Linear Algebra

Mada

Introduction to System of Linear Equation

What is a linear Equation?

The linear equation has the following properties:

لاتحتوي على حاصل ضرب متغيرات او جذور للمتغيرات

(1) Doesn't involve any products or roots of variables.

جميع المتغيرات تكون من الدرجه الاولى

(2) All variables occure only to the first power.

لاتحتوي على دوال مثلثيه او لوغاريتمية او اسيه

(3) Doesn't involve arguments such as trigonometric, logarithmic or exponential functions.

Example of linear equations

$x + 3y = 7$ $\qquad$ $x_1 + x_2 + \cdots + x_n = 1$

$y + \frac{1}{2}x + 3z = 1$ $\qquad$ $x_1 - 3x_2 + x_4 = 7$.

Example of nonlinear equations

$x + 3y^2 = 7$ $\qquad$ $3x + 2y - z + xz = 4$

$y - \sin x = 0$ $\qquad$ $\sqrt{x} + 2x_2 + x_3 = 1$

System of Linear Equations

النظام الغير متجانس

Non homogenous linear system

$$\begin{aligned} x+y+2z &= 9 \\ 2x+4y-3z &= 1 \\ 3x+6y-5z &= 0 \end{aligned}$$

$$Ax=b$$

النظام المتجانس

Homogenous Linear System

$$\begin{aligned} 2x+2y+2z &= 0 \\ -2x+5y+2z &= 0 \\ -7x+7y+z &= 0 \end{aligned}$$

$$Ax=0$$

How to solve system of linear Equations

Gauss Elimanation

نكتب المصفوفه الموسعة وهي على الشكل

$$\left[\begin{array}{c|c} \text{المعاملات الموجوده في المعادلات} & \text{النواتج} \end{array}\right]$$

نعمل العمليات الأولية على الصفوف حتى نصل الى الشكل

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

نقوم بعمل تعويض خلفي كي نحصل على الحل

ملاحظه يسمى شكل المصفوفه في الخطوه الثانية ب
Reduced Row Form (RRF)

Gauss-Jordan Elimantion

نكتب المصفوفه الموسعة وهي على الشكل

$$\left[\begin{array}{c|c} \text{المعاملات الموجوده في المعادلات} & \text{النواتج} \end{array}\right]$$

نعمل العمليات الأولية على الصفوف حتى نصل الى الشكل

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

نقوم بكتابة الحل

ملاحظه يسمى شكل المصفوفه في الخطوة الثانية ب
Reduced Row Echelon Form (RREF)

Elementary Row Operations

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

العمليات الأولية على الصفوف :

١. ضرب صف بعدد غير الصفر.
٢. تبديل صفين مكان الاخر.
٣. إضافة ماينتج من ضرب صف إلى صف آخر.

RRF RREF

Row-Echelon Form and Reduced Row-Echelon Form

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

١ـالصف الذي لايحوي بكامله أصفار فان ١ هو العنصر الاول ويسمى الواحد عنصر متقدم
٢ـالصف الذي جميع عناصره صفر يكون في أسفل المجموعة
٣ـفي الصفوف المتتالية يكون الواحد في الصف الأسفل يمين الواحد في الصف الأعلى
٤ـيكون العمود المحتوي على الواحد أصفار في كل مكان عدا هذا العنصر

Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

REEF

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

ملاحظه اذا تحققت الشروط الثلاث الاولى تكون REF وإذا تحققت جميع الشروط تكون RREF

The following matrices are in row echelon form but not reduced row echelon form.

REF

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Nonhomogenous Linear System

Type of solutions:

unique solution

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 1 & c \end{array}\right]$$

no solution

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 0 & c \end{array}\right]$$

Infinite solution

$$\left[\begin{array}{ccc|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 1: Solve

$$x+y+2z=9$$
$$2x+4y-3z=1$$
$$3x+6y-5z=0$$

by Gauss elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \quad -2R_1+R_2$$

$$\begin{array}{cccc} -2 & -2 & -4 & -18 \\ 2 & 4 & -3 & 1 \\ \hline 0 & 2 & -7 & -17 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right] \quad -3R_1+R_3$$

$$\begin{array}{cccc} -3 & -3 & -6 & -27 \\ 3 & 6 & -5 & 0 \\ \hline 0 & 3 & -11 & -27 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \quad \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right] \quad -3R_2 + R_3$$

0	-3	21/2	51/2
0	3	-11	-27
0	0	-1/2	-3/2

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{array}\right] \quad -2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right] \text{ RRF}$$

من آخر سطر
بالمصفوفة يتضح بأن
النظام له حل وحيد

Back Substitution (التعويض الخلفي)

$$x + y + 2z = 9$$
$$y - 7/2\, z = -17/2$$
$$z = 3$$

$$\because z = 3 \Rightarrow y - \frac{7}{2}(3) = \frac{-17}{2}$$

$$\Rightarrow y - \frac{21}{2} = \frac{-17}{2}$$

$$\Rightarrow y = \frac{-17}{2} + \frac{21}{2}$$

$$\Rightarrow y = \frac{4}{2}$$

$$\Rightarrow y = 2$$

$\because z=3,\ y=2 \Rightarrow x+2+2(3)=9$

$\Rightarrow x+8=9$

$\Rightarrow \boxed{x=1}$

Example 2: Solve

$$x+2y-3z=4$$
$$3x-y+5z=2$$
$$4x+y+2z=-2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & 2 & -2 \end{array}\right] \quad -3R_1+R_2$$

$$\begin{array}{cccc} -3 & -6 & +9 & -12 \\ 3 & -1 & 5 & 2 \\ \hline 0 & -7 & 14 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 4 & 1 & 2 & -2 \end{array}\right] \quad -4R_1+R_2$$

$$\begin{array}{cccc} -4 & -8 & 12 & -16 \\ 4 & 1 & 2 & -2 \\ \hline 0 & -7 & 14 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & 14 & -18 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & 14 & -18 \end{array}\right] \quad -2R_2 + R_1$$

$$\begin{array}{cccc} 0 & -2 & 4 & -\frac{20}{7} \\ 1 & 2 & -3 & 4 \\ \hline 1 & 0 & 1 & \frac{8}{7} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & 14 & -18 \end{array}\right] \quad 7R_2 + R_3$$

$$\begin{array}{cccc} 0 & 7 & -14 & \frac{70}{7} \\ 0 & -7 & 14 & -18 \\ \hline 0 & 0 & 0 & -8 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & -8 \end{array}\right] \rightarrow \text{The system has no solution}$$

Example 3:

Solve

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + 2z = 6$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & 2 & 6 \end{array}\right] \quad -3R_1 + R_2$$

$$\begin{array}{cccc} -3 & -6 & +9 & -12 \\ 3 & -1 & 5 & 2 \\ \hline 0 & -7 & 14 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 4 & 1 & 2 & 6 \end{array}\right] \quad -4R_1 + R_3$$

$$\begin{array}{cccc} -4 & -8 & 12 & -16 \\ 4 & 1 & 2 & 6 \\ \hline 0 & -7 & 14 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & 14 & -10 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & 14 & -10 \end{array}\right] \quad -2R_2 + R_1$$

$$\begin{array}{cccc} 0 & -2 & 4 & -\frac{20}{7} \\ 1 & 2 & -3 & 4 \\ \hline 1 & 0 & 1 & \frac{8}{7} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & 14 & -10 \end{array}\right] \quad 7R_2 + R_3$$

$$\begin{array}{cccc} 0 & 7 & -14 & \frac{70}{7} \\ 0 & -7 & 14 & -10 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has infinite solution.

$$x + z = 8/7 \Rightarrow x = 8/7 - z$$

$$y - 2z = 10/7 \Rightarrow y = 10/7 + 2z$$

Homogenous Linear System

Types of solutions

حل وحيد
Unique solution or
Trivial Solution
حل بديهي

عدد لانهائي من الحلول
Infinitely many solution or
Nontrivial solution
حل غير بديهي

Example 1 : Solve

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 0 \\ -7x_1 + 7x_2 + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{bmatrix} \quad 2R_1 + R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ -7 & 7 & 1 & 0 \end{bmatrix} \quad 7R_1 + R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 14 & 8 & 0 \end{bmatrix} \quad \frac{1}{7}R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 14 & 18 & 0 \end{bmatrix} \quad -R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & 3/7 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 14 & 8 & 0 \end{bmatrix} \quad -14R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & 3/7 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → The system has Infinite solution.

Free variable

$$\left.\begin{aligned} x_1 + \frac{3}{7}x_3 &= 0 \\ x_2 + 4/7\, x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= -\frac{3}{7}x_3 \\ x_2 &= 4/7\, x_3 \end{aligned}$$

ملاحظه يسمى
x_3 هنا متغير
حر وذلك لانه
من الممكن
فرض أي قيمه
له للحصول
على قيمة لx

let $x_3 = t$ then we have

$x_1 = -\frac{3}{7}t$, $x_2 = 4/7\, t$.

Number of Free Variable:

عدد المجاهيل في REEF

عدد الصفوف الغير صفريه في REEF

$$\# \text{ Free variable} = \# \text{ unknowns} - \# \text{ nonzero row}$$
$$= n - r$$

From last example we have:

$$\begin{bmatrix} 1 & 0 & 3/7 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\# \text{ Free variable} = n - r = 3 - 2 = 1$$

Example:

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

ملاحظه : عدد المتغيرات الحرة = عدد الأ عمده التي لا تحتوي على الواحد

Free variables = 3 ∴

الحل بالقانون

عدد المجاهيل   عدد المعادلات (الصفوف الغير صفريه)

$$\# \text{ Free variable} = n - r$$
$$= 6 - 3 = 3$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = 0$$

$\therefore x_1 = -3x_2 - 4x_4 - 2x_5$

$x_3 = -2x_4$

$x_6 = 3$

let $x_2 = r$, $x_4 = s$, $x_5 = t$ then

$x_1 = -3r - 4s - 2t$, $x_4 = -2s$, $x_6 = 0$

THEOREM
A homogeneous linear system with more unknowns than equations has infinitely many solutions.

النظام المتجانس الذي يحتوي على عدد مجاهيل اكبر من عدد المعادلات يكون له عدد لانهائي من الحلول

Example : Solve:

$$x_1 + 3x_2 + 4x_5 = 0$$
$$x_1 + 3x_3 + 5x_4 = 0$$

check !!

$\because$ عدد المجاهيل > عدد المعادلات
$\therefore$ النظام له عدد لانهائي من الحلول

Matrices and Matrix operation

Addition and subtraction: If $A = [a_{ij}]$, $B = [b_{ij}]$ then:

$$A \pm B = [a_{ij} \pm b_{ij}]$$

EXAMPLE 3 Addition and Subtraction

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined.

Condition: both matrix have the same size.

Scaler multiple: If $A = [a_{ij}]$ then

$$(cA) = c\,(a_{ij}).$$

EXAMPLE 4 Scalar Multiples

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

Multiplication:

AB=(RowA)(Col A)

EXAMPLE 5 Multiplying Matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{12} & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned}
(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\
(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\
(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\
(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\
(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\
(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12
\end{aligned}
\qquad
AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined.

لكي تكون عملية الضرب معرفه في المصفوفات لابد أن يكون عدد الأعمدة في المصفوفة الاولى A يساوي عدد الصفوف في المصفوفة الثانية B. وإذا لم يتحقق الشرط فإن الضرب غير معرف.

Condition:

$$\underset{m \times r}{A} \quad \underset{r \times n}{B} = \underset{m \times n}{AB}$$

Inside

Outside

Properties of Matrix Addition and Scalar Multiplication

The following theorem lists the basic algebraic properties of the matrix operations.

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(*a*) $A + B = B + A$ **[Commutative law for matrix addition]**

(*b*) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(*c*) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(*d*) $A(B + C) = AB + AC$ **[Left distributive law]**

(*e*) $(B + C)A = BA + CA$ **[Right distributive law]**

(*f*) $A(B - C) = AB - AC$

(*g*) $(B - C)A = BA - CA$

(*h*) $a(B + C) = aB + aC$

(*i*) $a(B - C) = aB - aC$

(*j*) $(a + b)C = aC + bC$

(*k*) $(a - b)C = aC - bC$

(*l*) $a(bC) = (ab)C$

(*m*) $a(BC) = (aB)C = B(aC)$

EXAMPLE 1 Associativity of Matrix Multiplication

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so $(AB)C = A(BC)$, as guaranteed by Theorem 1.4.1(*c*).

Linear Combination of matrix

DEFINITION 6 If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_r$ with ***coefficients*** $c_1, c_2, \ldots, c_r$.

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and $\mathbf{x}$ an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Transpose of Matrix:

EXAMPLE 11 Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(a) $(A^T)^T = A$
(b) $(A + B)^T = A^T + B^T$
(c) $(A - B)^T = A^T - B^T$
(d) $(kA)^T = kA^T$
(e) $(AB)^T = B^T A^T$

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

اذا كانت A مصفوفه مربعه وكان من الممكن إيجاد مصفوفه B بحيث
يكون AB=I =BA يقال ان A قابله للانعكاس وتسمى B مصفوفه
عكسية للمصفوفه A

EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

The quantity $ad - bc$ in Theorem 1.4.5 is called the ***determinant*** of the 2×2 matrix A and is denoted by

$$\det(A) = ad - bc$$

or alternatively by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

▶ EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$\text{(a) } A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

▶ EXAMPLE 9 The Inverse of a Product

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6. ◀

Finding inverse of A:

$$[A \mid I] \xrightarrow[\text{operation}]{\text{Row}} [I \mid A^{-1}]$$

Example 1 :-

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$[I \mid A^{-1}]$$

The computations are as follows:

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$ ← We added −2 times the first row to the second and −1 times the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$ ← We added 2 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← We multiplied the third row by −1.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← We added 3 times the third row to the second and −3 times the third row to the first.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← We added −2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

THEOREM 1.6.2 *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

Example : Solve the system using A^{-1}.

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 5x_2 + 3x_3 = 3$$
$$x_1 + \quad\quad + 8x_3 = 17$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$Ax = b \Rightarrow x = A^{-1} b$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \therefore \quad x_1 = 1, \quad x_2 = -1, \quad x_3 = 2$$

Determinants

Determinants of 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$$

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$$|A| = ad - bc$$

or

$$\det(A) = ad - bc$$

Determinants of 3x3 Matrix

مفكوك المعاملات

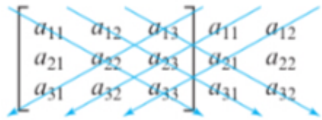
Cofactor Expansion

الأسهم

Arrows

نختار اَي صف أو عمود ثم نقوم بكتابة العنصر الاول الموجود في ذلك الصف او العمود مضروب في المحيديد والذي يأتي من إزاله الصف و العمود الذي يحوي ذلك العنصر ثم نكمل نفس ألطريقه على جميع العناصر مع مراعاة الإشارات كما هو موضح في المصفوفه

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$



Example 1: Comput $\det(A)$ where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

هنا اخترنا العمود الثالث

المحيديد المحيديد

$$\det(A) = -3 \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix}$$

$$= -3(12-5) - 2(-12+2)$$

$$= -3(7) - 2(-10)$$

$$= -21 + 20 = -1$$

Example 2 : compute det(A) where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

Solution : الحل بطريقة الأسهم

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right|$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

الحل بطريقة مفكوك المعاملات

$$|A| = \begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} + 4 \begin{vmatrix} 2 & 3 \\ -8 & 9 \end{vmatrix} + 7 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= (45 + 48) + 4(18 + 24) + 7(12 - 15)$$

$$= 93 + 168 - 21 = 240$$

Properties of Determinants

(1) $\det(A+B) \neq \det(A) + \det(B)$.

(2) $\det(AB) = \det(A)\det(B)$.

(3) $\det(A^{-1}) = \dfrac{1}{\det(A)}$, A is invertable.

Example 1: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Show that $\det(A+B) \neq \det(A) + \det(B)$.

$$A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

$\det(A) = 5-4 = 1$, $\det(B) = 9-1 = 8$

$\det(A+B) = 32-9 = 23$

$\therefore \det(A+B) \neq \det(A) + \det(B)$.

Example 2: let $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$

Show that $\det(AB) = \det(A) \cdot \det(B)$.

$\det(A) = 3 - 2 = 1$, $\det(B) = -8 - 15 = -23$

$\det(A) \cdot \det(B) = (1)(-23) = -23$

$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$, $\det(AB) = 28 - 51 = -23$

$\therefore \det(AB) = \det(A) \cdot \det(B)$.

Example 3 :

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$, compute $\det(A^{-1})$.

$|A| = -1 \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

$= -(12 - 12) - (4 - 4)$

$= 0$

$\therefore \det(A) = 0 \Rightarrow$ A is not invertable

we can't compute $\det(A^{-1})$.

DEFINITION 1 If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

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مصفوفة المحيديدات

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by adj(A). i.e $\text{adj}(A) = C^T$

Example: Find the adj (A) where

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} +\begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} & +\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -6 & -16 \\ -4 & 2 & -16 \\ 12 & 10 & 16 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$\therefore \operatorname{adj}(A) = C^T = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \tag{6}$$

Example: Find A^{-1} for $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A).$$

تم حسابها في المثال السابق

$$\det(A) = 2 \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} + 4 \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = 2(12) + 4(10) = 64$$

$$\therefore A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Cramer's Rule

THEOREM 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example: Use Cramer Rule to solve

$$\begin{aligned} x_1 + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \text{and}$$

$$|A| = 1\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}$$

$$= (12+12) + 2(6+4)$$

$$= 44$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \text{ and}$$

$$|A_1| = 6 \begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 30 & 4 \\ 8 & -2 \end{vmatrix}$$

$$= 6(12+12) + 2(-60-32)$$

$$= 144 - 184 = -40$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \text{ and}$$

$$|A_2| = 1 \begin{vmatrix} 30 & 6 \\ 8 & 3 \end{vmatrix} + 3 \begin{vmatrix} 6 & 2 \\ 8 & 3 \end{vmatrix} - 1 \begin{vmatrix} 6 & 2 \\ 30 & 6 \end{vmatrix}$$

$$= (90 - 48) + 3(18-16) - (36-60)$$

$$= 42 + 6 + 24 = 72$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix} \quad \text{and}$$

$$|A_3| = \begin{vmatrix} 4 & 30 \\ -2 & 8 \end{vmatrix} + 6 \begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}$$

$$= (32 + 60) + 6(6 + 4) = 152$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}.$$

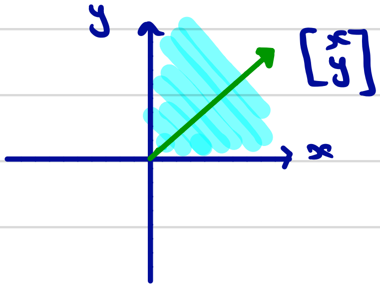
Vector Space

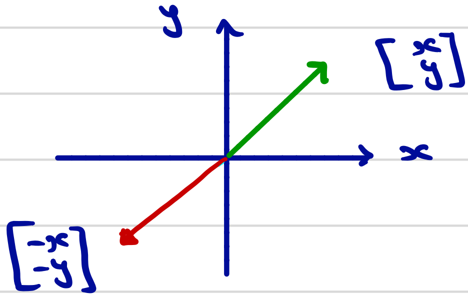
Definition: A vector space is a collection of vectors V and two opertaion $+, \cdot$ such that the following hold for $u, v, w \in V$ and $m, k \in \mathbb{R}$:

1) $u+v \in V$
2) $u+v = v+u$
3) $u+(v+w) = (u+v)+w$
4) $0 \in V$ where $0+u=u$
5) $\forall u \in V$, $u+(-u)=0$
6) $ku \in V$
7) $k(u+v) = ku+kv$
8) $(k+m)u = ku+mu$
9) $k(mu) = (km)u$
10) $1 \cdot u = u$

Example 1: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x \geq 0 \text{ and } y \geq 0 \right\}$ a vector space

No, because if we take $-1 \in \mathbb{R}$ then $-1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \notin V$





Example 2: Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$ a vector space.

No, because is not closed under addition since

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W \text{ and } \begin{bmatrix} -3 \\ -1 \end{bmatrix} \in W \text{ but } \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin W$$

$1 \cdot 2 \geq 0$ $\quad (-3)(-1) \geq 0$ $\quad (-2)(1) \not\geq 0$

Subspaces

Definition: $W \subseteq V$ is a subspace of V if

1) $u, v \in W \Rightarrow u+v \in W$
2) $k \in \mathbb{R},\ u \in W \Rightarrow ku \in W$
3) $0 \in W$

Example: Show that $W = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ is not a subspace of $\mathbb{R}^2$.

let $u = (0,1) \in W$ and $V = (1,0) \in W$ then

$u+w = (0,1) + (1,0) = (1,1) \notin W.$

Example 2: Show that $W = \{(x,y) \in \mathbb{R}^2 : ax+by=0\}$ is a subspace of $\mathbb{R}^2$.

$ax+by=0$ is homo. system as $Ax=0$

why?!

with $A = [a\ \ b]$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ has nontrivial solution

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اقل من عدد المجاهيل

$\therefore$ W is a subspace

THEOREM 4.2.4 *The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.*

هذه النظرية تعني ان حلول النظام المتجانس
تكون فضاء جزئي من R^n

DEFINITION Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in R^n are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \ldots, \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

▶ EXAMPLE 2 Equality of Vectors

$$(a, b, c, d) = (1, -4, 2, 7)$$

if and only if $a = 1$, $b = -4$, $c = 2$, and $d = 7$. ◀

DEFINITION If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \tag{10}$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \tag{11}$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \tag{12}$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n) \tag{13}$$

▶ EXAMPLE Algebraic Operations Using Components

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4)$$

$$-\mathbf{w} = (-4, -2, -1), \qquad \mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)$$ ◀

Definition:

If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ then

The length or norm of the vector u is:

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

The destance between u and v is :

$$d(u,v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + \dots + (u_n - v_n)^2}$$

The dot product

$u \cdot v = \|u\| . \|v\| \cos\theta$

$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

Example 1: Let $u = (-3, 2, 1)$. Find $\|u\|$

$$\|u\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

▶ EXAMPLE Calculating Distance in R^n

If

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between **u** and **v** is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$ ◀

Example: Let $u = (3, 2, 1)$, $v = (2, 1, 0)$. Find $u \cdot v$

$$u \cdot v = (3)(2) + (2)(1) + (1)(0)$$
$$= 6 + 2 + 0 = 8.$$

Example: Find the dot product between the vectors $u = (0, 2, 2)$, $v = (0, 0, 1)$ and $\theta = 45$

$$u \cdot v = \|u\| \, \|v\| \cos\theta.$$

$$\|u\| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\|v\| = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$$

$$u \cdot v = (2\sqrt{2})(1)\cos(45)$$

$$= 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2$$

Linear Combination of vectors

Definition : w is a linear combination of vectors $v_1, v_2, \dots, v_k$ iff $w = c_1 v_1 + \dots + c_k v_k$
where $c_1, c_2, \dots c_k \in \mathbb{R}$.

Example 1: Consider the vectors $v_1 = (1, 2, -1)$ and $v_2 = (6, 4, 2)$.

(a): Show that $w = (9, 2, 7)$ is a linear combination.

$$w = c_1 v_1 + c_2 v_2$$

$$(9, 2, 7) = c_1 (1, 2, -1) + c_2 (6, 4, 2)$$

$$= (c_1, 2c_1, -c_1) + (6c_2 + 4c_2 + 2c_2)$$

$$c_1 + 6c_2 = 9$$
$$2c_1 + 4c_2 = 2$$
$$-c_1 + 2c_2 = 7$$

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore c_1 = -3, \; c_2 = 2$ and

$$w = -3v_1 + 2v_2$$

(b) Show that $w' = (4, -1, 8)$ is not linear combination.

$$(4, -1, 8) = c_1(1, 2, -1) + c_2(6, 4, 2)$$

$$= (c_1, 2c_1, -c_1) + (6c_2, 4c_2, 2c_2)$$

$$c_1 + 6c_2 = 4$$
$$2c_1 + 4c_2 = -1$$
$$-c_1 + 2c_2 = 8$$

$$\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ 0 & 8 & 12 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 4 & -1 \\ 0 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & 9/8 \\ 0 & 8 & 12 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & 9/8 \\ 0 & 0 & 3 \end{array}\right]$$

The system has no solution.

$\therefore$ w' is not a linear combination of v_1 and v_2

Example 2: Show that $u_1=(2,1,4)$, $u_2=(1,-1,3)$ and $u_3=(3,2,5)$ is a linear combination of $w=(5,9,5)$

Solution: $w = c_1u_1 + c_2u_2 + c_3u_3$

$$(5,9,5) = c_1(2,1,4) + c_2(1,-1,3) + c_3(3,2,5)$$

$$= (2c_1, c_1, 4c_1) + (c_2, -c_2, 3c_2) + (3c_3, 2c_3, 5c_3)$$

$$2c_1 + c_2 + 3c_3 = 5$$

$$c_1 - c_2 + 2c_3 = 9$$

$$4c_1 + 3c_2 + 5c_3 = 5$$

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 1 & -1 & 2 & 9 \\ 4 & 3 & 5 & 5 \end{bmatrix} \qquad R_1 \longleftrightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 9 \\ 2 & 1 & 3 & 5 \\ 4 & 3 & 5 & 5 \end{bmatrix}$$

-2	2	-4	-18
2	1	3	5
0	3	-1	-13

$$\begin{bmatrix} 1 & -1 & 2 & 9 \\ 0 & 3 & -1 & -13 \\ 4 & 3 & 5 & 5 \end{bmatrix}$$

-4	4	-8	-36
4	3	5	5
0	7	-3	-31

$$\begin{bmatrix} 1 & -1 & 2 & 9 \\ 0 & 3 & -1 & -13 \\ 0 & 7 & -3 & -31 \end{bmatrix}$$

$R_2 / 3$

$$\begin{bmatrix} 1 & -1 & 2 & 9 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 7 & -3 & -31 \end{bmatrix}$$

$$\begin{array}{cccc} 1 & -1 & 2 & 9 \\ 0 & 1 & -1/3 & 13/3 \\ \hline 1 & 0 & 5/3 & 14/3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5/3 & 14/3 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 7 & -3 & -13 \end{bmatrix}$$

$$\begin{array}{cccc} 0 & -7 & 7/3 & 91/3 \\ 0 & 7 & -3 & -13 \\ \hline 0 & 0 & -2/3 & -2/3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5/3 & 14/3 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 0 & -2/3 & -2/3 \end{bmatrix}$$

$-3/2 \; R_3$

$$\begin{bmatrix} 1 & 0 & 5/3 & 14/3 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} 0 & 1 & -1/3 & -13/3 \\ 0 & 0 & 1/3 & 1/3 \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5/3 & 14/3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} 1 & 0 & 5/3 & 14/3 \\ 0 & 0 & -5/3 & -5/3 \\ \hline 1 & 0 & 0 & 3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\therefore c_1 = 3, \; c_2 = -4, \; c_3 = 1$

$w = 3u_1 - 4u_2 + 1u_3$

Spanning

Definition: If $v_1, v_2, \dots, v_k$ are vectors in V and if every vector in V is written as a linear combination of $v_1, v_2, \dots, v_k$ the we say that these vectors span V.

Remark:

let $S = \{v_1, v_2, \dots, v_k\}$ be the set of vectors in R^n then if

k=عدد المتجهات
n=عددالخانات

$k = n$

let $A = [v_1 \ v_2 \dots v_n]$ and compute
$|A| = |v_1 \ v_2 \dots v_n|$

$|A| \neq 0$
S span R^n

$|A| = 0$
S does n't span R^n

$k > n$

- let $w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$
- Solve the system.

ملاحظه w يفرض على
حساب خانات المتجهات
المعطاة في السؤال

has solution
span

has no solution
do n't span

$k < n$

S does not span R^n

Example 2: Determine if $v_1 = (1, 1, 2)$, $v_2 = (1, 0, 1)$ and $v_3 = (2, 1, 3)$ span R^3

Solution:

$$k = 3, \; n = 3$$

let $A = [v_1 \;\; v_2 \;\; v_3]$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -(3-2) - (1-2)$$

$$= -1 + 1 = 0$$

$|A| = 0 \Rightarrow v_1, v_2$ and v_3 don't span R^3

Example 1: Determine if $v_1 = (2, -1, 3)$, $v_2 = (4, 1, 2)$ and $v_3 = (8, -1, 8)$ span R^3

Solution:

$$k = 3, \; n = 3$$

let $A = [v_1 \;\; v_2 \;\; v_3]$

$$|A| = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 1 \begin{vmatrix} 4 & 8 \\ 2 & 8 \end{vmatrix} + 1 \begin{vmatrix} 2 & 8 \\ 3 & 8 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= (32-16) + (16-24) + (4-12)$$

$$= 16 - 8 - 8 = 16 - 16 = 0$$

$\because |A| = 0 \Rightarrow v_1, v_2$ and v_3 don't span R^3.

Example 3: Determine if $v_1 = (1,-1,0)$, $v_2 = (0,1,2)$
$v_3 = (2,0,1)$, $v_4 = (1,0,1)$ span R^3.

Let $w = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$

$$(a,b,c) = c_1(1,-1,0) + c_2(0,1,2) + c_3(2,0,1) + c_4(1,0,1)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & a \\ -1 & 1 & 0 & 0 & b \\ 0 & 2 & 1 & 1 & c \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 2 & 1 & a+b \\ 0 & 2 & 1 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 2 & 1 & a+b \\ 0 & 0 & -3 & -1 & -2a-2b+c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 2 & 1 & a+b \\ 0 & 0 & 1 & 1/3 & (2a+2b-c)/3 \end{bmatrix}$$

The system has infinitley many solution.

$\Rightarrow v_1, v_2, v_3$ and v_4 span R^3.

Example 4: Determine if $v_1 = (0,1,2)$, $v_2 = (4,1,2)$ and span $R3$

Solution:

$k = 2$, $n = 3$

$\because k < n$

$\therefore v_1$ and v_2 don't span R^3.

Linearly independent

Definition: Let $S = \{v_1, \ldots, v_k\}$ be the set of vectors in $\mathbb{R}^n$. We say that S is

Linearly Independent	Linearly Dependent
if $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ and $c_1 = c_2 = \cdots = c_k = 0$	if $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ and at least one c's $\neq 0$

Remark:

Let $S = \{v_1, \ldots, v_k\}$ be vectors in $\mathbb{R}^n$. Then if

$k = n$

let $A = [v_1 \ v_2 \ldots v_n]$ and compute $|A| = |v_1 \ v_2 \ldots v_n|$

- $|A| \neq 0$: S is LI
- $|A| = 0$: S is LD.

$k < n$

- let $c_1 v_1 + \cdots + c_k v_k = 0$
- homo. system
  - One solution: LI
  - Infinite solution: LD.

$k > n$

S is LD

Example 1 :- Determine wether the following sets are LD or LI.

(a) $S = \{(0,1,5),\ (1,2,8),\ (4,-1,0)\}$

$k = 3,\ n = 3$

$k = n$

$A = [v_1 \quad v_2 \quad v_3]$

$$|A| = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} = -1 \begin{vmatrix} 1 & 4 \\ 8 & 0 \end{vmatrix} + 5 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}$$

$$= -(0-32) + 5(-1-8)$$

$$= 32 - 45 = -13 \neq 0$$

$\because |A| = -13 \neq 0 \Rightarrow S$ is LI

(b)- $S = \{(2,2,1),\ (-4,6,5),\ (-2,8,6)\}$

$A = [v_1 \quad v_2 \quad v_3]$

$k = 3,\ n = 3$

$k = n$

$$|A| = \left|\begin{array}{ccc|cc} 2 & -4 & -2 & 2 & -4 \\ 2 & 6 & 8 & 2 & 6 \\ 1 & 5 & 6 & 1 & 5 \end{array}\right.$$

$$= (72 - 32 - 20) - (-12 + 80 - 48)$$

$$= (72 - 52) - (80 - 60)$$

$$= 20 - 20 = 0$$

$\because |A| = 0 \Rightarrow S$ is LD.

Example 2: Determine whether the vectors $v_1 = (1, 2, 2, -1)$
$v_2 = (4, 9, 9, -4)$, $v_3 = (5, 8, 9, -5)$ in R^4 are
LD or LI

$\because k = 3$, $n = 4$, $k < n$

Solution:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 (1, 2, 2, -1) + c_2 (4, 9, 9, -4) + c_3 (5, 8, 9, -5) = 0$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{bmatrix}$$

$R_1 + R_4$

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\because n = 3$
$r = 3$
$n - r = 0$
no free variable
$\therefore$ The system has solution

n=عدد المعادلات
r=عدد المعادلات

$\therefore v_1, v_2$ and v_3 are LI.

Example 3: Determine whether the polynomials
$P_1 = 1 - x$, $P_2 = 5 + 3x - 2x^2$
$P_3 = 1 + 3x - x^2$ are LI or LD in P_2

Solution:

$$P = [P_1 \ P_2 \ P_3]$$

$$|P| = \begin{vmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ -2 & -1 \end{vmatrix}$$

$$= (-3 - (-6)) + (-5 - (-2))$$

$$= 3 - 3 = 0$$

$\therefore P_1 , P_2$ and P_3 are LD.

Example 4: Determine whether the polynomials
$P_1 = 2 - x + 4x^2$, $P_2 = 3 + 6x + 2x^2$
$P_3 = 2 + 10x - 4x^2$. are LI or LD in P_2.

$$P = [P_1 \ P_2 \ P_3]$$

$$|P| \begin{vmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 6 & 10 & -4 \end{vmatrix} = 2 \begin{vmatrix} 6 & 2 \\ 10 & -4 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 6 & -4 \end{vmatrix} + 4 \begin{vmatrix} 3 & 6 \\ 6 & 10 \end{vmatrix}$$

$$= 2(-24 - 20) + (-12 - 12) + 4(30 - 36)$$

$= 2(44) + (-24) + 4(-6)$

$= 88 - 24 - 24 = -136$

$\because |P| = -136 \neq 0$

$\therefore P_1, P_2$ and P_3 are LI.

THEOREM 4.3.2

(a) *A finite set that contains* **0** *is linearly dependent.*

(b) *A set with exactly one vector is linearly independent if and only if that vector is not* **0**.

(c) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Determine which of the following lists of vectors are linearly independent:

(a) $\{(1, 2, 0, -1, 5), (0, 0, 0, 0, 0), (15, 6, 2, -17, 0)\}$ LD

(b) $\{(5, 7)\}$ one vector $\rightarrow$ L.I

(c) $\{(3, 1, 4), (-2, 2, 5), (3, 0, 4), (2, -1, -2)\}$ LD, $k = 4$, $n = 3$, $k > n$

(D) $\{(-3, 7), (5, 5)\}$, two vectors and neither vector is a scalar multiple of the other

Basis of Vector Space

Definition: A set of linearly independent vectors that spans all of V is called a basis

The number of vectors in a basis of R^n is called the dimention of R^n.

Remark: To check if $S = \{v_1, v_2, \ldots, v_n\}$ is a basis

لماذا ؟ هنا درسنا حالة واحده
فقط وهي عندما k=n

$|A| \neq 0$
S span R^n
S LI
S forms a basis

$|A| = 0$
S doesn't span R^n
S LD.
S doesn't form a basis

Example: Let $v_1 = (3, 1, -4)$, $v_2 = (2, 5, 6)$ and $v_3 = (1, 4, 8)$. Show that the set $S = \{v_1, v_2, v_3\}$ form a basis for R^3.

$$A = [v_1 \quad v_2 \quad v_3]$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 3\begin{vmatrix} 5 & 4 \\ 6 & 8 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 6 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 3(40-24) - (16-6) - 4(8-5)$$

$$= 3(16) - (10) - 4(3)$$

$$= 48 - 22 = 26$$

$\because |A| = 26 \neq 0$

$\therefore$ S forms a basis for R^3

dim $(R^3) = 3$.

Example 2: Show the vectors $v_1 = (1, 2, 1)$, $v_2 (2, 9, 0)$
$v_3 = (3, 3, 4)$ form a basis for R^3.

Solution:

$$A = [v_1 \ v_2 \ v_3]$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$$

$+ \quad - \quad +$

$$= (6-27) + 4(9-4)$$

$$= -21 + 20 = -1$$

$\because |A| \neq 0 \Rightarrow v_1, v_2, v_3$ form a basis for R^3

Standard Basis:

▶ **The Standard Basis for R^n**

Recall from Example 11 of Section 4.2 that the standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

span R^n and from Example 1 of Section 4.3 that they are linearly independent. Thus, they form a basis for R^n that we call the ***standard basis for R^n***. In particular,

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

is the standard basis for R^3.

▶ **The Standard Basis for P_n**

Show that $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials of degree n or less.

▶ **The Standard Basis for M_{mn}**

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Row, Column and null Space of a Matrix

null space	row space	column space
الفضاء الذي يجمع حلول النظام المتجانس	الفضاء الذي يجمع صفوف المصفوفة REEF	الفضاء الذي يجمع أعمدة المصفوفه REEF

Remark: To find the row or column space, we need first to put the matrix in RREF for example if we have the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ then

Row space

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

non zero row

$$\left\{ \begin{array}{l} \vec{u}_1 = (1\ -2\ 0\ -1\ 3) \\ \vec{u}_2 = (0\ 0\ 1\ 2\ -2) \end{array} \right\}$$

column space

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot column

$$\left\{ \vec{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

null space: Solutions of $Ax = 0$

$$x_1 - 2x_2 - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5 \quad , \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$x = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 0 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_5 \\ 0 \\ 2x_5 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Null space $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Rank and Nullity for a Matrix

الرتبه

The Rank of A = The number of nonzero wros in RREF

الرتبه: عبارة عن عدد الصفوف الغير صفريه في المصفوفه المختزلة للمصفوفه A

or = The number of pivote of the column in RREF.

الرتبه عبارة عن عدد الواحد في الأعمده في المصفوفه المختزلة للمصفوفه A

or = dim (row A) or dim (Col A)

الرتبة عبارة عن بعد فضاء الصفوف أو بعد فضاء الأعمده

الصفريه

The Nullity of A = The number of column without pivot. in REEF.

عبارة عن عدد الأعمده التي لا تحتوي على الواحد في المصفوفه المختزلة للمصفوفه A

or = The number of free variable in the solutions of $Ax=0$.

عبارة عن عدد المتغيرات الحرة في حل النظام المتجانس

or = dim (null A).

عبارة عن بعد فضاء النواه

العلاقه بين الرتبه و الصفرية

$$\text{Rank } A + \text{nullity } A = \overset{\text{عدد}}{\#} \text{ column of } A$$

Example 1: compute rank and nullity of

$$A = \begin{bmatrix} 1 & -3 & -3 & -1 \\ -1 & 4 & 6 & 4 \\ -1 & 3 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -3 & -1 \\ -1 & 4 & 6 & 4 \\ -1 & 3 & 3 & 2 \end{bmatrix} \quad \begin{matrix} R_2 - 3R_3 \\ R_1 + R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 0 & -3 & -1 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 + 3R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

الرتبه: عبارة عن عدد الصفوف
الغير صفريه في مصفوفه REEF
للمصفوفه A
أو
عدد الأعمده التي تحتوي على
الواحد في REEF للمصفوفه A

الصفريه عبارة عن
عدد الأعمده التي لا
تحتوي على الواحد

Rank = 3

Nullity = 2

check !!

Rank + nullity = # column A

3 + 2 = 5

Example 2 : Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix}$

a)- Find the row, column and null space.

b)- Compute the rank and nullity of A.

Solution:

a)- $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix}$ $\quad -2R_1 + R_2$

$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & 1 \\ -1 & 0 & 1 & -2 \end{bmatrix}$ $\quad R_1 + R_3$

$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

فضاء المصفوفه

row space $= \{(1,0,-1,2),\ (0,1,4,-1)\}$

فضاء الأعمدة

column space $= \{(1,2,-1),\ (0,1,0)\}$

فضاء النواه

null space : $Ax = 0$

$$x_1 - x_3 + 2x_4 = 0 \Rightarrow x_1 = x_3 - 2x_4$$

$$x_2 + 4x_3 - x_4 = 0 \Rightarrow x_2 = -4x_3 + x_4$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ -4x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

null space $= \{(1, -4, 1, 0), (-2, 1, 0, 1)\}$

b)_ Rank = dim (row A)

$= 2$

Nullity = dim (null A)

$= 2$

هنا أوجدنا الرتبه بالتعريف
الثالث وهو ان الرتبه عبارة
عن بعد فضاء الصفوف.
وبالمثل أوجدنا الصفريه وهي
عبارة عن بعد فضاء النواه
للتذكير البعد هو عبارة عن
عدد المتجهات في الفضاء

Definition: An inner product space is a vector V along with a function $\langle \, , \, \rangle$ and which satisfies:

1)- $\langle u, v \rangle = \langle v, u \rangle$.

2)- $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

3)- $\langle ku, v \rangle = k \langle u, v \rangle$.

4)- $\langle v, v \rangle \geq 0 \iff \langle v, v \rangle = 0$.

Example 1: Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$

a)- Show that $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$ is an inner product

b)- Compute $u = (2,-1)$, $v = (-1, 3)$.

Solution:-

a)-

1)- $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$

$= 3v_1u_1 + 2v_2u_2$

$= \langle v, u \rangle$.

2)- $\langle u+v, w\rangle = 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2$

$= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2)$

$= \langle u, w\rangle + \langle v, w\rangle .$

3)- $\langle ku, v\rangle = 3(ku_1)v_1 + 2(ku_2)v_2$

$= k(3u_1v_1 + 2u_2v_2)$

$= k\langle u, v\rangle .$

4)- $\langle v, v\rangle = 3v_1v_1 + 2v_2v_2$

$= 3v_1^2 + 2v_2^2 \geq 0$

$\therefore\ 3v_1^2 + 2v_2^2 = 0 \iff v_1 = v_2 = 0$

b)-

$\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$

$= 3.2(-1) + 2(-1)(3)$

$= -6 - 6 = -12 .$

Example 2: Let $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$.

a)- Show that $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$ is an inner product.

b)- Compute $u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $v = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$

Solution:-

1)- $\langle u, v \rangle = \langle v, u \rangle$

2)- $\langle u+v, w \rangle = (u_1 + v_1) w_1 + (u_2 + v_2) w_2 + (u_3 , v_3) w_3 + (u_4 + v_4) w_4$

$= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2 + u_3 w_3 + v_3 w_3 + u_4 w_4 + v_4 w_4$

$= (u_1 w_1 + u_2 w_2 + u_3 w_3 + u_4 w_4) + (v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4)$

$= \langle u, w \rangle + \langle v, w \rangle .$

3)- $\langle ku, v\rangle = (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3 + (ku_4)v_4$

$= k(u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4)$

$= k\langle u, v\rangle$.

4- $\langle v, v\rangle = v_1v_1 + v_2v_2 + v_3v_3 + v_4v_4$

$= v_1^2 + v_2^2 + v_3^2 + v_4^2 \geq 0$

$\therefore \langle v, v\rangle = 0 \iff v_1 = v_2 = v_3 = v_4 = 0$.

b)- $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$

$= (1)(-1) + (2)(0) + 3(3) + 4(2)$

$= 16$

Example 3: Let $p = p(x)$ and $q = q(x)$. Show that

$\langle p, q\rangle = \int_a^b p(x)\,q(x)\,dx$ is an inner product.

Solution :-

(1) $\langle p, q \rangle = \int_a^b p(x)\, q(x)\, dx$

$= \int_a^b q(x)\, p(x)\, dx$

$= \langle q, p \rangle.$

(2) $\langle p+q, s \rangle = \int_a^b (p(x) + q(x))\, s(x)\, dx$

$= \int_a^b \left[p(x)\, s(x) + q(x)\, s(x) \right] dx$

$= \int_a^b p(x)\, s(x)\, dx + \int_a^b q(x)\, s(x)\, dx$

$= \langle p, s \rangle + \langle q, s \rangle.$

(3) $\langle kp, q \rangle = \int_a^b k\, p(x)\, q(x)\, dx$

$= k \int_a^b p(x)\, q(x)\, dx$

$= k \langle p, q \rangle.$

(4) $\langle p, p\rangle = \int_a^b p(x)\,p(x)\,dx$

$$= \int_a^b p^2(x)\,dx \geq 0$$

$\therefore \int_a^b p^2(x)\,dx = 0 \Leftrightarrow p(x) = 0$

Norm, Distance, Angels in Inner product Space

If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ then:

(1) The norm or length of the vector u is:

$$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{u_1 + u_2 + \dots + u_n}$$

(2) The distance between u and v is

$$d(u, v) = \|u - v\| = \sqrt{\langle u - v, u - v\rangle}$$

$$= \sqrt{(u_1 - v_1)^2 + \dots + (u_n + v_n)^2}$$

(3) The angle between u and v is

$$\theta = \cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\,\|v\|}\right)$$

Example 1: Let R^2 have the inner product

$$\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$$

where $u=(1,0)$ and $v=(0,1)$ then

a)- Find $\|u\|$

b)- Find $d(u,v)$

Solution:-

$$\|u\| = \sqrt{\langle u,u\rangle}$$

$$\langle u,u\rangle = 3u_1u_1 + 2u_2u_2$$

$$= 3(1)(1) + 2(0)(0)$$

$$= 3$$

$$\therefore \|u\| = \sqrt{3}$$

$$d(u,v) = \sqrt{\langle u-v, u-v\rangle}$$

$$\langle u-v, u-v\rangle = 3(u_1-v_1)(u_1-v_1) + 2(u_2-v_2)(u_2-v_2)$$

$$= 3(1)(1) + 2(-1)(-1) =$$

$$= 5$$

$$\therefore d(u,v) = \sqrt{5}$$

Example 2: Let P_2 have the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$$

where $p = x$ and $q = x^2$. Find $\|p\|$ and $\|q\|$.

Solution:-

$$\|p\| = \sqrt{\langle p, p \rangle}$$

$$\langle p, p \rangle = \int_{-1}^{1} p(x)\, p(x)\, dx$$

$$= \int_{-1}^{1} x \; x \; dx$$

$$= \int_{-1}^{1} x^2 \, dx$$

$$= \left. \frac{x^3}{3} \right|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\therefore \|p\| = \sqrt{\langle p, p \rangle} = \sqrt{\frac{2}{3}}$$

$$\|q\| = \sqrt{\langle q, q\rangle}$$

$$\langle q, q\rangle = \int_{-1}^{1} q(x)\, q(x)\, dx$$

$$= \int_{-1}^{1} x^2 \cdot x^2 \, dx$$

$$= \int_{-1}^{1} x^4 \, dx$$

$$= \left.\frac{x^5}{5}\right|_{-1}^{1}$$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$\therefore \|q\| = \sqrt{\langle q, q\rangle} = \sqrt{\frac{2}{5}}$$

$\cos\theta$

Example 3:- Find the cosine of the angle θ between the vectors $u = (4, 3, 1, -2)$ and $v = (-2, 1, 2, 3)$

$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|}$$

$$u \cdot v = 4(-2) + 3(1) + (1)(2) + (-2)(3)$$

$$= -8 + 3 + 2 - 6 = -9$$

$$\|u\| = \sqrt{4^2 + 3^2 + 1^2 + (-2)^2} = \sqrt{30}$$

$$\|v\| = \sqrt{(-2)^2 + 1^2 + 2^2 + 3^2} = \sqrt{18}$$

$$\therefore \cos\theta = \frac{-9}{\sqrt{30}\sqrt{18}}$$

Definition: Let $T: R^n \longrightarrow R^m$, $\vec{u}, \vec{v} \in R^n$, we say tha T is is a linear transformation iff

1)- $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

2)- $T(c\vec{u}) = c\,T(\vec{u})$

Example 1:

Let $T: R^2 \longrightarrow R^2$ defined by $T(x_1, x_2) = (2x_1 - x_2,\ x_1 + x_2)$.

IS T a L.T

let $u_1 = (x_1, x_2)$, $u_2 = (y_1, y_2)$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2)$$

$$= \left(2(x_1+y_1) - (x_2+y_2),\ x_1+y_1+x_2+y_2\right)$$

$$= \left(2x_1 + 2y_1 - x_2 - y_2,\ x_1 + y_1 + x_2 + y_2\right)$$

$$= (2x_1 - x_2, x_1 + x_2) + (2y_1 - y_2, y_1 + y_2)$$

$$= T(x_1, x_2) + T(y_1, y_2)$$

$$= T(u_1) + T(u_2)$$

$$T(cu) = T(c(x_1, x_2)) = T(cx_1, cx_2)$$

$$= (2cx_1 - cx_2, cx_1 + cx_2)$$

$$= c(2x_1 - x_2, x_1 + x_2)$$

$$= c\,T(x_1, x_2) = c\,T(u)$$

Example 2 :- Let $T: R^2 \to R^2$ defined by $T(x_1, x_2) = (x_1^2, 0)$.
is T a LT ?

$$T(c\vec{u}) = ((cu)^2, 0) = (c^2u^2, 0) = c^2(u^2, 0) = c^2 (T(\vec{u}))$$

$\therefore$ T is not linear transformation.

Example 3 : Let $T: R^3 \to R^1$ be the linear transformation defined as $T(v_1) = (1,0)$, $T(u_2) = (2,-1)$
$T(u_3) = (4,3)$ where $u_1 = (1,1,1)$
$u_2 = (1,0,1)$, $u_3 = (1,0,0)$. Find $T(2,-3,5)$

$$(2,-3,5) = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$= c_1(1,1,1) + c_2(1,0,1) + c_3(1,0,0)$$

$$= (c_1 + c_2 + c_3, c_1, c_1 + c_2)$$

$$\therefore \left.\begin{array}{l} c_1 + c_2 + c_3 = 2 \\ c_1 = -3 \\ c_1 + c_2 = 5 \end{array}\right\} \Rightarrow -3 + c_2 = 5 \Rightarrow c_2 = 8$$

$$\Rightarrow -3 + 8 + c_3 = 2$$

$$\Rightarrow c_3 = 2 - 5 \Rightarrow c_3 = -3$$

$(2,-3,5) = -3v_1 + 8v_2 - 3v_3$

$T(2,-3,5) = T(-3v_1 + 8v_2 - 3v_3)$

$= -3\,T(v_1) + 8\,T(v_2) - 3\,T(v_3)$

$= -3(1,0) + 8(2,-1) - 3(4,3)$

$= (1,-17)$

Kernel and Range

kernel (T_A) = null space of A

Range (T_A) = Column space of A.

Example 4: Let T be the linear transformation from $R^2 \to R^3$ defined by

$T(x) = Ax$. with $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$

a)- Find the kernel and the range of T

Example: Let $T: R^3 \to R^1$ be a LT such that
$T(e_1) = (1,-2)$, $T(e_2) = (0,3)$, $T(e_3) = (1,1)$

a)- Find a matrix A such that $T(x) = Ax$.

$T(1,0,0) = (1,-2)$, $T(0,1,0) = (0,3)$, $T(0,0,1) = (1,1)$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 1 \end{bmatrix}$$

b)- Find $(1,2,3)$

$(1,2,3) = c_1 e_1 + c_2 e_2 + c_3 e_3$

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$= (c_1, c_2, c_3)$

$\therefore c_1 = 1$, $c_2 = 2$, $c_3 = 3$

$\therefore (1,2,3) = 1e_1 + 2e_2 + 3e_3$

$T(1,2,3) = T(e_1 + 2e_2 + 3e_3)$

$= T(e_1) + 2T(e_2) + 3T(e_3)$

$= (1,-2) + 2(0,3) + 3(1,1)$

$= (4,7)$.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a L.T such that
$T(1,2) = (1,-1,0)$ and $T(1,1) = (1,2,-1)$

a)_ Find $T(1,4)$

$$(1,4) = c_1(1,2) + c_2(1,1)$$

$$= (c_1 + c_2 \,,\, 2c_1 + c_2)$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + c_2 &= 4 \end{aligned}\right] \Rightarrow \left.\begin{aligned} -c_1 - c_2 &= -1 \\ 2c_1 + c_2 &= 4 \end{aligned}\right] \begin{aligned} c_1 &= 3 \text{ and} \\ c_2 &= -2 \end{aligned}$$

$$\therefore (1,4) = 3(1,2) - 2(1,1)$$

$$T(1,4) = 3T(1,2) - 2T(1,1)$$

$$= 3(1,0,-1) - 2(3,-3,0)$$

$$= (1,-7,2).$$

b)_ Find a matrix A s.t $T(x) = Ax$

$$T(x) = Ax$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A\begin{bmatrix}1&1\\2&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix}^{-1}=\begin{bmatrix}1&1\\-1&2\\0&-1\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix}^{-1}$$

$$A=\begin{bmatrix}1&1\\-1&2\\0&-1\end{bmatrix}\cdot\left(\frac{1}{-1}\right)\begin{bmatrix}1&-1\\-2&1\end{bmatrix}$$

$$A=\begin{bmatrix}1&1\\-1&2\\0&-1\end{bmatrix}\begin{bmatrix}-1&1\\2&-1\end{bmatrix}=\begin{bmatrix}1&0\\5&-3\\-2&1\end{bmatrix}$$

c)- Find the kernel, range, rank and nullity of T.

$$\left[\begin{array}{cc|c}1&1&0\\2&1&0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1&1&0\\0&-1&0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$$

ker(T) = null space of A

= Solution of $Ax=0$

$\therefore$ ker(T) $=\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

Nullity(T) = 0

Range(T) = column space of A

$=\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$

Rank(T) = 2.

Eigenvectors and Eigenvalues

DEFINITION 1 If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an ***eigenvector*** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of A (or of T_A), and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** $\boldsymbol{\lambda}$.

Example: Show that the vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

$$Ax = \lambda x$$

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Remark: To find the eigenvalue of an $n \times n$ matrix A we rewrite $Ax = \lambda x$ as

$$Ax = \lambda I x$$

$$(\lambda I - A)x = 0$$

THEOREM 5.1.1 *If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation*

$$\det(\lambda I - A) = 0 \qquad (1)$$

This is called the ***characteristic equation*** *of A.*

Example 1: Find the eigenvalue of the matrix.

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda - 3)(\lambda + 1)$$
$$= \lambda^2 + \lambda - 3\lambda - 3$$
$$= \lambda^2 - 2\lambda - 3$$
$$= (\lambda - 3)(\lambda + 1)$$

$$(\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda = 3 \text{ or } \lambda = -1$$

Example 2: Find the eigenvalue of the matrix.

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda + 2)(\lambda - 2) + 5$$

$$\det(\lambda I - A) = \lambda^2 - \cancel{2\lambda} + \cancel{2\lambda} - 4 + 5$$

$$= \lambda^2 + 1$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = i \text{ or } \lambda = -i$$

$\because \lambda$ complex number $\Rightarrow$ ther is no eigenvalue for A.

Example 8:- Find bases for the eigenspaces of

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 3 & 2 & 0 \\ 2 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 5 \end{bmatrix}$$

$$\begin{matrix} + & - & + \end{matrix}$$

$$\det(\lambda I - A) = \lambda - 5 \begin{vmatrix} \lambda - 3 & 2 \\ 2 & \lambda - 3 \end{vmatrix}$$

$$= \lambda - 5 \left[(\lambda - 3)(\lambda - 3) - 4 \right]$$

$$\det(\lambda I - A) = \lambda - 5\,[\lambda^2 - 3\lambda - 3\lambda + 9 - 4]$$

$$= \lambda - 5\,[\lambda^2 - 6\lambda + 5]$$

$$= (\lambda - 5)(\lambda - 5)(\lambda - 1)$$

$$= (\lambda - 5)^2(\lambda + 1)$$

$\therefore\ (\lambda - 5)^2(\lambda + 1) = 0 \Rightarrow \lambda = 5$ or $\lambda = 1$

To find a base we solve

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} \lambda - 3 & 2 & 0 \\ 2 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda = 5$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2x_1 + 2x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned}\right] \longrightarrow \left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}\right\} x_1 = -x_2$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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$\therefore \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are basis for the eigenspace corresponding to $\lambda = 5$

If $\lambda = 1$

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\left.\begin{aligned} -2x_1 + 2x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \end{aligned}\right] \begin{aligned} &\Rightarrow 0 = 0 \\ &\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2 \end{aligned}$$

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ونشتغل عليها

$$-4x_3 = 0 \quad \Longrightarrow \quad x_3 = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$