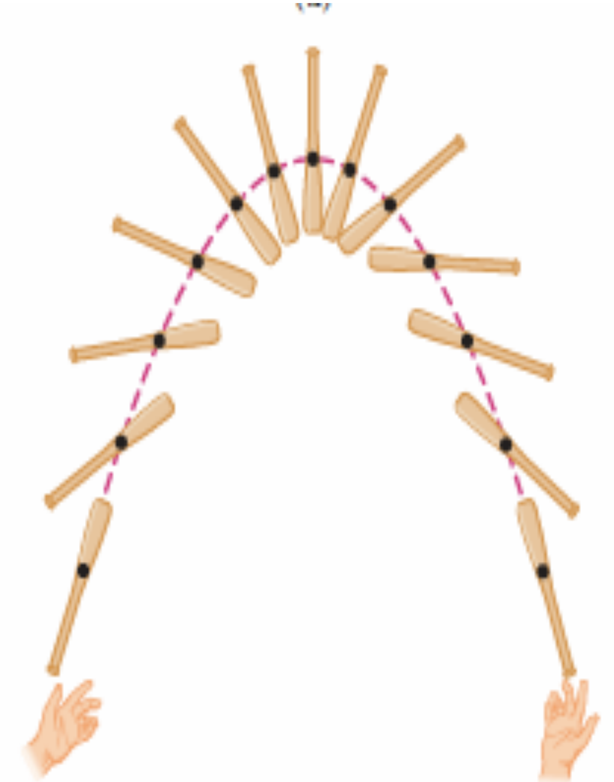
Chapter 9

- Center of mass
- Linear momentum
- Collision and impulse
- Conservation of linear momentum
- Momentum and kinetic energy in collision
- Inelastic collision in one dimension
- Elastic collision in one dimension

Center of mass for system of particles:



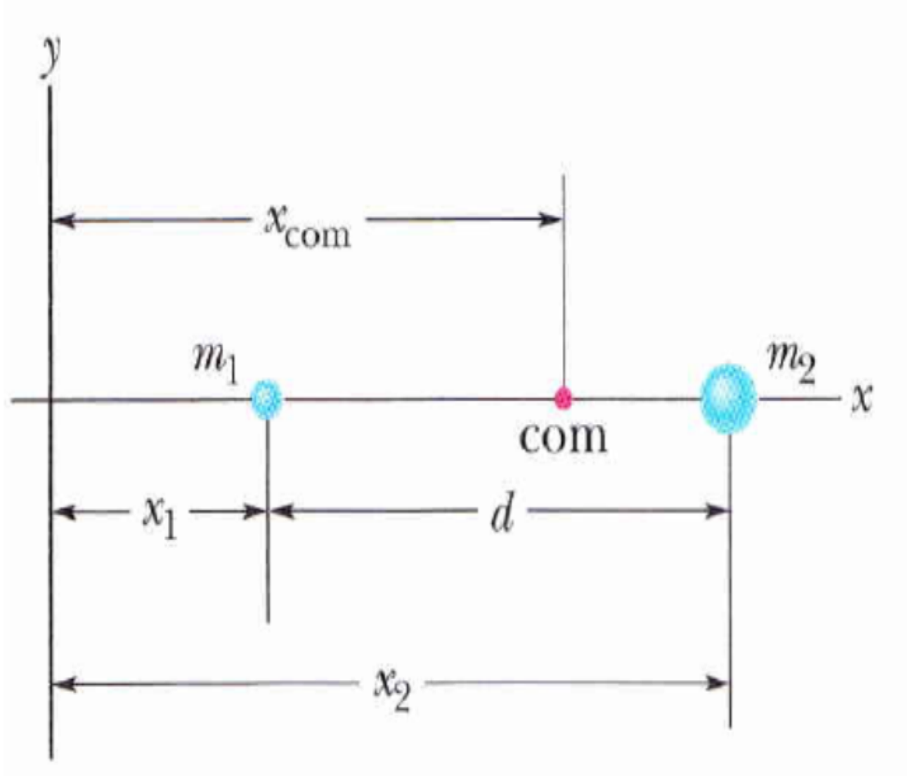
The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

The center of mass of a system of two particles is defined to be the point whose coordinates are given by

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{M},$$

where,
M$=m_1+m_2$



Then the center of mass of a system of n particles is generally defined to be:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i. \tag{9-5}$$

$$\vec{r}_i = x_i\hat{\mathrm{i}} + y_i\hat{\mathrm{j}} + z_i\hat{\mathrm{k}}.$$

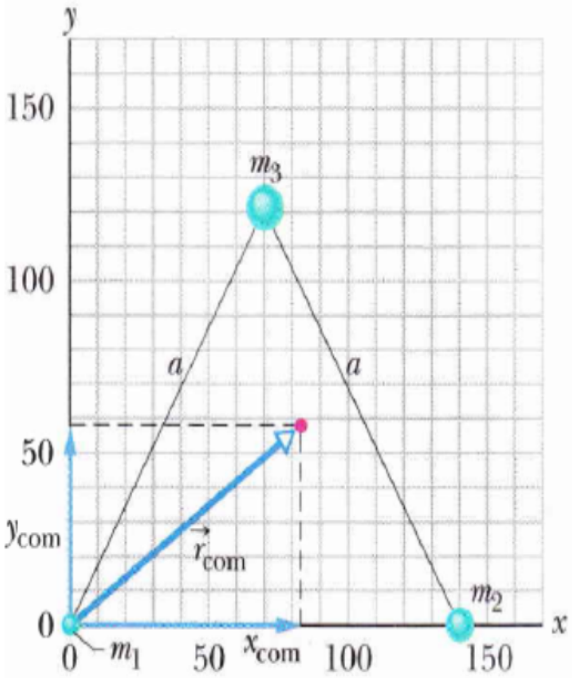
$$\vec{r}_{com} = x_{com}\hat{\mathrm{i}} + y_{com}\hat{\mathrm{j}} + z_{com}\hat{\mathrm{k}}.$$

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i,$$

Sample Problem

Three particles of masses $m_1 = 1.2$ kg, $m_2 = 2.5$ kg, and $m_3 = 3.4$ kg form an equilateral triangle of edge length $a = 140$ cm. Where is the center of mass of this system?

Particle	Mass (kg)	x (cm)	y (cm)
1	1.2	0	0
2	2.5	140	0
3	3.4	70	120



The total mass M of the system is 7.1 kg.

$$x_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M}$$
$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(140\text{ cm}) + (3.4\text{ kg})(70\text{ cm})}{7.1\text{ kg}}$$
$$= 83\text{ cm} \qquad \text{(Answer)}$$

$$y_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{M}$$
$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(0) + (3.4\text{ kg})(120\text{ cm})}{7.1\text{ kg}}$$
$$= 58\text{ cm}. \qquad \text{(Answer)}$$

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i,$$

$$M\vec{r}_{com} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n,$$

Differentiate this equation with respect to time gives

$$M\vec{v}_{com} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n.$$

Differentiate again with respect to time give

$$M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n.$$

From Newton's second law, $m_i\vec{a}_i$ is equal to the resultant force $\vec{F}_i$ that acts on the ith particle. Thus,

$$M\vec{a}_{com} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n.$$

$$\vec{F}_{net} = M\vec{a}_{com} \quad \text{(system of particles).}$$

Sample Problem 9-3

The three particles in Fig. 9-7*a* are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 6.0$ N, $F_2 = 12$ N, and $F_3 = 14$ N. What is the acceleration of the center of mass of the system, and in what direction does it move?

KEY IDEAS The position of the center of mass, calculated by the method of Sample Problem 9-1, is marked by a dot in the figure. We can treat the center of mass as if it were a real particle, with a mass equal to the system's total mass $M = 16$ kg. We can also treat the three external forces as if they act at the center of mass

Calculations: We can now apply Newton's second law ($\vec{F}_{net} = m\vec{a}$) to the center of mass, writing

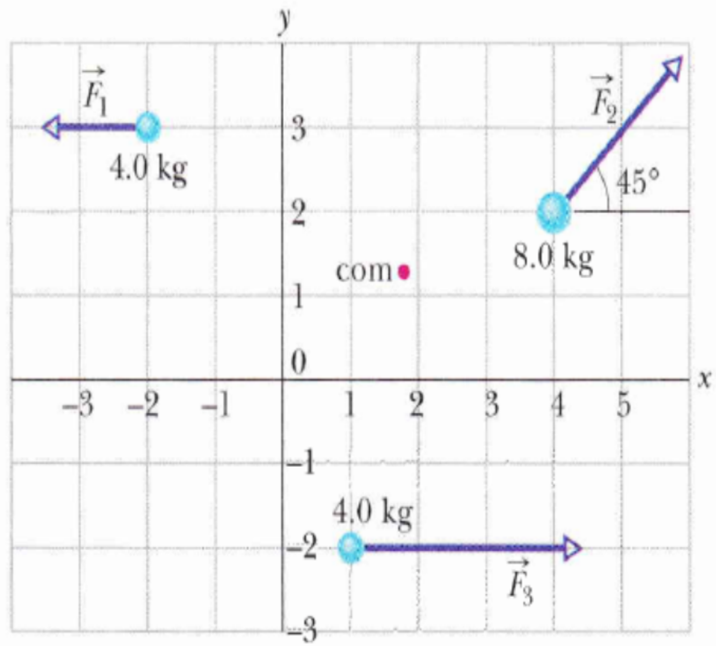
$$\vec{F}_{net} = M\vec{a}_{com} \tag{9-20}$$

or

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{com}$$

so

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}. \tag{9-21}$$



(*a*)

Sample Problem 9-3

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Calculations: We can now apply Newton's second law ($\vec{F}_{net} = m\vec{a}$) to the center of mass, writing

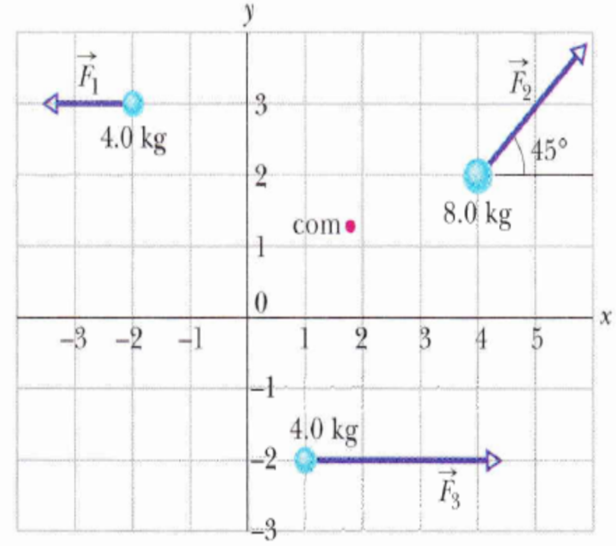
$$\vec{F}_{net} = M\vec{a}_{com} \qquad (9\text{-}20)$$

or

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{com}$$

so

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}. \qquad (9\text{-}21)$$



(*a*)

Along the x axis, we have

$$a_{com,x} = \frac{F_{1x} + F_{2x} + F_{3x}}{M}$$

$$= \frac{-6.0\text{ N} + (12\text{ N})\cos 45^\circ + 14\text{ N}}{16\text{ kg}} = 1.03\text{ m/s}^2.$$

Along the y axis, we have

$$a_{com,y} = \frac{F_{1y} + F_{2y} + F_{3y}}{M}$$

$$= \frac{0 + (12\text{ N})\sin 45^\circ + 0}{16\text{ kg}} = 0.530\text{ m/s}^2.$$

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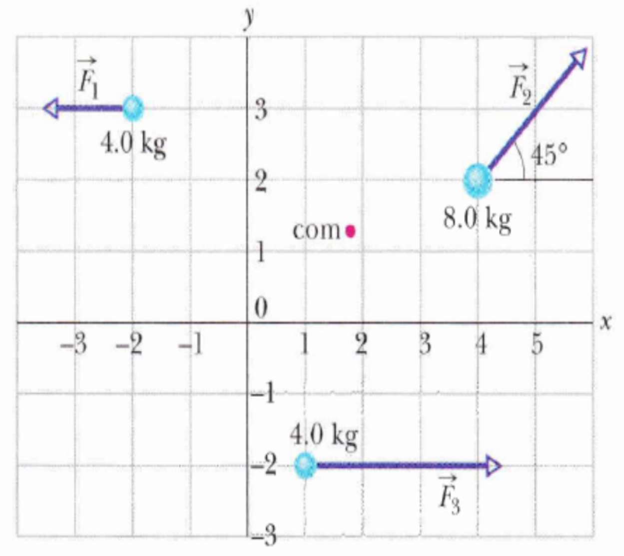
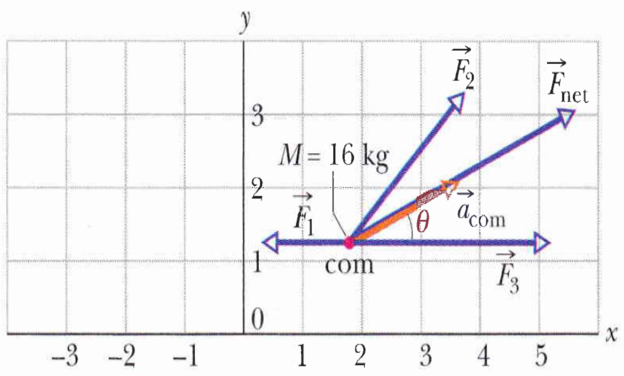
$$a_{com} = \sqrt{(a_{com,x})^2 + (a_{com,y})^2} = 1.16\text{ m/s}^2 \approx 1.2\text{ m/s}^2 \qquad \text{(Answer)}$$

$$\theta = \tan^{-1}\frac{a_{com,y}}{a_{com,x}} = 27°.$$



(a)



(b)

Linear Momentum:

Linear momentum *P* of a body of mass m and velocity *v* is defined as

$$\vec{p} = m\vec{v}$$

The Linear Momentum of a System of Particles

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n$$
$$= m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n.$$

$$\vec{P} = M\vec{v}_{com} \quad \text{(linear momentum, system of particles),}$$

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{com}}{dt} = M\vec{a}_{com}.$$

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \quad \text{(system of particles),}$$

Sample Problem 9-8

Two-dimensional explosion: A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor. An overhead view is shown in Fig. 9-14*a*. Piece C, with mass $0.30M$, has final speed $v_{fC} = 5.0$ m/s.

(a) What is the speed of piece B, with mass $0.20M$?

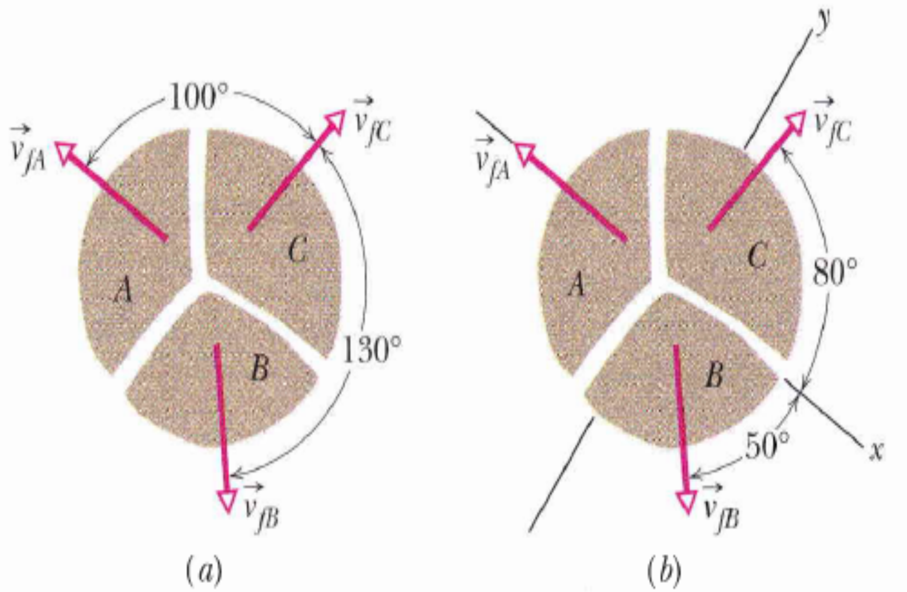


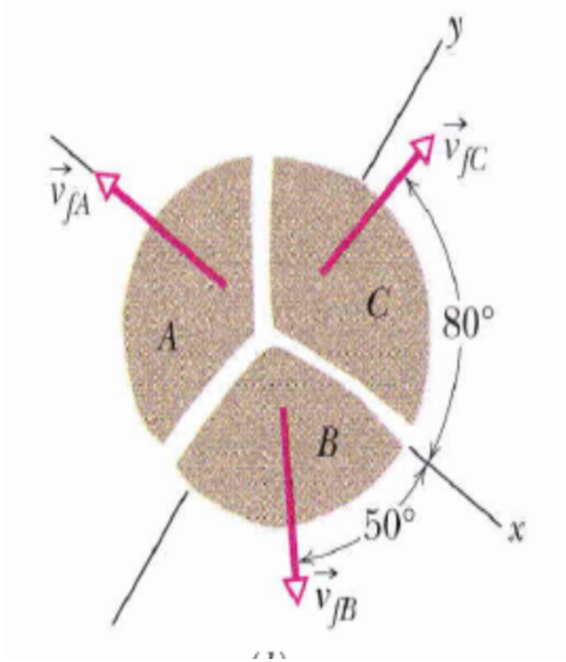
FIG. 9-14 Three pieces of an exploded coconut move off in three directions along a frictionless floor. (*a*) An overhead view of the event. (*b*) The same with a two-dimensional axis system imposed.

KEY IDEA First we need to see whether linear momentum is conserved. We note that (1) the coconut and its pieces form a closed system, (2) the explosion forces are internal to that system, and (3) no net external force acts on the system. Therefore, the linear momentum of the system is conserved.

Linear momentum is conserved separately along each axis. Let's use the y axis and write

$$P_{iy} = P_{fy},$$

$$P_{iy} = P_{fy} = p_{fA,y} + p_{fB,y} + p_{fC,y}.$$



$$p_{fA,y} = 0,$$

$$p_{fB,y} = -0.20Mv_{fB,y} = -0.20Mv_{fB}\sin 50°,$$

$$p_{fC,y} = 0.30Mv_{fC,y} = 0.30Mv_{fC}\sin 80°.$$

Then, with $v_{fC} = 5.0$ m/s, we have

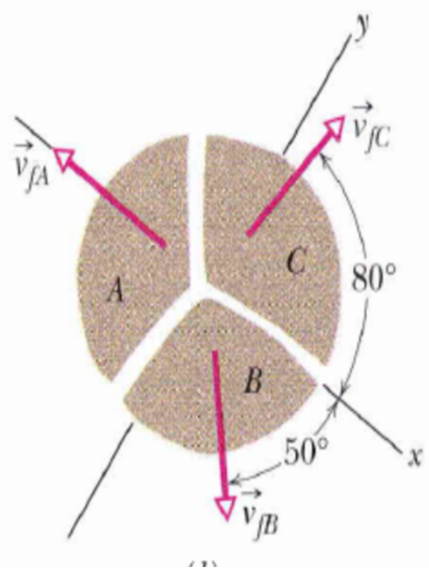
$$0 = 0 - 0.20Mv_{fB}\sin 50° + (0.30M)(5.0\text{ m/s})\sin 80°,$$

$$v_{fB} = 9.64\text{ m/s} \approx 9.6\text{ m/s}. \qquad \text{(Answer)}$$

(b) What is the speed of piece A?

Calculations: Because linear momentum is also conserved along the x axis, we have

$$P_{ix} = P_{fx}, \qquad (9\text{-}49)$$



$$p_{fA,x} = -0.50Mv_{fA},$$
$$p_{fB,x} = 0.20Mv_{fB,x} = 0.20Mv_{fB}\cos 50°,$$
$$p_{fC,x} = 0.30Mv_{fC,x} = 0.30Mv_{fC}\cos 80°.$$

$$P_{ix} = P_{fx} = p_{fA,x} + p_{fB,x} + p_{fC,x}.$$

$$0 = -0.50Mv_{fA} + 0.20M(9.64\text{ m/s})\cos 50° + 0.30M(5.0\text{ m/s})\cos 80°,$$

$$v_{fA} = 3.0\text{ m/s}. \qquad \text{(Answer)}$$

Collision and impulse:

Let us study the time rate of change of linear momentum: $\frac{d\vec{p}}{dt}$

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

But *m* is constant

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt}$$

$$\frac{d\vec{p}}{dt} = m\vec{a}$$

Newton's second law tells us

$$\vec{F}_{net} = m\vec{a}$$

Then

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

Newton expressed his second law of motion in terms of momentum:

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

A force F acts on a particle in time interval Δt (from t_1 to t_2), we define the Impulse J

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{net} dt$$

If force F acts on a particle is not known we can estimate the average force from the relation

$$\vec{J} = \vec{F}_{ave} \Delta t$$

If force F acts on a particle is constant then

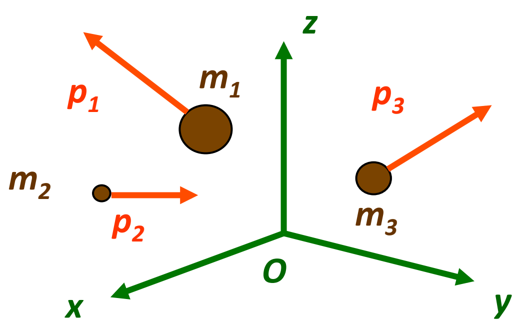
$$\vec{J} = \vec{F}_{net} \Delta t$$ Constant force only

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{net}\, dt$$

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

Thus, the change in an object’s momentum is equal to the impulse on the object

* SI unit for impulse is: N·s



Conservation of Linear Momentum :

Consider a system of particles for which $\vec{F}_{\text{net}} = 0$

$$\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}} = 0 \rightarrow \vec{P} = \text{ Constant}$$

If no net external force acts on a system of particles, the total linear momentum $\vec{P}$ cannot change.

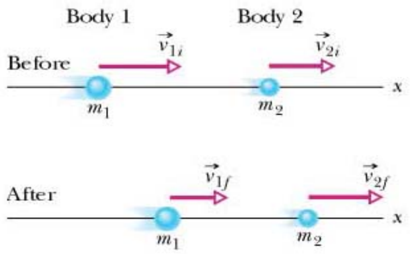
$$\begin{bmatrix} \text{total linear momentum} \\ \text{at some initial time } t_i \end{bmatrix} = \begin{bmatrix} \text{total linear momentum} \\ \text{at some later time } t_f \end{bmatrix}$$

The conservation of linear momentum is an important principle in physics. It also provides a powerful rule we can use to solve problems in mechanics such as collisions.

Note 1: In systems in which $\vec{F}_{\text{net}} = 0$ we can always apply conservation of linear momentum even when the internal forces are very large as in the case of colliding objects.

Note 2: We will encounter problems (e.g., inelastic collisions) in which the energy is not conserved but the linear momentum is.

Momentum and Kinetic Energy in Collisions



Consider two colliding objects with masses m_1 and m_2, initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$, and final velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$, respectively.

If the system is isolated, i.e., the net force $\vec{F}_{\text{net}} = 0$, linear momentum is conserved. The conservation of linear momentum is true regardless of the collision type. This is a powerful rule that allows us to determine the results of a collision without knowing the details. Collisions are divided into two broad classes: elastic and inelastic.

A collision is elastic if there is no loss of kinetic energy, i.e., $K_i = K_f$.

A collision is inelastic if kinetic energy is lost during the collision due to conversion into other forms of energy. In this case we have $K_f < K_i$.

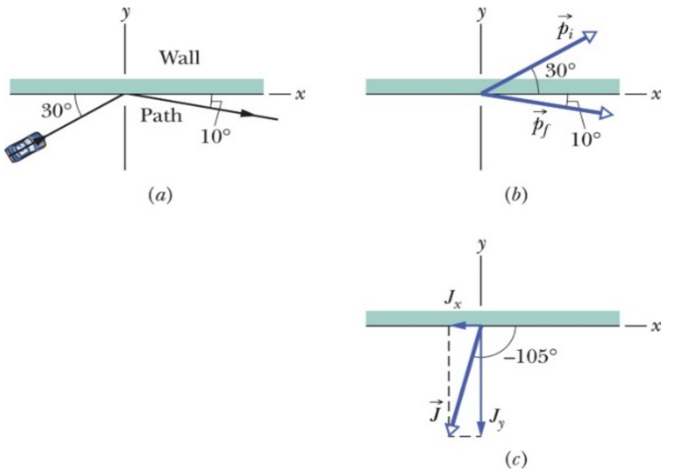
A special case of inelastic collisions are known as completely inelastic. In these collisions the two colliding objects stick together and they move as a single body. In these collisions the loss of kinetic energy is maximum.

Sample Problem:

Race-car wall collision. Figure 9-12a is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is travelling at speed v_i= 70 m/s along a straight line at 30° from the wall. Just after the collision, he is travelling t speed v_f 50m/s along a straight line at 10° from the wall. His mass m is 80 kg.

(a) What is the impulse J on the driver due to the collision?

(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?



$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i).$$

***x* component:** Along the x axis we have

$$\begin{aligned} J_x &= m(v_{fx} - v_{ix}) \\ &= (80\text{ kg})[(50\text{ m/s})\cos(-10°) - (70\text{ m/s})\cos 30°] \\ &= -910\text{ kg}\cdot\text{m/s}. \end{aligned}$$

***y* component:** Along the y axis,

$$\begin{aligned} J_y &= m(v_{fy} - v_{iy}) \\ &= (80\text{ kg})[(50\text{ m/s})\sin(-10°) - (70\text{ m/s})\sin 30°] \\ &= -3495\text{ kg}\cdot\text{m/s} \approx -3500\text{ kg}\cdot\text{m/s}. \end{aligned}$$

Impulse: The impulse is then

$$\vec{J} = (-910\hat{i} - 3500\hat{j})\text{ kg}\cdot\text{m/s}, \quad \text{(Answer)}$$

which means the impulse magnitude is

$$J = \sqrt{J_x^2 + J_y^2} = 3616\text{ kg}\cdot\text{m/s} \approx 3600\text{ kg}\cdot\text{m/s}.$$

The angle of $\vec{J}$ is given by

$$\theta = \tan^{-1}\frac{J_y}{J_x}, \qquad \text{(Answer)}$$

which a calculator evaluates as 75.4°. Recall that the

(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

Calculations: We have

$$F_{avg} = \frac{J}{\Delta t} = \frac{3616 \text{ kg} \cdot \text{m/s}}{0.014 \text{ s}}$$

$$= 2.583 \times 10^5 \text{ N} \approx 2.6 \times 10^5 \text{ N}. \quad \text{(Answer)}$$

Sample problem:

One-dimensional explosion: A ballot box with mass m = 6.0 kg slides with speed v = 4.0 m/s across a friction- less floor in the positive direction of an x axis The box explodes into two pieces. One piece, with mass m_1 = 2.0 kg, moves in the positive direction of the x axis at v_1 8.0 m/s. What is the velocity of the second piece, with mass m_2?

$$\vec{P}_i = m\vec{v}.$$

$$\vec{P}_f = \vec{P}_{f1} + \vec{P}_{f2} = m_1\vec{v}_1 + m_2\vec{v}_2.$$

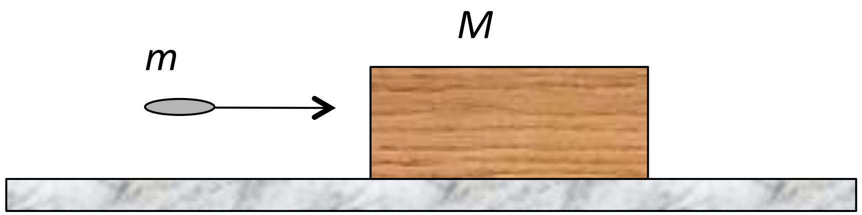
$$P_i = P_f$$

$$mv = m_1 v_1 + m_2 v_2.$$

$$(6.0\text{ kg})(4.0\text{ m/s}) = (2.0\text{ kg})(8.0\text{ m/s}) + (4.0\text{ kg})v_2$$

v_2 = 2 m/s

A 3.50 g bullet is fired horizontally at a block at rest on a frictionless table. The bullet embeds itself in the block (mass 2.5 kg). The block ends up with speed 1.5 m/s. Find the speed of the bullet as enters the block.



$p_{ix} = p_{bullet,i} + p_{block,i}$

$p_{ix} = m\,v + 0$

$p_{ix} = m\,v$

$p_{fx} = p_{bullet,f} + p_{block,f}$

$p_{fx} = mV + MV$

$p_{fx} = (m + M)V$

conservation of linear momentum

$$\boldsymbol{P}_i = \boldsymbol{P}_f$$

$$m\,v = (m + M)V$$

$$v = \frac{m + M}{m} V$$

v = ????? m/s

Example: A cart with mass 340 g moving on a frictionless linear air track at an initial speed of 1.2 m/s undergoes an elastic collision with an initially stationary cart of unknown mass. After the collision, the first cart continues in its original direction at 0.66 m/s. (a) What is the mass of the second cart? (b) What is its speed after impact?

Conservation of momentum

$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$ x - components

$0.34\text{ kg} \times (+1.2\text{ m/s}) + 0 = 0.34\text{ kg} \times (+0.66\text{ m/s}) + m_2 v_{2f}$

$$m_2 v_{2f} = 0.1836\text{ kg} \cdot \text{m/s}$$

Because the collision is ***elastic*** then the kinetic energy is conserved.

$\frac{1}{2} m_1 (v_{1i})^2 + \frac{1}{2} m_2 (v_{2i})^2 = \frac{1}{2} m_1 (v_{1f})^2 + \frac{1}{2} m_2 (v_{2f})^2$

$m_1 (v_{1i})^2 + m_2 (v_{2i})^2 = m_1 (v_{1f})^2 + m_2 (v_{2f})^2$

$0.34\text{ kg} \times (+1.2\text{ m/s})^2 + 0 = 0.34\text{ kg} \times (+0.66\text{ m/s})^2 + m_2 (v_{2f})^2$

$$m_2 (v_{2f})^2 = 0.341496\text{ J}$$

•61 A cart with mass 340 g moving on a frictionless linear air track at an initial speed of 1.2 m/s undergoes an elastic collision with an initially stationary cart of unknown mass. After the collision, the first cart continues in its original direction at 0.66 m/s. (a) What is the mass of the second cart? (b) What is its speed after impact?

$m_2\, v_{2f}$ = 0.1836 kg· m/s   eq 1

$m_2\, (v_{2f})^2$= 0.341496 J   eq 2

divide eq2 by eq1

$$\frac{m_2\,(v_{2f})^2}{m_2\,v_{2f}} = \frac{0.341496}{0.1836}$$

v_{2f} = 1.86 m/s

now we use eq1 to calculate the mass

$m_2 \times$ 1.86 m/s = 0.1836 kg· m/s

m_2 = 0.0987 kg

m_2 = 98.7 g

Conservation of total momentum

$$m_1 V_1 = m_1 v_1 + m_2 v_2$$

$$m_1 V_1 - m_1 v_1 = m_2 v_2$$

$$m_1 (V_1 - v_1) = m_2 v_2$$

Conservation of total kinetic energy

$$\frac{1}{2} m_1 V_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$m_1 V_1^2 = m_1 v_1^2 + m_2 v_2^2$$

$$m_1 V_1^2 - m_1 v_1^2 = m_2 v_2^2$$

$$m_1 (V_1^2 - v_1^2) = m_2 v_2^2$$

$$m_1 (V_1 - v_1)(V_1 + v_1) = m_2 v_2^2$$

$$\frac{m_1 (V_1 - v_1)(V_1 + v_1)}{m_1 (V_1 - v_1)} = \frac{m_2 v_2^2}{m_2 v_2}$$

$$\frac{m_1(V_1-v_1)(V_1+v_1)}{m_1(V_1-v_1)} = \frac{m_2 v_2^2}{m_2 v_2}$$

$$V_1 + v_1 = v_2$$

$$m_1(V_1-v_1) = m_2 v_2$$

$$m_1(V_1-v_1) = m_2(V_1+v_1)$$

$$m_1 V_1 - m_1 v_1 = m_2 V_1 + m_2 v_1$$

$$m_1 V_1 - m_2 V_1 = m_2 v_1 + m_1 v_1$$

$$(m_1-m_2)V_1 = (m_2+m_1)v_1$$

$$v_1 = V_1 \frac{m_1 - m_2}{m_1 + m_2}$$

$$v_2 = V_1 + v_1$$

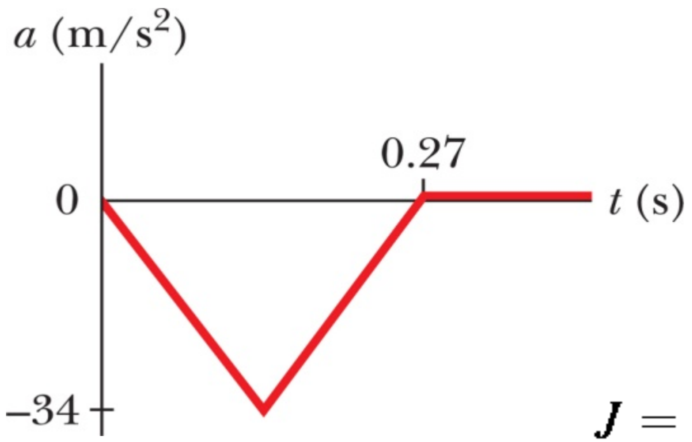
$$v_2 = V_1 + V_1 \frac{m_1 - m_2}{m_1 + m_2}$$

$$v_2 = V_1 \left[1 + \frac{m_1 - m_2}{m_1 + m_2}\right]$$

$$v_2 = V_1 \left[\frac{2m_1}{m_1 + m_2}\right]$$

Sample Problem 9-4

When a male bighorn sheep runs head-first into another male, the rate at which its speed drops to zero is dramatic. Figure 9-11 gives a typical graph of the acceleration a versus time t for such a collision, with the acceleration taken as negative to correspond to an initially positive velocity. The peak acceleration has magnitude 34 m/s^2 and the duration of the collision is 0.27 s. Assume that the sheep's mass is 90.0 kg. What are the magnitudes of the impulse and average force due to the collision?



$$J = \text{area} = \tfrac{1}{2}(0.27\ \text{s})(90.0\ \text{kg})(34.0\ \text{m/s}^2)$$
$$= 4.13 \times 10^2\ \text{N}\cdot\text{s} \approx 4.1 \times 10^2\ \text{N}\cdot\text{s}. \quad \text{(Answer)}$$

For the magnitude of the average force, we can write

$$F_{avg} = \frac{J}{\Delta t} = \frac{4.13 \times 10^2\ \text{N}\cdot\text{s}}{0.27\ \text{s}}$$
$$= 1.5 \times 10^3\ \text{N}. \quad \text{(Answer)}$$