Assignment 1

Mada Altiary

1: Subrings and Ideals

(a) Prove the set of all real numbers of the form $m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$ is a subring of the real numbers.

(b) Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of $\mathbb{Z}$.

(c) Find all subring of $\mathbb{Z}_8$

(d) prove that $I = \{0, 3, 6\}$is an ideal of $\mathbb{Z}_9$ and compute the cosets of $\mathbb{Z}_9 \backslash I$

(e) Find all principal ideal in $\mathbb{Z}_{15}$

2: Units, Zero divisors, idempotent and nilpotent

(a) Find an integer n that shows that the rings Z_n need not have the following properties that the ring of integers has.
a. $a^2 = a$ implies $a = 0$ or $a = 1$.
b. $ab = 0$ implies $a = 0$ or $b = 0$.
c. $ab = ac$ and $a \neq 0$ imply $b = c$.
Is the n you found prime?

(b) List all zero divisors and units in , $\mathbb{Z}_{10}, \mathbb{Z}_{11}$.

(c) Find all idempotent and nilpotent elements in $\mathbb{Z}_8$

3: Ring, Integral domain and Field

Give an example of the following or explain why it is not possible to do so. Justify your answers.

(a) An integral domain that is not a field.

(b) A field that is not a commutative ring.

(c) A commutative ring with unity that is not an integral domain.

(d) A subring of a ring R that is not an ideal in R.

(e) Zero divisors in $n\mathbb{Z}$.

(f) A subset of a ring that is a subgroup under addition but not a subring

(g) A commutative ring without zero-divisors that is not an integral domain.

4: Ring and Field

(a) Let $R = \{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10. Prove that R is a field.
(b) Let S be the subring of $\mathbb{Z}_{16}$ given by $S = \{0, 2, 4, 6, 8, 10, 12, 14\}$. What is the characteristic of this ring s

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5: True or False

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(1) Addition in every ring is commutative (T)

(2) The characteristic of $2\mathbb{Z}$ is zero (T)

Not **(3)** Every ring has a multiplicative identity (F) $2\mathbb{Z}$ is ring but $1 \notin 2\mathbb{Z}$

1

(4) $\mathbb{Z}_n$ is not integral domain if n is not prime T

(5) Multiplication in a field is commutative T

(6) $2\mathbb{Z}$ is not an integral domaim T

(7) $\mathbb{Z}_6$ is a subring of $\mathbb{Z}_{12}$ F

(8) A field is a commutative division ring T

(9) There is exactly one element $n \in \mathbb{Z}_8$ such that $n \neq 0$ but $n^2 = 0$. T

(10) Every integral domain is a field F

(11) The set of odd integers is a subrig of $\mathbb{Z}$ F , $O_{odd} = \{1, 3, 5, 7, ---\}$ and $1+3 = 4 \notin O$ even

(12) The integral domain $\mathbb{Z}[i]$ is not a field T

(13) 5 is a zero divisor in $\mathbb{Z}_6$ F unit

(14) $M_2(\mathbb{R})$ is a commutative ring with unity F not

(15) The element $3 + i$ is a zero in $\mathbb{Z}_5[i]$ T $\longrightarrow$ $(3+i)(2+i) = 6 + 3i + 2i + i^2$

$= 6 + 5i - 1$

$= 5 + 5i$

$= 0 + 0i \mod 5$

(16) Multiplication in a field is commutative T

(17) $\mathbb{R}$ have no zero divisors T

(18) The only units in $\mathbb{Z}$ are 1 and -1 T

(19) Every ideal is a subring T

(20) $n\mathbb{Z} \trianglerighteq \mathbb{Z}$ T

1

(a) Prove the set of all real numbers of the form $m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$ is a subring of the real numbers.

Prove $\mathbb{Z}\sqrt{2} = \{m+n\sqrt{2} \mid m,n \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$.

Solutions

Let $x = m+n\sqrt{2}$, $y = s+t\sqrt{2}$

for $m, n, s, t \in \mathbb{Z}$

1) $x - y = (m+n\sqrt{2}) - (s+t\sqrt{2})$

$= m+n\sqrt{2} - s - t\sqrt{2}$

$= (m-s) + (n-t)\sqrt{2}$

$= a\sqrt{2} - b\sqrt{2}$

$\because a, b \in \mathbb{Z} \Rightarrow x-y \in \mathbb{Z}\sqrt{2}$.

2)- $x \cdot y = (m+n\sqrt{2})(s-t\sqrt{2})$

$= ms - mt\sqrt{2} + ns\sqrt{2} - nt$

$= (ms-nt) + (ns-mt\sqrt{2})$

$= c + d\sqrt{2}$

$\because c, d \in \mathbb{Z} \Rightarrow x \cdot y \in \mathbb{Z}\sqrt{2}$

$\therefore$ by subring test, $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of $\mathbb{Z}$.

$2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \ldots\}$ and

$3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \pm 12, \ldots\}$

$2\mathbb{Z} \cup 3\mathbb{Z} = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots\}$ is not a subring because $2+3=5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$

(c) Find all subring of $\mathbb{Z}_8$

The divisors of $\mathbb{Z}_8$ are 1, 2, 4 and 8. Thus the subrings are:

$1\mathbb{Z}_8 = \mathbb{Z}_8$

$2.\mathbb{Z}_8 = \{0, 2, 4, 6\}$

$4\mathbb{Z}_8 = \{0, 4\}$

$8\mathbb{Z}_8 = \{0\}$

(d) prove that $I = \{0, 3, 6\}$is an ideal of $\mathbb{Z}_9$ and compute the cosets of $\mathbb{Z}_9 \backslash I$

+	0	3	6
0	0	3	6
3	3	6	0
6	6	0	3

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
3	0	3	6	0	3	6	0	3
6	0	6	3	0	6	3	0	6

closed under ×.

- closed under +
- $0 \in S$

$0 + I = \{0, 3, 6\}$

$1 + I = \{1, 4, 7\}$

$2 + I = \{2, 5, 8\}$

$3 + I = \{3, 6, 0\}$

$4 + I = \{4, 7, 1\}$

$\therefore \mathbb{Z}_9 / I = \{0 + I, 1 + I, 3 + I\}$

2

(a) Find an integer n that shows that the rings Z_n need not have the following properties that the ring of integers has.
a. $a^2 = a$ implies $a = 0$ or $a = 1$.
b. $ab = 0$ implies $a = 0$ or $b = 0$.
c. $ab = ac$ and $a \neq 0$ imply $b = c$.
Is the n you found prime?

Solution: In Z_6

a) $3^2 = 3$ but $3 \neq 0$ or $3 \neq 1$

b) $2.3 = 6 = 0$ but $2 \neq 0$ and $b \neq 0$

c). For $a = 2$, $b = 4$, $c = 1$, we have $\underset{a\ b}{2.4} = 2 = \underset{b\ c}{2.1}$
but $4 \neq 2$

(b) List all zero divisors and units in , $\mathbb{Z}_{10}, \mathbb{Z}_{11}$.

All zero divisors of $Z_{10} = \{ a \in Z_{10} \mid \gcd(a, 10) \neq 1 \}$

$= \{2, 4, 5, 6, 8\}$

$U_{10} = \{ a \in Z_{10} \mid \gcd(a, 10) = 1 \}$

$= \{1, 3, 7, 9\}$

In Z_{11} : There is no zerodivisors.

All units $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(c) Find all idempotent and nilpotent elements in $\mathbb{Z}_8$

All Idempotent:

$0^2 = 0$

$1^2 = 1$

All nilpotent:

$2^3 = 8 = 0 \pmod 8$

$4^2 = 16 = 0 \pmod 8$

$6^3 = 64 = 0 \pmod 8$.

3 Give an example of the following or explain why it is not possible to do so. Justify your answers.

(a) An integral domain that is not a field.

$\mathbb{Z}$ is an integral domain but not a field

because $2 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

(b) A field that is not a commutative ring.

Impossible. A field is a commutative ring with unity in which each element has an multiplicative inverse

(c) A commutative ring with unity that is not an integral domain.

$\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_{12}$. In general:

Any $\mathbb{Z}_m$, m is not prime will be an example.

(d) A subring of a ring R that is not an ideal in R.

$\mathbb{Z}$ is a subring of $\mathbb{Q}$ but is not ideal of $\mathbb{Q}$ because is not closed under multiplication.

let $3 \in \mathbb{Z}$ and $\frac{1}{4} \in \mathbb{Q}$ the $3 \cdot \frac{1}{4} = \frac{3}{4} \notin \mathbb{Z}$.

(e) Zero divisors in $n\mathbb{Z}$.

Impossible.

(f) A subset of a ring that is a subgroup under addition but not a subring

Let $R = \mathbb{C}$ and $S = \{ix : x \in \mathbb{R}\}$

$$\text{then} \quad \left.\begin{array}{l} 0 \in S \\ a - b \in S \end{array}\right\} \text{subgroup}$$

$(S,+)$ is a subgroup of $(R,+)$ $\begin{cases} \text{When } a \in S \text{ and } b \in S \text{ then } a+b \in S. \\ \text{When } a \in S \text{ then } -a \in S \ (0 \in S) \end{cases}$
When $a \in S$ and $b \in S$ then $a \cdot b \in S$

but $i \cdot i = i^2 = -1 \notin S$ not subring

(g) A commutative ring without zero-divisors that is not an integral domain. $\mathbb{Z}$.

4

(a) Let $R = \{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10. Prove that R is a field.

Solutions:-

+	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

×	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

- closed under addation
- $0 \in R$
- Addation is asso
- commutative
- Satisfy distributive Law
- each element has an inverse

- closed under multiplication
- $6 \in R$
- multiplication is asso
- Commutative
- Satisfy distributive Law
- each element has an inverse

a	0	2	4	6	8
a^{-1}	0	8	6	4	2

a	2	4	6	8
a^{-1}	8	4	6	2

(b) Let S be the subring of $\mathbb{Z}_{16}$ given by $S = \{0, 2, 4, 6, 8, 10, 12, 14\}$. What is the characteristic of this ring s

$2 \cdot 1 = 2 \neq 0$

$2 \cdot 2 = 4 \neq 0$

$2 \cdot 3 = 6 \neq 0$

$2 \cdot 4 = 8 \neq 0$

$2 \cdot 5 = 10 \neq 0$

$2 \cdot 6 = 12 \neq 0$

$2 \cdot 7 = 14 \neq 0$

$2 \cdot 8 = 16 = 0$

$\therefore$ The $\mathrm{char}(S) = 8$.