المملكة العربية السعودية

وزارة التعليم

MINISTRY OF EDUCATION

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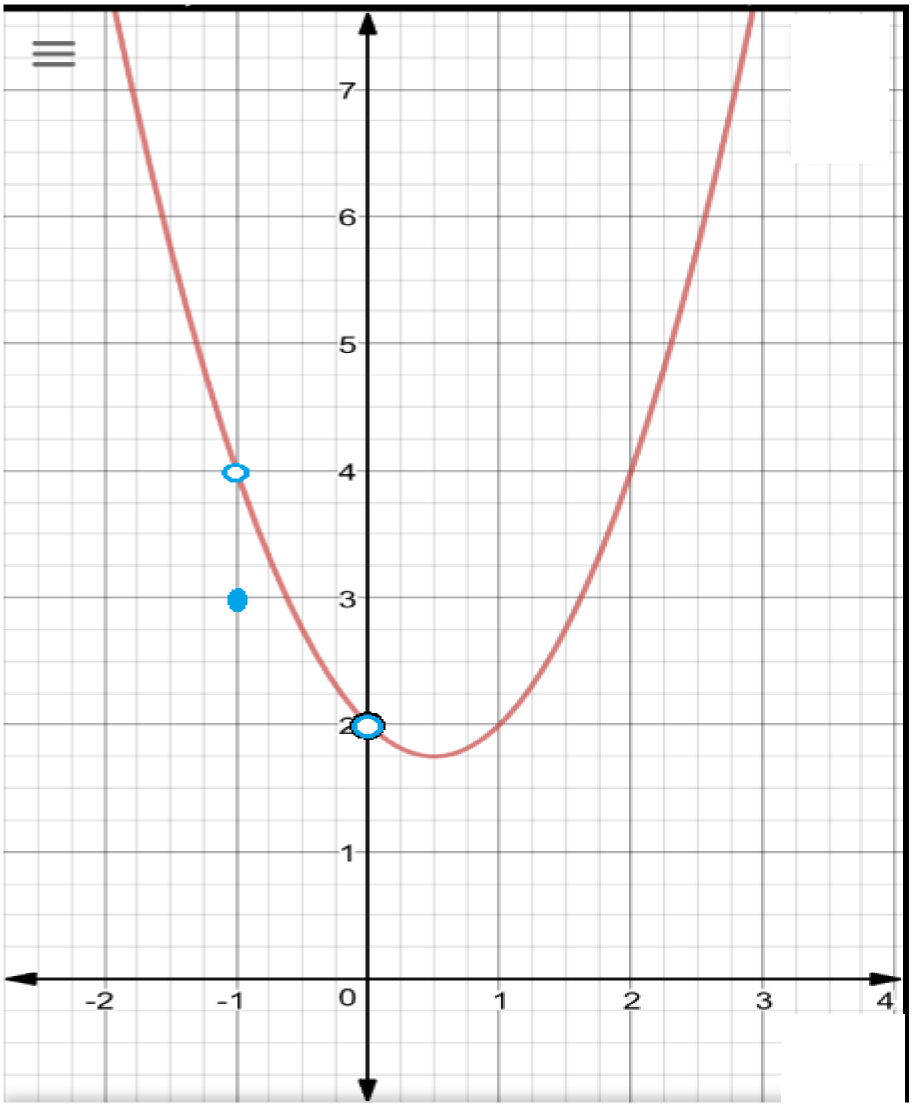
(2.2) The Limit Of A Function

$\lim_{x \to a} f(x) = L$ is $f(x) \to L$ as $x \to a$

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$

Example (1)

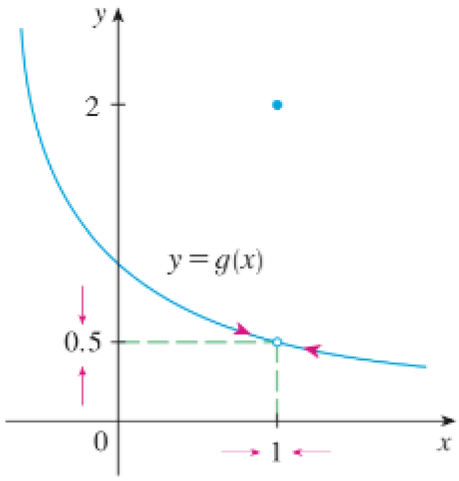


$a)\ \lim_{x \to 2} f(x) = 4$ and $f(2) = 4$

$b)\ \lim_{x \to -1} f(x) = 4$ and $f(-1) = 3$

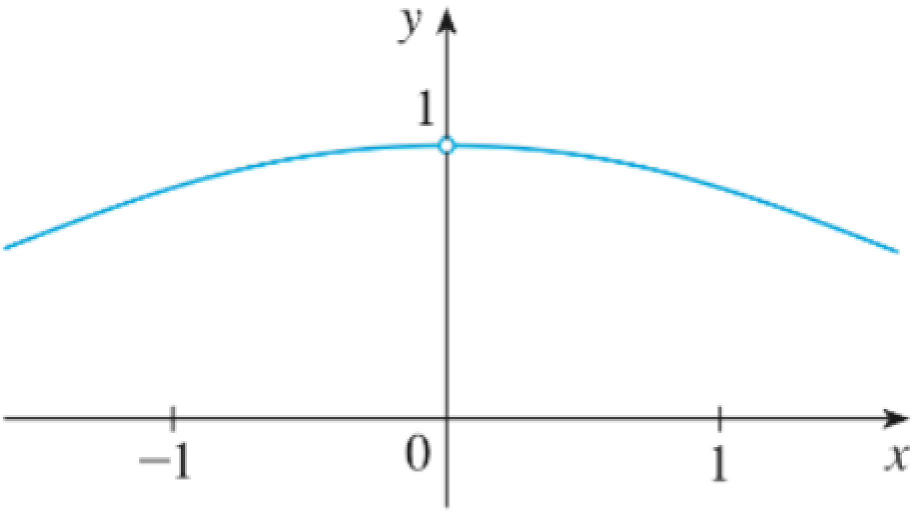
$c)\ \lim_{x \to 0} f(x) = 2$ and $f(0) =$ undefind or not defind

Example (2)



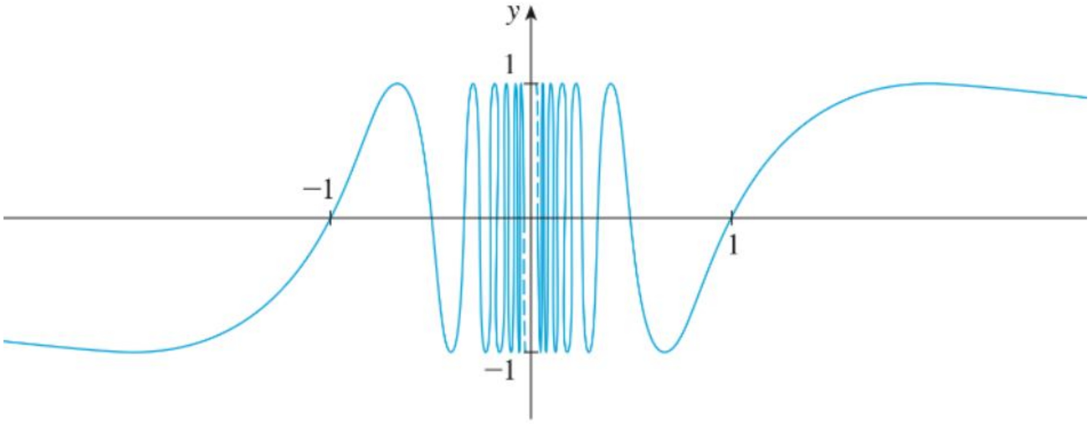
$\lim_{x\to 1} g(x) = 0.5$ and $f(1) = 2$

Example (3)



$\lim_{x\to 0} f(x) = 1$ and $f(0) =$ undefind or not defind

Example (4)

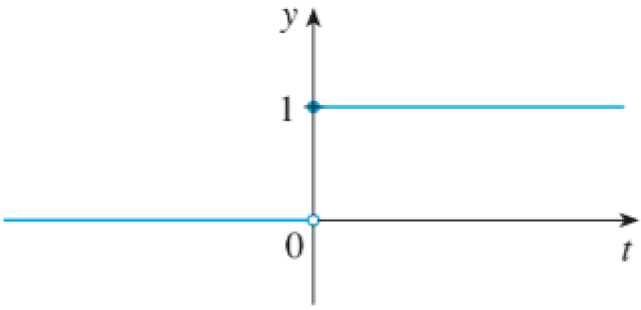


$a)\ \lim_{x \to 0} f(x) = D.N.E$ and $f(0) =$ undefind or not defind

$b)\ \lim_{x \to 1} f(x) = 0$ and $f(1) = 0$

One side limits

Example (5)



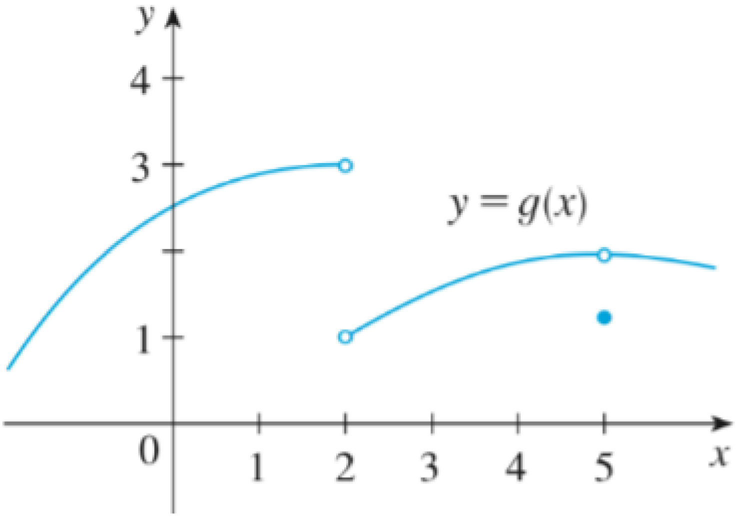
$\lim_{x \to 0^+} f(t) = 1$

$\lim_{x \to 0^-} f(t) = 0$

$\because \lim_{x \to 0^+} f(t) \neq \lim_{x \to 0^-} f(t)$

$\therefore \lim_{x \to 0} f(t) = D.N.E$

Example (6)



$a)\ \lim_{x\to 0^+} g(x) = 2.5$

$\lim_{x\to 0^-} g(x) = 2.5$

$\lim_{x\to 0} g(x) = 2.5 \ \text{ since}: \lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x)$

$g(0) = 2.5$

$b)\ \lim_{x\to 2^+} g(x) = 1$

$\lim_{x\to 2^-} g(x) = 3$

$\lim_{x\to 2} g(x) = D.N.E \ \text{ since}: \lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$

$g(2) = \text{undefind}$

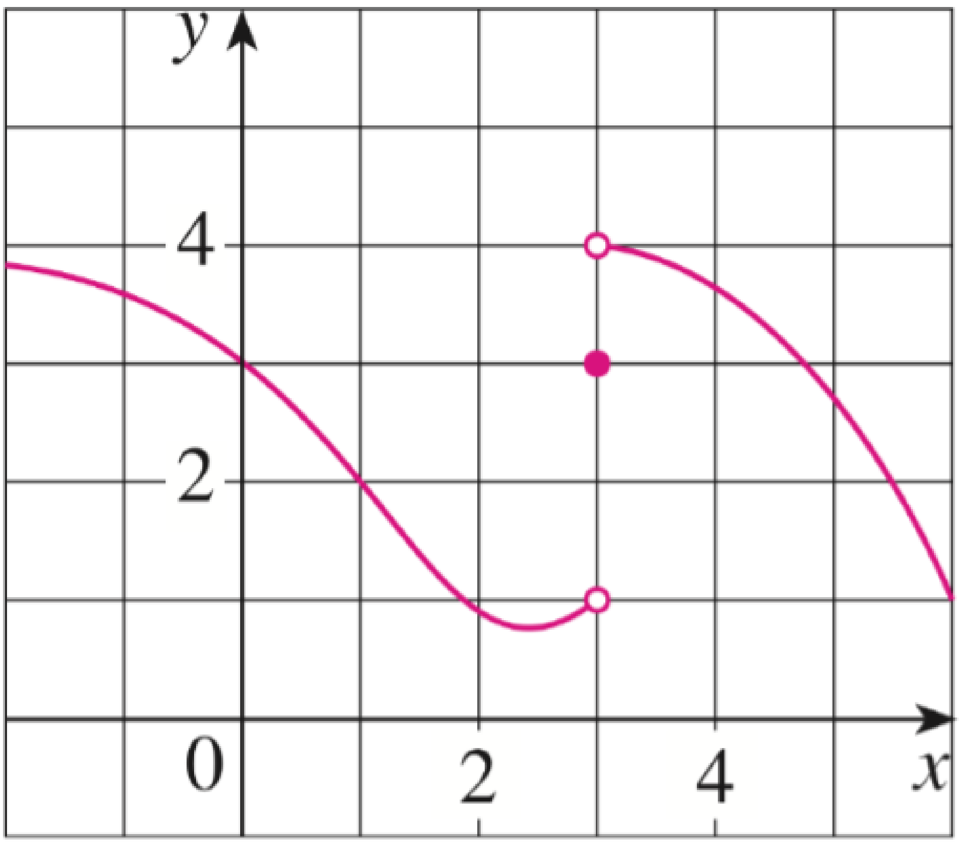
$c)\ \lim_{x\to 5^+} g(x) = 2$

$\lim_{x\to 5^-} g(x) = 2$

$\lim_{x\to 5} g(x) = 2 \ \text{ since}: \lim_{x\to 5^+} g(x) = \lim_{x\to 5^-} g(x)$

$g(5) = 1$

Example (7)



$a)\ \lim_{x\to 0} g(x) = 3$ and $g(0) = 3$

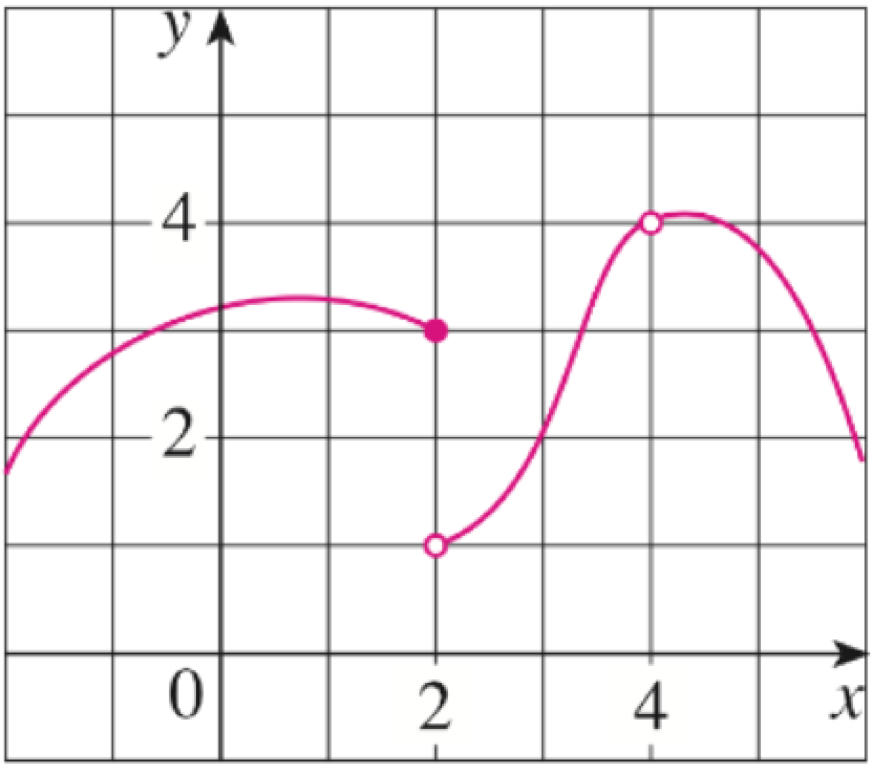
$b)\ \lim_{x\to 3^-} g(x) = 1$

$\lim_{x\to 3^+} g(x) = 4$

$\lim_{x\to 3} g(x) = D.N.E$ since : $\lim_{x\to 3^+} g(x) \neq \lim_{x\to 3^-} g(x)$

$c)\ \ g(3) = 3$

Example (8)



$a)\ \lim_{x\to 4} g(x) = 4$

$b)\ \lim_{x\to 2^-} g(x) = 3$

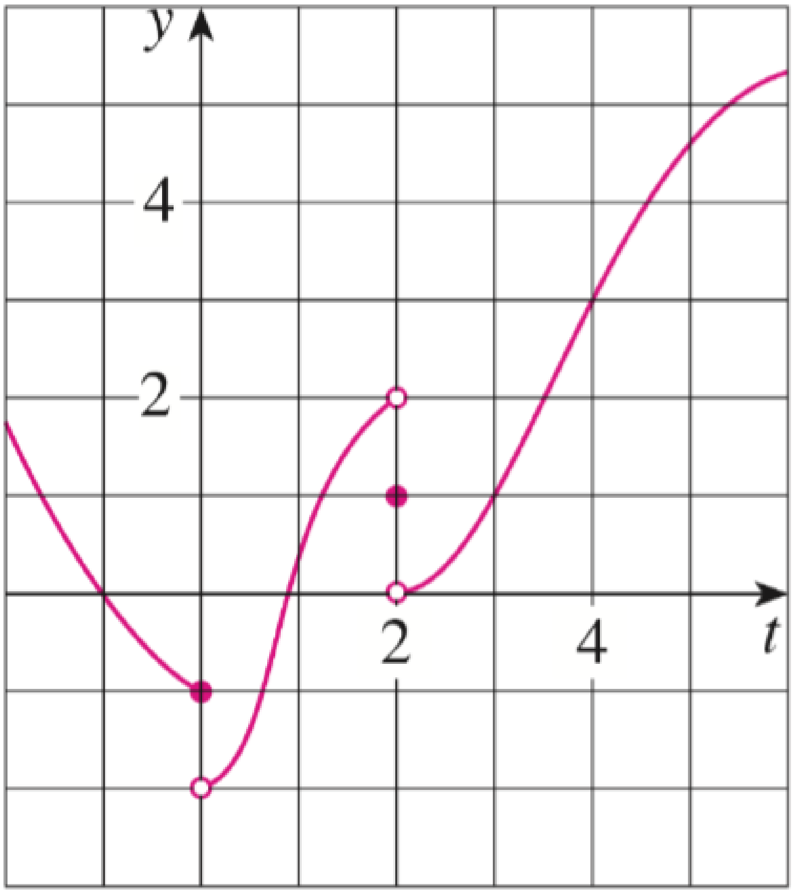
$\lim_{x\to 2^+} g(x) = 1$

$\lim_{x\to 2} g(x) = D.N.E$ since : $\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$

$c)\ g(2) = 3$

$g(4)$ is not defind

Example (9)



$a)$ $\lim_{x\to 0^+} g(x) = -2$

$\lim_{x\to 0^-} g(x) = -1$

$\because \lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^-} g(x)$

$\therefore \lim_{x\to 0} g(x) = D.N.E$

$b)$ $\lim_{x\to 2^-} g(x) = 2$

$\lim_{x\to 2^+} g(x) = 0$

$\lim_{x\to 2} g(x) = D.N.E$ since : $\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$

$c)$ $g(2) = 1$

$g(0) = -1$

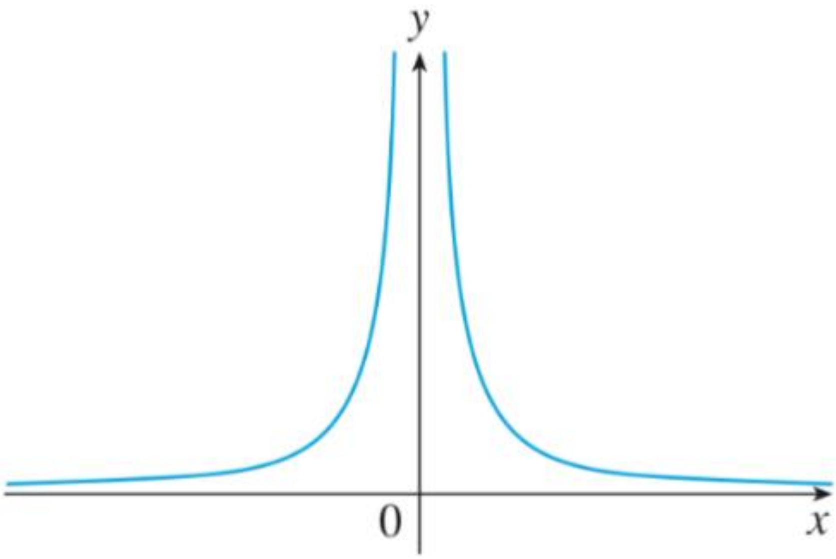
Infinite limits

$\lim_{x \to a} f(x) = \pm\infty$ if and only if $x = a$ is a vertical asymptote

$\lim_{x \to a^+} f(x) = \pm\infty$ if and only if $x = a$ is a vertical asymptote

$\lim_{x \to a^-} f(x) = \pm\infty$ if and only if $x = a$ is a vertical asymptote

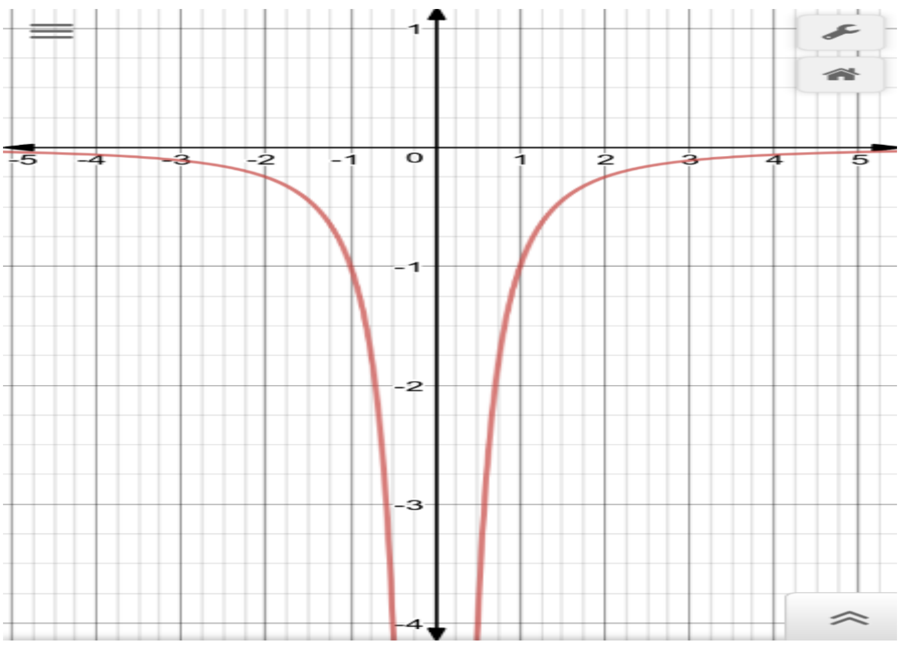
Example (10)



$$\lim_{x \to 0} f(x) = \infty$$

$\therefore$ $x = 0$ is a vertical asymptote

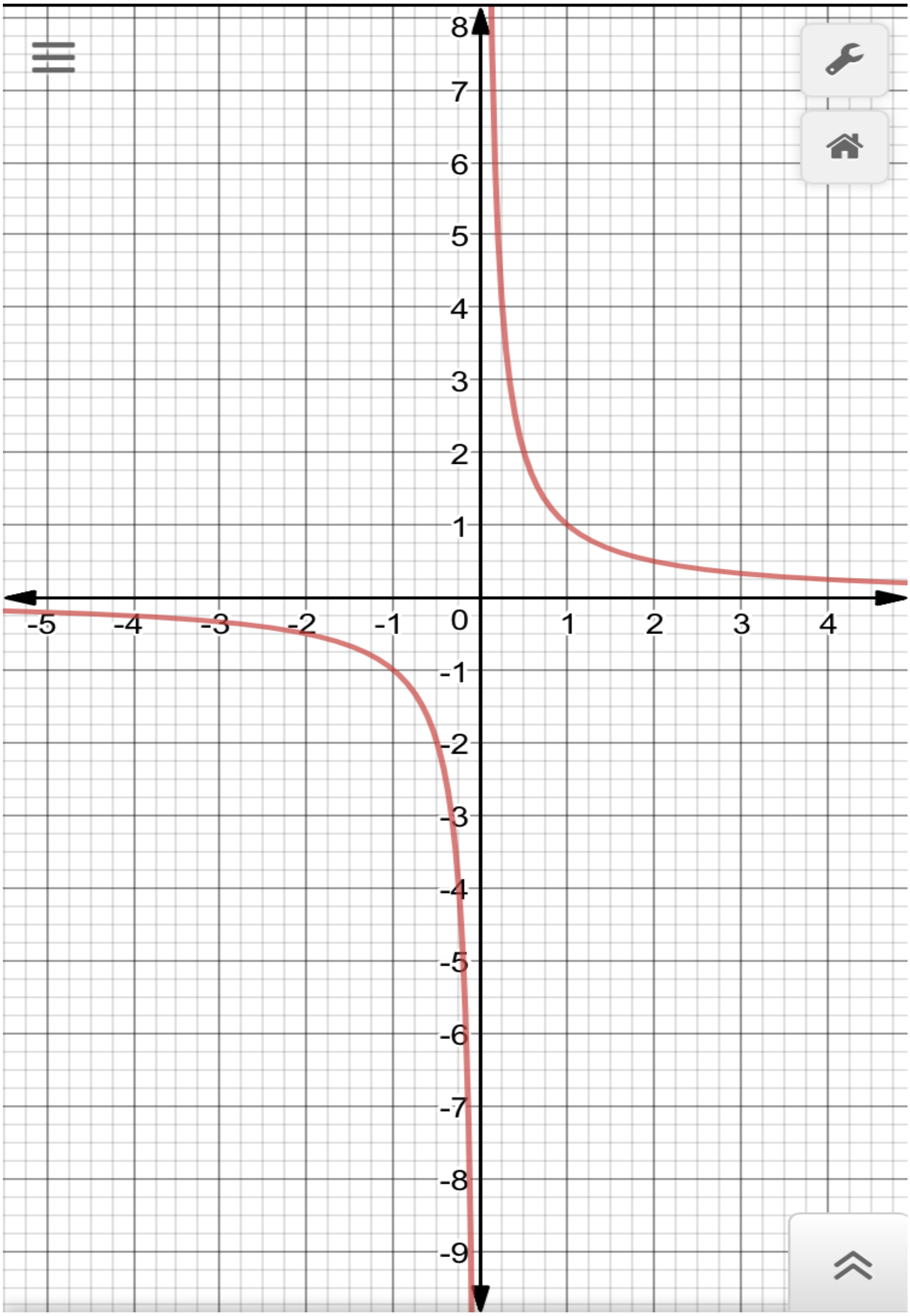
Example (11)



$$\lim_{x \to 0} f(x) = -\infty$$

$\therefore$ $x = 0$ is a vertical asymptote

Example (12)



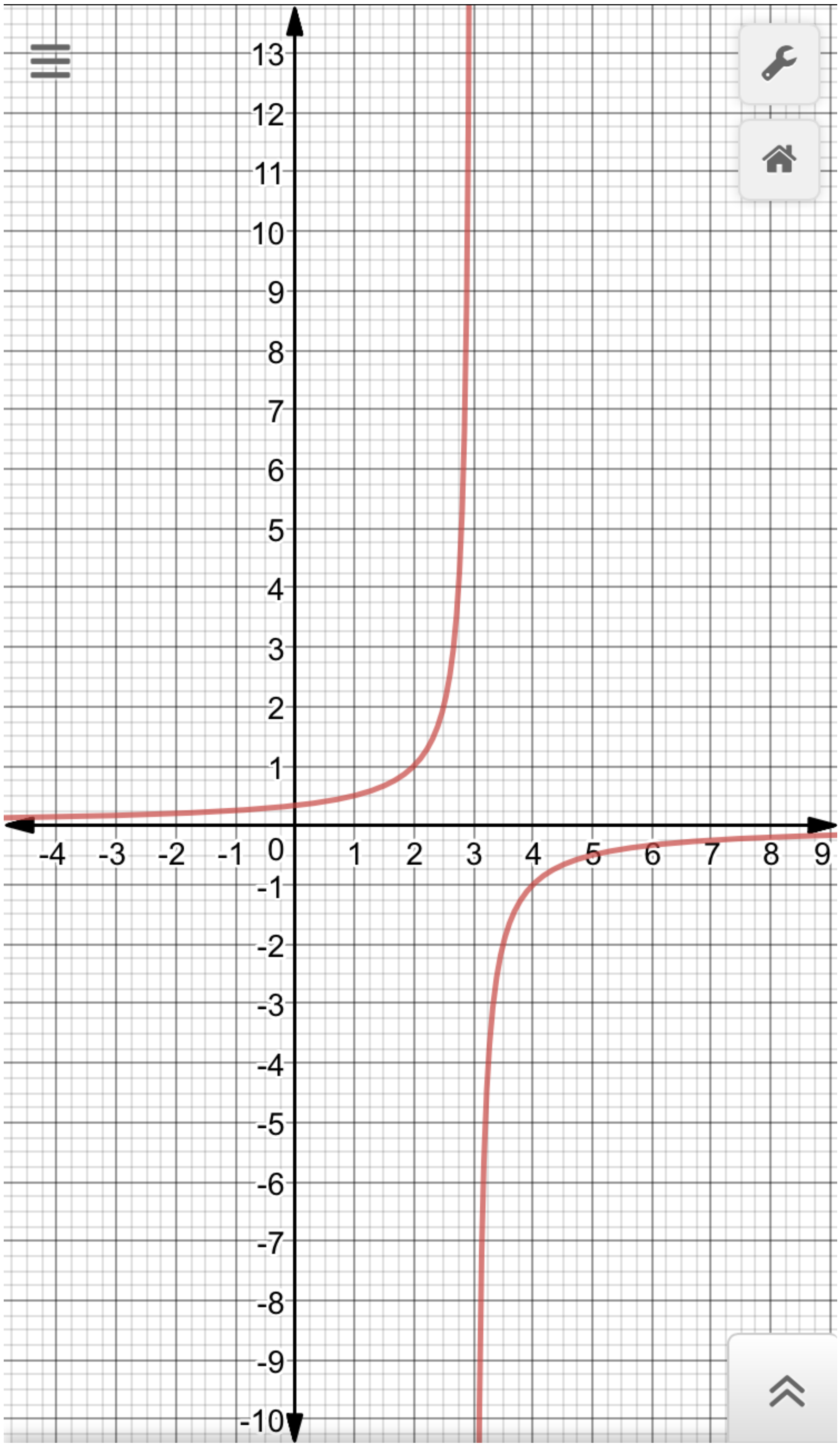
$$\lim_{x\to 0^+} f(x) = \infty$$

$$\lim_{x\to 0^-} f(x) = -\infty$$

$\lim_{x\to 0} f(x) = D.N.E$ since : $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

$\therefore$ $x = 0$ is a vertical asymptote

Example (13)



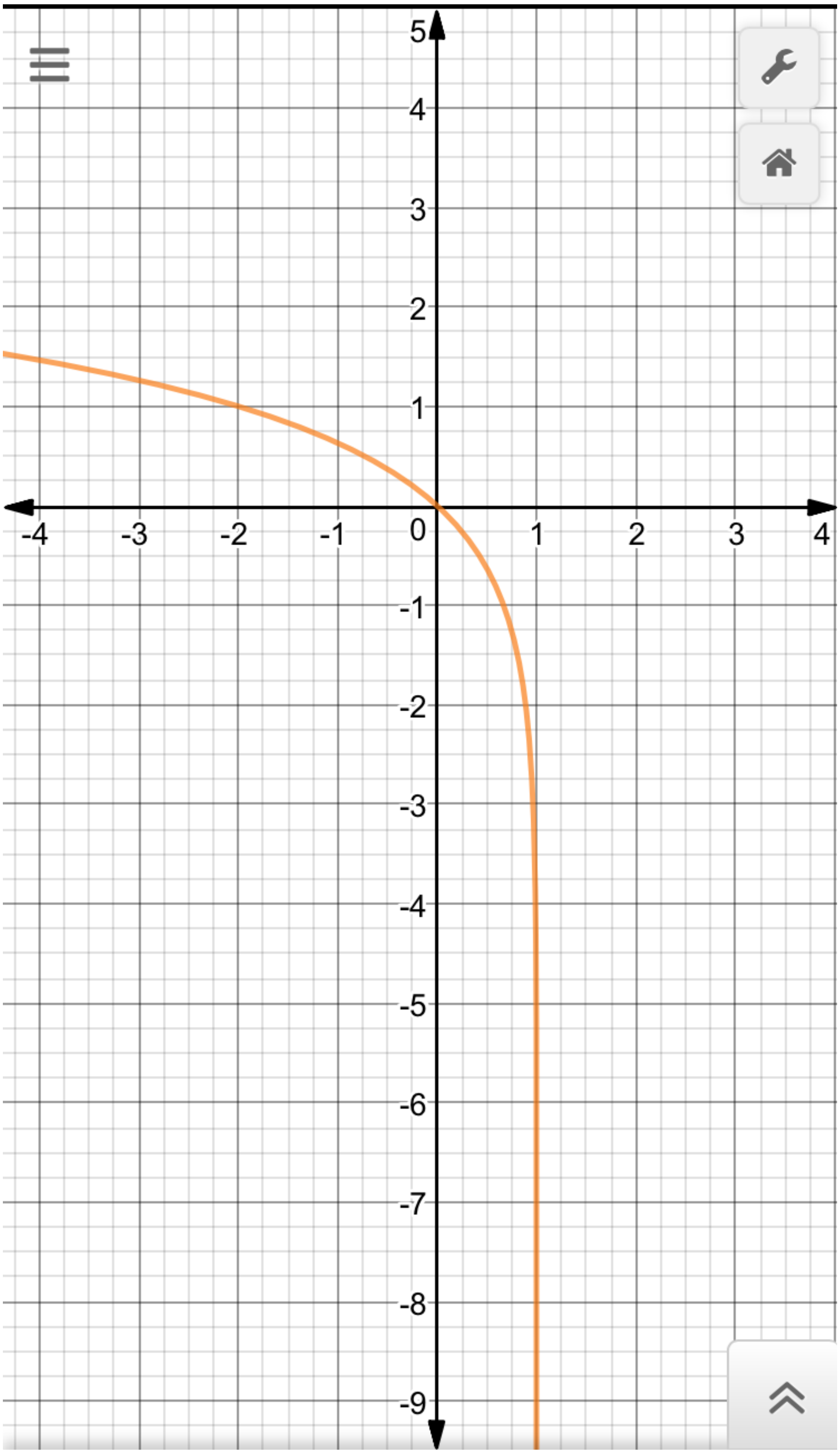
$\lim_{x \to 3^+} f(x) = -\infty$

$\lim_{x \to 3^-} f(x) = \infty$

$\lim_{x \to 3} f(x) = D.N.E$ since : $\lim_{x \to 3^+} f(x) \neq \lim_{x \to 3^-} f(x)$

$\therefore x = 3$ is a vertical asymptote

Example (14)



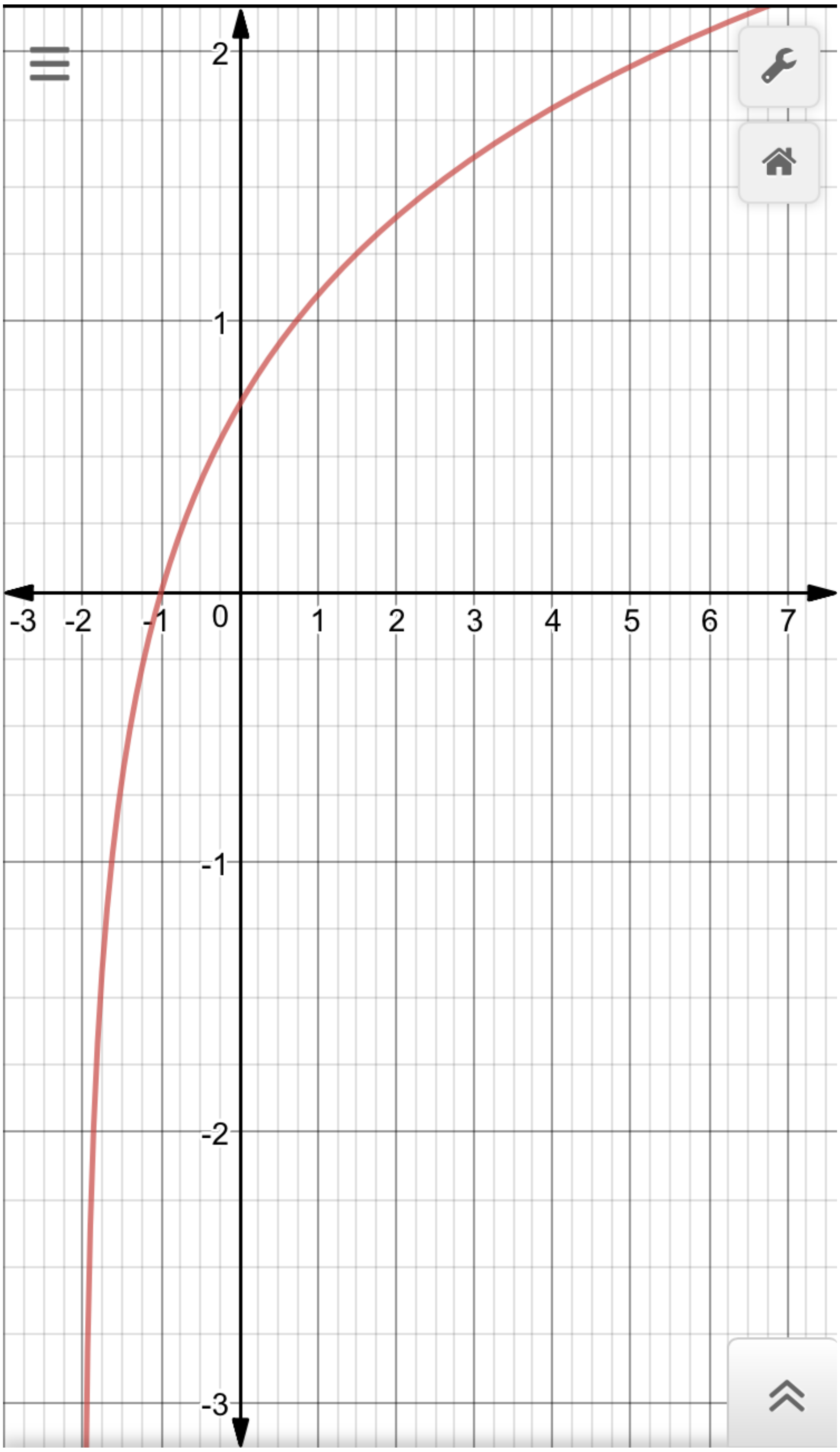
$\lim\limits_{x \to 1^-} f(x) = -\infty$

$\lim\limits_{x \to 1^+} f(x) = D.N.E$

$\lim\limits_{x \to 1} f(x) = D.N.E$ since : $\lim\limits_{x \to 1^+} f(x) \neq \lim\limits_{x \to 1^-} f(x)$

$\therefore x = 1$ is a vertical asymptote

Example (15)



$\lim_{x \to -2^+} f(x) = -\infty$

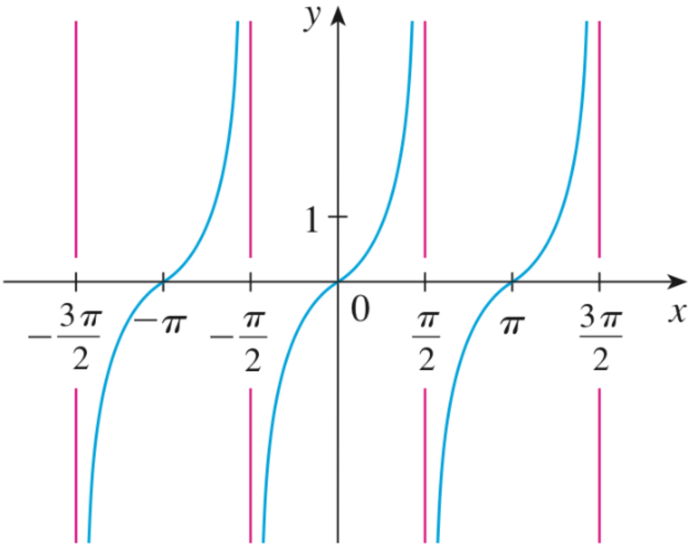
$\lim_{x \to -2^-} f(x) = D.N.E$

$\lim_{x \to -2} f(x) = D.N.E$ since : $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$

$\therefore$ $x = -2$ is a vertical asymptote

Example (16)

Find the vertical asymptotes of $f(x) = \tan(x)$



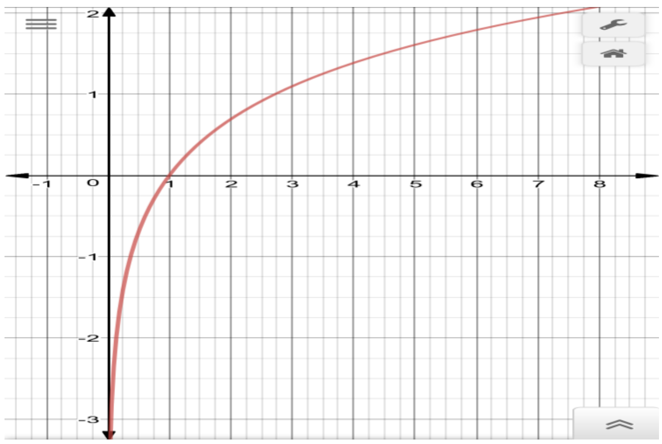
$\lim_{x \to \left(\pm\frac{n\pi}{2}\right)^{\pm}} \tan x = \mp\infty$ for all n is an odd number

$\lim_{x \to \pm\frac{n\pi}{2}} \tan x = D.N.E$ since : $\lim_{x \to \left(\pm\frac{n\pi}{2}\right)^{+}} \tan x \neq \lim_{x \to \left(\pm\frac{n\pi}{2}\right)^{-}} \tan x$

$\therefore x = \pm\frac{n\pi}{2}$ are a vertical asymptotes for all n is an odd number

Example (17)

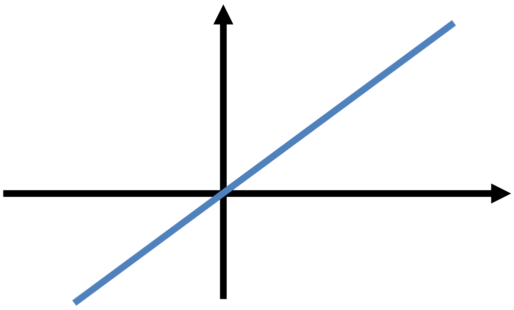
Find the vertical asymptotes of $f(x) = \ln(x)$



$x = 0$ is a vertical asymptote since : $\lim_{x \to 0^{+}} \ln x = -\infty$

Example (18)

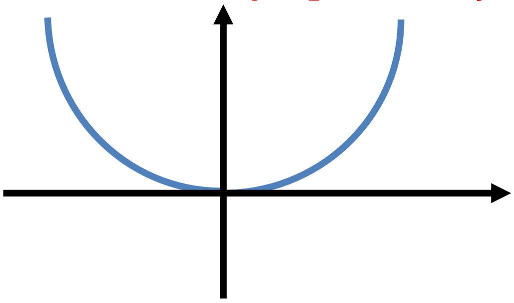
Find the vertical asymptotes of $f(x) = x$



$f(x)$ has no vertical asymptotes

Example (19)

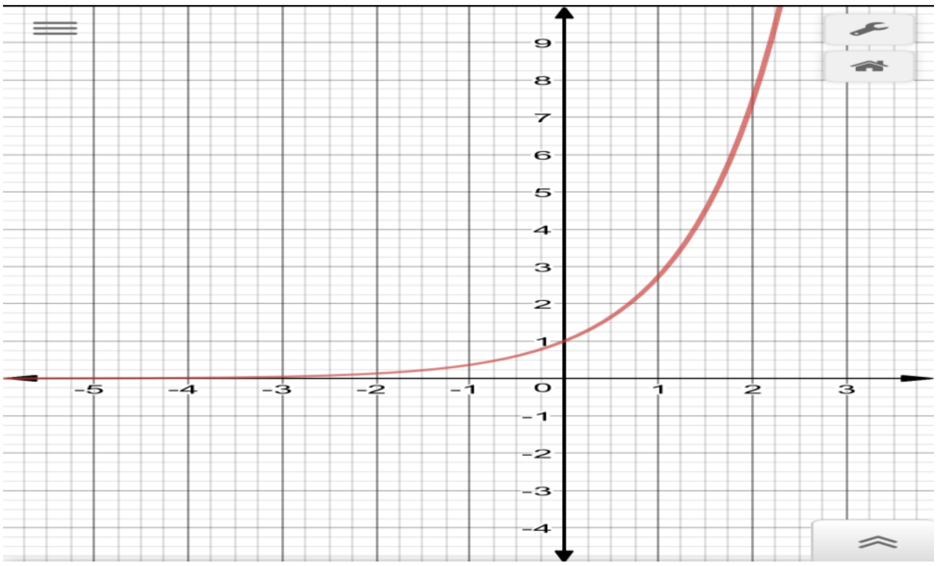
Find the vertical asymptotes of $f(x) = x^2$



$f(x)$ has no vertical asymptotes

Example (20)

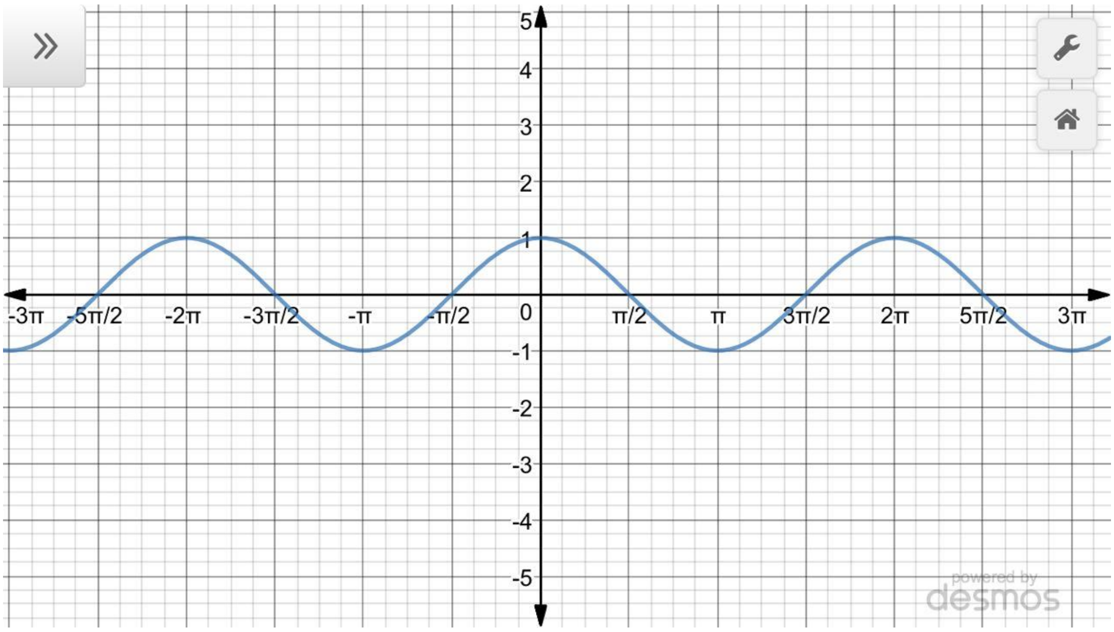
Find the vertical asymptotes of $f(x) = e^x$



$f(x)$ has no vertical asymptotes

Example (21)

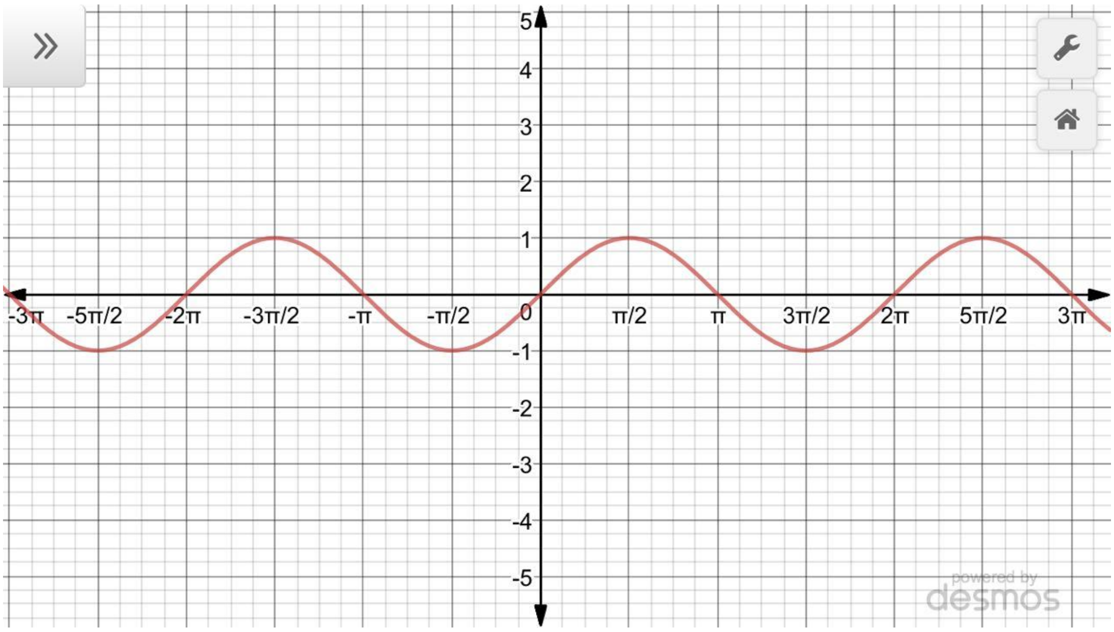
Find the vertical asymptotes of $f(x) = \cos x$



$f(x)$ has no vertical asymptotes

Example (22)

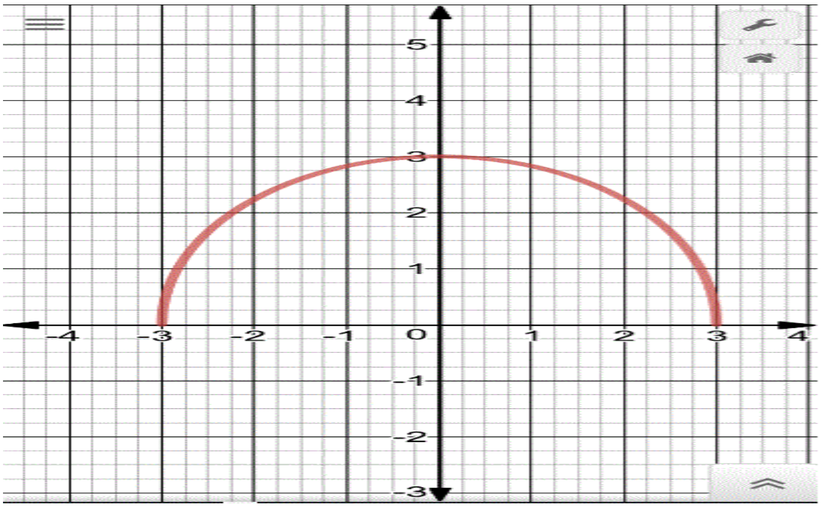
Find the vertical asymptotes of $f(x) = \sin x$



$f(x)$ has no vertical asymptotes

Example (23)

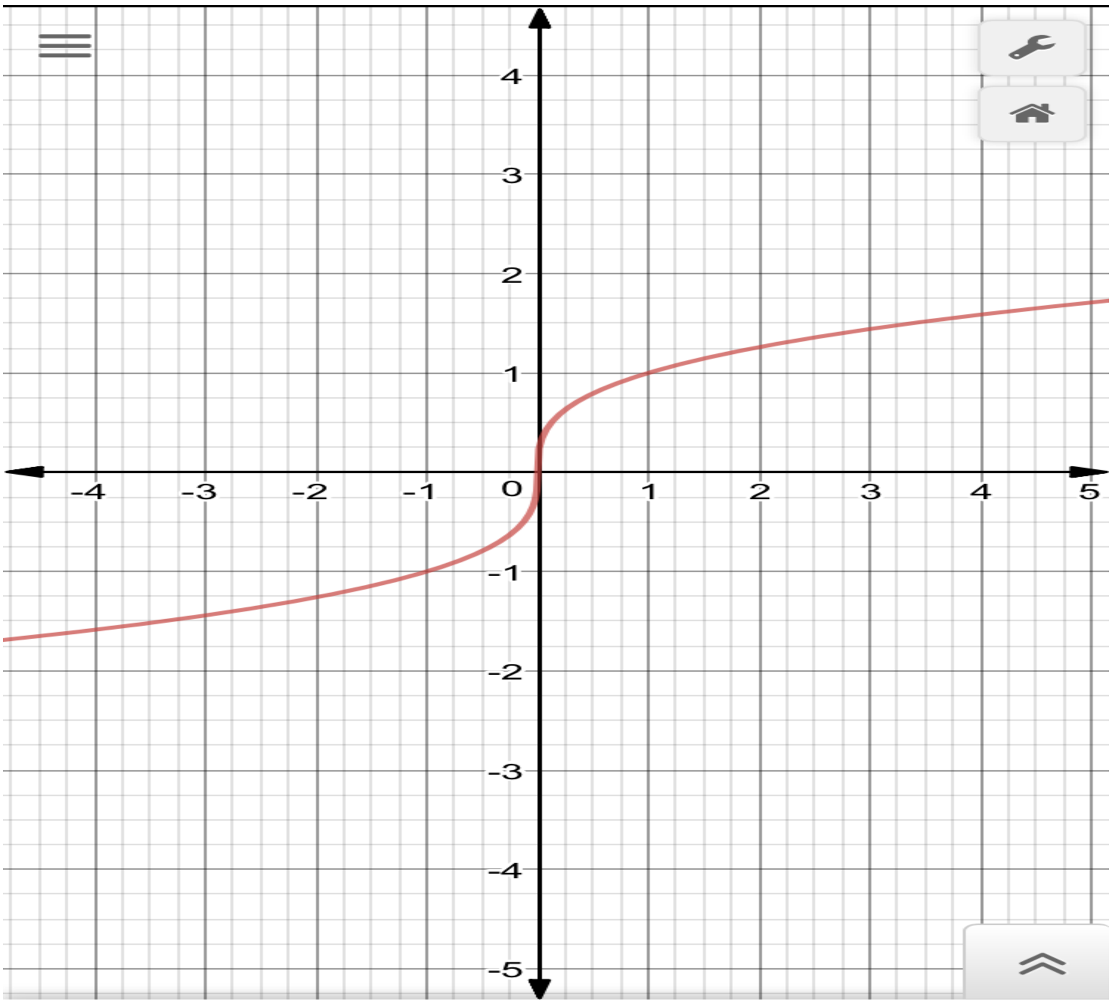
Find the vertical asymptotes of $f(x) = \sqrt{9 - x^2}$



$f(x)$ has no vertical asymptotes

Example (24)

Find the vertical asymptotes of $f(x) = \sqrt[3]{x}$

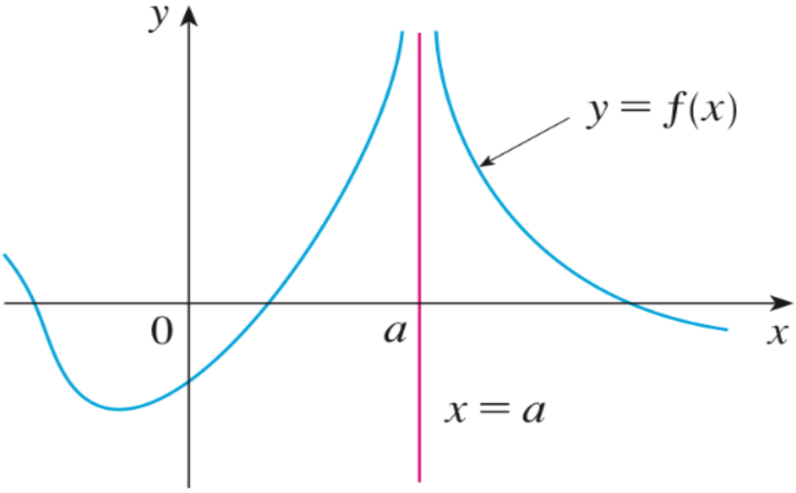


$f(x)$ has no vertical asymptotes

Note

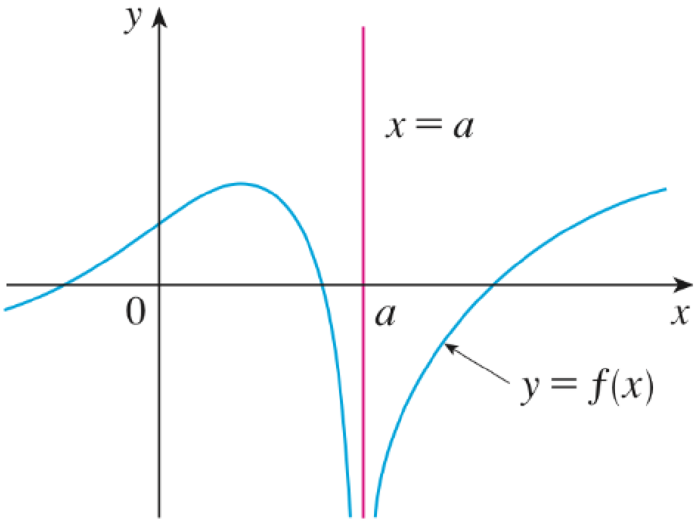
1. Any polynomial function has no vertical asymptote
2. Any exponintial function has no vertical asymptote
3. Any radical function has no vertical asymptote
4. Only $\sin x$ and $\cos x$ has no vertical asymptote

Summary of infinte limits



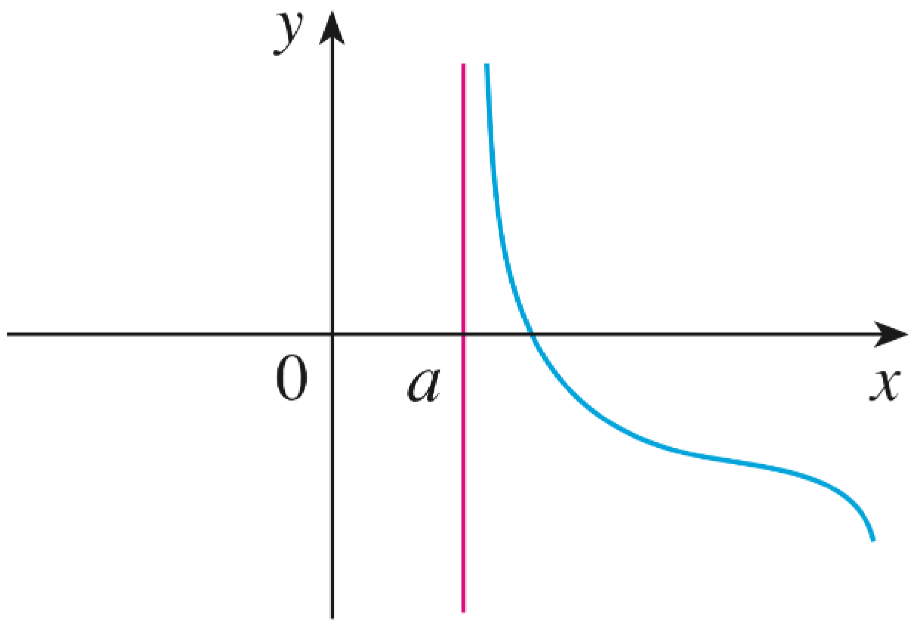
$$\lim_{x \to a} f(x) = \infty$$

$\therefore$ $x = a$ is a vertical asymptote



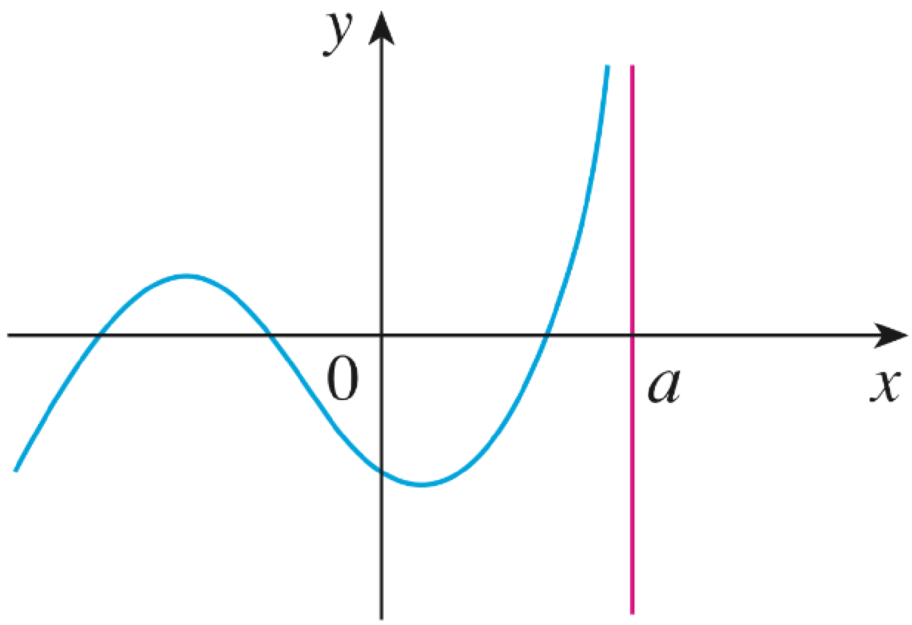
$$\lim_{x \to a} f(x) = -\infty$$

$\therefore$ $x = a$ is a vertical asymptote



$$\lim_{x \to a^+} f(x) = \infty$$

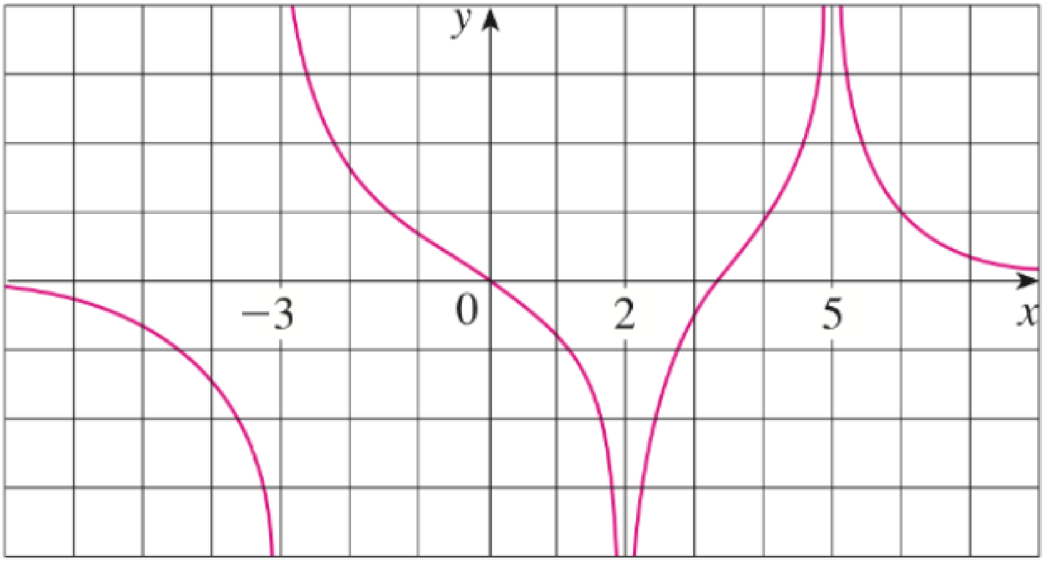
$\therefore\ x = a$ is a vertical asymptote



$$\lim_{x \to a^-} f(x) = \infty$$

$\therefore\ x = a$ is a vertical asymptote

Example(25)



$a)\ \lim_{x\to 5} f(x) = \infty$

$\therefore\ x = 5$ is a vertical asymptote

$b)\ \lim_{x\to 2} f(x) = -\infty$

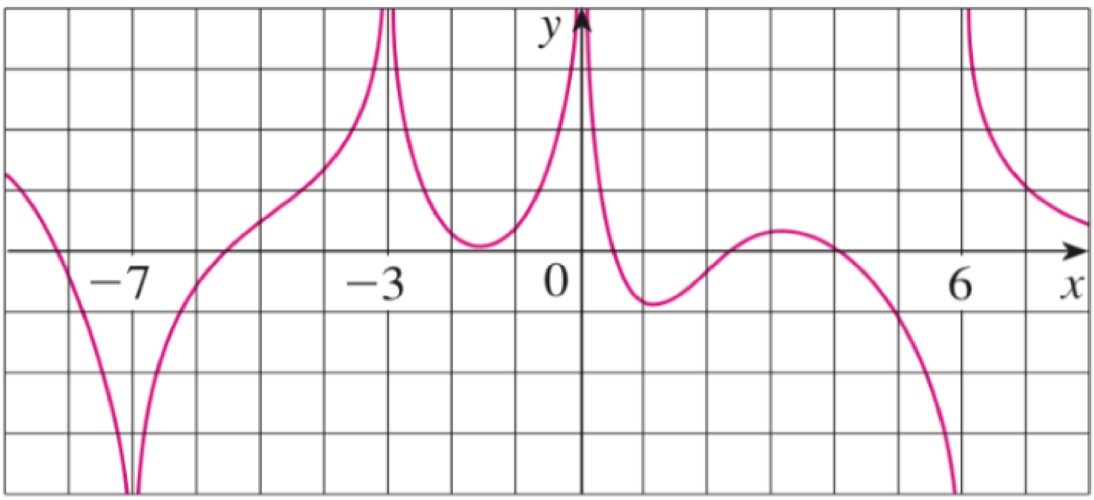
$\therefore\ x = 2$ is a vertical asymptote

$c)\ \lim_{x\to -3^+} f(x) = \infty$

$\lim_{x\to -3^-} f(x) = -\infty$

$\therefore\ x = -3$ is a vertical asymptote

Example(26)



$a)\ \lim_{x \to 0} f(x) = \infty$

$\therefore x = 0$ is a vertical asymptote

$b)\ \lim_{x \to -3} f(x) = \infty$

$\therefore x = -3$ is a vertical asymptote

$b)\ \lim_{x \to -7} f(x) = -\infty$

$\therefore x = -7$ is a vertical asymptote

$c)\ \lim_{x \to 6^+} f(x) = \infty$

$\lim_{x \to 6^-} f(x) = -\infty$

$\therefore x = 6$ is a vertical asymptote

Example(27)

Find the vertical asymptotes of the following functions

$$a)\ f(x) = \frac{2x}{x-3}$$

Zeros of the denominator : $x - 3 = 0 \Longrightarrow x = 3$

Let $g(x) = 2x$

$g(3) = 2(3) = 6 \neq 0$

$\therefore x = 3$ is vertical asymptote

$$b)\ f(x) = \frac{x+3}{x^2-4}$$

Zeros of the denominator : $x^2 - 4 = 0 \Longrightarrow x = \pm 2$

Let $g(x) = x + 3$

$g(2) = 2 + 3 = 5 \neq 0$

$g(-2) = -2 + 3 = 1 \neq 0$

$\therefore x = -2$ and $x = 2$ are vertical asymptote

$c)\ f(x) = \dfrac{x^2+1}{3x-2x^2}$

Zeros of the denominator : $3x - 2x^2 = 0 \Longrightarrow x(3-2x) = 0$

$x = 0$ or $3 - 2x = 0 \Longrightarrow x = \frac{3}{2}$

Let $g(x) = x^2 + 1$

$g(0) = 0 + 1 = 1 \neq 0$

$g\left(\dfrac{3}{2}\right) = \dfrac{9}{4} + 1 = \dfrac{9+4}{4} = \dfrac{13}{4} \neq 0$

$\therefore x = 0$ and $x = \dfrac{3}{2}$ are vertical asymptote

$d)\ f(x) = \dfrac{x^2-3x-10}{x^2-6x+5}$

Zeros of the denominator : $x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0 \Longrightarrow x = 1$ or $x = 5$

Let $g(x) = x^2 - 3x - 10$

$g(1) = 1 - 3 - 10 = -12 \neq 0$

$x = 1$ is a vertical asymptote

$g(5) = 25 - 15 - 10 = 0$

$x = 5$ is not vertical asymptote

$e)\ f(x) = \csc x$

$= \dfrac{1}{\sin x}$

Zeros of the denominator : $\sin x = 0 \Longrightarrow x = n\pi\ \forall\ n \in Z$

Let $g(x) = 1$

$g(n\pi) = 1 \neq 0$

$\therefore x = n\pi$ is a vertical asymptoe

$f)\ f(x) = \sec x$

$$= \frac{1}{\cos x}$$

Zeros of the denominator : $\cos x = 0 \Rightarrow x = \frac{(2n+1)\pi}{2} \quad \forall\, n \in Z$

$$\text{Let } g(x) = 1$$
$$g\left(\frac{(2n+1)\pi}{2}\right) = 1 \neq 0$$
$$\therefore x = \frac{(2n+1)\pi}{2} \text{ is a vertical asymptote}$$

$g)\ f(x) = \cot x$

$$= \frac{1}{\tan x}$$

Zeros of the denominator : $\tan x = 0 \Rightarrow x = n\pi \quad \forall\, n \in Z$

$$\text{Let } g(x) = 1$$
$$g(n\pi) = 1 \neq 0$$
$$\therefore x = n\pi \text{ is a vertical asymptoe}$$

$h)\ f(x) = \log_2(1 - x^2)$

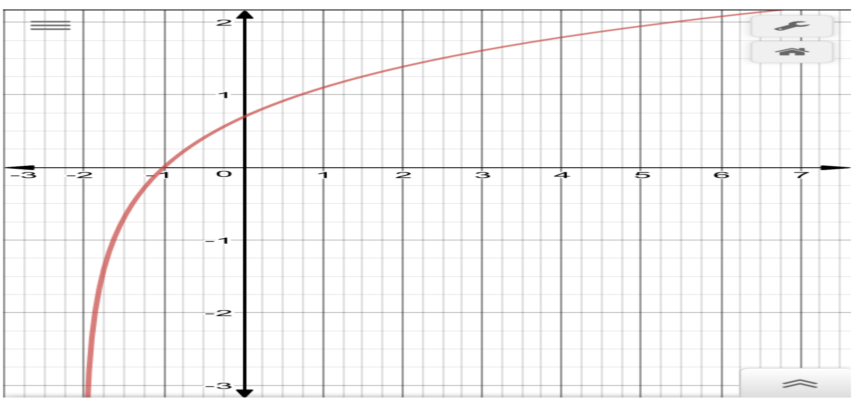
$1 - x^2 = 0 \Rightarrow x = \pm 1$

$\therefore x = \pm 1$ are a vertical asymptoe

$i)\ f(x) = \log_2(x + 2)$

$x + 2 = 0 \Rightarrow x = -2$

$\therefore x = -2$ are a vertical asymptoe since: $\lim_{x \to -2^+} \log_2(x + 2) = -\infty$



2.3 _ Calculating Limits Using the Limit Laws.

Limit Laws.

Suppose that c is a constant and the limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist Then

1) $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

2) $\lim\limits_{x \to a} [f(x) . g(x)] = \lim\limits_{x \to a} f(x) . \lim\limits_{x \to a} g(x)$

3) $\lim\limits_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

4) $\lim\limits_{x \to a} x = a$

5) $\lim\limits_{x \to a} x^n = a^n \quad \forall\, n \in \mathbb{Z}^+$

لكل

6) $\lim\limits_{x \to a} [f(x)]^n = [\lim\limits_{x \to a} f(x)]^n \quad \forall\, n \in \mathbb{Z}^+$

7] $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \forall$ n is an odd number

$\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ Such that $a > 0$ $\forall$ n is an even number

بحيث

8] $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} \quad \forall$ n is an odd number

$\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ Such that $\lim\limits_{x \to a} f(x) > 0$ $\forall$ n is an even number.

9] $\lim\limits_{x\to a} c f(x) = c \lim\limits_{x\to a} f(x)$

10] $\lim\limits_{x\to a} c = c$

Example (1)

Given that :

$\lim\limits_{x\to 2} f(x) = 4$; $\lim\limits_{x\to 2} g(x) = -2$; $\lim\limits_{x\to 2} h(x) = 0$

a) $\lim\limits_{x\to 2}\left[3f(x) - \frac{5}{2}g(x)\right] = \lim\limits_{x\to 2} 3f(x) - \lim\limits_{x\to 2} \frac{5}{2} g(x)$

$= 3\lim\limits_{x\to 2} f(x) - \frac{5}{2}\lim\limits_{x\to 2} g(x)$

$= 3(4) - \frac{5}{\cancel{2}}(-\cancel{2})$

$= 12 + 5$

$= 17$

b) $\lim\limits_{x\to 2} \frac{3f(x)}{g(x)} = \frac{\lim\limits_{x\to 2} 3f(x)}{\lim\limits_{x\to 2} g(x)} = \frac{3\lim\limits_{x\to 2} f(x)}{\lim\limits_{x\to 2} g(x)}$

$= \frac{3(4)}{-2} = \frac{12}{-2} = -6$

c) $\lim\limits_{x\to 2}\sqrt{f(x)} = \sqrt{\lim\limits_{x\to 2} f(x)} = \sqrt{4} = 2$

d) $\lim\limits_{x\to 2}[h(x)]^3 = \left[\lim\limits_{x\to 2} h(x)\right]^3 = 0^3 = 0$

e) $\lim\limits_{x\to 2} \frac{e^c}{\ln(c)} = \frac{e^c}{\ln(c)}$; $\lim\limits_{x\to 2} \frac{c^2}{\sqrt{2}} =$ H.W

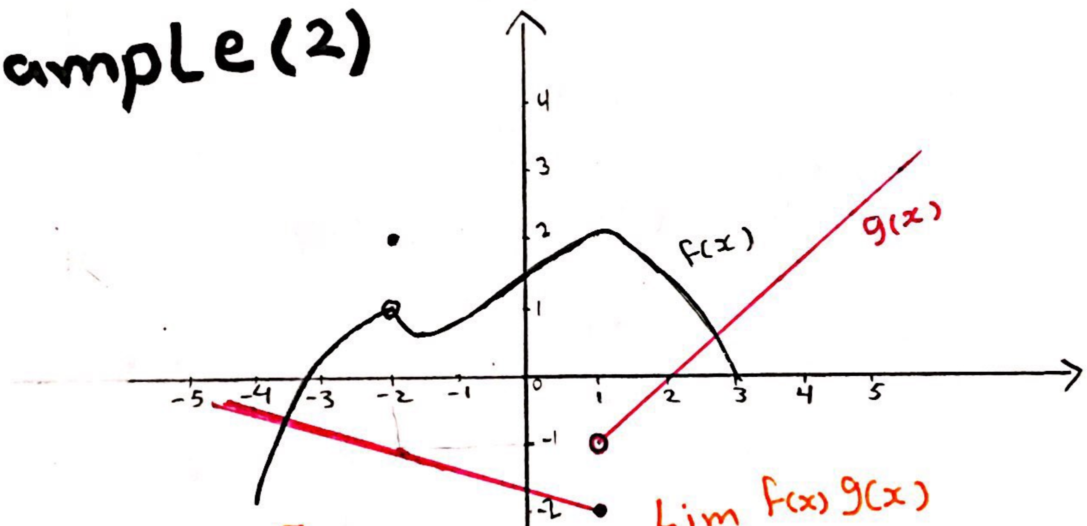
f) $\lim\limits_{x\to 2} \sin(\pi/12) = \sin(\pi/12)$

g) $\lim\limits_{x\to 2} \cos(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

h) $\lim\limits_{x\to 2} \tan^{-1}(-1) = \tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}$

(i) $\lim\limits_{x\to 2} a\sqrt[3]{b} = a\sqrt[3]{b}$

Example (2)



$\lim\limits_{x\to -2} [f(x) + 5g(x)]$

$\lim\limits_{x\to -2} f(x) + \lim\limits_{x\to -2} 5g(x)$

$\lim\limits_{x\to -2} f(x) + 5\lim\limits_{x\to -2} g(x)$

$1 + 5(-1)$

$1 - 5$

-4

$\lim\limits_{x\to 1} f(x)g(x)$

$\lim\limits_{x\to 1^+} f(x)g(x) = \lim\limits_{x\to 1^+} f(x) \cdot \lim\limits_{x\to 1^+} g(x)$

$= (+2)(-1)$

$= -2$

$\lim\limits_{x\to 1^-} f(x)g(x) = \lim\limits_{x\to 1^-} f(x) \lim\limits_{x\to 1^-} g(x)$

$= 2(-2) = -4$

$\because \lim\limits_{x\to 1^+} f(x)g(x) \neq \lim\limits_{x\to 1^-} f(x)g(x)$

$\therefore \lim\limits_{x\to 1} f(x)g(x) =$ D.N.E

Example (3)

a) $$\lim_{x\to 9} \frac{x-9}{\sqrt{x}+9} = \frac{9-9}{\sqrt{9}+9} = \frac{0}{3+9} = \frac{0}{12} = 0$$

b) $$\lim_{x\to 5} (2x^2-3x+4) = 2(5)^2-3(5)+4 = 2(25)-3(5)+4$$
$$= 50-15+4 = 35+4$$
$$= 39$$

c) $$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \frac{-8+2(4)-1}{5+6}$$
$$= \frac{-8+8-1}{11} = -\frac{1}{11}$$

Example (4)

a) $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{(1)^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$$ كمية غير محددة

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x\to 1}(x+1) = 1+1 = 2$$

b) $$\lim_{x\to -6} \frac{x+6}{x^2-36} = \frac{-6+6}{(-6)^2-36} = \frac{-6+6}{36-36} = \frac{0}{0}$$ كمية غير محددة

$$\lim_{x\to -6} \frac{x+6}{x^2-36} = \lim_{x\to -6} \frac{\cancel{(x+6)}}{(x-6)\cancel{(x+6)}} = \lim_{x\to -6} \frac{1}{x-6} = \frac{1}{-6-6} = -\frac{1}{12}$$

c) $$\lim_{x\to 2} \frac{x-2}{x^3-8} = \frac{2-2}{(2)^3-8} = \frac{2-2}{8-8} = \frac{0}{0}$$ كمية غير محددة

$$\lim_{x\to 2} \frac{x-2}{x^3-8} = \lim_{x\to 2} \frac{x-2}{x^3-2^3} = \lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x^2+2x+4)}$$

$$\lim_{x\to 2} \frac{1}{x^2+2x+4} = \frac{1}{(2)^2+2(2)+4} = \frac{1}{4+4+4} = \frac{1}{12}$$

d) $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \frac{(-4)^2+5(-4)+4}{(-4)^2+3(-4)-4} = \frac{16-20+4}{16-12-4}$

$= \frac{-4+4}{4-4} = \frac{0}{0}$ كمية غير محددة

$\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x \to -4} \frac{(x+1)\cancel{(x+4)}}{\cancel{(x+4)}(x-1)}$

$= \lim_{x \to -4} \frac{x+1}{x-1} = \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$

e) $\lim_{x \to -1} \frac{2x^2+3x+1}{x^2-2x-3} = \cancel{\lim_{x \to -1}} \frac{2(-1)^2+3(-1)+1}{(-1)^2-2(-1)-3}$

$= \frac{2(1)+3(-1)+1}{1-2(-1)-3} = \frac{2-3+1}{1+2-3} = \frac{-1+1}{3-3}$

$= \frac{0}{0}$ كمية غير محددة

$\lim_{x \to -1} \frac{2x^2+3x+1}{x^2-2x-3} = \lim_{x \to -1} \frac{(2x+1)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}} = \lim_{x \to -1} \frac{2x+1}{x-3}$

$= \frac{2(-1)+1}{-1-3} = \frac{-2+1}{-1-3} = \frac{-1}{-4} = \frac{1}{4}$

طريقة التحليل: $2x^2+3x+1$

$2x$	+	1	$+1x$
$1x$	+	1	$+2x$
			$+3x$ ✓

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$\therefore 2x^2+3x+1 = (2x+1)(x+1)$

f) $\lim\limits_{t \to 3} \dfrac{t^2-9}{2t^2-7t+3} = \dfrac{(3)^2-9}{2(3)^2-7(3)+3} = \dfrac{9-9}{2(9)-7(3)+3}$

$= \dfrac{9-9}{18-21+3} = \dfrac{9-9}{-3+3}$

$= \dfrac{0}{0}$ كمية غيرمحددة

$\lim\limits_{t \to 3} \dfrac{t^2-9}{2t^2-7t+3} = \lim\limits_{t \to 3} \dfrac{\cancel{(t-3)}(t+3)}{\cancel{(t-3)}(2t-1)}$

$= \lim\limits_{t \to 3} \dfrac{t+3}{2t-1} = \dfrac{3+3}{2(3)-1} = \dfrac{3+3}{6-1}$

$= \dfrac{6}{5}$

طريقة تحليل المقدار الجبري : $2t^2-7t+3$

$2t$	$-$	1	$-1t$
$1t$	$-$	3	$-6t$
			$+$
			$-7t$ ✓

ضرب

ضرب

g) $\lim\limits_{x \to 2} \dfrac{x^2+x-6}{x-2} = \dfrac{(2)^2+2-6}{2-2} = \dfrac{4+2-6}{2-2} = \dfrac{6-6}{2-2} = \dfrac{0}{0}$

كمية غيرمحددة

$\lim\limits_{x \to 2} \dfrac{x^2+x-6}{x-2} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}} = \lim\limits_{x \to 2} (x+3) = 2+3 = 5$

h) $\lim_{x \to 1} \frac{x^4-1}{x^3-1} = \frac{(1)^4-1}{(1)^3-1} = \frac{1-1}{1-1} = \frac{0}{0}$ كمية غير محددة

$\lim_{x \to 1} \frac{x^4-1}{x^3-1} = \lim_{x \to 1} \frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)}$

$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{(x-1)}(x^2+x+1)}$

$= \lim_{x \to 1} \frac{(x+1)(x^2+1)}{(x^2+x+1)}$

$= \frac{(1+1)(1+1)}{1+1+1} = \frac{2(2)}{3} = \frac{4}{3}$

ملاحظة

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$a^2-b^2 = (a+b)(a-b)$

$a^2+b^2 =$ لايمكن تحليله

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

i) $\lim_{x \to \frac{5}{3}} \frac{3x-5}{6x^2+5x-25} = \frac{3(\frac{5}{3})-5}{6(\frac{25}{9})+5(\frac{5}{3})-25} = \frac{5-5}{2(\frac{25}{3})+\frac{25}{3}-25}$

$= \frac{0}{\frac{50}{3}+\frac{25}{3}-25} = \frac{0}{\frac{75}{3}-25} = \frac{0}{25-25} = \frac{0}{0}$

$\lim_{x \to \frac{5}{3}} \frac{\cancel{(3x-5)}}{\cancel{(3x-5)}(2x+5)} = \lim_{x \to \frac{5}{3}} \frac{1}{2x+5} = \frac{1}{2(\frac{5}{3})+5} = \frac{1}{\frac{10}{3}+5}$

$= \frac{1}{\frac{10+15}{3}} = \frac{1}{\frac{25}{3}} = \frac{3}{25}$

طريقة تحليل المقدار الجبري : $6x^2+5x-25$

$$\begin{array}{ccc|c} 3x & - & 5 & -10x \\ 2x & + & 5 & +15x \\ \hline & & & +5x \end{array}$$

ضرب

Example (5)

a) $\lim\limits_{h\to 0} \dfrac{(3+h)^2-9}{h} = \dfrac{(3+0)^2-9}{0} = \dfrac{3^2-9}{0} = \dfrac{9-9}{0}$

$= \dfrac{0}{0}$ كمية غير محددة

$\lim\limits_{h\to 0} \dfrac{\cancel{9}+2(3)h+h^2\cancel{-9}}{h} = \lim\limits_{h\to 0} \dfrac{6h+h^2}{h} = \dfrac{0}{0}$ كمية غير محددة

$\lim\limits_{h\to 0} \dfrac{6h+h^2}{h} = \lim\limits_{h\to 0} \dfrac{\cancel{h}(6+h)}{\cancel{h}} = \lim\limits_{h\to 0}(6+h) = 6+0 = 6$

b) $\lim\limits_{h\to 0} \dfrac{(2+h)^3-8}{h} = \dfrac{(2+0)^3-8}{0} = \dfrac{2^3-8}{0} = \dfrac{8-8}{0}$

$= \dfrac{0}{0}$ كمية غير محددة

$\lim\limits_{h\to 0} \dfrac{(2+h)^3-8}{h} = \lim\limits_{h\to 0} \dfrac{\cancel{8}+3(4)h+3(2)h^2+h^3\cancel{-8}}{h}$

$\lim\limits_{h\to 0} \dfrac{(12h+6h^2+h^3)}{h} = \dfrac{0}{0} \Rightarrow \lim\limits_{h\to 0} \dfrac{\cancel{h}(12+6h+h^2)}{\cancel{h}}$

$\lim\limits_{h\to 0} (12+6h+h^2) = 12+6(0)+0^2 = 12$

Example (6)

a) $\lim_{h \to 0} \frac{(3+h)^{-1} - 3^{-1}}{h} = \frac{3^{-1} - 3^{-1}}{0} = \frac{0}{0}$ كمية غير محددة

$$\lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{\left(\frac{h}{1}\right)}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{3}-\cancel{3}-h}{3(3+h)}}{\left(\frac{h}{1}\right)}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{3(3+h)}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{-h}{3(3+h)} \div \frac{h}{1}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{3(3+h)} \times \frac{1}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{3(3+h)}$$

$$= \frac{-1}{3(3)}$$

$$= \frac{-1}{9}$$

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b) $\lim\limits_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \dfrac{\frac{1}{4} - \frac{1}{4}}{4 - 4} = \dfrac{0}{0}$ كمية غير محددة

$$\lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \lim_{x \to -4} \frac{\left(\frac{x+4}{4x}\right)}{\left(\frac{4+x}{1}\right)}$$

$$= \lim_{x \to -4} \frac{(x+4)}{4x} \div \frac{(4+x)}{1}$$

$$= \lim_{x \to -4} \frac{(x+4)}{4x} \times \frac{1}{(4+x)}$$

$$= \lim_{x \to -4} \frac{\cancel{(x+4)}}{4x} \times \frac{1}{\cancel{(x+4)}}$$

$$= \lim_{x \to -4} \frac{1}{4x}$$

$$= \frac{1}{4(-4)}$$

$$= -\frac{1}{16}$$

c) $\lim\limits_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right) = \dfrac{1}{0} - \dfrac{1}{0}$ كمية غير محددة

$$\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right) = \lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t(t+1)} \right)$$

$$= \lim_{t \to 0} \left(\frac{t + \cancel{1} - \cancel{1}}{t(t+1)} \right) = \lim_{t \to 0} \frac{\cancel{t}}{\cancel{t}(t+1)}$$

$$= \lim_{t \to 0} \frac{1}{t+1} = \frac{1}{0+1} = \frac{1}{1} = 1$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin^2 x - 1}{\sin x - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{x \to \pi/2} \frac{\cancel{(\sin x - 1)}(\sin x + 1)}{\cancel{(\sin x - 1)}}$$

$$\lim_{x \to \frac{\pi}{2}} (\sin x + 1) = \sin\left(\frac{\pi}{2}\right) + 1 = 1 + 1 = 2$$

$$\lim_{x \to -\frac{\pi}{4}} \frac{\sin x + \cos x}{\cos^2 x - \sin^2 x} = \frac{\sin\left(-\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)}{\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(-\frac{\pi}{4}\right)} = \frac{-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\frac{1}{2} - \frac{1}{2}}$$

$$= \frac{0}{0}$$

$$\lim_{x \to -\frac{\pi}{4}} \frac{\cancel{(\sin x + \cos x)}}{\cancel{(\cos x + \sin x)}(\cos x - \sin x)}$$

$$\lim_{x \to -\frac{\pi}{4}} \frac{1}{\cos x - \sin x} = \frac{1}{\cos\left(-\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)} = \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\frac{2}{\sqrt{2}}} = \frac{\sqrt{2}}{2}$$

$$\lim_{x \to \frac{\pi}{6}} \frac{2\sin x - 1}{6\sin^2 x + 10\sin x - 5} = \frac{2\sin(\frac{\pi}{6}) - 1}{6\sin^2(\frac{\pi}{6}) + 10\sin(\frac{\pi}{6}) - 5}$$

$$= \frac{2(\frac{1}{2}) - 1}{6(\frac{1}{2})^2 + 10(\frac{1}{2}) - 5}$$

$$= \frac{1 - 1}{\frac{6}{4} + 5 - 5}$$

$$= \frac{0}{\frac{6}{4}} = 0$$

$$\lim_{x \to \frac{\pi}{6}} \frac{2\sin x - 1}{6\sin^2 x + 7\sin x - 5} = \frac{2(\frac{1}{2}) - 1}{6(\frac{1}{2})^2 + 7(\frac{1}{2}) - 5}$$

$$= \frac{1 - 1}{\frac{6}{4} + \frac{7}{2} - 5}$$

$$= \frac{0}{\frac{3}{2} + \frac{7}{2} - 5} = \frac{0}{\frac{10}{2} - 5}$$

$$= \frac{0}{5 - 5} = \frac{0}{0}$$

$$\lim_{x \to \pi/6} \frac{\cancel{(2\sin x - 1)}}{(3\sin x + 5)\cancel{(2\sin x - 1)}} = \lim_{x \to \frac{\pi}{6}} \frac{1}{3\sin x + 5}$$

$$= \frac{1}{3(\frac{1}{2}) + 5} = \frac{1}{\frac{3}{2} + 5} = \frac{1}{\frac{3+10}{2}}$$

$$= \frac{2}{13}$$

طريقة تحليل المقدار : $6\sin^2 x + 7\sin x - 5$

$2\sin x$	$-$	1	$-3\sin x$
$3\sin x$	$+$	5	$+10\sin x$ +
			$+7\sin x$

Example(7)

a) $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{\sqrt{9}-3}{0^2} = \frac{3-3}{0} = \frac{0}{0}$ كمية غيرمحددة

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{(\sqrt{t^2+9}-3)}{t^2} \times \frac{(\sqrt{t^2+9}+3)}{(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+9})^2-(3)^2}{t^2\cdot(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{t^2+9-9}{t^2\cdot(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}\cdot(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\sqrt{9}+3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

b) $\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = \frac{\sqrt{9}-3}{0} = \frac{3-3}{0} = \frac{0}{0}$ كمية غير محددة

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h\to 0} \frac{(\sqrt{9+h}-3)}{h} \times \frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h\cdot(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{(\sqrt{9+h})^2-(3)^2}{h\cdot(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{\cancel{9}+h-\cancel{9}}{h\cdot(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\cdot(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3}$$

$$= \frac{1}{\sqrt{9}+3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

c) $\lim\limits_{u \to 2} \frac{\sqrt{4u+1} - 3}{u-2} = \frac{\sqrt{4(2)+1} - 3}{2-2} = \frac{\sqrt{9} - 3}{0}$

$= \frac{3-3}{0} = \frac{0}{0}$ كمية غير محددة

$$\lim_{u \to 2} \frac{\sqrt{4u+1} - 3}{u-2} = \lim_{u \to 2} \frac{(\sqrt{4u+1} - 3)}{(u-2)} \times \frac{(\sqrt{4u+1} + 3)}{(\sqrt{4u+1} + 3)}$$

$$= \lim_{u \to 2} \frac{(\sqrt{4u+1} - 3)(\sqrt{4u+1} + 3)}{(u-2)(\sqrt{4u+1} + 3)}$$

$$= \lim_{u \to 2} \frac{(\sqrt{4u+1})^2 - (3)^2}{(u-2)(\sqrt{4u+1} + 3)}$$

$$= \lim_{u \to 2} \frac{4u+1-9}{(u-2)(\sqrt{4u+1} + 3)}$$

$$= \lim_{u \to 2} \frac{4u-8}{(u-2)(\sqrt{4u+1} + 3)}$$

$$= \lim_{u \to 2} \frac{4\cancel{(u-2)}}{\cancel{(u-2)}(\sqrt{4u+1} + 3)}$$

$$= \lim_{u \to 2} \frac{4}{\sqrt{4u+1} + 3} = \frac{4}{\sqrt{9} + 3}$$

$$= \frac{4}{3+3} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

d) $\lim_{t \to 0} \left[\frac{\sqrt{1+t} - \sqrt{1-t}}{t} \right] = \frac{\sqrt{1} - \sqrt{1}}{0} = \frac{1-1}{0} = \frac{0}{0}$

$$\lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} = \lim_{t \to 0} \frac{(\sqrt{1+t} - \sqrt{1-t})}{t} \times \frac{(\sqrt{1+t} + \sqrt{1-t})}{(\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \to 0} \frac{(\sqrt{1+t} - \sqrt{1-t})(\sqrt{1+t} + \sqrt{1-t})}{t \cdot (\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \to 0} \frac{(\sqrt{1+t})^2 - (\sqrt{1-t})^2}{t \cdot (\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \to 0} \frac{(1+t) - (1-t)}{t \cdot (\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \to 0} \frac{\cancel{1} + t - \cancel{1} + t}{t \cdot (\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \to 0} \frac{2\cancel{t}}{\cancel{t} \cdot (\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \to 0} \frac{2}{\sqrt{1+t} + \sqrt{1-t}}$$

$$= \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{1+1} = \frac{2}{2}$$

$$= 1$$

d) $\lim_{x \to 0} \dfrac{\sqrt{3+x} - \sqrt{3}}{x} = \dfrac{\sqrt{3} - \sqrt{3}}{0} = \dfrac{0}{0}$

$$\lim_{x \to 0} \frac{(\sqrt{3+x} - \sqrt{3})}{x} \times \frac{(\sqrt{3+x} + \sqrt{3})}{(\sqrt{3+x} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{(\sqrt{3+x} - \sqrt{3})(\sqrt{3+x} + \sqrt{3})}{x(\sqrt{3+x} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{(\sqrt{3+x})^2 - (\sqrt{3})^2}{x(\sqrt{3+x} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{\cancel{3} + x - \cancel{3}}{x(\sqrt{3+x} + \sqrt{3})} = \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x} \cdot (\sqrt{3+x} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} = \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2(3)} = \frac{\sqrt{3}}{6}$$

e) $\lim_{x \to 0} \frac{x}{\sqrt{1+3x} - 1} = \frac{0}{\sqrt{1} - \sqrt{1}} = \frac{0}{0}$

$$\lim_{x \to 0} \frac{x}{(\sqrt{1+3x} - 1)} \times \frac{(\sqrt{1+3x} + 1)}{(\sqrt{1+3x} + 1)}$$

$$\lim_{x \to 0} \frac{x(\sqrt{1+3x} + 1)}{(\sqrt{1+3x} - 1)(\sqrt{1+3x} + 1)}$$

$$\lim_{x \to 0} \frac{x \cdot (\sqrt{1+3x} + 1)}{(\sqrt{1+3x})^2 - (1)^2}$$

$$\lim_{x \to 0} \frac{x \cdot (\sqrt{1+3x} + 1)}{\cancel{1} + 3x \cancel{- 1}}$$

$$\lim_{x \to 0} \frac{\cancel{x} \cdot (\sqrt{1+3x} + 1)}{3\cancel{x}}$$

$$\lim_{x \to 0} \frac{(\sqrt{1+3x} + 1)}{3} = \frac{\sqrt{1} + 1}{3}$$

$$= \frac{1+1}{3}$$

$$= \frac{2}{3}$$

f) $\lim_{x \to 16} \frac{4-\sqrt{x}}{16x - x^2} = \frac{4-\sqrt{16}}{16(16)-(16)^2} = \frac{4-4}{256-256}$

$= \frac{0}{0}$

$\lim_{x \to 16} \frac{(4-\sqrt{x})}{(16x-x^2)} \times \frac{(4+\sqrt{x})}{(4+\sqrt{x})}$

$\lim_{x \to 16} \frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}$

$\lim_{x \to 16} \frac{(4)^2-(\sqrt{x})^2}{(16x-x^2)(4+\sqrt{x})}$

$\lim_{x \to 16} \frac{(16-x)}{(16x-x^2)(4+\sqrt{x})}$

$\lim_{x \to 16} \frac{\cancel{(16-x)}}{x.\cancel{(16-x)}(4+\sqrt{x})}$

$\lim_{x \to 16} \frac{1}{x(4+\sqrt{x})} = \frac{1}{16(4+\sqrt{16})}$

$= \frac{1}{16(4+4)}$

$= \frac{1}{16(8)} = \frac{1}{128}$

9) $\lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4} = \frac{\sqrt{16+9} - 5}{-4+4} = \frac{\sqrt{25} - 5}{0}$

$= \frac{5-5}{0} = \frac{0}{0}$

$$\lim_{x \to -4} \frac{(\sqrt{x^2+9} - 5)}{(x+4)} \times \frac{(\sqrt{x^2+9} + 5)}{(\sqrt{x^2+9} + 5)}$$

$$\lim_{x \to -4} \frac{(\sqrt{x^2+9} - 5)(\sqrt{x^2+9} + 5)}{(x+4) \cdot (\sqrt{x^2+9} + 5)}$$

$$\lim_{x \to -4} \frac{(\sqrt{x^2+9})^2 - (5)^2}{(x+4) \cdot (\sqrt{x^2+9} + 5)}$$

$$\lim_{x \to -4} \frac{x^2+9-25}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$\lim_{x \to -4} \frac{(x^2-16)}{(x+4)(\sqrt{x^2+9} + 5)} = \lim_{x \to -4} \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9} + 5}$$

$$= \frac{-4-4}{\sqrt{16+9} + 5} = \frac{-8}{\sqrt{25} + 5}$$

$$= \frac{-8}{5+5} = \frac{-8 \div 2}{10 \div 2}$$

$$= -\frac{4}{5}$$

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i) $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) = \frac{1}{0} - \frac{1}{0}$ كمية غير محددة

$$\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) = \lim_{t \to 0} \left(\frac{1 - \sqrt{1+t}}{t\sqrt{1+t}} \right) = \frac{0}{0}$$

$$= \lim_{t \to 0} \frac{(1 - \sqrt{1+t})(1 + \sqrt{1+t})}{t\sqrt{1+t}\,(1 + \sqrt{1+t})}$$

$$= \lim_{t \to 0} \frac{(1)^2 - (\sqrt{1+t})^2}{t\sqrt{1+t}\,(1 + \sqrt{1+t})}$$

$$= \lim_{t \to 0} \frac{1 - (1+t)}{t\sqrt{1+t}\,(1 + \sqrt{1+t})}$$

$$= \lim_{t \to 0} \frac{\cancel{1} - \cancel{1} - t}{t\sqrt{1+t}\,(1 + \sqrt{1+t})}$$

$$= \lim_{t \to 0} \frac{-\cancel{t}}{\cancel{t}\sqrt{1+t}\,(1 + \sqrt{1+t})}$$

$$= \lim_{t \to 0} \frac{-1}{\sqrt{1+t}\,(1 + \sqrt{1+t})}$$

$$= \frac{-1}{\sqrt{1}\,(1+\sqrt{1})} = \frac{-1}{1(1+1)} = \frac{-1}{1(2)}$$

$$= -\frac{1}{2}$$

j) $\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} = \frac{\sqrt{6-2} - 2}{\sqrt{3-2} - 1} = \frac{2-2}{1-1} = \frac{0}{0}$

$$\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} \times \frac{\sqrt{6-x} + 2}{\sqrt{6-x} + 2}$$

$$\lim_{x \to 2} \frac{(\sqrt{6-x} - 2)(\sqrt{6-x} + 2)}{(\sqrt{6-x} + 2)(\sqrt{3-x} - 1)} = \lim_{x \to 2} \frac{(\sqrt{6-x})^2 - 2^2}{(\sqrt{6-x} + 2)(\sqrt{3-x} - 1)}$$

$$\lim_{x \to 2} \frac{6-x-4}{(\sqrt{6-x} + 2)(\sqrt{3-x} - 1)} = \lim_{x \to 2} \frac{2-x}{(\sqrt{6-x} + 2)(\sqrt{3-x} - 1)}$$

$$\lim_{x \to 2} \frac{(2-x)}{(\sqrt{6-x} + 2)(\sqrt{3-x} - 1)} \times \frac{\sqrt{3-x} + 1}{\sqrt{3-x} + 1}$$

$$\lim_{x \to 2} \frac{(2-x)(\sqrt{3-x} + 1)}{(\sqrt{6-x} + 2)(\sqrt{3-x} - 1)(\sqrt{3-x} + 1)}$$

$$\lim_{x \to 2} \frac{(2-x)(\sqrt{3-x} + 1)}{(\sqrt{6-x} + 2)((\sqrt{3-x})^2 - 1^2)}$$

$$\lim_{x \to 2} \frac{(2-x)(\sqrt{3-x} + 1)}{(\sqrt{6-x} + 2)(3-x-1)}$$

$$\lim_{x \to 2} \frac{\cancel{(2-x)}(\sqrt{3-x} + 1)}{(\sqrt{6-x} + 2)\cancel{(2-x)}} = \frac{\sqrt{3-2} + 1}{\sqrt{6-2} + 2} = \frac{1+1}{2+2}$$

$$= \frac{2}{4} = \frac{1}{2}$$

Example (8)

a) $\lim\limits_{x \to 0} \sqrt[3]{8-x}$

$D_{\sqrt[3]{8-x}} = \mathbb{R}$

$\lim\limits_{x \to 0} \sqrt[3]{8-x} = \sqrt[3]{8} = \sqrt[3]{2^3} = 2^{3/3} = 2$

b) $\lim\limits_{x \to 0} \sqrt[5]{x-32}$

$D_{\sqrt[5]{x-32}} = \mathbb{R}$

$\lim\limits_{x \to 0} \sqrt[5]{x-32} = \sqrt[5]{-32} = \sqrt[5]{(-2)^5} = (-2)^{5/5} = -2$

c) $\lim\limits_{x \to 7} \sqrt[11]{x^2-49}$

$D_{\sqrt[11]{x^2-49}} = \mathbb{R}$

$\lim\limits_{x \to 7} \sqrt[11]{x^2-49} = \sqrt[11]{49-49} = \sqrt[11]{0} = 0$

d) $\lim\limits_{x \to 2} \sqrt[7]{x^3-9}$

$D_{\sqrt[7]{x^3-9}} = \mathbb{R}$

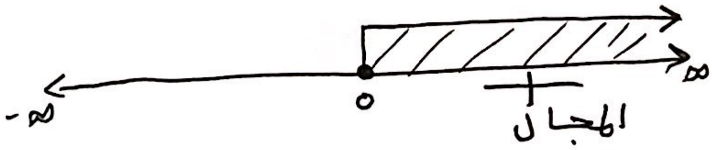
$\lim\limits_{x \to 2} \sqrt[7]{x^3-9} = \sqrt[7]{2^3-9} = \sqrt[7]{8-9} = \sqrt[7]{-1} = -1$

Example (9)

a) $\lim\limits_{x \to 0} \sqrt{x}$

Domain of $\sqrt{x}$: $x > 0 \Rightarrow D_{\sqrt{x}} = [0, \infty)$

$\lim\limits_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$



$\lim\limits_{x \to 0^-} \sqrt{x} = D.N.E$

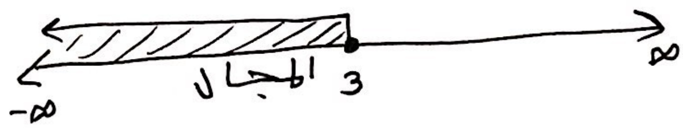
$\because \lim\limits_{x \to 0^+} \sqrt{x} \neq \lim\limits_{x \to 0^-} \sqrt{x}$

$\therefore \lim\limits_{x \to 0} \sqrt{x} = D.N.E$

b) $\lim\limits_{x \to 3} \sqrt[4]{3 - x}$

Domain of $\sqrt[4]{3-x}$: $3 - x \geq 0 \Rightarrow -x \geq -3 \Rightarrow x \leq 3$

$\Rightarrow D_{\sqrt[4]{3-x}} = (-\infty, 3]$



$\lim\limits_{x \to 3^-} \sqrt[4]{3-x} = \sqrt[4]{3-3} = \sqrt[4]{0} = 0$

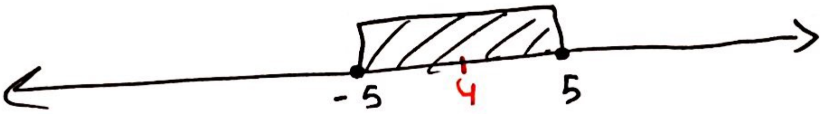
$\lim\limits_{x \to 3^+} \sqrt[4]{3-x} = D.N.E$

$\because \lim\limits_{x \to 3^+} \sqrt[4]{3-x} \neq \lim\limits_{x \to 3^-} \sqrt[4]{3-x}$

$\therefore \lim\limits_{x \to 3} \sqrt[4]{3-x} = D.N.E$

c) $\lim\limits_{x \to 4} \sqrt{25 - x^2}$

Domain of $\sqrt{25 - x^2} = [-5, 5]$



$\because 4 \in [-5, 5]$

$\therefore \lim\limits_{x \to 4} \sqrt{25 - x^2} = \sqrt{25 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$

d) $\lim\limits_{x \to 2} \sqrt[6]{x^2 - 6x + 5}$

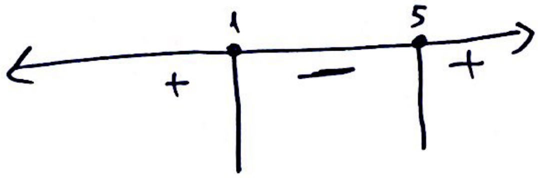
Domain of $\sqrt[6]{x^2 - 6x + 5}$: $x^2 - 6x + 5 \geqslant 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x - 1 = 0$ or $x - 5 = 0$

$x = 1$ $\qquad$ $x = 5$

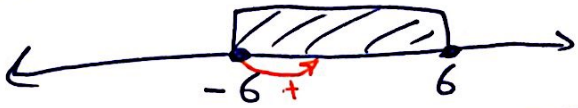


$D_{\sqrt[6]{x^2 - 6x + 5}} = (-\infty, 1] \cup [5, \infty)$

$\because 2 \notin D_{\sqrt[6]{x^2 - 6x + 5}}$ $\quad$ $\therefore \lim\limits_{x \to 2} \sqrt[6]{x^2 - 6x + 5} = \text{D.N.E}$

e) $\lim\limits_{x \to -6^+} \sqrt{36 - x^2}$

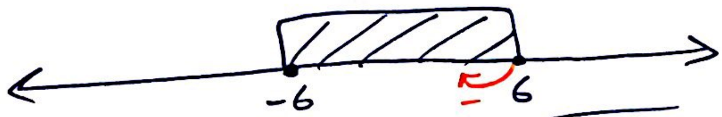
Domain of $\sqrt{36-x^2} = [-6, 6]$



$\lim\limits_{x \to -6^+} \sqrt{36 - x^2} = \sqrt{36-(-6)^2} = \sqrt{36-36} = \sqrt{0} = 0$

f) $\lim\limits_{x \to 6^-} \sqrt{36 - x^2}$

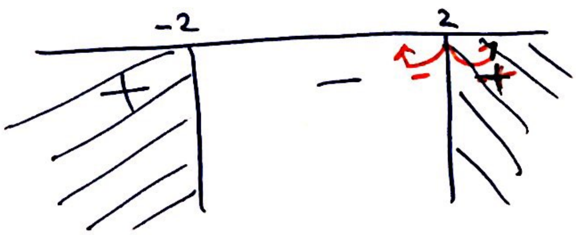
Domain of $\sqrt{36-x^2} = [-6, 6]$



$\lim\limits_{x \to 6^-} \sqrt{36 - x^2} = \sqrt{36-6^2} = \sqrt{36-36} = \sqrt{0} = 0$

g) $\lim\limits_{x \to 2} \sqrt{x^2 - 4}$

$D_{\sqrt{x^2-4}} = (-\infty, -2] \cup [2, \infty)$



$\lim\limits_{x \to 2^+} \sqrt{x^2 - 4} = \sqrt{4-4} = 0$

$\lim\limits_{x \to 2^-} \sqrt{x^2 - 4} = D.N.E$

$\because \lim\limits_{x \to 2^+} \sqrt{x^2-4} \neq \lim\limits_{x \to 2^-} \sqrt{x^2-4} \Rightarrow \lim\limits_{x \to 2} \sqrt{x^2-4} = D.N.E$

Example (10)

a) $\lim\limits_{x \to 2} |x+3| = |2+3| = |5| = 5$

b) $\lim\limits_{x \to 4} |4-x| = |4-4| = |0| = 0$

c) $\lim\limits_{x \to 2} |x^2-5| = |2^2-5| = |4-5| = |-1| = 1$

d) $\lim\limits_{x \to 0} \frac{|x|}{x} = \frac{|0|}{0} = \frac{0}{0}$ كمية غير محددة

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{\substack{x \to 0^+ \\ (x>0)}} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{\substack{x \to 0^- \\ (x<0)}} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$$

$$\because \lim_{x \to 0^+} \frac{|x|}{x} \neq \lim_{x \to 0^-} \frac{|x|}{x}$$

$$\therefore \lim_{x \to 0} \frac{|x|}{x} = \text{D.N.E}$$

e) $\lim_{x \to 3} \frac{|3-x|}{2x-6} = \frac{|3-3|}{2(3)-6} = \frac{|0|}{6-6} = \frac{0}{0}$

$$|3-x| = \begin{cases} 3-x & \text{if} \quad 3-x \geqslant 0 \\ -(3-x) & \text{if} \quad 3-x < 0 \end{cases}$$

$$= \begin{cases} 3-x & \text{if} \quad -x \geqslant -3 \\ -(3-x) & \text{if} \quad -x < -3 \end{cases}$$

$$= \begin{cases} 3-x & \text{if} \quad x \leqslant 3 \\ -(3-x) & \text{if} \quad x > 3 \end{cases}$$

$$\lim_{x \to 3^+} \frac{|3-x|}{2x-6} = \lim_{\substack{x \to 3^+ \\ (x>3)}} \frac{-(3-x)}{2x-6} = \frac{0}{0}$$

$$= \lim_{x \to 3^+} \frac{\cancel{(x-3)}}{2\cancel{(x-3)}} = \lim_{x \to 3^+} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x \to 3^-} \frac{|3-x|}{2x-6} = \lim_{\substack{x \to 3^- \\ (x<3)}} \frac{3-x}{2x-6} = \frac{0}{0}$$

$$\lim_{x \to 3^-} \frac{3-x}{2(x-3)}$$

$$\lim_{x \to 3^-} \frac{-\cancel{(x-3)}}{2\cancel{(x-3)}} = \lim_{x \to 3^-} -\frac{1}{2} = -\frac{1}{2}$$

$$\because \lim_{x \to 3^+} \frac{|3-x|}{2x-6} \neq \lim_{x \to 3^-} \frac{|3-x|}{2x-6}$$

$$\therefore \lim_{x \to 3} \frac{|3-x|}{2x-6} = \text{D.N.E}$$

f) $\lim_{x\to -6^-} \frac{2x+12}{|x+6|} = \frac{2(-6)+12}{|-6+6|} = \frac{-12+12}{|0|} = \frac{0}{0}$

$$|x+6| = \begin{cases} x+6 & \text{if} \quad x+6 \geq 0 \\ -(x+6) & \text{if} \quad x+6<0 \end{cases} = \begin{cases} x+6 & \text{if } x \geq -6 \\ -(x+6) & \text{if } x<-6 \end{cases}$$

$$\lim_{x\to -6^-} \frac{2x+12}{|x+6|} = \lim_{\substack{x\to -6^- \\ (x<-6)}} \frac{2x+12}{-(x+6)} = \frac{0}{0}$$

$$= \lim_{x\to -6^-} \frac{2\cancel{(x+6)}}{-\cancel{(x+6)}} = \lim_{x\to -6^-} -2 = -2$$

g) $\lim_{x\to -2^-} \frac{2-|x|}{2+x} = \frac{2-|-2|}{2-2} = \frac{2-2}{0} = \frac{0}{0}$

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ \boxed{-x \quad \text{if} \quad x<0} \end{cases}$$

$$\lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{\substack{x\to -2 \\ (x<0) \\ -2<0}} \frac{2-(-x)}{2+x} = \lim_{x\to -2} \frac{\cancel{(2+x)}}{\cancel{(2+x)}}$$

$$= \lim_{x\to -2} 1 = 1$$

h) $\lim_{x\to 3} (2x+|x-3|) = 2(3)+|3-3| = 6+0 = 6$

$$\lim_{x \to 2} \frac{x^2+x-6}{|x-2|} = \frac{0}{0}$$

$$\lim_{x \to 2^+} \frac{x^2+x-6}{x-2} = \frac{0}{0}$$

$$\lim_{x \to 2^+} \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}} = \lim_{x \to 2^+} (x+3) = 2+3 = 5$$

$$\lim_{x \to 2^-} \frac{x^2+x-6}{-(x-2)} = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+3)}{-\cancel{(x-2)}}$$

$$= \lim_{x \to 2^-} -(x+3) = -(2+3)$$

$$= -5$$

$$\therefore \lim_{x \to 2^+} \frac{x^2+x-6}{|x-2|} \neq \lim_{x \to 2^-} \frac{x^2+x-6}{|x-2|}$$

$$\therefore \lim_{x \to 2} \frac{x^2+x-6}{|x-2|} = \text{D.N.E}$$

Example (11)

a) if $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x \geq 4 \\ 8-2x & \text{if } x \leq 4 \end{cases}$

then find $\lim\limits_{x \to 2} f(x)$ $= \lim\limits_{x \to 2} 8-2x$

$= 8-2(2)$

$= 8-4$

$= 4$

$\lim\limits_{x \to 13} f(x) = \lim\limits_{x \to 13} \sqrt{x-4} = \sqrt{13-4} = \sqrt{9} = 3$

$\lim\limits_{x \to 4} f(x)$

$\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = \sqrt{0} = 0$

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (8-2x) = 8-2(4) = 8-8 = 0$

$\because \lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^-} f(x)$

$\therefore \lim\limits_{x \to 4} f(x) = 0$

b) $f(x) = \begin{cases} 1+x & \text{if } \underline{x < -1} \\ x^2 & \text{if } -1 \leq x < 1 \\ 2-x & \text{if } \underline{\underline{x \geq 1}} \end{cases}$

$$\lim_{x \to -3} f(x) = \lim_{\substack{x \to -3 \\ -3 < -1}} \underline{1+x} = 1-3 = -2$$

$$\lim_{x \to 5} f(x) = \lim_{\substack{x \to 5 \\ 5 > 1}} \underline{\underline{2-x}} = 2-5 = -3$$

$$\lim_{x \to \frac{1}{3}} f(x) = \lim_{\substack{x \to 1/3 \\ -1 < 1/3 < 1}} x^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\lim_{x \to -1} f(x)$$

$$\lim_{x \to -1^+} f(x) = \lim_{\substack{x \to -1^+ \\ x > -1}} x^2 = (-1)^2 = 1$$

$$\lim_{x \to -1^-} f(x) = \lim_{\substack{x \to -1^- \\ x < -1}} 1+x = 1-1 = 0$$

$$\because \lim_{x \to -1^+} f(x) \neq \lim_{x \to -1^-} f(x)$$

$$\therefore \lim_{x \to -1} f(x) = \text{D.N.E}$$

c) $f(x) = \begin{cases} 1+\sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \sin x & \text{if } x > \frac{\pi}{2} \end{cases}$

$$\lim_{x \to -\frac{\pi}{4}} f(x) = \lim_{x \to -\frac{\pi}{4}} (1+\sin x) = 1 + \sin(-\tfrac{\pi}{4}) = 1 - \sin\tfrac{\pi}{4}$$
$$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \cos x = \cos(\tfrac{\pi}{2}) = 0$$

$$\lim_{x \to \frac{3\pi}{2}} f(x) = \lim_{x \to \frac{3\pi}{2}} \sin x = \sin(\tfrac{3\pi}{2}) = -1$$

$$\lim_{x \to 0} f(x)$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \cos x = \cos(0) = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1+\sin x) = 1 + \sin(0) = 1+0 = 1$$

$$\therefore \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\therefore \lim_{x \to 0} f(x) = 1$$

$$\lim_{x \to \pi} f(x)$$

$$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \sin x = \sin(\pi) = 0$$

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} \cos x = \cos(\pi) = -1$$

$$\therefore \lim_{x \to \pi^+} f(x) \neq \lim_{x \to \pi^-} f(x)$$

$$\therefore \lim_{x \to \pi} f(x) = \text{D.N.E}$$

d) if $g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$

Then $g(1) = 3$

$\lim_{x \to 1} g(x)$

$\lim_{x \to 1^+} g(x) = \lim_{\substack{x \to 1^+ \\ x > 1}} (2 - x^2) = 2 - 1 = 1$

$\lim_{x \to 1^-} g(x) = \lim_{\substack{x \to 1^- \\ x < 1}} (x) = 1$

$\because \lim_{x \to 1^+} g(x) = \lim_{x \to 1^-} g(x) \quad \therefore \lim_{x \to 1} g(x) = 1$

$\lim_{x \to 2} g(x)$

$\lim_{x \to 2^+} g(x) = \lim_{\substack{x \to 2^+ \\ x > 2}} (x - 3) = 2 - 3 = -1$

$\lim_{x \to 2^-} g(x) = \lim_{\substack{x \to 2^- \\ x < 2}} (2 - x^2) = 2 - 4 = -2$

$\because \lim_{x \to 2^+} g(x) \neq \lim_{x \to 2^-} g(x)$

$\therefore \lim_{x \to 2} g(x) = \text{D.N.E}$

$$\lim_{x \to 3} g(x) = \lim_{\substack{x \to 3 \\ 3>2}} (x-3) = 3-3 = 0$$

$$\lim_{x \to -4} g(x) = \lim_{\substack{x \to -4 \\ -4<1}} x = -4$$

$$\lim_{x \to \frac{3}{2}} g(x) = \lim_{\substack{x \to \frac{3}{2} \\ 1<\frac{3}{2}<2}} (2-x^2) = 2-\left(\frac{3}{2}\right)^2$$

$$= 2 - \frac{9}{4}$$

$$= \frac{8-9}{4} = -\frac{1}{4}$$

e) if $g(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$

$$\lim_{\substack{x \to 1 \\ x \neq 1}} g(x) = \lim_{x \to 1} (x+1) = 1+1 = 2$$

$$g(1) = \pi$$

f) if $h(x) = \begin{cases} x^2+3 & \text{if } x \neq 3 \\ 5x-3 & \text{if } x = 3 \end{cases}$

$$\lim_{x \to 3} h(x) = \lim_{x \to 3} (x^2+3) = 3^2+3 = 9+3 = 12.$$

$$h(3) = 5(3)-3 = 15-3 = 12$$

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Example (12)

a) $\lim\limits_{x \to 1} \ln(2-x)$

Domain of $\ln(2-x)$: $2 - x > 0$

$-x > -2$

$x < 2$

$D_{\ln(2-x)} = (-\infty, 2)$

$\because 1 \in D_{\ln(2-x)}$ $\quad \therefore \lim\limits_{x \to 1} \ln(2-x) = \ln(2-1)$

$= \ln(1)$

$= 0$

b) $\lim\limits_{x \to 3} \log_3(9-x^2)$

Domain of $\log_3(9-x^2)$: $9 - x^2 > 0$

$-x^2 > -9$

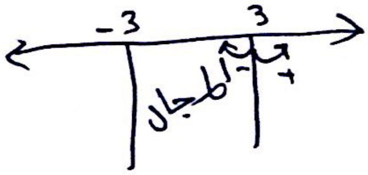
$x^2 < 9$

$\sqrt{x^2} < \sqrt{9}$

$|x| < 3$

$-3 < x < 3$

$D_{\log_3(9-x^2)} = (-3, 3)$



$\lim\limits_{x \to 3^+} \log_3(9-x^2) = D.N.E$

$\lim\limits_{x \to 3^-} \log_3(9-x^2) = \log_3(9-9) = \log_3(0) = -\infty$

$\because \lim_{x \to 3^+} \log_3 (9 - x^2) \neq \lim_{x \to 3^-} \log_3 (9 - x^2)$

$\therefore \lim_{x \to 3} \log_3 (9 - x^2) = D.N.E$

c) $\lim_{x \to 5} \operatorname{Log}_5 (x)$

$D_{\log_5 (x)} = (0, \infty)$

$\because 5 \in (0, \infty)$

$\therefore \lim_{x \to 5} \log_5 x = \log_5 5 = 1$

Example 13

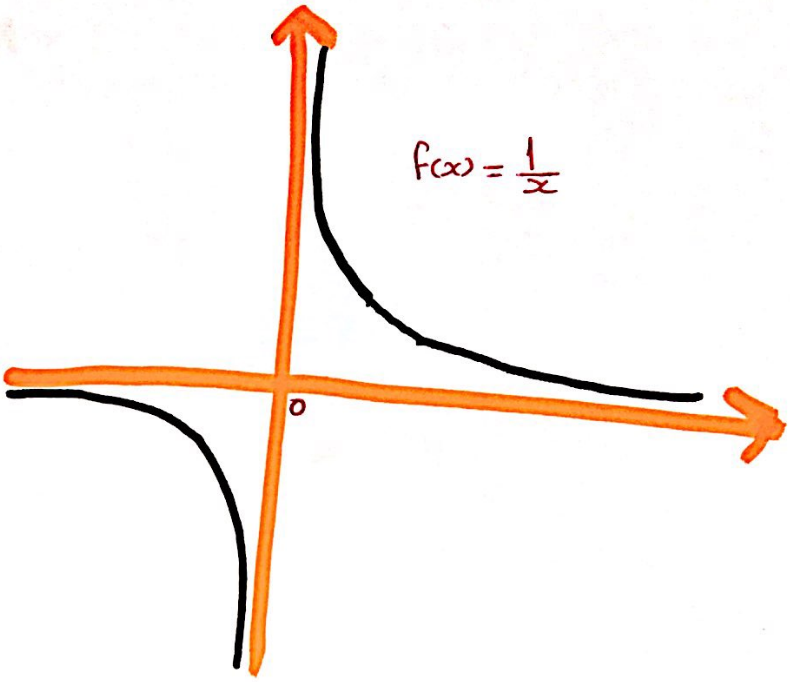
a) $\lim_{x \to 0} \frac{1}{x} = \frac{1}{0}$

$\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0} = \infty$

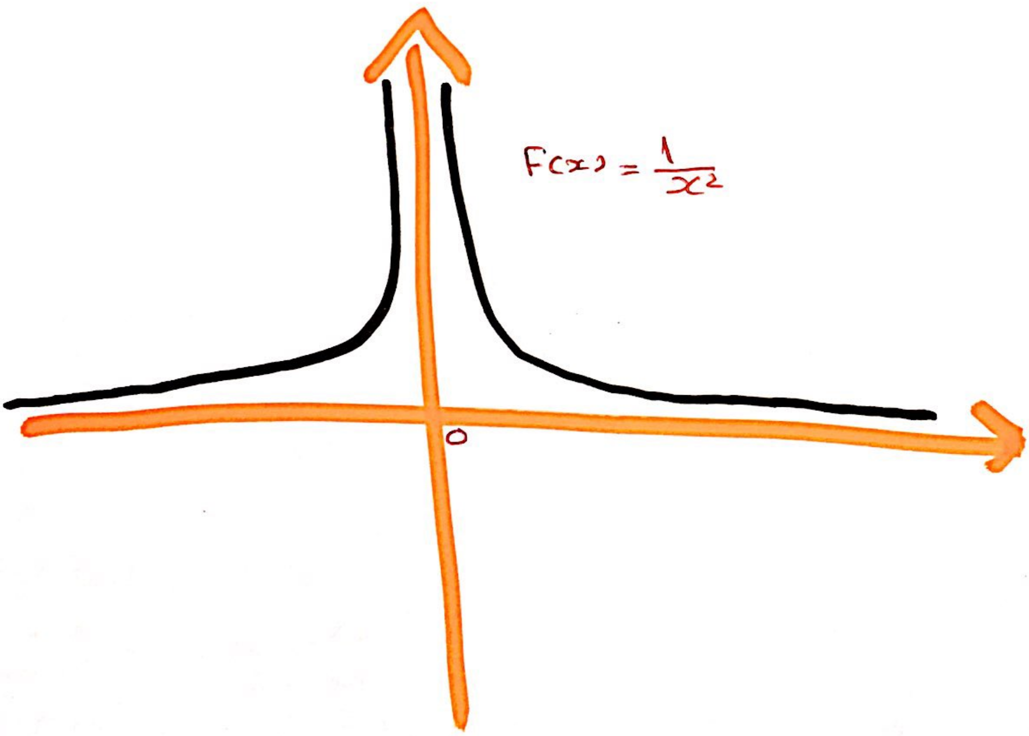
$\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{0} = -\infty$

$\because \lim_{x \to 0^-} \frac{1}{x} \neq \lim_{x \to 0^+} \frac{1}{x}$

$\therefore \lim_{x \to 0} \frac{1}{x} = \text{D.N.E}$



b) $\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0} = \infty$



c) $\lim\limits_{x \to 4^+} \dfrac{3-x}{4-x}$

الطريقة الأولى

$$\lim_{x \to 4^+} \frac{3-x}{4-x} = \frac{3-4}{4-4}$$

$x > 4$

$x = 4.1$

نعوضها في المقام

$4 - 4.1 = -0.1$

$\boxed{\text{الاشارة سالبة}}$

$$= \frac{-1}{0}$$

$$= \frac{-}{\boxed{-}} = +\infty = \infty$$

الطريقة الثانية

$$\lim_{x \to 4^+} \frac{3-x}{4-x} = \frac{3-4}{4-4}$$

$$= \frac{-1}{0}$$

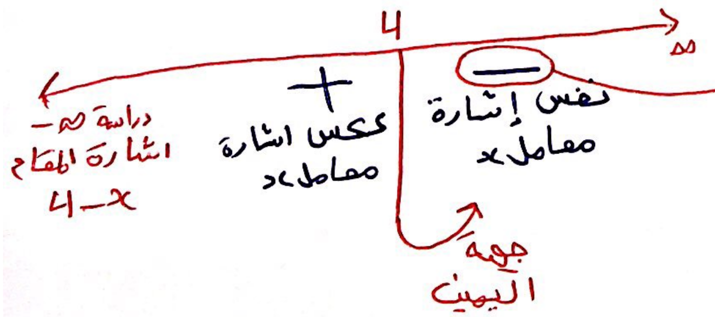
$$= \frac{-}{\boxed{-}} = +\infty = \infty$$

✓ نوجد أصفار المقام :

$4 - x = 0$

$\boxed{x = 4}$

✓ نضع صفر المقام على خط الأعداد



d) $\lim_{x \to 5^+} \frac{x+5}{25-x^2}$

الطريقة الأولى

$$\lim_{x \to 5^+} \frac{x+5}{25-x^2} = \frac{5+5}{25-25}$$

$x > 5$

$$= \frac{10}{0}$$

$x = 5.1$

$$= \frac{+}{\boxed{-}} = -\infty$$

نعوضها في المقام

$25 - (5.1)^2$

$25 - 26.01 = -1.01$

الإشارة سالبة

الطريقة الثانية

$$\lim_{x \to 5^+} \frac{x+5}{25-x^2} = \frac{5+5}{25-25}$$

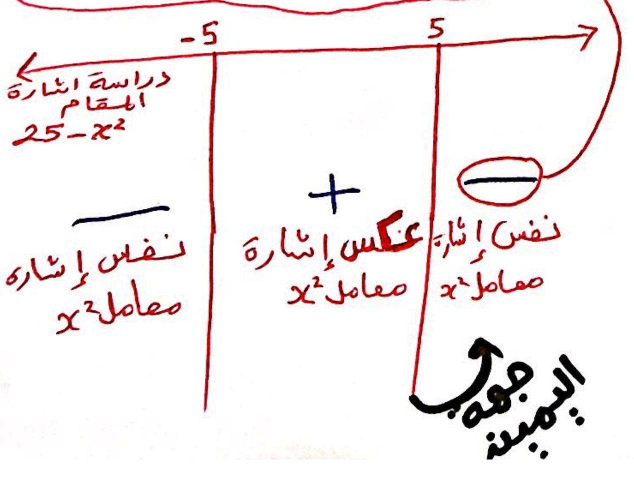
$$= \frac{10}{0}$$

$$= \frac{+}{\boxed{-}} = -\infty$$

✓ نوجد أصفار المقام:

$25 - x^2 = 0 \Rightarrow -x^2 = -25 \Rightarrow x^2 = 25 \Rightarrow \sqrt{x^2} = \sqrt{25} \Rightarrow |x| = 5 \Rightarrow \boxed{x = \pm 5}$

✓ نضع أصفار المقام على خط الأعداد



e) $\lim\limits_{x\to1}\dfrac{x-2}{(x-1)^2}=\dfrac{1-2}{(1-1)^2}$

$=\dfrac{-1}{0}$

بما أنه المقام كله أسه زوجي فإن المقام دائماً موجب
فلا نحتاج دراسة نهاية الدالة من جهة اليمين واليسار
إذن :

$\therefore \lim\limits_{x\to1}\dfrac{x-2}{(x-1)^2}=\dfrac{-1}{0}$

$=\dfrac{-}{+}$

$=-\infty$

f) $\lim\limits_{x\to5}\dfrac{e^x}{(x-5)^3}$

الطريقة الأولى

$\lim\limits_{x\to5^+}\dfrac{e^x}{(x-5)^3}=\dfrac{e^5}{(5-5)^3}$

$x>5$

$x=5.1$

نعوضها في المقام

$(5.1-5)^3=(0.1)^3$

الإشارة موجبة

$=\dfrac{e^5}{0}$

$=\dfrac{+}{\boxed{+}}=+\infty$

$\lim\limits_{x\to5^-}\dfrac{e^x}{(x-5)^3}=\dfrac{e^5}{(5-5)^3}$

$x<5$

$x=4.9$

نعوضها في المقام

$(4.9-5)^3=(-0.1)^3$

$=-(0.1)^3$

الإشارة سالبة

$=\dfrac{e^5}{0}$

$=\dfrac{+}{\boxed{-}}=-\infty$

$\because \lim\limits_{x\to5^+}\dfrac{e^x}{(x-5)^3}\neq\lim\limits_{x\to5^-}\dfrac{e^x}{(x-5)^3}$

$\therefore \lim\limits_{x\to5}\dfrac{e^x}{(x-5)^3}=D.N.E$ #

الطريقة الثانية

$$\lim_{x\to5^+}\frac{e^x}{(x-5)^3}=\frac{e^5}{(5-5)^3}$$

$$=\frac{e^5}{0}$$

$$=\frac{+}{\boxed{+}}$$

$$=+\infty$$

$$\lim_{x\to5^-}\frac{e^x}{(x-5)^3}=\frac{e^5}{(5-5)^3}$$

$$=\frac{e^5}{0}$$

$$=\frac{+}{\boxed{-}}$$

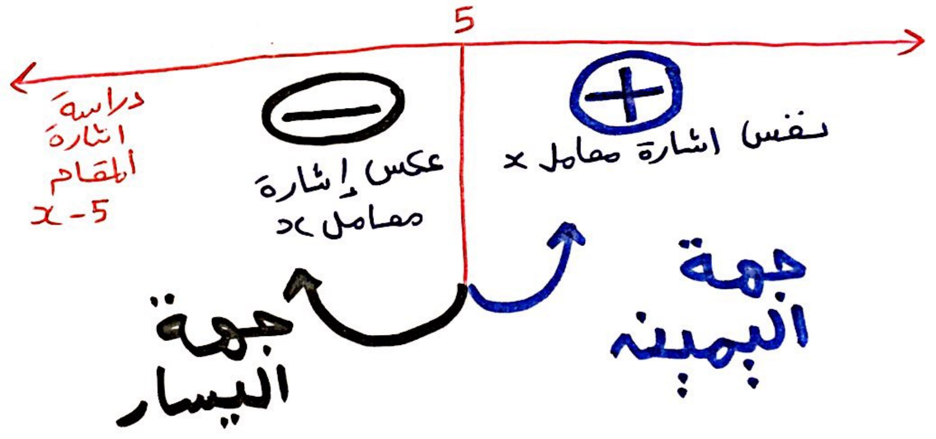
$$=-\infty$$

نوجد أصفار المقام:

$$x-5=0$$

$$x=5$$

نضع أصفار المقام على خط الأعداد



$$\because \lim_{x\to5^+}\frac{e^x}{(x-5)^3}\neq\lim_{x\to5^-}\frac{e^x}{(x-5)^3}$$

$$\therefore \lim_{x\to5}\frac{e^x}{(x-5)^3}=D.N.E$$

h) $\lim\limits_{x \to 2^+} \dfrac{x^2 - 2x - 8}{x^2 - 5x + 6}$

الطريقة الأولى

$$\lim_{x \to 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6} = \frac{4 - 4 - 8}{4 - 10 + 6}$$

$$= \frac{-8}{0}$$

$$= \frac{-}{\boxed{-}} = +\infty$$

$x > 2$

$x = 2.1$

نعوضها في المقام

$x^2 - 5x + 6 = (x-2)(x-3)$

$= (2.1 - 2)(2.1 - 3)$

$= (0.1)(-0.9)$

$= -(0.1)(0.9)$

الاشارة بالسالب

الطريقة الثانية

$$\lim_{x \to 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6} = \frac{4 - 4 - 8}{4 - 10 + 6}$$

$$= \frac{-8}{0}$$

$$= \frac{-}{\boxed{-}} = +\infty$$

نوجد أصفار المقام :

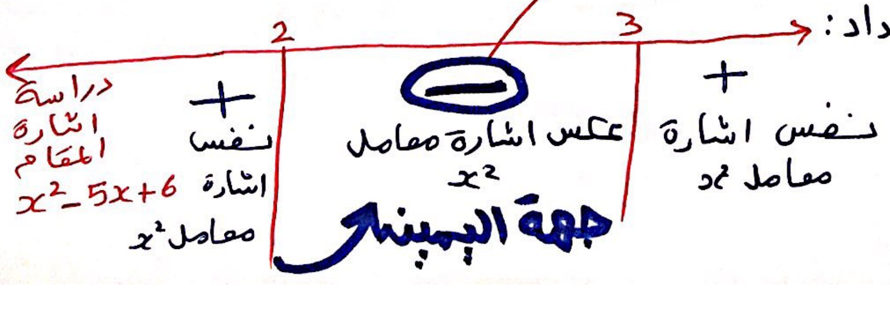
$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x - 2 = 0$ or $x - 3 = 0$

$x = 2$ $\qquad$ $x = 3$

نضع أصفار المقام على خط الأعداد :



$x \to 2^-$ $\quad x^2 - 4x + 4$

الطريقة 1)

$$\lim_{x \to 2^-} \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x-2)} = \lim_{x \to 2^-} \frac{x}{x-2} = \frac{2}{0}$$

$x < 2$

$x = 1.9$

نعوضها في المقام

$x - 2 = 1.9 - 2$

$= -0.1$

الإشارة سالبة

$$= \frac{+}{\boxed{-}} = -\infty$$

الطريقة 2)

$$\lim_{x \to 2^-} \frac{x(x-2)}{(x-2)(x-2)} = \lim_{x \to 2^-} \frac{x}{x-2} = \frac{2}{0}$$

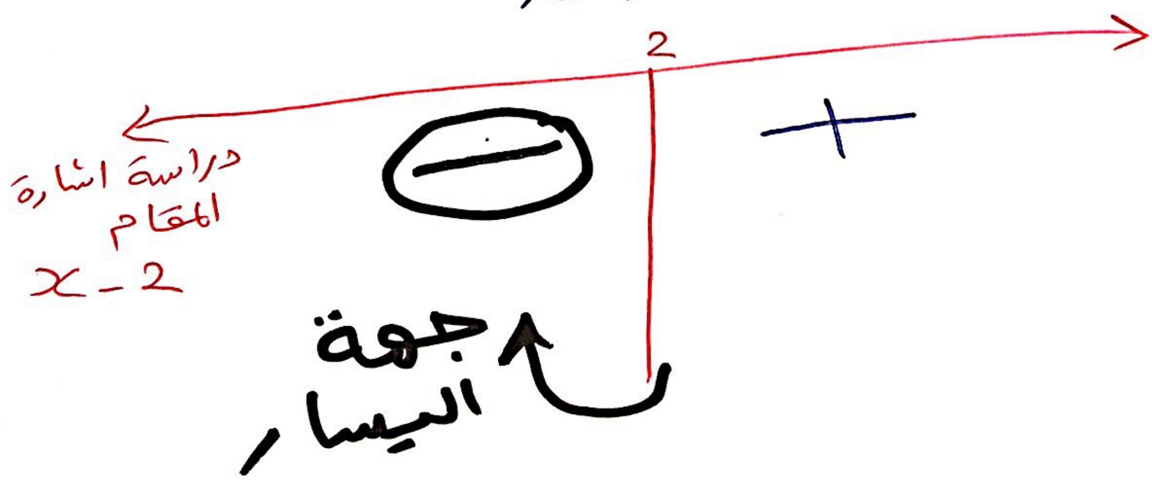
$$= \frac{+}{\boxed{-}} = -\infty$$

✓ نوجد أصفار المقام بعد الإختصار

$x - 2 = 0$

$\boxed{x = 2}$

✓ نضع أصفار المقام على خط الأعداد



i) $\lim\limits_{x \to 2\pi^-} x\csc x = \lim\limits_{x \to 2\pi^-} \dfrac{x}{\sin x}$

$x < 2\pi$

$= \dfrac{2\pi^-}{\sin(2\pi)}$

$x < 360$

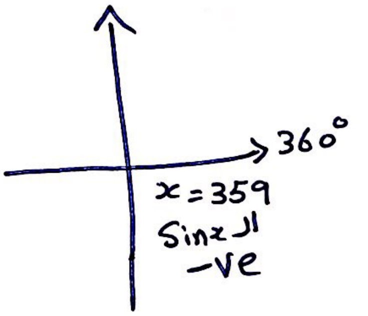
$x = 359$

تقع في ربع الرابع

$= \dfrac{2\pi}{0}$

وإشارة الـ $\sin x$ في المقام سالبة

$= \dfrac{+}{\boxed{-}}$



$= -\infty$

j) $\lim\limits_{x \to \pi^-} \cot x = \lim\limits_{x \to \pi^-} \dfrac{1}{\tan x}$

$x < \pi$

$= \dfrac{1}{\tan(\pi)}$

$x < 180$

$x = 179$

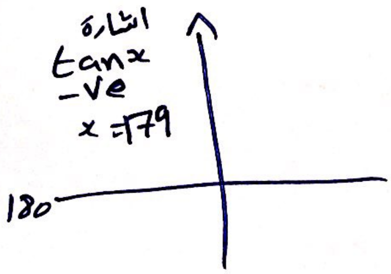
$= \dfrac{1}{0}$

تقع في الربع الثاني

وإشارة الـ $\tan x$ سالبة

$= \dfrac{+}{\boxed{-}}$

$= -\infty$



k) $\lim\limits_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \frac{1}{0} - \frac{1}{0}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ \boxed{-x \quad \text{if } x < 0} \end{cases}$$

$$\lim_{\substack{x \to 0^- \\ x<0}} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{\substack{x \to 0^- \\ x<0}} \left(\frac{1}{x} - \frac{1}{-x} \right)$$

$$= \lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{x} \right)$$

$$= \lim_{\substack{x \to 0^- \\ x<0}} \left(\frac{2}{x} \right)$$

$x=-0.1$ نعوضها في مقام : عدد سالب -٠.١

$$= \frac{+2}{0}$$

$$= \frac{+}{\boxed{-}}$$

$$= -\infty$$

Example(14)

if $\lim\limits_{x \to 5} \dfrac{f(x)-8}{x-1} = 10$ then $\lim\limits_{x \to 5} f(x) = ?$

$$\lim_{x \to 5} \left[\frac{f(x)-8}{x-1}\right] = 10$$

$$\frac{\lim\limits_{x \to 5} (f(x)-8)}{\lim\limits_{x \to 5} (x-1)} = 10$$

$$\frac{\lim\limits_{x \to 5} f(x) - \lim\limits_{x \to 5} 8}{\lim\limits_{x \to 5} x - \lim\limits_{x \to 5} 1} = 10$$

$$\frac{\lim\limits_{x \to 5} f(x) - 8}{5-1} = 10$$

$$\frac{\lim\limits_{x \to 5} f(x) - 8}{4} = 10$$

$$\lim_{x \to 5} f(x) - 8 = 4(10) \Rightarrow \lim_{x \to 5} f(x) = 40 + 8 = 48$$

Theorem

If $f(x) \leq g(x)$ when x is near a

and $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ are exist

then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Note

If $f(x) = g(x)$ when $x \neq a$ and $\lim_{x \to a} f(x)$,

$\lim_{x \to a} g(x)$ are ~~exists~~ exist then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

Theorem: "The Squeeze Theorem"

if $f(x) \leq g(x) \leq h(x)$ ~~then~~ and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} g(x) = L$

Example (15)

a) if $4x-9 \leq f(x) \leq x^2-4x+7 \quad \forall x \geq 0$

then find $\lim\limits_{x\to 4} f(x)$?

$$\lim_{x\to 4}(4x-9) = 4(4)-9 = 16-9 = 7$$

$$\lim_{x\to 4}(x^2-4x+7) = 4^2-4(4)+7 = 16-16+7 = 7$$

$$\because \lim_{x\to 4}(4x-9) = \lim_{x\to 4}(x^2-4x+7) = 7$$

$$\therefore \lim_{x\to 4} f(x) = 7$$

b) if $\log_9 x \leq f(x) \leq \frac{1}{6}x$ then find $\lim\limits_{x\to 3} f(x)$?

" section 1.6 "

$$\lim_{x\to 3}\log_9 x = \log_9 3 = \frac{1}{2}$$

$$\lim_{x\to 3}\frac{1}{6}x = \frac{1}{6}(3) = \frac{\cancel{3}(1)}{\cancel{3}(2)} = \frac{1}{2}$$

$$\because \lim_{x\to 3}\log_9 x = \lim_{x\to 3}\frac{1}{6}x = \frac{1}{2}$$

$$\therefore \lim_{x\to 3} f(x) = \frac{1}{2}$$

c) if $\sin x \leq f(x) \leq \frac{1}{\sqrt{2}}$ then find $\lim_{x \to \frac{\pi}{4}} f(x)$

$$\lim_{x \to \frac{\pi}{4}} \sin(x) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to \frac{\pi}{4}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\because \lim_{x \to \frac{\pi}{4}} \sin(x) = \lim_{x \to \frac{\pi}{4}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{\sqrt{2}}$$

Example (16)

a) $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \cdot \sin\left(\frac{1}{0}\right)$ كمية غير محددة

$\therefore$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\lim_{x \to 0} -x^2 = 0$$

$$\lim_{x \to 0} x^2 = 0$$

$$\therefore \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$$

$$\therefore \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

2.5- Continuity

Definition

A function f is continuous at a number $\boxed{a}$

if $\lim_{x \to a} f(x) = f(a)$

Note

Notice that Definition implicitly requires three things $\boxed{\text{if } f \text{ is continuous at } a}$:

1) $f(a)$ is defined

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

Example (1)

Explain why the function is continuous at a number $\boxed{a}$

1) $f(x) = x^2 + \sqrt{7-x}$ $a = 4$

① $f(4) = 4^2 + \sqrt{7-4} = 4^2 + \sqrt{3} = 16 + \sqrt{3}$

$\therefore f(4)$ is defined

② $\lim_{x \to 4} f(x) = \lim_{x \to 4} [x^2 + \sqrt{7-x}]$

$= \lim_{x \to 4} x^2 + \lim_{x \to 4} \sqrt{7-x}$

$= (4)^2 + \sqrt{7-4}$

$= 16 + \sqrt{3}$

③ $\lim_{x \to 4} f(x) = f(4)$

$\therefore$ from ①, ② and ③ we get $f(x)$ is continuous at 4

2) $f(x) = (x + 2x^3)^4$ $a = -1$

① $f(-1) = (-1 + 2(-1)^3)^4 = (-1 - 2)^4 = (-3)^4 = 81$

$\therefore f(-1)$ is defined

② $\lim_{x \to -1} f(x) = \lim_{x \to -1} (x + 2x^3)^4 = (-1 + 2(-1)^3)^4 = (-1-2)^4 = (-3)^4 = 81$

③ $\lim_{x \to -1} f(x) = 81 = f(-1)$

$\therefore$ from ①, ② and ③ we get $f(x)$ is continuous at -1

3) $h(t) = \dfrac{2t - 3t^2}{1 + t^3}$ $\quad a = 1$

① $h(1) = \dfrac{2(1) - 3(1)^2}{1 + (1)^3} = \dfrac{2 - 3}{1 + 1} = \dfrac{-1}{2}$ is defined

② $\lim\limits_{t \to 1} h(t) = \lim\limits_{t \to 1} \dfrac{2t - 3t^2}{1 + t^3} = \dfrac{2(1) - 3(1)^2}{1 + (1)^3} = \dfrac{2 - 3}{1 + 1} = -\dfrac{1}{2}$

③ $\lim\limits_{t \to 1} h(t) = -\dfrac{1}{2} = h(1)$

$\therefore$ from ①, ② and ③ we get $h(t)$ is continuous at $a = 1$

4) $G(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$ $\quad a = 3$

① $G(3) = 7$ is defined

② $\lim\limits_{x \to 3} G(x) = \lim\limits_{x \to 3} \dfrac{2x^2 - 5x - 3}{x - 3} = \dfrac{0}{0}$

$= \lim\limits_{x \to 3} \dfrac{(2x+1)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim\limits_{x \to 3} (2x+1) = 2(3) + 1$

$= 6 + 1 = 7$

③ $\lim\limits_{x \to 3} G(x) = 7 = G(3)$

$\therefore$ from ①, ② and ③ we get $G(x)$ is continuous at $a = 3$

5) $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 1 - x^2 & \text{if } x \geq 0 \end{cases}$ $a = 0$

① $f(0) = 1 - (0)^2 = 1 - 0 = 1$ is defind

② $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1 - x^2) = 1 - 0 = 1 = f(0)$

$\therefore$ $f(x)$ is continuous at $a = 0$ from the right

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x = \cos(0) = 1 = f(0)$

$\therefore$ $f(x)$ is continuous at $a = 0$ from the left

$\because \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

$\therefore \lim\limits_{x \to 0} f(x) = 1$ exist

③ $\lim\limits_{x \to 0} f(x) = 1 = f(0)$

from ①, ② and ③ we get $f(x)$ is continuous at $a = 0$

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Example (2)

Explain why the function is discontinuous at number $\boxed{a}$

1) $f(x) = \dfrac{1}{x+2}$ $\quad a=-2$

① $f(-2) = \dfrac{1}{-2+2} = \dfrac{1}{0}$

$\therefore$ $f(-2)$ is not defind

$\because$ $f(-2)$ is undefined.

$\therefore$ $f(x)$ is discontinuous at $a=-2$

2) $f(x) = \dfrac{x^2-x-2}{x-2}$ $\quad a=2$

① $f(2) = \dfrac{2^2-2-2}{2-2} = \dfrac{4-4}{2-2} = \dfrac{0}{0}$

$\therefore$ $f(+2)$ is not defined.

$\because$ $f(+2)$ is undefined.

$\therefore$ $f(x)$ is discontinuous at $a=2$

3) $f(x) = \sqrt[6]{49-x^2}$ $\quad a=8$

$f(8) = \sqrt[6]{49-64} = \sqrt[6]{-15} \notin \mathbb{R}$

$\therefore$ $f(8)$ is undefined

$\Rightarrow$ $f(x)$ is discontinuous at $a=8$

4) $f(x) = \sqrt{x-5}$ at $a = 5$

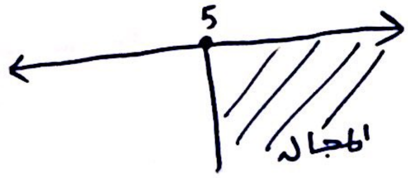
① $f(5) = \sqrt{5-5} = \sqrt{0} = 0$ is defined.

② $\lim\limits_{x \to 5} \sqrt{x-5}$

Domain of $\sqrt{x-5}$: $x - 5 \geq 0$

$x \geq 5$

$\therefore D_{\sqrt{x-5}} = [5, \infty)$



✓ $\lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5^+} \sqrt{x-5} = \sqrt{5-5} = \sqrt{0} = 0 = f(5)$

$\therefore$ $f(x)$ is continuous at 5 from the right

✓ $\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} \sqrt{x-5} = D.N.E$

$\therefore$ $f(x)$ is discontinuous at 5 from the left

$\because \lim\limits_{x \to 5^+} f(x) \neq \lim\limits_{x \to 5^-}$

$\therefore \lim\limits_{x \to 5} f(x) = D.N.E$

$\Longrightarrow$ $f(x)$ is discontinuous at 5

5) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ $a=0$

① $f(0) = 1$ defined

② $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0} = \frac{+}{+} = +\infty$ D.N.E

$\Rightarrow$ $f(x)$ is discont at $a=0$

6) $f(x) = \begin{cases} \frac{-1}{(x-4)^3} & \text{if } x \neq 4 \\ \frac{1}{2} & \text{if } x = 4 \end{cases}$

① $f(4) = \frac{1}{2}$ defined

② $\lim_{x \to 4^+} f(x) = \lim_{\substack{x \to 4^+ \\ x>4 \\ x=4.1}} \frac{-1}{(x-4)^3} = \frac{-1}{0} = \frac{-}{+} = -\infty$

نعوضها في المقام: $(4.1-4)^3$ الناتج $+$

$\lim_{x \to 4^-} f(x) = \lim_{\substack{x \to 4^- \\ x<4 \\ x=3.9}} \frac{-1}{(x-4)^3} = \frac{-1}{0} = \frac{-}{-} = +\infty$

نعوضها في المقام: $(3.9-4)^3$ الناتج $-$

$\therefore \lim_{x \to 4} f(x) = $ D.N.E " Since $\lim_{x \to 4^+} f(x) \neq \lim_{x \to 4^-} f(x)$ "

$\Rightarrow$ $f(x)$ is discont at $x=4$

$\Longrightarrow f(x)$ is discontinuous at $a=4$

7) $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $a=1$

① $f(1) = 1$ is defined

② $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2-x}{x^2-1} = \dfrac{0}{0}$

$= \lim\limits_{x\to 1} \dfrac{x(x-1)}{(x-1)(x+1)} = \lim\limits_{x\to 1} \dfrac{x}{x+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\therefore \lim\limits_{x\to 1} f(x) = \dfrac{1}{2}$ exist

③ $\lim\limits_{x\to 1} f(x) = \dfrac{1}{2} \neq f(1)$

$\Longrightarrow f(x)$ is discontinuous at $a=1$

8) $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ $a=0$

① $f(0) = 0^2 = 0$ is defined

② $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 = 0^2 = 0 = f(0)$

$\therefore f(x)$ is continuous at $a=0$ from the right

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} e^x = e^0 = 1 \neq f(0)$

$\therefore f(x)$ is discontinuous at $a=0$ from the left

$\because \lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x) \Longrightarrow f(x)$ is discontinuous at $a=0$

$\therefore \lim\limits_{x\to 0} f(x) = D.N.E$

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Note

1) $f(x)$ is continuous at a from the right
if $\lim_{x \to a^+} f(x) = f(a)$.

2) $f(x)$ is continuous from the left at number a
if $\lim_{x \to a^-} f(x) = f(a)$

3) if $f(x)$ is continuous at then $f(x)$ is continuous at a from the left and from the right

i.e. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

4) if $f(x)$ is discontinuous at a then

* $f(a)$ is not defined.

or

* $\lim_{x \to a} f(x) \neq f(a)$

or

* $\lim_{x \to a} f(x) = D.N.E$

or

* $f(x)$ is discontinuous at a from the left ~~only~~

or

* $f(x)$ is discontinuous at a from the right

or

* $f(x)$ is discontinuous at a from the left and from the righ.

Continuity on the Interval

1) $f(x)$ is continuous on $[a,b]$ if

- $f(x)$ is continuous on (a,b)
  i.e :- $f(x)$ is continuous at every number in the interval (a,b)
- $f(x)$ is continuous at a number $\boxed{a}$ from the right but $f(x)$ is discontinuous at $\boxed{a}$
- $f(x)$ is continuous at a number $\boxed{b}$ from the left but $f(x)$ is discontinuous at $\boxed{b}$

2) $f(x)$ is continuous on (a,b) if

- $f(x)$ is continuous at every number in the interval (a,b)

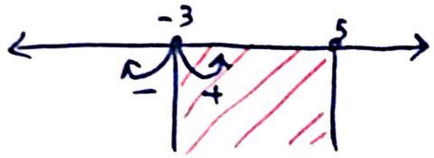
Note

- $f(x)$ is discontinuous at a since: $f(a)$ is not defined or $a \notin (a,b)$
- $f(x)$ is discontinuous at b since: $f(b)$ is not defined or $b \notin (a,b)$

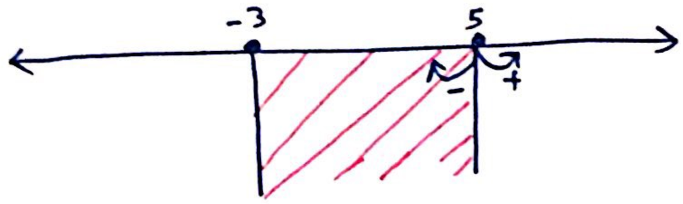
Example (3)

① if $f(x)$ is continuous on $[-3, 5]$ then

ⓐ $f(x)$ is continuous at number -3 from the right but discontinuous from the left



ⓑ $f(x)$ is continuous at number 5 from the left but discontinuous from the right



ⓒ From ⓐ we get $f(x)$ is discontinuous at number -3.

ⓓ from ⓑ we get $f(x)$ is discontinuous at number 5

ⓔ $f(x)$ is continuous at every number in the interval $(-3, 5)$

For example. $f(x)$ is continuous at 2 since $2 \in (-3, 5)$

$f(x)$ is continuous at -1 since $-1 \in (-3, 5)$

$f(x)$ is continuous at 0 since $0 \in (-3, 5)$

$f(x)$ is discontinuous at 6 since $6 \notin (-3, 5)$

$f(x)$ is discontinuous at -4 since $-4 \notin (-3, 5)$

② if f(x) is continuous on (-5, 0) then
f(x) is continuous at -5 from the right

✓ X

③ if f(x) is continuous on (-5, 0) then
f(x) is discontinuous at 0

✓ X

④ if f(x) is continuous on (-5, 0) then
f(x) is continuous at +1

✓ X

⑤ if f(x) is continuous on (-5, 0) then
f(x) is discontinuous at -8

✓ X

6) if f(x) is continuous on (-5, 0) then
f(x) is continuous at -3 ✓ X

Theorem

If f and g are Continuous at a and c is constant then the following functions are Continuous at a

1] $f+g$ 2] $f-g$ 3] cf 4] fg

5) $\frac{f}{g}$ if $g(a) \neq 0$

Note

If f and g are continuous on interval I then the following functions are continuous on interval I

1] $f+g$ 2] $f-g$ 3] cf $\forall c \in \mathbb{R}$ 4] fg

5] $\frac{f}{g}$ if $g(x) \neq 0$

Theorem

If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Theorem

a) Any polynomial is continuous everywhere

i.e Any polynomial is continuous on $\mathbb{R} = (-\infty, \infty)$

b) Any rational function is continuous on the Domain.

c) The following Types of functions are Continuous at every number in their Domains:

- Polynomails functions
- rational functions
- root functions
- Radical functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions
- algabric functions
- not algabric function

Example (4)

① $\lim_{x \to 2} \tan^{-1}\left(\frac{x^2-4}{3x^2-6x}\right)$

$$\tan^{-1}\left(\lim_{x \to 2} \frac{x^2-4}{3x^2-6x}\right)$$

$$\tan^{-1}\left(\lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{3x\cancel{(x-2)}}\right)$$

$$\tan^{-1}\left(\lim_{x \to 2} \frac{x+2}{3x}\right)$$

$$\tan^{-1}\left(\frac{2+2}{3(2)}\right).$$

$$\tan^{-1}\left(\frac{4}{3(2)}\right)$$

$$\tan^{-1}\left(\frac{2}{3}\right)$$

② $\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x\to} \frac{1-\sqrt{x}}{1-x}\right)$$

$$\sin^{-1}\left(\lim_{x\to 1} \frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{(1)^2-(\sqrt{x})^2}{(1-x)(1+\sqrt{x})}\right)$$

$$\sin^{-1}\left(\lim_{x\to 1} \frac{(1-x)}{(1-x)(1+\sqrt{x})}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right)$$

$$\sin^{-1}\left(\frac{1}{1+\sqrt{1}}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

3] $\lim_{x\to 1} e^{x^2-x} = e^{\lim_{x\to 1}(x^2-x)} = e^{1^2-1} = e^{1-1}$

$= e^0 = 1$

4] $\lim_{x\to 0^+} \left(\frac{2}{3}\right)^{\frac{1}{x}} = \left(\frac{2}{3}\right)^{\lim_{x\to 0^+}\frac{1}{x}} = \left(\frac{2}{3}\right)^{\infty} = 0$

5] $\lim_{x\to 0^-} \left(\frac{2}{3}\right)^{\frac{1}{x}} = \left(\frac{2}{3}\right)^{\lim_{x\to 0^-}\frac{1}{x}} = \left(\frac{2}{3}\right)^{-\infty} = \infty$

6] $\lim_{x\to 1^+} 3^{\frac{-1}{x-1}} = 3^{\lim_{x\to 1^+}\frac{-1}{x-1}} = 3^{-\infty} = \frac{1}{3^{\infty}} = \frac{1}{\infty} = 0$

7] $\lim_{x\to 1^-} 3^{\frac{-1}{x-1}} = 3^{\lim_{x\to 1^-}\frac{-1}{x-1}} = 3^{\infty} = \infty$

$\boxed{8}$ $\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$

$$\lim_{x \to 2^+} \tan^{-1}\left(\frac{1}{x-2}\right)$$

$$\tan^{-1}\left(\lim_{x \to 2^+} \frac{1}{x-2}\right) = \tan^{-1}\left(\frac{1}{0}\right)$$

$$\tan^{-1}(\infty) = \frac{\pi}{2}$$

Note

(1) $\frac{\text{عدد}}{\pm\infty} = 0$

(2) if $a > 1$ then $a^{\infty} = \infty$

$a^{-\infty} = 0$

(3) if $0 < a < 1$ then $a^{\infty} = 0$

$a^{-\infty} = \infty$

Example(5)

Where are the following functions Continuous?

$h(x) = \sin(x^2)$

① let $h_1(x) = \sin(x)$ and $h_2(x) = x^2$

② $D_{h_1(x)} = \mathbb{R}$ and $D_{h_2(x)} = \mathbb{R}$

③ $D_{h(x)} = D_{h_1(x)} \cap D_{h_2(x)} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

④ $h(x)$ is continuous on $\mathbb{R}$

$h(x) = \sin^{-1}(2t+1)$

① $-1 \leq 2t+1 \leq 1$

$-1-1 \leq 2t \leq 1-1$

$-\frac{2}{2} \leq \frac{2t}{2} \leq \frac{0}{2}$

$-1 \leq t \leq 0$

② $D_{h(x)} = [-1, 0]$

③ $h(x)$ is cont on $[-1, 0]$

④ $h(x)$ is cont at $x=-1$ from the right and discont at $x=-1$ from the left

⑤ $h(x)$ is cont at $x=0$ from the left and discont at $x=0$ from the right

6) $f(x)$ is discont at $x=0$ and $x=-1$

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$$G(x) = \frac{x}{x^2+5x+6}$$

$D_{G(x)}$: $x^2+5x+6=0$

$(x+2)(x+3)=0$

$x+2=0$ or $x+3=0$

$x=-2$ $\quad$ $x=-3$

$D_{G(x)} = \mathbb{R} - \{-2,-3\}$

$= (-\infty,-3) \cup (-3,-2) \cup (-2,\infty)$

$G(x)$ is continuous on $\mathbb{R} - \{-2,-3\}$

$G(x)$ is discont at $x=-2$ and $x=-3$

$$F(x) = \sqrt[3]{x}\,(1+x^2)$$

$D_{F(x)} = D_{\sqrt[3]{x}} \cap D_{(1+x^2)}$

$= \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

$F(x)$ is continuous on $\mathbb{R}$

$$f(x) = \frac{\sin x}{2 + \cos x}$$

let $f_1(x) = \sin x \Rightarrow D_{f_1(x)} = \mathbb{R}$

$f_2(x) = 2 + \cos x \Rightarrow D_{f_2(x)} = \mathbb{R}$

أصفار المقام :

$$2 + \cos x = 0$$

$$\cos x = -2$$

مستحيل لأنه $-1 \leq \cos x \leq 1$

اذنه لا يوجد أصفار مقام

$D_{f(x)} = D_{f_1(x)} \cap D_{f_2(x)} - \{$ أصفار المقام $\}$

$= \mathbb{R} \cap \mathbb{R}$

$= \mathbb{R}$

$f(x)$ is cont on $\mathbb{R}$

$$f(x) = \frac{\ln(x) + \tan^{-1}(x)}{x^2 - 1}$$

✓ let $f_1(x) = \ln(x) + \tan^{-1}(x)$

$$D_{f_1(x)} = D_{\ln(x)} \cap D_{\tan^{-1}(x)}$$

$$= (0,\infty) \cap \mathbb{R}$$

$$= (0,\infty)$$

✓ let $f_2(x) = x^2 - 1 \Rightarrow D_{f_2(x)} = \mathbb{R}$

✓ أصفار المقام:

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$\sqrt{x^2} = \sqrt{1}$$

$$|x| = 1$$

$$x = \pm 1$$

✓ $D_{f(x)} = D_{f_1(x)} \cap D_{f_2(x)} - \{\text{أصفار المقام}\}$

$$= (0,\infty) \cap \mathbb{R} - \{-1, 1\}$$

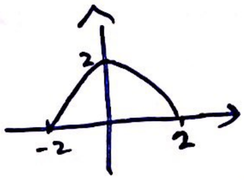
$$= (0,\infty) - \{-1, 1\}$$

$= (0,1) \cup (1,\infty) \Rightarrow f(x)$ is cont on $(0,1) \cup (1,\infty)$

$$h(x) = \frac{\cos x}{\sqrt{4 - x^2}}$$

① Let $h_1(x) = \cos x$ and $h_2(x) = \sqrt{4 - x^2}$

② $D_{h_1(x)} = \mathbb{R}$ and $D_{h_2(x)} = [-2, 2]$



③ أصفار المقام

$4 - x^2 = 0$

$-x^2 = -4$

$x^2 = 4$

$\sqrt{x^2} = \sqrt{4}$

$|x| = 2$

$x = \pm 2$

④ $D_{h(x)} = D_{h_1(x)} \cap D_{h_2(x)} - \{\text{أصفار المقام}\}$

$= \mathbb{R} \cap [-2, 2] - \{-2, 2\}$

$= [-2, 2] - \{-2, 2\}$

$= (-2, 2)$

5) $h(x)$ is cont on $(-2, 2)$

but $h(x)$ is discont at $x = -2$ and $x = 2$

$$f(x) = \frac{\sin x}{x - 1}$$

① Let $f_1(x) = \sin x$ and $f_2(x) = x - 1$

② $D_{f_1(x)} = \mathbb{R}$ and $D_{f_2(x)} = \mathbb{R}$

③ أصفار المقام : $x - 1 = 0 \implies x = 1$

④ $D_{f(x)} = D_{f_1(x)} \cap D_{f_2(x)} - \{\text{أصفار المقام}\} = \mathbb{R} \cap \mathbb{R} - \{1\} = \mathbb{R} - \{1\}$

⑤ $f(x)$ is cont on $(-\infty, 1) \cup (1, \infty)$ but $f(x)$ is discont at 1

$$R(x) = x^2 + \sqrt{2x-1}$$

$$D_{R(x)} = D_{x^2} \cap D_{\sqrt{2x-1}}$$

$$= \mathbb{R} \cap [1/2, \infty)$$

$$= [1/2, \infty)$$

$R(x)$ is continuous on $[1/2, \infty)$

$$g(x) = \tan^{-1}(1 + \sqrt{x})$$

$$D_{g(x)} = D_{\tan^{-1}(x)} \cap D_{1+\sqrt{x}}$$

$$= \mathbb{R} \cap [0, \infty)$$

$$= [0, \infty)$$

$g(x)$ is continuous on $[0, \infty)$

$$f(x) = \sqrt{2x-10}$$

$$2x - 10 \geqslant 0$$

$$2x \geqslant 10$$

$$\frac{2x}{2} \geqslant \frac{10}{2}$$

$$x \geqslant 5$$

$$D_{f(x)} = [5, \infty)$$

$f(x)$ is Cont on $[5, \infty)$

$f(x)$ is discont at $x = 5$ but

$f(x)$ is Cont at 5 from the right and

discont at 5 from the left

$$f(x) = \sqrt{2 + \cos x}$$

$$2 + \cos x \geqslant 0$$

$$\cos x \geqslant -2$$

مستحيل لأن

$-1 \leqslant \cos x \leqslant 1$

$$\therefore D_{f(x)} = \mathbb{R} = (-\infty, \infty)$$

$f(x)$ is Cont on $\mathbb{R}$

$$f(x) = \sqrt{1+\cos x}$$

$$1 + \cos x \geqslant 0$$

$$\cos x \geqslant -1$$

$$-1 \leq \cos x \leq 1$$

$\therefore D_{f(x)} = \mathbb{R}$

$f(x)$ is cont on $\mathbb{R}$

$$F(x) = \ln(1+\cos x)$$

① $1 + \cos x > 0$

$\cos x > -1$

$1 < \cos x \leq 1$

② $1 + \cos x = 0$

$\cos x = -1$

$x = \pm\pi, \pm 3\pi, \pm 5\pi, \dots$

③ $D_{f(x)} = \mathbb{R} - \{\pm\pi, \pm 3\pi, \pm 5\pi, \dots\}$

④ $f(x)$ is continuous on $\mathbb{R} - \{\pm\pi, \pm 3\pi, \pm 5\pi, \dots\}$

but $f(x)$ is discont at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \dots$

$f(x) = \tan x$

$f(x) = \sec x$

$f(x) = \tan x$ and $f(x) = \sec x$

are cont on $\mathbb{R} - \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\}$

but $f(x) = \tan x$ and $f(x) = \sec x$ are

discont at $x = \pm\pi/2, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

$f(x) = \cot x$

$f(x) = \csc x$

$f(x) = \cot x$ and $f(x) = \csc x$ are

cont on $\mathbb{R} - \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \dots\}$

but $f(x) = \cot x$ and $f(x) = \csc x$ are

discont at $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \dots$

Example. $f(x) = \sec x$ is discont at $x = \frac{21\pi}{4}$ (T - F)

$f(x) = \csc x$ is discont at $x = \dots$

$\pi/2$ $\quad$ $\pi/4$ $\quad$ $\pi/6$ $\quad$ 0

$$f(x)=\begin{cases} \dfrac{-1}{(x-6)^4} & \text{if } x\neq 6 \\ 6 & \text{if } x=6 \end{cases}$$

$D_{f(x)} = \mathbb{R}$

$\boxed{x=6}$

① $f(6)=6$ defind

② $\lim\limits_{x\to 6} f(x) = \lim\limits_{x\to 6} \dfrac{-1}{(x-6)} = \dfrac{-1}{0} = \dfrac{-}{\boxed{+}} = -\infty$ "D.N.E"

$\therefore$ $f(x)$ is discontinuous at $x=6$

$\Rightarrow$ $f(x)$ is continuous on $\mathbb{R}-\{6\}$

$$f(x)=\begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x\neq 2 \\ 4 & \text{if } x=2 \end{cases}$$

$D_{f(x)} = \mathbb{R}$

$\boxed{x=2}$

① $f(2)=4$ defined

② $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \dfrac{0}{0}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)} = \lim\limits_{x\to 2}(x+2) = 2+2=4$

③ $\lim\limits_{x\to 2} f(x) = f(2) = 4$ $\Rightarrow$ $f(x)$ is continuous at $x=2$

$\Rightarrow$ $f(x)$ is continuous on $\mathbb{R}$

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$$f(x) = \begin{cases} 1+x^2 & \text{if } x \leq 0 \\ 2-x & \text{if } 0 < x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$

$D_{f(x)} = (-\infty, 0] \cup (0, 2] \cup (2, \infty) = \mathbb{R}$

$\boxed{x = 0}$

✓ $f(0) = 1 + (0)^2 = 1 + 0 = 1$ is defined.

✓ $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (2 - x) = 2 - 0 = 2 \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at 0 from the right

$\Rightarrow$ $f(x)$ is discontinuous at 0

$\boxed{x = 2}$

✓ $f(2) = 2 - 2 = 0$ is defined

✓ $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x - 2)^2 = (2-2)^2 = 0^2 = 0 = f(2)$

$\therefore$ $f(x)$ is continuous at 2 from the right

✓ $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2 - x) = 2 - 2 = 0 = f(2)$

$\therefore$ $f(x)$ is continuous at 2 from the left

$\Rightarrow$ $f(x)$ is continous at 2

$\Rightarrow$ $f(x)$ is continuous on $\mathbb{R} - \{0\}$

$$f(x) = \begin{cases} \tan^{-1}(x) & \text{if } x \leq 1 \\ -\frac{\pi}{4}x + \frac{\pi}{2} & \text{if } x > 1 \end{cases}$$

$D_{f(x)} = (-\infty, 1] \cup (1, \infty) = \mathbb{R}$

$\boxed{x = 1}$

✓ $f(1) = \tan^{-1}(1) = \frac{\pi}{4}$

✓ $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(-\frac{\pi}{4}x + \frac{\pi}{2}\right) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{-\pi + 2\pi}{4} = \frac{\pi}{4}$

$\therefore \lim_{x \to 1^+} f(x) = \frac{\pi}{4} = f(1)$

$\Rightarrow f(x)$ is continuous at $\boxed{1}$ from the right

✓ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \tan^{-1}(x) = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \lim_{x \to 1^-} f(x) = \frac{\pi}{4} = f(1)$

$\Rightarrow f(x)$ is continuous at $\boxed{1}$ from the left

$\because \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = \frac{\pi}{4}$ exist

✓ $\therefore \lim_{x \to 1} f(x) = \frac{\pi}{4} = f(1)$

$\Rightarrow f(x)$ is continuous at $\boxed{1}$

$\Rightarrow f(x)$ is continuous on $\mathbb{R}$.

Examp (6)

For what Value of c is the function

$f(x)$ is Continuous on $\mathbb{R} = (-\infty, \infty)$

① $f(x) = \begin{cases} cx^2 + x^3 & \text{if } x < 2 \\ x^2 - cx & \text{if } x \geq 2 \end{cases}$

$\because$ $f(x)$ is Continuous on $\mathbb{R}$

$\therefore$ $f(x)$ is continuous at $x = 2$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

$$(2)^2 - 2c = (2)^2 c + (2)^3$$

$$4 - 2c = 4c + 8$$

$$-2c - 4c = 8 - 4$$

$$-6c = 4$$

$$\frac{-6c}{-6} = \frac{4 \div 2}{-6 \div 2}$$

$$c = -\frac{2}{3}$$

② $f(x) = \begin{cases} K^2x - 4 & \text{if } x > 1 \\ 12x & \text{if } x \leq 1 \end{cases}$

$\because$ $f(x)$ is continuous on $\mathbb{R}$

$\therefore$ $f(x)$ is continuous at $x = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

$$\lim_{x \to 1^+} (K^2x - 4) = \lim_{x \to 1^-} 12x$$

$$K^2 - 4 = 12$$

$$K^2 = 12 + 4$$

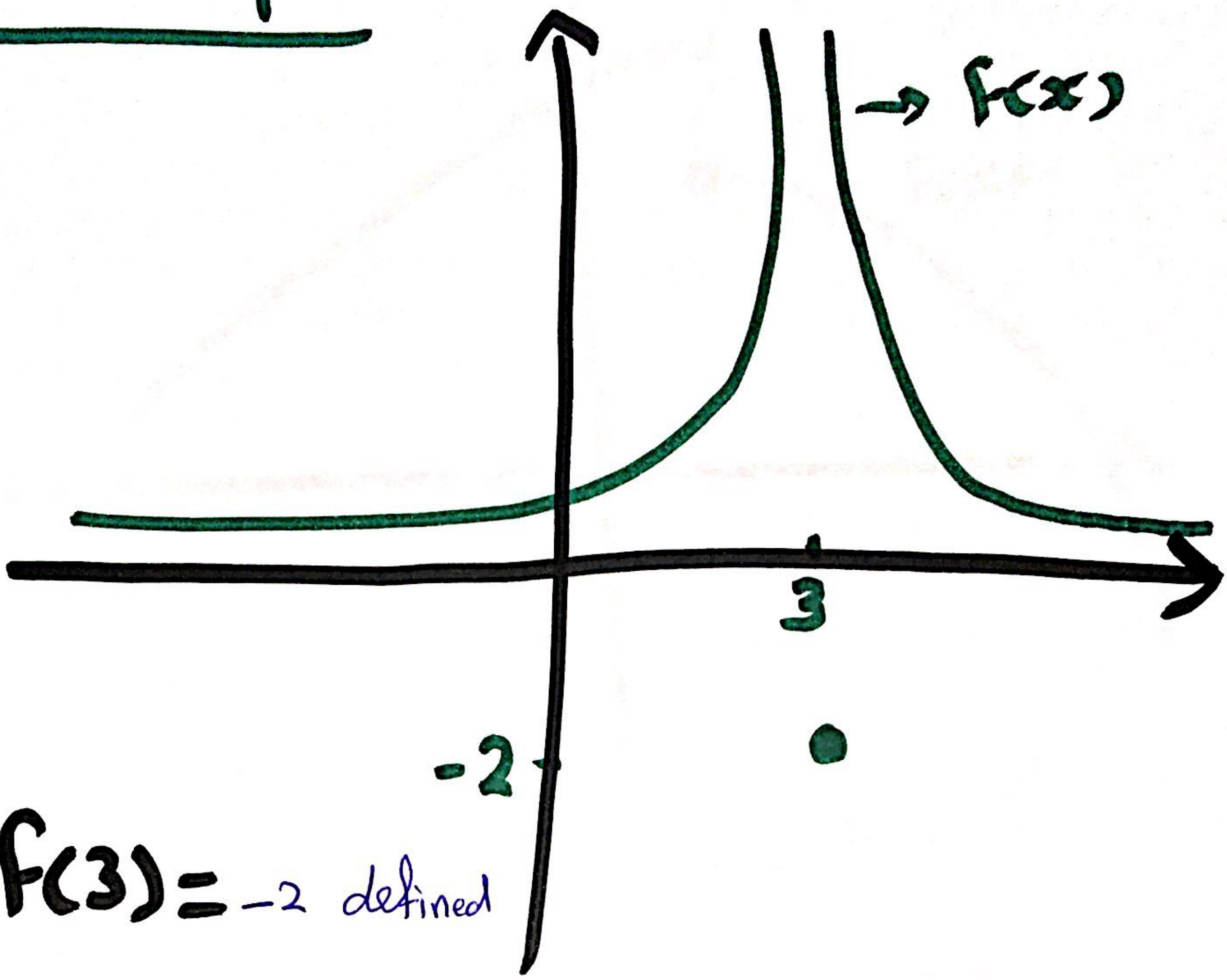
$$K^2 = 16$$

$$\sqrt{K^2} = \sqrt{16}$$

$$|K| = 4$$

$$\boxed{K = \pm 4}$$

Example



① $f(3) = -2$ defined

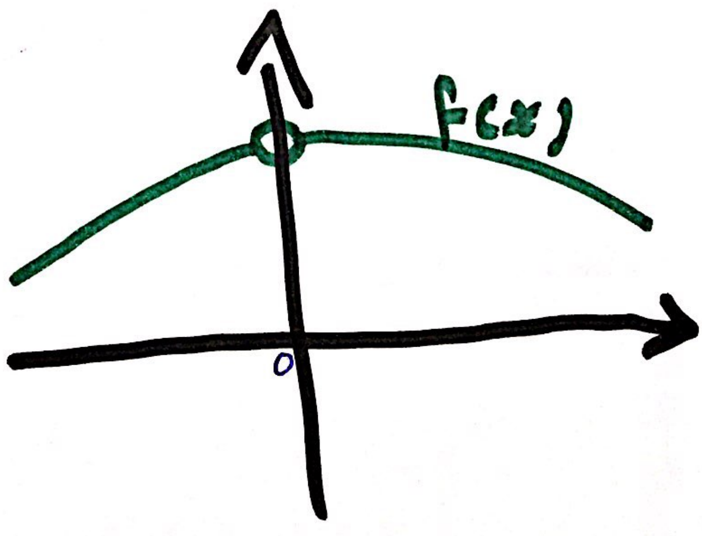
② $\lim_{x \to 3} f(x) = \infty$ d.N.E

From ① and ② we get: f(x) is discont at $x=3$

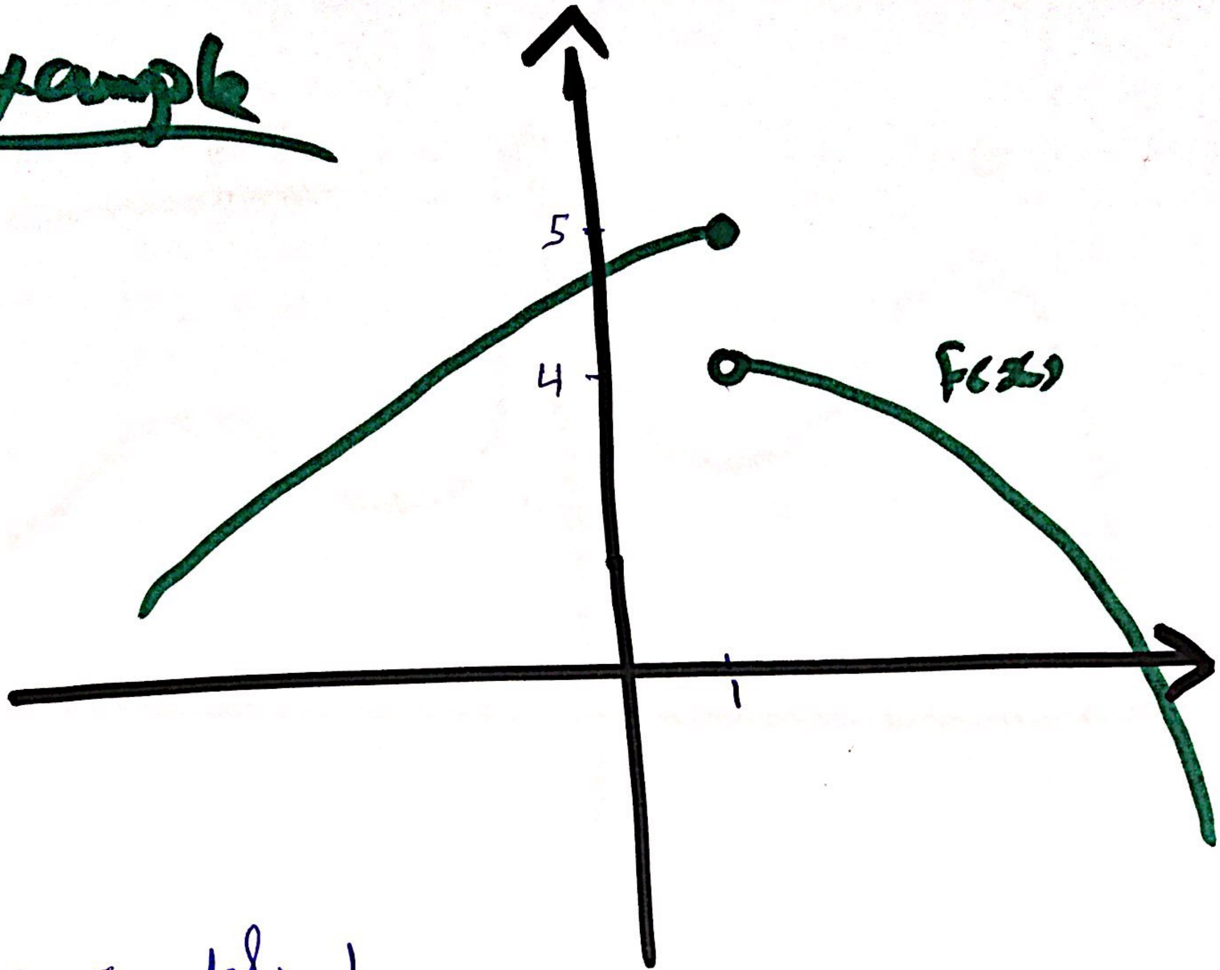
Example

f(0) = Undefined

∴ f(x) is discont at $x=0$



Example



$f(1) = 5$ defind.

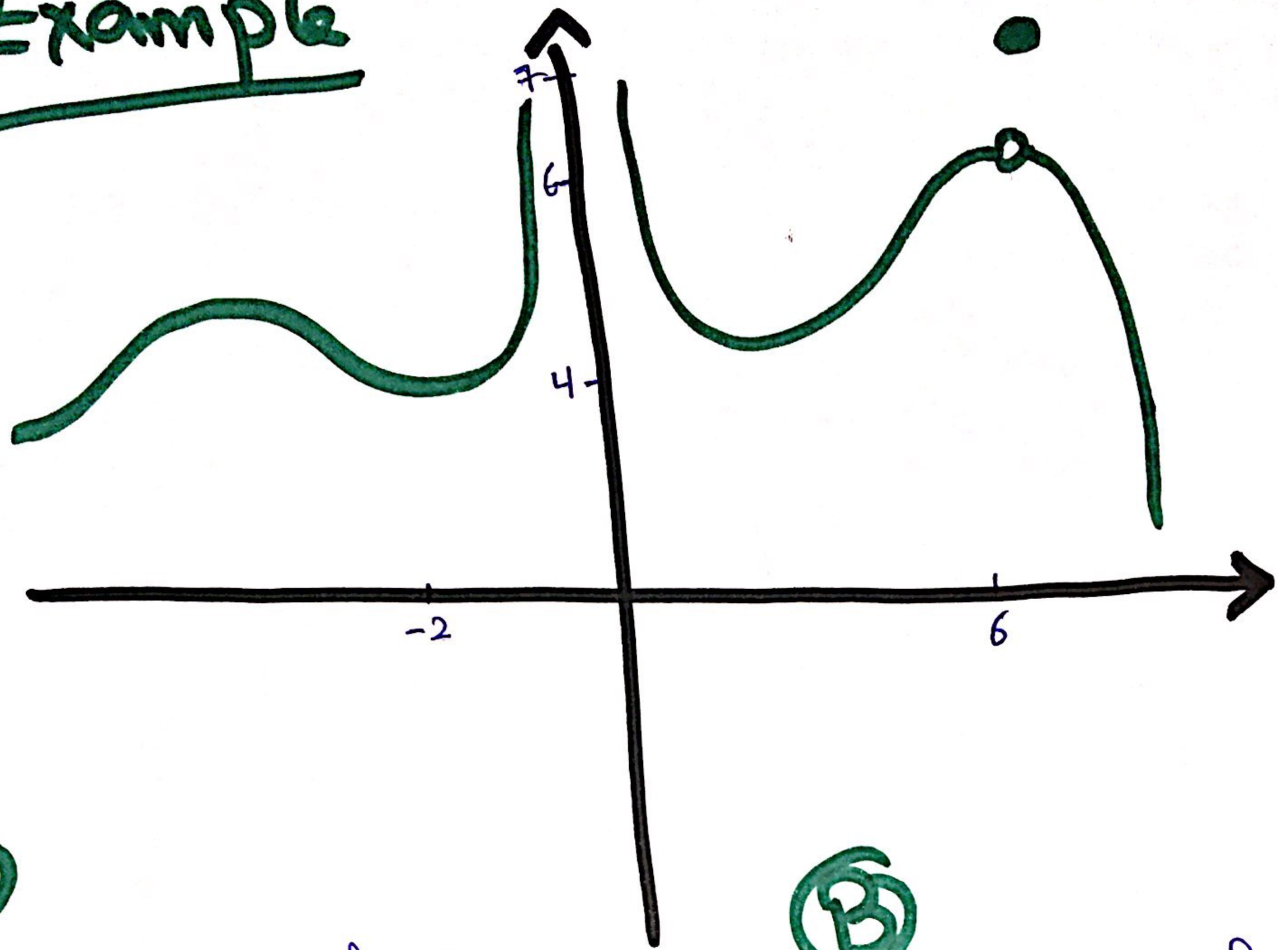
$\lim_{x \to 1^+} f(x) = 4 \neq f(1) \Rightarrow$ f(x) is discont at $x = 1$ from the right

$\lim_{x \to 1^-} f(x) = 5 = f(1) \Rightarrow$ f(x) is cont at $x = 1$ from the left

$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x) \Rightarrow \lim_{x \to 1} f(x) =$ D.N.E

$\therefore$ f(x) is discont at $x = 1$

Example



A

① $f(6) = 7$ defind

② $\lim_{x \to 6} f(x) = 6$ exist

③ $\lim_{x \to 6} f(x) \neq f(6)$

$\therefore f(x)$ is discont at $x = 6$

B

① $f(-2) = 4$ defind

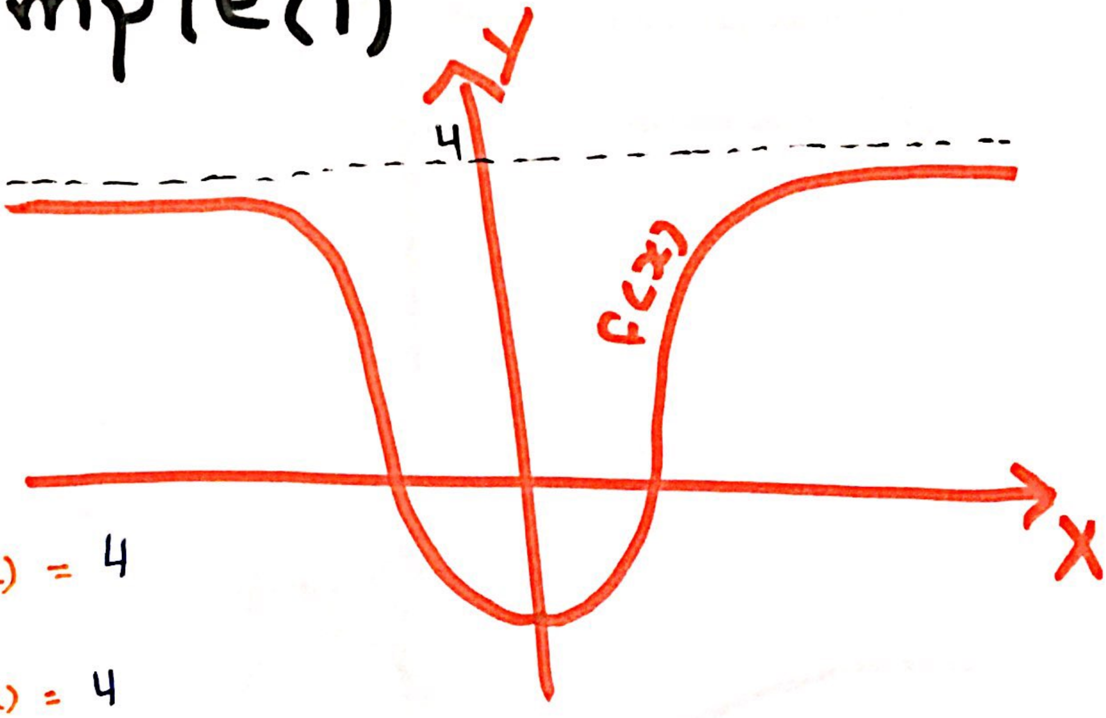
② $\lim_{x \to -2} f(x) = 4$

③ $\lim_{x \to -2} f(x) = f(-2)$

$\therefore f(x)$ is cont at $x = -2$

2.6 :. Limit at infinity and Horizontal Asymptotes.

Example(1)



$\lim_{x \to \infty} f(x) = 4$

$\lim_{x \to -\infty} f(x) = 4$

$\because \lim_{x \to \infty} f(x) = 4$ or $\lim_{x \to -\infty} f(x) = 4$

$\therefore y = 4$ is Horizontal Asymptote

if $\lim_{x \to +\infty} f(x) = L_1$ then $y = L_1$ is H.A

if $\lim_{x \to -\infty} f(x) = L_2$ then $y = L_2$ is H.A

if $y = L$ is H.A then $\lim_{x \to \pm\infty} f(x) = L$

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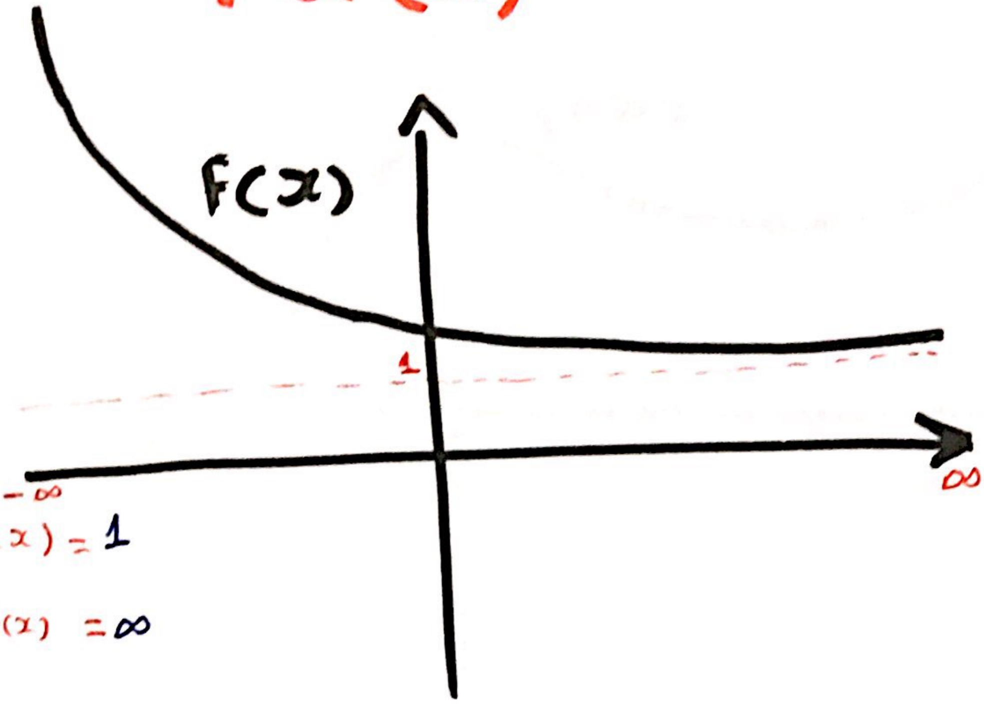
Example (2)

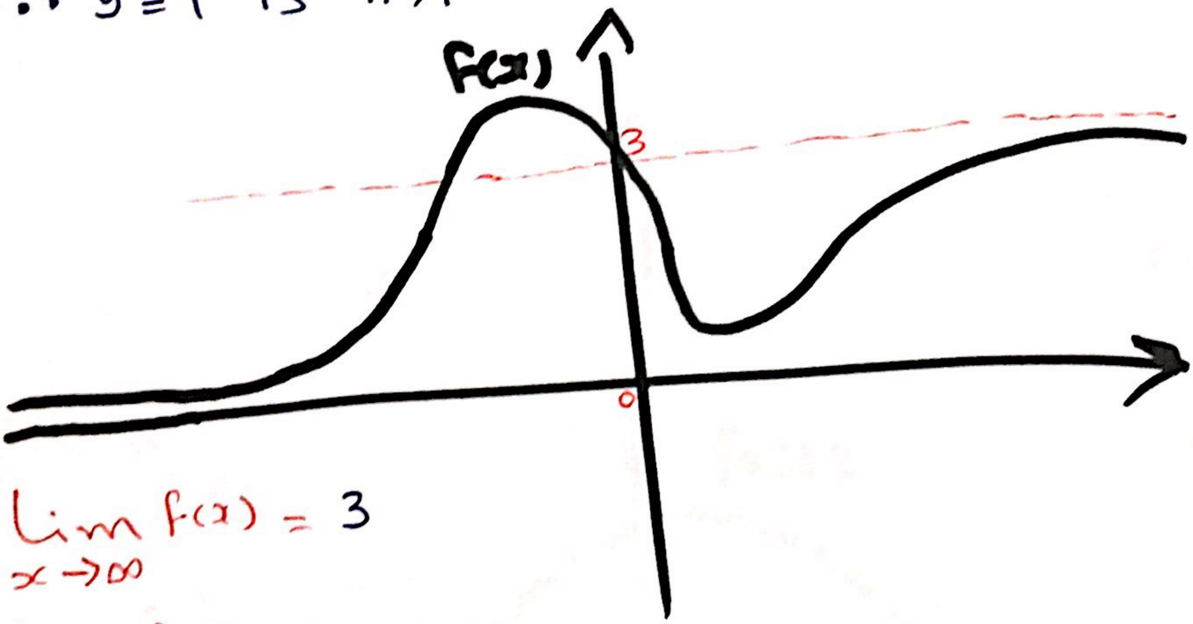


$\lim\limits_{x \to \infty} f(x) = 1$

$\lim\limits_{x \to \infty} f(x) = \infty$

$\because \lim\limits_{x \to \infty} f(x) = 1$

$\therefore y = 1$ is H.A



$\lim\limits_{x \to \infty} f(x) = 3$

$\lim\limits_{x \to -\infty} f(x) = 0$

$\because \lim\limits_{x \to \infty} f(x) = 3$ and $\lim\limits_{x \to -\infty} f(x) = 0$

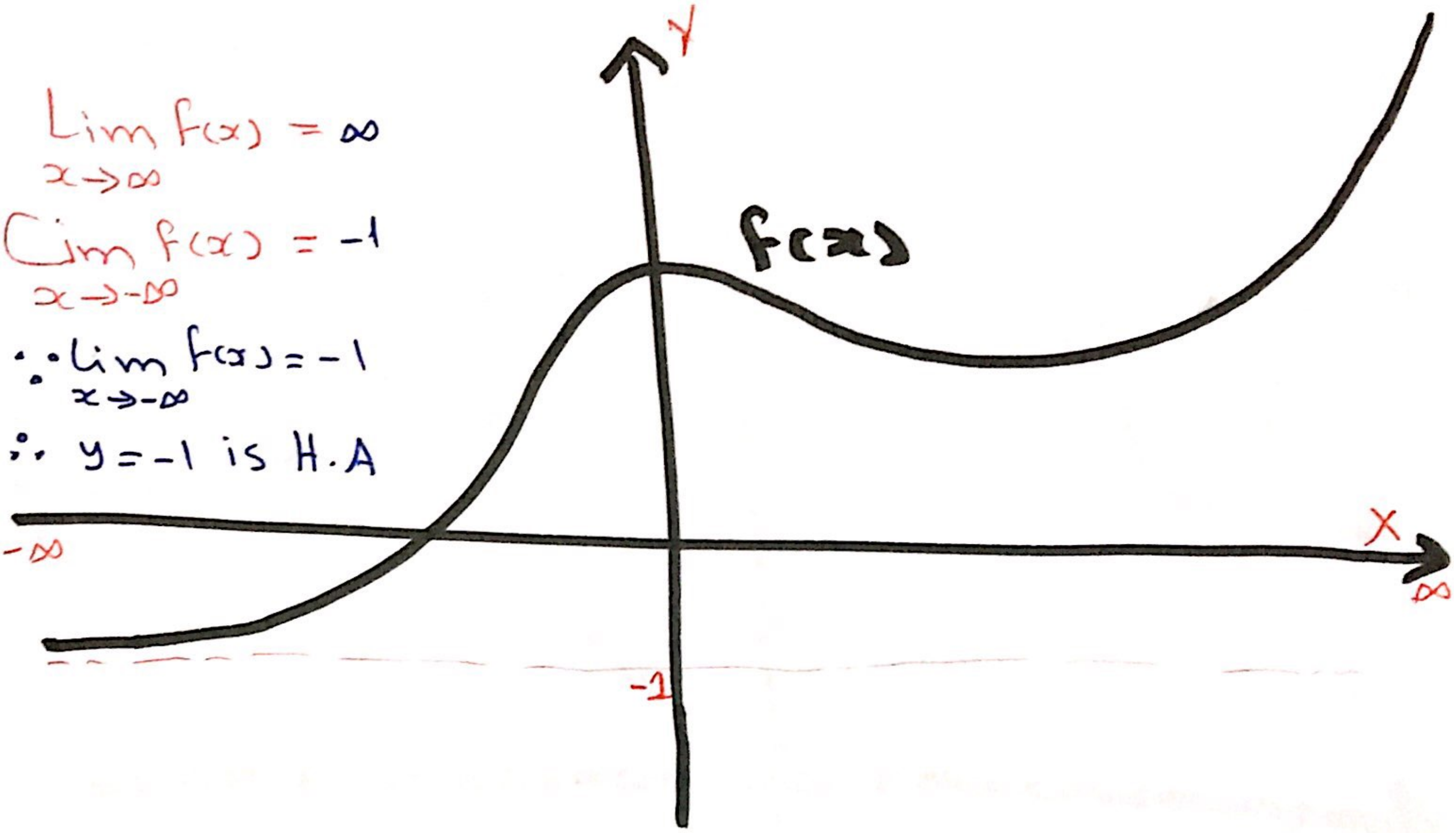
$\therefore y = 3$ and $y = 0$ are H.A

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -1$

$\because \lim_{x \to -\infty} f(x) = -1$

$\therefore y = -1$ is H.A

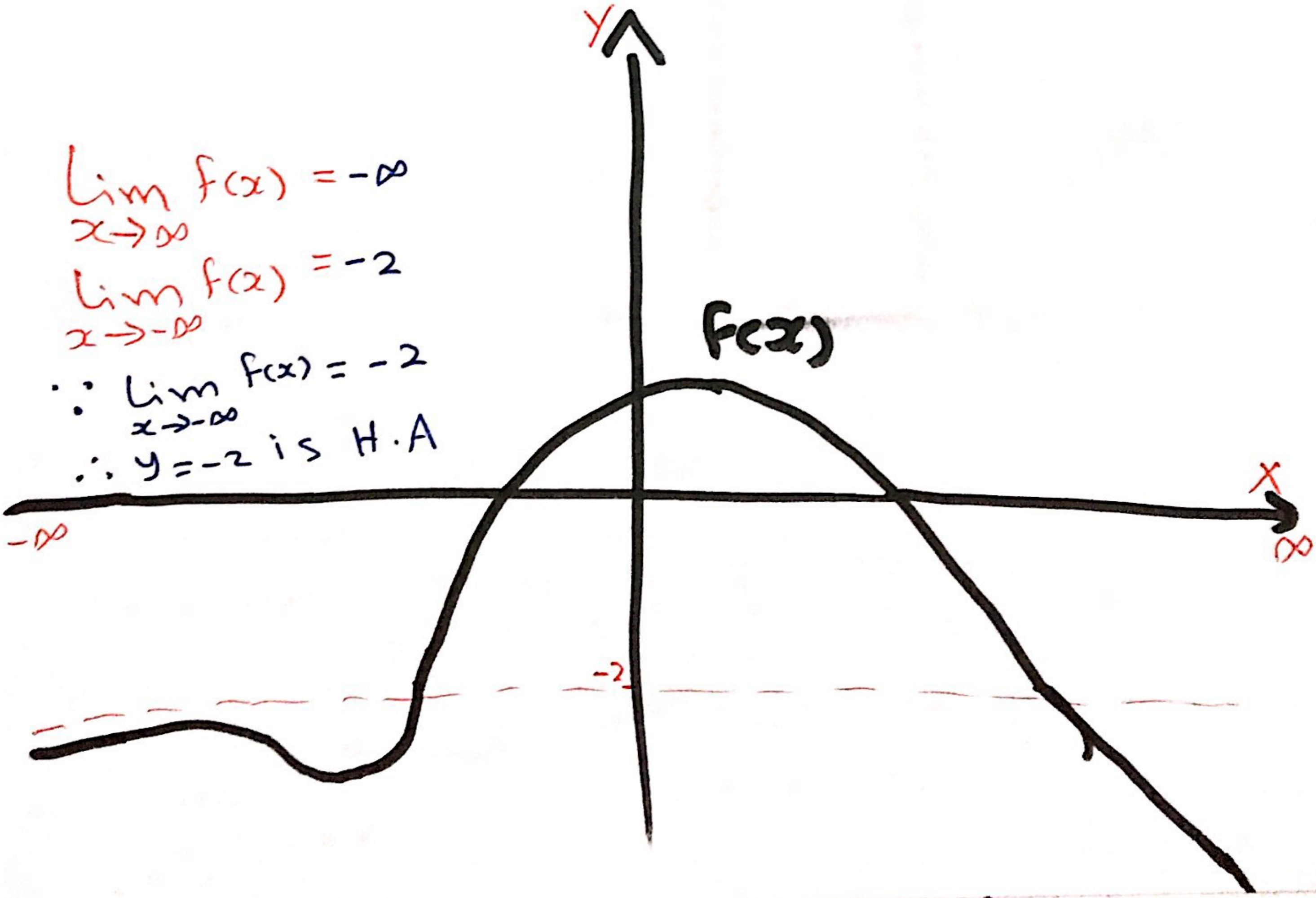


$\lim_{x \to \infty} f(x) = -\infty$

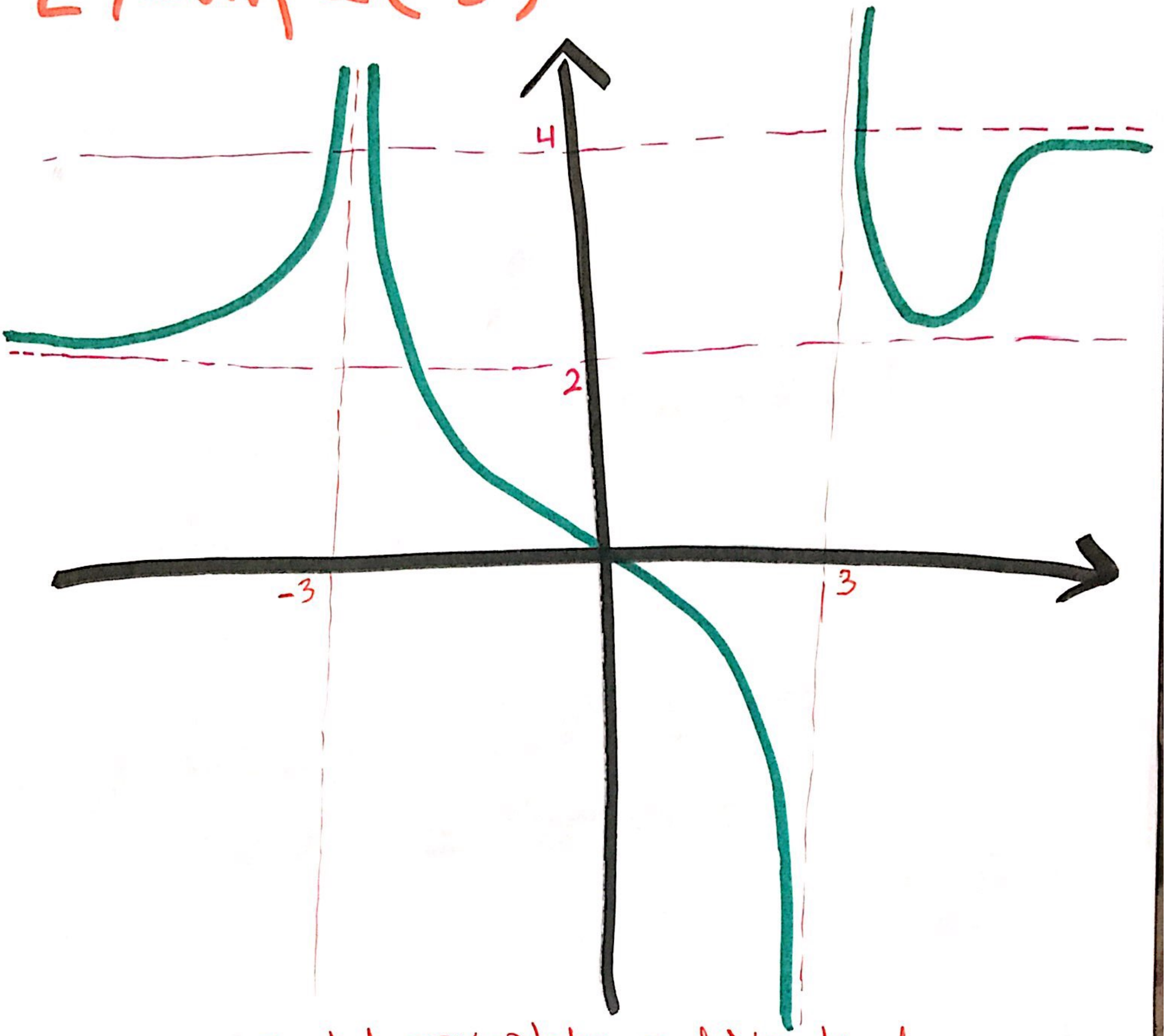
$\lim_{x \to -\infty} f(x) = -2$

$\because \lim_{x \to -\infty} f(x) = -2$

$\therefore y = -2$ is H.A



Example(3)



Find Horizontal asymptote and Vertical asymptote

H.A	V.A
$y=2$ and $y=4$ are H.A since: $\lim_{x\to\infty} f(x)=4$ and $\lim_{x\to-\infty} f(x)=2$	$x=3$ and $x=-3$ are V.A since: $\lim_{x\to3^+} f(x)=\infty$, $\lim_{x\to3^-} f(x)=-\infty$, $\lim_{x\to-3} f(x)=\infty$

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Example (4)

Find the Horizontal Asymptote and Vertical Asymptote of the following functions.

① $f(x) = 2x^2 + 3x + 1$

f(x) has no Vertical and Horizontal Asymptotes.

② $f(x) = \cos x$ or $f(x) = \sin x$

f(x) has no Vertical and Horizontal Asymptotes.

③ $f(x) = e^x$ or $f(x) = \left(\frac{1}{2}\right)^x$ or $f(x) = 3^x$ or $f(x) = \pi^{-x}$

f(x) has no Vertical Asymptote but f(x) has Horizontal asymptote $(y=0)$

④ $f(x) = 4^{x} + 2$

✓ $y = 2$ is H.A of $f(x)$

✓ $f(x)$ has no Vertical Asymptote.

⑤ $f(x) = 3^{-x} - 1$

✓ $y = -1$ is H.A of $f(x)$

✓ $f(x)$ has no Vertical asymptote.

⑥ $f(x) = \ln(x+5)$

or $f(x) = \log_3(x+5)$

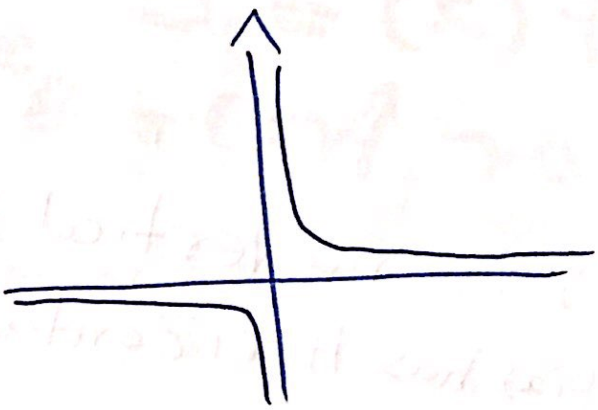
✓ $f(x)$ has no Horizontal Asymptote

$x = -5$ is V.A

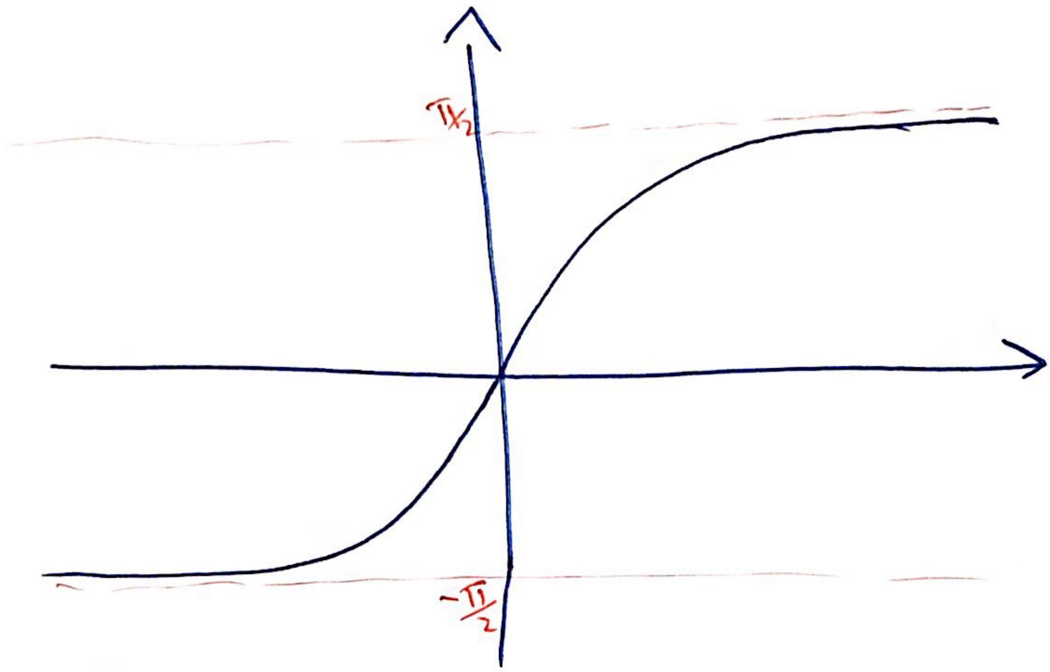
⑦ $f(x) = \frac{1}{x}$

$x = 0$ is V.A

$y = 0$ is H.A



8) $f(x) = \tan^{-1} x$



$y = \pi/2$ and $y = -\pi/2$ are H.A

i.e $\boxed{\lim_{x \to \infty} \tan^{-1} x = \pi/2 \qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}}$

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Note :- 1) $\lim_{x\to\pm\infty} \frac{c}{x^n} = 0$ for all $n>0$

2) $\lim_{x\to\infty} x^n = \infty$ for all $n>0$

3) $\lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is an even} \\ -\infty & \text{if } n \text{ is an odd.} \end{cases}$

4) $\lim_{x\to\pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) = \lim_{x\to\pm\infty} a_n x^n$

Example (5)

1) $\lim_{x\to\pm\infty} \frac{3}{x^3} = 0$ $\qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0$

$\lim_{x\to\pm\infty} 3x^{-5} = \lim_{x\to\pm\infty} \frac{3}{x^5} = 0$

$\lim_{x\to\infty} \frac{-3}{\sqrt{x}} = \lim_{x\to\infty} \frac{-3}{x^{1/2}} = 0$

2) $\lim_{x\to\infty} x^3 = \infty$ $\qquad \lim_{x\to-\infty} x^3 = -\infty$

$\lim_{x\to\infty} x^4 = \infty$ $\qquad \lim_{x\to-\infty} x^6 = \infty$

$\lim_{x\to\infty} -5x^7 = -5 \lim_{x\to\infty} x^7$
$= -5(+\infty)$
$= -5(\infty) = -\infty$

$\lim_{x\to-\infty} -5x^7 = -5 \lim_{x\to-\infty} x^7$
$= -5(-\infty)$
$= 5(\infty)$
$= \infty$

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3) $\lim\limits_{x\to\infty} \frac{1}{2}x^{-4} = \frac{1}{2}\lim\limits_{x\to\infty} x^{-4}$

$= \frac{1}{2}\lim\limits_{x\to\infty} \frac{1}{x^4}$

$= \frac{1}{2}(0)$

$= 0$

$\lim\limits_{x\to-\infty} \frac{e^2}{x^{-3}} = e^2 \lim\limits_{x\to-\infty} \frac{1}{x^{-3}}$

$= e^2 \lim\limits_{x\to-\infty} x^3$

$= e^2(-\infty)$

$= -e^2(\infty)$

$= -\infty$

$\lim\limits_{x\to\infty} \frac{5}{x^{-5/7}} = 5\lim\limits_{x\to\infty} \frac{1}{x^{-5/7}}$

$= 5\lim\limits_{x\to\infty} x^{5/7}$

$= 5(\infty)$

$= \infty$

4) $\lim\limits_{x\to\infty} (x^3 - x^7) = \lim\limits_{x\to\infty} -x^7 = -\lim\limits_{x\to\infty} x^7$

$= -\infty$

$\lim\limits_{x\to-\infty} (2x^4 + 3x^2 + 1) = \lim\limits_{x\to-\infty} 2x^4 = 2\lim\limits_{x\to-\infty} x^4$

$= 2(\infty) = \infty$

Note

If $f(x) = \dfrac{P(x)}{Q(x)}$ is a Rational function

then

1. $\lim\limits_{x \to \pm\infty} f(x) = 0$

إذا كانت درجة البسط أصغر من درجة المقام

2. $\lim\limits_{x \to \pm\infty} f(x) = \infty$ or $-\infty$

إذا كانت درجة البسط أكبر من درجة المقام

3. $\lim\limits_{x \to \pm\infty} f(x) = \dfrac{\text{معامل أكبر أس في البسط}}{\text{معامل أكبر أس في المقام}}$

إذا كانت درجة البسط تساوي درجة المقام

Example (6)

$\lim\limits_{x \to \pm\infty} \dfrac{1}{2x+3} = 0$ لأنه درجة البسط أقل من درجة المقام

$\lim\limits_{x \to \pm\infty} \dfrac{5x^3}{x^7+1} = 0 \Rightarrow y = 0$ is H.A

$\lim\limits_{x \to \infty} \dfrac{1-x^2-2x^4}{3x^4-2} = \dfrac{-2}{3}$ لأنه درجة البسط تساوي درجة المقام

$y = -\frac{2}{3}$ is H.A

$\lim\limits_{x \to -\infty} \dfrac{3x+5}{15x-4} = \dfrac{3 \div 3}{15 \div 3} = \dfrac{1}{5} \Rightarrow y = \frac{1}{5}$ is H.A

$$\lim_{x\to\infty} \frac{1+x^6}{x^4+1} = \frac{\infty}{\infty}$$

درجة البسط أكبر
من درجة المقام

$$\lim_{x\to\infty} \frac{x^6}{x^4} = \lim_{x\to\infty} x^2 = \infty$$

$f(x) = \dfrac{1+x^6}{x^4+1}$ has no H.A

$$\lim_{x\to\infty} \frac{x^2+x}{3-x} = \frac{\infty}{-\infty}$$

$$\lim_{x\to\infty} \frac{x^2}{-x} = \lim_{x\to\infty} -x = -\infty$$

$f(x) = \dfrac{x^2+x}{3-x}$ has no H.A

$$\lim_{x\to-\infty} \frac{2+x^3}{1-x^2} = \frac{\infty}{\infty}$$

$$\lim_{x\to-\infty} \frac{x^3}{-x^2} = \lim_{x\to-\infty} -x = +\infty$$

$f(x) = \dfrac{2+x^3}{1-x^2}$ has no H.A

Example (7)

1

① $\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \infty - \infty$ كمية غير محددة

$$\lim_{x \to \infty} \frac{\left(\sqrt{x^2+1} - x\right)}{1} \times \frac{\left(\sqrt{x^2+1} + x\right)}{\left(\sqrt{x^2+1} + x\right)}$$

$$\lim_{x \to \infty} \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\left(\sqrt{x^2+1} + x\right)}$$

$$\lim_{x \to \infty} \frac{\left(\sqrt{x^2+1}\right)^2 - (x)^2}{\left(\sqrt{x^2+1} + x\right)}$$

$$\lim_{x \to \infty} \frac{\cancel{x^2} + 1 - \cancel{x^2}}{\left(\sqrt{x^2+1} + x\right)}$$

$$\lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} = \frac{1}{\infty + \infty} = \frac{1}{\infty} = 0$$

$\Longrightarrow y = 0$ is H.A

Not $\lim_{x \to \infty} \left(\sqrt{x^2+1} + x\right) = \infty + \infty = 2\infty = \infty$

has no H.A

② $$\lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\infty}{\infty}$$

بالقسمة على أكبر أس في المقام : x

$$\lim_{x \to \infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x}{x} - \frac{5}{x}}$$

$$\lim_{x \to \infty} \frac{\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}}{3 - \frac{5}{x}}$$

$$\lim_{x \to \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{\sqrt{2 + \lim_{x \to \infty} \frac{1}{x^2}}}{3 - \lim_{x \to \infty} \frac{5}{x}}$$

$$= \frac{\sqrt{2+0}}{3-0}$$

$$= \frac{\sqrt{2}}{3}$$

$\Rightarrow y = \frac{\sqrt{2}}{3}$ is H.A

③ $$\lim_{x \to -\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} = \frac{\infty}{-\infty}$$

$$\lim_{x \to -\infty} \frac{\dfrac{\sqrt{9x^6 - x}}{-x^3}}{\dfrac{x^3}{-x^3} + \dfrac{1}{-x^3}} = \lim_{x \to -\infty} \frac{\sqrt{\dfrac{9x^6}{(x^3)^2} - \dfrac{x}{(x^3)^2}}}{-1 + \dfrac{1}{x^3}}$$

$$\lim_{x \to -\infty} \frac{\sqrt{\dfrac{9x^6}{x^6} - \dfrac{x}{x^6}}}{-1 + \dfrac{1}{x^3}} = \lim_{x \to -\infty} \frac{\sqrt{9 - \dfrac{1}{x^5}}}{-1 + \dfrac{1}{x^3}}$$

$$= \frac{\sqrt{9 - \lim_{x \to -\infty} \frac{1}{x^5}}}{-1 + \lim_{x \to -\infty} \frac{1}{x^3}}$$

$$= \frac{\sqrt{9 - 0}}{-1 + 0}$$

$$= \frac{\sqrt{9}}{-1}$$

$$= \frac{3}{-1} = -3$$

$\Longrightarrow y = -3$ is H.A

Example (8)

Find Vertical Asymptote and Horizontal Asymptote of functions.

① $y = \dfrac{2x^2+x-1}{x^2+x-2}$

H.A

$$\lim_{x \to \pm\infty} \frac{2x^2+x-1}{x^2+x-2} = 2$$

$\Rightarrow y = 2$ is H.A of $f(x)$

V.A

① نوجد أصفار المقام

$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$

$x+2=0$ or $x-1=0$

$x=-2$ $\qquad$ $x=1$

② $g(x) = 2x^2+x-1$

$g(1) = 2(1)^2+(1)-1 = 2(1)+1-1 = 2+1-1 = 2 \neq 0$

$g(-2) = 2(-2)^2+(-2)-1 = 2(4)-2-1 = 8-2-1 = 5 \neq 0$

③ $x=-2$ and $x=1$ are V.A

② $f(x) = \dfrac{\sqrt{x^6 - 1}}{x^3 - 1}$

H.A

$$\lim_{x \to \infty} \frac{\sqrt{x^6 - 1}}{x^3 - 1} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{\dfrac{\sqrt{x^6 - 1}}{x^3}}{\dfrac{x^3}{x^3} - \dfrac{1}{x^3}}$$

$$\lim_{x \to \infty} \frac{\sqrt{\dfrac{x^6}{x^6} - \dfrac{1}{x^6}}}{1 - \dfrac{1}{x^3}}$$

$$\frac{\sqrt{1 - \lim_{x \to \infty} \frac{1}{x^6}}}{1 - \lim_{x \to \infty} \frac{1}{x^3}}$$

$$\frac{\sqrt{1 - 0}}{1 - 0} = \frac{\sqrt{1}}{1} = 1$$

$\therefore y = 1$ is H.A

$$\lim_{x \to -\infty} \frac{\sqrt{x^6 - 1}}{x^3 - 1} = \frac{\infty}{-\infty}$$

$$\lim_{x \to -\infty} \frac{\dfrac{\sqrt{x^6 - 1}}{-x^3}}{\dfrac{x^3}{-x^3} - \dfrac{1}{-x^3}}$$

$$\lim_{x \to -\infty} \frac{\sqrt{\dfrac{x^6}{x^6} - \dfrac{1}{x^6}}}{-1 + \dfrac{1}{x^3}}$$

$$\frac{\sqrt{1 - \lim_{x \to \infty} \frac{1}{x^6}}}{-1 + \lim_{x \to -\infty} \frac{1}{x^3}}$$

$$\frac{\sqrt{1 - 0}}{-1 + 0} = \frac{\sqrt{1}}{-1} = -1$$

$\therefore y = -1$ is H.A

$\Rightarrow y = 1$ and $y = -1$ are H.A.

V.A

① أصفار المقام

$x^3 - 1 = 0$

$x^3 = 1$

$\sqrt[3]{x^3} = \sqrt[3]{1}$

$\boxed{x = 1}$

② $g(x) = \sqrt{x^6 - 1}$

$g(1) = \sqrt{1 - 1} = \sqrt{0} = 0$

③ $x = 1$ is not V.A

$\Rightarrow f(x)$ has no V.A

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$$2)\ f(x) = \frac{\sqrt{4x^2+1}}{x+1}$$

H.A

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{x+1}$$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+1}}{x}}{\frac{x}{x}+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{4x^2}{x^2}+\frac{1}{x^2}}}{1+\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt{4+\frac{1}{x^2}}}{1+\frac{1}{x}}$$

$$= \frac{\sqrt{4+\lim_{x\to\infty}\frac{1}{x^2}}}{1+\lim_{x\to\infty}\frac{1}{x}}$$

$$= \frac{\sqrt{4+0}}{1+0}$$

$$= \frac{\sqrt{4}}{1} = 2$$

$y=2$ is H.A

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x+1}$$

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2+1}}{-x}}{\frac{x}{-x}+\frac{1}{-x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2}{x^2}+\frac{1}{x^2}}}{-1-\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{4+\frac{1}{x^2}}}{-1-\frac{1}{x}}$$

$$= \frac{\sqrt{4+\lim_{x\to-\infty}\frac{1}{x^2}}}{-1-\lim_{x\to-\infty}\frac{1}{x}}$$

$$= \frac{\sqrt{4+0}}{-1-0} = \frac{\sqrt{4}}{-1} = -2$$

$y=-2$ is H.A

V.A

① أصفار المقام

$x+1=0 \Rightarrow x=-1 <$ $\boxed{V.A}$ / not V.A

② $g(x)=\sqrt{4x^2+1}$

$g(-1)=\sqrt{4(-1)^2+1}=\sqrt{4(1)+1}=\sqrt{4+1}=\sqrt{5}\neq 0$

③ $x=-1$ is V.A of $f(x)$

$3x - 5$

H.A

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x}{x} - \frac{5}{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3 - \frac{5}{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}}{3 - \frac{5}{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}}$$

$$\frac{\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x^2}}}{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \frac{5}{x}}$$

$$\frac{\sqrt{2+0}}{3+0} = \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\infty}{-\infty}$$

$$-\lim_{x\to-\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x}{+x} - \frac{5}{+x}}$$

$$-\lim_{x\to-\infty} \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3 - \frac{5}{x}}$$

$$-\lim_{x\to-\infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}}$$

$$-\frac{\sqrt{\lim\limits_{x\to-\infty} 2 + \lim\limits_{x\to-\infty} \frac{1}{x^2}}}{\lim\limits_{x\to-\infty} 3 - \lim \frac{5}{x}}$$

$$-\frac{\sqrt{2+0}}{3} = -\frac{\sqrt{2}}{3}$$

$\therefore$ f(x) have H.A at

$y = \frac{\sqrt{2}}{3}$ and $y = -\frac{\sqrt{2}}{3}$

V.A

$3x - 5 = 0$

$3x = 5$

$x = 5/3$

Let $g(x) = \sqrt{2x^2 + 1}$

$g(5/3) = \sqrt{2\left(\frac{25}{9}\right) + 1}$

$= \sqrt{\frac{50}{9} + 1}$

$= \sqrt{\frac{59}{9}}$

$= \frac{\sqrt{59}}{3} \neq 0$

$\therefore\ x = 5/3$ is V.A of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

Example (9)

$$\lim_{x \to \infty} \tan^{-1}(e^x)$$

$$\tan^{-1}\left(\lim_{x \to \infty} e^x\right) = \tan^{-1}(e^{\infty}) = \tan^{-1}(\infty)$$

$$= \pi/2$$

$\Rightarrow y = \pi/2$ is H.A

$$\lim_{x \to 0^+} \tan^{-1}(\ln x) = \tan^{-1}\left(\lim_{x \to 0^+} \ln(x)\right)$$

$$= \tan^{-1}(\ln(0))$$

$$= \tan^{-1}(-\infty)$$

$$= -\pi/2$$

$$\lim_{x \to \infty} \sin x = \sin(\infty)$$ "D.N.E"

$$\lim_{t\to\infty} e^{-2t}\cos(t) = e^{-\infty}\cos(\infty) = 0 \cdot (D.N.E)$$

$$-1 \leq \cos(t) \leq 1$$

$$-e^{-2t} \leq e^{-2t}\cos(t) \leq e^{-2t}$$

$$\lim_{t\to\infty} -e^{-2t} = -e^{-\infty} = 0$$

$$\lim_{t\to\infty} e^{-2t} = +e^{-\infty} = 0$$

$$\because \lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty} -e^{-2t} = 0$$

$$\therefore \lim_{t\to\infty} e^{-2t}\cos t = 0$$

$$\lim_{x\to\infty} \sqrt{x^2+1} = \sqrt{\lim_{x\to\infty}(x^2+1)}$$

$$= \sqrt{\lim_{x\to\infty} x^2}$$

$$= \sqrt{\infty}$$

$$= \infty$$

Example (10)

$$\lim_{x \to -\infty} \left(\sqrt{4x^2+3x} + 2x\right) = \infty - \infty$$

$$\lim_{x \to -\infty} \frac{\left(\sqrt{4x^2+3x} + 2x\right)}{1} \cdot \frac{\left(\sqrt{4x^2+3x} - 2x\right)}{\left(\sqrt{4x^2+3x} - 2x\right)}$$

$$\lim_{x \to -\infty} \frac{\left(\sqrt{4x^2+3x} + 2x\right)\left(\sqrt{4x^2+3x} - 2x\right)}{1 \cdot \left(\sqrt{4x^2+3x} - 2x\right)}$$

$$\lim_{x \to -\infty} \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x} - 2x} = \lim_{x \to -\infty} \frac{3x}{\sqrt{4x^2+3x} - 2x} = \frac{-\infty}{\infty}$$

$$\lim_{x \to -\infty} \frac{\frac{3x}{-x}}{\frac{\sqrt{4x^2+3x}}{-x} - \frac{2x}{-x}} = \lim_{x \to -\infty} \frac{-3}{\sqrt{\frac{4x^2}{x^2} + \frac{3x}{x^2}} + 2}$$

$$\lim_{x \to -\infty} \frac{-3}{\sqrt{4+\frac{3}{x}} + 2} = \frac{-3}{\sqrt{4+0} + 2} = \frac{-3}{2+2} = \frac{-3}{4}$$

$$\lim_{x\to\infty}\left[\ln(1+x^2)-\ln(1+x)\right]=\infty-\infty$$

$$\lim_{x\to\infty}\left[\ln\left[\frac{1+x^2}{1+x}\right]\right]=\ln\left(\lim_{x\to\infty}\frac{1+x^2}{1+x}\right)$$

$$=\ln(\infty)$$

$$=\infty$$

$$\lim_{x\to\infty}\left[\ln(2+x)-\ln(1+x)\right]=\infty-\infty$$

$$\lim_{x\to\infty}\ln\left(\frac{2+x}{1+x}\right)=\ln\left(\lim_{x\to\infty}\frac{2+x}{1+x}\right)$$

$$=\ln\left(\frac{1}{1}\right)$$

$$=\ln(1)$$

$$=0$$

$$\lim_{x\to\infty}\left[\ln(x+3)-\ln(x^2-1)\right]=\infty-\infty$$

$$\lim_{x\to\infty}\ln\left(\frac{x+3}{x^2-1}\right)=\ln\left(\lim_{x\to\infty}\frac{x+3}{x^2-1}\right)=\ln(0)$$

$$=-\infty$$

$$\lim_{x\to\infty} \frac{\sin^2 x}{1+x^2} = \frac{\text{D.N.E}}{\infty}$$

$$-1 \le \sin x \le 1$$

$$0 \le \sin^2 x \le 1$$

$$\frac{0}{1+x^2} \le \frac{\sin^2 x}{1+x^2} \le \frac{1}{x^2+1}$$

$$0 \le \frac{\sin^2 x}{1+x^2} \le \frac{1}{x^2+1}$$

$$\lim_{x\to\infty} 0 = 0$$

$$\lim_{x\to\infty} \frac{1}{1+x^2} = 0$$

$$\therefore \lim_{x\to\infty} \frac{\sin^2 x}{1+x^2} = 0$$

$$\lim_{x \to \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \frac{e^{\infty} - e^{-\infty}}{e^{\infty} + e^{-\infty}} = \frac{\infty - 0}{\infty + 0} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{\left(\frac{e^{3x}}{e^{3x}} - \frac{e^{-3x}}{e^{3x}}\right)}{\left(\frac{e^{3x}}{e^{3x}} + \frac{e^{-3x}}{e^{3x}}\right)} = \lim_{x \to \infty} \frac{1 - e^{-6x}}{1 + e^{-6x}}$$

$$= \frac{1 - e^{-\infty}}{1 + e^{-\infty}}$$

$$= \frac{1 - 0}{1 + 0}$$

$$= \frac{1}{1} = 1$$

$$\lim_{x \to -\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \frac{e^{-\infty} - e^{\infty}}{e^{-\infty} + e^{\infty}} = \frac{0 - \infty}{0 + \infty} = -\frac{\infty}{\infty}$$

$$\lim_{x \to -\infty} \frac{\left(\frac{e^{3x}}{e^{-3x}} - \frac{e^{-3x}}{e^{-3x}}\right)}{\left(\frac{e^{3x}}{e^{-3x}} + \frac{e^{-3x}}{e^{-3x}}\right)} = \lim_{x \to -\infty} \frac{e^{6x} - 1}{e^{6x} + 1}$$

$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = \frac{0 - 1}{0 + 1} = -\frac{1}{1} = -$$

$\therefore f(x) = \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$ have H.A

the H.A are $y = \pm 1$

$$\lim_{x \to \infty} \frac{1-e^x}{1+2e^x} = \frac{-\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{\frac{1}{e^x} - \frac{e^x}{e^x}}{\frac{1}{e^x} + \frac{2e^x}{e^x}} = \lim_{x \to \infty} \frac{e^{-x}-1}{e^{-x}+2}$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+2} = \frac{0-1}{0+2}$$

$$= -\frac{1}{2}$$

$$\lim_{x \to -\infty} \frac{1-e^x}{1+2e^x} = \frac{1-e^{-\infty}}{1+2e^{-\infty}} = \frac{1-0}{1+0} = \frac{1}{1} = 1$$

$$\boxed{\text{the H.A of } f(x) = \frac{1-e^x}{1+2e^x} : \begin{matrix} y = -\frac{1}{2} \\ y = 1 \end{matrix}}$$

Find the H.A and V.A of $y = \dfrac{2e^x}{e^x - 5}$

V.A

① $e^x - 5 = 0 \Rightarrow e^x = 5 \Rightarrow \ln e^x = \ln 5$

$\Rightarrow \boxed{x = \ln 5}$

② let $g(x) = 2e^x$

$g(\ln(5)) = 2e^{\ln 5} = 2(5) = 10 \neq 0$

$\therefore x = \ln(5)$ is V.A of $y = \dfrac{2e^x}{e^x - 5}$

H.A

✓ $\lim\limits_{x \to \infty} \dfrac{2e^x}{e^x - 5} = \dfrac{\infty}{\infty}$

$\lim\limits_{x \to \infty} \left[\dfrac{\frac{2e^x}{e^x}}{\frac{e^x}{e^x} - \frac{5}{e^x}} \right] = \lim\limits_{x \to \infty} \dfrac{2}{1 - \frac{5}{e^x}} = \dfrac{2}{1 - \frac{5}{e^\infty}} = \dfrac{2}{1 - \frac{5}{\infty}}$

$= \dfrac{2}{1 - 0} = \dfrac{2}{1} = 2$

✓ $\lim\limits_{x \to -\infty} \dfrac{2e^x}{e^x - 5} = \dfrac{2e^{-\infty}}{e^{-\infty} - 5} = \dfrac{2(0)}{0 - 5} = \dfrac{0}{-5} = 0$

$\therefore y = 2$ and $y = 0$ are H.A of $y = \dfrac{2e^x}{e^x - 5}$

2.7 :- The Derivatives of the Functions at number a

1) The Derivative of $f(x)$ at number a is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

or

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2) The Derivative of $f(x)$ at number a is a Slope of the tangent line at number a i.e $m = f'(a)$

3) The equation of the tangent line to the curve $y = f(x)$ at the Point $(a, f(a))$ is $y - f(a) = m(x - a)$

$$\boxed{y - f(a) = f'(a)(x - a)}$$

Example:

If $f(x) = x^3$ then find $f'(2)$

a) $f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^3 - 2^3}{x - 2} = \lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

b) $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^3 - 2^3}{h} = \lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$

c) $f(x) = x^3$

$f'(x) = 3x^{3-1} = 3x^2$

$f'(2) = 3(2)^2 = 3(4) = 12$

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Example:

If $f(x) = \sqrt{x}$ then $f'(9) = \ldots\ldots$

a) $f'(9) = \lim\limits_{x \to 9} \dfrac{f(x) - f(9)}{x - 9} = \lim\limits_{x \to 9} \dfrac{\sqrt{x} - \sqrt{9}}{x - 9} = \lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

b) $f'(9) = \lim\limits_{h \to 0} \dfrac{f(9+h) - f(9)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{9+h} - \sqrt{9}}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{9+h} - 3}{h}$

c) $f(x) = \sqrt{x}$

$= x^{1/2}$

$f'(x) = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-1/2} = \dfrac{1}{2x^{1/2}} = \dfrac{1}{2\sqrt{x}}$

$f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{2(3)} = \dfrac{1}{6}$

Example:

If $f(x) = \dfrac{3}{x}$ then find the slope of the tangent line at 2

a) $m = f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{\frac{3}{x} - \frac{3}{2}}{x - 2} = \lim\limits_{x \to 2} \dfrac{6 - 3x}{2x(x-2)}$

b) $m = f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0} \dfrac{\frac{3}{2+h} - \frac{3}{2}}{h} = \lim\limits_{h \to 0} \dfrac{6 - 3(2+h)}{2h(2+h)}$

c) $f(x) = \dfrac{3}{x}$

$= 3x^{-1}$

$f'(x) = 3(-1)x^{-1-1} = -3x^{-2} = \dfrac{-3}{x^2}$

$\therefore m = f'(2) = \dfrac{-3}{(2)^2} = \dfrac{-3}{4}$

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Example

Find the equation of tangent line to the curve $y = x^2 - 8x + 9$ at $(4,-7)$

① $y' = 2x - 8$

② $m = y'(a)$

$= y'(4)$

$= 2(4) - 8$

$= 8 - 8$

$= 0$

③ if $m = 0$ then the tangent line is Horizontal

④ the equation of tangent line is

$y = f(a)$

$\boxed{y = -7}$

Example

Find the equation of tangent line to the curve $y = x^2$ at $(1,1)$

① $y' = 2x$

② $m = y'(a) = y'(1)$

$= 2(1)$

$= 2$

③ the equation of tangent line

is $y - f(a) = m(x - a)$

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$y - 2x - 1 + 2 = 0$

$\boxed{y - 2x + 1 = 0}$

or $\boxed{y - 2x = -1}$

or $\boxed{y = -1 + 2x}$

Example

Find the equation of normal line of $f(x) = \sqrt{x}$ at $x = 4$

(1) $f(a) = f(4) = \sqrt{4} = 2$

(2) $f(x) = \sqrt{x} = x^{1/2}$

(3) $f'(x) = \frac{1}{2} x^{\frac{1}{2}-1}$

$= \frac{1}{2} x^{-1/2}$

$= \frac{1}{2x^{1/2}}$

$= \frac{1}{2\sqrt{x}}$

(4) $m = f'(a)$

$= f'(4)$

$= \frac{1}{2\sqrt{4}}$

$= \frac{1}{2(2)}$

$= \frac{1}{4}$

(5) $m_{\perp} = \frac{-1}{m}$

$= \frac{(-1)}{(\frac{1}{4})}$

$= -1 \div \frac{1}{4}$

$= (-1) \times (4)$

$= -4$

⑥ The equation of the normal line is $y - f(a) = m_{\perp}(x - a)$

$$y - 2 = -4(x - 4)$$

$$y - 2 = -4x + 16$$

$$y + 4x - 2 - 16 = 0$$

$$\boxed{y + 4x - 18 = 0}$$

or $\boxed{y + 4x = 18}$

or $\boxed{y = 18 - 4x}$

Example

Find the points on the curve $y = x^4 - 6x^2 + 2$ where the tangent line is Horizontal

If The tangent line is Horizontal

then $m = 0$

$y' = 0$

$4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$4x = 0$ or $x^2 - 3 = 0$

$\frac{4x}{4} = \frac{0}{4}$

$x = 0$

$x^2 = 3$

$\sqrt{x^2} = \sqrt{3}$

$|x| = \sqrt{3}$

$\boxed{x = \pm\sqrt{3}}$

$\therefore$ the Curve $y = x^4 - 6x^2 + 2$ have Horizontal tangent line when $x = 0$ and $x = \pm\sqrt{3}$

or

the curve $y = x^4 - 6x^2 + 2$
have Horizontal tangent at
Points : $(0, y(0)) = (0, 0^4 - 6(0)^2 + 2) = (0, 2)$

$(\sqrt{3}, y(\sqrt{3})) = (\sqrt{3}, (\sqrt{3})^4 - 6(\sqrt{3})^2 + 2)$

$= (\sqrt{3}, (3^{1/2})^4 - 6(3^{1/2})^2 + 2)$

$= (\sqrt{3}, 3^{4/2} - 6(3^{2/2}) + 2)$

$= (\sqrt{3}, 3^2 - 6(3) + 2)$

$= (\sqrt{3}, 9 - 18 + 2)$

$= (\sqrt{3}, -7)$

$(-\sqrt{3}, y(-\sqrt{3})) = (-\sqrt{3}, (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 2)$

$= (-\sqrt{3}, 9 - 6(3) + 2)$

$= (-\sqrt{3}, 9 - 18 + 2)$

$= (-\sqrt{3}, -7)$

2.8 - The Derivative as the Function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow$$ التفاضل بالتعريف

Example:

If $f(x) = x^3 - x$ then find $f'(x)$

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$

b) $f'(x) = 3x^{3-1} - 1$

$= 3x^2 - 1$

Note

① other notation of $f'(x)$

$f'(x)$ or y' or $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $\frac{d}{dx}[f(x)]$

or $D_{f(x)}$ or $D_x f(x)$ or $D[f(x)]$

② A function f is differentiable at number a if $f'(a)$ exists

③ A function f is differentiable on (a,b) if it is differentiable at every number in the (a,b)

④ $D_{f'(x)} \subseteq D_{f(x)}$

Theorem

IF $F(x)$ is differentiable at a then $f(x)$ is continuous at a "the converse not true"

Note

$F(x)$ is discontinuous at a the $F(x)$ is not differentiable at a

Example

where is the following functions differentiable.

① $f(x) = \frac{1}{x+1}$ is cont on $\mathbb{R} - \{-1\}$

$\therefore$ $f(x)$ is discont at $x=-1$

$\Rightarrow$ $f(x)$ is not differentiable at $x=-1$

$\Rightarrow$ $f(x)$ is differentiable on $\mathbb{R} - \{-1\}$

② $f(x) = \sqrt{x-4}$ is cont on $[4, \infty)$

$\Rightarrow$ $F(x)$ is discont at $x=4$

$\Rightarrow$ $F(x)$ is not differentiable at $x=4$

$\Rightarrow$ $F(x)$ is differentiable on $(4, \infty)$.

* $f(x) = \sqrt[3]{x-4}$ is cont on $\mathbb{R}$

ولكن الدالة غير قابلة للتفاضل عند أصفار المقدار الذي بداخل الجذر

أصفار المقدار: $x-4=0 \Rightarrow x=4$

$f(x)$ is not differentiable at 4

$F(x)$ is differentiable on $\mathbb{R} - \{4\}$

③ $f(x) = |x+5|$ is cont on $\mathbb{R}$

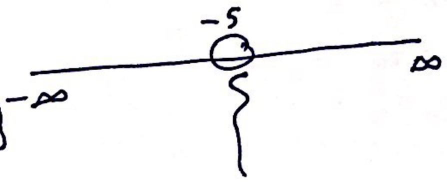
دالة القيمة المطلقة غير قابلة للتفاضل عند نقاط الإنكسار
وهي أصفار المقدار الذي داخل القيمة المطلقة

أصفار المقدار:

$$x+5=0$$
$$x=-5$$

$f(x)$ is not differentiable at $x=-5$

$f(x)$ is differentiable on $\mathbb{R}-\{-5\}$



④ $f(x) = x^2+2x+3$ is cont on $\mathbb{R}$ and differentiable on $\mathbb{R}$

Example

If $f(x) = 2-3x+5x^2-2x^3+\underline{\underline{10x^4}}$
then find of the following:

a) $f'(x)$, $f''(x)$

$$f'(x) = -3+10x-6x^2+40x^3$$

$$f''(x) = 10-12x+120x^2$$

b) $f^{(100)}_{(x)}$

درجة كثيرة الحدود (4) أقل من درجة التفاضل

$$\Rightarrow f^{(100)}_{(x)} = 0$$

c) $f^{(4)}_{(x)} \rightarrow$ درجة كثيرة الحدود

$$f^{(4)}_{(x)} = 10(4!)$$

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Example

If $f(x) = |12 - 4x|$ then

find $f'(3)$, $f'(7)$, $f'(2)$

$$f(x) = |12 - 4x| = \begin{cases} 12 - 4x & \text{if } 12 - 4x \geq 0 \\ -(12 - 4x) & \text{if } 12 - 4x < 0 \end{cases}$$

$$= \begin{cases} 12 - 4x & \text{if } -4x \geq -12 \\ 4x - 12 & \text{if } -4x < -12 \end{cases}$$

$$= \begin{cases} 12 - 4x & \text{if } x \leq 3 \\ 4x - 12 & \text{if } x > 3 \end{cases}$$

$$f'(x) = \begin{cases} -4 & \text{if } x < 3 \\ 4 & \text{if } x > 3 \end{cases}$$

$f'(3) = D.N.E$

السبب

$[f'(3)]^+ = 4$

$[f'(3)]^- = -4$

$\because [f'(3)]^+ \neq [f'(3)]^-$

$\therefore f'(3)$ D.N.E

$f'(7) = 4$

$f'(2) = -4$

3.1 - Derivatives of Polynomials and Exponential Functions

1) Derivative of a Constant function

$$\frac{d}{dx}[c] = 0 \text{ for all } c \in \mathbb{R}$$

Example:-

$\frac{d}{dx}[\pi^2] = 0$ $\frac{d}{dx}[5^c] = 0$ $\frac{d}{dy}[18.5] = 0$

$\frac{d}{dx}[\sqrt{36}] = 0$ $\frac{d}{dx}[\ln(9)] = 0$

$\frac{d}{dx}[\sin(\frac{\pi}{2})] = 0$ $\frac{d}{dx}[\cos^2(5)] = 0$

if $f(x) = \sqrt{4+c^2}$ then $f'(x) = 0$...

2) if $f(x) = ax$ for all $a \in \mathbb{R}$ then $f'(x) = a$

Example:

$\frac{d}{dx}[10x] = 10$

if $f(x) = \frac{-3}{4}x$ then $f'(x) = \frac{-3}{4}$...

if $f(x) = -x$ then $f'(x) = -1$.

$\frac{d}{dt}[2t] = 2$

if $f(\emptyset) = 18.5\emptyset$ then $f'(\emptyset) = 18.5$

3) if $f(x) = x^n$ then $f'(x) = n x^{n-1}$

Example:

$\frac{d}{dx}[x^2] = 2x$ $\quad \frac{d}{dx}[x^3] = 3x^2$ $\quad \frac{d}{dx}[x^4] = 4x^3$

$$\frac{d}{dx}\left[\frac{1}{x^5}\right] = \frac{d}{dx}[x^{-5}]$$
$$= -5x^{-5-1}$$
$$= -5x^{-6}$$
$$= \frac{-5}{x^6}$$

$$\frac{d}{dx}\left[\sqrt[3]{x^2}\right] = \frac{d}{dx}\left[(x^2)^{1/3}\right]$$
$$= \frac{d}{dx}\left[x^{2/3}\right]$$
$$= \frac{2}{3}x^{2/3-1}$$
$$= \frac{2}{3}x^{-\frac{1}{3}}$$
$$= \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

$$\frac{d}{dx}\left[x^2\sqrt{x}\right] = \frac{d}{dx}\left[x^2 \cdot x^{1/2}\right]$$
$$= \frac{d}{dx}\left[x^{2+\frac{1}{2}}\right]$$
$$= \frac{d}{dx}\left[x^{5/2}\right]$$
$$= \frac{5}{2}x^{5/2-1}$$
$$= \frac{5}{2}x^{3/2}$$
$$= \frac{5}{2}\sqrt{x^3}$$

4] $\frac{d}{dx}\left[c f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$

$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$

Example:

a) $\frac{d}{dx}\left[x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + \sqrt{\frac{2}{5}}\right]$

$8x^7 + 12(5)x^4 - 4(4)x^3 + 10(3)x^2 - 6 + 0$

$8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

b) If $f(x) = (3x-2)^2$ then $f'(x) = \dots$

$f(x) = 9x^2 - 2(3x)(2) + 4$

$= 9x^2 - 12x + 4$

$f'(x) = 18x - 12$

c) $\frac{d}{dx}\left[x^2(1-2x)\right] = \frac{d}{dx}\left[x^2 - 2x^3\right]$

$= 2x - 6x^2$

d) $\frac{d}{dt}\left[\sqrt{t}\,(t-1)\right] = \frac{d}{dt}\left[t^{1/2}(t-1)\right]$

$= \frac{d}{dt}\left[t^{1/2} \cdot t^1 - t^{1/2}\right] = \frac{d}{dt}\left[t^{3/2} - t^{1/2}\right]$

$= \frac{3}{2}t^{3/2 - 1} - \frac{1}{2}t^{1/2 - 1}$

$= \frac{3}{2}t^{1/2} - \frac{1}{2}t^{-1/2} = \frac{3}{2}\sqrt{t} - \frac{1}{2t^{1/2}}$

$= \frac{3}{2}\sqrt{t} - \frac{1}{2\sqrt{t}} = \frac{3\sqrt{t}\cdot\sqrt{t} - 1}{2\sqrt{t}} = \frac{3t-1}{2\sqrt{t}}$

e) $\dfrac{d}{dx}\left[(2x+3)(4x-5)\right]$

$\dfrac{d}{dx}\left[2x(4x-5)+3(4x-5)\right]$

$\dfrac{d}{dx}\left[8x^2-10x+12x-15\right]$

$\dfrac{d}{dx}\left[8x^2+2x-15\right] = 16x+2$

f) $\dfrac{d}{dx}\left[(x-2)^3\right] = \dfrac{d}{dx}\left[x^3-3(2)x^2+3(4)x-2^3\right]$

$= \dfrac{d}{dx}\left[x^3-6x^2+12x-8\right]$

$= 3x^2-2(6)x+12$

$= 3x^2-12x+12$

g) $\dfrac{d}{dx}\left[x(2x+3)^2\right] = \dfrac{d}{dx}\left[x(4x^2+12x+9)\right]$

$= \dfrac{d}{dx}\left[4x^3+12x^2+9x\right]$

$= 12x^2+24x+9$

h) $f(t) = (3x^2+2)(x^3-5)$

$f'(t) =$ H.W

if $G(x) = \frac{5x^2+4x+3}{x^2}$ then $G'(x) =$ ———

$$G(x) = \frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{3}{x^2}$$

$$= 5 + \frac{4}{x} + \frac{3}{x^2}$$

$$= 5 + 4x^{-1} + 3x^{-2}$$

$$G'(x) = 0 + 4(-1)x^{-1-1} + 3(-2)x^{-2-1}$$

$$= -4x^{-2} - 6x^{-3}$$

$$= \frac{-4}{x^2} - \frac{6}{x^3}$$

$$= \frac{-4x}{x^2 \cdot x} - \frac{6}{x^3}$$

$$= \frac{-4x}{x^3} - \frac{6}{x^3}$$

$$= \frac{-4x-6}{x^3}$$

if $y = \frac{\sqrt{x}+x}{x^2}$ then $y' =$ – – .

$$y = \frac{x^{1/2}+x^1}{x^2} = \frac{x^{1/2}}{x^2} + \frac{x^1}{x^2} = x^{1/2-2} + x^{1-2}$$

$$= x^{-3/2} + x^{-1}$$

$$y' = -\frac{3}{2}x^{-\frac{3}{2}-1} - x^{-1-1} = -\frac{3}{2}x^{-5/2} - x^{-2} = \frac{-3}{2x^{5/2}} - \frac{1}{x^2} = \frac{-3}{2\sqrt{x^5}} - \frac{1}{x^2}$$

5] $\frac{d}{dx}[a^x] = a^x . \ln a$

$\frac{d}{dx}[e^x] = e^x$

Example

$\frac{d}{dx}[\pi^x] = \pi^x . \ln\pi = \ln(\pi) \cdot (\pi)^x$

$\frac{d}{dx}[\sqrt{2^x}] = \frac{d}{dx}[(\sqrt{2})^x]$

$= (\sqrt{2})^x \cdot \ln\sqrt{2}$

$= (\sqrt{2})^x . \ln 2^{1/2}$

$= (\sqrt{2})^x . \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln 2 \cdot (\sqrt{2})^x$

$\frac{d}{dx}[3^x + x^3] = \frac{d}{dx}[3^x] + \frac{d}{dx}[x^3]$

$= 3^x . \ln(3) + 3x^2$

$\frac{d}{dx}[e^x - x^e] = \frac{d}{dx}[e^x] - \frac{d}{dx}[x^e]$

$= e^x - e x^{e-1}$

$= e(e^{x-1} - x^{e-1})$

if $y = e^{x+1} + x^2$ then find ① y''' or $\frac{d^3y}{dx^3}$

② $y^{(100)}$

$y' = e^{x+1} + 2x$

$y'' = e^{x+1} + 2$

$y''' = e^{x+1}$

$y^{(4)} = e^{x+1}$

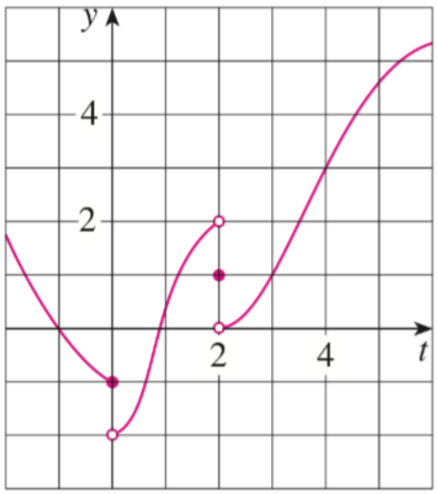
$y^{(5)} = e^{x+1}$

$\vdots$

$y^{(100)} = e^{x+1}$

SECOND EXAM-MATH 110
FROM SECTION 2.2 TO SECTION 3.1

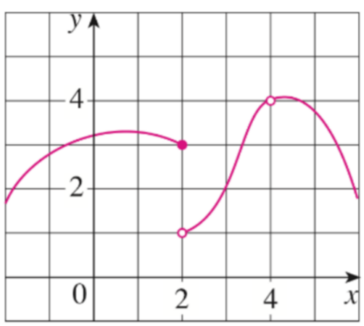
1. If $f(x)$ is a function whose graph is shown



then $\lim_{x\to 0} f(x) =$ ....

a) 0 b) -1 c) -2 d) does not exist

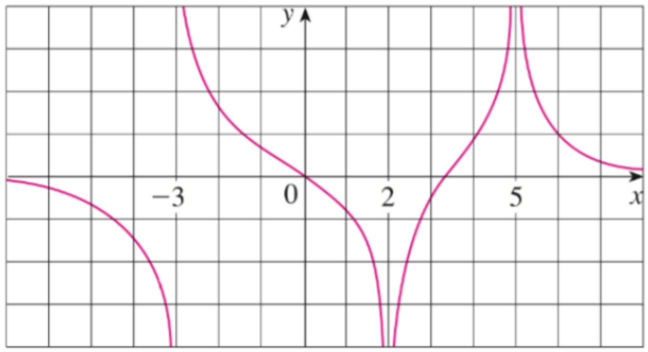
2. If $f(x)$ is a function whose graph is shown



then $\lim_{x\to 2^-} f(x) =$ ....

a) 1 b) 3 c) 2 d) does not exist

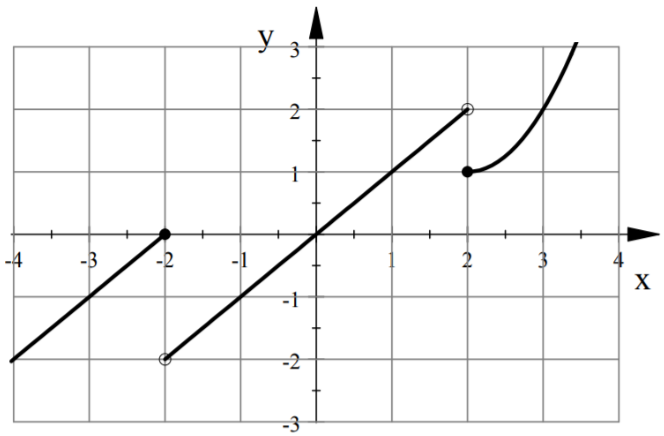
3. If $f(x)$ is a function whose graph is shown



then $\lim\limits_{x \to -3^+} f(x) = -\infty$

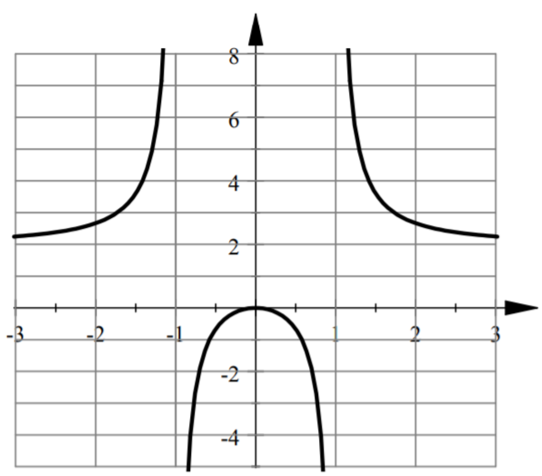
a) True   b) False

4. If $f(x)$ is a function whose graph is shown is discontinuous at …………….



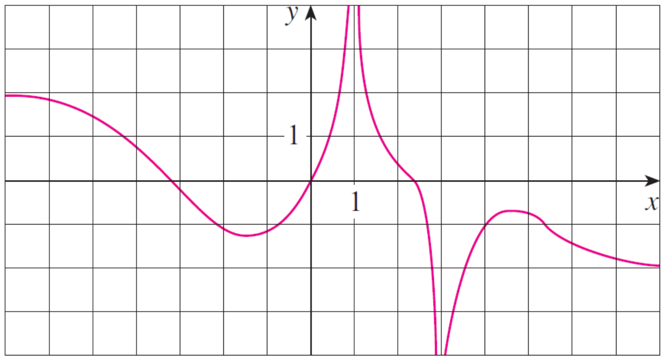
a) $x = -3$ b) $x = -1$ c) $x = -2$ d) $x = 0$

5. The vertical asymptote(s) of the function whose graph is shown below is (are)…………………



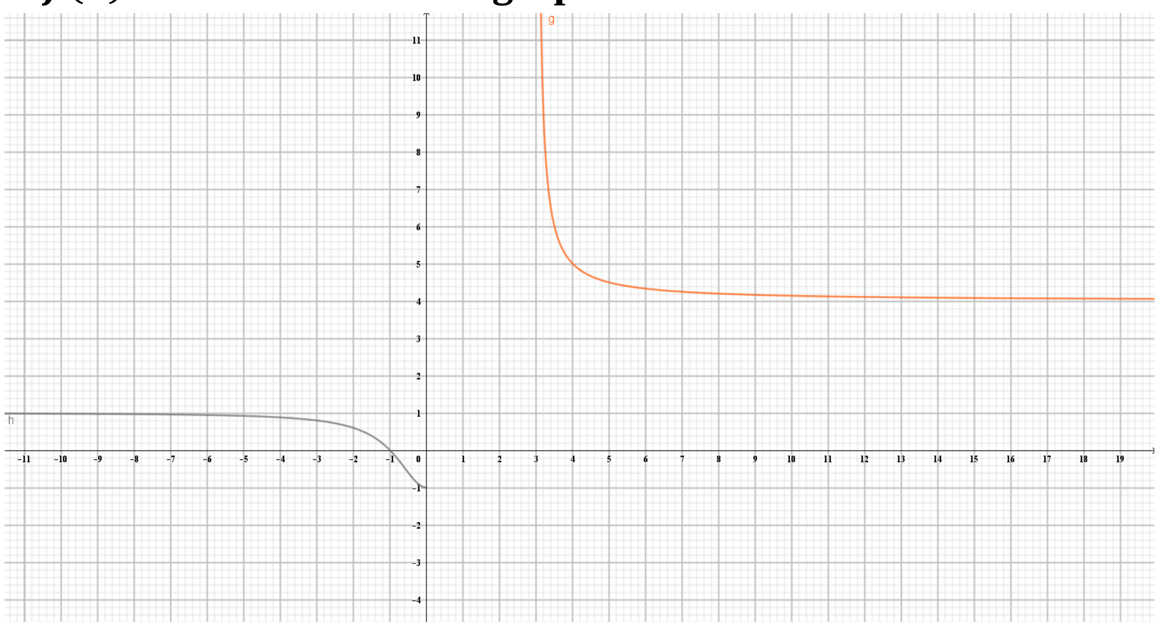
a) $y = 2$
b) $x = 2$
c) $y = -1$ and $y = 1$
d) $x = -1$ and $x = 1$

6. **The horizontal asymptote(s) of the function whose graph is shown below is (are)…………………**



a) $y = 1$
b) $x = 1$
c) $y = -2$ and $y = 2$
d) $x = -2$ and $x = 2$

7. **If $f(x)$ is a function whose graph is shown**



then $\lim_{x\to\infty} f(x) = 4$

a) **True** b) **False**

8. $\lim_{x\to1} \frac{\sqrt{x}-4}{x^2-4x} = \ldots\ldots.$

a) 1 b) $\frac{1}{5}$ c) -1 d) $-\frac{1}{5}$

9. $\lim_{h\to0} \frac{(h+5)^2-25}{h} = \ldots\ldots.$

a) 0 b) 1 c) 10 d) 5

10. $\lim_{x \to -3} \dfrac{x^2 + 3x}{x^2 - x - 12} = \ldots\ldots.$

a) 3 b) -3 c) $\frac{3}{7}$ d) $-\frac{3}{7}$

11. $\lim_{x \to 5} \dfrac{\frac{1}{5} - \frac{1}{x}}{5 - x} = \ldots\ldots.$

a) $-\frac{1}{25}$ b) $\frac{1}{25}$ c) $\frac{1}{5}$ d) $-\frac{1}{5}$

12. $\lim_{u \to 2} \dfrac{u - 2}{\sqrt{2u^2 + 1} - 3} = \ldots\ldots.$

b) 1 b) 0 c) $\frac{3}{4}$ d) $\frac{3}{2}$

13. $\lim_{t \to 1^-} \ln(1 - t) = \ldots\ldots.$

a) 1 b) 0 c) $-\infty$ d) $\ln(2)$

14. $\lim_{x \to 4} \dfrac{e^c}{\sqrt{c}} = \dfrac{e^4}{2}$

a) True b) False

15. $\lim_{x \to 7^-} \dfrac{x^2 - 49}{|x - 7|} = \ldots\ldots.$

a) 14 b) -14 c) does not exist d) 0

16. $\lim_{x \to 8} \dfrac{6 - x}{(x - 8)^2} = -\infty$

a) True b) False

17. If $\lim\limits_{x\to 4} \dfrac{10f(x)-6}{3x+4f(x)} = 2$ then $\lim\limits_{x\to 4} f(x) = \ldots\ldots.$

a) 15 b) 14 c) 30 d) 28

18. If $f(x) = \begin{cases} \dfrac{\tan 5x}{\sin 3x} & \text{if} \quad x \neq 0 \\ 2x+10 & \text{if} \quad x = 0 \end{cases}$ then $\lim\limits_{x\to 0} f(x) = \ldots\ldots.$

a) $\frac{5}{3}$ b) 10 c) $\frac{3}{5}$ d) 1

19. If $2\sin x \leq f(x) \leq \sec x$ then $\lim\limits_{x\to \frac{\pi}{4}} f(x) = \ldots\ldots.$

a) $\frac{1}{\sqrt{2}}$ b) does not exist c) 2 d) $\sqrt{2}$

20. If $\lim\limits_{x\to 2} f(x) = 4$ then $\lim\limits_{x\to 2}\left(2f(x) - \dfrac{1}{x}\right) = \dfrac{15}{2}$

a) True b) False

21. $\lim\limits_{x\to\sqrt{\pi}} \left(\dfrac{\cos(x^2)-1}{x^2}\right) = \ldots\ldots.$

a) 0 b) 1 c) $\frac{-2}{\pi}$ d) $\frac{2}{\pi}$

22. $\lim\limits_{x\to\infty} \dfrac{6-x-14x^2}{2x^2-x-12} = \ldots\ldots.$

b) 1 b) 7 c) -7 d) 3

23. $\lim_{x\to\infty} \dfrac{\sqrt{3x^2 - x}}{1 - 4x} = \ldots\ldots.$

a) $\frac{\sqrt{3}}{4}$ b) 0 c) $-\frac{\sqrt{3}}{4}$ d) ∞

24. $\lim_{x\to\infty} \sqrt{4 + 5x^{-2}} = \ldots\ldots.$

a) ∞ b) 2 c) $-\infty$ d) 3

25. $\lim_{x\to-\infty} (x^2 - 5x^7) = \ldots\ldots.$

a) ∞ b) -4 c) $-\infty$ d) -5

26. The vertical asymptote(s) of the function

$f(x) = \frac{4-x^2}{3x^2-5x-2}$ is (are) … … ….

a) $x = 2$ and $x = -\frac{1}{3}$ b) $x = -\frac{1}{3}$ c) $x = 2$

d) $y = 2$ e) $y = -\frac{1}{3}$

27. The horizontal asymptote(s) of the function

$f(x) = \frac{2e^x}{3e^x-5}$ is (are) … … ….

a) $x = \frac{2}{3}$ and $x = -\frac{2}{3}$ b) $x = \frac{2}{3}$ and $x = 0$

c) $y = \frac{2}{3}$ and $y = 0$ d) $y = \frac{2}{3}$ and $y = -\frac{2}{3}$

28. $f(x) = \tan(x)$ is discontinuous at……

a) $x = \frac{7\pi}{4}$ b) $x = \frac{7\pi}{3}$ c) $x = \frac{7\pi}{2}$ d) $x = 0$

29. If $f(x) = \begin{cases} cx^2 + 2x & \text{if} \quad x \geq 3 \\ x^3 - cx & \text{if} \quad x < 3 \end{cases}$ is continuous on $\mathbb{R}$

then $c = \ldots\ldots$

a) $\frac{7}{4}$ b) $\frac{1}{3}$ c) $\frac{7}{2}$ d) 1

30. $f(x) = \ln(x) - \sqrt{3 - x}$ is continuous on……………

a) $(0, \infty)$ b) $(0, 3]$ c) $[0, 3]$ d) $(-\infty, 3]$

31. $f(x) = \dfrac{x - 2}{x^3 + 9x}$ is discontinuous at……

a) $x = 2$ b) $x = 0$ c) $x = 0$ and $x = \pm 3$

32. $f(x) = \begin{cases} x^2 - 3x - 8 & \text{if} \quad x \geq 3 \\ \dfrac{\sin(x - 3)}{(x - 3)} & \text{if} \quad x < 3 \end{cases}$ is continuous on $\mathbb{R}$

a) True b) False

33. If $f(x) = |3x - 6|$ then $f(x)$ is not differentiable at

a) $x = 2$ b) $x = -2$ c) $x = 3$ d) $x = 6$

34. If $y = \sqrt{\pi}$ then $y' = \dfrac{1}{2\sqrt{\pi}}$

a) True b) False

35. The equation of the tangent line of the curve $f(x) = 4x - 3x^2$ at $x = 2$ is ....

a) $y = 12 - 8x$ b) $y = \frac{1}{8}x - \frac{17}{4}$

c) $x = 12 - 8y$ d) $x = \frac{1}{8}x - \frac{17}{4}$

36. If $g(x) = e^x + x^e$ then $g'(1) =$ ....

a) 2 b) e^2 c) 2e d) 1

37. If $g(x) = \dfrac{15x^6 - 12x^4 + 6x^2}{3x^2}$ then $g''(x) =$ ....

a) $5x^4 - 4x^2 + 2$ b) $120x$

c) $60x^2 - 8$ d) $20x^3 - 8x$

38. If $h(x) = \sqrt{1+2x}$ then $h'(2) =$ ....

$a)$ $\lim\limits_{x\to 2} \dfrac{\sqrt{1+2x} - \sqrt{5}}{x-2}$ $b)$ $\lim\limits_{x\to 2} \dfrac{\sqrt{5} - \sqrt{1+2x}}{x-2}$

$c)$ $\lim\limits_{h\to 0} \dfrac{\sqrt{1+2h} - \sqrt{5}}{h}$ $d)$ $\lim\limits_{h\to 2} \dfrac{\sqrt{4+2h} - \sqrt{5}}{h}$

39. If $f(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x$ then $f(x)$ has horizontal tangents when ... ...

$a)$ $x = 5, -2$ $b)$ $x = -5, 2$

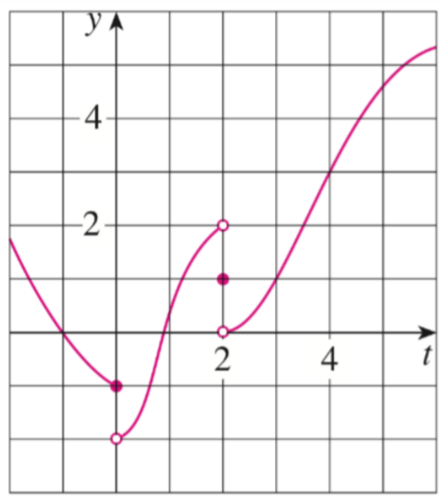
$c)$ $x = 5, 2$ $d)$ $x = -5, -2$

40. If f is differentiable at a , then f is continuous at a

a) True b) False

SECOND EXAM-MATH 110
FROM SECTION 2.2 TO SECTION 3.1

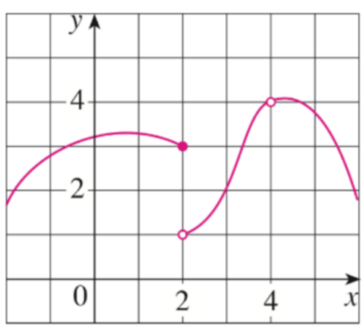
1. If $f(x)$ is a function whose graph is shown



then $\lim_{x\to 0} f(x) = \dots\dots$

a) 0 b) -1 c) -2 d) does not exist

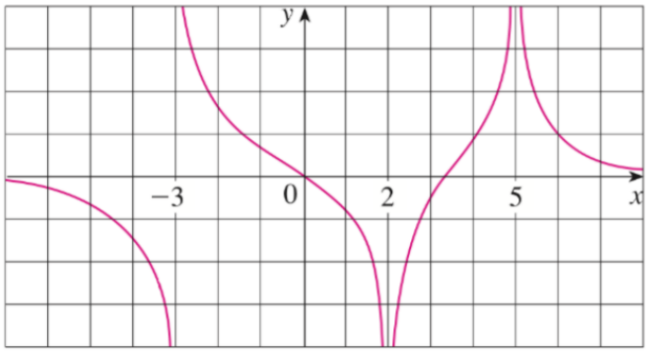
2. If $f(x)$ is a function whose graph is shown



then $\lim_{x\to 2^-} f(x) = \dots\dots$

a) 1 b) 3 c) 2 d) does not exist

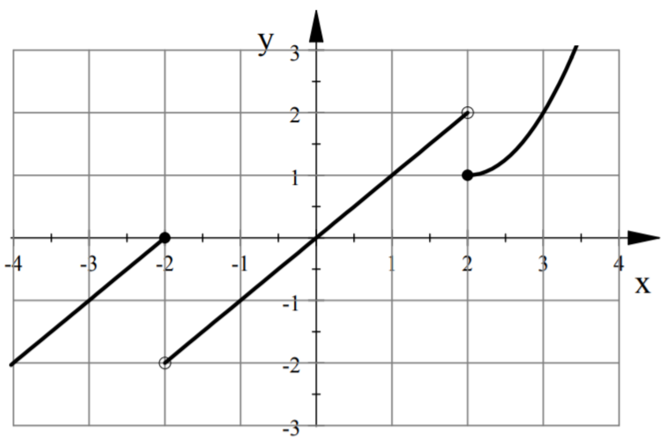
3. **If $f(x)$ is a function whose graph is shown**



then $\lim_{x\to -3^+} f(x) = -\infty$

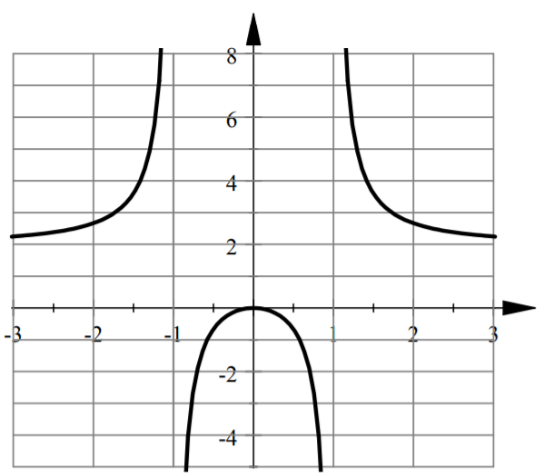
a) True **b) False**

4. **If $f(x)$ is a function whose graph is shown is discontinuous at …………….**



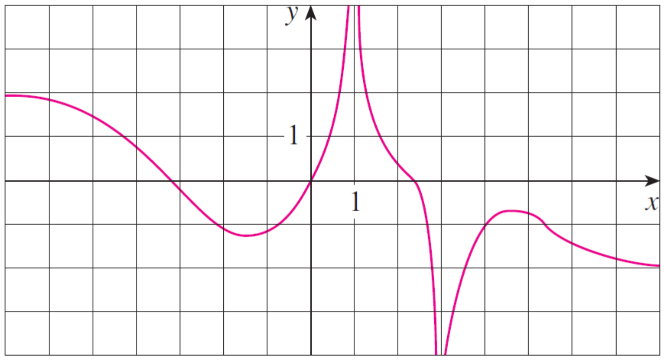
a) $x = -3$ **b)** $x = -1$ **c)** $x = -2$ **d)** $x = 0$

5. **The vertical asymptote(s) of the function whose graph is shown below is (are)…………………**



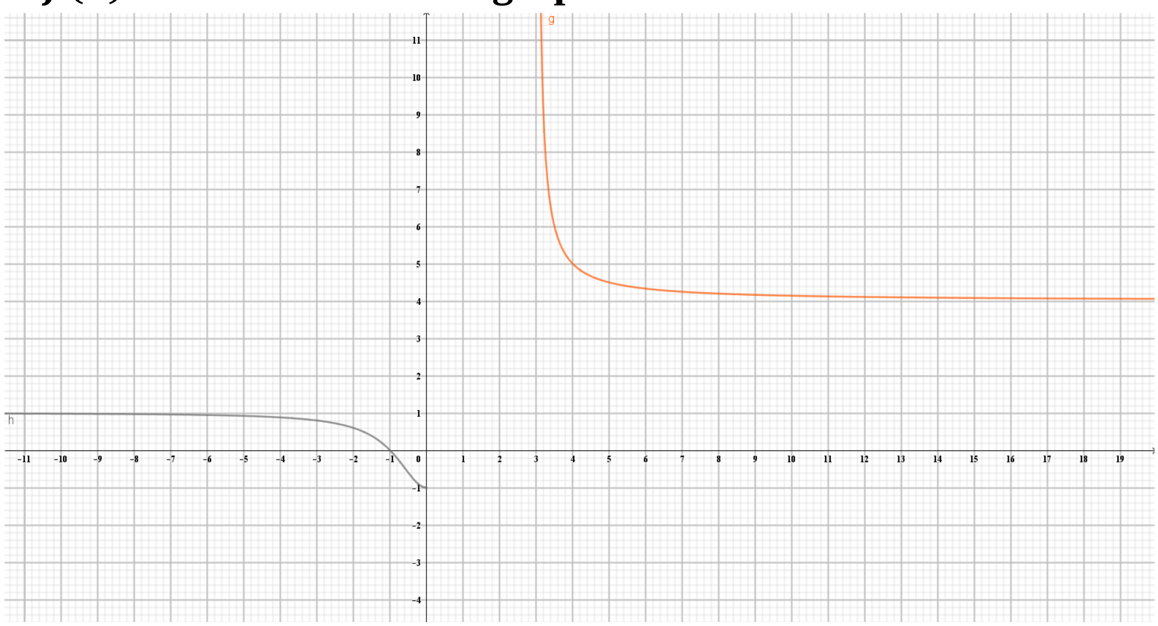
a) $y = 2$
b) $x = 2$
c) $y = -1$ and $y = 1$
d) $x = -1$ and $x = 1$

6. **The horizontal asymptote(s) of the function whose graph is shown below is (are)…………………**



a) $y = 1$
b) $x = 1$
c) $y = -2$ and $y = 2$
d) $x = -2$ and $x = 2$

7. **If $f(x)$ is a function whose graph is shown**



then $\lim_{x \to \infty} f(x) = 4$

a) **True** b) **False**

8. $\lim_{x \to 1} \dfrac{\sqrt{x} - 4}{x^2 - 4x} = \ldots\ldots.$

a) 1 b) $\frac{1}{5}$ c) -1 d) $-\frac{1}{5}$

9. $\lim_{h \to 0} \dfrac{(h+5)^2 - 25}{h} = \ldots\ldots.$

a) 0 b) 1 c) 10 d) 5

10. $\lim\limits_{x\to -3} \dfrac{x^2+3x}{x^2-x-12} = \ldots\ldots.$

a) 3 b) -3 c) $\frac{3}{7}$ d) $-\frac{3}{7}$

11. $\lim\limits_{x\to 5} \dfrac{\frac{1}{5}-\frac{1}{x}}{5-x} = \ldots\ldots.$

a) $-\frac{1}{25}$ b) $\frac{1}{25}$ c) $\frac{1}{5}$ d) $-\frac{1}{5}$

12. $\lim\limits_{u\to 2} \dfrac{u-2}{\sqrt{2u^2+1}-3} = \ldots\ldots.$

b) 1 b) 0 c) $\frac{3}{4}$ d) $\frac{3}{2}$

13. $\lim\limits_{t\to 1^-} \ln(1-t) = \ldots\ldots.$

a) 1 b) 0 c) $-\infty$ d) $\ln(2)$

14. $\lim\limits_{x\to 4} \dfrac{e^c}{\sqrt{c}} = \dfrac{e^4}{2}$

a) True b) False

15. $\lim\limits_{x\to 7^-} \dfrac{x^2-49}{|x-7|} = \ldots\ldots.$

a) 14 b) -14 c) does not exist d) 0

16. $\lim\limits_{x\to 8} \dfrac{6-x}{(x-8)^2} = -\infty$

a) True b) False

17. If $\lim\limits_{x\to 4} \dfrac{10f(x)-6}{3x+4f(x)} = 2$ then $\lim\limits_{x\to 4} f(x) = \ldots\ldots.$

a) 15 b) 14 c) 30 d) 28

18. If $f(x) = \begin{cases} \dfrac{\tan 5x}{\sin 3x} & \text{if} \quad x \neq 0 \\ 2x+10 & \text{if} \quad x = 0 \end{cases}$ then $\lim\limits_{x\to 0} f(x) = \ldots\ldots.$

a) $\frac{5}{3}$ b) 10 c) $\frac{3}{5}$ d) 1

19. If $2\sin x \leq f(x) \leq \sec x$ then $\lim\limits_{x\to \frac{\pi}{4}} f(x) = \ldots\ldots.$

a) $\frac{1}{\sqrt{2}}$ b) does not exist c) 2 d) $\sqrt{2}$

20. If $\lim\limits_{x\to 2} f(x) = 4$ then $\lim\limits_{x\to 2} \left(2f(x) - \dfrac{1}{x}\right) = \dfrac{15}{2}$

a) True b) False

21. $\lim\limits_{x\to \sqrt{\pi}} \left(\dfrac{\cos(x^2)-1}{x^2}\right) = \ldots\ldots.$

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22. $\lim\limits_{x\to \infty} \dfrac{6-x-14x^2}{2x^2-x-12} = \ldots\ldots.$

b) 1 b) 7 c) -7 d) 3

23. $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2 - x}}{1 - 4x} = \text{.......}$

a) $\frac{\sqrt{3}}{4}$ b) 0 c) $-\frac{\sqrt{3}}{4}$ d) ∞

24. $\lim\limits_{x\to\infty} \sqrt{4 + 5x^{-2}} = \text{.......}$

a) ∞ b) 2 c) $-\infty$ d) 3

25. $\lim\limits_{x\to-\infty} (x^2 - 5x^7) = \text{.......}$

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26. The vertical asymptote(s) of the function

$f(x) = \frac{4-x^2}{3x^2-5x-2}$ is (are) ... ...

a) $x = 2$ and $x = -\frac{1}{3}$ b) $x = -\frac{1}{3}$ c) $x = 2$

d) $y = 2$ e) $y = -\frac{1}{3}$

27. The horizontal asymptote(s) of the function

$f(x) = \frac{2e^x}{3e^x-5}$ is (are) ... ...

a) $x = \frac{2}{3}$ and $x = -\frac{2}{3}$ b) $x = \frac{2}{3}$ and $x = 0$

c) $y = \frac{2}{3}$ and $y = 0$ d) $y = \frac{2}{3}$ and $y = -\frac{2}{3}$

28. $f(x) = \tan(x)$ is discontinuous at......

a) $x = \frac{7\pi}{4}$ b) $x = \frac{7\pi}{3}$ c) $x = \frac{7\pi}{2}$ d) $x = 0$

29. If $f(x) = \begin{cases} cx^2 + 2x & \text{if} \quad x \geq 3 \\ x^3 - cx & \text{if} \quad x < 3 \end{cases}$ is continuous on $\mathbb{R}$

then $c =$ ....

a) $\frac{7}{4}$ b) $\frac{1}{3}$ c) $\frac{7}{2}$ d) 1

30. $f(x) = \ln(x) - \sqrt{3-x}$ is continuous on................

a) $(0, \infty)$ b) $(0, 3]$ c) $[0, 3]$ d) $(-\infty, 3]$

31. $f(x) = \dfrac{x-2}{x^3 + 9x}$ is discontinuous at......

a) $x = 2$ b) $x = 0$ c) $x = 0$ and $x = \pm 3$

32. $f(x) = \begin{cases} x^2 - 3x - 8 & \text{if} \quad x \geq 3 \\ \dfrac{\sin(x-3)}{(x-3)} & \text{if} \quad x < 3 \end{cases}$ is continuous on $\mathbb{R}$

a) True b) False

33. If $f(x) = |3x - 6|$ then $f(x)$ is not differentiable at

a) $x = 2$ b) $x = -2$ c) $x = 3$ d) $x = 6$

34. If $y = \sqrt{\pi}$ then $y' = \dfrac{1}{2\sqrt{\pi}}$

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35. The equation of the tangent line of the curve $f(x) = 4x - 3x^2$ at $x = 2$ is ....

a) $y = 12 - 8x$ b) $y = \frac{1}{8}x - \frac{17}{4}$

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36. If $g(x) = e^x + x^e$ then $g'(1) =$ ....

a) 2 b) e^2 c) 2e d) 1

37. If $g(x) = \dfrac{15x^6 - 12x^4 + 6x^2}{3x^2}$ then $g''(x) =$ ....

a) $5x^4 - 4x^2 + 2$ b) $120x$

c) $60x^2 - 8$ d) $20x^3 - 8x$

If $h(x) = \sqrt{1 + 2x}$ then $h'(2) =$ ....

$a)\ \lim\limits_{x\to 2} \dfrac{\sqrt{1+2x} - \sqrt{5}}{x - 2}$ $b)\ \lim\limits_{x\to 2} \dfrac{\sqrt{5} - \sqrt{1+2x}}{x - 2}$

$c)\ \lim\limits_{h\to 0} \dfrac{\sqrt{1+2h} - \sqrt{5}}{h}$ $d)\ \lim\limits_{h\to 2} \dfrac{\sqrt{4+2h} - \sqrt{5}}{h}$

39. If $f(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x$ then $f(x)$ has horizontal tangents when … … … …

$a)\ x = 5, -2$ $b)\ x = -5, 2$

$c)\ x = 5, 2$ $d)\ x = -5, -2$

40. If f is differentiable at a , then f is continuous at a

a) True b) False

Workshop Solutions to Sections 3.4 and 3.5 (2.2 & 2.5)

1) $\lim\limits_{x\to 3^+} \dfrac{2}{x-3} =$

Solution:

If $x \to 3^+$, then $x > 3 \implies x - 3 > 0$

$$\therefore \lim_{x\to 3^+} \frac{2}{x-3} = \infty$$

2) $\lim\limits_{x\to 3^-} \dfrac{2}{x-3} =$

Solution:

If $x \to 3^-$, then $x < 3 \implies x - 3 < 0$

$$\therefore \lim_{x\to 3^-} \frac{2}{x-3} = -\infty$$

3) $\lim\limits_{x\to 3^+} \dfrac{-2}{x-3} =$

Solution:

If $x \to 3^+$, then $x > 3 \implies x - 3 > 0$

$$\therefore \lim_{x\to 3^+} \frac{-2}{x-3} = -\infty$$

4) $\lim\limits_{x\to 3^-} \dfrac{-2}{x-3} =$

Solution:

If $x \to 3^-$, then $x < 3 \implies x - 3 < 0$

$$\therefore \lim_{x\to 3^-} \frac{2}{x-3} = \infty$$

5) $\lim\limits_{x\to -3^+} \dfrac{2}{x+3} =$

Solution:

If $x \to -3^+$, then $x > -3 \implies x + 3 > 0$

$$\therefore \lim_{x\to -3^+} \frac{2}{x+3} = \infty$$

6) $\lim\limits_{x\to -3^-} \dfrac{2}{x+3} =$

Solution:

If $x \to -3^-$, then $x < -3 \implies x + 3 < 0$

$$\therefore \lim_{x\to -3^-} \frac{2}{x+3} = -\infty$$

7) $\lim\limits_{x\to 2^+} \dfrac{3x-1}{x-2} =$

Solution:

If $x \to 2^+$, then $x > 2 \implies x - 2 > 0$ and $3x - 1 > 0$

$$\therefore \lim_{x\to 2^+} \frac{3x-1}{x-2} = \infty$$

8) $\lim\limits_{x\to 2^-} \dfrac{3x-1}{x-2} =$

Solution:

If $x \to 2^-$, then $x < 2 \implies x - 2 < 0$ and $3x - 1 > 0$

$$\therefore \lim_{x\to 2^-} \frac{3x-1}{x-2} = -\infty$$

9) $\lim\limits_{x\to -2^+} \dfrac{1-x}{(x+2)^2} =$

Solution:

If $x \to -2^+$, then $x > -2$

$\implies 1 - x > 0$ and $(x+2)^2 > 0$

$$\therefore \lim_{x\to -2^+} \frac{1-x}{(x+2)^2} = \infty$$

10) $\lim\limits_{x\to -2^-} \dfrac{1-x}{(x+2)^2} =$

Solution:

If $x \to -2^-$, then $x < -2$

$\implies 1 - x > 0$ and $(x+2)^2 > 0$

$$\therefore \lim_{x\to -2^+} \frac{1-x}{(x+2)^2} = \infty$$

11) $\lim\limits_{x\to -2^+} \dfrac{x-1}{(x+2)^2} =$

Solution:

If $x \to -2^+$, then $x > -2$

$\implies x - 1 < 0$ and $(x+2)^2 > 0$

$$\therefore \lim_{x\to -2^+} \frac{x-1}{(x+2)^2} = -\infty$$

12) $\lim\limits_{x\to -2^-} \dfrac{x-1}{(x+2)^2} =$

Solution:

If $x \to -2^-$, then $x < -2$

$\implies x - 1 < 0$ and $(x+2)^2 > 0$

$$\therefore \lim_{x\to -2^-} \frac{x-1}{(x+2)^2} = -\infty$$

13) $\lim\limits_{x\to 2^+} \dfrac{6x-1}{x^2-4} =$

Solution:

If $x \to 2^+$, then $x^2 > 4$

$\implies x^2 - 4 > 0$ and $6x - 1 > 0$

$$\therefore \lim_{x\to 2^+} \frac{6x-1}{x^2-4} = \infty$$

14) $\lim\limits_{x\to 2^-} \dfrac{6x-1}{x^2-4} =$

Solution:

If $x \to 2^-$, then $x^2 < 4$

$\implies x^2 - 4 < 0$ and $6x - 1 > 0$

$$\therefore \lim_{x\to 2^+} \frac{6x-1}{x^2-4} = -\infty$$

15) $\lim\limits_{x\to -2^+} \dfrac{6x-1}{x^2-4} =$

Solution:

If $x \to -2^+$, then $x^2 < 4$

$$\Longrightarrow\ x^2-4<0 \text{ and } 6x-1<0$$

$$\therefore\ \lim_{x\to 2^+} \frac{6x-1}{x^2-4} = \infty$$

16) $\lim\limits_{x\to -2^-} \dfrac{6x-1}{x^2-4} =$

Solution:

If $x \to -2^-$, then $x^2 > 4$

$$\Longrightarrow\ x^2-4>0 \text{ and } 6x-1<0$$

$$\therefore\ \lim_{x\to 2^+} \frac{6x-1}{x^2-4} = -\infty$$

17) $\lim\limits_{x\to -2^-} \dfrac{6x-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{6x-1}{x^2-x-6} = \frac{6x-1}{(x-3)(x+2)}$$

If $x \to -2^-$, then $x < -2$

$$\Longrightarrow\ x-3<0,\ x+2<0 \text{ and } 6x-1<0$$

$$\therefore\ \lim_{x\to -2^-} \frac{6x-1}{x^2-x-6} = -\infty$$

18) $\lim\limits_{x\to -2^+} \dfrac{6x-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{6x-1}{x^2-x-6} = \frac{6x-1}{(x-3)(x+2)}$$

If $x \to -2^+$, then $x > -2$

$$\Longrightarrow\ x-3<0,\ x+2>0 \text{ and } 6x-1<0$$

$$\therefore\ \lim_{x\to -2^+} \frac{6x-1}{x^2-x-6} = \infty$$

19) $\lim\limits_{x\to 3^+} \dfrac{-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{-1}{x^2-x-6} = \frac{-1}{(x-3)(x+2)}$$

If $x \to 3^+$, then $x > 3$

$$\Longrightarrow\ x-3>0,\ x+2>0 \text{ and } -1<0$$

$$\therefore\ \lim_{x\to 3^+} \frac{-1}{x^2-x-6} = -\infty$$

20) $\lim\limits_{x\to 3^-} \dfrac{-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{-1}{x^2-x-6} = \frac{-1}{(x-3)(x+2)}$$

If $x \to 3^-$, then $x < 3$

$$\Longrightarrow\ x-3<0,\ x+2>0 \text{ and } -1<0$$

$$\therefore\ \lim_{x\to 3^-} \frac{-1}{x^2-x-6} = \infty$$

21) $\lim\limits_{x\to (\pi/2)^+} \tan x =$

Solution:

$$\lim_{x\to (\pi/2)^+} \tan x = -\infty$$

22) $\lim\limits_{x\to (\pi/2)^-} \tan x =$

Solution:

$$\lim_{x\to (\pi/2)^-} \tan x = \infty$$

23) The vertical asymptote of $f(x) = \frac{1-x}{2x+1}$ is

Solution:

We see that the function $f(x)$ is not defined when $2x+1=0 \ \Longrightarrow\ x=-\frac{1}{2}$. Since

$$\lim_{x\to \left(-\frac{1}{2}\right)^+} \frac{1-x}{2x+1} = \infty$$

and

$$\lim_{x\to \left(-\frac{1}{2}\right)^-} \frac{1-x}{2x+1} = -\infty$$

then, $x=-\frac{1}{2}$ is a vertical asymptote.

24) The vertical asymptote of $f(x) = \frac{3-x}{x^2-4}$ is

Solution:

We see that the function $f(x)$ is not defined when $x^2-4=0 \ \Longrightarrow\ x=\pm 2$. Since

$$\lim_{x\to 2^+} \frac{3-x}{x^2-4} = \infty, \qquad \lim_{x\to 2^-} \frac{3-x}{x^2-4} = -\infty$$

and

$$\lim_{x\to -2^+} \frac{3-x}{x^2-4} = -\infty, \qquad \lim_{x\to -2^-} \frac{3-x}{x^2-4} = \infty$$

then, $x=\pm 2$ are vertical asymptotes.

25) The vertical asymptote of $f(x)=\frac{3-x}{x^2-x-6}$ is

Solution:

$$f(x)=\frac{3-x}{x^2-x-6}=\frac{3-x}{(x-3)(x+2)}=\frac{-(x-3)}{(x-3)(x+2)}$$
$$=-\frac{1}{x+2}$$

We see that the function $f(x)$ is not defined when $x^2-x-6=0 \implies (x-3)(x+2)=0 \implies x=3$ or $x=-2$. Since

$$\lim_{x\to 3}\frac{3-x}{x^2-x-6}=\lim_{x\to 3}\frac{3-x}{(x-3)(x+2)}$$
$$=\lim_{x\to 3}\frac{-(x-3)}{(x-3)(x+2)}=\lim_{x\to 3}\frac{-1}{x+2}=-\frac{1}{5}$$

then, $x=3$ is a removable discontinuity.

$$\lim_{x\to -2^+}\frac{3-x}{x^2-x-6}=\lim_{x\to -2^+}\frac{3-x}{(x-3)(x+2)}=-\infty$$

and

$$\lim_{x\to -2^-}\frac{3-x}{x^2-x-6}=\lim_{x\to -2^-}\frac{3-x}{(x-3)(x+2)}=-\infty \quad +$$

then, $x=-2$ is a vertical asymptote only.

26) The vertical asymptote of $f(x)=\frac{7-x}{x^2-5x+6}$ is

Solution:

$$f(x)=\frac{7-x}{x^2-5x+6}=\frac{7-x}{(x-3)(x-2)}$$

We see that the function $f(x)$ is not defined when $x-3=0$ or $x-2=0 \implies x=3$ or $x=2$. Since

$$\lim_{x\to 3^+}\frac{7-x}{x^2-5x+6}=\lim_{x\to 3^+}\frac{7-x}{(x-3)(x-2)}=\infty$$
$$\lim_{x\to 3^-}\frac{7-x}{x^2-5x+6}=\lim_{x\to 3^-}\frac{7-x}{(x-3)(x-2)}=-\infty$$

and

$$\lim_{x\to 2^+}\frac{7-x}{x^2-5x+6}=\lim_{x\to 2^+}\frac{7-x}{(x-3)(x-2)}=-\infty$$
$$\lim_{x\to 2^-}\frac{7-x}{x^2-5x+6}=\lim_{x\to 2^-}\frac{7-x}{(x-3)(x-2)}=\infty$$

then, $x=3$ and $x=2$ are vertical asymptotes.

27) The vertical asymptote of $f(x)=\frac{x-7}{x^2+5x+6}$ is

Solution:

$$f(x)=\frac{x-7}{x^2+5x+6}=\frac{x-7}{(x+3)(x+2)}$$

We see that the function $f(x)$ is not defined when $x+3=0$ or $x+2=0 \implies x=-3$ or $x=-2$. Since

$$\lim_{x\to -3^+}\frac{x-7}{x^2+5x+6}=\lim_{x\to -3^+}\frac{x-7}{(x+3)(x+2)}=\infty$$
$$\lim_{x\to -3^-}\frac{x-7}{x^2+5x+6}=\lim_{x\to -3^-}\frac{x-7}{(x+3)(x+2)}=-\infty$$

and

$$\lim_{x\to -2^+}\frac{x-7}{x^2+5x+6}=\lim_{x\to -2^+}\frac{x-7}{(x+3)(x+2)}=-\infty$$
$$\lim_{x\to -2^-}\frac{x-7}{x^2+5x+6}=\lim_{x\to -2^-}\frac{x-7}{(x+3)(x+2)}=\infty$$

then, $x=-3$ and $x=-2$ are vertical asymptotes.

28) The vertical asymptote of $f(x)=\frac{x-7}{x^2+3x}$ is

Solution:

$$f(x)=\frac{x-7}{x^2+3x}=\frac{x-7}{x(x+3)}$$

We see that the function $f(x)$ is not defined when $x=0$ or $x+3=0 \implies x=0$ or $x=-3$. Since

$$\lim_{x\to -3^+}\frac{x-7}{x^2+3x}=\lim_{x\to -3^+}\frac{x-7}{x(x+3)}=\infty$$
$$\lim_{x\to -3^-}\frac{x-7}{x^2+3x}=\lim_{x\to -3^-}\frac{x-7}{x(x+3)}=-\infty$$

and

$$\lim_{x\to 0^+}\frac{x-7}{x^2+3x}=\lim_{x\to 0^+}\frac{x-7}{x(x+3)}=-\infty$$
$$\lim_{x\to 0^-}\frac{x-7}{x^2+3x}=\lim_{x\to 0^-}\frac{x-7}{x(x+3)}=\infty$$

then, $x=-3$ and $x=0$ are vertical asymptotes.

29) The vertical asymptote of $f(x)=\frac{x-7}{x^2-3x}$ is

Solution:

$$f(x)=\frac{x-7}{x^2-3x}=\frac{x-7}{x(x-3)}$$

We see that the function $f(x)$ is not defined when $x=0$ or $x-3=0 \implies x=0$ or $x=3$. Since

$$\lim_{x\to 3^+}\frac{x-7}{x^2-3x}=\lim_{x\to 3^+}\frac{x-7}{x(x-3)}=-\infty$$
$$\lim_{x\to 3^-}\frac{x-7}{x^2-3x}=\lim_{x\to 3^-}\frac{x-7}{x(x-3)}=\infty$$

and

$$\lim_{x\to 0^+}\frac{x-7}{x^2-3x}=\lim_{x\to 0^+}\frac{x-7}{x(x-3)}=\infty$$
$$\lim_{x\to 0^-}\frac{x-7}{x^2-3x}=\lim_{x\to 0^-}\frac{x-7}{x(x-3)}=-\infty$$

then, $x=3$ and $x=0$ are vertical asymptotes.

30) The vertical asymptotes of $f(x)=\frac{2x^2+1}{x^2-9}$ are

Solution:

$$f(x)=\frac{2x^2+1}{x^2-9}=\frac{2x^2+1}{(x+3)(x-3)}$$

We see that the function $f(x)$ is not defined when $x^2-9=0 \implies x=\pm 3$. Since

$$\lim_{x\to 3^+}\frac{2x^2+1}{x^2-9}=\lim_{x\to 3^+}\frac{2x^2+1}{(x+3)(x-3)}=\infty$$
$$\lim_{x\to 3^-}\frac{2x^2+1}{x^2-9}=\lim_{x\to 3^-}\frac{2x^2+1}{(x+3)(x-3)}=-\infty$$

and

$$\lim_{x\to -3^+}\frac{2x^2+1}{x^2-9}=\lim_{x\to -3^+}\frac{2x^2+1}{(x+3)(x-3)}=-\infty$$
$$\lim_{x\to -3^-}\frac{2x^2+1}{x^2-9}=\lim_{x\to -3^-}\frac{2x^2+1}{(x+3)(x-3)}=\infty$$

then, $x=\pm 3$ are vertical asymptotes.

31) The function $f(x) = \frac{x+1}{x^2-9}$ is continuous at $a = 2$ because

$1 - \quad f(2) = \dfrac{(2)+1}{(2)^2-9} = \dfrac{3}{-5} = -\dfrac{3}{5}$

$2 - \quad \lim\limits_{x\to 3^-} \dfrac{x+1}{x^2-9} = \lim\limits_{x\to 2} \dfrac{(2)+1}{(2)^2-9} = \dfrac{3}{-5} = -\dfrac{3}{5}$

$3 - \quad \lim\limits_{x\to 2} \dfrac{x+1}{x^2-9} = f(2)$

OR

We know that $D_f = \mathbb{R} \setminus \{\pm 3\}$, so $\{2\} \in D_f$.

Note: Any function is continuous on its domain.

32) The function $f(x) = \frac{x+1}{x^2-9}$ is discontinuous at $a = \pm 3$ because we know that $D_f = \mathbb{R} \setminus \{\pm 3\}$, so $\{\pm 3\} \notin D_f$.

33) The function $f(x) = \frac{x+1}{x^2-9}$ is discontinuous at ± 3 because $\{\pm 3\} \notin D_f$.

34) The function $f(x) = \frac{x+1}{x^2-9}$ is continuous on its domain which is $D_f = \mathbb{R} \setminus \{\pm 3\}$.

35) The function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$ is continuous at $a = 0$ because

1- $f(0) = 3$

2- $\lim\limits_{x\to 0} \frac{\sin 3x}{x} = 3 \lim\limits_{x\to 0} \frac{\sin 3x}{3x} = 3(1) = 3$

3- $\lim\limits_{x\to 0} f(x) = f(0)$

36) The function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & x \neq 0 \\ 5, & x = 0 \end{cases}$ is discontinuous at $a = 0$ because

1- $f(0) = 5$

2- $\lim\limits_{x\to 0} \frac{\sin 3x}{x} = 3 \lim\limits_{x\to 0} \frac{\sin 3x}{3x} = 3(1) = 3$

3- $\lim\limits_{x\to 0} f(x) \neq f(0)$

37) The function $f(x) = \begin{cases} \frac{2x^2-3x+1}{x-1}, & x \neq 1 \\ 7, & x = 1 \end{cases}$ is discontinuous at $a = 1$ because

1- $f(1) = 7$

2- $\lim\limits_{x\to 1} \frac{2x^2-3x+1}{x-1} = \lim\limits_{x\to 1} \frac{(2x-1)(x-1)}{x-1} = \lim\limits_{x\to 1}(2x-1) = 1$

3- $\lim\limits_{x\to 1} f(x) \neq f(1)$

38) The function $f(x) = \begin{cases} \frac{2x^2-3x+1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ is continuous at $a = 1$ because

1- $f(1) = 1$

2- $\lim\limits_{x\to 1} \frac{2x^2-3x+1}{x-1} = \lim\limits_{x\to 1} \frac{(2x-1)(x-1)}{x-1} = \lim\limits_{x\to 1}(2x-1) = 1$

3- $\lim\limits_{x\to 1} f(x) = f(1)$

39) The function $f(x) = \frac{x^2-x-2}{x-2}$ is discontinuous at $a = 2$ because $\{2\} \notin D_f$.

40) The function $f(x) = \begin{cases} 2x+3, & x > 2 \\ 3x+1, & x \leq 2 \end{cases}$ is continuous at $a = 2$ because

1- $f(2) = 3(2) + 1 = 7$

2- $\lim\limits_{x\to 2^+} (2x+3) = 2(2) + 3 = 7$

$\lim\limits_{x\to 2^-} (3x+1) = 3(2) + 1 = 7$

$\therefore \lim\limits_{x\to 2} f(x) = 7$

3- $\lim\limits_{x\to 2} f(x) = f(2)$

41) The function $f(x) = \frac{x+3}{\sqrt{x^2-4}}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x^2 - 4 > 0 \implies x^2 > 4 \implies \sqrt{x^2} > \sqrt{4}$$
$$\implies |x| > 2 \iff x > 2 \text{ or } x < -2$$

Hence,

$D_f = (-\infty, -2) \cup (2, \infty)$.

42) The function $f(x) = \sqrt{x^2-4}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x^2 - 4 \geq 0 \implies x^2 \geq 4 \implies \sqrt{x^2} \geq \sqrt{4}$$
$$\implies |x| \geq 2 \iff x \geq 2 \text{ or } x \leq -2$$

Hence,

$D_f = (-\infty, -2] \cup [2, \infty)$.

43) The function $f(x) = \sqrt{4-x^2}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$4 - x^2 \geq 0 \implies -x^2 \geq -4 \implies x^2 \leq 4$$
$$\implies \sqrt{x^2} \leq \sqrt{4} \implies |x| \leq 2 \iff -2 \leq x \leq 2$$

Hence,

$D_f = [-2,2]$.

44) The function $f(x) = \frac{x+3}{\sqrt{4-x^2}}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$4 - x^2 > 0 \implies -x^2 > -4 \implies x^2 < 4$$
$$\implies \sqrt{x^2} < \sqrt{4} \implies |x| < 2 \iff -2 < x < 2$$

Hence,

$D_f = (-2,2)$.

45) The function $f(x) = \frac{x+1}{x^2-4}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x^2 - 4 \neq 0 \implies x^2 \neq 4 \implies x \neq \pm 2$$

Hence,

$D_f = \mathbb{R} \setminus \{\pm 2\}$

$= (-\infty, -2) \cup (-2,2) \cup (2, \infty) = \{x \in \mathbb{R} : x \neq \pm 2\}$.

46) The function $f(x) = \log_2(x+2)$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x + 2 > 0 \implies x > -2$$

Hence,

$D_f = (-2, \infty)$.

47) The function $f(x) = \sqrt{x-1} + \sqrt{x+4}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x - 1 \geq 0 \text{ and } x + 4 \geq 0 \implies x \geq 1 \ \cap \ x \geq -4$$

Hence,

$D_f = [1, \infty)$.

48) The function $f(x) = 5^x$ is continuous on its domain .

Hence,

$D_f = \mathbb{R} = (-\infty, \infty)$.

49) The function $f(x) = e^x$ is continuous on its domain .

Hence,

$D_f = \mathbb{R} = (-\infty, \infty)$.

50) The function $f(x) = \sin^{-1}(3x-5)$ is continuous on its domain where $f(x)$ is defined, we mean that

$$-1 \leq 3x - 5 \leq 1 \quad \Leftrightarrow \quad 4 \leq 3x \leq 6 \quad \Leftrightarrow \quad \tfrac{4}{3} \leq x \leq 2 .$$

Hence,

$D_f = \left[\frac{4}{3}, 2\right]$.

51) The function $f(x) = \cos^{-1}(3x+5)$ is continuous on its domain where $f(x)$ is defined, we mean that

$$-1 \leq 3x + 5 \leq 1 \Leftrightarrow -6 \leq 3x \leq -4 \Leftrightarrow -2 \leq x \leq -\tfrac{4}{3} .$$

Hence,

$D_f = \left[-2, -\frac{4}{3}\right]$.

52) The number c that makes $f(x) = \begin{cases} c + x , & x > 2 \\ 2x - c, & x \leq 2 \end{cases}$ is continuous at $x = 2$ is

Solution:

$\lim\limits_{x\to 2} f(x)$ exists if

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$
$$\lim_{x\to 2^+} (c + x) = \lim_{x\to 2^-} (2x - c)$$
$$c + 2 = 4 - c$$
$$c + c = 4 - 2$$
$$2c = 2$$
$$c = 1$$

53) The number c that makes $f(x) = \begin{cases} cx^2 - 2x + 1, & x \leq -1 \\ 3x + 2 \quad , & x > -1 \end{cases}$ is continuous at -1 is

Solution:

$\lim\limits_{x\to -1} f(x)$ exists if

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x)$$
$$\lim_{x\to -1^+} (3x + 2\ ) = \lim_{x\to -1^-} (cx^2 - 2x + 1)$$
$$3(-1) + 2 = c(-1)^2 - 2(-1) + 1$$
$$-1 = c + 3$$
$$c = -1 - 3$$
$$c = -4$$

54) The number c that makes

$f(x) = \begin{cases} \frac{\sin cx}{x} + 2x - 1, & x < 0 \\ 3x + 4 \quad , & x \geq 0 \end{cases}$ is continuous at 0 is

Solution:

$\lim\limits_{x\to 0} f(x)$ exists if

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$
$$\lim_{x\to 0^+} (3x + 4) = \lim_{x\to 0^-} \left(\frac{\sin cx}{x} + 2x - 1\right)$$
$$3(0) + 4 = c(1) + 2(0) - 1$$
$$4 = c - 1$$
$$c = 4 + 1$$
$$c = 5$$

55) The value c that makes $f(x) = \begin{cases} cx^2 + 2x, & x \leq 2 \\ x^3 - cx \ , & x > 2 \end{cases}$ is continuous at 2 is

Solution:

$\lim\limits_{x\to 2} f(x)$ exists if

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$
$$\lim_{x\to 2^+} (x^3 - cx\ ) = \lim_{x\to 2^-} (cx^2 + 2x)$$
$$(2)^3 - c(2) = c(2)^2 + 2(2)$$
$$8 - 2c = 4c + 4$$
$$-2c - 4c = 4 - 8$$
$$-6c = -4$$
$$c = \frac{-4}{-6}$$
$$c = \frac{2}{3}$$

56) The number c that makes $f(x) = \begin{cases} c^2x^2 - 1, & x \leq 3 \\ x + 5 \quad , & x > 3 \end{cases}$ is continuous at 3 is

Solution:

$\lim\limits_{x\to 3} f(x)$ exists if

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$$
$$\lim_{x\to 3^+} (x + 5) = \lim_{x\to 3^-} (c^2x^2 - 1)$$
$$(3) + 5 = c^2(3)^2 - 1$$
$$8 = 9c^2 - 1$$
$$9c^2 = 8 + 1$$
$$c^2 = 1$$
$$c = \pm 1$$

57) The number c that makes $f(x) = \begin{cases} x - 2 , & x > 5 \\ cx - 3, & x \leq 5 \end{cases}$ is continuous at 5 is

Solution:

$\lim\limits_{x\to 5} f(x)$ exists if

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^-} f(x)$$
$$\lim_{x\to 5^+} (x - 2) = \lim_{x\to 5^-} (cx - 3)$$
$$(5) - 2 = c(5) - 3$$
$$3 = 5c - 3$$
$$5c = 3 + 3$$
$$5c = 6$$
$$c = \frac{6}{5}$$

58) The number c that makes $f(x) = \begin{cases} x+3\,, & x > -1 \\ 2x - c, & x \le -1 \end{cases}$ is continuous at -1 is <u>Solution:</u> $\lim_{x \to -1} f(x)$ exists if $$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$$ $$\lim_{x \to -1^+} (x+3) = \lim_{x \to -1^-} (2x - c)$$ $$(-1) + 3 = 2(-1) - c$$ $$2 = -2 - c$$ $$c = -2 - 2$$ $$c = -4$$	

Workshop Solutions to Section 3.3 (2.6 & page 192,193)

1) If $f(x)=\begin{cases}2x+3; & x\ge -2\\ 2x+5; & x<-2\end{cases}$ then $\lim\limits_{x\to(-2)^-} f(x)=$ Solution: $\lim\limits_{x\to(-2)^-} f(x)=\lim\limits_{x\to(-2)^-}(2x+5)=2(-2)+5=-4+5$ $=1$	2) If $f(x)=\begin{cases}2x+3; & x\ge -2\\ 2x+5; & x<-2\end{cases}$ then $\lim\limits_{x\to(-2)^+} f(x)=$ Solution: $\lim\limits_{x\to(-2)^+} f(x)=\lim\limits_{x\to(-2)^+}(2x+3)=2(-2)+3=-4+3$ $=-1$
3) If $f(x)=\begin{cases}2x+3; & x\ge -2\\ 2x+5; & x<-2\end{cases}$ then $\lim\limits_{x\to-2} f(x)=$ Solution: $\lim\limits_{x\to-2} f(x)$ does not exist because $\lim\limits_{x\to(-2)^-} f(x)\neq\lim\limits_{x\to(-2)^+} f(x)$	4) If $f(x)=\begin{cases}x^2-2x+3; & x\ge 3\\ x^3-3x-12; & x<3\end{cases}$ then $\lim\limits_{x\to3} f(x)=$ Solution: $\lim\limits_{x\to3^-} f(x)=\lim\limits_{x\to3^-}(x^3-3x-12)=(3)^3-3(3)-12$ $=27-9-12=6$ $\lim\limits_{x\to3^+} f(x)=\lim\limits_{x\to3^+}(x^2-2x+3)=(3)^2-2(3)+3$ $=9-6+3=6$ $\therefore\ \lim\limits_{x\to3} f(x)=6$
5) If $f(x)=\begin{cases}x^2-7x; & x<1\\ 5; & 1\le x\le 3\\ 3x+1; & x>3\end{cases}$ then $\lim\limits_{x\to1^-} f(x)=$ Solution: $\lim\limits_{x\to1^-} f(x)=\lim\limits_{x\to1^-}(x^2-7x)=(1)^2-7(1)=1-7=-6$	6) If $f(x)=\begin{cases}x^2-7x; & x<1\\ 5; & 1\le x\le 3\\ 3x+1; & x>3\end{cases}$ then $\lim\limits_{x\to1^+} f(x)=$ Solution: $\lim\limits_{x\to1^+} f(x)=\lim\limits_{x\to1^+}(5)=5$
7) If $f(x)=\begin{cases}x^2-7x; & x<1\\ 5; & 1\le x\le 3\\ 3x+1; & x>3\end{cases}$ then $\lim\limits_{x\to3^-} f(x)=$ Solution: $\lim\limits_{x\to3^-} f(x)=\lim\limits_{x\to3^-}(5)=5$	8) If $f(x)=\begin{cases}x^2-7x; & x<1\\ 5; & 1\le x\le 3\\ 3x+1; & x>3\end{cases}$ then $\lim\limits_{x\to3^+} f(x)=$ Solution: $\lim\limits_{x\to3^+} f(x)=\lim\limits_{x\to3^+}(3x+1)=3(3)+1=9+1=10$
9) If $f(x)=\begin{cases}\frac{x^2+x-6}{x^2-4}; & x^2-4>0\\ \frac{x^2+x-6}{4-x^2}; & x^2-4<0\end{cases}$ then $\lim\limits_{x\to2^+} f(x)=$ Solution: $f(x)=\begin{cases}\dfrac{x^2+x-6}{x^2-4}; & x^2-4>0\\ \dfrac{x^2+x-6}{4-x^2}; & x^2-4<0\end{cases}$ $=\begin{cases}\dfrac{x^2+x-6}{x^2-4}; & x^2>4\\ \dfrac{x^2+x-6}{-(x^2-4)}; & x^2<4\end{cases}$ $=\begin{cases}\dfrac{(x+3)(x-2)}{(x-2)(x+2)}; & \lvert x\rvert>4\\ \dfrac{(x+3)(x-2)}{-(x-2)(x+2)}; & \lvert x\rvert<4\end{cases}$ $=\begin{cases}\dfrac{x+3}{x+2}; & x>2 \text{ or } x<-2\\ -\dfrac{x+3}{x+2}; & -2<x<2\end{cases}$ then $\therefore\ \lim\limits_{x\to2^+} f(x)=\lim\limits_{x\to2^+}\left(\dfrac{x+3}{x+2}\right)=\dfrac{(2)+3}{(2)+2}=\dfrac{5}{4}$	10) If $f(x)=\begin{cases}\frac{x^2+x-6}{x^2-4}; & x^2-4>0\\ \frac{x^2+x-6}{4-x^2}; & x^2-4<0\end{cases}$ then $\lim\limits_{x\to2^-} f(x)=$ Solution: $f(x)=\begin{cases}\dfrac{x^2+x-6}{x^2-4}; & x^2-4>0\\ \dfrac{x^2+x-6}{4-x^2}; & x^2-4<0\end{cases}$ $=\begin{cases}\dfrac{x^2+x-6}{x^2-4}; & x^2>4\\ \dfrac{x^2+x-6}{-(x^2-4)}; & x^2<4\end{cases}$ $=\begin{cases}\dfrac{(x+3)(x-2)}{(x-2)(x+2)}; & \lvert x\rvert>4\\ \dfrac{(x+3)(x-2)}{-(x-2)(x+2)}; & \lvert x\rvert<4\end{cases}$ $=\begin{cases}\dfrac{x+3}{x+2}; & x>2 \text{ or } x<-2\\ -\dfrac{x+3}{x+2}; & -2<x<2\end{cases}$ then $\therefore\ \lim\limits_{x\to2^-} f(x)=\lim\limits_{x\to2^-}\left(-\dfrac{x+3}{x+2}\right)=-\dfrac{(2)+3}{(2)+2}=-\dfrac{5}{4}$

11)

$$\lim_{x\to a^-}\frac{|x-a|}{x-a}=$$

Solution:

$$f(x)=\frac{|x-a|}{x-a}=\begin{cases}\dfrac{x-a}{x-a} & ;\ x-a>0\\ \dfrac{-(x-a)}{x-a} & ;\ x-a<0\end{cases}=\begin{cases}1; & x>a\\ -1; & x<a\end{cases}$$

$$\therefore\quad \lim_{x\to a^-}\frac{|x-a|}{x-a}=\lim_{x\to a^-}\frac{-(x-a)}{x-a}=\lim_{x\to a^-}(-1)=-1$$

12)

$$\lim_{x\to a^+}\frac{|x-a|}{x-a}=$$

Solution:

$$f(x)=\frac{|x-a|}{x-a}=\begin{cases}\dfrac{x-a}{x-a} & ;\ x-a>0\\ \dfrac{-(x-a)}{x-a} & ;\ x-a<0\end{cases}=\begin{cases}1; & x>a\\ -1; & x<a\end{cases}$$

$$\therefore\quad \lim_{x\to a^+}\frac{|x-a|}{x-a}=\lim_{x\to a^+}\frac{(x-a)}{x-a}=\lim_{x\to a^+}(1)=1$$

13)

$$\lim_{x\to a}\frac{|x-a|}{x-a}=$$

Solution:

$\lim_{x\to a}\frac{|x-a|}{x-a}$ does not exist because

$$\lim_{x\to a^-}\frac{|x-a|}{x-a}\neq\lim_{x\to a^+}\frac{|x-a|}{x-a}$$

It is clearly obvious from questions (11) and (12) above.

14)

$$\lim_{x\to a^+}\frac{|a-x|}{x-a}=$$

Solution:

$$f(x)=\frac{|a-x|}{x-a}=\begin{cases}\dfrac{a-x}{x-a} & ;\ a-x>0\\ \dfrac{-(a-x)}{x-a} & ;\ a-x<0\end{cases}$$

$$=\begin{cases}\dfrac{-(x-a)}{x-a} & ;\ a>x\\ \dfrac{(x-a)}{x-a} & ;\ a<x\end{cases}=\begin{cases}-1; & x<a\\ 1; & x>a\end{cases}$$

$$\therefore\quad \lim_{x\to a^+}\frac{|a-x|}{x-a}=\lim_{x\to a^+}(1)=1$$

15)

$$\lim_{x\to a^-}\frac{|a-x|}{x-a}=$$

Solution:

$$f(x)=\frac{|a-x|}{x-a}=\begin{cases}\dfrac{a-x}{x-a} & ;\ a-x>0\\ \dfrac{-(a-x)}{x-a} & ;\ a-x<0\end{cases}$$

$$=\begin{cases}\dfrac{-(x-a)}{x-a} & ;\ a>x\\ \dfrac{(x-a)}{x-a} & ;\ a<x\end{cases}=\begin{cases}-1; & x<a\\ 1; & x>a\end{cases}$$

$$\therefore\quad \lim_{x\to a^-}\frac{|a-x|}{x-a}=\lim_{x\to a^-}(-1)=-1$$

16)

$$\lim_{x\to a}\frac{|a-x|}{x-a}=$$

Solution:

$\lim_{x\to a}\frac{|a-x|}{x-a}$ does not exist because

$$\lim_{x\to a^-}\frac{|a-x|}{x-a}\neq\lim_{x\to a^+}\frac{|a-x|}{x-a}$$

It is clearly obvious from questions (14) and (15) above.

17)

$$\lim_{x\to(-a)^-}\frac{|x+a|}{x+a}=$$

Solution:

$$f(x)=\frac{|x+a|}{x+a}=\begin{cases}\dfrac{x+a}{x+a} & ;\ x+a>0\\ \dfrac{-(x+a)}{x+a} & ;\ x+a<0\end{cases}=\begin{cases}1; & x>-a\\ -1; & x<-a\end{cases}$$

$$\therefore\quad \lim_{x\to(-a)^-}\frac{|x+a|}{x+a}=\lim_{x\to(-a)^-}(-1)=-1$$

18)

$$\lim_{x\to(-a)^+}\frac{|x+a|}{x+a}=$$

Solution:

$$f(x)=\frac{|x+a|}{x+a}=\begin{cases}\dfrac{x+a}{x+a} & ;\ x+a>0\\ \dfrac{-(x+a)}{x+a} & ;\ x+a<0\end{cases}=\begin{cases}1; & x>-a\\ -1; & x<-a\end{cases}$$

$$\therefore\quad \lim_{x\to(-a)^+}\frac{|x+a|}{x+a}=\lim_{x\to(-a)^+}(1)=1$$

19)

$$\lim_{x\to -a}\frac{|x+a|}{x+a}=$$

Solution:

$\lim_{x\to -a}\frac{|x+a|}{x+a}$ does not exist because

$$\lim_{x\to(-a)^-}\frac{|x+a|}{x+a}\neq\lim_{x\to(-a)^+}\frac{|x+a|}{x+a}$$

It is clearly obvious from questions (17) and (18) above.

20)

$$\lim_{x\to 0^+}\frac{2x-|x|}{x^2+|x|}=$$

Solution:

$$f(x)=\frac{2x-|x|}{x^2+|x|}=\begin{cases}\dfrac{2x-(x)}{x^2+(x)} & ;\ x>0\\[2mm] \dfrac{2x-(-x)}{x^2+(-x)} & ;\ x<0\end{cases}$$
$$=\begin{cases}\dfrac{2x-x}{x^2+x} & ;\ x>0\\[2mm] \dfrac{2x+x}{x^2-x} & ;\ x<0\end{cases}=\begin{cases}\dfrac{x}{x^2+x} & ;\ x>0\\[2mm] \dfrac{3x}{x^2-x} & ;\ x<0\end{cases}$$
$$=\begin{cases}\dfrac{x}{x(x+1)} & ;\ x>0\\[2mm] \dfrac{3x}{x(x-1)} & ;\ x<0\end{cases}$$
$$=\begin{cases}\dfrac{1}{x+1} & ;\ x>0\\[2mm] \dfrac{3}{x-1} & ;\ x<0\end{cases}$$
$$\therefore\quad \lim_{x\to 0^+}\frac{2x-|x|}{x^2+|x|}=\lim_{x\to 0^+}\frac{1}{x+1}=\frac{1}{0+1}=1$$

21)

$$\lim_{x\to 0^-}\frac{2x-|x|}{x^2+|x|}=$$

Solution:

$$f(x)=\frac{2x-|x|}{x^2+|x|}=\begin{cases}\dfrac{2x-(x)}{x^2+(x)} & ;\ x>0\\[2mm] \dfrac{2x-(-x)}{x^2+(-x)} & ;\ x<0\end{cases}$$
$$=\begin{cases}\dfrac{2x-x}{x^2+x} & ;\ x>0\\[2mm] \dfrac{2x+x}{x^2-x} & ;\ x<0\end{cases}=\begin{cases}\dfrac{x}{x^2+x} & ;\ x>0\\[2mm] \dfrac{3x}{x^2-x} & ;\ x<0\end{cases}$$
$$=\begin{cases}\dfrac{x}{x(x+1)} & ;\ x>0\\[2mm] \dfrac{3x}{x(x-1)} & ;\ x<0\end{cases}$$
$$=\begin{cases}\dfrac{1}{x+1} & ;\ x>0\\[2mm] \dfrac{3}{x-1} & ;\ x<0\end{cases}$$
$$\therefore\quad \lim_{x\to 0^-}\frac{2x-|x|}{x^2+|x|}=\lim_{x\to 0^-}\frac{3}{x-1}=\frac{3}{0-1}=-3$$

22)

$$\lim_{x\to 0}\frac{2x-|x|}{x^2+|x|}=$$

Solution:

$\displaystyle\lim_{x\to 0}\frac{2x-|x|}{x^2+|x|}$ does not exist because

$$\lim_{x\to 0^-}\frac{2x-|x|}{x^2+|x|}\neq\lim_{x\to 0^+}\frac{2x-|x|}{x^2+|x|}$$

It is clearly obvious from questions (20) and (21) above.

23)

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}=$$

Solution:

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}=\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{(\cos x-\sin x)(\cos x+\sin x)}$$
$$=\lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x+\sin x}=\frac{1}{\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)}$$
$$=\frac{1}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}=\frac{1}{\frac{2}{\sqrt{2}}}=\frac{\sqrt{2}}{2}$$

24)

$$\lim_{x\to 0}\frac{\cos^2 x+2\cos x-3}{2\cos^2 x-\cos x-1}=$$

Solution:

$$\lim_{x\to 0}\frac{\cos^2 x+2\cos x-3}{2\cos^2 x-\cos x-1}=\lim_{x\to 0}\frac{(\cos x+3)(\cos x-1)}{(2\cos x+1)(\cos x-1)}$$
$$=\lim_{x\to 0}\frac{\cos x+3}{2\cos x+1}=\frac{\cos(0)+3}{2\cos(0)+1}$$
$$=\frac{1+3}{2(1)+1}=\frac{4}{3}$$

25)

$$\lim_{x\to 0}(\sin^2 x+3\tan x-4)=$$

Solution:

$$\lim_{x\to 0}(\sin^2 x+3\tan x-4)=\sin^2(0)+3\tan(0)-4$$
$$=0+3(0)-4=-4$$

26) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{\sin(nx)}{mx}=$$

Solution:

$$\lim_{x\to 0}\frac{\sin(nx)}{mx}=\frac{n}{m}\lim_{x\to 0}\frac{\sin(nx)}{nx}=\frac{n}{m}(1)=\frac{n}{m}$$

27) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{\tan(nx)}{mx}=$$

Solution:

$$\lim_{x\to 0}\frac{\tan(nx)}{mx}=\frac{n}{m}\lim_{x\to 0}\frac{\tan(nx)}{nx}=\frac{n}{m}(1)=\frac{n}{m}$$

28) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{nx}{\sin(mx)}=$$

Solution:

$$\lim_{x\to 0}\frac{nx}{\sin(mx)}=\frac{n}{m}\lim_{x\to 0}\frac{mx}{\sin(mx)}=\frac{n}{m}(1)=\frac{n}{m}$$

29) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{nx}{\tan(mx)}=$$

Solution:

$$\lim_{x\to 0}\frac{nx}{\tan(mx)}=\frac{n}{m}\lim_{x\to 0}\frac{mx}{\tan(mx)}=\frac{n}{m}(1)=\frac{n}{m}$$

30) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\sin(nx)}{\sin(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(nx)}{\sin(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\sin(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\sin(mx)}\right)$$

$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

31) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\sin(nx)}{\tan(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(nx)}{\tan(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\sin(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\tan(mx)}\right)$$

$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

32) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\tan(nx)}{\tan(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\tan(nx)}{\tan(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\tan(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\tan(mx)}\right)$$

$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

33) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\tan(nx)}{\sin(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\tan(nx)}{\sin(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\tan(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\sin(mx)}\right)$$

$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

34)

$$\lim_{x\to 0}\frac{\sin(1-\cos x)}{1-\cos x} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(1-\cos x)}{1-\cos x} = 1$$

35)

$$\lim_{x\to 0}\frac{\sin(\sin(2x))}{\sin(2x)} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(\sin(2x))}{\sin(2x)} = 1$$

36)

$$\lim_{x\to 0}\frac{1-\cos(2x)}{x^2} =$$

Solution:

$$\lim_{x\to 0}\frac{1-\cos(2x)}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$= 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$$

37)

$$\lim_{x\to\infty}\sqrt{\frac{1}{x^2}-\frac{3}{x}+4} =$$

Solution:

$$\lim_{x\to\infty}\sqrt{\frac{1}{x^2}-\frac{3}{x}+4} = \sqrt{\lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{3}{x}+4\right)} = \sqrt{0-0+4}$$

$$= 2$$

38)

$$\lim_{x\to\infty}\left(\frac{1}{x^{2/5}}+2\right) =$$

Solution:

$$\lim_{x\to-\infty}\left(\frac{1}{x^{2/5}}+2\right) = 0+2 = 2$$

39)

$$\lim_{x\to\infty}\frac{3x+15}{9x^2+4x-13} =$$

Solution:

$$\lim_{x\to\infty}\frac{3x+15}{9x^2+4x-13} = \lim_{x\to\infty}\frac{\frac{3x}{x^2}+\frac{15}{x^2}}{\frac{9x^2}{x^2}+\frac{4x}{x^2}-\frac{13}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{3}{x}+\frac{15}{x^2}}{9+\frac{4}{x}-\frac{13}{x^2}} = \frac{0+0}{9+0+0} = 0$$

40)

$$\lim_{x\to\infty}\frac{3x^2-8x+15}{9x^2+4x-13} =$$

Solution:

$$\lim_{x\to\infty}\frac{3x^2-8x+15}{9x^2+4x-13} = \lim_{x\to\infty}\frac{\frac{3x^2}{x^2}-\frac{8x}{x^2}+\frac{15}{x^2}}{\frac{9x^2}{x^2}+\frac{4x}{x^2}-\frac{13}{x^2}}$$

$$= \lim_{x\to\infty}\frac{3-\frac{8}{x}+\frac{15}{x^2}}{9+\frac{4}{x}-\frac{13}{x^2}} = \frac{3-0+0}{9+0+0} = \frac{1}{3}$$

41)

$$\lim_{x\to-\infty}\frac{3x^2-8x+15}{9x^2+4x-13} =$$

Solution:

$$\lim_{x\to-\infty}\frac{3x^2-8x+15}{9x^2+4x-13} = \lim_{x\to-\infty}\frac{\frac{3x^2}{-x^2}-\frac{8x}{-x^2}+\frac{15}{-x^2}}{\frac{9x^2}{-x^2}+\frac{4x}{-x^2}-\frac{13}{-x^2}}$$

$$= \lim_{x\to-\infty}\frac{-3+\frac{8}{x}-\frac{15}{x^2}}{-9-\frac{4}{x}+\frac{13}{x^2}} = \frac{-3+0-0}{-9-0+0} = \frac{1}{3}$$

42)

$$\lim_{x\to\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=$$

Solution:

$$\lim_{x\to\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=\lim_{x\to\infty}\frac{\frac{3x^5}{x^2}-\frac{8x}{x^2}+\frac{15}{x^2}}{\frac{9x^2}{x^2}+\frac{4x}{x^2}-\frac{13}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3x^3-\frac{8}{x}+\frac{15}{x^2}}{9+\frac{4}{x}-\frac{13}{x^2}}=\frac{3(\infty)-0+0}{9+0+0}=\infty$$

43)

$$\lim_{x\to-\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=$$

Solution:

$$\lim_{x\to-\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=\lim_{x\to-\infty}\frac{\frac{3x^5}{-x^2}-\frac{8x}{-x^2}+\frac{15}{-x^2}}{\frac{9x^2}{-x^2}+\frac{4x}{-x^2}-\frac{13}{-x^2}}$$

$$=\lim_{x\to-\infty}\frac{-3x^3+\frac{8}{x}-\frac{15}{x^2}}{-9-\frac{4}{x}+\frac{13}{x^2}}=\frac{-3(-\infty)+0-0}{-9-0+0}=-\infty$$

44)

$$\lim_{x\to\infty}\left(\sqrt{x^2-3x+7}-x\right)=$$

Solution:

$$\lim_{x\to\infty}\left(\sqrt{x^2-3x+7}-x\right)$$

$$=\lim_{x\to\infty}\left[\left(\sqrt{x^2-3x+7}-x\right)\times\frac{\left(\sqrt{x^2-3x+7}+x\right)}{\left(\sqrt{x^2-3x+7}+x\right)}\right]$$

$$=\lim_{x\to\infty}\left(\frac{(x^2-3x+7)-x^2}{\sqrt{x^2-3x+7}+x}\right)=\lim_{x\to\infty}\left(\frac{-3x+7}{\sqrt{x^2-3x+7}+x}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{-3x}{x}+\frac{7}{x}}{\frac{\sqrt{x^2-3x+7}}{x}+\frac{x}{x}}$$

$$=\lim_{x\to\infty}\frac{-3+\frac{7}{x}}{\sqrt{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{7}{x^2}}+1}$$

$$=\lim_{x\to\infty}\frac{-3+\frac{7}{x}}{\sqrt{1-\frac{3}{x}+\frac{7}{x^2}}+1}$$

$$=\frac{-3+0}{\sqrt{1-0+0}+1}=\frac{-3}{1+1}=-\frac{3}{2}$$

45)

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=$$

Solution:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$$

$$=\lim_{x\to\infty}\left[\left(\sqrt{x^2+x}-x\right)\times\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right]$$

$$=\lim_{x\to\infty}\left(\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x}{\sqrt{x^2+x}+x}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{\sqrt{x^2+x}}{x}+\frac{x}{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}}+1}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

46)

$$\lim_{x\to\infty}(x^2-5x+4)=$$

Solution:

$$\lim_{x\to\infty}(x^2-5x+4)=\lim_{x\to\infty}x^2\left(\frac{x^2}{x^2}-\frac{5x}{x^2}+\frac{4}{x^2}\right)$$

$$=\lim_{x\to\infty}x^2\left(1-\frac{5}{x}+\frac{4}{x^2}\right)=(\infty)^2(1-0+0)=\infty$$

OR

$$\lim_{x\to\infty}(x^2-5x+4)=\lim_{x\to\infty}(x^2)=\infty$$

47)

$$\lim_{x\to-\infty}(x^4-2x^3+9)=$$

Solution:

$$\lim_{x\to-\infty}(x^4-2x^3+9)=\lim_{x\to-\infty}x^4\left(\frac{x^4}{x^4}-\frac{2x^3}{x^4}+\frac{9}{x^4}\right)$$

$$=\lim_{x\to-\infty}x^4\left(1-\frac{2}{x}+\frac{9}{x^4}\right)=(-\infty)^4(1-0+0)=\infty$$

OR

$$\lim_{x\to-\infty}(x^4-2x^3+9)=\lim_{x\to-\infty}(x^4)=\infty$$

48)

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=$$

Solution:

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=\lim_{x\to-\infty}\frac{\frac{\sqrt{3x^2-8}}{-x}+\frac{2}{-x}}{\frac{x}{-x}+\frac{5}{-x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{3x^2-8}{x^2}}-\frac{2}{x}}{-1-\frac{5}{x}}=\lim_{x\to-\infty}\frac{\sqrt{\frac{3x^2}{x^2}-\frac{8}{x^2}}-\frac{2}{x}}{-1-\frac{5}{x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{3-\frac{8}{x^2}}-\frac{2}{x}}{-1-\frac{5}{x}}=\frac{\sqrt{3-0}-0}{-1-0}=-\sqrt{3}$$

49)

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=$$

Solution:

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=\lim_{x\to\infty}\frac{\frac{\sqrt{3x^2-8}}{x}+\frac{2}{x}}{\frac{x}{x}+\frac{5}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{3x^2-8}{x^2}}+\frac{2}{x}}{1+\frac{5}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{3x^2}{x^2}-\frac{8}{x^2}}+\frac{2}{x}}{1+\frac{5}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3-\frac{8}{x^2}}+\frac{2}{x}}{1+\frac{5}{x}}=\frac{\sqrt{3-0}+0}{1+0}=\sqrt{3}$$

50) The horizontal asymptotes of

$$f(x)=\frac{\sqrt{3x^2-8}+2}{x+5}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{\sqrt{3x^2-8}+2}{x+5}$$

It is clear from the previous questions (48) and (49) that

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=\sqrt{3}$$

and

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=-\sqrt{3}$$

Thus, the horizontal asymptotes are

$$y=\pm\sqrt{3}$$

51) The horizontal asymptote of

$$f(x)=\frac{1-x}{2x+1}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{1-x}{2x+1}$$

$$\lim_{x\to\infty}\frac{1-x}{2x+1}=\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{x}{x}}{\frac{2x}{x}+\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}-1}{2+\frac{1}{x}}=\frac{0-1}{2+0}=-\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{1-x}{2x+1}=\lim_{x\to-\infty}\frac{\frac{1}{-x}-\frac{x}{-x}}{\frac{2x}{-x}+\frac{1}{-x}}=\lim_{x\to-\infty}\frac{\frac{1}{-x}+1}{-2-\frac{1}{x}}=\frac{0+1}{-2-0}$$

$$=-\frac{1}{2}$$

Thus, the horizontal asymptote is

$$y=-\frac{1}{2}$$

52) The horizontal asymptote of

$$f(x)=\frac{7x^2+5}{3x^2+2}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{7x^2+5}{3x^2+2}$$

$$\lim_{x\to\infty}\frac{7x^2+5}{3x^2+2}=\lim_{x\to\infty}\frac{\frac{7x^2}{x^2}+\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}}=\lim_{x\to\infty}\frac{7+\frac{5}{x^2}}{3+\frac{2}{x^2}}=\frac{7+0}{3+0}=\frac{7}{3}$$

$$\lim_{x\to-\infty}\frac{7x^2+5}{3x^2+2}=\lim_{x\to-\infty}\frac{\frac{7x^2}{-x^2}+\frac{5}{-x^2}}{\frac{3x^2}{-x^2}+\frac{2}{-x^2}}$$

$$=\lim_{x\to-\infty}\frac{-7-\frac{5}{x^2}}{-3-\frac{2}{x^2}}=\frac{-7-0}{-3-0}=\frac{7}{3}$$

Thus, the horizontal asymptote is

$$y=\frac{7}{3}$$

53) The horizontal asymptote of

$$f(x)=\frac{\sqrt{x^2+2x-3}}{2x+7}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{\sqrt{x^2+2x-3}}{2x+7}$$

$$\lim_{x\to\infty}\frac{\sqrt{x^2+2x-3}}{2x+7}=\lim_{x\to\infty}\frac{\frac{\sqrt{x^2+2x-3}}{x}}{\frac{2x}{x}+\frac{7}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{x^2+2x-3}{x^2}}}{2+\frac{7}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}-\frac{3}{x^2}}}{2+\frac{7}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{1+\frac{2}{x}-\frac{3}{x^2}}}{2+\frac{7}{x}}=\frac{\sqrt{1+0-0}}{2+0}=\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2x-3}}{2x+7}=\lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+2x-3}}{-x}}{\frac{2x}{-x}+\frac{7}{-x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{x^2+2x-3}{x^2}}}{-2-\frac{7}{x}}=\lim_{x\to-\infty}\frac{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}-\frac{3}{x^2}}}{-2-\frac{7}{x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{1+\frac{2}{x}-\frac{3}{x^2}}}{-2-\frac{7}{x}}=\frac{\sqrt{1+0-0}}{-2-0}=-\frac{1}{2}$$

Thus, the horizontal asymptotes are

$$y=\pm\frac{1}{2}$$

54) The horizontal asymptote of

$$f(x)=\frac{\sqrt{2x-3}}{2x^2+7x-1}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{\sqrt{2x-3}}{2x^2+7x-1}$$

$$\lim_{x\to\infty}\frac{\sqrt{2x-3}}{2x^2+7x-1}=\lim_{x\to\infty}\frac{\frac{\sqrt{2x-3}}{x^2}}{\frac{2x^2}{x^2}+\frac{7x}{x^2}-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{2x-3}{x^4}}}{2+\frac{7}{x}-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\sqrt{\frac{2x}{x^4}-\frac{3}{x^4}}}{2+\frac{7}{x}-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{2}{x^3}-\frac{3}{x^4}}}{2+\frac{7}{x}-\frac{1}{x^2}}=\frac{\sqrt{0-0}}{2+0-0}=\frac{0}{2}=0$$

$$\lim_{x\to-\infty}\frac{\sqrt{2x-3}}{2x^2+7x-1}=\lim_{x\to-\infty}\frac{\frac{\sqrt{2x-3}}{-x^2}}{\frac{2x^2}{-x^2}+\frac{7x}{-x^2}-\frac{1}{-x^2}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{2x-3}{x^4}}}{-2-\frac{7}{x}+\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{\sqrt{\frac{2x}{x^4}-\frac{3}{x^4}}}{-2-\frac{7}{x}+\frac{1}{x^2}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{2}{x^3}-\frac{3}{x^4}}}{-2-\frac{7}{x}+\frac{1}{x^2}}=\frac{\sqrt{0-0}}{-2-0+0}=\frac{0}{-2}=0$$

Thus, the horizontal asymptote is

$$y=0$$

55)

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-8}+3}{x+1}=$$

Solution:

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-8}+3}{x+1}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2-8}}{-x}+\frac{3}{-x}}{\frac{x}{-x}+\frac{1}{-x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{4x^2-8}{x^2}}-\frac{3}{x}}{-1-\frac{1}{x}}=\lim_{x\to-\infty}\frac{\sqrt{\frac{4x^2}{x^2}-\frac{8}{x^2}}-\frac{3}{x}}{-1-\frac{1}{x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{4-\frac{8}{x^2}}-\frac{3}{x}}{-1-\frac{1}{x}}=\frac{\sqrt{4-0}-0}{-1-0}=-2$$

56)

$$\lim_{x\to\infty}\frac{\sqrt{4x^2-8}+3}{x+1}=$$

Solution:

$$\lim_{x\to\infty}\frac{\sqrt{4x^2-8}+3}{x+1}=\lim_{x\to\infty}\frac{\frac{\sqrt{4x^2-8}}{x}+\frac{3}{x}}{\frac{x}{x}+\frac{1}{x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{4x^2-8}{x^2}}+\frac{3}{x}}{1+\frac{1}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{4x^2}{x^2}-\frac{8}{x^2}}+\frac{3}{x}}{1+\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{4-\frac{8}{x^2}}+\frac{3}{x}}{1+\frac{1}{x}}=\frac{\sqrt{4-0}+0}{1+0}=2$$

Workshop Solutions to Chapter 4 (chapter 3)

1) If $f(x)$ is a differentiable function, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2) If $f(x) = 4x^2$, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h}$$

3) If $f(x) = x^2 - 3$, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[(x+h)^2 - 3] - [x^2 - 3]}{h}$$

4) If $f(x) = \sqrt{x}$, $x \geq 0$, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

5) If f is a differentiable function at a, then f is a continuous function at a.

6) If f is a continuous function at a, then f is a differentiable function at a.

Solution:

False

7) If $y = x^4 + 5x^2 + 3$, then $y' =$

Solution:

$$y' = 4x^3 + 10x$$

8) If $y = x^4 - 5x^2 + 3$, then $y' =$

Solution:

$$y' = 4x^3 - 10x$$

9) If $y = x^{-5/2}$, then $y' =$

Solution:

$$y' = -\frac{5}{2} x^{-\frac{5}{2}-1} = -\frac{5}{2} x^{-7/2}$$

10) If $y = \frac{1}{3x^3} + 2\sqrt{x} = \frac{1}{3} x^{-3} + 2x^{1/2}$, then $y' =$

Solution:

$$y' = (-3)\left(\frac{1}{3}\right) x^{-3-1} + \left(\frac{1}{2}\right)(2) x^{\frac{1}{2}-1}$$
$$= -x^{-4} + x^{-1/2} = -\frac{1}{x^4} + \frac{1}{x^{1/2}} = -\frac{1}{x^4} + \frac{1}{\sqrt{x}}$$

11) If $y = (x-3)(x-2)$, then $y' =$

Solution:

$$y = (x-3)(x-2) = x^2 - 5x + 6$$
$$y' = 2x - 5$$

12) If $y = (x^3 + 3)(x^2 - 1)$, then $y' =$

Solution:

$$y = (x^3 + 3)(x^2 - 1) = x^5 - x^3 + 3x^2 - 3$$
$$y' = 5x^4 - 3x^2 + 6x$$

13) If $y = \sqrt{x}(2x+1)$, then $y' =$

Solution:

$$y = \sqrt{x}(2x+1) = 2x\sqrt{x} + \sqrt{x} = 2x^{\frac{3}{2}} + x^{\frac{1}{2}}$$
$$y' = \left(\frac{3}{2}\right)(2) x^{\frac{3}{2}-1} + \left(\frac{1}{2}\right) x^{\frac{1}{2}-1} = 3x^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{1}{2}}$$
$$= 3\sqrt{x} + \frac{1}{2\sqrt{x}}$$

OR

Use the rule $(f.g)' = f'g + fg'$

$$y' = (2)(\sqrt{x}) + \left(\frac{1}{2\sqrt{x}}\right)(2x+1) = 2\sqrt{x} + \frac{2x+1}{2\sqrt{x}}$$

14) If $y = \frac{x+3}{x-2}$, then $y' =$

Solution:

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x-2) - (x+3)(1)}{(x-2)^2} = \frac{x-2-x-3}{(x-2)^2} = \frac{-5}{(x-2)^2}$$
$$= -\frac{5}{(x-2)^2}$$

15) If $y = \frac{x+3}{x-2}$, then $y'|_{x=4} =$

Solution:

$$y' = \frac{(1)(x-2) - (x+3)(1)}{(x-2)^2} = \frac{x-2-x-3}{(x-2)^2}$$
$$= \frac{-5}{(x-2)^2} = -\frac{5}{(x-2)^2}$$
$$y'|_{x=4} = -\frac{5}{(4-2)^2} = -\frac{5}{4}$$

16) If $y = \frac{x-1}{x+2}$, then $y' =$

Solution:

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x+2) - (x-1)(1)}{(x+2)^2} = \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

17) If $y = \sqrt{3x^2 + 6x}$, then $y' =$

<u>Solution:</u>

Use the rule $\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y' = \frac{6x+6}{2\sqrt{3x^2+6x}} = \frac{6(x+1)}{2\sqrt{3x^2+6x}} = \frac{3(x+1)}{\sqrt{3x^2+6x}}$$

18) If $y = \sqrt{3x^2 + 6x}$, then $y'|_{x=1} =$

<u>Solution:</u>

$$y' = \frac{6x+6}{2\sqrt{3x^2+6x}} = \frac{6(x+1)}{2\sqrt{3x^2+6x}} = \frac{3(x+1)}{\sqrt{3x^2+6x}}$$

$$y'|_{x=1} = \frac{3((1)+1)}{\sqrt{3(1)^2+6(1)}} = \frac{6}{\sqrt{9}} = \frac{6}{3} = 2$$

19) The tangent line equation to the curve $y = x^2 + 2$ at the point $(1,3)$ is

<u>Solution:</u>

First, we have to find the slope of the curve which is

$$y' = 2x$$

Thus, the slope at $x = 1$ is

$$y'|_{x=1} = 2(1) = 2$$

Hence, the tangent line equation passing through the point $(1,3)$ with slope $m = 2$ is

$$y - 3 = 2(x-1)$$
$$y - 3 = 2x - 2$$
$$y = 2x - 2 + 3$$
$$y = 2x + 1$$

20) The tangent line equation to the curve $y = \frac{2x}{x+1}$ at the point $(0,0)$ is

<u>Solution:</u>

First, we have to find the slope of the curve which is

$$y' = \frac{(2)(x+1) - (2x)(1)}{(x+1)^2} = \frac{2x+2-2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Thus, the slope at $x = 0$ is

$$y'|_{x=0} = \frac{2}{(0+1)^2} = 2$$

Hence, the tangent line equation passing through the point $(0,0)$ with slope $m = 2$ is

$$y - 0 = (2)(x-0)$$
$$y = 2x$$

21) The tangent line equation to the curve $y = 3x^2 - 13$ at the point $(2,-1)$ is

<u>Solution:</u>

First, we have to find the slope of the curve which is

$$y' = 6x$$

Thus, the slope at $x = 2$ is

$$y'|_{x=2} = 6(2) = 12$$

Hence, the tangent line equation passing through the point $(2,-1)$ with slope $m = 12$ is

$$y - (-1) = 12(x-2)$$
$$y + 1 = 12x - 24$$
$$y = 12x - 24 - 1$$
$$y = 12x - 25$$

22) The tangent line equation to the curve $y = 3x^2 + 2x + 5$ at the point $(0,5)$ is

<u>Solution:</u>

First, we have to find the slope of the curve which is

$$y' = 6x + 2$$

Thus, the slope at $x = 2$ is

$$y'|_{x=0} = 6(0) + 2 = 2$$

Hence, the tangent line equation passing through the point $(0,5)$ with slope $m = 2$ is

$$y - 5 = 2(x-0)$$
$$y - 5 = 2x$$
$$y = 2x + 5$$

23) If $y = xe^x$, then $y' =$

<u>Solution:</u>

Use the rules $(f.g)' = f'g + fg'$ and $(e^u) = e^u.u'$

$$y' = (1)(e^x) + (x)(e^x) = e^x + xe^x = e^x(1+x)$$

24) If $y = x - e^x$, then $y'' =$

<u>Solution:</u>

Use the rules $(f-g)' = f' - g'$ and $(e^u) = e^u.u'$

$$y' = 1 - e^x$$
$$y'' = -e^x$$

25) If $x^2 - y^2 = 4$, then $y' =$

<u>Solution:</u>

$$2x - 2yy' = 0$$
$$-2yy' = -2x$$
$$y' = \frac{-2x}{-2y}$$
$$y' = \frac{x}{y}$$

26) If $x^2 + y^2 = 4$, then $y' =$

<u>Solution:</u>

$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$y' = \frac{-2x}{2y}$$
$$y' = -\frac{x}{y}$$

27) If $y = \frac{x+1}{x+2}$, then $y' =$

<u>Solution:</u>

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x+2) - (x+1)(1)}{(x+2)^2} = \frac{x+2-x-1}{(x+2)^2}$$
$$= \frac{1}{(x+2)^2}$$

28) If $y = \frac{1}{\sqrt[2]{x^5}} + \sec x$, then $y' =$

<u>Solution:</u>

Use the rules

$$(f+g)' = f' + g' \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y = \frac{1}{\sqrt[2]{x^5}} + \sec x = x^{-\frac{5}{2}} + \sec x$$

$$y' = \left(-\frac{5}{2}\right)x^{-\frac{5}{2}-1} + \sec x \tan x = -\frac{5}{2}x^{-7/2} + \sec x \tan x$$

29) If $y = \tan^{-1}(x^3)$, then $y' =$

Solution:

Use the rule $\quad (\tan^{-1} u)' = \frac{u'}{1+u^2}$

$$y' = \frac{1}{1+(x^3)^2} . (3x^2) = \frac{3x^2}{1+x^6}$$

30) If $y = \tan x - x$, then $y' =$

Solution:

Use the rules

$$(f-g)' = f' - g' \quad \text{and} \quad (\tan u)' = \sec^2 u \ .u'$$

$$y' = \sec^2 x - 1$$

31) If $y = \sec^2 x - 1$, then $y' =$

Solution:

Use the rules $(f-g)' = f' - g'$, $\quad (u)^n = n(u)^{n-1}.u'$
and $\quad (\sec u)' = \sec u \tan u \, .u'$

$$y' = 2\sec x \, . \, \sec x \tan x = 2\sec^2 x \tan x$$

32) If $y = x^{\sin x}$, then $y' =$

Solution:

Use the rule $\quad (\sin u)' = \cos u \ .u'$

$$y = x^{\sin x}$$
$$\ln y = \ln x^{\sin x}$$
$$\ln y = \sin x \, . \ln x$$
$$\frac{y'}{y} = \cos x \, . \, \ln x + \sin x \, . \frac{1}{x} = \cos x \, . \, \ln x + \frac{\sin x}{x}$$
$$y' = y\left(\cos x \, . \, \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \, . \, \ln x + \frac{\sin x}{x}\right)$$

33) If $y = x^{\cos x}$, then $y' =$

Solution:

Use the rule $\quad (\cos u)' = -\sin u \ .u'$

$$y = x^{\cos x}$$
$$\ln y = \ln x^{\cos x}$$
$$\ln y = \cos x \, . \ln x$$
$$\frac{y'}{y} = -\sin x \, . \, \ln x + \cos x \, . \frac{1}{x} = -\sin x \, . \, \ln x + \frac{\cos x}{x}$$
$$y' = y\left(-\sin x \, . \, \ln x + \frac{\cos x}{x}\right)$$
$$= x^{\cos x}\left(\frac{\cos x}{x} - \sin x \, . \, \ln x\right)$$

34) If $y = (2x^2 + \csc x)^9$, then $y' =$

Solution:

Use the rules

$$(u)^n = n(u)^{n-1}.u' \quad \text{and} \quad (\csc u)' = -\csc u \cot u \, .u'$$

$$y' = 9(2x^2 + \csc x)^8.(4x - \csc x \cot x)$$

35) If $y = \frac{5^x}{\cot x}$, then $y' =$

Solution:

Use the rules

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} , \quad (a^u)' = a^u . \ln a \, .u'$$
and $\quad (\csc u)' = -\csc u \cot u \, .u'$

$$y' = \frac{(5^x \ln 5)(\cot x) - (5^x)(-\csc^2 x)}{(\cot x)^2}$$
$$= \frac{5^x(\ln 5 \cot x + \csc^2 x)}{\cot^2 x}$$

36) If $y = e^{2x}$, then $y^{(6)} =$

Solution:

Use the rule $\quad (e^u)' = e^u.u'$

$$y' = 2e^{2x}$$
$$y'' = 4e^{2x}$$
$$y''' = 8e^{2x}$$
$$y^{(4)} = 16e^{2x}$$
$$y^{(5)} = 32e^{2x}$$
$$y^{(6)} = 64e^{2x}$$

37) If $y = x^{-2}e^{\sin x}$, then $y' =$

Solution:

Use the rules $\quad (f.g)' = f'g + fg'$, $\quad (e^u) = e^u.u'$
and $\ (\sin u)' = \cos u \ .u'$

$$y' = (-2x^{-3})(e^{\sin x}) + (x^{-2})(e^{\sin x}.\cos x)$$
$$= -2x^{-3}e^{\sin x} + x^{-2}\cos x \, e^{\sin x}$$
$$= x^{-3}e^{\sin x}(-2 + x\cos x)$$
$$= x^{-3}e^{\sin x}(x\cos x - 2)$$

38) If $y = 5^{\tan x}$, then $y' =$

Solution:

Use the rules

$$(a^u)' = a^u . \ln a \, .u' \quad \text{and} \quad (\tan u)' = \sec^2 u \, .u'$$

$$y' = 5^{\tan x} . \, \ln 5 \, . \sec^2 x$$

39) If $x^2 + y^2 = 3xy + 7$, then $y' =$

Solution:

$$2x + 2yy' = 3y + 3xy'$$
$$2yy' - 3xy' = 3y - 2x$$
$$y'(2y - 3x) = 3y - 2x$$
$$y' = \frac{3y - 2x}{2y - 3x}$$

40) If $y = \sin^3(4x)$, then $y^{(6)} =$ y'=

Solution:

Use the rules

$$(u)^n = n(u)^{n-1}.u' \quad \text{and} \quad (\sin u)' = \cos u \, .u'$$

$$y' = 3\sin^2(4x).\cos(4x).(4)$$
$$= 12\sin^2(4x).\cos(4x)$$

41) If $y = 3^x \cot x$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$, $(a^u)' = a^u . \ln a . u'$ and $(\cot u)' = -\csc^2 u \ .u'$

$$y' = (3^x . \ln 3)(\cot x) + (3^x)(-\csc^2 x)$$
$$= 3^x \ln 3 \ \cot x - 3^x \csc^2 x$$
$$= 3^x(\ln 3 \ \cot x - \csc^2 x)$$

42) If $y = (2x^2 + \sec x)^7$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sec u)' = \sec u \tan u . u'$

$$y' = 7(2x^2 + \sec x)^6.(4x + \sec x \tan x)$$

43) If $f(x) = \cos x$, then $f^{(45)}(x) =$

Solution:

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$

Note: $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4.

Hence,

$$f^{(44)}(x) = \cos x$$
$$f^{(45)}(x) = -\sin x$$

44) If $D^{47}(\sin x) =$

Solution:

$$D(\sin x) = \cos x$$
$$D^2(\sin x) = -\sin x$$
$$D^3(\sin x) = -\cos x$$
$$D^4(\sin x) = \sin x$$

Note: $D^n(\sin x) = \sin x$ whenever n is a multiple of 4.

Hence,

$$D^{44}(\sin x) = \sin x$$
$$D^{45}(\sin x) = \cos x$$
$$D^{46}(\sin x) = -\sin x$$
$$D^{47}(\sin x) = -\cos x$$

45) If $y = x^x$, then $y' =$

Solution:

Use the rule $(\ln u)' = \frac{u'}{u}$

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{y'}{y} = (1)(\ln x) + (x)\left(\frac{1}{x}\right)$$
$$\frac{y'}{y} = \ln x + 1$$
$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

46) If $f(x) = \frac{\ln x}{x^2}$, then $f'(1) =$

Solution:

Use the rules $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ and $(\ln u)' = \frac{u'}{u}$

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x^2) - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x \ln x}{x^4}$$
$$= \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3}$$

$$\therefore \quad f'(1) = \frac{1 - 2\ln(1)}{(1)^3} = \frac{1 - 2(0)}{1} = 1$$

47) If $y = \cot^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\cot^{-1} u)' = -\frac{u'}{1+u^2}$ and $(e^u) = e^u.u'$

$$y' = -\frac{1}{1 + (e^x)^2} . e^x = -\frac{e^x}{1 + e^{2x}}$$

48) If $y = \tan^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\tan^{-1} u)' = \frac{u'}{1+u^2}$ and $(e^u) = e^u.u'$

$$y' = \frac{1}{1 + (e^x)^2} . e^x = \frac{e^x}{1 + e^{2x}}$$

49) If $y = \sin^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$ and $(e^u) = e^u.u'$

$$y' = \frac{1}{\sqrt{1 - (e^x)^2}} . e^x = \frac{e^x}{\sqrt{1 - e^{2x}}}$$

50) If $y = \cos^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$ and $(e^u) = e^u.u'$

$$y' = -\frac{1}{\sqrt{1 - (e^x)^2}} . e^x = -\frac{e^x}{\sqrt{1 - e^{2x}}}$$

51) If $y = \cos(2x^3)$, then $y' =$

Solution:

Use the rule $(\cos u)' = -\sin u \ .u'$

$$y' = -\sin(2x^3).(6x^2) = -6x^2 \sin(2x^3)$$

52) If $y = \csc x \cot x$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$,
$(\csc u)' = -\csc u \cot u . u'$ and $(\cot u)' = -\csc^2 u \ .u'$

$$y' = (-\csc x \cot x)(\cot x) + (\csc x)(-\csc^2 x)$$
$$= -\csc x \cot^2 x - \csc^3 x = -\csc x(\cot^2 x + \csc^2 x)$$

53) If $y = \sqrt{x^2 - 2\sec x}$, then $y' =$
Solution:
Use the rules

$$\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}} \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = \frac{2x - 2\sec x \tan x}{2\sqrt{x^2 - 2\sec x}} = \frac{2(x - \sec x \tan x)}{2\sqrt{x^2 - 2\sec x}}$$
$$= \frac{x - \sec x \tan x}{\sqrt{x^2 - 2\sec x}}$$

54) If $y = (3x^2 + 1)^6$, then $y' =$
Solution:
Use the rule $\quad (u)^n = n(u)^{n-1}.u'$

$$y' = 6(3x^2 + 1)^5.(6x) = 36x(3x^2 + 1)^5$$

55) If $xy + \tan x = 2x^3 + \sin y$, then $y' =$
Solution:

$$[(1)(y) + (x)(y')] + \sec^2 x = 6x^2 + \cos y \,.\, y'$$
$$y + xy' + \sec^2 x = 6x^2 + y' \cos y$$
$$xy' - y' \cos y = 6x^2 - y - \sec^2 x$$
$$y'(x - \cos y) = 6x^2 - y - \sec^2 x$$
$$y' = \frac{6x^2 - y - \sec^2 x}{x - \cos y}$$

56) If $y = x^{-1} \sec x$, then $y' =$
Solution:
Use the rules
$(f.g)' = f'g + fg'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = (-x^{-2})(\sec x) + (x^{-1})(\sec x \tan x)$$
$$= x^{-1} \sec x \tan x - x^{-2} \sec x$$
$$= x^{-2} \sec x\,(x \tan x - 1)$$

57) If $y = \sin^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $\quad (\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$

$$y' = \frac{1}{\sqrt{1-(x^3)^2}}\,.3x^2 = \frac{3x^2}{\sqrt{1-x^6}}$$

58) If $y = \cos^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $\quad (\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$

$$y' = -\frac{1}{\sqrt{1-(x^3)^2}}\,.3x^2 = -\frac{3x^2}{\sqrt{1-x^6}}$$

59) If $y = \sec^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $\quad (\sec^{-1} u)' = \frac{u'}{|u|\sqrt{u^2-1}}$

$$y' = \frac{1}{x^3\sqrt{(x^3)^2-1}}\,.3x^2 = \frac{3x^2}{x^3\sqrt{x^6-1}} = \frac{3}{x\sqrt{x^6-1}}$$

60) If $y = \csc^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $\quad (\csc^{-1} u)' = -\frac{u'}{|u|\sqrt{u^2-1}}$

$$y' = -\frac{1}{x^3\sqrt{(x^3)^2-1}}\,.3x^2 = -\frac{3x^2}{x^3\sqrt{x^6-1}} = -\frac{3}{x\sqrt{x^6-1}}$$

61) If $y = \ln(x^3 - 2\sec x)$, then $y' =$
Solution:
Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = \frac{1}{x^3 - 2\sec x}\,.\,(3x^2 - 2\sec x \tan x)$$
$$= \frac{3x^2 - 2\sec x \tan x}{x^3 - 2\sec x}$$

62) If $y = \ln(\cos x)$, then $y' =$
Solution:
Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\cos u)' = -\sin u \,.\, u'$$

$$y' = \frac{1}{\cos x}\,.\,(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

63) If $y = \ln(\sin x)$, then $y' =$
Solution:
Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\sin u)' = \cos u \,.\, u'$$

$$y' = \frac{1}{\sin x}\,.\,(\cos x) = \frac{\cos x}{\sin x} = \cot x$$

64) If $y = \ln\sqrt{3x^2 + 5x}$, then $y' =$
Solution:
Use the rules $\quad (\ln u)' = \frac{u'}{u} \quad$ and $\quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y' = \frac{1}{\sqrt{3x^2 + 5x}}\,.\left(\frac{6x + 5}{2\sqrt{3x^2 + 5x}}\right) = \frac{6x + 5}{2(3x^2 + 5x)}$$

65) If $y = \log_5(x^3 - 2\csc x)$, then $y' =$

Solution:

Use the rules

$$(\log_a u)' = \frac{u'}{u \ln a} \quad \text{and} \quad (\csc u)' = -\csc u \cot u \,.\, u'$$

$$y' = \frac{1}{(x^3 - 2\csc x)(\ln 5)} \,.\, [3x^2 - 2\,(-\csc x \cot x)]$$

$$= \frac{3x^2 + 2\csc x \cot x}{(x^3 - 2\csc x)(\ln 5)}$$

66) If $y = \ln \frac{x-1}{\sqrt{x+2}}$, then $y' =$

Solution:

Use the rules

$$(\ln u)' = \frac{u'}{u} \;,\; \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \text{and} \quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

$$y' = \frac{1}{\frac{x-1}{\sqrt{x+2}}} \cdot \left(\frac{(1)\left(\sqrt{x+2}\right) - (x-1)\left(\frac{1}{2\sqrt{x+2}}\right)}{\left(\sqrt{x+2}\right)^2} \right)$$

$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\sqrt{x+2} - \frac{x-1}{2\sqrt{x+2}}}{x+2} \right)$$

$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\frac{2(x+2) - (x-1)}{2\sqrt{x+2}}}{x+2} \right)$$

$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\frac{x+5}{2\sqrt{x+2}}}{x+2} \right)$$

$$= \frac{\sqrt{x+2}}{x-1} \left(\frac{x+5}{2(x+2)\sqrt{x+2}} \right)$$

$$= \frac{x+5}{2(x-1)(x+2)}$$

67) If $y = 2x^3 - \sin x$, then $y' =$

Solution:

Use the rule $\quad (\sin u)' = \cos u \,.\, u'$

$$y' = 6x^2 - \cos x$$

68) If $y = x^3 \cos x$, then $y' =$

Solution:

Use the rules

$(f.g)' = f'g + fg'$ and $(\cos u)' = -\sin u \,.\, u'$

$$y' = (3x^2)(\cos x) + (x^3)(-\sin x)$$
$$= 3x^2 \cos x - x^3 \sin x$$

69) If $y = x^{\sqrt{x}}$, then $y' =$

Solution:

Use the rule $\quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y = x^{\sqrt{x}}$$
$$\ln y = \ln x^{\sqrt{x}}$$
$$\ln y = \sqrt{x} \ln x$$
$$\frac{y'}{y} = \left(\frac{1}{2\sqrt{x}}\right)(\ln x) + \left(\sqrt{x}\right)\left(\frac{1}{x}\right)$$
$$\frac{y'}{y} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{x \ln x + 2x}{2x\sqrt{x}} = \frac{x(\ln x + 2)}{2x\sqrt{x}}$$
$$= \frac{\ln x + 2}{2\sqrt{x}}$$
$$y' = y\left(\frac{\ln x + 2}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)$$

70) If $y = (\sin x)^x$, then $y' =$

Solution:

Use the rule $\quad (\sin u)' = \cos u \,.\, u'$

$$y = (\sin x)^x$$
$$\ln y = \ln(\sin x)^x$$
$$\ln y = x \ln(\sin x)$$
$$\frac{y'}{y} = (1)(\ln(\sin x)) + (x)\left(\frac{\cos x}{\sin x}\right)$$
$$\frac{y'}{y} = \ln(\sin x) + \frac{x \cos x}{\sin x} = \ln(\sin x) + x \cot x$$
$$y' = y(\ln(\sin x) + x \cot x)$$
$$= (\sin x)^x(\ln(\sin x) + x \cot x)$$

71) If $y = \log_7(x^3 - 2)$, then $y' =$

Solution:

Use the rule $\quad (\log_a u)' = \frac{u'}{u \ln a}$

$$y' = \frac{1}{(x^3 - 2)(\ln 7)} \,.\, (3x^2) = \frac{3x^2}{(x^3 - 2)(\ln 7)}$$

72) If $y = \cos(x^5)$, then $y' =$

Solution:

Use the rule $\quad (\cos u)' = -\sin u \,.\, u'$

$$y' = -\sin(x^5)\,.\,(5x^4) = -5x^4 \sin(x^5)$$

73) If $y = \sec x \tan x$, then $y' =$

Solution:

$(f.g)' = f'g + fg'$, $(\sec u)' = \sec u \tan u \,.\, u'$ and $(\tan u)' = \sec^2 u \,.\, u'$

$$y' = (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x)$$
$$= \sec x \tan^2 x + \sec^3 x = \sec x(\tan^2 x + \sec^2 x)$$

74) If $D^{99}(\cos x) =$

Solution:

$$D(\cos x) = -\sin x$$
$$D^2(\cos x) = -\cos x$$
$$D^3(\cos x) = \sin x$$
$$D^4(\cos x) = \cos x$$

Note: $D^n(\cos x) = \cos x$ whenever n is a multiple of 4.
Hence,

$$D^{96}(\cos x) = \cos x$$
$$D^{97}(\cos x) = -\sin x$$
$$D^{98}(\cos x) = -\cos x$$
$$D^{99}(\cos x) = \sin x$$

75) If $y = (x + \sec x)^3$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = 3(x + \sec x)^2.(1 + \sec x \tan x)$$

76) If $x^2 = 5y^2 + \sin y$, then $y' =$

Solution:

$$2x = 10yy' + \cos y \,.\, y'$$
$$y'(10y + \cos y) = 2x$$
$$y' = \frac{2x}{10y + \cos y}$$

77) If $x^2 - 5y^2 + \sin y = 0$, then $y' =$

Solution:

$$2x - 10yy' + \cos y \,.\, y' = 0$$
$$y'(-10y + \cos y) = -2x$$
$$y' = \frac{-2x}{-10y + \cos y} = \frac{2x}{10y - \cos y}$$

78) If $y = \sin x \sec x$, then $y' =$

Solution:

$(f.g)' = f'g + fg'$, $(\sin u)' = \cos u \,.\, u'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = (\cos x)(\sec x) + (\sin x)(\sec x \tan x)$$
$$= 1 + \sin x \,.\, \frac{1}{\cos x} \,.\, \frac{\sin x}{\cos x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$
$$= \sec^2 x$$

79) If $f(x) = \sin^2(x^3 + 1)$, then $f'(x) =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sin u)' = \cos u \,.\, u'$

$$f'(x) = 2\sin(x^3 + 1) \,.\, (\cos(x^3 + 1)) \,.\, (3x^2)$$
$$= 6x^2 \sin(x^3 + 1)\cos(x^3 + 1)$$

80) If $y = (x + \cot x)^3$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\cot u)' = -\csc^2 u \,.\, u'$

$$y' = 3(x + \cot x)^2.(1 - \csc^2 x)$$

81) If $y = \tan^{-1}\left(\frac{x}{2}\right)$, then $y' =$

Solution:

Use the rule $(\tan^{-1} u)' = \frac{u'}{1+u^2}$

$$y' = \frac{1}{1 + \left(\frac{x}{2}\right)^2} \,.\, \frac{1}{2} = \frac{1}{2\left(1 + \frac{x^2}{4}\right)} = \frac{1}{2\left(\frac{4 + x^2}{4}\right)} = \frac{2}{4 + x^2}$$

82) If $y = \cot^{-1}\left(\frac{x}{2}\right)$, then $y' =$

Solution:

Use the rule $(\cot^{-1} u)' = -\frac{u'}{1+u^2}$

$$y' = -\frac{1}{1 + \left(\frac{x}{2}\right)^2} \,.\, \frac{1}{2} = -\frac{1}{2\left(1 + \frac{x^2}{4}\right)} = -\frac{1}{2\left(\frac{4 + x^2}{4}\right)}$$
$$= -\frac{2}{4 + x^2}$$

83) If $y = \sin^{-1}\left(\frac{x}{3}\right)$, then $y' =$

Solution:

Use the rule $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$

$$y' = \frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \,.\, \frac{1}{3} = \frac{1}{3\sqrt{1 - \frac{x^2}{9}}} = \frac{1}{3\sqrt{\frac{9 - x^2}{9}}}$$
$$= \frac{1}{\sqrt{9 - x^2}}$$

84) If $y = \cos^{-1}\left(\frac{x}{3}\right)$, then $y' =$

Solution:

Use the rule $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$

$$y' = -\frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \,.\, \frac{1}{3} = -\frac{1}{3\sqrt{1 - \frac{x^2}{9}}} = -\frac{1}{3\sqrt{\frac{9 - x^2}{9}}}$$
$$= -\frac{1}{\sqrt{9 - x^2}}$$

85) If $D^{99}(\sin x) =$ <u>Solution:</u> $$D(\sin x) = \cos x$$ $$D^2(\sin x) = -\sin x$$ $$D^3(\sin x) = -\cos x$$ $$D^4(\sin x) = \sin x$$ **Note:** $D^n(\sin x) = \sin x$ whenever n is a multiple of 4. Hence, $$D^{96}(\sin x) = \sin x$$ $$D^{97}(\sin x) = \cos x$$ $$D^{98}(\sin x) = -\sin x$$ $$D^{99}(\sin x) = -\cos x$$	

Workshop Solutions to Sections 3.1 and 3.2 (2.3)

1) $$\begin{aligned}\lim_{x\to-2}(x^3-2x+1) &= (-2)^3-2(-2)+1\\ &= -8+4+1=-3\end{aligned}$$

2) $$\begin{aligned}\lim_{x\to2}(3x^2+x-4) &= 3(2)^2+(2)-4\\ &= 12+2-4=10\end{aligned}$$

3) $$\begin{aligned}\lim_{x\to1}(x^2+3x-5)^3 &= ((1)^2+3(1)-5)^3\\ &= (1+3-5)^3=(-1)^3=-1\end{aligned}$$

4) $$\begin{aligned}\lim_{x\to-2}(2x^3+3x^2+5) &= 2(-2)^3+3(-2)^2+5\\ &= 2(-8)+3(4)+5\\ &= -16+12+5=1\end{aligned}$$

5) $$\lim_{x\to-2}\frac{x^2-2}{x-2}=\frac{(-2)^2-2}{(-2)-2}=\frac{4-2}{-2-2}=\frac{2}{-4}=-\frac{1}{2}$$

6) $$\lim_{x\to2}\frac{x^3+5}{x^2+1}=\frac{(2)^3+5}{(2)^2+1}=\frac{8+5}{4+1}=\frac{13}{5}$$

7) $$\begin{aligned}\lim_{x\to0}\frac{x^2+3x+5}{x^2-3}&=\frac{(0)^2+3(0)+5}{(0)^2-3}=\frac{0+0+5}{0-3}\\ &=\frac{5}{-3}=-\frac{5}{3}\end{aligned}$$

8) $$\lim_{x\to1}\frac{x-1}{x^2+x-5}=\frac{(1)-1}{(1)^2+(1)-5}=\frac{1-1}{1+1-5}=\frac{0}{-3}=0$$

9) $$\begin{aligned}\lim_{x\to-1}\sqrt{x^3-10x+7}&=\sqrt{(-1)^3-10(-1)+7}\\ &=\sqrt{-1+10+7}=\sqrt{16}=4\end{aligned}$$

10) $$\begin{aligned}\lim_{x\to-1}\frac{1-(x+4)^{-2}}{x-2}&=\frac{1-\left((-1)+4\right)^{-2}}{(-1)-2}\\ &=\frac{1-(-1+4)^{-2}}{-1-2}=\frac{1-(3)^{-2}}{-3}=\frac{1-\frac{1}{3^2}}{-3}\\ &=\frac{1-\frac{1}{9}}{-3}=\frac{\frac{8}{9}}{-3}=\frac{8}{9}\times\frac{1}{-3}=\frac{8}{-27}=-\frac{8}{27}\end{aligned}$$

11) $$\begin{aligned}\lim_{x\to-1}\frac{x^3+2x}{8-2x}&=\frac{(-1)^3+2(-1)}{8-2(-1)}=\frac{-1-2}{8+2}=\frac{-3}{10}\\ &=-\frac{3}{10}\end{aligned}$$

12) $$\lim_{x\to4}\frac{x^2-3x}{5+x}=\frac{(4)^2-3(4)}{5+(4)}=\frac{16-12}{5+4}=\frac{4}{9}$$

13) $$\lim_{x\to4}\frac{x^2-4x}{5+x}=\frac{(4)^2-4(4)}{5+(4)}=\frac{16-16}{5+4}=\frac{0}{9}=0$$

15) $$\begin{aligned}\lim_{x\to0}\frac{x^3-5x^2}{x^2}&=\lim_{x\to0}\frac{x^2(x-5)}{x^2}\\ &=\lim_{x\to0}(x-5)=(0)-5=-5\end{aligned}$$

14) $$\begin{aligned}\lim_{x\to4}\frac{3^{-1}-(2x-5)^{-1}}{4-x}&=\lim_{x\to4}\frac{\frac{1}{3}-\frac{1}{2x-5}}{4-x}\\ &=\lim_{x\to4}\frac{\frac{2x-5-3}{3(2x-5)}}{4-x}\\ &=\lim_{x\to4}\frac{2x-8}{3(2x-5)(4-x)}\\ &=\lim_{x\to4}\frac{2(x-4)}{3(2x-5)(4-x)}\\ &=\lim_{x\to4}\frac{-2(4-x)}{3(2x-5)(4-x)}=\lim_{x\to4}\frac{-2}{3(2x-5)}\\ &=\frac{-2}{3(2(4)-5)}=\frac{-2}{3(8-5)}=\frac{-2}{9}=-\frac{2}{9}\end{aligned}$$

16) $$\begin{aligned}\lim_{x\to6}\frac{x-6}{x^2-36}&=\lim_{x\to6}\frac{x-6}{(x-6)(x+6)}=\lim_{x\to6}\frac{1}{x+6}\\ &=\frac{1}{(6)+6}=\frac{1}{12}\end{aligned}$$

17) $$\begin{aligned}\lim_{x\to6}\frac{x^2-36}{x-6}&=\lim_{x\to6}\frac{(x-6)(x+6)}{x-6}=\lim_{x\to6}(x+6)\\ &=(6)+6=12\end{aligned}$$

18) $$\begin{aligned}\lim_{x\to-6}\frac{x+6}{x^2-36}&=\lim_{x\to-6}\frac{x+6}{(x-6)(x+6)}=\lim_{x\to-6}\frac{1}{x-6}\\ &=\frac{1}{(-6)-6}=\frac{1}{-12}=-\frac{1}{12}\end{aligned}$$

19) $$\begin{aligned}\lim_{x\to3}\frac{x^3-27}{x-3}&=\lim_{x\to3}\frac{(x-3)(x^2+3x+9)}{x-3}\\ &=\lim_{x\to3}(x^2+3x+9)=(3)^2+3(3)+9\\ &=9+9+9=27\end{aligned}$$

20) $$\begin{aligned}\lim_{x\to3}\frac{x-3}{x^3-27}&=\lim_{x\to3}\frac{x-3}{(x-3)(x^2+3x+9)}\\ &=\lim_{x\to3}\frac{1}{x^2+3x+9}=\frac{1}{(3)^2+3(3)+9}\\ &=\frac{1}{9+9+9}=\frac{1}{27}\end{aligned}$$

21) $$\lim_{x\to-2}\frac{x+2}{x^3+8}=\lim_{x\to-2}\frac{x+2}{(x+2)(x^2-2x+4)}$$
$$=\lim_{x\to-2}\frac{1}{x^2-2x+4}$$
$$=\frac{1}{(-2)^2-2(-2)+4}=\frac{1}{4+4+4}=\frac{1}{12}$$

22) $$\lim_{x\to-2}\frac{x^3+8}{x+2}=\lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{x+2}$$
$$=\lim_{x\to-2}(x^2-2x+4)=(-2)^2-2(-2)+4$$
$$=4+4+4=12$$

23) $$\lim_{x\to4}\frac{x^2-3x-4}{x-4}=\lim_{x\to4}\frac{(x-4)(x+1)}{x-4}=\lim_{x\to4}(x+1)$$
$$=(4)+1=5$$

24) $$\lim_{x\to3}\frac{x^2+4x-21}{x^2-8x+15}=\lim_{x\to3}\frac{(x+7)(x-3)}{(x-5)(x-3)}=\lim_{x\to3}\frac{x+7}{x-5}$$
$$=\frac{(3)+7}{(3)-5}=\frac{10}{-2}=-5$$

25) $$\lim_{x\to0}\frac{x}{1-(1-x)^2}=\lim_{x\to0}\frac{x}{1-(1-2x+x^2)}$$
$$=\lim_{x\to0}\frac{x}{1-1+2x-x^2}$$
$$=\lim_{x\to0}\frac{x}{2x-x^2}=\lim_{x\to0}\frac{x}{x(2-x)}$$
$$=\lim_{x\to0}\frac{1}{2-x}=\frac{1}{2-(0)}=\frac{1}{2}$$

26) $$\lim_{x\to2}\frac{\sqrt[3]{x+6}-2}{x-2}=\lim_{x\to2}\frac{\sqrt[3]{x+6}-2}{(x+6)-8}=\lim_{x\to2}\frac{\sqrt[3]{x+6}-2}{\left(\sqrt[3]{x+6}\right)^3-8}$$
$$=\lim_{x\to2}\frac{\sqrt[3]{x+6}-2}{\left(\sqrt[3]{x+6}-2\right)\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)}$$
$$=\lim_{x\to2}\frac{1}{\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4}$$
$$=\frac{1}{\left(\sqrt[3]{(2)+6}\right)^2+2\sqrt[3]{(2)+6}+4}=\frac{1}{4+4+4}=\frac{1}{12}$$ deleted

27) $$\lim_{x\to0}\frac{\sqrt{x+25}-5}{x}$$
$$=\lim_{x\to0}\left[\frac{\sqrt{x+25}-5}{x}\times\frac{\sqrt{x+25}+5}{\sqrt{x+25}+5}\right]$$
$$=\lim_{x\to0}\frac{(x+25)-25}{x\left(\sqrt{x+25}+5\right)}$$
$$=\lim_{x\to0}\frac{x}{x\left(\sqrt{x+25}+5\right)}$$
$$=\lim_{x\to0}\frac{1}{\sqrt{x+25}+5}=\frac{1}{\sqrt{(0)+25}+5}$$
$$=\frac{1}{5+5}=\frac{1}{10}$$

28) $$\lim_{x\to0}\frac{x}{\sqrt{x+25}-5}=\lim_{x\to0}\left[\frac{x}{\sqrt{x+25}-5}\times\frac{\sqrt{x+25}+5}{\sqrt{x+25}+5}\right]$$
$$=\lim_{x\to0}\frac{x\left(\sqrt{x+25}+5\right)}{(x+25)-25}$$
$$=\lim_{x\to0}\frac{x\left(\sqrt{x+25}+5\right)}{x}$$
$$=\lim_{x\to0}\left(\sqrt{x+25}+5\right)=\sqrt{(0)+25}+5$$
$$=5+5=10$$

29) $$\lim_{x\to2}\frac{x-2}{2-\sqrt{6-x}}=\lim_{x\to2}\left[\frac{x-2}{2-\sqrt{6-x}}\times\frac{2+\sqrt{6-x}}{2+\sqrt{6-x}}\right]$$
$$=\lim_{x\to2}\frac{(x-2)\left(2+\sqrt{6-x}\right)}{4-(6-x)}$$
$$=\lim_{x\to2}\frac{(x-2)\left(2+\sqrt{6-x}\right)}{4-6+x}$$
$$=\lim_{x\to2}\frac{(x-2)\left(2+\sqrt{6-x}\right)}{x-2}$$
$$=\lim_{x\to2}\left(2+\sqrt{6-x}\right)=2+\sqrt{6-(2)}$$
$$=2+2=4$$

30) $$\lim_{x\to2}\frac{2-\sqrt{6-x}}{x+2}=\frac{2-\sqrt{6-(2)}}{(2)+2}=\frac{2-2}{4}=0$$

31) $$\lim_{x\to3}\frac{1-\sqrt{x-2}}{2-\sqrt{x+1}}$$
$$=\lim_{x\to3}\left[\frac{1-\sqrt{x-2}}{2-\sqrt{x+1}}\times\frac{1+\sqrt{x-2}}{1+\sqrt{x-2}}\times\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right]$$
$$=\lim_{x\to3}\left[\frac{1-(x-2)}{4-(x+1)}\times\frac{2+\sqrt{x+1}}{1+\sqrt{x-2}}\right]$$
$$=\lim_{x\to3}\left[\frac{3-x}{3-x}\times\frac{2+\sqrt{x+1}}{1+\sqrt{x-2}}\right]$$
$$=\lim_{x\to3}\frac{2+\sqrt{x+1}}{1+\sqrt{x-2}}=\frac{2+\sqrt{(3)+1}}{1+\sqrt{(3)-2}}=\frac{2+2}{1+1}$$
$$=\frac{4}{2}=2$$

32) If $2x \le f(x) \le 3x^2 - 8$, then
$$\lim_{x\to 2} f(x) =$$
Solution:
$$\lim_{x\to 2} 2x = 2(2) = 4$$
and
$$\lim_{x\to 2}(3x^2 - 8) = 3(2)^2 - 8 = 12 - 8 = 4$$
It follows from the Sandwich Theorem that
$$\lim_{x\to 2} f(x) = 4$$

33) $\lim_{x\to 0}\left[x\cos\left(x+\frac{1}{x}\right)\right] =$

We know that the cosine of any angle is between -1 and 1. So,
$$-1 \le \cos\left(x+\frac{1}{x}\right) \le 1$$
Now, multiply throughout by x, we get
$$-x \le x\cos\left(x+\frac{1}{x}\right) \le x$$
But $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(-x) = 0$.
It follows from the Sandwich Theorem that
$$\lim_{x\to 0}\left[x\cos\left(x+\frac{1}{x}\right)\right] = 0$$

34) $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right] =$

We know that the sine of any angle is between -1 and 1. So,
$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$
Now, multiply throughout by x, we get
$$-x \le x\sin\left(\frac{1}{x}\right) \le x$$
But $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(-x) = 0$.
It follows from the Sandwich Theorem that
$$\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right] = 0$$

35) If $\frac{x^2+1}{x-1} \le f(x) \le x - 1$, then
$$\lim_{x\to 0} f(x) =$$
Solution:
$$\lim_{x\to 0}\frac{x^2+1}{x-1} = \frac{(0)^2+1}{(0)-1} = \frac{1}{-1} = -1$$
and
$$\lim_{x\to 0}(x-1) = (0) - 1 = -1$$
It follows from the Sandwich Theorem that
$$\lim_{x\to 0} f(x) = -1$$

36) If $4(x-1) \le f(x) \le x^3 + x - 2$, then
$$\lim_{x\to 1} f(x) =$$
Solution:
$$\lim_{x\to 1}\big(4(x-1)\big) = 4\big((1)-1\big) = 4\times 0 = 0$$
and
$$\lim_{x\to 1}(x^3 + x - 2) = (1)^3 + (1) - 2 = 1 + 1 - 2 = 0$$
It follows from the Sandwich Theorem that
$$\lim_{x\to 1} f(x) = 0$$

37) If
$$\lim_{x\to 3}\frac{f(x)+4}{x-1} = 3,$$
then
$$\lim_{x\to 3} f(x) =$$
Solution:
$$\lim_{x\to 3}\frac{f(x)+4}{x-1} = \frac{\lim_{x\to 3}(f(x)+4)}{\lim_{x\to 3}(x-1)} = \frac{\lim_{x\to 3} f(x) + \lim_{x\to 3}(4)}{\lim_{x\to 3}(x) - \lim_{x\to 3}(1)}$$
$$= \frac{\lim_{x\to 3} f(x) + 4}{3-1} = \frac{\lim_{x\to 3} f(x) + 4}{2}$$
Now
$$\frac{\lim_{x\to 3} f(x) + 4}{2} = 3$$
$$\lim_{x\to 3} f(x) + 4 = 6 \quad \Leftrightarrow \quad \lim_{x\to 3} f(x) = 2$$

38) $\lim_{x\to2}\dfrac{2^{-1}-(3x-4)^{-1}}{2-x}$

$$= \lim_{x\to2}\frac{\frac{1}{2}-\frac{1}{3x-4}}{2-x}$$

$$= \lim_{x\to2}\frac{\frac{3x-4-2}{2(3x-4)}}{2-x}$$

$$= \lim_{x\to2}\frac{\frac{3x-6}{2(3x-4)}}{2-x}$$

$$= \lim_{x\to2}\frac{\frac{3(x-2)}{2(3x-4)}}{2-x}$$

$$= \lim_{x\to2}\frac{3(x-2)}{2(3x-4)(2-x)}$$

$$= \lim_{x\to2}\frac{-3(2-x)}{2(3x-4)(2-x)} = \lim_{x\to2}\frac{-3}{2(3x-4)}$$

$$= \frac{-3}{2(3(2)-4)} = \frac{-3}{2\times2} = -\frac{3}{4}$$

39) $\lim_{x\to0}\dfrac{(x+1)^3-1}{x} = \lim_{x\to0}\dfrac{(x^3+3x^2+3x+1)-1}{x}$

$$= \lim_{x\to0}\frac{x^3+3x^2+3x}{x}$$

$$= \lim_{x\to0}\frac{x(x^2+3x+3)}{x} = \lim_{x\to0}(x^2+3x+3)$$

$$= (0)^2+3(0)+3 = 3$$

40) If

$$\lim_{x\to1}\frac{f(x)+3x}{x^2-5f(x)} = 1,$$

then

$$\lim_{x\to1}f(x) =$$

<u>Solution:</u>

$$\lim_{x\to1}\frac{f(x)+3x}{x^2-5f(x)} = \frac{\lim_{x\to1}(f(x)+3x)}{\lim_{x\to1}(x^2-5f(x))}$$

$$= \frac{\lim_{x\to1}f(x)+\lim_{x\to1}(3x)}{\lim_{x\to1}(x^2)-\lim_{x\to1}(5f(x))}$$

$$= \frac{\lim_{x\to1}f(x)+3(1)}{(1)^2-5\lim_{x\to1}f(x)} = \frac{\lim_{x\to1}f(x)+3}{1-5\lim_{x\to1}f(x)}$$

Now

$$\frac{\lim_{x\to1}f(x)+3}{1-5\lim_{x\to1}f(x)} = 1$$

$$\lim_{x\to1}f(x)+3 = (1)\left(1-5\lim_{x\to1}f(x)\right)$$

$$\Leftrightarrow \quad \lim_{x\to1}f(x)+3 = 1-5\lim_{x\to1}f(x)$$

$$\Leftrightarrow \quad \lim_{x\to1}f(x)+5\lim_{x\to1}f(x) = 1-3$$

$$\Leftrightarrow \quad 6\lim_{x\to1}f(x) = -2$$

$$\Leftrightarrow \quad \lim_{x\to1}f(x) = \frac{-2}{6} = -\frac{1}{3}$$

41) $\lim_{x\to4}\dfrac{x^2-6x+8}{x^2+x-20}$

$$= \lim_{x\to4}\frac{(x-2)(x-4)}{(x-4)(x+5)}$$

$$= \lim_{x\to4}\frac{x-2}{x+5} = \frac{(4)-2}{(4)+5} = \frac{2}{9}$$

42) $\lim_{x\to-2}\dfrac{x^3+8}{x^2-x-6}$

$$= \lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{(x-3)(x+2)}$$

$$= \lim_{x\to-2}\frac{x^2-2x+4}{x-3} = \frac{(-2)^2-2(-2)+4}{(-2)-3}$$

$$= \frac{4+4+4}{-5} = \frac{12}{-5} = -\frac{12}{5}$$

43) $\lim_{x\to1}\left[\dfrac{x^2-2}{x+4}+x^2-2x\right] = \dfrac{(1)^2-2}{(1)+4}+(1)^2-2(1)$

$$= \frac{1-2}{1+4}+1-2 = \frac{-1}{5}-1 = \frac{-1-5}{5} = -\frac{6}{5}$$

44) $\lim_{x\to -2}\frac{4x^2+6x-4}{2x^2-8}$ $= \lim_{x\to -2}\frac{2(2x^2+3x-2)}{2(x^2-4)}$ $= \lim_{x\to -2}\frac{2x^2+3x-2}{x^2-4}$ $= \lim_{x\to -2}\frac{(2x-1)(x+2)}{(x-2)(x+2)}$ $= \lim_{x\to -2}\frac{2x-1}{x-2} = \frac{2(-2)-1}{(-2)-2} = \frac{-4-1}{-2-2}$ $= \frac{-5}{-4} = \frac{5}{4}$	45) $\lim_{x\to -1}\frac{x^2-2x-3}{x^5-x^3}$ $= \lim_{x\to -1}\frac{(x-3)(x+1)}{x^3(x^2-1)}$ $= \lim_{x\to -1}\frac{(x-3)(x+1)}{x^3(x-1)(x+1)}$ $= \lim_{x\to -1}\frac{x-3}{x^3(x-1)} = \frac{(-1)-3}{(-1)^3\big((-1)-1\big)}$ $= \frac{-1-3}{(-1)(-2)} = \frac{-4}{2} = -2$
46) $\lim_{x\to 3}\frac{\sqrt{2x+1}(x^2-9)}{(2x+3)(x-3)}$ $= \lim_{x\to 3}\frac{\sqrt{2x+1}(x-3)(x+3)}{(2x+3)(x-3)}$ $= \lim_{x\to 3}\frac{\sqrt{2x+1}(x+3)}{2x+3} = \frac{\sqrt{2(3)+1}\big((3)+3\big)}{2(3)+3}$ $= \frac{6\sqrt{7}}{9} = \frac{2\sqrt{7}}{3}$	47) $\lim_{x\to 1}\frac{\sqrt{3-2x}-1}{x-1} = \lim_{x\to 1}\left[\frac{\sqrt{3-2x}-1}{x-1}\times\frac{\sqrt{3-2x}+1}{\sqrt{3-2x}+1}\right]$ $= \lim_{x\to 1}\frac{(3-2x)-1}{(x-1)\big(\sqrt{3-2x}+1\big)}$ $= \lim_{x\to 1}\frac{2-2x}{(x-1)\big(\sqrt{3-2x}+1\big)}$ $= \lim_{x\to 1}\frac{2(1-x)}{(x-1)\big(\sqrt{3-2x}+1\big)} =$ $= \lim_{x\to 1}\frac{-2(x-1)}{(x-1)\big(\sqrt{3-2x}+1\big)} =$ $= \lim_{x\to 1}\frac{-2}{\sqrt{3-2x}+1} = \frac{-2}{\sqrt{3-2(1)}+1}$ $= \frac{-2}{\sqrt{3-2}+1} = \frac{-2}{2} = -1$
48) $\lim_{x\to 0}\frac{(x+1)^2-1}{x} = \lim_{x\to 0}\frac{(x^2+2x+1)-1}{x}$ $= \lim_{x\to 0}\frac{x^2+2x}{x} = \lim_{x\to 0}\frac{x(x+2)}{x}$ $= \lim_{x\to 0}(x+2) = (0)+2 = 2$	49) $\lim_{x\to 1}\frac{\sqrt{2x+2}-2}{\sqrt{3x-2}-1}$ $= \lim_{x\to 1}\left[\frac{\sqrt{2x+2}-2}{\sqrt{3x-2}-1}\times\frac{\sqrt{2x+2}+2}{\sqrt{2x+2}+2}\times\frac{\sqrt{3x-2}+1}{\sqrt{3x-2}+1}\right]$ $= \lim_{x\to 1}\left[\frac{(2x+2)-4}{(3x-2)-1}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]$ $= \lim_{x\to 1}\left[\frac{2x-2}{3x-3}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]$ $= \lim_{x\to 1}\left[\frac{2(x-1)}{3(x-1)}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]$ $= \lim_{x\to 1}\left[\frac{2}{3}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right] = \frac{2}{3}\times\frac{\sqrt{3(1)-2}+1}{\sqrt{2(1)+2}+2}$ $= \frac{2}{3}\times\frac{\sqrt{1}+1}{\sqrt{4}+2} = \frac{2}{3}\times\frac{2}{4} = \frac{1}{3}$

50) $\lim_{x\to 2}\frac{3-\sqrt{2x+5}}{x-2}$

$$= \lim_{x\to 2}\left[\frac{3-\sqrt{2x+5}}{x-2}\times\frac{3+\sqrt{2x+5}}{3+\sqrt{2x+5}}\right]$$
$$= \lim_{x\to 2}\frac{9-(2x+5)}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$= \lim_{x\to 2}\frac{4-2x}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$= \lim_{x\to 2}\frac{2(2-x)}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$= \lim_{x\to 2}\frac{-2(x-2)}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$= \lim_{x\to 2}\frac{-2}{3+\sqrt{2x+5}} = \frac{-2}{3+\sqrt{2(2)+5}}$$
$$= \frac{-2}{3+\sqrt{9}} = \frac{-2}{6} = -\frac{1}{3}$$

51) $\lim_{x\to -1}\frac{x^2+3x+2}{x^2+1} = \frac{(-1)^2+3(-1)+2}{(-1)^2+1} = \frac{1-3+2}{1+1}$
$$= \frac{0}{2} = 0$$

52) If
$$\lim_{x\to k} f(x) = -\frac{1}{2}$$
and
$$\lim_{x\to k} g(x) = \frac{2}{3}$$
Then
$$\lim_{x\to k}\frac{f(x)}{g(x)} = \frac{-\frac{1}{2}}{\frac{2}{3}} = -\frac{1}{2}\times\frac{3}{2} = -\frac{3}{4}$$

53) $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} = \lim_{x\to 0}\left[\frac{\sqrt{x+4}-2}{x}\times\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right]$
$$= \lim_{x\to 0}\frac{(x+4)-4}{x\left(\sqrt{x+4}+2\right)}$$
$$= \lim_{x\to 0}\frac{x}{x\left(\sqrt{x+4}+2\right)}$$
$$= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{(0)+4}+2}$$
$$= \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

54) $\lim_{x\to -1}\frac{x^2-5x-6}{x+1} = \lim_{x\to -1}\frac{(x-6)(x+1)}{x+1} = \lim_{x\to -1}(x-6)$
$$= (-1)-6 = -7$$

55) $\lim_{x\to 0}\frac{(x+3)^{-1}-3^{-1}}{x} = \lim_{x\to 0}\frac{\frac{1}{x+3}-\frac{1}{3}}{x} = \lim_{x\to 0}\frac{\frac{3-(x+3)}{3(x+3)}}{x}$
$$= \lim_{x\to 0}\frac{-x}{3x(x+3)} = \lim_{x\to 0}\frac{-1}{3(x+3)}$$
$$= \frac{-1}{3\left((0)+3\right)} = \frac{-1}{9} = -\frac{1}{9}$$

56) If
$$\lim_{x\to 1} f(x) = 3$$
$$\lim_{x\to 1} g(x) = -4$$
and
$$\lim_{x\to 1} h(x) = -1$$
then
$$\lim_{x\to 1}\left[\frac{5f(x)}{2g(x)}+h(x)\right] = \frac{\lim_{x\to 1} 5f(x)}{\lim_{x\to 1} 2g(x)}+\lim_{x\to 1} h(x)$$
$$= \frac{5\lim_{x\to 1} f(x)}{2\lim_{x\to 1} g(x)}+\lim_{x\to 1} h(x)$$
$$= \frac{5(3)}{2(-4)}+(-1) = \frac{15}{-8}-1 = -\frac{15}{8}-1$$
$$= \frac{-15-8}{8} = -\frac{23}{8}$$

57) If
$$\lim_{x\to 1} g(x) = -4$$
and
$$\lim_{x\to 1} h(x) = -1$$
then
$$\lim_{x\to 1}\sqrt{g(x)h(x)} = \sqrt{\left[\lim_{x\to 1} g(x)\right]\left[\lim_{x\to 1} h(x)\right]} = \sqrt{(-4)(-1)}$$
$$= \sqrt{4} = 2$$

58) If
$$\lim_{x\to 1} f(x) = 3$$
$$\lim_{x\to 1} g(x) = -4$$
and
$$\lim_{x\to 1} h(x) = -1$$
then
$$\lim_{x\to 1}[2f(x)g(x)h(x)] = 2\left[\lim_{x\to 1} f(x)\right]\left[\lim_{x\to 1} g(x)\right]\left[\lim_{x\to 1} h(x)\right]$$
$$= 2(3)(-4)(-1) = 24$$

Part from Section 3.3

① $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

$\lim\limits_{\theta \to 0} \dfrac{\theta}{\sin\theta} = 1$

② $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$

$\lim\limits_{\theta \to 0} \dfrac{\theta}{\tan\theta} = 1$

③ $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

Example (17)

$\lim\limits_{x \to 0} x\cot x = \lim\limits_{x \to 0} \dfrac{x}{\tan x} = 1$

$\lim\limits_{x \to 0} \dfrac{\sin 7x}{4x} = \dfrac{0}{0}$

$\lim\limits_{x \to 0} \dfrac{\sin 7x}{4x} = \dfrac{1}{4} \lim\limits_{7x \to 0} \dfrac{7 \sin 7x}{7x} = \dfrac{7}{4} \lim\limits_{7x \to 0} \dfrac{\sin 7x}{7x}$

$= \dfrac{7}{4}(1)$

$= \dfrac{7}{4}$

Note

$$\lim_{x\to 0} \frac{\sin mx}{nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\sin 6x}{2x} = \frac{6}{2} = 3$$

$$\lim_{x\to 0} \frac{mx}{\sin nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{8x}{\sin 6x} = \frac{8\div 2}{6\div 2} = \frac{4}{3}$$

$$\lim_{x\to 0} \frac{\tan mx}{nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\tan 7x}{10x} = \frac{7}{10}$$

$$\lim_{x\to 0} \frac{mx}{\tan nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\frac{3}{2}x}{\tan\left(\frac{5}{12}\right)x} = \frac{\left(\frac{3}{2}\right)}{\left(\frac{5}{12}\right)} = \frac{3}{\cancel{2}} \times \frac{\cancel{12}\,6}{5} = \frac{18}{5}$$

$$\lim_{x\to 0} \frac{\sin(mx)}{\sin(nx)} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(20x)} = \frac{4\div 4}{20\div 4} = \frac{1}{5}$$

$$\lim_{x\to 0} \frac{\tan(mx)}{\tan(nx)} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)} = \frac{3}{5}$$

$$\lim_{x\to 0} \frac{\sin(mx)}{\tan(nx)} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\sin(14x)}{\tan(7x)} = \frac{14}{7} = 2$$

$$\lim_{x\to 0} \frac{\tan mx}{\sin nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\tan(10x)}{\sin(2x)} = \frac{10}{2} = 5$$

Example (18)

a) $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\sin\theta} = \dfrac{0}{0}$

$$\lim_{\theta \to 0} \frac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta}}{\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}} = \frac{0}{1} = 0$$

b) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{5x^3 - 4x} = \dfrac{0}{0}$

$$\lim_{x \to 0} \frac{\sin 3x}{x(5x^2 - 4)} = \lim_{x \to 0} \frac{\sin 3x}{x} \cdot \frac{1}{5x^2 - 4}$$

$$= \lim_{x \to 0} \left(\frac{\sin 3x}{x}\right) \cdot \lim_{x \to 0} \left(\frac{1}{5x^2 - 4}\right)$$

$$= \left(\frac{3}{1}\right)\left(\frac{1}{5(0)^2 - 4}\right) = 3\left(\frac{1}{0-4}\right)$$

$$= 3\left(-\frac{1}{4}\right) = -\frac{3}{4}$$

c) $\lim\limits_{x \to 0} \dfrac{\sin 3x \sin 5x}{4x^2} = \dfrac{1}{4} \lim \dfrac{\sin 3x \cdot \sin 5x}{x \cdot x}$

$$= \frac{1}{4}\left[\lim_{x \to 0} \frac{\sin 3x}{x} \cdot \lim_{x \to 0} \frac{\sin 5x}{x}\right]$$

$$= \frac{1}{4}\left[\frac{3}{1}\left(\frac{5}{1}\right)\right] = \frac{15}{4}$$

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d) $\lim_{\theta \to 0} \frac{15\theta + \tan(3\theta)}{\sin(10\theta)} = \frac{0}{0}$

$$\lim_{\theta \to 0} \left[\frac{15\theta}{\sin(10\theta)} + \frac{\tan(3\theta)}{\sin(10\theta)} \right]$$

$$\lim_{\theta \to 0} \frac{15\theta}{\sin(10\theta)} + \lim_{\theta \to 0} \frac{\tan(3\theta)}{\sin(10\theta)}$$

$$\frac{15}{10} + \frac{3}{10} = \frac{18 \div 2}{10 \div 2} = \frac{9}{5}$$

e) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta + \tan\theta} = \frac{0}{0}$

$$\lim_{\theta \to 0} \left[\frac{\frac{\sin\theta}{\theta}}{\frac{\theta}{\theta} + \frac{\tan\theta}{\theta}} \right] = \frac{\lim_{\theta \to 0} \frac{\sin\theta}{\theta}}{\lim_{\theta \to 0} 1 + \lim_{\theta \to 0} \frac{\tan\theta}{\theta}}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

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f) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1-\tan x}{\sin x - \cos x} = \dfrac{1-1}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} = \dfrac{0}{0}$

$$\lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x} = \lim_{x \to \frac{\pi}{4}} \frac{1 - \frac{\sin x}{\cos x}}{\sin x - \cos x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\sin x - \cos x}{1}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} \div \frac{\sin x - \cos x}{1}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} \cdot \frac{1}{\sin x - \cos x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-\cancel{(\sin x - \cos x)}}{\cos x} \cdot \frac{1}{\cancel{(\sin x - \cos x)}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-1}{\cos x} = \frac{-1}{\cos\left(\frac{\pi}{4}\right)}$$

$$= \frac{-1}{\frac{1}{\sqrt{2}}} = -1 \div \frac{1}{\sqrt{2}}$$

$$= -1 \cdot \frac{\sqrt{2}}{1}$$

$$= -\sqrt{2}$$

g) $\displaystyle \lim_{x\to 0} \frac{\sin(x^2)}{x} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{x.\sin(x^2)}{x.x} = \lim_{x\to 0} \frac{x\sin(x^2)}{x^2}$$

$$= \lim_{x\to 0} x \,.\, \lim_{\substack{x\to 0\\ x^2\to 0}} \frac{\sin(x^2)}{x^2}$$

$$= 0 \,.\, 1$$

$$= 0$$

h) $\displaystyle \lim_{x\to 3} \frac{\sin(x-3)}{(x-3)} = \frac{0}{0}$

$$\lim_{x-3\to 0} \frac{\sin(x-3)}{(x-3)} = 1$$

$*$ $\displaystyle \lim_{x\to 0} \cancel{\cos}\sin(\sin x)$

$$\lim_{x\to 0} \frac{\sin(\sin x)}{(\sin x)} = \frac{0}{0}$$

$$\lim_{\sin x\to \sin(0)} \frac{\sin(\sin x)}{(\sin x)}$$

$$\lim_{\sin x\to 0} \frac{\sin(\sin x)}{(\sin x)} = 1$$

g) $\displaystyle \lim_{x\to \frac{\pi}{2}} \frac{\sin(\cos x)}{\cos(x)} = \frac{0}{0}$

$$\lim_{x\to \frac{\pi}{2}} \frac{\sin(\cos x)}{\cos x} = 1$$

$\cos x \to \cos(\frac{\pi}{2})$

$\cos x \to 0$

$*$ $\displaystyle \lim_{x\to 0} \frac{\cos(x^2)-1}{x^2} = \frac{0}{0}$

$$\lim_{x^2\to 0^2} \frac{\cos(x^2)-1}{x^2}$$

$$\lim_{x^2\to 0} \frac{\cos(x^2)-1}{x^2} = 0$$

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h) $\lim\limits_{x \to 1} \dfrac{\sin(x-1)}{x^2+x-2} = \dfrac{0}{0}$

$$\lim_{x \to 1} \frac{\sin(x-1)}{(x-1)(x+2)}$$

$$\lim_{x \to 1} \frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{x+2}$$

$$\lim_{\substack{x \to 1 \\ x-1 \to 0}} \frac{\sin(x-1)}{(x-1)} \cdot \lim_{x \to 1} \frac{1}{x+2}$$

$$1 \cdot \left(\frac{1}{1+2}\right) = 1 \cdot \frac{1}{3} = \frac{1}{3}$$

Note

$$\lim_{x \to a} \frac{\tan(x-a)}{(x-a)} = 1$$

$$\lim_{x \to a} \frac{(x-a)}{\tan(x-a)} = 1$$

$$\lim_{x \to a} \frac{(x-a)}{\sin(x-a)} = 1$$

$$\lim_{x \to a} \frac{\sin(x-a)}{(x-a)} = 1$$

$$\lim_{x \to a} \frac{\cos(x-a) - 1}{(x-a)} = 0$$

$$\lim_{f(x) \to 0} \frac{\sin(f(x))}{f(x)} = 1$$

$$\lim_{f(x) \to 0} \frac{f(x)}{\sin(f(x))} = 1$$

$$\lim_{f(x) \to 0} \frac{\tan(f(x))}{f(x)} = 1$$

$$\lim_{f(x) \to 0} \frac{f(x)}{\tan(f(x))} = 1$$

$$\lim_{f(x) \to 0} \frac{\cos(f(x)) - 1}{f(x)} = 0$$

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Example (19)

$$\lim_{x \to 2} \frac{\tan(x^2-4)}{3x^2-12} = \frac{0}{0}$$

$$\lim_{x \to 2} \frac{\tan(x^2-4)}{3(x^2-4)} = \frac{1}{3}\lim_{x \to 2} \frac{\tan(x^2-4)}{(x^2-4)} = \frac{1}{3}(1) = \frac{1}{3}$$

$x^2 \to 2^2$
$x^2 \to 4$
$(x^2-4) \to 0$

$$\lim_{t \to 2} \frac{5t^2-10t}{\tan(t-2)} = \frac{0}{0}$$

$$\lim_{t \to 2} \frac{5t(t-2)}{\tan(t-2)} = \lim_{t \to 2} \frac{(5t)(t-2)}{(1)\tan(t-2)}$$

$$= \lim_{t \to 2}\left(\frac{5t}{1}\right) \cdot \lim_{t \to 2} \frac{(t-2)}{\tan(t-2)}$$

$$= 5(2) \cdot \lim_{(t-2) \to 0} \frac{(t-2)}{\tan(t-2)} = 10(1) = 10$$

$$\lim_{\theta \to -\frac{5}{2}} \frac{\sin(2\theta+5)}{12\theta+30} = \frac{0}{0}$$

$$\lim_{\theta \to -5/2} \frac{\sin(2\theta+5)}{6(2\theta+5)}$$

$$\frac{1}{6} \lim_{\theta \to -\frac{5}{2}} \frac{\sin(2\theta+5)}{(2\theta+5)}$$

$$\frac{1}{6} \lim_{\substack{2\theta \to -5 \\ (2\theta+5) \to 0}} \frac{\sin(2\theta+5)}{(2\theta+5)} = \frac{1}{6}(1) = \frac{1}{6}$$

$$\lim_{\theta \to 0} \frac{1-\cos^2\theta}{\sin(\sin^2\theta)} = \frac{0}{0}$$

$$\lim_{\theta \to 0} \frac{(\sin^2\theta)}{\sin(\sin^2\theta)} = 1$$

$\sin\theta \to \sin(0) = 0$

$\sin^2\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin(1-\cos\theta)}{1-\cos^2\theta} = \frac{0}{0}$$

$$\lim_{\theta \to 0} \frac{\sin(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$\lim_{\theta \to 0} \frac{\sin(1-\cos\theta)}{(1-\cos\theta)} \cdot \frac{1}{1+\cos\theta}$$

$$\lim_{\theta \to 0} \frac{\sin(1-\cos\theta)}{(1-\cos\theta)} \cdot \lim_{\theta \to 0} \frac{1}{1+\cos\theta}$$

$\cos\theta \to \cos(0) = 1$

$\cos\theta \to 1$

$$= 1 \cdot \left(\frac{1}{1+\cos\theta}\right) = 1\left(\frac{1}{1+1}\right) = 1\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

$$\lim_{\theta \to 0} \frac{\cos(\theta)-1}{2\theta^2} = \frac{\cos(0)-1}{2(0)^2} = \frac{1-1}{0} = \frac{0}{0}$$

$$\lim_{\theta \to 0} \frac{\cos(\theta)-1}{2\theta^2} \cdot \frac{\cos(\theta)+1}{\cos(\theta)+1}$$

$$\lim_{\theta \to 0} \frac{(\cos(\theta)-1)(\cos(\theta)+1)}{2\theta^2(\cos(\theta)+1)}$$

$$\lim_{\theta \to 0} \frac{\cos^2(\theta)-1^2}{2\theta^2(\cos(\theta)+1)}$$

$$\lim_{\theta \to 0} \frac{\cos^2(\theta)-1}{2\theta^2(\cos(\theta)+1)} = \lim_{\theta \to 0} \frac{-\sin^2(\theta)}{2\theta^2(\cos\theta+1)}$$

$$\frac{1}{2} \lim_{\theta \to 0} \frac{-\sin^2(\theta)}{\theta^2} \cdot \frac{1}{\cos\theta+1}$$

$$\frac{1}{2} \lim_{\theta \to 0} \frac{-\sin^2(\theta)}{\theta^2} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta+1}$$

$$\frac{-1}{2} \lim_{\theta \to 0} \left[\frac{\sin(\theta)}{\theta}\right]^2 \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta+1} = \frac{-1}{2}(1) \cdot \frac{1}{\cos(0)+1}$$

$$= \frac{-1}{2} \cdot \frac{1}{2} = \frac{-1}{4}$$

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