Chapter #3

(BMT-222) [redacted]

Chapter 3 Systems of Linear Equations and Introduction to Determinants (Summary)

Exercises / Section 3.1 (page 84)

- Solve the following systems of equations graphically.

Problem # 1. $\begin{matrix} 2x-y=1 \\ x-y=2 \end{matrix}$, Problem # 5. $\begin{matrix} -x-3y=4 \\ 2x+2y=5 \end{matrix}$ Problem # 7. $\begin{matrix} 2x+3y=2 \\ 3x+2y=1 \end{matrix}$

Problem # 9. $\begin{matrix} 5x-2y=9 \\ 4x-3y=4 \end{matrix}$ Problem # 11. $\begin{matrix} 3x+y=0 \\ x-2y=20 \end{matrix}$

(Problems solved in class # 1, 11) HW: Problem # 5, Problem # 7, Problem # 9

Exercises / Section 3.2 (page 90-91)

- Solve the following systems of equations by the method of addition or subtraction.

Problem # 1. $\begin{matrix} x+y=4 \\ 2x-y=5 \end{matrix}$ Problem # 5. $\begin{matrix} 3x-2y=21 \\ 4x-5y=42 \end{matrix}$

Problem # 7. $\begin{matrix} 4x-3y=-11 \\ 12x+25y=69 \end{matrix}$ Problem # 9. $\begin{matrix} 2x+2y=1 \\ 5x-5y=1 \end{matrix}$

- Solve the following systems of equations by the method of substitution.

Problem # 15. $\begin{matrix} 2x+y=1 \\ x+3y=8 \end{matrix}$ Problem # 17. $\begin{matrix} 8x-10y=-13 \\ x+2y=0 \end{matrix}$ Problem # 19. $\begin{matrix} 5x+2y=3 \\ 6x+3y=2 \end{matrix}$

- Solve the following systems of equations by either method. Problem # 23. $\begin{matrix} 3x-2y=1 \\ 6x-4y=5 \end{matrix}$

Problem # 27. $\begin{matrix} \frac{2}{x}-\frac{3}{y}=1 \\ \frac{3}{x}-\frac{2}{y}=2 \end{matrix}$ Problem # 35. $\begin{matrix} 2w-3z=5 \\ 4w-6z=10 \end{matrix}$ Problem # 37. $\begin{matrix} -2v+5w=10 \\ 4v-10w=15 \end{matrix}$

(Problems solved in class # 1, 9, 15, 23, 35) HW: Problem # 5, 7, 17, 19, 27, 37

Exercises / Section 3.3 (page 95-96)

- Expand each determinant. Problem # 3. $\begin{vmatrix} -2 & 4 \\ 4 & -8 \end{vmatrix}$

Problem # 9. $\begin{vmatrix} -2 & -1 \\ 12 & 5 \end{vmatrix}$ Problem # 13. $\begin{vmatrix} 32 & 21 \\ -17 & 16 \end{vmatrix}$ Problem # 15. $\begin{vmatrix} 18 & -6 \\ 75 & 0 \end{vmatrix}$

- Solve the following systems of equations by using Cramer rule. Problem # 17. $\begin{aligned}3x+4y&=1\\2x+3y&=4\end{aligned}$

Problem # 25. $\begin{aligned}\frac{2}{x}-\frac{3}{y}&=7\\\frac{1}{x}+\frac{5}{y}&=3\end{aligned}$ Problem # 31. $\begin{aligned}F_1+2F_2&=5\\2F_1+F_2&=6\end{aligned}$ Problem # 33. $\begin{aligned}3R_1+4R_2&=20\\4R_1+2R_2&=15\end{aligned}$

(**Problems** solved in class # 3, 15, 17, 31) HW: Problem # 9, 13, 25, 33

Exercises / Section 3.4 (page 99-101)

Problem # 1.

In figure 3.10 the moment of weight W is 5. The lever balances when $d_1 = 2\,ft$ and $d_2 = 1\,ft$ and when $d_1 = 1\,ft$ and $d_2 = 3\,ft$. Determine the weights w_1 and w_2.

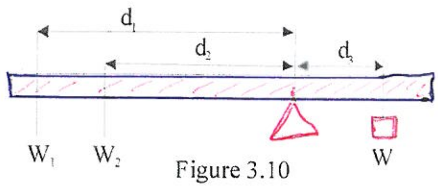


Figure 3.10

Problem # 7.

Two resistors connected in series have a combined resistance of 150 Ω. If the resistance of one resistor is 10 Ω less than the other, find the resistance of each.

Problem # 15.

The sum of the voltages across two resistors is 55.1 V. It was found that 3 times the first voltage is 9.7 V less than 4 times the second. What are the two voltages?

Problem # 17.

Tickets for an industrial exhibit cost $5.00 for regular admission and $4.00 for senior citizens. On one day 215 tickets were sold for total intake of $1050. How many tickets of each type were sold?

Problem # 21.

Two machines have a total of 62 moving parts. If one machine has 2 more than 3 times as many moving parts as the other, how many moving parts does each machine have?

Problem # 25.

One consultant to a firm charges $200 per day, and another consultant charges $250 per day. After 13 days the total charged by the two consultants comes to $2950. Assuming that only one of the two consultants was called in on any one day, how many days did each one work?

P-# 21

(Problems solved in class # 1, 17) HW: Problem # 7, Problem # 15, Problem # 25

Exercises / Section 3.5 (page 103)

- Solve the following systems of equations

Problem # 3. $\begin{aligned} 3x+2z&=-1 \\ 4x-y-2z&=7 \\ x+y&=2 \end{aligned}$

Problem # 7. $\begin{aligned} 2x-y+3z&=16 \\ 3x+4y+2z&=7 \\ 5x-6y+8z&=47 \end{aligned}$

Problem # 11. $\begin{aligned} \frac{2}{x}-\frac{1}{y}+\frac{2}{z}&=2 \\ -\frac{4}{x}+\frac{5}{y}-\frac{3}{z}&=1 \\ \frac{3}{x}-\frac{4}{y}+\frac{1}{z}&=3 \end{aligned}$

(Problems solved in class # 11) HW: Problem # 3, Problem # 7

Exercises / Section 3.6 (page 111-114)

Problem # 5. $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 10 & 5 & -10 \end{vmatrix}$

Problem # 7. $\begin{vmatrix} 2 & 3 & 8 \\ -1 & 3 & -2 \\ 5 & -6 & -12 \end{vmatrix}$

Problem # 11. $\begin{vmatrix} -3 & -4 & -7 \\ 3 & 0 & -6 \\ 10 & 15 & 18 \end{vmatrix}$

- Solve the system of equation by Cramer's rule:

Problem # 19. $\begin{aligned} 2x-y+3z&=16 \\ 3x+4y+2z&=7 \\ 5x-6y+8z&=47 \end{aligned}$

Problem # 21 $\begin{aligned} 2x-3y+z&=1 \\ x-2y-3z&=1, \\ x-4y+2z&=2 \end{aligned}$

Problem # 25

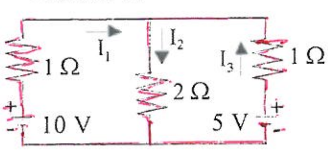
A portion of $ 5950 was invested at 8 %, another portion at 10 %, and the rest at 12 %. The total interest income was $ 635. If the sum of the second investment and twice the first investment was $ 750 more than the third investment, find the amount invested in each rate.

Problem # 27

Three machine parts cost a total of $ 40. The first part costs as much as the other two together, while the cost of 6 times the second is $ 2 more than the total cost of the other two. Find the cost of each part.

Problem # 31 Find the currents of the circuits by solving the system of equations given

Problem 31



$\begin{aligned} I_1-I_2+I_3&=0 \\ I_1+2I_2&=10 \\ -2I_2-I_3&=-5 \end{aligned}$

Problem # 33 $\begin{aligned} -I_1+I_2+I_3&=0 \\ -I_1-3I_2&=-10 \\ 3I_2-5I_3&=-6 \end{aligned}$

(Problems solved in class # 7, 27, 33). HW: Problem # 5, 11, 19, 21, 25, 31

BMT-222
Chapter #3

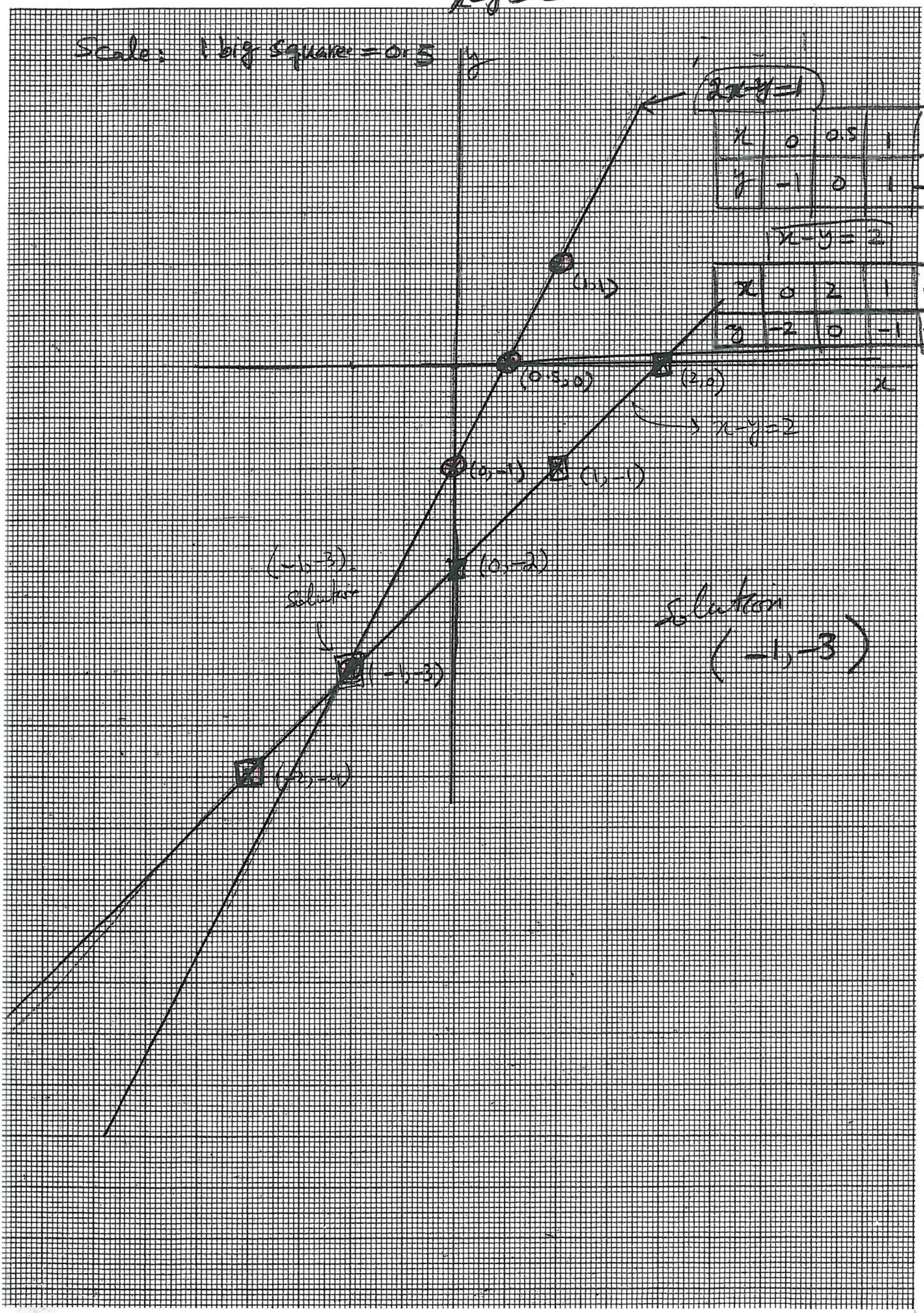
Exercise 3-1 Problem #1

$2x - y = 1$

$x - y = 2$

Pr

Scale: 1 big square = 0.5



2x − y = 1

x	0	0.5	1	−1
y	−1	0	1	−3

x − y = 2

x	0	2	1	−1	−2
y	−2	0	−1	−3	−4

Solution
(−1, −3)

Exercise 3.1

Q #9

$5x - 2y = 7 \rightarrow ①$

$4x - 3y = 4 \rightarrow ②$

Scale:

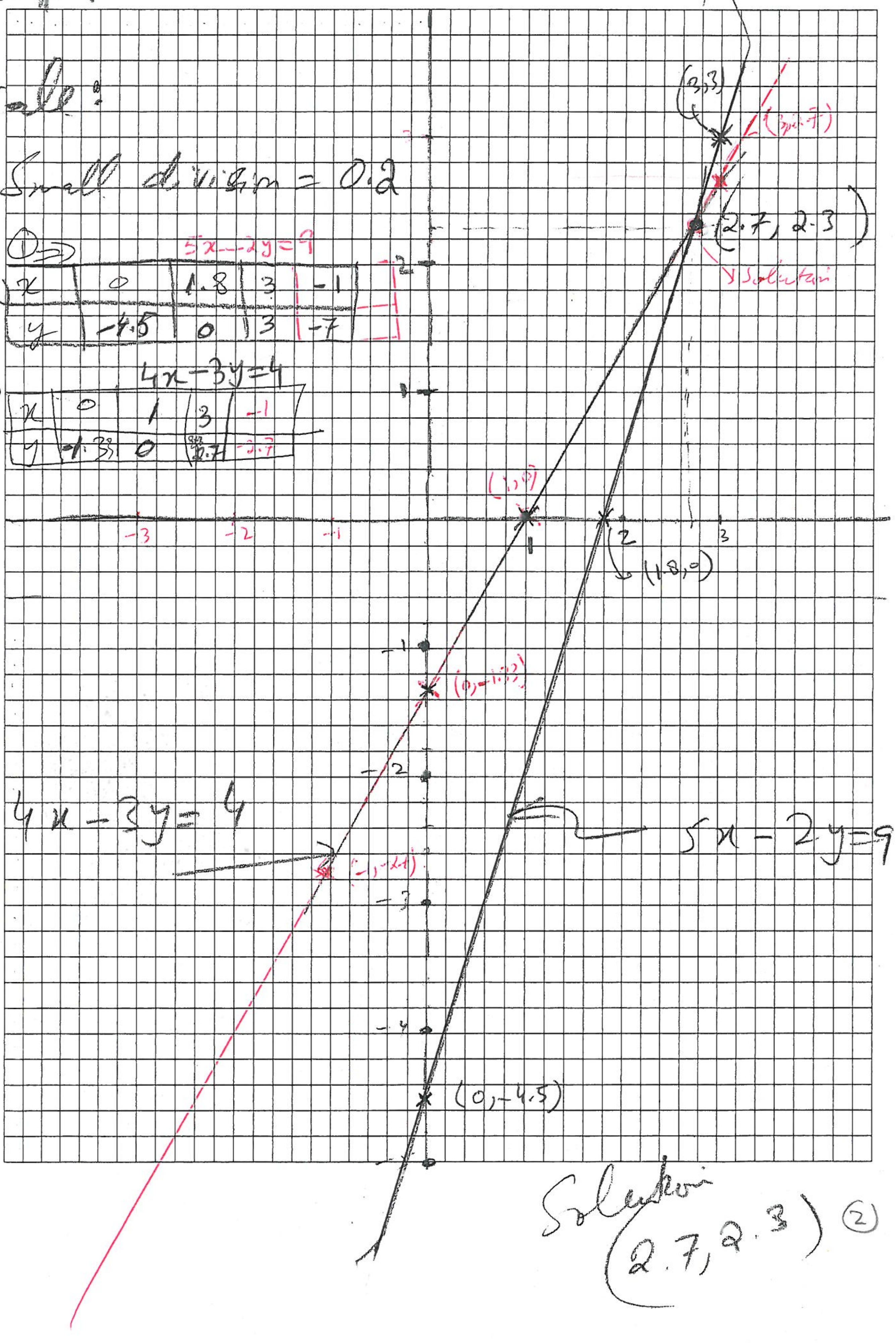
1 Small division = 0.2

① $5x - 2y = 9$

x	0	1.8	3	-1
y	-4.5	0	3	-7

② $4x - 3y = 4$

x	0	1	3	-1
y	-1.33	0	2.7	-2.7



Solution

(2.7, 2.3) ②

Exercise 3.1 (Page 84)

Problem #11 $3x + y = 0$

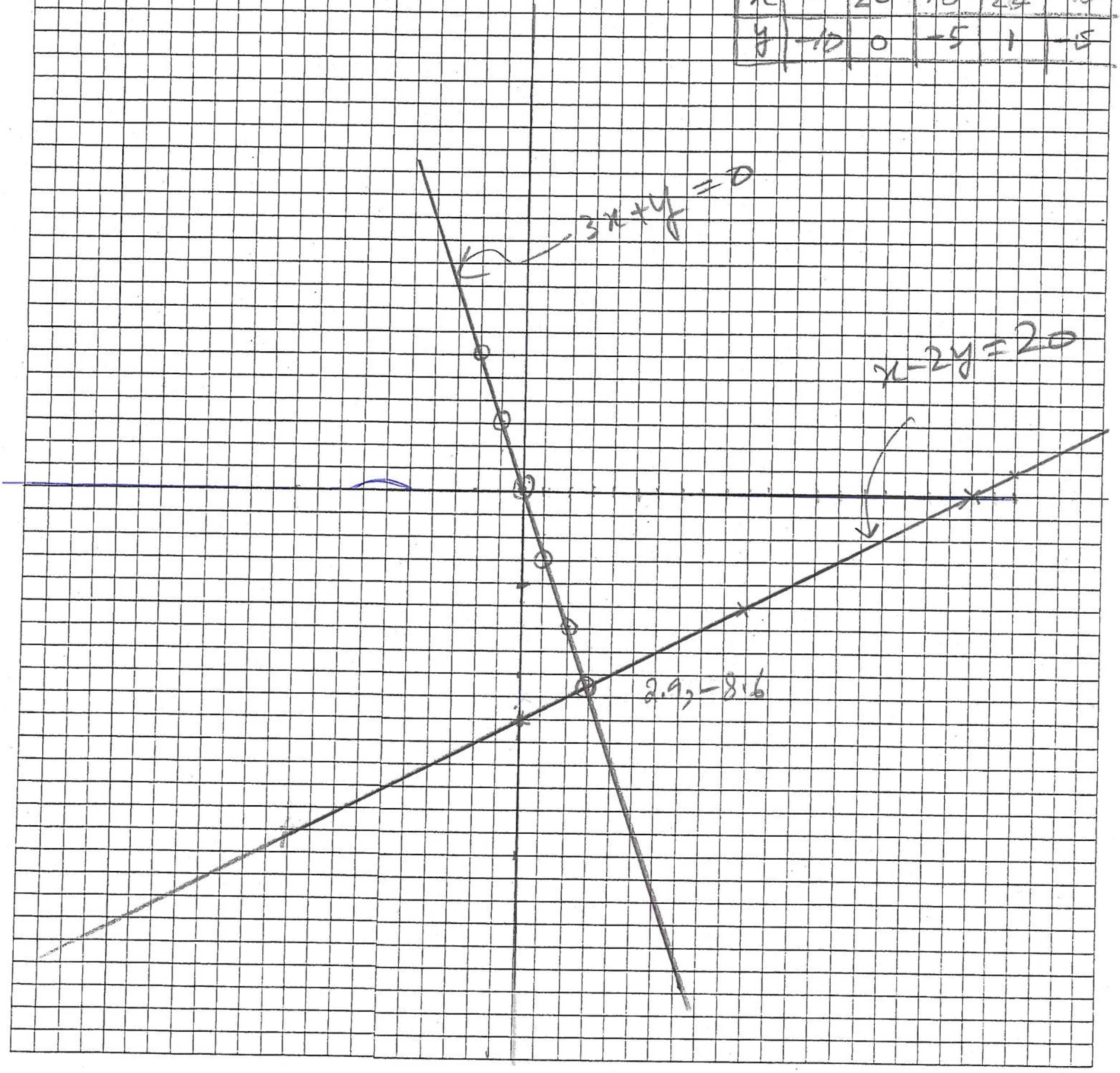
$x - 2y = 20$

$3x + y = 0$

X	0	1	-1	2	-2
y	0	-3	3	-6	6

$x - 2y = 20$

x	0	20	10	22	-10
y	-10	0	-5	1	-15



③

Exercise 3.2 Page (90-91) Ch. #3 (BMT-222)

Problem #1

$x + y = 4 \rightarrow (1)$
$2x - y = 5 \rightarrow (2)$

Adding equation (1) & (2)

$x + y = 4$
$2x - y = 5$
$+$
$3x = 9 \Rightarrow x = \frac{9}{3} \Rightarrow \boxed{x = 3}$

- Substituting value of x in equation (1)

$x + y = 4$
$3 + y = 4$
$\Rightarrow y = 4 - 3$
$\Rightarrow \boxed{y = 1}$

Solution
$\therefore \{3, 1\}$

Problem # 9

$2x + 2y = 1 \rightarrow (1)$
$5x - 5y = 1 \rightarrow (2)$

multiply equation (1) with 5 and equation (2) with 2 and then add them

$10x + 10y = 5$
$10x - 10y = 2$
$+$
$20x = 7 \Rightarrow \boxed{x = \frac{7}{20}}$

- Substituting value of x in equation (1)

$2x + 2y = 1$
$2\left(\frac{7}{20}\right) + 2y = 1$
$\Rightarrow \frac{7}{10} + 2y = 1$
$\Rightarrow 2y = 1 - \frac{7}{10} \Rightarrow 2y = \frac{10-7}{10}$
$\Rightarrow 2y = \frac{3}{10}$
$\Rightarrow \boxed{y = \frac{3}{20}}$

So solution $\left\{\frac{7}{20}, \frac{3}{20}\right\}$

Problem #15

$2x + y = 1 \rightarrow (1)$
$x + 3y = 8 \rightarrow (2)$

From equation (1)
$2x + y = 1$
$\Rightarrow y = 1 - 2x$

Substituting value of y in equation (2)

$x + 3y = 8$
$\Rightarrow x + 3(1 - 2x) = 8$
$\Rightarrow x + 3 - 6x = 8$
$\Rightarrow -5x = 8 - 3 \Rightarrow -5x = 5 \Rightarrow x = -\frac{5}{5}$
$\Rightarrow \boxed{x = -1}$

Substituting value of "x" in equat (1)

$2x + y = 1$
$\Rightarrow 2(-1) + y = 1$
$\Rightarrow -2 + y = 1 \Rightarrow y = 1 + 2$
$\Rightarrow \boxed{y = 3}$

Solution $\{-1, 3\}$

Problem #23

$3x - 2y = 1 \rightarrow (1)$
$6x - 4y = 5 \rightarrow (2)$

- By addition subtraction method Multiplying of (1) with 2 and subtract the equation.

$6x - 4y = 2$
$6x - 4y = 5$
$- \quad + \quad -$
$0 + 0 = -3$

$0 = -3 \qquad 0 \neq -3$

which is not possible
So equation are inconsistent
no solution

(4)

inconsistent = lacking agreement
lacking in harmony between the different parts

Exercise 3.2 Page 90-91

Problem # 35

$2w - 3z = 5 \rightarrow (1)$

$4w - 6z = 10 \rightarrow (2)$

By addition & subtraction method.

Multiply equation (1) with 2

$4w - 6z = 10$

$4w - 6z = 10$

So we have two variables w and z but only one equation.

We can not solve it independently

$4w - 6z = 10$

$\Rightarrow 2(2w - 3z) = 10$

$\Rightarrow 2w - 3z = 5$

$\Rightarrow 2w = 5 + 3z$

$\Rightarrow w = \frac{5+3z}{2}$

dependent

w depend on z value

Exercise 3.3 Page 95-96

Problem #3

$$\begin{vmatrix} -2 & 4 \\ 4 & -8 \end{vmatrix}$$

$= (-2)(-8) - (4)(4)$

$= 16 - 16$

$= 0$

Exercise 3.3

Problem #15

$$\begin{vmatrix} 18 & -6 \\ 75 & 0 \end{vmatrix}$$

$= 18(0) - (75)(-6)$

$= 0 - (-450)$

$= 450$

Problem # 17

$3x + 4y = 1$

$2x + 3y = 4$

Using Cramer's rule

$$x = \frac{\begin{vmatrix} 1 & 4 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}} = \frac{3-16}{9-8}$$

$\Rightarrow x = \frac{-13}{1} \Rightarrow x = -13$

$$y = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}} = \frac{12-2}{9-8}$$

$\Rightarrow y = \frac{10}{1} \Rightarrow y = 10$

Solution $\{-13, 10\}$

Problem # 31

$F_1 + 2F_2 = 5$

$2F_1 + F_2 = 6$

Using Cramer's rule

$$F_1 = \frac{\begin{vmatrix} 5 & 2 \\ 6 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{5-12}{1-4}$$

$\Rightarrow F_1 = \frac{-7}{-3} \Rightarrow F_1 = \frac{7}{3}$

$$F_2 = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{6-10}{1-4} = \frac{-4}{-3}$$

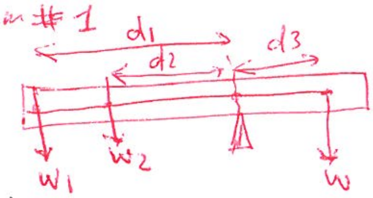
$\Rightarrow F_2 = \frac{4}{3}$

Solution $\{\frac{7}{3}, \frac{4}{3}\}$

Exercise 3.4

Problems solved in class

Problem # 1



Clockwise moment = $W \times d3 = 5$

The system is balanced when $d_1 = 2ft$ and $d_2 = 1ft$

also balance when $d_1 = 1ft$ and $d_2 = 3ft$

$W = ?$, $w_2 = ?$

system is balanced when

Clockwise torque (moment) = anticlockwise torqe (or moment)

Clockwise torq (moment) $= W \times d_3 = 5$

Anticlockwise torque $= w_1 \times d_1 + w_2 \times d_2$

$\therefore \quad w_1 \times 2 + w_2 \times 1 = 5 \Rightarrow 2w_1 + w_2 = 5 \rightarrow (1)$

and $\quad w_1 \times 1 + w_2 \times 3 = 5 \Rightarrow w_1 + 3w_2 = 5 \rightarrow (2)$

Using Cramer's rule

$$w_1 = \frac{\begin{vmatrix} 5 & 1 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}} = \frac{15-5}{6-1} \Rightarrow w_1 = \frac{10}{5}$$

$\Rightarrow w_1 = 2 \qquad \therefore w_1 = 2$ lb (Pound)

Substituting the value of w_1 in equation (1)

That is $\quad 2w_1 + w_2 = 5$

$\Rightarrow 2(2) + w_2 = 5$

$\Rightarrow 4 + w_2 = 5$

$\Rightarrow w_2 = 5 - 4$

$\Rightarrow w_2 = 1$ lb (Pound)

{ 2 lb, 1 lb }

Problem # 17

Let regular tickets sold be $= x$

senior citizen " " $= y$

$x + y = 215 \rightarrow (1)$

$5x + 4y = 1050 \rightarrow (2)$

Addition subtraction method.

Multiply equation (1) with -4 and then add both equations

$$\begin{aligned} -4x - 4y &= -860 \\ + \quad 5x + 4y &= 1050 \\ \hline x &= 190 \end{aligned}$$

Substituting value of x in equation (1)

That is $\quad x + y = 215$

$190 + y = 215$

$\Rightarrow y = 215 - 190$

$\Rightarrow y = 25$

{ 190, 25 }

Check $x + y = 215$

L.H.S $x + y$

$= 190 + 25$

$= 215 =$ R.H.S

$\therefore$ L.H.S = R.H.S

Q# 3

$3w_1 + 4w_2 = 4 \times 17.25 = 69$

$2 \times (5w_1 + 2w_2 = 4 \times 18.25 = 73)$

$3w_1 + 4w_2 = 69$

$10w_1 + 4w_2 = 146$

$w_1 = 11, \quad w_2 = 9$

Q# 5

$s = a - 3T$

Exercise 3.5 (Page 103)

Problem # 11

$\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 2 \rightarrow (1)$

$-\frac{4}{x} + \frac{5}{y} - \frac{3}{z} = 1 \rightarrow (2)$

$\frac{3}{x} - \frac{4}{y} + \frac{1}{z} = 3 \rightarrow (3)$

Let $\frac{1}{x} = u$, $\frac{1}{y} = v$, $\frac{1}{z} = w$

$2u - v + 2w = 2 \rightarrow (4)$

$-4u + 5v - 3w = 1 \rightarrow (5)$

$3u - 4v + w = 3 \rightarrow (6)$

We want to eliminate w multiply equation (4) with 3 and equation (5) with 2 and add them

$6u - 3v + 6w = 6$

$-8u + 10v - 6w = 2$

$+$

$-2u + 7v = 8 \rightarrow (7)$

multiply equation (6) with 3 and adding equation (5) and (6)

$-4u + 5v - 3w = 1$

$9u - 12v + 3w = 9$

$+$

$5u - 7v = 10 \rightarrow (8)$

Adding equation (7) and (8)

$-2u + 7v = 8 \rightarrow (7)$

$5u - 7v = 10 \rightarrow (8)$

$+$

$3u = 18$

$\Rightarrow \boxed{u = 6} \Rightarrow x = \frac{1}{6}$

Substituting value of u in equation (7)

$-2u + 7v = 8$

$-2(6) + 7v = 8$

$-12 + 7v = 8$

$\Rightarrow 7v = 20$

$\Rightarrow \boxed{v = \frac{20}{7}} \Rightarrow y = \frac{7}{20}$

Substituting value of u and v in equation (4)

$2u - v + 2w = 2$

$\Rightarrow 2(6) - \frac{20}{7} + 2w = 2$

$\Rightarrow 2w = 2 + \frac{20}{7} - 12$

$\Rightarrow 2w = \frac{14 + 20 - 84}{7}$

$\Rightarrow 2w = -\frac{50}{7}$

$\Rightarrow \boxed{w = -\frac{25}{7}} \Rightarrow \boxed{z = -\frac{7}{25}}$

Solution: $\left\{\frac{1}{6}, \frac{7}{20}, \frac{-7}{25}\right\}$

Exercise 3.6

Page # 111 – 114

Q # 7

$$\begin{vmatrix} 2 & 3 & 8 \\ -1 & 3 & -2 \\ 5 & -6 & -12 \end{vmatrix}$$

Expanding with respect to Row 1 (R_1)

$$= 2\begin{vmatrix} 3 & -2 \\ -6 & -12 \end{vmatrix} - 3\begin{vmatrix} -1 & -2 \\ 5 & -12 \end{vmatrix} + 8\begin{vmatrix} -1 & 3 \\ 5 & -6 \end{vmatrix}$$

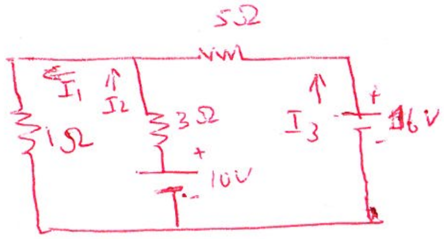
$= 2(-36-12) - 3(12+10) + 8(6-15)$

$= 2(-48) - 3(22) + 8(-9)$

$= -96 - 66 - 72$

$= -234$

Q # 33



$-I_1 + I_2 + I_3 = 0 \quad \rightarrow (1)$

$-I_1 - 3I_2 = -10 \quad \rightarrow (2)$

$3I_2 - 5I_3 = -6 \quad \rightarrow (3)$

Using Cramers rule

$$I_1 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ -10 & -3 & 0 \\ -6 & 3 & -5 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ -1 & -3 & 0 \\ 0 & 3 & -5 \end{vmatrix}} = \frac{N}{D}$$

where $N = \begin{vmatrix} 0 & 1 & 1 \\ -10 & -3 & 0 \\ -6 & 3 & -5 \end{vmatrix}$

expanding with respect to R_1

$$= 0\begin{vmatrix} -3 & 0 \\ 3 & -5 \end{vmatrix} - 1\begin{vmatrix} -10 & 0 \\ -6 & -5 \end{vmatrix} + 1\begin{vmatrix} -10 & -3 \\ -6 & 3 \end{vmatrix}$$

$= 0 - 1(50-0) + 1(-30-18)$

$= -50 - 48 \Rightarrow N = -98$

and $D = \begin{vmatrix} -1 & 1 & 1 \\ -1 & -3 & 0 \\ 0 & 3 & -5 \end{vmatrix}$

Expanding with respect to column 1 (C_1)

$$= -1\begin{vmatrix} -3 & 0 \\ 3 & -5 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix}$$

$= (-1)(15-0) + 1(-5-3) + 0$

$= -15 - 8$

$D = -23$

$\therefore I_1 = \frac{-98}{-23} \Rightarrow I_1 = \frac{98}{23} A$

Substituting the value of I_1 in equation (2)

$-I_1 - 3I_2 = -10$

$\Rightarrow -\frac{98}{23} - 3I_2 = -10$

$\Rightarrow 3I_2 = -\frac{98}{23} + 10$

$\Rightarrow 3I_2 = \frac{-98+230}{23}$

$\Rightarrow 3I_2 = \frac{132}{23}$

$\Rightarrow I_2 = \frac{44}{23} A$

Substituting the value of I_1 and I_2 in equation (1)

$-I_1 + I_2 + I_3 = 0$

$\Rightarrow I_3 = I_1 - I_2$

$\Rightarrow I_3 = \frac{98}{23} - \frac{44}{23}$

$\Rightarrow I_3 = \frac{98-44}{23}$

$\Rightarrow I_3 = \frac{54}{23} A$

Problem # 25 Exercise 3.6 (Page 112) Home work (9)

Let "first" 8% be x

second 10% be y

and third 12% be z

$x+y+z=5950 \rightarrow (1)$

$0.08x+0.1y+0.12z=635 \rightarrow (2)$

$2x+y-z=750 \rightarrow (3)$

Using Cramer's rule

$$x = \frac{\begin{vmatrix} 5950 & 1 & 1 \\ 635 & 0.1 & 0.12 \\ 750 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0.08 & 0.1 & 0.12 \\ 2 & 1 & 1 \end{vmatrix}} = \frac{N}{D}$$

Where $N = \begin{vmatrix} 5950 & 1 & 1 \\ 635 & 0.1 & 0.12 \\ 750 & 1 & -1 \end{vmatrix}$

expand w.r.t R_1

$= 5950\begin{vmatrix} 0.1 & 0.12 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 635 & 0.12 \\ 750 & -1 \end{vmatrix} + 1\begin{vmatrix} 635 & 0.1 \\ 750 & 1 \end{vmatrix}$

$= 5950(-0.1-0.12) - 1(-635-90) + 1(635-75)$

$= 5950(-0.22) - 1(-725) + 1(560)$

$= -1309 + 725 + 560$

$N = -1309 + 1285$

$\Rightarrow N = -24$

$D = \begin{vmatrix} 1 & 1 & 1 \\ .08 & 0.1 & 0.12 \\ 2 & 1 & -1 \end{vmatrix}$

$= 1\begin{vmatrix} 0.1 & 0.12 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0.08 & 0.12 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} .08 & 0.1 \\ 2 & 1 \end{vmatrix}$

$= 1(-0.1-0.12) - 1(-0.08-0.24) + 1(.08-0.2)$

$D = -0.22 + 0.32 - 0.12 \Rightarrow D = -0.02$

$\therefore x = \frac{N}{D} = \frac{-24}{-0.02} \Rightarrow x = 1200$ \$ for 8%

For y

$$y = \frac{\begin{vmatrix} 1 & 5950 & 1 \\ 0.08 & 635 & 0.12 \\ 2 & 750 & -1 \end{vmatrix}}{D} = \frac{M}{D}$$

(Where $D = -0.02$)

$M = \begin{vmatrix} 1 & 5950 & 1 \\ .08 & 635 & 0.12 \\ 2 & 750 & -1 \end{vmatrix}$

$= 1\begin{vmatrix} 635 & 0.12 \\ 750 & -1 \end{vmatrix} - 5950\begin{vmatrix} 0.08 & 0.12 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 0.08 & 635 \\ 2 & 750 \end{vmatrix}$

$= 1(-635-90) - 5950(-0.08-0.24) + 1(60-1270)$

$= -725 + 1904 - 1210$

$= -1935 + 1904$

$M = -31$

$\therefore y = \frac{-31}{-0.02} \Rightarrow y = 1550$

$\therefore y = 1550$ \$ for 10%

Putting value of x, and y in equation no. (1)

$x+y+z=5950$

$1200+1550+z=5950$

$\Rightarrow z = 5950-2750$

$\Rightarrow z = 3200$

$\therefore z = 3200$ \$ for 12%

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CHAPTER 3

Systems of Linear Equations and Introduction to Determinants

Basic Technical Mathematics with Calculus
Peter K. F. Kuhfitting

Objectives Upon completion of this chapter, you should be able to:

1. Solve systems of two linear equations graphically.
2. Solve systems of two linear equations algebraically by means of:
   a. Addition or subtraction.
   b. Substitution.
   c. Determinants.
3. Solve word problems leading to systems of equations.
4. Solve systems of three or four equations algebraically.
5. Expand third-order determinants by minors.
6. Solve systems of three equations by Cramer's rule.

3.1 Simultaneous Linear Equations

In Chapter 2 we studied the solution of first-order equations. However, many applications in technology require the solution of systems containing two or more equations. In this section we will study the geometric basis of systems containing two equations, as well as how to solve such equations by drawing graphs.

It is revealed through analytic geometry that the general equation of a straight line is

$$ax + by = c \tag{3.1}$$

(A detailed discussion of the line is given in Chapter 19.) Since equation (3.1) represents a straight line, it is called a **linear equation** or a **linear equation in two variables.** For example, the equation $3x - 2y = 6$ fits form (3.1) and

linear equation

therefore represents a line. Suppose we graph this line from the followi[ng] table of values:

following

x:	0	1	2
y:	−3	$-\frac{3}{2}$	0

The graph is shown in Figure 3.1. Note that the coordinates of the poi[nts] $(0, -3)$, $(1, -\frac{3}{2})$, and $(2, 0)$ in the table satisfy the equation. Moreover, t[he] coordinates of *every* point on the line satisfy the equation.

Now consider another line, $x + y = 2$, shown in Figure 3.2. As befo[re] the coordinates of *every* point on this line satisfy the equation $x + y = 2$

Points the before

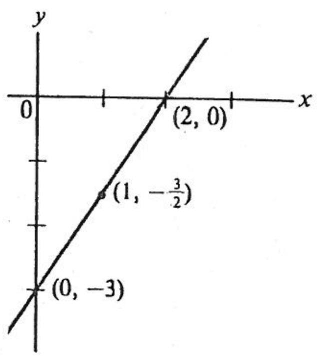

Figure 3.1

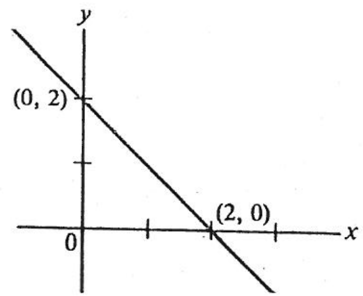

Figure 3.2

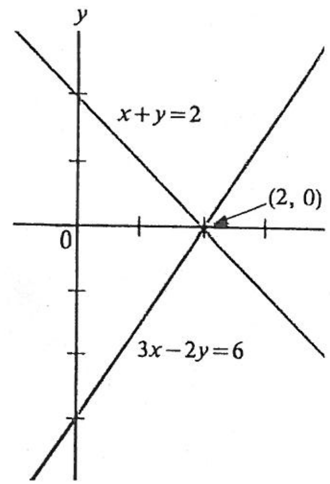

Figure 3.3

According to Figure 3.3, which shows both lines, the two lines interse[ct] at (2, 0). So the coordinates of the point (2, 0) satisfy *both* equations. Cons[e]quently, $x = 2$ and $y = 0$ is called the **common solution** of the system

intersect conse

$$3x - 2y = 6$$
$$x + y = 2$$

In general, two linear equations in two unknowns are referred to as a system of two **simultaneous linear equations.**

Simultaneous linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2 \tag{3.2}$$

Common solution

Any pair (x, y) of values that satisfies both equations is called a **common solution** of the system, or simply a **solution.** If a common solution exists, then it follows from the foregoing discussion that the point (x, y) is the intersection of the two lines. If the lines are distinct, then this point, and hence the solution, is necessarily unique. On the other hand, since two lines may be parallel, a system may not have any solution.

> **Graphical solution:** To find the common solution of a system of two linear equations graphically, draw the two lines and determine the point of intersection from the graph.

To be able to sketch the graphs of linear equations rapidly, we need the notion of **intercept,** which is defined next.

> **Definition of intercept:** An **intercept** is a point at which the graph crosses a coordinate axis. A point $(a, 0)$ is called an ***x*-intercept,** and a point $(0, b)$ is called a ***y*-intercept.**

To find the x-intercept, we let $y = 0$ and solve the resulting equation for x. To find the y-intercept, we let $x = 0$ and solve the resulting equation for y.

For example, in $3x - 2y = 6$, if $y = 0$, then $x = 2$. So the point **(2, 0)** is the x-intercept. (See Figure 3.4.) If $x = 0$, then $y = -3$, so that **(0, −3)** is the y-intercept. Since two distinct points determine a straight line, these are the only points needed to draw the graph. A third point, $(1, -\frac{3}{2})$, is included only as a check. (See Figure 3.4.)

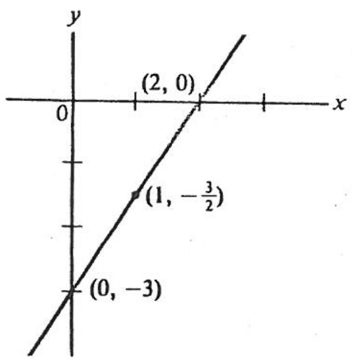

Figure 3.4

Example 1 Determine the common solution of the system

$$2x + y = 5$$
$$x + 3y = 5$$

by drawing the graph of each line and estimating the coordinates of the po of intersection.

Solution. As indicated earlier, the simplest way to draw a line is to find t intercepts. Letting $x = 0$ in the first equation, we find that $y = 5$. Similarly we let $y = 0$, then $x = \frac{5}{2}$. (See Figure 3.5.) For the second equation, if $x =$ then $y = \frac{5}{3}$; and if $y = 0$, then $x = 5$. (See Figure 3.5.) To get a check point the first equation, we let $x = 1$, so that $y = 3$. A check point for the seco line is $(-1, 2)$; both points are shown in Figure 3.5.

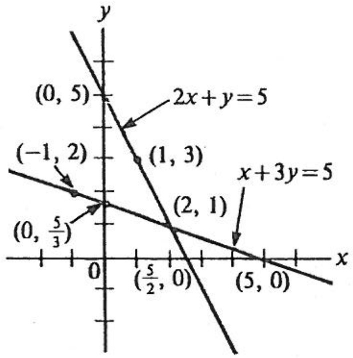

Figure 3.5

We now draw the two lines and observe that they appear to cross (2, 1), at least as closely as can be determined from the graph. As a check, us substitute the coordinates of this point into the given equations:

$$2(\mathbf{2}) + \mathbf{1} = 5$$
$$\mathbf{2} + 3(\mathbf{1}) = 5$$

So the common solution is indeed given by

$$x = 2 \quad \text{and} \quad y = 1$$

Example 2 Determine the solution of the system

$$2x - y = -8$$
$$x - 3y = 3$$

graphically to the nearest tenth of a unit.

Solution. To obtain the intercepts for the first equation, let $x = 0$, so that $y = 8$. If $y = 0$, then $x = -4$. The intercepts are $(-4, 0)$ and $(0, 8)$.

Now assign some arbitrary value to x, such as $x = 1$. Then $2(1) - y = -8$, and $y = 10$. So the check point for the first equation is $(1, 10)$.

Similarly, for the second equation we find that the intercepts are $(3, 0)$ and $(0, -1)$. A check point is $(6, 1)$.

Now we draw the two lines and estimate the coordinates of the point of intersection. To the nearest tenth the coordinates appear to be $(-5.4, -2.8)$. (See Figure 3.6.)

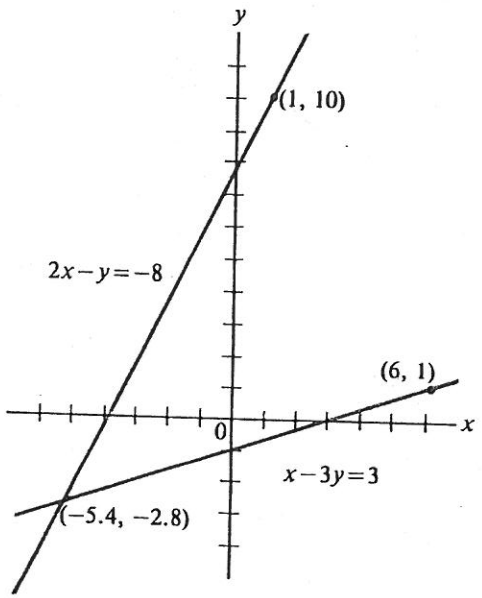

Figure 3.6

We know from geometry that two distinct lines are either parallel or intersecting. If they are parallel, then a common solution cannot exist. Whether two lines really are parallel cannot be determined graphically, since two lines may look parallel and yet intersect at some distant point. Moreover, Example 2 shows that the graphical method for solving simultaneous equations is awkward at best. Fortunately, there are algebraic methods for solving systems directly, which we will study in the next section. However, the graphical approach has provided us with the necessary geometric background.

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Exercises / Section 3.1

Solve the following systems of equations graphically. Estimate the answers to the nearest tenth of a unit.

1. $2x - y = 1$, $x - y = 2$

2. $x - y = -3$, $x + y = -1$

3. $x - 2y = -4$, $x + 3y = 11$

4. $2x - y = 2$, $x + y = 1$

5. $-x - 3y = 4$, $2x + 2y = 5$

6. $x + 2y = 13$, $x - 3y = 2$

7. $2x + 3y = 2$, $3x + 2y = 1$

8. $3x + 3y = 19$, $-x + 2y = 1$

9. $5x - 2y = 9$, $4x - 3y = 4$

10. $6x + 2y = 1$, $3x + 4y = 10$

11. $3x + y = 0$, $x - 2y = 20$

12. $x - 3y = 10$, $3x - y = 3$

3.2 Algebraic Solutions

In the last section we considered the graphical solution of two simultaneous linear equations. In this section we shall turn our attention to solving such systems algebraically.

The real difficulty in solving two equations simultaneously is that two different unknowns are involved. Algebraic solutions resolve this difficulty by eliminating one of the unknowns, thereby reducing the problem to solving a single equation with one unknown.

Addition or Subtraction

The first method to be considered is the **method of addition or subtraction.**

Method of addition or subtraction

1. Multiply both sides of the equations (if necessary) by constants so chosen that the coefficients of one of the unknowns are numerically equal.
2. If the coefficients have opposite signs, add the corresponding members of the equations. If the coefficients have like signs, subtract the corresponding members of the equations.
3. Solve the resulting equation in one unknown for the unknown.
4. Substitute the value of the unknown in either of the original equations and solve for the second unknown.
5. Check the solution in the original system.

To see how addition or subtraction can eliminate one of the unknowns, consider the system

$$2x + 3y = 1$$
$$x + 3y = 2$$

Note that the y-coefficients are the same. If the second equation is subtracted from the first, y is eliminated:

$$\begin{array}{rl} 2x + 3y = & 1 \\ x + 3y = & 2 \\ \hline x \quad = & -1 \quad \text{subtracting} \end{array}$$

We conclude that $x = -1$ From the second equation $(x + 3y = 2)$ we get $(-1) + 3y = 2$, so that $y = 1$. The common solution is therefore $x = -1$ and $y = 1$.

Consider another example.

Example 1 Solve the following system:

$$2x - y = 1$$
$$x - 3y = -2$$

Solution. We can eliminate x as follows: Multiply both sides of the second equation by 2, thereby making the coefficients the same, and then subtract the second equation from the first. Thus

$$\begin{array}{lrl} & 2x - y = & 1 & \\ \text{Step 1.} & 2x - 6y = & -4 & 2(x - 3y) = 2(-2) \\ \hline \text{Step 2.} & 0 + 5y = & 5 & \text{subtracting} \end{array}$$

Step 3. Solving the resulting equation, we get $y = 1$

Step 4. To find the corresponding x-value, substitute $y = 1$ into either of the given equations and solve for x. Using the second equation, we get

$$x - 3(1) = -2$$

which yields $x = 1$. The solution is therefore given by $(1, 1)$.

Step 5. As a check, let us substitute these values into the given equations:

$$2(1) - 1 = 1 \quad \text{and} \quad 1 - 3(1) = -2$$

The solution checks.

If the coefficients of one of the two variables are numerically equal but have opposite signs, this variable is eliminated by adding the two equations.

Example 2 Solve the system of equations

$$3x - 2y = 5$$
$$5x + 2y = 1$$

Solution. Since the coefficients of y are numerically equal but have opposite signs, we can eliminate y by adding the equations. Thus

$$\begin{array}{rl} 3x - 2y = & 5 \\ 5x + 2y = & 1 \\ \hline 8x \quad = & 6 \quad \text{adding} \end{array}$$

and $x = \frac{3}{4}$. Substituting into the first equation, we get

$$3\left(\frac{3}{4}\right) - 2y = 5$$

$$-2y = 5 - \frac{9}{4} \qquad \text{transposing}$$

$$-2y = \frac{11}{4} \qquad \text{simplifying}$$

$$y = -\frac{11}{8} \qquad \text{dividing by } -2$$

As a check, substitute $(\frac{3}{4}, -\frac{11}{8})$ into the second equation:

$$5\left(\frac{3}{4}\right) + 2\left(-\frac{11}{8}\right) = \frac{15}{4} - \frac{11}{4} = 1$$

in agreement with the right side.

In some cases both equations have to be multiplied by a constant before one of the variables can be eliminated.

Example 3 Solve the system

$$3x + 2y = 1$$
$$4x - 3y = 7$$

Solution. In this example, direct addition or subtraction will not eliminate either variable. The simplest approach is to eliminate y by multiplying the first equation by 3 and the second by 2. Thus

$$\begin{array}{rll} 9x + 6y = & 3 & 3(3x + 2y) = 3(1) \\ 8x - 6y = & 14 & 2(4x - 3y) = 2(7) \\ \hline 17x \quad = & 17 & \text{adding} \end{array}$$

It follows that $x = 1$ and $y = -1$.

Sometimes a system of equations does not have any solution, as we can see in the next example.

Example 4 Solve the system

$$-5x + 2y = 7$$
$$10x - 4y = 5$$

Solution. In this example y appears to be the easier of the two unknowns to eliminate:

$$-10x + 4y = 14 \quad \text{multiplying by 2}$$
$$10x - 4y = 5$$
$$0 = 19 \quad \text{adding}$$

There is something wrong, since 0 cannot be equal to 19. It follows that the system has no solution. Geometrically, the equations represent two parallel lines. (See Figure 3.7.)

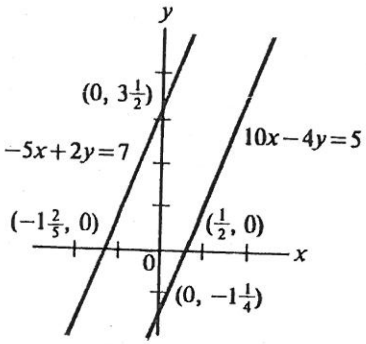

Figure 3.7

Example 5 Compare the system

$$-5x + 2y = -\frac{5}{2}$$
$$10x - 4y = 5$$

to that in Example 4.

Solution.

$$-10x + 4y = -5 \quad \text{multiplying by 2}$$
$$10x - 4y = 5$$
$$0 = 0 \quad \text{adding}$$

This time no contradiction results. In fact, we have merely shown that the two equations represent exactly the same line. As a result, the coordinates of any point on this line satisfy the given system. (See Figure 3.8.)

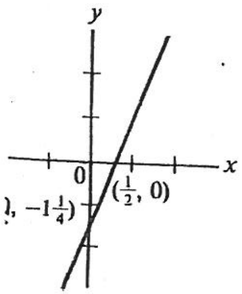

Figure 3.8

The system in Example 4 is said to be **inconsistent,** which means that the system has no solution. Geometrically, an inconsistent system consists of two parallel lines. The system in Example 5 is said to be **dependent;** that is, the lines coincide, and the system has infinitely many solutions.

In both cases the coefficients of the respective unknowns are multiples of each other. These ideas are summarized next.

A **dependent system** has the form

$$ax + by = c$$
$$kax + kby = kc \qquad (k \neq 0)$$

An **inconsistent system** has the form

$$ax + by = c$$
$$kax + kby = d, \quad d \neq kc$$

Substitution

At this point we seem to have covered all cases. Indeed, the method of addition or subtraction works with any system of linear equations. However, if one equation is easily solved for one of the unknowns, it may be more convenient to solve by the **method of substitution.**

Method of substitution

1. Solve one of the equations for one of the unknowns in terms of the other.
2. Substitute the expression obtained into the other equation.
3. Solve the resulting equation in one unknown for the unknown.
4. Substitute the value of the unknown in either of the original equations and solve for the second unknown.
5. Check the solution in the original system.

To see how the method of substitution can be used to eliminate one of the unknowns, consider the system

$$2x - y = 2$$
$$6x + 2y = 1$$

Note that the first equation is readily solved for y to yield

Step 1. $y = 2x - 2$

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Substituting this expression for y in the second equation results in an equation containing only one unknown:

$6x + 2y = 1$ second equation

Step 2. $6x + 2(2x - 2) = 1$ substituting $2x - 2$ for y

Step 3. Solve for x:

$$6x + 4x - 4 = 1$$
$$10x = 5$$
$$x = \frac{1}{2}$$

Step 4. From the first equation, rewritten as $y = 2x - 2$, we get

$$y = 2\left(\frac{1}{2}\right) - 2 = -1$$

The solution is therefore given by $(\frac{1}{2}, -1)$.

Step 5. Check:

$$2\left(\frac{1}{2}\right) - (-1) = 2 \qquad 6\left(\frac{1}{2}\right) + 2(-1) = 1$$

This example shows that the method of substitution is most convenient if one of the variables has a coefficient of 1. (Otherwise solving for one of the unknowns may lead to an expression involving fractions.)

Some equations have unknowns occurring in the denominator:

$$\frac{1}{z} + \frac{2}{w} = 1$$
$$\frac{3}{z} - \frac{2}{w} = 7$$

Such systems can be solved by the usual method if we let $x = 1/z$ and $y = 1/w$. Consider the following example:

EXAMPLE

Example 6 Use the method of substitution to solve the system

$$\frac{3}{s_1} - \frac{2}{s_2} = 1$$
$$\frac{16}{s_1} - \frac{12}{s_2} = 5$$

Solution. This system can be written in the usual form by letting $x = 1/s_1$ and $y = 1/s_2$:

$$3x - 2y = 1$$
$$16x - 12y = 5$$

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Suppose we solve the first equation for y in terms of x. Then

$$-2y = 1 - 3x$$
$$y = -\frac{1}{2}(1 - 3x) \qquad (3.3)$$

Substituting into the second equation, we get

$$16x - 12\left[-\frac{1}{2}(1 - 3x)\right] = 5 \qquad \text{substituting } -\frac{1}{2}(1 - 3x) \text{ for } y$$
$$16x + 6(1 - 3x) = 5$$
$$16x + 6 - 18x = 5$$
$$-2x = -1$$
$$x = \frac{1}{2}$$

Since $y = -\frac{1}{2}(1 - 3x)$, we now get

$$y = -\frac{1}{2}\left(1 - \frac{3}{2}\right) = \frac{1}{4}$$

Finally, since $x = \frac{1}{s_1}$, it follows that $s_1 = 1/x = 2$. Similarly, $s_2 = 1/y = 4$.

Remark. The method of substitution is particularly important for solving equations of the second degree, where the variables have the form x^2 and y^2. Since these will be taken up in Chapter 13, you need to become familiar with the method of substitution.

Exercises / Section 3.2

In Exercises 1–10, solve each system of equations by the method of addition or subtraction.

1. $x + y = 4$, $2x - y = 5$
2. $3x - 2y = 6$, $3x - 4y = -2$
3. $3x - 2y = 1$, $4x - 3y = 4$
4. $2x + 7y = 0$, $3x - 2y = 25$
5. $3x - 2y = 21$, $4x - 5y = 42$
6. $3x - 2y = 7$, $4x - 6y = 11$
7. $4x - 3y = -11$, $12x + 25y = 69$
8. $4x + 3y = 1$, $5x + 8y = 10$
9. $2x + 2y = 1$, $5x - 5y = 1$
10. $3x + 5y = 10$, $5x + 3y = 10$

In Exercises 11–20, solve each system of equations by the method of substitution.

11. $x - 3y = 4$, $2x - y = 3$
12. $x + 2y = 12$, $x - 3y = 2$
13. $3x + 4y = 21$, $-x + 2y = 3$
14. $x + 3y = 1$, $2x - y = -5$

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15. $2x + y = 1$
$x + 3y = 8$

16. $x + 2y = 13$
$3x - y = -31$

17. $8x - 10y = -13$
$x + 2y = 0$

18. $-x - 2y = 3$
$2x + y = 4$

19. $5x + 2y = 3$
$6x + 3y = 2$

20. $2x - 3y = 3$
$5x - 4y = 1$

n Exercises 21–40, solve each system of equations by either method.

21. $6x - 7y = 49$
$8x - 9y = 63$

22. $3x + 5y = 0$
$x + 4y = 0$

23. $3x - 2y = 1$
$6x - 4y = 5$

24. $2x + 4y = 1$
$6x + 12y = -1$

5. $\frac{4}{x} - \frac{3}{y} = 1$
$\frac{5}{x} - \frac{4}{y} = 1$

26. $\frac{5}{x} - \frac{25}{y} = 51$
$\frac{10}{x} - \frac{55}{y} = 112$

27. $\frac{2}{x} - \frac{3}{y} = 1$
$\frac{3}{x} - \frac{2}{y} = 2$

28. $\frac{2}{x} - \frac{1}{y} = 2$
$\frac{4}{x} + \frac{5}{y} = 6$

9. $\frac{2}{p} - \frac{1}{q} = 1$
$\frac{4}{p} - \frac{3}{q} = 4$

30. $\frac{2}{R} - \frac{3}{P} = 12$
$\frac{6}{R} - \frac{5}{P} = 15$

31. $3F_1 - 2F_2 = -7$
$3F_1 - 12F_2 = -37$

32. $12s - 8t = 19$
$4s - 12t = 25$

3. $3A_1 + 2A_2 = 2$
$5A_1 + 3A_2 = 3$

34. $-2y + 4w = 1$
$-3y + 5w = 2$

35. $2w - 3z = 5$
$4w - 6z = 10$

36. $7m_1 + 2m_2 = 20$
$9m_1 + 3m_2 = -15$

'. $-2v + 5w = 10$
$4v - 10w = 15$

38. $2v_0 - 6v_1 = 3$
$-3v_0 + 3v_1 = 1$

39. $3I + 2I_0 = 7$
$2I + I_0 = 4$

40. $-2r + 4s = -6$
$r - 2s = 3$

3.3 Introduction to Determinants

So far we have considered three methods for solving systems of equations. In this section we are going to study yet another, the method of **determinants.** This method is not only elegant but also quite easy to use and to apply to larger systems.

briel Cramer

Determinants were discovered by Gottfried Leibniz, the codiscoverer of the calculus, and promptly forgotten. They were rediscovered by the Swiss mathematician Gabriel Cramer (1704–1752) in 1750. Some time later the English algebraist Arthur Cayley (1821–1895) undertook a systematic study of determinants and matrices. Cayley's work eventually led to a separate branch of mathematics called *linear algebra.*

To see the value of determinants, consider the general solution of the class of systems

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \tag{3.4}$$

We can eliminate y by multiplying the first equation by b_2 and the second by b_1 to obtain

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 \\ a_2b_1x + b_2b_1y &= c_2b_1 \end{aligned}$$

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Subtracting, we get

$$a_1b_2x - a_2b_1x = c_1b_2 - c_2b_1$$

It now follows from the distributive law that

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

and therefore

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \tag{3.5}$$

By eliminating x, one can show that

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \tag{3.6}$$

We have found the solution of the general system (3.4). Thus anyone who memorizes formulas (3.5) and (3.6) would never again have to solve such a system. Unfortunately, memorizing such formulas is probably more trouble than solving the system unless, of course, some kind of simple pattern can be discovered. Such a pattern is provided by a square array of numbers called a **determinant.** Thus $a_1b_2 - a_2b_1$, which appears in each of the denominators, is a quantity denoted by the symbol given next.

Definition of a 2 × 2 determinant

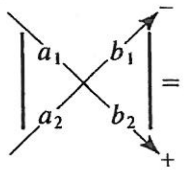
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \tag{3.7}$$

The determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called a 2 × 2 (two-by-two) determinant (larger determinants will be taken up later). The entries are called **elements.** The elements a_1 and b_1 form the first **row,** and the elements a_1 and a_2 the first **column.** (The second row and column are then defined in a similar way.) Writing the square array as a number, called the **expansion** of the determinant, can best be remembered by means of the following diagram:

Expansion of a 2×2 determinant



$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \tag{3.8}$$

The arrows indicate which elements are to be multiplied, and the sign at the

Example 1 Expand the determinants

a. $\begin{vmatrix} 0 & 2 \\ -3 & 5 \end{vmatrix}$ **b.** $\begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix}$

Solution. By expansion (3.8)

a. $\begin{vmatrix} 0 & 2 \\ -3 & 5 \end{vmatrix} = (0)(5) - (-3)(2) = 0 + 6 = 6$

b. $\begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix} = (2)(-4) - (-6)(-1) = -8 - 6 = -14$

Let us now return to the system of equations (3.4):

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \tag{3.9}$$

Observe that the solution, given in statements (3.5) and (3.6), can be written

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Now notice the pattern: The entries in the two denominators are the coefficients of the unknowns arranged as in the original system (3.9). The entries in the numerator of the x-value are obtained by replacing the coefficients of x by the constants on the right side of the system. Similarly, the entries in the numerator of the y-value are obtained by replacing the coefficients of y by the constants on the right. This method of solution is known as **Cramer's rule** and can be extended to larger systems of equations.

Before continuing with an example, let's summarize the solution of systems of two equations by Cramer's rule.

> **Cramer's rule:** The solution of the system
>
> $$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$
>
> is given by
>
> $$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Example 2 Solve the system

$$\begin{aligned} 5x - 2y &= \mathbf{10} \\ -3x + 7y &= \mathbf{0} \end{aligned}$$

by Cramer's rule.

Solution. By Cramer's rule, the determinant in both denominators consists of the coefficients of the unknowns arranged as in the given system:

$$\begin{vmatrix} 5 & -2 \\ -3 & 7 \end{vmatrix}$$

The numerator for x can be constructed from this determinant by replacing the first column, the coefficients of x, by the constants on the right:

$$\begin{vmatrix} \mathbf{10} & -2 \\ \mathbf{0} & 7 \end{vmatrix}$$

Similarly, to find y we construct the determinant in the numerator by replacing the second column, the coefficients of y, by the constants on the right.

$$\begin{vmatrix} 5 & \mathbf{10} \\ -3 & \mathbf{0} \end{vmatrix}$$

The solution is now written as follows:

$$x = \frac{\begin{vmatrix} \mathbf{10} & -2 \\ \mathbf{0} & 7 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -3 & 7 \end{vmatrix}} = \frac{(10)(7) - (0)(-2)}{(5)(7) - (-3)(-2)} = \frac{70 - 0}{35 - 6} = \frac{70}{29}$$

$$y = \frac{\begin{vmatrix} 5 & \mathbf{10} \\ -3 & \mathbf{0} \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -3 & 7 \end{vmatrix}} = \frac{(5)(0) - (-3)(10)}{29} = \frac{30}{29}$$

Returning to Cramer's rule, note that the determinant occurring in each denominator has to be different from 0 to avoid division by 0. If the determinant is different from 0, then the solution of system (3.9) is necessarily unique, since the value of a determinant is unique. If

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

then the system does not have a unique solution. In that case the system is either dependent or inconsistent.

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CALCULATOR COMMENT If the arithmetic involved in expanding a determinant is awkward a calculator can make the computation more convenient. Consid Consider the determinant

$$\begin{vmatrix} -3.59 & 5.26 \\ 7.35 & 3.98 \end{vmatrix} = -3.59 \times 3.98 - 7.35 \times 5.26$$

Since most scientific calculators perform multiplication and div division before addition and subtraction, the sequence is

3.59 [+/−] [×] 3.98 [−] 7.35 [×] 5.26 [=]

Display: −52.9492

For calculators that do not automatically perform the oper operation in the above order, an alternate sequence is

3.59 [+/−] [×] 3.98 [=] [STO] 7.35 [×] 5.26 [=] [+/−] [+] [MR] [=]

Display: −52.9492

Exercises / Section 3.3

In Exercises 1–16, expand each determinant.

1. $\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & -2 \\ 4 & -8 \end{vmatrix}$ 3. $\begin{vmatrix} -2 & 4 \\ 4 & -8 \end{vmatrix}$

4. $\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix}$ 5. $\begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix}$ 6. $\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix}$ 8. $\begin{vmatrix} -2 & 6 \\ 8 & -4 \end{vmatrix}$ 9. $\begin{vmatrix} -2 & -1 \\ 12 & 5 \end{vmatrix}$

10. $\begin{vmatrix} -3 & -5 \\ -6 & -4 \end{vmatrix}$ 11. $\begin{vmatrix} -6 & -17 \\ 3 & 20 \end{vmatrix}$ 12. $\begin{vmatrix} 12 & -3 \\ -15 & 6 \end{vmatrix}$

13. $\begin{vmatrix} 32 & 21 \\ -17 & 16 \end{vmatrix}$ 14. $\begin{vmatrix} 21 & 5 \\ 20 & -8 \end{vmatrix}$ 15. $\begin{vmatrix} 18 & -6 \\ 75 & 0 \end{vmatrix}$

6. $\begin{vmatrix} -12 & -11 \\ 5 & -18 \end{vmatrix}$

n Exercises 17–41, solve each system of equations by using Cramer's rule.

7. $3x + 4y = 1$, $2x + 3y = 4$ 18. $-x + 3y = 5$, $-2x - y = 0$ 19. $3x - 2y = 4$, $-7x + 5y = 1$

0. $-2x - 6y = 4$, $5x + 10y = 5$ 21. $-5x + 6y = 7$, $4x - 5y = 8$ 22. $3x - 4y = 20$, $5x - 6y = 8$

3. $6x + 3y = -1$, $5x + 3y = 2$ 24. $-3x + 4y = 11$, $6x - 8y = 8$ 25. $\frac{2}{x} - \frac{3}{y} = 7$, $\frac{1}{x} + \frac{5}{y} = 3$

* we have seen the method of substitution is not always as convenient as the method of addition or subtraction or the method of determinants.

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26. $\frac{4}{x} - \frac{4}{y} = 9$, $\frac{3}{x} + \frac{5}{y} = 8$ 27. $\frac{8}{x} - \frac{7}{y} = 20$, $\frac{3}{x} - \frac{5}{y} = 16$ 28. $\frac{2}{x} + \frac{5}{y} = 3$, $\frac{5}{x} - \frac{7}{y} = 6$

29. $4T_1 - 7T_2 = 10$, $5T_1 - 8T_2 = 20$ 30. $2w_1 + w_2 = 5$, $w_1 + 3w_2 = 5$ 31. $F_1 + 2F_2 = 5$, $2F_1 + F_2 = 6$

32. $2W_1 + 3W_2 = 12$, $2W_1 + 4W_2 = 15$ 33. $3R_1 + 4R_2 = 20$, $4R_1 + 2R_2 = 15$ 34. $2I_1 + 8I_2 = 25$, $4I_1 + 7I_2 = 30$

35. $3C_1 - 7C_2 = 3$, $-9C_1 + 21C_2 = 5$ 36. $2p - 7q = 10$, $5p - q = 15$

(The remaining problems should be solved with a calculator.)

37. $2.73x - 1.52y = 5.02$, $0.130x + 2.49y = 2.98$ 38. $0.980x + 0.730y = 1.21$, $-1.32x - 5.21y = -1.11$ 39. $-2.10x + 3.64y = 1.32$, $1.00x + 1.78y = -4.05$

40. $6.52x + 3.98y = -1.25$, $1.35x - 1.44y = 2.73$ 41. $-7.63x - 5.02y = 1.31$, $2.84x - 1.54y = 3.87$

3.4 Applications of Systems of Linear Equations

While systems of equations have varied applications in technology, the type of problems already discussed in Section 2.3 can be done more conveniently by using systems of equations. As a typical example, suppose that the sum of two numbers is 35, and one of the numbers is 3 more than the other. Let x be the smaller number; thus $x + 3$ represents the larger. It follows that

$$x + (x + 3) = 35$$

and

$$2x + 3 = 35 \qquad (3.10)$$

From this we get $x = 16$ and $x + 3 = 19$.

Now consider another way to solve this problem by using systems of equations. Let x be the first number and y the second. Then

$$x + y = 35 \qquad (3.11)$$

From the other piece of information, we have

$$y = x + 3 \qquad (3.12)$$

Substituting equation (3.12) into equation (3.11), we get equation (3.10). The first method, then, is the same as solving the system by substitution. But as we have seen, the method of substitution ...

ple 1 One alloy contains 20% copper, and another 25% copper. How many pounds of each must be combined to form 60 lb of an alloy containing 22% copper?

Solution. Let x be the number of pounds of the first alloy and y the number of pounds of the second. Then $x + y = 60$. Recall that it is best to work with the quantities directly: For example, $(0.22)(60) = 13.2$ lb, the weight of copper in the 22% alloy. Thus

$$0.20x + 0.25y = (0.22)(60)$$
$$x + y = 60$$

are the resulting equations, which can be solved by addition or subtraction:

$$\begin{aligned} 20x + 25y &= (22)(60) && \text{multiplying first equation by 100} \\ 20x + 20y &= 1200 && \text{multiplying second equation by 20} \\ \hline 5y &= 120 && \text{subtracting} \\ y &= 24 \text{ lb} \\ x &= 36 \text{ lb} \end{aligned}$$

ple 2 Two resistors are connected in series. The resistance of the second is 5.6 Ω less than that of the first. The total resistance is 50.2 Ω. Find the resistance of each resistor.

Solution. Recall that the total resistance of two or more resistors in series is equal to the sum of the individual resistances. Letting R_1 and R_2 be the individual resistances, we get

$$\begin{aligned} R_1 - R_2 &= 5.6 \\ R_1 + R_2 &= 50.2 \\ \hline 2R_1 \quad &= 55.8 && \text{adding} \\ R_1 &= 27.9\ \Omega \\ R_2 &= 22.3\ \Omega \end{aligned}$$

ple 3 A portion of \$13,580 was invested at 8% interest and the rest at 10%. If the total interest income was \$1,253, how much was invested at each rate?

Solution. Let

x = amount invested at 8%
y = amount invested at 10%

Then

$$x + y = \$13{,}580$$
$$0.08x + 0.10y = \$1{,}253$$

are the equations to be solved.

$$\begin{aligned} 8x + 8y &= 108{,}640 && \text{multiplying first equation by 8} \\ 8x + 10y &= 125{,}300 && \text{multiplying second equation by 100} \\ \hline -2y &= -16{,}660 && \text{subtracting} \\ y &= \$8{,}330 \\ x &= \$5{,}250 \end{aligned}$$

Of course, the solution of a problem can always be checked against the given information. Thus 8% of \$5,250 equals $(0.08)(\$5{,}250) = \$420$ and 10% of \$8,330 is \$833, for a total of \$1,253.

Many problems in technology involve the lever. A **lever** is a rigid bar supported at a point called the **fulcrum,** which is usually positioned between the ends of the bar. Neglecting the weight of the lever, a weight w at a distance d from the fulcrum has a **moment** given by weight times distance, or wd. (The **moment** is a measure of the tendency of the weight to rotate about the fulcrum.) If two weights w_1 and w_2 are placed on opposite sides of the fulcrum at distances d_1 and d_2, respectively, then

$$w_1d_1 = w_2d_2$$

whenever the weights are balanced on the lever. The distance from the fulcrum is called the **moment arm.**

One of the fundamental principles of levers is the fact that moments are additive; that is, $w_1d_1 + w_2d_2$ is equal to the total moment. Consider, for example, the weights on the lever in Figure 3.9. For the weights to be balanced on the lever, the moment on the left must be equal to the moment on the right, or

$$w_1d_1 + w_2d_2 = w_3d_3$$

This formula can be extended to any number of weights.

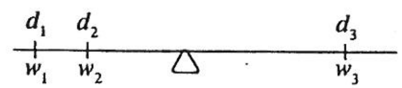


Figure 3.9

Example 4 A weight of 1 lb and a lever are to be used to determine two other weights. Referring to Figure 3.9, the following measurements were taken: given $w_3 = 1$ lb, the lever balances when $d_3 = 31$ in., $d_1 = 5$ in., and $d_2 = 4$ in. Another balance is obtained when $d_3 = 33$ in., $d_1 = 3$ in., and $d_2 = 6$ in.

Solution. From the relationship

$$w_1d_1 + w_2d_2 = w_3d_3$$

and the given measurements, we get the system

$$5w_1 + 4w_2 = 31 \cdot 1$$
$$3w_1 + 6w_2 = 33 \cdot 1$$

By Cramer's rule

$$w_1 = \frac{\begin{vmatrix} 31 & 4 \\ 33 & 6 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix}} \quad \text{and} \quad w_2 = \frac{\begin{vmatrix} 5 & 31 \\ 3 & 33 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix}}$$

Hence

$$w_1 = \frac{186 - 132}{30 - 12} = \frac{54}{18} = 3 \text{ lb}$$

and

$$w_2 = \frac{165 - 93}{18} = \frac{72}{18} = 4 \text{ lb}$$

xercises / Section 3.4

. In Figure 3.10 the moment of weight W is 5. The lever balances when $d_1 = 2$ ft and $d_2 = 1$ ft and
= 1 ft and $d_2 = 3$ ft. Determine the weights w_1 and w_2.

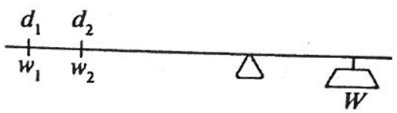


Figure 3.10

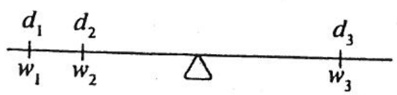


Figure 3.11

Exercises 2–4, refer to Figure 3.11 and find w_1 and w_2 in each case.

$w_3 = 2.0$ N; a balance is obtained if $d_1 = 2.0$ m, $d_2 = 2.0$ m, and $d_3 = 3.5$ m and if $d_1 = 2.0$ m, $d_2 =$
$d_3 = 2.0$ m.

$w_3 = 4.000$ lb; a balance is obtained if $d_1 = 3.000$ in., $d_2 = 4.000$ in., and $d_3 = 17.25$ in. and if $d_1 =$
$d_2 = 2.000$ in., and $d_3 = 18.25$ in.

$w_3 = \frac{1}{3}$ lb; a balance is obtained if $d_1 = 3$ ft, $d_2 = 4$ ft, and $d_3 = 9$ ft and if $d_1 = 1$ ft, $d_2 = 2$ ft, an

The relationship between the tensile strength S (measured in pounds) of a certain metal
temperature T (in degrees Celsius) has the form $S = a - bT$. Experimenters found that if $T =$
$S = 565.9$ lb; if $T = 100$°C, then $S = 565.8$ lb. Find the relationship.

6. The relationship between the length of a certain bar (measured in centimeters) and its temperatur in degrees Celsius) is known to be $L = aT + b$. Tests show that if $T = 15$°C, then L = 50.0 cm; if $T = 60$°C, then $L = 50.8$ cm. Find the relationship.

7. Two resistors connected in series have a combined resistance of 150 Ω. If the resistance of one resist is 10 Ω less than that of the other, find the resistance of each.

8. The combined resistance of two resistors in series is 130 Ω. If the resistance of one resistor is 20 Ω less than that of the other, find the resistance of each.

9. A portion of $8,500 is invested at 12% interest and the remainder at 11%. The total interest incon is $976.10. Determine the amount invested at each rate.

10. A woman invests a certain amount of money at 10% interest and the rest at 8%. If the first investme is $2,000 more than the second and her total interest income is $740, find how much was invested at each

11. The foreman of a machine shop ordered two sets of machine parts costing $35 per dozen and $50 per dozen. If twice the number in the first set was one dozen more than the number in the second set and i the total bill came to $625, find the number of parts in each set.

12. A machinist has an order for a rectangular metal plate with the following specifications: The length is 1. less than 3 times the width and the perimeter is 24.4 in. Find the dimensions of the plate.

13. The manager of a shop spends $189 in his budget to buy 80 small castings. Some cost $1.95 apiece, anc the rest cost $2.50 apiece. How many of each can he buy?

14. Two separate squares are to be made from a piece of wire 54.0 cm long. If the perimeter of one square to be 3.0 cm larger than that of the other, how must the wire be cut?

15. The sum of the voltages across two resistors is 55.1 V. It was found that 3 times the first voltage is 9.7 V less than 4 times the second. What are the two voltages?

16. Measurements of the tension, in pounds, of two supporting cables produced the following equations:

$$0.37T_1 - 0.47T_2 = 0$$
$$0.52T_1 + 0.87T_2 = 120.49$$

Find T_1 and T_2.

17. Tickets for an industrial exhibit cost $5.00 for regular admission and $4.00 for senior citizens. On one day 215 tickets were sold for a total intake of $1,050. How many tickets of each type were sold?

18. A technician needs 100 mL (milliliters) of a 16% nitric acid solution (by volume). He has a 20% and a 10% solution (by volume) in stock. How many milliliters of each must he mix to obtain the required soluti

19. How many liters of a 5% solution (by volume) must be added to a 10% solution to obtain 20 L of an 8% solution?

20. One alloy contains 6% brass (by weight) and another 12% brass (by weight). How many pounds of each r must be combined to form 50 lb of an alloy containing 10% brass?

21. Two machines have a total of 62 moving parts. If one machine has 2 more than 3 times as many moving p parts as the other, how many moving parts does each machine have?

22. John has $6.10 in dimes and quarters. If he has five more dimes than quarters, how many of each doe he have?

23. Jane has $3.30 in nickels and dimes. If she has nine more nickels than dimes, determine the number of e of each

24. An office building has 20 offices. The smaller offices rent for $300 per month, and the larger offices for $ 420 per month. If the rental income is $7,440 per month, how many of each type of office are there?

5. One consultant to a firm charges \$200 per day, and another consultant charges \$250 per day. After 13 days the total charged by the two consultants came to \$2,950. Assuming that only one of the two consultants was called in on any one day, how many days did each one work?

3.5 Systems of Linear Equations with More Than Two Unknowns

Our method for solving systems of equations can be readily extended to systems of three or more equations. We shall concentrate on the **method of addition or subtraction** in this section and return to the method of determinants in Section 3.6.

To solve a system of three equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad (3.13)$$

algebraically, we eliminate one of the unknowns between any two of the equations. Then, taking a different pair of equations, we eliminate the same unknown. The resulting system of two equations can then be solved by one of the earlier methods. Consider the following example.

Example 1 Solve the system

(1) $x - 3y - z = -2$
(2) $4x - y - 2z = 8$
(3) $3x + 2y + 2z = 1$

Solution. A glance at the different coefficients tells us that z is the easiest of the three unknowns to eliminate.

(4)	$2x - 6y - 2z = -4$	multiplying equation (1) by 2
(5)	$4x - y - 2z = 8$	repeating (2) and (3)
(6)	$3x + 2y + 2z = 1$	
(7)	$-2x - 5y = -12$	subtracting (5) from (4)
(8)	$7x + y = 9$	adding (5) and (6)
(9)	$-2x - 5y = -12$	repeating (7)
(10)	$35x + 5y = 45$	multiplying (8) by 5
(11)	$33x = 33$	adding (9) and (10)
(12)	$x = 1$	
(13)	$7(1) + y = 9$	substituting in (8)
(14)	$y = 2$	

(15)	$1 - 3(2) - z = -2$	substituting in (1)
(16)	$z = -3$	

The solution is therefore given by $x = 1$, $y = 2$, and $z = -3$. As a check, substitute these values into equations (2) and (3). Then

$$4(1) - 2 - 2(-3) = 8$$

and

$$3(1) + 2(2) + 2(-3) = 1$$

which checks.

Example 2 Solve the system

(1) $4R_1 - 2R_2 + R_3 = 8$
(2) $3R_1 - 3R_2 + 4R_3 = 8$
(3) $R_1 + R_2 + 2R_3 = 6$

Solution. We eliminate R_2 as follows:

(4)	$12R_1 - 6R_2 + 3R_3 = 24$	multiplying (1) by 3
(5)	$6R_1 - 6R_2 + 8R_3 = 16$	multiplying (2) by 2
(6)	$6R_1 + 6R_2 + 12R_3 = 36$	multiplying (3) by 6
(7)	$6R_1 - 5R_3 = 8$	subtracting (5) from (4)
(8)	$12R_1 + 20R_3 = 52$	adding (5) and (6)
(9)	$24R_1 - 20R_3 = 32$	multiplying (7) by 4
(10)	$12R_1 + 20R_3 = 52$	repeating (8)
(11)	$36R_1 = 84$	adding (9) and (10)
(12)	$R_1 = \frac{84}{36} = \frac{7}{3}$	
(13)	$6\left(\frac{7}{3}\right) - 5R_3 = 8$	substituting into (7)
(14)	$R_3 = \frac{6}{5}$	
(15)	$\frac{7}{3} + R_2 + 2\left(\frac{6}{5}\right) = 6$	substituting into (3)
(16)	$R_2 = \frac{19}{15}$	

As a check, we substitute the values $R_1 = \frac{7}{3}$, $R_2 = \frac{19}{15}$, and $R_3 = \frac{6}{5}$ into equation (1):

$$4\left(\frac{7}{3}\right) - 2\left(\frac{19}{15}\right) + \frac{6}{5} = \frac{140}{15} - \frac{38}{15} + \frac{18}{15} = \frac{120}{15} = 8$$

Equation (2) is checked similarly.

To solve a system with four unknowns, proceed by eliminating one of the unknowns among three different pairs of equations. The resulting system of three equations can then be solved by the methods of this section.

Exercises / Section 3.5

Solve the following systems of equations.

1. $x + y + 2z = 9$; $x - z = -2$; $2x - y = 0$

2. $x - y = 2$; $x + 2y - z = -3$; $-y + z = 3$

3. $3x + 2z = -1$; $4x - y - 2z = 7$; $x + y = 2$

4. $2x - 3y = 8$; $3x - y + 2z = 8$; $2x + 3y = -4$

5. $x - 2y - z = 1$; $2x - y - 2z = -1$; $3x + y + 2z = 6$

6. $3x - 3y - 2z = 1$; $-x + y - 6z = 3$; $x + y + 2z = 3$

7. $2x - y + 3z = 16$; $3x + 4y + 2z = 7$; $5x - 6y + 8z = 47$

8. $x - 2y + z = 1$; $2x + y + 2z = 2$; $3x + 3y - 3z = 2$

9. $3R_1 - 3R_2 + 7R_3 = 35$; $6R_1 + 3R_2 - R_3 = -5$; $9R_1 - 12R_2 + 3R_3 = 38$

$2F_1 - 5F_2 + 10F_3 = 21$; $3F_1 + 5F_2 + 10F_3 = 6$; $4F_1 - 10F_2 + 40F_3 = 64$

11. $\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 2$; $-\frac{4}{x} + \frac{5}{y} - \frac{3}{z} = 1$; $\frac{3}{x} - \frac{4}{y} + \frac{1}{z} = 3$

12. $\frac{2}{x} + \frac{1}{y} - \frac{1}{z} = 0$; $\frac{8}{x} - \frac{2}{y} + \frac{1}{z} = 1$; $\frac{4}{x} - \frac{4}{y} - \frac{1}{z} = -1$

$2y + 3z + w = 1$; $2x - 2y + w = 1$; $3x - 2z + 2w = 13$; $x + 3y - z = 9$

14. $x + 2z - 3w = 12$; $3x - 2y + z = -3$; $3y - 3z - 5w = 12$; $2x - y - 2w = 3$

15. $2x - y - 2z - 2w = -7$; $4x - 2y + 3z = 3$; $6x + 2y + 4z - 3w = 16$; $8x - 2z - 4w = -6$

3.6 More on Determinants

The method of **determinants** can be extended to systems of more than two unknowns. In this section we shall confine ourselves to systems of three equations. Larger systems will be discussed in Chapter 15.

The easiest way to see the extension of the determinant method is to solve the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \tag{3.14}$$

by the method of addition or subtraction. Although straightforward, the calculation is quite lengthy. The expression for x turns out to be

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 - d_2b_1c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1} \tag{3.15}$$

A determinant of the third order is defined with Cramer's rule in mind: If Cramer's rule is to carry over, then the denominator of solution (3.15) should be a determinant whose elements are the coefficients of the unknowns. In other words,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \tag{3.16}$$

The last expression may look like a jumble of symbols without any discernible pattern, but if you look more closely, you will see that *every product consists of exactly one element from each row and one from each column.* (The same holds true for higher-order determinants.) This observation enables us to express a third-order determinant in terms of second-order determinants. For example,

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) = a_1b_2c_3 - a_1b_3c_2$$

which are the first two terms in expansion (3.16). The determinant

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

is called the **minor** of the element a_1.

> **Definition of a minor:** The **minor** of a given element is the determinant formed by deleting all the elements in the row and column in which the element lies.

Thus in determinant (3.16) the minor of a_2 is

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

and the minor of b_2 is

$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

Now observe that, in terms of minors, the determinant (3.16) can be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

In other words, the determinant is expanded by forming the products of the elements in the first row with their corresponding minors and affixing either a plus or minus sign. Moreover, the same expression on the right side of expansion (3.16) can be obtained by using the elements in some other row or even in a column. For example, using the second column, we get the expansion given next.

Typical expansion by minors

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1a_2c_3 + b_1a_3c_2 + b_2a_1c_3 - b_2a_3c_1 - b_3a_1c_2 + b_3a_2c_1 \quad (3.17)$$

ule for signs

We still need a rule for affixing the sign. It turns out that the sign depends only on the position of the element. Consider the row and column in which the element lies. *If the sum of the number of the row and the number of the column is even, affix a plus sign; if the sum is odd, affix a minus sign.*

ample 1 Expand the determinant

$$\begin{vmatrix} 2 & -3 & 1 \\ -4 & 0 & -7 \\ -3 & -1 & 1 \end{vmatrix}$$

Solution. As already noted, we can expand the determinant along any row or column. Suppose we arbitrarily choose the first column. Then we get

$$\begin{vmatrix} 2 & -3 & 1 \\ -4 & 0 & -7 \\ -3 & -1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -7 \\ -1 & 1 \end{vmatrix} - (-4)\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} -3 & 1 \\ 0 & -7 \end{vmatrix}$$

Take a closer look at how the signs were determined. The first element, 2, lies in row 1, column 1, and 1 + 1 = 2, which is even. Hence the element is given a plus sign. The next element, −4, lies in row 2, column 1, and 2 + 1 = 3, which is odd. So the element −4 is given a minus sign to become −(−4). Finally, the third element, −3, lies in row 3, column 1, and 3 + 1 = 4, which is even. So the element is given a plus sign.

No particular row or column offers any obvious advantage with one notable exception: A row or a column containing one or more zeros reduces the number of calculations required. For example, expanding along the second row yields

$$\begin{vmatrix} 2 & -3 & 1 \\ -4 & 0 & -7 \\ -3 & -1 & 1 \end{vmatrix}$$

$$= -(-4)\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} + (0)\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} - (-7)\begin{vmatrix} 2 & -3 \\ -3 & -1 \end{vmatrix}$$

$$= 4(-3 + 1) + 0 + 7(-2 - 9) = -85$$

Example 2 Expand the determinant

$$\begin{vmatrix} -3 & 2 & 1 \\ 4 & -2 & 3 \\ -3 & 1 & 0 \end{vmatrix}$$

Solution. Because of the 0, we expand along the third row, starting with the element −3. This element is situated in row 3, column 1. Since 3 + 1 = 4, we affix a plus sign.

$$\begin{vmatrix} -3 & 2 & 1 \\ 4 & -2 & 3 \\ -3 & 1 & 0 \end{vmatrix}$$

$$= +(-3)\begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} - (1)\begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} + (0)\begin{vmatrix} -3 & 2 \\ 4 & -2 \end{vmatrix}$$

$$= -3(6 + 2) - (-9 - 4) + 0 = -11$$

(Note that the signs necessarily alternate, so that only the first one needs t be determined.)

Example 3 Expand the determinant

$$\begin{vmatrix} 7 & -2 & -3 \\ -11 & 0 & 5 \\ 2 & 0 & 2 \end{vmatrix}$$

Solution. If we expand along the second column, then only one minor needs to be evaluated. Note also that the element −2 lies in row 1, column 2, and is therefore given a minus sign.

$$\begin{vmatrix} 7 & -2 & -3 \\ -11 & 0 & 5 \\ 2 & 0 & 2 \end{vmatrix} = -(-2)\begin{vmatrix} -11 & 5 \\ 2 & 2 \end{vmatrix} = 2(-22-10) = -64$$

Having defined a third-order determinant, we can now return to Cramer's rule and the solution of equations.

Cramer's rule: The solution of the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \tag{3.18}$$

is given by

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \tag{3.19}$$

As in the case of two equations, the denominators are the same for all the unknowns. The determinants in the numerators are found by replacing the column of coefficients of the unknown to be found by the column of numbers on the right side of the equation. We shall see in Chapter 15 that the rule can be applied to systems of equations of any size.

If the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

in (3.19) is different from zero, then the system has a unique solution, since the values of the determinants are unique. Such a system is called *independent*. If the determinant is zero, then the system is not independent and no unique solution exists. (The system is either inconsistent, having no solution, or dependent, having infinitely many solutions.)

Independent system

Example 4 Solve the given system by determinants.

$$\begin{aligned} 2x - y + 3z &= 2 \\ x + 2y - z &= 1 \\ 3x - 2y &= 4 \end{aligned}$$

Solution. First we evaluate the determinant whose elements consist of the coefficients of the unknowns; this determinant will occur in all the denominators. Expanding along the third row:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 3 & -2 & 0 \end{vmatrix} = 3\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 3(1-6) + 2(-2-3) + 0 = -25$$

So by Cramer's rule,

$$x = \frac{\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 4 & -2 & 0 \end{vmatrix}}{-25}$$

Expanding along the third column, we get

$$\begin{aligned} x &= -\frac{1}{25}\left[3\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} + 0\right] \\ &= -\frac{1}{25}[3(-2-8) + 1(-4+4)] \\ &= -\frac{1}{25}(-30) \\ &= \frac{6}{5} \end{aligned}$$

Next,

$$\begin{aligned} y &= -\frac{1}{25}\begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & -1 \\ 3 & 4 & 0 \end{vmatrix} \\ &= -\frac{1}{25}\left[3\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 0\right] \end{aligned}$$ expanding along thirc column

$$= -\frac{1}{25}[3(4-3) + 1(8-6)]$$

$$= -\frac{1}{25}(5)$$

$$= -\frac{1}{5}$$

Finally,

$$z = -\frac{1}{25}\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & 1 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= -\frac{1}{25}\left[2\begin{vmatrix} 2 & 1 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}\right] \quad \text{expanding along first row}$$

$$= -\frac{1}{25}[2(8+2) + 1(4-3) + 2(-2-6)]$$

$$= -\frac{1}{25}(5)$$

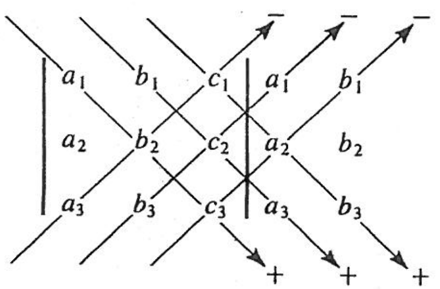
$$= -\frac{1}{5}$$

As a check, suppose we substitute $x = \frac{6}{5}$, $y = -\frac{1}{5}$, and $z = -\frac{1}{5}$ in the second equation. Then

$$\frac{6}{5} + 2\left(-\frac{1}{5}\right) - \left(-\frac{1}{5}\right) = \frac{5}{5} = 1$$

The other equations are checked similarly.

The expansion of a 3×3 determinant can also be accomplished by rewriting the first two columns to the right of the determinant and then forming products of elements along the resulting full diagonals, as shown in the following scheme:



(3.20)

The leftmost arrow pointing upward yields the product $-a_3b_2c_1$, and the leftmost arrow pointing downward yields $+a_1b_2c_3$, and so forth. Comparing

the resulting six products to the expansion by minors shows that the terms are identical. Unfortunately, *this scheme does not work with higher-order determinants.*

Example 5 Expand the determinant in Example 2 by using the expansion scheme (3.20).

Solution. Rewriting the first two columns, we get

$$\begin{vmatrix} -3 & 2 & 1 \\ 4 & -2 & 3 \\ -3 & 1 & 0 \end{vmatrix}\begin{matrix} -3 & 2 \\ 4 & -2 \\ -3 & 1 \end{matrix} = \begin{matrix} +(-3)(-2)(0) + (2)(3)(-3) + (1)(4)(1) \\ -(-3)(-2)(1) - (1)(3)(-3) - (0)(4)(2) \end{matrix}$$

$$= 0 - 18 + 4 - 6 + 9 - 0$$

$$= -11$$

You will have to judge whether this method is more convenient than expansion by minors.

An important application of systems of equations is the analysis of basic electrical circuits by means of Kirchhoff's laws. Since a discussion of setting up systems of equations using Kirchhoff's laws is too lengthy to consider here, we will work with the given system of equations corresponding to a particular circuit in the next example and the exercises.

Example 6 Consider the circuit in Figure 3.12. If the directions of the currents are as indicated in the diagram, then from Kirchhoff's laws the system of equations is given by

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 10I_1 - 3I_2 &= 4 \\ 3I_2 + 5I_3 &= 2 \end{aligned}$$

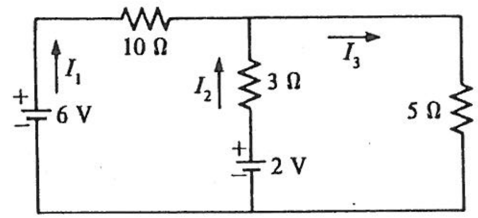


Figure 3.12

By Cramer's rule the solution is found to be $I_1 = \frac{38}{95}$ A, $I_2 = 0$ A, and $I_3 = \frac{38}{95}$ A. (See Exercise 36 in the following exercise set.)

Since the calculated values of I_1 and I_3 are positive, the directions of these currents agree with the directions originally assigned. (If a calculated current is negative, its direction is actually opposite to the direction originally assigned.)

Exercises / Section 3.6

In Exercises 1–12, evaluate each determinant.

1. $\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 3 \\ 1 & -4 & 4 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & -1 & 0 \\ 3 & 0 & -6 \\ -7 & 3 & -4 \end{vmatrix}$ **3.** $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 5 & 0 \\ 3 & 2 & -1 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 3 & -5 \\ 3 & -1 & 4 \\ 0 & -2 & 0 \end{vmatrix}$ **5.** $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 10 & 5 & -10 \end{vmatrix}$ **6.** $\begin{vmatrix} -5 & 12 & 3 \\ 7 & -2 & 1 \\ -3 & 0 & 2 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 3 & 8 \\ -1 & 3 & -2 \\ 5 & -6 & -12 \end{vmatrix}$ **8.** $\begin{vmatrix} -15 & 20 & 10 \\ 3 & 1 & 7 \\ 9 & -3 & 15 \end{vmatrix}$ **9.** $\begin{vmatrix} -3 & 2 & 4 \\ 10 & 18 & -20 \\ -11 & -15 & 8 \end{vmatrix}$

10. $\begin{vmatrix} -5 & 0 & 20 \\ 30 & 3 & 40 \\ 5 & -40 & 10 \end{vmatrix}$ **11.** $\begin{vmatrix} -3 & -4 & -7 \\ 3 & 0 & -6 \\ 10 & 15 & 18 \end{vmatrix}$ **12.** $\begin{vmatrix} -6 & 13 & -3 \\ -2 & 17 & 6 \\ 0 & 19 & 2 \end{vmatrix}$

n Exercises 13–22, solve each system by Cramer's rule. (Exercises 13–20 are the same as Exerci as exercises 1–8 ection 3.5, respectively.)

3. $x + y + 2z = 9$
$x - z = -2$
$2x - y = 0$

14. $x - y = 2$
$x + 2y - z = -3$
$-y + z = 3$

15. $3x + 2z = -1$
$4x - y - 2z = 7$
$x + y = 2$

5. $2x - 3y = 8$
$3x - y + 2z = 8$
$2x + 3y = -4$

17. $x - 2y - z = 1$
$2x - y - 2z = -1$
$3x + y + 2z = 6$

18. $3x - 3y - 2z = 1$
$-x + y - 6z = 3$
$x + y + 2z = 3$

). $2x - y + 3z = 16$
$3x + 4y + 2z = 7$
$5x - 6y + 8z = 47$

20. $x - 2y + z = 1$
$2x + y + 2z = 2$
$3x + 3y - 3z = 2$

21. $2x - 3y + z = 1$
$x - 2y - 3z = 1$
$x - 4y + 2z = 2$

. $3x - y + 4z = 4$
$-x + 2y - 3z = 6$
$-2x - 3y + z = 10$

lve Exercises 23 and 24; refer to Figure 3.13.

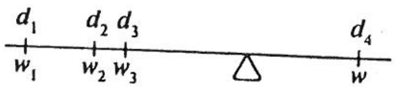


Figure 3.13

23. Find the weights w_1, w_2, and w_3 given the following sets of measurements: $w = 2$ lb and
(1) $d_1 = 4$ ft, $d_2 = 3$ ft, $d_3 = 2$ ft, $d_4 = 5.5$ ft
(2) $d_1 = 3$ ft, $d_2 = 2$ ft, $d_3 = 1$ ft, $d_4 = 3.5$ ft
(3) $d_1 = 5$ ft, $d_2 = 4$ ft, $d_3 = 1$ ft, $d_4 = 5.5$ ft

24. Find the weights w_1, w_2, and w_3 given the following sets of measurements: $w = 11$ N
(1) $d_1 = 2$ m, $d_2 = 1$ m, $d_3 = 3$ m, $d_4 = 2$ m
(2) $d_1 = 4$ m, $d_2 = 3$ m, $d_3 = 3$ m, $d_4 = 3\frac{6}{11}$ m
(3) $d_1 = 6$ m, $d_2 = 2$ m, $d_3 = 6$ m, $d_4 = 4\frac{7}{11}$ m

25. A portion of \$5,950 was invested at 8%, another portion at 10%, and the rest at 12%. The total int-rest income was \$635. If the sum of the second investment and twice the first investment was \$750 more tha The third investment, find the amount invested at each rate.

26. A woman has \$16,750 to invest. She decides to invest the largest portion at 8.25% in a safe investment She invests the smallest portion at 12% in a high-risk investment and the rest at 10%. In fact, the safe invest-ment is only \$750 less than the other two combined. Determine the amount invested at each rate, given the the total interest income is \$1,605.

27. Three machine parts cost a total of \$40. The first part costs as much as the other two together, while the Cost of 6 times the second is \$2 more than the total cost of the other two. Find the cost of each part.

28. The combined ages of three brothers, Richard, Paul, and Craig, total 24 years. Three years ago Rich-ard's age was one sixth of his brothers' combined present ages, while Paul's age was one fourth of wha his brothers' combined ages will be two years from now. Find their respective ages.

In Exercises 29–35, find the currents in each of the circuits by solving the system of equations given in each case.

29. $I_1 + I_2 - I_3 = 0$
$I_1 - 2I_2 = 4$
$2I_2 + 3I_3 = 2$

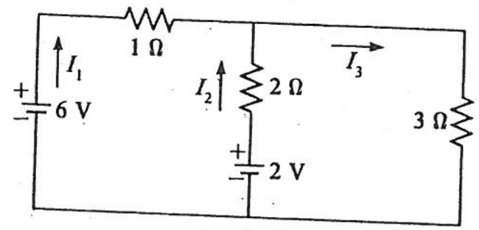


Figure 3.14

30. $I_1 - I_2 + I_3 = 0$
$2I_1 + 3I_2 = 2$
$-3I_2 - I_3 = -4$

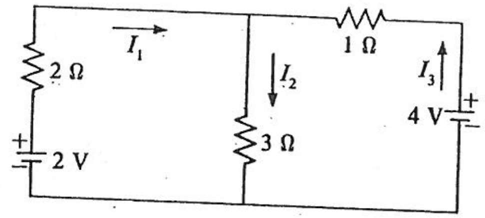


Figure 3.15

31. $I_1 - I_2 + I_3 = 0$
$I_1 + 2I_2 = 10$
$-2I_2 - I_3 = -5$

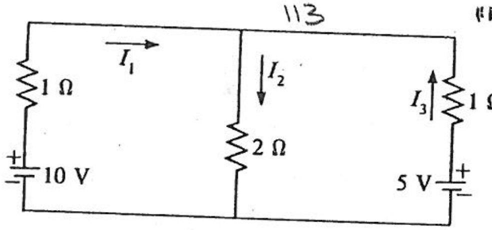


Figure 3.16

32. $-I_1 + I_2 + I_3 = 0$
$-2I_1 - 5I_2 = -20$
$5I_2 - 10I_3 = 10$

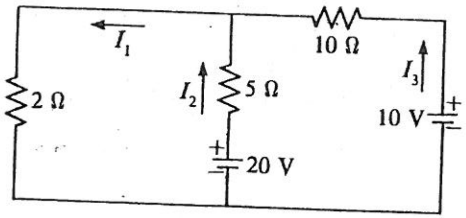


Figure 3.17

3. $-I_1 + I_2 + I_3 = 0$
$-I_1 - 3I_2 = -10$
$3I_2 - 5I_3 = -6$

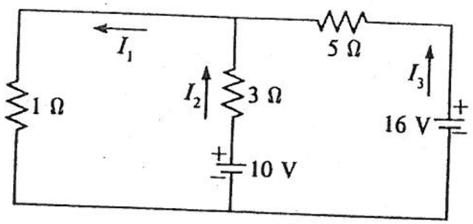


Figure 3.18

. $I_1 - I_2 - I_3 = 0$
$3I_1 + 4I_2 = -2$
$-4I_2 + 3I_3 = 2$

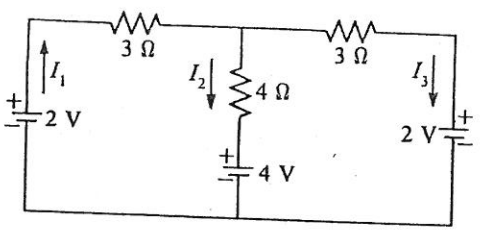


Figure 3.19

35. $I_1 + I_2 + I_3 - I_4 = 0$
$2I_1 - I_2 = 1$
$I_2 - 3I_3 = 3$
$3I_3 + I_4 = -2$

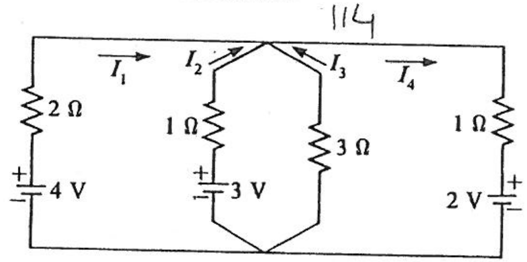


Figure 3.20

36. Solve the system of equations in Example 6.

Review Exercises / Chapter 3

In Exercises 1–8, evaluate each determinant.

1. $\begin{vmatrix} -2 & 1 \\ 5 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 10 \\ -5 & -3 \end{vmatrix}$

3. $\begin{vmatrix} -8 & 0 \\ 30 & -20 \end{vmatrix}$

4. $\begin{vmatrix} 1 & -7 \\ -10 & 1 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 3 & -2 \\ 0 & 0 & -3 \\ 1 & 4 & 5 \end{vmatrix}$

6. $\begin{vmatrix} 2 & 7 & 1 \\ 3 & -2 & 0 \\ 7 & 8 & -10 \end{vmatrix}$

7. $\begin{vmatrix} 5 & 6 & -1 \\ -4 & 7 & 2 \\ -3 & 0 & 5 \end{vmatrix}$

8. $\begin{vmatrix} 7 & -2 & -11 \\ 10 & 4 & 13 \\ 12 & 6 & 15 \end{vmatrix}$

In Exercises 9–12, solve each system of equations graphically, giving the values to the nearest tenth of a

9. $2x - y = -5$
$2x + 3y = 3$

10. $3x - 6y = 1$
$6x - 3y = 5$

11. $2x - y = 3$
$2x - 5y = -5$

12. $2x - y = 7$
$4x - y = 11$

In Exercises 13–16, solve each system by the method of substitution.

13. $x - 3y = 1$
$3x + 2y = 4$

14. $2x - y = 9$
$x + 2y = 7$

15. $-4x + 2y = 7$
$x - 3y = 2$

16. $5x - 4y = 5$
$2x - y = 1$

In Exercises 17–20, solve each system by the method of addition or subtraction.

17. $3x + 2y = 5$
$x - y = 2$

18. $2x - 4y = 2$
$3x - 5y = 4$

19. $4x + 3y = 9$
$5x + 6y = 12$

20. $2x + 6y = 17$
$-4x + 7y = 23$

115 116

In Exercises 21–24, solve each system by the method of determinants.

21. $5x - 7y = 8$
$4x + 2y = 3$

22. $-x + 3y = 2$
$2x + 4y = 3$

23. $2x - 2y = -7$
$2x + 3y = 6$

24. $x - 3y = 5$
$3x + 2y = 2$

In Exercises 25–36, solve the given systems by any method.

25. $x - 3y = 5$
$2x - y = 5$

26. $\frac{1}{x} - \frac{3}{y} = 4$
$\frac{2}{x} - \frac{1}{y} = 5$

27. $-x + 2y = 3$
$2x + 4y = 5$

28. $3x - 4y = 1$
$-2x + 6y = 1$

29. $\frac{5}{x} - \frac{4}{y} = 9$
$\frac{2}{x} - \frac{1}{y} = 11$

30. $\frac{3}{x} + \frac{2}{y} = 3$
$\frac{4}{x} + \frac{7}{y} = 5$

11. $s_1 + 3s_2 = 14$
$2s_1 - s_2 = 0$

32. $2T_1 - T_2 = 0$
$3T_1 + 2T_2 = 7$

33. $2d_1 + 3d_2 = 13$
$d_1 + 2d_2 = 8$

14. $2C_1 + 2C_2 = 9$
$4C_1 + 3C_2 = 14$

35. $2m + 3n = 8$
$4m + 3n = 7$

36. $r + 2s = 0$
$12r - 6s = -5$

n Exercises 37–40, solve the given systems algebraically.

7. $x - y = 2$
$2x + 3z = 11$
$3x + 2y + 4z = 13$

38. $2w_1 + 2w_2 = 3$
$-w_2 + w_3 = 1$
$4w_1 + 2w_3 = 4$

39. $\frac{1}{y} + \frac{2}{z} = 7$
$\frac{2}{x} + \frac{3}{z} = 5$
$\frac{3}{x} + \frac{4}{y} - \frac{1}{z} = -5$

). $\frac{3}{x} - \frac{1}{y} = 3$
$-\frac{4}{y} + \frac{2}{z} = 7$
$\frac{1}{y} - \frac{1}{z} = -1$

Exercises 41–44, solve the given systems by using Cramer's rule.

• $V_1 - V_2 + 4V_3 = -12$
$3V_1 + 2V_2 = 3$
$-2V_1 + 3V_2 - 3V_3 = 17$

42. $a + b + c = 1$
$2a - b + 2c = 2$
$a - b - 4c = 3$

43. $\frac{1}{x} - \frac{1}{y} - \frac{2}{z} = 3$
$\frac{2}{x} - \frac{4}{z} = 5$
$\frac{1}{x} - \frac{3}{y} + \frac{2}{z} = 2$

44. $T_1 - T_2 = 6$
$2T_1 - 2T_2 + T_3 = 5$
$3T_1 - T_2 + 2T_3 = 1$

45. Determine the value of a that makes the system

$2x + ay = 3$
$4x - 2y = 5$

inconsistent.

46. Show that the system

$x + 2y + 3z = 3$
$4x + 5y + 6z = 1$
$7x + 8y + 9z = 2$

cannot have a unique solution.

47. A portion of \$1,014.80 is invested at 10%, another at 8%, and the rest at 12½%. The 12½% investment yields as much interest as the other two investments combined. Find the amount invested at each rate if the total interest income is \$103.20.

48. Experimenters found that the velocity (in feet per second) of two moving parts in a machine satisfy the relation

$\frac{1}{v_1} + \frac{3}{v_2} = 5$
$\frac{2}{v_1} + \frac{1}{v_2} = 6$

Determine v_1 and v_2.

49. Two kinds of milk containing 1% butterfat and 4% butterfat by volume, respectively, are to be mixed to obtain 90 gal of milk containing 2% butterfat. Determine the number of gallons of each required.

50. The perimeter of a triangle is 14 in. The longest side is twice as long as the shortest side and 2 in. less than the sum of the two shorter sides. Find the length of each side.

In Exercises 51–53, find the currents in the given diagrams.

51. $I_1 - I_2 - I_3 = 0$
$2I_1 + I_2 = 3$
$-I_2 + 3I_3 = -1$

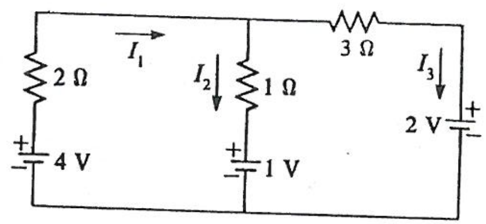


Figure 3.21

2. $I_1 - I_2 + I_3 = 0$
$2I_1 + 4I_2 = -2$
$-4I_2 - 2I_3 = 1$

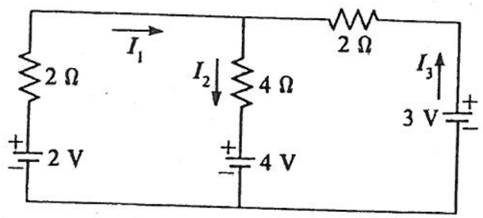


Figure 3.22

. $I_1 + I_2 + I_3 - I_4 = 0$
$I_1 - 3I_2 = 3$
$3I_2 - 4I_3 = -3$
$4I_3 + I_4 = -1$

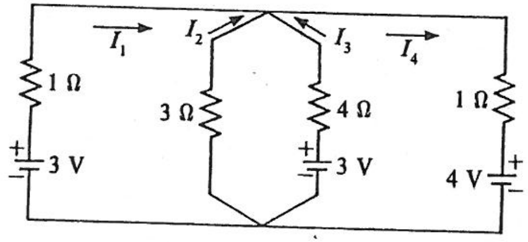


Figure 3.23

. An experimenter determined that the lever in the figure balances if the weights are positioned as fo[llows] as follows:
(1) $d_1 = 3$ in., $d_2 = 2$ in., $d_3 = 1$ in., $d = 2\frac{1}{4}$ in.
(2) $d_1 = 2$ in., $d_2 = 1$ in., $d_3 = 3$ in., $d = 2\frac{1}{2}$ in.
(3) $d_1 = 3$ in., $d_2 = 2$ in., $d_3 = 2$ in., $d = 2\frac{3}{4}$ in.
Find the weights.

Figure 3.24

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The distance to the moon is approximately 239,000 miles. Write the distance in scientific notation, using significant figures.

A rectangular computer chip is 3 times as long as it is wide. If the perimeter is 9.6 mm, find the dimensi[ons] dimensions.

The force exerted by a spring is directly proportional to the extension. If a force of 2.72 lb stretches a sp[ring] 1.70 in., determine the force required to stretch it 4.10 in.

Cumulative Review Exercises / Chapters 1–3 118

1. Subtract $5T_a + 2T_b - 3T_c$ from the sum of $-9T_a - 2T_b + 6T_c$ and $7T_a - 6T_b - 7T_c$.
2. Simplify: $-\{L - [(-L - C) - C] - 4L\}$
3. Write $\sqrt{150}$ in its simplest radical form.
4. Simplify by rationalizing the denominator: $\frac{L}{2\sqrt{\pi}}$
5. Simplify: $\frac{3p^{-2}q^{-1}}{p^{-6}q^{-4}}$
6. Simplify: $\frac{28A^2(-AB)^3}{-7A^3B^3}$
7. Perform the following multiplication:
$(a - 2b) \cdot (a^2 - 3ab - 2b^2)$
8. Perform the following division:
$(2x^3 - 2x - 12) \div (2x - 4)$
9. Convert the numbers to scientific notation before multiplying: $(0.000000721) \cdot (0.000000089)$
10. Solve for x: $2(2 - 3x) = 5(x - 5)$
11. Solve for x: $\frac{1}{4}x - \frac{1}{2} = \frac{1}{3}x - 2$
12. Solve for t_1: $L = L_1[1 + \beta(t_2 - t_1)]$
13. State the domain of the function $f(x) = \sqrt{4 - x}$.
14. If $f(x) = \sqrt{x^2 + 2}$, find $f(0)$ and $f(\sqrt{2})$.
15. Solve the following system graphically:
$2x - 3y = 4$
$x + 2y = 2$
16. Solve the following system algebraically:
$2r_1 - 3r_2 = 7$
$3r_1 - 5r_2 = 3$

Solve the following system by determinants:
$3x + 2y - z = 16$
$2x + 3y + 4z = 7$
$8x + 5y - 6z = 47$

18. The respective currents in milliamperes (mA) in the two loops of a certain circuit are $1.00I_1 -$ and $5.00I_1 - 3.00I_2$ are $1.00I_1 - 2.00I_2$. Find the sum of the currents.
19. Simplify the following expression from a problem in statics:
$-[F_1 + 2F_2 - 4(F_1 + F_2)]$

Chapter 3

Section 3.1 (page 84)

1. (−1.0, −3.0) **3.** (2.0, 3.0) **5.** (5.8, −3.3) **7.** (−0.2, 0.8) **9.** (2.7, 2.3) **11.** (2.9, −8.6)

Section 3.2 (page 90)

1. (3, 1) **3.** (−5, −8) **5.** (3, −6) **7.** $\left(-\frac{1}{2}, 3\right)$ **9.** $\left(\frac{7}{20}, \frac{3}{20}\right)$ **11.** (1, −1) **13.** (3, 3) **15.** (−1, 3) **17.** $\left(-1, \frac{1}{2}\right)$ **19.** $\left(\frac{5}{3}, -\frac{8}{3}\right)$ **21.** (0, −7) **23.** inconsistent **25.** (1, 1) **27.** $\left(\frac{5}{4}, 5\right)$ **29.** $\left(-2, -\frac{1}{2}\right)$ **31.** $\left(-\frac{1}{3}, 3\right)$ **33.** (0, 1) **35.** dependent **37.** inconsistent **39.** (1, 2)

Section 3.3 (page 95)

1. 0 **3.** 0 **5.** 1 **7.** −10 **9.** 2 **11.** −69 **13.** 869 **15.** 450 **17.** (−13, 10) **19.** (22, 31) **21.** (−83, −68) **23.** $\left(-3, \frac{17}{3}\right)$ **25.** $\left(\frac{13}{44}, -13\right)$ **27.** $\left(-\frac{19}{12}, -\frac{19}{68}\right)$ **29.** (20, 10) **31.** $\left(\frac{7}{3}, \frac{4}{3}\right)$ **33.** $\left(2, \frac{7}{2}\right)$ **35.** inconsistent **37.** (2.43, 1.07) **39.** (−2.32, −0.974) **41.** (0.669, −1.28)

Section 3.4 (page 99)

1. $w_1 = 2$ lb, $w_2 = 1$ lb **3.** $w_1 = 11.00$ lb, $w_2 = 9.000$ lb **5.** $S = 566 + 0.002007T$ **7.** 70 Ω, 80 Ω **9.** $4,110 at 12%, $4,390 at 11% **11.** 5 dozen at $35/dozen, 9 dozen at $50/dozen **13.** 20 at $1.95, 60 at $2.50 **15.** 30.1 V, 25.0 V **17.** 190, 25 **19.** 8 L of 5% solution **21.** 47, 15 **23.** 19 dimes, 28 nickels **25.** 7 days at $250, 6 days at $200

Section 3.5 (page 103)

1. (1, 2, 3) **3.** (1, 1, −2) **5.** (1, −1, 2) **7.** $\left(2, -\frac{3}{2}, \frac{7}{2}\right)$ **9.** $\left(\frac{2}{3}, -\frac{5}{3}, 4\right)$ **11.** $\left(\frac{1}{6}, \frac{7}{20}, -\frac{7}{25}\right)$ **13.** (1, 2, −2, 3) **15.** $\left(-\frac{1}{2}, 2, 3, -1\right)$

Section 3.6 (page 111)

1. 4 **3.** −55 **5.** 115 **7.** −234 **9.** 940 **11.** −129 **13.** (1, 2, 3) **15.** (1, 1, −2) **17.** (1, −1, 2) **19.** $\left(2, -\frac{3}{2}, \frac{7}{2}\right)$ **21.** $\left(-\frac{8}{19}, -\frac{12}{19}, -\frac{1}{19}\right)$ **23.** $w_1 = 1$ lb, $w_2 = 1$ lb, $w_3 = 2$ lb **25.** $1,200 at 8%, $1,550 at 10%, $3,200 at 12% **27.** $20, $6, $14 **29.** $I_1 = \frac{24}{11}$ A, $I_2 = -\frac{10}{11}$ A, $I_3 = \frac{14}{11}$ A **31.** $I_1 = 4$ A, $I_2 = 3$ A, $I_3 = -1$ A **33.** $I_1 = \frac{98}{23}$ A, $I_2 = \frac{44}{23}$ A, $I_3 = \frac{54}{23}$ A **35.** $I_1 = \frac{13}{17}$ A, $I_2 = \frac{9}{17}$ A, $I_3 = -\frac{14}{17}$ A, $I_4 = \frac{8}{17}$ A

Review Exercises for Chapter 3 (page 114)

1. −7 **3.** 160 **5.** 3 **7.** 238 **9.** (−1.5, 2.0) **11.** (2.5, 2.0) **13.** $\left(\frac{14}{11}, \frac{1}{11}\right)$ **15.** $\left(-\frac{5}{2}, -\frac{3}{2}\right)$ **17.** $\left(\frac{9}{5}, -\frac{1}{5}\right)$ **19.** $\left(2, \frac{1}{3}\right)$ **21.** $\left(\frac{37}{38}, -\frac{17}{38}\right)$ **23.** $\left(-\frac{9}{10}, \frac{13}{5}\right)$ **25.** (2, −1)

27. $\left(-\frac{1}{4}, \frac{11}{8}\right)$ **29.** $\left(\frac{3}{35}, \frac{3}{37}\right)$ **31.** $(2, 4)$ **33.** $(2, 3)$ **35.** $\left(-\frac{1}{2}, 3\right)$ **37.** $(1, -1, 3)$
39. $\left(-\frac{1}{2}, 1, \frac{1}{3}\right)$ **41.** $(-1, 3, -2)$ **43.** $\left(\frac{2}{3}, -2, -2\right)$ **45.** $a = -1$
47. \$172 at 10%, \$430 at 8%, \$412.80 at $12\frac{1}{2}$% **49.** 60 gal of 1% milk
51. $I_1 = 1$ A, $I_2 = 1$ A, $I_3 = 0$ A **53.** $I_1 = 0$ A, $I_2 = -1$ A, $I_3 = 0$ A, $I_4 = -1$ A

Cumulative Review Exercises for Chapters 1–3 (page 118)

1. $-7T_a - 10T_b + 2T_c$ **2.** $2L - 2C$ **3.** $5\sqrt{6}$ **4.** $L\sqrt{\pi}/(2\pi)$ **5.** $3p^4q^3$ **6.** $4A^2$
7. $a^3 - 5a^2b + 4ab^2 + 4b^3$ **8.** $x^2 + 2x + 3$ **9.** 6.4×10^{-14} **10.** $x = \frac{29}{11}$ **11.** $x = 18$
12. $t_1 = \frac{L_1 + L_1\beta t_2 - L}{L_1\beta}$ **13.** $x \le 4$ **14.** $\sqrt{2}, 2$ **15.** $(2, 0)$ **16.** $(26, 15)$ **17.** $\left(\frac{7}{2}, 2, -\frac{3}{2}\right)$
18. $6.00I_1 - 5.00I_2$ **19.** $3F_1 + 2F_2$ **20.** 2.4×10^5 miles **21.** 1.2 mm by 3.6 mm **22.** 6.56 lb

Chapter 4

Section 4.2 (page 123)

3. 205° **5.** 287°20′ **7.** 180°3′ **9.** 213°55′ **11.** 178°6′ **13.** 34°19′ **15.** 355°

Section 4.3 (page 129)

For Exercises 1–19, the trigonometric functions are given in the following order: sin θ, cos θ, tan θ, csc θ, sec θ, and cot θ.

1. $\frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}, \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, 2, \frac{\sqrt{5}}{2}, \sqrt{5}, \frac{1}{2}$ **3.** $\frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}, \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}, \frac{2}{5}, \frac{\sqrt{29}}{2}, \frac{\sqrt{29}}{5}, \frac{5}{2}$
5. $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}, \frac{1}{2}, \sqrt{5}, \frac{\sqrt{5}}{2}, 2$ **7.** $\frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}, \frac{1}{3}, \frac{4}{\sqrt{2}} = 2\sqrt{2}, \frac{3\sqrt{2}}{4}, 3, \frac{\sqrt{2}}{4}$
9. $\frac{\sqrt{6}}{\sqrt{42}} = \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}, \frac{6}{\sqrt{42}} = \frac{\sqrt{42}}{7}, \frac{\sqrt{6}}{6}, \sqrt{7}, \frac{\sqrt{42}}{6}, \sqrt{6}$
11. $\frac{\sqrt{10}}{\sqrt{59}} = \frac{\sqrt{590}}{59}, \frac{7\sqrt{59}}{59}, \frac{\sqrt{10}}{7}, \frac{\sqrt{590}}{10}, \frac{\sqrt{59}}{7}, \frac{7\sqrt{10}}{10}$ **13.** $\frac{\sqrt{3}}{6}, \frac{\sqrt{33}}{6}, \frac{\sqrt{11}}{11}, 2\sqrt{3}, \frac{2\sqrt{33}}{11}, \sqrt{11}$
15. $\frac{\sqrt{10}}{4}, \frac{\sqrt{6}}{4}, \frac{\sqrt{15}}{3}, \frac{2\sqrt{10}}{5}, \frac{2\sqrt{6}}{3}, \frac{\sqrt{15}}{5}$ **17.** $\frac{\sqrt{7}}{5}, \frac{3\sqrt{2}}{5}, \frac{\sqrt{14}}{6}, \frac{5\sqrt{7}}{7}, \frac{5\sqrt{2}}{6}, \frac{3\sqrt{14}}{7}$
19. $\frac{2\sqrt{142}}{71}, \frac{3\sqrt{497}}{71}, \frac{2\sqrt{14}}{21}, \frac{\sqrt{142}}{4}, \frac{\sqrt{497}}{21}, \frac{3\sqrt{14}}{4}$ **21.** $\frac{3}{4}$ **23.** $\frac{\sqrt{3}}{2}$ **25.** $\frac{\sqrt{34}}{5}$ **27.** $\frac{4}{9}$ **29.** $\frac{6}{7}$
31. $\frac{4\sqrt{19}}{19}$ **33.** $\frac{\sqrt{6}}{3}$ **35.** $\frac{\sqrt{33}}{6}$ **37.** $\sqrt{10}$ **39.** $\sqrt{10}$

Section 4.4 (page 137)

1. $\frac{\sqrt{3}}{3}$ **3.** $\frac{\sqrt{3}}{2}$ **5.** 1 **7.** 2 **9.** 1 **11.** $\frac{2\sqrt{3}}{3}$ **13.** 1 **15.** $\sqrt{2}$ **17.** $\frac{1}{2}$
19. $\frac{\sqrt{2}}{2}$ **21.** 45° **23.** 45° **25.** 30° **27.** 45° **29.** 0° **31.** 30° **33.** 60° **35.** 90°
37. 45° **39.** 30° **41.** 0.1851 **43.** 1.319 **45.** 11.10 **47.** 1.217 **49.** 1.047
51. 3.030 **53.** 0.3169 **55.** 0.9960 **57.** 0.7950 **59.** 4.843 **61.** 20°20′ **63.** 51°10′
65. 85°38′ **67.** 81°25′ **69.** 6°5′ **71.** 38°9′ **73.** 18°38′ **75.** 46°15′ **77.** 45°38′
79. 82°28′ **81.** **a.** $W = 16.1$ g **b.** $W = 17.1$ g **c.** $W = 19.3$ g **d.** $W = 23.4$ g **e.** $W = 25.1$ g
83. 46.1 V