical line test, the curve in Figure (1-4) does not represent a function, but the curve in Figure (1-5) represents a function.

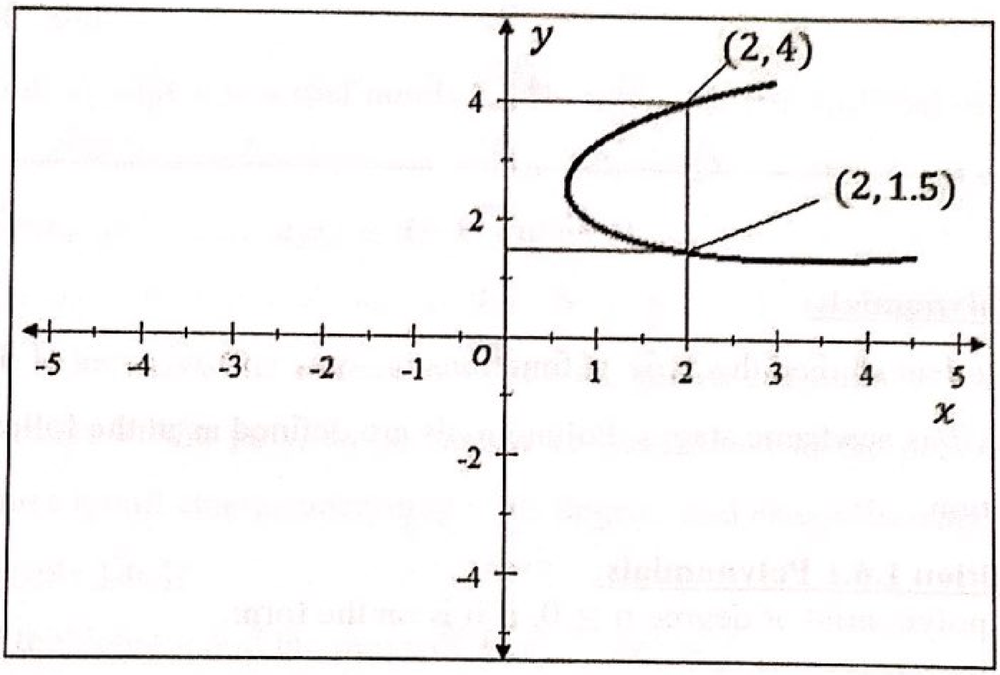


Figure (1-4)

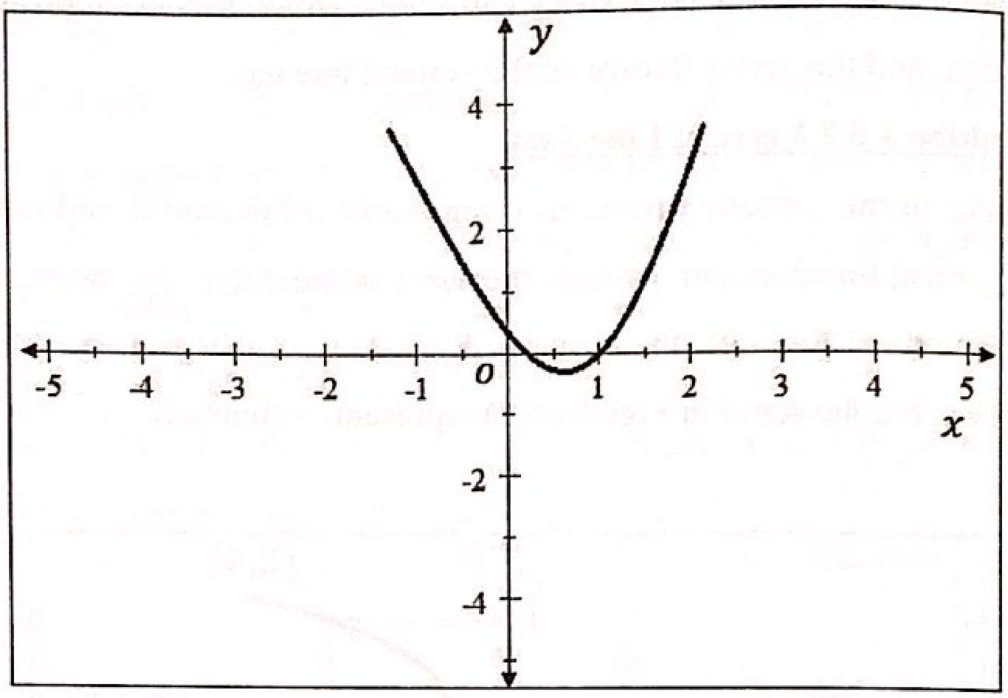


Figure (1-5)

1.6 Polynomials:

The student studied this type of functions or some special cases of them in previous academic stages. Polynomials are defined as in the following definition.

Definition 1.6.1 Polynomials:

P is a polynomial of degree n ≥ 0, if it is on the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

$$a_n \neq 0, a_i \in \mathbb{R} \; \forall \; i = 0,1,2, \ldots, n.$$

From the previous definition, $P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$, $a_4 \neq 0$ is a polynomial of the fourth degree. A polynomial of the first degree is called linear polynomial, of the second degree is called square polynomial, and of the third degree is called cubic polynomial.

Here a group of questions appear: do polynomials represent functions? What are their domain and range? The answer is in the following example.

Example 1.6.1:

Find the domain and the range of the polynomial?

Solution:

To find the domain of the polynomial we need to answer the question is there a real value of x that can make the value of $P(x)$ unreal or undefined?

Assuming that x is a real number. From the properties of real numbers, x^n, x^{n-1},..., $x^2 \in \mathbb{R}$, and since $a_i \in \mathbb{R}\ \forall\ i = 0,1,2,\dots,n$, then $a_n x^n$,$a_{n-1}x^{n-1}$,..., $a_2x^2 \in \mathbb{R}$. Thus, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{R}$. We now can say that for all the real values of x, $P(x)$ is real, so the domain of the polynomial is $\mathbb{R}$, and its codomain is also $\mathbb{R}$.

The range of the polynomial cannot be determined in the general case because it will change according to its degree, and its coefficients.

Example 1.6.2:

Find the domain and the range of $f(x) = 2x^2 - 3$.

Solution:

The function f is a polynomial of the second degree so its domain is the interval $(-\infty, \infty)$.

Let us discuss the range of the function. To determine the range, we need to find all possible values of $f(x)$, which change according to the change of values of x. If x is negative then the value of $2x^2$ is positive and the lowest value for $2x^2 - 3$ occurs where $x = 0$. that is $f(0) = -3$, so the range of the function is the interval $[-3, \infty)$. We will discuss algebraic methods to find the range in advanced stage of this course. See the curve of the function in Figure (1-6)

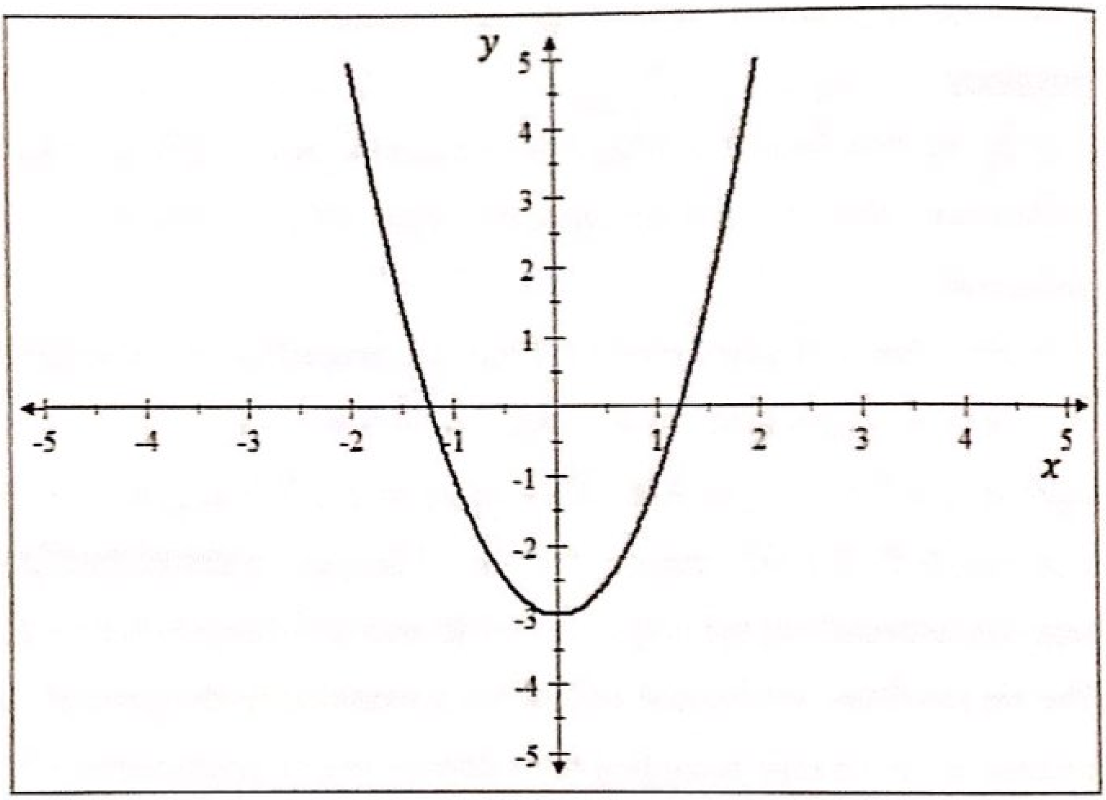


Figure (1-6)

Example 1.6.3:

Find the domain and the range of $f(x) = 3x^3 + 5x^2 - 3$.

Solution:

The function f is a polynomial of the third degree, so its domain is the interval $(-\infty, \infty)$.

We will discuss the change of the value of the function according to the change of the value of x. Where x is negative approaching $-\infty$, the value of $3x^3$ is also negative approaching $-\infty$, while the quantity $5x^2$ is a positive quantity and approaches ∞ but $3x^3$ is faster, so the quantity $3x^3 + 5x^2$ approaches $-\infty$, and so $3x^3 + 5x^2 - 3$ also approaches $-\infty$. Similarly, the quantity $3x^3 + 5x^2 - 3$ approaches ∞ when x approaches ∞, so the range of this function is the interval $(-\infty, \infty)$. See Figure (1-7)

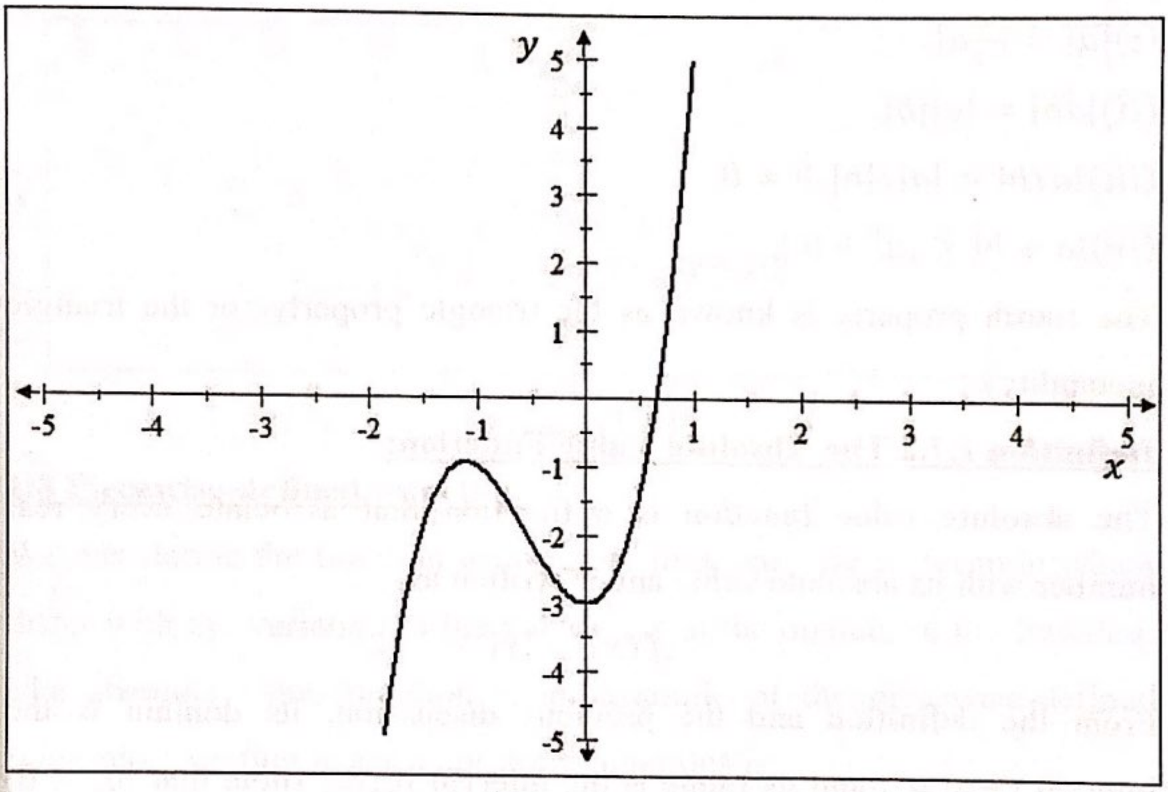


Figure (1-7)

1.7 The Absolute Value Function:

Before discussing the absolute value function, we will discuss the absolute value of a real number. The absolute value is the positive value of the number. For example, the absolute value of 3 is the same number 3, but the absolute value of -3 is the number 3. Based on this, the positive number is left as it is when finding its absolute value, but it is converted into a positive number if the original number is negative as if we are multiplying the negative number by -1 (negative sign). The absolute value of a number x will be denoted by $|x|$, and is read as "the absolute value of x".

Definition 1.7.1Properties of the Absolute Value:

If a and b are two real numbers, then

$(i)|a| = |-a|$,

$(ii)|ab| = |a||b|$,

$(iii)|a/b| = |a|/|b|, b \neq 0$,

$(iv)|a + b| \leq |a| + |b|$,

The fourth property is known as the triangle property (or the triangle inequality).

Definition 1.7.2 The Absolute Value Function:

The absolute value function is a function that associate every real number with its absolute value and is written as

$$.f(x) = |x|$$

From the definition and the previous discussion, its domain is the interval $(-\infty, \infty)$ and its range is the interval $[0, \infty)$ (note that $|0| = 0$) and it may be defined by

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Its curve is given in Figure (1-8).

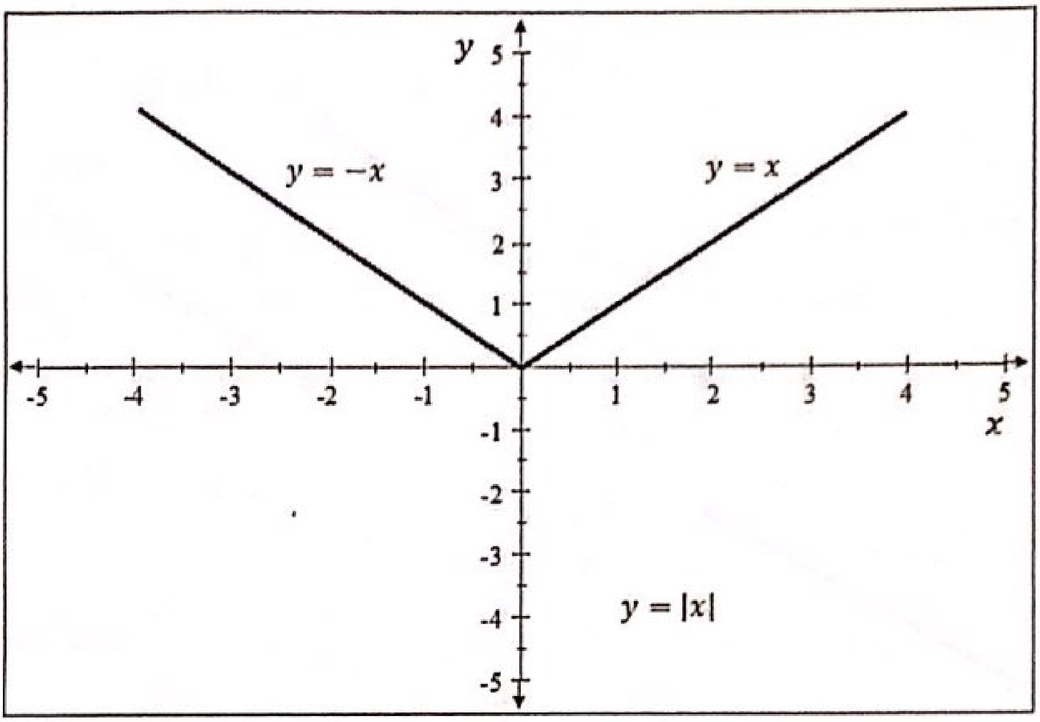


Figure (1-8)

1.8 Piecewise-defined Function:

We can define the function using more than one rule or formula which differ with the variation in the value of x in the domain of the function. The absolute value function is an example of the piecewise-defined functions. Another example of these functions is:

$$g(x) = \begin{cases} x, & x \leq -2 \\ x+2, & -2 < x < 2 \\ x-2, & x \geq 2. \end{cases}$$

The values where the function formula changes are known as the break points. In the previous example, $x = -2$ and $x = 2$ are breakpoints for the function g. The curve of the function g is given in Figure (1-9).

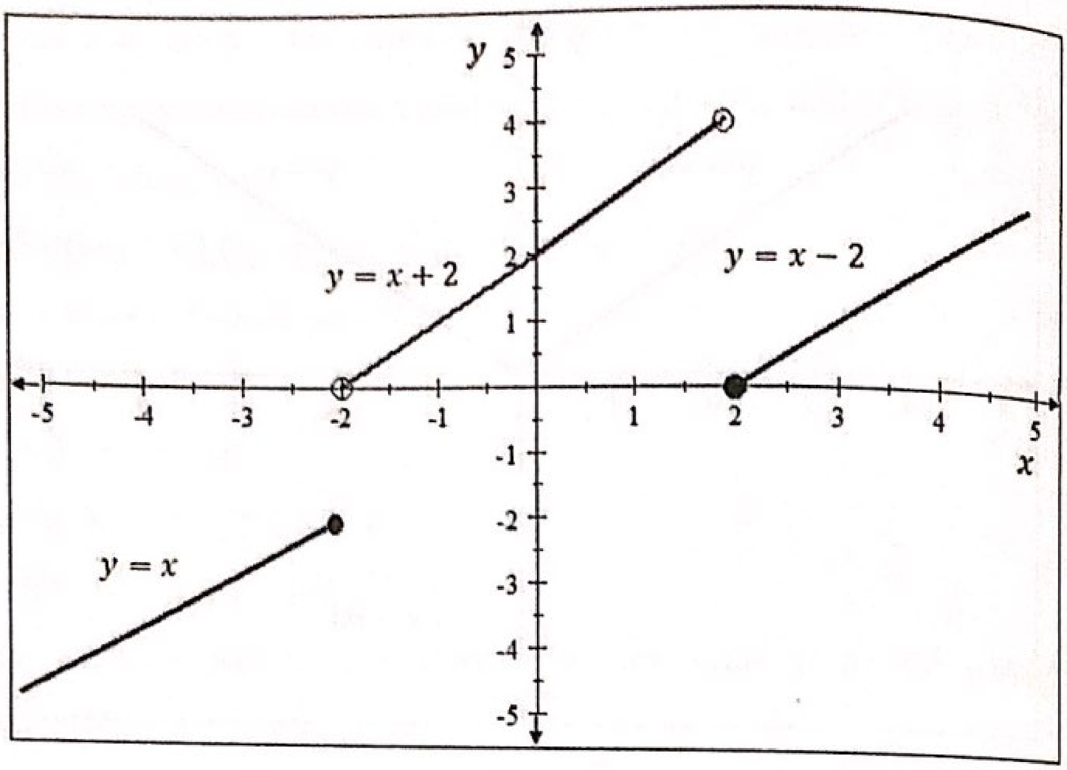


Figure (1-9)

1.9 Rational Function:

Using polynomials, we can define what is known as the rational function and is given in the following definition.

Definition 1.9.1 Rational Function:

If $P(x)$ and $Q(x)$ are two polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is called a rational function.

We note that if $Q(x) = 0$, then $f(x)$ will be undefined and so the natural domain of this function is all the values of x except the values that make $Q(x) = 0$ that is known as the set of real numbers except the zeros of the denominator i.e.,

$$D_f = \mathbb{R} - \{x: Q(x) = 0\}.$$

Example 1.9.1:

Find the domain of the function $f(x) = \frac{x+5}{x^2-4}$.

Solution:

Once we determine that the function is a rational function, the problem turns into finding the values of x that make the denominator of the function equal to zero; i.e., solving the equation:

$$x^2 - 4 = 0$$

$$\Rightarrow x^2 = 4,$$

$$\Rightarrow x = \pm 2.$$

Therefore:

$$D_f = \mathbb{R} - \{-2, 2\}.$$

We can represent the domain graphically on the real line as shown in Figure (1-10).

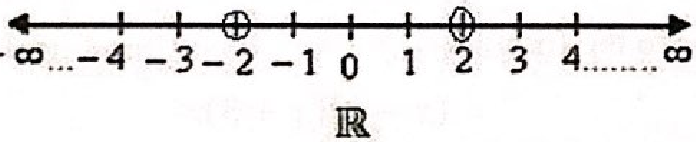


Figure (1-10)

We can also write it as a union of intervals:

$$D_f = (-\infty, -2) \cup (-2,2) \cup (2, \infty).$$

Example 1.9.2:

Find the domain of the function $g(x) = \frac{x^2-9}{x-3}$.

Solution:

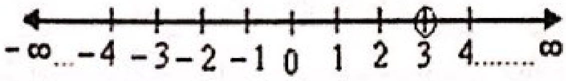
The function is a rational function, so what is needed is to find the values of x that make the denominator of the function equal to zero; i.e. solving the equation:

$$x - 3 = 0,$$

$$\Rightarrow x = 3.$$

Therefore:

$$D_g = \mathbb{R} - \{3\} = (-\infty, 3) \cup (3, \infty).$$



$\mathbb{R}$

Let us do some mathematical operations on this function. We can factor the numerator to the form:

$$(x - 3)(x + 3).$$

And then cancel $(x - 3)$ from the numerator and the denominator to get $(x + 3)$. Here a question arise which is, can we write the function in the form of $g(x) = x + 3$?

We can note that the domain of the function $x + 3$ is the set $\mathbb{R}$ while the domain of the function g is $\mathbb{R} - \{3\}$ and therefore we cannot write it exactly on this form, but we can write it in the form:

$$g(x) = x + 3, x \neq 3.$$

From the discussion in the previous example, it became clear that performing operations on the function would change its properties and an important concept is shown here which is equality of two functions that is given in the following definition.

Definition 1.9.2 Equality of Two Functions:

Two functions f and g are said to be equal and are denoted by $f = g$ if and only if $f(x) = g(x)$ for all values of x and they have the same domain $D_f = D_g$.

Example 1.9.3:

Are the two functions $f(x) = x - 2$ and $g(x) = \frac{x^2-4}{x+2}$ equal?

Solution:

We note that the quantity $\frac{x^2-4}{x+2}$ can be simplified to the form $x - 2$ but $x = -2$ does not belong to the domain of the function g, while it belongs to the domain of function f. Therefore, $D_f \neq D_g$ and hence $f \neq g$.

If we delete $x = -2$ from the domain of the function f, then we can say that the two functions are equal.

1.10 Power Function:

In this section we will define the power function and mention some of its different cases.

Definition 1.10.1 Power Function:

The function $f(x) = x^a$ is known as power function, where a is constant.

1. If a is a non-negative integer, then $f(x)$ is a polynomial with a single term.
2. If $a = \frac{1}{n}$, where $n > 0$, then $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is called a root function. When $n = 2$ it is called the square root function and its domain is $[0, \infty)$ and its range is $[0, \infty)$, see Figure (1-11). When $n = 3$ it is called the cubic root function and its domain is $(-\infty, \infty)$ and its range $(-\infty, \infty)$, see Figure (1-12).
3. If $a = -1$ then $f(x) = \frac{1}{x}$ is called the inverse function and its domain is $\mathbb{R} - \{0\}$ and its range is $\mathbb{R} - \{0\}$, see Figure (1-13).

Note that, in order to get a real value of a square root function the quantity under the root sign must be positive or zero.

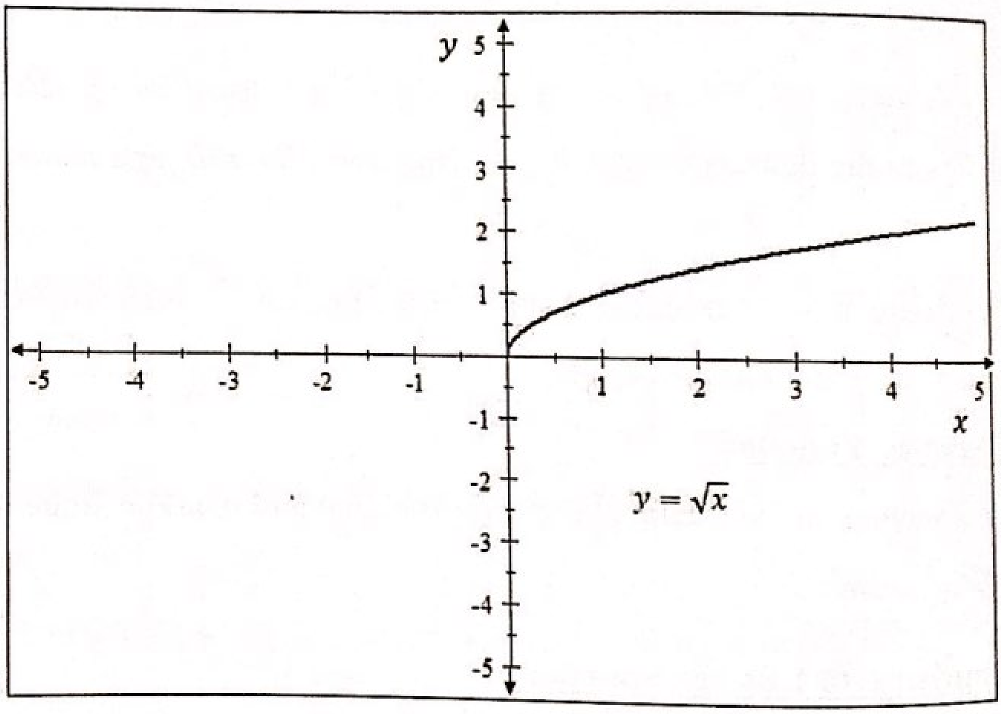


Figure (1-11)

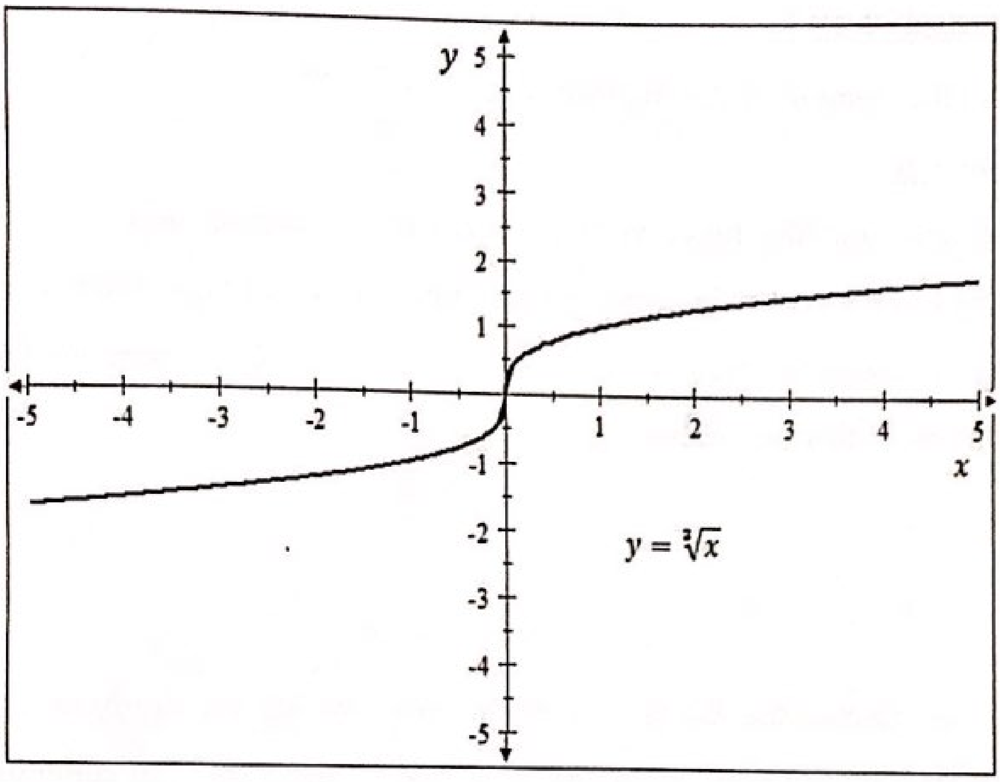


Figure (1-12)

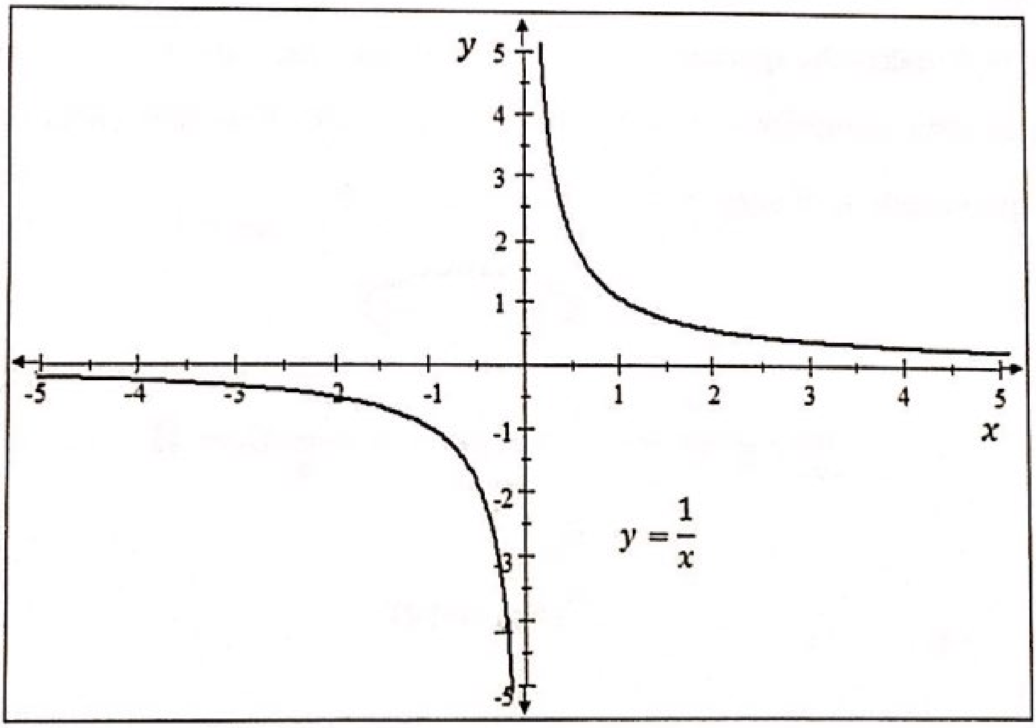


Figure (1-13)

Example 1.10.1:

Find the domain of the function $f(x) = \sqrt{x-2}$.

Solution:

We note that the function is a square root function, and therefore for $f(x)$ to be a real value, the quantity under the root sign must be greater than or equal to zero, i.e., $x - 2 \geq 0$ and the problem turns to find the solution of this inequality.

$$x - 2 \geq 0$$

$$\Rightarrow x \geq 2.$$

$$\therefore D_f = [2, \infty).$$

We can discuss the solution in another way, which is to study the sign of the quantity under the root (find what makes under the root equal to zero and then study the sign of the quantity on the right and left of this value). $x = 2$ makes the quantity $x - 2 = 0$ and hence the values to its right make the quantity$x - 2 > 0$ and so: $D_f = [2, \infty)$. Note the graphical representation of sign of $x - 2$ in Figure (1-14).

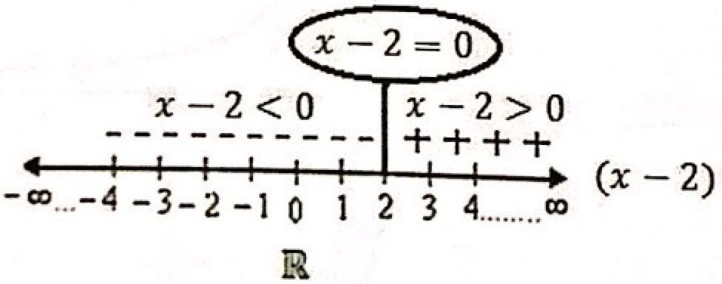


Figure (1-14)

Example 1.10.2:

Find the domain and the range of the function $g(x) = \sqrt{x^2 - 4}$

Solution:

The domain is the solution of the inequality:

$$x^2 - 4 \geq 0$$

$$\Rightarrow x^2 \geq 4$$

$$\Rightarrow |x| \geq 2$$

$$\therefore D_g = (-\infty, -2] \cup [2, \infty).$$

$|x| > a$ are the values to the right of a union of values to the left of $-a$, $(-\infty, -a) \cup (a, \infty)$ while $|x| < a$ are the values that are between $-a$ and a, i.e., the interval $(-a, a)$.

We can also find the domain by studying sign of the quantity $x^2 - 4$.

$$x^2 - 4 = 0$$

$$\Rightarrow (x-2)(x+2) = 0.$$

We will study the sign of quantity$(x + 2)$ and quantity $(x - 2)$, then the product of $(x-2)(x+2)$ as shown in figure (1-15).

From Figure (1-15), then: $D_g = (-\infty, -2] \cup [2, \infty)$.

The range of the function is $[0, \infty)$.

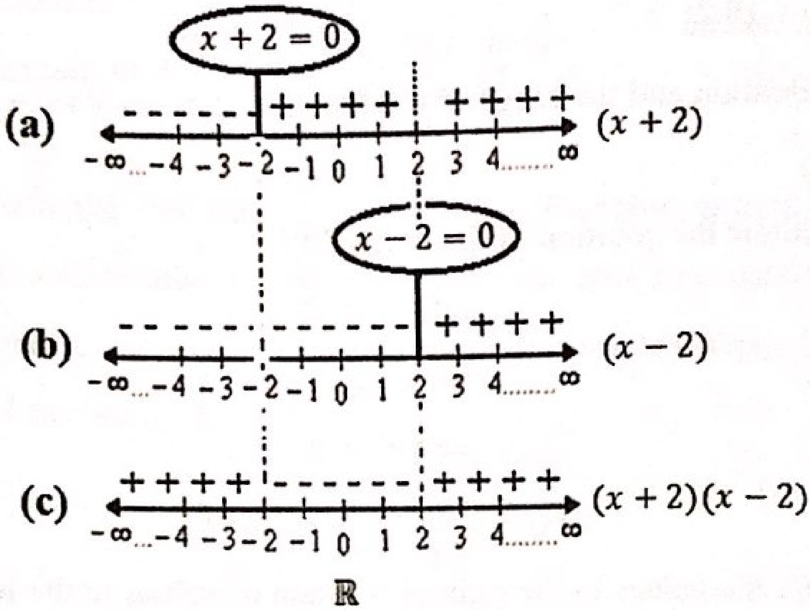


Figure (1-15)

Example 1.10.3:

Find the domain and the range of the function $h(x) = \sqrt{x^2 - 4x - 5}$.

Solution:

The domain is the solution of the inequality:

$$x^2 - 4x - 5 \geq 0$$

$$\Rightarrow (x+1)(x-5) \geq 0$$

	or	
$\Rightarrow x+1 \geq 0$ and $x-5 \geq 0$		$\Rightarrow x+1 \leq 0$ and $x-5 \leq 0$
$\Rightarrow x \geq -1$ and $x \geq 5$		$\Rightarrow x \leq -1$ and $x \leq 5$
$\Rightarrow x \geq 5$		$\Rightarrow x \leq -1$
$[5, \infty)$		$(-\infty, -1]$

$$D_h = (-\infty, -1] \cup [5, \infty).$$

We can solve this example by studying the sign of the magnitude $(x+1)(x-5)$ as shown in Figure (1-16).

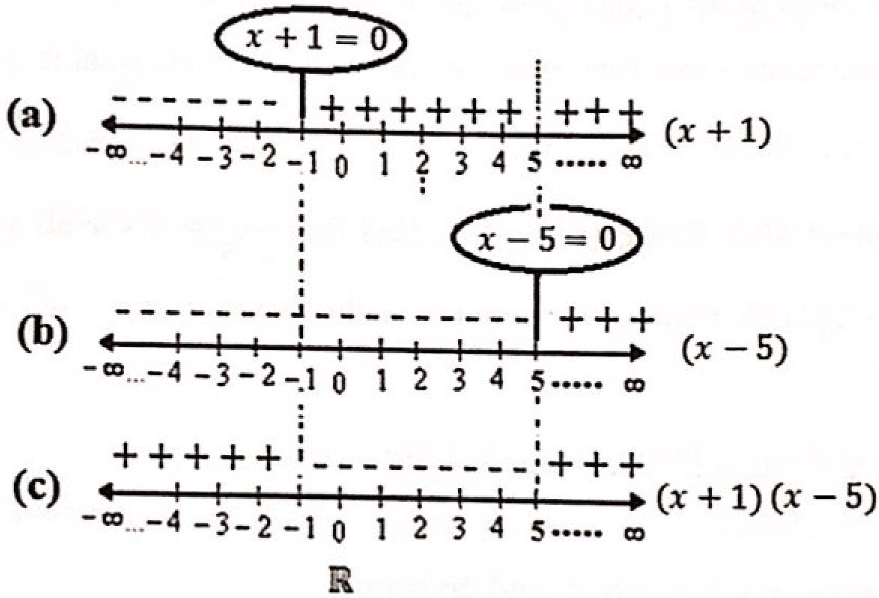


Figure (1-16)

The range of the function is $[0, \infty)$.

Example 1.10.4:

Find the domain of the function $h(x) = \sqrt{x^2 - 2x + 5}$

Solution:

The domain is the solution of the inequality $x^2 - 2x + 5 \geq 0$.

Note that we cannot factor this quantity, and therefore the method in the previous examples is not suitable for solving this example (has no real roots). It is positive for all real values x, so the domain of the function is $\mathbb{R}$.

Example 1.10.5:

Find the domain of the function $f(x) = \sqrt{\frac{x+1}{x-5}}$.

Solution:

The function here is a rational under the square root, and therefore its natural range is all x that make the rational function greater than or equal zero, and at the same time make its denominator not equal to zero.

Hence it is the solution of the inequality $\frac{x+1}{x-5} \geq 0$ and $x \neq 5$. Referring to Example 1.10.3 figure (1-16), we find that $\frac{x+1}{x-5} \geq 0$ for all values x in $(-\infty, -1] \cup (5, \infty)$, so the domain of the function is $D_f = (-\infty, -1] \cup (5, \infty)$.

1.11 Arithmetic Operations on Function:

Using two functions, we can create new functions by addition, subtraction, multiplication, and division.

Definition 1.11.1:

If f and g are functions with domains D_f and D_g respectively, then

1. Their summation is denoted by $f + g$, where $(f + g)(x) = f(x) + g(x)$ and its domain is $D_f \cap D_g$.
2. Their subtraction is denoted by the $f - g$, where $(f - g)(x) = f(x) - g(x)$ and its domain is $D_f \cap D_g$.
3. Their multiplication is denoted by fg, where $(fg)(x) = f(x)g(x)$ and its domain is $D_f \cap D_g$.
4. Their division is denoted by f/g, where $(f/g)(x) = f(x)/g(x)$ and its domain is $(D_f \cap D_g) - \{x: g(x) = 0\}$.

Definition 1.11.2 Algebraic Functions:

The functions produced from finite algebraic operations on polynomials (addition - subtraction - multiplication - division - root) are called algebraic functions.

Example 1.11.1:

Let $f(x) = \sqrt{x-3}$ and $g(x) = x + 2$. Find the following functions and their domains:

$f + g,\ f - g,\ fg,\ f/g,\ 3f.$

Solution:

$(i)(f + g)(x) = f(x) + g(x) = \sqrt{x-3} + x + 2.$

$\because D_f = [3, \infty), D_g = (-\infty, \infty)$

$\therefore D_{f+g} = [3, \infty) \cap (-\infty, \infty) = [3, \infty).$

$(ii)(f - g)(x) = f(x) - g(x) = \sqrt{x-3} - x - 2.$

$\therefore D_{f-g} = [3, \infty).$

$(iii)(fg)(x) = f(x)g(x) = (x + 2)\sqrt{x-3}.$

$\therefore D_{fg} = [3, \infty).$

$(iv)(f/g)(x) = f(x)/g(x) = \sqrt{x-3}/(x + 2).$

$\because D_{f/g} = (D_f \cap D_g) - \{x: g(x) = 0\}$

$\therefore D_{f/g} = [3, \infty).$

$(v)(3f)(x) = 3f(x) = 3(x + 2).$

$$\therefore D_{3f} = (-\infty, \infty).$$

Example 1.11.2:

If $(x) = \sqrt{x}$, $g(x) = \sqrt{x}$ and $h(x) = x$, then can we say that $h = fg$?

Solution:

$\because (fg)(x) = \sqrt{x}\sqrt{x} = x$

$\therefore h(x) = (fg)(x)$.

$\because D_{fg} = [0,\infty) \cap [0,\infty) = [0,\infty) \neq D_h = (-\infty,\infty)$.

Therefore, we can not say that $h = fg$.

However, we can say that $h = fg$ in the interval $[0,\infty)$.

Example 1.11.3:

If $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{x+3}$, find the following functions and their domain:

$f + g,\ f - g,\ fg,\ g/f,\ 5f.$

Solution:

$(i)(f+g)(x) = f(x) + g(x) = \sqrt{x-3} + \sqrt{x+3}$.

$\because D_f = [3,\infty), D_g = [-3,\infty)$

$\therefore D_{f+g} = [3,\infty) \cap [-3,\infty) = [3,\infty)$.

$(ii)(f-g)(x) = f(x) - g(x) = \sqrt{x-3} - \sqrt{x+3}$.

$\therefore D_{f-g} = [3,\infty)$.

$(iii)(fg)(x) = f(x)g(x) = \sqrt{x-3}\sqrt{x+3}$.

$\therefore D_{fg} = [3,\infty)$.

$(iv)(g/f)(x) = g(x)/f(x) = \sqrt{x+3}/\sqrt{x-3}$.

$\because D_{g/f} = (D_f \cap D_g) - \{x: f(x) = 0\}$

$\therefore D_{g/f} = (3,\infty)$.

$(v)(5f)(x) = 5f(x) = 5\sqrt{x-3}.$

$$\therefore D_{5f} = [3, \infty).$$

Definition 1.11.3 Composition of Functions

If f and g are functions with domains D_f and D_g respectively, then the composition of f with g which is denoted by $f \circ g$ is defined as $(f \circ g)(x) = f(g(x))$ and its domain is all the values x in the domain of g such that $g(x)$ in the domain of f. (If $x \in D_g$ and $g(x) \in D_f$ then $x \in D_{f \circ g}$, i.e., $D_{f \circ g} = \{x : x \in D_g \, and \; g(x) \in D_f\}$), see figure (1-17).

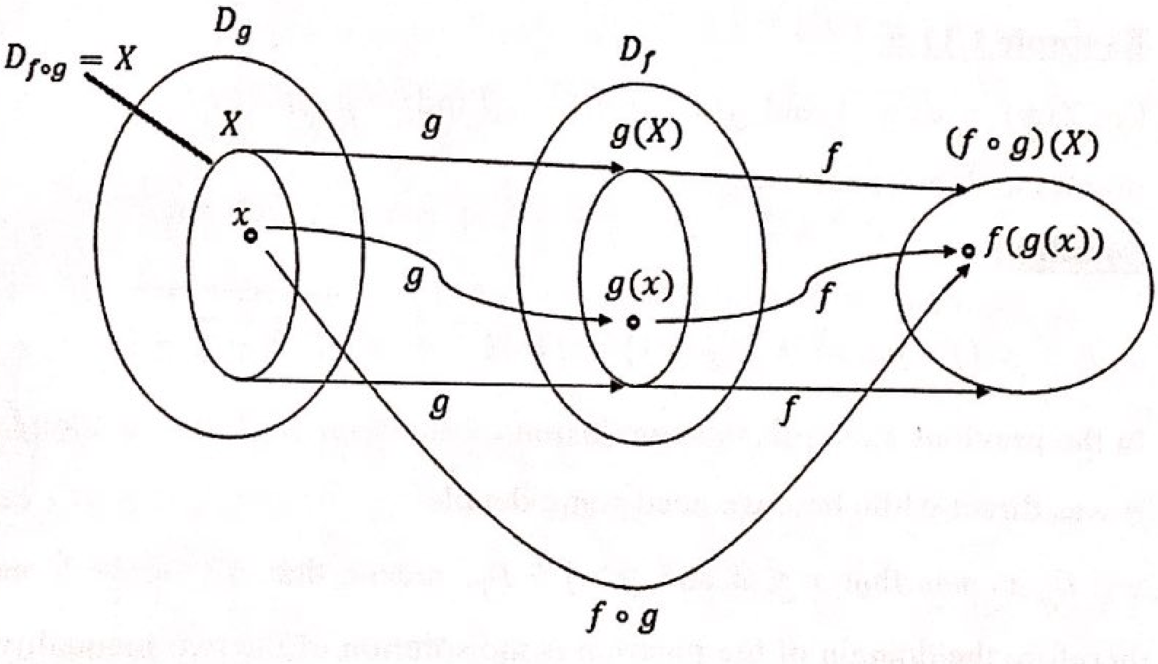


Figure (1-17)

Example 1.11.4:

Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find a formula for the functions $f \circ g$ and $g \circ f$, and the domain for each. If $h(x) = x$, can we say that $f \circ g = h$?

Solution:

Where,

$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x.$

$\therefore D_{f \circ g} = [0, \infty).$

Thus, we cannot say that $f \circ g = h$.

Whereas,

$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|.$

$$\therefore D_{g \circ f} = (-\infty, \infty).$$

Example 1.11.5:

Let $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{3-x}$.Find$f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and the domain for each.

Solution:

$$\because (f \circ g)(x) = f(g(x)) = f(\sqrt{3-x}) = \sqrt{\sqrt{3-x} - 1}.$$

In the previous example, the conclusion of the domain of the function$f \circ g$ was direct while here we need some details.

$x \in D_g$ means that $x \leq 3$ and $g(x) \in D_f$, means that $\sqrt{3-x} \geq 1$ and therefore the domain of the function is the solution of the two inequality:

$\sqrt{3-x} \geq 1$ and $x \leq 3$

$\Longrightarrow 2 - x \geq 0$ and $x \leq 3$

$\Longrightarrow x \leq 2$

$\therefore D_{f \circ g} = (-\infty, 2].$

$$\because (g \circ f)(x) = g(f(x)) = g(\sqrt{x-1}) = \sqrt{3-\sqrt{x-1}}.$$

$x \in D_f$ means that $x \geq 1$ and $f(x) \in D_g$, means that $\sqrt{x-1} \leq 3$ and hence, the domain of the function is the solution of the two inequalities

$x \geq 1$ and $\sqrt{x-1} \leq 3$

$\Rightarrow x \geq 1$ and $x \leq 10$

$\Rightarrow 1 \leq x \leq 10$

$\therefore D_{g \circ f} = [1,10]$.

$$\because (f \circ f)(x) = f(f(x)) = f(\sqrt{x-1}) = \sqrt{\sqrt{x-1}-1}.$$

$\therefore D_{f \circ f} = [2, \infty)$.

$$\because (g \circ g)(x) = g(g(x)) = g(\sqrt{3-x}) = \sqrt{3-\sqrt{3-x}}.$$

The domain of the function is the solution of the two inequalities:

$\sqrt{3-x} \leq 3$ and $x \leq 3$

$\Rightarrow 3-x \leq 9$ and $x \leq 3$

$\Rightarrow x \geq -6$ and $x \leq 3$

$\Rightarrow -6 \leq x \leq 3$

$$\therefore D_{g \circ g} = [-6,3].$$

Example 1.11.6:

Let $h(x) = \frac{x^2}{x^2+4}$. Write h as a composition of two functions $f \circ g$.

Solution:

Let $f(x) = \frac{x}{x+4}$ and $g(x) = x^2$. Then

--

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \frac{x^2}{x^2 + 4}.$$

In similar cases, we can always do this by finding the function f that is the same as h, but only for the value x (replacing all the similar quantities with x) and then finding the function g that represents this quantity.

In this example, we got the function f by replacing x^2 in the function h with x and making g the quantity x^2.

Example 1.11.7:

Let $h(x) = \sqrt{4 - 3x}$. Write h as a composition of two functions $f \circ g$.

Solution:

Let $f(x) = \sqrt{x}$ and $g(x) = 4 - 3x$ then

$(f \circ g)(x) = f(g(x)) = f(4 - 3x) = \sqrt{4 - 3x}.$

The constant function is a special case of functions, i.e., when $f(x) = c$. So the addition, multiplication and division of a function with a constant are special cases of the arithmetic operations on functions. What will be the geometric effect of these operations on the curve of the function?

Definition 1.11.4 Translation:

Assuming the function $y = f(x)$ and the positive constant c, when adding the constant to the independent variable x, then the curve of $f(x + c)$ is the curve of $f(x)$ shifted to the left by c units, while when subtracting the constant from the independent variable, the curve of the function is the curve of the function $f(x)$ shifted to the right by c units.

When adding the constant to the dependent variable, the curve of the function $f(x) + c$ is the curve of the function $f(x)$ shifted up by $c$units.

When subtracting the constant from the dependent variable, the function curve is the curve of the function$f(x)$ shifted down by c units.

Example 1.11.8:

Draw the graph of $h(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 2$.

Solution:

The function $h(x)$ can be written as $h(x) = (x + 1)^2$, thus the curve of this function is the curve of the function $f(x) = x^2$, shifted left by one unit.

The function $g(x)$ is the function $f(x)$ plus the constant 2, so its curve is the curve of the function $f(x)$, shifted up by 2 units. As shown in Figure (1-18).

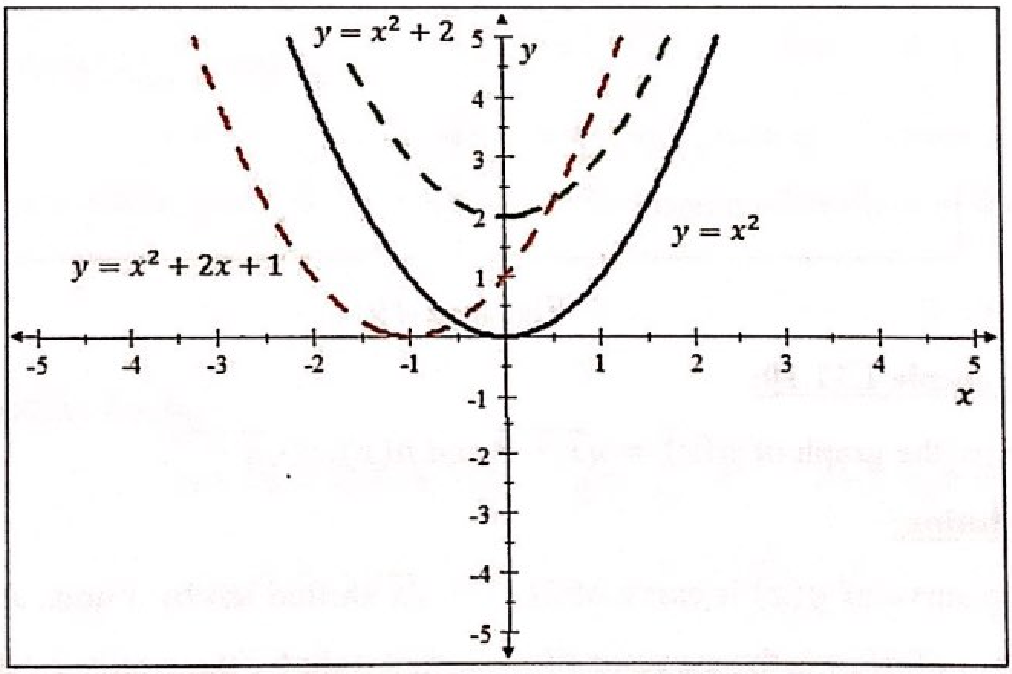


Figure (1-18)

Example 1.11.9:

Draw the graph of $g(x) = x^2 - 4x + 3$

Solution:

The function $g(x)$ can be written as $g(x) = (x-2)^2 - 1$, thus the curve of this function is the curve of the function $f(x) = x^2$, shifted to right by two units and shifted down by one unit. As shown in Figure (1-19).

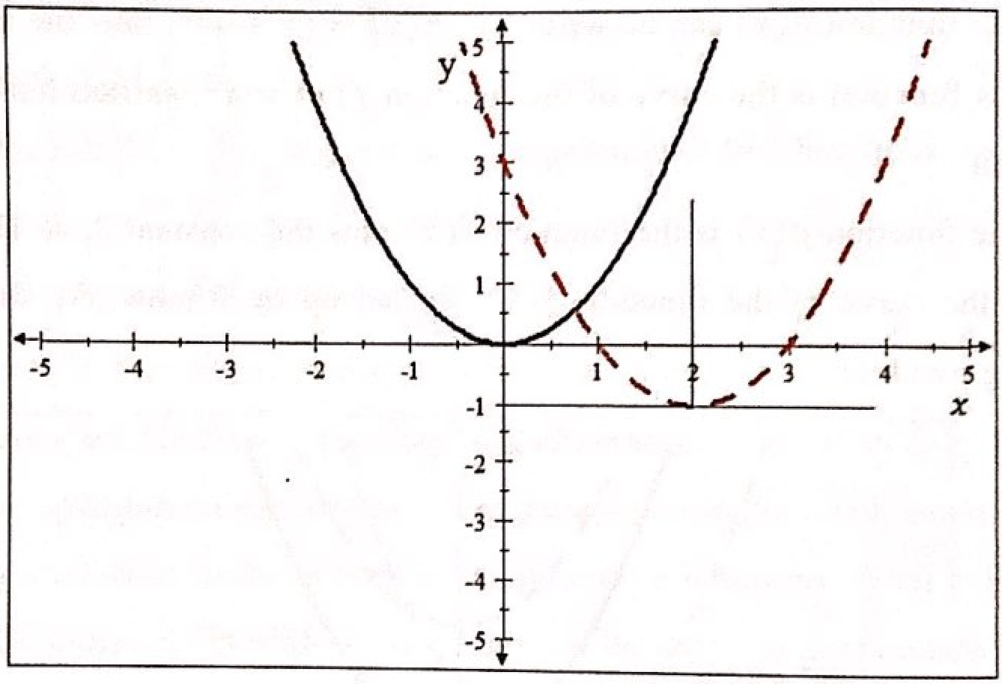


Figure (1-19)

Example 1.11.10:

Draw the graph of $g(x) = \sqrt{x+3}$ and $h(x) = \sqrt{x-3}$.

Solution:

The curve of $g(x)$ is curve of $f(x) = \sqrt{x}$ shifted left by 3 units and the curve of $h(x)$ is the curve of $f(x)$ shifted right by three units as shown in the figure (1-20).

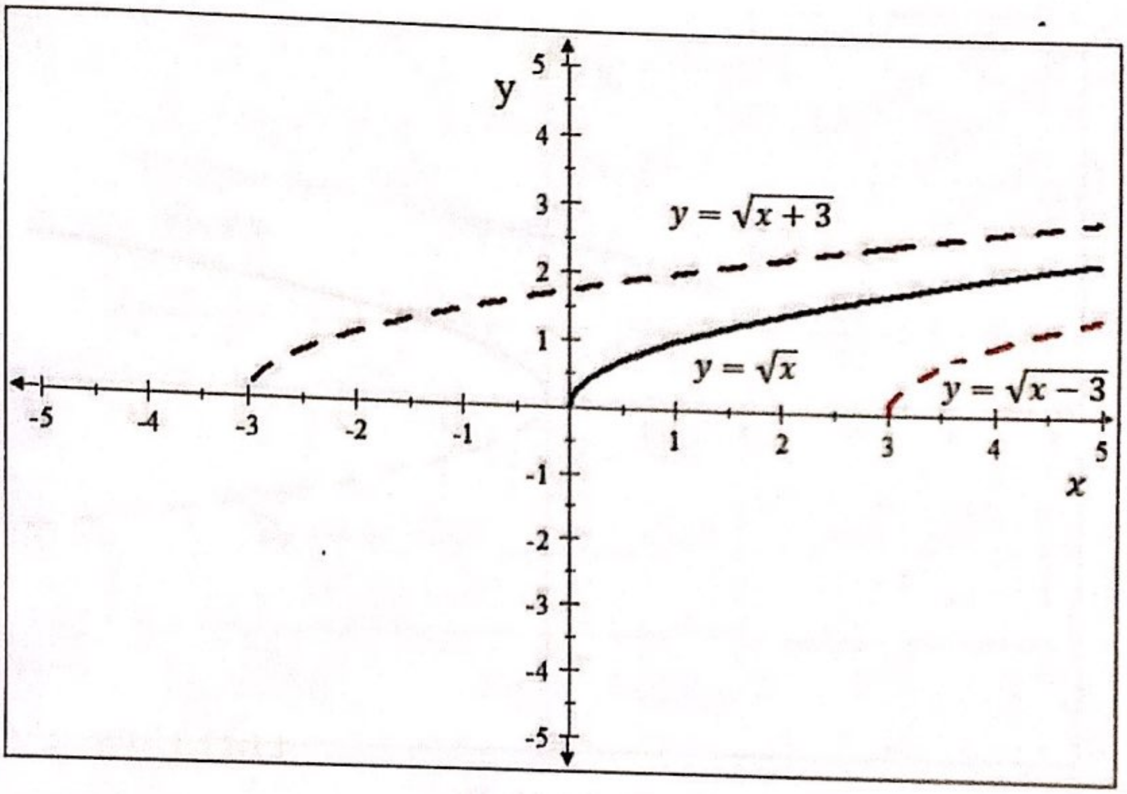


Figure (1-20)

Definition 1.11.5 Reflection:

Let $y = f(x)$, then $f(-x)$ is the reflection of the graph of $f(x)$ about the y axis, while $-f(x)$ is the reflection of the graph of $f(x)$about the x axis.

Example 1.11.11:

Figure (1-21) shows the graph of $f(x) = \sqrt{x}$, $f(-x) = \sqrt{-x}$ and $-f(x) = -\sqrt{x}$.

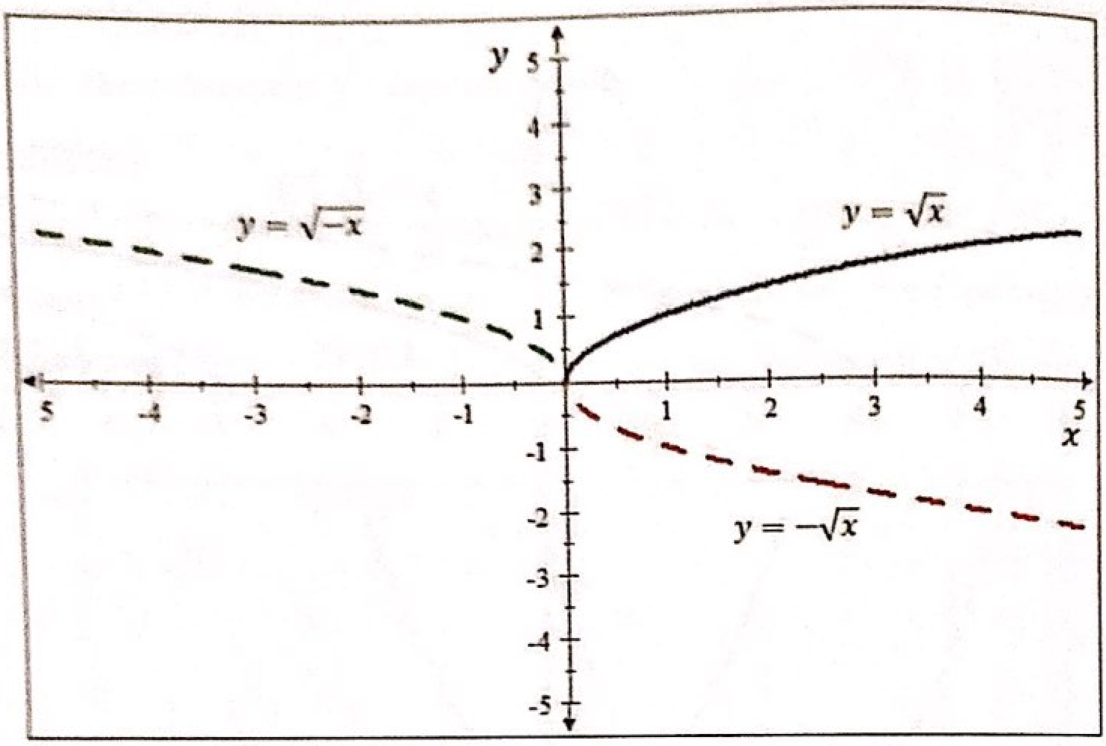


Figure (1-21)

Example 1.11.12:

Draw the graph of $y=\sqrt{2-x}$.

Solution:

The graph is the reflection of the graph of $f(x)=\sqrt{x}$ about the y axis, shifted right by 2 units. As shown in Figure (1-22).

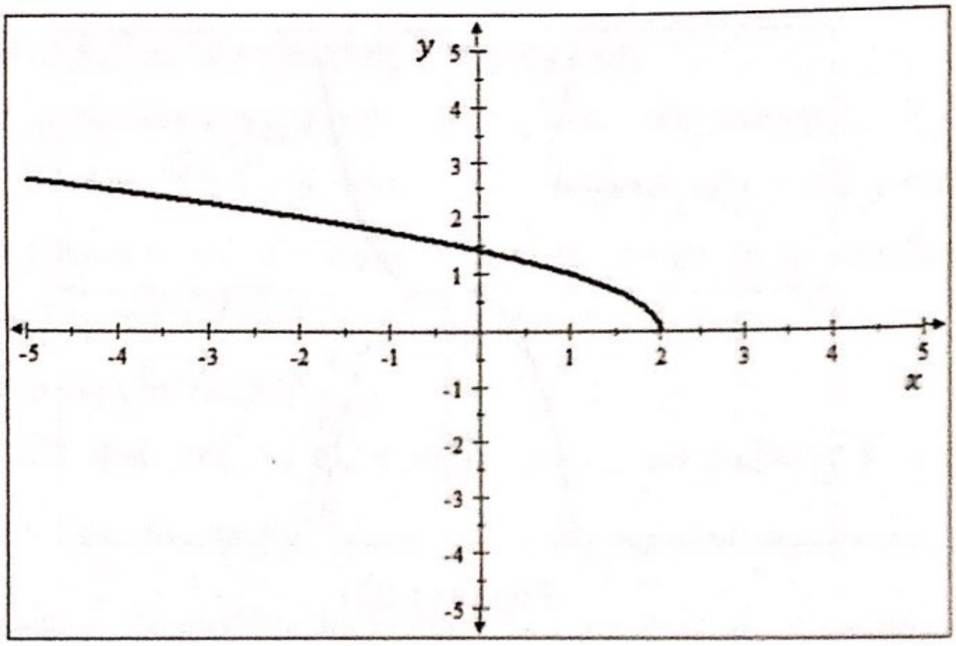


Figure (1-22)

Example 1.11.13:

Draw the graph of $y = 2 - x^3$.

Solution:

The graph is the reflection of the graph of $f(x) = x^3$ about the x axis, shifted up by 2 units. As shown in Figure (1-23).

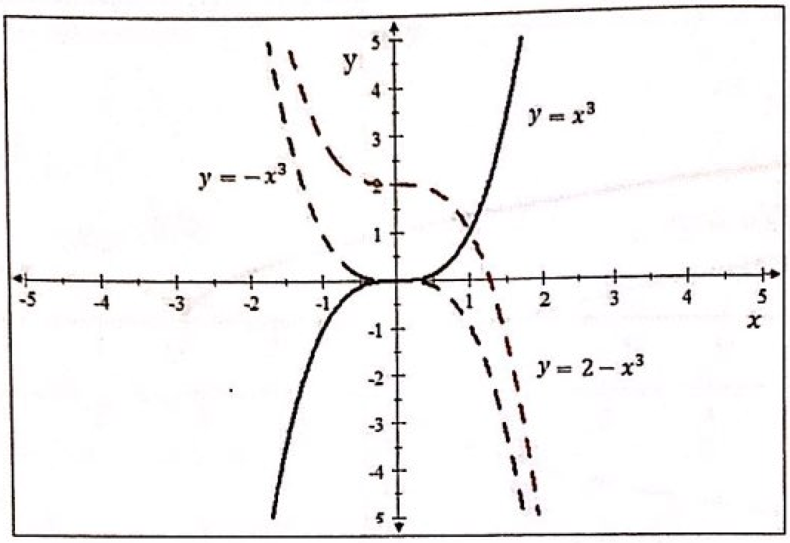


Figure (1-23)

Example 1.11.14:

Draw the graph of $y = 4 - |x - 2|$.

Solution:

The graph is the reflection of the graph of $f(x) = |x|$ about the x axis, shifted right by 2 units and followed by shifting to up by 4 units. As shown in Figure (1-24).

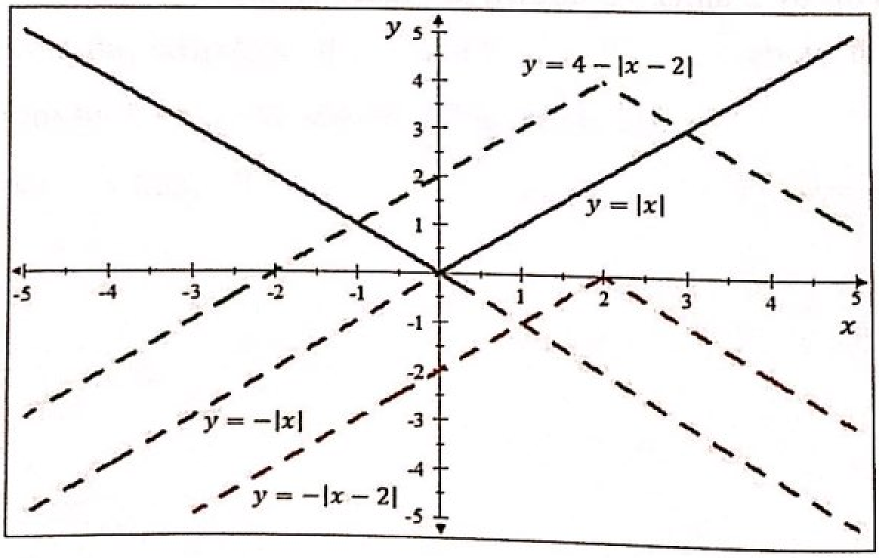


Figure (1-24)

Definition 1.11.6 Stretches and Compression

Assuming the function $y = f(x)$ and positive real constant c.

1. If $c > 1$ then $cf(x)$ is a stretching of the graph of f in the y-direction with the factor c, i.e., for each point of the graph, its y-coordinates is multiplied by the factor c and $f(cx)$ is the compression of the graph in the x-direction by factor c.

2. If $c < 1$ then $cf(x)$is the compression of the graph of f in the y-direction with the factor $\frac{1}{c}$, i.e., for each point of the graph, its y-coordinates is divided by the factor $\frac{1}{c}$ and $f(cx)$ is the stretching of the curve in the x-direction by factor $\frac{1}{c}$.

The definition is clear in the case of stretching and compression in the y-direction, while confusion may occur in the case of the$x$-direction, so we will discuss the definition in some detail. Let $c = 2$, then at $x = 4$ the value of $f(x)$ is $f(4)$ whereas $f(cx)$ has the same value at $x = 2$. Then we can say that the graph of $f(cx)$ will be closer to the y axis than that of $f(x)$, i.e., compressed. Let $c = \frac{1}{2}$, then at $x = 4$ the value of $f(x)$ is $f(4)$ whereas $f(cx)$ has the same value at $x = 8$. Then we can say that the graph of $f(cx)$ will be more far from the y axis than that of $f(x)$i.e., stretched.

Example 1.11.15:

The graph of $f(x) = 2sinx$ is the stretch of the graph of $y = sinx$by factor 2 in the y-direction i.e. the points of the graph of $f(x)$ are $(x, 2y)$.

While the graph of $g(x) = \frac{1}{2}sinx$ is the compression of the graph of $y = sinx$ by factor 2 in the y-direction i.e., the points of the graph of $f(x)$ are $(x, \frac{1}{2}y)$.

Note that if an arrow goes vertically from a point on the ox axis in the direction of y, it intersects $g(x) = \frac{1}{2}sinx$, then $y = sinx$ and then $f(x) = 2sinx$. (Stretching in the y-direction). See figure (1-25).

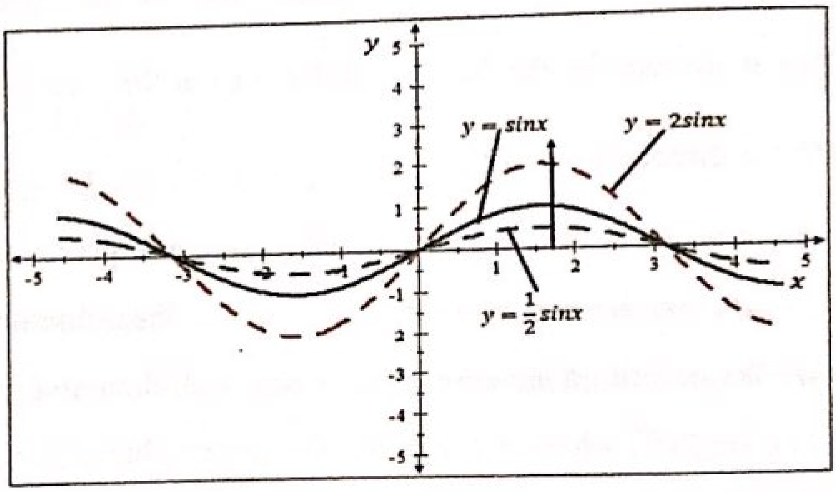


Figure (1-25)

The graph of $f(x) = sin2x$ is the compression of the graph of $y = sinx$ by the factor 2 in the x-direction i.e. the points of the graph of the $f(x)$ are $(2x, y)$.

While the graph of $g(x) = sin\frac{1}{2}x$ is a stretch of the graph of $y = sinx$ by the factor $c = 2$ in the x-direction i.e., the points of the curve of $f(x)$ are $(\frac{1}{2}x, y)$.

Note that if an arrow goes vertically from a point on the y axis in the direction of x it intersects $f(x) = sin2x$ then $y = sinx$ and then $g(x) = sin\frac{1}{2}x$. (Stretching in the x-direction). See figure (1-26).

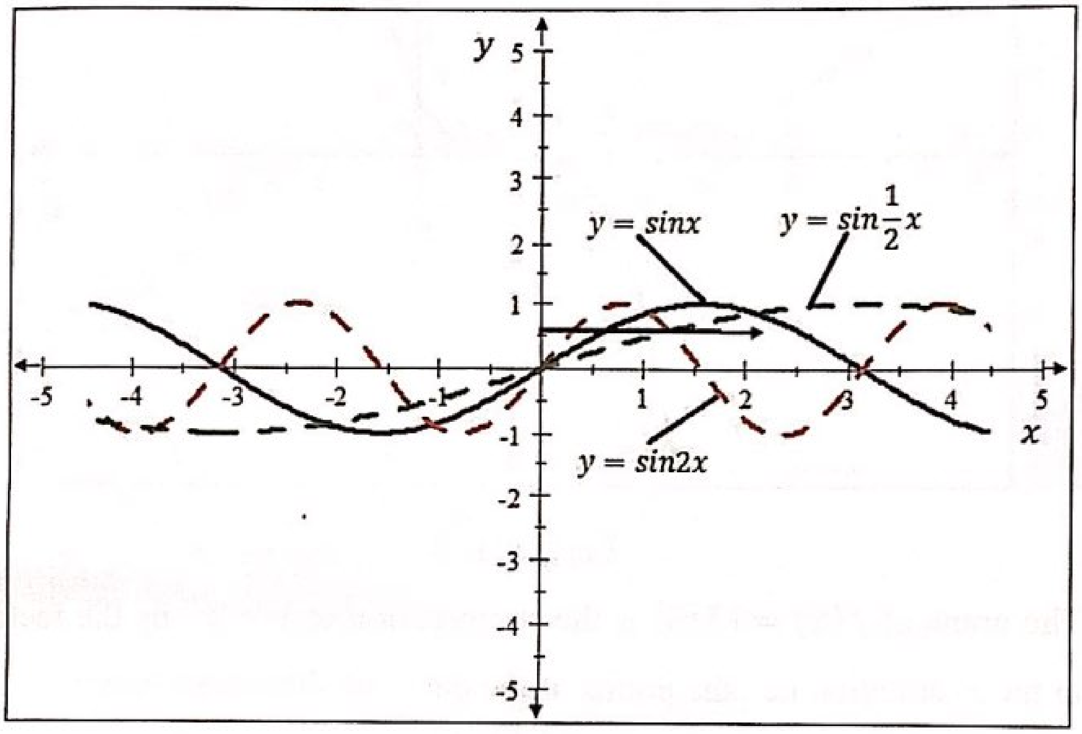


Figure (1-26)

Example 1.11.16:

The graph of $f(x) = 3x^3$ is the stretch of $y = x^3$by factor 3 in the y-direction i.e. the points of the curve of $f(x)$ are $(x, 3y)$.

While the graph of $g(x) = \frac{1}{3}x^3$ is the compression of $y = x^3$by the factor $c = 3$ in the y-direction i.e., the points of the curve of $f(x)$ are $(x, \frac{1}{3}y)$. See figure (1-27).

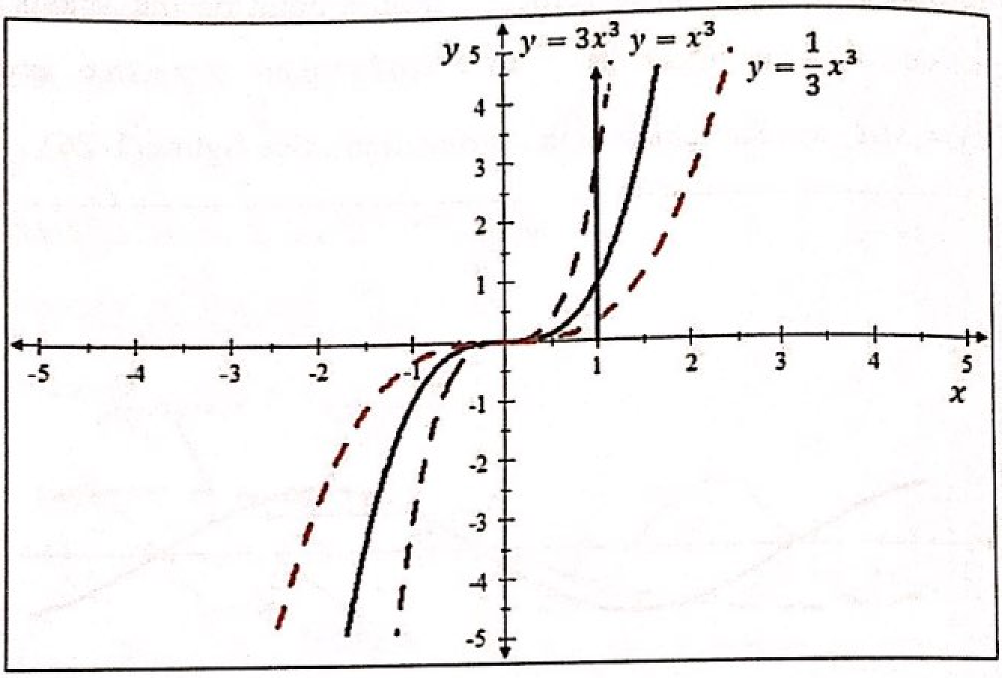


Figure (1-27)

The graph of $f(x) = (3x)^3$ is the compression of $y = x^3$ by the factor 3 in the x-direction i.e., the points of the curve of $f(x)$ are $(3x, y)$.

While the graph of $g(x) = (\frac{1}{3}x)^3$ is the stretch of $y = x^3$ by the factor 3 in the x-direction i.e., the points of the curve of $f(x)$are $(\frac{1}{3}x, y)$. See figure (1-28).

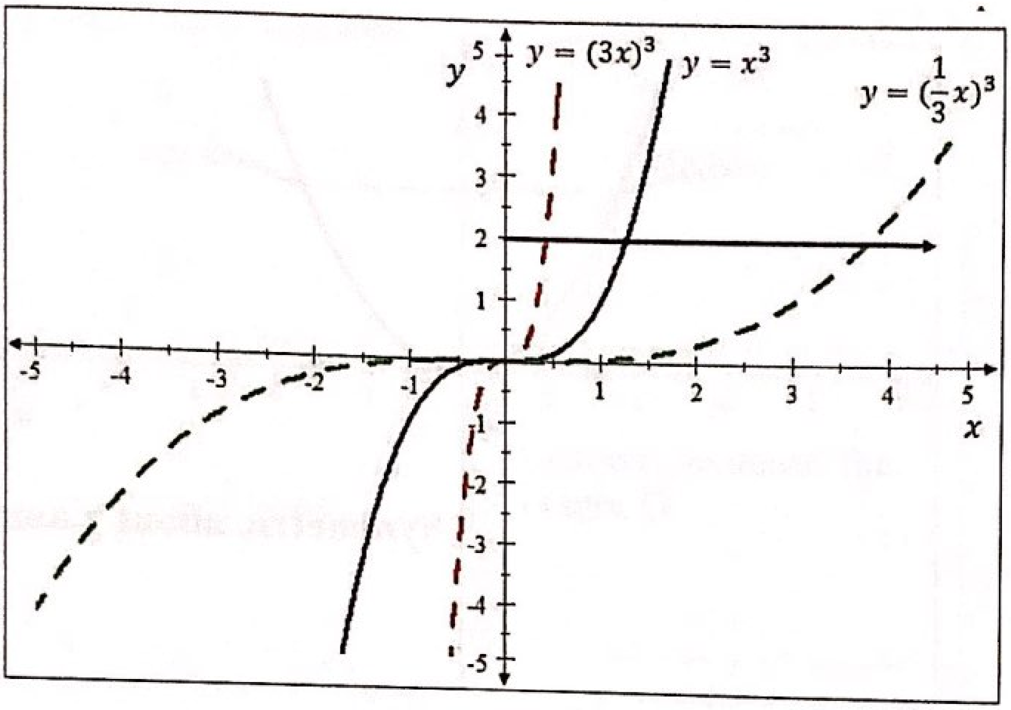


Figure (1-28)

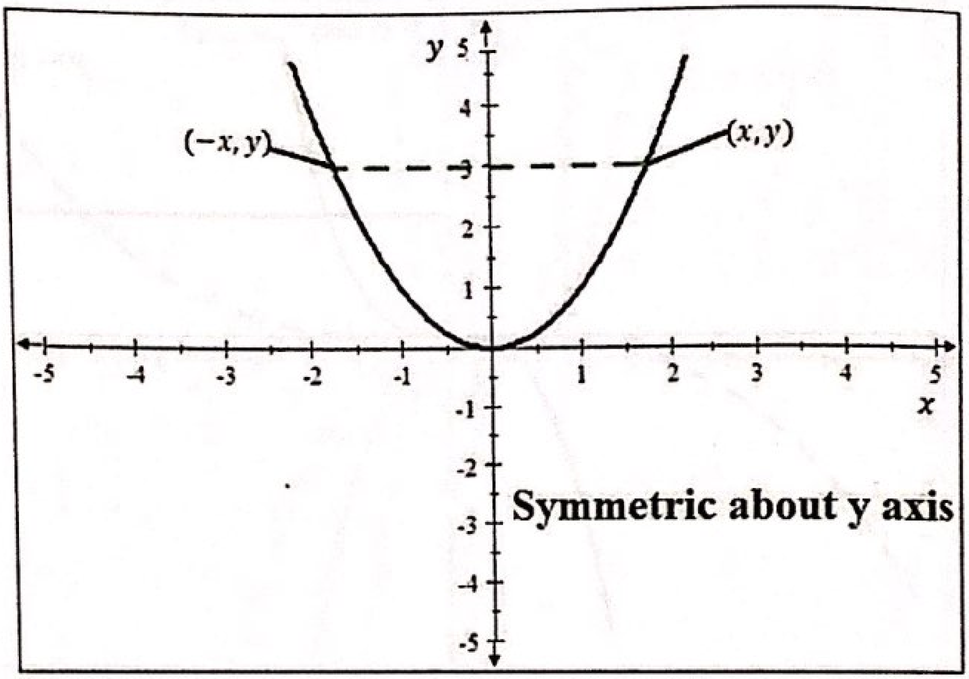
Definition 1.11.7 Symmetry:

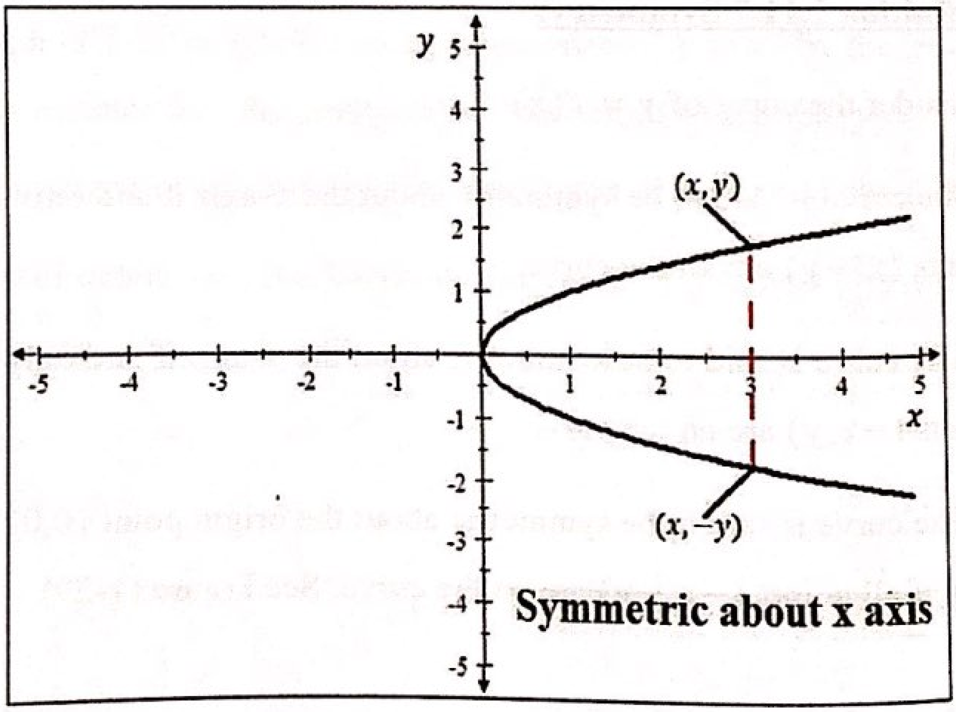
Consider the curve of $y = f(x)$.

1.The curve is said to be symmetric about the x-axis if and only if all points $(x, -y)$ are on the curve.

2. The curve is said to be symmetric about the y-axis if and only if all points $(-x, y)$ are on the curve.

3. The curve is said to be symmetric about the origin point $(0,0)$ if and only if all points $(-x, -y)$ are on the curve. See Figure (1-29).


y
x
(−x, y)
(x, y)
Symmetric about y axis


y
x
(x, y)
(x, −y)
Symmetric about x axis

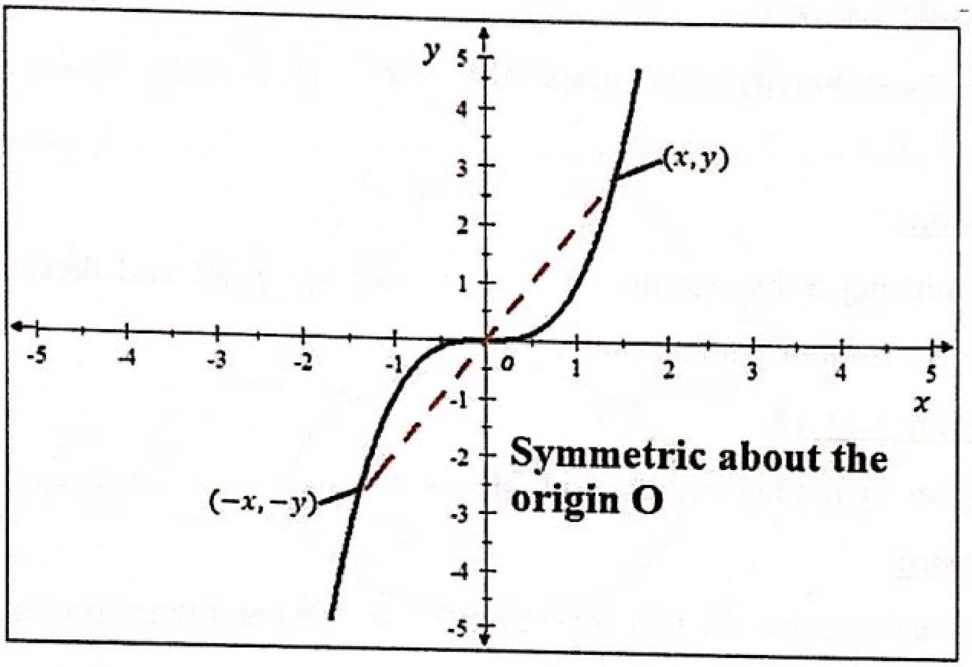


Figure (1-29)

Theory 1.11.1 Symmetry Test:

Consider the curve of $y = f(x)$.

1. The curve is symmetric about the y-axis, if and only if by replacing x by $-x$, we get the same curve, i.e., $f(x) = f(-x)$.
2. The curve is symmetric about the x-axis, if and only if by replacing y by $-y$, we get the same curve, i.e., $y = \pm f(x)$.
3. The curve is symmetric about the origin point $(0,0)$, if and only if by replacing x by $-x$ and y by $-y$, we get the same curve i.e., $f(-x) = -f(x)$.

Example 1.11.17:

Study the symmetry of the graph of $y = x^3$.

Solution:

By replacing x by $-x$ then $f(-x) = -x^3 = -f(x)$ and therefore the curve is symmetric about the origin.

Example 1.11.18:

Study the symmetry of the graph of $y = x^2$.

Solution:

By replacing x by $-x$ then $f(-x) = x^2 = f(x)$ and therefore the curve is symmetric about the y-axis.

Example 1.11.19:

Study the symmetry of the graph of $y^2 = x$.

Solution:

By replacing y by $-y$ then $(-y)^2 = x$ and then we get the same equation and hence the curve is symmetric about the x-axis or from the equation directly then, $y = \pm\sqrt{x}$.

Example 1.11.20:

Study the symmetry of the graph of $x^2 + y^2 = 4$.

Solution:

We can write the equation in the form $y = \pm\sqrt{4 - x^2}$ and therefore the curve is symmetric about the x-axis. Since$f(x) = f(-x)$, then it is symmetric about the y-axis. Since $f(-x) = -f(x)$, then it is symmetric about the origin. (It is an equation of a circle with center at the origin and radius 2) (see Figure 1-30).

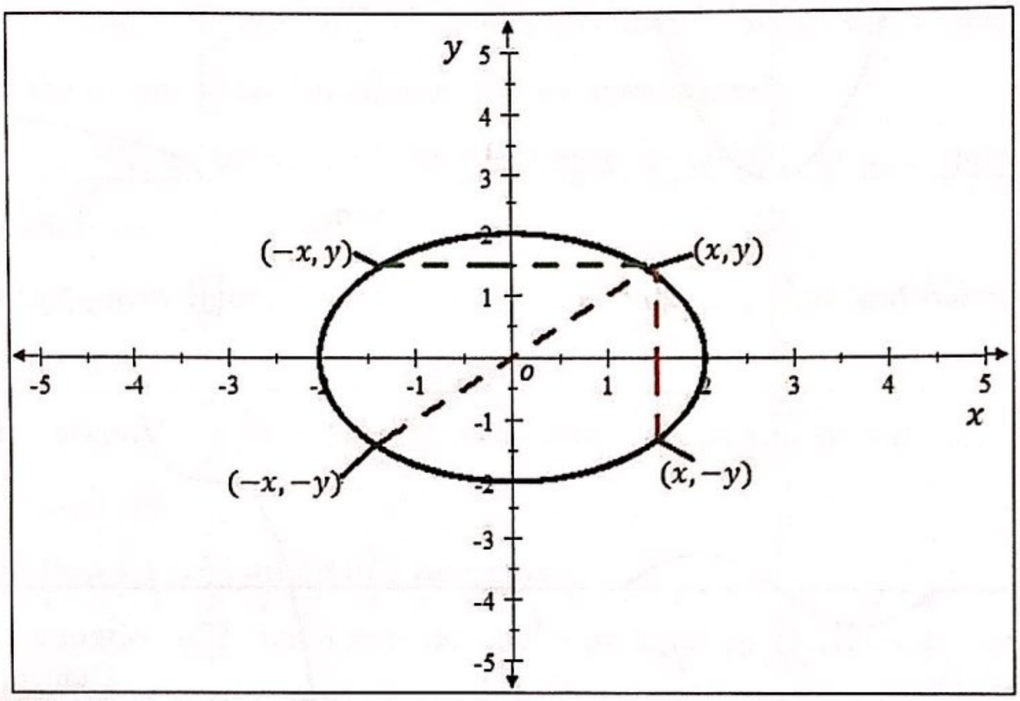


Figure (1-30)

Definition 1.11.8 Even and Odd Functions:

Assume the function $y = f(x)$.

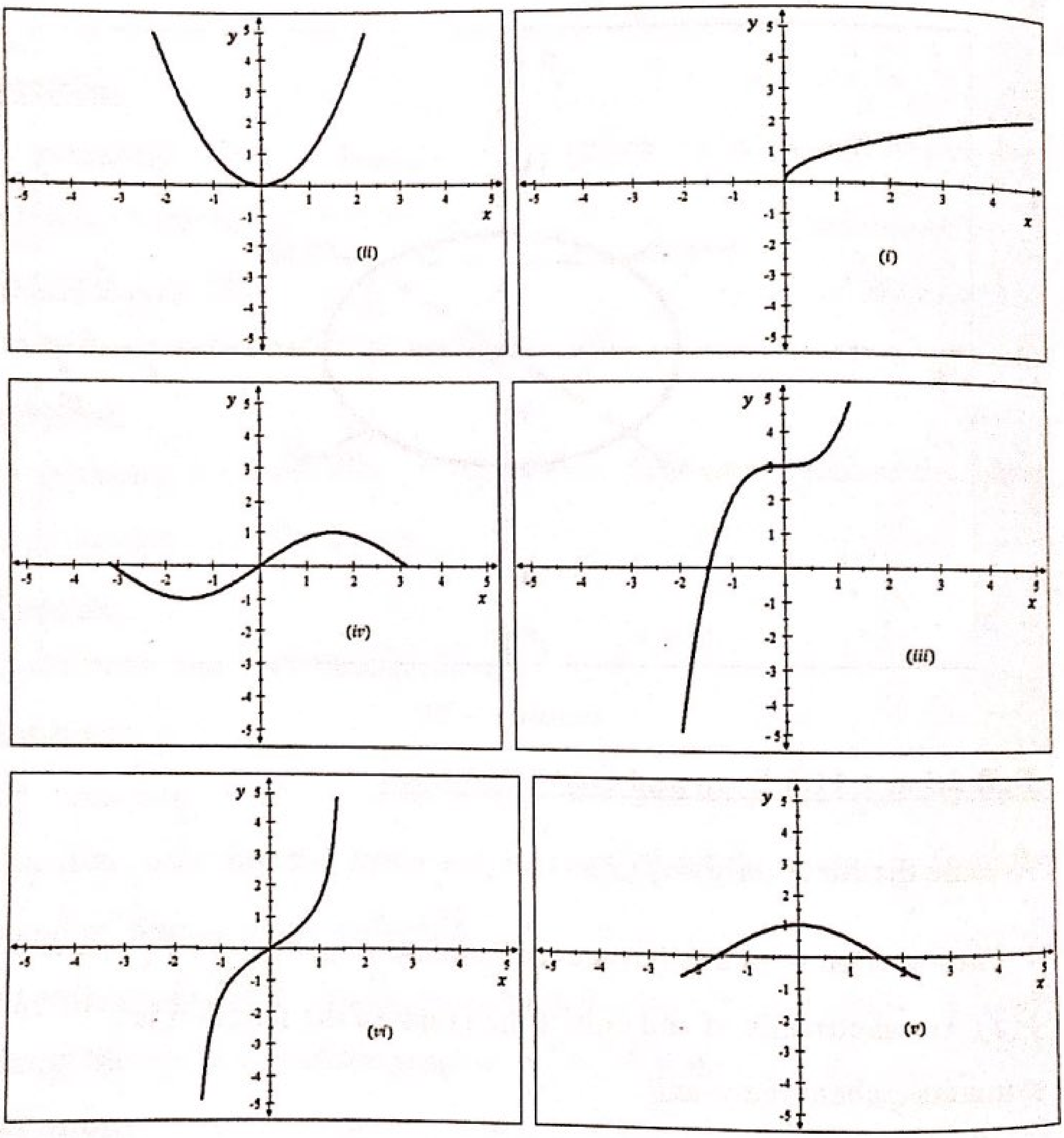
1. The function is said to be an even function if and only if $f(-x) = f(x)$. Geometrically, if and only if the graph of the function is symmetric about the y-axis.

2. The function is said to be an odd if and only if $f(-x) = -f(x)$. Geometrically, if and only if the graph of the function is symmetric about the origin.

Otherwise, the function is neither even nor odd.

Example 1.11.21:

In the following figure, identify whether the function is even or odd?



Solution:

- The graph in figure (i) is not symmetric and therefore the function is neither even nor odd.

- The graph in figure(ii) is symmetric about the y-axis and hence the function is even.
- The graph in figure (iii) is neither symmetric about the y-axis nor about the origin so the function is neither even nor odd.
- The graph in figure (iv) is symmetric about the origin, thus the function is odd.
- The graph in figure (v) is symmetric about the y-axis and hence the function is even.
- The graph in figure (vi) is symmetric about the origin, thus the function is odd.

Definition 1.11.9 Family of Functions:

If the equation of a function is depended on a parameter, then it is called an equation of family of functions.

In the previous definition, we used a concept we had previously studied, but we will give it a part of the discussion, which is the parameter. The constant is a constant quantity that cannot be changed, we can say Planck's constant. We can also say that the velocity of light is a constant quantity, while the parameter is a quantity that does not depend on the independent variable, and therefore it is a constant value, but it can change (not specified value). We say that the $y = c$, the value of c does not depend on the independent variable while it may take the constant 1, 2 or any other number and thus it is a parameter. Thus, $y = c$ is a family of functions, and it is geometrically represented by a family of curves (see Figure (1-31)).

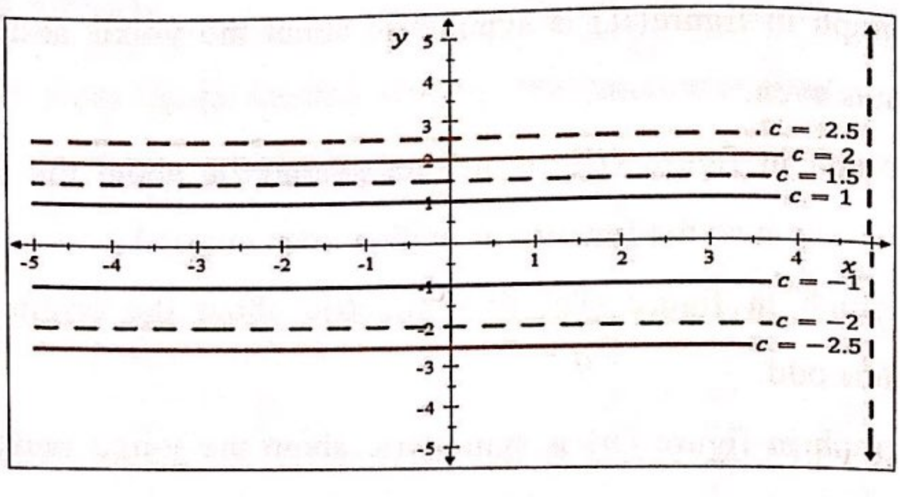


Figure (1-31)

Example 1.11.22:

The equation $y = \frac{k}{x}$ where k is a parameter, represents a family of functions. (here we can say that the variable y is inversely proportional to x). (see Figure 1-32).

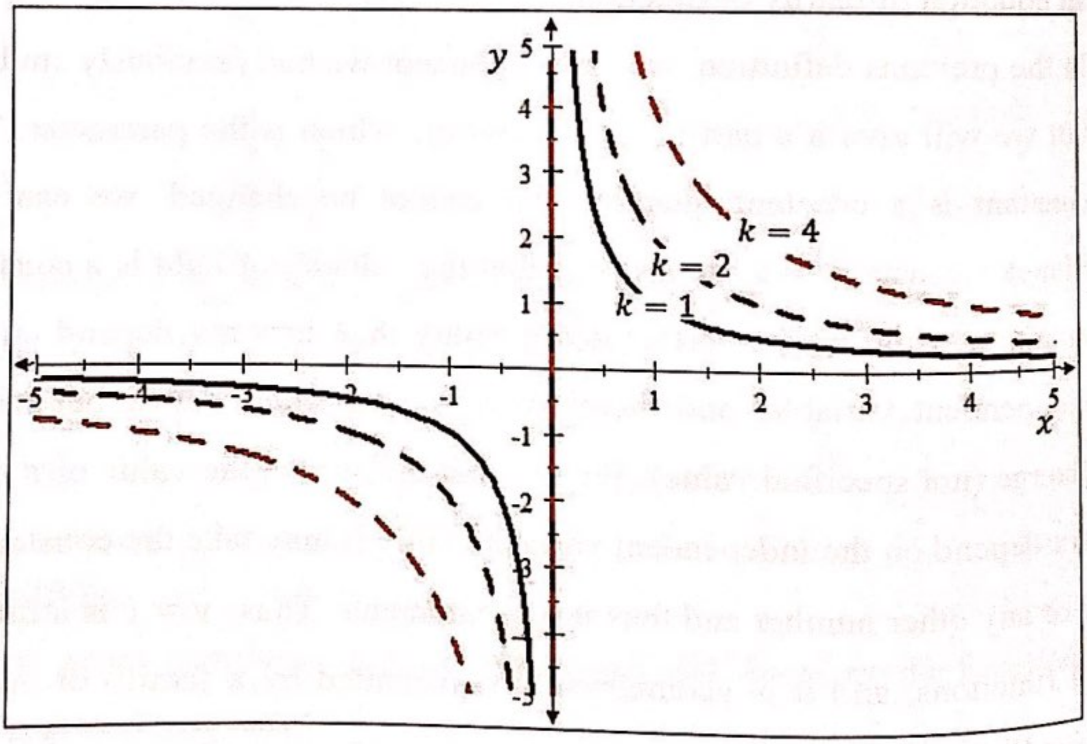


Figure (1-32)

Definition 1.11.10 Identity Function:

The function I where $I(x) = x$ is called identity function.

Definition 1.11.11 Inverse of Function:

The function g is said to be the inverse of the function f and is denoted by f^{-1}, if and only if $f \circ g = g \circ f = I$.

Now, we will give steps to find the inverse of the function, let $y = f(x)$. be a function. First, we write the function in the form $x = g(y)$ and then replace each y by x to get the inverse of the function f.

Example 1.11.23:

Find the inverse of the function $f(x) = \frac{x-1}{x-2}$.

Solution:

First, we write the function in the form of $x = g(y)$ as following:

$$y = f(x) = \frac{x-1}{x-2}$$

$$\Rightarrow y(x-2) = (x-1) \Rightarrow xy - 2y - x = -1$$

$$\Rightarrow x(y-1) - 2y = -1 \Rightarrow x = \frac{2y-1}{y-1}.$$

$$\therefore g(y) = \frac{2y-1}{y-1}.$$

By replacing each $y$by x, we get

$$\therefore g(x) = \frac{2x-1}{x-1}.$$

Therefore

$$\therefore f^{-1}(x) = \frac{2x-1}{x-1}, \quad x \neq 1.$$

Note that:

$$(i)(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{2x-1}{x-1}\right) = \frac{\frac{2x-1}{x-1} - 1}{\frac{2x-1}{x-1} - 2}$$

$$\Rightarrow (f \circ f^{-1})(x) = \frac{2x-1-x+1}{2x-1-2x+2} = \frac{x}{1} = x.$$

$$\therefore f \circ f^{-1} = I.$$

$$(ii)(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{x-1}{x-2}\right) = \frac{2(\frac{x-1}{x-2}) - 1}{\frac{x-1}{x-2} - 1}$$

$$\Rightarrow (f^{-1} \circ f)(x) = \frac{2x-2-x+2}{x-1-x+2} = \frac{x}{1} = x.$$

$$\therefore f^{-1} \circ f = I.$$

We mentioned previously that we will give an algebraic method to find the range of the function. We can use the same previous procedure to get the range of the function $y = f(x)$, where its range is the domain of its inverse. In the previous example, the domain of the function $f^{-1}(x) = \frac{2x-1}{x-1}$ is $\mathbb{R} - \{1\}$, so the range of the function $y = \frac{x-1}{x-2}$ is also $\mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$. See Figure (1-33).

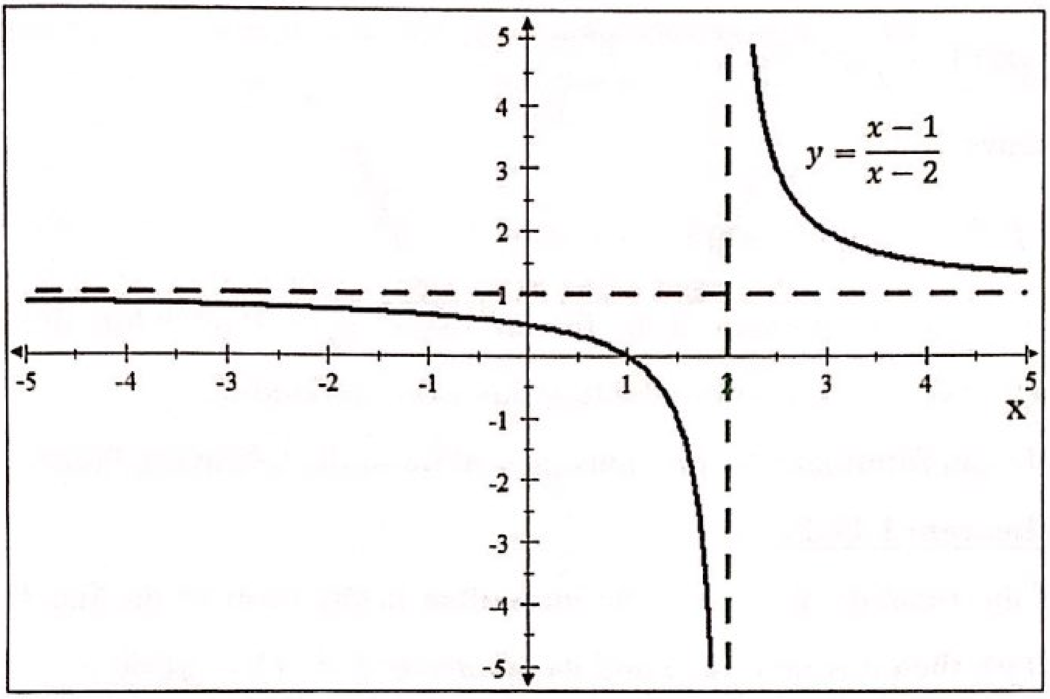


Figure (1-33)

Example 1.11.24:

Find the inverse of the function $y = \sqrt{2x-1}$ and its domain.

Solution:

First, we write the function in the form of $x = g(y)$, as shown in the following:

$$y = f(x) = \sqrt{2x-1}$$

$$\Rightarrow y^2 = 2x - 1 \Rightarrow x = \frac{y^2+1}{2}.$$

$$\therefore g(y) = \frac{1}{2}(y^2+1)$$

By replacing each y by x, we get

$$\therefore g(x) = \frac{1}{2}(x^2 + 1).$$

Hence

$$\therefore f^{-1}(x) = \frac{1}{2}(x^2 + 1),$$

Note that, the domain of the function $g(x)$ is $(-\infty, \infty)$while the domain of $f^{-1}$is $[0, \infty)$, in order the function to be defined.

We can formulate the previous procedure in the following theory.

Theorem 1.11.2:

If the function $y = f(x)$ can be written in the form of the function $x = g(y)$, then it is invertible and its inverse is $f^{-1}(x) = g(x)$.

Definition 1.11.12 One- to-One Function:

It is said that the function $y = f(x)$ is a one-to-one function if and only if:

$$x_1 = x_2 \ \forall \ f(x_1) = f(x_2),\ x_1, x_2 \in D_f.$$

(i.e. if the images are equal then the originals are equal)

Geometrically, any horizontal line intersects the curve at only one point at most.

Example 1.11.25:

Determine whether the function $y = x^2$ is one-to-one.

Solution:

Let $x_1,\ x_2 \in D_f$where $f(x_1) = f(x_2)$. Then

$$x_1{}^2 = x_2{}^2 \Rightarrow x_1 = \pm x_2.$$

Therefore, the function is not one-to-one. See Figure (1-34).

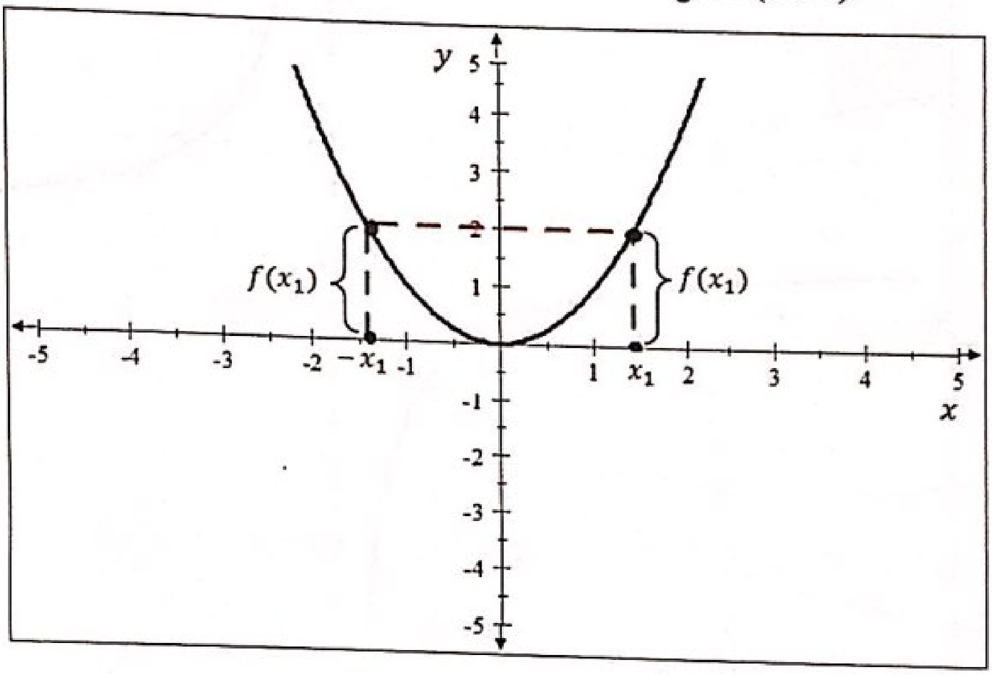


Figure (1-34)

Example 1.11.26:

Determine whether the function $f(x) = \frac{x+1}{x-1}$ is one-to-one.

Solution:

Let $x_1,\ x_2 \in D_f$ where $f(x_1) = f(x_2)$. Then

$$\frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1} \Rightarrow (x_1+1)(x_2-1) = (x_2+1)(x_1-1)$$

$$\Rightarrow x_1x_2 + x_2 - x_1 - 1 = x_1x_2 - x_2 + x_1 - 1$$

$$\Rightarrow 2x_2 = 2x_1, \quad \therefore x_1 = x_2.$$

Therefore, the function is one-to-one. See Figure (1-35).

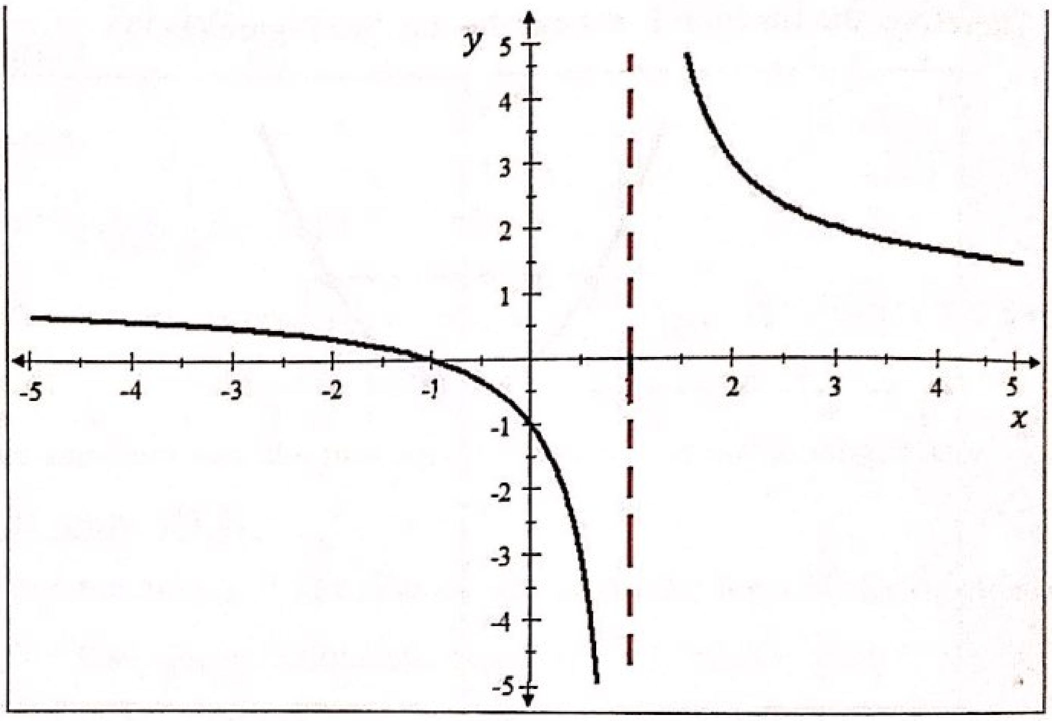


Figure (1-35)

Theorem 1.11.3:

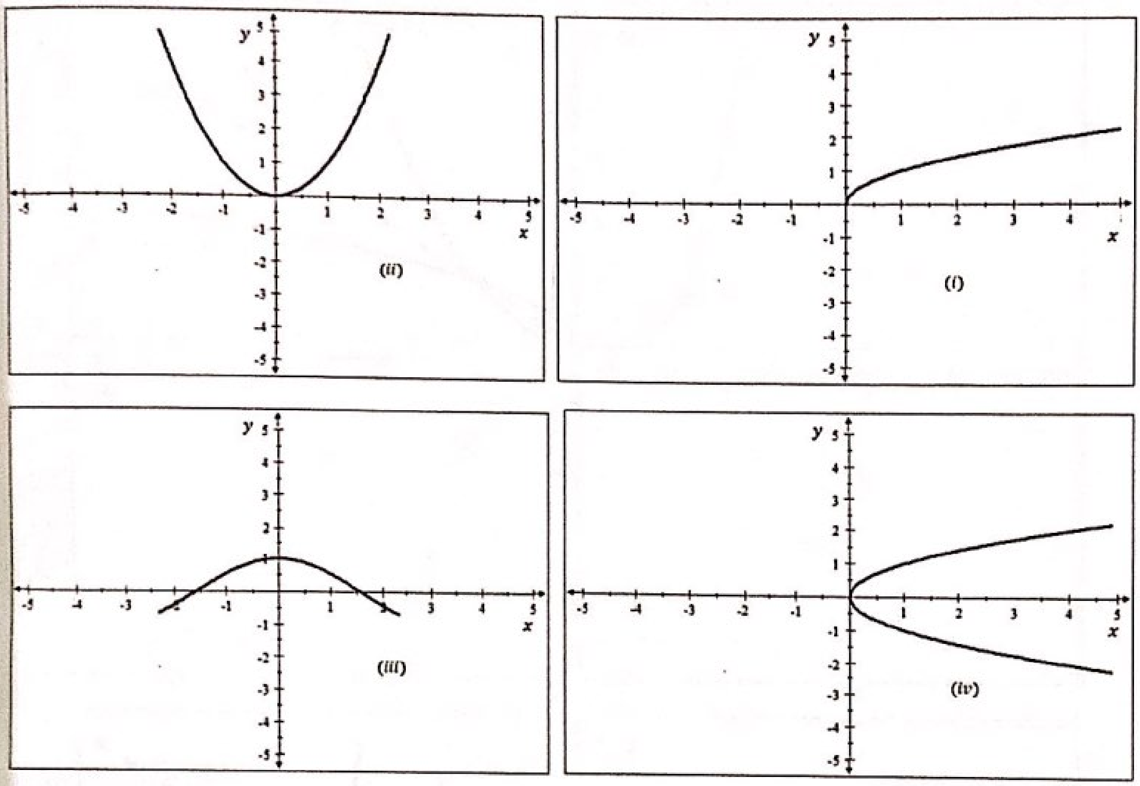
A function is invertible (has an inverse) if and only if it is one-to-one.

Theorem 1.11.4 (Horizontal Line Test):

A function is invertible (has inverted) if and only if any horizontal line cuts its curve at one point at most.

Example 1.11.27:

In the following Figure, identify whether the graph represents an invertible function.



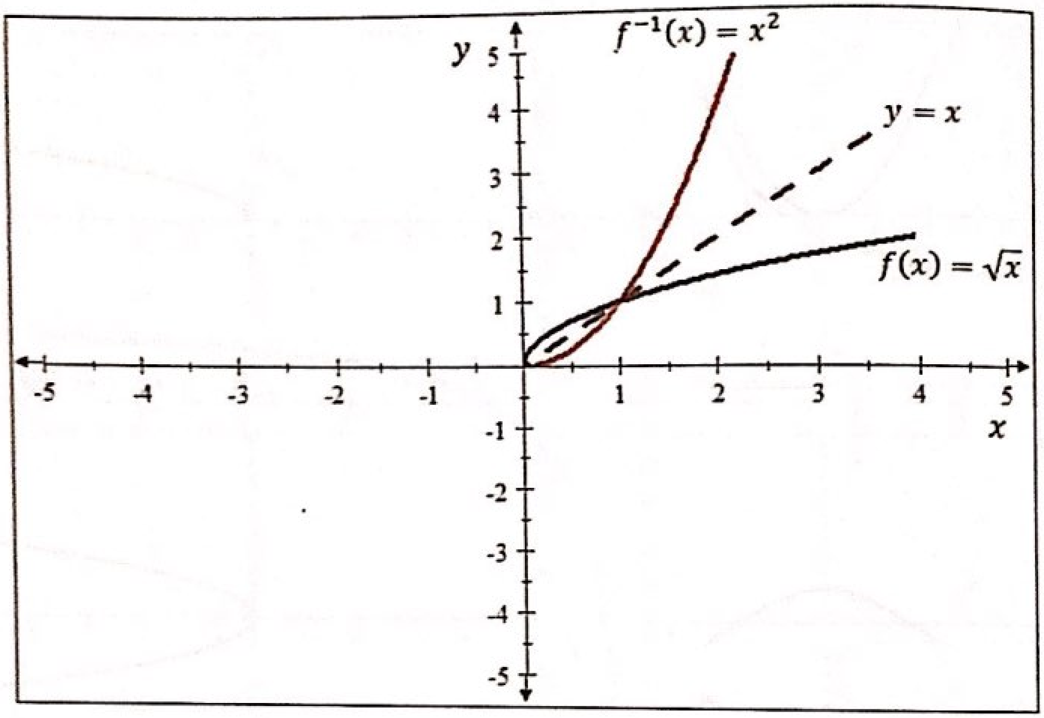
Solution:

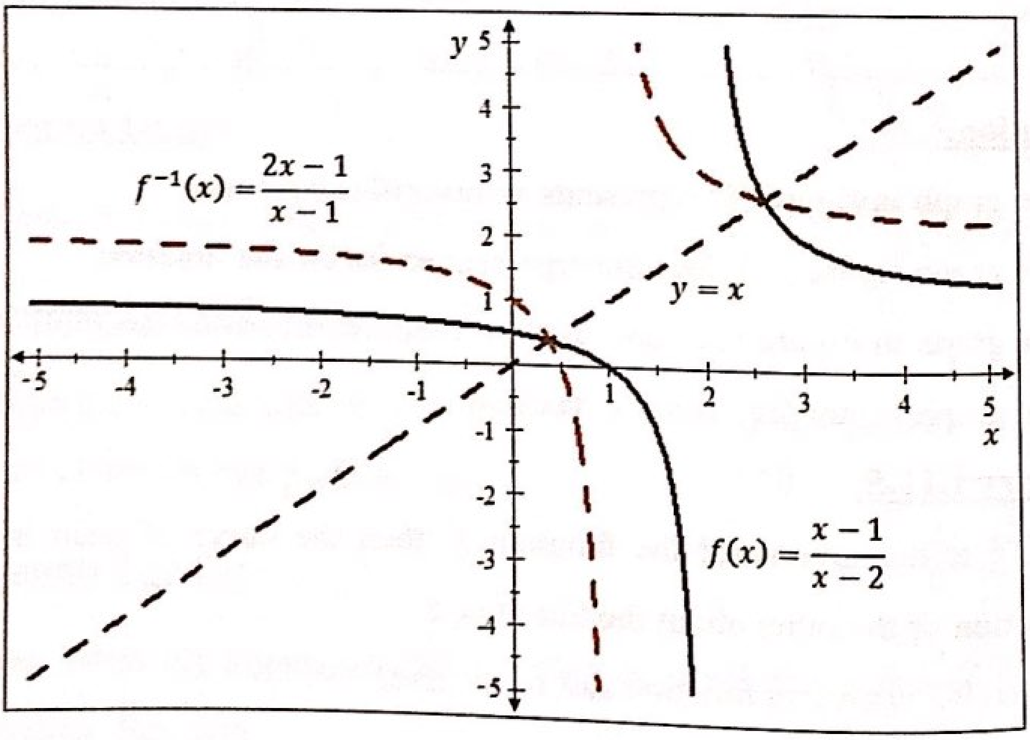
- The graph in figure (i) represents an invertible function.
- The graph figure (ii) does not represent an invertible function.
- The graph in Figure (iii) does not represent an invertible function.
- The graph figure (iv) is not a function.

Theory 1.11.5:

If f^{-1} is the inverse of the function f, then the curve of each is the reflection of the other about the line $y = x$.

Figure (1-36) gives a function and its inverse.


$f^{-1}(x) = x^2$
$y = x$
$f(x) = \sqrt{x}$
y
x
5
4
3
2
1
-1
-2
-3
-4
-5


$f^{-1}(x) = \frac{2x-1}{x-1}$
$y = x$
$f(x) = \frac{x-1}{x-2}$
y
x
5
4
3
2
1
-1
-2
-3
-4
-5

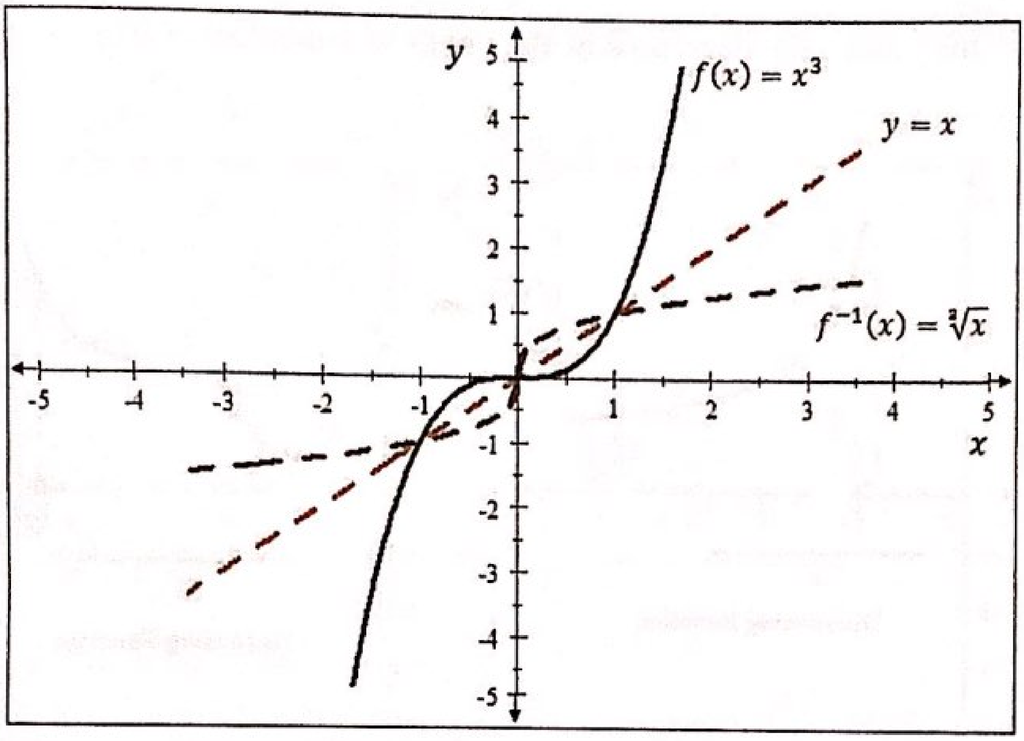


Figure (1-36)

An increase, decrease, and constant of a function is a description of its behavior when we move over the graph of the function from left to right and it is defined in the following definition.

Definition 1.11.13 Increasing and Decreasing Functions:

Let $y = f(x)$ be a function.

1.It is said that the function is increasing if and only if

$$f(x_2) > f(x_1)\ \forall\, x_2 > x_1, x_1, x_2 \in D_f.$$

(i.e., the value of y increases as the value of x increases)

2. It is said that the function is decreasing if and only if

$$f(x_2) < f(x_1)\ \forall\, x_2 > x_1, x_1, x_2 \in D_f.$$

(i.e.,the value of y decreases as the value of x increases). See Figure (1-37).

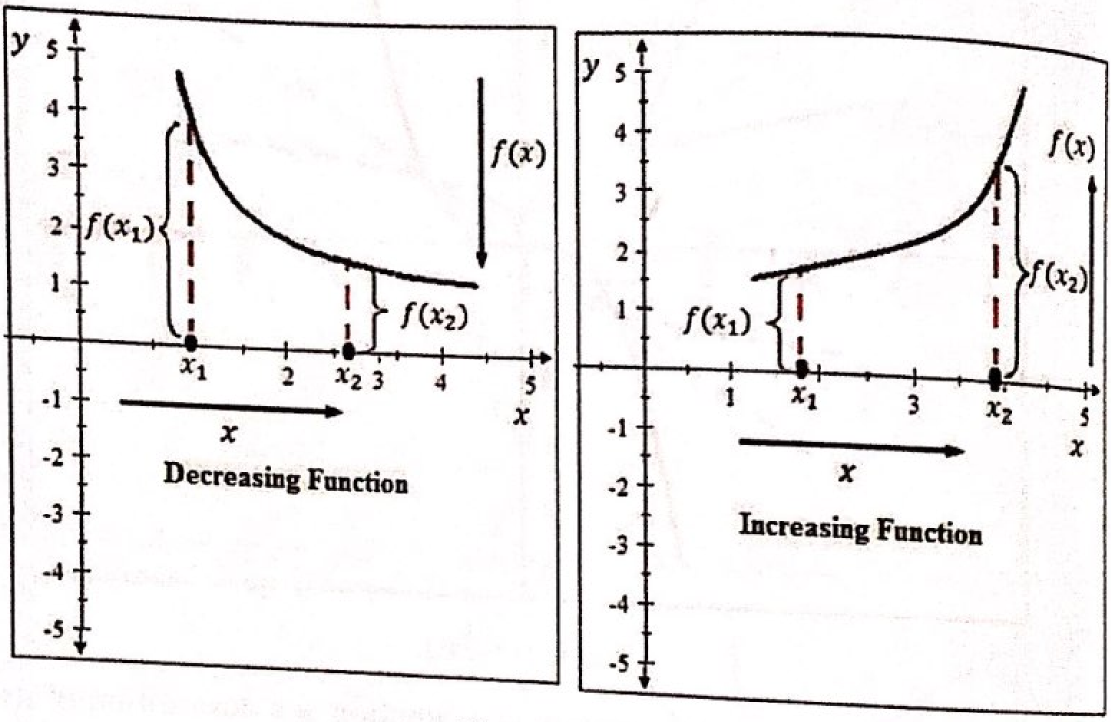


Figure (1-37)

Example 1.11.28:

Study the increase and decrease of the function $f(x) = x^2$.

Solution:

Assume that $x_1, x_2 \in D_f$, where

$$x_2 > x_1, \qquad x_1, x_2 > 0 \Rightarrow {x_2}^2 > {x_1}^2 \Rightarrow f(x_2) > f(x_1).$$

Hence the function is increasing in the interval $[0, \infty)$.

Assume that $x_1, x_2 \in D_f$ where:

$$x_2 > x_1, \qquad x_1, x_2 < 0 \Rightarrow {x_2}^2 < {x_1}^2 \Rightarrow f(x_2) < f(x_1).$$

Hence the function is decreasing function in the interval $(-\infty, 0]$. See Figure (1-38)

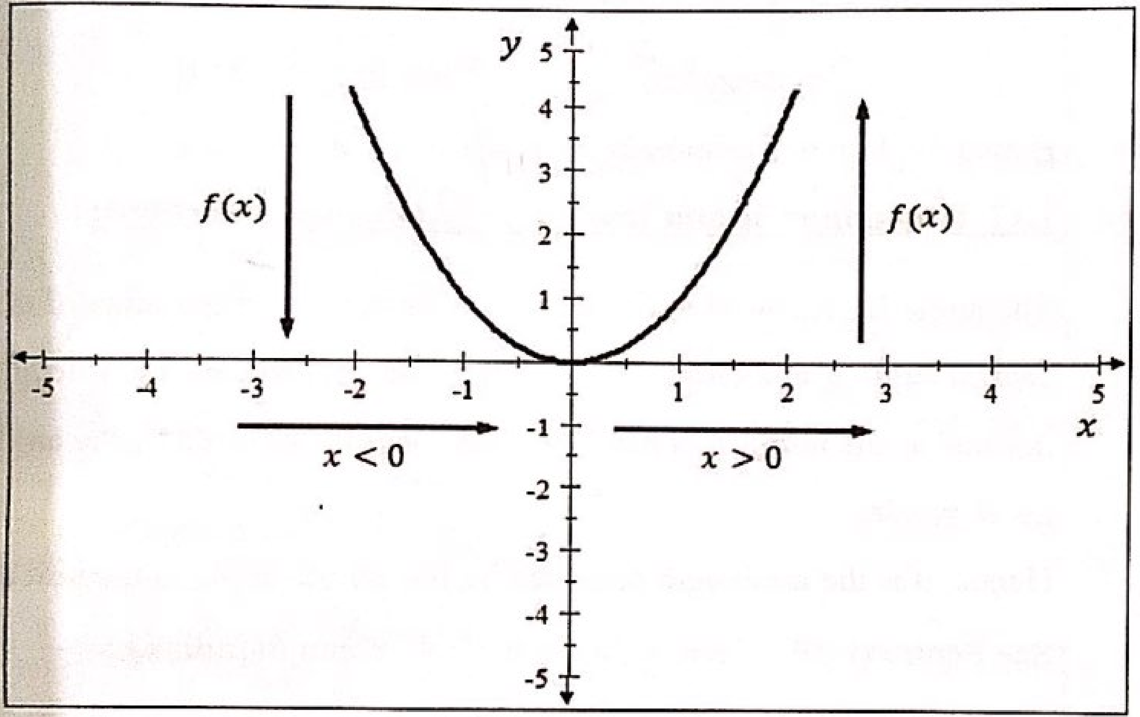


Figure (1-38)

Example 1.11.29:

Study the increase and decrease of the function $g(x) = \sqrt{x}$.

Solution:

Assume that $x_1, x_2 \in D_g$, where

$$x_2 > x_1 \Rightarrow \sqrt{x_2} > \sqrt{x_1} \Rightarrow g(x_2) > g(x_1).$$

Hence the function is increasing in the interval $[0, \infty)$.

Example 1.11.30:

Study the increase and decrease of the function $h(x) = x^3$.

Solution:

Assume that $x_1, x_2 \in D_h$ where

$$x_2 > x_1 \Rightarrow {x_2}^3 > {x_1}^3 \Rightarrow h(x_2) > h(x_1).$$

Hence the function is increasing in the interval $(-\infty, \infty)$.

1.12 Trigonometric and Inverse Trigonometric Functions:

The angle is measured in either radians or degrees. The radian of the central angle θ in a circle with radius r and enclosed an arc of length s is defined as the number of radii in the arc length enclosed by the angle i.e. $\theta = s/r$.

Hence, it is the arc length enclosed by the central angle in the unit circle. See Figure (1-39). Then, π (radian) $= 180°$and $\theta(\text{radian}) = \frac{\pi}{180°} C°$, from which we get the following table:

Angles in radians and degrees											
Degree	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Unless otherwise indicated, radian will be used throughout this book.

The angle in the Cartesian plane oxy in its standard form be such that its initial ray coincide with the positive ox axis and it is measured in positive units if the direction of rotation is counter-clockwise and in negative units if the rotation is in the direction of clockwise rotation. See figure (1-40).

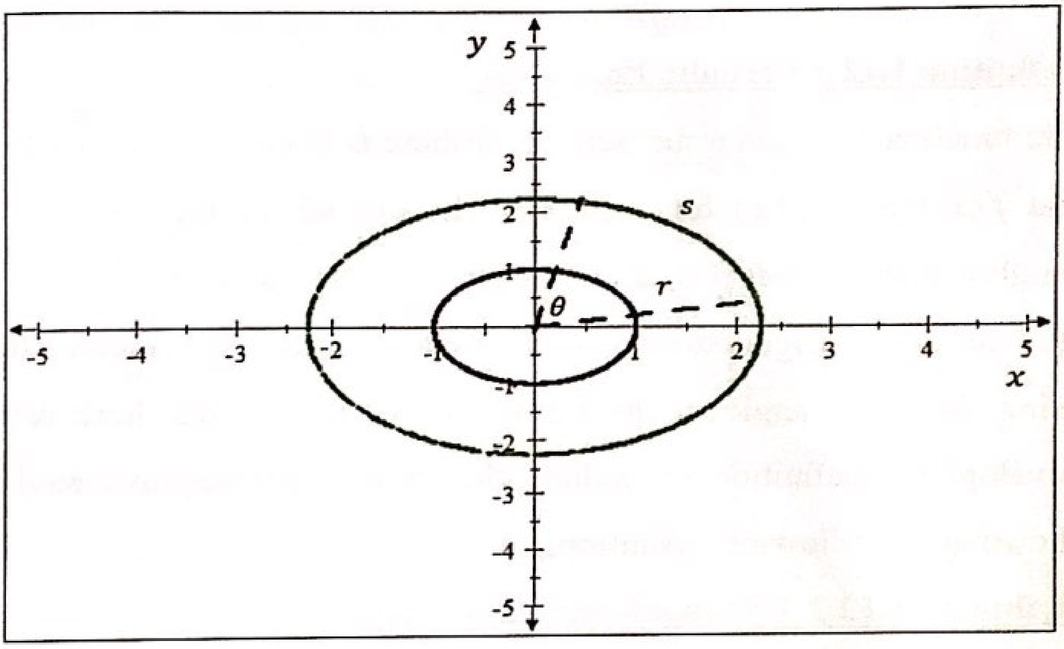


Figure (1-39)

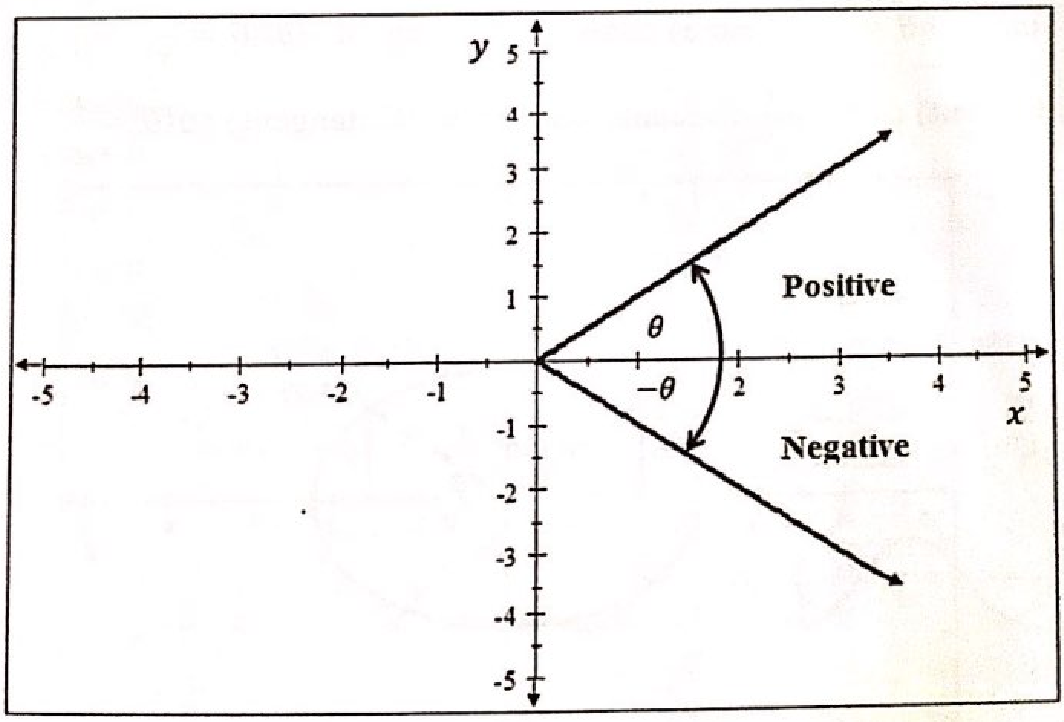


Figure (1-40)

Definition 1.12.1 Periodic Functions:

The function f is said to be periodic if there is a positive number p such that $f(x+p) = f(x)$ for each x in the domain of the function. The smallest such number p is called the period of the function.

We can define trigonometric functions as we have previously studied using the acute angle in the right-angled triangle. But here we will develop this definition to include the obtuse and negative angles as shown in the following definition:

Definition 1.12.2 Trigonometric Functions:

Let θ be an angle such that its terminal ray intersects a circle with radius r at the point $P(x, y)$, from the Figure (1-41), then

(Sine) $sin\theta = \frac{y}{r}$, (Cosine) $cos\theta = \frac{x}{r}$, (Tangent) $tan\theta = \frac{y}{x}$,

(Cosecant) $csc\theta = \frac{r}{y}$, (Secant) $sec\theta = \frac{r}{x}$, (Cotangent) $cot\theta = \frac{x}{y}$.

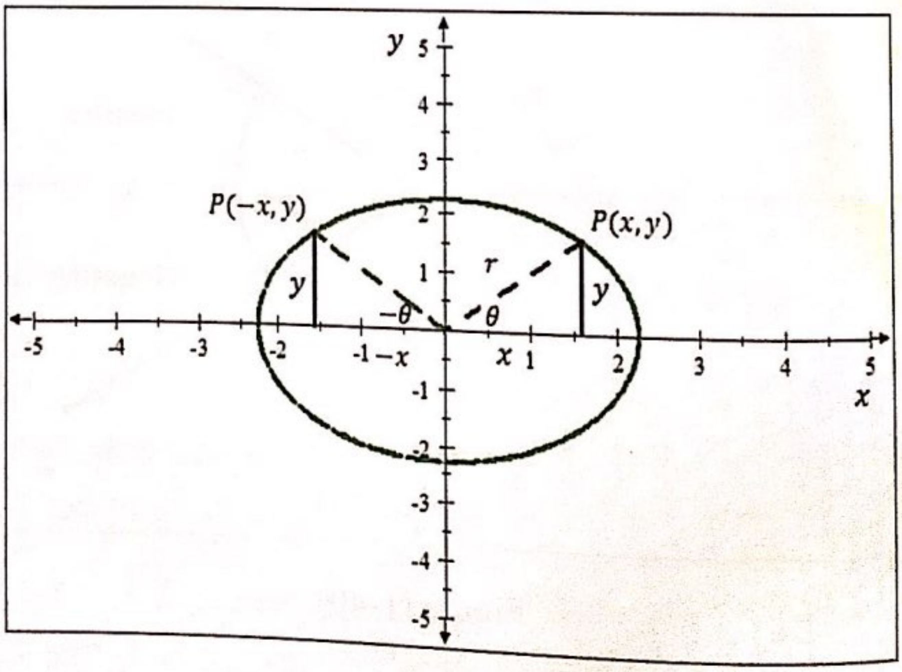
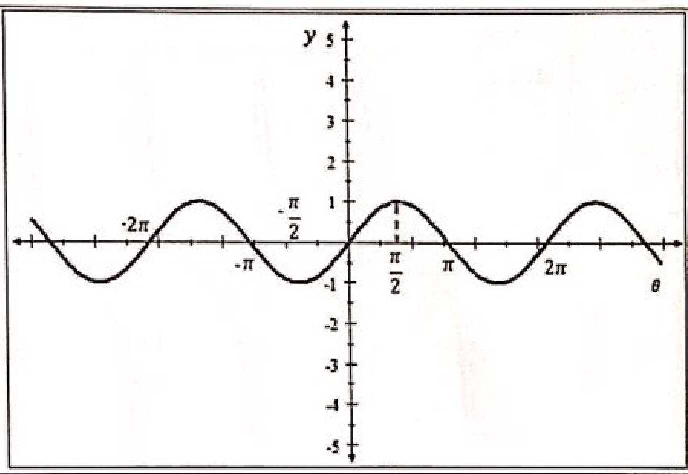
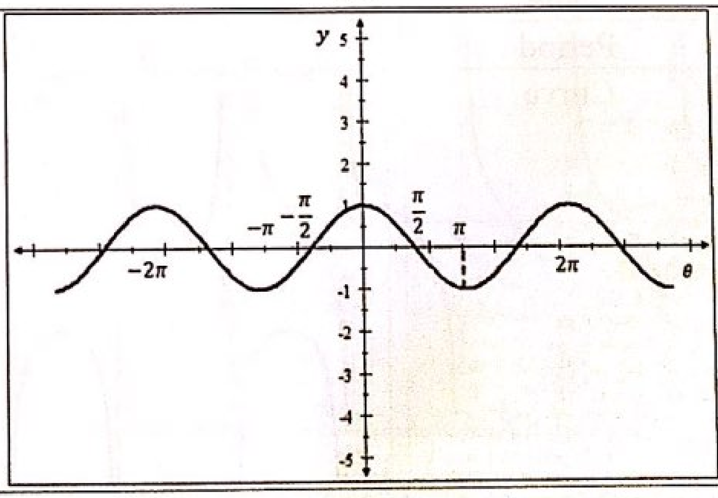
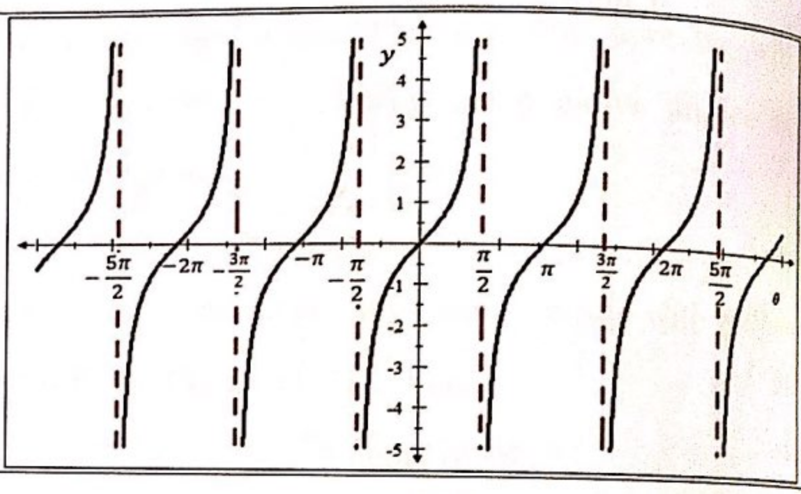


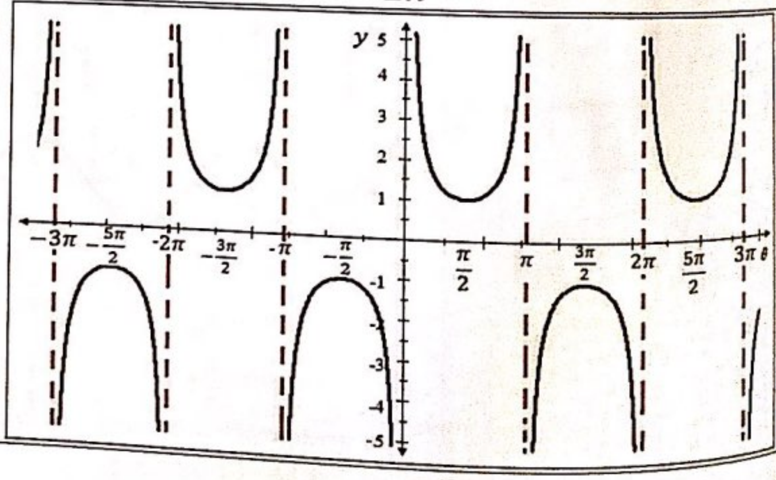
Figure (1-41)

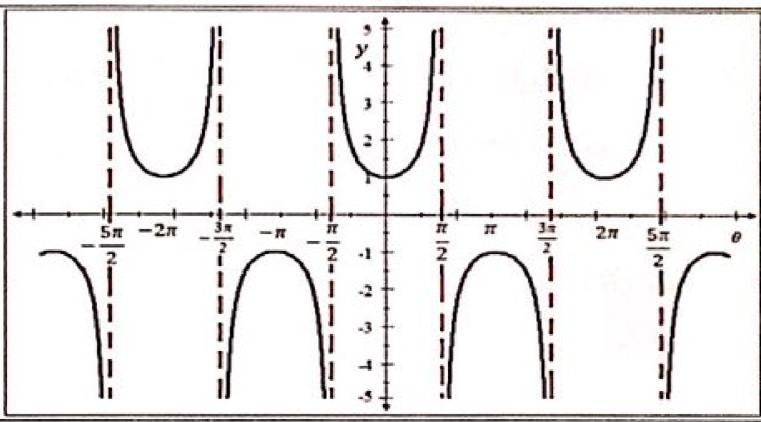
The following tables give the properties of trigonometric functions.

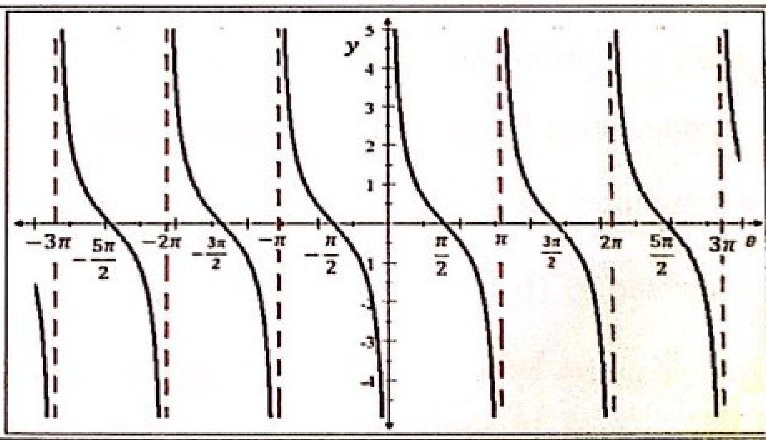
Function	$sin\theta$
Domain	$(-\infty, \infty)$
Range	$[-1,1]$
Type	Odd
Period	2π
Curve	

Function	$cos\theta$
Domain	$(-\infty, \infty)$
Range	$[-1,1]$
Type	Even
Period	2π
Curve	

Function	$tan\theta = \frac{sin\theta}{cos\theta}$
Domain	$\mathbb{R} - \{\frac{n\pi}{2}, n = \pm1, \pm3, \ldots\}$
Range	$(-\infty, \infty)$
Type	Odd
Period	π
Curve	

Function	$csc\theta = \frac{1}{sin\theta}$
Domain	$\mathbb{R} - \{n\pi, n = \pm1, \pm2, \ldots\}$
Range	$(-\infty, -1] \cup [1, \infty)$
Type	Odd
Period	2π
Curve	

Function	$sec\theta = \frac{1}{cos\theta}$
Domain	$\mathbb{R} - \{\frac{n\pi}{2}, n = \pm1, \pm3, \dots\}$
Range	$(-\infty, -1] \cup [1, \infty)$
Type	Even
Period	2π
Curve	

Function	$cot\theta = \frac{1}{tan\theta}$
Domain	$\mathbb{R} - \{n\pi, n = \pm1, \pm2, \dots\}$
Range	$(-\infty, \infty)$
Type	Odd
period	π
Curve	

Using the coordinates of the point $P(x, y)$, where $x = rcos\theta$ and $y = rsin\theta$ in the unit circle we get the following identities. (see Figure 1-39)

Trigonometric Identities:

$(i)\ sin^2\theta + cos^2\theta = 1,$

$(ii) 1 + tan^2\theta = sec^2\theta,$

$(iii) 1 + cot^2\theta = csc^2\theta,$

$(iv)\ \cos(\theta_1 + \theta_2) = \cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2,$

$(v)\ \sin(\theta_1 + \theta_2) = \sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2,$

$(vi)\ \cos(2\theta) = \cos^2\theta - sin^2\theta,$

$(vii)\ \sin(2\theta) = 2sin\theta cos\theta,$

$(viii)\ cos^2\theta = \frac{1}{2}(1 + cos2\theta),$

$(ix)\ sin^2\theta = \frac{1}{2}(1 - cos2\theta).$

If a, b, c are the lengths of the sides of the triangle ABC and the side coppositethe angle θ, then:

$c^2 = a^2 + b^2 - 2abcos\theta$, which is known as the law of cosines.

From the shapes of the curves of the trigonometric functions, we see that they are not one-to-one and so we cannot find their inverses directly, so its domain must be restricted to certain intervals to find their inverses. The domain of the function $sin\theta$ can be restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $cos\theta$ can be restricted to $(0, \pi)$, $tan\theta$ can be restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$, $csc\theta$ can be restricted to $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, $sec\theta$ can be restricted to $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, and $cot\theta$ can be restricted to $(0,\pi)$.The functions become as in Figure (1-42).

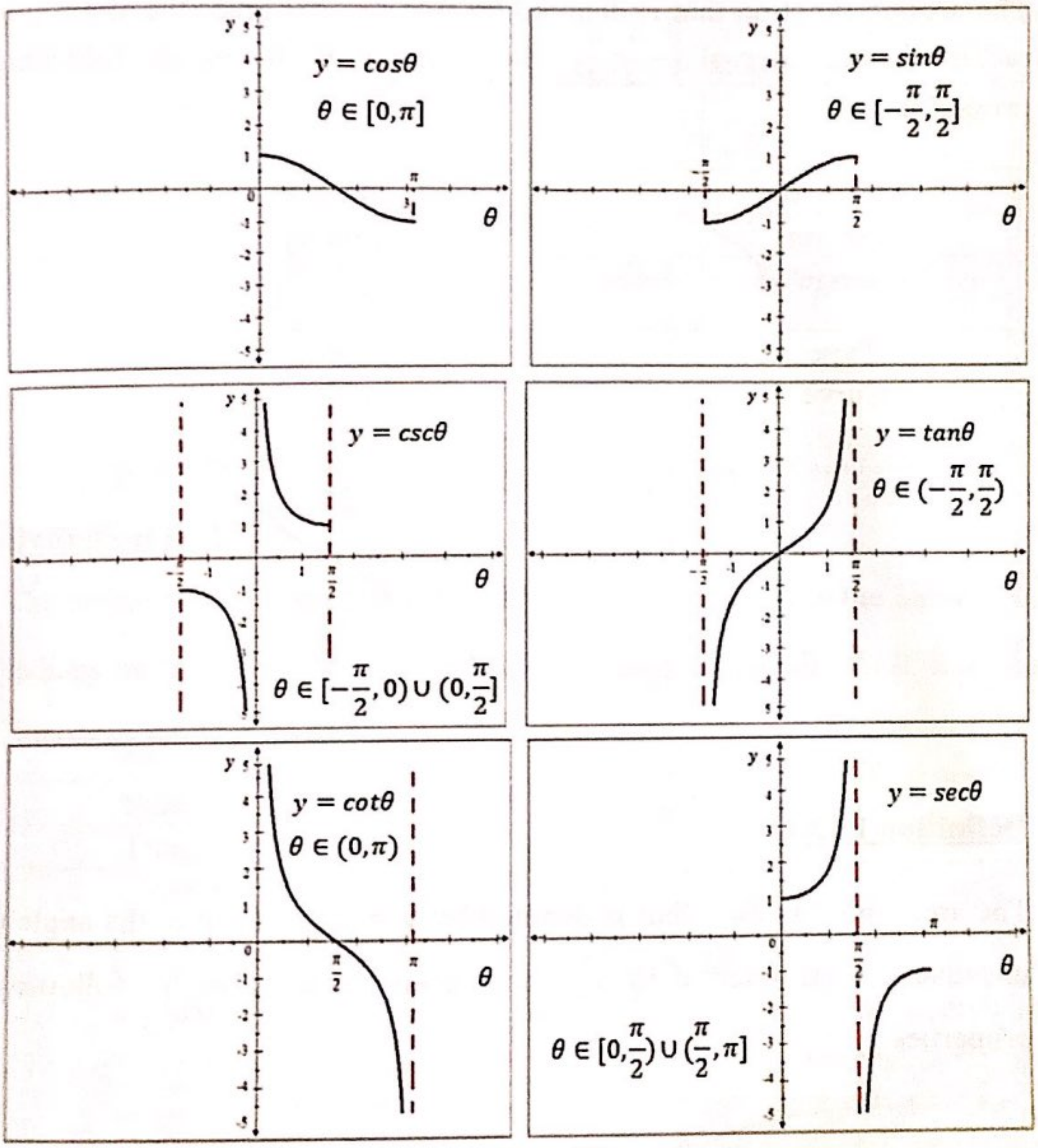
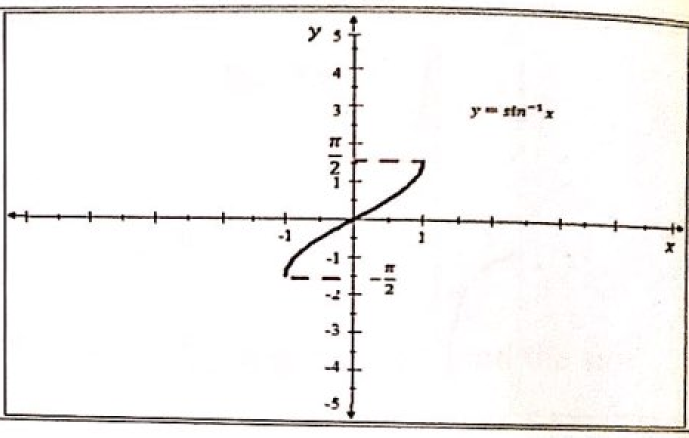


Figure (1-42)

Now, we can define the inverses of the trigonometric functions that are called inverse trigonometric functions as in the following definitions.

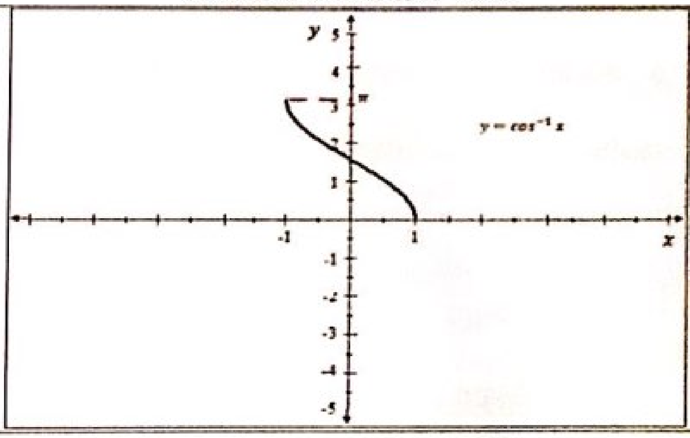
Definition 1.12.3:

The arcsine function that is denoted by $y = sin^{-1}\theta$ gives the angle y in radians in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is θ. It has the following properties:

Function	$sin^{-1}x$
Domain	$[-1,1]$
Range	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
Type	Odd
Curve	

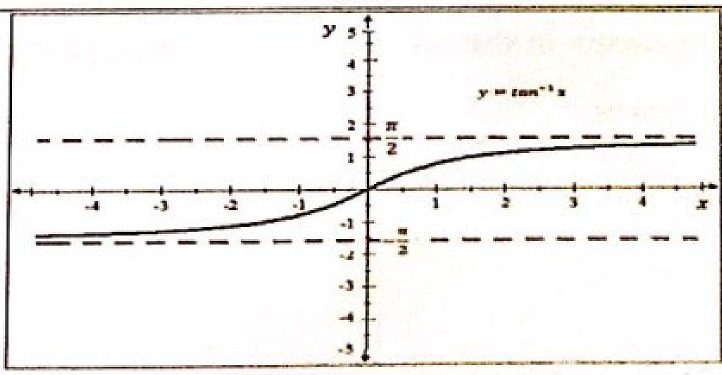
Definition 1.12.4:

The arccosine function that is denoted by $y = cos^{-1}\theta$ gives the angle y in radians in the interval $[0, \pi]$ whose cosine is θ. It has the following properties:

Function	$cos^{-1}x$
Domain	$[-1,1]$
Range	$[0,\pi]$
Type	Neither even nor odd
Curve	

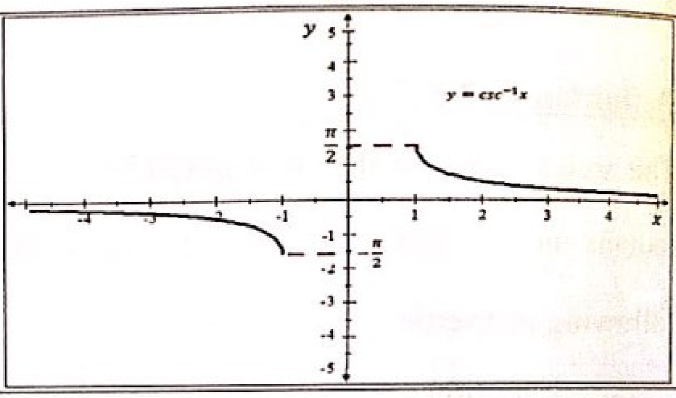
Definition 1.12.5:

The arctan function that is denoted by $y = tan^{-1}\theta$ gives the angle y in radians in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ whose tangent equals θ. It has the following properties:

Function	$tan^{-1}x$
Domain	$(-\infty,\infty)$
Range	$(-\frac{\pi}{2},\frac{\pi}{2})$
Type	Odd
Curve	

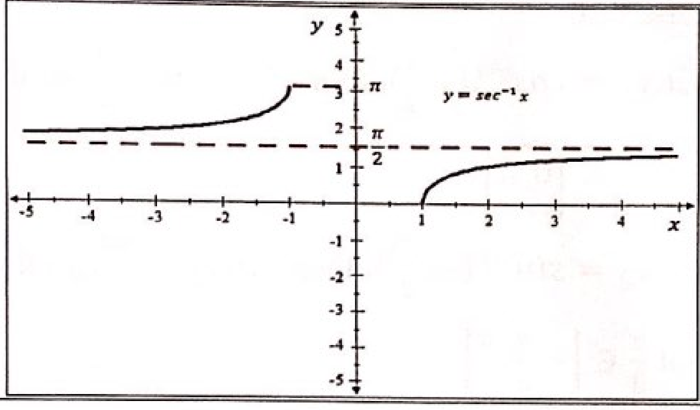
Definition 1.12.6:

The arc-cosecant function that is denoted by $y = csc^{-1}\theta$ gives the angle y in radians in the interval $\left[-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right]$ whose cosecant is θ. It has the following properties:

Function	$csc^{-1}x$
Domain	$(-\infty,-1] \cup [1,\infty)$
Range	$[-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}]$
Type	Odd
Curve	

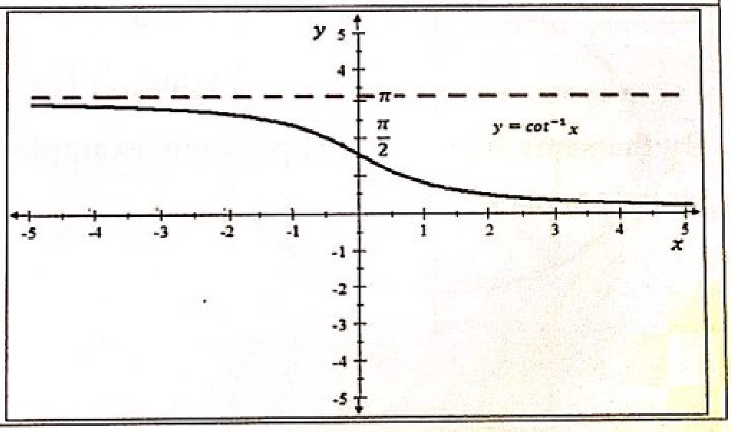
Definition 1.12.7:

The arc-secant function that is denoted by $y = sec^{-1}\theta$ gives the angle y in radians in the interval $[0,\pi]$ whose secant is θ. It has the following properties:

Function	$sec^{-1}x$
Domain	$(-\infty,-1]\cup[1,\infty)$
Range	$[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
Type	Neither even nor odd
Curve	

Definition 1.12.8:

The arc-cotangent function that is denoted by $y = cot^{-1}\theta$ gives the angle y in radians in the interval $(0,\pi)$ whose cotangent equals θ. It has the following properties:

Function	$cot^{-1}x$
Domain	$(-\infty,\infty)$
Range	$(0,\pi)$
Type	Neither even nor odd
Curve	

Example 1.12.1:

Calculate $cos^{-1}(-\frac{1}{2})$ and $sin^{-1}(\frac{\sqrt{3}}{2})$.

Solution:

Let $y_1 = cos^{-1}(-\frac{1}{2})$, then $cosy_1 = -\frac{1}{2}$, so the angle is $y_1 = \frac{2\pi}{3}$. Note that $\frac{2\pi}{3} \in [0, \pi]$.

Let $y_2 = sin^{-1}(-\frac{\sqrt{3}}{2})$, then $siny_2 = \frac{\sqrt{3}}{2}$, so the angle is $y_2 = \frac{\pi}{3}$. Note that $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

See Figure (1-43).

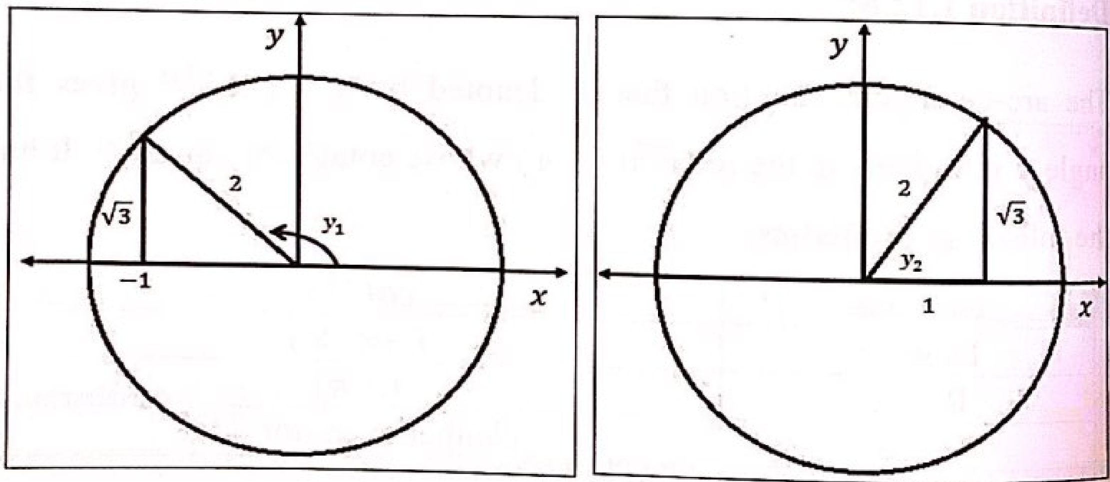


Figure (1-43)

By the same way as in the previous example we can obtain the following table for special angles:

x	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$
$sin^{-1}x$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$
$cos^{-1}x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$

Example 1.12.1:

Calculate $tan^{-1}(-\sqrt{3})$ and $tan^{-1}(\frac{1}{\sqrt{3}})$.

Solution:

Let $y_1 = tan^{-1}(-\sqrt{3})$, then $tany_1 = -\sqrt{3}$, so the angle is $y_1 = -\frac{\pi}{3}$.

Note that $-\frac{\pi}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Let $y_2 = tan^{-1}(\frac{1}{\sqrt{3}})$, then $tany_2 = \frac{1}{\sqrt{3}}$, so the angle is $y_2 = \frac{\pi}{6}$. Note that $\frac{\pi}{6} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

See Figure (1-44).

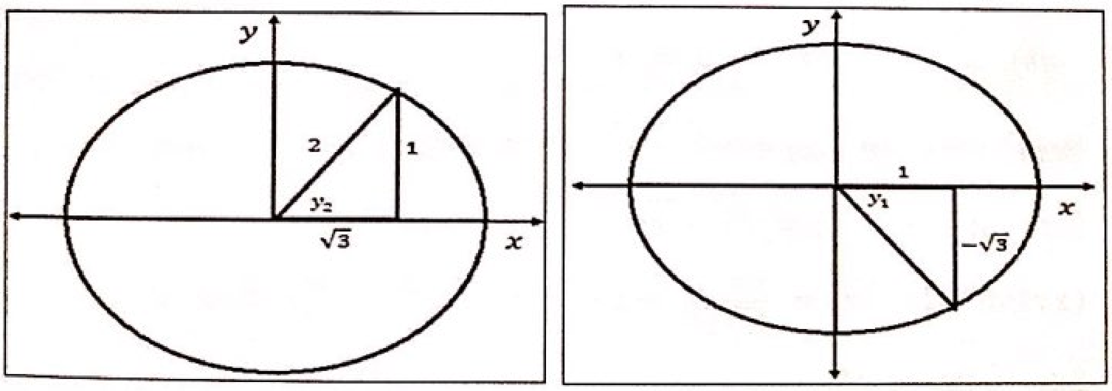


Figure (1-44)

By the same way in the previous example we can obtain the following table for special angles:

x	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$
$tan^{-1}x$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$

Using the unit circle we can obtain the following identities:

<u>The Inverse Trigonometric Identities:</u>

As long as the inverse trigonometric functions are defined, we have

$(i)\ sin^{-1}x + cos^{-1}x = \frac{\pi}{2},\ \ -1 \le x \le 1,$

$(ii)\ cos^{-1}x + cos^{-1}(-x) = \pi, -1 \le x \le 1,$

$(iii) tan^{-1}x + cot^{-1}x = \frac{\pi}{2}, -\infty < x < \infty,$

$(iv)\ csc^{-1}x + sec^{-1}x = \frac{\pi}{2}, -1 \le x\ or\ x \ge 1,$

$(v) csc^{-1}x = sin^{-1}\frac{1}{x}, -1 \le x\ or\ x \ge 1,$

$(vi) sec^{-1}x = cos^{-1}\frac{1}{x}, -1 \le x\ or\ x \ge 1,$

$(vii) cot^{-1}x = tan^{-1}\frac{1}{x}, x \ne 0,$

$(viii)\ cos(\ sin^{-1}x) = \sqrt{1-x^2}, -1 \le x \le 1,$

$(ix)\ \sin(\ cos^{-1}x) = \sqrt{1-x^2}, -1 \le x \le 1,$

$(x)\ \tan(\ sin^{-1}x) = \frac{x}{\sqrt{1-x^2}},\ \ -1 \le x \le 1.$

See Figure (1-45).

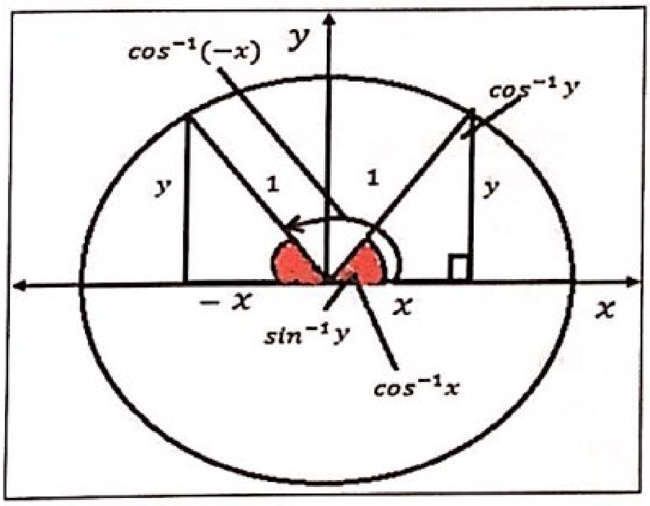


Figure (1-45)

1.13 Exponential and Logarithmic Functions:

The exponential function has a lot of applications in science and mathematics, we will, in this section, recognize the exponential function and its inverse and some of their properties. The exponential function is defined as in the following definition:

Definition 1.13.1 Exponential Function:

For any real number $b > 0$, the function $f(x) = b^x$ is called the exponential function for the base b.

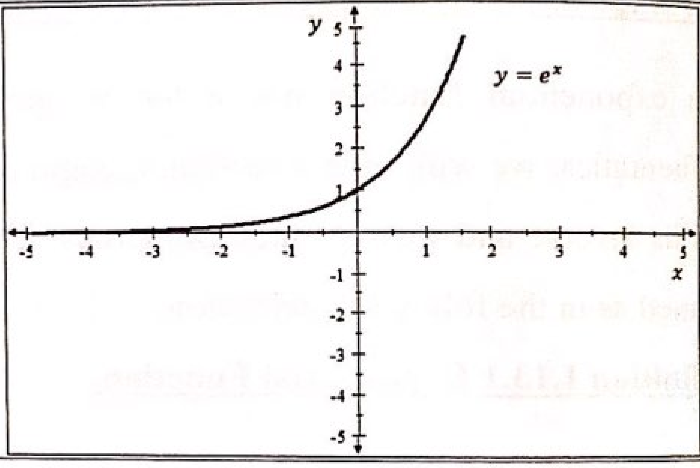
From the previous definition we can find that the power function is not an exponential function but the following functions are exponential functions:

$f(x) = 3^x, g(x) = (\frac{1}{3})^x, \quad h(x) = \pi^x, \ldots.$

The natural base is an irrational number and is referred to e. The value of e, approximated to six decimal places, is $e \approx 2.718282$ and is used to define the exponential function for the natural base as in the following definition:

Definition 1.13.2 Natural Exponential Function:

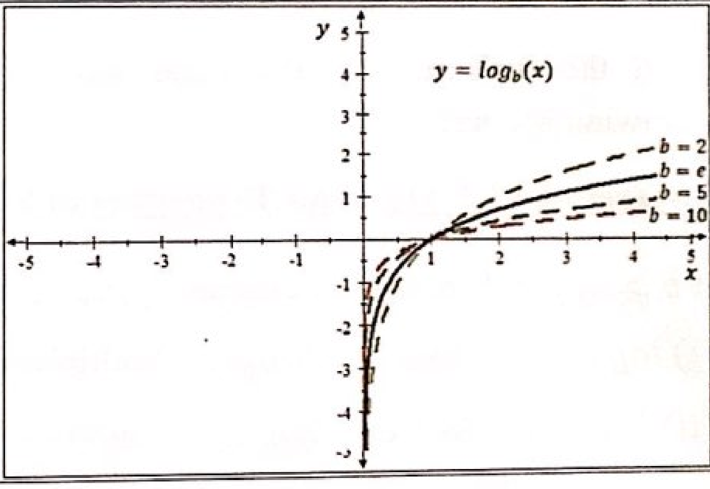
The function $f(x) = e^x$ is called the natural exponential function for the natural base e. It is sometimes written as $f(x) = exp(x)$. Its properties are in the following table:

Function	e^x
Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
Type	Neither even nor odd but increasing
Curve	

From the shape of the curve of the natural exponential function we can observe that it is an increasing function and increases extremely rapidly. The range of the function is $(0, \infty)$, i.e., the function increases without bound as x increases. It is said that a function f increases without bound as x increases, if for any number M regardless of how large it is, $f(x) > M$ where x increases indefinitely. Indeed, if $x > lnM$, then $e^x > M$, so it increases without bound as x increases.

Definition 1.13.3 Logarithmic Function:

The logarithmic function is denoted by $y = log_b(x)$, where $b > 0, b \neq 1$ and is read as the logarithm to base b of $x$and be if and only if $b^y = x$. It has the following properties.

Function	$log_b(x)$
Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
Type	Neither even nor odd but increasing
Curve	

From the shape of the logarithmic function curve, we notice that it is an increasing function but increases extremely slowly. The range of the function is the interval $(-\infty, \infty)$ i.e., the function increases without boundas$x$ increases. Indeed, if $x > e^M$, then $lnx > M$, so it increases without boundas$x$ increases.

Theorem 1.13.1:

The logarithmic function $y = log_b x$ where $b > 0, b \neq 1$ is the inverse of the exponential function $f(x) = b^x$.

When the base of the logarithmic function is 10,that is called the common base, we do not need to write the base and the function is written as $y = log(x)$ instead of $y = log_{10}(x)$.

Definition 1.13.4 Natural Logarithmic Function:

The logarithmic function with base e is called the natural logarithmic function or the logarithmic function for the natural base and is denoted by $y = ln(x)$.

From the definition of the logarithmic function we can prove the following properties:

Theorem 1.13.2 Algebraic Properties of Logarithmic Function:

If $b > 0, b \neq 1, a > 0$, $c > 0$ and r is a real number, then

$(i)\ log_b(ac) = log_b a + log_b c$, Multiplication property

$(ii)\ log_b(\frac{a}{c}) = log_b a - log_b c$, Quotient property

$(iii)\ log_b a^r = r log_b a$, Power property

$(iv)\ log_b a = \frac{lna}{lnb}$, Change of the base formula

Example 1.13.1:

Find x such that,

$(i)\ logx = 3$, $(ii)\ ln(x+2) = 6$, $(iii)\ 3^x = 8$.

Solution:

(i) Let $logx = 3$, then $x = 10^3 = 1000$.

(ii) Let $ln(x+2) = 6$

$\Rightarrow x + 2 = e^6 \Rightarrow x = e^6 - 2 \approx 401.43.$

(iii) Let $3^x = 8$

$\Rightarrow ln3^x = ln8 \Rightarrow xln3 = ln8 \Rightarrow x = \frac{ln8}{ln3} = 1.89.$

Example 1.13.2:

Solve $e^x - e^{-x} = 2$ for x.

Solution:

Assuming

$e^x - e^{-x} = 2$ ....(1)

Multiplying the two sides of equation (1) by e^x, then

$$e^{2x} - 1 = 2e^x,$$

$\Rightarrow e^{2x} - 2e^x - 1 = 0$ ....(2)

Replacing e^x in equation (2) by u, we obtain the following equation

$u^2 - 2u - 1 = 0,$

And it is a quadratic equation whose solution is:

$u_1 = \frac{2+\sqrt{4+4}}{2}, \;\; u_2 = \frac{2-\sqrt{4+4}}{2},$

$\Rightarrow u_1 = 1 + \sqrt{2}, \;\; u_2 = 1 - \sqrt{2}.$

Since $u = e^x > 0$, then the required solution is only $u_1 = 1 + \sqrt{2}$ and

so

$e^x = 1 + \sqrt{2} \Rightarrow x = ln(1 + \sqrt{2}) \approx 0.88.$

Note: The solutions of the quadratic equation on the form:

$$ax^2 + bx + c = 0, \;\; a \neq 0$$

are

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Example 1.13.3:

Solve $e^{2x-6} = 4$ for x.

Solution:

Let

$e^{2x-6} = 4$..(1)

Taking ln of the two sides of equation (1), then

$$ln(e^{2x-6}) = ln4$$

$$\Rightarrow (2x-6)lne = ln4 \Rightarrow 2x-6 = ln4 \Rightarrow x = \frac{ln4+6}{2} \approx 3.69.$$

Exercises

(1) If $f(x) = \sqrt{x+1} + 4$, complete the following statements:

- the domain of the function f is ....
- $f(3) =$ ....
- $f(t^2 - 1) =$ ....
- If $f(x) = 7$, then $x =$ ....
- the range of the function f is ....

(2) If the curve of the function $y = f(x)$ is given by the figure E-1, complete the following:

- the domain of the function f is ....
- the range of the function f is ....
- $f(-3) =$ ....
- $f\left(\frac{1}{2}\right) =$................................
- the solutions to $f(x) = -\frac{3}{2}$ are $x =$ .... and $x =$.............

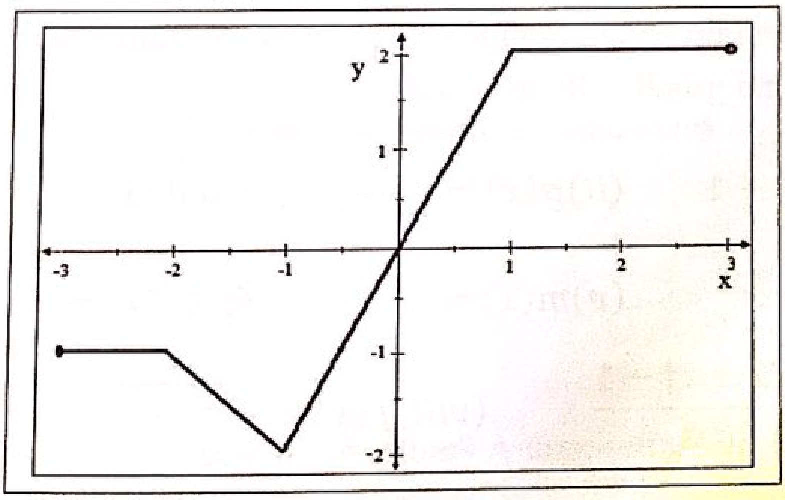
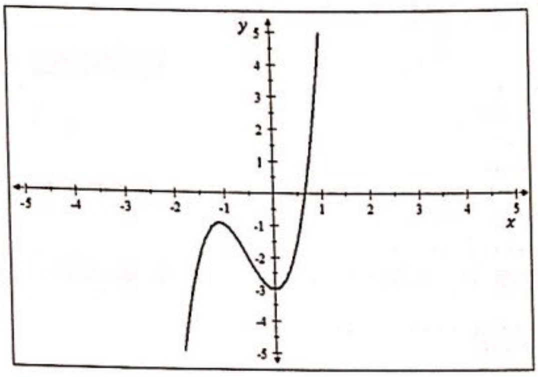
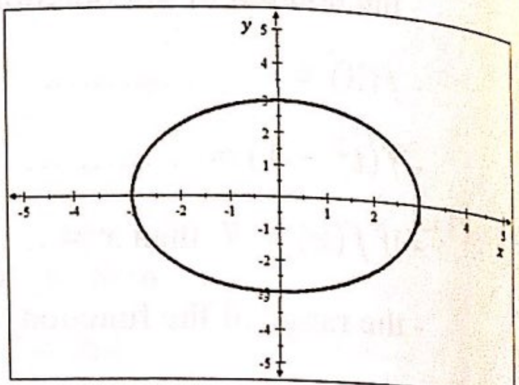


Figure E-1

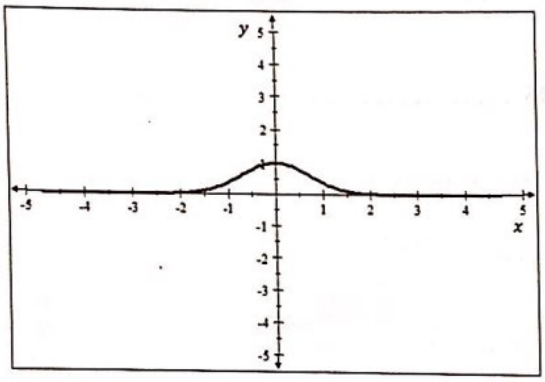
(3) Which of the following graphs defines y as a function of x.



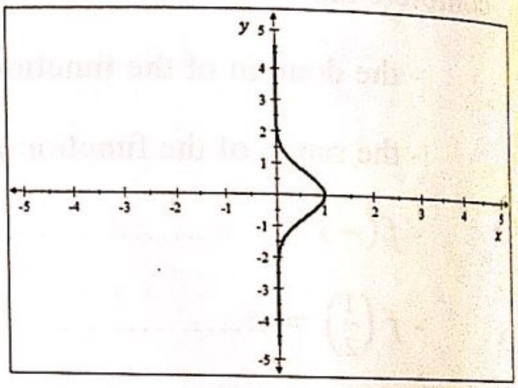
(1)



(2)



(3)



(4)

(4) Find the domain and the range of the following functions (analytically and graphically as possible):

$(i)\ f(x) = x^2 + 1$ $\quad (ii)\ g(x) = \dfrac{1}{x-2}$ $\quad (iii)\ h(x) = |x+1|$

$(iv)\ l(x) = \sqrt{2x}$ $\quad (v)\ m(x) = |x| - 1$ $\quad (vi)\ f(x) = \sqrt{1 - |x-2|}$

$(vii)\ f(x) = \dfrac{|x+1| - 1}{x}$ $\quad (viii)\ f(x) = \sqrt{x^2 - 3}$

(ix) $g(x) = \sqrt{x^2 - 2x + 5}$ (x) $f(x) = \sqrt{\frac{x^2 - 1}{x - 1}}$

(xi) $g(x) = \sqrt{4 - x^2}$ (xii) $h(x) = \sqrt{3 - x}$

(5) The greatest integer function is denoted by $f(x) = [x]$ and it returns the largest integer that is less than or equal to x. Find the domain and range of f.

(6) Determine whether each of the following statements is true or false explaining why:

(a) The curve that intersects the x axis at two different points cannot be a function.

(b) The domain of the real valued function consists of all the real numbers for which the value of the function is real.

(c) The range of the absolute value function is all the positive real numbers.

(d) If $g(x) = \frac{1}{f(x)}$, then the domain of the function g consists of all the real numbers x for which $f(x) \neq 0$.

(7) If $y = x^2 - 2x + 5$, answer the following questions:

(a) For what value of x is $y = 0$?

(b) For what value of x is $y = -10$?

(c) For what value of x is $y > 0$?

(d) Does y have a minimum value? A maximum value? If so, find them.

(8) If $y = 1 + \sqrt{x}$, answer the following questions:

(a) For what value of x is $y = 4$?

(b) For what value of x is $y = 0$?

(c) For what value of x is $y \geq 6$?

(d) Does y have a minimum value? A maximum value? If so, find them.

(9) Write the following functions as piecewise functions (i.e., with no absolute values)

$(i)\ f(x) = |x| + 3x - 1$ $(ii) g(x) = 3 + |2x - 5|$

$(iii)\ h(x) = |x| + |x - 1|$ $(iv)\ g(x) = 3|x - 2| - |x + 5|$.

(10) If $f(x) = 3\sqrt{x} - 2$ and $g(x) = |x|$, complete the following:

(a) $(f + g)(x) =$ .... and its domain is ....

(b) $(f - g)(x) =$ .... and its domain is ....

(c) $(fg)(x) =$ .... and its domain is ....

(d) $(f/g)(x) =$ .... and its domain is ....

(11) If $f(x) = 2 - x^2$ and $g(x) = \sqrt{x}$, complete the following.

(a) $(f \circ g)(x) =$ .... and its domain is ....

(b) $(g \circ f)(x) =$ .... and its domain is ....

(12) Using the curve of the function f given in Figure E-2, plot the following equations

$(i) y = f(x) - 1$ $(ii) y = f(x - 1)$ $(iii) y = \frac{1}{2} f(x)$

$(iv) y = f\left(-\frac{1}{2}x\right)$.

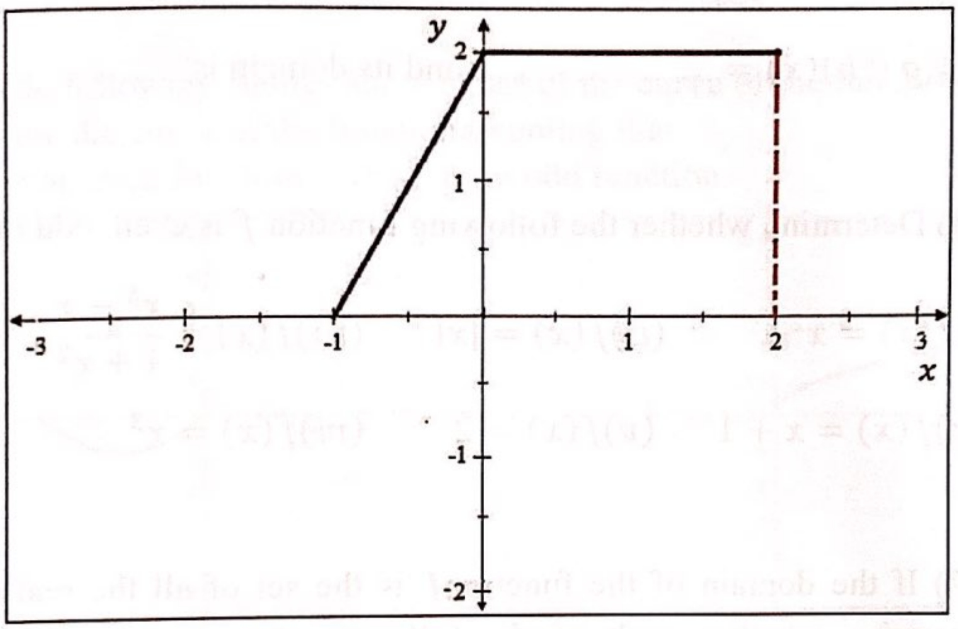


Figure E-2

(13) Write the function f as a composition of two other functions, i.e., find two functions g and h such that $f = g \circ h$.

$(i) f(x) = \sqrt{x+2}$ $(ii) f(x) = |x^2 - 3x + 5|$ $(iii) f(x) = \dfrac{1}{1-x^2}$

$(iv) f(x) = |2x + 5|$ $(v) f(x) = \dfrac{2}{x-3}$ $(vi) f(x) = x^2 + 1.$

(14) Use the data in the following table to plot $y = f(g(x))$, then find the domain of $g \circ f$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-4	-3	-2	-1	0	1	2
$g(x)$	-1	0	1	2	3	-2	-3

(15) If $f(x) = \frac{x}{x-1}$, $g(x) = \frac{1}{x}$, and , $h(x) = x^2 - 1$, then

$(f \circ g \circ h)(x) =$ .... and its domain is ....

(16) Determine whether the following function f is even, odd or neither.

$(i) f(x) = x^2$ $\quad (ii) f(x) = |x|$ $\quad (iii) f(x) = \frac{x^5 - x}{1 + x^2}$

$(iv) f(x) = x + 1$ $\quad (v) f(x) = 2$ $\quad (vi) f(x) = x^3$.

(17) If the domain of the function f is the set of all the real numbers, determine whether each of the following functions is even or odd. Explain.

$(i) g(x) = \frac{f(x)+f(-x)}{2}$ $\quad (ii) h(x) = \frac{f(x)-f(-x)}{2}$.

(18) Discuss the truthfulness of the following statement: the function f is odd if and only if $f(0) = 0$.

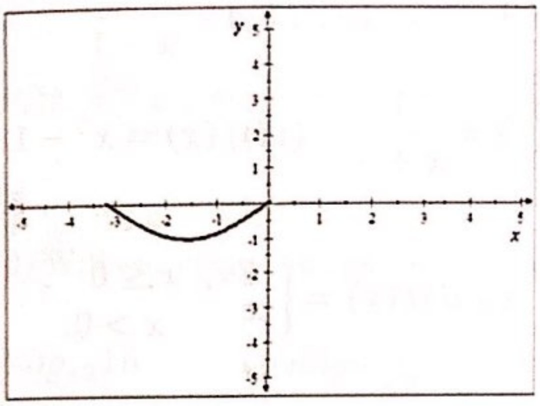
(19) If the domain of the function f is the set of all the real numbers, prove that it can be written as a sum of two functions one of them is even and the other is odd.

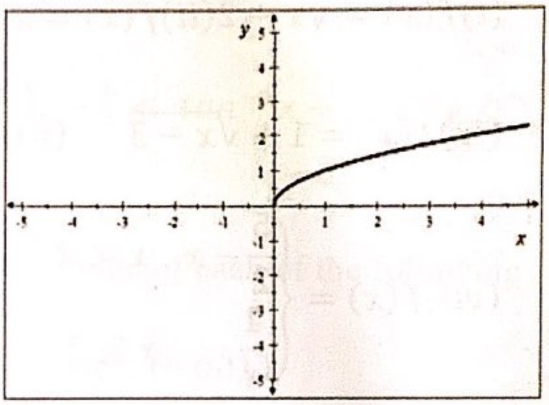
(20) Use the symmetry test to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin.
$(i) x = 5y^2 + 9$ $\quad (ii) x^2 - 2y^2 = 3$ $\quad (iii) xy = 5$

$(iv) y^2 = |x| - 5$ $(v) x^4 = 2y^3 + y$ $(vi)\ y = \frac{x}{3+x^2}$.

(21) The following figures show a part of the curve of the function f, complete the curve of the functionassuming that

(a) f is an even function (b) f is an odd function.





(22) Use the proper translations to plot each of the following equations.

$(i)\ y = 1 - \sqrt{x+2}$ $(ii)\ y = 2(x+1)^2$ $(iii)\ y = x^2 + 2x$

$(iv)\ y = \sqrt{|x|}$ $(v)\ y = \frac{-3}{(x+1)^2}$ $(vi)\ y = -3(x-2)^3$.

(23) Determine whether f is a one to one function.

$(i)\ f(x) = \sqrt{x+2}$ $(ii)\ f(x) = x^2 - 9$ $(iii)\ f(x) = \frac{1}{x-1}$

$(iv)\ f(x) = |x-5|$ $(v)\ f(x) = \frac{1}{x+3}$ $(vi)\ f(x) = x^3 - 1$.

(24) Show that each of the functions f and g is the inverse of the other. Discuss this using the curves of the two functions (in the same graph).

$(i) f(x) = 2x + 1,$ $g(x) = \frac{x-1}{2}$

$(ii) f(x) = x^2 - 9, \quad g(x) = \sqrt{x+9}, x > -9$

$(iii) f(x) = \frac{1}{x-1}, x \neq 1, \; g(x) = 1 + \frac{1}{x}, x \neq 0.$

(25) Show that f is a one to one function and find its inverse, domain and range:

$(i) f(x) = \sqrt{x+2}$ $(ii) f(x) = x - 9$ $\quad (iii) f(x) = \dfrac{1}{x-1}$

$(iv) f(x) = 1 + \sqrt{x-3} \quad (v) f(x) = \dfrac{1}{x+3} \quad (vi) f(x) = x^3 - 1$

$$(vii) f(x) = \begin{cases} \frac{5}{2} - x, & x < 2 \\ \frac{1}{x}, & x \geq 2 \end{cases} \qquad (viii) f(x) = \begin{cases} 2x, & x \leq 0 \\ x^2, & x > 0. \end{cases}$$

(26) Let $\theta = tan^{-1}\left(\frac{4}{3}\right)$. Find the values of

$sin\theta, \quad cos\theta, \quad cot\theta, \quad sec\theta, \quad csc\theta.$

(27) For which values of $x$is true that?

$(i) cos^{-1}(cosx) = x \qquad (ii) \cos(cos^{-1}x) = x$

$(iii) tan^{-1}(tanx) = x \qquad (iv) \tan(tan^{-1}x) = x.$

(28) Complete the following identities.

(i) $sin(cos^{-1}x) =$...................... (ii) $tan(cos^{-1}x) =$ ....

$(iii) csc(tan^{-1}x) =$................. $(iv) sin(tan^{-1}x) =$...................

$(v) cos(tan^{-1}x) =$ .... (vi) $tan(cos^{-1}x) =$ ....

$(vii) sin\ (sec^{-1}x) =$.............. $(viii) cot\ (sec^{-1}x) =$....................

(29) Prove each of the following:

(i) $\sin^{-1}(-x) = -\sin^{-1}x$ (ii) $\tan^{-1}(-x) = -\tan^{-1}x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$ (iv) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(v) $\sin^{-1}(x) = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}, |x| < 1$

(vi) $\cos^{-1}(x) = \dfrac{\pi}{2} - \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}, |x| < 1$

(vii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), -\frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \frac{\pi}{2}$.

(30) Without using the calculator find the value of each of the following.

(i) $\log_{10}16$ (ii) $\log_2\dfrac{1}{32}$ (iii) $\log_{10}(0.001)$

(iv) $\log_4 4$ (v) $\log_9 3$ (vi) $\log_{10}(10)^4$

(vii) $\ln(e^3)$ (viii) $\ln\sqrt{e}$.

(31) Without using the calculator find the value of x.

(i) $\log_{10}(1+x) = 3$ (ii) $\log_{10}\sqrt{x} = -1$ (iii) $\ln(x^2) = 4$

(iv) $\ln\left(\dfrac{1}{x}\right) = -2$ (v) $\log_3(3^x) = 7$ (vi) $\log_5(5^{2x}) = 8$

(vii) $\ln(4x) - 3\ln(x^2) = \ln 2$ (viii) $\ln\left(\dfrac{1}{x}\right) + \ln(2x^3) = \ln 3$

(ix) $3^x = 2$ (x) $5^{-2x} = 3$ (xi) $3e^{-2x} = 5$ (xii) $2e^{3x} = 7$

(xiii) $e^x - 2xe^x = 0$ (xiiii) $xe^{-x} - 2e^{-x} = 0$

(xv) $e^{-2x} - 3e^{-x} = -2$

(32) Expand the logarithm in terms of sums, differences, andproductsof simpler logarithms.

$(i) log\left(10x\sqrt{x-3}\right) = 3$ $(ii) \ln(\frac{x^2 sin^3 x}{\sqrt{x^2+1}})$

$(iii) log\left(\frac{\sqrt[3]{x+2}}{cos5x}\right)$ $(iv)\ ln(\sqrt{\frac{x^2+1}{x^3+5}}) = -2.$

(33) Prove each of the following:

$(i) log_b x = \frac{log_a x}{log_b x}$ $(ii)\ log(xy) = logx + logy$

$(iii)\ log\left(\frac{x}{y}\right) = logx - logy (iv)\ log\, x^y = ylogx.$

(34) Write the following function as a rational function of x.

$3\, ln(e^{2x}(e^x)^3) + 2e^{ln1}.$

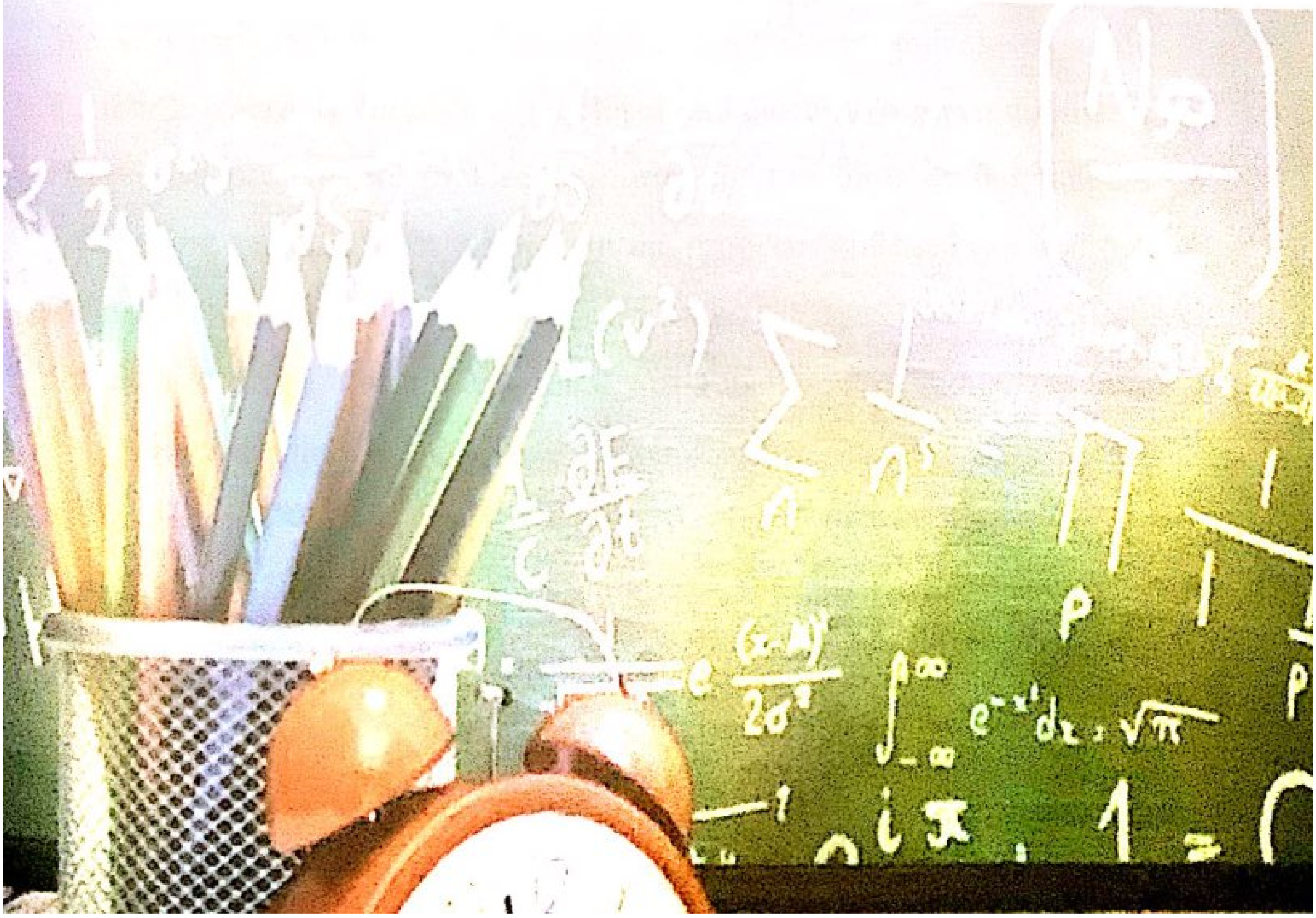
(35) Discuss the truthfulness of the following statements.

$lne^x = x,\ e^{lnx} = x,\ e^{xlnb} = b^x$, and $e^{nlnx} = x^n$.

Chapter 2

Limits and Continuity

2



Chapter 2 : Limits and Continuity

Learning Outcomes:

By completing this chapter, it is expected that the student will be able to:

- Define and find limit and limit from the right from the left.
- Define and find limit at the infinity.
- State the properties of limits.
- Apply limit properties.
- Define and study continuity of some functions.
- Determine the points of discontinuity of the function and redefine the function to be continuous.

In this chapter, we will present a basic concept of the limit, which is extremely important and one of the most important part of calculus.

We will explain how the value of $f(x)$ get closer and closer to the number L where its variable x get closer and closer to a given number a. Hence, we present the precise definition of the limit of the function $f(x)$ using ε and δ. Then we present the properties of the limit that help to find the limit of a function in an easy way.

Our main concern with the limit is the establishment of the definition of the continuous function, laying the technical basis for the definition of differentiation, by the end of this chapter; we will show the importance of limits in defining continuous functions that are widely used in calculus.

2.1 Limits

In the first chapter, we discuss the concept of the function and find the value of the function at a point in its domain, and we stopped at the points where the function is undefined, as an example, for function $f(x) = \frac{1}{x-1}$ that undefined at $x = 1$ (it does not belong to the function domain) and therefore we cannot examine the properties of the function at that point and so we need another concept to use it to examine the function at this point which is to examine the behavior of the function near the point (in the vicinity of the point) $x = 1$ that is known as the limit of the function, and therefore we can define the limit as preliminary as an examination of the behavior of the function where its independent variable gets closer and closer to a certain value as in the following definition.

Definition 2.1.1: (Preliminary Definition of the Limit)

If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as $x$approaches a." This expression can also be written as

$$f(x) \to L \text{ where } x \to a.$$

We will discuss this concept in the following example.

Example 2.1.1:

Examine the behavior of the function $f(x) = x + 1$ as x approaches 1 and give an appropriate conjecture to its limit.

Solution:

We will examine the behavior of the function as x gets closer and closer to 1 using some different numerical values of x as in the following table: (see Figure 1-2)

x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$f(x)$	1	1.5	1.9	1.99	1.999	?	2.001	2.01	2.1	2.5	3

From the table we note that, as x gets closer and closer to 1 from the left (values are less than 1) the function gets closer and closer to 2 and as x gets closer and closer to 1 from the right side (the values are greater than 1) the function gets closer and closer to 2. Thus 2 is an acceptable conjecture of the limit of the function as x approaches 1 and we can write:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2.$$

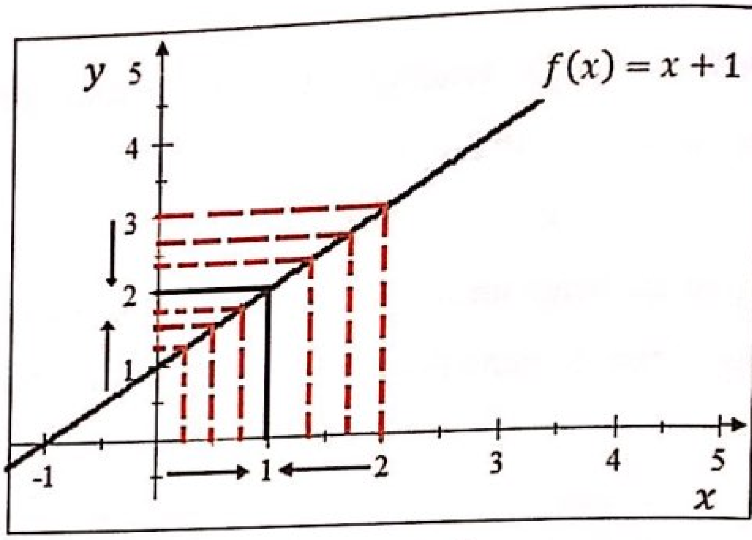


Figure (2-1)

Example 2.1.2:

Examine the behavior of the function $g(x) = \sqrt{x}$ as x approaches 9 and give an appropriate conjecture to its limit.

Solution:

By the same method in the previous example we get the following table.

x	8.9	8.95	8.98	8.99	8.999	9	9.001	9.01	9.1	9.5	10
$g(x)$	2.98	2.99	2.996	2.998	2.999	?	3.0001	3.001	3.01	3.08	3.2

From the table we note that, as x gets closer and closer to 9 from the left (the values are less than 9) the function approaches 3 and as x gets closer and closer to 9 from the right (the values are greater than 9) the function gets closer and closer to 3.Thus 3 is an acceptable conjecture as the limit of the function where x approaches 9, and we can write:

$$\lim_{x\to 9} g(x) = \lim_{x\to 9} \sqrt{x} = 3.$$

In the previous examples we have studied the behavior of a function near a point in its domain, what will be the case where the point is not in the domain of the function, this is what we will discuss in the following example.

Example 2.1.13:

Examine the behavior of the function $f(x) = \frac{x^2-16}{x-4}$ as x approaches 4 and give an appropriate conjecture to its limit.

Solution:

Note that: $x = 4$ does not belong to the domain of the function, although the limit of the function exists. By the same method in the previous examples we get the following table:

x	3.9	3.95	3.98	3.99	3.999	4	4.001	4.01	4.1	4.5	4.6
$f(x)$	7.9	7.95	7.98	7.99	7.999	?	8.001	8.01	8.1	8.5	8.6

From the table, we notice that, as x gets closer and closer to 4 from the left (the values are less than 4) the function approaches 8 and as x gets closer and closer to 4 on the right (the values are greater than 4) the function get closer and closer to 8, Thus 8 is an acceptable conjecture as the limit of the function where x approaches 4 and we can write:

$$\lim_{x\to4} f(x) = \lim_{x\to4} \frac{x^2-16}{x-4} = 8.$$

See figure (2-2)

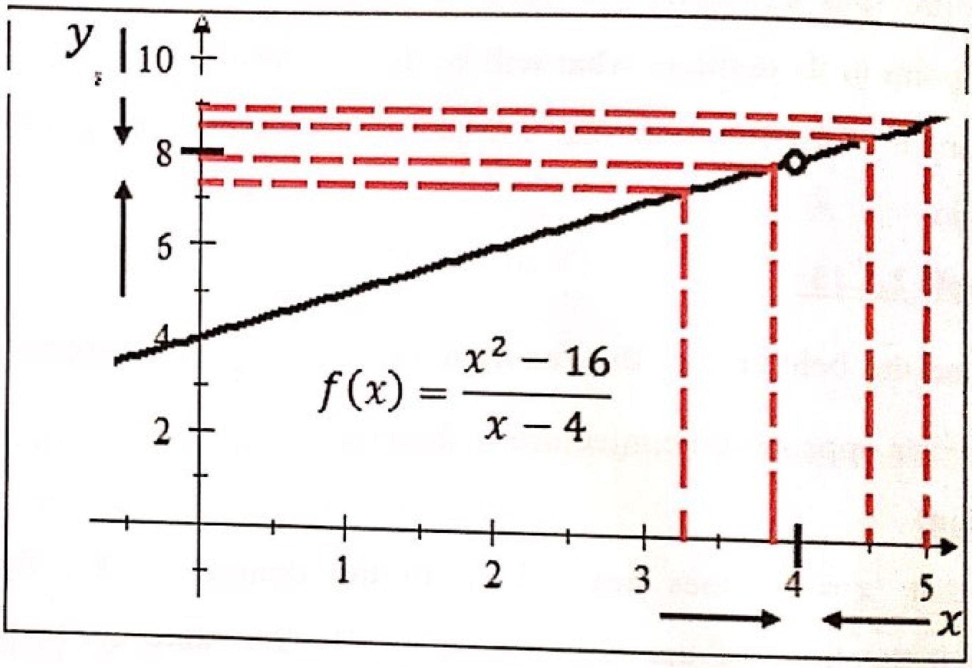


Figure (2-2)

Note:

The function f is not required to be defined at the point a (i.e., it is not necessary to be a in the domain of the function f), but function f must be defined in the vicinity of the point a (that means in an open interval containing a).

Example 2.1.4:

Let $f(x) = \frac{1}{4}x + 1$. Find $\lim_{x\to8} f(x)$.

Solution:

Through the following brief table:

x	7.5	7.9	7.99	7.999	8	8.001	8.01	8.1	8.5
$f(x)$	2.875	2.975	2.9975	2.99975	3	3.00025	3.0025	3.025	3.125

We conclude that $\lim_{x\to 8}(\frac{1}{4}x + 1) = 3$.

In this example we note the following.

If $7.5 < x < 8.5$, then $2.875 < f(x) < 3.125$.

If $7.9 < x < 8.1$, then $2.975 < f(x) < 3.025$.

If $7.99 < x < 8.01$, then $2.9975 < f(x) < 3.0025$.

If $7.999 < x < 8.001$, then $2.99975 < f(x) < 3.00025$.

This means, if $\varepsilon > 0, \delta > 0$ (two positive and small real numbers), then we conclude if:

$8 - \delta < x < 8 + \delta$, then $3 - \varepsilon < f(x) < 3 + \varepsilon$

In the first case: $\delta = 0.5$ and $\varepsilon = \frac{\delta}{4}$.

In the fourth case: $\delta = 0.001$ and $\varepsilon = \frac{\delta}{4}$.

It can be written in another form as follows:

If $-\delta < x - 8 < \delta$, then $-\varepsilon < f(x) - 3 < \varepsilon$.

That is, if $|x - 8| < \delta$, then $|f(x) - 3| < \varepsilon$.

From the previous example, we can define the limit as follows.

Definition 2.1.2: (Limit)

Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a.

We will write

$$\lim_{x \to a} f(x) = L$$

if given any number $\varepsilon > 0$ we can find a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Example 2.1.5:

Using the definition of the limit, prove that $\lim_{x \to 2} \frac{9x-2}{4} = 4$.

Solution:

We try to find $\delta > 0$ for all $\varepsilon > 0$, so that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Assuming $\varepsilon > 0$ where $|f(x) - L| < \varepsilon$

$$\Rightarrow \left|\frac{9x-2}{4} - 4\right| < \varepsilon$$

$$\Rightarrow \left|\frac{9x-2-16}{4}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{9(x-2)}{4}\right| < \varepsilon$$

or

$$\Rightarrow |x-2| < \frac{4}{9}\varepsilon$$

If $\delta = \frac{4}{9}\varepsilon$ ، so for all $|x-2| < \delta$, then $\left|\frac{9x-2}{4} - 4\right| < \varepsilon$.

In all the previous examples, we discussed the approach of x to apoint $(x \to a)$, but we have not discuss yet: Is this approach from the right or from the left? Is the limit of the function where the approach from the right side necessarily equals the limit where the approach from the left? Wherever does the limit of the function exist? The following example and definition illustrate this idea.

Example 2.1.6:

Find $\lim_{x\to 0} f(x)$if

$$f(x) = \begin{cases} 0 \ , & x < 0 \\ 2 \ , & x \geq 0 \end{cases}$$

Solution:

As x approaches zero from the right, we find that:

$$\lim_{x\to 0} f(x) = 2$$

Asxapproacheszero from the left, we find that:

$$\lim_{x\to 0} f(x) = 0$$

This means that $\lim_{x\to 0} f(x)$is not unique and therefore we say that the limit of the function $f(x)$ does not exist as x approaches zero (see Figure 2-3).

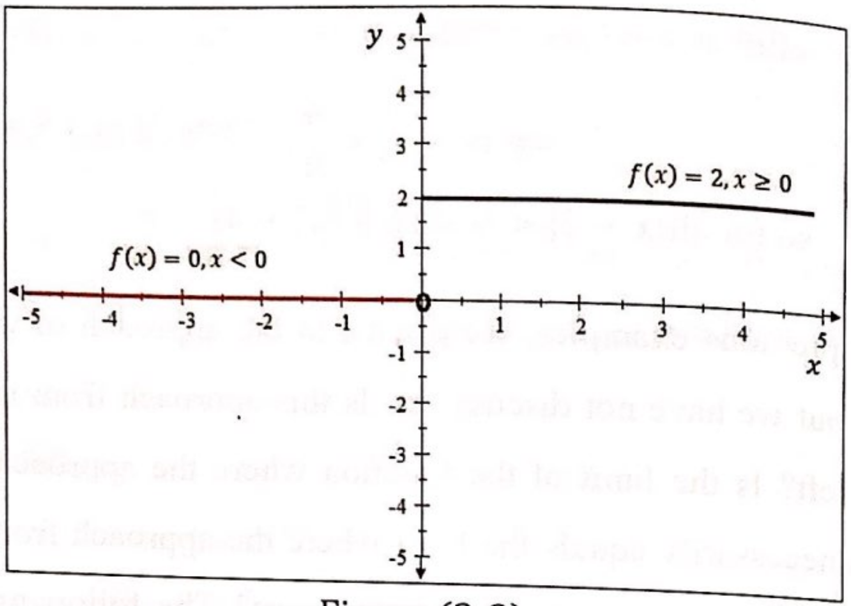


Figure (2-3)

2.2 One-Sided Limit

Sometimes we are interested in examining the behavior of a function as the variable x approaches one side- for example, if we have a physical system and only have data during a specific time interval and we want to say things about the behavior of the system at the end of the time interval. This motivates us to define the limit from one side as follows.

Definition 2.2.1:

Let L be a real number.

(1) The function $f(x)$ is said to have a right limit L when x approaches a from the right $(x > a)$, that is written $\lim_{x \to a^+} f(x) = L$, if for each $\varepsilon > 0$ (it does not matter how small the number ε), there is $\delta > 0$ (depends on ε) so that for each $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.

(2) The function $f(x)$ is said to have a left limit L as x approaches a from the left $(x < a)$, that is written $\lim_{x\to a^-} f(x) = L$, if for each $\varepsilon > 0$ (it does not matter how small the number ε) there is $\delta > 0$ (depends on ε) so that for each $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$.

Notes:

(1) The symbol $x \to a^+$ means that x is greater than a and approaches from the right, and the symbol $x \to a^-$ means that x is smaller than a and approaches from the left.

(2) The only difference in the above definition is the restriction on the x axis: the limit from two sides has $0 < |x - a| < \delta$, while the limit on the right side has $a < x < a + \delta$, and the limit from the left side has $a - \delta < x < a$

(3) The concept limit as defined in 2.1.2 is sometimes called "two-sided limits".

(4) For the limit of the function to be exist, the right limit and the left limit must be exist and equal at a.

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$$

(5) If one or both of the one-sided limits fail to exist or are not equal we say that the limit does not exist.

Example 2.2.1:

If $f(x) = \begin{cases} x^2 - 1 & , x > 2 \\ 5 & , x = 2 \\ x + 1 & , x < 2. \end{cases}$

Find

$\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $\lim_{x\to 2} f(x)$.

Solution:

- For $x > 2$, we have $f(x) = x^2 - 1$, thus $\lim_{x\to 2^+}(x^2 - 1) = 3$.
- For $x < 2$, we have $f(x) = x + 1$, so $f(x) = x + 1$.

  Since the limit from the right side equals the limit from the side, then the limit of the function exists and equals $\lim_{x\to 2} f(x) = 3$.

Example 2.2.2:

Let

$$f(x) = \frac{|x-3|}{x-3}.$$

Find

$$\lim_{x\to 3^+} f(x), \quad \lim_{x\to 3^-} f(x), \text{ and } \lim_{x\to 3} f(x).$$

Solution:

From the definition of the absolute value, we have

$$|x-3| = \begin{cases} x-3, & x-3 \geq 0 \\ -(x-3), & x-3 < 0 \end{cases}$$

or

$$|x-3| = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3. \end{cases}$$

So

$$\frac{|x-3|}{x-3} = \begin{cases} 1, & x > 3 \\ -1, & x < 3 \end{cases}$$

Thus $\lim_{x\to 3^+} \frac{|x-3|}{x-3} = 1$, and $\lim_{x\to 3^-} \frac{|x-3|}{x-3} = -1$,

therefor $\lim_{x\to 3} \frac{|x-3|}{x-3}$ does not exist.

2.3 Techniques for Computing Limits

In this section we will give some important properties and theorems for computing limits that cannot be computed by direct techniques.

Theorem 2.3.1: Uniqueness of limit of function

If the limit of the function $f(x)$ at a point a exists, then it is unique.

Proof:

We will proof the theorem using the definition of the limit.

Let L, and M be two limits of the function $f(x)$ at a.

i.e., $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} f(x) = M$ such that $L \neq M$.

We will show that, it is a contradiction.

Let $\varepsilon > 0$,

since $\lim_{x\to a} f(x) = L$ then for every $\varepsilon_1 = \frac{\epsilon}{2}$ there exists $\delta_1 > 0$ such that

if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon_1 = \frac{\varepsilon}{2}$.

Likewise if $\lim_{x\to a} f(x) = M$ then for every $\varepsilon_2 = \frac{\epsilon}{2}$ there exists $\delta_2 > 0$

such that if $0 < |x-a| < \delta_2$then $|f(x) - M| < \varepsilon_2 = \frac{\varepsilon}{2}$.

Let $\delta = min\{\delta_1, \delta_2\}$ (the smallest number betweenδ_1, δ_2) such that $0 < |x-a| < \delta$.

Then $|L - M| = |L - f(x) + f(x) - M| \leq |L - f(x)| + |f(x) - M|$

$$< \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is an unspecified value, then it can be chosen very close to zero, that means that $|L - M| = 0$, implies $L = M$ which is a contradiction.

So our assumption that $L \neq M$ was wrong, thus, if $\lim_{x \to a} f(x) = L$ then the limit L is unique.

Theorem 2.3.2:

The limit of the constant function is the same as the constantfunction, i.e., if $f(x) = L$ (where L is a constant), then

$$\lim_{x \to a} f(x) = \lim_{x \to a}(L) = L$$

Proof:

For every $\varepsilon > 0$ there is $\delta > 0$ such that, If $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. We can choose any small positive number $\delta > 0$ because the function f does not depend on x $(f(x) = L)$,

then $|f(x) - L| = |L - L| = 0 < \varepsilon$.

Theorem 2.3.3:

If $f(x) = \frac{1}{x}$, then the limit of the function where x approaches zero does not exist, i.e., $\lim_{x \to 0} \left(\frac{1}{x}\right)$ does not exist.

Proof:

We assume, by the way of contradiction, that $\lim\limits_{x\to 0}\left(\frac{1}{x}\right) = M$ i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that if $|x - 0| < \delta$ then $\left|\frac{1}{x} - M\right| < \varepsilon$, or in other words if $-\delta < x < \delta$ then $M - \varepsilon < \frac{1}{x} < M + \varepsilon$.

The first case: If $M > 0$, then $\frac{1}{M+\varepsilon} > 0$, i.e., if x so that $x < \frac{1}{M+\varepsilon}$, and $0 < x < \delta$ then for all $x < \delta$ then $\frac{1}{x} > M + \varepsilon$, which contradicts the assumption.

The second case: If $M < 0$, then $\frac{1}{M-\varepsilon} < 0$ i.e. if x so that $x > \frac{1}{M-\varepsilon}$, and $x > -\delta$ then for all $-\delta < x < 0$ then $\frac{1}{x} < M - \varepsilon$ which contradicts the assumption.

In the first case the positive number M cannot be found such that, $\lim\limits_{x\to 0}\left(\frac{1}{x}\right) = M$.

In the second case, the negative number M cannot be found such that, $\lim\limits_{x\to 0}\left(\frac{1}{x}\right) = M$.

Therefore, $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)$ does not exist.

2.4 Limits Properties

Suppose that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, and if c is a real constant then;

(1) The limit of the constant function is the same as the constant function $\lim_{x \to a} c = c$.

Example 2.4.1:

If $f(x) = 3$, then $\lim_{x \to a} f(x) = 3$.

(2) $\lim_{x \to a}[cf(x)] = c \lim_{x \to a}[f(x)]$.

(3) If $f(x)$, and $g(x)$ are polynomial such that

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$
$$g(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n,$$

where $b_0, b_1, b_2, \dots, b_n, c_0, c_1, c_2, \dots, c_n$ are real constants and n is a positive integer, then

$$\lim_{x \to a} f(x) = f(a) = b_0 + b_1a + b_2a^2 + \cdots + b_na^n.$$
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \quad if\ g(a) \neq 0.$$

(4) The limit of the algebraic sum of the functions $f(x)$ *and* $g(x)$ equals the algebraic sum of the two limits of the function, i.e.,

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

In general, the limit of the algebraic sum of a finite number of functions equals the algebraic sum of the limits of these functions.

Example 2.4.2:

Let $f(x) = x^3 - 3$ and $g(x) = x + 1$. Find $\lim_{x \to 2}[f(x) + g(x)]$.

Solution:

Since $\lim_{x \to 2} f(x) = \lim_{x \to 2}(x^3 - 3) = 2^3 - 3 = 5$, and

$\lim_{x \to 2} g(x) = \lim_{x \to 2}(x + 1) = 2 + 1 = 3.$

So

$$\lim_{x \to 2}[f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 5 + 3 = 8\,.$$

Example 2.4.3:

Let $f(x) = 3x^2 + 5x - 9$. Find $\lim_{x \to -2} f(x)$.

Solution:

$$\lim_{x \to -2} f(x) = \lim_{x \to -2}(3x^2 + 5x - 9) = 3(-2)^2 + 5(-2) - 9 = -7.$$

(5) $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x).$

Suppose the existence of these limits, the limit of the product of two functions equals the product of the limits of the two functions.

Generally, the limit of the product of a finite number of functions equals the product of the limits of these functions.

Example 2.4.4:

Find $\lim_{x \to 1}(x + 2)(x - 3)$.

Solution:

Since $\lim_{x \to 1}(x + 2) = 3$, $\lim_{x \to 1}(x - 3) = -2$, then

$$\lim_{x\to1}(x+2)(x-3) = \lim_{x\to1}(x+2).\lim_{x\to1}(x-3) = 3(-2) = -6.$$

(6) Let $\lim_{x\to a}[g(x)] \neq 0$. Then $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a}[f(x)]}{\lim_{x\to a}[g(x)]}$.

Suppose the existence of these limits, the limit of the quotient of two functions equals the quotient of the limits of the two functions.

Example 2.4.5:

Find

$$\lim_{x\to1}\left[\frac{6-3x+10x^2}{-2x^4+7x^3+1}\right].$$

Solution:

$$\lim_{x\to1}(6-3x+10x^2) = 13, \text{and } \lim_{x\to1}(-2x^4+7x^3+1) = 6.$$

So that

$$\lim_{x\to1}\left[\frac{6-3x+10x^2}{-2x^4+7x^3+1}\right] = \frac{\lim_{x\to1}(6-3x+10x^2)}{\lim_{x\to1}(-2x^4+7x^3+1)} = \frac{13}{6}.$$

(7) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a}f(x)\right]^n$, where n is a real number and $\lim_{x\to a}f(x) \neq 0$ at $n = 0$. If n is an integer, then it will be completely similar to the property number (5).

For example, if $n = 2$ then

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a}[f(x)f(x)].$$

Using the property No. (5)

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a} f(x) \lim_{x\to a} f(x) = \lim_{x\to a}[f(x)]^2$$

The same can be done for any integer n.

(8) $\lim_{x\to a} \sqrt[n]{f(x)} = \lim_{x\to a}[f(x)]^{\frac{1}{n}} = \left[\lim_{x\to a} f(x)\right]^{\frac{1}{n}} = \sqrt[n]{\lim_{x\to a} f(x)}.$

(9) $\lim_{x\to a} x^n = a^n$, where a and n are not zero in the same time, this is a special case of the property (7).

(10) if the direct substitution results in $f(a)$ in the indeterminate form $\frac{0}{0}$, then algebraic operations must be performed to compute $\lim_{x\to a} f(x)$.

Example 2.4.6:

Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

Solution:

By direct substitution we get the indeterminate form $\frac{0}{0}$, and therefore an algebraic operation is required (factoring the difference of two cubes).

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1} = x^2 + x + 1,$$

so that

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2 + x + 1) = 3.$$

Example 2.4.7:

Find $\lim_{x\to 0} x(1+x^{-1})$.

Solution:

Where

$$\lim_{x\to 0} x(1+x^{-1}) = \lim_{x\to 0} x \lim_{x\to 0}(1+x^{-1}).$$

By direct substitution we get the indeterminate form of $0.\infty$.

So we have to perform an algebraic operation.

$$\lim_{x\to 0} x(1+x^{-1}) = \lim_{x\to 0}(x+1) = 1.$$

Example 2.4.8:

Find $\lim_{x\to 2} \frac{x-\sqrt{2+x}}{x-2}$.

Solution:

By direct substitution we get the indeterminate form $\frac{0}{0}$, and therefore an algebraic operation is required.

$$\lim_{x\to 2} \frac{x-\sqrt{2+x}}{x-2} = \lim_{x\to 2} \frac{(x-\sqrt{2+x})(x+\sqrt{2+x})}{(x-2).(x+\sqrt{2+x})}$$

$$= \lim_{x\to 2} \frac{x^2-x-2}{(x-2).(x+\sqrt{2+x})}$$

$$= \lim_{x\to 2} \frac{(x-2).(x+1)}{(x-2).(x+\sqrt{2+x})}$$

$$= \lim_{x \to 2} \frac{(x+1)}{(x+\sqrt{2+x})} = \frac{3}{4}.$$

Theorem:2.4.1

$$\lim_{x \to a} \left[\frac{x^n - a^n}{x - a}\right] = na^{n-1},$$

Where n is a positive integer.

Proof:

Let $x = a + h$. Then $h \to 0$ where $x \to a$.

$$\lim_{x \to a} \left[\frac{x^n - a^n}{x - a}\right] = \lim_{x \to a} \frac{(a+h)^n - a^n}{a + h - a}$$

$$= \lim_{h \to 0} \frac{\left(a^n + \frac{n}{1!}a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \cdots + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{\left(a^n + \frac{n}{1!}a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \cdots + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h\left(\frac{n}{1!}a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \cdots + h^{n-1}\right)}{h}$$

$$= \lim_{h \to 0} \left(\frac{n}{1!}a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \cdots + h^{n-1}\right)$$

$$= na^{n-1} + 0 + 0 + \cdots + 0$$

$$\therefore \lim_{x \to a} \left[\frac{x^n - a^n}{x - a}\right] = na^{n-1}.$$

Theorem 2.4.2: Some Important Limits

i) $\lim_{x\to 0}\frac{\sin x}{x}=1$ ii) $\lim_{x\to 0}\frac{(1-\cos x)}{x}=0$ iii) $\lim_{x\to 0}\frac{\tan x}{x}=1$

iv) $\lim_{x\to 0}\sin x=0$ v) $\lim_{x\to 0}\cos x=1$

iv) $\lim_{x\to\infty}(1+x)^{\frac{1}{x}}=e=2.71828$ or $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^{x}=e.$

vii) $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$ viii) $\lim_{x\to 0}\frac{e^x-1}{x}=1.$

We will prove some of these limits.

Proof:

1- Proof of the relation (ii)

$$\lim_{x\to 0}\frac{(1-\cos x)}{x}=\lim_{x\to 0}\frac{(1-\cos x)}{x}\frac{(1+\cos x)}{(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

Using the identity $\sin^2 x+\cos^2 x=1$ then

$$\lim_{x\to 0}\frac{(1-\cos x)}{x}=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin x}{x(1+\cos x)}$$

$$=1.\left(\frac{0}{1+1}\right)=0$$

2- Proof of the relation (iii)

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\frac{1}{\cos x}\right)\text{ since, }\tan x=\frac{\sin x}{\cos x}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{1}{\cos x}\right)=1.1=1.$$

We will prove the relations $(vi)$and $(viii)$ only for reading, and we will illustrate them in more advanced courses.

3- The proof of the relation (vi), by using the binomial theorem where$|x|<1$ we get

$$(1+x)^{\frac{1}{x}}=\left[1+\frac{1}{x}x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3+\cdots\right]$$

$$=\left[1+1+\frac{(1-x)}{2!}x+\frac{(1-x)(1-2x)}{3!}+\cdots\right]$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to 0}\left[1+1+\frac{(1-x)}{2!}+\frac{(1-x)(1-2x)}{3!}+\cdots\right],$$

where

$$e^x=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{2!}+\cdots\right)\ldots\ldots\ldots\ldots\ldots(1)$$

So that

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}==\left[1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right]=e$$

4- Proof of the relation $(viii)$

$$\lim_{x\to 0}\frac{log(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}.\,log(1+x)$$

$$=\lim_{x\to 0} log(1+x)^{\frac{1}{x}}$$

$$=log\, e=1$$

5- Proof of the relation $(viii)$

From (1), then

$$e^x-1=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{2!}+\cdots-1\right)$$

$$e^x-1=\left(x+\frac{x^2}{2!}+\frac{x^3}{2!}+\cdots\right)$$

dividing by x we get

$$\frac{e^x-1}{x}=\frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{2!}+\cdots\right)}{x}$$

$$=\frac{x\left(1+\frac{x^1}{2!}+\frac{x^2}{2!}+\cdots\right)}{x}$$

$$=\left(1+\frac{x^1}{2!}+\frac{x^2}{2!}+\cdots\right).$$

$$\therefore \lim_{x\to 0}\frac{e^{x}-1}{x}=\lim_{x\to 0}\left(1+\frac{x^{1}}{2!}+\frac{x^{2}}{2!}+\cdots\right)$$

$$=1+0+0+\cdots=1$$

Example 2.4.9

Using the limits in the theorem 2.4.2, find the following limits

$(a)\ \lim_{x\to 0}\frac{sin(3x)}{9x}$ $\qquad$ $(b)\ \lim_{x\to 0}\frac{sin(x^2)}{x\,sin(x)}$

$(c)\ \lim_{\theta\to 0}\frac{1-cos(4\theta)}{1-cos(6\theta)}$ $\qquad$ $(d)\ \lim_{x\to 0}\frac{1-cos(x)}{x^2}$

$(e)\ \lim_{x\to 0}\frac{x+3sin(x)}{\sqrt{x^2+4\,sin(x)+1}-\sqrt{sin^2(x)-x+1}}$

$(f)\ \lim_{h\to 0}(1+3h)^{\frac{1}{h}}$ $\qquad$ $(g)\ \lim_{x\to 0}\frac{e^{x}-e^{x}}{x}$

$(h)\ \lim_{x\to 1}\frac{e^{x}-e}{x-1}$

Solution:

$$(a)\ \lim_{x\to 0}\frac{sin(3x)}{9x}=\lim_{x\to 0}\left(\frac{1}{3}.\frac{sin(3x)}{3x}\right)$$

$$=\lim_{x\to 0}\frac{1}{3}.\lim_{x\to 0}\frac{sin(3x)}{3x}$$

$$=\frac{1}{3}.1=\frac{1}{3}.$$

Where we used the property $\lim_{x\to 0}\frac{sin(x)}{x}=1$.

$$(b)\ \lim_{x\to 0}\frac{sin(x^2)}{x\,sin(x)} = \lim_{x\to 0}\left(\frac{sin(x^2)}{x^2}.\frac{x}{sin(x)}\right)$$

$$= \lim_{x\to 0}\frac{sin(x^2)}{x^2}.\lim_{x\to 0}\frac{x}{sin(x)}$$

$$= \lim_{x\to 0}\frac{sin(x^2)}{x^2}.\lim_{x\to 0}\frac{1}{\frac{sin(x)}{x}}$$

$$= 1.1 = 1.$$

$$(c)\ \lim_{\theta\to 0}\frac{1-\cos(4\theta)}{1-cos(6\theta)} = \lim_{\theta\to 0}\frac{2\,sin^2 2\theta}{2\,sin^2 3\theta}$$

$$= \lim_{\theta\to 0}\left(\left(\frac{sin\,2\theta}{2\theta}.2\theta\right)^2.\left(\frac{3\theta}{sin\,3\theta}.\frac{1}{3\theta}\right)^2\right)$$

$$= \lim_{\theta\to 0}\left(\left(\frac{sin\,2\theta}{2\theta}\right)^2.\left(\frac{3\theta}{sin\,3\theta}\right)^2.\frac{4\theta^2}{9\theta^2}\right)$$

$$= \frac{4}{9}.\lim_{2\theta\to 0}\left(\frac{sin\,2\theta}{2\theta}\right)^2.\lim_{3\theta\to 0}\left(\frac{3\theta}{sin\,3\theta}\right)^2$$

$$= \frac{4}{9}.1.1 = \frac{4}{9}.$$

$$(d)\ \lim_{x\to 0}\frac{1-cos(x)}{x^2} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{4.\left(\frac{x}{2}\right)^2}$$

$$= \frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)\left(\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right) = \frac{1}{2}.$$

Where we used the identity $1 - cos(x) = 2\sin^2\left(\frac{x}{2}\right)$.

(e) $\lim_{x\to 0}\frac{x + 3sin(x)}{\sqrt{x^2 + 4\,sin(x) + 1} - \sqrt{sin^2(x) - x + 1}}$

$$= \lim_{x\to 0}\frac{x + 3sin(x)}{\sqrt{x^2 + 4\,sin(x) + 1} - \sqrt{sin^2(x) - x + 1}}$$

$$\times \frac{\sqrt{x^2 + 4\,sin(x) + 1} + \sqrt{sin^2(x) - x + 1}}{\sqrt{x^2 + 4\,sin(x) + 1} + \sqrt{sin^2(x) - x + 1}}$$

$$= \lim_{x\to 0}\frac{(x + 3sin(x))(\sqrt{x^2 + 4\,sin(x) + 1} + \sqrt{sin^2(x) - x + 1})}{(x^2 + 4\,sin(x) + 1) - (sin^2(x) - x + 1)}$$

$$= \lim_{x\to 0}\frac{(x + 3sin(x))(\sqrt{x^2 + 4\,sin(x) + 1} + \sqrt{sin^2(x) - x + 1})}{x^2 - sin^2(x) + 4\,sin(x) + x},$$

divide by x

$$= \lim_{x\to 0}\frac{(1 + 3\frac{sin(x)}{x})(\sqrt{x^2 + 4\,sin(x) + 1} + \sqrt{sin^2(x) - x + 1})}{x - sin(x)\frac{sin(x)}{x} + 4\frac{sin(x)}{x} + 1}$$

$$= \frac{4(2)}{4 + 1} = \frac{8}{5}.$$

(f) $\lim_{h\to 0}(1 + 3h)^{\frac{1}{h}} = \lim_{h\to 0}(1 + 3h)^{\frac{3}{3h}}$

$$= \lim_{h\to 0}\left[(1 + 3h)^{\frac{1}{3h}}\right]^3 = e^3.$$

In order to compute the limit of the function in (g) we use the relationship $\lim_{x\to 0}\frac{e^{x}-1}{x}=1$ for $x=-x$ we have $\lim_{x\to 0}\frac{e^{-x}-1}{-x}=1$, then the required limit can be written as,

$$\lim_{x\to 0}\frac{e^{x}-e^{-x}}{x}=\lim_{x\to 0}\frac{e^{x}-1+1-e^{-x}}{x}=\lim_{x\to 0}\left[\frac{e^{x}-1}{x}+\frac{1-e^{-x}}{x}\right]$$

$$=\lim_{x\to 0}\left[\frac{e^{x}-1}{x}+\frac{e^{-x}-1}{(-x)}\right]$$

$$=\lim_{x\to 0}\left[\frac{e^{x}-1}{x}\right].\lim_{x\to 0}\left[\frac{e^{-x}-1}{(-x)}\right]=1+1=2$$

$$\therefore \lim_{x\to 0}\frac{e^{x}-e^{-x}}{x}=2.$$

In order to compute the limit of the function in (f), we assume $x=\mathrm{h}+1$ so $(x\to 1)\Leftrightarrow(h\to 0)$

$$\lim_{x\to 1}\frac{e^{x}-e}{x-1}=\lim_{h\to 1}\frac{e^{1+h}-e}{h}=\lim_{h\to 1}\frac{e^{1}e^{h}-e}{h}$$

$$=\lim_{h\to 1}\frac{e(e^{h}-e)}{h}=e\lim_{h\to 1}\frac{(e^{h}-1)}{h}$$

$$=e.1=e$$

$$\therefore \lim_{x\to 1}\frac{e^{x}-e}{x-1}=e.$$

<u>Theorem 2.4.3:</u>

If $f(x)\leq g(x)$ for all x in an interval containing a (except for the point itself) then $\lim_{x\to a}f(x)\leq \lim_{x\to a}g(x)$.

Theorem 2.4.4: Squeeze Theorem

If $f(x) \leq h(x) \leq g(x)$ for all x in an interval containing a (except for the point itself) and if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$,

Then $h(x)$ has the same limit as $f(x)$, and $g(x)$. (See Figure 2-4)

Proof:

Since $\lim_{x\to a} f(x) = L$,from the definition of the limit, for each $\varepsilon > 0$, there is $\delta_1 > 0$ so that if $|x-a| < \delta_1$ then $|f(x) - L| < \varepsilon$.

Also $\lim_{x\to a} g(x) = L$, then for each $\varepsilon > 0$ there is $\delta_2 > 0$ so that if $|x-a| < \delta_2$ then $|g(x) - L| < \varepsilon$.

Now by choosing $\delta = \min\{\delta_1, \delta_2\}$ (the smallest number between(δ_1, δ_2) if $|x-a| < \delta$ then $g(x)$, and $f(x)$are both lie between $L - \varepsilon$, and $L + \varepsilon$ as well as $f(x) \leq h(x) \leq g(x)$ implies that $L - \varepsilon < h(x) < L + \varepsilon$ meaning that $-\varepsilon < h(x) - L < \varepsilon$ or $|h(x) - L| < \varepsilon$ i.e.,

$$\lim_{x\to a} h(x) = L.$$

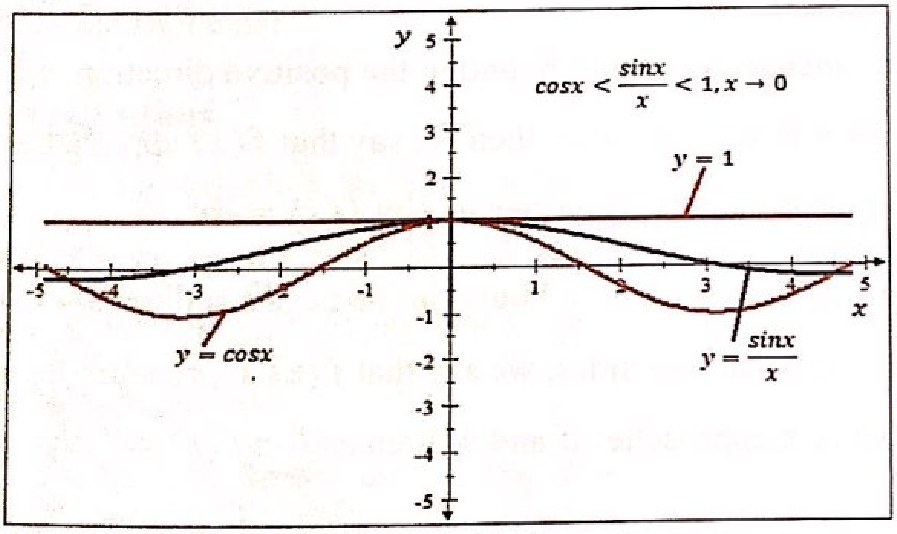


Figure (2-4)

Theorem 2.4.5:

Let a be a real number in the domain of the given trigonometric function. Then

(a) $\lim_{x \to a} \sin x = \sin a$ (b) $\lim_{x \to a} \cos x = \cos a$

(c) $\lim_{x \to a} \tan x = \tan a$ (d) $\lim_{x \to a} \cot x = \cot a$

(e) $\lim_{x \to a} \sec x = \sec a$ (f) $\lim_{x \to a} \csc x = \csc a.$

2.5 Infinity Limits

When we try to find the limit of the function $\frac{1}{x}$ where the value of the variable x approaches zero from the right, we find that $\frac{1}{x}$ increases without an upper bound, i.e., $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Noting that ∞ expresses the behavior of the function and is not a number on the real line.

Definition 2.5.1:

(1) If $f(x)$ increases without bound in the positive direction, where x approaches a from both sides, then we say that $f(x)$ approaches infinity where x approaches a and written as $\lim_{x \to a} f(x) = \infty$.

(2) If $f(x)$ decreases without bound in the negative direction where x approaches a from both sides, we say that $f(x)$ approaches the negative infinity when x approaches a and written as $\lim_{x \to a} f(x) = -\infty$.

Example 2.5.1:

Find $\lim_{x\to 1}\frac{1}{(x-1)^2}$.

Solution:

We note that where x gets closer and closer to1 from both sides, $(x-1)^2$ is positive and gets closer to zero, so $\frac{1}{(x-1)^2}$ increases without an upper bound, i.e.,

$$\lim_{x\to 1}\frac{1}{(x-1)^2}=\infty.$$

Example 2.5.2:

Find $\lim_{x\to 0}\frac{1}{x}$, if it exists.

Solution:

Note that $\lim_{x\to 0^+}\frac{1}{x}=\infty$ and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$ so that.

$$\lim_{x\to 0^-}\frac{1}{x}\neq\lim_{x\to 0^+}\frac{1}{x}$$

So the limit does not exist.

2.6 Limits at Infinity

If the value of the variable x increases without bound then we say that x approaches $+\infty$ and write $x\to+\infty$ and where the value of the variable x decreases without bound we say that x approaches $-\infty$ and write $x\to-\infty$. For example we have,

$$\lim_{x\to-\infty}\frac{1}{x}=0 \text{ or } \lim_{x\to+\infty}\frac{1}{x}=0.$$

Which is clear from the following table and figure:

		Values for the variable x					The result
x	-1	-10	-100	-1000	-10,000	...	When x approaches $-\infty$, the value of the function $1/x$ increases to zero
$1/x$	-1	-0.1	-0.01	-0.001	-0.0001	...	
x	1	10	100	1000	10,000	...	When x approaches $+\infty$, the value of the function $1/x$ decreases to zero
$1/x$	1	0.1	0.01	0.001	0.0001	...	

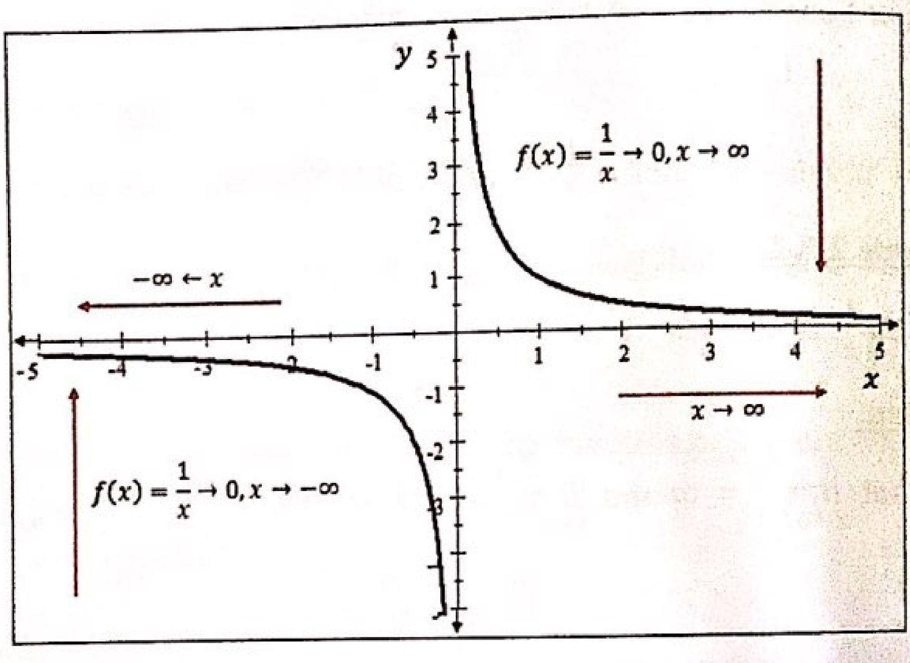


Figure (2-5)

Definition 2.6.1 : Limits at Infinity

If the value of the function $f(x)$ approaches L where the value of the variable x increases without bound we say that $f(x) \to L$ as $x \to +\infty$ or $\lim_{x \to +\infty} f(x) = L$.

Likewise if the value of the function $f(x)$ approaches L where the value of the variable x decreases without bound then we say that $f(x) \to L$ as $x \to -\infty$ or $\lim_{x \to -\infty} f(x) = L$.

General Rules for Limits of the Function at Infinity:

a) $\lim\limits_{x\to\pm\infty} (f(x))^n = \left(\lim\limits_{x\to\pm\infty} f(x)\right)^n$, where $\lim\limits_{x\to\pm\infty} f(x) \neq 0$ at $n = 0$.

b) $\lim\limits_{x\to\pm\infty} kf(x) = k \lim\limits_{x\to\pm\infty} f(x)$, where k is constant.

c) $\lim\limits_{x\to\pm\infty} k = k$

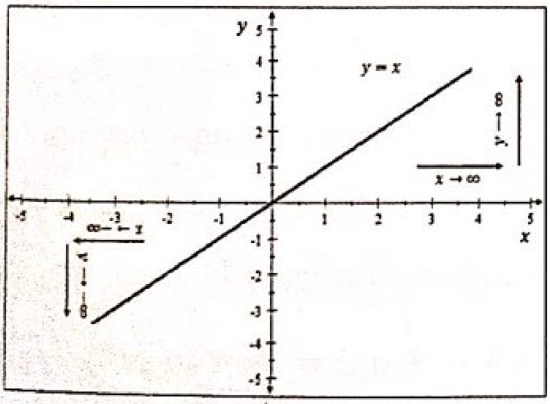
d) $\lim\limits_{x\to\pm\infty} \frac{1}{x^n} = \lim\limits_{x\to\pm\infty} \left(\frac{1}{x}\right)^n = 0$, where n is a positive number.

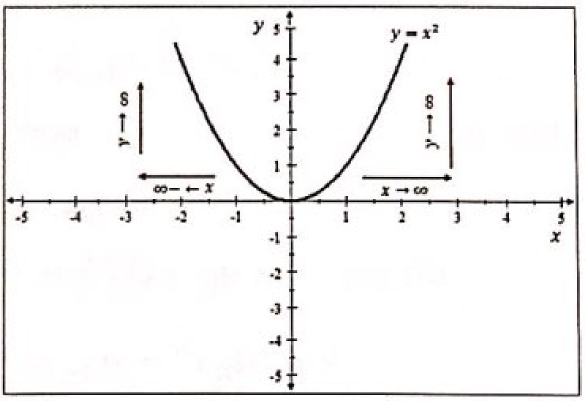
e) $\lim\limits_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = e.$

Limits of Functions as $x \to \pm\infty$

1) The power function $y = x^n$, we note that the limit of this function as $x \to +\infty$ is $+\infty$, and as $x \to -\infty$, the limit depends on the value of the number n as being an even or odd number as follows, (see figure 2-6)

$$\lim_{x\to+\infty} x^n = +\infty, \text{ and } \lim_{x\to-\infty} x^n = \begin{cases} -\infty, & n = 1,3,5,\dots \\ +\infty, & n = 2,4,6,\dots . \end{cases}$$





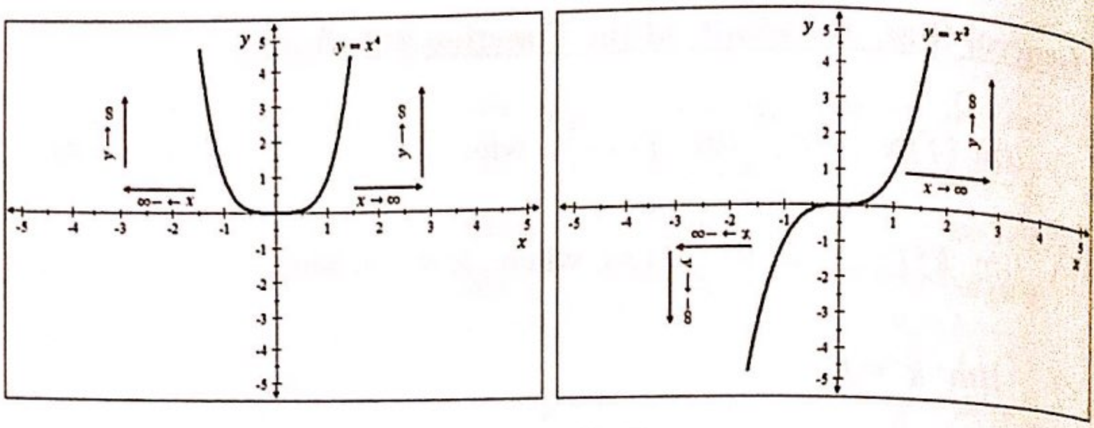


Figure (2-6)

Example 2.6.1:

Find the limits of the following functions.

a) $\lim\limits_{x \to +\infty} 6x^2$ b) $\lim\limits_{x \to -\infty} 6x^2$ c) $\lim\limits_{x \to +\infty} -3x^7$ d) $\lim\limits_{x \to -\infty} -3x^7$

Solution:

a) $\lim\limits_{x \to +\infty} 6x^2 = +\infty$ b) $\lim\limits_{x \to -\infty} 6x^2 = +\infty$ c) $\lim\limits_{x \to +\infty} -3x^7 = -\infty$

d) $\lim\limits_{x \to -\infty} -3x^7 = +\infty$.

2) Polynomials

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$.

For computing the limit of a polynomial we compute only the limit of the term with the highest power of x, i.e.,

$$\lim_{x \to +\infty} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) = \lim_{x \to +\infty} (a_n x^n)$$

$$\lim_{x \to -\infty} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) = \lim_{x \to -\infty} (a_n x^n)$$

Example 2.6.2:

a) $\lim_{x\to+\infty}(6x^3+7x^2-2x^2+3)=\lim_{x\to+\infty}6x^3=+\infty.$

b) $\lim_{x\to-\infty}(10x^7+6x^2-2)=\lim_{x\to-\infty}10x^7=-\infty.$

c) $\lim_{x\to+\infty}(1+x-3x^7)=-\infty.$

d) $\lim_{x\to-\infty}(-3x^9+2x+4)=+\infty.$

3) Rational Functions

Let the rational function as

$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\cdots+b_1x+b_0}$, where m,n are two positive integers and we wanted to find the limit of this function as $x\to\pm\infty$, both the numerator and denominator approaches $\pm\infty$ and the function becomes in the form $\frac{\pm\infty}{\pm\infty}$, which is one of the indeterminate forms.

Note: The indeterminate forms are $1^\infty, (\infty)^0, (0)^\infty, 0.\infty, \infty-\infty, \frac{\infty}{\infty}, \frac{0}{0}$.

There are three cases that can be studied as follows.

First case: If the degree of the numerator n is equal to the degree of the denominator m (i.e., $n=m$) then the limit is equal to the coefficient of x^n in the numerator divided by the coefficient of x^m in the denominator (the coefficient of x with the highest power in the numerator divided by the coefficient of x with the highest power in the denominator).

Example 2.6.3:

$$a)\ \lim_{x\to+\infty}\frac{4x^2+2x+1}{2x^2-1}=\lim_{x\to+\infty}\frac{4\frac{x^2}{x^2}+2\frac{x}{x^2}+\frac{1}{x^2}}{2\frac{x^2}{x^2}-\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{4+2\frac{1}{x}+\frac{1}{x^2}}{2-\frac{1}{x^2}}$$

$$=\frac{4}{2}=2.$$

Since, $\lim_{x\to\pm\infty}\frac{1}{x^n}=0.$

$$b)\ \lim_{x\to-\infty}\frac{1+2x-6x^3}{2x^3+1}=\lim_{x\to-\infty}\frac{\frac{1}{x^3}+2\frac{x}{x^3}-6\frac{x^3}{x^3}}{2\frac{x^3}{x^3}+\frac{1}{x^3}}=\frac{-6}{2}=-3.$$

The second case: If the degree of the numerator n is less than the degree of the denominator m (i.e., $n<m$) then the limit is equal to zero.

Example 2.6.4:

$$a)\ \lim_{x\to+\infty}\frac{4x^2-2x}{5x^3-8}=\lim_{x\to+\infty}\frac{4\frac{x^2}{x^3}-2\frac{x}{x^3}}{5\frac{x^3}{x^3}-\frac{8}{x^3}}=\lim_{x\to+\infty}\frac{4\frac{1}{x}-2\frac{1}{x}}{5-\frac{8}{x^3}}=\frac{0}{5}=0.$$

$$b)\ \lim_{x\to-\infty}\frac{1-3x}{2x^2+9}=\lim_{x\to-\infty}\frac{\frac{1}{x^2}-3\frac{x}{x^2}}{2\frac{x^2}{x^2}+\frac{9}{x^2}}=\lim_{x\to-\infty}\frac{\frac{1}{x^2}-3\frac{1}{x}}{2+\frac{9}{x^2}}=\frac{0}{2}=0.$$

The third case: if the degree of the numerator n is greater than the degree of the denominator m (i.e., $n>m$), the limit is equal to $+\infty$ or $-\infty$.

Example 2.6.5:

$$a)\ \lim_{x\to+\infty}\frac{6x^4-2x}{3x^3-8}=\lim_{x\to+\infty}\frac{6\frac{x^4}{x^3}-2\frac{x}{x^3}}{3\frac{x^3}{x^3}-\frac{8}{x^3}}$$

$$=\lim_{x\to+\infty}\frac{4x-2\frac{1}{x}}{3-\frac{8}{x^3}}=+\infty.$$

$$b)\ \lim_{x\to-\infty}\frac{1-5x^5}{2x^2+3}=\lim_{x\to-\infty}\frac{\frac{1}{x^2}-5\frac{x^5}{x^2}}{2\frac{x^2}{x^2}+\frac{3}{x^2}}=\lim_{x\to-\infty}\frac{\frac{1}{x^2}-5x^3}{2+\frac{3}{x^2}}=+\infty.$$

These three results can be summarized as follows:

$$\lim_{x\to+\infty}\left[\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}\right]=\begin{cases}0, & n<m\\ \frac{a_n}{b_m}, & n=m.\\ \pm\infty, & n>m\end{cases}$$

4 : (Limits of trigonometric, exponential and logarithmic functions)

In general, the limit of $\sin x$ and $\cos x$ as $x\to+\infty$ or $x\to-\infty$ fail to exist, because these functions are periodic and their range is $[-1,1]$ and therefore they are not constant and have no limit.

As for the limits of the exponential and logarithmic functions, they are as follows:

$\lim_{x\to+\infty} e^x=+\infty,\quad \lim_{x\to+\infty} \ln x=+\infty$

$\lim_{x\to0^+} \ln x=-\infty,\quad \lim_{x\to-\infty} e^x=0$

$\lim_{x\to+\infty} e^{-x}=0,\quad \lim_{x\to-\infty} e^{-x}=+\infty$

As shown in Figure 2-7.

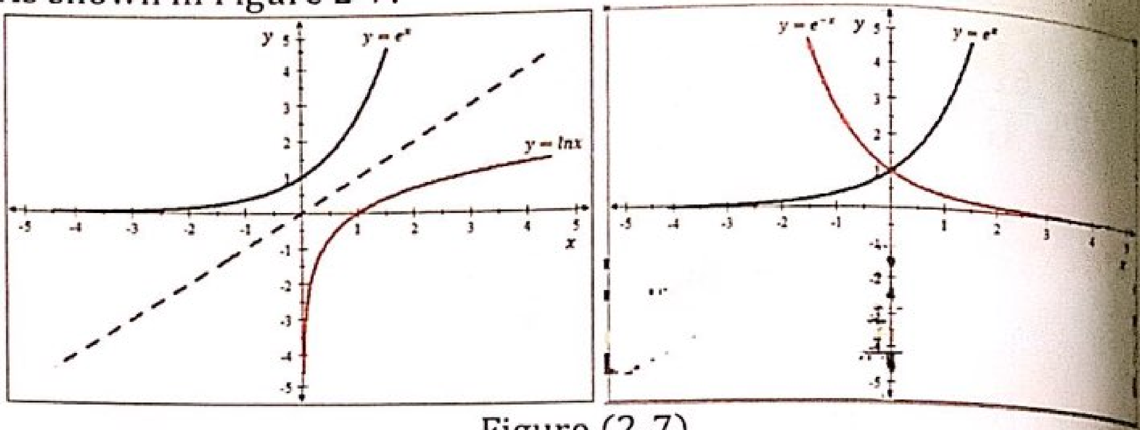


Figure (2-7)

Exercises

1) Find the following limits, if possible.

(i) $\lim\limits_{x\to1}(4x-3)$ (ii) $\lim\limits_{x\to3} 2x$ (iii) $\lim\limits_{x\to2}\dfrac{x^2-2}{x-2}$ (iv) $\lim\limits_{x\to2}\dfrac{3}{x}$

(v) $\lim\limits_{x\to3}\dfrac{x^3-7x^2}{x-7}$ (vi) $\lim\limits_{x\to5}\dfrac{x^2-5x+6}{x-3}$ (vii) $\lim\limits_{x\to3}\dfrac{x^2-9}{x^2-3x}$

(viii) $\lim\limits_{x\to3}\dfrac{x^3-7x^2}{x-7}$ (ix) $\lim\limits_{x\to\sqrt{5}}\dfrac{5-x^2}{\sqrt{5}-x}$ (x) $\lim\limits_{x\to4}|x-4|$

(xi) $\lim\limits_{x\to0^+}\dfrac{x-\sqrt{x}}{\sqrt{x}}$ (xii) $\lim\limits_{h\to0}\dfrac{\frac{1}{x+h}-\frac{1}{h}}{h}$

2) Using the definition of the limit, prove that

(a) $\lim\limits_{x\to2}\dfrac{x^2-3}{x-3}=5$ (b) $\lim\limits_{x\to3}x^2=9$

(c) $\lim\limits_{x\to1}(\dfrac{1}{x^2}+2)=3$ (d) $\lim\limits_{x\to2}\dfrac{1}{x^3}=\dfrac{1}{8}$

3) Let

$$f(x)=\begin{cases}4x^2-1, & x<1\\ 3x+2, & x\geq1.\end{cases}$$

Find $\lim\limits_{x\to1^+}f(x)$.

4) Let

$$f(x)=\begin{cases}-8, & x\leq-6\\ 3x+10, & -6<x<-2\\ -5, & x=-2\\ x^2, & -2<x\leq3\\ -2x+9, & x>3.\end{cases}$$

a) Show that $\lim\limits_{x\to-6}f(x)=f(-6)$.

b) Show that $\lim\limits_{x\to-2}f(x)=f(-2)$.

c) Show that $\lim_{x\to 3} f(x)$ does not exist.

5) Find the following limits.

(a) $\lim_{t\to 0}(14 - 6t + t^3)$

(b) $\lim_{x\to 6}(3x^2 + 7x - 16)$

(c) $\lim_{z\to 0}\frac{z^2 - 8z}{4 - 7z}$

(d) $\lim_{x\to -5}\frac{x + 7}{x^2 + 3x - 10}$

(e) $\lim_{x\to 0}\sqrt{x^2 + 6}$

(f) $\lim_{z\to 10}(4z + \sqrt[3]{z - 2})$

(g) $\lim_{x\to -1}(x - (x^2 + 3)^2)$

(h) $\lim_{x\to 0}\frac{2x}{\sqrt{2x^2 + x + 1} - \sqrt{x^2 - 3x + 1}}$

(i) $\lim_{w\to 1}\frac{\sqrt{w - 1}}{w - 1}$

(j) $\lim_{x\to 3}\frac{\sqrt{12 - x} - x}{\sqrt{6 + x} - 3}$

(k) $\lim_{x\to 0}\frac{\sin(4x)}{x}$

(l) $\lim_{x\to 3}\frac{1 - \cos(x)}{2x^2}$

6) Find the following Limits.

(a) $\lim_{x\to c}\frac{\sin(x - c)}{x^2 - c^2}$

(b) $\lim_{x\to 0}\left(\frac{1}{x} + 5\right)\sin x$

(c) $\lim_{x\to 0}\frac{\sin(3x)}{\tan(4x)}$

(d) $\lim_{x\to 0}\left(\frac{e^{2x} - 1}{x}\right)$

7) Prove that.

(a) $\lim_{x\to \frac{\pi}{2}}\frac{1 + \cos 2x}{(\pi - 2x)^2} = \frac{1}{2}$

(b) $\lim_{x\to 3}\frac{\cos(\frac{\pi}{x})}{x - 3} = \frac{\pi}{9}$

8) Let $\lim_{x\to 8} f(x) = -9$, $\lim_{x\to 8} g(x) = -9$, and $\lim_{x\to 8} h(x) = 4$.

Compute the following.

(a) $\lim_{x\to 8}[2f(x) - 12h(x)]$ (b) $\lim_{x\to 8}[3h(x) - 6]$

(c) $\lim_{x\to 8}[g(x)h(x) - f(x)]$ (d) $\lim_{x\to 8}[f(x) - g(x) + h(x)]$

9) Let $\lim_{x\to 0} f(x) = 6$, $\lim_{x\to 0} g(x) = -4$, and $\lim_{x\to 0} h(x) = -1$.

Compute the following.

(a) $\lim_{x\to 0}[f(x) + h(x)]^3$ (b) $\lim_{x\to 0} \sqrt{g(x)h(x)}$

(c) $\lim_{x\to 0} \sqrt[3]{11 + [g(x)]^2}$ (d) $\lim_{x\to 0} \sqrt{\dfrac{f(x)}{h(x) - g(x)}}$.

10) Find the following limits.

(a) $\lim_{x\to 0} \dfrac{tan2x}{sinx}$ (b) $\lim_{x\to 2} \dfrac{x - 2}{sin(x - 2)}$

(c) $\lim_{x\to 0} \dfrac{sinx - cosxsinx}{x^2}$ (d) $\lim_{x\to 0} \dfrac{x}{sin2x}$

(e) $\lim_{x\to 0} \dfrac{tan4x}{2x}$ (f) $\lim_{x\to 0} \dfrac{sin\, {}^{x}/_{2}}{x}$.

11) Find the following limits.

(a) $\lim_{x\to -\infty} \dfrac{1}{x}$ (b) $\lim_{x\to -\infty} \dfrac{1}{x^3}$

(c) $\lim_{x\to -\infty} \dfrac{1}{x^2}$ (d) $\lim_{x\to +\infty} \dfrac{1}{x^2}$

(e) $\lim_{x\to \infty} \dfrac{1 + 9x}{-2 + 3x}$ (f) $\lim_{z\to \infty} \dfrac{z^3 + 2z + 6}{z^5 + 3z + 9}$

(g) $\lim_{x\to-\infty} \frac{-2}{(x-1)^2}$

(h) $\lim_{x\to+\infty} \frac{1}{(x-1)^4}$

(l) $\lim_{x\to-\infty} \frac{3}{(x-2)^3}$

(m) $\lim_{t\to\infty} \frac{\sqrt{4t^3+7}}{2t+9}$

(n) $\lim_{x\to\infty} \frac{1}{2+sinx}$

(o) $\lim_{x\to\infty} \frac{1}{x^2}$

(p) $\lim_{x\to-\infty} \frac{3-e^x}{3+e^x}$

(q) $\lim_{x\to+\infty} \frac{3-e^x}{3+e^x}$

10) Find the following limits.

(a) $\lim_{x\to\infty} \frac{x-3}{x^2-9}$

(b) $\lim_{x\to\infty} \frac{x^2-3x+1}{x-1}$

(c) $\lim_{x\to\infty} \sqrt{\frac{1+x}{x}}$

(d) $\lim_{x\to\infty} \frac{4x}{\sqrt{2+x^2}}$

13) Find the following Limits.

(a) $\lim_{t\to3} \frac{1}{t^2-9}$

(b) $\lim_{x\to0} \frac{1}{x^3}$

(c) $\lim_{x\to1} \frac{2}{x-1}$

(d) $\lim_{x\to3} \frac{1}{(x-3)^2}$

(e) $\lim_{x\to5} \frac{1}{(x-5)^3}$

(f) $\lim_{z\to0} \frac{1}{z(z-1)}$

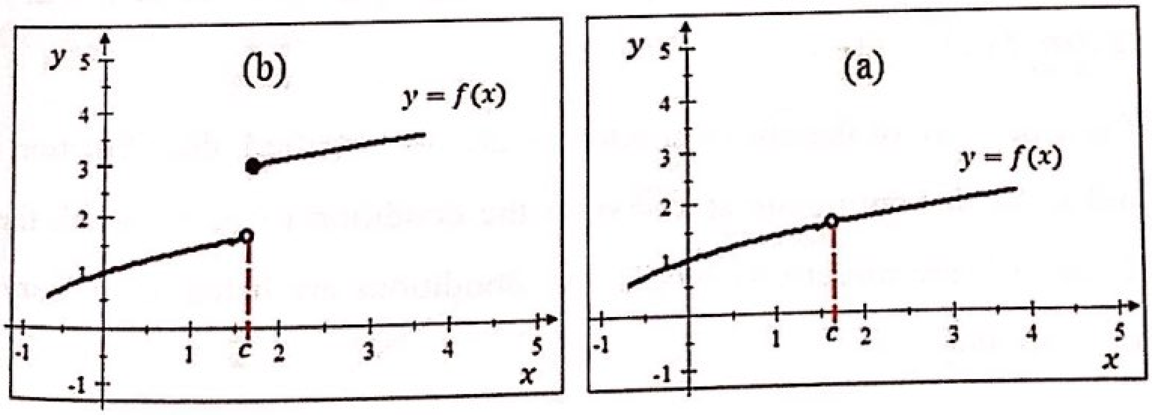
2.7 Continuity

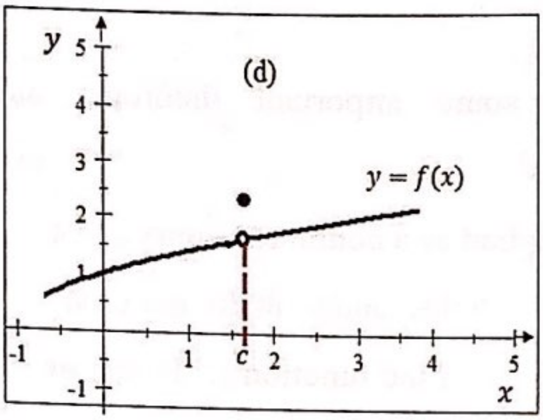
In this section we will present some important theorems and characteristics of continuous functions.

The graph of the function can be described as a continuous curve, we mean that it does not contain any cuts or holes, and to make the idea clearer we must understand the properties of the function with cuts or holes.

From Figure (2-8), we see that the curve of $f(x)$ has cuts or holes if any of the following conditions occur.

- $f(x)$ is undefined at c (Figure 2-8a).
- The limit of $f(x)$ does not exist as x approaches c(Fig. 2-8b, 2-8c).
- The value of the function and the limit of the function at $c$are not equal (are different) (Figure 2-8d).





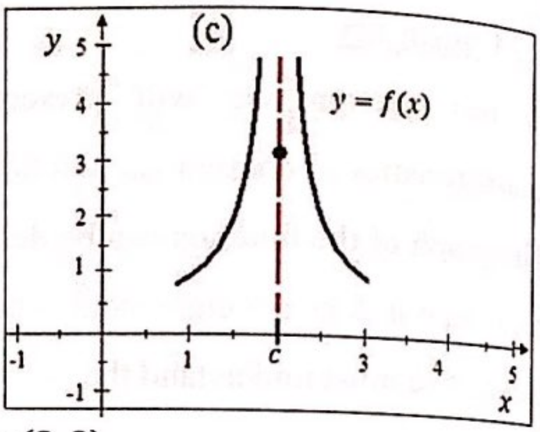


Figure (2-8)

Mathematically, continuity can be defined as follows.

Definition 2.7.1 Continuity

It is said that the function $f(x)$ is continuous at $x = a$ if the following conditions are satisfied.

(1) The function f is defined at $x = a$, i.e.,$f(a)$ is defined.

(2) The function f has a limit at $x = a$.

(3) The limit of the function equals the value of the function at $x = a$, i.e.,$\lim_{x \to a} f(x) = f(a)$.

If one or more of the above conditions are not satisfied, the f function is said to be discontinuous at $x = a$. If the condition (3) is satisfied, then (1) and (2) are directly satisfied. The conditions are listed in such style for simplicity.

Example 2.7.1:

Determine whether the following function $f(x) = x + 2$ is continuous at $x = 3$.

Solution:

We compute the value of the function at $x = 3$,

$$f(3) = 3 + 2 = 5.$$

We compute the limit of the function as $x \to 3$,

$$\lim_{x \to 3} x + 2 = 3 + 2 = 5.$$

We note that,

$$\lim_{x \to 3} f(x) = f(a) = 5$$

Then, the function is continuous at $x = 3$.

Example 2.7.2

Determine whether the function $f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = 2$.

Solution:

We compute the limit of the function at the point $x \to 2$,

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = 4.$$

But the function is undefined at $x = 2$, so the function is not continuous at $x = 2$. (See Figure 9-2).

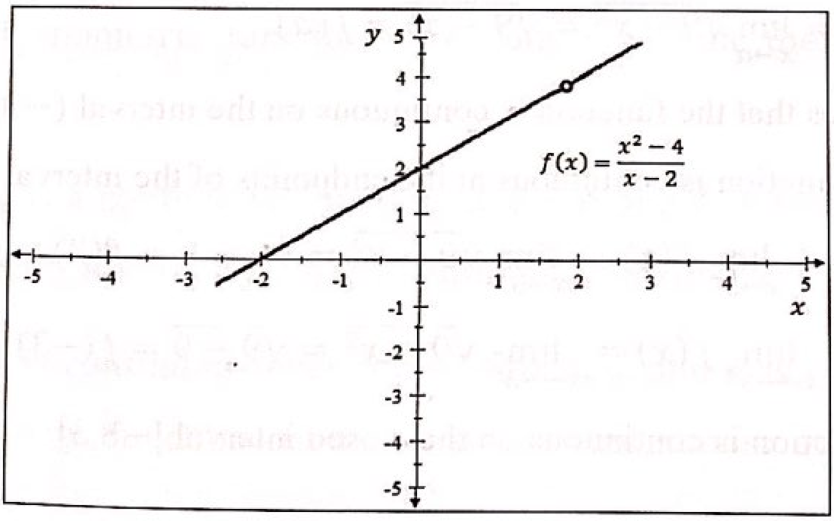


Figure (2-9)

2.8 Continuity on an Interval

The function $f(x)$ is said to be continuous on the open interval (a,b) if it is continuous at each number in the interval (a,b).

Definition 2.8.1: Continuity on an Interval

The function f is said to be continuous on the interval $[a,b]$ if the following conditions are satisfied.

(1) The function f is continuous on the interval(a,b).

(2) The function f is continuous from the right at a.

(3) The function f is continuous from the left at b.

Example 2.8.1:

Discuss the continuity of the function $f(x) = \sqrt{9-x^2}$.

Solution:

It is clear that the domain of the function is the closed interval $[-3,3]$, so we first examine the continuity of this function on the interval $(-3,3)$. Note that for any number $a \in (-3,3)$, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{9-x^2} = \sqrt{9-a^2} = f(a).$$

This proves that the function is continuous on the interval $(-3,3)$. Also the function is continuous at the endpoints of the interval where

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9-x^2} = \sqrt{9-9} = f(3),$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9-x^2} = \sqrt{9-9} = f(-3).$$

So the function is continuous on the closed interval $[-3,3]$.

2 .9 Properties of Continuous Functions

Theorem 2.9.1:

Let f and g be continuous functions at $x = a$. Then

(1) $f + g$ is a continuous function at $x = a$.

(2) $f - g$ is a continuous function at $x = a$.

(3) fg is a continuous function at $x = a$.

(4) For any real constant k, then kf is a continuous function at $x = a$.

(5) f/g is a continuous function at $x = a$ if $g(a) \neq 0$ and discontinuous if $g(a) = 0$.

General Rules for Continuity:

(1) Polynomials are continuous functions on $\mathbb{R}$.

(2) A rational function is continuous on its natural domain, and has discontinuities at the values where the denominator is zero.

(3) Trigonometric functions are continuous on their natural domain.

(4) If g is a continuous function at $x = a$ and f is a continuous function at $g(a)$ then $f \circ g$ is a continuous function at $x = a$.

(5) If g is continuous everywhere function, and f is continuous everywhere function then $f \circ g$ is continuous everywhere function.

Theorem 2.10.2: The Intermediate Value Theorem

If f is a continuous function on the closed interval $[a, b]$ and $f(a)$ and $f(b)$ are nonzero with opposite signs (i.e., $f(a).f(b) < 0$) then there is at least a number $x \in (a, b)$ such that $f(x) = 0$.

This theorem is very useful in computing roots of polynomials.

Example 2.9.1:

Show that the function $f(x) = x^3 + x^2 + x - 1$ has a root in the interval $[0,1]$.

Solution:

Since $f(x)$ is continuous on the closed interval $[0,1]$ and $f(0) = -1, and\ f(1) = 2$, so that, there is $x \in (0,1)$ such that $f(x) = 0$.

2.10 Types of Discontinuity

Let $y = f(x)$ be a function defined on the interval (a, b) and $\lim_{x \to c^+} f(x) = L_1$ and $\lim_{x \to c^-} f(x) = L_2$ where $c \in (a, b)$. Then

(1) If $L_1 \neq L_2$ (i.e., $\lim_{x \to c} f(x)$ does not exist) then the type of discontinuity is not removable.

(2) If $L_1 = L_2 \neq f(c)$ i.e., $\lim_{x \to c} f(x)$ exists and equal to L, but not equal to the value of the function at $x = c$, then the type of discontinuity is removable where the function can be redefined in another way to be continuous.

Example 2.10.1:

Examine the continuity of the following functions.

(a) $f(x) = 2x + 3 \ \forall x \in \mathbb{R}$

(b) $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(c) $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$ at $x = 2$

(d) $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ at $x = 0$

(e) $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$ at $x = 3$

(f) $f(x) = \begin{cases} 5+x, & x \leq 3 \\ 9-x, & x > 3 \end{cases}$ at $x = 3$

Solution:

Examination of the continuity of (a):

We note that the function $f(x) = 2x + 3$ is a polynomial and since polynomials are continuous on $\mathbb{R}$, then, the function $f(x) = 2x + 3$ is continuous at any number $x \in \mathbb{R}$.

Examination of the continuity of (b):

The function is defined at $x = 0$, since $f(0) = 0$,

and $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ thus the function is continuous at $x = 0$.

Examination of the continuity of (c):

The function is defined at $x = 2$ and equal to 3, $f(2) = 3$,

and, $\lim\limits_{x\to2}\frac{x^2-4}{x-2} = \lim\limits_{x\to2}\frac{(x-2)(x+2)}{x-2} = 4$.

We note that the limit of the function is not equal to the value of the function at the $x = 2$, and therefore the function is not continuous at $x = 2$, but this type of discontinuity is removable, so the function can be redefined to be continuous as follows:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2. \end{cases}$$

Examination of the continuity of (d):

The function is defined at $x = 0$ where $f(0) = 0$. But $\lim\limits_{x\to0^+} f(x) = 1$ and $\lim\limits_{x\to0^-} f(x) = -1$, this means that the left and the right limits are not equal, so the function has no limit.

So, the function is not continuous at $x = 0$, and this type is not removable.

Examination of the continuity of (e):

The function is defined at $x = 3$ and equal to 6, $f(3) = 6$,

and $\lim\limits_{x\to3}\frac{x^2-9}{x-3} = \lim\limits_{x\to3}\frac{(x-3)(x+3)}{x-3} = 6$.

Note that the limit of the function is equal to the value of the function at $x = 3$ and therefore it is continuous at $x = 3$.

Examination the continuity of (f):

The function is defined at $x = 3$, $f(3) = 5 + 3 = 8$. But

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (9 - x) = 9 - 3 = 6$, and

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (5 + x) = 8$.

This means that the left and the right limits are not equal therefore the function has no limit. So, the function is not continuous at $x = 3$. This type of discontinuity is not removable.

Theorem 2.10.1:

If f is a continuous one-to-one function on its natural domain then its inverse f^{-1}will be a continuous function on its natural domain.

Theorem 2.10.2:

Assume that $b > 0$ and $b \neq 1$, then

(1) The exponential function b^x is a continuous function on $(-\infty, +\infty)$.

(2) The logarithmic function $log_b\, x$ a continuous function on $(0, +\infty)$.

Example 2.10.2

Determine the intervals on which the function $f(x) = \frac{tan^{-1} x + ln\, x}{x^2 - 4}$ is continuous.

Solution:

We know that the quotient function is continuous if both the numerator and the denominator are continuous functions and the denominator is not equal to zero, and since $\tan^{-1} x$ is continuous on $(-\infty, +\infty)$, and $ln\, x$ is continuous on $(0\ ,+\infty)$, the numerator is a continuous function at the intersection of these two intervals i.e., on the interval $(0, +\infty)$. The denominator, is a continuous function onset of real numbers, except at -2 and 2 which make the denominator equal to zero, i.e., on $(-\infty, -2) \cup (-2,2) \cup (2, +\infty)$. Thus, the function f is continuous on $(0,2) \cup (2, +\infty)$.

Exercises

1) Examine continuity of the following functions.

(a) $f(t) = (14 - 2t + t^2)$ (b) $f(x) = (3x^2 + 7x - 16)$

(c) $f(x) = \begin{cases} \dfrac{2x^2 - x + 6}{x^2 - 3x + 2}, & x \neq 2 \\ 8, & x = 2 \end{cases}$ (d) $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 3 \\ 3, & x = 3 \end{cases}$

(e) $f(x) = \begin{cases} x^2 + 2x, & x \leq -2 \\ x^3 - 6x, & x > -2 \end{cases}$ (f) $f(x) = \begin{cases} \dfrac{\sqrt{x^2 + 1} - 1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

2) Find the interval or intervals on which the following functions are continuous?

(a) $f(x) = \dfrac{x + 2}{x - 2}$ (b) $f(x) = \sqrt{\dfrac{x - 3}{x + 2}}$ (c) $f(x) = \dfrac{x + 1}{x^2 - 1}$

(d) $f(x) = \dfrac{x^2 + 3x + 5}{x^2 + 3x - 4}$ (e) $f(x) = \dfrac{1}{\sin x - 1}$ (f) $f(x) = \dfrac{x^2}{x^2 + 1}$

3) Find the value of a that makes the function continuous.

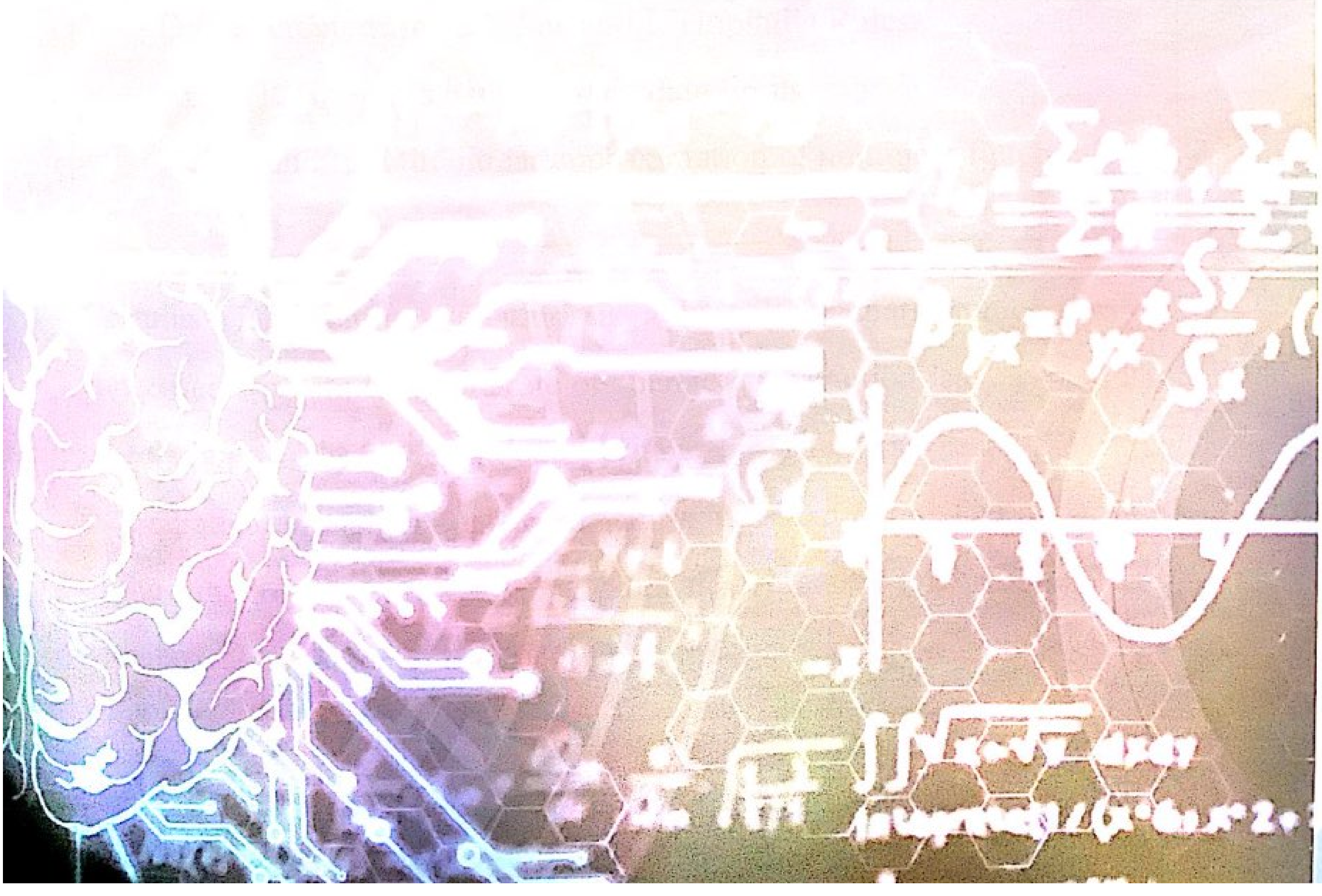
(a) $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ a, & x = 1 \end{cases}$ (b) $f(x) = \begin{cases} ax + 7, & x \neq 2 \\ -3, & x = 2 \end{cases}$

(c) $f(x) = \begin{cases} a + x, & x < 2 \\ 1 + x^2, & x \geq 2 \end{cases}$ (d) $f(x) = \begin{cases} ax, & x \leq 3 \\ 5, & x > 3 \end{cases}$

Chapter 3

Differentiation

3



Chapter 3 : Differentiation

Learning Outcomes:

By completing the study of this chapter, it is expected that the student will be able to:

- Define the differentiation of a function and calculate the first derivative and derivatives of higher orders.
- State and apply the chain rule.
- List the rules of differentiating the trigonometric, logarithmic, and exponential functions.
- Apply the rules of differentiating the trigonometric, logarithmic, and exponential functions.
- Define indeterminate forms and L'Hopital's Rules.
- Apply L'Hopital's Rules to calculate limits.
- Define and find the linear approximation of nonlinear functions.

Calculus (Differential Calculus) is a branch of mathematics that is concerned with studying the rate of change of a function (say; $y = f(x)$) in relation to the independent variable x. The first issue that this mathematical branch is concerned with is differentiation. The derivative of the function $y = f(x)$ at some point describes the mathematical and geometrical behavior of the function at this point or at points very close to it, and the first derivative of the function at a given point equals the value of the slope of the tangent of the function at this point, and in general the first derivative of the function at a certain point represents the best "linear approximation" of the function is at this point.

Differentiation has many applications. In physics, for example: the rate of change in the displacement of a moving particle with respect to time is the speed of the particle which is exactly the derivative the displacement with respect to time, whereas the differentiation of the velocity with time gives the acceleration.

The differentiation is also important in Newton's laws, as the second law states that force is the time rate of change in the amount of movement (i.e. differentiation of the amount of movement in relation to time). Also, differentiation is in finding the reaction rate for a chemical reaction and in operation research the derivatives or differentials determines the supplies for designing factories and transporting materials or raw materials or products.

Derivatives are used to find the maximum and minimum values of a function. Equations that include differentials (derivatives) are called differential equations and are among the basic and important equations for characterizing natural phenomena. Derivatives appear in many areas of mathematics such as numerical analysis, functional analysis, differential geometry, measure theory and abstract algebra.

3.1 Differentiability and Tangent Line

In this section we will be show how to derive the derivative of the function geometrically. Let $y = f(x)$ is a continuous function, and P is a point on the function curve with coordinates $(x, f(x))$.

Suppose that x changes by Δx, so the new x -coordinate of point $Q$is $x + \Delta x$, see figure (1-3).

But when the value of x changes, there is a change of Δy in the value of y, that is, in the value of $f(x)$ and it has the new value $f(x + \Delta x)$. The coordinates of point Q are $(x + \Delta x, y + \Delta y)$.

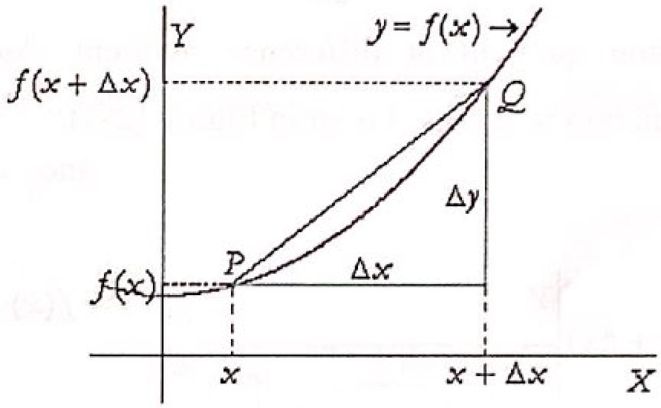


Figure (3-1)

so

$$Slop\ mPQ = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We now have the definition of the slope of the tangent line at P.

The slope of the tangent line at P is the Limit of the change in the function divided by the change in the independent variable when this change approaches zero.

$$f'(x)=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$f'(x)$ is called the derivative of the function $f(x)$, and sometimes the derivative of f is written by the following symbols:

$$f',\quad \frac{df}{dx},\quad \frac{d}{dx}f(x),\quad ,f^{(1)},\ Df,\quad y'(x),$$

$$y',\quad \frac{dy}{dx},\quad \frac{d}{dx}y,\quad Dy$$

The value

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

is called Newton quotient or difference quotient. Again, difference quotient is a function in Δx, as shown in Figure (2-3).

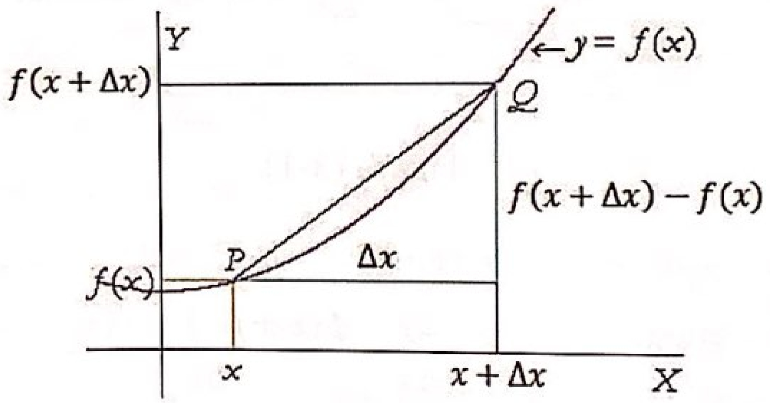


Figure (3-2)

$$\Delta y=f(x+\Delta x)-f(x)$$

Difference quotient becomes

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Now we will express the definition of the derivative as follows:

Definition 3.1.1: (Derivative)

$f'(x)$ is called the derivative of the function $f(x)$ if the following limit exists:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

In this case, we say that the function f is differentiable at x that is, that is; f has a derivative.

Example 3.1.1:

By using the definition of the derivative, find the first derivative of the following functions:

1) $f(x) = x.$

Solution:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)-x}{\Delta x} = \lim_{h \to 0} \frac{x+\Delta x - x}{\Delta x} = \lim_{h \to 0} \frac{\Delta x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} 1 = 1$$

2) $f(x) = x^2.$

Solution:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2(\Delta x)x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2(\Delta x)x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Example 3.1.2:

By using the definition of the derivative, find the first derivative of the following functions:

$$y = 3x^2 - x.$$

Solution:

$$y' = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - (x+\Delta x) - 3x^2 + x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(6x-1)\Delta x + 3(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} ((6x-1) + 3(\Delta x)) = 6x - 1.$$

Example 3.1.3:

In Example 3.1.1; find $f'(1)$.

Solution:

$$f'(x) = 2x$$

$$\therefore f'(1) = 2$$

Example 3.1.4

By using the definition, find $f'(4)$ if $f(x) = \sqrt{x}$.

Solution:

$$f'(4) = \lim_{x\to 4}\frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4}\frac{\sqrt{x} - 2}{x - 4} =$$

$$\lim_{x\to 4}\frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{1}{2+2} = \frac{1}{4}.$$

Example 3.1.5:

Find the first derivative of the function $y = x^n$

Solution:

Using Theorem 2.4.1,

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{x+\Delta x\to x}\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x} = nx^{n-1}.$$

Example 3.1.6:

Find the first derivative of the function $y = sinx$.

Solution:

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{sin(x+\Delta x) - sin(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{2cos\left(x+\frac{\Delta x}{2}\right)sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} cos\left(x+\frac{\Delta x}{2}\right). \lim_{\Delta x\to 0}\frac{sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = cosx.$$

Example 3.1.7:

Find the first derivative of the function $y = cosx.$

Solution:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{cos(x + \Delta x) - cos(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2sin\left(\frac{\Delta x}{2}\right) sin\left(x + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$= -\lim_{\Delta x \to 0} \frac{sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \lim_{\Delta x \to 0} sin\left(x + \frac{\Delta x}{2}\right) = -sinx.$$

Example 3.1.8:

Using the elementary principles, find the derivative of the function $y = \sqrt{x}$.

Solution:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} . \left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}\right)$$

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x) - (x)}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \frac{1}{\left(\sqrt{x} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

Theorem 3.1.1:

If f is differentiable at x_0 then f is continuous at x_0.

Proof:

Since f is differentiable at x_0 then $f'(x_0)$ exists, that is, the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Exists. Therefore, $f(x_0)$ must exist as if it does not exist, the limit will have no meaning, and thus the function f is defined at x_0. But

$$\lim_{x\to x_0}[f(x)-f(x_0)] = \lim_{x\to x_0}\left[(x-x_0)\frac{f(x)-f(x_0)}{x-x_0}\right]$$

$$\lim_{x\to x_0}(x-x_0)\,.\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right] = 0.f'(x_0).$$

Since

$$\lim_{x\to x_0}[f(x)-f(x_0)] = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} f(x_0)$$

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Then, the function f is continuous at x_0

3.2. One Sided Derivative:

Definition 3.2.1: (Right Hand Derivative)

If the function f is defined at x_0 then the derivative of f at x_0 from the right, which is denoted by the symbol, $f'_+(x_0)$ is defined as follows:

$$f'_+(x_0) = \lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

Definition 3.2.2: (Left Hand Derivative)

Similarly, the derivative can be defined from the left, which is denoted by $f'_-(x_0)$ and is defined as follows:

$$f'_-(x_0) = \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

It can be written as:

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}.$$

Theorem 3.2.1:

The necessary and sufficient condition for $f'_-(x_0) = f'_+(x_0) \neq \infty$ is that f has a derivative f' at the point x_0.

Example 3.2.1:

Examine the existence of the derivative of the function $f(x) = |x|$ at $x = 0$.

Solution:

First, we find $f'_-(x_0), f'_+(x_0)$, then we compare their values.

$$f'_+(x_0) = \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \frac{|\Delta x|-|0|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1.$$

$$f'_-(x_0) = \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{f(0+\Delta x)-f(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{|\Delta x|-|0|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1.$$

Since $f'_-(x_0)(x_0) \neq f'_+(x_0)$, the function f is not differentiable at $x = 0$.

Example 3.2.2:

Examine the differentiability of the function

$$f(x) = \begin{cases} x^2 & , \quad x \geq 2 \\ 3x\text{-}4 & , \quad x < 2, \end{cases}$$

at $x = 2$.

Solution:

We find $f'_-(2)$, $f'_+(2)$, then we compare their values.

$$f'_+(2) = \lim_{\Delta x \to 0^+} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(2 + \Delta x)^2 - f(2)}{\Delta x}$$

$$= \lim_{\Delta x \to 2^+} \frac{\Delta x(2 + \Delta x)}{\Delta x} = 4.$$

$$f'_-(2) = \lim_{\Delta x \to 0^-} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{3(2 + \Delta x) - f(2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{3\Delta x}{\Delta x} = 3.$$

Since $f'_-(2) \neq f'_+(2)$, the function f is not differentiable at $x = 2$.

3.3 Differentiation Formulas of Some Algebraic Functions:

Theorem 3.3.1:

(a) If a is a constant and$f(x) = a$ for all x values, then $f'(x) = 0$.

Proof:

Suppose that f is a constant function, that is, $f(x) = a$ for all x in the function domain. According to the definition of the derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a-a}{\Delta x} = 0.$$

(b) If $f(x) = x$, then $f'(x) = 1$.

Proof

Suppose that $f(x) = x$. According to the definition of the derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = 1.$$

(c) If n is a positive integer and $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Proof

Suppose that $f(x) = x^n$. According to the definition of a derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

Expanding $(x+\Delta x)^n$ using the binomial formula, we obtain

$$f'(x) =$$

$$= \lim_{\Delta x \to 0} \frac{\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right] - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right]}{\Delta x}$$

$$= nx^{n-1} + 0 + 0 + \cdots + 0 = nx^{n-1}.$$

Example 3.3.1:

Find $f'(x)$ if $f(x) = 4x^5 - 2x^2 + 7^3$.

Solution:

$$f(x) = 4x^5 - 2x^2 + 7^3$$

$$f'(x) = 20x^4 + 4x + 0 = 20x^4 + 4x.$$

Example 3.3.2:

Find $f'(x)$ if $f(x) = 12x^3 - 6x^2 + 5x + 8$.

Solution:

$$f'(x) = 36x^2 - 12x + 5 + 0 = 36x^2 - 12x + 5.$$

Example 3.3.3:

Find $f'(x)$ if $f(x) = \frac{3}{x^2} - 2\sqrt{x} + 7$.

Solution:

$f(x) = 3x^{-2} - 2x^{\frac{1}{2}} + 7$

$f'(x) = -6x^{-3} - x^{\frac{-1}{2}} + 0 = -\dfrac{6}{x^3} - \dfrac{1}{\sqrt{x}}.$

Example 3.3.4:

Find the derivative of each of the following functions:

1) $f(x) = \pi x$ 2) $g(y) = 5y^3$ 3) $y = -\dfrac{8}{x^2}$ 4) $v(t) = \sqrt{32}t^{\sqrt{2}}$

Solution:

1) $f'(x) = \pi$

2) $g'(y) = 15y^2$

3) $y' = 16x^{-3} = \dfrac{16}{x^3}$

4) $\acute{v}(t) = \sqrt{32}\sqrt{2}(t)^{\sqrt{2}-1} = \sqrt{64}(t)^{\sqrt{2}-1} = 8(t)^{\sqrt{2}-1}.$

The following theorem gives the differentiation rules of the sum, product and quotient of two functions.

Theorem 3.3.2:

(1) If $h(x) = f(x) \pm g(x)$, then $h'(x) = f'(x) \pm g'(x)$.

Proof

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) \pm g(x + \Delta x) - [f(x) \pm g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \pm \frac{g(x + \Delta x) - g(x)}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$h'(x) = f'(x) \pm g'(x).$$

(2) If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + g(x)f'(x)$.

(3) If $h(x) = \frac{f(x)}{(x)}$, $g(x) \neq 0$, then $h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

Where $f(x)$ and $g(x)$ are two differentiable functions.

Proof

2) Left as an exercise.

3)

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left\{\frac{f(x + \Delta x)g(x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)}\right\}$$

$$= \lim_{\Delta x \to 0} \left\{\frac{f(x + \Delta x)g(x) - f(x)g(x) - [f(x)g(x + \Delta x) - f(x)g(x)]}{\Delta x g(x)g(x + \Delta x)}\right\}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] - f(x)\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]}{g(x)g(x+\Delta x)}$$

$$h'(x) = \frac{\lim_{\Delta x \to 0} g(x) \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] - \lim_{\Delta x \to 0} f(x) \lim_{\Delta x \to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]}{\lim_{\Delta x \to 0} g(x) \lim_{\Delta x \to 0} g(x+\Delta x)}$$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)g(x)}$$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Example 3.3.5:

Find y' in each of the following cases:

1) $y(x) = \frac{5x}{x^2+1}$.

Solution:

By applying quotient rule:

$$y'(x) = \frac{(x^2+1)(5) - (5x)(2x)}{(x^2+1)^2} = \frac{5x^2+5-10x^2}{(x^2+1)^2} = \frac{5-5x^2}{(x^2+1)^2}.$$

2) $y(x) = \frac{x^2-1}{x^2+1}$.

Solution:

By applying quotient rule,

$$y'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}.$$

3) $y(x) = (3x^2 - 2)(3x^3 - 5x)$.

Solution:

By applying the product rule:

$$y'(x) = (3x^2 - 2)(9x^2 - 5) + (3x^3 - 5x)(6x).$$

4) $y(x) = x^{-n}$.

Solution:

By applying quotient rule:

$$y(x) = x^{-n} = \frac{1}{x^n}$$

$$y'(x) = \frac{x^n.0 - 1.nx^{n-1}}{[x^n]^2} = \frac{-nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}}.$$

3.4 Derivatives of Trigonometric Functions:

Derivatives of trigonometric functions are given in the following table:

Function	Derivative
$f(x) = sinx$	$f'(x) = cosx$
$f(x) = cosx$	$f'(x) = -sinx$
$f(x) = tanx$	$f'(x) = sec^2x$
$f(x) = cotx$	$f'(x) = -csc^2x$
$f(x) = secx$	$f'(x) = secx\ tanx$
$f(x) = cscx$	$f'(x) = -cscx\ cotx$

Example 3.4.1:

Find y' in each of the following:

1) $y = tanx = \frac{sinx}{cosx}$.

Solution:

Applying the quotient rule of the derivatives,

$$y'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$f(x) = sinx,\ \ g(x) = cosx \quad ,f'(x) = cosx \qquad g'(x) = -sinx$

$$y'(x) = \frac{cosx.cosx - sinx.(-sinx)}{[cosx]^2} = \frac{cos^2x + sin^2x}{[cosx]^2} = \frac{1}{[cosx]^2}$$

$$= sec^2x$$

2) $y = cotx = \frac{cosx}{sinx}$.

Solution:

Applying the quotient rule of the derivatives,

$$y'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$f(x) = cosx,\ \ g(x) = sinx \quad ,f'(x) = -sinx \qquad g'(x) = cosx$

$$y'(x) = \frac{sinx.(-sinx) - cosx.(cosx)}{[sinx]^2} = -\frac{sin^2x + cos^2x}{[sinx]^2} = \frac{-1}{[sinx]^2}$$

$$= -csc^2x.$$

3) $y = cscx = \frac{1}{sinx}$.

Solution:

Applying the quotient rule of the derivatives,

$$y'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$f(x) = 1,\ g(x) = sinx \quad ,f'(x) = 0 \qquad g'(x) = cosx$

$$y'(x) = \frac{sinx.(0) - 1.(cosx)}{[sinx]^2} = -\frac{cosx}{[sinx]^2} = \frac{-1}{sinx}.\frac{cosx}{sinx}$$

$$= -cscx\ cotx.$$

4) $y = secx = \frac{1}{cosx}$.

Solution:

Applying the quotient rule of the derivatives,

$$y'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$f(x) = 1,\ g(x) = cosx \quad ,f'(x) = 0 \qquad g'(x) = -sinx$

$$y'(x) = \frac{cosx.0 - 1.(-sinx)}{[cosx]^2} = \frac{sinx}{[cosx]^2} = \frac{1}{cosx}.\frac{sinx}{cosx} = secx\ tanx.$$

Example 3.4.2:

Find the first derivative for each of the following functions:

1) $y = x^2 sinx$.

Solution:

Applying the product rule of two functions, we get:

$$y' = x^2 cosx + sinx\ .2x = x^2 cosx + 2xsinx.$$

2) $y = \frac{cosx}{1 + sinx}$.

Solution:

By applying quotient rule, we obtain:

$$y' = \frac{(1+sinx)(-sinx) - cosx(0+cosx)}{(1+sinx)^2}$$

$$\frac{-sinx - sin^2x - cos^2x}{(1+sinx)^2} = -\frac{sinx+1}{(1+sinx)^2} = \frac{-1}{1+sinx}.$$

3) $y = secx\ tanx.$

Solution:

Applying the product rule of two functions we get:

$$y' = secxs\ ec^2x + tanx\ secx\ tanx = sec^3x + secx\ tan^2x.$$

Example 3.4.3:

Find the derivative for the following functions:

1) $y = 3\sin x - 4\cos x$ 2) $y = x^3\tan x$

Solution

1) $y' = 3\cos x + 4\sin x.$

2) $y' = x^3 sec^2x + 3x^2\tan x = x^2(x\, sec^2x + 3\tan x).$

3.5 Derivative of Composite Functions: The Chain Rule

Theorem 3.5.1:

Let the function g be differentiable at x, and the function f be differentiable at $g(x)$. Then the composition fog is differentiable at x, and the derivative of the composite function is given by the relationship

$$(fog)'(x) = f'(g(x))g'(x).$$

Let $u = g(x)$, $y = f(u)$. Then the chain rule takes the following equivalent form

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Proof:

Let the independent variable x has a small displacement Δx around the point x_0, accordingly a displacement occurs in u equals Δu and then a change in y of Δy occurs and

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Since g is a continuous function. So $\Delta u \to 0$ where $\Delta x \to 0$, so

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du}\frac{du}{dx}.$$

Example 3.5.1:

If $y = (2x^4 + 3x + 1)^5$, **find** y'.

Solution:

Let $u = 2x^4 + 3x + 1$. Then $y = u^5$ and hence

$$\frac{dy}{du} = 5u^4 \ , \ \frac{du}{dx} = 8x^3 + 3.$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4(8x^3 + 3) = 5(2x^4 + 3x + 1)^4(8x^3 + 3)$$

$$= 5(8x^3 + 3)(2x^4 + 3x + 1)^4.$$

Note: When applying the chain rule, we can apply this rule starting from outside (outside the brackets) and heading inwards until we finish finding the derivative.

For example, in the previous example,

$$y = (2x^4 + 3x + 1)^5$$

$$y' = 5(8x^3 + 3)(2x^4 + 3x + 1)^4.$$

Example 3.5.2

Find the derivative of the following functions:

1) $y = sin5x.$

Solution:

Let $u = 5x$. Then $y = sinu$ and hence

$$\frac{dy}{du} = cosu, \ \ \frac{du}{dx} = 5.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = cosu.5 = 5cos5x.$$

2) $y = sin(x^2)$.

Solution:

Let $u = x^2$. Then $y = sinu$ and

$$\frac{dy}{du} = cosu \ , \ \frac{du}{dx} = 2x.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = cosu.2x = 2xcosx^2.$$

3) $y = sin(\sqrt{x})$.

Solution:

Let $u = \sqrt{x}$. Then $y = sinu$ and hence

$$\frac{dy}{du} = cosu \ , \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}.$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = cosu.\frac{1}{2\sqrt{x}} = \frac{cos\sqrt{x}}{2\sqrt{x}}.$$

4) $y = sin\left(\sqrt{x^2+3x+1}\right)$.

Solution:

$$\frac{dy}{dx} = cos\left(\sqrt{x^2+3x+1}\right)\left(\frac{1}{2\sqrt{x^2+3x+1}}\right)(2x+3).$$

5) $y = \frac{1+sin2x}{1-sin2x}$.

Solution:

$$\frac{dy}{dx} = \frac{(1-sin2x)2cos2x - (1+sin2x)(-2cos2x)}{(1-sin2x)^2}$$

$$\frac{dy}{dx} = \frac{4cos2x}{(1-sin2x)^2}.$$

Example 3.5.3:

Find the derivative of the following functions:

$$(a)\ y = sin^2(4x+1).$$

Solution:

The function is made from three functions, which, from inside to outside, are $4x + 1$, sin, sin^2. That is

$$y = sin^2(4x + 1) = [sin(4x + 1)]^2$$

So, we start by finding the derivative of this function from outside as follows:

$$y' = 2[sin(4x + 1)]cosx(4x + 1) \cdot 4$$
$$= 8cosx(4x + 1) \cdot sin(4x + 1).$$

(b) $y = tan^3(x^2 - 1)$.

Solution:

It is made up of three functions, which are, from inside to outside, $x^2 - 1$, tan, tan^3, i.e.,

$$y = tan^3(x^2 - 1) = [tan(x^2 - 1)]^3$$

So, we start by finding the derivative of this function from outside as follows:

$$y' = 3[tan(x^2 - 1)]^2 . sec^2(x^2 - 1). 2x$$
$$= 6x\, sec^2(x^2 - 1)tan^2(x^2 - 1).$$

(c) $y = (1 + x^2cscx)^{-5}$.

Solution:

Applying the chain rule, we obtain,

$y' = -5(1 + x^2cscx)^{-6}(0 + x^2. -cscx\ cotx + cscx. 2x)$

$= -5(2xcscx - x^2. cscx\ cotx)(1 + x^2cscx)^{-6}$.

(d) $y = sin(tan^2 4x^2)$.

Solution:

Again applying the chain rule, we obtain,

$$\frac{dy}{dx} = cos\ (tan^2 4x^2).2tan(4x^2).sec^2(4x^2).8x$$

$$= 16x\ sec^2(4x^2)\ tan(4x^2)\ cos(tan^2 4x^2).$$

Example 3.5.4:

If $f(x) = cos^3 x$, solve the equation $f'(x) = 0$, where $x \in [0,2\pi]$.

Solution:

To find the derivative of f, we use the chain rule,

$f'(x) = 3cos^2 x.\frac{d}{dx}(cos\,x) = 3cos^2 x(-\,sin\,x) = -3\,sin\,x\ cos^2 x =$ 0, then

$sin\,x = 0 \quad or\ cos\,x = 0.$

If $sin\,x = 0$, the only solutions $x \in [0,2\pi]$ are

$x = 0,\ x = \pi,\ x = 2\pi.$

If $cos\,x = 0$, the only solutions $x \in [0,2\pi]$ are

$x = \frac{\pi}{2},\ x = \frac{3\pi}{2}.$

The only solutions $x \in [0,2\pi]$ to the equation $f'(x) = 0$ are:

$$x = 0, \quad x = \frac{\pi}{2}, \quad x = \pi, \quad x = \frac{3\pi}{2}.$$

Example 3.5.5:

Find the derivative of the following functions:

1) $y = cos(2x) + sin^2 x$ 2) $y = tan(sin\,x)$

3) $y = \frac{sin(3x)}{4 + 5\,cos(2x)}$ 4) $y = xsec^2(\pi x)$

5) $y = cos^3(tan(3x))$ 6) $y = \frac{x\,sec(x)}{3\,csc(x)}.$

Solution:

1) $y' = -\sin(2x).(2) + 2\sin x \cos x = -2\sin(2x) + \sin(2x)$

$= -\sin(2x).$

2) $y' = sec^2(\sin x)(\cos x) = sec^2(\sin x)\cos x.$

3) $y'(x)$

$$= \frac{(4+5\cos(2x))(\cos(3x).(3)) - \sin(3x).(-5\sin(2x).(2))}{(4+5\cos(2x))^2}$$

$$= \frac{12\cos(3x) + 15\cos(2x)\cos(3x) + 10\sin(2x)\sin(3x)}{(4+5\cos(2x))^2}.$$

4) $y'(x) = x\, 2\sec(\pi x)\sec(\pi x)\tan(\pi x)(\pi) + sec^2(\pi x)$

$= 2\pi x\, sec^2(\pi x)\tan(\pi x) + sec^2(\pi x)$

$= sec^2(\pi x)[2\pi x \tan(\pi x) + 1].$

5) $y'(x) = 3(\cos(\tan(3x)))^2 - \sin(tann(3x))\, sec^2(3x)(3)$

$= -9(\cos(\tan(3x)))^2 \sin(tann(3x))\; sec^2(3x).$

6)$y'(x)$

$$= \frac{3\csc(x)[x.\sec(x)\tan(x) + \sec(x)] - x\sec(x)(-3\csc(x)\cot(x))}{(3\csc(x))^2}$$

$\Rightarrow y'(x)$

$$= \frac{3x\sec(x)\tan(x)\csc(x) + 3\sec(x)\csc(x) + 3x\sec(x)\cot(x)\csc(x)}{(3\csc(x))^2}$$

$$= \frac{3x\sec(x)\tan(x)\csc(x) + 3\sec(x)\csc(x) + 3x\sec(x)\cot(x)\csc(x)}{(3\csc(x))^2}.$$

3.6 The Derivatives of Logarithmic and Exponential Functions

First: The derivative of logarithmic functions

In this section we will find the derivative of the logarithmic function $y = log_b x, \quad x > 0$, using the following fact:

$$\lim_{v \to 0^+} (1+v)^{\frac{1}{v}} = e$$

$$\frac{d}{dx}[\, log_b x] = \lim_{\Delta x \to 0} \frac{log_b(x+\Delta x) - log_b x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\, log_b \left(\frac{x+\Delta x}{x}\right)\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\, log_b \left(1+\frac{\Delta x}{x}\right)\right]$$

Let $v = \frac{\Delta x}{x}$. Then $\frac{1}{\Delta x} = \frac{1}{xv}$ so, if $\Delta x \to 0$, then $v \to 0$ and hence,

$$\frac{d}{dx}[\, log_b x] = \lim_{v \to 0} \frac{1}{vx} log_b(1+v)$$

$$= \lim_{v \to 0} \frac{1}{x} log_b(1+v)^{\frac{1}{v}} = \frac{1}{x} log_b \lim_{v \to 0}(1+v)^{\frac{1}{v}}.$$

This is because the function $log_b x$ is continues, so limits order can be swapped with the function to get:

$$\frac{d}{dx}[\, log_b x] = \frac{1}{x}\, log_b\, e\,, \qquad \dots.(1)$$

From the properties of logarithms, we know that $log_b x = \frac{lnx}{lnb}$, and therefore:

$$log_b\, e = \frac{lne}{lnb} = \frac{1}{lnb}.$$

Therefore, (1) becomes:

$$\frac{d}{dx}[\log_b x] = \frac{1}{x\ln b} \quad , x > 0.$$

If $b = e$, then $\ln e = 1$. Therefore

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad , x > 0. \quad ....(2)$$

Example 3.6.1:

Find $\frac{d}{dx}[\ln|x|]$.

Solution:

Since the function $\ln|x|$ is defined for all real numbers except at $x = 0$, so we will consider the cases $x > 0$ and $x < 0$ (see Figure 3-3)

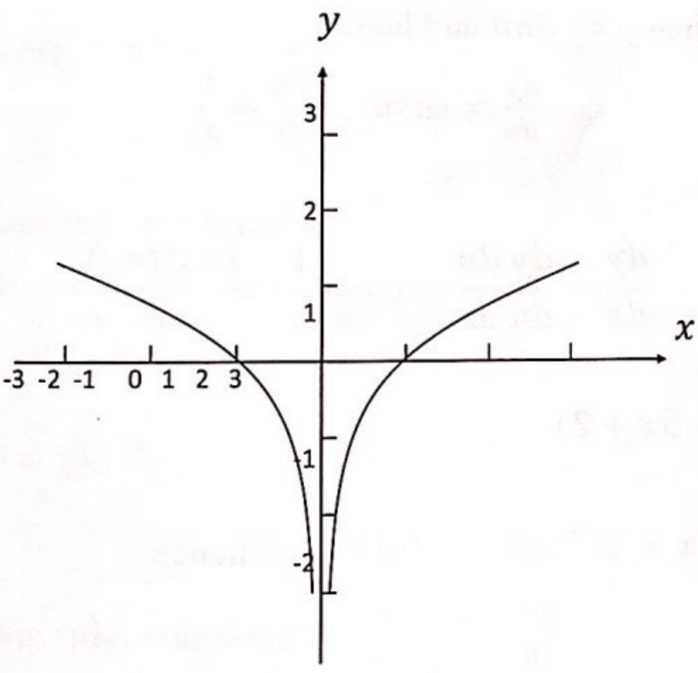


Figure (3-3)

In the case $x > 0$; we find that $ln|x| = lnx$. Therefore $\frac{d}{dx} ln|x| = \frac{d}{dx} lnx = \frac{1}{x}$.

In the case $x > 0$; we find that $ln|x| = ln(-x)$. So,

$$\frac{d}{dx} ln|x| = \frac{d}{dx} ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Therefore, we find in both cases that, $\frac{d}{dx} ln|x| = \frac{1}{x}, x \neq 0$.

Example 3.6.2:

Find $\frac{dy}{dx}$ in each of the following cases,

1) $y = sin(ln(x))$.

Solution:

Let $u = lnx$. Then $y = sinu$ and hence

$$\frac{dy}{du} = cosu \ , \ \frac{du}{dx} = \frac{1}{x}.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = cosu\frac{1}{x} = \frac{cos(lnx)}{x}.$$

2) $y = ln(x^2 + 5x + 2)$.

Solution:

Let $u = x^2 + 5x + 2$. Then $y = ln(u)$ and hence

$$\frac{dy}{du} = \frac{1}{u} \ , \ \frac{du}{dx} = 2x + 5.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2x + 5)\frac{1}{u} = \frac{2x + 5}{x^2 + 5x + 2}.$$

3) $y = ln\left(\frac{x^3 sinx}{\sqrt{2+x}}\right)$.

Solution:

First, we simplify the given function by using the properties of the logarithmic function as follows:

$$y = ln\left(\frac{x^3 sinx}{\sqrt{2+x}}\right) = ln(x^3 sinx) - ln(2+x)^{\frac{1}{2}}$$

$$= ln(x^3) + ln(sinx) - ln(2+x)^{\frac{1}{2}}$$

$$= 3lnx + ln(sinx) - \frac{1}{2}ln(2+x)$$

$$\frac{dy}{dx} = \frac{3}{x} + \frac{cosx}{sinx} - \frac{1}{2}\frac{1}{x+2} = \frac{3}{x} + cotx - \frac{1}{2(x+2)}.$$

4) $y = ln|cosx|$.

Solution:

Using the chain rule, we find that:

$$\frac{dy}{dx} = \frac{1}{cosx}\frac{d}{dx}(cosx) = \frac{-sinx}{cosx} = -tanx.$$

5) $y = ln(x^2 + 1)$.

Solution:

Using the chain rule, we find that

$$\frac{dy}{dx} = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}.$$

Note

Logarithmic differentiation is a way to simplify finding the derivatives of functions, that are in the form of a product or quotient or power of functions. This is explained in the following examples.

Example 3.6.3:

Find $\frac{dy}{dx}$ if

$$y = \frac{x^2\sqrt[3]{4x-12}}{(x^2+1)^3}.$$

Solution:

First note that if we try to find the derivative directly, the resulting algebraic operations will be complicated given the fact that there are two product functions in the numerator and a function of power 3 in the first place. Therefore, we first take the natural logarithm of both sides of the previous equation, then we use the properties of the logarithm to simplify the result and finally we differentiate the two sides with respect to the independent variable x.

$$lny = ln\left(\frac{x^2\sqrt[3]{4x-12}}{(x^2+1)^3}\right) = lnx^2 + ln\sqrt[3]{4x-12} - ln(x^2+1)^3$$

$$= 2lnx + \frac{1}{3}ln(4x-12) - 3ln(x^2+1).$$

Differentiate both sides with respect to x,

$$\frac{y'}{y} = \frac{2}{x} + \frac{4}{3(4x-12)} - \frac{3(2x)}{x^2+1} = \frac{2}{x} + \frac{4}{3(4x-12)} - \frac{6x}{x^2+1}$$

$$y' = y\left[\frac{2}{x} + \frac{4}{3(4x-12)} - \frac{6x}{x^2+1}\right]$$

$$y' = \left(\frac{x^2\sqrt[3]{4x-12}}{(x^2+1)^3}\right)\left[\frac{2}{x} + \frac{4}{3(4x-12)} - \frac{6x}{x^2+1}\right].$$

Example 3.6.4:

Find $\frac{dy}{dx}$ if

(1) $y = x^r,\ r \in \mathbb{R}$ (2) $y = x^x, \quad x > 0.$

Solution:

(1) Taking the logarithm of both sides, we get $lny = lnx^r = rlnx$.

Differentiate both sides with respect to$x$, then $\frac{y'}{y} = \frac{r}{x}$

$$y' = \frac{ry}{x} = \frac{rx^r}{x} = rx^{r-1}.$$

(2) Taking the logarithm of both sides we get $lny = lnx^x = xlnx$

Differentiate both sides with respect to x,

$$\frac{y'}{y} = \frac{x}{x} + lnx.1$$

$$y' = y(1 + lnx) = x^x(1 + lnx).$$

Second: Derivative of Exponential Functions

We turn now to find the derivatives of the exponential functions. Let $y = b^x$. We start taking the logarithm of both sides for the base $b > 0$,

$$log_b\, y = log_b\, b^x = x\, log_b\, b = x.$$

Then we differentiate both sides with respect to the variable x,so we get $\frac{y'}{ylnb} = 1$, which implies that$y' = b^x lnb$.

That is: $\frac{d}{dx}[b^x] = b^x lnb$. If $b = e$, then $lnb = 1$ and hence

$$\frac{d}{dx}[e^x] = e^x.$$

Example 3.6.5:

Find y' in each of the following cases,

1) $y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$

Solution:

$$y' = 0 + 1 + x + + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

The function e^x is the only function whose differentiation for x is equal to it, and here is the value of the function e^x.

2) $y = e^{5x}.$

Solution:

Let $u = 5x$. Then $y = e^u$ and hence

$$\frac{dy}{du} = e^u, \quad \frac{du}{dx} = 5.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u.5 = 5e^{5x}.$$

3) $y = e^{-x^2}.$

Solution:

Let $u = -x^2$. Then $y = e^u$. Therefore

$$\frac{dy}{du} = e^u \ , \quad \frac{du}{dx} = -2x.$$

Therefore

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=e^{u}.-2x=-2xe^{-x^2}.$$

$$4)\ y=e^{sinx^2}.$$

Solution:

$$y'=e^{sinx^2}\frac{d}{dx}(sinx^2)=e^{sinx^2}.2xcosx^2=2x\ cosx^2\ e^{sinx^2}.$$

5) $y=3^{tanx}$.

Solution:

$$y'=3^{tanx}\ ln3\frac{d}{dx}(tanx)=ln3.3^{tanx}sec^2x.$$

Remember that:

If $y=f(u)$, $u=u(x)$, the derivatives of the exponential and logarithmic functions are given in the following table:

	Function $y=$	**Derivative** $y'=$
1	$ln(u)$	$\frac{1}{u}.\frac{du}{dx}$
2	$log_a(u)$	$\frac{1}{u.ln(a)}.\frac{du}{dx}$
3	e^u	$e^u.\frac{du}{dx}$
4	a^u	$ln(a)\ a^u.\frac{du}{dx}$
5	u^v	$vu^{v-1}\frac{du}{dx}+u^v\ ln(u)\frac{dv}{dx}$

Note that:

$$\frac{d}{dx}(u^v) = \frac{d}{dx}(e^{v\ln u}) = e^{v\ln u}\frac{d}{dx}(v\ln u) = vu^{v-1}\frac{du}{dx} + u^v\ln(u)\frac{dv}{dx},$$

The logarithmic function with $a > 0$ is defined as:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}, a \neq 1.$$

3.7 Derivative of Inverse Trigonometric Functions

If $y = f(u)$ and $u = u(x)$, then the derivative of trigonometric functions is given in the following table:

Function $y =$	**Derivative** $y' =$
$\sin^{-1}(u)$	$\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}, \lvert u\rvert < 1$
$\cos^{-1}(u)$	$-\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}, \lvert u\rvert < 1$
$\tan^{-1}(u)$	$\frac{1}{1+u^2}\cdot\frac{du}{dx}$
$\cot^{-1}(u)$	$-\frac{1}{1+u^2}\cdot\frac{du}{dx}$
$\sec^{-1}(u)$	$\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\cdot\frac{du}{dx}, \lvert u\rvert > 1$
$\csc^{-1}(u)$	$-\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\cdot\frac{du}{dx}, \lvert u\rvert > 1$

Example 3.7.1:

Prove that

$$\frac{d}{dx}(sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Proof:

Let

$$y = sin^{-1} x$$
$$sin\, y = x$$

Implicitly differentiate both sides with respect to x

$$\frac{d}{dx}(sin\, y) = \frac{d}{dx}(x)$$
$$cos\, y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{cos\, y} = \frac{1}{\sqrt{1-sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Example 3.7.2:

Prove that

$$\frac{d}{dx}(cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$$

Proof:

Let

$$y = cos^{-1} x.$$

$$\cos y = x$$

Implicitly differentiae both sides with respect to x,

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$$

Example 3.7.3:

Prove that

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Proof:

Let

$$y = \tan^{-1} x$$

$$\tan y = x$$

Implicitly differentiate both sides with respect to x,

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(tan^{-1} x) = \frac{1}{1+x^2}.$$

Example 3.7.4:

Prove that

$$\frac{d}{dx}(sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}.$$

Proof:

Let

$$y = sec^{-1} x$$

$$sec\, y = x$$

Differentiate, implicitly, both sides with respect to x

$$\frac{d}{dx}(sec\, y) = \frac{d}{dx}(x)$$

$$sec\, y\, tan\, y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{sec\, y\, tan\, y} = \frac{1}{sec\, y\sqrt{sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}.$$

Exercise

Prove that:

(1) $\frac{d}{dx}(cot^{-1} x) = \frac{-1}{1+x^2}$ (2) $\frac{d}{dx}(csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}.$

Example 3.7.5:

Find the derivative of each of the following functions:

(1) $y = sin^{-1}(2x)$ (2) $y = tan^{-1}(3x)$ (3) $y = sec^{-1}(3x)$

(4) $y = sin^{-1}\sqrt{x}$ (5) $y = sin^{-1}x^3$ (6) $y = sec^{-1}(x^2+1)$

(7) $y = x^2 tan^{-1}x^2$.

Solution:

$$(1)\ \frac{dy}{dx} = \frac{d}{dx}[sin^{-1}(2x)] = \frac{2}{\sqrt{1-(2x)^2}}.$$

$$(2)\ \frac{dy}{dx} = \frac{3}{1+(3x)^2}.$$

$$(3)\ \frac{dy}{dx} = \frac{3}{3x\sqrt{(3x)^2-1}} = \frac{1}{x\sqrt{9x^2-1}}.$$

$$(4)\ \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}}.$$

$$(5)\ \frac{dy}{dx} = \frac{(3x^2)}{\sqrt{1-(x^3)^2}} = \frac{(3x^2)}{\sqrt{1-x^6}}.$$

$$(6)\ \frac{dy}{dx} = \frac{2x}{(x^2+1)\sqrt{(x^2+1)^2-1}} = \frac{2x}{(x^2+1)\sqrt{x^4+2x^2}}$$

$$= \frac{2}{(x^2+1)\sqrt{x^2+2}}.$$

$$(7)\ \frac{dy}{dx} = 2x tan^{-1}x^2 + x^2 \frac{2x}{1+(x^2)^2} = 2x tan^{-1}x^2 + \frac{2x^3}{1+x^4}.$$

Example 3.7.6:

Find the derivative of each of the following functions:

(1) $y = tan^{-1}\dfrac{-1}{x+1}$ (2) $y = x^2 cot^{-1}\left(\dfrac{x}{2}\right)$

(3) $y = xcsc^{-1}\dfrac{1}{x}$.

Solution:

(1) $\dfrac{dy}{dx} = \dfrac{1}{1+\left[\frac{-1}{(x+1)}\right]^2}\cdot\dfrac{1}{(x+1)^2} = \dfrac{(x+1)^2}{(x+1)^2+1}\cdot\dfrac{1}{(x+1)^2}$

$= \dfrac{1}{x^2+2x+2}$.

(2) $\dfrac{dy}{dx} = x^2.\dfrac{-1}{1+\left(\frac{x}{2}\right)^2}\cdot\dfrac{1}{2} + cot^{-1}\left(\dfrac{x}{2}\right).2x = 2xcot^{-1}\left(\dfrac{x}{2}\right) - \dfrac{2x^2}{4+x^2}$.

(3) $\dfrac{dy}{dx} = x.\dfrac{-1}{\left|\frac{1}{x}\right|\sqrt{\left(\frac{1}{x}\right)^2-1}}\cdot\dfrac{-1}{x^2} + csc^{-1}\dfrac{1}{x}.1 = \dfrac{|x|}{\sqrt{1-x^2}} + csc^{-1}\dfrac{1}{x}$.

3.8 Derivative of the Hyperbolic Functions

	Function $y =$	Derivative $y' =$
(a)	$sinhx$	$coshx$
(b)	$coshx$	$sinhx$
(c)	$tanhx$	$sech^2x$
(d)	$cothx$	$-csch^2x$
(e)	$sechx$	$-sechx\ tanhx$
(f)	$cschx$	$-cschx\ cothx$

Proof:

We will prove (a), (c), (e) and leave the rest of the cases as exercises for the student to prove.

$$(a)\frac{d}{dx}(sinhx) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = coshx.$$

$$(c)\frac{d}{dx}(tanhx) = \frac{d}{dx}\left(\frac{sinhx}{coshx}\right) = \frac{coshx.coshx - sinhx.sinhx}{cosh^2x}$$

$$= \frac{cosh^2x - sinh^2x}{cosh^2x} = \frac{1}{cosh^2x} = sech^2x.$$

$$(e)\frac{d}{dx}(sechx) = \frac{d}{dx}\left(\frac{1}{coshx}\right) = \frac{coshx(0) - sinhx}{cosh^2x}$$

$$= \frac{-sinhx}{cosh^2x} = \frac{-1}{coshx}\frac{sinhx}{coshx} = -sechx\ tanhx.$$

Example 3.8.1:

Find y' in each of the following cases:

(1) $y = ln(sinhx^3)$ (2) $y = tan^{-1}(coshx^2)$ (3) $y = (coshx)^{\frac{1}{x}}$.

Solution:

$$(1)y' = \frac{1}{sinh(x^3)}.coshx^3.3x^2 = 3x^2cothx^3.$$

$$(2)\ y' = \frac{1}{1+cosh^2x^2}.sinhx^2.2x = \frac{2xsinhx^2}{1+cosh^2x^2}.$$

(3) $y = (coshx)^{\frac{1}{x}}$,

take the logarithm of both sides:

$$lny = ln(coshx)^{\frac{1}{x}} = \frac{1}{x}ln(coshx),$$

take the derivative of both sides for the variable x:

$$\frac{1}{y}y' = \frac{1}{x}\frac{1}{coshx}sinhx + ln(coshx)\frac{-1}{x^2} = \frac{tanhx}{x} - \frac{ln(coshx)}{x^2}$$

$$y' = y\left[\frac{tanhx}{x} - \frac{ln(coshx)}{x^2}\right] = (coshx)^{\frac{1}{x}}\left[\frac{tanhx}{x} - \frac{ln(coshx)}{x^2}\right].$$

Example 3.8.2:

If $u = x^2$ ، $y = sinh\,u$ ، Findy'.

Solution:

Using the chain rule,

$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = cosh\,u\,(2x) = 2x\,cosh\,u = 2x\,cosh(x^2).$$

Example 3.8.3:

If $y = coth\left(\frac{1}{x}\right)$, find y'.

Solution:

Using the chain rule,

$$y' = \left(coth\left(\frac{1}{x}\right)\right)' = -csch^2\left(\frac{1}{x}\right)\left(\left(\frac{1}{x}\right)\right)' = -csch^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= \frac{csch^2\left(\frac{1}{x}\right)}{x^2}.$$

Example 3.8.4:

If $y = sech^2(ln\, x)$, find y'.

Solution:

Using the chain rule,

$$y' = (sech^2(ln\, x))' = 2sech((ln\, x))\left(sech((ln\, x))\right)'$$

$$= 2sech((ln\, x))(-sech((ln\, x))tanh((ln\, x))(ln\, x)')$$

$$= -\frac{2}{x}sech^2(ln\, x)\tanh(lnx).$$

Example 3.8.5:

Find the first derivative of the function,

$y = (sinh\, x)^2$.

Solution:

$$y' = 2(sinh\, x)(cosh\, x).$$

Another Solution:

Let $u = x$, $y = u^2$. Using the chain rule to find the derivative of y as follows:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u \cosh x = 2 \sinh x \cosh x.$$

Example 3.8.6:

Find the first derivative of the function:

(1) $f(x) = \sinh x^3$.

(2) $g(x) = -\sinh x + 4\cosh(x+2)$.

(3) $h(x) = \dfrac{\cosh x^2}{\sinh x} + 4\tanh(x^2+2)$.

Solution:

(1) $f'(x) = (3x^2)\cosh x^3$.

(2) $g'(x) = -\cosh x + 4\sinh(x+2)$.

(3) $h'(x) = \dfrac{\sinh x\,[(2x)\sinh x^2] - (\cosh x^2)(\cosh x)}{(\sinh x)^2}$

$+ 4\,\text{sech}^2(x^2+2)(2x)$.

3.9 Derivative of the Inverse Hyperbolic Functions

The following table gives the derivative of the inverse hyperbolic functions.

If the function $u = u(x)$ is differentiable with respect to x then:

Function	Derivation
$\frac{d}{dx}[\sinh^{-1}u]$	$\frac{u'}{\sqrt{1+u^2}}$
$\frac{d}{dx}[\cosh^{-1}u]$	$\frac{u'}{\sqrt{u^2-1}}, u > 1$
$\frac{d}{dx}[\tanh^{-1}u]$	$\frac{u'}{1-u^2}, \lvert u\rvert < 1$
$\frac{d}{dx}[\coth^{-1}u]$	$\frac{-u'}{u^2-1}, \lvert u\rvert > 1$
$\frac{d}{dx}[\operatorname{sech}^{-1}u]$	$\frac{-u'}{u\sqrt{1-u^2}}, 0 < u < 1$
$\frac{d}{dx}[\operatorname{csch}^{-1}u]$	$\frac{-u'}{\lvert u\rvert\sqrt{1+u^2}}, u \neq 0$

Example 3.9.1:

Prove that

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}.$$

Proof:

Let

$$y = sinh^{-1} x$$

$$sinh\, y = x$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{cosh\, y} = \frac{1}{\sqrt{1 + sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

$$\frac{d}{dx}(sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}.$$

Example 3.9.2:

Prove that

$$\frac{d}{dx}(cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

Proof:

Let

$$y = cosh^{-1} x$$

$$cosh\, y = x$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{sinh\, y} = \frac{1}{\sqrt{cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

Example 3.9.3:

Prove that

$$\frac{d}{dx}(tanh^{-1} x) = \frac{1}{1 - x^2}$$

Proof:

Let

$$y = tanh^{-1} x$$

$$tanh\, y = x$$

$$sech^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{sech^2 y} = \frac{1}{1 - tanh^2 y} = \frac{1}{1 - x^2}$$

$$\frac{d}{dx}(tanh^{-1} x) = \frac{1}{1 - x^2}.$$

Example 3.9.4:

Let

$$\frac{d}{dx}(sech^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}$$

Proof:

Let

$$y = sech^{-1} x$$

$$sech\, y = x$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-sech\, y\, tanh\, y} = \frac{-1}{sech\, y\sqrt{1 - sec^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(sech^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}.$$

Exercise:

Prove that

$$(1)\ \frac{d}{dx}(coth^{-1}x) = \frac{-1}{x^2-1}.$$

$$(2)\ \frac{d}{dx}(csch^{-1}x) = \frac{-1}{|x|\sqrt{1+x^2}}.$$

Example 3.9.5:

Find the derivative of each of the following functions:

(1) $y = sinh^{-1}x^2$ (2) $y = x^2cosh^{-1}x^2$ (3) $y = coth^{-1}(coshx)$

(4) $y = ln\sqrt{1+x^2} - x\,tanh^{-1}x.$

Solution:

$$(1)\ \frac{dy}{dx} = \frac{1}{\sqrt{x^4+1}}2x = \frac{2x}{\sqrt{x^4+1}}.$$

$$(2)\ \frac{dy}{dx} = x^2.\frac{1}{\sqrt{x^4-1}}.2x + cosh^{-1}x^2.2x = \frac{2x^3}{\sqrt{x^4-1}} + 2x\,cosh^{-1}x^2.$$

$$(3)\ \frac{dy}{dx} = \frac{1}{1-cosh^2x}.sinhx = \frac{sinhx}{-sinh^2x} = \frac{-1}{sinhx}.$$

$$(4)\ y = \frac{1}{2}ln(1+x^2) - xtanh^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{2x}{1+x^2} - \left[\frac{x}{1-x^2} - tanh^{-1}x\right]$$

$$= \frac{x}{1+x^2} - \left(\frac{x}{1-x^2} - tanh^{-1}x\right).$$

Example 3.9.6:

If $y = tanh^{-1}\left(\frac{1}{x}\right)$, Find y'.

Solution:

$$y'(x) = \frac{d}{dx}\left(tanh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{1-\left(\frac{1}{x}\right)^2}\left(\frac{1}{x}\right)'$$

$$= \frac{1}{1-\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = -\frac{x^2}{x^2-1}.\frac{1}{x^2} = \frac{1}{1-x^2}.$$

3.10 Derivative of the Parametric Functions

If $x\,, y$ are two continues functions in the variable t, that is

$$\begin{cases} x = \emptyset(t), \\ y = \psi(t). \end{cases}$$

Then these two equations are called parametric equations and if we can remove t from them, we get a direct relationship between $x\,, y$.

Let $\emptyset$ has a continuous inverse function or ψ has a continuous inverse function.

If $\emptyset$ has a continuous inverse function, $\frac{d}{dx}$ can be obtained for parametric functions as follows:

Let Δt is a slight change in t, accordingly a slight change Δx occurs in x and a slight change Δy in yand we notice that from the continuity of $\emptyset^{-1}$ when $\Delta x \to 0$ then $\Delta t \to 0$. Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Example 3.10.1:

Find $\frac{dy}{dx}$ if

$$\begin{cases} x = a\cos t \\ y = a\sin t, \end{cases}$$

where a is constant.

Solution:

We find

$$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{-a\sin t} = -\cot t.$$

Example 3.10.2:

Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ if $x = \ sint$ and $y = \ cos\, t$.

Solution:

$$y = \ cos\, t \Rightarrow \frac{dy}{dt} = -sint$$

$$x = \ sint \Rightarrow \frac{dx}{dt} = cost$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-sint}{cost} = -tant$$

$$\therefore \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -tan\frac{\pi}{4} = -1.$$

Example 3.10.3:

Find $\frac{dy}{dx}$ if $x = at - a\ sint$ and $y = a - a\ cos\,t$ at $t = \pi$, where a is a constant.

Solution:

$$y = a - a\ cos\,t \Rightarrow \frac{dy}{dt} = a\ sint$$

$$x = at - a\ sint \Rightarrow \frac{dx}{dt} = a - acost$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{asint}{a - acost} = \frac{sint}{1 - cost} = \frac{2sin\frac{t}{2}cos\frac{t}{2}}{2sin^2\frac{t}{2}} = cot\frac{t}{2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = cot\frac{\pi}{2} = 0.$$

3.11 Derivative of the Implicit Functions

If the relationship between x, y is given implicitly

$$F(x, y) = 0, \qquad (1)$$

to obtain the derivative of this in simple cases, it suffices to differentiate the left hand side of the relationship (1) with respect to the variable x, considering y as a function of x, equate this differential with zero. That is, put

$$\frac{d}{dx}F(x, y) = 0, \qquad (2)$$

then solve the resulting equation.

Example 3.11.1:

Find the derivative of the function

$$x^3 + y^3 - 3axy = 0.$$

Solution:

Differentiate the left hand side of the equation and equate to zero, we get

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left(x\frac{dy}{dx} + y\right) = 0$$

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}.$$

Example 3.11.2:

Find $\frac{dy}{dx}$ if $x^3 + y^3 = 9xy$.

Solution:

Take the derivative of both sides with respect to x,

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y$$

$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{(3y^2 - 9x)} = \frac{3y - x^2}{y^2 - 3x}.$$

Example 3.11.3:

Find $\frac{dy}{dx}$ if $y^3 - 3x^2y + 1 = 0$.

Solution:

Differentiating both sides with respect to x,

$$3y^2\frac{dy}{dx} - 6xy - 3x^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} = 6xy$$

$$\frac{dy}{dx}(3y^2 - 3x^2) = 6xy$$

$$\frac{dy}{dx} = \frac{6xy}{3y^2 - 3x^2} = \frac{2xy}{y^2 - x^2}.$$

Example 3.11.4:

Find $\frac{dy}{dx}$ for the following functions:

1) $sin(xy) = xy + x^2$ 2) $xy - \sqrt{xy} - 3x^2 = 0.$

Solution:

1) $sin(xy) = xy + x^2$

Take the derivative with respect to x for both sides,

$$cos(xy)\left[y + x\frac{dy}{dx}\right] = y + x\frac{dy}{dx} + 2x$$

$$ycos(xy) + x\frac{dy}{dx}cos(xy) = y + x\frac{dy}{dx} + 2x$$

$$\frac{dy}{dx}[xcos(xy) - x] = y + 2x - ycos(xy).$$

2) $xy - \sqrt{xy} - 3x^2 = 0.$

Solution:

Take the derivative with respect to x for both sides,

$$x\frac{dy}{dx}+y-\frac{1}{2\sqrt{xy}}\left(y+x\frac{dy}{dx}\right)-6x=0$$

$$\frac{dy}{dx}\left(x-\frac{x}{2\sqrt{xy}}\right)=6x-y+\frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx}(2x\sqrt{xy}-x)=2\sqrt{xy}(6x-y)+y$$

$$\frac{dy}{dx}=\frac{2\sqrt{xy}(6x-y)+y}{2x\sqrt{xy}-x}.$$

3.12 Higher Order Derivatives

Let us assume that the first derivative $\frac{dy}{dx}$ of $y = f(x)$ exist. If the derivative of the function $\frac{dy}{dx}$ exists, it is called the second derivative of the function $y = f(x)$ and if the derivative of the function $\frac{d^2y}{dx^2}$ exists then it is called the third derivative of the function $y = f(x)$ and derivatives are denoted by one of the following symbols

The second derivative:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f^{(2)}(x) = y^{(2)} = f''.$$

The third derivative:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f^{(3)}(x) = y^{(3)} = f'''.$$

For the n^{th} derivative

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}.$$

Some books denote the n^{th} derivative by the symbol

$$D^n(y) = \frac{d^ny}{dx^n}.$$

Example 3.12.1:

Find the second derivative of the function

$$y = ln(1 - x).$$

Solution:

$$y' = \frac{-1}{1-x}, \quad y'' = \left(\frac{-1}{1-x}\right)' = \frac{1}{(1-x)^2}.$$

Example 3.12.2:

Find the second derivative of the function

1) $y = 4x^2 - 5x + 8 - \frac{3}{x}$ 2) $y = \frac{x^2}{x^2+4}$

3) $y^4 + 3y - 4x^3 = 5x + 1.$

Solution:

(1) Take derivative with respect to x for both sides,

$$\frac{dy}{dx} = 8x - 5 + \frac{3}{x^2} = 8x + 3x^{-2} - 5$$

$$\frac{d^2y}{dx^2} = 8 - \frac{6}{x^3}.$$

(2) Take derivative with respect to x for both sides,

$$\frac{dy}{dx} = \frac{(x^2+4)(2x) - x^2(2x)}{(x^2+4)^2} = \frac{2x^3 + 8x - 2x^3}{(x^2+4)^2} = \frac{8x}{(x^2+4)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x^2+4)[8(x^2+4) - 32x^2]}{(x^2+4)^4} = \frac{32 - 24x^2}{(x^2+4)^3}.$$

(3) Take derivative with respect to x for both sides,

$$4y^3\frac{dy}{dx} + 3\frac{dy}{dx} - 12x^2 = 5$$

$$\frac{dy}{dx}(4y^3 + 3) = 12x^2 + 5$$

$$\frac{dy}{dx} = \frac{12x^2 + 5}{4y^3 + 3}$$

$$\frac{d^2y}{dx^2} = \frac{24x(4y^3+3) - (12x^2+5)\left(12y^2\frac{dy}{dx}\right)}{(4y^3+3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{24x(4y^3+3) - 12y^2(12x^2+5)\left(\frac{12x^2+5}{4y^3+3}\right)}{(4y^3+3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{24x(4y^3+3)^2 - 12y^2(12x^2+5)^2}{(4y^3+3)^3}.$$

Example 3.12.3

If $x^3 + y^3 = 1$, prove that,

$\frac{d^2y}{dx^2} = -\frac{2x}{y^5}.$

Solution:

Take derivative with respect to x for both sides,

$$3x^2 + 3y^2\frac{dy}{dx} = 0,$$

then $3y^2\frac{d}{dx} = -3x^2$. Therefore $\frac{dy}{dx} = \frac{-x^2}{y^2}$.

Take derivative with respect to x for both sides,

$$y'' = \frac{-[y^2.2x - x^2 2yy']}{y^4} = \frac{-y\left[y.2x - x^2 2\frac{-x^2}{y^2}\right]}{y^4} = \frac{-[2xy^3 + 2x^4]}{y^5}$$

$$= \frac{-2x[y^3 + x^3]}{y^5} = \frac{-2x}{y^5}.$$

If $u = \emptyset(x)$ and $v = \psi(x)$ have derivatives of order n, the Leibnitz's rule can be used to find the value of the n[th] derivative of the product of these two functions.

Higher Derivative of the Product (Leibnitz's Rule)

$$\frac{d^n}{dx^n}(f.g) = \frac{d^n}{dx^n}(f)g + \binom{n}{1}\frac{d^{n-1}}{dx^{n-1}}(f)\frac{d}{dx}(g)$$
$$+\binom{n}{2}\frac{d^{n-2}}{dx^{n-2}}(f)\frac{d^2}{dx^2}(g) + \cdots + \binom{n}{n-1}\frac{d}{dx}(f)\frac{d^{n-1}}{dx^{n-1}}(g)$$
$$+ f\frac{d^n}{dx^n}(g),$$

where, $\binom{n}{n-1}$ is a coefficients of the polynomial. For example, we obtain:

$$\frac{d^2}{dx^2}(fg) = f\frac{d^2g}{dx^2} + 2\frac{df}{dx}\frac{dg}{dx} + g\frac{d^2f}{dx^2},$$
$$\frac{d^3}{dx^3}(fg) = f\frac{d^3g}{dx^3} + 3\frac{df}{dx}.\frac{d^2g}{dx^2} + 3\frac{dg}{dx}.\frac{d^2f}{dx^2} + g\frac{d^3f}{dx^3}.$$

Higher Order Derivatives of Parametrically Represented Functions

If

$$\begin{cases} x = \emptyset(t) \\ y = \psi(t). \end{cases}$$

The derivatives

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \ldots, \quad \frac{d^ny}{dx^n}$$

can be calculated using formulas

$$\frac{dy}{dx} = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}, \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \quad \frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}}, \ldots.$$

Example 3.12.4:

Find the second derivative of the function

$$\begin{cases} x = a\ \cos t \\ y = b\ \sin t. \end{cases}$$

Solution:

Since

$$\frac{dx}{dt} = -a\ \sin t, \quad \frac{dy}{dt} = b\ \cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\ \cos t}{-a\ \sin t} = -\frac{b}{a}\cot t.$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)}{-a\ \sin t} = -\frac{b}{a^2}\frac{\csc^2 t}{\sin t} = -\frac{b}{a^2 \sin^3 t}.$$

Example 3.12.5

If $x = 4\cos^2\theta$, $y = 2\sin\theta$, prove that these equations represent the parabola $x + y^2 = 4$. Then find $\frac{dy}{dx}$.

Solution:

Substituting the values of x, y, we get

$$x + y^2 = 4\cos^2\theta + (2\sin\theta)^2 = 4\cos^2\theta + 4\sin^2\theta$$

$$= 4\cos^2\theta + 4\sin^2\theta = 4(\cos^2\theta + \sin^2\theta) = 4.$$

Therefore, the two equations satisfy the parabolic equation.

$$x = 4\cos^2\theta \Rightarrow \frac{dx}{d\theta} = -8\cos\theta\sin\theta$$

$$y = 2sin\theta \Rightarrow \frac{dy}{d\theta} = 2cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2cos\theta}{-8cos\theta\ sin\theta} = -\frac{1}{4sin\theta} = -\frac{1}{4}csc\theta.$$

Example 3.12.6:

If $x = \sqrt{t}, y = t - \frac{1}{\sqrt{t}}$, Find $\frac{d^2y}{dx^2}$.

Solution:

$$\because x = \sqrt{t} = t^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$y = t - t^{-\frac{1}{2}} \Rightarrow \frac{dy}{dt} = 1 - \left(-\frac{1}{2}\right)t^{\frac{-3}{2}} = 1 + \frac{1}{2t\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{2t\sqrt{t}}}{\frac{1}{2\sqrt{t}}} = \frac{2\sqrt{t}(2t\sqrt{t}+1)}{2t\sqrt{t}} = \frac{2t\sqrt{t}+1}{t} = 2\sqrt{t} + \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\sqrt{t} + t^{-1}\right) = \frac{d}{dt}\left(2\sqrt{t} + t^{-1}\right)\frac{dt}{dx}$$

$$= \left(\frac{2}{2\sqrt{t}} - \frac{1}{t^2}\right).\frac{1}{dx/_{dt}} = \left(\frac{1}{\sqrt{t}} - \frac{1}{t^2}\right).\frac{1}{1/_{2\sqrt{t}}}$$

$$= 2\sqrt{t}\left(\frac{1}{\sqrt{t}} - \frac{1}{t^2}\right) = 2 - \frac{2}{t\sqrt{t}}.$$

3.13 Indeterminate Forms and L' Hopital's Rule

We know, from the limit properties studied previously, that if $\lim_{x\to a} g(x) \neq 0$ then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}.$$

If $\lim\limits_{x\to a}f(x)=0,\ \lim\limits_{x\to a}g(x)=0$, then it is said that the limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ not specified and write it as $\frac{0}{0}$ for example:

$\lim\limits_{x\to 1}\frac{x^2-x}{x-1}$ and $\lim\limits_{x\to 0}\frac{\sin x}{x}$.

We also learned to find such limits by using some mathematical tricks such as factorizing algebraic terms, multiplying by conjugate, or using some trigonometric identities.

Here we will study a method called the L'Hopital's Rule due to a French nobility known by this name and published a book on differentiation at the end of the seventeenth century and this rule appeared in that book. As we will see later, the application of this rule facilitates finding the value of limits that may be difficult or impossible to find using the methods previously studied.

Theorem 3.13.1:

Let the functions f, g be differentiable in an open interval I, with the possibility that this will not be achieved at the number $a\in I$. Also, let $x\in I$, and $g'(x)\neq 0$. If $\lim\limits_{x\to a}f(x)=0,\ \lim\limits_{x\to a}g(x)=0$, and $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=L$ then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=L$.

Example 3.13.1:

Find the value of each of the following limits:

1) $\lim_{x\to1}\frac{x^2-x}{x-1}$ 2) $\lim_{x\to0}\frac{sinx}{x}$ 3) $\lim_{x\to0}\frac{e^x-1}{sin2x}$.

Solution:

If the direct substitution, leads to the form $\frac{0}{0}$, then we differentiate both the numerator and denominator separately, then we find the limits provided that the derivative of the denominator is not equal to zero. This is L' Hopital's Rule

1) $\lim_{x\to1}\frac{x^2-x}{x-1}=\lim_{x\to1}\frac{2x-1}{1}=2-1=1.$

2) $\lim_{x\to0}\frac{sinx}{x}=\lim_{x\to0}\frac{cosx}{1}=cos0=1.$

3) $\lim_{x\to0}\frac{e^x-1}{sin2x}=\lim_{x\to0}\frac{e^x}{2cos2x}=\frac{1}{2}.$

Remark (1)

When the limit of $\frac{f'(x)}{g'(x)}$ is the indeterminate form $\frac{0}{0}$ then we use the L'Hopital's Rule again and again as shown in the following example.

Example 3.13.2:

Find the value of the following limit:

$$\lim_{x\to1}\frac{1-x+lnx}{1+cos\pi x}.$$

Solution:

With direct substitution, we again obtain the indeterminate form of type $\frac{0}{0}$. Then we differentiate both the numerator and the denominator again and substitute for the value of x so we get

$$\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\pi\, sin\pi x},$$

with direct substitution, we again obtain the indeterminate form of type $\frac{0}{0}$. Then we differentiate both the numerator and the denominator again and substitute for the value of x so we get

$$\lim_{x\to 1}\frac{-\frac{1}{x^2}}{-\pi^2 cos\pi x}=\frac{-1}{\pi^2}.$$

Therefore,

$$\lim_{x\to 1}\frac{1-x+lnx}{1+cos\pi x}=\frac{-1}{\pi^2}.$$

Remark (2)

L' Hopital's Rule remains valid if the value of x increases or decreases without a limit according to the following theorem.

Theorem 3.13.2:

Let the functions f and g be differentiable for all values of $N < x$ where N is a positive constant number and also let for all values $N < x$, $g'(x) \neq 0$ if If $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to\infty} g(x) = 0$, and $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L$ then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$.

Example 3.13.3:

Find $\lim\limits_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\tan^{-1}\left(\frac{1}{x}\right)}$.

Solution:

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\tan^{-1}\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right).\left(\frac{-1}{x^2}\right)}{\frac{1}{1+\frac{1}{x^2}}.\left(\frac{-1}{x^2}\right)} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)}{\frac{1}{1+\frac{1}{x^2}}} = \frac{1}{1} = 1.$$

Remark (3)

L' Hopital's Rule also remains valid if the limit is from the right or the left side exists as shown in the following example.:

Example 3.13.4:

Find $\lim\limits_{x\to 0^+} \frac{\tan x}{x}$.

Solution:

$$\lim_{x\to 0^+} \frac{\tan x}{x} = \lim_{x\to 0^+} \frac{\sec^2 x}{1} = 1.$$

Remark (4)

The Theorem 2.13.3 remains valid if we substitute $+\infty$ by $-\infty$.

Example 3.13.5:

Find $\lim\limits_{x\to -\infty} \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}$.

Solution:

$$\lim_{x\to-\infty}\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}\quad \lim_{x\to-\infty}\frac{\cos\left(\frac{2}{x}\right).\left(\frac{-2}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to-\infty}2\cos\left(\frac{2}{x}\right)=2.$$

Theorem 3.13.3

Let f and g be differentiable functions over the open interval$I$ with the possibility that this will not be satisfied at $a\in I$. Let's also assume $x\neq a\in I$, $g'(x)\neq 0$, $\lim_{x\to a}f(x)=\infty$, $\lim_{x\to a}g(x)=\infty$ also $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$.

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=L$.

Example 3.13.6:

Find $\lim_{x\to-\frac{-\pi}{2}}\frac{sec^2x}{sec^2 3x}$.

Solution:

With direct substitution, we again obtain the indeterminate form of type $\frac{0}{0}$. Applying L' Hopital's Rule and then direct substitution we get,

$$\lim_{x\to\frac{-\pi}{2}}\frac{2secx.secx.tanx}{2sec3xsec3xtan3x.3}=\lim_{x\to\frac{-\pi}{2}}\frac{sec^2x.tanx}{3sec^2 3xtan3x}.$$

If we try to apply the previous theorem, we will get the form $\frac{\infty}{\infty}$, no matter how many times the theorem is applied, so this direction will not help in obtaining the required limit except that if we rewrite the original function in another way, we get the following:

$$\lim_{x\to\frac{-\pi}{2}} \frac{sec^2x}{sec^2 3x} = \lim_{x\to\frac{-\pi}{2}} \frac{cos^2 3x}{cos^2 x},$$

by applying the L' Hopital's Rule and then by direct substitution we get

$$\lim_{x\to\frac{-\pi}{2}} \frac{-2cos3x.sin3x.3}{-2cosx.sinx},$$

using the identity of double-angle for both the numerator and denominator, we obtain the indeterminate form of type $\frac{0}{0}$.

$$\lim_{x\to\frac{-\pi}{2}} \frac{sec^2x}{sec^2 3x} = \lim_{x\to\frac{-\pi}{2}} \frac{3sin6x}{sin2x},$$

by applying the L' Hopital's Rule one more time, we obtain

$$\lim_{x\to\frac{-\pi}{2}} \frac{3cos6x.6}{cos2x.2} = \lim_{x\to\frac{-\pi}{2}} \frac{9cos6x}{cos2x} = \frac{9cos3\pi}{cos2\pi} = \frac{9(-1)}{-1} = 9.$$

The following theorem gives another form for L' Hopital's Rule when limits of both the numerator and denominator tend to ∞.

Theorem 3.13.4

Let f and g be differentiable for all values $N < x$, $0 < N$. Also, let $g'(x) \neq 0$ for all $N < x$. If $\lim_{x\to\infty} f(x) = \infty(-\infty)$, $\lim_{x\to a} g(x) = \infty(-\infty)$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Example 3.13.7:

Find $\lim_{x\to\infty} \frac{ln(2+e^x)}{3x}$.

Solution:

By direct substitution, we obtain the indeterminate form of type $\frac{\infty}{\infty}$, and by differentiating the numerator and denominator separately, we obtain

$$\lim_{x\to\infty} \frac{ln(2+e^x)}{3x} = \lim_{x\to\infty} \frac{\frac{e^x}{2+e^x}}{3} = \lim_{x\to\infty} \frac{e^x}{3(2+e^x)}.$$

By differentiating the numerator and denominator separately one more time, we obtain

$$\lim_{x\to\infty} \frac{e^x}{3e^x} = \frac{1}{3}.$$

Remark (5)

If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty$, then it is said that the product of $f(x).g(x)$ has the indeterminate form $0.\infty$ at $x = a$ to find the limit of the product $f(x).g(x)$ at $x = a$, rewrite the problem to take the indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing the product of the product as:

$$f(x).g(x) = \frac{g(x)}{1/f(x)} \text{ or } f(x)g(x) = \frac{f(x)}{1/g(x)}.$$

Thus, L' Hopital's Rule can be applied as shown in the following example.

Example 3.13.8:

Find $\lim_{x\to 0} \sin^{-1}x \csc x$.

Solution:

By direct substitution, we get the indeterminate form of type $0.\infty$, and by rewriting the function as:

$$\lim_{x\to 0} \frac{\sin^{-1}x}{\sin x}$$

By direct substitution, we obtain the indeterminate form of type $\frac{0}{0}$, by applying the L'Hopital's Rule we get

$$\lim_{x\to 0} \frac{1/\sqrt{1-x^2}}{\cos x} = 1.$$

Remark (6)

If $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = \infty$ it is said that $f(x) - g(x)$ has the indeterminate form of type $\infty - \infty$ at $x = a$. To find the limit of the amount $f(x) - g(x)$ we rewrite the problem to take the indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by unifying the denominators if the amount $f(x) - g(x)$ is in the form of fractions and we may need to factorize and take the common multiple as shown in the following example.

Example 3.13.9:

Find $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution:

By direct substitution, we obtain the indeterminate form of type $\infty - \infty$. By unified the denominators and then direct substitution, we obtain the indeterminate form of type $\frac{0}{0}$.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{sinx}\right) = \lim_{x\to 0}\frac{sinx - x}{xsinx},$$

by applying the L'Hopital's Rule we obtain the indeterminate form of type $\frac{0}{0}$,

$$\lim_{x\to 0}\frac{cosx - 1}{xcosx + sinx},$$

by applying the L' Hopital's Rule one more time we obtain

$$\lim_{x\to 0}\frac{cosx - 1}{xcosx + sinx} = \lim_{x\to 0}\frac{-sinx}{-xsinx + cosx + cosx} = \frac{0}{2} = 0.$$

Remark (7)

Any of the following is indeterminate cases:

$$(1)^{\pm\infty}, (\pm\infty)^0, (0)^0$$

We can solve it as shown in the following example.

Example 3.13.10:

Find $\lim_{x\to 0^+} x^x$.

Solution:

By direct substitution, we get the indeterminate form of type 0^0, let $y = x^x$ and taking the natural logarithm of the two sides. Thus, we get

$$lny = lnx^x = x\, lnx.$$

Then we take limit of the two sides and apply L' Hopital's Rule we ge:

$$\lim_{x\to 0^+} lny = \lim_{x\to 0^+} \frac{lnx}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = 0.$$

As the exponential function is continues then:

$$\lim_{x\to 0^+} y = e^0 = 1.$$

Example 3.13.11:

Prove that $\lim_{x\to\infty} x^{1/x} = 1.$

Solution:

By direct substitution, we obtain the indeterminate form of type ∞^0, let

$$y = x^{1/x}.$$

Then we take the natural logarithm of the two sides, we get $lny = \frac{1}{x} lnx$,

then we take limit of the two sides and apply L' Hopital's Rule we get

$$\lim_{x\to\infty} lny = \lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

As the exponential function is continues then

$$\lim_{x\to\infty} y = e^0 = 1.$$

Example 3.13.12:

Find $\lim_{x\to 1^+} (x)^{\frac{1}{ln(2x-1)}}$.

Solution:

By direct substitution, we obtain the indeterminate form of type 1^∞, let

$$y = (x)^{\frac{1}{ln(2x-1)}}.$$

As the exponential function is continues then

$$lny = \frac{1}{ln(2x-1)} lnx.$$

Then we take the limit of the two sides and apply L' Hopital's Rule we get

$$\lim_{x \to 1^+} lny = \lim_{x \to 1^+} \frac{lnx}{ln(2x-1)},$$

which gives the indeterminate form of type $\frac{0}{0}$ and by applying the L'Hopital's Rule again we get

$$\lim_{x \to 1^+} lny = \lim_{x \to 1^+} \frac{1/x}{2/_{2x-1}} = \lim_{x \to 1^+} \frac{2x-1}{2x} = \frac{1}{2}.$$

As the exponential function is continues then

$$\lim_{x \to 1^+} y = e^{\frac{1}{2}}.$$

3.14 Linear Approximation:

In this section, we will discuss how to add a nonlinear function to a linear function. Let f be a differential function at x_0. The best linear approximation of the function curve in the vicinity of point $(x_0, f(x_0))$ is the tangent line of the function curve at x_0 which is given by:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Thus, for the values x whichare close to x_0, we can approximate it by using the formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

This equation is called the local linear approximation of the function f at x_0. Let $\Delta x = x - x_0$. Then

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x.$$

Example 3.14.1:

Find the linear approximation of the function $f(x) = \sqrt[4]{x}$at $x_0 = 1$, then use it to find an approximate value of $\sqrt[4]{1.1}$and then compare your approximation to the result produced directly by a calculator.

Solution:

Since $f'(x) = \frac{1}{4}(\sqrt[4]{x})^{-3}$ then $f'(1) = \frac{1}{4}$ and put $x_0 = 1$, $\Delta x = 0.1$ in the linear approximation equation we obtain

$$\sqrt[4]{1.1} = f(1 + 0.1) \approx f(1) + f'(1)(0.1).$$

The approximate value of $\sqrt[4]{1.1}$ is 1.025.

Using the calculator, $\sqrt[4]{1.1} = 1.02411$.

Example 3.14.2:

Find the linear approximation of the function $f(x) = \sin x$ at $x_0 = 0$, then use it to find an approximate value of $sin2°$and then compare your approximation to the result produced directly by a calculator.

Solution:

Since $f'(x) = cosx$, then

$$sinx \approx sin(0) + cos(0)(x - 0).$$

Therefore, $sinx \approx x$.

We can say that when x is close enough to zero, then $sinx \approx x$

Since $2°$ in radians is $2 \cdot \left(\frac{\pi}{180}\right) \approx 0.03492$, then $sin2° \approx 0.03492$.

Using the calculator, $sin2° \approx 0.03489$.

It is clear that the accuracy of the local linear approximation of the function f at x_0 will decrease with the value of x_0 goes away from the value of x_0, that is the absolute error as the absolute value of the difference between the function and its approximate value if we denote it by the symbol $E(x)$ then its value in the previous example is $E(x) = |sinx - x|$. Noting the curve of this function, we find that the further the value of x exceeds zero, the greater the value of the absolute error, see Figure (4-3).

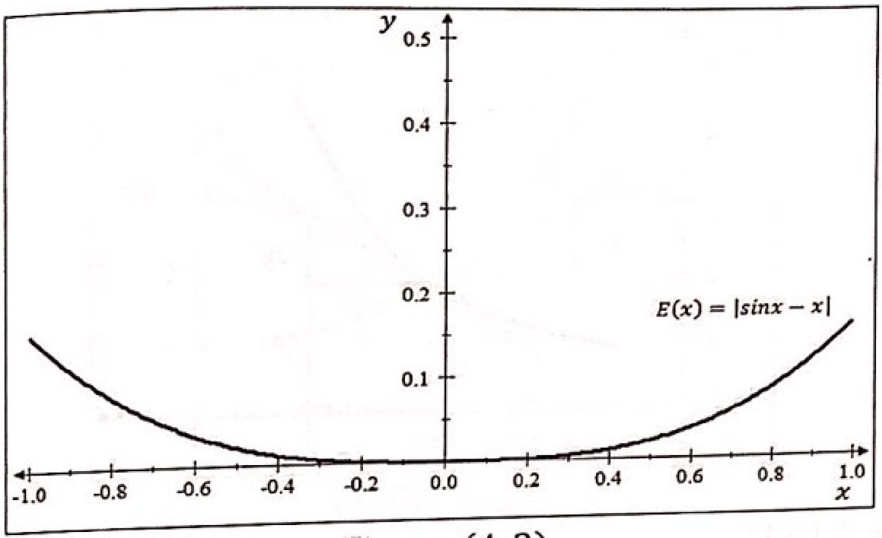


Figure (4-3)

3.15 Differentials

We previously used the symbol $\frac{dy}{dx}$ to denote the derivative of the function $y = f(x)$. In this case, what we will call the differential coefficients dy and dx have no separate meaning for the other. In this section, we will try to find meaning in this case. Let dx is an independent variable that can take any real value. Let dy can be written as:

$dy = f'(x)dx$ ....(i)

Let $dx \neq 0$. We can divide both sides of the equation (i) by dx and get the image:

$\frac{dy}{dx} = f'(x)$ ....(ii)

In this case, the first derivative of the function is the ratio between dy and dx (see Figure (5-3)). The equation (i) is called the differential form.

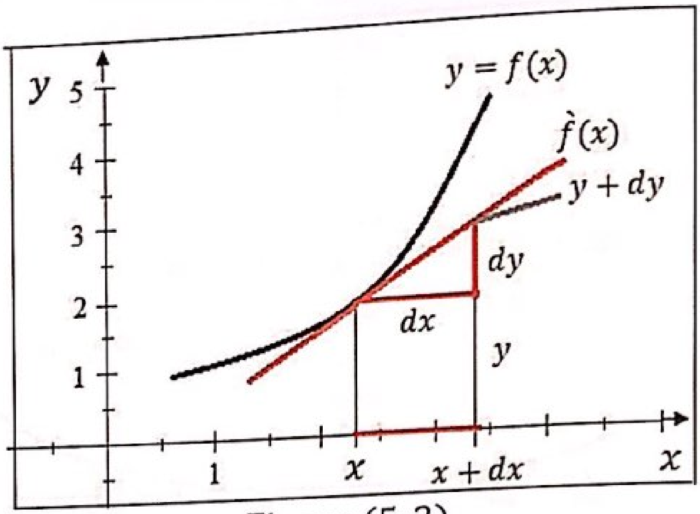


Figure (5-3)

Example 3.15.1:

Express the differentiation with respect to x for the function $y = x^3$ in the differential form and discuss the relationship between dy and dx at $x = 1$.

Solution:

Since $\frac{dy}{dx} = 3x^2$ so $y = 3x^2 dx$, hence $dy = 3dx$ at $x = 1$ which means that whenever x is changes by dx, y changes by $3dx$ along the tangent of the curve of the function $y = x^3$ at $x = 1$, see Figure(3-5).

It should be noted that there is a difference between the increase Δy and the differential coefficient dy .To note this difference,let $dx = \Delta x$, therefore the amount of variation dx results in an amount of variation whose value is dy which is a change to the tangent and Δy is the change to the function curve, see Figure(3-6).

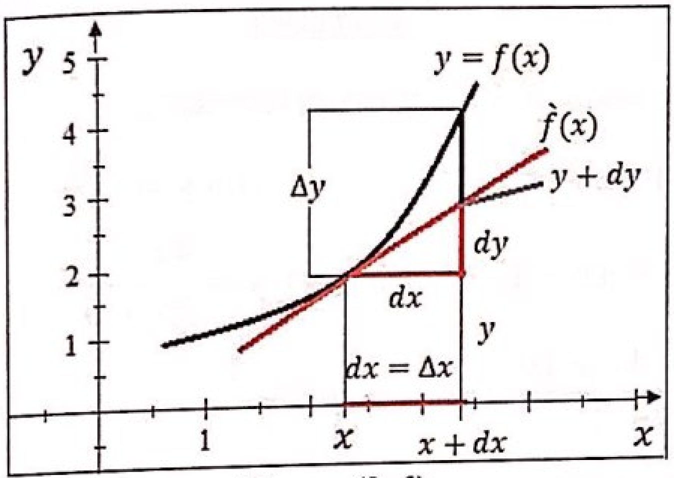


Figure (3-6)

Example 3.15.2:

If $y = \sqrt{x}$, find an equation for Δy and one for dy, then calculate their values at $x = 4$ and $dx = \Delta x = 3$.

Solution:

Since $y = \sqrt{x}$ and $\Delta y = f(x + \Delta x) - f(x)$ so, $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$ at $x = 4$, $\Delta x = 3$ so $\Delta y \approx 0.65$.

While $dy = \frac{dx}{2\sqrt{x}}$ at $x = 4$, $\Delta x = 3$ so $dy = 0.75$, see Figure (3-7).

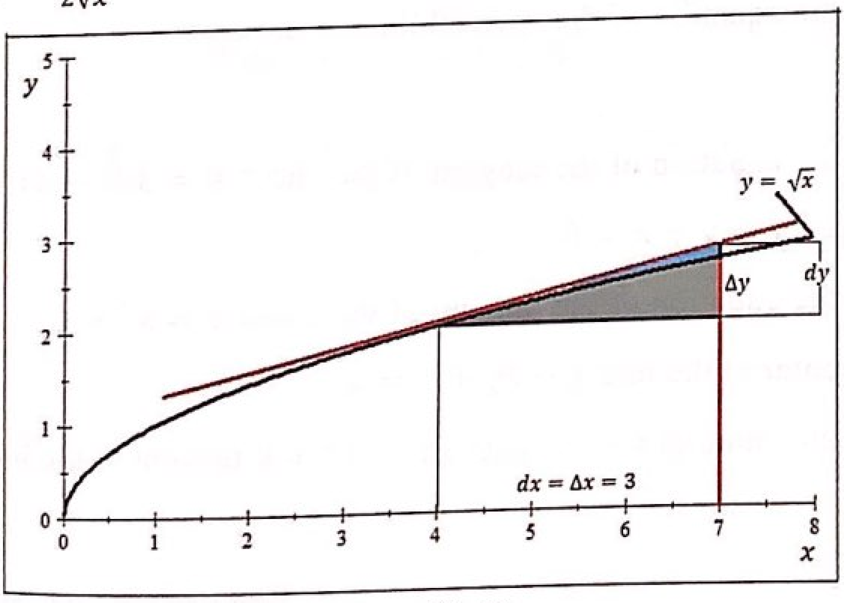


Figure (3-7)

Exercises

(1) Find $\frac{dy}{dx}$ for each of the following functions:

$(i)\ y = 10x^2 + 9x - 4$ $\qquad$ $(ii)\ y = (x^3 - 7)(2x^2 + 3)$

$(iii)\ y = (2x^2 - 4x + 1)(6x + 3)$ $(iv)\ y = \dfrac{4x - 5}{3x + 5}$

$(v)\ y = \dfrac{8x^2 - 4x + 10}{x - 2}$ $(vi)\ y = x^2 + \dfrac{1}{x^2}$

$(vii)\ y = 2x^2 + \sqrt{x}$

(2) For each of the following functions find f' at $x = a$

$(i)\ f(x) = x^3 + 5x - 2\sqrt{x}$, $at\ x = 4$

$(ii)\ f(x) = (x^3 - 5)(2x - 5)$, $at\ x = 2$

$(iii)\ f(x) = \dfrac{3}{x + 2}$, $at\ x = -5.$

(3) Find the equation of the tangent of the curve $y = x^3 - 4$ at point (2,4).

(4) Find the equation of the vertical line on the curve $y = \frac{10}{14-x^2}$ at point (4,-5).

(5) Find the equation of the tangent of the curve $y = 3x^2 - 4x$ parallel to the line $2x - y + 3 = 0$.

(6) Find the equation of the tangent of the curve $y = x^4 - 6x$ perpendicular to the line $x - 2y + 6 = 0$.

(7) Find the value of k if the curve $y = x^2 + k$ tangent to the line $y = 2x$.

(8) If $f(4) = 3$ and $f'(4) = -5$, then find $g'(4)$ in the following two cases:

(1) $g(x) = \sqrt{x}\, f(x)$ (2) $g(x) = \dfrac{f(x)}{x}$.

(9) Find $F'(2)$ if $f(2) = -1$and $f'(2) = 4$, $g(2) = 1$and $f'(2) = -5$ in the following cases:

(i) $F(x) = 5\,f(x) + 2g(x)$ (ii) $F(x) = f(x) - 3g(x)$

(iii) $F(x) = f(x)g(x)$(iv) $F(x) = f(x)/g(x)$

(10) Find y' of each of the following functions

(i) $y = \dfrac{sinx}{x}$ (ii) $y = sinx\ cosx$ (iii) $y = 3cosx - 2sinx$

(iv) $y = \dfrac{tanx}{1 + secx}$ (v) $y = cosecx\ cotx$ (vi) $y = \dfrac{cosx}{xsinx}$

(vii) $y = \dfrac{sinxsecx}{1 + xtanx}$ (viii) $y = \dfrac{cosecx}{tanx}$ (ix) $y = \dfrac{secx}{1 + tanx}$

(x) $y = (x - sinx)(x + cosx)$ (xi) $y = \dfrac{1 + tanx}{1 - tanx}$

(11) Find y' of each of the following functions

(i) $y = (x^2 - 3x + 8)^3$ (ii) $y = (8x - 8)^{-3}$ (iii) $y = \dfrac{x}{(x^2 - 1)^3}$

(iv) $y = (6x - 7)^3(8x^2 + 9)^2$ (v) $y = \dfrac{1}{(8 - 5x + 7x^2)^{10}}$

(12) Find $\frac{dy}{dx}$ if,

(i) $y = \sqrt{u}$, $u = 2x^2 - 3$ (ii) $y = u\sqrt{u + 1}$, $u = x^2 + 1$

(13) At any point, the slope of the tangent of the function $y = 2x^3 - 2x^2 + 1$ is equal to 12.

(14) Find y' at x = 2 where $y = \frac{(x^2-3)^2}{(3x^2-1)^3}$.

(15) Find f' of the following:

(i) $f(x) = sinx^3$ (ii) $f(x) = sin^3x$ (iii) $f(x) = tan^4x^3$

(iv) $f(x) = \sqrt{4+\sqrt{2x}}$ (v) $f(x) = \sqrt{3x - sin^2 4x}$

(vi) $f(x) = x^3 sin^2 3x$ (vii) $f(x) = \frac{sinx}{sec(3x+1)}$

(viii) $f(x) = [x^2 - sec(4x^2 - 2)]^{-4}$

(16) Find $\frac{dy}{dx}$ in each of the following functions

(i) $y = \frac{x}{lnx}$ (ii) $ln\frac{y}{x} + xy = 1$ (iii) $lnxy + x + y = 2$

(iv) $x = ln(x+y+1)$ (v) $y = e^{-3x^2}$ (vi) $y = e^{x}(x^2 + e^{x})$

(vii) $y = ln(e^{x} + e^{-x})$ (viii) $e^{y} = ln(x^3 + 3y)$

(ix) $y = 2^{\sqrt{x}}$(x) $ye^{2x} + xe^{2y} = 1$ (xi) $y = 3^{\sqrt{1-x^2}}$

(xii) $y = log_3(2^x)$ (xiii) $y = log_{10}(e^x)$

(17) By using the logarithmic differentiation; find y' for the following functions

(i) $y = \frac{\sqrt[3]{3-x^2}}{\sqrt[4]{x^4+1}}$ (ii) $y = x^{lnx}$ (iii) $y = (1+x)^{1/x}$

(iv) $y = \sqrt{lnx}$ (v) $y = \sqrt{x+2}\sqrt[3]{x+2}\sqrt[4]{x^2+2}$

(vi) $y = x^2(x^2-3)^3(x+1)^4$ (vii) $y = \frac{x(x-1)(x+2)}{(x-1)^2(x-4)^3}$

(viii) $y = \sqrt[3]{\frac{(x+1)(x+2)}{(x^2+1)(x^2+2)}}$ (ix) $y = cos(lnx)$

(x) $y = ln|tanx|$ (xi) $y = \frac{x^2}{1 + logx}$ (xii) $y = ln(sinx)$

(xiii) $y = (lnx)^{tanx}$ (xiv) $y = x^{sinx}$

(xv) $y = \sqrt{2 + ln^2 x}$ (xvi) $y = sin^2(lnx)$

(xvii) $y = x^3 e^x$ (xviii) $y = e^{1/x^2}$ (xix) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(xx) $y = (sine^x)$ (xxi) $y = exp\left(\sqrt{1 + 5x^3}\right)$

(xxii) $y = e^{xtanx}$ (xxiii) $y = ln(1 - xe^{-x})$

(xxiv) $y = ln(cose^x)$ (xxv) $y = \frac{sinxcosxtan^3 x}{\sqrt{x}}$

(xxvi) $y = log_x e$ (xxvii) $y = log_x 2$

(18) Find $\frac{dy}{dx}$ for each of the following functions

(i) $x^2 + y^2 = 9$ (ii) $\frac{1}{x^2} + \frac{x}{y} = 1$ (iii) $x^2 = \frac{x + 2y}{x - 2y}$

(iv) $x^2 y^2 = x^2 + y^2$ (v) $\sqrt{xy} + 2x = 1$ (vi) $\frac{1}{x^2} + \frac{1}{y^2} = 1$

(vii) $(y^2 - 9)^4 = (4x^2 + 3x - 1)^2$ (viii) $(x - y)^2 = 4$

(ix) $xsiny + ycosx = 1$ (x) $\frac{xy^2}{1 + secy} = 1 + y^4$

(xi) $sin(x^2 y^2) = x$ (xii) $tan^3(xy^2 + y) = x$

(xiii) $x^2 = \frac{coty}{1 + secy}$ (xiv) $x^4 + 4x^2 y^2 - 3xy^3 + 2x = 0$

(19) Find $\frac{dy}{dx}$ for each of the following functions, considering y as the independent variable.

(i) $x^4 + y^4 = 12x^2y$ (ii) $y = 2x^3 - 5x$ (iii) $y^2 = 2x - 3$

(20) In each of the following functions, use the implicit differentiation to obtain the slope of the tangent to the curve at the given point.

(i) $x^4 + y^4 = 16$, $(1, \sqrt[4]{15})$

(ii) $y^3 + yx^2 + x^2 - 3y^2 = 0, (0,3)$

(iii) $x^{2/3} + y^{2/3} = 4,$ $(1,3\sqrt{3})$

(iv) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3,1)$

(21) Find the value of both a and b in the equation for the curve $x^2y + ay^2 = b$ if point (1,1) is on the curve and the tangent line at this point is 4x + 3y = 7.

(22) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the followings.

(i) $x = t^2 + t$, $y = t + 1$ (ii) $x = t^2 + t$, $y = t^2 - t$

(iii) $x = 1 + cost$, $y = -2 + sint$ (iv) $x = 2 + cosht$, $y = -1 + sinht$

(v) $x = e^t + t$, $y = e^t + e^{-t}$ (vi) $x = 3cost$, $y = 2sint$

(vii) $x = 4t$, $y = 3\sqrt{1 - t^2}$ (viii) $x = 3sin^2t$, $y = 3cott$

(ix) $x = e + t$, $y = ln(t + e^t)$ (x) $x = 4e^{-t}$, $y = 2e^t$

(xi) $x = ln(1 + t)$, $y = t - tan^{-1}t$

(xii) $x = asin^3t$, $y = acos^3t$, $a \equiv const$

(23) Using the L' Hopital's Rule, calculate the value of the following limits.

(i) $\lim_{x\to1} \frac{2x^2 + x - 3}{4x^2 - 5x + 1}$ (ii) $\lim_{x\to0} \frac{2x + sinx}{tan4x}$ (iii) $\lim_{x\to1} \frac{x - sinx}{(xsinx)^{3/2}}$

(iv) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\sin 3x}$ (v) $\lim\limits_{x\to 0}\dfrac{\sin^{-1}2x-2\sin^{-1}x}{x^3}$ (vi) $\lim\limits_{x\to 0}\dfrac{\tan^2 x}{\sec 2x-1}$

(vii) $\lim\limits_{x\to 0}\dfrac{\ln(1+2x^2)}{\ln(1+3x)}$ (viii) $\lim\limits_{x\to\infty}\dfrac{2x^3+x^2-4x-3}{3x^3+8x^2+7x+2}$ (ix) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{\ln x}$

(x) $\lim\limits_{x\to\infty}\dfrac{\ln(\ln x)}{\ln(x+1)}$ (xi) $\lim\limits_{x\to\infty} xe^{-x}$ (xii) $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{4}{x^2+2x-3}\right)$

(xiii) $\lim\limits_{x\to 0}\left(\frac{1}{x\sin x}-\frac{1}{x^2}\right)$ (xiv) $\lim\limits_{x\to 0^+} x^{\sin x}$ (xv) $\lim\limits_{x\to 0^+} x^{1/\ln x}$

(xvii) $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$ (xviii) $\lim\limits_{x\to 0}(1+x^2)^{1/x}$

(xix) $\lim\limits_{x\to 1}\frac{x^x-1}{x\ln x}$ (xx) $\lim\limits_{x\to 1}\frac{e^{2x}-e^2}{\sqrt{x}-1}$ (xxi) $\lim\limits_{x\to 0}(1+\sinh x)^{2/x}$

(xxii) $\lim\limits_{x\to 0^+}(\sinh x)^{\tan x}$ (xxiii) $\lim\limits_{x\to\pm\infty}\tanh x$

(xxiv) $\lim\limits_{x\to\pm\infty}(\cosh x-\sinh x)$.

(24) If $\lim\limits_{n\to\infty}\left(\frac{nx+1}{nx-1}\right)^x = 9$ ، find the value of n .

(25) In each of the following exercises, show that the L' Hopital's Rule cannot be applied to calculate the limit, then calculate the limit value, if exists, using any other method.

(i) $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x}$ (ii) $\lim\limits_{x\to\infty}\dfrac{2x-\sin x}{3x+\sin x}$

(iii) $\lim\limits_{x\to\infty}\dfrac{x(2+\sin 2x)}{x+1}$ (iv) $\lim\limits_{x\to\infty}\dfrac{x(2+\sin 2x)}{x^2+1}$

(26) If

$$f(x)=\begin{cases}(x+1)^{\frac{k}{x}}, & x\neq 0\\ 5\quad , & x=0\end{cases}$$

Find the value of k so that f is continuous at $x=0$.

Chapter 4

Applications of Differentiation

4

Chapter 4 : Differentiation

Learning Outcomes:

By completing the study of this chapter, it is expected that the student will be able to:

- Define increasing and decreasing functions and determines the intervals of increase and intervals of decrease.
- Define concavity and define intervals of concave up and concave down.
- Define the relative maximum values of the function and determine it, if there were any.
- Define the absolute maximum values of the function and determine it if there were any.
- Analyze some functions and plot their curves.
- Establish a mathematical model for some applied problems, discuss it, and find its optimum value.
- Discuss and use Newton's method to find an approximate value for the root of the equation.

Differentiation has many important applications. We will consider some of such applications in this chapter such as analyzing the function and drawing its curve, as well as some applications on the maximum and minimum values.

4.1 Increasing or Decreasing Functions and Differentiation

In the first chapter of this book, we dealt with the concept of increasing and decreasing functions, as well as studying the behavior of some simple functions. In this part, we will study how to use the first derivative of a function to study its increasing, decreasing, fixed behaviors, and specifying their intervals. We identified the increasing and decreasing function from Figure 4-1.

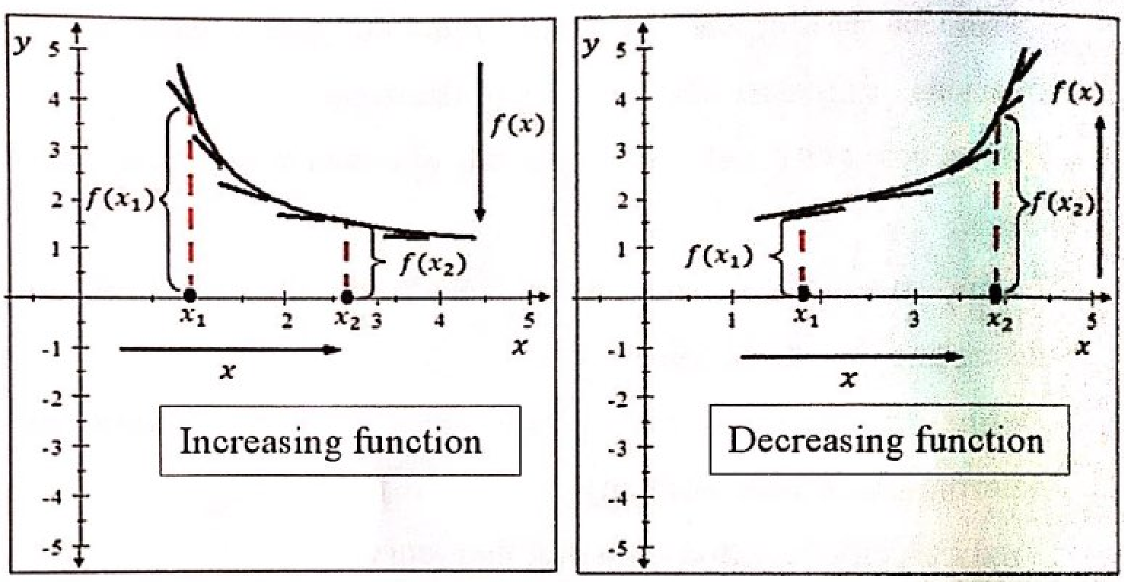


Figure (4-1)

By drawing the tangents of the curve at any point, we can say that the slope of the tangent to the increasing function at any point in the interval of increase is positive, whereas in the case of the decreasing function, the slope of the tangent at any point in the interval of decrease is negative and its value is equal to zero in the case of the constant function. We can formulate this intuitive conclusion in the following theory.

Theorem 4.1.1:

Let the function f be continuous on the closed interval $[a, b]$ and is differentiable over the open interval (a, b).Then:

(1) If $f'(x) > 0$ for all values of x in the interval (a, b), then the function f is increasing over the closed interval $[a, b]$.

(2) If $f'(x) < 0$ for all values of x in the interval (a, b), then the function f is decreasing over the closed interval $[a, b]$.

(3) If $f'(x) = 0$ for all x values in the interval (a, b), then the function f is constant over the closed interval $[a, b]$.

Example 4.1.1:

Find the intervals of increase and decrease of the following function, $f(x) = x^2 - 5x + 4$.

Solution:

The function f is continuous and differentiable on its domain, and its first derivative is

$f'(x) = 2x - 5$.

We study the first derivative sign by finding the solution to the equation $f'(x) = 0$,

$$\Rightarrow 2x - 5 = 0 \Rightarrow x = \frac{5}{2}.$$

Thus $f'(x) > 0$ for all values $x > \frac{5}{2}$ and the function increases on the interval $(\frac{5}{2}, \infty)$, And $f'(x) < 0$ for all values of $x < \frac{5}{2}$. That is the function decreases on the interval $(-\infty, \frac{5}{2})$.

Remark:

The problem of finding the intervals of increase and decrease turns into the question of determining the first derivative sign of the function, and we can do this using the same previous method to determine the sign of the function by drawing the line of real numbers (domain of the function).

In the previous example, we can draw line of real numbers and define intervals in which the function increases and decreases as shown in Figure (4-2)

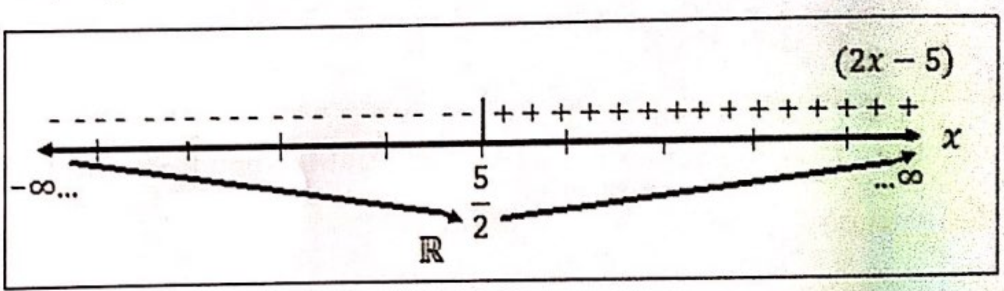


Figure (4-2)

To study the signal $f'(x)$, we the follow steps:

1. We find the roots of $f'(x)$ by solving equation $f'(x) = 0$.
2. If it has one root (linear function), then $f'(x)$ on to the right of the root has the same sign of the coefficient of x and an opposite sign to its left.

3. If it has two real different roots (a quadratic function), then $f'(x)$ has the same sign as the coefficient of x^2 on both limits of the number line and a different sign between the two roots.
4. If it has two equal roots, then $f'(x)$ is positive.
5. If $f'(x)$ has more than two roots, the sign of each linear (or non-linear) is studied individually, then the sign of the product.

Example 4.1.2:

Find the intervals on which the following function is increasing and decreasing,

$f(x) = -\frac{1}{3}x^3 - x^2 + 3x - 1.$

Solution:

The function is continuous and differentiable on its domain, and its first derivative is

$f'(x) = -x^2 - 2x + 3.$

We study the first derivative sign by finding the solution of the equation

$f'(x) = 0,$

$\Rightarrow -x^2 - 2x + 3 = 0 \Rightarrow (-x + 1)(x + 3) = 0 \Rightarrow x_1 = -3, x_2 = 1.$

Using rule no. (3), in the previous example, we get the following plot,

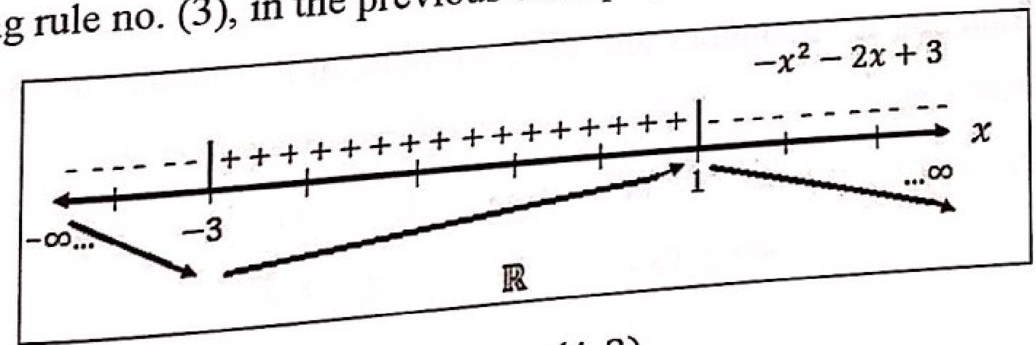


Figure (4-3)

That is, the function f is decreasing on the interval $(-\infty, -3)$ and the interval $(1, \infty)$ and increasing on the interval $(-3,1)$.

See the function curve in Figure (4-4).

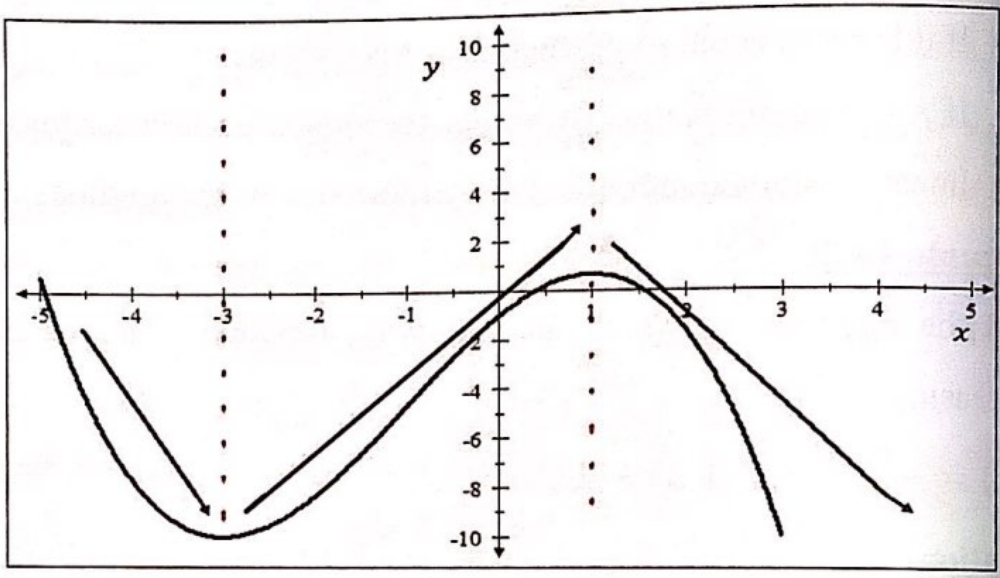


Figure (4-4)

Example 4.1.3:

Find the intervals on which the following function is increasing and decreasing,

$f(x) = 3x^4 + 4x^3 - 12x^2 + 2.$

Solution:

The function is continuous and differentiable on its domain, and its first derivative is

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2),$$

$$\Rightarrow f'(x) = 12x(x-1)(x+2).$$

We study the sign of the first derivative by finding the solution of the equation $f'(x) = 0$,

$\Rightarrow x_1 = 0, x_2 = 1, x_3 = -2.$

Using rule no. (3), in the previous example, we get the following drawing in Figure (4-5):

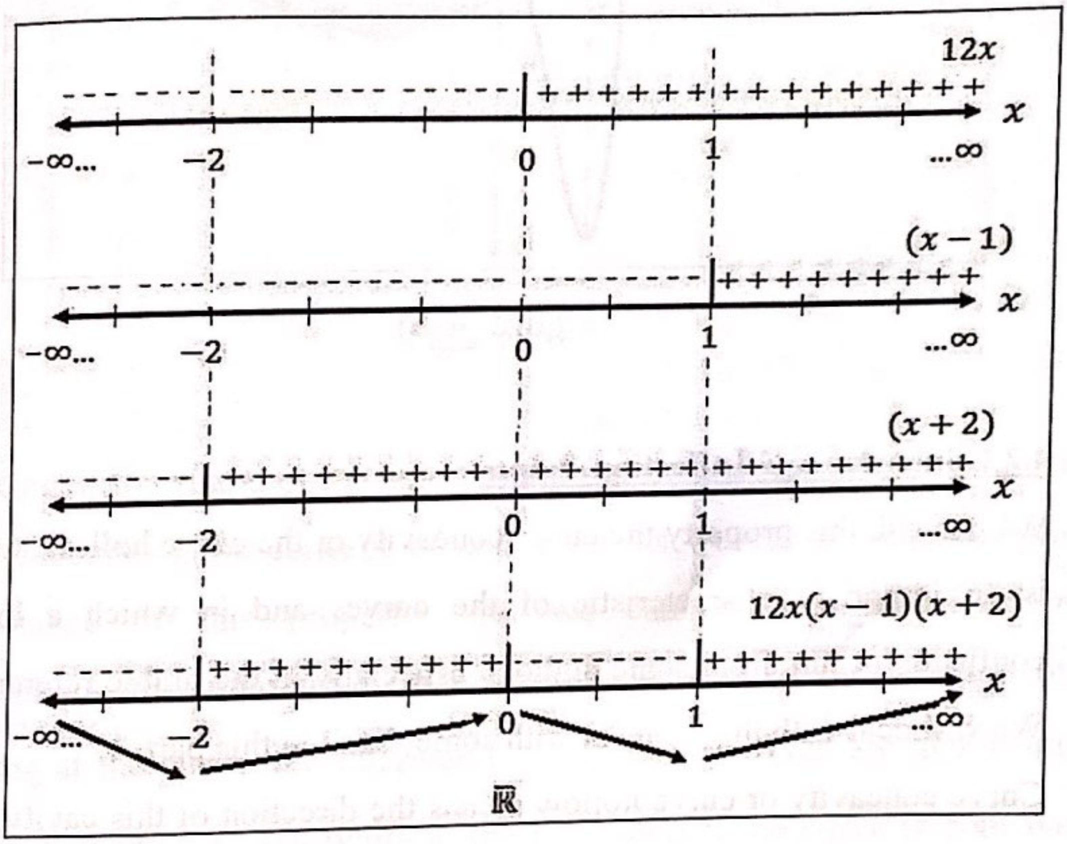


Figure (4-5)

That is, the function f is increasing on the intervals $(-2,0)$, $(1,\infty)$ and decreasing on the intervals $(-\infty,-2)$, $(0,1)$.

Compare this result with the function curve in Figure (4-6).

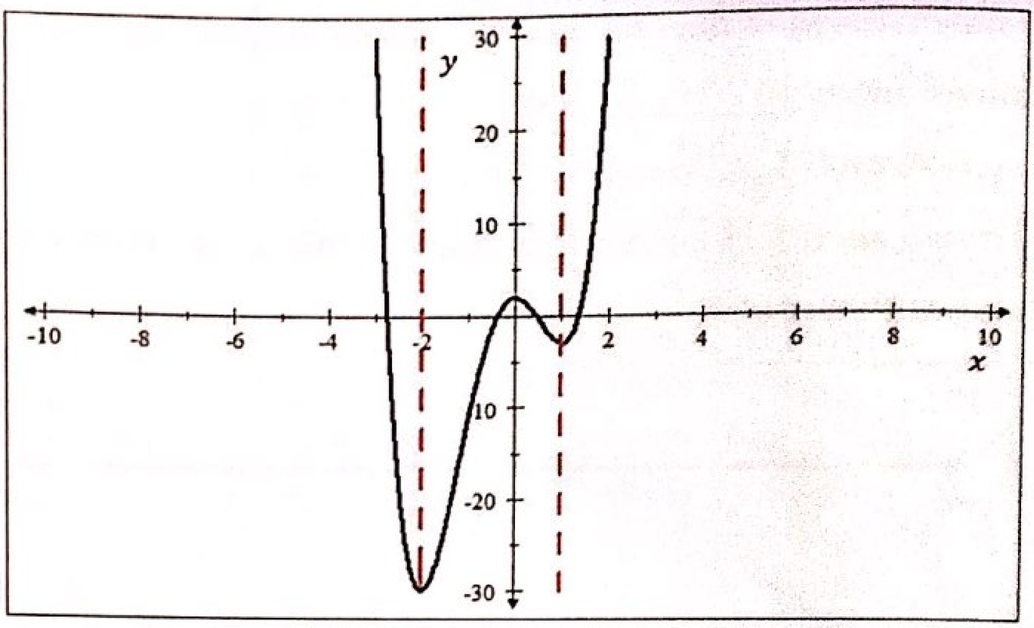


Figure (4-6)

4.2 Concavity and Inflection Points

We can call this property the curve concavity or the curve hollow, which is an important characteristic of the curves and in which a lot of confusion occurs from some authors, especially in the arabic references. We will deal with this concept with some detail in this part.

Curve concavity or curve hollow means the direction of this cavity, is it up or down? In the case of concavity up, we can visualize the curve as a vessel in its correct position, opened it upwards, and it can hold the water, while in the case of concavity down we can visualize it as an inverted vessel (hollowing it down) and cannot carry the water (see Figure 4-7).

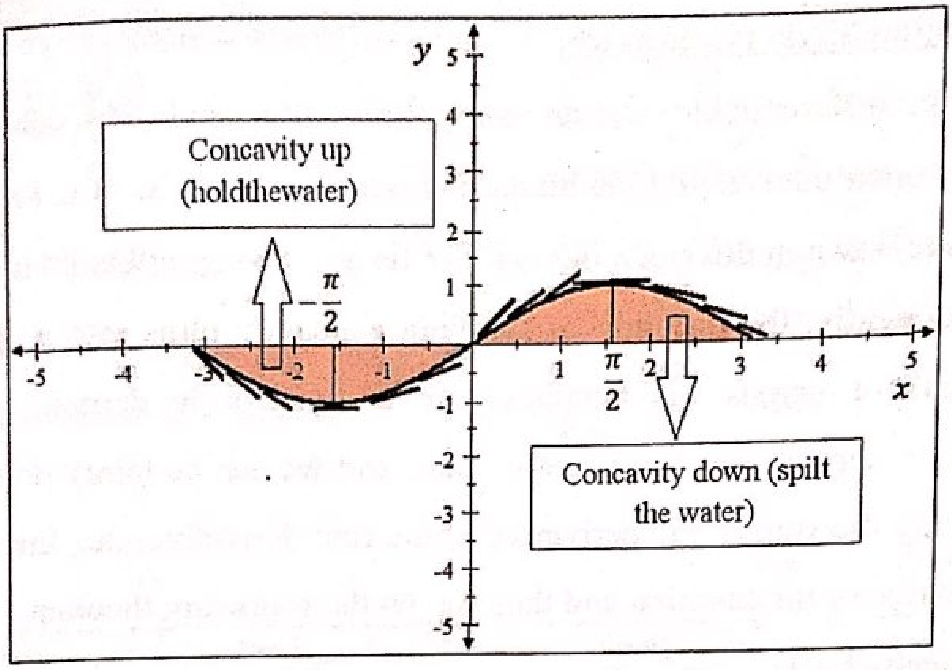


Figure (4-7)

Looking at the shape of the sine function in the previous figure, we can now understand the concept of concavity, as we can also deduce the importance of this property, and we cannot limit ourselves to determining the intervals of increase and decrease of the function by looking at the curve, the intervals $(-\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$ which are increasing intervals of the function while in the first interval the curve is concave upward while in the second interval it is concave down. From the shape of the curve, we also notice that in the concavity region to the top of the tangent value of the function increases by moving from left to right on the curve while in the concavity area to the bottom the value of the tangent diminishes by moving from left to right and we use this phenomenon to define concavity as follows:

Definition 4.2.1: (Concavity)

Let f be differentiable over an open interval, it is said to be concave up in this open interval if f' is increasing over this interval. It is said to be concave down in this open interval if f' is decreasing in this interval. Consequently, the question of studying concavity turns into a study of intervals of increase and decrease, but this time for the derivative of the function and not for the function itself, and we can certainly do that by studying the sign of the derivative of the first derivative, i.e., the second derivative of the function and thus we get the following theorem.

Theorem 4.2.1:

Let f be a continuous function on the closed interval $[a, b]$ and is twice differentiable over the open interval (a, b). Then

1. If $f''(x) > 0$ for all values of x on the interval (a, b), then the function f is concave up on the interval (a, b).
2. If $f''(x) < 0$ for all values of x on the interval (a, b), then the function f is concave down on the interval (a, b).

Example 4.2.1:

Study the concavity of the function in the Example 4.1.3.

Solution:

To study concavity using previous theorem, we only need to study the second derivative of the function.

$\because f'(x) = 12x^3 + 12x^2 - 24x$

$\Rightarrow f''(x) = 36x^2 + 24x - 24.$

We study the second derivative sign by finding the solution to the equation $f''(x) = 0$,

$$\Rightarrow 36x^2 + 24x - 24 = 0 \Rightarrow 3x^2 + 2x - 2 = 0$$

Using the Theorem, $x_1 = \frac{1}{3}(-1 + \sqrt{7}) \approx 0.55$, $x_2 = \frac{1}{3}(-1 - \sqrt{7}) \approx -1.21$.

The sign of the second derivative is shown in Figure (4-8).

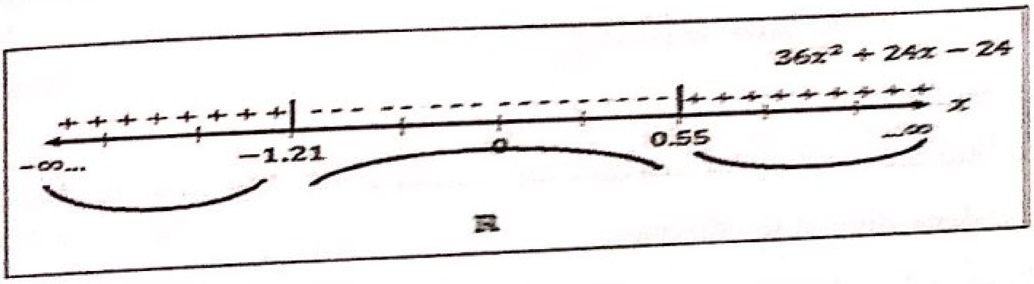


Figure (4-8)

From the previous figure, the function is concave up on the interval $(-\infty, -1.21)$ and the interval $(-\infty, -1.21)$, while concave down on the interval $(-1.21, 0.55)$. See the curve of the function in Example 4.1.3 to compare results.

There is a specific point at which the behavior of the function changes from concavity up to concavity down or vice versa. This point has maximum importance and is called the inflection point and is defined as follows.

Definition 4.2.2: (Inflection Points)

Let f be a continuous function on the open interval (a, b) that contains x_0 and f changes its concavity direction at x_0 from concavity upward to concavity down or vice versa, then we say that the function has an inflection point at x_0 which is the point $(x_0, f(x_0))$.

Referring to the sin curve in Figure (4-7), the inflection point is the point (0,0). In the previous example, the points are $(-1.21, -16.22)$, and $(0.55, -0.6)$.

Example 4.2.2:

Find the intervals in which the function $f(x) = x + 2sinx$ increases, decreases on the interval $[0,2\pi]$.

Solution:

To find the intervals of increase and decrease, we only need to study the first derivative of the function.

$f'(x) = 1 + 2cosx.$

We study the second derivative sign by finding the solution to the equation$f'(x) = 0$,

$$\Rightarrow 1 + 2cosx = 0 \Rightarrow cosx = -\frac{1}{2} \Rightarrow x = cos^{-1}\left(-\frac{1}{2}\right).$$

The solutions of this equation are $\left\{\frac{n\pi}{3} : n = \pm 2, \pm 4, \dots\right\}$ and on the interval $(0,2\pi)$ they are only $x_1 = \frac{2\pi}{3}$, and $x_2 = \frac{4\pi}{3}$.

Using the cosine function curve on the interval $[0,2\pi]$, we obtain Figure (4-9).

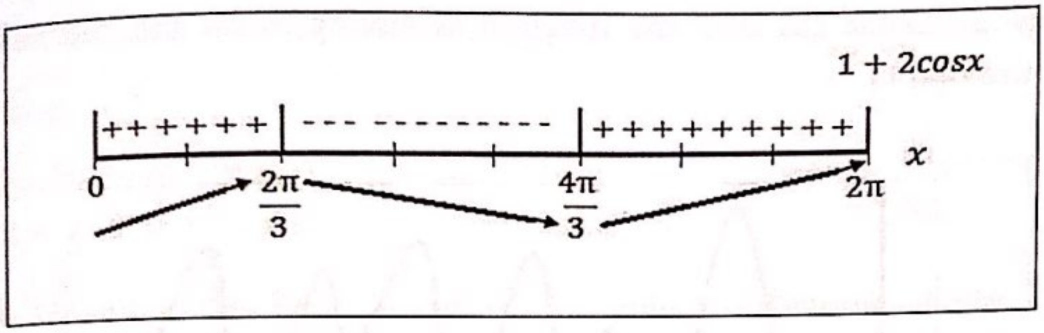


Figure (4-9)

Thus, the function is increasing on the intervals $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, while decreasing on the interval $\left(\frac{4\pi}{3}, 2\pi\right)$.

We are now studying the second derivative sign to determine the inflection points.

$f''(x) = -2sinx.$

$f''(x) = 0 \Rightarrow sinx = 0.$

Thus, the solutions to this equation are $\{n\pi: n = 0, \pm 1, \pm 2, \dots\}$ and on the interval $(0,2\pi)$ it is only $x = \pi$. Using the sin curve, we obtain Figure (4-10).

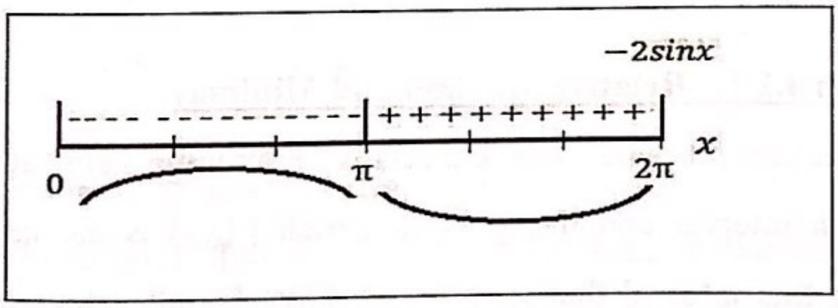


Figure (4-10)

Thus, the function is concave down on the interval $(0, \pi)$ and concave up on the interval $(\pi, 2\pi)$ and the inflection point is $(\pi, -1)$.

Looking at the curve of the function as having peaks and bottoms see Figure (4-11).

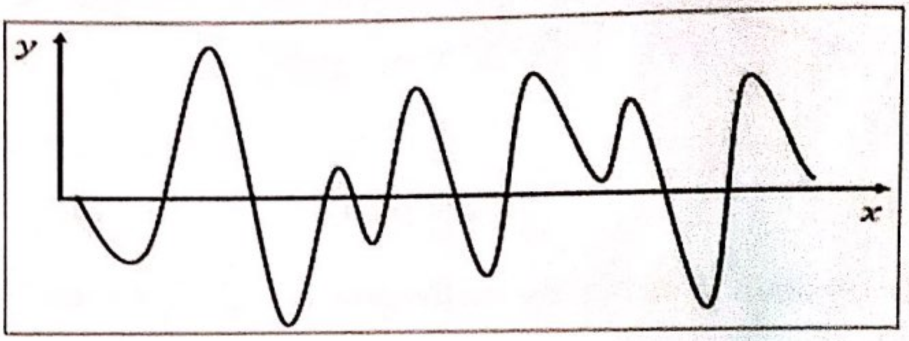


Figure (4-11)

We can give the name the relative maximum or local value of the peaks and the minimum relative or local value of the bottoms in relation to a close neighborhood of this top or bottom, and therefore the relative maximum value does not have to be the largest value along the curve as well as the relative minimum value does not have to be the smallest value along the curve and it is defined in the following definition.

Definition 4.2.3: (Relative Maxima and Minima)

- The function f is said to have a relative maximum value at x_0 if there is an open interval containing x_0 at which $f(x_0)$ is the largest of all values on this interval that is, $f(x_0) \geq f(x)$ for all values of x on the interval.

- The function f is said to have a relative minimum value at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest of all values on this interval that is, $f(x_0) \leq f(x)$ for all values x on the interval.

If the function has a relative minimum or a relative maximum value at x_0 it is said to have a relative extremum at x_0.

The function curves in Figure (4-12) all contain extremum values.

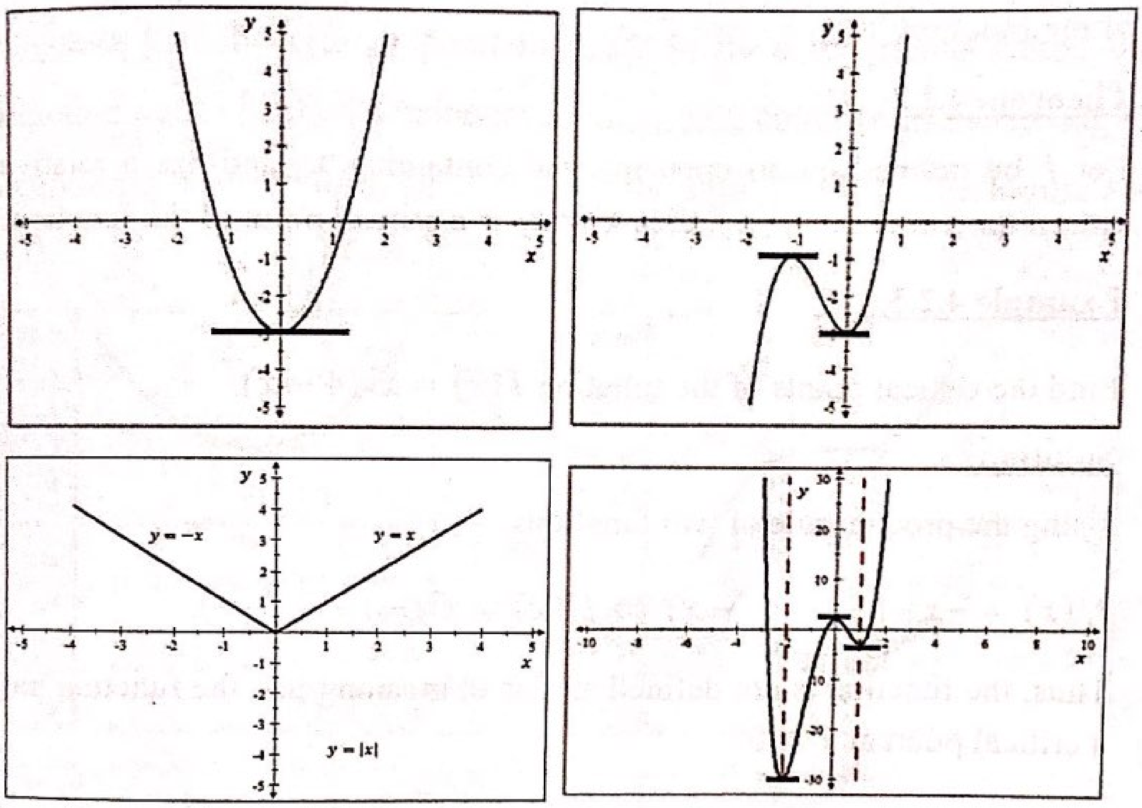


Figure (4-12)

We note that the extremum values always occur when the tangents of the curve are horizontal, that is, $f'(x) = 0$. Looking at the absolute value function in the figure, we also note that they occur when the function is not differentiable. Such points are known as critical points of a function and are defined by the following definition.

Definition 4.2.4: (Critical Points)

The critical points of the function f are the points in the domain of the function at which the graphs of the functions have horizontal tangent lines (the first derivative of the function is zero) or the function is not differentiable. If the tangent is horizontal, it is called the stationary point of the function.

Theorem 4.2.2:

Let f be defined on an open interval containing x_0 and has a relative extremum value at $x = x_0$, then $x = x_0$ is a critical point of the function.

Example 4.2.3

Find the critical points of the function $f(x) = x^{\frac{3}{5}}(4 - x)$.

Solution:

Using the product rule of two functions,

$$f'(x) = -x^{\frac{3}{5}} + \frac{3}{5}x^{\frac{-2}{5}}(4 - x) \Rightarrow f'(x) = x^{\frac{3}{5}}(-1 + \frac{3}{5}\frac{(4-x)}{x}).$$

Thus, the function is not defined at $x = 0$, meaning that the function has a critical point at $x = 0$.

By solving the equation $f'(x) = 0$, we get $x = \frac{3}{2}$ so $x = \frac{3}{2}$ is a critical point of the function (stationary point).

The previous theorem says that; the fact that a point is a critical point is a necessary but not sufficient condition for the maximum values. "It is necessary that the maximum point be a critical point and it is not sufficient that the critical point be a maximum point or in other form a critical point is not a maximum point but every maximum point is a critical point. "

For example, by studying the critical points of the function $f(x) = x^3$, we find that $x = 0$ is a critical point of the function while it is not a maximum value, see the curve of the cubic function (Figure 1-27). This suggests that the critical point in order to be a maximum value, the function must change its behavior at that point, either from increasing to decreasing or vice versa, i.e., changing the sign of $f'(x)$, see Figure (4-13).

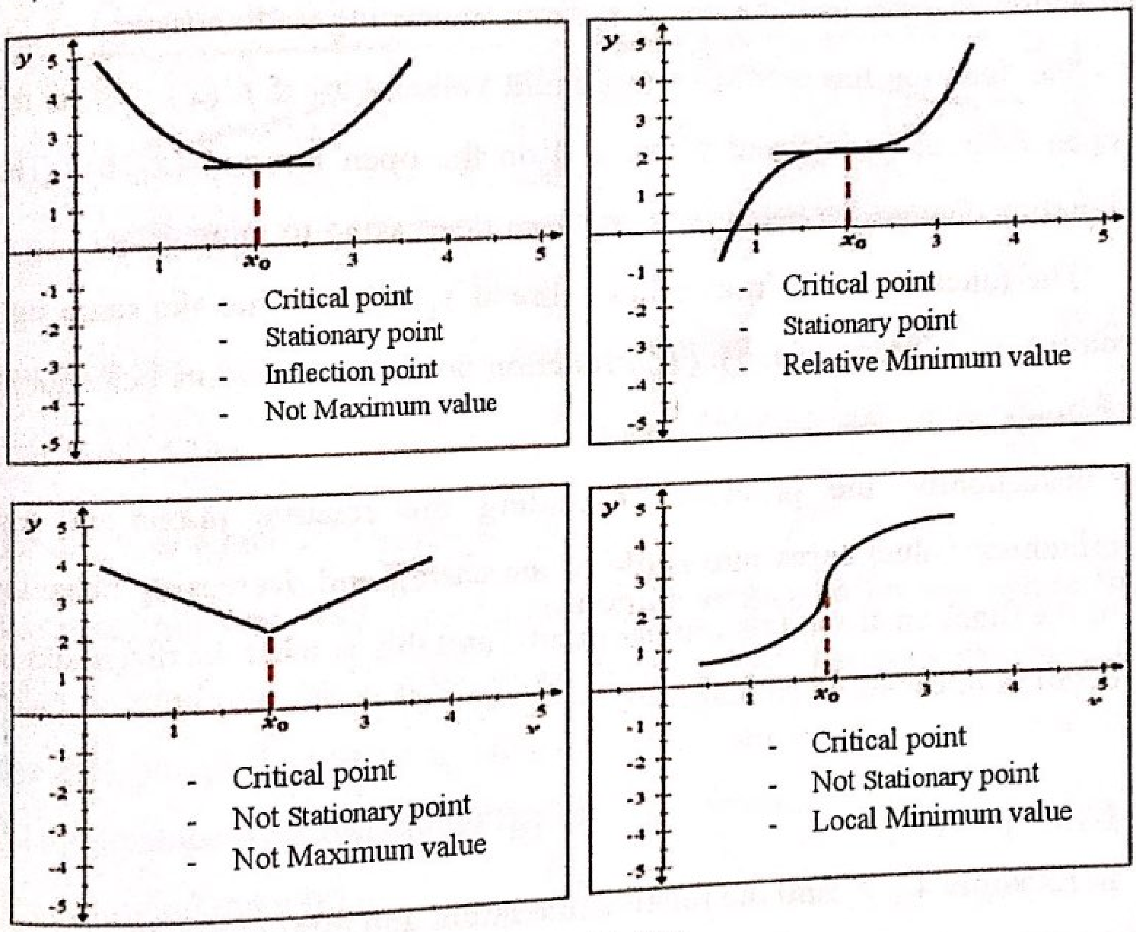


Figure (4-13)

We can also say that they are relative maximum if the behavior of the function changes from increasing to decreasing and it is relatively minimum if it changes from decreasing to increasing and this is formulated in a rule called the first derivative test as follows.

Theorem of 4.2.3: (First Derivative Test)

Let f be a continuous function on the open interval (a, b) which contains the critical point x_0. Then

- The function has a relative maximum value at x_0 if $f'(x) > 0$ on the open interval (a, x_0) and $f'(x) < 0$ on the open interval (x_0, b). (The function changes its behavior at x_0 from increasing to decreasing).
- The function has a relative minimum value at x_0 if $f'(x) < 0$ on the open interval (a, x_0) and $f'(x) > 0$ on the open interval (x_0, b). (The function changes its behavior at x_0 from decreasing to increasing)
- The function has no maximum value at x_0 if $f'(x)$ has the same sign on the open interval (a, b). (The function does not change its behavior at x_0).

Consequently, the problem of finding the relative maximum and minimum values turns into study of increasing and decreasing behavior of the function around its critical points, and this is what we discussed in detail in intervals of increase and decrease.

Example 4.2.4:

In Example 4.2.3, find the relative maximum and minimum values of the function.

Solution:

The critical points of the function are $x_1 = 0$ and $x_2 = \frac{3}{2}$ we study the sign of the first derivative around these points.

$f'(x) = \frac{4}{5}x^{\frac{-2}{5}}(-2x + 3).$

Since the value $\frac{4}{5}x^{\frac{-2}{5}}$ is always positive, we only need to study the sign of the quantity $(-2x + 3)$, see Figure (4-14).

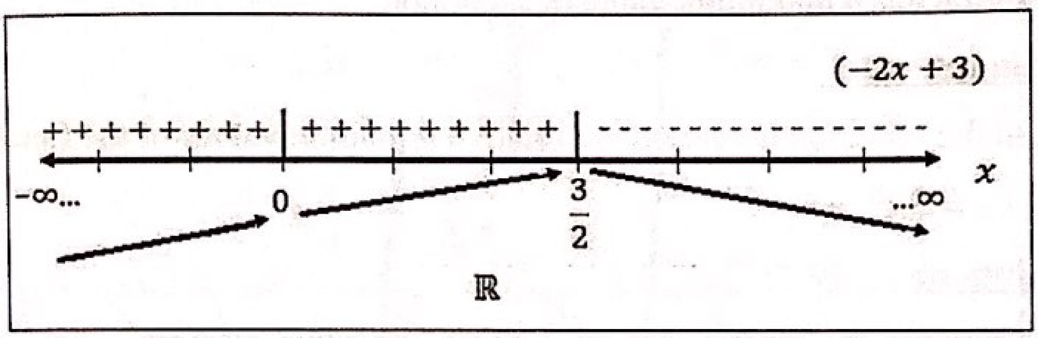


Figure (4-14)

So that the point $x_1 = 0$ is only a critical point and there is no maximum value at it, while $x_2 = \frac{3}{2}$ has a relative maximum value for the function which is $f\left(\frac{3}{2}\right) = 0$.

We note that the relative maximum value is located on top, while the relative minimum value falls on the bottom, and therefore we can also use the second derivative to find the maximum relative and minimum relative values by a test called the second derivative test and is given in the following theorem.

Theorem 4.2.4: (Second Derivative Test)

Let f be a twice differentiable function at the point x_0. Then

- The function has a relative maximum value at x_0 if $f'(x_0) = 0$ and $f''(x_0) < 0$.
- The function has a relative minimum value at x_0 if $f'(x_0) = 0$ and $f''(x_0) > 0$.
- If $f'(x_0) = 0$ and $f''(x_0) = 0$ we cannot determine whether the function has a maximum value of x_0 or not.

Example 4.2.5:

Find the relative maximum and relative minimum values of the function,

$f(x) = 3x^5 - 5x^3$.

Solution:

We will use the second derivative test to solve this example.

The first derivative of the function is

$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$.

By solving the equation $f'(x) = 0$, then the stationary points of the function are,

$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1$.

The second derivative of the function is

$f''(x) = 60x^3 - 30x^2 = 30x^2(2x - 1)$.

Since, $f''(-1) = -90 < 0$,

then the function has a relative maximum value at $x = -1$.

Since, $f''(0) = 0$.

The test does not give a result, and we cannot determine whether the function has a maximum point at $x = 0$ or not.

Referring to the first derivative, the function does not change its behavior at $x = 0$ and therefore has no maximum value.

Since, $f''(1) = 30 > 0$, then the function has a relative minimum value at $x = 1$, see the function curve in Figure (4-15).

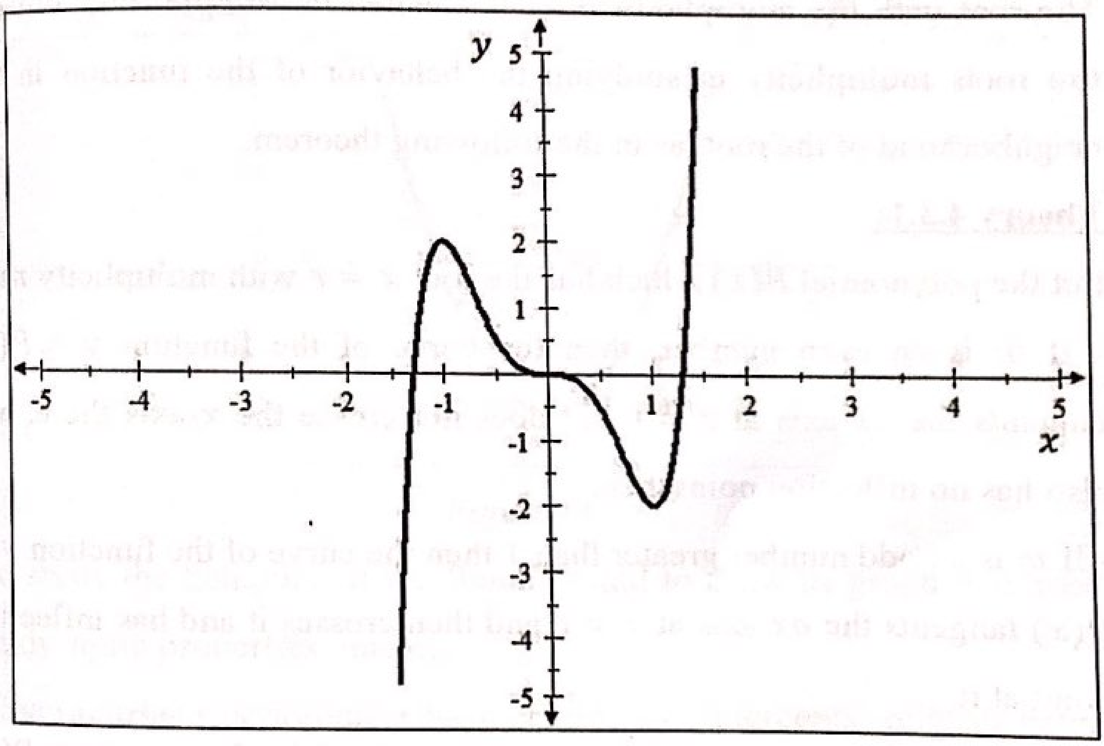


Figure (4-15)

4.3 Analysis and Graphing of Functions

In this section, we aim to use the results obtained in the previous sections to develop a methodology for analyzing and drawing some fundamental functions, including polynomials and fractional functions. Before we go into polynomial analysis, we will discuss an important characteristic, which is the relationship between the shape of the function curve and the multiplicity of roots.

Definition 4.3.1: Multiplicity of Root:

Let the polynomial $P(x)$ has the root $x = r$ ($P(x) = 0$ at $x = r$). It is said that the multiplicity of the root is m if and only if $(x - r)^m$ divides $P(x)$ while $(x - r)^{m+1}$ does not divide it.

The root with the multiplicity $m = 1$ is called the simple root. We can use roots multiplicity in studying the behavior of the function in the neighborhood of the root, as in the following theorem.

Theory 4.3.1:

Let the polynomial $P(x)$ which has the root $x = r$ with multiplicity m.

- If m is an even number, then the curve of the function $y = P(x)$ tangents the ox axis at $x = r$ but does not crosse the x-axis there, and also has no inflection point there.
- If m is an odd number greater than 1 then the curve of the function $y = P(x)$ tangents the ox axis at $x = r$ and then crosses it and has inflection point at it.
- If $m = 1$ (the root is simple), then the graph of the function $y = P(x)$ is not tangent to the x -axis at $x = r$, crosses the x -axis there, and may or may not have an inflection point there, see figure (4-16).

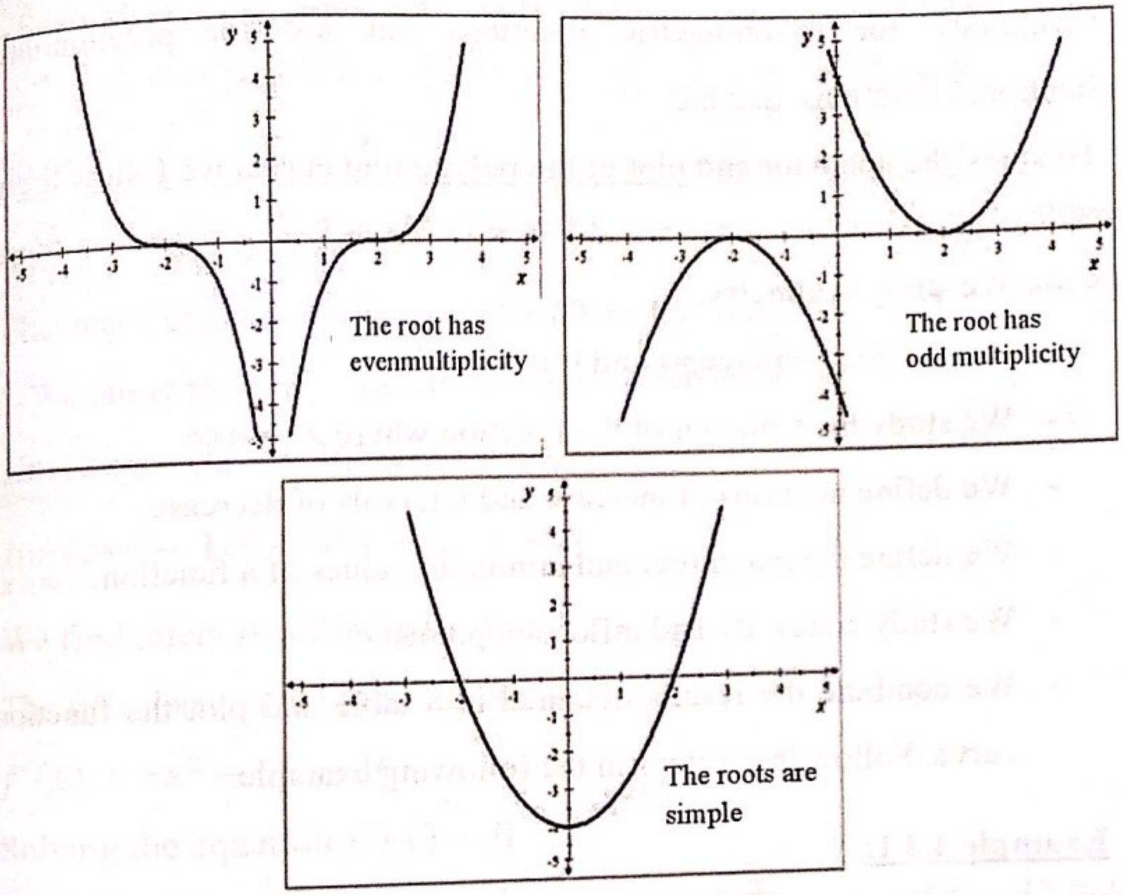


Figure (4-16)

To study the behavior of the function and to draw its graph, we need to study some properties, namely:

• Symmetries • periodicity• x -intercepts • y -intercepts• relative extrema • concavity • intervals of increase and decrease • inflection points • affinity lines • asymptotes • behavior as $x \to \pm\infty$.

It is not necessary to study all these properties since some of which are specific to certain functions and not to other functions, for example asymptotes are specific to fractional functions but not to polynomials, periodicity for trigonometric functions but not for polynomials, fractional functions, and etc.

To study the behavior and plot of the polynomial curve, we follow these steps:

- We study symmetry.
- We define x -intercepts and y-intercepts
- We study the behavior of the function where $x \to \pm\infty$
- We define intervals of increase and intervals of decrease.
- We define the maximum and minimum values of a function.
- We study concavity and inflection points.
- We combine the results obtained in a table and plot the function curve. Follow these steps in the following example.

Example 4.3.1:

Plot the curve of the flowing the function.

$f(x) = 2x^3 - 3x^2 - 36x + 5.$

Solution:

- We study the function symmetry.

$\because f(-x) = -2x^3 - 3x^2 + 36x + 5,$

so, the function is not symmetric.

- We determine x -intercepts and y-intercepts,

- First, y-intercepts.

When $x = 0$, $y = 5$, then y-intercepts is $(0,5)$.

- Second, x -intercepts, let $y = 0$. Then

$2x^3 - 3x^2 - 36x + 5 = 0 \Longrightarrow (x-5)(2x^2 + 7x - 1) = 0.$

So that,

$x_1 = 5,\ \ x_2 = \frac{1}{4}\left(-7 + \sqrt{57}\right) \approx 0.14, \quad x_3 = \frac{1}{4}\left(-7 - \sqrt{57}\right) \approx -3.64.$

The curve crosses x-axis at $(5,0)$, $(0.14,0)$, and $(-3.64,0)$.

- We study the behavior of the function where $x \to \pm\infty$.

$\lim_{x\to\infty}(2x^3 - 3x^2 - 36x + 5) = \infty,$

$\lim_{x\to\infty}(2x^3 - 3x^2 - 36x + 5) = -\infty.$

We find intervals of increase and decrease.

The first derivative of the function is

$f'(x) = 6x^2 - 6x - 36.$

Solving the equation $f'(x) = 0$,

$$6x^2 - 6x - 36 = 0 \Longrightarrow x^2 - x - 6 = 0 \Longrightarrow (x+2)(x-3) = 0$$

$\Longrightarrow x_1 = -2,\ x_2 = 3.$

Its sign is given from Figure (4-17).

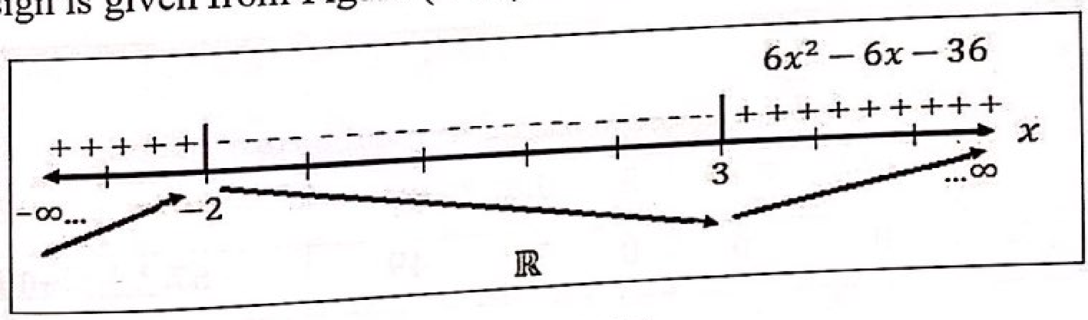


Figure (4-17)

Thus, the function is increasing on the interval $(-\infty, -2)$ and $(3, \infty)$ and decreasing on the interval $(-2,3)$.

It has a relative maximum value at $x = -2$ and a relative minimum value at $x = 3$.

We study concavity and inflection points.

The second derivative of the function is

$f''(x) = 12x - 6.$

By solving the equation $f''(x) = 0$, then $x = \frac{1}{2}$ and the second derivative sign is given in Figure (4-18).

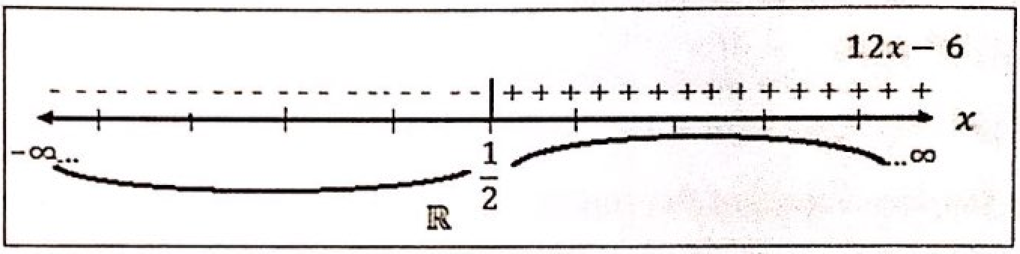


Figure (4-18)

Thus, the function is concave up on the interval $\left(-\infty, \frac{1}{2}\right)$ and concave down on the interval $(\frac{1}{2}, \infty)$ and has an inflection point at $x = \frac{1}{2}$.

We collect the results obtained in the following table.

					Relative Maximum	*Relative Minimum*	*Inflection*
x	-3.64	0.14	5	0	-2	3	0.5
$f(x)$	0	0	0	5	49	-67	-13.5

We draw the points in the previous table, bearing in mind that the relative maximum point represents a peak, while the minimum point represents a bottom, and we get the figure (4-19).

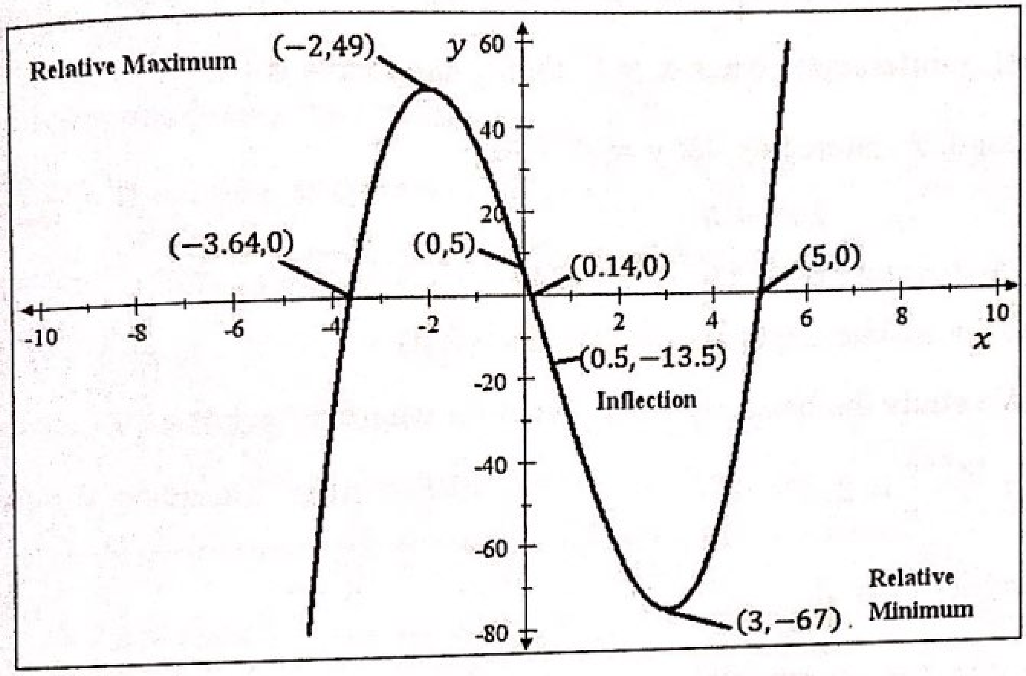


Figure (4-19)

When analyzing and drawing fractional functions, we follow the same previous steps in addition to finding vertical asymptote lines. Let the fractional function $f(x) = \frac{P(x)}{Q(x)}$. Thevertical asymptotes are the straight lines that have $Q(x) = 0$. We also study the behavior of the function on these lines, as in the following example.

Example 4.3.2:

Plot the curve of the flowing the function, $f(x) = \frac{2x^2-8}{x^2-16}$.

Solution:

- We study the function symmetry.

$\because f(-x) = \frac{2x^2-8}{x^2-16} = f(x).$

So, the function is symmetric about oy axis.

- We fined x-intercepts and y-intercepts,

First, y-intercepts, when $x = 0$, then y-intercepts is $(0, \frac{1}{2})$.

Second, x -intercepts, let $y = 0$. Then

$$\frac{2x^2 - 8}{x^2 - 16} = 0 \Longrightarrow 2x^2 - 8 = 0 \Longrightarrow x = \pm 2.$$

So that, x -intercepts are $(-2,0)$, and $(2,0)$

- We study the behavior of the function where $x \to \pm\infty$.

$\lim_{x\to\infty} \frac{2x^2-8}{x^2-16} = 2,$

$\lim_{x\to-\infty} \frac{2x^2-8}{x^2-16} = 2.$

So $y = 2$ is a horizontal asymptote.

We study vertical asymptotes

The vertical asymptotes are at $x^2 - 16 = 0$ thus at the values of $x =$

$\pm 4.$

-We study the verticals asymptotes

$\lim_{x\to 4^-} \frac{2x^2-8}{x^2-16} = -\infty,$

$\lim_{x\to 4^+} \frac{2x^2-8}{x^2-16} = \infty,$

$\lim_{x\to -4^-} \frac{2x^2-8}{x^2-16} = \infty,$

$\lim_{x\to -4^+} \frac{2x^2-8}{x^2-16} = -\infty,$

We find intervals of increase and decrease.

The first derivative of the function is

$f'(x) = \frac{-48x}{(x^2-16)^2}$.

By solving equation $f'(x) = 0$,

$\frac{48x}{(x^2-16)^2} = 0 \Rightarrow 48x = 0 \Rightarrow x = 0.$

And $f'(x)$ is not defined at the values $x = \pm 4$, so the function has critical values at

$x_1 = -4,\ x_2 = 0, x_3 = 4.$

Its sign is given in Figure (4-20).

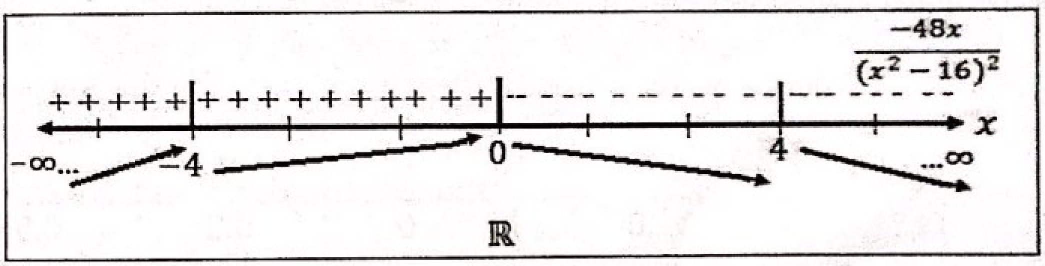


Figure (4-20)

Thus, the function is increasing on the intervals (-∞, -4) and $(-4,0)$, and decreasing on the intervals $(0,4)$ and $(4, \infty)$.

It has a relative maximum value at $x = 0$.

We study concavity and inflection points.

The second derivative of the function is

$f''(x) = \frac{144x^2+768}{(x^2-16)^3}$.

$f''(x) \neq 0$ and the second derivative sign is determined from the sign of $(x^2-16)^3$ given in Fig. (4-21).

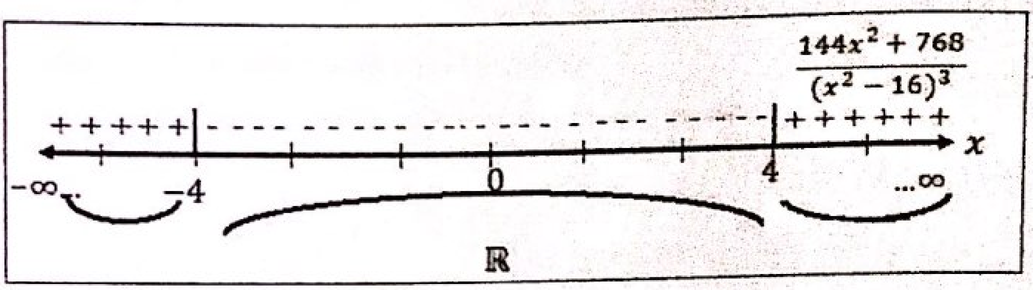


Figure (4-21)

Thus, the function is concave up on the intervals $(-\infty,-4)$ and $(-\infty,-4)$ and concave down on the interval $(-4,4)$ and there are no inflection points. We collect the results obtained in the following table:

				Relative Maximum
x	-2	2	0	0
$f(x)$	0	0	0.5	0.5

	Vertical Asymptotes		$x=4$		$x=-4$	
$x\to$	∞	$-\infty$	4^-	4^+	-4^-	-4^+
$f(x)\to$	2	2	$-\infty$	∞	∞	$-\infty$

We draw the points in the previous table and the asymptotes, so we get the figure (4-22).

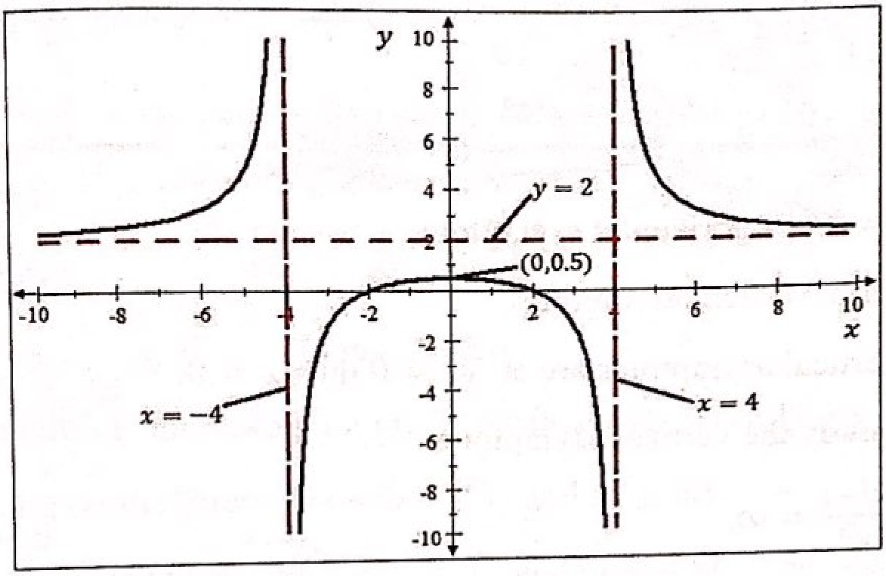


Figure (4-22)

Example 4.3.3:

Plot the curve of the flowing the function, $f(x) = \frac{x^2-1}{x^3}$.

Solution:

- We study the function symmetry.

$\because f(-x) = -\frac{x^2-1}{x^3} = -f(x)$,

so, the function is symmetric about the origin point.

- We find x-intercepts and y-intercepts.
- First, y-intercepts, let $x = 0$. Then there is no y-intercepts
- Second, x-intercepts, let $y = 0$ then

$$\frac{x^2-1}{x^3} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

x-intercepts are $(-1,0)$, and $(1,0)$.

- We study the behavior of the function where $x \to \pm\infty$.

$\lim_{x \to \infty} \frac{x^2-1}{x^3} = 0,$

$\lim_{x \to -\infty} \frac{x^2-1}{x^3} = 0.$

So $y = 0$ is a horizontal asymptote.

We study vertical asymptotes.

The vertical asymptotes are at $x^3 = 0$ thus $x = 0$.

-We study the vertical asymptotes

$\lim_{x \to 0^-} \frac{x^2-1}{x^3} = \infty,$

$\lim_{x \to 0^+} \frac{x^2-1}{x^3} = -\infty.$

We find intervals of increase and decrease.

The first derivative of the function is

$f'(x) = \frac{-x^2+3}{x^4}.$

By solving equation $f'(x) = 0$,

$\Rightarrow \frac{-x^2+3}{x^4} = 0 \Rightarrow -x^2 + 3 = 0 \Rightarrow x = \pm\sqrt{3},$

and $f'(x)$ is not defined at the values $x = 0$, so the function has critical values at

$x_1 = -\sqrt{3},\ x_2 = 0, x_3 = \sqrt{3}.$

Its sign is given in Figure (4-23).

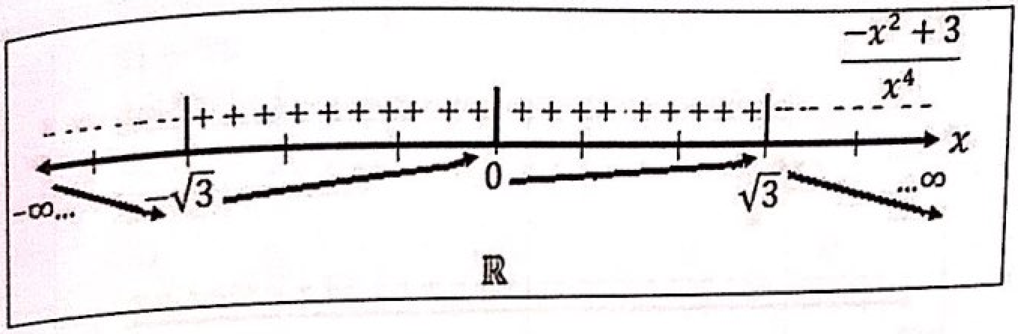


Figure (4-23)

Thus the function is increasing on the intervals $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$,

and decreasing on the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

It has a relative maximum value at $x = \sqrt{3}$ and minimum value at $x =$

$-\sqrt{3}$.

We study concavity and inflection points.

The second derivative of the function is

$f''(x) = \frac{2x^2-12}{x^5}$.

By solving equation $f''(x) \neq 0$ so $x = \pm\sqrt{6}$ and the second derivative sign is determined from the sign of $(x^2 - 16)^3$ and is given in Figure. (4-24).

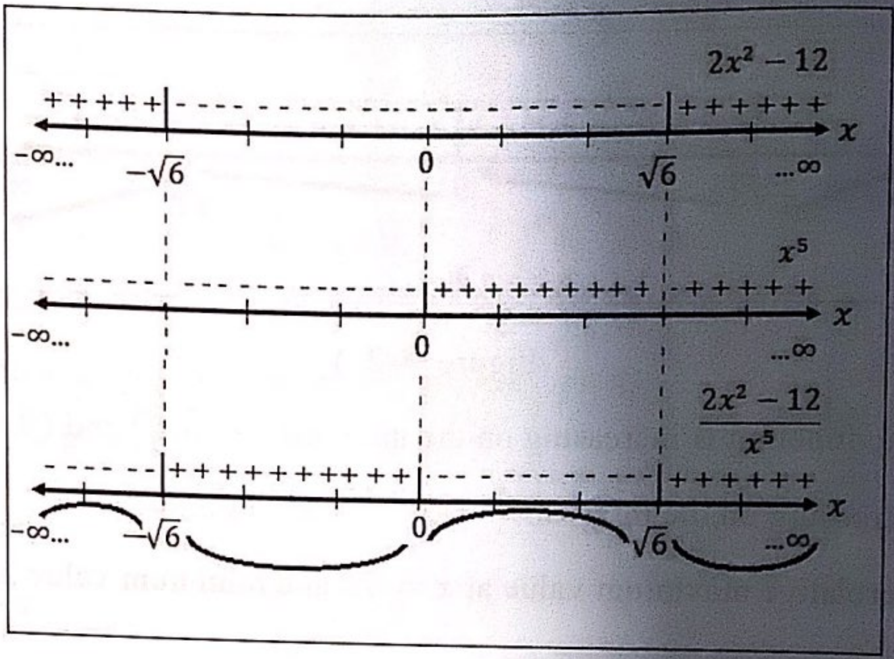


Figure (4-24)

Thus, the function is concave up on the intervals $(-\sqrt{6}, 0)$ and $(\sqrt{6}, \infty)$ and concave down on the intervals $(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$ and there is inflection points at $x = \pm\sqrt{6}$.

We collect the results obtained in the following table:

			Relative Maximum	*Relative Minimum*	*Inflection*	*Inflection*
x	-1	1	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{6}$	$\sqrt{6}$
$f(x)$	0	0	-0.38	0.38	-0.34	0.34

	Vertical Asymptotes		$x = 0$	
$x \to$	∞	$-\infty$	0^-	0^+
$f(x) \to$	0	0	∞	$-\infty$

We draw the points in the previous table and the asymptotes, so we get the Figure (4-25).

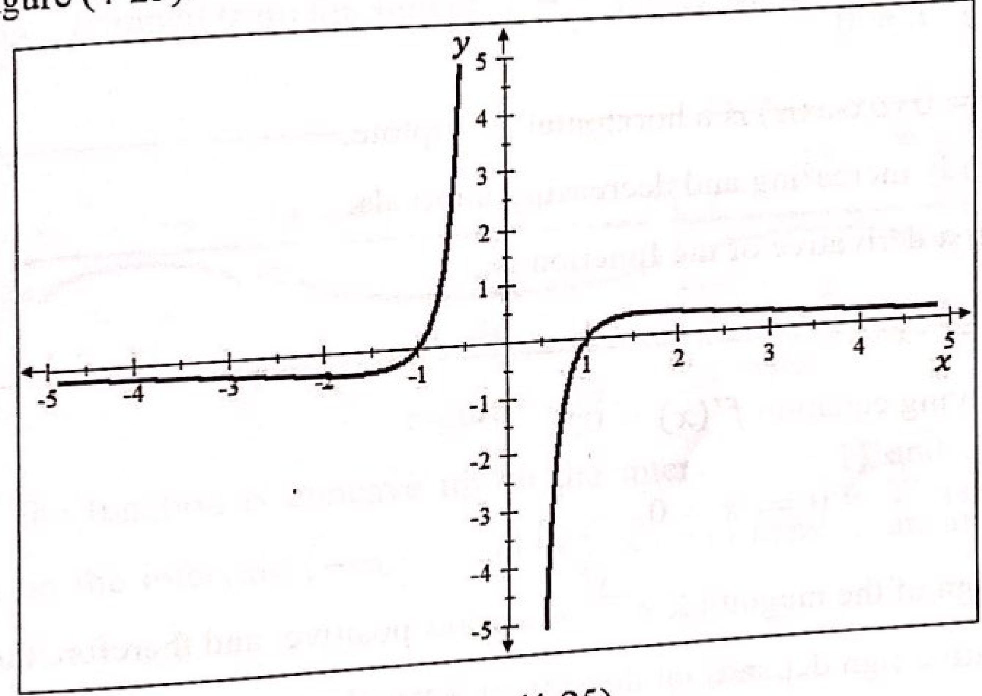


Figure (4-25)

Example 4.3.4

Plot the curve of the flowing the function, $f(x) = e^{-\frac{x^2}{2}}$.

Solution:

- We study the function symmetry.

$\because f(-x) = e^{-\frac{x^2}{2}} = f(x)$.

So the function is symmetric about oy axis.

- We determine x-intercepts and y-intercepts,

First, x-intercepts; let $x = 0$, so $y = 1$ then the x-intercepts is $(0,1)$.

Second, y -intercepts; there is no y-intercepts.

We study the behavior of the function where $x \to \pm\infty$.

$$\lim_{x\to\infty} e^{-\frac{x^2}{2}} = 0,$$

$$\lim_{x\to-\infty} e^{-\frac{x^2}{2}} = 0.$$

So, $x = 0$ (ox-axis) is a horizontal asymptote.

We study increasing and decreasing intervals.

The first derivative of the function is,

$$\grave{f}(x) = -xe^{-\frac{x^2}{2}}.$$

By solving equation $f'(x) = 0$,

$$\Rightarrow\ -xe^{-\frac{x^2}{2}} = 0 \Rightarrow x = 0.$$

The sign of the magnitude $e^{-\frac{x^2}{2}}$ is always positive, and therefore the first derivative sign depends on the sign of $-x$ and is given in Figure (4-26).

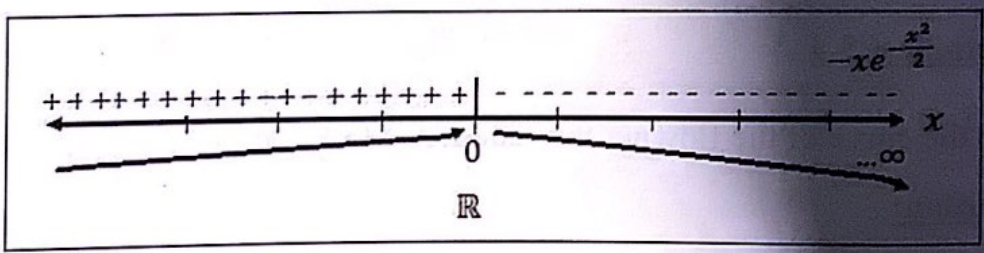


Figure (4-26)

Thus, the function is increasing on the interval $(-\infty, 0)$ and decreasing on the interval $(0, \infty)$) and it has maximum value at $x = 0$.

We study concavity and inflection points.

The second derivative of the function is

$\grave{f}(x) = (1 - x^2)e^{-\frac{x^2}{2}}$.

By solving equation $f''(x) \neq 0$ so $x = \pm 1$ and the second derivative sign is determined from the sign of$(1 - x^2)$ as given in Figure (4-27).

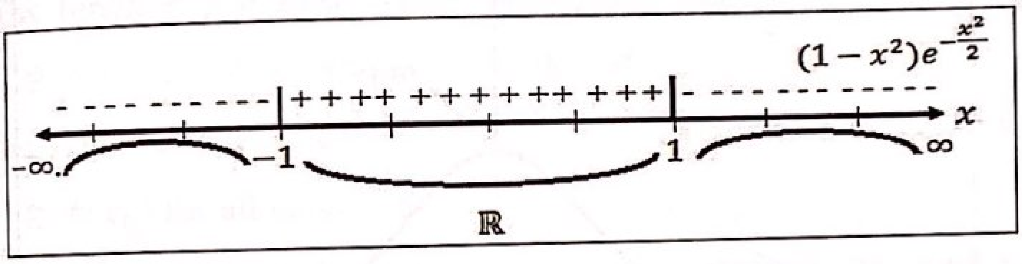


Figure (4-27)

Thus, the function is concave up on the interval $(-1,1)$ and concave down on the intervals $(-\infty, -1)$ and $(1, \infty)$ and there is an inflection points at $x = \pm 1$.

We collect the results obtained in the following table:

		Relative Maximum	*Inflection*	*Inflection*
x	0	0	1	-1
$f(x)$	1	1	0.61	0.61

Horizontal asymptotes	$y=0$	
$x \to$	∞	$-\infty$
$f(x) \to$	0	0

We draw the points in the previous table and the asymptotes, so we get the figure (28-4).

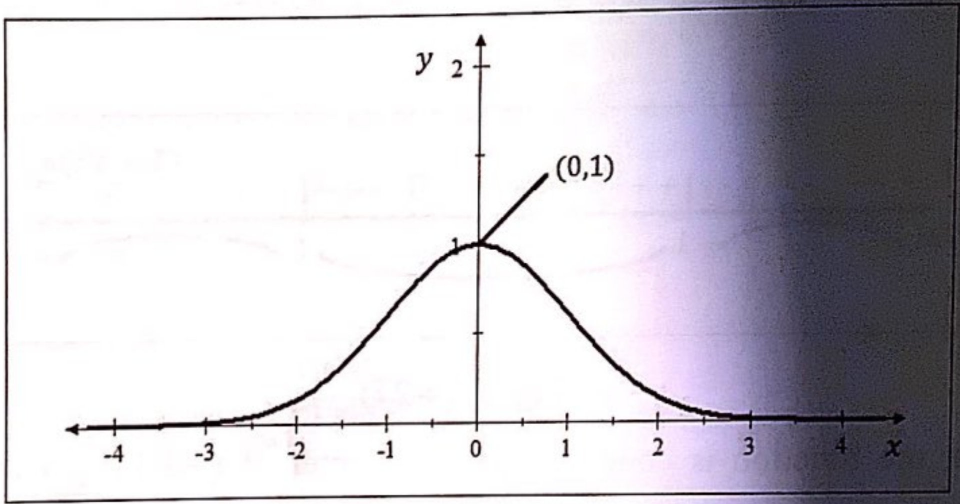


Figure (4-28)

4.4 Absolute Maxima and Minima

In the previous section, we examined the relative maximum and minimum values and determine these values compared to the points in their neighborhood, while the absolute value, is compared to all points in the function range or in some interval of the domain, and it is defined as in the following definition.

Definition 4.4.1: Absolute Maxima and Minimum

If an interval in the domain of the function f contains the point x_0, then

- The function f is said to have an absolute maximum value at x_0 if $f(x) \leq f(x_0)$ for all x values in this interval.

- The function f is said to have an absolute minimum value at x_0 if $f(x) \geq f(x_0)$ for all values of x in this interval.

- The function is said to have an absolute extreme value at the point x_0 if the function has an absolute minimum or maximum value.

From the previous definition, we can say that the absolute maximum value of a function in an interval is the largest value of the function in this interval, and the absolute minimum value is the smallest value of the function in this interval.

Theorem 4.4.1:

If the function f is continuous on the closed interval $[a, b]$, $-\infty < a < b < \infty$ then f has an absolute maximum value and an absolute minimum value within this interval.

From the previous theorem, we can conclude that there is a value in the closed interval at which the function has an absolute maximum value there and another value at which the function has an absolute minimum value if this function is continuous on this interval. However, with this theorem, we cannot find these values. Let us divide the closed interval $[a, b]$ into two parts, which are the points of the limits and the open interval (a, b). Thus, the absolute maximum values either lie at the points of the limits or within the open interval (a, b) and this is what we will discuss in the following theorem.

Theorem 4.4.2:

If f has an absolute maximum value in the open interval (a, b) then it must be located at a critical point of the function.

From the previous theorem, we can use the following steps to determine the absolute maximum values for a function.

Step 1. Find the critical points of f in (a, b).

Step 2. Evaluate f at all critical points and at the endpoints a and b.

Step 3. The largest of the values in Step 2 is the absolute maximum value of f on [a, b] and the smallest value is the absolute minimum, as in the following example.

Example 4.4.1:

Find the absolute maximum and minimum values of $f(x) = \sqrt{4 - x^2}$ on the interval $[-2,0]$.

Solution:

- We determine the critical points of the function.

The first derivative of the function is

$f'(x) = \frac{-x}{\sqrt{4-x^2}}$.

The critical points are at $f'(x) = 0$ and then at $x = 0$ there is a critical point of the function, or when $f'(x)$ is undefined and then at $x = -2$ exists a critical points of the function.

Since, $f(-2) = 0$ and $f(0) = 2$, the absolute maximum value of the function is 2 and occurs at $x = 0$, and an absolute minimum value at $x = -2$ and its value is 0.

Example 4.4.2:

Find the absolute maximum and minimum values of $f(x) = 4x^2 - 12x + 10$ on the interval [1,2].

Solution:

-We find the critical points of the function.

The first derivative of the function is

$f'(x) = 8x - 12$.

The critical points are where $f'(x) = 0$, then at $x = \frac{3}{2}$ there is a critical point of the function.

Since $f(1) = 2$, $f\left(\frac{3}{2}\right) = 1$,and $f(2) = 2$, the absolute maximum value of the function on the interval [1,2] is 2 and it occurs at the values of $x = 1, 2$. The absolute minimum value for it on the same interval is 1 and its value is at $x = \frac{3}{2}$.

To determine the absolute maximum and minimum values in an infinite interval, we only need to study the behavior of the function where $x \to \pm\infty$. There are four different situations.

- If $\lim_{x\to\pm\infty} f(x) = \infty$, then there is an absolute minimum value for the function while there is no absolute maximum value.
- If $\lim_{x\to\pm\infty} f(x) = -\infty$, then there is an absolute maximum value of the function while there is no absolute minimum value.
- If $\lim_{x\to\pm\infty} f(x) = \pm\infty$ or $\lim_{x\to\pm\infty} f(x) = \mp\infty$, then there is no absolute minimum value and there is no absolute maximum value, see Figure (4-29).

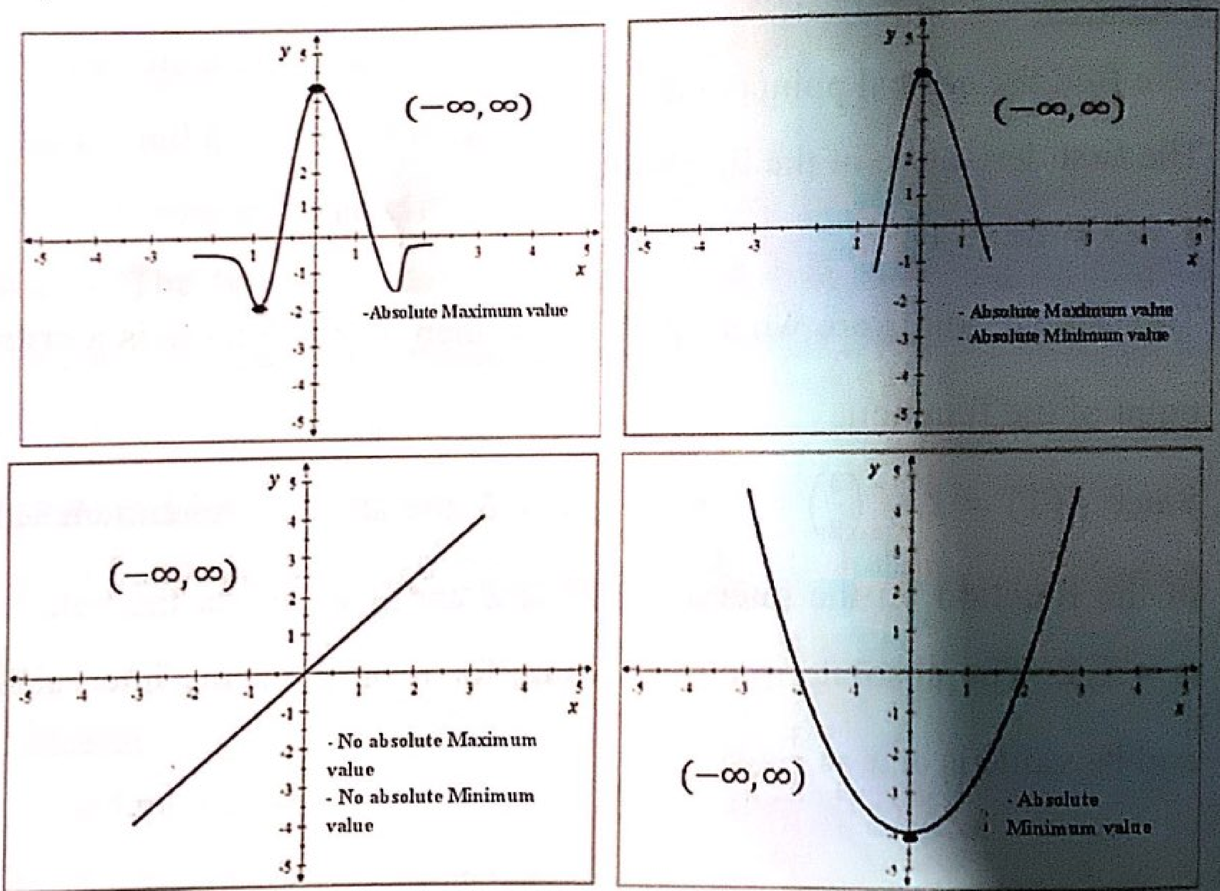


Figure (4-29)

Example 4.4.3:

In the previous example, determine whether the function has maximum or minimum values on the interval $(-\infty, \infty)$. Find those values, if any.

Solution:

Since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} (4x^2 - 12x + 10) = \infty.$$

Thus, the function has an absolute minimum value, while it has no absolute maximum value, and this value occurs at the critical points of the function, therefore it is 1 and it occurs at $x = \frac{3}{2}$. See the function curve in Figure (4-30).

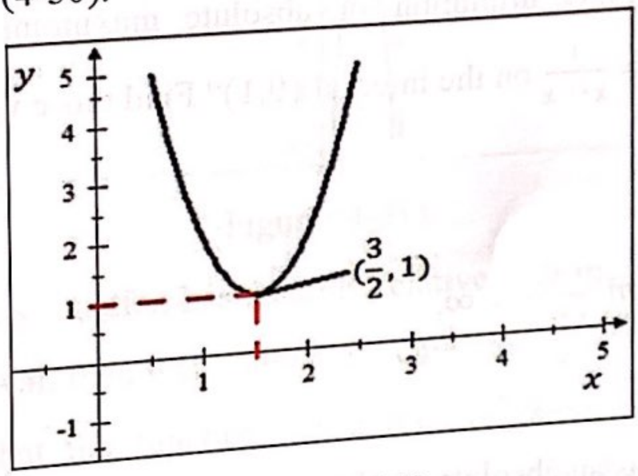


Figure (4-30)

We can also conclude that the absolute maximum and minimum values of the function exist on an open interval (a, b) by studying the behavior of the function where $x \to a^+$ and $x \to b^-$, we can also use that for intervals $(-\infty, a)$ and (b, ∞). It has the following cases.

- If $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to b^-} f(x) = \infty$, then the function has an absolute minimum value on the interval (a, b) and it has no absolute maximum value on (a, b).
- If $\lim_{x\to a^+} f(x) = -\infty$ and $\lim_{x\to b^-} f(x) = -\infty$, then the function has an absolute maximum value on interval (a, b) and it has no absolute minimum value on (a, b).
- If $\lim_{x\to a^+} f(x) = \pm\infty$ and $\lim_{x\to b^-} f(x) = \mp\infty$, then the function has neither an absolute maximum nor an absolute minimum on (a, b).

Example 4.4.4

Are there absolute minimum or absolute maximum values of the function $f(x) = \frac{1}{x^2-x}$ on the interval $(0,1)$? Find those values, if any.

Solution:

Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2-x} = \infty,$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x^2-x} = \infty.$$

The function has an absolute minimum value on the interval (0,1) and it has no absolute maximum value.

The first derivative of the function is

$$f'(x) = \frac{-2x+1}{(x^2-x)^2}.$$

The critical points of the function on the interval $(0,1)$ are $x = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = -4$, so the absolute minimum value of the function is -4 which occurs at $x = \frac{1}{2}$, see the Figure (4-31).

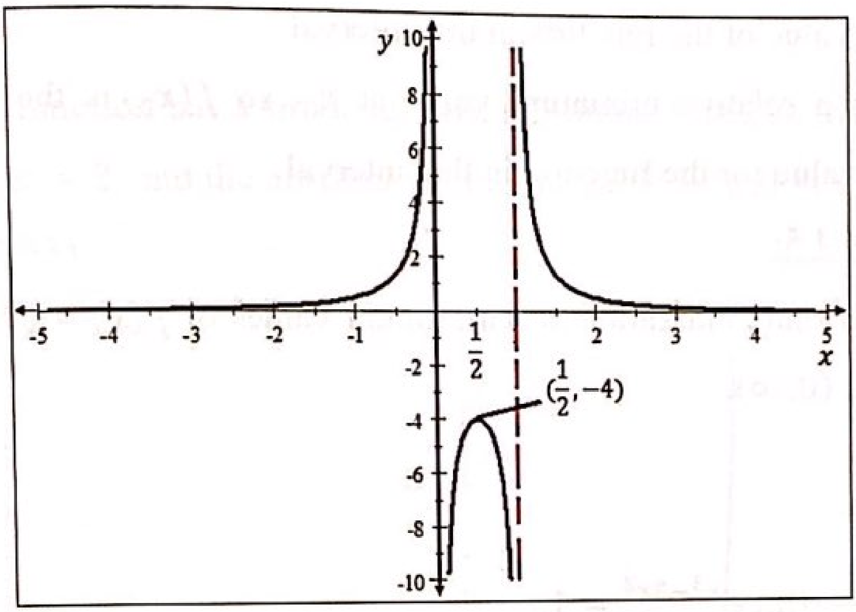


Figure (4-31)

If the continuous function has a single relative extreme value in a finite or infinite interval, then this relative extreme value must be the absolute extreme. Let that the function has a relative maximum value in an interval and that it is at x_0 . Let that $f(x_0)$ is not an absolute maximum value of the function. Therefore there is another relative maximum value at which the function is greater than $f(x_0)$ therefore there is another maximum locale point and this is contradicts the assumption that a single value exists and therefore the absolute maximum value of the function is at x_0.

Theorem 4.4.3:

Let f be a continuous function and has only one relative extreme value in an interval and it is at x_0, then

-If f has a relative maximum value at x_0 so $f(x_0)$ is the absolute maximum value of the function in this interval.

-If f has a relative minimum value at x_0, so $f(x_0)$ is the absolute minimum value for the function in this interval.

Example 4.4.5:

Find the absolute maximum and minimum values of $f(x) = e^{x^3-3x^2}$ on the interval $(0, \infty)$.

Solution:

Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x^3-3x^2} = 1,$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} e^{x^3-3x^2} = \infty.$$

The function does not have an absolute maximum value on the interval $(0, \infty)$.

The first derivative of the function is

$$f'(x) = (3x^2 - 6x)e^{x^3-3x^2}.$$

So, the critical point of the function on the interval $(0, \infty)$ is $x = 2$.

Since the value$e^{x^3-3x^2}$ is always positive, the second derivative sign depends on the sign of $(3x^2 - 6x)$ as shown in Figure (4-32).

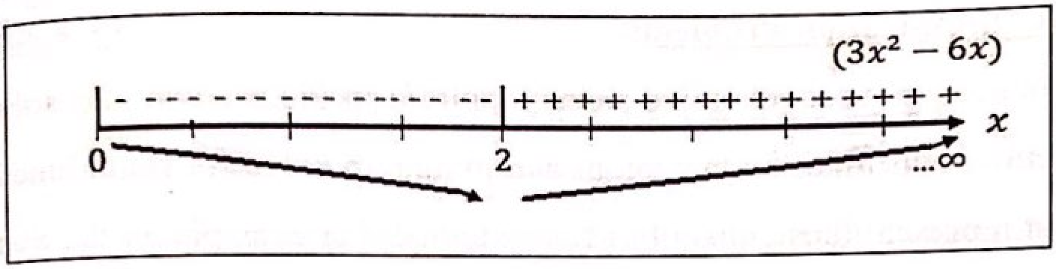


Figure (4-32)

Thus, the function has a single relative minimum value on the interval $(0, \infty)$ at $x = 2$, and the absolute minimum value is $f(2) = e^{-4}$, see Figure (4-33).

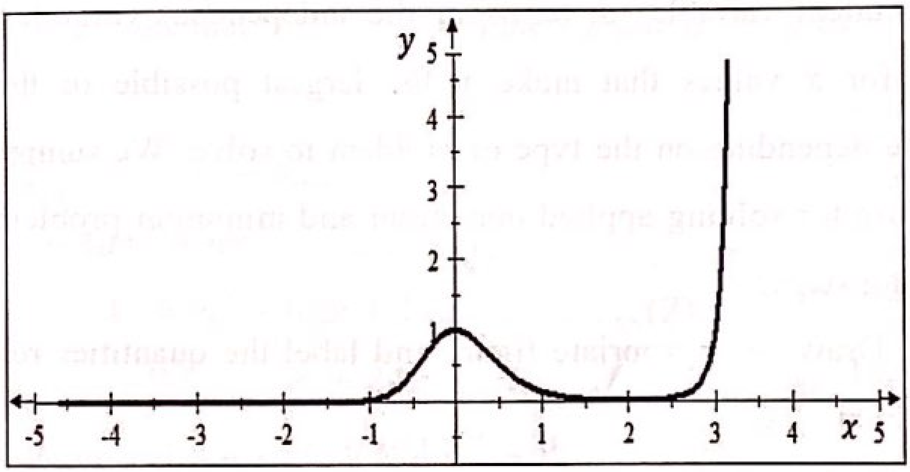


Figure (4-33)

5.4 Optimization Problems:

Calculus is used to solve many applied problems, and the solution method is to find the maximum and minimum values of some functions that represent those quantities to be studied. For example, in the domain of industry, the necessary conditions that achieve the largest possible production at the necessary costs can be examined. We will focus our attention on those applications that need to be solved by developing a mathematical model of the problem to be solved by finding the function y (dependent variable) in terms of the independent variable x. Then search for x values that make y the largest possible or the lowest possible depending on the type of problem to solve. We summarize the procedure for solving applied maximum and minimum problems in the following steps:

Step 1. Draw an appropriate figure and label the quantities relevant to the problem.

Step 2. Find a formula for the quantity to be maximized or minimized.

Step 3. Use the conditions stated in the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.

Step 4. Find the interval of possible values for this variable from the physical restrictions in the problem.

Step 5. If applicable, use the techniques of the previous section to obtain the maximum or minimum values.

Example 4.5.1:

An open box is to be made from a card board by cutting out squares of equal size from the four corners and bending up the sides (Figure 4-34). What size should the squares be to obtain a box with the largest volume?

Solution:

We Suppose that the length of the side of the cut out part is x, and the volume of the box is V, so we have the length of the base of the box equal to $(a-2x)$ and the height of the box equal to x as shown in the following figure. Note that $0 \le x \le \frac{a}{2}$ and the volume is given by:

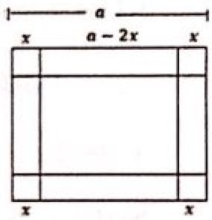


Figure (4-34)

$$V = x(a-2x)^2 = x(a^2 - 4ax + 4x^2)$$
$$= a^2x - 4ax^2 + 4x^3 \;\ldots\ldots\ldots\ldots(1)$$
$$V' = a^2 - 8ax + 12x^2 \;\ldots\ldots\ldots\ldots(2)$$
$$V'' = -8a + 24x \;\ldots\ldots\ldots\ldots(3)$$

V has a maximum value if $V' = 0$ and $V'' < 0$

From equation (2) we put $V' = 0$.

Then,

$$a^2 - 8ax + 12x^2 = (a-2x)(a-6x) = 0.$$

Therefore, $x = \frac{a}{2}$ or $x = \frac{a}{6}$

Since,

$$V''\left(\frac{a}{2}\right) = -8a + 24\left(\frac{a}{2}\right) = -8a + 12a = 4a > 0$$

and

$$V''\left(\frac{a}{6}\right) = -8a + 24\left(\frac{a}{6}\right) = -8a + 4a = -4a < 0$$

Then, the variable V has a maximum value (i.e., the volume of the box as large as possible) if $x = \frac{a}{6}$.

So, the base $a - 2x = a - \frac{a}{3} = \frac{2a}{3}$

The height of the box is equal to $(\frac{a}{6})$.

(Note that the other solution gives the length of thesquareof the base equal to zero.)

Example 4.5.2:

Find a largest rectangular area whose lower base on the x-axis and its upper side limits are on the curve

$y = 12 - x^2$

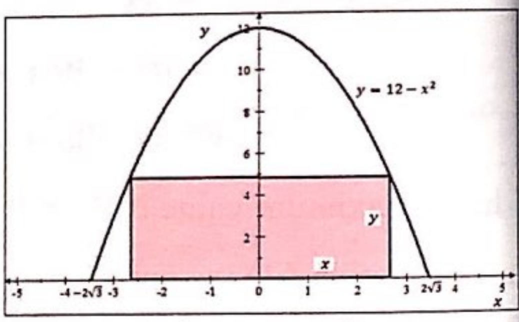


Figure (4-35)

Solution:

Suppose that

Rectangle length $=2x$

Rectangle width $= y$

Rectangle area $=A$

As shown in Figure (4-35)

Note that $0 \leq x \leq 2\sqrt{3}$.

Then $A = 2xy$. Suppose that the point $P(x, y)$ is on the curve $y = 12 - x^2$.

Thus, the coordinate of the point P satisfies the equation of this curve and hence the variable y in the relation of the area of the rectangle can be eliminated as follows:

$$A = 2x(12 - x^2) = 24x - 2x^3 \ldots\ldots\ldots\ldots(1)$$

$$\frac{dA}{dx} = 24 - 6x^2 \ldots\ldots\ldots\ldots(2)$$

$$\frac{d^2A}{dx^2} = -12x \ldots\ldots\ldots\ldots(3)$$

The area A is a maximum value if $A' = 0$.

So that $24 - 6x^2 = 0$

$$x^2 = 4 \therefore x = \pm 2.$$

The negative answer is rejected because the length is always positive.

So, the length of the rectangle is 4 and its width is $12 - x^2 = 12 - 4 = 8$ and the area is 32.

Example 4.5.3:

Prove that the size of the largest right circular cylinder that can be drawn inside an existing circular cone equals to $\frac{4}{9}$ of the volume of the cone.

Solution:

Let the cone's base radius be a, its height is b, and its size is V_1 so that we have $V_1 = \frac{\pi a^2 b}{3}$. Suppose also that the cylinder base radius x length, y height, and V_2 size. By this we have,

$$V_2 = \pi x^2 y.$$

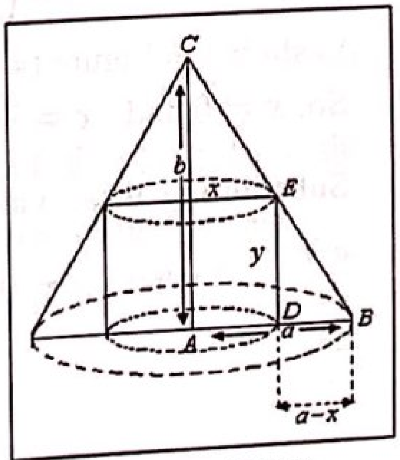


Figure (4-36)

Note that $0 \leq x \leq a$.

From the similarity of the two triangles in ABC and BED in Figure (4-36), we find that:

$$\frac{y}{b} = \frac{a-x}{a}$$

$y = \frac{b}{a}(a-x) = b(1-\frac{x}{a})$.

Substituting the value of y in V_2 we get (1):

$$V_2 = \pi x^2 b\left(1-\frac{x}{a}\right) = \pi b\left(x^2 - \frac{x^3}{a}\right) \ldots\ldots\ldots (1)$$

$$\frac{dV_2}{dx} = \pi b\left(2x - \frac{3x^2}{a}\right) \ldots\ldots\ldots (2)$$

$$\frac{d^2V_2}{dx^2} = \pi b\left(2 - \frac{6x}{a}\right) \ldots\ldots\ldots (3)$$

So V_2 has a maximum value if $\frac{dV_2}{dx} = 0$ and $\frac{d^2V_2}{dx^2} < 0$.

From equation (2) we obtain

$$\pi b\left(2x - \frac{3x^2}{a}\right) = \pi b x\left(2 - \frac{3x}{a}\right) = 0$$

So, $x = 0$ and $x = \frac{2a}{3}$.

Substituting these values for the variable x into equation (3), we get $\frac{d^2V_2}{x^2} > 0$ where $x = 0$ (neglected. Why?) Whereas $\frac{d^2V_2}{dx^2} < 0$ where $x = \frac{2a}{3}$.

So, the cylinder size is the largest when $x = \frac{2a}{3}$ and $y = \frac{b}{3}$ and therefore its size is

$$V_2 = \pi x^2 y = \pi.\frac{4a^2}{9}.\frac{b}{3} = \frac{4}{9}V_1$$
$$\frac{V_2}{V_1} = \frac{4}{9}$$

That is, the volume of the cylinder equals $\frac{4}{9}$ of the size of the cone.

Example 4.5.4:

A water tank in the form of parallel rectangles with a square base with an open top of capacity 32 m^3 of water. Find the dimensions of the tank that make the costs of making it is the minimum.

Solution:

Let the length of the base side be x, height y, and its volume V.

As shown in Figure (37-4), notice that $0 < x < \infty$.

The volume of the tank is given by $V = x^2y$ such that $V = 32$.

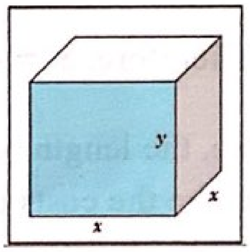


Figure (4-37)

Accordingly:

$$y = \frac{32}{x^2} \ldots\ldots\ldots (1)$$

Cost is the minimum when the surface area of the tank S is the minimum. Therefore, we assume that the surface area of the tank is

Tank surface area = base area + area of the four sides.

That is,

$$S = 4xy + x^2 \dots\dots\dots(2)$$

Substituting from (1) for the value of y, we get

$$S = 4x\left(\frac{32}{x^2}\right) + x^2 = \frac{128}{x} + x^2$$

$$S' = \frac{-128}{x^2} + 2x \dots\dots\dots(3)$$

$$S'' = \frac{256}{x^3} + 2.\dots\dots\dots(4)$$

So S has a minimum value where $S' = 0$ and $S'' > 0$. Using equation (3), we obtain

$$\frac{-128}{x^2} + 2x = 0$$

$$2x^3 = 128 \qquad \rightarrow \quad x^3 = 64.$$

Therefore, $x = 4$ and $y = \frac{32}{16} = 2$

So, the length of the tank side should be 4m and height 2 in order to make the costs of making it the minimum.

Example 4.5.5:

A square floor room whose walls are to be painted with a square meter cost of 40 riyals and whose ceiling is to be painted with another type of paint the cost of one square meter of which is 50 riyals. Find the dimensions of the room so that the costs are the minimum, where the room size is 216 cubic meters.

Solution:

Let the length of the room be$x$ meters and its height y meters. Since the room is a square floor, then we have $x^2y = 216$, so

$$y = \frac{216}{x^2} \dots\dots\dots (1)$$

Let that the cost of painting is S,

$$S = 4xy(40) + x^2(50) = (160)\left(\frac{216}{x}\right) + 50x^2 \dots\dots\dots.(2)$$

$$S' = -4(40)\left(\frac{216}{x^2}\right) + 100x \dots\dots.(3)$$

$$S'' = 2(160)\left(\frac{216}{x^3}\right) + 100.\dots\dots.(4)$$

The cost of the painting is the minimum when $S' = 0$and $S'' > 0$, and using equation (3) we obtain:

$$x^3 = \frac{160 \times 216}{100} = 345.6$$

So, $x \approx 7.018m$ and $y = \frac{216}{(7.018)(7.018)} \approx 4.446m.$

Then $y = \frac{216}{(7.018)(7.018)} \approx 4.446m$ and $x \approx 7.018m$

Example 4.5.6:

A cylindrical cane is made of tin is intended to be filled with honey, provided that the can contain $250\pi cm^3$ of honey. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?

Solution:

Let the cylinder base radius be x, its height y, its volume V and its surface area S.

As shown in Figure (4-38).

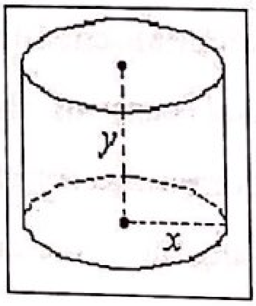


Figure (4-38)

Since $V = x^2y = 250\pi$,

then

$$y = \frac{250}{x^2} \ldots\ldots\ldots\ldots\ldots (1)$$

The surface area S of the cane is given by

$$S = 2\pi xy + 2\pi x^2 = 2\pi\left(\frac{250}{x}\right) + 2\pi x^2 \ldots\ldots\ldots (2)$$

$$\frac{dS}{dx} = -\pi\left(\frac{500}{x^2}\right) + 4\pi x \ldots\ldots\ldots (3)$$

$$\frac{d^2S}{dx^2} = \pi\left(\frac{1000}{x^3}\right) + 4\pi \ldots\ldots\ldots (4)$$

Fabrication costs are minimal when the surface area is minimized.

That is, when $\frac{dS}{dx} = 0$ and $\frac{d^2S}{dx^2} > 0$, using equation (3) we obtain

$$\frac{-500\pi}{x^2} + 4\pi x = 0$$

$$x^3 = 125.$$

Then $y = \frac{250}{25} = 10cm$ and $x = 5cm$

Therefore, the required cylinder radius is 5cm, while the height is 10cm.

4.6 Mean Value Theorem

Mean value Theorem is one of the most important theorems in differential calculus. The mathematician Michel Rolle developed a special case of this result which states that if a continuous curve intercepts the ox axis, then there is at least a point on the curve where the tangent of the curve is horizontally, see Figure (4-39).

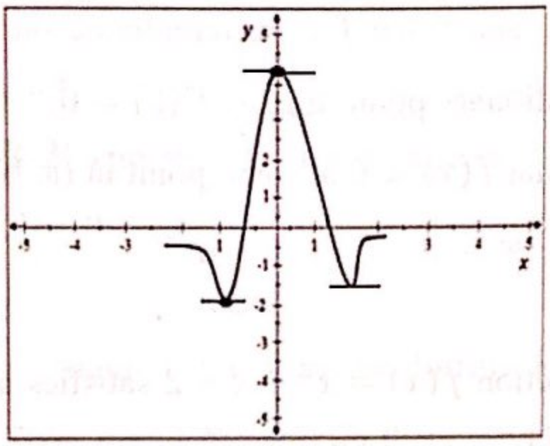


Figure (4-391)

Theorem 4.6.1: (Rolle's Theorem)

Let f be a continuous function on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b) = 0$. Then there is at least a point c on the interval (a, b) such that $f'(c) = 0$.

Proof:

We will divide the proof into three cases, the case where $f(x) = 0$ for all x in (a, b), the case where $f(x) > 0$ at some point in (a, b), and the case where $f(x) < 0$ at some point in (a, b).

Case 1: If $f(x) = 0$ for all x in (a, b), then $f' = 0$ at every point $c \in (a, b)$ because f is a constant function on that interval.

Case 2: Suppose that $f(x) > 0$ at some point in (a, b). Since f is continuous on [a, b], it follows from the Extreme-Value Theorem (4.4.2) that f has an absolute maximum on [a, b]. The absolute maximum value cannot occur at an endpoint of [a, b] because we have supposed that $f(a) = f(b) = 0$, and that $f(x) > 0$ at some point in (a, b). Thus, the absolute maximum must occur at some point c in (a, b). So that c is a critical point of f, and since f is differentiable on (a, b), this critical point must be a stationary point; that is, $f'(c) = 0$.

Case 3: Suppose that $f(x) < 0$ at some point in (a, b). The proof of this case is similar as Case 2.

Example 4.6.1:

Show that the function $f(x) = x^2 + x - 2$ satisfies a Roll's theorem on the closed interval $[-2, 1]$ and then find a number c on the open interval $(-2,1)$ so that $f'(c) = 0$.

Solution:

Note that the function f is continues on the closed interval $[-2, 1]$ because it is a polynomial of the second order (in fact any polynomial of degree n where n is a positive integer is a continuous function).

We also note that the function f is differentiable in the open interval $(-2,1)$ so,

$$f'(x) = 2x + 1.$$

Finally, we note that $f(-2) = f(1) = 0$.

This indicates that the conditions have been satisfied. That is, the function f satisfies Roll's theorem.

So, there is a number c in the open interval $(-2,1)$ so that, $f'(c) = 2c + 1 = 0$.

So, $c = -\frac{1}{2}$ this number actually is in the open interval $(-2,1)$.

Example 4.6.2

Show that the function $f(x) = x^3 - 4x$ satisfies Roll's theorem on the closed interval $[-2, 2]$ and then find a number c in the open interval $(-2,2)$ so that $f'(c) = 0$.

Solution:

As in the previous example, the function fulfills Roll's theorem and $f'(x) = 3x^2 - 4$. Thus $c = \pm 2/\sqrt{3}$, is on the interval $(-2,2)$.

Corollary 4.6.1:

If f is a continuous function on the closed interval $[a, b]$, is differentiable on the open interval (a, b) and $f(a), f(b)$ is of different sign then there is at least one root of the function f on the interval (a, b). That is, the function curve intersects the ox axis at least one point.

Example 4.6.3:

Prove that the function curve $f(x) = x^3 + 3x + 1$ intersects ox axis at one point on the closed interval $[-1,0]$ without solving the equation $f(x) = 0$.

Solution:

Since, $f(x) = x^3 + 3x + 1$, then $f'(x) = 3x^2 + 3$.

So $f'(x) > 0$ for all real x and this indicates that the function is always increasing.

Since, $f(0) = 1 > 0$ and $f(-1) = -3 < 0$.

For the equation $f(x) = 0$, there is only one root in the open interval $(-1,0)$, so the curve of the function $f(x) = x^3 + 3x + 1$ intersects ox axis in only one point.

Theorem 4.6.2 (Mean Value Theorem):

Let f be a continuous function of the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there is at least a point c on the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof:

Since, the equation of the straight line connecting the two points $(a, f(a))$ and $(b, f(b))$ is

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \text{ .... } (1)$$

Thus, the distance between any point (x, y) on the curve of the function f and the line (1) is $g(x)$ where

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \text{ .... } (2)$$

Note the Figure (4-40).

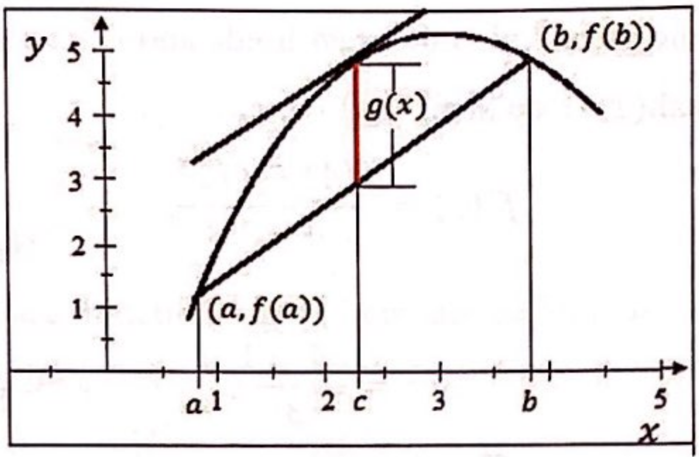


Figure (4-40)

Since the function f is continues on the closed interval $[a, b]$ and is differentiable on the interval (a, b) and $g(a) = 0$, $g(b) = 0$ then the function g fulfills Roll's theorem and thus there is a number c in the open interval (a, b) so that,

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

Then

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 4.6.4:

Apply the mean value theory for the function $f(x) = x^3$, and then find a number c on the open interval $(1,4)$ which satisfies the theorem.

Solution:

It is clear that $f(x) = x^3$ is a continuous function on the closed interval $[1,4]$ and that it is differentiable on the open interval $(1,4)$ and $f'(x) =$

$3x^2$ and thus the mean value theorem holds and there is a number c on the open interval (1,4), so that,

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Then

$$3c^2 = \frac{64 - 1}{3},$$

which implies that $c = \pm\sqrt{7}$.

Since, $-\sqrt{7} \notin (1,4)$, then we conclude that $c = \sqrt{7}$.

<u>Corollary 4.6.2:</u>

If $f'(x) = 0$ for all values on the open interval (a, b) then the function $f(x)$ is constant over the whole interval.

<u>Proof:</u>

Let $f'(x) = 0$ for all values $x \in (a, b)$ and x_1, x_2 be any two points on the open interval (a, b) so that $a < x_1 < x_2 < b$. Since the function is differentiable for all the x between x_1, x_2 that is, $x_1 < x < x_2$, and this function is continuous on the closed interval $[a, b]$, by applying the mean value theorem we can say that there is at least a point c between x_1, x_2 so that,

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

that is,

$(x_2 - x_1)f'(c) = f(x_2) - f(x_1)$.

Since $f'(x) = 0$, then $f(x_2) - f(x_1) = 0$.

Therefore $f(x_2) = f(x_1)$, that is, $f(x)$ is a constant value.

Corollary 4.6.3:

If f_1 and f_2 are functions and their derivatives are equal to $f(x)$ where a $< x < b$ i.e.,

$$\frac{d}{dx}f_1(x) = \frac{d}{dx}f_2(x) = f(x)$$

For all values of x that such that a $< x < b$, then $f_1(x) - f_2(x)$ equals a constant value on the interval (a, b).

Proof:

Suppose that $f(x) = f_1(x) - f_2(x)$ and so we have $\frac{df}{dx} = \frac{df_1}{dx} - \frac{df_2}{dx} = 0$ and using the result 2.6.4 we find that $f(x)$ is a constant value, and from it $f_1(x) - f_2(x)$ equals to a constant value.

Corollary 4.6.4:

Let f be a continuous function on the closed interval $[a, b]$ and is differentiable on the open interval (a, b). If $f'(x)$ is positive on (a, b) then f is an increasing function on the closed interval $[a, b]$. If $f'(x)$ is negative on the open interval (a, b) then f is a decreasing function on the closed interval [a, b].

Proof:

Suppose that x_1 , x_2 are any two numbers in the closed interval $[\mathrm{a}, \mathrm{b}]$ such that $x_1 < x_2$ and applying the mean value theorem, we find

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

where $c \in (a, b)$. Since the right side is positive because $f'(c)$ is supposed positive.

Then the value of $f(x_2) - f(x_1)$ must also be positive. That is

$$f(x_2) > f(x_1).$$

Consequently, $f(x)$ is an increasing function on the closed interval $[a, b]$. Similarly, if $f'(x)$ is negative then $f(x_2) - f(x_1)$ is also negative. Therefore, $f(x_2) < f(x_1))$ and hence $f(x)$ is a decreasing function on the closed interval $[a, b]$.

4.7 Newton's Method

If we can solve some equations algebraically, there are many of them that are difficult and even impossible to solve by known algebraic methods. Therefore, approximate methods are used to find such solutions, including the Newton method, which is one of the first applications of the derivative of function.

Let the equation $y = f(x)$ and x_1 be an approximate value for the root of the equation r.This value can be determined by drawing the equation or by finding an interval that contains this root which is an interval within it the equation sign changes. If $f(x_1) = 0$ then $x_1 = r$ and if $f(x_1) \neq 0$ then the tangent of the function curve at x_1 intersects the ox axis at a point x_2 which represents a better approximation to the root values. If $f(x_2) = 0$ then $x_2 = r$ and if $f(x_2) \neq 0$, then the tangent of the function curve at x_2 intersects the ox axis at a point; say x_3, which represents a better approximation of the root values. Thus, we can repeat this until a certain precision is obtained and this method is called the Newton's method, see Figure (4-41).

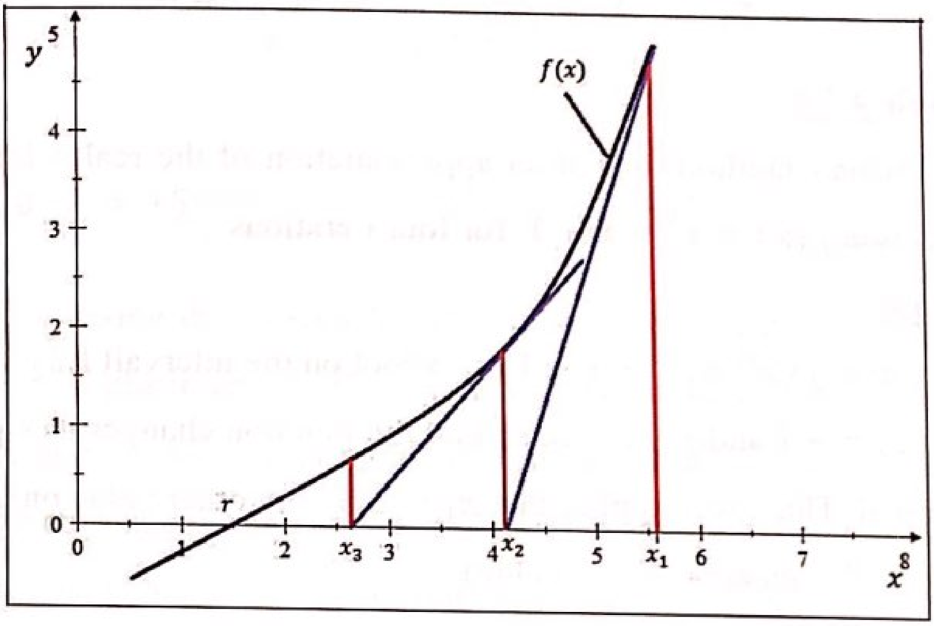


Figure (4-41)

We can formulate the previous method in the form of an iterative equation as follows.

Let x_1 be the starting value, then the tangent equation is

$$y - f(x_1) = f'(x_1)(x - x_1)\text{..........................}(1)$$

Let the tangent intersects the horizontal axis at the point $(x_2, 0)$. Then it satisfies equation (1), and hence

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad f'(x_1) \neq 0.$$

Repeating this for x_2, we obtain

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \qquad f'(x_2) \neq 0.$$

Generally, if x_n is the n[th] approximation, then the x_{n+1} approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0.$$

Example 4.7.1

Use Newton's method to find an approximation of the real solutions of the function $f(x) = x^3 - x - 1$, for four iterations.

Solution:

The function $f(x) = x^3 - x - 1$ has a root on the interval$(1,2)$ since$f(1) = -1$ and $f(2) = 5$, so that the function changes its signon the interval. Thus, we suppose that $x_1 = 1.5$ (any other value on the interval is also an appropriate value).

Since $f'(x) = 3x^2 - 1$, Newton's iterative equation is

$$x_{n+1} = x_n - \frac{{x_n}^3 - x_n - 1}{3{x_n}^2 - 1}.$$

Then we get:

$x_1 = 1.5, x_2 = 1.3478, x_3 = 1.3252,$ and $x_4 = 1.3247.$

Example 4.7.2

Use Newton's method to find an approximation to the solution of the equation $cosx = x$ where x is measured in radians with an initial value $x_1 = 1$ (four iterations).

Solution:

Let the equation $f(x) = x - cosx$. Since $f'(x) = 1 + sinx$ then Newton's iterative equation is

$$x_{n+1} = x_n - \frac{x_n - cosx_n}{1 + sinx_n}.$$

Then we get

$x_1 = 1.0, x_2 = 0.5704x_3 = 0.7391,$ and $x_4 = 0.7391.$

When discussing the Newton's method, we supposed that the tangent at the point would intersects the ox axis. It is possible that this may not happen, and the tangent is parallel to the axis. In this case $f'(x_n) = 0$ for a given value n, and therefore Newton's method failed to find the approximate value x_{n+1}. Also, sometimes Newton's method fails to find the required root and converges to another root of the function, or it does not converge at all.

Exercises

(1) Find the intervals of increase, decrease, and concavity up and down and the inflection points, if any, for the function f.

$(i) f(x) = x^3 - 12x + 1$ $(ii) f(x) = (x^2 - 1)^2$

$(iii) f(x) = \dfrac{1}{x^2 + 2}$ $(iv) f(x) = x^3(4 - x)^2$

$(v) f(x) = x^3 \ln x$ $(vi) f(x) = e^{-x^2/2}$

$(vii) f(x) = x - \sin x$ $(viii) f(x) = \ln\sqrt{x^2 + 4}$

$(ix) f(x) = x^{4/3} - x^{1/3}$ $(x) f(x) = x^{2/3} - x$

$(xi) f(x) = xe^{x^2}$ $(xii) f(x) = \sqrt[3]{x^2 + x + 1}$

(2) Plot the curve of the continuous function $y = f(x)$ which has properties:

$(i) f(2) = 4, \quad f'(2) = 0, \; f''(x) > 0 \; \forall x.$

$(ii) f(2) = 4, \quad f'(2) = 0, \; f'(x) < 0, x < 2, f''(x) > 0, x > 2.$

$(iii) f(2) = 4, \quad f'(2) = 0, \; f''(x) < 0, x \neq 2, \lim_{x \to 2^+} f''(x) = \infty,$

$\lim_{x \to 2^-} f'(x) = -\infty.$

$(iv) f(2) = 4, f'(2) = 0, \; f''(x) < 0 \; \forall x.$

$(v) f(2) = 4, \quad f'(2) = 0, \; f''(x) < 0, x < 2, f''(x) < 0, x > 2.$

$(vi) f(2) = 4, \quad f'(2) = 0, \; f''(x) > 0, x \neq 2, \lim_{x \to 2^+} f'(x) = -\infty,$

$\lim_{x \to 2^-} f'(x) = \infty.$

(3) Show, by using the properties of increasing functions, that

$(i)\sqrt[3]{x+1} < 1+\frac{1}{3}x, x > 0$ $\quad$ $(ii)\ x < tanx, 0 < x < \frac{\pi}{2}$

$(iii)\ ln(x+1) < x, x > 0$ $\quad$ $(iv)\ ln(x+1) > x - \frac{1}{2}x^2$

$(v) e^x > 1 + x, x > 0$ $\quad$ $(vi) e^x \geq 1 + x + \frac{1}{2}x^2, x \geq 0.$

(4) Determine whether the following statements are true or false, with explanation

(i) If f, gareincreasing functions on an interval, then $f + g$ is also increasing on this interval.

(ii) If f, g are increasing functions on an interval, then $fg$is also increasing on this interval.

(5) Find two increasing functions f, g on $(-\infty, \infty)$, such that $f - g$ has the following property,

$(i) f - g$ is decreasing on the interval $(-\infty, \infty)$.

$(ii) f - g$ is constant on the interval $(-\infty, \infty)$.

$(iii) f - g$ is increasing on the interval $(-\infty, \infty)$.

(6) Find the increasing functions f, g on $(-\infty, \infty)$ such that f/g has the property,

(i) f/g is decreasing on the interval $(-\infty, \infty)$.

(ii) f/g is constant on the interval $(-\infty, \infty)$.

$(iii) f/g$ is increasing on the interval $(-\infty, \infty)$.

(7) Plot the function $y = f(x)$. Use all the information you can get from the function and its first and second derivatives.

$(i) y = (x^2 - 4)^3$ $(ii) y = x(x^2 - 4)^2$ $(iii) y = \frac{1-x}{x}$

$(iv) y = \frac{x-4}{x+2}$ $(v) y = \frac{x^3}{x+2}$ $(vi) y = \frac{1}{x^2+1}$

$(vii) y = \frac{x}{x^2-4}$ $(viii) y = \frac{x^2}{x^2-9}$ $(ix) y = e^{-x} \sin x, x \geq 0$

$(x) y = e^{-x^2}$ $(xi) y = \frac{x}{x^2+x-2}$ $(xii) y = xe^x$.

(8) Find the local and absolute maximum and minimum values of the function f whenever possible.

$(i) f(x) = x + 1, [-1,1]$ $(ii) f(x) = x + 2, (-\infty, 0]$

$(iii) f(x) = x^2 - 1, [-2,3]$ $(iv) f(x) = x^3 + x - 2, [a, b]$

$(v) f(x) = \frac{1}{x-1}, [2,3]$ $(vi) f(x) = |x - 1|, [-2,2]$.

(9) Find the maximum and minimum local (relative) values of f.

$(i) f(x) = \frac{x}{x^2+1}$ $(ii) f(x) = x\sqrt{2-x}$ $(iii) f(x) = e^{-x^2/2}$

$(iv) f(x) = x2^{-x}$ $(v) f(x) = |x^2 - 1|$

$(vi) f(x) = (x-1)^{2/3} + (x+1)^{2/3}$.

References

- R. A. Adams and C. Essex "Calculus a Complete Course" 7th ed. (2010), Pearson Canada Inc.
- S. M. Adel, "Calculus (1) Using Maple: Part One," First Edition (2011), Al-Mutanabbi Library. (In Arabic)
- T. M. Al-Adawi and M. Z. Abdullah, "Calculus: Part One, Calculus" (2003), Al-Rushd Library. (In Arabic)
- I. K. Al-Husseini, M. Abu Al-Ezz and E.Al Zahrani, "Calculus: Part One "(2008). (In Arabic)
- H. Anton, I. Bivens and S. Davis "Calculus Early Transcendentals" 10th ed, (2012), John Wiley & Sons, Inc.
- J. L. Bartlett "Calculus from the Ground Up" (2018), BP Learning.
- M. Boelkins, D. Austin and S. Schlicker "Active Calculus 2018: Single Variable" (2018), CreateSpace Independent Publishing Platform.

- W. L. Briggs, L. Cochran, et al. “Calculus: Early Transcendentals” 3ed ed. (2018), Pearson Education, Inc.
- J. R. Hass, M. D. Weir and J. B.Thomas Jr. “University Calculus: Early Transcendentals " 3ed ed. (2016), Pearson Education, Inc.
- D. Hughes-Hallett et al. “Applied Calculus” 5th ed. (2013), Wilely.
- R. M. Juhaima and A. A. Hab Al-Reeh, "Differentiation and Integration: Part One" (2002), The New Book. (In Arabic)
- J. W. Kitchen “Calculus of One Variable” (2020), Dover Publications.
- R. Larson and B. H. Edwards” Calculus” 11th ed. (2017), Cengage Learning.
- P. P. Mikusinski and J. Mikusinski “An Introduction to Analysis” (2017), World Scientific Publishing Company.
- C. McMullen “50 Challenging Calculus Problems (Fully Solved)” (2018), Zishka Publishing.
- C. McMullen "Essential Calculus Skills Practice Workbook with Full Solutions"(2018), Zishka Publishing.

- M. Ryan "Calculus Essentials For Dummies"1st ed., (2019), For Dummies.
- J. Stewart "Calculus" 5th ed. (2002) Brooks Cole.
- M. D. Weir and J. Hass "Thomas' Calculus: Early Transcendentals" 12th ed. (2010), Pearson Education, Inc.
- M. Wiso "Learn Calculus Quickly: The Complete Guide To Easily Master Calculus in 100 Solved Equations!" (2019), Independently published.