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هام

لكل المهتمين و المهتمات
بدروس و مراجع الجامعية

Chapter Two

Matrices

2.1 Operations with Matrices

Definition 1. (**Equality of Matrices**)

Two matrices $A = [a_{ij}]_{mn}$ and $B = [b_{ij}]_{mn}$ are ***equal*** if they have the same size $m \times n$ and $a_{ij} = b_{ij}$, for $1 \le i \le m$ and $1 \le j \le n$.

Example 1. Consider the four matrices

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \text{ and } B = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \text{. Then } A = B .$$

$$\text{Also, } A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \ne B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}_{23} .$$

$$\text{If } B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}_{23} = \begin{bmatrix} x & 0 & 3 \\ 2 & -2 & y \end{bmatrix}_{23} \text{. Then}$$

$x = -1,$ and $y = 1.$

Definition 2. Let $A = [a_{ij}]_{mn}$ and $B = [b_{ij}]_{mn}$. Then the sum is given by $A + B = [a_{ij} + b_{ij}]_{mn}$, for $1 \le i \le m$ and $1 \le j \le n$.

Example 2. Consider the four matrices

$$\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} + \begin{bmatrix} 3 & -2 \\ 5 & -6 \\ 0 & 11 \end{bmatrix}_{32} = \begin{bmatrix} -1+3 & 2-2 \\ 0+5 & -2-6 \\ 3+0 & 1+11 \end{bmatrix}_{32} = \begin{bmatrix} 2 & 0 \\ 5 & -8 \\ 3 & 12 \end{bmatrix}_{32}$$

Also, if $A=\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}$, and $B=\begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}_{23}$, the $A+B$ is undefined.

Definition 3. Let $A=\left[a_{ij}\right]_{mn}$ and $k\in\mathbb{R}$. Then the scalar multiple of A by a scalar $k\in\mathbb{R}$ is defined by $kA=\left[ka_{ij}\right]_{mn}$, for $1\le i\le m$ and $1\le j\le n$.

You can use $-A=(-1)A$. If $A=\left[a_{ij}\right]_{mn}$ and $B=\left[b_{ij}\right]_{mn}$, then $A-B=\left[a_{ij}-b_{ij}\right]_{mn}$,for $1\le i\le m$ and $1\le j\le n$.

Example 3.

$$3\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}=\begin{bmatrix}-3 & 6\\ 0 & -6\\ 9 & 3\end{bmatrix}_{32}$$

$$-2\begin{bmatrix}3 & -2\\ 5 & -6\\ 0 & 11\end{bmatrix}_{32}=\begin{bmatrix}-6 & 4\\ -10 & 12\\ 0 & -22\end{bmatrix}_{32}$$

$$3\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}-2\begin{bmatrix}3 & -2\\ 5 & -6\\ 0 & 11\end{bmatrix}_{32}=\begin{bmatrix}-3 & 6\\ 0 & -6\\ 9 & 3\end{bmatrix}_{32}+\begin{bmatrix}-6 & 4\\ -10 & 12\\ 0 & -22\end{bmatrix}_{32}$$

$$=\begin{bmatrix}-3-6 & 6+4\\ 0-10 & -6+12\\ 9+0 & 3-22\end{bmatrix}_{32}=\begin{bmatrix}-9 & 10\\ -10 & 6\\ 9 & -19\end{bmatrix}_{32}$$

Definition 4. (Matrix Multiplication)

Let $A=\left[a_{ij}\right]_{mn}$ and $B=\left[b_{ij}\right]_{np}$. Then the ***product*** AB is given by $AB=\left[c_{ij}\right]_{mp}$, where

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

for $1\le i\le m$ and $1\le j\le p$.

Example 3.

$$\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}_{22} = \begin{bmatrix} (-1)(2)+(2)(1) & (-1)(-3)+(2)(5) \\ (0)(2)+(-2)(1) & (0)(-3)+(-2)(5) \\ (3)(2)+(1)(1) & (3)(-3)+(1)(5) \end{bmatrix}_{32}$$

$$= \begin{bmatrix} -2+2 & 3+10 \\ 0-2 & 0-10 \\ 6+1 & -9+5 \end{bmatrix}_{32} = \begin{bmatrix} 0 & 13 \\ -2 & -10 \\ 7 & -4 \end{bmatrix}_{32}$$

Example 5.

$$AB = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2-9 & 8-3 \\ -1-6 & 4-2 \end{bmatrix} = \begin{bmatrix} -11 & 5 \\ -7 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+4 & 3-8 \\ 6+1 & -9-2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 7 & -11 \end{bmatrix}$$

$$AB \neq BA$$

Matrix multiplication is not commutative in general

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{31} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix}_{13} = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -6 & 0 \\ 6 & -9 & 0 \end{bmatrix}_{33}$$

$$\begin{bmatrix} 2 & -3 & 0 \end{bmatrix}_{13} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{31} = \begin{bmatrix} -2-6+0 \end{bmatrix} = \begin{bmatrix} -8 \end{bmatrix}$$

Systems of Linear Equations

One practical application of matrix multiplication is representing a system of linear equations.

Consider the system of m linear equations in n variables

$$\begin{array}{ccccc} a_{11}x_1 & +a_{12}x_2+ & \cdots & +a_{1n}x_n = & b_1 \\ a_{21}x_1 & +a_{22}x_2+ & \cdots & +a_{2n}x_n = & b_2 \\ a_{31}x_1 & +a_{32}x_2+ & \cdots & +a_{3n}x_n = & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2+ & \cdots & +a_{mn}x_n = & b_m \end{array}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn} \quad X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n1} \quad \text{and } B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m1}$$

Then

$$AX=B$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m1}$$

Example 6. Solve the matrix equation $AX=\underline{0}$, where $A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$.

Solution. We have $AX=\underline{0}$. Then

$$A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \; X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{31} \quad \text{and } B=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{31}.$$

Hence the system is

$$\begin{array}{rrr} x_1 - & 2x_2 + & x_3 = 0 \\ 2x_1 + & 3x_2 - & 2x_3 = 0 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 2 & 3 & -2 & | & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 7 & -4 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{7}R_2} \begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -\frac{4}{7} & | & 0 \end{bmatrix}$$

$$\xrightarrow{R1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{7} & | & 0 \\ 0 & 1 & -\frac{4}{7} & | & 0 \end{bmatrix}$$

$$x_1 \quad - \frac{1}{7}x_3 = 0$$

$$x_2 - \frac{4}{7}x_3 = 0$$

Then $x_3 = t \in \mathbb{R}, x_2 = \frac{4}{7}t$, and $x_3 = \frac{1}{7}t$. The solution is

$$X = \begin{bmatrix} \frac{1}{7}t \\ \frac{4}{7}t \\ t \end{bmatrix}_{31} = \frac{1}{7}t \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}_{31} = s \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}_{31} ; s \in \mathbb{R}.$$

Example 7. Solve the matrix equation $AX = \underline{0}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}.$$

Solution. We have $AX = \underline{0}$. Then

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Thus}$$

$$\left[A|0\right] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ -1 & 1 & 1 & -3 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 3 & 2 & -3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the system is

$$\begin{aligned} x_1 \quad - x_3 + 2x_4 &= 0 \\ x_2 \quad - x_4 &= 0 \end{aligned}$$

Then

$x_4 = t \in \mathbb{R}, x_3 = s \in \mathbb{R}, x_1 = x_3 - 2x_4 = s - 2t$, and $x_2 = x_4 = t$.

The solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 2t \\ 0s + t \\ s + 0t \\ 0s + t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Definition 5.

- A square matrix

$$A=\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & \vdots \\ \vdots & \vdots & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}_n$$

is called a **diagonal matrix** if all entries that are not on the main diagonal are zero.

- The **trace** of a square matrix $A=\left[a_{ij}\right]_n$ is the sum of the main diagonal entries. That is, $\text{Tr}(A)=a_{11}+a_{22}+\cdots+a_{nn}$.

Example 8.

1) The matrix $=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ a diagonal matrix.

2) Consider the matrix $A=\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}$. Then

$\text{Tr}(A)=a_{11}+a_{22}+a_{33}+a_{44}=1+5+(-1)+4=9$.

Homework. 4, 7, 20-21,27, 44, 47, 48, (page (55-59)

2.2 Properties of Matrix Operations

In this section, we begin to develop the algebra of matrices.

Theorem 1. Let A, B and C be $m \times n$ matrices, and let c, and d be scalars. Then the following is true.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + \mathrm{C}$
3. $(cd)A = c(dA)$
4. $1 \cdot A = A$
5. $c(A + B) = cA + cB$
6. $(c + d)A = cA + dA$

Proof. Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$. Then

1.
$$\begin{aligned} A + B &= \left[a_{ij}\right] + \left[b_{ij}\right] = \left[a_{ij} + b_{ij}\right] \\ &= \left[b_{ij} + a_{ij}\right] = \left[b_{ij}\right] + \left[a_{ij}\right] \\ &= B + A \end{aligned}$$

$$\begin{aligned} (cd)A &= (cd)\left[a_{ij}\right] = \left[(cd)a_{ij}\right] \\ &= \left[c(da_{ij})\right] = c\left[(da_{ij})\right] = c\left(d\left[a_{ij}\right]\right) = c(dA) \end{aligned}$$

$$\begin{aligned} c(A + B) &= c\left(\left[a_{ij}\right] + \left[b_{ij}\right]\right) = c\left[a_{ij} + b_{ij}\right] = \left[c\left(a_{ij} + b_{ij}\right)\right] \\ &= \left[ca_{ij} + cb_{ij}\right] = \left[ca_{ij}\right] + \left[cb_{ij}\right] = c\left[a_{ij}\right] + c\left[b_{ij}\right] \\ &= cA + cB \end{aligned}$$

Example 1. (Addition of More than Two Matrices)

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

Theorem 2. Let A be an $m \times n$ matrix, and let c be a scalar. Then

1. $A_{mn} + O_{mn} = A = O_{mn} + A_{mn}$
2. $A_{mn} + (-A_{mn}) = O_{mn} = (-A_{mn}) + A_{mn}$ ($-A_{mn}$ the additive inverse of A_{mn}).
3. If $cA_{mn} = O_{mn}$, then $c = 0$ or $A_{mn} = O_{mn}$

Example 2. (Solving a Matrix Equation).
Solve for X , the equation $3X + A = B$, for
$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$. Then

$$3X + A = B \Rightarrow 3X = B - A$$

$$\Rightarrow X = \frac{1}{3}(B - A)$$

$$= \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Properties of Matrix Multiplication

Theorem 3. Let A, B and C be $m \times n$ matrices, and let c be a scalar. Then the following.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$ (product defined)
3. $(B + C)A = BA + CA$ $(cd)A = c(dA)$ (product defined)
4. $c(AB) = (cA)B$

Remark. If $AC = BC$, the is not necessarily $A = B$

Example 3.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$AC = BC, \text{ but } A \neq B$$

$$AB = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ 2 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 2 & 9 \end{bmatrix}$$

$$AB \neq BA$$

Remark. The identity matrix is $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem 4. Let A_{mn} be an $m \times n$ matrix. Then

1. $A_{mn} I_n = A_{mn}$
2. $I_m A_{mn} = A_{mn}$

Remark. If $A = A_n$, then

1. $A_n I_n = I_n A_n = A_n$.
2. $A^k = \underbrace{A \times A \times \cdots \times A}_{k\text{–factors}}$, where $0 \neq k \in \mathbb{N}$, and $A^0 = I_n$.
3. $A^j A^k = A^{j+k}; 0 \neq j, 0 \neq k \in \mathbb{N}$.
4. $\left(A^k\right)^j = A^{kj}; 0 \neq j, 0 \neq k \in \mathbb{N}$

Example 4. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, then

$$A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}$$

$$AI_2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = A$$

The Transpose of a Matrix

Definition 1. The ***transpose*** of a matrix

$$A_{mn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn}$$

is

$$A^T_{nm} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{nm}$$

A matrix A_n is called symmetric if $A_n^T = A_n$.

Example 5.

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 5 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ 2 & -3 & 1 & 4 \end{bmatrix}$$

$$\left(A^T\right)^T = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix} = A$$

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 5 & 1 & -3 \\ -1 & 1 & -1 & 1 \\ 2 & -3 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow B^T = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 5 & 1 & -3 \\ -1 & 1 & -1 & 1 \\ 2 & -3 & 1 & 4 \end{bmatrix} = B \Rightarrow B \text{ symmetric}$$

Theorem 5. Let A, and B be $m \times n$ matrices, and let c be a scalar (with order such that the operations is defined. Then.

1. $\left(A^T\right)^T = A$
2. $\left(A+B\right)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $\left(AB\right)^T = B^T A^T$

Example 6.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\left(AB\right)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\left(AB\right)^T = B^T A^T$$

Example 7.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$$

$$\left(AA^T\right)^T = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix} = AA^T$$

- A square matrix A_n is called ***skew-symmetric*** if $A_n^T = -A_n$.

Example 8. The matrix

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = -A$$

Then A_n is skew-symmetric.

Home work.

1, 5, 8, 13-14, 16,19-22, 32, 59
7, 29,30, 39, 40

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Also, $A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \ne B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}_{23}$.

If $B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}_{23} = \begin{bmatrix} x & 0 & 3 \\ 2 & -2 & y \end{bmatrix}_{23}$. Then $x = -1$, and $y = 1$.

Definition 2. Let $A = \left[a_{ij}\right]_{mn}$ and $B = \left[b_{ij}\right]_{mn}$. Then the sum is given by $A + B = \left[a_{ij} + b_{ij}\right]_{mn}$, for $1 \le i \le m$ and $1 \le j \le n$.

Example 2. Consider the four matrices

$$\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} + \begin{bmatrix} 3 & -2 \\ 5 & -6 \\ 0 & 11 \end{bmatrix}_{32} = \begin{bmatrix} -1+3 & 2-2 \\ 0+5 & -2-6 \\ 3+0 & 1+11 \end{bmatrix}_{32} = \begin{bmatrix} 2 & 0 \\ 5 & -8 \\ 3 & 12 \end{bmatrix}_{32}$$

Also, if $A=\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}$, and $B=\begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}_{23}$, the $A+B$ is undefined.

Definition 3. Let $A=\left[a_{ij}\right]_{mn}$ and $k\in\mathbb{R}$. Then the scalar multiple of A by a scalar $k\in\mathbb{R}$ is defined by $kA=\left[ka_{ij}\right]_{mn}$, for $1\le i\le m$ and $1\le j\le n$.

You can use $-A=(-1)A$. If $A=\left[a_{ij}\right]_{mn}$ and $B=\left[b_{ij}\right]_{mn}$, then $A-B=\left[a_{ij}-b_{ij}\right]_{mn}$,for $1\le i\le m$ and $1\le j\le n$.

Example 3.

$$3\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}=\begin{bmatrix}-3 & 6\\ 0 & -6\\ 9 & 3\end{bmatrix}_{32}$$

$$-2\begin{bmatrix}3 & -2\\ 5 & -6\\ 0 & 11\end{bmatrix}_{32}=\begin{bmatrix}-6 & 4\\ -10 & 12\\ 0 & -22\end{bmatrix}_{32}$$

$$3\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}-2\begin{bmatrix}3 & -2\\ 5 & -6\\ 0 & 11\end{bmatrix}_{32}=\begin{bmatrix}-3 & 6\\ 0 & -6\\ 9 & 3\end{bmatrix}_{32}+\begin{bmatrix}-6 & 4\\ -10 & 12\\ 0 & -22\end{bmatrix}_{32}$$

$$=\begin{bmatrix}-3-6 & 6+4\\ 0-10 & -6+12\\ 9+0 & 3-22\end{bmatrix}_{32}=\begin{bmatrix}-9 & 10\\ -10 & 6\\ 9 & -19\end{bmatrix}_{32}$$

Definition 4. (Matrix Multiplication)

Let $A=\left[a_{ij}\right]_{mn}$ and $B=\left[b_{ij}\right]_{np}$. Then the ***product*** AB is given by $AB=\left[c_{ij}\right]_{mp}$, where

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

for $1\le i\le m$ and $1\le j\le p$.

Example 3.

$$\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}_{22} = \begin{bmatrix} (-1)(2)+(2)(1) & (-1)(-3)+(2)(5) \\ (0)(2)+(-2)(1) & (0)(-3)+(-2)(5) \\ (3)(2)+(1)(1) & (3)(-3)+(1)(5) \end{bmatrix}_{32}$$

$$= \begin{bmatrix} -2+2 & 3+10 \\ 0-2 & 0-10 \\ 6+1 & -9+5 \end{bmatrix}_{32} = \begin{bmatrix} 0 & 13 \\ -2 & -10 \\ 7 & -4 \end{bmatrix}_{32}$$

Example 5.

$$AB = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2-9 & 8-3 \\ -1-6 & 4-2 \end{bmatrix} = \begin{bmatrix} -11 & 5 \\ -7 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+4 & 3-8 \\ 6+1 & -9-2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 7 & -11 \end{bmatrix}$$

$AB \neq BA$

Matrix multiplication is not commutative in general

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{31} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix}_{13} = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -6 & 0 \\ 6 & -9 & 0 \end{bmatrix}_{33}$$

$$\begin{bmatrix} 2 & -3 & 0 \end{bmatrix}_{13} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{31} = \begin{bmatrix} -2-6+0 \end{bmatrix} = \begin{bmatrix} -8 \end{bmatrix}$$

Systems of Linear Equations

One practical application of matrix multiplication is representing a system of linear equations.

Consider the system of m linear equations in n variables

$$\begin{array}{ccccc} a_{11}x_1 & +a_{12}x_2 + & \cdots & +a_{1n}x_n = & b_1 \\ a_{21}x_1 & +a_{22}x_2 + & \cdots & +a_{2n}x_n = & b_2 \\ a_{31}x_1 & +a_{32}x_2 + & \cdots & +a_{3n}x_n = & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2 + & \cdots & +a_{mn}x_n = & b_m \end{array}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n1} \quad \text{and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m1}$$

Then

$$AX = B$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m1}$$

Example 6. Solve the matrix equation $AX = \underline{0}$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$.

Solution. We have $AX = \underline{0}$. Then

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{31} \quad \text{and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{31} .$$

Hence the system is

$$\begin{array}{cccc} x_1 - & 2x_2 + & x_3 = 0 \\ 2x_1 + & 3x_2 - & 2x_3 = 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 3 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 7 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{array}\right]$$

$$\xrightarrow{R1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{array}\right]$$

$$x_1 \qquad - \frac{1}{7}x_3 = 0$$

$$x_2 - \frac{4}{7}x_3 = 0$$

Then $x_3 = t \in \mathbb{R}, x_2 = \frac{4}{7}t$, and $x_3 = \frac{1}{7}t$. The solution is

$$X = \begin{bmatrix} \frac{1}{7}t \\ \frac{4}{7}t \\ t \end{bmatrix}_{31} = \frac{1}{7}t \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}_{31} = s \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}_{31} ; s \in \mathbb{R}.$$

Example 7. Solve the matrix equation $AX = \underline{0}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}.$$

Solution. We have $AX = \underline{0}$. Then

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Thus}$$

$$\left[A|0\right] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ -1 & 1 & 1 & -3 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 3 & 2 & -3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the system is

$$\begin{aligned} x_1 \quad - x_3 + 2x_4 &= 0 \\ x_2 \quad - x_4 &= 0 \end{aligned}$$

Then

$x_4 = t \in \mathbb{R}, x_3 = s \in \mathbb{R}, x_1 = x_3 - 2x_4 = s - 2t$, and $x_2 = x_4 = t$.

The solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 2t \\ 0s + t \\ s + 0t \\ 0s + t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Definition 5.

- A square matrix

$$A=\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & \vdots \\ \vdots & \vdots & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}_n$$

is called a **diagonal matrix** if all entries that are not on the main diagonal are zero.

- The **trace** of a square matrix $A=\left[a_{ij}\right]_n$ is the sum of the main diagonal entries. That is, $\operatorname{Tr}(A)=a_{11}+a_{22}+\cdots+a_{nn}$.

Example 8.

1) The matrix $=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ a diagonal matrix.

2) Consider the matrix $A=\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}$. Then

$\operatorname{Tr}(A)=a_{11}+a_{22}+a_{33}+a_{44}=1+5+(-1)+4=9$.

Homework. 4, 7, 20-21,27, 44, 47, 48, (page (55-59)

Chapter Two

Matrices

2.1 Operations with Matrices

Definition 1. (Equality of Matrices)

Two matrices $A=\left[a_{ij}\right]_{mn}$ and $B=\left[b_{ij}\right]_{mn}$ are ***equal*** if they have the same size $m\times n$ and $a_{ij}=b_{ij}$, for $1\le i\le m$ and $1\le j\le n$.

Example 1. Consider the four matrices

$$A=\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32} \text{ and } B=\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}. \text{ Then } A=B.$$

Also, $A=\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32} \ne B=\begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}_{23}$.

If $B=\begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}_{23}=\begin{bmatrix}x & 0 & 3\\ 2 & -2 & y\end{bmatrix}_{23}$. Then $x=-1,$ and $y=1.$

Definition 2. Let $A=\left[a_{ij}\right]_{mn}$ and $B=\left[b_{ij}\right]_{mn}$. Then the sum is given by $A+B=\left[a_{ij}+b_{ij}\right]_{mn}$, for $1\le i\le m$ and $1\le j\le n$.

Example 2. Consider the four matrices

$$\begin{bmatrix}-1 & 2\\ 0 & -2\\ 3 & 1\end{bmatrix}_{32}+\begin{bmatrix}3 & -2\\ 5 & -6\\ 0 & 11\end{bmatrix}_{32}=\begin{bmatrix}-1+3 & 2-2\\ 0+5 & -2-6\\ 3+0 & 1+11\end{bmatrix}_{32}=\begin{bmatrix}2 & 0\\ 5 & -8\\ 3 & 12\end{bmatrix}_{32}$$

Also, if $A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32}$, and $B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}_{23}$, the $A + B$ is undefined.

Definition 3. Let $A = \left[a_{ij} \right]_{mn}$ and $k \in \mathbb{R}$. Then the scalar multiple of A by a scalar $k \in \mathbb{R}$ is defined by $kA = \left[ka_{ij} \right]_{mn}$, for $1 \le i \le m$ and $1 \le j \le n$.

You can use $-A = (-1)A$. If $A = \left[a_{ij} \right]_{mn}$ and $B = \left[b_{ij} \right]_{mn}$, then $A - B = \left[a_{ij} - b_{ij} \right]_{mn}$,for $1 \le i \le m$ and $1 \le j \le n$.

Example 3.

$$3\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} = \begin{bmatrix} -3 & 6 \\ 0 & -6 \\ 9 & 3 \end{bmatrix}_{32}$$

$$-2\begin{bmatrix} 3 & -2 \\ 5 & -6 \\ 0 & 11 \end{bmatrix}_{32} = \begin{bmatrix} -6 & 4 \\ -10 & 12 \\ 0 & -22 \end{bmatrix}_{32}$$

$$3\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} - 2\begin{bmatrix} 3 & -2 \\ 5 & -6 \\ 0 & 11 \end{bmatrix}_{32} = \begin{bmatrix} -3 & 6 \\ 0 & -6 \\ 9 & 3 \end{bmatrix}_{32} + \begin{bmatrix} -6 & 4 \\ -10 & 12 \\ 0 & -22 \end{bmatrix}_{32}$$

$$= \begin{bmatrix} -3-6 & 6+4 \\ 0-10 & -6+12 \\ 9+0 & 3-22 \end{bmatrix}_{32} = \begin{bmatrix} -9 & 10 \\ -10 & 6 \\ 9 & -19 \end{bmatrix}_{32}$$

Definition 4. (Matrix Multiplication)

Let $A = \left[a_{ij} \right]_{mn}$ and $B = \left[b_{ij} \right]_{np}$. Then the ***product*** AB is given by $AB = \left[c_{ij} \right]_{mp}$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

for $1 \le i \le m$ and $1 \le j \le p$.

Example 3.

$$\begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}_{32} \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}_{22} = \begin{bmatrix} (-1)(2)+(2)(1) & (-1)(-3)+(2)(5) \\ (0)(2)+(-2)(1) & (0)(-3)+(-2)(5) \\ (3)(2)+(1)(1) & (3)(-3)+(1)(5) \end{bmatrix}_{32}$$

$$= \begin{bmatrix} -2+2 & 3+10 \\ 0-2 & 0-10 \\ 6+1 & -9+5 \end{bmatrix}_{32} = \begin{bmatrix} 0 & 13 \\ -2 & -10 \\ 7 & -4 \end{bmatrix}_{32}$$

Example 5.

$$AB = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2-9 & 8-3 \\ -1-6 & 4-2 \end{bmatrix} = \begin{bmatrix} -11 & 5 \\ -7 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+4 & 3-8 \\ 6+1 & -9-2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 7 & -11 \end{bmatrix}$$

$$AB \neq BA$$

Matrix multiplication is not commutative in general

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{31} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix}_{13} = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -6 & 0 \\ 6 & -9 & 0 \end{bmatrix}_{33}$$

$$\begin{bmatrix} 2 & -3 & 0 \end{bmatrix}_{13} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{31} = \begin{bmatrix} -2-6+0 \end{bmatrix} = \begin{bmatrix} -8 \end{bmatrix}$$

Systems of Linear Equations

One practical application of matrix multiplication is representing a system of linear equations.

Consider the system of m linear equations in n variables

$$\begin{array}{ccccc} a_{11}x_1 & +a_{12}x_2+ & \cdots & +a_{1n}x_n = & b_1 \\ a_{21}x_1 & +a_{22}x_2+ & \cdots & +a_{2n}x_n = & b_2 \\ a_{31}x_1 & +a_{32}x_2+ & \cdots & +a_{3n}x_n = & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2+ & \cdots & +a_{mn}x_n = & b_m \end{array}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn} \quad X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n1} \quad \text{and } B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m1}$$

Then

$$AX=B$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m1}$$

Example 6. Solve the matrix equation $AX=\underline{0}$, where $A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$.

Solution. We have $AX=\underline{0}$. Then

$$A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \quad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{31} \quad \text{and } B=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{31} .$$

Hence the system is

$$\begin{array}{rrr} x_1- & 2x_2+ & x_3=0 \\ 2x_1+ & 3x_2- & 2x_3=0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 3 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 7 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{array}\right]$$

$$\xrightarrow{R1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{array}\right]$$

$$x_1 \qquad - \frac{1}{7}x_3 = 0$$

$$x_2 - \frac{4}{7}x_3 = 0$$

Then $x_3 = t \in \mathbb{R}, x_2 = \frac{4}{7}t$, and $x_3 = \frac{1}{7}t$. The solution is

$$X = \begin{bmatrix} \frac{1}{7}t \\ \frac{4}{7}t \\ t \end{bmatrix}_{31} = \frac{1}{7}t \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}_{31} = s \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}_{31} ; s \in \mathbb{R}.$$

Example 7. Solve the matrix equation $AX = \underline{0}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}.$$

Solution. We have $AX = \underline{0}$. Then

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Thus}$$

$$\left[A|0\right] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ -1 & 1 & 1 & -3 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 3 & 2 & -3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the system is

$$x_1 \qquad - x_3 + 2x_4 = 0$$

$$x_2 \qquad - x_4 = 0$$

Then

$x_4 = t \in \mathbb{R}, x_3 = s \in \mathbb{R}, x_1 = x_3 - 2x_4 = s - 2t$, and $x_2 = x_4 = t$.

The solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 2t \\ 0s + t \\ s + 0t \\ 0s + t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Definition 5.

- A square matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & \vdots \\ \vdots & \vdots & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}_n$$

is called a **diagonal matrix** if all entries that are not on the main diagonal are zero.

- The **trace** of a square matrix $A = \left[a_{ij}\right]_n$ is the sum of the main diagonal entries. That is, $\text{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Example 8.

1) The matrix $= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ a diagonal matrix.

2) Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}$. Then

$\text{Tr}(A) = a_{11} + a_{22} + a_{33} + a_{44} = 1 + 5 + (-1) + 4 = 9$.

Homework. 4, 7, 20-21,27, 44, 47, 48, (page (55-59)

2.2 Properties of Matrix Operations

In this section, we begin to develop the algebra of matrices.

Theorem 1. Let A, B and C be $m \times n$ matrices, and let c, and d be scalars. Then the following is true.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + \mathrm{C}$
3. $(cd)A = c(dA)$
4. $1 \cdot A = A$
5. $c(A + B) = cA + cB$
6. $(c + d)A = cA + dA$

Proof. Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$. Then

1.

$$\begin{aligned} A + B &= \left[a_{ij}\right] + \left[b_{ij}\right] = \left[a_{ij} + b_{ij}\right] \\ &= \left[b_{ij} + a_{ij}\right] = \left[b_{ij}\right] + \left[a_{ij}\right] \\ &= B + A \end{aligned}$$

$$\begin{aligned} (cd)A &= (cd)\left[a_{ij}\right] = \left[(cd)a_{ij}\right] \\ &= \left[c(da_{ij})\right] = c\left[(da_{ij})\right] = c\left(d\left[a_{ij}\right]\right) = c(dA) \end{aligned}$$

$$\begin{aligned} c(A + B) &= c\left(\left[a_{ij}\right] + \left[b_{ij}\right]\right) = c\left[a_{ij} + b_{ij}\right] = \left[c\left(a_{ij} + b_{ij}\right)\right] \\ &= \left[ca_{ij} + cb_{ij}\right] = \left[ca_{ij}\right] + \left[cb_{ij}\right] = c\left[a_{ij}\right] + c\left[b_{ij}\right] \\ &= cA + cB \end{aligned}$$

Example 1. (Addition of More than Two Matrices)

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

Theorem 2. Let A be an $m \times n$ matrix, and let c be a scalar. Then

1. $A_{mn} + O_{mn} = A = O_{mn} + A_{mn}$
2. $A_{mn} + (-A_{mn}) = O_{mn} = (-A_{mn}) + A_{mn}$ ($-A_{mn}$ the additive inverse of A_{mn}).
3. If $cA_{mn} = O_{mn}$, then $c = 0$ or $A_{mn} = O_{mn}$

Example 2. (Solving a Matrix Equation).
Solve for X , the equation $3X + A = B$, for

$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$. Then

$$3X + A = B \Rightarrow 3X = B - A$$

$$\Rightarrow X = \frac{1}{3}(B - A)$$

$$= \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Properties of Matrix Multiplication

Theorem 3. Let A, B and C be $m \times n$ matrices, and let c be a scalar. Then the following.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$ (product defined)
3. $(B + C)A = BA + CA$ $(cd)A = c(dA)$ (product defined)
4. $c(AB) = (cA)B$

Remark. If $AC = BC$, the is not necessarily $A = B$

Example 3.

$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$AC = BC$, but $A \neq B$

$$AB=\begin{bmatrix}1&3\\0&1\end{bmatrix}\begin{bmatrix}2&4\\2&3\end{bmatrix}=\begin{bmatrix}8&13\\2&3\end{bmatrix}$$

$$BA=\begin{bmatrix}2&4\\2&3\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}=\begin{bmatrix}2&10\\2&9\end{bmatrix}$$

$$AB\neq BA$$

Remark. The identity matrix is $I_n=\begin{bmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&1\end{bmatrix}$.

Theorem 4. Let A_{mn} be an $m\times n$ matrix. Then

1. $A_{mn}I_n=A_{mn}$
2. $I_mA_{mn}=A_{mn}$

Remark. If $A=A_n$, then

1. $A_nI_n=I_nA_n=A_n$.
2. $A^k=\underbrace{A\times A\times\cdots\times A}_{k-\text{factors}}$, where $0\neq k\in\mathbb{N}$, and $A^0=I_n$.
3. $A^jA^k=A^{j+k};0\neq j,0\neq k\in\mathbb{N}$.
4. $\left(A^k\right)^j=A^{kj};0\neq j,0\neq k\in\mathbb{N}$

Example 4. If $A=\begin{bmatrix}1&3\\0&1\end{bmatrix}$, then

$$A^3=\left(\begin{bmatrix}2&-1\\3&0\end{bmatrix}\begin{bmatrix}2&-1\\3&0\end{bmatrix}\right)\begin{bmatrix}2&-1\\3&0\end{bmatrix}$$

$$=\begin{bmatrix}1&-2\\6&-3\end{bmatrix}\begin{bmatrix}2&-1\\3&0\end{bmatrix}=\begin{bmatrix}-4&-1\\3&-6\end{bmatrix}$$

$$AI_2=\begin{bmatrix}1&3\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}1&3\\0&1\end{bmatrix}=A$$

The Transpose of a Matrix

Definition 1. The ***transpose*** of a matrix

$$A_{mn}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mn}$$

is

$$A^T_{nm}=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{nm}$$

A matrix A_n is called symmetric if $A_n^T = A_n$.

Example 5.

$$A=\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix} \Rightarrow A^T=\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 5 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ 2 & -3 & 1 & 4 \end{bmatrix}$$

$$\left(A^T\right)^T=\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 5 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}=A$$

$$B=\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 5 & 1 & -3 \\ -1 & 1 & -1 & 1 \\ 2 & -3 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow B^T=\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 5 & 1 & -3 \\ -1 & 1 & -1 & 1 \\ 2 & -3 & 1 & 4 \end{bmatrix}=B \Rightarrow B \text{ symmetric}$$

Theorem 5. Let A, and B be $m \times n$ matrices, and let c be a scalar (with order such that the operations is defined. Then.

1. $\left(A^T\right)^T = A$
2. $\left(A + B\right)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $\left(AB\right)^T = B^T A^T$

Example 6.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\left(AB\right)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\left(AB\right)^T = B^T A^T$$

Example 7.

$$A=\begin{bmatrix}1 & 3\\ 0 & -2\\ -2 & -1\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}1 & 0 & -2\\ 3 & -2 & -1\end{bmatrix}$$

$$AA^T=\begin{bmatrix}1 & 3\\ 0 & -2\\ -2 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ 3 & -2 & -1\end{bmatrix}=\begin{bmatrix}10 & -6 & -5\\ -6 & 4 & 2\\ -5 & 2 & 5\end{bmatrix}$$

$$\left(AA^T\right)^T=\begin{bmatrix}10 & -6 & -5\\ -6 & 4 & 2\\ -5 & 2 & 5\end{bmatrix}=AA^T$$

- A square matrix A_n is called ***skew-symmetric*** if $A_n^T=-A_n$.

Example 8. The matrix

$$A=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$$

$$A^T=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=-\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}=-A$$

Then A_n is skew-symmetric.

Home work.

1, 5, 8, 13-14, 16,19-22, 32, 59
7, 29,30, 39, 40

2.3 The inverse of a Matrix

Definition 1. A square matrix A of order n is called ***invertible (non-singular)*** if there exists a square matrix C of order n such that $AB = BA = I_n$. The matrix A is called the inverse of A , and denoted by $B = A^{-1}$.

If A has no inverse, then A is called non-invertible (singular) matrix.

Theorem 1. If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

Proof. Because A is invertible, you know it has at least one inverse B such that

$$AB = BA = I_n$$

Suppose C has another inverse such that

$$AC = CA = I_n$$

Then

$$AB = I_n$$

$$C(AB) = CI_n$$

$$(CA)B = C$$

$$I_nB = C$$

$$B = C$$

Hence, the inverse is unique.

Thus

$$AA^{-1} = A^{-1}A = I_n$$

Example 1. Let $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, B is the inverse of A .

Example 2. Find the inverse of $A=\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}$. We solve $AX=I_2$. Then

$$\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x+4z & y+4w\\ -x-3z & -y-3w\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$x+4z=1$$

$$y+4w=0$$

$$-x-3z=0$$

$$-y-3w=1$$

Then $x=-3, z=1, y=-4, w=1$

$$\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}\begin{bmatrix}-3 & -4\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Then $A^{-1}=\begin{bmatrix}-3 & -4\\ 1 & 1\end{bmatrix}$

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n

1. Write the matrix $\left[A_n \middle| I_n\right]_{n\times 2n}$ that consists of the given matrix A_n on the left and the identity matrix I_n on the right. Note that you separate the matrices by a line. This process is called **adjoining** matrix I_n to matrix A_n.

2. If possible, row reduce A_n to I_n using elementary row operations on the *entire* matrix $\left[A_n \middle| I_n\right]_{n\times 2n}$. The result will be the matrix $\left[I_n \middle| A_n^{-1}\right]_{n\times 2n}$. If this is not possible, then A_n is non-invertible (or singular).

3. Check your work by multiplying AA^{-1} and $A^{-1}A$ to see that $AA^{-1}=A^{-1}A=I$.

Example 3. Find the inverse of

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$= [I|A^{-1}] \Rightarrow A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Example 4. Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

Solution.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 6R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 4R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right].$$

$$= \left[I \,|\, A^{-1} \right]$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

Example 5. Show that the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$ does not exists.

Solution.

$$\left[A \,|\, I \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 3R_1} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Thus A has no inverse, since we can't get $\left[I \,|\, A^{-1} \right]$. Thus A is non-invertible (singular).

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible (non-singular) if and only if $\delta = ad - bc \neq 0, \delta$ is the determinant of A

. Then $A^{-1}=\dfrac{1}{\delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\begin{bmatrix} d/\delta & -b/\delta \\ -c/\delta & a/\delta \end{bmatrix}$.

Example 6.

$$A=\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$$\delta=(3)(2)-(-1)(-2)=6-2=4\neq 0$$

$$A^{-1}=\begin{bmatrix} \frac{2}{4} & \frac{1}{4} \\ \frac{2}{4} & \frac{3}{4} \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$B=\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$$

$$\delta=(3)(2)-(-1)(-6)=0$$

Then B is singular.

Theorem 2. If A is an invertible matrix with inverse A^{-1}, $k>0$ be a positive integer, and $0\neq c\in\mathbb{R}$, then

1) $\left(A^{-1}\right)^{-1}=A$.

2) $\left(A^{k}\right)^{-1}=\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k\text{-times}}$.

3) $\left(cA\right)^{-1}=\dfrac{1}{c}A^{-1}; c\neq 0$.

4) $\left(A^{T}\right)^{-1}=\left(A^{-1}\right)^{T}$.

Proof.

1) We have $AA^{-1}=I$ and $A^{-1}A=I$. Then $\left(A^{-1}\right)^{-1}=A$.

$$A^k \cdot \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k\text{-}times} = A^{k-1}\cdot\left(AA^{-1}\right)\cdot\underbrace{A^{-1}\cdots A^{-1}}_{k-1\text{-}times} = A^{k-1}\cdot\left(I\right)\cdot\underbrace{A^{-1}\cdots A^{-1}}_{k-1\text{-}times}$$

$$= A^{k-1}\cdot\underbrace{A^{-1}\cdots A^{-1}}_{k-1\text{-}times} = A^{k-2}\cdot\left(AA^{-1}\right)\cdot\underbrace{A^{-1}\cdots A^{-1}}_{k-2\text{-}times}$$

$$= A^{k-2}\cdot\left(I\right)\cdot\underbrace{A^{-1}\cdots A^{-1}}_{k-2\text{-}times} = A^{k-2}\underbrace{A^{-1}\cdots A^{-1}}_{k-2\text{-}times}$$

$$\vdots$$

$$= AA^{-1} = I$$

$$\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k\text{-}times}\cdot A^k = \underbrace{A^{-1}\cdots A^{-1}}_{k-1\text{-}times}\cdot\left(A^{-1}A\right)\cdot A^{k-1} = \underbrace{A^{-1}\cdots A^{-1}}_{k-1\text{-}times}\cdot\left(I\right)\cdot A^{k-1}$$

$$= \underbrace{A^{-1}\cdots A^{-1}}_{k-1\text{-}times}\cdot A^{k-1} = \underbrace{A^{-1}\cdots A^{-1}}_{k-2\text{-}times}\cdot\left(A^{-1}A\right)\cdot A^{k-2}$$

$$= \underbrace{A^{-1}\cdots A^{-1}}_{k-2\text{-}times}\cdot\left(I\right)\cdot A^{k-2} = \underbrace{A^{-1}\cdots A^{-1}}_{k-2\text{-}times}\cdot A^{k-2}$$

$$\vdots$$

$$= A^{-1}A = I$$

Then $\left(A^k\right)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k\text{-times}}$.

3) Let $0 \neq c \in \mathbb{R}$. Then

$$\left(cA\right)\left(\frac{1}{c}A^{-1}\right) = c\cdot\frac{1}{c}\left(AA^{-1}\right) = \left(1\right)\cdot I = I$$

$$\left(\frac{1}{c}A^{-1}\right)\left(cA\right) = c\cdot\frac{1}{c}\left(A^{-1}A\right) = \left(1\right)\cdot I = I$$

Then $\left(cA\right)^{-1} = \frac{1}{c}A^{-1}; c \neq 0$.

4)

$$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I^T = I$$

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$$

Then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Remark. If A is an invertible matrix with inverse A^{-1}, and $j,k \in \mathbb{Z}$, then

1) $\left(A^{j}\right)^{k} = A^{jk}$

2) $A^{j}A^{k} = A^{j+k}$

Example 6. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$, then

$$|A| = 4 - 2 = 2 \neq 0$$

$$A^{2} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

$$|A| = (3)(18) - (5)(10) = 54 - 50 = 4 \neq 0$$

$$\left(A^{2}\right)^{-1} = \frac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\left(A^{-1}\right)^{2} = \left(\frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix}$$

$$\left(A^{-1}\right)^{2} = \left(A^{2}\right)^{-1}$$

Theorem 3. If A, and B are invertible matrices, then

$$\left(AB\right)^{-1} = B^{-1}A^{-1}$$

Proof.

$$\left(AB\right)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = A\left(I\right)A^{-1}$$

$$= \left(AI\right)A^{-1} = AA^{-1} = I$$

$$\left(B^{-1}A^{-1}\right)\left(AB\right) = B^{-1}\left(A^{-1}A\right)B = B^{-1}\left(I\right)B$$

$$= B^{-1}\left(IB\right) = B^{-1}B = I$$

Then $\left(AB\right)^{-1} = B^{-1}A^{-1}$.

Example 7. If $A=\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}$, and $B=\begin{bmatrix}-1 & -1\\ 3 & 4\end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}\begin{bmatrix}-1 & -1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}2 & 3\\ 9 & 13\end{bmatrix}$$

$$\left|AB\right|=(2)(13)-(3)(9)=26-27=-1$$

$$\left(AB\right)^{-1}=\frac{1}{(-1)}\begin{bmatrix}13 & -3\\ -9 & 2\end{bmatrix}=\begin{bmatrix}-13 & 3\\ 9 & -2\end{bmatrix}$$

$$\left|A\right|=4-3=1; \left|B\right|=-4+3=-1$$

$$A^{-1}=\frac{1}{1}\begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}=\begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}$$

$$B^{-1}=\frac{1}{(-1)}\begin{bmatrix}4 & 1\\ -3 & -1\end{bmatrix}=\begin{bmatrix}-4 & -1\\ 3 & 1\end{bmatrix}$$

$$B^{-1}A^{-1}=\begin{bmatrix}-4 & -1\\ 3 & 1\end{bmatrix}\begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}=\begin{bmatrix}-13 & 3\\ 9 & -2\end{bmatrix}$$

Theorem 4. If C is an invertible matrix with inverse C^{-1}, then

1) If $AC=BC$, then $A=B$ (right cancellation)(the product defined).
2) If $CA=CB$, then $A=B$ (right cancellation)(product defined)

Proof.1)

If $AC=BC$, then

$$\left(AC\right)C^{-1}=\left(BC\right)C^{-1}$$

$$A\left(CC^{-1}\right)=B\left(CC^{-1}\right)$$

$$A\left(I\right)=B\left(I\right)$$

$$AI=BI$$

$$A=B$$

$$C^{-1}(CA) = C^{-1}(CB)$$
$$(C^{-1}C)A = (C^{-1}C)B$$
$$(I)A = (I)B$$
$$IA = IB$$
$$A = B$$

Theorem 4. If A is an invertible matrix with inverse A^{-1}, then the system $AX = B$ has a unique solution given by $X = A^{-1}B$.

Proof. We have $AX = B$. Then

$$(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$(I)X = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

To show that the solution is unique, we let X_1, and X_2 be two solutions for $AX = B$. Then $AX_1 = B$ and $AX_2 = B$. By cancellation law, we get $X_1 = X_2$.

Example 8. Solve the system

$$\begin{aligned} x_1 \quad -x_2 \quad\quad &= 1 \\ x_1 \quad\quad -x_3 &= 2 \\ -6x_1 \quad +2x_2 \quad +3x_3 &= -1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$AX = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-6+1 \\ -3-6+1 \\ -2-8+1 \end{bmatrix} = \begin{bmatrix} -7 \\ -8 \\ -9 \end{bmatrix}$$

$$x_1 = -7, x_2 = -8, \text{ and } x_3 = -9$$

Home work. Page 77-79
(2,4, 9,15,18,20,25, 27,32,33,38,40,41,45, 46,47,48, 52)

1.1

1) Solve

$$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \quad E_1 \\ -2x + \frac{4y}{3} &= -4 \quad E_2 \end{aligned}$$

Solution.

$$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \quad E_1 \\ -2x + \frac{4y}{3} &= -4 \quad E_2 \end{aligned}$$

$$\xrightarrow{E_1 \to 4E_1} \begin{aligned} 2x - \frac{4y}{3} &= 4 \quad E_3 \\ -2x + \frac{4y}{3} &= -4 \quad E_4 \end{aligned}$$

$$\xrightarrow{E_2 \to E_2 + E_1} 2x - \frac{4y}{3} = 4 \to 2x = \frac{4y}{3} + 4 \to x = \frac{2y}{3} + 2$$

$$\to x = \frac{2t}{3} + 2; t \in \mathbb{R}$$

2) Solve

$$\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \quad E_1 \\ x - 2y &= 5 \quad E_2 \end{aligned}$$

Solution.

$$\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \quad E_1 \\ x - 2y &= 5 \quad E_2 \end{aligned}$$

$$\begin{aligned} \frac{6(x-1)}{2} + \frac{6(y+2)}{3} &= 4(6) \\ x - 2y &= 5 \end{aligned}$$

$$\begin{aligned} 3x - 3 + 2y + 4 &= 24 \\ x - 2y &= 5 \end{aligned}$$

$$3x - 3 + 2y + 4 = 24$$
$$x - 2y = 5$$
$$3x + 2y = 23$$
$$x - 2y = 5$$
$$4x = 28 \Rightarrow x = 7$$
$$x - 2y = 5 \Rightarrow 7 - 2y = 5 \Rightarrow y = 1$$

3) Use Gaussian-Jordan to solve:

$$\begin{aligned} 2x \quad + 2z &= 2 \\ 5x + 3y \quad &= 4 \\ 3y - 4z &= 4 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 2 \\ 5 & 3 & 0 & 4 \\ 0 & 3 & -4 & 4 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 5 & 3 & 0 & 4 \\ 0 & 3 & -4 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & -5 & -1 \\ 0 & 3 & -4 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + 5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 3 & 0 & 24 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x = -4, y = 8, z = 5$

4) Solve

$$(\cos\alpha)x + (\sin\alpha)y = 1 \quad (1)$$
$$(-\sin\alpha)x + (\cos\alpha)y = 0 \quad (2)$$

Solution.

$$(\cos\alpha)x + (\sin\alpha)y = 1 \quad (1)$$
$$(-\sin\alpha)x + (\cos\alpha)y = 0 \quad (2)$$

$(\sin\alpha)(1)$ and $(\cos\alpha)(2)$

$$(\cos\alpha\sin\alpha)x+(\sin^2\alpha)y=\sin\alpha \quad (3)$$
$$(-\cos\alpha\sin\alpha)x+(\cos^2\alpha)y=0 \quad (4)$$
$$(3)+(4)$$
$$(\sin^2\alpha+\cos^2\alpha)y=\sin\alpha$$
$$y=\sin\alpha$$
$$(-\sin\alpha)x+(\cos\alpha)y=0$$
$$(-\sin\alpha)x+\cos\alpha\sin\alpha=0$$
$$(-\sin\alpha)x=-(\cos\alpha)\sin\alpha$$
$$x=\frac{-\cos\alpha\sin\alpha}{-\sin\alpha}=\cos\alpha$$

5) Solve

$$\frac{12}{x}-\frac{12}{y}=7$$
$$\frac{3}{x}+\frac{4}{y}=0$$

Solution.

$$\frac{12}{x}-\frac{12}{y}=7$$
$$\frac{3}{x}+\frac{4}{y}=0$$
$$\text{Let } \frac{1}{x}=X\ ;\frac{1}{y}=Y$$
$$12X-12Y=7$$
$$3X+4Y=0$$
$$12X-12Y=7$$
$$9X+12Y=0$$
$$21X=7\Rightarrow X=\frac{7}{21}=\frac{1}{3}$$

$$12Y = -9X \Rightarrow Y = -\frac{3}{4}X = -\frac{3}{4}\left(\frac{1}{3}\right) = -\frac{1}{4}$$

$$X = \frac{1}{x} \Rightarrow x = \frac{1}{X} = \frac{1}{\left(\frac{1}{3}\right)} = 3$$

$$Y = \frac{1}{y} \Rightarrow y = \frac{1}{y} = \frac{1}{\left(-\frac{1}{4}\right)} = -4$$

6) Find the value of k such that the system has one solution

$$x + ky = 0$$
$$kx + y = 0$$

Solution

$$x + ky = 0$$
$$kx + y = 0$$
$$x + ky = 0$$
$$-k^2x - ky = 0$$
$$x - k^2x = 0$$
$$(1-k^2)x = 0$$

The system has one solution if

$$1 - k^2 \neq 0$$
$$1 \neq k^2$$
$$k \neq \pm 1$$

1.2

1) Solve

$$\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & 6 \\ 3 & -2 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 6 \\ 0 & -8 & 8 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2; R_3 \to -\frac{1}{8} R_3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -6 \\ 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{array}\right]$$

Thus the system has no solution.

2) Solve the system

$$\begin{aligned} x + y + z &= 0 \\ 2x + 3y + z &= 0 \\ 3x + 5y + z &= 0 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 1 & 0 \end{array}\right]$$

$$R_2 \to R_2 + (-2)R_1$$

$$R_3 \to R_3 + (-3)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 \to R_3 + (-2)R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$
$$R_3 \to R_3 + (-3)R_1$$
$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$y - z = 0 \Rightarrow y = z = t \in \mathbb{R}$$
$$x + 2z = 0 \Rightarrow x = -2z = -2t$$

3) When the system has a consistent or inconsistent

$$\begin{aligned} x + y + z &= 2 \\ y + z &= 2 \\ x \quad + z &= 2 \\ ax + by + cz &= 0 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ a & b & c & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 + (-a)R_1$$
$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & -a & c & -2a \end{array}\right]$$

$$R_3 \to R_3 + R_2$$
$$R_4 \to R_4 - (b-a)R_2$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & a-b+c & -2b \end{array}\right]$$
$$R_3 \to \frac{1}{2}R_3$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & a-b+c & -2b \end{array}\right]$$

$$R_4 \to R_4 - (\mathrm{a}-b+c)R_3$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$
$$R_2 \to R_2 - R_3$$
$$R_1 \to R_1 - R_2$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

i)The system has only one solution $x = y = z = 1$, when $a + b + c = 0$.

ii) The system has no solution, when $a + b + c \neq 0$

iii) We can not have an infinite solution.

4) Find the value of k such that the system has non-trivial solution

$$(k-2)x + y = 0$$
$$x + (k-2)y = 0$$

Solution

$$\left[\begin{array}{cc|c} k-2 & 1 & 0 \\ 1 & k-2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ k-2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_2 - (k-2)R_1} \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & 1-(k-2)^2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & 1-(k^2-4k+4) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & 1-k^2+4k-4 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & k^2-4k+3 & 0 \end{array}\right]$$

The system has a non-trivial solution if

$$k^2 - 4k + 3 = 0 \Rightarrow (k-1)(k-3) = 0$$
$$\Rightarrow x = 1 \text{ or } x = 3$$

The system has trivial solution if

$$k - 2 = 0 \Rightarrow k = 2$$

1.1

1) Solve

$$\frac{x}{2} - \frac{y}{3} = 1 \quad E_1$$
$$-2x + \frac{4y}{3} = -4 \quad E_2$$

Solution.

$$\frac{x}{2} - \frac{y}{3} = 1 \quad E_1$$
$$-2x + \frac{4y}{3} = -4 \quad E_2$$

$$\xrightarrow{E_1 \to 4E_1} \begin{array}{l} 2x - \frac{4y}{3} = 4 \quad E_3 \\ -2x + \frac{4y}{3} = -4 \quad E_4 \end{array}$$

$$\xrightarrow{E_2 \to E_2 + E_1} 2x - \frac{4y}{3} = 4 \to 2x = \frac{4y}{3} + 4 \to x = \frac{2y}{3} + 2$$

$$\to x = \frac{2t}{3} + 2; t \in \mathbb{R}$$

2) Solve

$$\frac{x-1}{2} + \frac{y+2}{3} = 4 \quad E_1$$
$$x - 2y = 5 \quad E_2$$

Solution.

$$\frac{x-1}{2} + \frac{y+2}{3} = 4 \quad E_1$$
$$x - 2y = 5 \quad E_2$$

$$\frac{6(x-1)}{2} + \frac{6(y+2)}{3} = 4(6)$$
$$x - 2y = 5$$

$$3x - 3 + 2y + 4 = 24$$
$$x - 2y = 5$$

$$3x - 3 + 2y + 4 = 24$$
$$x - 2y = 5$$
$$3x + 2y = 23$$
$$x - 2y = 5$$
$$4x = 28 \Rightarrow x = 7$$
$$x - 2y = 5 \Rightarrow 7 - 2y = 5 \Rightarrow y = 1$$

3) Use Gaussian-Jordan to solve:

$$\begin{aligned} 2x \quad\quad + 2z &= 2 \\ 5x + 3y \quad\quad &= 4 \\ 3y - 4z &= 4 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 2 \\ 5 & 3 & 0 & 4 \\ 0 & 3 & -4 & 4 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 5 & 3 & 0 & 4 \\ 0 & 3 & -4 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & -5 & -1 \\ 0 & 3 & -4 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + 5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 3 & 0 & 24 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x = -4, y = 8, z = 5$

4) Solve

$$(\cos\alpha)x + (\sin\alpha)y = 1 \quad (1)$$
$$(-\sin\alpha)x + (\cos\alpha)y = 0 \quad (2)$$

Solution.

$$(\cos\alpha)x + (\sin\alpha)y = 1 \quad (1)$$
$$(-\sin\alpha)x + (\cos\alpha)y = 0 \quad (2)$$

$(\sin\alpha)(1)$ and $(\cos\alpha)(2)$

$$(\cos\alpha\sin\alpha)x+(\sin^2\alpha)y=\sin\alpha \quad (3)$$

$$(-\cos\alpha\sin\alpha)x+(\cos^2\alpha)y=0 \quad (4)$$

$$(3)+(4)$$

$$(\sin^2\alpha+\cos^2\alpha)y=\sin\alpha$$

$$y=\sin\alpha$$

$$(-\sin\alpha)x+(\cos\alpha)y=0$$

$$(-\sin\alpha)x+\cos\alpha\sin\alpha=0$$

$$(-\sin\alpha)x=-(\cos\alpha)\sin\alpha$$

$$x=\frac{-\cos\alpha\sin\alpha}{-\sin\alpha}=\cos\alpha$$

5) Solve

$$\frac{12}{x}-\frac{12}{y}=7$$

$$\frac{3}{x}+\frac{4}{y}=0$$

Solution.

$$\frac{12}{x}-\frac{12}{y}=7$$

$$\frac{3}{x}+\frac{4}{y}=0$$

$$\text{Let } \frac{1}{x}=X\ ;\frac{1}{y}=Y$$

$$12X-12Y=7$$

$$3X+4Y=0$$

$$12X-12Y=7$$

$$9X+12Y=0$$

$$21X=7\Rightarrow X=\frac{7}{21}=\frac{1}{3}$$

$$12Y = -9X \Rightarrow Y = -\frac{3}{4}X = -\frac{3}{4}\left(\frac{1}{3}\right) = -\frac{1}{4}$$

$$X = \frac{1}{x} \Rightarrow x = \frac{1}{X} = \frac{1}{\left(\frac{1}{3}\right)} = 3$$

$$Y = \frac{1}{y} \Rightarrow y = \frac{1}{Y} = \frac{1}{\left(-\frac{1}{4}\right)} = -4$$

6) Find the value of k such that the system has one solution

$$x + ky = 0$$
$$kx + y = 0$$

Solution

$$x + ky = 0$$
$$kx + y = 0$$
$$x + ky = 0$$
$$-k^2 x - ky = 0$$
$$x - k^2 x = 0$$
$$(1 - k^2)x = 0$$

The system has one solution if

$$1 - k^2 \neq 0$$
$$1 \neq k^2$$
$$k \neq \pm 1$$

1.2

1) Solve

$$\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & 6 \\ 3 & -2 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 6 \\ 0 & -8 & 8 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2;\, R_3 \to -\frac{1}{8}R_3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -6 \\ 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{array}\right]$$

Thus the system has no solution.

2) Solve the system

$$x + y + z = 0$$

$$2x + 3y + z = 0$$

$$3x + 5y + z = 0$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 1 & 0 \end{array}\right]$$

$$R_2 \to R_2 + (-2)R_1$$

$$R_3 \to R_3 + (-3)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 \to R_3 + (-2)R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$R_3 \to R_3 + (-3)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$y - z = 0 \Rightarrow y = z = t \in \mathbb{R}$$

$$x + 2z = 0 \Rightarrow x = -2z = -2t$$

3) When the system has a consistent or inconsistent

$$\begin{aligned} x + y \quad &= 2 \\ y + z &= 2 \\ x \quad + z &= 2 \\ ax + by + cz &= 0 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ a & b & c & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_1$$

$$R_4 \to R_4 + (-a)R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & -a & c & -2a \end{array}\right]$$

$$R_3 \rightarrow R_3 + R_2$$
$$R_4 \rightarrow R_4 - (b-a)R_2$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 2 & | & 2 \\ 0 & 0 & a-b+c & | & -2b \end{bmatrix}$$
$$R_3 \rightarrow \frac{1}{2}R_3$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & a-b+c & | & -2b \end{bmatrix}$$

$$R_4 \rightarrow R_4 - (\mathrm{a}-b+c)R_3$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & a+b+c \end{bmatrix}$$
$$R_2 \rightarrow R_2 - R_3$$
$$R_1 \rightarrow R_1 - R_2$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & a+b+c \end{bmatrix}$$

i)The system has only one solution $x = y = z = 1$, when $a + b + c = 0$.

ii) The system has no solution, when $a + b + c \neq 0$

iii) We can not have an infinite solution.

4) Find the value of k such that the system has non-trivial solution

$$(k-2)x + y = 0$$

$$x + (k-2)y = 0$$

Solution

$$\left[\begin{array}{cc|c} k-2 & 1 & 0 \\ 1 & k-2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ k-2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_2 - (k-2)R_1} \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & 1-(k-2)^2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & 1-(k^2-4k+4) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & 1-k^2+4k-4 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & k-2 & 0 \\ 0 & k^2-4k+3 & 0 \end{array}\right]$$

The system has a non-trivial solution if

$$k^2 - 4k + 3 = 0 \Rightarrow (k-1)(k-3) = 0$$

$$\Rightarrow x = 1 \text{ or } x = 3$$

The system has trivial solution if

$$k - 2 = 0 \Rightarrow k = 2$$

21)

$$\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ 5 & 4 & 2 & 4 \end{array}\right]$$

$$R_1 \to -R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 2 & 3 & 1 & -2 \\ 5 & 4 & 2 & 4 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$

$$R_3 \to R_3 - 5R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 5 & 5 & 0 \\ 0 & 9 & 12 & 9 \end{array}\right]$$

$$R_2 \to \frac{1}{5}R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & 12 & 9 \end{array}\right]$$

$$R_3 \to R_3 + (-9)R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$R_3 \to \tfrac{1}{3} R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_2 \to R_2 - R_3$$

$$R_1 \to R_1 + 2R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \to R_1 + R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now, converting back to a system of linear equations, you have the solution is $x = 2, y = -3$ and $z = 3$.

27)

$$x + 2y + 6z = 1$$

$$2x + 3y + 15z = 4$$

$$3x + y + 3z = 6$$

Then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 1 \\ 2 & 5 & 15 & 4 \\ 3 & 1 & 3 & -6 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & -5 & -15 & -9 \end{array}\right]$$
$$R_3 \to R_3 + 5R_2$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system has no solution.

35) when the system is consistent and inconsistent

$$x + ky = 1$$
$$kx + y = 0$$

Solution. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 0 \end{array}\right]$$
$$R_2 \to R_2 - kR_1$$
$$\Downarrow$$
$$\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & -k^2+1 & -k \end{array}\right]$$
$$R_2 \to \frac{1}{1-k^2} R_2$$
$$\Downarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1-\dfrac{-k^2}{k^2-1} \\ 0 & 1 & \dfrac{k}{k^2-1} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1-\dfrac{1-k^2}{1-k^2} \\ 0 & 1 & \dfrac{-k}{k^2-1} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \dfrac{1+k^2-k^2}{1-k^2} \\ 0 & 1 & \dfrac{-k}{k^2-1} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \dfrac{1}{1-k^2} \\ 0 & 1 & \dfrac{-k}{k^2-1} \end{array}\right]$$

If $k^2-1\neq 0 \Rightarrow k \neq \pm 1$, then the system is consistent. If $k=\pm 1$ is inconsistent.

Find the inverse of $A=\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

Solution.

$$[A|I]=\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3+R_1]{R_2\to R_2-3R_1}\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3\to R_3+2R_2}\left[\begin{array}{ccc|ccc}1&2&2&1&0&0\\0&1&3&-3&1&0\\0&0&1&-5&2&1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-2R_2}\left[\begin{array}{ccc|ccc}1&0&-4&7&-2&0\\0&1&3&-3&1&0\\0&0&1&-5&2&1\end{array}\right]$$

$$\xrightarrow[R_1\to R+4R_3]{R_2\to R_2-3R_3}\left[\begin{array}{ccc|ccc}1&0&0&-13&6&4\\0&1&0&12&-5&-3\\0&0&1&-5&2&1\end{array}\right]=\left[I\,|\,A^{-1}\right]$$

Then

$$A^{-1}=\begin{bmatrix}-13&6&4\\12&-5&-3\\-5&2&1\end{bmatrix}$$

Find the inverse of $A=\begin{bmatrix}1&2&-1\\3&7&-10\\7&16&-21\end{bmatrix}$

Solution.

$$\left[A\,|\,I\right]=\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\3&7&-10&0&1&0\\7&16&-21&0&0&1\end{array}\right]$$

$$\xrightarrow[R_3\to R_3-7R_1]{R_2\to R_2-3R_1}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&-7&-3&1&0\\0&2&-14&7&0&1\end{array}\right]$$

$$\xrightarrow{R_3\to R_3+2R_2}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&-7&-3&1&0\\0&0&0&-1&-2&1\end{array}\right]$$

$$\xrightarrow[R_1 \to R+4R_3]{R_2 \to R_2 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right] = \left[I \middle| A^{-1} \right]$$

Then $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{bmatrix}$ has no inverse.

49) If $A^2 = A$, then $A = I$ or A is singular.

Solution. Let $A^2 = A$, and A be non-singular. Then A^{-1} exist , and

$$\begin{aligned} A^2 = A &\Rightarrow A^2 - A = \underline{0} \\ &\Rightarrow A(A - I) = \underline{0} \\ &\Rightarrow A^{-1}\left(A(A - I)\right) = A^{-1}\underline{0} \\ &\Rightarrow \left(A^{-1}A\right)\left(A - I\right) = \underline{0} \\ &\Rightarrow \left(I\right)\left(A - I\right) = \underline{0} \\ &\Rightarrow A - I = \underline{0} \\ &\Rightarrow A = I \end{aligned}$$