Mada Altiary

Infinite sequence: $a_1, a_2, \ldots, a_n$ متتابعة لا نهائية

Infinite Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ متسلسلة لا نهائية

Remark: To determine if an infinite series converges or diverges we could consider the sequance of partial sum.

معنى الملاحظة هو انه كي نحدد ما اذا
كانت المتسلسله اللا نهائيه متقاربة أو
متباعدة فإننا ننظر الى متتابعة المجموع
الجزئي اذا تقترب من عدد معين فإن
المتسلسله تكون تقاربية وإذا لايوجد نمط
محدد للاعداد فإن المتسلسله تكون تباعديه.

Example:

Determine wether the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ converges or diverges. If it converges find the sum.

Solution:-

نسمي S_1, S_2, S_3, S_4

Partial Sum [المجموع الجزئي].

نسمي النواتج :

متتابعة المجموع الجزئي.

$S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$S_4 = 1 - 1 + 1 - 1 = 0$

The sequance of partial sum:

$1, 0, 1, 0, \ldots$

converge or diverge ? $\xrightarrow{\text{why}}$ لإنها لا تقترب من قيمة معينة

Types of series

أنواع المتسلسلات

1. Geometric Series:

Form: $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)$

convergent $|r| < 1$

divergent $|r| \geq 1$

Sum: $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

Example: In each part determine whether the series converges and if so find its sum

a) $\sum_{k=0} \frac{5}{4^k}$

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^k} + \cdots$$

is a geometric series with $a = 5$, $r = \frac{1}{4}$

$\because |r| = |\frac{1}{4}| = \frac{1}{4} < 1$

$\therefore$ the series is convergent and the sum

$$\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{5}{3/4} = 5 \cdot \frac{4}{3} = \frac{20}{3}$$

b)- $\sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$

$$\sum_{k=1}^{\infty} 3^{2k} 5^{1-k} = \sum_{k=1}^{\infty} \frac{(3^2)^k}{5^{-(1-k)}}$$

$$= \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} \frac{9 . 9^k . 9^{-1}}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} 9 . \frac{9^{k-1}}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} 9 \left(\frac{9}{5}\right)^{k-1}$$

geometric series : $a = 9$ and $r = 9/5$

$\because |r| = |9/5| > 1 \Rightarrow$ the given series is divergent.

2- Harmonic Series:

is a divergent series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

3- P- Series:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \underbrace{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{k^p} + \cdots}$$

convergent $p > 1$

divergent $p \leq 1$

Example 2: Determine the whether the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ converges or divrges.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

or :

$$\sum_{k=1}^{\infty} \frac{1}{(k)^{1/3}} = 1 + \frac{1}{(2)^{1/3}} + \frac{1}{(3)^{1/3}} + \cdots + \frac{1}{(k)^{1/3}} + \cdots$$

P series with $P = 1/3 < 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ divergent.

Testing for Convergence or Divergence of a series

1. The divergence test

- If $\lim_{k\to\infty} u_k \neq 0$ then $\sum u_k$ is divergent
- If $\lim_{k\to\infty} u_k = 0$, then $\sum u_k$ may either converge or diverge. (Fail)

Example: Determine wether the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$ converges or diverges.

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{k/k}{\frac{k}{k}+\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{1+1/k}$$

$$= \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1 \neq 0$$

$\therefore \sum_{k=1}^{\infty} \frac{k}{k+1}$ is divergent.

2. The Integral Test:

If $u_k = f(k)$ $\forall n$ an $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$ then

- If $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} u_k$ converges.
- If $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} u_k$ diverges.

Example : Use the integral test to determine whether the following series converge or diverge.

(a) $\sum_{k=1}^{\infty} \frac{1}{k}$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \left[\ln|x|\right]_1^b$$

$$= \lim_{b\to\infty} \left[\ln b - \ln 1\right]$$

$$= \ln\infty - 0$$

$$= \infty \rightarrow \text{diverges.}$$

$\therefore \int_1^{\infty} \frac{1}{x}\,dx$ diverges $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{b} + 1\right]$$

$$= \frac{-1}{\infty} + 1 = 0 + 1 = 1 \rightarrow \text{converges.}$$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent

More Exercise s on diverges or converges of series

Determine wether the following series converges or diverges

1). $\sum_{k=0}^{\infty} 2^k$

$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + \cdots + 2^k$

Type: Geometric , $a = 1$, $r = 2 > 1$

$\therefore \sum_{k=0}^{\infty} 2^k$ divergent

2. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots$

Type: P-Series , $P = 2 > 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent

3. $\sum_{k=1}^{\infty} \frac{1}{k}$

Type: Harmonic

$\therefore \sum_{k=1}^{\infty} \frac{1}{k}$ divergent

4. $\sum_{k=0}^{\infty} \frac{3}{10^k}$

$\sum_{k=0}^{\infty} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots$

Type: Geometric , $a = \frac{3}{10}$, $r = \frac{1}{10} < 1$

$\therefore \sum_{k=0}^{\infty} \frac{3}{10^k}$ convergent and $\displaystyle\sum_{k=0}^{\infty} \frac{3}{10^k} = \frac{3/10}{1-1/10} = \frac{3/10}{9/10} = 3/9$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{4k-1}{7k+4}$

$\displaystyle\lim_{k\to\infty} \frac{4k-1}{7k+4} = \frac{4}{7} \neq 0$

في النهايات إذا كانت درجة
البسط = درجة المقام
النهاية = المعاملات.

$\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{4k-1}{7k+4}$ divergent by divergent test

6. $\displaystyle\sum_{k=0}^{\infty} x^k$.

$\displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$

Type : Geometric , $a = 1$, $r = x$ $\rightarrow$ متغير

- If $x < 1 \Rightarrow \displaystyle\sum_{k=0}^{\infty} x^k$ convergent and
$\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

- If $x > 1 \Rightarrow \displaystyle\sum_{k=0}^{\infty} x^k$ divergent.

Maclaurin and Taylor polynomials

1-Maclaurin polynomials

Linear Approximation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

at $x_0 = 0$

$$f(x) = f(0) + f'(x_0)\, x$$

Quadratic Approximation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

at $x_0 = 0$

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2} x^2$$

Example:- Find the local linear and quadratic approximation of e^x at $x = 0$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$
$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$$
$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$$

linear: $f(x) = f(0) + f'(0)\, x$

$$e^x = 1 + x$$

Quadratic: $f(x) \simeq f(0) + f'(0)\, x + \frac{f''(0)}{2} x^2$

$$e^x \approx 1 + x + \frac{1}{2} x^2$$

nth Maclaurin polynomial

We define the **nth Maclaurin Polynomial** for f about $x=0$ to be:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n}(0)}{n!}x^n$$

Example: Find the Maclaurin polynomial P_0, P_1, P_2, P_3 and P_n for e^x

$f(x) = e^x$ $\qquad$ $f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad$ $f'(0) = e^0 = 1$

$f''(x) = e^x$ $\qquad$ $f''(0) = e^0 = 1$

$f'''(x) = e^x$ $\qquad$ $f'''(0) = e^0 = 1$

$\vdots$ $\qquad$ $\vdots$

$\therefore$ $f^{(n)}(x) = e^x$ $\qquad$ $f^{(n)}(0) = e^0 = 1$

$P_0(x) = f(0) = 1$

$P_1(x) = f(0) + f'(0)x = 1 + x$

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

$= 1 + x + \frac{1}{2!}x^2$

$$P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(x)}{3!}\,x^3$$

$$= 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3$$

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(x)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n$$

$$= 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n$$

2-Tylor Polynomials

We define the nth Taylor Polynomial for f about $x = x_0$ to be:

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

ملاحظه: كثيرة الحدود ماكلورين ماهي
الإحالة خاصه من تايلور

ماكلورين $\leftarrow$ $x_0 = 0$

تايلور $\leftarrow$ $x_0 =$ أي نقطة أخرى

الحدود الاربعه الاولى

Example: Find the first four Taylor polynomial for $\ln x$ about $x = 2$

$f(x) = \ln x$ $\qquad$ $f(2) = \ln 2$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(2) = \frac{1}{2}$

$f''(x) = -1/x^2$ $\qquad$ $f''(2) = -\frac{1}{4}$

$f'''(x) = \frac{2}{x^3}$ $\qquad$ $f'''(2) = \frac{2}{8} = \frac{1}{4}$

Then:

$P_0(x) = f(2) = \ln 2.$

$$P_1(x) = f(2) + f'(2)(x-2)$$
$$= \ln 2 + \frac{1}{2}(x-2)$$

$$P_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$
$$= \ln 2 + \frac{1}{2}\ln(x-2) - \frac{1}{8}(x-2)^2$$

$$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$
$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1/4}{6}(x-2)^3$$
$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3.$$

HW: Find the nth Maclaurin polynomials for

a) $\sin x$

b) $\cos x$