Kenneth H. Rosen

Discrete Mathematics and Its Applications

SEVENTH EDITION

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Seventh Edition

Kenneth H. Rosen

Monmouth University
(and formerly AT&T Laboratories)

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Contents

About the Author

Kenneth H. Rosen has had a long career as a Distinguished Member of the Technical Staff at AT&T Laboratories in Monmouth County, New Jersey. He currently holds the position of Visiting Research Professor at Monmouth University, where he teaches graduate courses in computer science.

Dr. Rosen received his B.S. in Mathematics from the University of Michigan, Ann Arbor (1972), and his Ph.D. in Mathematics from M.I.T. (1976), where he wrote his thesis in the area of number theory under the direction of Harold Stark. Before joining Bell Laboratories in 1982, he held positions at the University of Colorado, Boulder; The Ohio State University, Columbus; and the University of Maine, Orono, where he was an associate professor of mathematics. While working at AT&T Labs, he taught at Monmouth University, teaching courses in discrete mathematics, coding theory, and data security. He currently teaches courses in algorithm design and in computer security and cryptography.

Dr. Rosen has published numerous articles in professional journals in number theory and in mathematical modeling. He is the author of the widely used *Elementary Number Theory and Its Applications*, published by Pearson, currently in its sixth edition, which has been translated into Chinese. He is also the author of *Discrete Mathematics and Its Applications*, published by McGraw-Hill, currently in its seventh edition. *Discrete Mathematics and Its Applications* has sold more than 350,000 copies in North America during its lifetime, and hundreds of thousands of copies throughout the rest of the world. This book has also been translated into Spanish, French, Greek, Chinese, Vietnamese, and Korean. He is also co-author of *UNIX: The Complete Reference*; *UNIX System V Release 4: An Introduction*; and *Best UNIX Tips Ever*, all published by Osborne McGraw-Hill. These books have sold more than 150,000 copies, with translations into Chinese, German, Spanish, and Italian. Dr. Rosen is also the editor of the *Handbook of Discrete and Combinatorial Mathematics*, published by CRC Press, and he is the advisory editor of the CRC series of books in discrete mathematics, consisting of more than 55 volumes on different aspects of discrete mathematics, most of which are introduced in this book. Dr. Rosen serves as an Associate Editor for the journal *Discrete Mathematics*, where he works with submitted papers in several areas of discrete mathematics, including graph theory, enumeration, and number theory. He is also interested in integrating mathematical software into the educational and professional environments, and worked on several projects with Waterloo Maple Inc.'s Maple$^{\text{TM}}$ software in both these areas. Dr. Rosen has also worked with several publishing companies on their homework delivery platforms.

At Bell Laboratories and AT&T Laboratories, Dr. Rosen worked on a wide range of projects, including operations research studies, product line planning for computers and data communications equipment, and technology assessment. He helped plan AT&T's products and services in the area of multimedia, including video communications, speech recognition, speech synthesis, and image networking. He evaluated new technology for use by AT&T and did standards work in the area of image networking. He also invented many new services, and holds more than 55 patents. One of his more interesting projects involved helping evaluate technology for the AT&T attraction that was part of EPCOT Center.

Preface

In writing this book, I was guided by my long-standing experience and interest in teaching discrete mathematics. For the student, my purpose was to present material in a precise, readable manner, with the concepts and techniques of discrete mathematics clearly presented and demonstrated. My goal was to show the relevance and practicality of discrete mathematics to students, who are often skeptical. I wanted to give students studying computer science all of the mathematical foundations they need for their future studies. I wanted to give mathematics students an understanding of important mathematical concepts together with a sense of why these concepts are important for applications. And most importantly, I wanted to accomplish these goals without watering down the material.

For the instructor, my purpose was to design a flexible, comprehensive teaching tool using proven pedagogical techniques in mathematics. I wanted to provide instructors with a package of materials that they could use to teach discrete mathematics effectively and efficiently in the most appropriate manner for their particular set of students. I hope that I have achieved these goals.

I have been extremely gratified by the tremendous success of this text. The many improvements in the seventh edition have been made possible by the feedback and suggestions of a large number of instructors and students at many of the more than 600 North American schools, and at any many universities in parts of the world, where this book has been successfully used.

This text is designed for a one- or two-term introductory discrete mathematics course taken by students in a wide variety of majors, including mathematics, computer science, and engineering. College algebra is the only explicit prerequisite, although a certain degree of mathematical maturity is needed to study discrete mathematics in a meaningful way. This book has been designed to meet the needs of almost all types of introductory discrete mathematics courses. It is highly flexible and extremely comprehensive. The book is designed not only to be a successful textbook, but also to serve as valuable resource students can consult throughout their studies and professional life.

Goals of a Discrete Mathematics Course

A discrete mathematics course has more than one purpose. Students should learn a particular set of mathematical facts and how to apply them; more importantly, such a course should teach students how to think logically and mathematically. To achieve these goals, this text stresses mathematical reasoning and the different ways problems are solved. Five important themes are interwoven in this text: mathematical reasoning, combinatorial analysis, discrete structures, algorithmic thinking, and applications and modeling. A successful discrete mathematics course should carefully blend and balance all five themes.

1. *Mathematical Reasoning:* Students must understand mathematical reasoning in order to read, comprehend, and construct mathematical arguments. This text starts with a discussion of mathematical logic, which serves as the foundation for the subsequent discussions of methods of proof. Both the science and the art of constructing proofs are addressed. The technique of mathematical induction is stressed through many different types of examples of such proofs and a careful explanation of why mathematical induction is a valid proof technique.

2. *Combinatorial Analysis:* An important problem-solving skill is the ability to count or enumerate objects. The discussion of enumeration in this book begins with the basic techniques of counting. The stress is on performing combinatorial analysis to solve counting problems and analyze algorithms, not on applying formulae.
3. *Discrete Structures:* A course in discrete mathematics should teach students how to work with discrete structures, which are the abstract mathematical structures used to represent discrete objects and relationships between these objects. These discrete structures include sets, permutations, relations, graphs, trees, and finite-state machines.
4. *Algorithmic Thinking:* Certain classes of problems are solved by the specification of an algorithm. After an algorithm has been described, a computer program can be constructed implementing it. The mathematical portions of this activity, which include the specification of the algorithm, the verification that it works properly, and the analysis of the computer memory and time required to perform it, are all covered in this text. Algorithms are described using both English and an easily understood form of pseudocode.
5. *Applications and Modeling:* Discrete mathematics has applications to almost every conceivable area of study. There are many applications to computer science and data networking in this text, as well as applications to such diverse areas as chemistry, biology, linguistics, geography, business, and the Internet. These applications are natural and important uses of discrete mathematics and are not contrived. Modeling with discrete mathematics is an extremely important problem-solving skill, which students have the opportunity to develop by constructing their own models in some of the exercises.

Changes in the Seventh Edition

Although the sixth edition has been an extremely effective text, many instructors, including longtime users, have requested changes designed to make this book more effective. I have devoted a significant amount of time and energy to satisfy their requests and I have worked hard to find my own ways to make the book more effective and more compelling to students.

The seventh edition is a major revision, with changes based on input from more than 40 formal reviewers, feedback from students and instructors, and author insights. The result is a new edition that offers an improved organization of topics making the book a more effective teaching tool. Substantial enhancements to the material devoted to logic, algorithms, number theory, and graph theory make this book more flexible and comprehensive. Numerous changes in the seventh edition have been designed to help students more easily learn the material. Additional explanations and examples have been added to clarify material where students often have difficulty. New exercises, both routine and challenging, have been added. Highly relevant applications, including many related to the Internet, to computer science, and to mathematical biology, have been added. The companion website has benefited from extensive development activity and now provides tools students can use to master key concepts and explore the world of discrete mathematics, and many new tools under development will be released in the year following publication of this book.

I hope that instructors will closely examine this new edition to discover how it might meet their needs. Although it is impractical to list all the changes in this edition, a brief list that highlights some key changes, listed by the benefits they provide, may be useful.

More Flexible Organization

- Applications of propositional logic are found in a new dedicated section, which briefly introduces logic circuits.
- Recurrence relations are now covered in Chapter 2.
- Expanded coverage of countability is now found in a dedicated section in Chapter 2.

- Separate chapters now provide expanded coverage of algorithms (Chapter 3) and number theory and cryptography (Chapter 4).
- More second and third level heads have been used to break sections into smaller coherent parts.

Tools for Easier Learning

- Difficult discussions and proofs have been marked with the famous Bourbaki "dangerous bend" symbol in the margin.
- New marginal notes make connections, add interesting notes, and provide advice to students.
- More details and added explanations, in both proofs and exposition, make it easier for students to read the book.
- Many new exercises, both routine and challenging, have been added, while many existing exercises have been improved.

Enhanced Coverage of Logic, Sets, and Proof

- The satisfiability problem is addressed in greater depth, with Sudoku modeled in terms of satisfiability.
- Hilbert's Grand Hotel is used to help explain uncountability.
- Proofs throughout the book have been made more accessible by adding steps and reasons behind these steps.
- A template for proofs by mathematical induction has been added.
- The step that applies the inductive hypothesis in mathematical induction proof is now explicitly noted.

Algorithms

- The pseudocode used in the book has been updated.
- Explicit coverage of algorithmic paradigms, including brute force, greedy algorithms, and dynamic programing, is now provided.
- Useful rules for big-O estimates of logarithms, powers, and exponential functions have been added.

Number Theory and Cryptography

- Expanded coverage allows instructors to include just a little or a lot of number theory in their courses.
- The relationship between the **mod** function and congruences has been explained more fully.
- The sieve of Eratosthenes is now introduced earlier in the book.
- Linear congruences and modular inverses are now covered in more detail.
- Applications of number theory, including check digits and hash functions, are covered in great depth.
- A new section on cryptography integrates previous coverage, and the notion of a cryptosystem has been introduced.
- Cryptographic protocols, including digital signatures and key sharing, are now covered.

Graph Theory

- A structured introduction to graph theory applications has been added.
- More coverage has been devoted to the notion of social networks.
- Applications to the biological sciences and motivating applications for graph isomorphism and planarity have been added.
- Matchings in bipartite graphs are now covered, including Hall's theorem and its proof.
- Coverage of vertex connectivity, edge connectivity, and n-connectedness has been added, providing more insight into the connectedness of graphs.

Enrichment Material

- Many biographies have been expanded and updated, and new biographies of Bellman, Bézout Bienyamé, Cardano, Catalan, Cocks, Cook, Dirac, Hall, Hilbert, Ore, and Tao have been added.
- Historical information has been added throughout the text.
- Numerous updates for latest discoveries have been made.

Expanded Media

- Extensive effort has been devoted to producing valuable web resources for this book.
- Extra examples in key parts of the text have been provided on companion website.
- Interactive algorithms have been developed, with tools for using them to explore topics and for classroom use.
- A new online ancillary, *The Virtual Discrete Mathematics Tutor*, available in fall 2012, will help students overcome problems learning discrete mathematics.
- A new homework delivery system, available in fall 2012, will provide automated homework for both numerical and conceptual exercises.
- Student assessment modules are available for key concepts.
- Powerpoint transparencies for instructor use have been developed.
- A supplement *Exploring Discrete Mathematics* has been developed, providing extensive support for using Maple$^{\mathrm{TM}}$ or Mathematica$^{\mathrm{TM}}$ in conjunction with the book.
- An extensive collection of external web links is provided.

Features of the Book

ACCESSIBILITY This text has proved to be easily read and understood by beginning students. There are no mathematical prerequisites beyond college algebra for almost all the content of the text. Students needing extra help will find tools on the companion website for bringing their mathematical maturity up to the level of the text. The few places in the book where calculus is referred to are explicitly noted. Most students should easily understand the pseudocode used in the text to express algorithms, regardless of whether they have formally studied programming languages. There is no formal computer science prerequisite.

Each chapter begins at an easily understood and accessible level. Once basic mathematical concepts have been carefully developed, more difficult material and applications to other areas of study are presented.

FLEXIBILITY This text has been carefully designed for flexible use. The dependence of chapters on previous material has been minimized. Each chapter is divided into sections of approximately the same length, and each section is divided into subsections that form natural blocks of material for teaching. Instructors can easily pace their lectures using these blocks.

WRITING STYLE The writing style in this book is direct and pragmatic. Precise mathematical language is used without excessive formalism and abstraction. Care has been taken to balance the mix of notation and words in mathematical statements.

MATHEMATICAL RIGOR AND PRECISION All definitions and theorems in this text are stated extremely carefully so that students will appreciate the precision of language and rigor needed in mathematics. Proofs are motivated and developed slowly; their steps are all carefully justified. The axioms used in proofs and the basic properties that follow from them are explicitly described in an appendix, giving students a clear idea of what they can assume in a proof. Recursive definitions are explained and used extensively.

WORKED EXAMPLES Over 800 examples are used to illustrate concepts, relate different topics, and introduce applications. In most examples, a question is first posed, then its solution is presented with the appropriate amount of detail.

APPLICATIONS The applications included in this text demonstrate the utility of discrete mathematics in the solution of real-world problems. This text includes applications to a wide variety of areas, including computer science, data networking, psychology, chemistry, engineering, linguistics, biology, business, and the Internet.

ALGORITHMS Results in discrete mathematics are often expressed in terms of algorithms; hence, key algorithms are introduced in each chapter of the book. These algorithms are expressed in words and in an easily understood form of structured pseudocode, which is described and specified in Appendix 3. The computational complexity of the algorithms in the text is also analyzed at an elementary level.

HISTORICAL INFORMATION The background of many topics is succinctly described in the text. Brief biographies of 83 mathematicians and computer scientists are included as footnotes. These biographies include information about the lives, careers, and accomplishments of these important contributors to discrete mathematics and images, when available, are displayed.

In addition, numerous historical footnotes are included that supplement the historical information in the main body of the text. Efforts have been made to keep the book up-to-date by reflecting the latest discoveries.

KEY TERMS AND RESULTS A list of key terms and results follows each chapter. The key terms include only the most important that students should learn, and not every term defined in the chapter.

EXERCISES There are over 4000 exercises in the text, with many different types of questions posed. There is an ample supply of straightforward exercises that develop basic skills, a large number of intermediate exercises, and many challenging exercises. Exercises are stated clearly and unambiguously, and all are carefully graded for level of difficulty. Exercise sets contain special discussions that develop new concepts not covered in the text, enabling students to discover new ideas through their own work.

Exercises that are somewhat more difficult than average are marked with a single star *; those that are much more challenging are marked with two stars **. Exercises whose solutions require calculus are explicitly noted. Exercises that develop results used in the text are clearly identified with the right pointing hand symbol ☞. Answers or outlined solutions to all odd-numbered exercises are provided at the back of the text. The solutions include proofs in which most of the steps are clearly spelled out.

REVIEW QUESTIONS A set of review questions is provided at the end of each chapter. These questions are designed to help students focus their study on the most important concepts and techniques of that chapter. To answer these questions students need to write long answers, rather than just perform calculations or give short replies.

SUPPLEMENTARY EXERCISE SETS Each chapter is followed by a rich and varied set of supplementary exercises. These exercises are generally more difficult than those in the exercise sets following the sections. The supplementary exercises reinforce the concepts of the chapter and integrate different topics more effectively.

COMPUTER PROJECTS Each chapter is followed by a set of computer projects. The approximately 150 computer projects tie together what students may have learned in computing and in discrete mathematics. Computer projects that are more difficult than average, from both a mathematical and a programming point of view, are marked with a star, and those that are extremely challenging are marked with two stars.

COMPUTATIONS AND EXPLORATIONS A set of computations and explorations is included at the conclusion of each chapter. These exercises (approximately 120 in total) are designed to be completed using existing software tools, such as programs that students or instructors have written or mathematical computation packages such as Maple$^{\text{TM}}$ or Mathematica$^{\text{TM}}$. Many of these exercises give students the opportunity to uncover new facts and ideas through computation. (Some of these exercises are discussed in the *Exploring Discrete Mathematics* companion workbooks available online.)

WRITING PROJECTS Each chapter is followed by a set of writing projects. To do these projects students need to consult the mathematical literature. Some of these projects are historical in nature and may involve looking up original sources. Others are designed to serve as gateways to new topics and ideas. All are designed to expose students to ideas not covered in depth in the text. These projects tie mathematical concepts together with the writing process and help expose students to possible areas for future study. (Suggested references for these projects can be found online or in the printed *Student's Solutions Guide*.)

APPENDIXES There are three appendixes to the text. The first introduces axioms for real numbers and the positive integers, and illustrates how facts are proved directly from these axioms. The second covers exponential and logarithmic functions, reviewing some basic material used heavily in the course. The third specifies the pseudocode used to describe algorithms in this text.

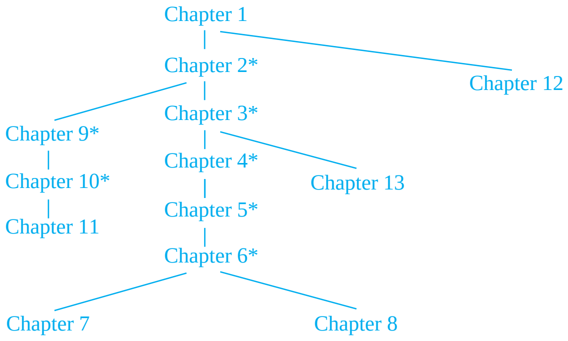
SUGGESTED READINGS A list of suggested readings for the overall book and for each chapter is provided after the appendices. These suggested readings include books at or below the level of this text, more difficult books, expository articles, and articles in which discoveries in discrete mathematics were originally published. Some of these publications are classics, published many years ago, while others have been published in the last few years.

How to Use This Book

This text has been carefully written and constructed to support discrete mathematics courses at several levels and with differing foci. The following table identifies the core and optional sections. An introductory one-term course in discrete mathematics at the sophomore level can be based on the core sections of the text, with other sections covered at the discretion of the instructor. A two-term introductory course can include all the optional mathematics sections in addition to the core sections. A course with a strong computer science emphasis can be taught by covering some or all of the optional computer science sections. Instructors can find sample syllabi for a wide range of discrete mathematics courses and teaching suggestions for using each section of the text can be found in the *Instructor's Resource Guide* available on the website for this book.

Chapter	*Core*	*Optional CS*	*Optional Math*
1	1.1–1.8 (as needed)		
2	2.1–2.4, 2.6 (as needed)		2.5
3		3.1–3.3 (as needed)	
4	4.1–4.4 (as needed)	4.5, 4.6	
5	5.1–5.3	5.4, 5.5	
6	6.1–6.3	6.6	6.4, 6.5
7	7.1	7.4	7.2, 7.3
8	8.1, 8.5	8.3	8.2, 8.4, 8.6
9	9.1, 9.3, 9.5	9.2	9.4, 9.6
10	10.1–10.5		10.6–10.8
11	11.1	11.2, 11.3	11.4, 11.5
12		12.1–12.4	
13		13.1–13.5	

Instructors using this book can adjust the level of difficulty of their course by choosing either to cover or to omit the more challenging examples at the end of sections, as well as the more challenging exercises. The chapter dependency chart shown here displays the strong dependencies. A star indicates that only relevant sections of the chapter are needed for study of a later chapter. Weak dependencies have been ignored. More details can be found in the Instructor Resource Guide.



Ancillaries

STUDENT'S SOLUTIONS GUIDE This student manual, available separately, contains full solutions to all odd-numbered problems in the exercise sets. These solutions explain why a particular method is used and why it works. For some exercises, one or two other possible approaches are described to show that a problem can be solved in several different ways. Suggested references for the writing projects found at the end of each chapter are also included in this volume. Also included are a guide to writing proofs and an extensive description of common mistakes students make in discrete mathematics, plus sample tests and a sample crib sheet for each chapter designed to help students prepare for exams.

(ISBN-10: 0-07-735350-1) *(ISBN-13: 978-0-07-735350-6)*

INSTRUCTOR'S RESOURCE GUIDE This manual, available on the website and in printed form by request for instructors, contains full solutions to even-numbered exercises in the text. Suggestions on how to teach the material in each chapter of the book are provided, including the points to stress in each section and how to put the material into perspective. It also offers sample tests for each chapter and a test bank containing over 1500 exam questions to choose from. Answers to all sample tests and test bank questions are included. Finally, several sample syllabi are presented for courses with differing emphases and student ability levels.

(ISBN-10: 0-07-735349-8) *(ISBN-13: 978-0-07-735349-0)*

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I would like to thank the many instructors and students at a variety of schools who have used this book and provided me with their valuable feedback and helpful suggestions. Their input has made this a much better book than it would have been otherwise. I especially want to thank Jerrold Grossman, Jean-Claude Evard, and Georgia Mederer for their technical reviews of the seventh edition and their "eagle eyes," which have helped ensure the accuracy of this book. I also appreciate the help provided by all those who have submitted comments via the website.

I thank the reviewers of this seventh and the six previous editions. These reviewers have provided much helpful criticism and encouragement to me. I hope this edition lives up to their high expectations.

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I would like to thank Bill Stenquist, Executive Editor, for his advocacy, enthusiasm, and support. His assistance with this edition has been essential. I would also like to thank the original editor, Wayne Yuhasz, whose insights and skills helped ensure the book's success, as well as all the many other previous editors of this book.

I want to express my appreciation to the staff of RPK Editorial Services for their valuable work on this edition, including Rose Kernan, who served as both the developmental editor and the production editor, and the other members of the RPK team, Fred Dahl, Martha McMaster, Erin Wagner, Harlan James, and Shelly Gerger-Knecthl. I thank Paul Mailhot of PreTeX, Inc., the compositor, for the tremendous amount to work he devoted to producing this edition, and for his intimate knowledge of LaTeX. Thanks also to Danny Meldung of Photo Affairs, Inc., who was resourceful obtaining images for the new biographical footnotes.

The accuracy and quality of this new edition owe much to Jerry Grossman and Jean-Claude Evard, who checked the entire manuscript for technical accuracy and Georgia Mederer, who checked the accuracy of the answers at the end of the book and the solutions in the *Student's Solutions Guide* and *Instructor's Resource Guide*. As usual, I cannot thank Jerry Grossman enough for all his work authoring these two essential ancillaries.

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Kenneth H. Rosen

The Companion Website

The extensive companion website accompanying this text has been substantially enhanced for the seventh edition This website is accessible at www.mhhe.com/rosen. The homepage shows the *Information Center*, and contains login links for the site's *Student Site* and *Instructor Site*. Key features of each area are described below:

THE INFORMATION CENTER

The Information Center contains basic information about the book including the expanded table of contents (including subsection heads), the preface, descriptions of the ancillaries, and a sample chapter. It also provides a link that can be used to submit errata reports and other feedback about the book.

STUDENT SITE

The Student site contains a wealth of resources available for student use, including the following, tied into the text wherever the special icons displayed below are found in the text:

Extra Examples

- ***Extra Examples*** You can find a large number of additional examples on the site, covering all chapters of the book. These examples are concentrated in areas where students often ask for additional material. Although most of these examples amplify the basic concepts, more-challenging examples can also be found here.

Demo

- ***Interactive Demonstration Applets*** These applets enable you to interactively explore how important algorithms work, and are tied directly to material in the text with linkages to examples and exercises. Additional resources are provided on how to use and apply these applets.

Assessment

- ***Self Assessments*** These interactive guides help you assess your understanding of 14 key concepts, providing a question bank where each question includes a brief tutorial followed by a multiple-choice question. If you select an incorrect answer, advice is provided to help you understand your error. Using these Self Assessments, you should be able to diagnose your problems and find appropriate help.

Links

- ***Web Resources Guide*** This guide provides annotated links to hundreds of external websites containing relevant material such as historical and biographical information, puzzles and problems, discussions, applets, programs, and more. These links are keyed to the text by page number.

Additional resources in the Student site include:

- ***Exploring Discrete Mathematics*** This ancillary provides help for using a computer algebra system to do a wide range of computations in discrete mathematics. Each chapter provides a description of relevant functions in the computer algebra system and how they are used, programs to carry out computations in discrete mathematics, examples, and exercises that can be worked using this computer algebra system. Two versions, *Exploring Discrete Mathematics with Maple*$^{\text{TM}}$ and *Exploring Discrete Mathematics with Mathematica*$^{\text{TM}}$ will be available.
- ***Applications of Discrete Mathematics*** This ancillary contains 24 chapters—each with its own set of exercises—presenting a wide variety of interesting and important applications

covering three general areas in discrete mathematics: discrete structures, combinatorics, and graph theory. These applications are ideal for supplementing the text or for independent study.

- ***A Guide to Proof-Writing*** This guide provides additional help for writing proofs, a skill that many students find difficult to master. By reading this guide at the beginning of the course and periodically thereafter when proof writing is required, you will be rewarded as your proof-writing ability grows. (Also available in the *Student's Solutions Guide*.)
- ***Common Mistakes in Discrete Mathematics*** This guide includes a detailed list of common misconceptions that students of discrete mathematics often have and the kinds of errors they tend to make. You are encouraged to review this list from time to time to help avoid these common traps. (Also available in the *Student's Solutions Guide*.)
- ***Advice on Writing Projects*** This guide offers helpful hints and suggestions for the Writing Projects in the text, including an extensive bibliography of helpful books and articles for research; discussion of various resources available in print and online; tips on doing library research; and suggestions on how to write well. (Also available in the *Student's Solutions Guide*.)
- ***The Virtual Discrete Mathematics Tutor*** This extensive ancillary provides students with valuable assistance as they make the transition from lower-level courses to discrete mathematics. The errors students have made when studying discrete mathematics using this text has been analyzed to design this resource. Students will be able to get many of their questions answered and can overcome many obstacles via this ancillaries. The *Virtual Discrete Mathematics Tutor* is expected to be available in the fall of 2012.

INSTRUCTOR SITE

This part of the website provides access to all of the resources on the Student Site, as well as these resources for instructors:

- ***Suggested Syllabi*** Detailed course outlines are shown, offering suggestions for courses with different emphases and different student backgrounds and ability levels.
- ***Teaching Suggestions*** This guide contains detailed teaching suggestions for instructors, including chapter overviews for the entire text, detailed remarks on each section, and comments on the exercise sets.
- ***Printable Tests*** Printable tests are offered in TeX and Word format for every chapter, and can be customized by instructors.
- ***PowerPoints Lecture Slides and PowerPoint Figures and Tables*** An extensive collection of PowerPoint slides for all chapters of the text are provided for instructor use. In addition, images of all figures and tables from the text are provided as PowerPoint slides.
- ***Homework Delivery System*** An extensive homework delivery system, under development for availability in fall 2012, will provide questions tied directly to the text, so that students will be able to do assignments on-line. Moreover, they will be able to use this system in a tutorial mode. This system will be able to automatically grade assignments, and deliver free-form student input to instructors for their own analysis. Course management capabilities will be provided that will allow instructors to create assignments, automatically assign and grade homework, quiz, and test questions from a bank of questions tied directly to the text, create and edit their own questions, manage course announcements and due dates, and track student progress.

To the Student

W*hat is discrete mathematics?* Discrete mathematics is the part of mathematics devoted to the study of discrete objects. (Here *discrete* means consisting of distinct or unconnected elements.) The kinds of problems solved using discrete mathematics include:

- How many ways are there to choose a valid password on a computer system?
- What is the probability of winning a lottery?
- Is there a link between two computers in a network?
- How can I identify spam e-mail messages?
- How can I encrypt a message so that no unintended recipient can read it?
- What is the shortest path between two cities using a transportation system?
- How can a list of integers be sorted so that the integers are in increasing order?
- How many steps are required to do such a sorting?
- How can it be proved that a sorting algorithm correctly sorts a list?
- How can a circuit that adds two integers be designed?
- How many valid Internet addresses are there?

You will learn the discrete structures and techniques needed to solve problems such as these.

More generally, discrete mathematics is used whenever objects are counted, when relationships between finite (or countable) sets are studied, and when processes involving a finite number of steps are analyzed. A key reason for the growth in the importance of discrete mathematics is that information is stored and manipulated by computing machines in a discrete fashion.

WHY STUDY DISCRETE MATHEMATICS? There are several important reasons for studying discrete mathematics. First, through this course you can develop your mathematical maturity: that is, your ability to understand and create mathematical arguments. You will not get very far in your studies in the mathematical sciences without these skills.

Second, discrete mathematics is the gateway to more advanced courses in all parts of the mathematical sciences. Discrete mathematics provides the mathematical foundations for many computer science courses including data structures, algorithms, database theory, automata theory, formal languages, compiler theory, computer security, and operating systems. Students find these courses much more difficult when they have not had the appropriate mathematical foundations from discrete math. One student has sent me an e-mail message saying that she used the contents of this book in every computer science course she took!

Math courses based on the material studied in discrete mathematics include logic, set theory, number theory, linear algebra, abstract algebra, combinatorics, graph theory, and probability theory (the discrete part of the subject).

Also, discrete mathematics contains the necessary mathematical background for solving problems in operations research (including many discrete optimization techniques), chemistry, engineering, biology, and so on. In the text, we will study applications to some of these areas.

Many students find their introductory discrete mathematics course to be significantly more challenging than courses they have previously taken. One reason for this is that one of the primary goals of this course is to teach mathematical reasoning and problem solving, rather than a discrete set of skills. The exercises in this book are designed to reflect this goal. Although there are plenty of exercises in this text similar to those addressed in the examples, a large percentage of the exercises require original thought. This is intentional. The material discussed in the text provides the tools needed to solve these exercises, but your job is to successfully apply these tools using your own creativity. One of the primary goals of this course is to learn how to attack problems that may be somewhat different from any you may have previously seen. Unfortunately, learning how to solve only particular types of exercises is not sufficient for success in developing the problem-solving skills needed in subsequent courses and professional work. This text addresses many different topics, but discrete mathematics is an extremely diverse and large area of study. One of my goals as an author is to help you develop the skills needed to master the additional material you will need in your own future pursuits.

THE EXERCISES I would like to offer some advice about how you can best learn discrete mathematics (and other subjects in the mathematical and computing sciences). You will learn the most by actively working exercises. I suggest that you solve as many as you possibly can. After working the exercises your instructor has assigned, I encourage you to solve additional exercises such as those in the exercise sets following each section of the text and in the supplementary exercises at the end of each chapter. (Note the key explaining the markings preceding exercises.)

Key to the Exercises

no marking	A routine exercise
*	A difficult exercise
**	An extremely challenging exercise
☞	An exercise containing a result used in the book (Table 1 on the following page shows where these exercises are used.)
(*Requires calculus*)	An exercise whose solution requires the use of limits or concepts from differential or integral calculus

The best approach is to try exercises yourself before you consult the answer section at the end of this book. Note that the odd-numbered exercise answers provided in the text are answers only and not full solutions; in particular, the reasoning required to obtain answers is omitted in these answers. The *Student's Solutions Guide*, available separately, provides complete, worked solutions to all odd-numbered exercises in this text. When you hit an impasse trying to solve an odd-numbered exercise, I suggest you consult the *Student's Solutions Guide* and look for some guidance as to how to solve the problem. The more work you do yourself rather than passively reading or copying solutions, the more you will learn. The answers and solutions to the even-numbered exercises are intentionally not available from the publisher; ask your instructor if you have trouble with these.

WEB RESOURCES You are ***strongly*** encouraged to take advantage of additional resources available on the Web, especially those on the companion website for this book found at *www.mhhe.com/rosen*. You will find many Extra Examples designed to clarify key concepts; Self Assessments for gauging how well you understand core topics; Interactive Demonstration Applets exploring key algorithms and other concepts; a Web Resources Guide containing an extensive selection of links to external sites relevant to the world of discrete mathematics; extra explanations and practice to help you master core concepts; added instruction on writing proofs and on avoiding common mistakes in discrete mathematics; in-depth discussions of important applications; and guidance on utilizing Maple$^{\text{TM}}$ software to explore the computational aspects of discrete mathematics. Places in the text where these additional online resources are available are identified in the margins by special icons. You will also find (after fall 2012) the *Virtual Discrete Mathematics Tutor*, an on-line resource that provides extra support to help you make the transition from lower level courses to discrete mathematics. This tutorial should help answer many of your questions and correct errors that you may make, based on errors other students using this book, have made. For more details on these and other online resources, see the description of the companion website immediately preceding this "To the Student" message.

TABLE 1 Hand-Icon Exercises and Where They Are Used

Section	*Exercise*	*Section Where Used*	*Pages Where Used*
1.1	40	1.3	31
1.1	41	1.3	31
1.3	9	1.6	71
1.3	10	1.6	70, 71
1.3	15	1.6	71
1.3	30	1.6	71, 74
1.3	42	12.2	820
1.7	16	1.7	86
2.3	72	2.3	144
2.3	79	2.5	170
2.5	15	2.5	174
2.5	16	2.5	173
3.1	43	3.1	197
3.2	72	11.2	761
4.2	36	4.2	270
4.3	37	4.1	239
4.4	2	4.6	301
4.4	44	7.2	464
6.4	17	7.2	466
6.4	21	7.4	480
7.2	15	7.2	466
9.1	26	9.4	598
10.4	59	11.1	747
11.1	15	11.1	750
11.1	30	11.1	755
11.1	48	11.2	762
12.1	12	12.3	825
A.2	4	8.3	531

THE VALUE OF THIS BOOK My intention is to make your substantial investment in this text an excellent value. The book, the associated ancillaries, and companion website have taken many years of effort to develop and refine. I am confident that most of you will find that the text and associated materials will help you master discrete mathematics, just as so many previous students have. Even though it is likely that you will not cover some chapters in your current course, you should find it helpful—as many other students have—to read the relevant sections of the book as you take additional courses. Most of you will return to this book as a useful tool throughout your future studies, especially for those of you who continue in computer science, mathematics, and engineering. I have designed this book to be a gateway for future studies and explorations, and to be comprehensive reference, and I wish you luck as you begin your journey.

Kenneth H. Rosen

CHAPTER

1 The Foundations: Logic and Proofs

The rules of logic specify the meaning of mathematical statements. For instance, these rules help us understand and reason with statements such as "There exists an integer that is not the sum of two squares" and "For every positive integer n, the sum of the positive integers not exceeding n is $n(n + 1)/2$." Logic is the basis of all mathematical reasoning, and of all automated reasoning. It has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study.

To understand mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Once we prove a mathematical statement is true, we call it a theorem. A collection of theorems on a topic organize what we know about this topic. To learn a mathematical topic, a person needs to actively construct mathematical arguments on this topic, and not just read exposition. Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situations.

Everyone knows that proofs are important throughout mathematics, but many people find it surprising how important proofs are in computer science. In fact, proofs are used to verify that computer programs produce the correct output for all possible input values, to show that algorithms always produce the correct result, to establish the security of a system, and to create artificial intelligence. Furthermore, automated reasoning systems have been created to allow computers to construct their own proofs.

In this chapter, we will explain what makes up a correct mathematical argument and introduce tools to construct these arguments. We will develop an arsenal of different proof methods that will enable us to prove many different types of results. After introducing many different methods of proof, we will introduce several strategies for constructing proofs. We will introduce the notion of a conjecture and explain the process of developing mathematics by studying conjectures.

1.1 Propositional Logic

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Because a major goal of this book is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic.

Besides the importance of logic in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing some, but not all, types of proofs automatically. We will discuss these applications of logic in this and later chapters.

Propositions

Our discussion begins with an introduction to the basic building blocks of logic—propositions. A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE 1 All the following declarative sentences are propositions.

Extra Examples

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false. ◀

Some sentences that are not propositions are given in Example 2.

EXAMPLE 2 Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables. We will also discuss other ways to turn sentences such as these into propositions in Section 1.4. ◀

We use letters to denote **propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The

Links

ARISTOTLE (384 B.C.E.–322 B.C.E.) Aristotle was born in Stagirus (Stagira) in northern Greece. His father was the personal physician of the King of Macedonia. Because his father died when Aristotle was young, Aristotle could not follow the custom of following his father's profession. Aristotle became an orphan at a young age when his mother also died. His guardian who raised him taught him poetry, rhetoric, and Greek. At the age of 17, his guardian sent him to Athens to further his education. Aristotle joined Plato's Academy, where for 20 years he attended Plato's lectures, later presenting his own lectures on rhetoric. When Plato died in 347 B.C.E., Aristotle was not chosen to succeed him because his views differed too much from those of Plato. Instead, Aristotle joined the court of King Hermeas where he remained for three years, and married the niece of the King. When the Persians defeated Hermeas, Aristotle moved to Mytilene and, at the invitation of King Philip of Macedonia, he tutored Alexander, Philip's son, who later became Alexander the Great. Aristotle tutored Alexander for five years and after the death of King Philip, he returned to Athens and set up his own school, called the Lyceum.

Aristotle's followers were called the peripatetics, which means "to walk about," because Aristotle often walked around as he discussed philosophical questions. Aristotle taught at the Lyceum for 13 years where he lectured to his advanced students in the morning and gave popular lectures to a broad audience in the evening. When Alexander the Great died in 323 B.C.E., a backlash against anything related to Alexander led to trumped-up charges of impiety against Aristotle. Aristotle fled to Chalcis to avoid prosecution. He only lived one year in Chalcis, dying of a stomach ailment in 322 B.C.E.

Aristotle wrote three types of works: those written for a popular audience, compilations of scientific facts, and systematic treatises. The systematic treatises included works on logic, philosophy, psychology, physics, and natural history. Aristotle's writings were preserved by a student and were hidden in a vault where a wealthy book collector discovered them about 200 years later. They were taken to Rome, where they were studied by scholars and issued in new editions, preserving them for posterity.

conventional letters used for propositional variables are $p, q, r, s, \ldots$. The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

Links

We now turn our attention to methods for producing new propositions from those that we already have. These methods were discussed by the English mathematician George Boole in 1854 in his book *The Laws of Thought*. Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

DEFINITION 1 Let p be a proposition. The *negation of* p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

EXAMPLE 3 Find the negation of the proposition

"Michael's PC runs Linux"

Extra Examples

and express this in simple English.

Solution: The negation is

"It is not the case that Michael's PC runs Linux."

This negation can be more simply expressed as

"Michael's PC does not run Linux." ◀

EXAMPLE 4 Find the negation of the proposition

"Vandana's smartphone has at least 32GB of memory"

and express this in simple English.

Solution: The negation is

"It is not the case that Vandana's smartphone has at least 32GB of memory."

This negation can also be expressed as

"Vandana's smartphone does not have at least 32GB of memory"

or even more simply as

"Vandana's smartphone has less than 32GB of memory." ◀

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Table 1 displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

DEFINITION 2 Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Table 2 displays the truth table of $p \wedge q$. This table has a row for each of the four possible combinations of truth values of p and q. The four rows correspond to the pairs of truth values TT, TF, FT, and FF, where the first truth value in the pair is the truth value of p and the second truth value is the truth value of q.

Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining." (In natural language, there is a subtle difference in meaning between "and" and "but"; we will not be concerned with this nuance here.)

EXAMPLE 5 Find the conjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.◀

DEFINITION 3 Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Table 3 displays the truth table for $p \vee q$.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, as an **inclusive or**. A disjunction is true when at least one of the two propositions is true. For instance, the inclusive or is being used in the statement

> "Students who have taken calculus or computer science can take this class."

Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects. On the other hand, we are using the **exclusive or** when we say

> "Students who have taken calculus or computer science, but not both, can enroll in this class."

Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Similarly, when a menu at a restaurant states, "Soup or salad comes with an entrée," the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, rather than an inclusive, or.

EXAMPLE 6 What is the disjunction of the propositions p and q where p and q are the same propositions as in Example 5?

Extra Examples

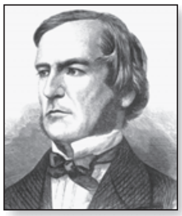
Solution: The disjunction of p and q, $p \vee q$, is the proposition

> "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."

This proposition is true when Rebecca's PC has at least 16 GB free hard disk space, when the PC's processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebecca's PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower. ◀

As was previously remarked, the use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, in an inclusive way. Thus, a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use *or* in an exclusive sense. When the exclusive or is used to connect the propositions p and q, the proposition "p or q (but not both)" is obtained. This proposition is true when p is true and q is false, and when p is false and q is true. It is false when both p and q are false and when both are true.

Links



GEORGE BOOLE (1815–1864) George Boole, the son of a cobbler, was born in Lincoln, England, in November 1815. Because of his family's difficult financial situation, Boole struggled to educate himself while supporting his family. Nevertheless, he became one of the most important mathematicians of the 1800s. Although he considered a career as a clergyman, he decided instead to go into teaching, and soon afterward opened a school of his own. In his preparation for teaching mathematics, Boole—unsatisfied with textbooks of his day—decided to read the works of the great mathematicians. While reading papers of the great French mathematician Lagrange, Boole made discoveries in the calculus of variations, the branch of analysis dealing with finding curves and surfaces by optimizing certain parameters.

In 1848 Boole published *The Mathematical Analysis of Logic*, the first of his contributions to symbolic logic. In 1849 he was appointed professor of mathematics at Queen's College in Cork, Ireland. In 1854 he published *The Laws of Thought*, his most famous work. In this book, Boole introduced what is now called *Boolean algebra* in his honor. Boole wrote textbooks on differential equations and on difference equations that were used in Great Britain until the end of the nineteenth century. Boole married in 1855; his wife was the niece of the professor of Greek at Queen's College. In 1864 Boole died from pneumonia, which he contracted as a result of keeping a lecture engagement even though he was soaking wet from a rainstorm.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

DEFINITION 4

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The truth table for the exclusive or of two propositions is displayed in Table 4.

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5

Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Assessment

The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

The truth table for the conditional statement $p \rightarrow q$ is shown in Table 5. Note that the statement $p \rightarrow q$ is true when both p and q are true and when p is false (no matter what truth value q has).

Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is

"If I am elected, then I will lower taxes."

If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \to q$.

Similarly, consider a statement that a professor might make:

"If you get 100% on the final, then you will get an A."

If you manage to get a 100% on the final, then you would expect to receive an A. If you do not get 100% you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

Of the various ways to express the conditional statement $p \to q$, the two that seem to cause the most confusion are "p only if q" and "q unless $\neg p$." Consequently, we will provide some guidance for clearing up this confusion.

To remember that "p only if q" expresses the same thing as "if p, then q," note that "p only if q" says that p cannot be true when q is not true. That is, the statement is false if p is true, but q is false. When p is false, q may be either true or false, because the statement says nothing about the truth value of q. Be careful not to use "q only if p" to express $p \to q$ because this is incorrect. To see this, note that the true values of "q only if p" and $p \to q$ are different when p and q have different truth values.

You might have trouble understanding how "unless" is used in conditional statements unless you read this paragraph carefully.

To remember that "q unless $\neg p$" expresses the same conditional statement as "if p, then q," note that "q unless $\neg p$" means that if $\neg p$ is false, then q must be true. That is, the statement "q unless $\neg p$" is false when p is true but q is false, but it is true otherwise. Consequently, "q unless $\neg p$" and $p \to q$ always have the same truth value.

We illustrate the translation between conditional statements and English statements in Example 7.

EXAMPLE 7 Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Extra Examples

Solution: From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job," $p \to q$ represents the statement

"If Maria learns discrete mathematics, then she will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are:

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

and

"Maria will find a good job unless she does not learn discrete mathematics." ◀

Note that the way we have defined conditional statements is more general than the meaning attached to such statements in the English language. For instance, the conditional statement in Example 7 and the statement

"If it is sunny, then we will go to the beach."

are statements used in normal language where there is a relationship between the hypothesis and the conclusion. Further, the first of these statements is true unless Maria learns discrete mathematics, but she does not get a good job, and the second is true unless it is indeed sunny, but we do not go to the beach. On the other hand, the statement

"If Juan has a smartphone, then $2 + 3 = 5$"

is true from the definition of a conditional statement, because its conclusion is true. (The truth value of the hypothesis does not matter then.) The conditional statement

"If Juan has a smartphone, then $2 + 3 = 6$"

is true if Juan does not have a smartphone, even though $2 + 3 = 6$ is false. We would not use these last two conditional statements in natural language (except perhaps in sarcasm), because there is no relationship between the hypothesis and the conclusion in either statement. In mathematical reasoning, we consider conditional statements of a more general sort than we use in English. The mathematical concept of a conditional statement is independent of a cause-and-effect relationship between hypothesis and conclusion. Our definition of a conditional statement specifies its truth values; it is not based on English usage. Propositional language is an artificial language; we only parallel English usage to make it easy to use and remember.

The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as **if** p **then** S, where p is a proposition and S is a program segment (one or more statements to be executed). When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false, as illustrated in Example 8.

EXAMPLE 8 What is the value of the variable x after the statement

if $2 + 2 = 4$ **then** $x := x + 1$

if $x = 0$ before this statement is encountered? (The symbol := stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x.)

Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◀

CONVERSE, CONTRAPOSITIVE, AND INVERSE We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$ is called the **converse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$. We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

We first show that the contrapositive, $\neg q \to \neg p$, of a conditional statement $p \to q$ always has the same truth value as $p \to q$. To see this, note that the contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. We now show that neither the converse, $q \to p$, nor the inverse, $\neg p \to \neg q$, has the same truth value as $p \to q$ for all possible truth values of p and q. Note that when p is true and q is false, the original conditional statement is false, but the converse and the inverse are both true.

Remember that the contrapositive, but neither the converse or inverse, of a conditional statement is equivalent to it.

When two compound propositions always have the same truth value we call them **equivalent**, so that a conditional statement and its contrapositive are equivalent. The converse and the inverse of a conditional statement are also equivalent, as the reader can verify, but neither is equivalent to the original conditional statement. (We will study equivalent propositions in Section 1.3.) Take note that one of the most common logical errors is to assume that the converse or the inverse of a conditional statement is equivalent to this conditional statement.

We illustrate the use of conditional statements in Example 9.

EXAMPLE 9 What are the contrapositive, the converse, and the inverse of the conditional statement

> "The home team wins whenever it is raining?"

Extra Examples

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as

> "If it is raining, then the home team wins."

Consequently, the contrapositive of this conditional statement is

> "If the home team does not win, then it is not raining."

The converse is

> "If the home team wins, then it is raining."

The inverse is

> "If it is not raining, then the home team does not win."

Only the contrapositive is equivalent to the original statement. ◀

BICONDITIONALS We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

DEFINITION 6 Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

The truth table for $p \leftrightarrow q$ is shown in Table 6. Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \to q$ and $q \to p$ are true and is false otherwise. That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols $\to$ and $\leftarrow$. There are some other common ways to express $p \leftrightarrow q$:

> "p is necessary and sufficient for q"
> "if p then q, and conversely"
> "p iff q."

The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \to q) \wedge (q \to p)$.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 10 Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

"You can take the flight if and only if you buy a ticket."

Extra Examples

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

IMPLICIT USE OF BICONDITIONALS You should be aware that biconditionals are not always explicit in natural language. In particular, the "if and only if" construction used in biconditionals is rarely used in common language. Instead, biconditionals are often expressed using an "if, then" or an "only if" construction. The other part of the "if and only if" is implicit. That is, the converse is implied, but not stated. For example, consider the statement in English "If you finish your meal, then you can have dessert." What is really meant is "You can have dessert if and only if you finish your meal." This last statement is logically equivalent to the two statements "If you finish your meal, then you can have dessert" and "You can have dessert only if you finish your meal." Because of this imprecision in natural language, we need to make an assumption whether a conditional statement in natural language implicitly includes its converse. Because precision is essential in mathematics and in logic, we will always distinguish between the conditional statement $p \to q$ and the biconditional statement $p \leftrightarrow q$.

Truth Tables of Compound Propositions

Demo

We have now introduced four important logical connectives—conjunctions, disjunctions, conditional statements, and biconditional statements—as well as negations. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions, as Example 11 illustrates. We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up. The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

EXAMPLE 11 Construct the truth table of the compound proposition

$$(p \vee \neg q) \to (p \wedge q).$$

Solution: Because this truth table involves two propositional variables p and q, there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q, respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column. The fifth column gives the truth value of $p \wedge q$. Finally, the truth value of $(p \vee \neg q) \to (p \wedge q)$ is found in the last column. The resulting truth table is shown in Table 7.

TABLE 7 The Truth Table of $(p \vee \neg q) \to (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

TABLE 8 Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg(p \wedge q)$.

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Finally, it is an accepted rule that the conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than the conjunction and disjunction operators, $\wedge$ and $\vee$. Consequently, $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$. We will use parentheses when the order of the conditional operator and biconditional operator is at issue, although the conditional operator has precedence over the biconditional operator. Table 8 displays the precedence levels of the logical operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

Logic and Bit Operations

Truth Value	*Bit*
T	1
F	0

Computers represent information using bits. A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from *b*inary dig*it*, because zeros and ones are the digits used in binary representations of numbers. The well-known statistician John Tukey introduced this terminology in 1946. A bit can be used to represent a truth value, because there are two truth values, namely, *true* and *false*. As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false). A variable is called a **Boolean variable** if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Links

Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators $\wedge$, $\vee$, and $\oplus$, the tables shown in Table 9 for the corresponding bit operations are obtained. We will also use the notation *OR*, *AND*, and *XOR* for the operators $\vee$, $\wedge$, and $\oplus$, as is done in various programming languages.

Links

JOHN WILDER TUKEY (1915–2000) Tukey, born in New Bedford, Massachusetts, was an only child. His parents, both teachers, decided home schooling would best develop his potential. His formal education began at Brown University, where he studied mathematics and chemistry. He received a master's degree in chemistry from Brown and continued his studies at Princeton University, changing his field of study from chemistry to mathematics. He received his Ph.D. from Princeton in 1939 for work in topology, when he was appointed an instructor in mathematics at Princeton. With the start of World War II, he joined the Fire Control Research Office, where he began working in statistics. Tukey found statistical research to his liking and impressed several leading statisticians with his skills. In 1945, at the conclusion of the war, Tukey returned to the mathematics department at Princeton as a professor of statistics, and he also took a position at AT&T Bell Laboratories. Tukey founded the Statistics Department at Princeton in 1966 and was its first chairman. Tukey made significant contributions to many areas of statistics, including the analysis of variance, the estimation of spectra of time series, inferences about the values of a set of parameters from a single experiment, and the philosophy of statistics. However, he is best known for his invention, with J. W. Cooley, of the fast Fourier transform. In addition to his contributions to statistics, Tukey was noted as a skilled wordsmith; he is credited with coining the terms *bit* and *software*.

Tukey contributed his insight and expertise by serving on the President's Science Advisory Committee. He chaired several important committees dealing with the environment, education, and chemicals and health. He also served on committees working on nuclear disarmament. Tukey received many awards, including the National Medal of Science.

HISTORICAL NOTE There were several other suggested words for a binary digit, including *binit* and *bigit*, that never were widely accepted. The adoption of the word *bit* may be due to its meaning as a common English word. For an account of Tukey's coining of the word *bit*, see the April 1984 issue of *Annals of the History of Computing.*

TABLE 9 **Table for the Bit Operators *OR*, *AND*, and *XOR*.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Information is often represented using bit strings, which are lists of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 12 101010011 is a bit string of length nine. ◀

We can extend bit operations to bit strings. We define the **bitwise *OR***, **bitwise *AND***, and **bitwise *XOR*** of two strings of the same length to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$ to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 13.

EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

$$\begin{array}{ll} 01\ 1011\ 0110 & \\ \underline{11\ 0001\ 1101} & \\ 11\ 1011\ 1111 & \text{bitwise } OR \\ 01\ 0001\ 0100 & \text{bitwise } AND \\ 10\ 1010\ 1011 & \text{bitwise } XOR \end{array}$$

◀

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?
   - **a)** Boston is the capital of Massachusetts.
   - **b)** Miami is the capital of Florida.
   - **c)** $2 + 3 = 5$.
   - **d)** $5 + 7 = 10$.
   - **e)** $x + 2 = 11$.
   - **f)** Answer this question.
2. Which of these are propositions? What are the truth values of those that are propositions?
   - **a)** Do not pass go.
   - **b)** What time is it?
   - **c)** There are no black flies in Maine.
   - **d)** $4 + x = 5$.
   - **e)** The moon is made of green cheese.
   - **f)** $2^n \geq 100$.
3. What is the negation of each of these propositions?
   - **a)** Mei has an MP3 player.
   - **b)** There is no pollution in New Jersey.
   - **c)** $2 + 1 = 3$.
   - **d)** The summer in Maine is hot and sunny.
4. What is the negation of each of these propositions?
   - **a)** Jennifer and Teja are friends.
   - **b)** There are 13 items in a baker's dozen.
   - **c)** Abby sent more than 100 text messages every day.
   - **d)** 121 is a perfect square.

5. What is the negation of each of these propositions?

a) Steve has more than 100 GB free disk space on his laptop.
b) Zach blocks e-mails and texts from Jennifer.
c) $7 \cdot 11 \cdot 13 = 999$.
d) Diane rode her bicycle 100 miles on Sunday.

6. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone B has the most RAM of these three smartphones.
b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.
c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.
d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.
e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

7. Suppose that during the most recent fiscal year, the annual revenue of Acme Computer was 138 billion dollars and its net profit was 8 billion dollars, the annual revenue of Nadir Software was 87 billion dollars and its net profit was 5 billion dollars, and the annual revenue of Quixote Media was 111 billion dollars and its net profit was 13 billion dollars. Determine the truth value of each of these propositions for the most recent fiscal year.

a) Quixote Media had the largest annual revenue.
b) Nadir Software had the lowest net profit and Acme Computer had the largest annual revenue.
c) Acme Computer had the largest net profit or Quixote Media had the largest net profit.
d) If Quixote Media had the smallest net profit, then Acme Computer had the largest annual revenue.
e) Nadir Software had the smallest net profit if and only if Acme Computer had the largest annual revenue.

8. Let p and q be the propositions

p : I bought a lottery ticket this week.
q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$ **b)** $p \vee q$ **c)** $p \rightarrow q$
d) $p \wedge q$ **e)** $p \leftrightarrow q$ **f)** $\neg p \rightarrow \neg q$
g) $\neg p \wedge \neg q$ **h)** $\neg p \vee (p \wedge q)$

9. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.

a) $\neg q$ **b)** $p \wedge q$ **c)** $\neg p \vee q$
d) $p \rightarrow \neg q$ **e)** $\neg q \rightarrow p$ **f)** $\neg p \rightarrow \neg q$
g) $p \leftrightarrow \neg q$ **h)** $\neg p \wedge (p \vee \neg q)$

10. Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.

a) $\neg p$ **b)** $p \vee q$
c) $\neg p \wedge q$ **d)** $q \rightarrow p$
e) $\neg q \rightarrow \neg p$ **f)** $\neg p \rightarrow \neg q$
g) $p \leftrightarrow q$ **h)** $\neg q \vee (\neg p \wedge q)$

11. Let p and q be the propositions

p : It is below freezing.
q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.
b) It is below freezing but not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both).
e) If it is below freezing, it is also snowing.
f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

12. Let p, q, and r be the propositions

p : You have the flu.
q : You miss the final examination.
r : You pass the course.

Express each of these propositions as an English sentence.

a) $p \rightarrow q$ **b)** $\neg q \leftrightarrow r$
c) $q \rightarrow \neg r$ **d)** $p \vee q \vee r$
e) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$
f) $(p \wedge q) \vee (\neg q \wedge r)$

13. Let p and q be the propositions

p : You drive over 65 miles per hour.
q : You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour.
b) You drive over 65 miles per hour, but you do not get a speeding ticket.
c) You will get a speeding ticket if you drive over 65 miles per hour.
d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
f) You get a speeding ticket, but you do not drive over 65 miles per hour.
g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

14. Let p, q, and r be the propositions

p : You get an A on the final exam.
q : You do every exercise in this book.
r : You get an A in this class.

Write these propositions using p, q, and r and logical connectives (including negations).

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final.

d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

15. Let p, q, and r be the propositions

p : Grizzly bears have been seen in the area.
q : Hiking is safe on the trail.
r : Berries are ripe along the trail.

Write these propositions using p, q, and r and logical connectives (including negations).

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.

b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.

e) For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.

f) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

16. Determine whether these biconditionals are true or false.

a) $2 + 2 = 4$ if and only if $1 + 1 = 2$.

b) $1 + 1 = 2$ if and only if $2 + 3 = 4$.

c) $1 + 1 = 3$ if and only if monkeys can fly.

d) $0 > 1$ if and only if $2 > 1$.

17. Determine whether each of these conditional statements is true or false.

a) If $1 + 1 = 2$, then $2 + 2 = 5$.

b) If $1 + 1 = 3$, then $2 + 2 = 4$.

c) If $1 + 1 = 3$, then $2 + 2 = 5$.

d) If monkeys can fly, then $1 + 1 = 3$.

18. Determine whether each of these conditional statements is true or false.

a) If $1 + 1 = 3$, then unicorns exist.

b) If $1 + 1 = 3$, then dogs can fly.

c) If $1 + 1 = 2$, then dogs can fly.

d) If $2 + 2 = 4$, then $1 + 2 = 3$.

19. For each of these sentences, determine whether an inclusive or, or an exclusive or, is intended. Explain your answer.

a) Coffee or tea comes with dinner.

b) A password must have at least three digits or be at least eight characters long.

c) The prerequisite for the course is a course in number theory or a course in cryptography.

d) You can pay using U.S. dollars or euros.

20. For each of these sentences, determine whether an inclusive or, or an exclusive or, is intended. Explain your answer.

a) Experience with C++ or Java is required.

b) Lunch includes soup or salad.

c) To enter the country you need a passport or a voter registration card.

d) Publish or perish.

21. For each of these sentences, state what the sentence means if the logical connective or is an inclusive or (that is, a disjunction) versus an exclusive or. Which of these meanings of or do you think is intended?

a) To take discrete mathematics, you must have taken calculus or a course in computer science.

b) When you buy a new car from Acme Motor Company, you get \$2000 back in cash or a 2% car loan.

c) Dinner for two includes two items from column A or three items from column B.

d) School is closed if more than 2 feet of snow falls or if the wind chill is below -100.

22. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) It is necessary to wash the boss's car to get promoted.

b) Winds from the south imply a spring thaw.

c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

d) Willy gets caught whenever he cheats.

e) You can access the website only if you pay a subscription fee.

f) Getting elected follows from knowing the right people.

g) Carol gets seasick whenever she is on a boat.

23. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements.]

a) It snows whenever the wind blows from the northeast.

b) The apple trees will bloom if it stays warm for a week.

c) That the Pistons win the championship implies that they beat the Lakers.

d) It is necessary to walk 8 miles to get to the top of Long's Peak.

e) To get tenure as a professor, it is sufficient to be world-famous.

f) If you drive more than 400 miles, you will need to buy gasoline.

g) Your guarantee is good only if you bought your CD player less than 90 days ago.

h) Jan will go swimming unless the water is too cold.

24. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) I will remember to send you the address only if you send me an e-mail message.
b) To be a citizen of this country, it is sufficient that you were born in the United States.
c) If you keep your textbook, it will be a useful reference in your future courses.
d) The Red Wings will win the Stanley Cup if their goalie plays well.
e) That you get the job implies that you had the best credentials.
f) The beach erodes whenever there is a storm.
g) It is necessary to have a valid password to log on to the server.
h) You will reach the summit unless you begin your climb too late.

25. Write each of these propositions in the form "p if and only if q" in English.

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.

26. Write each of these propositions in the form "p if and only if q" in English.

a) For you to get an A in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.
b) If you read the newspaper every day, you will be informed, and conversely.
c) It rains if it is a weekend day, and it is a weekend day if it rains.
d) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him.

27. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows today, I will ski tomorrow.
b) I come to class whenever there is going to be a quiz.
c) A positive integer is a prime only if it has no divisors other than 1 and itself.

28. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows tonight, then I will stay at home.
b) I go to the beach whenever it is a sunny summer day.
c) When I stay up late, it is necessary that I sleep until noon.

29. How many rows appear in a truth table for each of these compound propositions?

a) $p \to \neg p$
b) $(p \vee \neg r) \wedge (q \vee \neg s)$
c) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$
d) $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$

30. How many rows appear in a truth table for each of these compound propositions?

a) $(q \to \neg p) \vee (\neg p \to \neg q)$
b) $(p \vee \neg t) \wedge (p \vee \neg s)$
c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$
d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

31. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$
b) $p \vee \neg p$
c) $(p \vee \neg q) \to q$
d) $(p \vee q) \to (p \wedge q)$
e) $(p \to q) \leftrightarrow (\neg q \to \neg p)$
f) $(p \to q) \to (q \to p)$

32. Construct a truth table for each of these compound propositions.

a) $p \to \neg p$
b) $p \leftrightarrow \neg p$
c) $p \oplus (p \vee q)$
d) $(p \wedge q) \to (p \vee q)$
e) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

33. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \to (p \oplus q)$
b) $(p \oplus q) \to (p \wedge q)$
c) $(p \vee q) \oplus (p \wedge q)$
d) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
f) $(p \oplus q) \to (p \oplus \neg q)$

34. Construct a truth table for each of these compound propositions.

a) $p \oplus p$
b) $p \oplus \neg p$
c) $p \oplus \neg q$
d) $\neg p \oplus \neg q$
e) $(p \oplus q) \vee (p \oplus \neg q)$
f) $(p \oplus q) \wedge (p \oplus \neg q)$

35. Construct a truth table for each of these compound propositions.

a) $p \to \neg q$
b) $\neg p \leftrightarrow q$
c) $(p \to q) \vee (\neg p \to q)$
d) $(p \to q) \wedge (\neg p \to q)$
e) $(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$
f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$

36. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \vee r$
b) $(p \vee q) \wedge r$
c) $(p \wedge q) \vee r$
d) $(p \wedge q) \wedge r$
e) $(p \vee q) \wedge \neg r$
f) $(p \wedge q) \vee \neg r$

37. Construct a truth table for each of these compound propositions.

a) $p \to (\neg q \vee r)$
b) $\neg p \to (q \to r)$
c) $(p \to q) \vee (\neg p \to r)$
d) $(p \to q) \wedge (\neg p \to r)$
e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$
f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

38. Construct a truth table for $((p \to q) \to r) \to s$.

39. Construct a truth table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$.

☞ **40.** Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when p, q, and r have the same truth value and it is false otherwise.

☞ **41.** Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false, but is false when all three variables have the same truth value.

42. What is the value of x after each of these statements is encountered in a computer program, if $x = 1$ before the statement is reached?

a) **if** $x + 2 = 3$ **then** $x := x + 1$
b) **if** $(x + 1 = 3)$ *OR* $(2x + 2 = 3)$ **then** $x := x + 1$
c) **if** $(2x + 3 = 5)$ *AND* $(3x + 4 = 7)$ **then** $x := x + 1$
d) **if** $(x + 1 = 2)$ *XOR* $(x + 2 = 3)$ **then** $x := x + 1$
e) **if** $x < 2$ **then** $x := x + 1$

43. Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of each of these pairs of bit strings.

a) 101 1110, 010 0001
b) 1111 0000, 1010 1010
c) 00 0111 0001, 10 0100 1000
d) 11 1111 1111, 00 0000 0000

44. Evaluate each of these expressions.

a) $1\,1000 \wedge (0\,1011 \vee 1\,1011)$
b) $(0\,1111 \wedge 1\,0101) \vee 0\,1000$
c) $(0\,1010 \oplus 1\,1011) \oplus 0\,1000$
d) $(1\,1011 \vee 0\,1010) \wedge (1\,0001 \vee 1\,1011)$

Fuzzy logic is used in artificial intelligence. In fuzzy logic, a proposition has a truth value that is a number between 0 and 1, inclusive. A proposition with a truth value of 0 is false and one with a truth value of 1 is true. Truth values that are between 0 and 1 indicate varying degrees of truth. For instance, the truth value 0.8 can be assigned to the statement "Fred is happy," because Fred is happy most of the time, and the truth value 0.4 can be assigned to the statement "John is happy," because John is happy slightly less than half the time. Use these truth values to solve Exercises 45–47.

45. The truth value of the negation of a proposition in fuzzy logic is 1 minus the truth value of the proposition. What are the truth values of the statements "Fred is not happy" and "John is not happy?"

46. The truth value of the conjunction of two propositions in fuzzy logic is the minimum of the truth values of the two propositions. What are the truth values of the statements "Fred and John are happy" and "Neither Fred nor John is happy?"

47. The truth value of the disjunction of two propositions in fuzzy logic is the maximum of the truth values of the two propositions. What are the truth values of the statements "Fred is happy, or John is happy" and "Fred is not happy, or John is not happy?"

***48.** Is the assertion "This statement is false" a proposition?

***49.** The nth statement in a list of 100 statements is "Exactly n of the statements in this list are false."

a) What conclusions can you draw from these statements?
b) Answer part (a) if the nth statement is "At least n of the statements in this list are false."
c) Answer part (b) assuming that the list contains 99 statements.

50. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

1.2 Applications of Propositional Logic

Introduction

Logic has many important applications to mathematics, computer science, and numerous other disciplines. Statements in mathematics and the sciences and in natural language often are imprecise or ambiguous. To make such statements precise, they can be translated into the language of logic. For example, logic is used in the specification of software and hardware, because these specifications need to be precise before development begins. Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems. Logic can be used to analyze and solve many familiar puzzles. Software systems based on the rules of logic have been developed for constructing some, but not all, types of proofs automatically. We will discuss some of these applications of propositional logic in this section and in later chapters.

Translating English Sentences

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is

1.3 Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

We begin our discussion with a classification of compound propositions according to their possible truth values.

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Tautologies and contradictions are often important in mathematical reasoning. Example 1 illustrates these types of compound propositions.

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction. ◀

Logical Equivalences

Demo

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2 The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions p and q are equivalent if and only if the columns

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE 2 De Morgan's Laws.

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Extra Examples

giving their truth values agree. Example 2 illustrates this method to establish an extremely important and useful logical equivalence, namely, that of $\neg(p \vee q)$ with $\neg p \wedge \neg q$. This logical equivalence is one of the two **De Morgan laws**, shown in Table 2, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.

EXAMPLE 2 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution: The truth tables for these compound propositions are displayed in Table 3. Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent. ◀

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

EXAMPLE 3 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4. Because the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree, they are logically equivalent. ◀

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We will now establish a logical equivalence of two compound propositions involving three different propositional variables p, q, and r. To use a truth table to establish such a logical equivalence, we need eight rows, one for each possible combination of truth values of these three variables. We symbolically represent these combinations by listing the truth values of p, q, and r, respectively. These eight combinations of truth values are TTT, TTF, TFT, TFF, FTT, FTF, FFT, and FFF; we use this order when we display the rows of the truth table. Note that we need to double the number of rows in the truth tables we use to show that compound propositions are equivalent for each additional propositional variable, so that 16 rows are needed to establish the logical equivalence of two compound propositions involving four propositional variables, and so on. In general, 2^n rows are required if a compound proposition involves n propositional variables.

TABLE 5 **A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

EXAMPLE 4 Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution: We construct the truth table for these compound propositions in Table 5. Because the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these compound propositions are logically equivalent. ◀

The identities in Table 6 are a special case of Boolean algebra identities found in Table 5 of Section 12.1. See Table 1 in Section 2.2 for analogous set identities.

Table 6 contains some important equivalences. In these equivalences, **T** denotes the compound proposition that is always true and **F** denotes the compound proposition that is always

TABLE 6 **Logical Equivalences.**

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 **Logical Equivalences Involving Conditional Statements.**

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

false. We also display some useful equivalences for compound propositions involving conditional statements and biconditional statements in Tables 7 and 8, respectively. The reader is asked to verify the equivalences in Tables 6–8 in the exercises.

The associative law for disjunction shows that the expression $p \vee q \vee r$ is well defined, in the sense that it does not matter whether we first take the disjunction of p with q and then the disjunction of $p \vee q$ with r, or if we first take the disjunction of q and r and then take the disjunction of p with $q \vee r$. Similarly, the expression $p \wedge q \wedge r$ is well defined. By extending this reasoning, it follows that $p_1 \vee p_2 \vee \cdots \vee p_n$ and $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ are well defined whenever $p_1, p_2, \ldots, p_n$ are propositions.

Furthermore, note that De Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

and

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

We will sometimes use the notation $\bigvee_{j=1}^{n} p_j$ for $p_1 \vee p_2 \vee \cdots \vee p_n$ and $\bigwedge_{j=1}^{n} p_j$ for $p_1 \wedge p_2 \wedge \cdots \wedge p_n$. Using this notation, the extended version of De Morgan's laws can be written concisely as $\neg\left(\bigvee_{j=1}^{n} p_j\right) \equiv \bigwedge_{j=1}^{n} \neg p_j$ and $\neg\left(\bigwedge_{j=1}^{n} p_j\right) \equiv \bigvee_{j=1}^{n} \neg p_j$. (Methods for proving these identities will be given in Section 5.1.)

Using De Morgan's Laws

When using De Morgan's laws, remember to change the logical connective after you negate.

The two logical equivalences known as De Morgan's laws are particularly important. They tell us how to negate conjunctions and how to negate disjunctions. In particular, the equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ tells us that the negation of a disjunction is formed by taking the conjunction of the negations of the component propositions. Similarly, the equivalence $\neg(p \wedge q) \equiv \neg p \vee \neg q$ tells us that the negation of a conjunction is formed by taking the disjunction of the negations of the component propositions. Example 5 illustrates the use of De Morgan's laws.

EXAMPLE 5 Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Assessment

Solution: Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer."

Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \vee s$. By the second of De Morgan's laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$. Consequently, we can express the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert." ◀

Constructing New Logical Equivalences

The logical equivalences in Table 6, as well as any others that have been established (such as those shown in Tables 7 and 8), can be used to construct additional logical equivalences. The reason for this is that a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition. This technique is illustrated in Examples 6–8, where we also use the fact that if p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent (see Exercise 56).

EXAMPLE 6 Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Extra Examples

Solution: We could use a truth table to show that these compound propositions are equivalent (similar to what we did in Example 4). Indeed, it would not be hard to do so. However, we want to illustrate how to use logical identities that we already know to establish new logical identities, something that is of practical importance for establishing equivalences of compound propositions with a large number of variables. So, we will establish this equivalence by developing a series of

Links

AUGUSTUS DE MORGAN (1806–1871) Augustus De Morgan was born in India, where his father was a colonel in the Indian army. De Morgan's family moved to England when he was 7 months old. He attended private schools, where in his early teens he developed a strong interest in mathematics. De Morgan studied at Trinity College, Cambridge, graduating in 1827. Although he considered medicine or law, he decided on mathematics for his career. He won a position at University College, London, in 1828, but resigned after the college dismissed a fellow professor without giving reasons. However, he resumed this position in 1836 when his successor died, remaining until 1866.

De Morgan was a noted teacher who stressed principles over techniques. His students included many famous mathematicians, including Augusta Ada, Countess of Lovelace, who was Charles Babbage's collaborator in his work on computing machines (see page 31 for biographical notes on Augusta Ada). (De Morgan cautioned the countess against studying too much mathematics, because it might interfere with her childbearing abilities!)

De Morgan was an extremely prolific writer, publishing more than 1000 articles in more than 15 periodicals. De Morgan also wrote textbooks on many subjects, including logic, probability, calculus, and algebra. In 1838 he presented what was perhaps the first clear explanation of an important proof technique known as *mathematical induction* (discussed in Section 5.1 of this text), a term he coined. In the 1840s De Morgan made fundamental contributions to the development of symbolic logic. He invented notations that helped him prove propositional equivalences, such as the laws that are named after him. In 1842 De Morgan presented what is considered to be the first precise definition of a limit and developed new tests for convergence of infinite series. De Morgan was also interested in the history of mathematics and wrote biographies of Newton and Halley.

In 1837 De Morgan married Sophia Frend, who wrote his biography in 1882. De Morgan's research, writing, and teaching left little time for his family or social life. Nevertheless, he was noted for his kindness, humor, and wide range of knowledge.

logical equivalences, using one of the equivalences in Table 6 at a time, starting with $\neg(p \to q)$ and ending with $p \wedge \neg q$. We have the following equivalences.

$$\begin{aligned}
\neg(p \to q) &\equiv \neg(\neg p \vee q) && \text{by Example 3}\\
&\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law}\\
&\equiv p \wedge \neg q && \text{by the double negation law}
\end{aligned}$$

◀

EXAMPLE 7 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. (*Note:* we could also easily establish this equivalence using a truth table.) We have the following equivalences.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law}\\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law}\\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F}\\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction}\\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent. ◀

EXAMPLE 8 Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (*Note:* This could also be done using a truth table.)

$$\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3}\\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law}\\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction}\\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction}\\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}$$

◀

Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**. Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called a **solution** of this particular satisfiability problem. However, to show that a compound proposition is unsatisfiable, we need to show that *every* assignment of truth values to its variables makes it false. Although we can always use a truth table to determine whether a compound proposition is satisfiable, it is often more efficient not to, as Example 9 demonstrates.

EXAMPLE 9 Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$, and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.

Solution: Instead of using truth table to solve this problem, we will reason about truth values. Note that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when the three variable p, q, and r have the same truth value (see Exercise 40 of Section 1.1). Hence, it is satisfiable as there is at least one assignment of truth values for p, q, and r that makes it true. Similarly, note that $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false (see Exercise 41 of Section 1.1). Hence, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable, as there is at least one assignment of truth values for p, q, and r that makes it true.

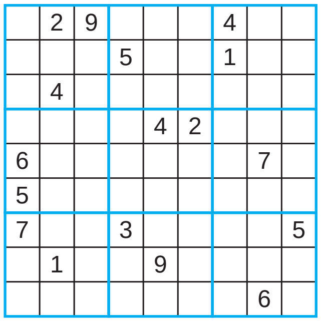
Finally, note that for $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ to be true, $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ and $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ must both be true. For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of three variables must be true and at least one must be false. However, these conditions are contradictory. From these observations we conclude that no assignment of truth values to p, q, and r makes $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ true. Hence, it is unsatisfiable. ◀

Links

AUGUSTA ADA, COUNTESS OF LOVELACE (1815–1852) Augusta Ada was the only child from the marriage of the famous poet Lord Byron and Lady Byron, Annabella Millbanke, who separated when Ada was 1 month old, because of Lord Byron's scandalous affair with his half sister. The Lord Byron had quite a reputation, being described by one of his lovers as "mad, bad, and dangerous to know." Lady Byron was noted for her intellect and had a passion for mathematics; she was called by Lord Byron "The Princess of Parallelograms." Augusta was raised by her mother, who encouraged her intellectual talents especially in music and mathematics, to counter what Lady Byron considered dangerous poetic tendencies. At this time, women were not allowed to attend universities and could not join learned societies. Nevertheless, Augusta pursued her mathematical studies independently and with mathematicians, including William Frend. She was also encouraged by another female mathematician, Mary Somerville, and in 1834 at a dinner party hosted by Mary Somerville, she learned about Charles Babbage's ideas for a calculating machine, called the Analytic Engine. In 1838 Augusta Ada married Lord King, later elevated to Earl of Lovelace. Together they had three children.

Augusta Ada continued her mathematical studies after her marriage. Charles Babbage had continued work on his Analytic Engine and lectured on this in Europe. In 1842 Babbage asked Augusta Ada to translate an article in French describing Babbage's invention. When Babbage saw her translation, he suggested she add her own notes, and the resulting work was three times the length of the original. The most complete accounts of the Analytic Engine are found in Augusta Ada's notes. In her notes, she compared the working of the Analytic Engine to that of the Jacquard loom, with Babbage's punch cards analogous to the cards used to create patterns on the loom. Furthermore, she recognized the promise of the machine as a general purpose computer much better than Babbage did. She stated that the "engine is the material expression of any indefinite function of any degree of generality and complexity." Her notes on the Analytic Engine anticipate many future developments, including computer-generated music. Augusta Ada published her writings under her initials A.A.L. concealing her identity as a woman as did many women at a time when women were not considered to be the intellectual equals of men. After 1845 she and Babbage worked toward the development of a system to predict horse races. Unfortunately, their system did not work well, leaving Augusta Ada heavily in debt at the time of her death at an unfortunately young age from uterine cancer.

In 1953 Augusta Ada's notes on the Analytic Engine were republished more than 100 years after they were written, and after they had been long forgotten. In his work in the 1950s on the capacity of computers to think (and his famous Turing Test), Alan Turing responded to Augusta Ada's statement that "The Analytic Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform." This "dialogue" between Turing and Augusta Ada is still the subject of controversy. Because of her fundamental contributions to computing, the programming language Ada is named in honor of the Countess of Lovelace.



	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

FIGURE 1 A 9 × 9 Sudoku puzzle.

Applications of Satisfiability

Many problems, in diverse areas such as robotics, software testing, computer-aided design, machine vision, integrated circuit design, computer networking, and genetics, can be modeled in terms of propositional satisfiability. Although most of these applications are beyond the scope of this book, we will study one application here. In particular, we will show how to use propositional satisfiability to model Sudoku puzzles.

SUDOKU A **Sudoku puzzle** is represented by a 9 × 9 grid made up of nine 3 × 3 subgrids, known as **blocks**, as shown in Figure 1. For each puzzle, some of the 81 cells, called **givens**, are assigned one of the numbers 1, 2, . . . , 9, and the other cells are blank. The puzzle is solved by assigning a number to each blank cell so that every row, every column, and every one of the nine 3 × 3 blocks contains each of the nine possible numbers. Note that instead of using a 9 × 9 grid, Sudoku puzzles can be based on $n^2 \times n^2$ grids, for any positive integer n, with the $n^2 \times n^2$ grid made up of n^2 $n \times n$ subgrids.

Links

The popularity of Sudoku dates back to the 1980s when it was introduced in Japan. It took 20 years for Sudoku to spread to rest of the world, but by 2005, Sudoku puzzles were a worldwide craze. The name Sudoku is short for the Japanese *suuji wa dokushin ni kagiru*, which means "the digits must remain single." The modern game of Sudoku was apparently designed in the late 1970s by an American puzzle designer. The basic ideas of Sudoku date back even further; puzzles printed in French newspapers in the 1890s were quite similar, but not identical, to modern Sudoku.

Sudoku puzzles designed for entertainment have two additional important properties. First, they have exactly one solution. Second, they can be solved using reasoning alone, that is, without resorting to searching all possible assignments of numbers to the cells. As a Sudoku puzzle is solved, entries in blank cells are successively determined by already known values. For instance, in the grid in Figure 1, the number 4 must appear in exactly one cell in the second row. How can we determine which of the seven blank cells it must appear? First, we observe that 4 cannot appear in one of the first three cells or in one of the last three cells of this row, because it already appears in another cell in the block each of these cells is in. We can also see that 4 cannot appear in the fifth cell in this row, as it already appears in the fifth column in the fourth row. This means that 4 must appear in the sixth cell of the second row.

Many strategies based on logic and mathematics have been devised for solving Sudoku puzzles (see [Da10], for example). Here, we discuss one of the ways that have been developed for solving Sudoku puzzles with the aid of a computer, which depends on modeling the puzzle as a propositional satisfiability problem. Using the model we describe, particular Sudoku puzzles can be solved using software developed to solve satisfiability problems. Currently, Sudoku puzzles can be solved in less than 10 milliseconds this way. It should be noted that there are many other approaches for solving Sudoku puzzles via computers using other techniques.

To encode a Sudoku puzzle, let $p(i, j, n)$ denote the proposition that is true when the number n is in the cell in the ith row and jth column. There are $9 \times 9 \times 9 = 729$ such propositions, as i, j, and n all range from 1 to 9. For example, for the puzzle in Figure 1, the number 6 is given as the value in the fifth row and first column. Hence, we see that $p(5, 1, 6)$ is true, but $p(5, j, 6)$ is false for $j = 2, 3, \ldots, 9$.

Given a particular Sudoku puzzle, we begin by encoding each of the given values. Then, we construct compound propositions that assert that every row contains every number, every column contains every number, every 3×3 block contains every number, and each cell contains no more than one number. It follows, as the reader should verify, that the Sudoku puzzle is solved by finding an assignment of truth values to the 729 propositions $p(i, j, n)$ with i, j, and n each ranging from 1 to 9 that makes the conjunction of all these compound propositions true. After listing these assertions, we will explain how to construct the assertion that every row contains every integer from 1 to 9. We will leave the construction of the other assertions that every column contains every number and each of the nine 3×3 blocks contains every number to the exercises.

- For each cell with a given value, we assert $p(i, j, n)$ when the cell in row i and column j has the given value n.
- We assert that every row contains every number:

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$$

- We assert that every column contains every number:

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

It is tricky setting up the two inner indices so that all nine cells in each square block are examined.

- We assert that each of the nine 3×3 blocks contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

- To assert that no cell contains more than one number, we take the conjunction over all values of n, n', i, and j where each variable ranges from 1 to 9 and $n \neq n'$ of $p(i, j, n) \rightarrow \neg p(i, j, n')$.

We now explain how to construct the assertion that every row contains every number. First, to assert that row i contains the number n, we form $\bigvee_{j=1}^{9} p(i, j, n)$. To assert that row i contains all n numbers, we form the conjunction of these disjunctions over all nine possible values of n, giving us $\bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$. Finally, to assert that every row contains every number, we take the conjunction of $\bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$ over all nine rows. This gives us $\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$. (Exercises 65 and 66 ask for explanations of the assertions that every column contains every number and that each of the nine 3×3 blocks contains every number.)

Given a particular Sudoku puzzle, to solve this puzzle we can find a solution to the satisfiability problems that asks for a set of truth values for the 729 variables $p(i, j, n)$ that makes the conjunction of all the listed assertions true.

Solving Satisfiability Problems

A truth table can be used to determine whether a compound proposition is satisfiable, or equivalently, whether its negation is a tautology (see Exercise 60). This can be done by hand for a compound proposition with a small number of variables, but when the number of variables grows, this becomes impractical. For instance, there are $2^{20} = 1{,}048{,}576$ rows in the truth table for a compound proposition with 20 variables. Clearly, you need a computer to help you determine, in this way, whether a compound proposition in 20 variables is satisfiable.

When many applications are modeled, questions concerning the satisfiability of compound propositions with hundreds, thousands, or millions of variables arise. Note, for example, that when there are 1000 variables, checking every one of the 2^{1000} (a number with more than 300 decimal digits) possible combinations of truth values of the variables in a compound proposition cannot be done by a computer in even trillions of years. No procedure is known that a computer can follow to determine in a reasonable amount of time whether an arbitrary compound proposition in such a large number of variables is satisfiable. However, progress has been made developing methods for solving the satisfiability problem for the particular types of compound propositions that arise in practical applications, such as for the solution of Sudoku puzzles. Many computer programs have been developed for solving satisfiability problems which have practical use. In our discussion of the subject of algorithms in Chapter 3, we will discuss this question further. In particular, we will explain the important role the propositional satisfiability problem plays in the study of the complexity of algorithms.

Exercises

1. Use truth tables to verify these equivalences.

a) $p \wedge \mathbf{T} \equiv p$ **b)** $p \vee \mathbf{F} \equiv p$
c) $p \wedge \mathbf{F} \equiv \mathbf{F}$ **d)** $p \vee \mathbf{T} \equiv \mathbf{T}$
e) $p \vee p \equiv p$ **f)** $p \wedge p \equiv p$

2. Show that $\neg(\neg p)$ and p are logically equivalent.

3. Use truth tables to verify the commutative laws

a) $p \vee q \equiv q \vee p$. **b)** $p \wedge q \equiv q \wedge p$.

4. Use truth tables to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

5. Use a truth table to verify the distributive law
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

6. Use a truth table to verify the first De Morgan law
$\neg(p \wedge q) \equiv \neg p \vee \neg q$.

7. Use De Morgan's laws to find the negation of each of the following statements.

a) Jan is rich and happy.

b) Carlos will bicycle or run tomorrow.

HENRY MAURICE SHEFFER (1883–1964) Henry Maurice Sheffer, born to Jewish parents in the western Ukraine, emigrated to the United States in 1892 with his parents and six siblings. He studied at the Boston Latin School before entering Harvard, where he completed his undergraduate degree in 1905, his master's in 1907, and his Ph.D. in philosophy in 1908. After holding a postdoctoral position at Harvard, Henry traveled to Europe on a fellowship. Upon returning to the United States, he became an academic nomad, spending one year each at the University of Washington, Cornell, the University of Minnesota, the University of Missouri, and City College in New York. In 1916 he returned to Harvard as a faculty member in the philosophy department. He remained at Harvard until his retirement in 1952.

Sheffer introduced what is now known as the Sheffer stroke in 1913; it became well known only after its use in the 1925 edition of Whitehead and Russell's *Principia Mathematica*. In this same edition Russell wrote that Sheffer had invented a powerful method that could be used to simplify the *Principia*. Because of this comment, Sheffer was something of a mystery man to logicians, especially because Sheffer, who published little in his career, never published the details of this method, only describing it in mimeographed notes and in a brief published abstract.

Sheffer was a dedicated teacher of mathematical logic. He liked his classes to be small and did not like auditors. When strangers appeared in his classroom, Sheffer would order them to leave, even his colleagues or distinguished guests visiting Harvard. Sheffer was barely five feet tall; he was noted for his wit and vigor, as well as for his nervousness and irritability. Although widely liked, he was quite lonely. He is noted for a quip he spoke at his retirement: "Old professors never die, they just become emeriti." Sheffer is also credited with coining the term "Boolean algebra" (the subject of Chapter 12 of this text). Sheffer was briefly married and lived most of his later life in small rooms at a hotel packed with his logic books and vast files of slips of paper he used to jot down his ideas. Unfortunately, Sheffer suffered from severe depression during the last two decades of his life.

c) Mei walks or takes the bus to class.
d) Ibrahim is smart and hard working.

8. Use De Morgan's laws to find the negation of each of the following statements.

a) Kwame will take a job in industry or go to graduate school.
b) Yoshiko knows Java and calculus.
c) James is young and strong.
d) Rita will move to Oregon or Washington.

9. Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$ **b)** $p \rightarrow (p \vee q)$
c) $\neg p \rightarrow (p \rightarrow q)$ **d)** $(p \wedge q) \rightarrow (p \rightarrow q)$
e) $\neg(p \rightarrow q) \rightarrow p$ **f)** $\neg(p \rightarrow q) \rightarrow \neg q$

10. Show that each of these conditional statements is a tautology by using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$
b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
c) $[p \wedge (p \rightarrow q)] \rightarrow q$
d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

11. Show that each conditional statement in Exercise 9 is a tautology without using truth tables.

12. Show that each conditional statement in Exercise 10 is a tautology without using truth tables.

13. Use truth tables to verify the absorption laws.

a) $p \vee (p \wedge q) \equiv p$ **b)** $p \wedge (p \vee q) \equiv p$

14. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.

15. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

Each of Exercises 16–28 asks you to show that two compound propositions are logically equivalent. To do this, either show that both sides are true, or that both sides are false, for exactly the same combinations of truth values of the propositional variables in these expressions (whichever is easier).

16. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

17. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

18. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

19. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.

20. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

21. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

22. Show that $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are logically equivalent.

23. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.

24. Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

25. Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.

26. Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.

27. Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.

28. Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.

29. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

30. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.

31. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.

32. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

33. Show that $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$ are not logically equivalent.

The **dual** of a compound proposition that contains only the logical operators $\vee$, $\wedge$, and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each **T** by **F**, and each **F** by **T**. The dual of s is denoted by s^*.

34. Find the dual of each of these compound propositions.

a) $p \vee \neg q$ **b)** $p \wedge (q \vee (r \wedge \mathbf{T}))$
c) $(p \wedge \neg q) \vee (q \wedge \mathbf{F})$

35. Find the dual of each of these compound propositions.

a) $p \wedge \neg q \wedge \neg r$ **b)** $(p \wedge q \wedge r) \vee s$
c) $(p \vee \mathbf{F}) \wedge (q \vee \mathbf{T})$

36. When does $s^* = s$, where s is a compound proposition?

37. Show that $(s^*)^* = s$ when s is a compound proposition.

38. Show that the logical equivalences in Table 6, except for the double negation law, come in pairs, where each pair contains compound propositions that are duals of each other.

****39.** Why are the duals of two equivalent compound propositions also equivalent, where these compound propositions contain only the operators $\wedge$, $\vee$, and $\neg$?

40. Find a compound proposition involving the propositional variables p, q, and r that is true when p and q are true and r is false, but is false otherwise. [*Hint:* Use a conjunction of each propositional variable or its negation.]

41. Find a compound proposition involving the propositional variables p, q, and r that is true when exactly two of p, q, and r are true and is false otherwise. [*Hint:* Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the compound proposition is true. Each conjunction should include each of the three propositional variables or its negations.]

42. Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction included for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in **disjunctive normal form**.

A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

43. Show that $\neg$, $\wedge$, and $\vee$ form a functionally complete collection of logical operators. [*Hint:* Use the fact that every compound proposition is logically equivalent to one in disjunctive normal form, as shown in Exercise 42.]

***44.** Show that $\neg$ and $\wedge$ form a functionally complete collection of logical operators. [*Hint:* First use a De Morgan law to show that $p \vee q$ is logically equivalent to $\neg(\neg p \wedge \neg q)$.]

***45.** Show that $\neg$ and $\vee$ form a functionally complete collection of logical operators.

The following exercises involve the logical operators *NAND* and *NOR*. The proposition p *NAND* q is true when either p or q, or both, are false; and it is false when both p and q are true. The proposition p *NOR* q is true when both p and q are false, and it is false otherwise. The propositions p *NAND* q and p *NOR* q are denoted by $p \mid q$ and $p \downarrow q$, respectively. (The operators | and $\downarrow$ are called the **Sheffer stroke** and the **Peirce arrow** after H. M. Sheffer and C. S. Peirce, respectively.)

46. Construct a truth table for the logical operator *NAND*.

47. Show that $p \mid q$ is logically equivalent to $\neg(p \wedge q)$.

48. Construct a truth table for the logical operator *NOR*.

49. Show that $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$.

50. In this exercise we will show that $\{\downarrow\}$ is a functionally complete collection of logical operators.

a) Show that $p \downarrow p$ is logically equivalent to $\neg p$.

b) Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \vee q$.

c) Conclude from parts (a) and (b), and Exercise 49, that $\{\downarrow\}$ is a functionally complete collection of logical operators.

***51.** Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator $\downarrow$.

52. Show that $\{|\}$ is a functionally complete collection of logical operators.

53. Show that $p \mid q$ and $q \mid p$ are equivalent.

54. Show that $p \mid (q \mid r)$ and $(p \mid q) \mid r$ are not equivalent, so that the logical operator | is not associative.

***55.** How many different truth tables of compound propositions are there that involve the propositional variables p and q?

56. Show that if p, q, and r are compound propositions such that p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.

57. The following sentence is taken from the specification of a telephone system: "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." This specification is hard to understand because it involves two conditional statements. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not conditional statements.

58. How many of the disjunctions $p \vee \neg q$, $\neg p \vee q$, $q \vee r$, $q \vee \neg r$, and $\neg q \vee \neg r$ can be made simultaneously true by an assignment of truth values to p, q, and r?

59. How many of the disjunctions $p \vee \neg q \vee s$, $\neg p \vee \neg r \vee s$, $\neg p \vee \neg r \vee \neg s$, $\neg p \vee q \vee \neg s$, $q \vee r \vee \neg s$, $q \vee \neg r \vee \neg s$, $\neg p \vee \neg q \vee \neg s$, $p \vee r \vee s$, and $p \vee r \vee \neg s$ can be made simultaneously true by an assignment of truth values to p, q, r, and s?

60. Show that the negation of an unsatisfiable compound proposition is a tautology and the negation of a compound proposition that is a tautology is unsatisfiable.

61. Determine whether each of these compound propositions is satisfiable.

a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

b) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$

c) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

62. Determine whether each of these compound propositions is satisfiable.

a) $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$

b) $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg r \vee \neg s)$

c) $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (\neg p \vee r \vee s) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee s) \wedge (\neg p \vee \neg r \vee \neg s)$

63. Show how the solution of a given 4 × 4 Sudoku puzzle can be found by solving a satisfiability problem.

64. Construct a compound proposition that asserts that every cell of a 9 × 9 Sudoku puzzle contains at least one number.

65. Explain the steps in the construction of the compound proposition given in the text that asserts that every column of a 9 × 9 Sudoku puzzle contains every number.

***66.** Explain the steps in the construction of the compound proposition given in the text that asserts that each of the nine 3 × 3 blocks of a 9 × 9 Sudoku puzzle contains every number.

1.4 Predicates and Quantifiers

Introduction

Propositional logic, studied in Sections 1.1–1.3, cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that

"Every computer connected to the university network is functioning properly."

No rules of propositional logic allow us to conclude the truth of the statement

"MATH3 is functioning properly,"

where MATH3 is one of the computers connected to the university network. Likewise, we cannot use the rules of propositional logic to conclude from the statement

"CS2 is under attack by an intruder,"

where CS2 is a computer on the university network, to conclude the truth of

"There is a computer on the university network that is under attack by an intruder."

In this section we will introduce a more powerful type of logic called **predicate logic**. We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects. To understand predicate logic, we first need to introduce the concept of a predicate. Afterward, we will introduce the notion of quantifiers, which enable us to reason with statements that assert that a certain property holds for all objects of a certain type and with statements that assert the existence of an object with a particular property.

Predicates

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"

and

"computer x is under attack by an intruder,"

and

"computer x is functioning properly,"

are often found in mathematical assertions, in computer programs, and in system specifications. These statements are neither true nor false when the values of the variables are not specified. In this section, we will discuss the ways that propositions can be produced from such statements.

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part—the **predicate**, "is greater than 3"—refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value. Consider Examples 1 and 2.

EXAMPLE 1 Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true. However, $P(2)$, which is the statement "$2 > 3$," is false. ◀

EXAMPLE 2 Let $A(x)$ denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of A(CS1), A(CS2), and A(MATH1)?

Solution: We obtain the statement A(CS1) by setting $x =$ CS1 in the statement "Computer x is under attack by an intruder." Because CS1 is not on the list of computers currently under attack, we conclude that A(CS1) is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that A(CS2) and A(MATH1) are true. ◀

We can also have statements that involve more than one variable. For instance, consider the statement "$x = y + 3$." We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate. When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

EXAMPLE 3 Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Extra Examples

Solution: To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true. ◀

Links

CHARLES SANDERS PEIRCE (1839–1914) Many consider Charles Peirce, born in Cambridge, Massachusetts, to be the most original and versatile American intellect. He made important contributions to an amazing number of disciplines, including mathematics, astronomy, chemistry, geodesy, metrology, engineering, psychology, philology, the history of science, and economics. Peirce was also an inventor, a lifelong student of medicine, a book reviewer, a dramatist and an actor, a short story writer, a phenomenologist, a logician, and a metaphysician. He is noted as the preeminent system-building philosopher competent and productive in logic, mathematics, and a wide range of sciences. He was encouraged by his father, Benjamin Peirce, a professor of mathematics and natural philosophy at Harvard, to pursue a career in science. Instead, he decided to study logic and scientific methodology. Peirce attended Harvard (1855–1859) and received a Harvard master of arts degree (1862) and an advanced degree in chemistry from the Lawrence Scientific School (1863).

In 1861, Peirce became an aide in the U.S. Coast Survey, with the goal of better understanding scientific methodology. His service for the Survey exempted him from military service during the Civil War. While working for the Survey, Peirce did astronomical and geodesic work. He made fundamental contributions to the design of pendulums and to map projections, applying new mathematical developments in the theory of elliptic functions. He was the first person to use the wavelength of light as a unit of measurement. Peirce rose to the position of Assistant for the Survey, a position he held until forced to resign in 1891 when he disagreed with the direction taken by the Survey's new administration.

While making his living from work in the physical sciences, Peirce developed a hierarchy of sciences, with mathematics at the top rung, in which the methods of one science could be adapted for use by those sciences under it in the hierarchy. During this time, he also founded the American philosophical theory of pragmatism.

The only academic position Peirce ever held was lecturer in logic at Johns Hopkins University in Baltimore (1879–1884). His mathematical work during this time included contributions to logic, set theory, abstract algebra, and the philosophy of mathematics. His work is still relevant today, with recent applications of this work on logic to artificial intelligence. Peirce believed that the study of mathematics could develop the mind's powers of imagination, abstraction, and generalization. His diverse activities after retiring from the Survey included writing for periodicals, contributing to scholarly dictionaries, translating scientific papers, guest lecturing, and textbook writing. Unfortunately, his income from these pursuits was insufficient to protect him and his second wife from abject poverty. He was supported in his later years by a fund created by his many admirers and administered by the philosopher William James, his lifelong friend. Although Peirce wrote and published voluminously in a vast range of subjects, he left more than 100,000 pages of unpublished manuscripts. Because of the difficulty of studying his unpublished writings, scholars have only recently started to understand some of his varied contributions. A group of people is devoted to making his work available over the Internet to bring a better appreciation of Peirce's accomplishments to the world.

EXAMPLE 4 Let $A(c, n)$ denote the statement "Computer c is connected to network n," where c is a variable representing a computer and n is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

Solution: Because MATH1 is not connected to the CAMPUS1 network, we see that A(MATH1, CAMPUS1) is false. However, because MATH1 is connected to the CAMPUS2 network, we see that A(MATH1, CAMPUS2) is true. ◀

Similarly, we can let $R(x, y, z)$ denote the statement ‘$x + y = z$." When values are assigned to the variables x, y, and z, this statement has a truth value.

EXAMPLE 5 What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution: The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement "$1 + 2 = 3$," which is true. Also note that $R(0, 0, 1)$, which is the statement "$0 + 0 = 1$," is false. ◀

In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by

$$P(x_1, x_2, \ldots, x_n).$$

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called an n**-place predicate** or a n**-ary predicate**.

Propositional functions occur in computer programs, as Example 6 demonstrates.

EXAMPLE 6 Consider the statement

if $x > 0$ **then** $x := x + 1$.

When this statement is encountered in a program, the value of the variable x at that point in the execution of the program is inserted into $P(x)$, which is "$x > 0$." If $P(x)$ is true for this value of x, the assignment statement $x := x + 1$ is executed, so the value of x is increased by 1. If $P(x)$ is false for this value of x, the assignment statement is not executed, so the value of x is not changed. ◀

PRECONDITIONS AND POSTCONDITIONS Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input. (Note that unless the correctness of a computer program is established, no amount of testing can show that it produces the desired output for all input values, unless every input value is tested.) The statements that describe valid input are known as **preconditions** and the conditions that the output should satisfy when the program has run are known as **postconditions**. As Example 7 illustrates, we use predicates to describe both preconditions and postconditions. We will study this process in greater detail in Section 5.5.

EXAMPLE 7 Consider the following program, designed to interchange the values of two variables x and y.

```
temp := x
x := y
y := temp
```

Find predicates that we can use as the precondition and the postcondition to verify the correctness of this program. Then explain how to use them to verify that for all valid input the program does what is intended.

Solution: For the precondition, we need to express that x and y have particular values before we run the program. So, for this precondition we can use the predicate $P(x, y)$, where $P(x, y)$ is the statement "$x = a$ and $y = b$," where a and b are the values of x and y before we run the program. Because we want to verify that the program swaps the values of x and y for all input values, for the postcondition we can use $Q(x, y)$, where $Q(x, y)$ is the statement "$x = b$ and $y = a$."

To verify that the program always does what it is supposed to do, suppose that the precondition $P(x, y)$ holds. That is, we suppose that the statement "$x = a$ and $y = b$" is true. This means that $x = a$ and $y = b$. The first step of the program, $temp := x$, assigns the value of x to the variable *temp*, so after this step we know that $x = a$, $temp = a$, and $y = b$. After the second step of the program, $x := y$, we know that $x = b$, $temp = a$, and $y = b$. Finally, after the third step, we know that $x = b$, $temp = a$, and $y = a$. Consequently, after this program is run, the postcondition $Q(x, y)$ holds, that is, the statement "$x = b$ and $y = a$" is true. ◀

Quantifiers

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called **quantification**, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications. We will focus on two types of quantification here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Assessment

THE UNIVERSAL QUANTIFIER Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the **domain**. Such a statement is expressed using universal quantification. The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain. Note that the domain specifies the possible values of the variable x. The meaning of the universal quantification of $P(x)$ changes when we change the domain. The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

Assessment

DEFINITION 1

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

The meaning of the universal quantifier is summarized in the first row of Table 1. We illustrate the use of the universal quantifier in Examples 8–13.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

EXAMPLE 8 Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Extra Examples

Solution: Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x)$$

is true. ◀

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. Note that if the domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.

Remember that the truth value of $\forall x P(x)$ depends on the domain!

Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," "for each," and "for any."

Remark: It is best to avoid using "for any x" because it is often ambiguous as to whether "any" means "every" or "some." In some cases, "any" is unambiguous, such as when it is used in negatives, for example, "there is not any reason to avoid studying."

A statement $\forall x P(x)$ is false, where $P(x)$ is a propositional function, if and only if $P(x)$ is not always true when x is in the domain. One way to show that $P(x)$ is not always true when x is in the domain is to find a counterexample to the statement $\forall x P(x)$. Note that a single counterexample is all we need to establish that $\forall x P(x)$ is false. Example 9 illustrates how counterexamples are used.

EXAMPLE 9 Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus

$$\forall x Q(x)$$

is false. ◀

EXAMPLE 10 Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$. ◀

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections of this book.

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

EXAMPLE 11 What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false. ◀

EXAMPLE 12 What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network." ◀

As we have pointed out, specifying the domain is mandatory when quantifiers are used. The truth value of a quantified statement often depends on which elements are in this domain, as Example 13 shows.

EXAMPLE 13 What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 \not\geq \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$. ◀

THE EXISTENTIAL QUANTIFIER Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the domain.

DEFINITION 2 The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.

Besides the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is." The existential quantification $\exists x P(x)$ is read as

"There is an x such that $P(x)$,"
"There is at least one x such that $P(x)$,"

or

"For some $x P(x)$."

The meaning of the existential quantifier is summarized in the second row of Table 1. We illustrate the use of the existential quantifier in Examples 14–16.

EXAMPLE 14 Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Extra Examples

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true. ◀

Observe that the statement $\exists x P(x)$ is false if and only if there is no element x in the domain for which $P(x)$ is true. That is, $\exists x P(x)$ is false if and only if $P(x)$ is false for every element of the domain. We illustrate this observation in Example 15.

EXAMPLE 15 Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false. ◀

Remember that the truth value of $\exists x P(x)$ depends on the domain!

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element x in the domain for which $Q(x)$ is true.

When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$— the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

EXAMPLE 16 What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true. ◀

It is sometimes helpful to think in terms of looping and searching when determining the truth value of a quantification. Suppose that there are n objects in the domain for the variable x. To determine whether $\forall x P(x)$ is true, we can loop through all n values of x to see whether $P(x)$ is always true. If we encounter a value x for which $P(x)$ is false, then we have shown that $\forall x P(x)$ is false. Otherwise, $\forall x P(x)$ is true. To see whether $\exists x P(x)$ is true, we loop through the n values of x searching for a value for which $P(x)$ is true. If we find one, then $\exists x P(x)$ is true. If we never find such an x, then we have determined that $\exists x P(x)$ is false. (Note that this searching procedure does not apply if there are infinitely many values in the domain. However, it is still a useful way of thinking about the truth values of quantifications.)

THE UNIQUENESS QUANTIFIER We have now introduced universal and existential quantifiers. These are the most important quantifiers in mathematics and computer science. However, there is no limitation on the number of different quantifiers we can define, such as "there are exactly two," "there are no more than three," "there are at least 100," and so on. Of these other quantifiers, the one that is most often seen is the **uniqueness quantifier**, denoted by $\exists!$ or $\exists_1$. The notation $\exists! x P(x)$ [or $\exists_1 x P(x)$] states "There exists a unique x such that $P(x)$ is true." (Other phrases for uniqueness quantification include "there is exactly one" and "there is one and only one.") For instance, $\exists! x(x - 1 = 0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x - 1 = 0$. This is a true statement, as $x = 1$ is the unique real number such that $x - 1 = 0$. Observe that we can use quantifiers and propositional logic to express uniqueness (see Exercise 52 in Section 1.5), so the uniqueness quantifier can be avoided. Generally, it is best to stick with existential and universal quantifiers so that rules of inference for these quantifiers can be used.

Quantifiers with Restricted Domains

An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, a condition a variable must satisfy is included after the quantifier. This is illustrated in Example 17. We will also describe other forms of this notation involving set membership in Section 2.1.

EXAMPLE 17 What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution: The statement $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \rightarrow x^2 > 0)$.

The statement $\forall y \neq 0\,(y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

Finally, the statement $\exists z > 0\,(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$. ◀

Note that the restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0\ (x^2 > 0)$ is another way of expressing $\forall x(x < 0 \rightarrow x^2 > 0)$. On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0\ (z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

Binding Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**. All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier. Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

EXAMPLE 18 In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1)$, x is bound, but y is free.

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement. Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$. That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$. Observe that we could have written our statement using two different variables x and y, as $\exists x(P(x) \wedge Q(x)) \vee \forall y R(y)$, because the scopes of the two quantifiers do not overlap. The reader should be aware that in common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap. ◀

Logical Equivalences Involving Quantifiers

In Section 1.3 we introduced the notion of logical equivalences of compound propositions. We can extend this notion to expressions involving predicates and quantifiers.

DEFINITION 3 Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example 19 illustrates how to show that two statements involving predicates and quantifiers are logically equivalent.

EXAMPLE 19 Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout). This logical equivalence shows that we can distribute a universal quantifier over a conjunction. Furthermore, we can also distribute an existential quantifier over a disjunction. However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction. (See Exercises 50 and 51.)

Solution: To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates P and Q are, and no matter which domain of discourse is used. Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent by doing two things. First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall x P(x) \wedge \forall x Q(x)$ is true. Second, we show that if $\forall x P(x) \wedge \forall x Q(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

So, suppose that $\forall x(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true. This means that $\forall x P(x) \wedge \forall x Q(x)$ is true.

Next, suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true. It follows that $\forall x P(x)$ is true and $\forall x Q(x)$ is true. Hence, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true [because $P(x)$ and $Q(x)$ are both true for all elements in the domain, there is no conflict using the same value of a here].

It follows that for all a, $P(a) \wedge Q(a)$ is true. It follows that $\forall x(P(x) \wedge Q(x))$ is true. We can now conclude that

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x).$$

◀

Negating Quantified Expressions

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely,

$$\forall x P(x),$$

Assessment

where $P(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class. The negation of this statement is "It is not the case that every student in your class has taken a course in calculus." This is equivalent to "There is a student in your class who has not taken a course in calculus." And this is simply the existential quantification of the negation of the original propositional function, namely,

$$\exists x \neg P(x).$$

This example illustrates the following logical equivalence:

$$\neg \forall x P(x) \equiv \exists x \neg P(x).$$

To show that $\neg \forall x P(x)$ and $\exists x P(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is, first note that $\neg \forall x P(x)$ is true if and only if $\forall x P(x)$ is false. Next, note that $\forall x P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false. This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true. Finally, note that there is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x \neg P(x)$ is true. Putting these steps together, we can conclude that $\neg \forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true. It follows that $\neg \forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent.

Suppose we wish to negate an existential quantification. For instance, consider the proposition "There is a student in this class who has taken a course in calculus." This is the existential quantification

$$\exists x Q(x),$$

where $Q(x)$ is the statement "x has taken a course in calculus." The negation of this statement is the proposition "It is not the case that there is a student in this class who has taken a course in calculus." This is equivalent to "Every student in this class has not taken calculus," which is just the universal quantification of the negation of the original propositional function, or, phrased in the language of quantifiers,

$$\forall x \neg Q(x).$$

This example illustrates the equivalence

$$\neg \exists x Q(x) \equiv \forall x \neg Q(x).$$

To show that $\neg \exists x Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is, first note that $\neg \exists x Q(x)$ is true if and only if $\exists x Q(x)$ is false. This is true if and

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

only if no x exists in the domain for which $Q(x)$ is true. Next, note that no x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain. Finally, note that $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x \neg Q(x)$ is true. Putting these steps together, we see that $\neg\exists x Q(x)$ is true if and only if $\forall x \neg Q(x)$ is true. We conclude that $\neg\exists x Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent.

The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**. These rules are summarized in Table 2.

Remark: When the domain of a predicate $P(x)$ consists of n elements, where n is a positive integer greater than one, the rules for negating quantified statements are exactly the same as De Morgan's laws discussed in Section 1.3. This is why these rules are called De Morgan's laws for quantifiers. When the domain has n elements $x_1, x_2, \ldots, x_n$, it follows that $\neg\forall x P(x)$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n))$, which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$. Similarly, $\neg\exists x P(x)$ is the same as $\neg(P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n))$, which by De Morgan's laws is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n)$, and this is the same as $\forall x \neg P(x)$.

We illustrate the negation of quantified statements in Examples 20 and 21.

EXAMPLE 20 What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution: Let $H(x)$ denote "x is honest." Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the domain consists of all politicians. The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed as "Every politician is dishonest." (*Note:* In English, the statement "All politicians are not honest" is ambiguous. In common usage, this statement often means "Not all politicians are honest." Consequently, we do not use this statement to express this negation.)

Extra Examples

Let $C(x)$ denote "x eats cheeseburgers." Then the statement "All Americans eat cheeseburgers" is represented by $\forall x C(x)$, where the domain consists of all Americans. The negation of this statement is $\neg\forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This negation can be expressed in several different ways, including "Some American does not eat cheeseburgers" and "There is an American who does not eat cheeseburgers." ◀

EXAMPLE 21 What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x \neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀

We use De Morgan's laws for quantifiers in Example 22.

EXAMPLE 22 Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Solution: By De Morgan's law for universal quantifiers, we know that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(\neg(P(x) \to Q(x)))$ are logically equivalent. By the fifth logical equivalence in Table 7 in Section 1.3, we know that $\neg(P(x) \to Q(x))$ and $P(x) \land \neg Q(x)$ are logically equivalent for every x. Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent. ◀

Translating from English into Logical Expressions

Translating sentences in English (or other natural languages) into logical expressions is a crucial task in mathematics, logic programming, artificial intelligence, software engineering, and many other disciplines. We began studying this topic in Section 1.1, where we used propositions to express sentences in logical expressions. In that discussion, we purposely avoided sentences whose translations required predicates and quantifiers. Translating from English to logical expressions becomes even more complex when quantifiers are needed. Furthermore, there can be many ways to translate a particular sentence. (As a consequence, there is no "cookbook" approach that can be followed step by step.) We will use some examples to illustrate how to translate sentences from English into logical expressions. The goal in this translation is to produce simple and useful logical expressions. In this section, we restrict ourselves to sentences that can be translated into logical expressions using a single quantifier; in the next section, we will look at more complicated sentences that require multiple quantifiers.

EXAMPLE 23 Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain:

"For every student in this class, that student has studied calculus."

Extra Examples

Next, we introduce a variable x so that our statement becomes

"For every student x in this class, x has studied calculus."

Continuing, we introduce $C(x)$, which is the statement "x has studied calculus." Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

However, there are other correct approaches; different domains of discourse and other predicates can be used. The approach we select depends on the subsequent reasoning we want to carry out. For example, we may be interested in a wider group of people than only those in this class. If we change the domain to consist of all people, we will need to express our statement as

"For every person x, if person x is a student in this class then x has studied calculus."

If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \to C(x))$. [*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \land C(x))$ because this statement says that all people are students in this class and have studied calculus!]

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier $Q(x, y)$ for the statement "student x has studied subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \to Q(x, \text{calculus}))$. ◀

In Example 23 we displayed different approaches for expressing the same statement using predicates and quantifiers. However, we should always adopt the simplest approach that is adequate for use in subsequent reasoning.

EXAMPLE 24 Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution: The statement "Some student in this class has visited Mexico" means that

> "There is a student in this class with the property that the student has visited Mexico."

We can introduce a variable x, so that our statement becomes

> "There is a student x in this class having the property that x has visited Mexico."

We introduce $M(x)$, which is the statement "x has visited Mexico." If the domain for x consists of the students in this class, we can translate this first statement as $\exists x M(x)$.

However, if we are interested in people other than those in this class, we look at the statement a little differently. Our statement can be expressed as

> "There is a person x having the properties that x is a student in this class and x has visited Mexico."

In this case, the domain for the variable x consists of all people. We introduce $S(x)$ to represent "x is a student in this class." Our solution becomes $\exists x(S(x) \wedge M(x))$ because the statement is that there is a person x who is a student in this class and who has visited Mexico. [*Caution!* Our statement cannot be expressed as $\exists x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either $\mathbf{F} \rightarrow \mathbf{T}$ or $\mathbf{F} \rightarrow \mathbf{F}$, both of which are true.]

Similarly, the second statement can be expressed as

> "For every x in this class, x has the property that x has visited Mexico or x has visited Canada."

(Note that we are assuming the inclusive, rather than the exclusive, or here.) We let $C(x)$ be "x has visited Canada." Following our earlier reasoning, we see that if the domain for x consists of the students in this class, this second statement can be expressed as $\forall x(C(x) \vee M(x))$. However, if the domain for x consists of all people, our statement can be expressed as

> "For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada."

In this case, the statement can be expressed as $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$.

Instead of using $M(x)$ and $C(x)$ to represent that x has visited Mexico and x has visited Canada, respectively, we could use a two-place predicate $V(x, y)$ to represent "x has visited country y." In this case, $V(x, \text{Mexico})$ and $V(x, \text{Canada})$ would have the same meaning as $M(x)$ and $C(x)$ and could replace them in our answers. If we are working with many statements that involve people visiting different countries, we might prefer to use this two-variable approach. Otherwise, for simplicity, we would stick with the one-variable predicates $M(x)$ and $C(x)$. ◀

Using Quantifiers in System Specifications

In Section 1.2 we used propositions to represent system specifications. However, many system specifications involve predicates and quantifications. This is illustrated in Example 25.

EXAMPLE 25 Use predicates and quantifiers to express the system specifications "Every mail message larger than one megabyte will be compressed" and "If a user is active, at least one network link will be available."

Extra Examples

Solution: Let $S(m, y)$ be "Mail message m is larger than y megabytes," where the variable x has the domain of all mail messages and the variable y is a positive real number, and let $C(m)$ denote "Mail message m will be compressed." Then the specification "Every mail message larger than one megabyte will be compressed" can be represented as $\forall m(S(m, 1) \to C(m))$.

Remember the rules of precedence for quantifiers and logical connectives!

Let $A(u)$ represent "User u is active," where the variable u has the domain of all users, let $S(n, x)$ denote "Network link n is in state x," where n has the domain of all network links and x has the domain of all possible states for a network link. Then the specification "If a user is active, at least one network link will be available" can be represented by $\exists u A(u) \to \exists n S(n, \text{available})$. ◀

Examples from Lewis Carroll

Lewis Carroll (really C. L. Dodgson writing under a pseudonym), the author of *Alice in Wonderland*, is also the author of several works on symbolic logic. His books contain many examples of reasoning using quantifiers. Examples 26 and 27 come from his book *Symbolic Logic;* other examples from that book are given in the exercises at the end of this section. These examples illustrate how quantifiers are used to express various types of statements.

EXAMPLE 26 Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

(In Section 1.6 we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a lion," "x is fierce," and "x drinks coffee," respectively. Assuming that the domain consists of all creatures, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.

Links

CHARLES LUTWIDGE DODGSON (1832–1898) We know Charles Dodgson as Lewis Carroll—the pseudonym he used in his literary works. Dodgson, the son of a clergyman, was the third of 11 children, all of whom stuttered. He was uncomfortable in the company of adults and is said to have spoken without stuttering only to young girls, many of whom he entertained, corresponded with, and photographed (sometimes in poses that today would be considered inappropriate). Although attracted to young girls, he was extremely puritanical and religious. His friendship with the three young daughters of Dean Liddell led to his writing *Alice in Wonderland*, which brought him money and fame.

Dodgson graduated from Oxford in 1854 and obtained his master of arts degree in 1857. He was appointed lecturer in mathematics at Christ Church College, Oxford, in 1855. He was ordained in the Church of England in 1861 but never practiced his ministry. His writings published under this real name include articles and books on geometry, determinants, and the mathematics of tournaments and elections. (He also used the pseudonym Lewis Carroll for his many works on recreational logic.)

Solution: We can express these statements as:

$$\forall x(P(x) \to Q(x)).$$
$$\exists x(P(x) \wedge \neg R(x)).$$
$$\exists x(Q(x) \wedge \neg R(x)).$$

Notice that the second statement cannot be written as $\exists x(P(x) \to \neg R(x))$. The reason is that $P(x) \to \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \to \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

$$\exists x(Q(x) \to \neg R(x)).$$ ◀

EXAMPLE 27 Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

“All hummingbirds are richly colored.”
“No large birds live on honey.”
“Birds that do not live on honey are dull in color.”
“Hummingbirds are small.”

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “x is a hummingbird,” “x is large,” “x lives on honey,” and “x is richly colored,” respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Solution: We can express the statements in the argument as

$$\forall x(P(x) \to S(x)).$$
$$\neg\exists x(Q(x) \wedge R(x)).$$
$$\forall x(\neg R(x) \to \neg S(x)).$$
$$\forall x(P(x) \to \neg Q(x)).$$

(Note we have assumed that “small” is the same as “not large” and that “dull in color” is the same as “not richly colored.” To show that the fourth statement is a valid conclusion of the first three, we need to use rules of inference that will be discussed in Section 1.6.) ◀

Logic Programming

Links

An important type of programming language is designed to reason using the rules of predicate logic. Prolog (from *Pro*gramming in *Log*ic), developed in the 1970s by computer scientists working in the area of artificial intelligence, is an example of such a language. Prolog programs include a set of declarations consisting of two types of statements, **Prolog facts** and **Prolog rules**. Prolog facts define predicates by specifying the elements that satisfy these predicates. Prolog rules are used to define new predicates using those already defined by Prolog facts. Example 28 illustrates these notions.

EXAMPLE 28 Consider a Prolog program given facts telling it the instructor of each class and in which classes students are enrolled. The program uses these facts to answer queries concerning the professors who teach particular students. Such a program could use the predicates $\mathit{instructor}(p, c)$ and *enrolled*(s, c) to represent that professor p is the instructor of course c and that student s is enrolled in course c, respectively. For example, the Prolog facts in such a program might include:

```
instructor(chan,math273)
instructor(patel,ee222)
instructor(grossman,cs301)
enrolled(kevin,math273)
enrolled(juana,ee222)
enrolled(juana,cs301)
enrolled(kiko,math273)
enrolled(kiko,cs301)
```

(Lowercase letters have been used for entries because Prolog considers names beginning with an uppercase letter to be variables.)

A new predicate *teaches*(p, s), representing that professor p teaches student s, can be defined using the Prolog rule

```
teaches(P,S) :- instructor(P,C), enrolled(S,C)
```

which means that *teaches*(p, s) is true if there exists a class c such that professor p is the instructor of class c and student s is enrolled in class c. (Note that a comma is used to represent a conjunction of predicates in Prolog. Similarly, a semicolon is used to represent a disjunction of predicates.)

Prolog answers queries using the facts and rules it is given. For example, using the facts and rules listed, the query

```
?enrolled(kevin,math273)
```

produces the response

```
yes
```

because the fact *enrolled*(kevin, math273) was provided as input. The query

```
?enrolled(X,math273)
```

produces the response

```
kevin
kiko
```

To produce this response, Prolog determines all possible values of X for which *enrolled*$(X$, math273) has been included as a Prolog fact. Similarly, to find all the professors who are instructors in classes being taken by Juana, we use the query

```
?teaches(X,juana)
```

This query returns

```
patel
grossman
```

◀

Exercises

1. Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values?

a) $P(0)$ **b)** $P(4)$ **c)** $P(6)$

2. Let $P(x)$ be the statement "the word x contains the letter a." What are these truth values?

a) $P(\text{orange})$ **b)** $P(\text{lemon})$
c) $P(\text{true})$ **d)** $P(\text{false})$

3. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?

a) Q(Denver, Colorado)
b) Q(Detroit, Michigan)
c) Q(Massachusetts, Boston)
d) Q(New York, New York)

4. State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$," if the value of x when this statement is reached is

a) $x = 0$. **b)** $x = 1$.
c) $x = 2$.

5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\exists x \neg P(x)$ **d)** $\forall x \neg P(x)$

6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$ **b)** $\forall x N(x)$ **c)** $\neg\exists x N(x)$
d) $\exists x \neg N(x)$ **e)** $\neg\forall x N(x)$ **f)** $\forall x \neg N(x)$

7. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$ **b)** $\forall x(C(x) \wedge F(x))$
c) $\exists x(C(x) \rightarrow F(x))$ **d)** $\exists x(C(x) \wedge F(x))$

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

a) $\forall x(R(x) \rightarrow H(x))$ **b)** $\forall x(R(x) \wedge H(x))$
c) $\exists x(R(x) \rightarrow H(x))$ **d)** $\exists x(R(x) \wedge H(x))$

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

b) There is a student at your school who can speak Russian but who doesn't know C++.

c) Every student at your school either can speak Russian or knows C++.

d) No student at your school can speak Russian or knows C++.

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.

b) All students in your class have a cat, a dog, or a ferret.

c) Some student in your class has a cat and a ferret, but not a dog.

d) No student in your class has a cat, a dog, and a ferret.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$
d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

12. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$
d) $\exists x Q(x)$ **e)** $\forall x Q(x)$ **f)** $\exists x \neg Q(x)$
g) $\forall x \neg Q(x)$

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$ **b)** $\exists n(2n = 3n)$
c) $\exists n(n = -n)$ **d)** $\forall n(3n \leq 4n)$

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$ **b)** $\exists x(x^4 < x^2)$
c) $\forall x((-x)^2 = x^2)$ **d)** $\forall x(2x > x)$

15. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$ **b)** $\exists n(n^2 = 2)$
c) $\forall n(n^2 \geq n)$ **d)** $\exists n(n^2 < 0)$

16. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\exists x(x^2 = 2)$ **b)** $\exists x(x^2 = -1)$
c) $\forall x(x^2 + 2 \geq 1)$ **d)** $\forall x(x^2 \neq x)$

17. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ **e)** $\neg\exists x P(x)$ **f)** $\neg\forall x P(x)$

18. Suppose that the domain of the propositional function $P(x)$ consists of the integers -2, -1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ **e)** $\neg\exists x P(x)$ **f)** $\neg\forall x P(x)$

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\neg\exists x P(x)$ **d)** $\neg\forall x P(x)$
e) $\forall x((x \neq 3) \to P(x)) \vee \exists x \neg P(x)$

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\forall x((x \neq 1) \to P(x))$
d) $\exists x((x \geq 0) \wedge P(x))$
e) $\exists x(\neg P(x)) \wedge \forall x((x < 0) \to P(x))$

21. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No two different people have the same grandmother.

22. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone speaks Hindi.
b) There is someone older than 21 years.
c) Every two people have the same first name.
d) Someone knows more than two other people.

23. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

a) Someone in your class can speak Hindi.
b) Everyone in your class is friendly.
c) There is a person in your class who was not born in California.
d) A student in your class has been in a movie.
e) No student in your class has taken a course in logic programming.

24. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

a) Everyone in your class has a cellular phone.
b) Somebody in your class has seen a foreign movie.
c) There is a person in your class who cannot swim.
d) All students in your class can solve quadratic equations.
e) Some student in your class does not want to be rich.

25. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

26. Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.

a) Someone in your school has visited Uzbekistan.
b) Everyone in your class has studied calculus and C++.
c) No one in your school owns both a bicycle and a motorcycle.
d) There is a person in your school who is not happy.
e) Everyone in your school was born in the twentieth century.

27. Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.

a) A student in your school has lived in Vietnam.
b) There is a student in your school who cannot speak Hindi.
c) A student in your school knows Java, Prolog, and C++.
d) Everyone in your class enjoys Thai food.
e) Someone in your class does not play hockey.

28. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

29. Express each of these statements using logical operators, predicates, and quantifiers.

a) Some propositions are tautologies.
b) The negation of a contradiction is a tautology.
c) The disjunction of two contingencies can be a tautology.
d) The conjunction of two tautologies is a tautology.

30. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\exists x\ P(x, 3)$ **b)** $\forall y\ P(1, y)$
c) $\exists y \neg P(2, y)$ **d)** $\forall x\ \neg P(x, 2)$

31. Suppose that the domain of $Q(x, y, z)$ consists of triples x, y, z, where $x = 0$, 1, or 2, $y = 0$ or 1, and $z = 0$ or 1. Write out these propositions using disjunctions and conjunctions.

a) $\forall y Q(0, y, 0)$ **b)** $\exists x Q(x, 1, 1)$
c) $\exists z \neg Q(0, 0, z)$ **d)** $\exists x \neg Q(x, 0, 1)$

32. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) All dogs have fleas.

b) There is a horse that can add.

c) Every koala can climb.

d) No monkey can speak French.

e) There exists a pig that can swim and catch fish.

33. Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) Some old dogs can learn new tricks.

b) No rabbit knows calculus.

c) Every bird can fly.

d) There is no dog that can talk.

e) There is no one in this class who knows French and Russian.

34. Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.

b) All Swedish movies are serious.

c) No one can keep a secret.

d) There is someone in this class who does not have a good attitude.

35. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x(x^2 \geq x)$

b) $\forall x(x > 0 \vee x < 0)$

c) $\forall x(x = 1)$

36. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x(x^2 \neq x)$ **b)** $\forall x(x^2 \neq 2)$

c) $\forall x(|x| > 0)$

37. Express each of these statements using predicates and quantifiers.

a) A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.

b) A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.

c) A student must take at least 60 course hours, or at least 45 course hours and write a master's thesis, and receive a grade no lower than a B in all required courses, to receive a master's degree.

d) There is a student who has taken more than 21 credit hours in a semester and received all A's.

Exercises 38–42 deal with the translation between system specification and logical expressions involving quantifiers.

38. Translate these system specifications into English where the predicate $S(x, y)$ is "x is in state y" and where the domain for x and y consists of all systems and all possible states, respectively.

a) $\exists x S(x, \text{open})$

b) $\forall x(S(x, \text{malfunctioning}) \vee S(x, \text{diagnostic}))$

c) $\exists x S(x, \text{open}) \vee \exists x S(x, \text{diagnostic})$

d) $\exists x \neg S(x, \text{available})$

e) $\forall x \neg S(x, \text{working})$

39. Translate these specifications into English where $F(p)$ is "Printer p is out of service," $B(p)$ is "Printer p is busy," $L(j)$ is "Print job j is lost," and $Q(j)$ is "Print job j is queued."

a) $\exists p(F(p) \wedge B(p)) \rightarrow \exists j L(j)$

b) $\forall p B(p) \rightarrow \exists j Q(j)$

c) $\exists j(Q(j) \wedge L(j)) \rightarrow \exists p F(p)$

d) $(\forall p B(p) \wedge \forall j Q(j)) \rightarrow \exists j L(j)$

40. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.

b) No directories in the file system can be opened and no files can be closed when system errors have been detected.

c) The file system cannot be backed up if there is a user currently logged on.

d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.

41. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.

b) Whenever there is an active alert, all queued messages are transmitted.

c) The diagnostic monitor tracks the status of all systems except the main console.

d) Each participant on the conference call whom the host of the call did not put on a special list was billed.

42. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) Every user has access to an electronic mailbox.

b) The system mailbox can be accessed by everyone in the group if the file system is locked.

c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

43. Determine whether $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ are logically equivalent. Justify your answer.

44. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\ P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

45. Show that $\exists x(P(x) \lor Q(x))$ and $\exists x P(x) \lor \exists x Q(x)$ are logically equivalent.

Exercises 46–49 establish rules for **null quantification** that we can use when a quantified variable does not appear in part of a statement.

46. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \lor A \equiv \forall x(P(x) \lor A)$

b) $(\exists x P(x)) \lor A \equiv \exists x(P(x) \lor A)$

47. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \land A \equiv \forall x(P(x) \land A)$

b) $(\exists x P(x)) \land A \equiv \exists x(P(x) \land A)$

48. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(A \to P(x)) \equiv A \to \forall x P(x)$

b) $\exists x(A \to P(x)) \equiv A \to \exists x P(x)$

49. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(P(x) \to A) \equiv \exists x P(x) \to A$

b) $\exists x(P(x) \to A) \equiv \forall x P(x) \to A$

50. Show that $\forall x P(x) \lor \forall x Q(x)$ and $\forall x(P(x) \lor Q(x))$ are not logically equivalent.

51. Show that $\exists x P(x) \land \exists x Q(x)$ and $\exists x(P(x) \land Q(x))$ are not logically equivalent.

52. As mentioned in the text, the notation $\exists! x P(x)$ denotes

"There exists a unique x such that $P(x)$ is true."

If the domain consists of all integers, what are the truth values of these statements?

a) $\exists! x(x > 1)$ **b)** $\exists! x(x^2 = 1)$

c) $\exists! x(x + 3 = 2x)$ **d)** $\exists! x(x = x + 1)$

53. What are the truth values of these statements?

a) $\exists! x P(x) \to \exists x P(x)$

b) $\forall x P(x) \to \exists! x P(x)$

c) $\exists! x \neg P(x) \to \neg \forall x P(x)$

54. Write out $\exists! x P(x)$, where the domain consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.

55. Given the Prolog facts in Example 28, what would Prolog return given these queries?

a) `?instructor(chan,math273)`

b) `?instructor(patel,cs301)`

c) `?enrolled(X,cs301)`

d) `?enrolled(kiko,Y)`

e) `?teaches(grossman,Y)`

56. Given the Prolog facts in Example 28, what would Prolog return when given these queries?

a) `?enrolled(kevin,ee222)`

b) `?enrolled(kiko,math273)`

c) `?instructor(grossman,X)`

d) `?instructor(X,cs301)`

e) `?teaches(X,kevin)`

57. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *sibling*(X, Y), which represents that X and Y are siblings (that is, have the same mother and the same father).

58. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *grandfather*(X, Y), which represents that X is the grandfather of Y. [*Hint:* You can write a disjunction in Prolog either by using a semicolon to separate predicates or by putting these predicates on separate lines.]

Exercises 59–62 are based on questions found in the book *Symbolic Logic* by Lewis Carroll.

59. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.

a) No professors are ignorant.

b) All ignorant people are vain.

c) No professors are vain.

d) Does (c) follow from (a) and (b)?

60. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

***d)** Does (c) follow from (a) and (b)?

61. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a baby," "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) Babies are illogical.

b) Nobody is despised who can manage a crocodile.

c) Illogical persons are despised.

d) Babies cannot manage crocodiles.

***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

62. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) No ducks are willing to waltz.

b) No officers ever decline to waltz.

c) All my poultry are ducks.

d) My poultry are not officers.

***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

1.5 Nested Quantifiers

Introduction

In Section 1.4 we defined the existential and universal quantifiers and showed how they can be used to represent mathematical statements. We also explained how they can be used to translate English sentences into logical expressions. However, in Section 1.4 we avoided **nested quantifiers**, where one quantifier is within the scope of another, such as

$$\forall x \exists y (x + y = 0).$$

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example,

$$\forall x \exists y (x + y = 0)$$

is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

Nested quantifiers commonly occur in mathematics and computer science. Although nested quantifiers can sometimes be difficult to understand, the rules we have already studied in Section 1.4 can help us use them. In this section we will gain experience working with nested quantifiers. We will see how to use nested quantifiers to express mathematical statements such as "The sum of two positive integers is always positive." We will show how nested quantifiers can be used to translate English sentences such as "Everyone has exactly one best friend" into logical statements. Moreover, we will gain experience working with the negations of statements involving nested quantifiers.

Understanding Statements Involving Nested Quantifiers

To understand statements involving nested quantifiers, we need to unravel what the quantifiers and predicates that appear mean. This is illustrated in Examples 1 and 2.

EXAMPLE 1 Assume that the domain for the variables x and y consists of all real numbers. The statement

$$\forall x \forall y (x + y = y + x)$$

Extra Examples

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers. ◀

1.6 Rules of Inference

Introduction

Later in this chapter we will study proofs. Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an **argument**, we mean a sequence of statements that end with a conclusion. By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument. That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements.

Before we study mathematical proofs, we will look at arguments that involve only compound propositions. We will define what it means for an argument involving compound propositions to be valid. Then we will introduce a collection of rules of inference in propositional logic. These rules of inference are among the most important ingredients in producing valid arguments. After we illustrate how rules of inference are used to produce valid arguments, we will describe some common forms of incorrect reasoning, called **fallacies**, which lead to invalid arguments.

After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements. We will describe how these rules of inference can be used to produce valid arguments. These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.

Finally, we will show how rules of inference for propositions and for quantified statements can be combined. These combinations of rule of inference are often used together in complicated arguments.

Valid Arguments in Propositional Logic

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

> "If you have a current password, then you can log onto the network."
>
> "You have a current password."
>
> Therefore,
>
> "You can log onto the network."

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion "You can log onto the network" must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Before we discuss the validity of this particular argument, we will look at its form. Use p to represent "You have a current password" and q to represent "You can log onto the network." Then, the argument has the form

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

where $\therefore$ is the symbol that denotes "therefore."

We know that when p and q are propositional variables, the statement $((p \to q) \land p) \to q$ is a tautology (see Exercise 10(c) in Section 1.3). In particular, when both $p \to q$ and p are true, we know that q must also be true. We say this form of argument is **valid** because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true. Now suppose that both "If you have a current password, then you can log onto the network" and "You have a current password" are true statements. When we replace p by "You have a current password" and q by "You can log onto the network," it necessarily follows that the conclusion "You can log onto the network" is true. This argument is **valid** because its form is valid. Note that whenever we replace p and q by propositions where $p \to q$ and p are both true, then q must also be true.

What happens when we replace p and q in this argument form by propositions where not both p and $p \to q$ are true? For example, suppose that p represents "You have access to the network" and q represents "You can change your grade" and that p is true, but $p \to q$ is false. The argument we obtain by substituting these values of p and q into the argument form is

$$\begin{array}{rl} & \text{"If you have access to the network, then you can change your grade."} \\ & \text{"You have access to the network."} \\ \hline \therefore & \text{"You can change your grade."} \end{array}$$

The argument we obtained is a valid argument, but because one of the premises, namely the first premise, is false, we cannot conclude that the conclusion is true. (Most likely, this conclusion is false.)

In our discussion, to analyze an argument, we replaced propositions by propositional variables. This changed an argument to an **argument form**. We saw that the validity of an argument follows from the validity of the form of the argument. We summarize the terminology used to discuss the validity of arguments with our definition of the key notions.

DEFINITION 1

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when $(p_1 \land p_2 \land \cdots \land p_n) \to q$ is a tautology.

The key to showing that an argument in propositional logic is valid is to show that its argument form is valid. Consequently, we would like techniques to show that argument forms are valid. We will now develop methods for accomplishing this task.

Rules of Inference for Propositional Logic

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic.

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (Modus ponens is Latin for *mode that affirms*.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol $\therefore$ denotes "therefore"):

$$\begin{array}{rl} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$$

Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion. In particular, modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true. Example 1 illustrates the use of modus ponens.

EXAMPLE 1 Suppose that the conditional statement "If it snows today, then we will go skiing" and its hypothesis, "It is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, "We will go skiing," is true. ◀

As we mentioned earlier, a valid argument can lead to an incorrect conclusion if one or more of its premises is false. We illustrate this again in Example 2.

EXAMPLE 2 Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.

"If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$."

Solution: Let p be the proposition "$\sqrt{2} > \frac{3}{2}$" and q the proposition "$2 > (\frac{3}{2})^2$." The premises of the argument are $p \rightarrow q$ and p, and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$. ◀

There are many useful rules of inference for propositional logic. Perhaps the most widely used of these are listed in Table 1. Exercises 9, 10, 15, and 30 in Section 1.3 ask for the verifications that these rules of inference are valid argument forms. We now give examples of arguments that use these rules of inference. In each argument, we first use propositional variables to express the propositions in the argument. We then show that the resulting argument form is a rule of inference from Table 1.

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
$\begin{array}{rl} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\begin{array}{rl} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$	$p \rightarrow (p \vee q)$	Addition
$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification
$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

EXAMPLE 3 State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

This is an argument that uses the addition rule. ◀

EXAMPLE 4 State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

This argument uses the simplification rule. ◀

EXAMPLE 5 State which rule of inference is used in the argument:

> If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism. ◀

Using Rules of Inference to Build Arguments

When there are many premises, several rules of inference are often needed to show that an argument is valid. This is illustrated by Examples 6 and 7, where the steps of arguments are displayed on separate lines, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

EXAMPLE 6 Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Extra Examples

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p, q, r, s, and t, such a truth table would have 32 rows. ◀

EXAMPLE 7 Show that the premises “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.”

Solution: Let p be the proposition “You send me an e-mail message,” q the proposition “I will finish writing the program,” r the proposition “I will go to sleep early,” and s the proposition “I will wake up feeling refreshed.” Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

◀

Resolution

Links

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution**. This rule of inference is based on the tautology

$$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r).$$

(Exercise 30 in Section 1.3 asks for the verification that this is a tautology.) The final disjunction in the resolution rule, $q \vee r$, is called the **resolvent**. When we let $q = r$ in this tautology, we obtain $(p \vee q) \wedge (\neg p \vee q) \to q$. Furthermore, when we let $r = \mathbf{F}$, we obtain $(p \vee q) \wedge (\neg p) \to q$ (because $q \vee \mathbf{F} \equiv q$), which is the tautology on which the rule of disjunctive syllogism is based.

EXAMPLE 8 Use resolution to show that the hypotheses “Jasmine is skiing or it is not snowing” and “It is snowing or Bart is playing hockey” imply that “Jasmine is skiing or Bart is playing hockey.”

Extra Examples

Solution: Let p be the proposition “It is snowing,” q the proposition “Jasmine is skiing,” and r the proposition “Bart is playing hockey.” We can represent the hypotheses as $\neg p \vee q$ and $p \vee r$, respectively. Using resolution, the proposition $q \vee r$, “Jasmine is skiing or Bart is playing hockey,” follows. ◀

Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied). Furthermore, it can be used to build automatic theorem proving systems. To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses**, where a clause is a disjunction of variables or negations of these variables. We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses. For example, suppose we have a statement of the form $p \vee (q \wedge r)$. Because $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, we can replace the single statement $p \vee (q \wedge r)$ by two statements $p \vee q$ and $p \vee r$, each of which is a clause. We can replace a statement of the form $\neg(p \vee q)$ by the two statements $\neg p$ and $\neg q$ because De Morgan’s law tells us that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. We can also replace a conditional statement $p \to q$ with the equivalent disjunction $\neg p \vee q$.

EXAMPLE 9 Show that the premises $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Solution: We can rewrite the premises $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$. Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conclude $p \vee s$. ◀

Fallacies

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies. These are discussed here to show the distinction between correct and incorrect reasoning.

The proposition $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. In other words, they treat the argument with premises $p \rightarrow q$ and q and conclusion p as a valid argument form, which it is not. This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.

EXAMPLE 10 Is the following argument valid?

> If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.
>
> Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \rightarrow q$ and q, then p. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.) ◀

The proposition $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis**.

EXAMPLE 11 Let p and q be as in Example 10. If the conditional statement $p \rightarrow q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \rightarrow q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. ◀

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

We summarize these rules of inference in Table 2. We will illustrate how some of these rules of inference for quantified statements are used in Examples 12 and 13.

EXAMPLE 12 Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”

Solution: Let $D(x)$ denote “x is in this discrete mathematics class,” and let $C(x)$ denote “x has taken a course in computer science.” Then the premises are $\forall x(D(x) \to C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

Extra Examples

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \to C(x))$	Premise
2. $D(\text{Marla}) \to C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

◀

EXAMPLE 13 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

◀

Combining Rules of Inference for Propositions and Quantified Statements

We have developed rules of inference both for propositions and for quantified statements. Note that in our arguments in Examples 12 and 13 we used both universal instantiation, a rule of inference for quantified statements, and modus ponens, a rule of inference for propositional logic. We will often need to use this combination of rules of inference. Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**. This rule tells us that if $\forall x(P(x) \rightarrow Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ is true. Then, by modus ponens, $Q(a)$ must also be true. We can describe universal modus ponens as follows:

$$\begin{array}{ll} & \forall x(P(x) \rightarrow Q(x)) \\ & \underline{P(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore & Q(a) \end{array}$$

Universal modus ponens is commonly used in mathematical arguments. This is illustrated in Example 14.

EXAMPLE 14 Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

Solution: Let $P(n)$ denote "$n > 4$" and $Q(n)$ denote "$n^2 < 2^n$." The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(100)$ is true, namely that $100^2 < 2^{100}$. ◀

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **universal modus tollens**. Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & \underline{\neg Q(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore & \neg P(a) \end{array}$$

The verification of universal modus tollens is left as Exercise 25. Exercises 26–29 develop additional combinations of rules of inference in propositional logic and quantified statements.

Exercises

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

$$\begin{array}{rl} & \text{If Socrates is human, then Socrates is mortal.} \\ & \underline{\text{Socrates is human.}} \\ \therefore & \text{Socrates is mortal.} \end{array}$$

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

$$\begin{array}{rl} & \text{If George does not have eight legs, then he is not a} \\ & \text{spider.} \\ & \underline{\text{George is a spider.}} \\ \therefore & \text{George has eight legs.} \end{array}$$

3. What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

4. What rule of inference is used in each of these arguments?

a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.

e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

5. Use rules of inference to show that the hypotheses “Randy works hard,” “If Randy works hard, then he is a dull boy,” and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job.”

6. Use rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,” and “The trophy was not awarded” imply the conclusion “It rained.”

7. What rules of inference are used in this famous argument? “All men are mortal. Socrates is a man. Therefore, Socrates is mortal.”

8. What rules of inference are used in this argument? “No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.”

9. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) “If I take the day off, it either rains or snows.” “I took Tuesday off or I took Thursday off.” “It was sunny on Tuesday.” “It did not snow on Thursday.”

b) “If I eat spicy foods, then I have strange dreams.” “I have strange dreams if there is thunder while I sleep.” “I did not have strange dreams.”

c) “I am either clever or lucky.” “I am not lucky.” “If I am lucky, then I will win the lottery.”

d) “Every computer science major has a personal computer.” “Ralph does not have a personal computer.” “Ann has a personal computer.”

e) “What is good for corporations is good for the United States.” “What is good for the United States is good for you.” “What is good for corporations is for you to buy lots of stuff.”

f) “All rodents gnaw their food.” “Mice are rodents.” “Rabbits do not gnaw their food.” “Bats are not rodents.”

10. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."

d) "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."

e) "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."

f) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

11. Show that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion $q \to r$ is valid if the argument form with premises $p_1, p_2, \ldots, p_n, q,$ and conclusion r is valid.

12. Show that the argument form with premises $(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, and $\neg s$ and conclusion $q \to r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.

13. For each of these arguments, explain which rules of inference are used for each step.

a) "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."

b) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."

c) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

d) "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."

14. For each of these arguments, explain which rules of inference are used for each step.

a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."

b) "Each of five roommates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

d) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

15. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.

b) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.

c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.

d) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

16. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.

b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

c) Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

17. What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.

18. What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

19. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

b) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.

c) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

20. Determine whether these are valid arguments.

a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

21. Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 26 of Section 1.4?

22. Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 27 of Section 1.4?

23. Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \wedge \exists x Q(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.

1. $\exists x P(x) \vee \exists x Q(x)$	Premise
2. $\exists x P(x)$	Simplification from (1)
3. $P(c)$	Existential instantiation from (2)
4. $\exists x Q(x)$	Simplification from (1)
5. $Q(c)$	Existential instantiation from (4)
6. $P(c) \wedge Q(c)$	Conjunction from (3) and (5)
7. $\exists x(P(x) \wedge Q(x))$	Existential generalization

24. Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.

1. $\forall x(P(x) \vee Q(x))$	Premise
2. $P(c) \vee Q(c)$	Universal instantiation from (1)
3. $P(c)$	Simplification from (2)
4. $\forall x P(x)$	Universal generalization from (3)
5. $Q(c)$	Simplification from (2)
6. $\forall x Q(x)$	Universal generalization from (5)
7. $\forall x(P(x) \vee \forall x Q(x))$	Conjunction from (4) and (6)

25. Justify the rule of universal modus tollens by showing that the premises $\forall x(P(x) \rightarrow Q(x))$ and $\neg Q(a)$ for a particular element a in the domain, imply $\neg P(a)$.

26. Justify the rule of **universal transitivity**, which states that if $\forall x(P(x) \rightarrow Q(x))$ and $\forall x(Q(x) \rightarrow R(x))$ are true, then $\forall x(P(x) \rightarrow R(x))$ is true, where the domains of all quantifiers are the same.

27. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

28. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

29. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

30. Use resolution to show the hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy."

31. Use resolution to show that the hypotheses "It is not raining or Yvette has her umbrella," "Yvette does not have her umbrella or she does not get wet," and "It is raining or Yvette does not get wet" imply that "Yvette does not get wet."

32. Show that the equivalence $p \wedge \neg p \equiv \mathbf{F}$ can be derived using resolution together with the fact that a conditional statement with a false hypothesis is true. [*Hint:* Let $q = r = \mathbf{F}$ in resolution.]

33. Use resolution to show that the compound proposition $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ is not satisfiable.

***34.** The Logic Problem, taken from *WFF'N PROOF, The Game of Logic*, has these two assumptions:

1. "Logic is difficult or not many students like logic."

2. "If mathematics is easy, then logic is not difficult."

By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

a) That mathematics is not easy, if many students like logic.

b) That not many students like logic, if mathematics is not easy.

c) That mathematics is not easy or logic is difficult.

d) That logic is not difficult or mathematics is not easy.

e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

***35.** Determine whether this argument, taken from Kalish and Montague [KaMo64], is valid.

> If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

1.7 Introduction to Proofs

Introduction

In this section we introduce the notion of a proof and describe methods for constructing proofs. A proof is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypotheses of the theorem, if any, axioms assumed to be true, and previously proven theorems. Using these ingredients and rules of inference, the final step of the proof establishes the truth of the statement being proved.

In our discussion we move from formal proofs of theorems toward more informal proofs. The arguments we introduced in Section 1.6 to show that statements involving propositions and quantified statements are true were formal proofs, where all steps were supplied, and the rules for each step in the argument were given. However, formal proofs of useful theorems can be extremely long and hard to follow. In practice, the proofs of theorems designed for human consumption are almost always **informal proofs**, where more than one rule of inference may be used in each step, where steps may be skipped, where the axioms being assumed and the rules of inference used are not explicitly stated. Informal proofs can often explain to humans why theorems are true, while computers are perfectly happy producing formal proofs using automated reasoning systems.

The methods of proof discussed in this chapter are important not only because they are used to prove mathematical theorems, but also for their many applications to computer science. These applications include verifying that computer programs are correct, establishing that operating systems are secure, making inferences in artificial intelligence, showing that system specifications are consistent, and so on. Consequently, understanding the techniques used in proofs is essential both in mathematics and in computer science.

Some Terminology

Formally, a **theorem** is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important. Less important theorems sometimes are called **propositions**. (Theorems can also be referred to as **facts** or **results**.) A theorem may be the universal quantification of a conditional statement with one or more premises and a conclusion. However, it may be some other type of logical statement, as the examples later in this chapter will show. We demonstrate that a theorem is true with a **proof**. A proof is a valid argument that establishes the truth of a theorem. The statements used in a proof can include **axioms** (or **postulates**), which are statements we assume to be true (for example, the axioms for the real numbers, given in Appendix 1, and the axioms of plane geometry), the premises, if any, of the theorem, and previously proven theorems. Axioms may be stated using primitive terms that do not require definition, but all other terms used in theorems and their proofs must be defined. Rules of inference, together with definitions of terms, are used to draw conclusions from other assertions, tying together the steps of a proof. In practice, the final step of a proof is usually just the conclusion of the theorem. However, for clarity, we will often recap the statement of the theorem as the final step of a proof.

Links

A less important theorem that is helpful in the proof of other results is called a **lemma** (plural *lemmas* or *lemmata*). Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually. A **corollary** is a theorem that can be established directly from a theorem that has been proved. A **conjecture** is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems.

Understanding How Theorems Are Stated

Extra Examples

Before we introduce methods for proving theorems, we need to understand how many mathematical theorems are stated. Many theorems assert that a property holds for all elements in a domain, such as the integers or the real numbers. Although the precise statement of such theorems needs to include a universal quantifier, the standard convention in mathematics is to omit it. For example, the statement

“If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$.”

really means

“For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$.”

Furthermore, when theorems of this type are proved, the first step of the proof usually involves selecting a general element of the domain. Subsequent steps show that this element has the property in question. Finally, universal generalization implies that the theorem holds for all members of the domain.

Methods of Proving Theorems

Assessment

Proving mathematical theorems can be difficult. To construct proofs we need all available ammunition, including a powerful battery of different proof methods. These methods provide the overall approach and strategy of proofs. Understanding these methods is a key component of learning how to read and construct mathematical proofs. One we have chosen a proof method, we use axioms, definitions of terms, previously proved results, and rules of inference to complete the proof. Note that in this book we will always assume the axioms for real numbers found in Appendix 1. We will also assume the usual axioms whenever we prove a result about geometry. When you construct your own proofs, be careful not to use anything but these axioms, definitions, and previously proved results as facts!

To prove a theorem of the form $\forall x(P(x) \to Q(x))$, our goal is to show that $P(c) \to Q(c)$ is true, where c is an arbitrary element of the domain, and then apply universal generalization. In this proof, we need to show that a conditional statement is true. Because of this, we now focus on methods that show that conditional statements are true. Recall that $p \to q$ is true unless p is true but q is false. Note that to prove the statement $p \to q$, we need only show that q is true if p is true. The following discussion will give the most common techniques for proving conditional statements. Later we will discuss methods for proving other types of statements. In this section, and in Section 1.8, we will develop a large arsenal of proof techniques that can be used to prove a wide variety of theorems.

When you read proofs, you will often find the words “obviously” or “clearly.” These words indicate that steps have been omitted that the author expects the reader to be able to fill in. Unfortunately, this assumption is often not warranted and readers are not at all sure how to fill in the gaps. We will assiduously try to avoid using these words and try not to omit too many steps. However, if we included all steps in proofs, our proofs would often be excruciatingly long.

Direct Proofs

A **direct proof** of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true. You will find that direct proofs of many results are quite straightforward, with a fairly obvious sequence of steps leading from the hypothesis to the conclusion. However, direct proofs sometimes require particular insights and can be quite tricky. The first direct proofs we present here are quite straightforward; later in the text you will see some that are less obvious.

We will provide examples of several different direct proofs. Before we give the first example, we need to define some terminology.

DEFINITION 1 The integer n is *even* if there exists an integer k such that $n = 2k$, and n is *odd* if there exists an integer k such that $n = 2k + 1$. (Note that every integer is either even or odd, and no integer is both even and odd.) Two integers have the *same parity* when both are even or both are odd; they have *opposite parity* when one is even and the other is odd.

EXAMPLE 1 Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Note that this theorem states $\forall n P((n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd." As we have said, we will follow the usual convention in mathematical proofs by showing that $P(n)$ implies $Q(n)$, and not explicitly using universal instantiation. To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer. ◀

Extra Examples

EXAMPLE 2 Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

Solution: To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares. By the definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that mn must also be a perfect square when m and n are; looking ahead we see how we can show this by substituting s^2 for m and t^2 for n into mn. This tells us that $mn = s^2t^2$. Hence, $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication. By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square. ◀

Proof by Contraposition

Direct proofs lead from the premises of a theorem to the conclusion. They begin with the premises, continue with a sequence of deductions, and end with the conclusion. However, we will see that attempts at direct proofs often reach dead ends. We need other methods of proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$. Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called **indirect proofs**.

An extremely useful type of indirect proof is known as **proof by contraposition**. Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true. In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow. We will illustrate proof by contraposition with two examples. These examples show that proof by contraposition can succeed when we cannot easily find a direct proof.

EXAMPLE 3 Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: We first attempt a direct proof. To construct a direct proof, we first assume that $3n + 2$ is an odd integer. This means that $3n + 2 = 2k + 1$ for some integer k. Can we use this fact

Extra Examples

to show that n is odd? We see that $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd." ◀

EXAMPLE 4 Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution: Because there is no obvious way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ directly from the equation $n = ab$, where a and b are positive integers, we attempt a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false. That is, we assume that the statement $(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false. Using the meaning of disjunction together with De Morgan's law, we see that this implies that both $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$ are false. This implies that $a > \sqrt{n}$ and $b > \sqrt{n}$. We can multiply these inequalities together (using the fact that if $0 < s < t$ and $0 < u < v$, then $su < tv$) to obtain $ab > \sqrt{n} \cdot \sqrt{n} = n$. This shows that $ab \neq n$, which contradicts the statement $n = ab$.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. ◀

VACUOUS AND TRIVIAL PROOFS We can quickly prove that a conditional statement $p \rightarrow q$ is true when we know that p is false, because $p \rightarrow q$ must be true when p is false. Consequently, if we can show that p is false, then we have a proof, called a **vacuous proof**, of the conditional statement $p \rightarrow q$. Vacuous proofs are often used to establish special cases of theorems that state that a conditional statement is true for all positive integers [i.e., a theorem of the kind $\forall n P(n)$, where $P(n)$ is a propositional function]. Proof techniques for theorems of this kind will be discussed in Section 5.1.

EXAMPLE 5 Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution: Note that $P(0)$ is "If $0 > 1$, then $0^2 > 0$." We can show $P(0)$ using a vacuous proof. Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true. ◀

Remark: The fact that the conclusion of this conditional statement, $0^2 > 0$, is false is irrelevant to the truth value of the conditional statement, because a conditional statement with a false hypothesis is guaranteed to be true.

We can also quickly prove a conditional statement $p \rightarrow q$ if we know that the conclusion q is true. By showing that q is true, it follows that $p \rightarrow q$ must also be true. A proof of $p \rightarrow q$ that uses the fact that q is true is called a **trivial proof**. Trivial proofs are often important when special cases of theorems are proved (see the discussion of proof by cases in Section 1.8) and in mathematical induction, which is a proof technique discussed in Section 5.1.

EXAMPLE 6 Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Solution: The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true. Hence, this conditional statement, which is $P(0)$, is true. This is an example of a trivial proof. Note that the hypothesis, which is the statement "$a \geq b$," was not needed in this proof. ◀

A LITTLE PROOF STRATEGY We have described two important approaches for proving theorems of the form $\forall x(P(x) \to Q(x))$: direct proof and proof by contraposition. We have also given examples that show how each is used. However, when you are presented with a theorem of the form $\forall x(P(x) \to Q(x))$, which method should you use to attempt to prove it? We will provide a few rules of thumb here; in Section 1.8 we will discuss proof strategy at greater length. When you want to prove a statement of the form $\forall x(P(x) \to Q(x))$, first evaluate whether a direct proof looks promising. Begin by expanding the definitions in the hypotheses. Start to reason using these hypotheses, together with axioms and available theorems. If a direct proof does not seem to go anywhere, try the same thing with a proof by contraposition. Recall that in a proof by contraposition you assume that the conclusion of the conditional statement is false and use a direct proof to show this implies that the hypothesis must be false. We illustrate this strategy in Examples 7 and 8. Before we present our next example, we need a definition.

DEFINITION 2 The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.

EXAMPLE 7 Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number r and every real number s, if r and s are rational numbers, then $r + s$ is rational.)

Extra Examples

Solution: We first attempt a direct proof. To begin, suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are integers p and q, with $q \neq 0$, such that $r = p/q$, and integers t and u, with $u \neq 0$, such that $s = t/u$. Can we use this information to show that $r + s$ is rational? The obvious next step is to add $r = p/q$ and $s = t/u$, to obtain

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu}.$$

Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$. Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu, where $qu \neq 0$. This means that $r + s$ is rational. We have proved that the sum of two rational numbers is rational; our attempt to find a direct proof succeeded. ◀

EXAMPLE 8 Prove that if n is an integer and n^2 is odd, then n is odd.

Solution: We first attempt a direct proof. Suppose that n is an integer and n^2 is odd. Then, there exists an integer k such that $n^2 = 2k + 1$. Can we use this information to show that n is odd? There seems to be no obvious approach to show that n is odd because solving for n produces the equation $n = \pm\sqrt{2k + 1}$, which is not terribly useful.

Because this attempt to use a direct proof did not bear fruit, we next attempt a proof by contraposition. We take as our hypothesis the statement that n is not odd. Because every integer is odd or even, this means that n is even. This implies that there exists an integer k such that $n = 2k$. To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that is, that n^2 is even. Can we use the equation $n = 2k$ to achieve this? By squaring both sides of this equation, we obtain $n^2 = 4k^2 = 2(2k^2)$, which implies that n^2 is also even because $n^2 = 2t$, where $t = 2k^2$. We have proved that if n is an integer and n^2 is odd, then n is odd. Our attempt to find a proof by contraposition succeeded. ◀

Proofs by Contradiction

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true. Because q is false, but $\neg p \to q$ is true, we can conclude that $\neg p$ is false, which means that p is true. How can we find a contradiction q that might help us prove that p is true in this way?

Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called **proofs by contradiction**. Because a proof by contradiction does not prove a result directly, it is another type of indirect proof. We provide three examples of proof by contradiction. The first is an example of an application of the pigeonhole principle, a combinatorial technique that we will cover in depth in Section 6.2.

EXAMPLE 9 Show that at least four of any 22 days must fall on the same day of the week.

Extra Examples

Solution: Let p be the proposition "At least four of 22 chosen days fall on the same day of the week." Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the premise that we have 22 days under consideration. That is, if r is the statement that 22 days are chosen, then we have shown that $\neg p \to (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week. ◀

EXAMPLE 10 Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition "$\sqrt{2}$ is irrational." To start a proof by contradiction, we suppose that $\neg p$ is true. Note that $\neg p$ is the statement "It is not the case that $\sqrt{2}$ is irrational," which says that $\sqrt{2}$ is rational. We will show that assuming that $\neg p$ is true leads to a contradiction.

If $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors (so that the fraction a/b is in lowest terms.) (Here, we are using the fact that every rational number can be written in lowest terms.) Because $\sqrt{2} = a/b$, when both sides of this equation are squared, it follows that

$$2 = \frac{a^2}{b^2}.$$

Hence,

$$2b^2 = a^2.$$

By the definition of an even integer it follows that a^2 is even. We next use the fact that if a^2 is even, a must also be even, which follows by Exercise 16. Furthermore, because a is even, by the definition of an even integer, $a = 2c$ for some integer c. Thus,

$$2b^2 = 4c^2.$$

Dividing both sides of this equation by 2 gives

$$b^2 = 2c^2.$$

By the definition of even, this means that b^2 is even. Again using the fact that if the square of an integer is even, then the integer itself must be even, we conclude that b must be even as well.

We have now shown that the assumption of $\neg p$ leads to the equation $\sqrt{2} = a/b$, where a and b have no common factors, but both a and b are even, that is, 2 divides both a and b. Note that the statement that $\sqrt{2} = a/b$, where a and b have no common factors, means, in particular, that 2 does not divide both a and b. Because our assumption of $\neg p$ leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b, $\neg p$ must be false. That is, the statement p, "$\sqrt{2}$ is irrational," is true. We have proved that $\sqrt{2}$ is irrational. ◀

Proof by contradiction can be used to prove conditional statements. In such proofs, we first assume the negation of the conclusion. We then use the premises of the theorem and the negation of the conclusion to arrive at a contradiction. (The reason that such proofs are valid rests on the logical equivalence of $p \to q$ and $(p \wedge \neg q) \to \mathbf{F}$. To see that these statements are equivalent, simply note that each is false in exactly one case, namely when p is true and q is false.)

Note that we can rewrite a proof by contraposition of a conditional statement as a proof by contradiction. In a proof of $p \to q$ by contraposition, we assume that $\neg q$ is true. We then show that $\neg p$ must also be true. To rewrite a proof by contraposition of $p \to q$ as a proof by contradiction, we suppose that both p and $\neg q$ are true. Then, we use the steps from the proof of $\neg q \to \neg p$ to show that $\neg p$ is true. This leads to the contradiction $p \wedge \neg p$, completing the proof. Example 11 illustrates how a proof by contraposition of a conditional statement can be rewritten as a proof by contradiction.

EXAMPLE 11 Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

Solution: Let p be "$3n + 2$ is odd" and q be "n is odd." To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n + 2$ is odd and that n is not odd. Because n is not odd, we know that it is even. Because n is even, there is an integer k such that $n = 2k$. This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Because $3n + 2$ is $2t$, where $t = 3k + 1$, $3n + 2$ is even. Note that the statement "$3n + 2$ is even" is equivalent to the statement $\neg p$, because an integer is even if and only if it is not odd. Because both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd. ◀

Note that we can also prove by contradiction that $p \to q$ is true by assuming that p and $\neg q$ are true, and showing that q must be also be true. This implies that $\neg q$ and q are both true, a contradiction. This observation tells us that we can turn a direct proof into a proof by contradiction.

PROOFS OF EQUIVALENCE To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology

$$(p \leftrightarrow q) \leftrightarrow (p \to q) \wedge (q \to p).$$

EXAMPLE 12 Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: This theorem has the form "p if and only if q," where p is "n is odd" and q is "n^2 is odd." (As usual, we do not explicitly deal with the universal quantification.) To prove this theorem, we need to show that $p \to q$ and $q \to p$ are true.

Extra Examples

We have already shown (in Example 1) that $p \to q$ is true and (in Example 8) that $q \to p$ is true.

Because we have shown that both $p \to q$ and $q \to p$ are true, we have shown that the theorem is true. ◀

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3, \ldots, p_n$ are equivalent. This can be written as

$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n,$$

which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \le i \le n$ and $1 \le j \le n$, p_i and p_j are equivalent. One way to prove these mutually equivalent is to use the tautology

$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n \leftrightarrow (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_n \to p_1).$$

This shows that if the n conditional statements $p_1 \to p_2, p_2 \to p_3, \ldots, p_n \to p_1$ can be shown to be true, then the propositions $p_1, p_2, \ldots, p_n$ are all equivalent.

This is much more efficient than proving that $p_i \to p_j$ for all $i \ne j$ with $1 \le i \le n$ and $1 \le j \le n$. (Note that there are $n^2 - n$ such conditional statements.)

When we prove that a group of statements are equivalent, we can establish any chain of conditional statements we choose as long as it is possible to work through the chain to go from any one of these statements to any other statement. For example, we can show that p_1, p_2, and p_3 are equivalent by showing that $p_1 \to p_3$, $p_3 \to p_2$, and $p_2 \to p_1$.

EXAMPLE 13 Show that these statements about the integer n are equivalent:

p_1: n is even.
p_2: $n - 1$ is odd.
p_3: n^2 is even.

Solution: We will show that these three statements are equivalent by showing that the conditional statements $p_1 \to p_2$, $p_2 \to p_3$, and $p_3 \to p_1$ are true.

We use a direct proof to show that $p_1 \to p_2$. Suppose that n is even. Then $n = 2k$ for some integer k. Consequently, $n - 1 = 2k - 1 = 2(k - 1) + 1$. This means that $n - 1$ is odd because it is of the form $2m + 1$, where m is the integer $k - 1$.

We also use a direct proof to show that $p_2 \to p_3$. Now suppose $n - 1$ is odd. Then $n - 1 = 2k + 1$ for some integer k. Hence, $n = 2k + 2$ so that $n^2 = (2k + 2)^2 = 4k^2 + 8k + 4 = 2(2k^2 + 4k + 2)$. This means that n^2 is twice the integer $2k^2 + 4k + 2$, and hence is even.

To prove $p_3 \to p_1$, we use a proof by contraposition. That is, we prove that if n is not even, then n^2 is not even. This is the same as proving that if n is odd, then n^2 is odd, which we have already done in Example 1. This completes the proof. ◀

COUNTEREXAMPLES In Section 1.4 we stated that to show that a statement of the form $\forall x P(x)$ is false, we need only find a **counterexample**, that is, an example x for which $P(x)$ is false. When presented with a statement of the form $\forall x P(x)$, which we believe to be false or which has resisted all proof attempts, we look for a counterexample. We illustrate the use of counterexamples in Example 14.

EXAMPLE 14 Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Extra Examples

Solution: To show that this statement is false, we look for a counterexample, which is a particular integer that is not the sum of the squares of two integers. It does not take long to find a counterexample, because 3 cannot be written as the sum of the squares of two integers. To show this is the case, note that the only perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. Furthermore, there is no way to get 3 as the sum of two terms each of which is 0 or 1. Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false. ◀

Mistakes in Proofs

There are many common errors made in constructing mathematical proofs. We will briefly describe some of these here. Among the most common errors are mistakes in arithmetic and basic algebra. Even professional mathematicians make such errors, especially when working with complicated formulae. Whenever you use such computations you should check them as carefully as possible. (You should also review any troublesome aspects of basic algebra, especially before you study Section 5.1.)

Each step of a mathematical proof needs to be correct and the conclusion needs to follow logically from the steps that precede it. Many mistakes result from the introduction of steps that do not logically follow from those that precede it. This is illustrated in Examples 15–17.

EXAMPLE 15 What is wrong with this famous supposed "proof" that $1 = 2$?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	**Reason**
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

Solution: Every step is valid except for one, step 5 where we divided both sides by $a - b$. The error is that $a - b$ equals zero; division of both sides of an equation by the same quantity is valid as long as this quantity is not zero. ◀

EXAMPLE 16 What is wrong with this "proof?"

"Theorem:" If n^2 is positive, then n is positive.

"Proof:" Suppose that n^2 is positive. Because the conditional statement "If n is positive, then n^2 is positive" is true, we can conclude that n is positive.

Solution: Let $P(n)$ be "n is positive" and $Q(n)$ be "n^2 is positive." Then our hypothesis is $Q(n)$. The statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \rightarrow Q(n))$. From the hypothesis $Q(n)$ and the statement $\forall n(P(n) \rightarrow Q(n))$ we cannot conclude $P(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of affirming the conclusion. A counterexample is supplied by $n = -1$ for which $n^2 = 1$ is positive, but n is negative. ◀

EXAMPLE 17 What is wrong with this "proof?"

"Theorem:" If n is not positive, then n^2 is not positive. (This is the contrapositive of the "theorem" in Example 16.)

"Proof:" Suppose that n is not positive. Because the conditional statement "If n is positive, then n^2 is positive" is true, we can conclude that n^2 is not positive.

Solution: Let $P(n)$ and $Q(n)$ be as in the solution of Example 16. Then our hypothesis is $\neg P(n)$ and the statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \to Q(n))$. From the hypothesis $\neg P(n)$ and the statement $\forall n(P(n) \to Q(n))$ we cannot conclude $\neg Q(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of denying the hypothesis. A counterexample is supplied by $n = -1$, as in Example 16. ◀

Finally, we briefly discuss a particularly nasty type of error. Many incorrect arguments are based on a fallacy called **begging the question**. This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning**.

EXAMPLE 18 Is the following argument correct? It supposedly shows that n is an even integer whenever n^2 is an even integer.

> Suppose that n^2 is even. Then $n^2 = 2k$ for some integer k. Let $n = 2l$ for some integer l. This shows that n is even.

Solution: This argument is incorrect. The statement "let $n = 2l$ for some integer l" occurs in the proof. No argument has been given to show that n can be written as $2l$ for some integer l. This is circular reasoning because this statement is equivalent to the statement being proved, namely, "n is even." Of course, the result itself is correct; only the method of proof is wrong. ◀

Making mistakes in proofs is part of the learning process. When you make a mistake that someone else finds, you should carefully analyze where you went wrong and make sure that you do not make the same mistake again. Even professional mathematicians make mistakes in proofs. More than a few incorrect proofs of important results have fooled people for many years before subtle errors in them were found.

Just a Beginning

We have now developed a basic arsenal of proof methods. In the next section we will introduce other important proof methods. We will also introduce several important proof techniques in Chapter 5, including mathematical induction, which can be used to prove results that hold for all positive integers. In Chapter 6 we will introduce the notion of combinatorial proofs.

In this section we introduced several methods for proving theorems of the form $\forall x(P(x) \to Q(x))$, including direct proofs and proofs by contraposition. There are many theorems of this type whose proofs are easy to construct by directly working through the hypotheses and definitions of the terms of the theorem. However, it is often difficult to prove a theorem without resorting to a clever use of a proof by contraposition or a proof by contradiction, or some other proof technique. In Section 1.8 we will address proof strategy. We will describe various approaches that can be used to find proofs when straightforward approaches do not work. Constructing proofs is an art that can be learned only through experience, including writing proofs, having your proofs critiqued, and reading and analyzing other proofs.

Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

2. Use a direct proof to show that the sum of two even integers is even.

3. Show that the square of an even number is an even number using a direct proof.

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

5. Prove that if $m + n$ and $n + p$ are even integers, where $m, n,$ and p are integers, then $m + p$ is even. What kind of proof did you use?

6. Use a direct proof to show that the product of two odd numbers is odd.

7. Use a direct proof to show that every odd integer is the difference of two squares.

8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

10. Use a direct proof to show that the product of two rational numbers is rational.

11. Prove or disprove that the product of two irrational numbers is irrational.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

13. Prove that if x is irrational, then $1/x$ is irrational.

14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

☞ **16.** Prove that if m and n are integers and mn is even, then m is even or n is even.

17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

18. Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

19. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

20. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?

21. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

23. Show that at least ten of any 64 days chosen must fall on the same day of the week.

24. Show that at least three of any 25 days chosen must fall in the same month of the year.

25. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]

26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

27. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

29. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.

30. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.

31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.

32. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, (*iii*) $3x - 1$ is rational.

33. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.

34. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (*2*) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.

35. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (*1*) $\sqrt{x + 3} = 3 - x$ is given; (*2*) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.

36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.

37. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \rightarrow p_4$, $p_3 \rightarrow p_1$, $p_4 \rightarrow p_2$, $p_2 \rightarrow p_5$, and $p_5 \rightarrow p_3$ are true.

38. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

39. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

40. Use Exercise 39 to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

41. Prove that if n is an integer, these four statements are equivalent: (*i*) n is even, (*ii*) $n + 1$ is odd, (*iii*) $3n + 1$ is odd, (*iv*) $3n$ is even.

42. Prove that these four statements about the integer n are equivalent: (*i*) n^2 is odd, (*ii*) $1 - n$ is even, (*iii*) n^3 is odd, (*iv*) $n^2 + 1$ is even.

1.8 Proof Methods and Strategy

Introduction

Assessment

In Section 1.7 we introduced many methods of proof and illustrated how each method can be used. In this section we continue this effort. We will introduce several other commonly used proof methods, including the method of proving a theorem by considering different cases separately. We will also discuss proofs where we prove the existence of objects with desired properties.

In Section 1.7 we briefly discussed the strategy behind constructing proofs. This strategy includes selecting a proof method and then successfully constructing an argument step by step, based on this method. In this section, after we have developed a versatile arsenal of proof methods, we will study some aspects of the art and science of proofs. We will provide advice on how to find a proof of a theorem. We will describe some tricks of the trade, including how proofs can be found by working backward and by adapting existing proofs.

When mathematicians work, they formulate conjectures and attempt to prove or disprove them. We will briefly describe this process here by proving results about tiling checkerboards with dominoes and other types of pieces. Looking at tilings of this kind, we will be able to quickly formulate conjectures and prove theorems without first developing a theory.

We will conclude the section by discussing the role of open questions. In particular, we will discuss some interesting problems either that have been solved after remaining open for hundreds of years or that still remain open.

Exhaustive Proof and Proof by Cases

Sometimes we cannot prove a theorem using a single argument that holds for all possible cases. We now introduce a method that can be used to prove a theorem, by considering different cases separately. This method is based on a rule of inference that we will now introduce. To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \to q$$

the tautology

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \to q] \leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)]$$

can be used as a rule of inference. This shows that the original conditional statement with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$ can be proved by proving each of the n conditional statements $p_i \to q$, $i = 1, 2, \ldots, n$, individually. Such an argument is called a **proof by cases**. Sometimes to prove that a conditional statement $p \to q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of p as the hypothesis of the conditional statement, where p and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent.

EXHAUSTIVE PROOF Some theorems can be proved by examining a relatively small number of examples. Such proofs are called **exhaustive proofs**, or **proofs by exhaustion** because these proofs proceed by exhausting all possibilities. An exhaustive proof is a special type of proof by cases where each case involves checking a single example. We now provide some illustrations of exhaustive proofs.

EXAMPLE 1 Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Extra Examples

Solution: We use a proof by exhaustion. We only need verify the inequality $(n+1)^3 \geq 3^n$ when $n = 1, 2, 3$, and 4. For $n = 1$, we have $(n+1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$; for $n = 2$, we have $(n+1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$; for $n = 3$, we have $(n+1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$; and for $n = 4$, we have $(n+1)^3 = 5^3 = 125$ and $3^n = 3^4 = 81$. In each of these four cases, we see that $(n+1)^3 \geq 3^n$. We have used the method of exhaustion to prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$. ◀

EXAMPLE 2 Prove that the only consecutive positive integers not exceeding 100 that are perfect powers are 8 and 9. (An integer is a **perfect power** if it equals n^a, where a is an integer greater than 1.)

Solution: We use a proof by exhaustion. In particular, we can prove this fact by examining positive integers n not exceeding 100, first checking whether n is a perfect power, and if it is, checking whether $n+1$ is also a perfect power. A quicker way to do this is simply to look at all perfect powers not exceeding 100 and checking whether the next largest integer is also a perfect power. The squares of positive integers not exceeding 100 are 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. The cubes of positive integers not exceeding 100 are 1, 8, 27, and 64. The fourth powers of positive integers not exceeding 100 are 1, 16, and 81. The fifth powers of positive integers not exceeding 100 are 1 and 32. The sixth powers of positive integers not exceeding 100 are 1 and 64. There are no powers of positive integers higher than the sixth power not exceeding 100, other than 1. Looking at this list of perfect powers not exceeding 100, we see that $n = 8$ is the only perfect power n for which $n+1$ is also a perfect power. That is, $2^3 = 8$ and $3^2 = 9$ are the only two consecutive perfect powers not exceeding 100. ◀

Proofs by exhaustion can tire out people and computers when the number of cases challenges the available processing power!

People can carry out exhaustive proofs when it is necessary to check only a relatively small number of instances of a statement. Computers do not complain when they are asked to check a much larger number of instances of a statement, but they still have limitations. Note that not even a computer can check all instances when it is impossible to list all instances to check.

PROOF BY CASES A proof by cases must cover all possible cases that arise in a theorem. We illustrate proof by cases with a couple of examples. In each example, you should check that all possible cases are covered.

EXAMPLE 3 Prove that if n is an integer, then $n^2 \geq n$.

Solution: We can prove that $n^2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -1$. We split the proof into three cases because it is straightforward to prove the result by considering zero, positive integers, and negative integers separately.

Case (i): When $n = 0$, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.

Extra Examples

Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive integer n, we obtain $n \cdot n \geq n \cdot 1$. This implies that $n^2 \geq n$ for $n \geq 1$.

Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.

Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$. ◀

EXAMPLE 4 Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|a|$, the absolute value of a, equals a when $a \geq 0$ and equals $-a$ when $a \leq 0$.)

Solution: In our proof of this theorem, we remove absolute values using the fact that $|a| = a$ when $a \geq 0$ and $|a| = -a$ when $a < 0$. Because both $|x|$ and $|y|$ occur in our formula, we will need four cases: *(i)* x and y both nonnegative, *(ii)* x nonnegative and y is negative, *(iii)* x negative and y nonnegative, and *(iv)* x negative and y negative. We denote by p_1, p_2, p_3, and p_4, the proposition stating the assumption for each of these four cases, respectively.

(Note that we can remove the absolute value signs by making the appropriate choice of signs within each case.)

Case (i): We see that $p_1 \to q$ because $xy \geq 0$ when $x \geq 0$ and $y \geq 0$, so that $|xy| = xy = |x||y|$.

Case (ii): To see that $p_2 \to q$, note that if $x \geq 0$ and $y < 0$, then $xy \leq 0$, so that $|xy| = -xy = x(-y) = |x||y|$. (Here, because $y < 0$, we have $|y| = -y$.)

Case (iii): To see that $p_3 \to q$, we follow the same reasoning as the previous case with the roles of x and y reversed.

Case (iv): To see that $p_4 \to q$, note that when $x < 0$ and $y < 0$, it follows that $xy > 0$. Hence, $|xy| = xy = (-x)(-y) = |x||y|$.

Because $|xy| = |x||y|$ holds in each of the four cases and these cases exhaust all possibilities, we can conclude that $|xy| = |x||y|$, whenever x and y are real numbers. ◀

LEVERAGING PROOF BY CASES The examples we have presented illustrating proof by cases provide some insight into when to use this method of proof. In particular, when it is not possible to consider all cases of a proof at the same time, a proof by cases should be considered. When should you use such a proof? Generally, look for a proof by cases when there is no obvious way to begin a proof, but when extra information in each case helps move the proof forward. Example 5 illustrates how the method of proof by cases can be used effectively.

EXAMPLE 5 Formulate a conjecture about the final decimal digit of the square of an integer and prove your result.

Solution: The smallest perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, and so on. We notice that the digits that occur as the final digit of a square are 0, 1, 4, 5, 6, and 9, with 2, 3, 7, and 8 never appearing as the final digit of a square. We conjecture this theorem: The final decimal digit of a perfect square is 0, 1, 4, 5, 6 or 9. How can we prove this theorem?

We first note that we can express an integer n as $10a + b$, where a and b are positive integers and b is 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. Here a is the integer obtained by subtracting the final decimal digit of n from n and dividing by 10. Next, note that $(10a + b)^2 = 100a^2 + 20ab + b^2 = 10(10a^2 + 2b) + b^2$, so that the final decimal digit of n^2 is the same as the final decimal digit of b^2. Furthermore, note that the final decimal digit of b^2 is the same as the final decimal digit of $(10 - b)^2 = 100 - 20b + b^2$. Consequently, we can reduce our proof to the consideration of six cases.

Case (i): The final digit of n is 1 or 9. Then the final decimal digit of n^2 is the final decimal digit of $1^2 = 1$ or $9^2 = 81$, namely 1.

Case (ii): The final digit of n is 2 or 8. Then the final decimal digit of n^2 is the final decimal digit of $2^2 = 4$ or $8^2 = 64$, namely 4.

Case (iii): The final digit of n is 3 or 7. Then the final decimal digit of n^2 is the final decimal digit of $3^2 = 9$ or $7^2 = 49$, namely 9.

Case (iv): The final digit of n is 4 or 6. Then the final decimal digit of n^2 is the final decimal digit of $4^2 = 16$ or $6^2 = 36$, namely 6.

Case (v): The final decimal digit of n is 5. Then the final decimal digit of n^2 is the final decimal digit of $5^2 = 25$, namely 5.

Case (vi): The final decimal digit of n is 0. Then the final decimal digit of n^2 is the final decimal digit of $0^2 = 0$, namely 0.

Because we have considered all six cases, we can conclude that the final decimal digit of n^2, where n is an integer is either 0, 1, 2, 4, 5, 6, or 9. ◀

Sometimes we can eliminate all but a few examples in a proof by cases, as Example 6 illustrates.

EXAMPLE 6 Show that there are no solutions in integers x and y of $x^2 + 3y^2 = 8$.

Solution: We can quickly reduce a proof to checking just a few simple cases because $x^2 > 8$ when $|x| \geq 3$ and $3y^2 > 8$ when $|y| \geq 2$. This leaves the cases when x equals $-2, -1, 0, 1$, or 2 and y equals $-1, 0$, or 1. We can finish using an exhaustive proof. To dispense with the remaining cases, we note that possible values for x^2 are 0, 1, and 4, and possible values for $3y^2$ are 0 and 3, and the largest sum of possible values for x^2 and $3y^2$ is 7. Consequently, it is impossible for $x^2 + 3y^2 = 8$ to hold when x and y are integers. ◀

WITHOUT LOSS OF GENERALITY In the proof in Example 4, we dismissed case (*iii*), where $x < 0$ and $y \geq 0$, because it is the same as case (*ii*), where $x \geq 0$ and $y < 0$, with the roles of x and y reversed. To shorten the proof, we could have proved cases (*ii*) and (*iii*) together by assuming, **without loss of generality**, that $x \geq 0$ and $y < 0$. Implicit in this statement is that we can complete the case with $x < 0$ and $y \geq 0$ using the same argument as we used for the case with $x \geq 0$ and $y < 0$, but with the obvious changes.

In general, when the phrase "without loss of generality" is used in a proof (often abbreviated as WLOG), we assert that by proving one case of a theorem, no additional argument is required to prove other specified cases. That is, other cases follow by making straightforward changes to the argument, or by filling in some straightforward initial step. Proofs by cases can often be made much more efficient when the notion of without loss of generality is employed. Of course, incorrect use of this principle can lead to unfortunate errors. Sometimes assumptions are made that lead to a loss in generality. Such assumptions can be made that do not take into account that one case may be substantially different from others. This can lead to an incomplete, and possibly unsalvageable, proof. In fact, many incorrect proofs of famous theorems turned out to rely on arguments that used the idea of "without loss of generality" to establish cases that could not be quickly proved from simpler cases.

In a proof by cases be sure not to omit any cases and check that you have proved all cases correctly!

We now illustrate a proof where without loss of generality is used effectively together with other proof techniques.

EXAMPLE 7 Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even.

Solution: We will use proof by contraposition, the notion of without loss of generality, and proof by cases. First, suppose that x and y are not both even. That is, assume that x is odd or that y is odd (or both). Without loss of generality, we assume that x is odd, so that $x = 2m + 1$ for some integer k.

To complete the proof, we need to show that xy is odd or $x + y$ is odd. Consider two cases: (i) y even, and (ii) y odd. In (i), $y = 2n$ for some integer n, so that $x + y = (2m + 1) + 2n = 2(m + n) + 1$ is odd. In (ii), $y = 2n + 1$ for some integer n, so that $xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$ is odd. This completes the proof by contraposition. (Note that our use of without loss of generality within the proof is justified because the proof when y is odd can be obtained by simply interchanging the roles of x and y in the proof we have given.) ◀

COMMON ERRORS WITH EXHAUSTIVE PROOF AND PROOF BY CASES A common error of reasoning is to draw incorrect conclusions from examples. No matter how many separate examples are considered, a theorem is not proved by considering examples unless every possible case is covered. The problem of proving a theorem is analogous to showing that a computer program always produces the output desired. No matter how many input values are tested, unless all input values are tested, we cannot conclude that the program always produces the correct output.

EXAMPLE 8 Is it true that every positive integer is the sum of 18 fourth powers of integers?

Solution: To determine whether a positive integer n can be written as the sum of 18 fourth powers of integers, we might begin by examining whether n is the sum of 18 fourth powers of integers for the smallest positive integers. Because the fourth powers of integers are 0, 1, 16, 81, ..., if we can select 18 terms from these numbers that add up to n, then n is the sum of 18 fourth powers. We can show that all positive integers up to 78 can be written as the sum of 18 fourth powers. (The details are left to the reader.) However, if we decided this was enough checking, we would come to the wrong conclusion. It is not true that every positive integer is the sum of 18 fourth powers because 79 is not the sum of 18 fourth powers (as the reader can verify). ◀

Another common error involves making unwarranted assumptions that lead to incorrect proofs by cases where not all cases are considered. This is illustrated in Example 9.

EXAMPLE 9 What is wrong with this "proof?"

"Theorem:" If x is a real number, then x^2 is a positive real number.

"Proof:" Let p_1 be "x is positive," let p_2 be "x is negative," and let q be "x^2 is positive." To show that $p_1 \rightarrow q$ is true, note that when x is positive, x^2 is positive because it is the product of two positive numbers, x and x. To show that $p_2 \rightarrow q$, note that when x is negative, x^2 is positive because it is the product of two negative numbers, x and x. This completes the proof.

Solution: The problem with this "proof" is that we missed the case of $x = 0$. When $x = 0$, $x^2 = 0$ is not positive, so the supposed theorem is false. If p is "x is a real number," then we can prove results where p is the hypothesis with three cases, p_1, p_2, and p_3, where p_1 is "x is positive," p_2 is "x is negative," and p_3 is "$x = 0$" because of the equivalence $p \leftrightarrow p_1 \vee p_2 \vee p_3$. ◀

Existence Proofs

Many theorems are assertions that objects of a particular type exist. A theorem of this type is a proposition of the form $\exists x P(x)$, where P is a predicate. A proof of a proposition of the form $\exists x P(x)$ is called an **existence proof**. There are several ways to prove a theorem of this type. Sometimes an existence proof of $\exists x P(x)$ can be given by finding an element a, called a **witness**, such that $P(a)$ is true. This type of existence proof is called **constructive**. It is also possible to give an existence proof that is **nonconstructive**; that is, we do not find an element a such that $P(a)$ is true, but rather prove that $\exists x P(x)$ is true in some other way. One common method of giving a nonconstructive existence proof is to use proof by contradiction and show that the negation of the existential quantification implies a contradiction. The concept of a constructive existence proof is illustrated by Example 10 and the concept of a nonconstructive existence proof is illustrated by Example 11.

EXAMPLE 10 **A Constructive Existence Proof** Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Extra Examples

Solution: After considerable computation (such as a computer search) we find that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Because we have displayed a positive integer that can be written as the sum of cubes in two different ways, we are done.

There is an interesting story pertaining to this example. The English mathematician G. H. Hardy, when visiting the ailing Indian prodigy Ramanujan in the hospital, remarked that 1729, the number of the cab he took, was rather dull. Ramanujan replied “No, it is a very interesting number; it is the smallest number expressible as the sum of cubes in two different ways.” ◀

EXAMPLE 11 **A Nonconstructive Existence Proof** Show that there exist irrational numbers x and y such that x^y is rational.

Solution: By Example 10 in Section 1.7 we know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, we have two irrational numbers x and y with x^y rational, namely, $x = \sqrt{2}$ and $y = \sqrt{2}$. On the other hand if $\sqrt{2}^{\sqrt{2}}$ is irrational, then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ so that $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2$.

This proof is an example of a nonconstructive existence proof because we have not found irrational numbers x and y such that x^y is rational. Rather, we have shown that either the pair $x = \sqrt{2}$, $y = \sqrt{2}$ or the pair $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$ have the desired property, but we do not know which of these two pairs works! ◀

Links

GODFREY HAROLD HARDY (1877–1947) Hardy, born in Cranleigh, Surrey, England, was the older of two children of Isaac Hardy and Sophia Hall Hardy. His father was the geography and drawing master at the Cranleigh School and also gave singing lessons and played soccer. His mother gave piano lessons and helped run a boardinghouse for young students. Hardy’s parents were devoted to their children’s education. Hardy demonstrated his numerical ability at the early age of two when he began writing down numbers into the millions. He had a private mathematics tutor rather than attending regular classes at the Cranleigh School. He moved to Winchester College, a private high school, when he was 13 and was awarded a scholarship. He excelled in his studies and demonstrated a strong interest in mathematics. He entered Trinity College, Cambridge, in 1896 on a scholarship and won several prizes during his time there, graduating in 1899.

Hardy held the position of lecturer in mathematics at Trinity College at Cambridge University from 1906 to 1919, when he was appointed to the Sullivan chair of geometry at Oxford. He had become unhappy with Cambridge over the dismissal of the famous philosopher and mathematician Bertrand Russell from Trinity for antiwar activities and did not like a heavy load of administrative duties. In 1931 he returned to Cambridge as the Sadleirian professor of pure mathematics, where he remained until his retirement in 1942. He was a pure mathematician and held an elitist view of mathematics, hoping that his research could never be applied. Ironically, he is perhaps best known as one of the developers of the Hardy–Weinberg law, which predicts patterns of inheritance. His work in this area appeared as a letter to the journal *Science* in which he used simple algebraic ideas to demonstrate errors in an article on genetics. Hardy worked primarily in number theory and function theory, exploring such topics as the Riemann zeta function, Fourier series, and the distribution of primes. He made many important contributions to many important problems, such as Waring’s problem about representing positive integers as sums of kth powers and the problem of representing odd integers as sums of three primes. Hardy is also remembered for his collaborations with John E. Littlewood, a colleague at Cambridge, with whom he wrote more than 100 papers, and the famous Indian mathematical prodigy Srinivasa Ramanujan. His collaboration with Littlewood led to the joke that there were only three important English mathematicians at that time, Hardy, Littlewood, and Hardy–Littlewood, although some people thought that Hardy had invented a fictitious person, Littlewood, because Littlewood was seldom seen outside Cambridge. Hardy had the wisdom of recognizing Ramanujan’s genius from unconventional but extremely creative writings Ramanujan sent him, while other mathematicians failed to see the genius. Hardy brought Ramanujan to Cambridge and collaborated on important joint papers, establishing new results on the number of partitions of an integer. Hardy was interested in mathematics education, and his book *A Course of Pure Mathematics* had a profound effect on undergraduate instruction in mathematics in the first half of the twentieth century. Hardy also wrote *A Mathematician’s Apology*, in which he gives his answer to the question of whether it is worthwhile to devote one’s life to the study of mathematics. It presents Hardy’s view of what mathematics is and what a mathematician does.

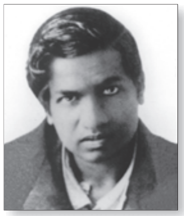
Hardy had a strong interest in sports. He was an avid cricket fan and followed scores closely. One peculiar trait he had was that he did not like his picture taken (only five snapshots are known) and disliked mirrors, covering them with towels immediately upon entering a hotel room.

Nonconstructive existence proofs often are quite subtle, as Example 12 illustrates.

EXAMPLE 12 **Chomp** is a game played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned, as shown in Figure 1(a). The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below it (see Figure 1(b), for example). The loser is the player who has no choice but to eat the poisoned cookie. We ask whether one of the two players has a winning strategy. That is, can one of the players always make moves that are guaranteed to lead to a win?

Solution: We will give a nonconstructive existence proof of a winning strategy for the first player. That is, we will show that the first player always has a winning strategy without explicitly describing the moves this player must follow.

First, note that the game ends and cannot finish in a draw because with each move at least one cookie is eaten, so after no more than $m \times n$ moves the game ends, where the initial grid is $m \times n$. Now, suppose that the first player begins the game by eating just the cookie in the bottom right corner. There are two possibilities, this is the first move of a winning strategy for the first player, or the second player can make a move that is the first move of a winning strategy for the second player. In this second case, instead of eating just the cookie in the bottom right corner, the first player could have made the same move that the second player made as the first



SRINIVASA RAMANUJAN (1887–1920) The famous mathematical prodigy Ramanujan was born and raised in southern India near the city of Madras (now called Chennai). His father was a clerk in a cloth shop. His mother contributed to the family income by singing at a local temple. Ramanujan studied at the local English language school, displaying his talent and interest for mathematics. At the age of 13 he mastered a textbook used by college students. When he was 15, a university student lent him a copy of *Synopsis of Pure Mathematics*. Ramanujan decided to work out the over 6000 results in this book, stated without proof or explanation, writing on sheets later collected to form notebooks. He graduated from high school in 1904, winning a scholarship to the University of Madras. Enrolling in a fine arts curriculum, he neglected his subjects other than mathematics and lost his scholarship. He failed to pass examinations at the university four times from 1904 to 1907, doing well only in mathematics. During this time he filled his notebooks with original writings, sometimes rediscovering already published work and at other times making new discoveries.

Without a university degree, it was difficult for Ramanujan to find a decent job. To survive, he had to depend on the goodwill of his friends. He tutored students in mathematics, but his unconventional ways of thinking and failure to stick to the syllabus caused problems. He was married in 1909 in an arranged marriage to a young woman nine years his junior. Needing to support himself and his wife, he moved to Madras and sought a job. He showed his notebooks of mathematical writings to his potential employers, but the books bewildered them. However, a professor at the Presidency College recognized his genius and supported him, and in 1912 he found work as an accounts clerk, earning a small salary.

Ramanujan continued his mathematical work during this time and published his first paper in 1910 in an Indian journal. He realized that his work was beyond that of Indian mathematicians and decided to write to leading English mathematicians. The first mathematicians he wrote to turned down his request for help. But in January 1913 he wrote to G. H. Hardy, who was inclined to turn Ramanujan down, but the mathematical statements in the letter, although stated without proof, puzzled Hardy. He decided to examine them closely with the help of his colleague and collaborator J. E. Littlewood. They decided, after careful study, that Ramanujan was probably a genius, because his statements "could only be written down by a mathematician of the highest class; they must be true, because if they were not true, no one would have the imagination to invent them."

Hardy arranged a scholarship for Ramanujan, bringing him to England in 1914. Hardy personally tutored him in mathematical analysis, and they collaborated for five years, proving significant theorems about the number of partitions of integers. During this time, Ramanujan made important contributions to number theory and also worked on continued fractions, infinite series, and elliptic functions. Ramanujan had amazing insight involving certain types of functions and series, but his purported theorems on prime numbers were often wrong, illustrating his vague idea of what constitutes a correct proof. He was one of the youngest members ever appointed a Fellow of the Royal Society. Unfortunately, in 1917 Ramanujan became extremely ill. At the time, it was thought that he had trouble with the English climate and had contracted tuberculosis. It is now thought that he suffered from a vitamin deficiency, brought on by Ramanujan's strict vegetarianism and shortages in wartime England. He returned to India in 1919, continuing to do mathematics even when confined to his bed. He was religious and thought his mathematical talent came from his family deity, Namagiri. He considered mathematics and religion to be linked. He said that "an equation for me has no meaning unless it expresses a thought of God." His short life came to an end in April 1920, when he was 32 years old. Ramanujan left several notebooks of unpublished results. The writings in these notebooks illustrate Ramanujan's insights but are quite sketchy. Several mathematicians have devoted many years of study to explaining and justifying the results in these notebooks.

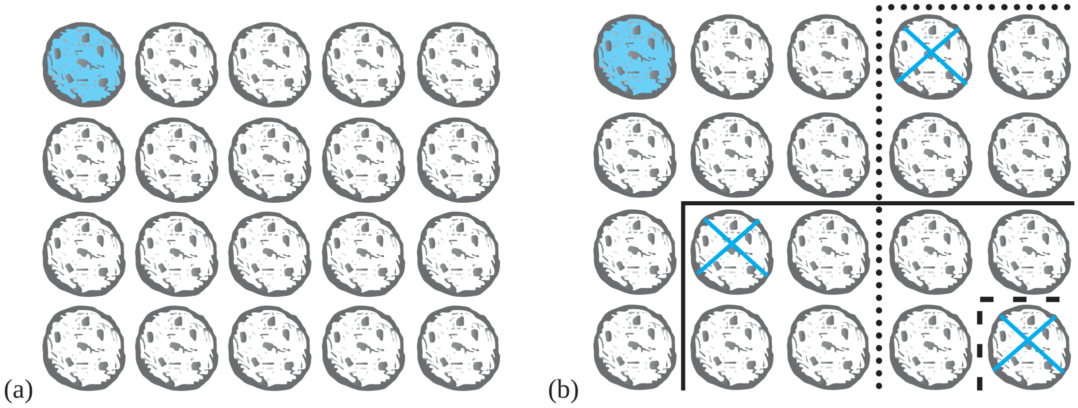


FIGURE 1 **(a) Chomp (Top Left Cookie Poisoned). (b) Three Possible Moves.**

move of a winning strategy (and then continued to follow that winning strategy). This would guarantee a win for the first player.

Note that we showed that a winning strategy exists, but we did not specify an actual winning strategy. Consequently, the proof is a nonconstructive existence proof. In fact, no one has been able to describe a winning strategy for that Chomp that applies for all rectangular grids by describing the moves that the first player should follow. However, winning strategies can be described for certain special cases, such as when the grid is square and when the grid only has two rows of cookies (see Exercises 15 and 16 in Section 5.2). ◀

Uniqueness Proofs

Some theorems assert the existence of a unique element with a particular property. In other words, these theorems assert that there is exactly one element with this property. To prove a statement of this type we need to show that an element with this property exists and that no other element has this property. The two parts of a **uniqueness proof** are:

Existence: We show that an element x with the desired property exists.
Uniqueness: We show that if $y \neq x$, then y does not have the desired property.

Equivalently, we can show that if x and y both have the desired property, then $x = y$.

Remark: Showing that there is a unique element x such that $P(x)$ is the same as proving the statement $\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$.

We illustrate the elements of a uniqueness proof in Example 13.

EXAMPLE 13 Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution: First, note that the real number $r = -b/a$ is a solution of $ar + b = 0$ because $a(-b/a) + b = -b + b = 0$. Consequently, a real number r exists for which $ar + b = 0$. This is the existence part of the proof.

Extra Examples

Second, suppose that s is a real number such that $as + b = 0$. Then $ar + b = as + b$, where $r = -b/a$. Subtracting b from both sides, we find that $ar = as$. Dividing both sides of this last equation by a, which is nonzero, we see that $r = s$. This means that if $s \neq r$, then $as + b \neq 0$. This establishes the uniqueness part of the proof. ◀

Proof Strategies

Finding proofs can be a challenging business. When you are confronted with a statement to prove, you should first replace terms by their definitions and then carefully analyze what the hypotheses and the conclusion mean. After doing so, you can attempt to prove the result using one of the available methods of proof. Generally, if the statement is a conditional statement, you should first try a direct proof; if this fails, you can try an indirect proof. If neither of these approaches works, you might try a proof by contradiction.

FORWARD AND BACKWARD REASONING Whichever method you choose, you need a starting point for your proof. To begin a direct proof of a conditional statement, you start with the premises. Using these premises, together with axioms and known theorems, you can construct a proof using a sequence of steps that leads to the conclusion. This type of reasoning, called *forward reasoning*, is the most common type of reasoning used to prove relatively simple results. Similarly, with indirect reasoning you can start with the negation of the conclusion and, using a sequence of steps, obtain the negation of the premises.

Unfortunately, forward reasoning is often difficult to use to prove more complicated results, because the reasoning needed to reach the desired conclusion may be far from obvious. In such cases it may be helpful to use *backward reasoning*. To reason backward to prove a statement q, we find a statement p that we can prove with the property that $p \to q$. (Note that it is not helpful to find a statement r that you can prove such that $q \to r$, because it is the fallacy of begging the question to conclude from $q \to r$ and r that q is true.) Backward reasoning is illustrated in Examples 14 and 15.

EXAMPLE 14 Given two positive real numbers x and y, their **arithmetic mean** is $(x + y)/2$ and their **geometric mean** is $\sqrt{xy}$. When we compare the arithmetic and geometric means of pairs of distinct positive real numbers, we find that the arithmetic mean is always greater than the geometric mean. [For example, when $x = 4$ and $y = 6$, we have $5 = (4 + 6)/2 > \sqrt{4 \cdot 6} = \sqrt{24}$.] Can we prove that this inequality is always true?

Solution: To prove that $(x + y)/2 > \sqrt{xy}$ when x and y are distinct positive real numbers, we can work backward. We construct a sequence of equivalent inequalities. The equivalent inequalities are

$$\begin{aligned}
(x + y)/2 &> \sqrt{xy},\\
(x + y)^2/4 &> xy,\\
(x + y)^2 &> 4xy,\\
x^2 + 2xy + y^2 &> 4xy,\\
x^2 - 2xy + y^2 &> 0,\\
(x - y)^2 &> 0.
\end{aligned}$$

Extra Examples

Because $(x - y)^2 > 0$ when $x \neq y$, it follows that the final inequality is true. Because all these inequalities are equivalent, it follows that $(x + y)/2 > \sqrt{xy}$ when $x \neq y$. Once we have carried out this backward reasoning, we can easily reverse the steps to construct a proof using forward reasoning. We now give this proof.

Suppose that x and y are distinct positive real numbers. Then $(x - y)^2 > 0$ because the square of a nonzero real number is positive (see Appendix 1). Because $(x - y)^2 = x^2 - 2xy + y^2$, this implies that $x^2 - 2xy + y^2 > 0$. Adding $4xy$ to both sides, we obtain $x^2 + 2xy + y^2 > 4xy$. Because $x^2 + 2xy + y^2 = (x + y)^2$, this means that $(x + y)^2 \geq 4xy$. Dividing both sides of this equation by 4, we see that $(x + y)^2/4 > xy$. Finally, taking square roots of both sides (which preserves the inequality because both sides are positive) yields

$(x+y)/2 > \sqrt{xy}$. We conclude that if x and y are distinct positive real numbers, then their arithmetic mean $(x+y)/2$ is greater than their geometric mean $\sqrt{xy}$. ◀

EXAMPLE 15 Suppose that two people play a game taking turns removing one, two, or three stones at a time from a pile that begins with 15 stones. The person who removes the last stone wins the game. Show that the first player can win the game no matter what the second player does.

Solution: To prove that the first player can always win the game, we work backward. At the last step, the first player can win if this player is left with a pile containing one, two, or three stones. The second player will be forced to leave one, two, or three stones if this player has to remove stones from a pile containing four stones. Consequently, one way for the first person to win is to leave four stones for the second player on the next-to-last move. The first person can leave four stones when there are five, six, or seven stones left at the beginning of this player's move, which happens when the second player has to remove stones from a pile with eight stones. Consequently, to force the second player to leave five, six, or seven stones, the first player should leave eight stones for the second player at the second-to-last move for the first player. This means that there are nine, ten, or eleven stones when the first player makes this move. Similarly, the first player should leave twelve stones when this player makes the first move. We can reverse this argument to show that the first player can always make moves so that this player wins the game no matter what the second player does. These moves successively leave twelve, eight, and four stones for the second player. ◀

ADAPTING EXISTING PROOFS An excellent way to look for possible approaches that can be used to prove a statement is to take advantage of existing proofs of similar results. Often an existing proof can be adapted to prove other facts. Even when this is not the case, some of the ideas used in existing proofs may be helpful. Because existing proofs provide clues for new proofs, you should read and understand the proofs you encounter in your studies. This process is illustrated in Example 16.

EXAMPLE 16 In Example 10 of Section 1.7 we proved that $\sqrt{2}$ is irrational. We now conjecture that $\sqrt{3}$ is irrational. Can we adapt the proof in Example 10 in Section 1.7 to show that $\sqrt{3}$ is irrational?

Extra Examples

Solution: To adapt the proof in Example 10 in Section 1.7, we begin by mimicking the steps in that proof, but with $\sqrt{2}$ replaced with $\sqrt{3}$. First, we suppose that $\sqrt{3} = d/c$ where the fraction c/d is in lowest terms. Squaring both sides tells us that $3 = c^2/d^2$, so that $3d^2 = c^2$. Can we use this equation to show that 3 must be a factor of both c and d, similar to how we used the equation $2b^2 = a^2$ in Example 10 in Section 1.7 to show that 2 must be a factor of both a and b? (Recall that an integer s is a factor of the integer t if t/s is an integer. An integer n is even if and only if 2 is a factor of n.) In turns out that we can, but we need some ammunition from number theory, which we will develop in Chapter 4. We sketch out the remainder of the proof, but leave the justification of these steps until Chapter 4. Because 3 is a factor of c^2, it must also be a factor of c. Furthermore, because 3 is a factor of c, 9 is a factor of c^2, which means that 9 is a factor of $3d^2$. This implies that 3 is a factor of d^2, which means that 3 is a factor of that d. This makes 3 a factor of both c and d, which contradicts the assumption that c/d is in lowest terms. After we have filled in the justification for these steps, we will have shown that $\sqrt{3}$ is irrational by adapting the proof that $\sqrt{2}$ is irrational. Note that this proof can be extended to show that $\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square. We leave the details of this to Chapter 4. ◀

A good tip is to look for existing proofs that you might adapt when you are confronted with proving a new theorem, particularly when the new theorem seems similar to one you have already proved.

Looking for Counterexamples

In Section 1.7 we introduced the use of counterexamples to show that certain statements are false. When confronted with a conjecture, you might first try to prove this conjecture, and if your attempts are unsuccessful, you might try to find a counterexample, first by looking at the simplest, smallest examples. If you cannot find a counterexample, you might again try to prove the statement. In any case, looking for counterexamples is an extremely important pursuit, which often provides insights into problems. We will illustrate the role of counterexamples in Example 17.

EXAMPLE 17 In Example 14 in Section 1.7 we showed that the statement "Every positive integer is the sum of two squares of integers" is false by finding a counterexample. That is, there are positive integers that cannot be written as the sum of the squares of two integers. Although we cannot write every positive integer as the sum of the squares of two integers, maybe we can write every positive integer as the sum of the squares of three integers. That is, is the statement "Every positive integer is the sum of the squares of three integers" true or false?

Extra Examples

Solution: Because we know that not every positive integer can be written as the sum of two squares of integers, we might initially be skeptical that every positive integer can be written as the sum of three squares of integers. So, we first look for a counterexample. That is, we can show that the statement "Every positive integer is the sum of three squares of integers" is false if we can find a particular integer that is not the sum of the squares of three integers. To look for a counterexample, we try to write successive positive integers as a sum of three squares. We find that $1 = 0^2 + 0^2 + 1^2$, $2 = 0^2 + 1^2 + 1^2$, $3 = 1^2 + 1^2 + 1^2$, $4 = 0^2 + 0^2 + 2^2$, $5 = 0^2 + 1^2 + 2^2$, $6 = 1^2 + 1^2 + 2^2$, but we cannot find a way to write 7 as the sum of three squares. To show that there are not three squares that add up to 7, we note that the only possible squares we can use are those not exceeding 7, namely, 0, 1, and 4. Because no three terms where each term is 0, 1, or 4 add up to 7, it follows that 7 is a counterexample. We conclude that the statement "Every positive integer is the sum of the squares of three integers" is false.

We have shown that not every positive integer is the sum of the squares of three integers. The next question to ask is whether every positive integer is the sum of the squares of four positive integers. Some experimentation provides evidence that the answer is yes. For example, $7 = 1^2 + 1^2 + 1^2 + 2^2$, $25 = 4^2 + 2^2 + 2^2 + 1^2$, and $87 = 9^2 + 2^2 + 1^2 + 1^2$. It turns out the conjecture "Every positive integer is the sum of the squares of four integers" is true. For a proof, see [Ro10]. ◀

Proof Strategy in Action

Mathematics is generally taught as if mathematical facts were carved in stone. Mathematics texts (including the bulk of this book) formally present theorems and their proofs. Such presentations do not convey the discovery process in mathematics. This process begins with exploring concepts and examples, asking questions, formulating conjectures, and attempting to settle these conjectures either by proof or by counterexample. These are the day-to-day activities of mathematicians. Believe it or not, the material taught in textbooks was originally developed in this way.

Extra Examples

People formulate conjectures on the basis of many types of possible evidence. The examination of special cases can lead to a conjecture, as can the identification of possible patterns. Altering the hypotheses and conclusions of known theorems also can lead to plausible conjectures. At other times, conjectures are made based on intuition or a belief that a result holds. No matter how a conjecture was made, once it has been formulated, the goal is to prove or disprove it. When mathematicians believe that a conjecture may be true, they try to find a proof. If they cannot find a proof, they may look for a counterexample. When they cannot find a counterexample, they may switch gears and once again try to prove the conjecture. Although many conjectures are quickly settled, a few conjectures resist attack for hundreds of years and lead to

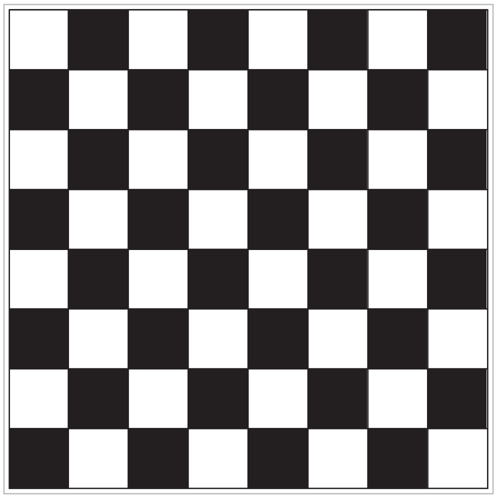


FIGURE 2 **The Standard Checkerboard.**

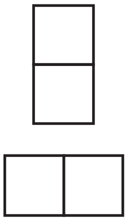


FIGURE 3 **Two Dominoes.**

the development of new parts of mathematics. We will mention a few famous conjectures later in this section.

Tilings

Links

We can illustrate aspects of proof strategy through a brief study of tilings of checkerboards. Looking at tilings of checkerboards is a fruitful way to quickly discover many different results and construct their proofs using a variety of proof methods. There are almost an endless number of conjectures that can be made and studied in this area too. To begin, we need to define some terms. A **checkerboard** is a rectangle divided into squares of the same size by horizontal and vertical lines. The game of checkers is played on a board with 8 rows and 8 columns; this board is called the **standard checkerboard** and is shown in Figure 2. In this section we use the term **board** to refer to a checkerboard of any rectangular size as well as parts of checkerboards obtained by removing one or more squares. A **domino** is a rectangular piece that is one square by two squares, as shown in Figure 3. We say that a board is **tiled** by dominoes when all its squares are covered with no overlapping dominoes and no dominoes overhanging the board. We now develop some results about tiling boards using dominoes.

EXAMPLE 18 Can we tile the standard checkerboard using dominoes?

Solution: We can find many ways to tile the standard checkerboard using dominoes. For example, we can tile it by placing 32 dominoes horizontally, as shown in Figure 4. The existence of one such tiling completes a constructive existence proof. Of course, there are a large number of other ways to do this tiling. We can place 32 dominoes vertically on the board or we can place some tiles vertically and some horizontally. But for a constructive existence proof we needed to find just one such tiling. ◀

EXAMPLE 19 Can we tile a board obtained by removing one of the four corner squares of a standard checkerboard?

Extra Examples

Solution: To answer this question, note that a standard checkerboard has 64 squares, so removing a square produces a board with 63 squares. Now suppose that we could tile a board obtained from the standard checkerboard by removing a corner square. The board has an even number of

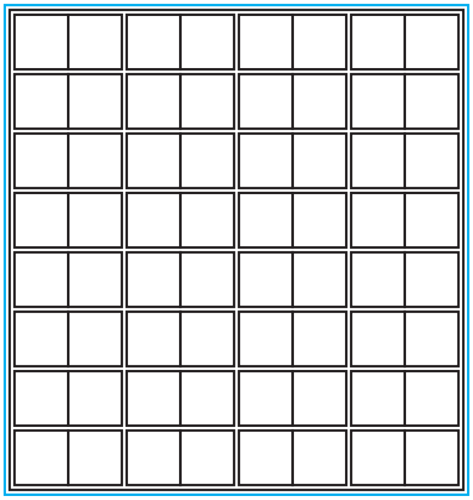


FIGURE 4 **Tiling the Standard Checkerboard.**

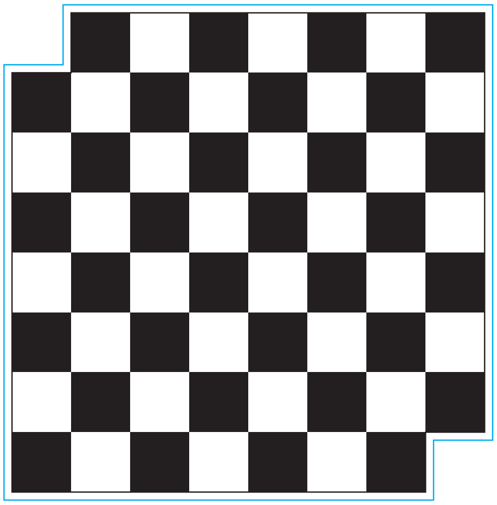


FIGURE 5 **The Standard Checkerboard with the Upper Left and Lower Right Squares Removed.**

squares because each domino covers two squares and no two dominoes overlap and no dominoes overhang the board. Consequently, we can prove by contradiction that a standard checkerboard with one square removed cannot be tiled using dominoes because such a board has an odd number of squares. ◀

We now consider a trickier situation.

EXAMPLE 20 Can we tile the board obtained by deleting the upper left and lower right corner squares of a standard checkerboard, shown in Figure 5?

Solution: A board obtained by deleting two squares of a standard checkerboard contains $64 - 2 = 62$ squares. Because 62 is even, we cannot quickly rule out the existence of a tiling of the standard checkerboard with its upper left and lower right squares removed, unlike Example 19, where we ruled out the existence of a tiling of the standard checkerboard with one corner square removed. Trying to construct a tiling of this board by successively placing dominoes might be a first approach, as the reader should attempt. However, no matter how much we try, we cannot find such a tiling. Because our efforts do not produce a tiling, we are led to conjecture that no tiling exists.

We might try to prove that no tiling exists by showing that we reach a dead end however we successively place dominoes on the board. To construct such a proof, we would have to consider all possible cases that arise as we run through all possible choices of successively placing dominoes. For example, we have two choices for covering the square in the second column of the first row, next to the removed top left corner. We could cover it with a horizontally placed tile or a vertically placed tile. Each of these two choices leads to further choices, and so on. It does not take long to see that this is not a fruitful plan of attack for a person, although a computer could be used to complete such a proof by exhaustion. (Exercise 45 asks you to supply such a proof to show that a 4×4 checkerboard with opposite corners removed cannot be tiled.)

We need another approach. Perhaps there is an easier way to prove there is no tiling of a standard checkerboard with two opposite corners removed. As with many proofs, a key observation can help. We color the squares of this checkerboard using alternating white and black squares, as in Figure 2. Observe that a domino in a tiling of such a board covers one white square and one black square. Next, note that this board has unequal numbers of white square and black squares. We can use these observations to prove by contradiction that a standard checkerboard with opposite corners removed cannot be tiled using dominoes. We now present such a proof.

Proof: Suppose we can use dominoes to tile a standard checkerboard with opposite corners removed. Note that the standard checkerboard with opposite corners removed contains $64 - 2 = 62$ squares. The tiling would use $62/2 = 31$ dominoes. Note that each domino in this tiling covers one white and one black square. Consequently, the tiling covers 31 white squares and 31 black squares. However, when we remove two opposite corner squares, either 32 of the remaining squares are white and 30 are black or else 30 are white and 32 are black. This contradicts the assumption that we can use dominoes to cover a standard checkerboard with opposite corners removed, completing the proof. ◀

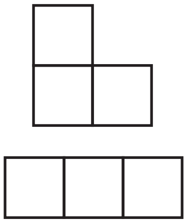


FIGURE 6 A Right Triomino and a Straight Triomino.

We can use other types of pieces besides dominoes in tilings. Instead of dominoes we can study tilings that use identically shaped pieces constructed from congruent squares that are connected along their edges. Such pieces are called **polyominoes**, a term coined in 1953 by the mathematician Solomon Golomb, the author of an entertaining book about them [Go94]. We will consider two polyominoes with the same number of squares the same if we can rotate and/or flip one of the polyominoes to get the other one. For example, there are two types of triominoes (see Figure 6), which are polyominoes made up of three squares connected by their sides. One type of triomino, the **straight triomino**, has three horizontally connected squares; the other type, **right triominoes**, resembles the letter L in shape, flipped and/or rotated, if necessary. We will study the tilings of a checkerboard by straight triominoes here; we will study tilings by right triominoes in Section 5.1.

EXAMPLE 21 Can you use straight triominoes to tile a standard checkerboard?

Solution: The standard checkerboard contains 64 squares and each triomino covers three squares. Consequently, if triominoes tile a board, the number of squares of the board must be a multiple of 3. Because 64 is not a multiple of 3, triominoes cannot be used to cover an 8×8 checkerboard. ◀

In Example 22, we consider the problem of using straight triominoes to tile a standard checkerboard with one corner missing.

EXAMPLE 22 Can we use straight triominoes to tile a standard checkerboard with one of its four corners removed? An 8×8 checkerboard with one corner removed contains $64 - 1 = 63$ squares. Any tiling by straight triominoes of one of these four boards uses $63/3 = 21$ triominoes. However, when we experiment, we cannot find a tiling of one of these boards using straight triominoes. A proof by exhaustion does not appear promising. Can we adapt our proof from Example 20 to prove that no such tiling exists?

Solution: We will color the squares of the checkerboard in an attempt to adapt the proof by contradiction we gave in Example 20 of the impossibility of using dominoes to tile a standard checkerboard with opposite corners removed. Because we are using straight triominoes rather than dominoes, we color the squares using three colors rather than two colors, as shown in Figure 7. Note that there are 21 blue squares, 21 black squares, and 22 white squares in this coloring. Next, we make the crucial observation that when a straight triomino covers three squares of the checkerboard, it covers one blue square, one black square, and one white square. Next, note that each of the three colors appears in a corner square. Thus without loss of generality, we may assume that we have rotated the coloring so that the missing square is colored blue. Therefore, we assume that the remaining board contains 20 blue squares, 21 black squares, and 22 white squares.

If we could tile this board using straight triominoes, then we would use $63/3 = 21$ straight triominoes. These triominoes would cover 21 blue squares, 21 black squares, and 21 white

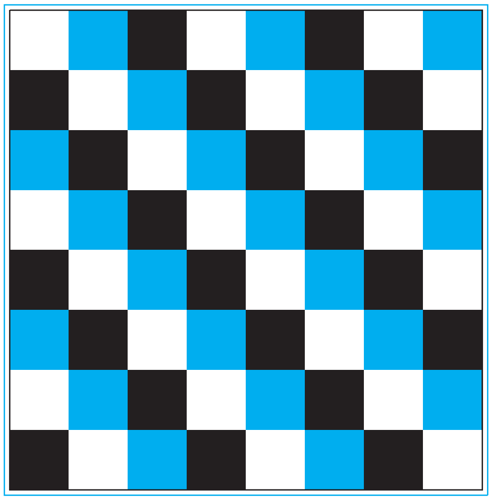


FIGURE 7 Coloring the Squares of the Standard Checkerboard with Three Colors.

squares. This contradicts the fact that this board contains 20 blue squares, 21 black squares, and 22 white squares. Therefore we cannot tile this board using straight triominoes. ◀

The Role of Open Problems

Many advances in mathematics have been made by people trying to solve famous unsolved problems. In the past 20 years, many unsolved problems have finally been resolved, such as the proof of a conjecture in number theory made more than 300 years ago. This conjecture asserts the truth of the statement known as **Fermat's last theorem**.

THEOREM 1

FERMAT'S LAST THEOREM The equation

$$x^n + y^n = z^n$$

has no solutions in integers x, y, and z with $xyz \neq 0$ whenever n is an integer with $n > 2$.

Links

Remark: The equation $x^2 + y^2 = z^2$ has infinitely many solutions in integers x, y, and z; these solutions are called Pythagorean triples and correspond to the lengths of the sides of right triangles with integer lengths. See Exercise 32.

This problem has a fascinating history. In the seventeenth century, Fermat jotted in the margin of his copy of the works of Diophantus that he had a "wondrous proof" that there are no integer solutions of $x^n + y^n = z^n$ when n is an integer greater than 2 with $xyz \neq 0$. However, he never published a proof (Fermat published almost nothing), and no proof could be found in the papers he left when he died. Mathematicians looked for a proof for three centuries without success, although many people were convinced that a relatively simple proof could be found. (Proofs of special cases were found, such as the proof of the case when $n = 3$ by Euler and the proof of the $n = 4$ case by Fermat himself.) Over the years, several established mathematicians thought that they had proved this theorem. In the nineteenth century, one of these failed attempts led to the development of the part of number theory called algebraic number theory. A correct proof, requiring hundreds of pages of advanced mathematics, was not found until the 1990s, when Andrew Wiles used recently developed ideas from a sophisticated area of number theory called the theory of elliptic curves to prove Fermat's last theorem. Wiles's quest to find a proof of Fermat's last theorem using this powerful theory, described in a program in the *Nova* series on public television, took close to ten years! Moreover, his proof was based on major contributions of many mathematicians. (The interested reader should consult [Ro10] for more information about Fermat's last theorem and for additional references concerning this problem and its resolution.)

We now state an open problem that is simple to describe, but that seems quite difficult to resolve.

EXAMPLE 23

Links

The* $\mathbf{3x+1}$ *Conjecture Let T be the transformation that sends an even integer x to $x/2$ and an odd integer x to $3x+1$. A famous conjecture, sometimes known as the **$\mathbf{3x+1}$ conjecture**, states that for all positive integers x, when we repeatedly apply the transformation T, we will eventually reach the integer 1. For example, starting with $x = 13$, we find $T(13) = 3 \cdot 13 + 1 = 40$, $T(40) = 40/2 = 20$, $T(20) = 20/2 = 10$, $T(10) = 10/2 = 5$, $T(5) = 3 \cdot 5 + 1 = 16$, $T(16) = 8$, $T(8) = 4$, $T(4) = 2$, and $T(2) = 1$. The $3x+1$ conjecture has been verified using computers for all integers x up to $5.6 \cdot 10^{13}$.

Watch out! Working on the $3x + 1$ problem can be addictive.

The $3x+1$ conjecture has an interesting history and has attracted the attention of mathematicians since the 1950s. The conjecture has been raised many times and goes by many other names, including the Collatz problem, Hasse's algorithm, Ulam's problem, the Syracuse problem, and Kakutani's problem. Many mathematicians have been diverted from their work to spend time attacking this conjecture. This led to the joke that this problem was part of a conspiracy to slow down American mathematical research. See the article by Jeffrey Lagarias [La10] for a fascinating discussion of this problem and the results that have been found by mathematicians attacking it. ◀

In Chapter 4 we will describe additional open questions about prime numbers. Students already familiar with the basic notions about primes might want to explore Section 4.3, where these open questions are discussed. We will mention other important open questions throughout the book.

Additional Proof Methods

Build up your arsenal of proof methods as you work through this book.

In this chapter we introduced the basic methods used in proofs. We also described how to leverage these methods to prove a variety of results. We will use these proof methods in all subsequent chapters. In particular, we will use them in Chapters 2, 3, and 4 to prove results about sets, functions, algorithms, and number theory and in Chapters 9, 10, and 11 to prove results in graph theory. Among the theorems we will prove is the famous halting theorem which states that there is a problem that cannot be solved using any procedure. However, there are many important proof methods besides those we have covered. We will introduce some of these methods later in this book. In particular, in Section 5.1 we will discuss mathematical induction, which is an extremely useful method for proving statements of the form $\forall n P(n)$, where the domain consists of all positive integers. In Section 5.3 we will introduce structural induction, which can be used to prove results about recursively defined sets. We will use the Cantor diagonalization method, which can be used to prove results about the size of infinite sets, in Section 2.5. In Chapter 6 we will introduce the notion of combinatorial proofs, which can be used to prove results by counting arguments. The reader should note that entire books have been devoted to the activities discussed in this section, including many excellent works by George Pólya ([Po61], [Po71], [Po90]).

Finally, note that we have not given a procedure that can be used for proving theorems in mathematics. It is a deep theorem of mathematical logic that there is no such procedure.

Exercises

1. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

2. Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

3. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. [*Hint:* Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.]

4. Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.

5. Prove using the notion of without loss of generality that $\min(x, y) = (x + y - |x - y|)/2$ and $\max(x, y) = (x + y + |x - y|)/2$ whenever x and y are real numbers.

6. Prove using the notion of without loss of generality that $5x + 5y$ is an odd integer when x and y are integers of opposite parity.

7. Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

8. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?

9. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?

10. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?

11. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

12. Show that the product of two of the numbers $65^{1000} - 8^{2001} + 3^{177}$, $79^{1212} - 9^{2399} + 2^{2001}$, and $24^{4493} - 5^{8192} + 7^{1777}$ is nonnegative. Is your proof constructive or nonconstructive? [*Hint:* Do not try to evaluate these numbers!]

13. Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

14. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

15. Show that each of these statements can be used to express the fact that there is a unique element x such that $P(x)$ is true. [Note that we can also write this statement as $\exists! x P(x)$.]

a) $\exists x \forall y (P(y) \leftrightarrow x = y)$

b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$

c) $\exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y))$

16. Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

17. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a - c| = |b - c|$.

18. Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is less than $1/2$.

19. Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

20. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n + \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

21. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n - \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

22. Use forward reasoning to show that if x is a nonzero real number, then $x^2 + 1/x^2 \geq 2$. [*Hint:* Start with the inequality $(x - 1/x)^2 \geq 0$ which holds for all nonzero real numbers x.]

23. The **harmonic mean** of two real numbers x and y equals $2xy/(x + y)$. By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

24. The **quadratic mean** of two real numbers x and y equals $\sqrt{(x^2 + y^2)/2}$. By computing the arithmetic and quadratic means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

***25.** Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

***26.** Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [*Hint:* Work backward, assuming that you did end up with nine zeros.]

27. Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.

28. Formulate a conjecture about the final two decimal digits of the square of an integer. Prove your conjecture using a proof by cases.

29. Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

30. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

31. Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.

32. Prove that there are infinitely many solutions in positive integers x, y, and z to the equation $x^2 + y^2 = z^2$. [*Hint:* Let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$, where m and n are integers.]

33. Adapt the proof in Example 4 in Section 1.7 to prove that if $n = abc$, where a, b, and c are positive integers, then $a \leq \sqrt[3]{n}$, $b \leq \sqrt[3]{n}$, or $c \leq \sqrt[3]{n}$.

34. Prove that $\sqrt[3]{2}$ is irrational.

35. Prove that between every two rational numbers there is an irrational number.

36. Prove that between every rational number and every irrational number there is an irrational number.

***37.** Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

- **a)** Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).
- **b)** Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

38. Prove or disprove that if you have an 8-gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, then you can measure 4 gallons by successively pouring some of or all of the water in a jug into another jug.

39. Verify the $3x + 1$ conjecture for these integers.

a) 6 **b)** 7 **c)** 17 **d)** 21

40. Verify the $3x + 1$ conjecture for these integers.

a) 16 **b)** 11 **c)** 35 **d)** 113

41. Prove or disprove that you can use dominoes to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite).

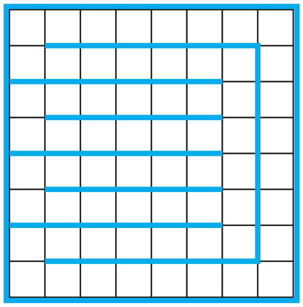
42. Prove or disprove that you can use dominoes to tile a standard checkerboard with all four corners removed.

43. Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares.

44. Prove or disprove that you can use dominoes to tile a 5×5 checkerboard with three corners removed.

45. Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist. [*Hint:* First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers squares 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

***46.** Prove that when a white square and a black square are removed from an 8×8 checkerboard (colored as in the text) you can tile the remaining squares of the checkerboard using dominoes. [*Hint:* Show that when one black and one white square are removed, each part of the partition of the remaining cells formed by inserting the barriers shown in the figure can be covered by dominoes.]



47. Show that by removing two white squares and two black squares from an 8×8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

***48.** Find all squares, if they exist, on an 8×8 checkerboard such that the board obtained by removing one of these square can be tiled using straight triominoes. [*Hint:* First use arguments based on coloring and rotations to eliminate as many squares as possible from consideration.]

***49.** **a)** Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.

b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard using these tetrominoes.

***50.** Prove or disprove that you can tile a 10×10 checkerboard using straight tetrominoes.

Key Terms and Results

TERMS

proposition: a statement that is true or false

propositional variable: a variable that represents a proposition

truth value: true or false

¬ *p* (negation of *p*): the proposition with truth value opposite to the truth value of p

logical operators: operators used to combine propositions

compound proposition: a proposition constructed by combining propositions using logical operators

truth table: a table displaying all possible truth values of propositions

$p \vee q$ (disjunction of *p* and *q*): the proposition "p or q," which is true if and only if at least one of p and q is true

$p \wedge q$ (conjunction of p and q): the proposition "p and q," which is true if and only if both p and q are true

$p \oplus q$ (exclusive or of p and q): the proposition "p *XOR* q," which is true when exactly one of p and q is true

$p \rightarrow q$ (p implies q): the proposition "if p, then q," which is false if and only if p is true and q is false

converse of $p \rightarrow q$: the conditional statement $q \rightarrow p$

contrapositive of $p \rightarrow q$: the conditional statement $\neg q \rightarrow \neg p$

inverse of $p \rightarrow q$: the conditional statement $\neg p \rightarrow \neg q$

$p \leftrightarrow q$ (biconditional): the proposition "p if and only if q," which is true if and only if p and q have the same truth value

bit: either a 0 or a 1

Boolean variable: a variable that has a value of 0 or 1

bit operation: an operation on a bit or bits

bit string: a list of bits

bitwise operations: operations on bit strings that operate on each bit in one string and the corresponding bit in the other string

logic gate: a logic element that performs a logical operation on one or more bits to produce an output bit

logic circuit: a switching circuit made up of logic gates that produces one or more output bits

tautology: a compound proposition that is always true

contradiction: a compound proposition that is always false

contingency: a compound proposition that is sometimes true and sometimes false

consistent compound propositions: compound propositions for which there is an assignment of truth values to the variables that makes all these propositions true

satisfiable compound proposition: a compound proposition for which there is an assignment of truth values to its variables that makes it true

logically equivalent compound propositions: compound propositions that always have the same truth values

predicate: part of a sentence that attributes a property to the subject

propositional function: a statement containing one or more variables that becomes a proposition when each of its variables is assigned a value or is bound by a quantifier

domain (or universe) of discourse: the values a variable in a propositional function may take

$\exists x\, P(x)$ (existential quantification of $P(x)$): the proposition that is true if and only if there exists an x in the domain such that $P(x)$ is true

$\forall x P(x)$ (universal quantification of $P(x)$): the proposition that is true if and only if $P(x)$ is true for every x in the domain

logically equivalent expressions: expressions that have the same truth value no matter which propositional functions and domains are used

free variable: a variable not bound in a propositional function

bound variable: a variable that is quantified

scope of a quantifier: portion of a statement where the quantifier binds its variable

argument: a sequence of statements

argument form: a sequence of compound propositions involving propositional variables

premise: a statement, in an argument, or argument form, other than the final one

conclusion: the final statement in an argument or argument form

valid argument form: a sequence of compound propositions involving propositional variables where the truth of all the premises implies the truth of the conclusion

valid argument: an argument with a valid argument form

rule of inference: a valid argument form that can be used in the demonstration that arguments are valid

fallacy: an invalid argument form often used incorrectly as a rule of inference (or sometimes, more generally, an incorrect argument)

circular reasoning or begging the question: reasoning where one or more steps are based on the truth of the statement being proved

theorem: a mathematical assertion that can be shown to be true

conjecture: a mathematical assertion proposed to be true, but that has not been proved

proof: a demonstration that a theorem is true

axiom: a statement that is assumed to be true and that can be used as a basis for proving theorems

lemma: a theorem used to prove other theorems

corollary: a proposition that can be proved as a consequence of a theorem that has just been proved

vacuous proof: a proof that $p \rightarrow q$ is true based on the fact that p is false

trivial proof: a proof that $p \rightarrow q$ is true based on the fact that q is true

direct proof: a proof that $p \rightarrow q$ is true that proceeds by showing that q must be true when p is true

proof by contraposition: a proof that $p \rightarrow q$ is true that proceeds by showing that p must be false when q is false

proof by contradiction: a proof that p is true based on the truth of the conditional statement $\neg p \rightarrow q$, where q is a contradiction

exhaustive proof: a proof that establishes a result by checking a list of all possible cases

proof by cases: a proof broken into separate cases, where these cases cover all possibilities

without loss of generality: an assumption in a proof that makes it possible to prove a theorem by reducing the number of cases to consider in the proof

counterexample: an element x such that $P(x)$ is false

constructive existence proof: a proof that an element with a specified property exists that explicitly finds such an element

nonconstructive existence proof: a proof that an element with a specified property exists that does not explicitly find such an element

rational number: a number that can be expressed as the ratio of two integers p and q such that $q \neq 0$

uniqueness proof: a proof that there is exactly one element satisfying a specified property

RESULTS

The logical equivalences given in Tables 6, 7, and 8 in Section 1.3.

De Morgan's laws for quantifiers.

Rules of inference for propositional calculus.

Rules of inference for quantified statements.

Review Questions

1. a) Define the negation of a proposition.

b) What is the negation of "This is a boring course"?

2. a) Define (using truth tables) the disjunction, conjunction, exclusive or, conditional, and biconditional of the propositions p and q.

b) What are the disjunction, conjunction, exclusive or, conditional, and biconditional of the propositions "I'll go to the movies tonight" and "I'll finish my discrete mathematics homework"?

3. a) Describe at least five different ways to write the conditional statement $p \to q$ in English.

b) Define the converse and contrapositive of a conditional statement.

c) State the converse and the contrapositive of the conditional statement "If it is sunny tomorrow, then I will go for a walk in the woods."

4. a) What does it mean for two propositions to be logically equivalent?

b) Describe the different ways to show that two compound propositions are logically equivalent.

c) Show in at least two different ways that the compound propositions $\neg p \vee (r \to \neg q)$ and $\neg p \vee \neg q \vee \neg r$ are equivalent.

5. *(Depends on the Exercise Set in Section 1.3)*

a) Given a truth table, explain how to use disjunctive normal form to construct a compound proposition with this truth table.

b) Explain why part (a) shows that the operators $\wedge$, $\vee$, and $\neg$ are functionally complete.

c) Is there an operator such that the set containing just this operator is functionally complete?

6. What are the universal and existential quantifications of a predicate $P(x)$? What are their negations?

7. a) What is the difference between the quantification $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$, where $P(x, y)$ is a predicate?

b) Give an example of a predicate $P(x, y)$ such that $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ have different truth values.

8. Describe what is meant by a valid argument in propositional logic and show that the argument "If the earth is flat, then you can sail off the edge of the earth," "You cannot sail off the edge of the earth," therefore, "The earth is not flat" is a valid argument.

9. Use rules of inference to show that if the premises "All zebras have stripes" and "Mark is a zebra" are true, then the conclusion "Mark has stripes" is true.

10. a) Describe what is meant by a direct proof, a proof by contraposition, and a proof by contradiction of a conditional statement $p \to q$.

b) Give a direct proof, a proof by contraposition and a proof by contradiction of the statement: "If n is even, then $n + 4$ is even."

11. a) Describe a way to prove the biconditional $p \leftrightarrow q$.

b) Prove the statement: "The integer $3n + 2$ is odd if and only if the integer $9n + 5$ is even, where n is an integer."

12. To prove that the statements p_1, p_2, p_3, and p_4 are equivalent, is it sufficient to show that the conditional statements $p_4 \to p_2$, $p_3 \to p_1$, and $p_1 \to p_2$ are valid? If not, provide another collection of conditional statements that can be used to show that the four statements are equivalent.

13. a) Suppose that a statement of the form $\forall x P(x)$ is false. How can this be proved?

b) Show that the statement "For every positive integer n, $n^2 \geq 2n$" is false.

14. What is the difference between a constructive and nonconstructive existence proof? Give an example of each.

15. What are the elements of a proof that there is a unique element x such that $P(x)$, where $P(x)$ is a propositional function?

16. Explain how a proof by cases can be used to prove a result about absolute values, such as the fact that $|xy| = |x||y|$ for all real numbers x and y.

Supplementary Exercises

1. Let p be the proposition "I will do every exercise in this book" and q be the proposition "I will get an "A" in this course." Express each of these as a combination of p and q.

a) I will get an "A" in this course only if I do every exercise in this book.

b) I will get an "A" in this course and I will do every exercise in this book.

c) Either I will not get an "A" in this course or I will not do every exercise in this book.

d) For me to get an "A" in this course it is necessary and sufficient that I do every exercise in this book.

2. Find the truth table of the compound proposition $(p \vee q) \rightarrow (p \wedge \neg r)$.

3. Show that these compound propositions are tautologies.

a) $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

b) $((p \vee q) \wedge \neg p) \rightarrow q$

4. Give the converse, the contrapositive, and the inverse of these conditional statements.

a) If it rains today, then I will drive to work.

b) If $|x| = x$, then $x \geq 0$.

c) If n is greater than 3, then n^2 is greater than 9.

5. Given a conditional statement $p \rightarrow q$, find the converse of its inverse, the converse of its converse, and the converse of its contrapositive.

6. Given a conditional statement $p \rightarrow q$, find the inverse of its inverse, the inverse of its converse, and the inverse of its contrapositive.

7. Find a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these propositional variables are true and is false otherwise.

8. Show that these statements are inconsistent: "If Sergei takes the job offer then he will get a signing bonus." "If Sergei takes the job offer, then he will receive a higher salary." "If Sergei gets a signing bonus, then he will not receive a higher salary." "Sergei takes the job offer."

9. Show that these statements are inconsistent: "If Miranda does not take a course in discrete mathematics, then she will not graduate." "If Miranda does not graduate, then she is not qualified for the job." "If Miranda reads this book, then she is qualified for the job." "Miranda does not take a course in discrete mathematics but she reads this book."

Teachers in the Middle Ages supposedly tested the realtime propositional logic ability of a student via a technique known as an **obligato game**. In an obligato game, a number of rounds is set and in each round the teacher gives the student successive assertions that the student must either accept or reject as they are given. When the student accepts an assertion, it is added as a commitment; when the student rejects an assertion its negation is added as a commitment. The student passes the test if the consistency of all commitments is maintained throughout the test.

10. Suppose that in a three-round obligato game, the teacher first gives the student the proposition $p \rightarrow q$, then the proposition $\neg(p \vee r) \vee q$, and finally the proposition q. For which of the eight possible sequences of three answers will the student pass the test?

11. Suppose that in a four-round obligato game, the teacher first gives the student the proposition $\neg(p \rightarrow (q \wedge r))$, then the proposition $p \vee \neg q$, then the proposition $\neg r$, and finally, the proposition $(p \wedge r) \vee (q \rightarrow p)$. For which of the 16 possible sequences of four answers will the student pass the test?

12. Explain why every obligato game has a winning strategy.

Exercises 13 and 14 are set on the island of knights and knaves described in Example 7 in Section 1.2.

13. Suppose that you meet three people Aaron, Bohan, and Crystal. Can you determine what Aaron, Bohan, and Crystal are if Aaron says "All of us are knaves" and Bohan says "Exactly one of us is a knave."?

14. Suppose that you meet three people, Anita, Boris, and Carmen. What are Anita, Boris, and Carmen if Anita says "I am a knave and Boris is a knight" and Boris says "Exactly one of the three of us is a knight"?

15. (Adapted from [Sm78]) Suppose that on an island there are three types of people, knights, knaves, and normals (also known as spies). Knights always tell the truth, knaves always lie, and normals sometimes lie and sometimes tell the truth. Detectives questioned three inhabitants of the island—Amy, Brenda, and Claire—as part of the investigation of a crime. The detectives knew that one of the three committed the crime, but not which one. They also knew that the criminal was a knight, and that the other two were not. Additionally, the detectives recorded these statements: Amy: "I am innocent." Brenda: "What Amy says is true." Claire: "Brenda is not a normal." After analyzing their information, the detectives positively identified the guilty party. Who was it?

16. Show that if S is a proposition, where S is the conditional statement "If S is true, then unicorns live," then "Unicorns live" is true. Show that it follows that S cannot be a proposition. (This paradox is known as *Löb's paradox*.)

17. Show that the argument with premises "The tooth fairy is a real person" and "The tooth fairy is not a real person" and conclusion "You can find gold at the end of the rainbow" is a valid argument. Does this show that the conclusion is true?

18. Suppose that the truth value of the proposition p_i is **T** whenever i is an odd positive integer and is **F** whenever i is an even positive integer. Find the truth values of $\bigvee_{i=1}^{100}(p_i \wedge p_{i+1})$ and $\bigwedge_{i=1}^{100}(p_i \vee p_{i+1})$.

***19.** Model 16×16 Sudoku puzzles (with 4×4 blocks) as satisfiability problems.

20. Let $P(x)$ be the statement "Student x knows calculus" and let $Q(y)$ be the statement "Class y contains a student who knows calculus." Express each of these as quantifications of $P(x)$ and $Q(y)$.

a) Some students know calculus.

b) Not every student knows calculus.

c) Every class has a student in it who knows calculus.

d) Every student in every class knows calculus.

e) There is at least one class with no students who know calculus.

21. Let $P(m, n)$ be the statement "m divides n," where the domain for both variables consists of all positive integers. (By "m divides n" we mean that $n = km$ for some integer k.) Determine the truth values of each of these statements.

a) $P(4, 5)$ **b)** $P(2, 4)$

c) $\forall m \, \forall n \, P(m, n)$ **d)** $\exists m \, \forall n \, P(m, n)$

e) $\exists n \, \forall m \, P(m, n)$ **f)** $\forall n \, P(1, n)$

22. Find a domain for the quantifiers in $\exists x \exists y (x \neq y \wedge \forall z((z = x) \vee (z = y)))$ such that this statement is true.

23. Find a domain for the quantifiers in $\exists x \exists y(x \neq y \wedge \forall z((z = x) \vee (z = y)))$ such that this statement is false.

24. Use existential and universal quantifiers to express the statement "No one has more than three grandmothers" using the propositional function $G(x, y)$, which represents "x is the grandmother of y."

25. Use existential and universal quantifiers to express the statement "Everyone has exactly two biological parents" using the propositional function $P(x, y)$, which represents "x is the biological parent of y."

26. The quantifier $\exists_n$ denotes "there exists exactly n," so that $\exists_n x P(x)$ means there exist exactly n values in the domain such that $P(x)$ is true. Determine the true value of these statements where the domain consists of all real numbers.

a) $\exists_0 x(x^2 = -1)$
b) $\exists_1 x(|x| = 0)$
c) $\exists_2 x(x^2 = 2)$
d) $\exists_3 x(x = |x|)$

27. Express each of these statements using existential and universal quantifiers and propositional logic where $\exists_n$ is defined in Exercise 26.

a) $\exists_0 x P(x)$
b) $\exists_1 x P(x)$
c) $\exists_2 x P(x)$
d) $\exists_3 x P(x)$

28. Let $P(x, y)$ be a propositional function. Show that $\exists x \forall y P(x, y) \rightarrow \forall y \exists x P(x, y)$ is a tautology.

29. Let $P(x)$ and $Q(x)$ be propositional functions. Show that $\exists x (P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \exists x Q(x)$ always have the same truth value.

30. If $\forall y \exists x P(x, y)$ is true, does it necessarily follow that $\exists x \forall y P(x, y)$ is true?

31. If $\forall x \exists y P(x, y)$ is true, does it necessarily follow that $\exists x \forall y P(x, y)$ is true?

32. Find the negations of these statements.

a) If it snows today, then I will go skiing tomorrow.

b) Every person in this class understands mathematical induction.

c) Some students in this class do not like discrete mathematics.

d) In every mathematics class there is some student who falls asleep during lectures.

33. Express this statement using quantifiers: "Every student in this class has taken some course in every department in the school of mathematical sciences."

34. Express this statement using quantifiers: "There is a building on the campus of some college in the United States in which every room is painted white."

35. Express the statement "There is exactly one student in this class who has taken exactly one mathematics class at this school" using the uniqueness quantifier. Then express this statement using quantifiers, without using the uniqueness quantifier.

36. Describe a rule of inference that can be used to prove that there are exactly two elements x and y in a domain such that $P(x)$ and $P(y)$ are true. Express this rule of inference as a statement in English.

37. Use rules of inference to show that if the premises $\forall x(P(x) \rightarrow Q(x))$, $\forall x(Q(x) \rightarrow R(x))$, and $\neg R(a)$, where a is in the domain, are true, then the conclusion $\neg P(a)$ is true.

38. Prove that if x^3 is irrational, then x is irrational.

39. Prove that if x is irrational and $x \geq 0$, then $\sqrt{x}$ is irrational.

40. Prove that given a nonnegative integer n, there is a unique nonnegative integer m such that $m^2 \leq n < (m + 1)^2$.

41. Prove that there exists an integer m such that $m^2 > 10^{1000}$. Is your proof constructive or nonconstructive?

42. Prove that there is a positive integer that can be written as the sum of squares of positive integers in two different ways. (Use a computer or calculator to speed up your work.)

43. Disprove the statement that every positive integer is the sum of the cubes of eight nonnegative integers.

44. Disprove the statement that every positive integer is the sum of at most two squares and a cube of nonnegative integers.

45. Disprove the statement that every positive integer is the sum of 36 fifth powers of nonnegative integers.

46. Assuming the truth of the theorem that states that $\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square, prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Computer Projects

Write programs with the specified input and output.

1. Given the truth values of the propositions p and q, find the truth values of the conjunction, disjunction, exclusive or, conditional statement, and biconditional of these propositions.

2. Given two bit strings of length n, find the bitwise *AND*, bitwise *OR*, and bitwise *XOR* of these strings.

***3.** Give a compound proposition, determine whether it is satisfiable by checking its truth value for all positive assignments of truth values to its propositional variables.

4. Given the truth values of the propositions p and q in fuzzy logic, find the truth value of the disjunction and the conjunction of p and q (see Exercises 46 and 47 of Section 1.1).

***5.** Given positive integers m and n, interactively play the game of Chomp.

***6.** Given a portion of a checkerboard, look for tilings of this checkerboard with various types of polyominoes, including dominoes, the two types of triominoes, and larger polyominoes.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Look for positive integers that are not the sum of the cubes of nine different positive integers.

2. Look for positive integers greater than 79 that are not the sum of the fourth powers of 18 positive integers.

3. Find as many positive integers as you can that can be written as the sum of cubes of positive integers, in two different ways, sharing this property with 1729.

***4.** Try to find winning strategies for the game of Chomp for different initial configurations of cookies.

5. Construct the 12 different pentominoes, where a pentomino is a polyomino consisting of five squares.

6. Find all the rectangles of 60 squares that can be tiled using every one of the 12 different pentominoes.

Writing Projects

Respond to these with essays using outside sources.

1. Discuss logical paradoxes, including the paradox of Epimenides the Cretan, Jourdain's card paradox, and the barber paradox, and how they are resolved.

2. Describe how fuzzy logic is being applied to practical applications. Consult one or more of the recent books on fuzzy logic written for general audiences.

3. Describe some of the practical problems that can be modeled as satisfiability problems.

4. Describe some of the techniques that have been devised to help people solve Sudoku puzzles without the use of a computer.

5. Describe the basic rules of *WFF'N PROOF, The Game of Modern Logic*, developed by Layman Allen. Give examples of some of the games included in *WFF'N PROOF*.

6. Read some of the writings of Lewis Carroll on symbolic logic. Describe in detail some of the models he used to represent logical arguments and the rules of inference he used in these arguments.

7. Extend the discussion of Prolog given in Section 1.4, explaining in more depth how Prolog employs resolution.

8. Discuss some of the techniques used in computational logic, including Skolem's rule.

9. "Automated theorem proving" is the task of using computers to mechanically prove theorems. Discuss the goals and applications of automated theorem proving and the progress made in developing automated theorem provers.

10. Describe how DNA computing has been used to solve instances of the satisfiability problem.

11. Look up some of the incorrect proofs of famous open questions and open questions that were solved since 1970 and describe the type of error made in each proof.

12. Discuss what is known about winning strategies in the game of Chomp.

13. Describe various aspects of proof strategy discussed by George Pólya in his writings on reasoning, including [Po62], [Po71], and [Po90].

14. Describe a few problems and results about tilings with polyominoes, as described in [Go94] and [Ma91], for example.

CHAPTER

2 Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines. These are some of the topics we will study in later chapters.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct. Functions play important roles throughout discrete mathematics. They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways. Useful structures such as sequences and strings are special types of functions. In this chapter, we will introduce the notion of a sequence, which represents ordered lists of elements. Furthermore, we will introduce some important types of sequences and we will show how to define the terms of a sequence using earlier terms. We will also address the problem of identifying a sequence from its first few terms.

In our study of discrete mathematics, we will often add consecutive terms of a sequence of numbers. Because adding terms from a sequence, as well as other indexed sets of numbers, is such a common occurrence, a special notation has been developed for adding such terms. In this chapter, we will introduce the notation used to express summations. We will develop formulae for certain types of summations that appear throughout the study of discrete mathematics. For instance, we will encounter such summations in the analysis of the number of steps used by an algorithm to sort a list of numbers so that its terms are in increasing order.

The relative sizes of infinite sets can be studied by introducing the notion of the size, or cardinality, of a set. We say that a set is countable when it is finite or has the same size as the set of positive integers. In this chapter we will establish the surprising result that the set of rational numbers is countable, while the set of real numbers is not. We will also show how the concepts we discuss can be used to show that there are functions that cannot be computed using a computer program in any programming language.

Matrices are used in discrete mathematics to represent a variety of discrete structures. We will review the basic material about matrices and matrix arithmetic needed to represent relations and graphs. The matrix arithmetic we study will be used to solve a variety of problems involving these structures.

2.1 Sets

Introduction

In this section, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set. Sets are used to group objects together. Often, but not always, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion. We now provide a definition of a set. This definition is an intuitive definition, which is not part of a formal theory of sets.

DEFINITION 1 A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$. ◀

EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$. ◀

EXAMPLE 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a$, 2, Fred, New Jersey$\}$ is the set containing the four elements a, 2, Fred, and New Jersey. ◀

Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (...) are used when the general pattern of the elements is obvious.

EXAMPLE 4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$. ◀

Extra Examples

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set $\mathbf{Q}^+$ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \tfrac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$$

Beware that mathematicians disagree whether 0 is a natural number. We consider it quite natural.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

(Note that some people do not consider 0 a natural number, so be careful to check how the term *natural numbers* is used when you read other books.)

Recall the notation for **intervals** of real numbers. When a and b are real numbers with $a < b$, we write

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}$$

Note that $[a, b]$ is called the **closed interval** from a to b and (a, b) is called the **open interval** from a to b.

Sets can have other sets as members, as Example 5 illustrates.

EXAMPLE 5 The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are $\mathbf{N}$, the set of natural numbers; $\mathbf{Z}$, the set of integers; $\mathbf{Q}$, the set of rational numbers; and $\mathbf{R}$, the set of real numbers. ◀

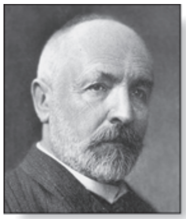
Remark: Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *boolean* is the name of the set $\{0, 1\}$ together with operators on one or more elements of this set, such as AND, OR, and NOT.

Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

DEFINITION 2 Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

EXAMPLE 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. ◀

Links



GEORG CANTOR (1845–1918) Georg Cantor was born in St. Petersburg, Russia, where his father was a successful merchant. Cantor developed his interest in mathematics in his teens. He began his university studies in Zurich in 1862, but when his father died he left Zurich. He continued his university studies at the University of Berlin in 1863, where he studied under the eminent mathematicians Weierstrass, Kummer, and Kronecker. He received his doctor's degree in 1867, after having written a dissertation on number theory. Cantor assumed a position at the University of Halle in 1869, where he continued working until his death.

Cantor is considered the founder of set theory. His contributions in this area include the discovery that the set of real numbers is uncountable. He is also noted for his many important contributions to analysis. Cantor also was interested in philosophy and wrote papers relating his theory of sets with metaphysics.

Cantor married in 1874 and had five children. His melancholy temperament was balanced by his wife's happy disposition. Although he received a large inheritance from his father, he was poorly paid as a professor. To mitigate this, he tried to obtain a better-paying position at the University of Berlin. His appointment there was blocked by Kronecker, who did not agree with Cantor's views on set theory. Cantor suffered from mental illness throughout the later years of his life. He died in 1918 from a heart attack.

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by ∅. The empty set can also be denoted by { } (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

{∅} has one more element than ∅.

A set with one element is called a **singleton set**. A common error is to confuse the empty set ∅ with the set {∅}, which is a singleton set. The single element of the set {∅} is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Links

NAIVE SET THEORY Note that the term *object* has been used in the definition of a set, Definition 1, without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated in 1895 by the German mathematician Georg Cantor. The theory that results from this intuitive definition of a set, and the use of the intuitive notion that for any property whatever, there is a set consisting of exactly the objects with this property, leads to **paradoxes**, or logical inconsistencies. This was shown by the English philosopher Bertrand Russell in 1902 (see Exercise 46 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory beginning with axioms. However, we will use Cantor's original version of set theory, known as **naive set theory**, in this book because all sets considered in this book can be treated consistently using Cantor's original theory. Students will find familiarity with naive set theory helpful if they go on to learn about axiomatic set theory. They will also find the development of axiomatic set theory much more abstract than the material in this text. We refer the interested reader to [Su72] to learn more about axiomatic set theory.

Venn Diagrams

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the **universal set** U, which contains all the objects under consideration, is represented by a rectangle. (Note that the universal set varies depending on which objects are of interest.) Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets. We show how a Venn diagram can be used in Example 7.

Assessment

EXAMPLE 7 Draw a Venn diagram that represents V, the set of vowels in the English alphabet.

Solution: We draw a rectangle to indicate the universal set U, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V. Inside this circle we indicate the elements of V with points (see Figure 1). ◀

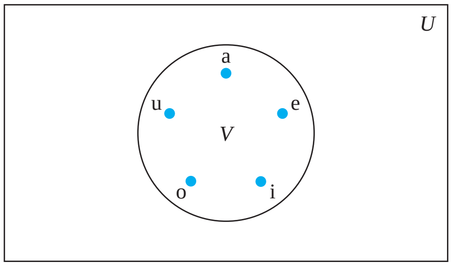


FIGURE 1 **Venn Diagram for the Set of Vowels.**

Subsets

It is common to encounter situations where the elements of one set are also the elements of a second set. We now introduce some terminology and notation to express such relationships between sets.

DEFINITION 3 The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \rightarrow x \in B)$$

is true. Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

We have these useful rules for determining whether one set is a subset of another:

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

EXAMPLE 8 The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself). Each of these facts follows immediately by noting that an element that belongs to the first set in each pair of sets also belongs to the second set in that pair. ◀

EXAMPLE 9 The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 100$], but not the later set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major. ◀

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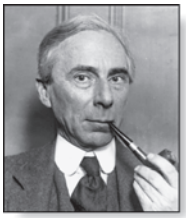

BERTRAND RUSSELL (1872–1970) Bertrand Russell was born into a prominent English family active in the progressive movement and having a strong commitment to liberty. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science. He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica*, written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for literature in 1950.

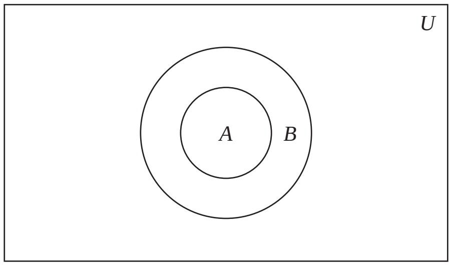

FIGURE 2 **Venn Diagram Showing that A Is a Subset of B.**

Theorem 1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

THEOREM 1 For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Proof: We will prove (i) and leave the proof of (ii) as an exercise.

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. This completes the proof of (i). Note that this is an example of a vacuous proof. ◁

When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true. Venn diagrams can be used to illustrate that a set A is a subset of a set B. We draw the universal set U as a rectangle. Within this rectangle we draw a circle for B. Because A is a subset of B, we draw the circle for A within the circle for B. This relationship is shown in Figure 2.

A useful way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. That is, $A = B$ if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$ or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$, which is what it means for the A and B to be equal. Because this method of showing two sets are equal is so useful, we highlight it here.

JOHN VENN (1834–1923) John Venn was born into a London suburban family noted for its philanthropy. He attended London schools and got his mathematics degree from Caius College, Cambridge, in 1857. He was elected a fellow of this college and held his fellowship there until his death. He took holy orders in 1859 and, after a brief stint of religious work, returned to Cambridge, where he developed programs in the moral sciences. Besides his mathematical work, Venn had an interest in history and wrote extensively about his college and family.

Venn's book *Symbolic Logic* clarifies ideas originally presented by Boole. In this book, Venn presents a systematic development of a method that uses geometric figures, known now as *Venn diagrams*. Today these diagrams are primarily used to analyze logical arguments and to illustrate relationships between sets. In addition to his work on symbolic logic, Venn made contributions to probability theory described in his widely used textbook on that subject.

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \qquad \text{and} \qquad B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}.$$

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

The Size of a Set

Sets are used extensively in counting problems, and for such applications we need to discuss the sizes of sets.

DEFINITION 4 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Remark: The term *cardinality* comes from the common usage of the term *cardinal number* as the size of a finite set.

EXAMPLE 10 Let A be the set of odd positive integers less than 10. Then $|A| = 5$. ◀

EXAMPLE 11 Let S be the set of letters in the English alphabet. Then $|S| = 26$. ◀

EXAMPLE 12 Because the null set has no elements, it follows that $|\emptyset| = 0$. ◀

We will also be interested in sets that are not finite.

DEFINITION 5 A set is said to be *infinite* if it is not finite.

EXAMPLE 13 The set of positive integers is infinite. ◀

Extra Examples

We will extend the notion of cardinality to infinite sets in Section 2.5, a challenging topic full of surprising results.

Power Sets

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set S, we build a new set that has as its members all the subsets of S.

DEFINITION 6 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

EXAMPLE 14 What is the power set of the set $\{0, 1, 2\}$?

Extra Examples

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets. ◀

EXAMPLE 15 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$ ◀

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

Cartesian Products

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

DEFINITION 7 The *ordered n-tuple* $(a_1, a_2, \dots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\dots$, and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \dots, n$. In particular, ordered 2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

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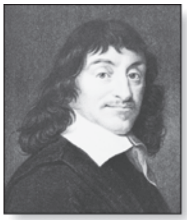


RENÉ DESCARTES (1596–1650) René Descartes was born into a noble family near Tours, France, about 200 miles southwest of Paris. He was the third child of his father's first wife; she died several days after his birth. Because of René's poor health, his father, a provincial judge, let his son's formal lessons slide until, at the age of 8, René entered the Jesuit college at La Flèche. The rector of the school took a liking to him and permitted him to stay in bed until late in the morning because of his frail health. From then on, Descartes spent his mornings in bed; he considered these times his most productive hours for thinking.

Descartes left school in 1612, moving to Paris, where he spent 2 years studying mathematics. He earned a law degree in 1616 from the University of Poitiers. At 18 Descartes became disgusted with studying and decided to see the world. He moved to Paris and became a successful gambler. However, he grew tired of bawdy living and moved to the suburb of Saint-Germain, where he devoted himself to mathematical study. When his gambling friends found him, he decided to leave France and undertake a military career. However, he never did any fighting. One day, while escaping the cold in an overheated room at a military encampment, he had several feverish dreams, which revealed his future career as a mathematician and philosopher.

After ending his military career, he traveled throughout Europe. He then spent several years in Paris, where he studied mathematics and philosophy and constructed optical instruments. Descartes decided to move to Holland, where he spent 20 years wandering around the country, accomplishing his most important work. During this time he wrote several books, including the *Discours,* which contains his contributions to analytic geometry, for which he is best known. He also made fundamental contributions to philosophy.

In 1649 Descartes was invited by Queen Christina to visit her court in Sweden to tutor her in philosophy. Although he was reluctant to live in what he called "the land of bears amongst rocks and ice," he finally accepted the invitation and moved to Sweden. Unfortunately, the winter of 1649–1650 was extremely bitter. Descartes caught pneumonia and died in mid-February.

Many of the discrete structures we will study in later chapters are based on the notion of the *Cartesian product* of sets (named after René Descartes). We first define the Cartesian product of two sets.

DEFINITION 8 Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE 16 Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Extra Examples

Solution: The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university. ◀

EXAMPLE 17 What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$ ◀

Note that the Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$ (see Exercises 31 and 38). This is illustrated in Example 18.

EXAMPLE 18 Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are as in Example 17.

Solution: The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 17. ◀

The Cartesian product of more than two sets can also be defined.

DEFINITION 9 The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 19 What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2),\\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.\end{aligned}$$

◀

Remark: Note that when A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$ (see Exercise 39).

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 20 Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. ◀

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$. A relation from a set A to itself is called a relation on A.

EXAMPLE 21 What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently, the ordered pairs in R are (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and (3, 3). ◀

We will study relations and their properties at length in Chapter 9.

Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \land P(x))$.

EXAMPLE 22 What do the statements $\forall x \in \mathbf{R}\ (x^2 \geq 0)$ and $\exists x \in \mathbf{Z}\ (x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the **truth set** of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

EXAMPLE 23 What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Solution: The truth set of P, $\{x \in \mathbf{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q, $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

The truth set of R, $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is $\mathbf{N}$, the set of nonnegative integers. ◀

Note that $\forall x P(x)$ is true over the domain U if and only if the truth set of P is the set U. Likewise, $\exists x P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Exercises

1. List the members of these sets.

a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$

b) $\{x \mid x \text{ is a positive integer less than } 12\}$

c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

2. Use set builder notation to give a description of each of these sets.

a) $\{0, 3, 6, 9, 12\}$

b) $\{-3, -2, -1, 0, 1, 2, 3\}$

c) $\{m, n, o, p\}$

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi

b) the set of people who speak English, the set of people who speak Chinese

c) the set of flying squirrels, the set of living creatures that can fly

4. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of people who speak English, the set of people who speak English with an Australian accent

b) the set of fruits, the set of citrus fruits

c) the set of students studying discrete mathematics, the set of students studying data structures

5. Determine whether each of these pairs of sets are equal.

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$

b) $\{\{1\}\}$, $\{1, \{1\}\}$ **c)** $\emptyset$, $\{\emptyset\}$

6. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.

7. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$

b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$

c) $\{2,\{2\}\}$ **d)** $\{\{2\},\{\{2\}\}\}$

e) $\{\{2\},\{2,\{2\}\}\}$ **f)** $\{\{\{2\}\}\}$

8. For each of the sets in Exercise 7, determine whether $\{2\}$ is an element of that set.

9. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$ **b)** $\emptyset \in \{0\}$

c) $\{0\} \subset \emptyset$ **d)** $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$ **f)** $\{0\} \subset \{0\}$

g) $\{\emptyset\} \subseteq \{\emptyset\}$

10. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$ **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$

c) $\{\emptyset\} \in \{\emptyset\}$ **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$

e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

11. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ **b)** $\{x\} \subseteq \{x\}$ **c)** $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$ **e)** $\emptyset \subseteq \{x\}$ **f)** $\emptyset \in \{x\}$

12. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.

13. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.

14. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

15. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

17. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

18. Find two sets A and B such that $A \in B$ and $A \subseteq B$.

19. What is the cardinality of each of these sets?

a) $\{a\}$ **b)** $\{\{a\}\}$

c) $\{a, \{a\}\}$ **d)** $\{a, \{a\}, \{a, \{a\}\}\}$

20. What is the cardinality of each of these sets?

a) $\emptyset$ **b)** $\{\emptyset\}$

c) $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

21. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

22. Can you conclude that $A = B$ if A and B are two sets with the same power set?

23. How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\emptyset))$

24. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$ **b)** $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ **d)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

25. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

26. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

27. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$. **b)** $B \times A$.

28. What is the Cartesian product $A \times B$, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

29. What is the Cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

30. Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

31. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

32. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$. **b)** $C \times B \times A$.

c) $C \times A \times B$. **d)** $B \times B \times B$.

33. Find A^2 if

a) $A = \{0, 1, 3\}$. **b)** $A = \{1, 2, a, b\}$.

34. Find A^3 if

a) $A = \{a\}$. **b)** $A = \{0, a\}$.

35. How many different elements does $A \times B$ have if A has m elements and B has n elements?

36. How many different elements does $A \times B \times C$ have if A has m elements, B has n elements, and C has p elements?

37. How many different elements does A^n have when A has m elements and n is a positive integer?

38. Show that $A \times B \neq B \times A$, when A and B are nonempty, unless $A = B$.

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

40. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

41. Translate each of these quantifications into English and determine its truth value.

a) $\forall x \in \mathbf{R}\ (x^2 \neq -1)$ **b)** $\exists x \in \mathbf{Z}\ (x^2 = 2)$

c) $\forall x \in \mathbf{Z}\ (x^2 > 0)$ **d)** $\exists x \in \mathbf{R}\ (x^2 = x)$

42. Translate each of these quantifications into English and determine its truth value.

a) $\exists x \in \mathbf{R}\ (x^3 = -1)$ **b)** $\exists x \in \mathbf{Z}\ (x + 1 > x)$

c) $\forall x \in \mathbf{Z}\ (x - 1 \in \mathbf{Z})$ **d)** $\forall x \in \mathbf{Z}\ (x^2 \in \mathbf{Z})$

43. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)\colon x^2 < 3$ **b)** $Q(x)\colon x^2 > x$

c) $R(x)\colon 2x + 1 = 0$

44. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)\colon x^3 \geq 1$ **b)** $Q(x)\colon x^2 = 2$

c) $R(x)\colon x < x^2$

***45.** The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [*Hint:* First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

***46.** This exercise presents **Russell's paradox**. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.

a) Show the assumption that S is a member of S leads to a contradiction.

b) Show the assumption that S is not a member of S leads to a contradiction.

By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

***47.** Describe a procedure for listing all the subsets of a finite set.

2.2 Set Operations

Introduction

Two, or more, sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

Links

DEFINITION 1 Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

The Venn diagram shown in Figure 1 represents the union of two sets A and B. The area that represents $A \cup B$ is the shaded area within either the circle representing A or the circle representing B.

We will give some examples of the union of sets.

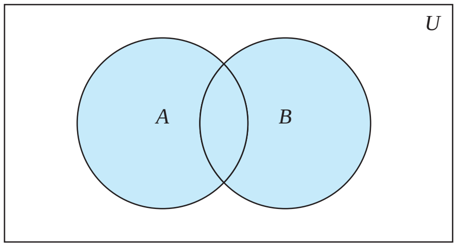
EXAMPLE 1 The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$. ◀

EXAMPLE 2 The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both). ◀

DEFINITION 2 Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

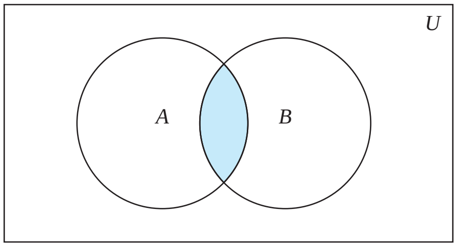
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram of the Union of A and B.



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram of the Intersection of A and B.

The Venn diagram shown in Figure 2 represents the intersection of two sets A and B. The shaded area that is within both the circles representing the sets A and B is the area that represents the intersection of A and B.

We give some examples of the intersection of sets.

EXAMPLE 3 The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

EXAMPLE 4 The intersection of the set of all computer science majors at your school and the set of all mathematics majors is the set of all students who are joint majors in mathematics and computer science. ◀

DEFINITION 3 Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE 5 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint. ◀

Be careful not to overcount!

We are often interested in finding the cardinality of a union of two finite sets A and B. Note that $|A| + |B|$ counts each element that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from $|A| + |B|$, elements in $A \cap B$ will be counted only once. Hence,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion–exclusion**. The principle of inclusion–exclusion is an important technique used in enumeration. We will discuss this principle and other counting techniques in detail in Chapters 6 and 8.

There are other important ways to combine sets.

DEFINITION 4 Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

Remark: The difference of sets A and B is sometimes denoted by $A \backslash B$.

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

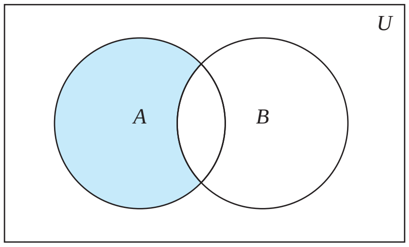
$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

The Venn diagram shown in Figure 3 represents the difference of the sets A and B. The shaded area inside the circle that represents A and outside the circle that represents B is the area that represents $A - B$.

We give some examples of differences of sets.

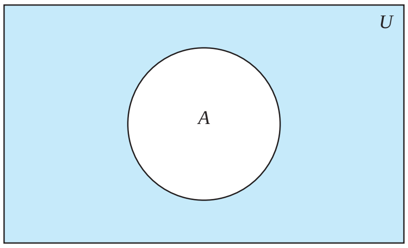
EXAMPLE 6 The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ◀

EXAMPLE 7 The difference of the set of computer science majors at your school and the set of mathematics majors at your school is the set of all computer science majors at your school who are not also mathematics majors. ◀



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B.



$\overline{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A.

Once the universal set U has been specified, the **complement** of a set can be defined.

DEFINITION 5 Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

An element belongs to $\overline{A}$ if and only if $x \notin A$. This tells us that

$$\overline{A} = \{x \in U \mid x \notin A\}.$$

In Figure 4 the shaded area outside the circle representing A is the area representing $\overline{A}$.

We give some examples of the complement of a set.

EXAMPLE 8 Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. ◀

EXAMPLE 9 Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ◀

It is left to the reader (Exercise 19) to show that we can express the difference of A and B as the intersection of A and the complement of B. That is,

$$A - B = A \cap \overline{B}.$$

Set Identities

Set identities and propositional equivalences are just special cases of identities for Boolean algebra.

Table 1 lists the most important set identities. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem. The proofs of the remaining identities will be left as exercises. The reader should note the similarity between these set identities and the logical equivalences discussed in Section 1.3. (Compare Table 6 of Section 1.6 and Table 1.) In fact, the set identities given can be proved directly from the corresponding logical equivalences. Furthermore, both are special cases of identities that hold for Boolean algebra (discussed in Chapter 12).

One way to show that two sets are equal is to show that each is a subset of the other. Recall that to show that one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set. We generally use a direct proof to do this. We illustrate this type of proof by establishing the first of De Morgan's laws.

TABLE 1 **Set Identities.**

Identity	***Name***
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

EXAMPLE 10 Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

This identity says that the complement of the intersection of two sets is the union of their complements.

Solution: We will prove that the two sets $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are equal by showing that each set is a subset of the other.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. We do this by showing that if x is in $\overline{A \cap B}$, then it must also be in $\overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$ is true.

By applying De Morgan's law for propositions, we see that $\neg(x \in A)$ or $\neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, we see that this implies that $x \in \overline{A}$ or $x \in \overline{B}$. Consequently, by the definition of union, we see that $x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Extra Examples

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. We do this by showing that if x is in $\overline{A} \cup \overline{B}$, then it must also be in $\overline{A \cap B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. By the definition of union, we know that $x \in \overline{A}$ or $x \in \overline{B}$. Using the definition of complement, we see that $x \notin A$ or $x \notin B$. Consequently, the proposition $\neg(x \in A) \vee \neg(x \in B)$ is true.

By De Morgan's law for propositions, we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$. We now use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved. ◀

We can more succinctly express the reasoning used in Example 10 using set builder notation, as Example 11 illustrates.

EXAMPLE 11 Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Note that besides the definitions of complement, union, set membership, and set builder notation, this proof uses the second De Morgan law for logical equivalences. ◀

Proving a set identity involving more than two sets by showing each side of the identity is a subset of the other often requires that we keep track of different cases, as illustrated by the proof in Example 12 of one of the distributive laws for sets.

EXAMPLE 12 Prove the second distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$. We conclude that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. ◀

Set identities can also be proved using **membership tables**. We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used. (The reader should note the similarity between membership tables and truth tables.)

EXAMPLE 13 Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid. ◀

Additional set identities can be established using those that we have already proved. Consider Example 14.

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

EXAMPLE 14 Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

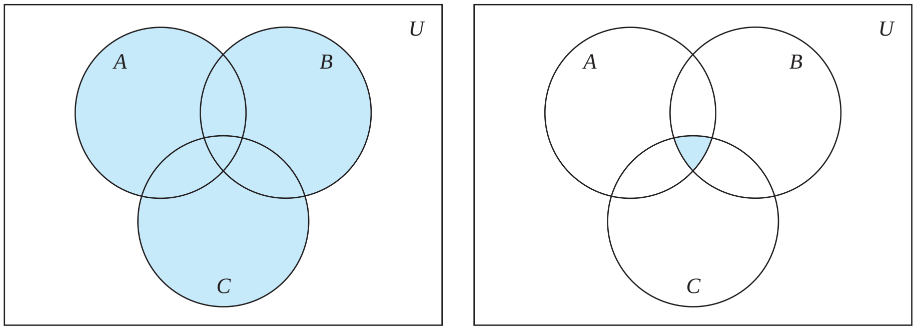
Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

◀

Generalized Unions and Intersections

Because unions and intersections of sets satisfy associative laws, the sets $A \cup B \cup C$ and $A \cap B \cap C$ are well defined; that is, the meaning of this notation is unambiguous when A, B, and C are sets. That is, we do not have to use parentheses to indicate which operation comes first because $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. Note that $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C, and that $A \cap B \cap C$ contains those elements that are in all of A, B, and C. These combinations of the three sets, A, B, and C, are shown in Figure 5.



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

FIGURE 5 The Union and Intersection of A, B, and C.

EXAMPLE 15 Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Solution: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,

$$A \cap B \cap C = \{0\}.$$ ◀

We can also consider unions and intersections of an arbitrary number of sets. We introduce these definitions.

DEFINITION 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

DEFINITION 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$. We illustrate generalized unions and intersections with Example 16.

EXAMPLE 16 For $i = 1, 2, \ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{n, n + 1, n + 2, \ldots\} = A_n.$$ ◀

We can extend the notation we have introduced for unions and intersections to other families of sets. In particular, we use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n, \ldots$. Similarly, the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I \, (x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I \, (x \in A_i)\}$.

EXAMPLE 17 Suppose that $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$

To see that the union of these sets is the set of positive integers, note that every positive integer n is in at least one of the sets, because it belongs to $A_n = \{1, 2, \ldots, n\}$, and every element of the sets in the union is a positive integer. To see that the intersection of these sets is the set $\{1\}$, note that the only element that belongs to all the sets $A_1, A_2, \ldots$ is 1. To see this note that $A_1 = \{1\}$ and $1 \in A_i$ for $i = 1, 2, \ldots$. ◀

Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A. Example 18 illustrates this technique.

EXAMPLE 18 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, {1, 3, 5, 7, 9}, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading.) Similarly, we represent the subset of all even integers in U, namely, {2, 4, 6, 8, 10}, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, {1, 2, 3, 4, 5}, is represented by the string

11 1110 0000. ◀

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin \overline{A}$. Note that this operation corresponds to taking the negation of each bit when we associate a bit with a truth value—with 1 representing true and 0 representing false.

EXAMPLE 19 We have seen that the bit string for the set {1, 3, 5, 7, 9} (with universal set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}) is

10 1010 1010.

What is the bit string for the complement of this set?

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set {2, 4, 6, 8, 10}. ◀

To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the ith position of the bit string of the union is 1 if either of the bits in the ith position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise *OR* of the bit strings for the two sets. The bit in the ith position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits is 0 (or both are). Hence, the bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.

EXAMPLE 20 The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set {1, 2, 3, 4, 5, 7, 9}. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set {1, 3, 5}. ◀

Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$ **b)** $A \cup B$

c) $A - B$ **d)** $B - A$

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school

b) the set of sophomores at your school who are not taking discrete mathematics

c) the set of students at your school who either are sophomores or are taking discrete mathematics

d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.

c) $A - B$. **d)** $B - A$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$. **b)** $A \cap B$.

c) $A - B$. **d)** $B - A$.

In Exercises 5–10 assume that A is a subset of some underlying universal set U.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that

a) $A \cup \emptyset = A$. **b)** $A \cap U = A$.

7. Prove the domination laws in Table 1 by showing that

a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

8. Prove the idempotent laws in Table 1 by showing that

a) $A \cup A = A$. **b)** $A \cap A = A$.

9. Prove the complement laws in Table 1 by showing that

a) $A \cup \overline{A} = U$. **b)** $A \cap \overline{A} = \emptyset$.

10. Show that

a) $A - \emptyset = A$. **b)** $\emptyset - A = \emptyset$.

11. Let A and B be sets. Prove the commutative laws from Table 1 by showing that

a) $A \cup B = B \cup A$.

b) $A \cap B = B \cap A$.

12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if A and B are sets, then $A \cap (A \cup B) = A$.

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.

b) using a membership table.

16. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$. **b)** $A \subseteq (A \cup B)$.

c) $A - B \subseteq A$. **d)** $A \cap (B - A) = \emptyset$.

e) $A \cup (B - A) = A \cup B$.

17. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.

b) using a membership table.

18. Let A, B, and C be sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$.

b) $(A \cap B \cap C) \subseteq (A \cap B)$.

c) $(A - B) - C \subseteq A - C$.

d) $(A - C) \cap (C - B) = \emptyset$.

e) $(B - A) \cup (C - A) = (B \cup C) - A$.

19. Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.

b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

20. Show that if A and B are sets with $A \subseteq B$, then

a) $A \cup B = B$.

b) $A \cap B = A$.

21. Prove the first associative law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

22. Prove the second associative law from Table 1 by showing that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

23. Prove the first distributive law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

25. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.

c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

26. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$

c) $(A - B) \cup (A - C) \cup (B - C)$

27. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

28. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

a) $(A \cap B) \cup (C \cap D)$ **b)** $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$

c) $A - (B \cap C \cap D)$

29. What can you say about the sets A and B if we know that

a) $A \cup B = A$? **b)** $A \cap B = A$?

c) $A - B = A$? **d)** $A \cap B = B \cap A$?

e) $A - B = B - A$?

30. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$? **b)** $A \cap C = B \cap C$?

c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

31. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

32. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

34. Draw a Venn diagram for the symmetric difference of the sets A and B.

35. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

36. Show that $A \oplus B = (A - B) \cup (B - A)$.

37. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. **b)** $A \oplus \emptyset = A$.

c) $A \oplus U = \overline{A}$. **d)** $A \oplus \overline{A} = U$.

38. Show that if A and B are sets, then

a) $A \oplus B = B \oplus A$. **b)** $(A \oplus B) \oplus B = A$.

39. What can you say about the sets A and B if $A \oplus B = A$?

***40.** Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

***41.** Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

42. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

43. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

44. Show that if A and B are finite sets, then $A \cup B$ is a finite set.

45. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

***46.** Show that if A, B, and C are sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| \\ - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

47. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

48. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

49. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

50. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{i, i + 1, i + 2, \ldots\}$.

b) $A_i = \{0, i\}$.

c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.

d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

51. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i + 1, \ldots, -1, 0, 1, \ldots, i - 1, i\}$.

b) $A_i = \{-i, i\}$.

c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \le x \le i$.

d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \ge i$.

52. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$

b) $\{1, 3, 6, 10\}$

c) $\{2, 3, 4, 7, 8, 9\}$

53. Using the same universal set as in the last problem, find the set specified by each of these bit strings.

a) 11 1100 1111

b) 01 0111 1000

c) 10 0000 0001

54. What subsets of a finite universal set do these bit strings represent?

a) the string with all zeros

b) the string with all ones

55. What is the bit string corresponding to the difference of two sets?

56. What is the bit string corresponding to the symmetric difference of two sets?

57. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ **b)** $A \cap B$

c) $(A \cup D) \cap (B \cup C)$ **d)** $A \cup B \cup C \cup D$

58. How can the union and intersection of n sets that all are subsets of the universal set U be found using bit strings?

The **successor** of the set A is the set $A \cup \{A\}$.

59. Find the successors of the following sets.

a) $\{1, 2, 3\}$ **b)** $\emptyset$

c) $\{\emptyset\}$ **d)** $\{\emptyset, \{\emptyset\}\}$

60. How many elements does the successor of a set with n elements have?

Sometimes the number of times that an element occurs in an unordered collection matters. **Multisets** are unordered collections of elements where an element can occur as a member more than once. The notation $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ denotes the multiset with element a_1 occurring m_1 times, element a_2 occurring m_2 times, and so on. The numbers m_i, $i = 1, 2, \ldots, r$ are called the **multiplicities** of the elements a_i, $i = 1, 2, \ldots, r$.

Let P and Q be multisets. The **union** of the multisets P and Q is the multiset where the multiplicity of an element is the maximum of its multiplicities in P and Q. The **intersection** of P and Q is the multiset where the multiplicity of an element is the minimum of its multiplicities in P and Q. The **difference** of P and Q is the multiset where the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0. The **sum** of P and Q is the multiset where the multiplicity of an element is the sum of multiplicities in P and Q. The union, intersection, and difference of P and Q are denoted by $P \cup Q$, $P \cap Q$, and $P - Q$, respectively (where these operations should not be confused with the analogous operations for sets). The sum of P and Q is denoted by $P + Q$.

61. Let A and B be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot d\}$, respectively. Find

a) $A \cup B$. **b)** $A \cap B$. **c)** $A - B$.
d) $B - A$. **e)** $A + B$.

62. Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset {107 · personal computers, 44 · routers, 6 · servers} and B could be the multiset {14 · personal computers, 6 · routers, 2 · mainframes}.

a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment?

b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment?

c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?

d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment?

Fuzzy sets are used in artificial intelligence. Each element in the universal set U has a **degree of membership**, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set S. The fuzzy set S is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write {0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita} for the set F (of famous people) to indicate that Alice has a 0.6 degree of membership in F, Brian has a 0.9 degree of membership in F, Fred has a 0.4 degree of membership in F, Oscar has a 0.1 degree of membership in F, and Rita has a 0.5 degree of membership in F (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that R is the set of rich people with R = {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita}.

63. The **complement** of a fuzzy set S is the set $\overline{S}$, with the degree of the membership of an element in $\overline{S}$ equal to 1 minus the degree of membership of this element in S. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

64. The **union** of two fuzzy sets S and T is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cup R$ of rich or famous people.

65. The **intersection** of two fuzzy sets S and T is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cap R$ of rich and famous people.

2.3 Functions

Introduction

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated in Figure 1.

This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science. For example, in discrete mathematics functions are used in the definition of such discrete structures as sequences and strings. Functions are also used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. Recursive functions,

CHAPTER

5 Induction and Recursion

Many mathematical statements assert that a property is true for all positive integers. Examples of such statements are that for every positive integer n: $n! \leq n^n$, $n^3 - n$ is divisible by 3; a set with n elements has 2^n subsets; and the sum of the first n positive integers is $n(n+1)/2$. A major goal of this chapter, and the book, is to give the student a thorough understanding of mathematical induction, which is used to prove results of this kind.

Proofs using mathematical induction have two parts. First, they show that the statement holds for the positive integer 1. Second, they show that if the statement holds for a positive integer then it must also hold for the next larger integer. Mathematical induction is based on the rule of inference that tells us that if $P(1)$ and $\forall k(P(k) \rightarrow P(k+1))$ are true for the domain of positive integers, then $\forall n P(n)$ is true. Mathematical induction can be used to prove a tremendous variety of results. Understanding how to read and construct proofs by mathematical induction is a key goal of learning discrete mathematics.

In Chapter 2 we explicitly defined sets and functions. That is, we described sets by listing their elements or by giving some property that characterizes these elements. We gave formulae for the values of functions. There is another important way to define such objects, based on mathematical induction. To define functions, some initial terms are specified, and a rule is given for finding subsequent values from values already known. (We briefly touched on this sort of definition in Chapter 2 when we showed how sequences can be defined using recurrence relations.) Sets can be defined by listing some of their elements and giving rules for constructing elements from those already known to be in the set. Such definitions, called *recursive definitions*, are used throughout discrete mathematics and computer science. Once we have defined a set recursively, we can use a proof method called structural induction to prove results about this set.

When a procedure is specified for solving a problem, this procedure must *always* solve the problem correctly. Just testing to see that the correct result is obtained for a set of input values does not show that the procedure always works correctly. The correctness of a procedure can be guaranteed only by proving that it always yields the correct result. The final section of this chapter contains an introduction to the techniques of program verification. This is a formal technique to verify that procedures are correct. Program verification serves as the basis for attempts under way to prove in a mechanical fashion that programs are correct.

5.1 Mathematical Induction

Introduction

Suppose that we have an infinite ladder, as shown in Figure 1, and we want to know whether we can reach every step on this ladder. We know two things:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we

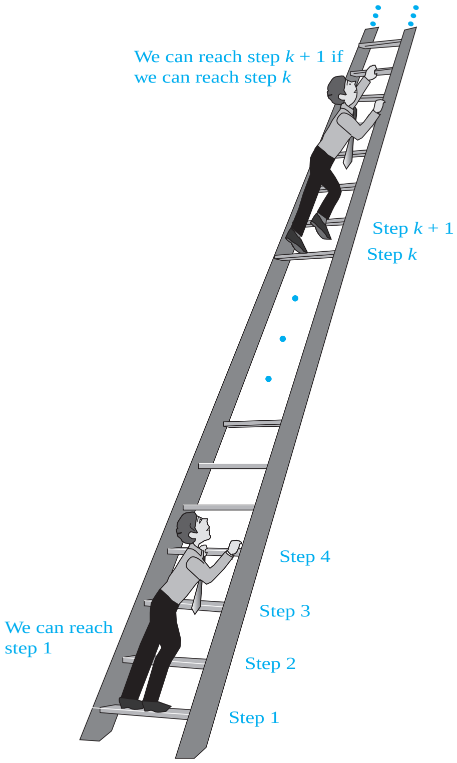


FIGURE 1 **Climbing an Infinite Ladder.**

can reach the fourth rung, the fifth rung, and so on. For example, after 100 uses of (2), we know that we can reach the 101st rung. But can we conclude that we are able to reach every rung of this infinite ladder? The answer is yes, something we can verify using an important proof technique called **mathematical induction**. That is, we can show that $P(n)$ is true for every positive integer n, where $P(n)$ is the statement that we can reach the nth rung of the ladder.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type. As we will see in this section and in subsequent sections of this chapter and later chapters, mathematical induction is used extensively to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In this section, we will describe how mathematical induction can be used and why it is a valid proof technique. It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way. It is *not* a tool for discovering formulae or theorems.

Mathematical Induction

Assessment

In general, mathematical induction * can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical

*Unfortunately, using the terminology "mathematical induction" clashes with the terminology used to describe different types of reasoning. In logic, **deductive reasoning** uses rules of inference to draw conclusions from premises, whereas **inductive reasoning** makes conclusions only supported, but not ensured, by evidence. Mathematical proofs, including arguments that use mathematical induction, are deductive, not inductive.

induction has two parts, a **basis step**, where we show that $P(1)$ is true, and an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**. Once we complete both steps in a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers, that is, we have shown that $\forall n P(n)$ is true where the quantification is over the set of positive integers. In the inductive step, we show that $\forall k(P(k) \to P(k+1))$ is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \land \forall k(P(k) \to P(k+1))) \to \forall n P(n),$$

when the domain is the set of positive integers. Because mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that $P(n)$ is true for all positive integers n is to show that $P(1)$ is true. This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true. Then we must show that $P(k) \to P(k+1)$ is true for every positive integer k. To prove that this conditional statement is true for every positive integer k, we need to show that $P(k+1)$ cannot be false when $P(k)$ is true. This can be accomplished by assuming that $P(k)$ is true and showing that *under this hypothesis* $P(k+1)$ must also be true.

Remark: In a proof by mathematical induction it is *not* assumed that $P(k)$ is true for all positive integers! It is only shown that *if it is assumed* that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

When we use mathematical induction to prove a theorem, we first show that $P(1)$ is true. Then we know that $P(2)$ is true, because $P(1)$ implies $P(2)$. Further, we know that $P(3)$ is true, because $P(2)$ implies $P(3)$. Continuing along these lines, we see that $P(n)$ is true for every positive integer n.

Links

HISTORICAL NOTE The first known use of mathematical induction is in the work of the sixteenth-century mathematician Francesco Maurolico (1494–1575). Maurolico wrote extensively on the works of classical mathematics and made many contributions to geometry and optics. In his book *Arithmeticorum Libri Duo*, Maurolico presented a variety of properties of the integers together with proofs of these properties. To prove some of these properties, he devised the method of mathematical induction. His first use of mathematical induction in this book was to prove that the sum of the first n odd positive integers equals n^2. Augustus De Morgan is credited with the first presentation in 1838 of formal proofs using mathematical induction, as well as introducing the terminology "mathematical induction." Maurolico's proofs were informal and he never used the word "induction." See [Gu11] to learn more about the history of the method of mathematical induction.

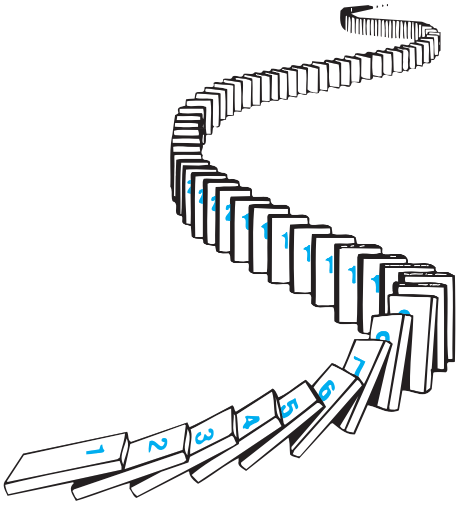


FIGURE 2 **Illustrating How Mathematical Induction Works Using Dominoes.**

WAYS TO REMEMBER HOW MATHEMATICAL INDUCTION WORKS Thinking of the infinite ladder and the rules for reaching steps can help you remember how mathematical induction works. Note that statements (1) and (2) for the infinite ladder are exactly the basis step and inductive step, respectively, of the proof that $P(n)$ is true for all positive integers n, where $P(n)$ is the statement that we can reach the nth rung of the ladder. Consequently, we can invoke mathematical induction to conclude that we can reach every rung.

Another way to illustrate the principle of mathematical induction is to consider an infinite row of dominoes, labeled $1, 2, 3, \ldots, n, \ldots$, where each domino is standing up. Let $P(n)$ be the proposition that domino n is knocked over. If the first domino is knocked over—i.e., if $P(1)$ is true—and if, whenever the kth domino is knocked over, it also knocks the $(k+1)$st domino over—i.e., if $P(k) \rightarrow P(k+1)$ is true for all positive integers k—then all the dominoes are knocked over. This is illustrated in Figure 2.

Why Mathematical Induction is Valid

Why is mathematical induction a valid proof technique? The reason comes from the well-ordering property, listed in Appendix 1, as an axiom for the set of positive integers, which states that every nonempty subset of the set of positive integers has a least element. So, suppose we know that $P(1)$ is true and that the proposition $P(k) \rightarrow P(k+1)$ is true for all positive integers k. To show that $P(n)$ must be true for all positive integers n, assume that there is at least one positive integer for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty. Thus, by the well-ordering property, S has a least element, which will be denoted by m. We know that m cannot be 1, because $P(1)$ is true. Because m is positive and greater than 1, $m - 1$ is a positive integer. Furthermore, because $m - 1$ is less than m, it is not in S, so $P(m-1)$ must be true. Because the conditional statement $P(m-1) \rightarrow P(m)$ is also true, it must be the case that $P(m)$ is true. This contradicts the choice of m. Hence, $P(n)$ must be true for every positive integer n.

The Good and the Bad of Mathematical Induction

An important point needs to be made about mathematical induction before we commence a study of its use. The good thing about mathematical induction is that it can be used to prove a conjecture once it is has been made (and is true). The bad thing about it is that it cannot be used to find new theorems. Mathematicians sometimes find proofs by mathematical induction unsatisfying because they do not provide insights as to why theorems are true. Many theorems can be proved in many ways, including by mathematical induction. Proofs of these theorems by methods other than mathematical induction are often preferred because of the insights they bring.

You can prove a theorem by mathematical induction even if you do not have the slightest idea why it is true!

Examples of Proofs by Mathematical Induction

Many theorems assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. Mathematical induction is a technique for proving theorems of this kind. In other words, mathematical induction can be used to prove statements of the form $\forall n\ P(n)$, where the domain is the set of positive integers. Mathematical induction can be used to prove an extremely wide variety of theorems, each of which is a statement of this form. (Remember, many mathematical assertions include an implicit universal quantifier. The statement "if n is a positive integer, then $n^3 - n$ is divisible by 3" is an example of this. Making the implicit universal quantifier explicit yields the statement "for every positive integer n, $n^3 - n$ is divisible by 3.)

Links

We will use how theorems are proved using mathematical induction. The theorems we will prove include summation formulae, inequalities, identities for combinations of sets, divisibility results, theorems about algorithms, and some other creative results. In this section and in later sections, we will employ mathematical induction to prove many other types of results, including the correctness of computer programs and algorithms. Mathematical induction can be used to prove a wide variety of theorems, not just summation formulae, inequalities, and other types of examples we illustrate here. (For proofs by mathematical induction of many more interesting and diverse results, see the *Handbook of Mathematical Induction* by David Gunderson [Gu11]. This book is part of the extensive CRC Series in Discrete Mathematics, many of which may be of interest to readers. The author is the Series Editor of these books).

Note that there are many opportunities for errors in induction proofs. We will describe some incorrect proofs by mathematical induction at the end of this section and in the exercises. To avoid making errors in proofs by mathematical induction, try to follow the guidelines for such proofs given at the end of this section.

SEEING WHERE THE INDUCTIVE HYPOTHESIS IS USED To help the reader understand each of the mathematical induction proofs in this section, we will note where the inductive hypothesis is used. We indicate this use in three different ways: by explicit mention in the text, by inserting the acronym IH (for inductive hypothesis) over an equals sign or a sign for an inequality, or by specifying the inductive hypothesis as the reason for a step in a multi-line display.

Look for the $\overset{\text{IH}}{=}$ symbol to see where the inductive hypothesis is used.

PROVING SUMMATION FORMULAE We begin by using mathematical induction to prove several summation formulae. As we will see, mathematical induction is particularly well suited for proving that such formulae are valid. However, summation formulae can be proven in other ways. This is not surprising because there are often different ways to prove a theorem. The major disadvantage of using mathematical induction to prove a summation formula is that you cannot use it to derive this formula. That is, you must already have the formula before you attempt to prove it by mathematical induction.

Examples 1–4 illustrate how to use mathematical induction to prove summation formulae. The first summation formula we will prove by mathematical induction, in Example 1, is a closed formula for the sum of the smallest n positive integers.

EXAMPLE 1 Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Extra Examples

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

If you are rusty simplifying algebraic expressions, this is the time to do some reviewing!

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n+1)/2$ for all positive integers n. ◀

As we noted, mathematical induction is not a tool for finding theorems about all positive integers. Rather, it is a proof method for proving such results once they are conjectured. In Example 2, using mathematical induction to prove a summation formula, we will both formulate and then prove a conjecture.

EXAMPLE 2 Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$\begin{array}{lll} 1 = 1, & 1 + 3 = 4, & 1 + 3 + 5 = 9, \\ 1 + 3 + 5 + 7 = 16, & 1 + 3 + 5 + 7 + 9 = 25. & \end{array}$$

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We need a method to *prove* that this *conjecture* is correct, if in fact it is.

Let $P(n)$ denote the proposition that the sum of the first n odd positive integers is n^2. Our conjecture is that $P(n)$ is true for all positive integers. To use mathematical induction to prove this conjecture, we must first complete the basis step; that is, we must show that $P(1)$ is true. Then we must carry out the inductive step; that is, we must show that $P(k + 1)$ is true when $P(k)$ is assumed to be true. We now attempt to complete these two steps.

BASIS STEP: $P(1)$ states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1. The basis step is complete.

INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(k) \to P(k + 1)$ is true for every positive integer k. To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that $P(k)$ is true for an arbitrary positive integer k, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

(Note that the kth odd positive integer is $(2k - 1)$, because this integer is obtained by adding 2 a total of $k - 1$ times to 1.) To show that $\forall k(P(k) \to P(k + 1))$ is true, we must show that if $P(k)$ is true (the inductive hypothesis), then $P(k + 1)$ is true. Note that $P(k + 1)$ is the statement that

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

So, assuming that $P(k)$ is true, it follows that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + \cdots + (2k - 1)] + (2k + 1) \\ &\overset{\text{IH}}{=} k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

This shows that $P(k + 1)$ follows from $P(k)$. Note that we used the inductive hypothesis $P(k)$ in the second equality to replace the sum of the first k odd positive integers by k^2.

We have now completed both the basis step and the inductive step. That is, we have shown that $P(1)$ is true and the conditional statement $P(k) \to P(k + 1)$ is true for all positive integers k. Consequently, by the principle of mathematical induction we can conclude that $P(n)$ is true for all positive integers n. That is, we know that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all positive integers n. ◀

Often, we will need to show that $P(n)$ is true for $n = b,\ b + 1,\ b + 2, \ldots,$ where b is an integer other than 1. We can use mathematical induction to accomplish this, as long as we change the basis step by replacing $P(1)$ with $P(b)$. In other words, to use mathematical induction to show that $P(n)$ is true for $n = b, b + 1, b + 2, \ldots,$ where b is an integer other than 1, we show that $P(b)$ is true in the basis step. In the inductive step, we show that the conditional statement $P(k) \to P(k + 1)$ is true for $k = b, b + 1, b + 2, \ldots$. Note that b can be negative, zero, or positive. Following the domino analogy we used earlier, imagine that we begin by knocking down the bth domino (the basis step), and as each domino falls, it knocks down the next domino (the inductive step). We leave it to the reader to show that this form of induction is valid (see Exercise 83).

We illustrate this notion in Example 3, which states that a summation formula is valid for all nonnegative integers. In this example, we need to prove that $P(n)$ is true for $n = 0, 1, 2, \ldots$. So, the basis step in Example 3 shows that $P(0)$ is true.

EXAMPLE 3 Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n. ◀

The formula given in Example 3 is a special case of a general result for the sum of terms of a geometric progression (Theorem 1 in Section 2.4). We will use mathematical induction to provide an alternative proof of this formula.

EXAMPLE 4 **Sums of Geometric Progressions** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

Solution: To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n + 1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r - 1)}{r - 1} = a.$$

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\begin{aligned}\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} &= \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1}\\ &= \frac{ar^{k+2} - a}{r - 1}.\end{aligned}$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction $P(n)$ is true for all nonnegative integers n. This shows that the formula for the sum of the terms of a geometric series is correct. ◀

As previously mentioned, the formula in Example 3 is the case of the formula in Example 4 with $a = 1$ and $r = 2$. The reader should verify that putting these values for a and r into the general formula gives the same formula as in Example 3.

PROVING INEQUALITIES Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a particular positive integer, as Examples 5–7 illustrate.

EXAMPLE 5 Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers n.

Extra Examples

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$. This completes the basis step.

INDUCTIVE STEP: We first assume the inductive hypothesis that $P(k)$ is true for anarbitrary positive integer k. That is, the inductive hypothesis $P(k)$ is the statement that $k < 2^k$. To complete the inductive step, we need to show that if $P(k)$ is true, then $P(k+1)$, which is the statement that $k + 1 < 2^{k+1}$, is true. That is, we need to show that if $k < 2^k$, then $k + 1 < 2^{k+1}$. To show that this conditional statement is true for the positive integer k, we first add 1 to both sides of $k < 2^k$, and then note that $1 \leq 2^k$. This tells us that

$$k + 1 \overset{\text{IH}}{<} 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

This shows that $P(k+1)$ is true, namely, that $k + 1 < 2^{k+1}$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n. ◀

EXAMPLE 6 Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2,$ and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

INDUCTIVE STEP: For the inductive step, we assume that $P(k)$ is true for an arbitrary integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$. We must show that under this hypothesis, $P(k+1)$ is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \geq 4$, then $2^{k+1} < (k+1)!$. We have

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\ &< 2 \cdot k! && \text{by the inductive hypothesis} \\ &< (k+1)k! && \text{because } 2 < k+1 \\ &= (k+1)! && \text{by definition of factorial function.} \end{aligned}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

An important inequality for the sum of the reciprocals of a set of positive integers will be proved in Example 7.

EXAMPLE 7 **An Inequality for Harmonic Numbers** The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots,$ are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is an arbitrary nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}$$

This establishes the inductive step of the proof.

We have completed the basis step and the inductive step. Thus, by mathematical induction $P(n)$ is true for all nonnegative integers n. That is, the inequality $H_{2^n} \geq 1 + \frac{n}{2}$ for the harmonic numbers holds for all nonnegative integers n. ◀

Remark: The inequality established here shows that the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is a divergent infinite series. This is an important example in the study of infinite series.

PROVING DIVISIBILITY RESULTS Mathematical induction can be used to prove divisibility results about integers. Although such results are often easier to prove using basic results in number theory, it is instructive to see how to prove such results using mathematical induction, as Examples 8 and 9 illustrate.

EXAMPLE 8 Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer. (Note that this is the statement with $p = 3$ of Fermat's little theorem, which is Theorem 3 of Section 4.4.)

Extra Examples

Solution: To construct the proof, let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

BASIS STEP: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true; that is, we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer k. To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k+1)$, the statement that $(k+1)^3 - (k+1)$ is divisible by 3, is also true. That is, we must show that $(k+1)^3 - (k+1)$ is divisible by 3. Note that

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1)\\ &= (k^3 - k) + 3(k^2 + k).\end{aligned}$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3. The second term is divisible by 3 because it is 3 times an integer. So, by part (i) of Theorem 1 in Section 4.1, we know that $(k+1)^3 - (k+1)$ is also divisible by 3. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer. ◀

The next example presents a more challenging proof by mathematical induction of a divisibility result.

EXAMPLE 9 Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Extra Examples

Solution: To construct the proof, let $P(n)$ denote the proposition: “$7^{n+2} + 8^{2n+1}$ is divisible by 57.”

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3}\\ &= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1}\\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1}\\ &= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n. ◀

PROVING RESULTS ABOUT SETS Mathematical induction can be used to prove many results about sets. In particular, in Example 10 we prove a formula for the number of subsets of a finite set and in Example 11 we establish a set identity.

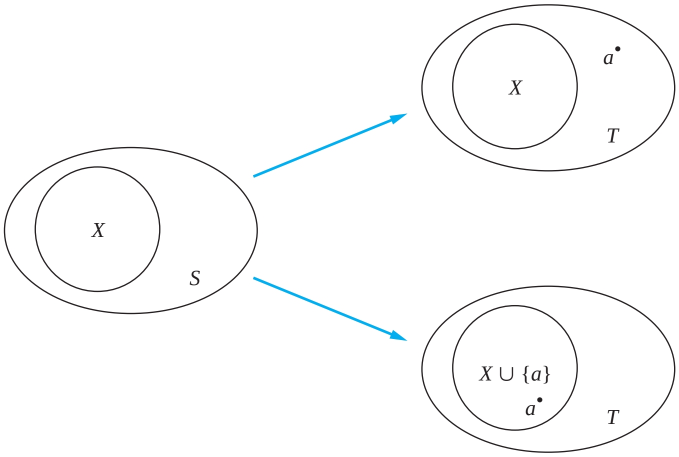


FIGURE 3 Generating Subsets of a Set with $k + 1$ Elements. Here $T = S \cup \{a\}$.

EXAMPLE 10 **The Number of Subsets of a Finite Set** Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets. (We will prove this result directly in several ways in Chapter 6.)

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

BASIS STEP: $P(0)$ is true, because a set with zero elements, the empty set, has exactly $2^0 = 1$ subset, namely, itself.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k, that is, we assume that every set with k elements has 2^k subsets. It must be shown that under this assumption, $P(k + 1)$, which is the statement that every set with $k + 1$ elements has 2^{k+1} subsets, must also be true. To show this, let T be a set with $k + 1$ elements. Then, it is possible to write $T = S \cup \{a\}$, where a is one of the elements of T and $S = T - \{a\}$ (and hence $|S| = k$). The subsets of T can be obtained in the following way. For each subset X of S there are exactly two subsets of T, namely, X and $X \cup \{a\}$. (This is illustrated in Figure 3.) These constitute all the subsets of T and are all distinct. We now use the inductive hypothesis to conclude that S has 2^k subsets, because it has k elements. We also know that there are two subsets of T for each subset of S. Therefore, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the inductive argument.

Because we have completed the basis step and the inductive step, by mathematical induction it follows that $P(n)$ is true for all nonnegative integers n. That is, we have proved that a set with n elements has 2^n subsets whenever n is a nonnegative integer. ◀

EXAMPLE 11 Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

Solution: Let $P(n)$ be the identity for n sets.

BASIS STEP: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's laws; it was proved in Example 11 of Section 2.2.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary integer with $k \geq 2$; that is, it is the statement that

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. To carry out the inductive step, we need to show that this assumption implies that $P(k+1)$ is true. That is, we need to show that if this equality holds for every collection of k subsets of U, then it must also hold for every collection of $k+1$ subsets of U. Suppose that $A_1, A_2, \ldots, A_k, A_{k+1}$ are subsets of U. When the inductive hypothesis is assumed to hold, it follows that

$$\begin{aligned}
\overline{\bigcap_{j=1}^{k+1} A_j} &= \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}} && \text{by the definition of intersection} \\
&= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} && \text{by De Morgan's law (where the two sets are } \textstyle\bigcap_{j=1}^{k} A_j \text{ and } A_{k+1}\text{)} \\
&= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} && \text{by the inductive hypothesis} \\
&= \bigcup_{j=1}^{k+1} \overline{A_j} && \text{by the definition of union.}
\end{aligned}$$

This completes the inductive step.

Because we have completed both the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true whenever n is a positive integer, $n \geq 2$. That is, we know that

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$. ◀

PROVING RESULTS ABOUT ALGORITHMS Next, we provide an example (somewhat more difficult than previous examples) that illustrates one of many ways mathematical induction is used in the study of algorithms. We will show how mathematical induction can be used to prove that a greedy algorithm we introduced in Section 3.1 always yields an optimal solution.

EXAMPLE 12 Recall the algorithm for scheduling talks discussed in Example 7 of Section 3.1. The input to this algorithm is a group of m proposed talks with preset starting and ending times. The goal is to schedule as many of these lectures as possible in the main lecture hall so that no two talks overlap. Suppose that talk t_j begins at time s_j and ends at time e_j. (No two lectures can proceed in the main lecture hall at the same time, but a lecture in this hall can begin at the same time another one ends.)

Without loss of generality, we assume that the talks are listed in order of nondecreasing ending time, so that $e_1 \leq e_2 \leq \cdots \leq e_m$. The greedy algorithm proceeds by selecting at each stage a talk with the earliest ending time among all those talks that begin no sooner than when the last talk scheduled in the main lecture hall has ended. Note that a talk with the earliest end time is always selected first by the algorithm. We will show that this greedy algorithm is optimal in the sense that it always schedules the most talks possible in the main lecture hall. To prove the optimality of this algorithm we use mathematical induction on the variable n, the number of talks scheduled by the algorithm. We let $P(n)$ be the proposition that if the greedy algorithm schedules n talks in the main lecture hall, then it is not possible to schedule more than n talks in this hall.

BASIS STEP: Suppose that the greedy algorithm managed to schedule just one talk, t_1, in the main lecture hall. This means that no other talk can start at or after e_1, the end time of t_1. Otherwise, the first such talk we come to as we go through the talks in order of nondecreasing end times could be added. Hence, at time e_1 each of the remaining talks needs to use the main lecture hall because they all start before e_1 and end after e_1. It follows that no two talks can be scheduled because both need to use the main lecture hall at time e_1. This shows that $P(1)$ is true and completes the basis step.

INDUCTIVE STEP: The inductive hypothesis is that $P(k)$ is true, where k is an arbitrary positive integer, that is, that the greedy algorithm always schedules the most possible talks when it selects k talks, where k is a positive integer, given any set of talks, no matter how many. We must show that $P(k+1)$ follows from the assumption that $P(k)$ is true, that is, we must show that under the assumption of $P(k)$, the greedy algorithm always schedules the most possible talks when it selects $k+1$ talks.

Now suppose that the greedy algorithm has selected $k+1$ talks. Our first step in completing the inductive step is to show there is a schedule including the most talks possible that contains talk t_1, a talk with the earliest end time. This is easy to see because a schedule that begins with the talk t_i in the list, where $i > 1$, can be changed so that talk t_1 replaces talk t_i. To see this, note that because $e_1 \leq e_i$, all talks that were scheduled to follow talk t_i can still be scheduled.

Once we included talk t_1, scheduling the talks so that as many as possible are scheduled is reduced to scheduling as many talks as possible that begin at or after time e_1. So, if we have scheduled as many talks as possible, the schedule of talks other than talk t_1 is an optimal schedule of the original talks that begin once talk t_1 has ended. Because the greedy algorithm schedules k talks when it creates this schedule, we can apply the inductive hypothesis to conclude that it has scheduled the most possible talks. It follows that the greedy algorithm has scheduled the most possible talks, $k+1$, when it produced a schedule with $k+1$ talks, so $P(k+1)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step. So, by mathematical induction we know that $P(n)$ is true for all positive integers n. This completes the proof of optimality. That is, we have proved that when the greedy algorithm schedules n talks, when n is a positive integer, then it is not possible to schedule more than n talks. ◀

CREATIVE USES OF MATHEMATICAL INDUCTION Mathematical induction can often be used in unexpected ways. We will illustrate two particularly clever uses of mathematical induction here, the first relating to survivors in a pie fight and the second relating to tilings with regular triominoes of checkerboards with one square missing.

EXAMPLE 13

Extra Examples

Odd Pie Fights An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at their nearest neighbor, hitting this person. Use mathematical induction to show that there is at least one survivor, that is, at least one person who is not hit by a pie. (This problem was introduced by Carmony [Ca79]. Note that this result is false when there are an even number of people; see Exercise 75.)

Solution: Let $P(n)$ be the statement that there is a survivor whenever $2n+1$ people stand in a yard at distinct mutual distances and each person throws a pie at their nearest neighbor. To prove this result, we will show that $P(n)$ is true for all positive integers n. This follows because as n runs through all positive integers, $2n+1$ runs through all odd integers greater than or equal to 3. Note that one person cannot engage in a pie fight because there is no one else to throw the pie at.

BASIS STEP: When $n = 1$, there are $2n + 1 = 3$ people in the pie fight. Of the three people, suppose that the closest pair are A and B, and C is the third person. Because distances between pairs of people are different, the distance between A and C and the distance between B and C are both different from, and greater than, the distance between A and B. It follows that A and B throw pies at each other, while C throws a pie at either A or B, whichever is closer. Hence, C is not hit by a pie. This shows that at least one of the three people is not hit by a pie, completing the basis step.

INDUCTIVE STEP: For the inductive step, assume that $P(k)$ is true for an arbitrary odd integer k with $k \geq 3$. That is, assume that there is at least one survivor whenever $2k + 1$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor. We must show that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$, the statement that there is at least one survivor whenever $2(k + 1) + 1 = 2k + 3$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor, is also true.

So suppose that we have $2(k + 1) + 1 = 2k + 3$ people in a yard with distinct distances between pairs of people. Let A and B be the closest pair of people in this group of $2k + 3$ people. When each person throws a pie at the nearest person, A and B throw pies at each other. We have two cases to consider, *(i)* when someone else throws a pie at either A or B and *(ii)* when no one else throws a pie at either A or B.

Case (i): Because A and B throw pies at each other and someone else throws a pie at either A and B, at least three pies are thrown at A and B, and at most $(2k + 3) - 3 = 2k$ pies are thrown at the remaining $2k + 1$ people. This guarantees that at least one person is a survivor, for if each of these $2k + 1$ people was hit by at least one pie, a total of at least $2k + 1$ pies would have to be thrown at them. (The reasoning used in this last step is an example of the pigeonhole principle discussed further in Section 6.2.)

Case (ii): No one else throws a pie at either A and B. Besides A and B, there are $2k + 1$ people. Because the distances between pairs of these people are all different, we can use the inductive hypothesis to conclude that there is at least one survivor S when these $2k + 1$ people each throws a pie at their nearest neighbor. Furthermore, S is also not hit by either the pie thrown by A or the pie thrown by B because A and B throw their pies at each other, so S is a survivor because S is not hit by any of the pies thrown by these $2k + 3$ people.

We have completed both the basis step and the inductive step, using a proof by cases. So by mathematical induction it follows that $P(n)$ is true for all positive integers n. We conclude that whenever an odd number of people located in a yard at distinct mutual distances each throws a pie at their nearest neighbor, there is at least one survivor. ◀

Links

In Section 1.8 we discussed the tiling of checkerboards by polyominoes. Example 14 illustrates how mathematical induction can be used to prove a result about covering checkerboards with right triominoes, pieces shaped like the letter "L."

EXAMPLE 14 Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, where these pieces cover three squares at a time, as shown in Figure 4.

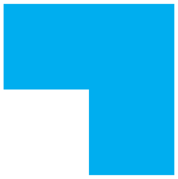


FIGURE 4 A Right Triomino.

Solution: Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. We can use mathematical induction to prove that $P(n)$ is true for all positive integers n.

BASIS STEP: $P(1)$ is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino, as shown in Figure 5.

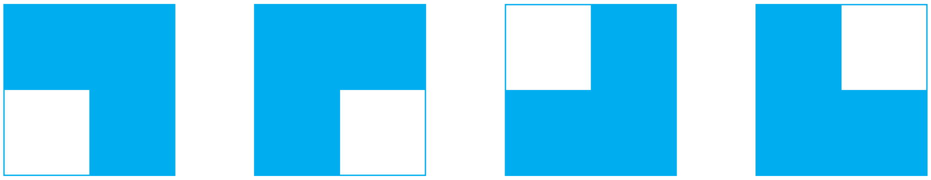


FIGURE 5 **Tiling 2 × 2 Checkerboards with One Square Removed.**

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(k)$ is true for the positive integer k; that is, it is the assumption that every $2^k \times 2^k$ checkerboard with one square removed can be tiled using right triominoes. It must be shown that under the assumption of the inductive hypothesis, $P(k+1)$ must also be true; that is, any $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

To see this, consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions. This is illustrated in Figure 6. No square has been removed from three of these four checkerboards. The fourth $2^k \times 2^k$ checkerboard has one square removed, so we now use the inductive hypothesis to conclude that it can be covered by right triominoes. Now temporarily remove the square from each of the other three $2^k \times 2^k$ checkerboards that has the center of the original, larger checkerboard as one of its corners, as shown in Figure 7. By the inductive hypothesis, each of these three $2^k \times 2^k$ checkerboards with a square removed can be tiled by right triominoes. Furthermore, the three squares that were temporarily removed can be covered by one right triomino. Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard can be tiled with right triominoes.

We have completed the basis step and the inductive step. Therefore, by mathematical induction $P(n)$ is true for all positive integers n. This shows that we can tile every $2^n \times 2^n$ checkerboard, where n is a positive integer, with one square removed, using right triominoes. ◀

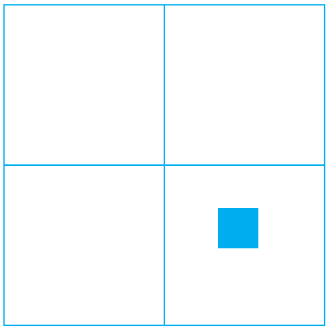


FIGURE 6 **Dividing a $2^{k+1} \times 2^{k+1}$ Checkerboard into Four $2^k \times 2^k$ Checkerboards.**

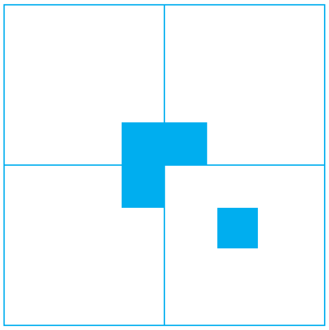


FIGURE 7 **Tiling the $2^{k+1} \times 2^{k+1}$ Checkerboard with One Square Removed.**

Mistaken Proofs By Mathematical Induction

Consult *Common Errors in Discrete Mathematics* on this book's website for more basic mistakes.

As with every proof method, there are many opportunities for making errors when using mathematical induction. Many well-known mistaken, and often entertaining, proofs by mathematical induction of clearly false statements have been devised, as exemplified by Example 15 and Exercises 49–51. Often, it is not easy to find where the error in reasoning occurs in such mistaken proofs.

To uncover errors in proofs by mathematical induction, remember that in every such proof, both the basis step and the inductive step must be done correctly. Not completing the basis step in a supposed proof by mathematical induction can lead to mistaken proofs of clearly ridiculous statements such as "$n = n + 1$ whenever n is a positive integer." (We leave it to the reader to show that it is easy to construct a correct inductive step in an attempted proof of this statement.) Locating the error in a faulty proof by mathematical induction, as Example 15 illustrates, can be quite tricky, especially when the error is hidden in the basis step.

EXAMPLE 15 Find the error in this "proof" of the clearly false claim that every set of lines in the plane, no two of which are parallel, meet in a common point.

"Proof:" Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. We will attempt to prove that $P(n)$ is true for all positive integers $n \geq 2$.

BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point (by the definition of parallel lines).

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer k, that is, it is the assumption that every set of k lines in the plane, no two of which are parallel, meet in a common point. To complete the inductive step, we must show that if $P(k)$ is true, then $P(k + 1)$ must also be true. That is, we must show that if every set of k lines in the plane, no two of which are parallel, meet in a common point, then every set of $k + 1$ lines in the plane, no two of which are parallel, meet in a common point. So, consider a set of $k + 1$ distinct lines in the plane. By the inductive hypothesis, the first k of these lines meet in a common point p_1. Moreover, by the inductive hypothesis, the last k of these lines meet in a common point p_2. We will show that p_1 and p_2 must be the same point. If p_1 and p_2 were different points, all lines containing both of them must be the same line because two points determine a line. This contradicts our assumption that all these lines are distinct. Thus, p_1 and p_2 are the same point. We conclude that the point $p_1 = p_2$ lies on all $k + 1$ lines. We have shown that $P(k + 1)$ is true assuming that $P(k)$ is true. That is, we have shown that if we assume that every k, $k \geq 2$, distinct lines meet in a common point, then every $k + 1$ distinct lines meet in a common point. This completes the inductive step.

We have completed the basis step and the inductive step, and supposedly we have a correct proof by mathematical induction.

Solution: Examining this supposed proof by mathematical induction it appears that everything is in order. However, there is an error, as there must be. The error is rather subtle. Carefully looking at the inductive step shows that this step requires that $k \geq 3$. We cannot show that $P(2)$ implies $P(3)$. When $k = 2$, our goal is to show that every three distinct lines meet in a common point. The first two lines must meet in a common point p_1 and the last two lines must meet in a common point p_2. But in this case, p_1 and p_2 do not have to be the same, because only the second line is common to both sets of lines. Here is where the inductive step fails. ◀

Guidelines for Proofs by Mathematical Induction

Examples 1–14 illustrate proofs by mathematical induction of a diverse collection of theorems. Each of these examples includes all the elements needed in a proof by mathematical induction. We have provided an example of an invalid proof by mathematical induction. Summarizing what we have learned from these examples, we can provide some useful guidelines for constructing correct proofs by mathematical induction. We now present these guidelines.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

It is worthwhile to revisit each of the mathematical induction proofs in Examples 1–14 to see how these steps are completed. It will be helpful to follow these guidelines in the solutions of the exercises that ask for proofs by mathematical induction. The guidelines that we presented can be adapted for each of the variants of mathematical induction that we introduce in the exercises and later in this chapter.

Exercises

1. There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

2. Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Prove that this golfer plays every hole on the course.

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

7. Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

9. **a)** Find a formula for the sum of the first n even positive integers.

b) Prove the formula that you conjectured in part (a).

10. **a)** Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

11. **a)** Find a formula for

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

12. Prove that

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

whenever n is a nonnegative integer.

13. Prove that $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1} n(n+1)/2$ whenever n is a positive integer.

14. Prove that for every positive integer n, $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

15. Prove that for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3.$$

16. Prove that for every positive integer n,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2)$$
$$= n(n+1)(n+2)(n+3)/4.$$

17. Prove that $\sum_{j=1}^{n} j^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$ whenever n is a positive integer.

Use mathematical induction to prove the inequalities in Exercises 18–30.

18. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

a) What is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

19. Let $P(n)$ be the statement that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

where n is an integer greater than 1.

a) What is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

20. Prove that $3^n < n!$ if n is an integer greater than 6.

21. Prove that $2^n > n^2$ if n is an integer greater than 4.

22. For which nonnegative integers n is $n^2 \leq n!$? Prove your answer.

23. For which nonnegative integers n is $2n + 3 \leq 2^n$? Prove your answer.

24. Prove that $1/(2n) \leq [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]/(2 \cdot 4 \cdot \cdots \cdot 2n)$ whenever n is a positive integer.

***25.** Prove that if $h > -1$, then $1 + nh \leq (1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

***26.** Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.

***27.** Prove that for every positive integer n,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

28. Prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.

In Exercises 29 and 30, H_n denotes the nth harmonic number.

***29.** Prove that $H_{2^n} \leq 1 + n$ whenever n is a nonnegative integer.

***30.** Prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

Use mathematical induction in Exercises 31–37 to prove divisibility facts.

31. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

32. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

33. Prove that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

34. Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

***35.** Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

***36.** Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

***37.** Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

Use mathematical induction in Exercises 38–46 to prove results about sets.

38. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j.$$

39. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

40. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B$$
$$= (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

41. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

42. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)$$
$$= (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

43. Prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}.$$

44. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B)$$
$$= (A_1 \cup A_2 \cup \cdots \cup A_n) - B.$$

45. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

***46.** Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

In Exercises 47 and 48 we consider the problem of placing towers along a straight road, so that every building on the road receives cellular service. Assume that a building receives cellular service if it is within one mile of a tower.

47. Devise a greedy algorithm that uses the minimum number of towers possible to provide cell service to d buildings located at positions $x_1, x_2, \ldots, x_d$ from the start of the road. [*Hint:* At each step, go as far as possible along the road before adding a tower so as not to leave any buildings without coverage.]

***48.** Use mathematical induction to prove that the algorithm you devised in Exercise 47 produces an optimal solution, that is, that it uses the fewest towers possible to provide cellular service to all buildings.

Exercises 49–51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in each exercise.

49. What is wrong with this "proof" that all horses are the same color?

Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k+1$ horses; number these as horses $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Because the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction.

50. What is wrong with this "proof"?

"Theorem" For every positive integer n, $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By the inductive hypothesis, $\sum_{i=1}^{n+1} i = (n + \frac{1}{2})^2/2 + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n+1) + \frac{1}{2}]^2/2$, completing the inductive step.

51. What is wrong with this "proof"?

"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

52. Suppose that m and n are positive integers with $m > n$ and f is a function from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

***53.** Use mathematical induction to show that n people can divide a cake (where each person gets one or more separate pieces of the cake) so that the cake is divided fairly, that is, in the sense that each person thinks he or she got at least $(1/n)$th of the cake. [*Hint:* For the inductive step, take a fair division of the cake among the first k people, have each person divide their share into what this person thinks are $k+1$ equal portions, and then have the $(k+1)$st person select a portion from each of the k people. When showing this produces a fair division for $k+1$ people, suppose that person $k+1$ thinks that person i got p_i of the cake where $\sum_{i=1}^{k} p_i = 1$.]

54. Use mathematical induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

***55.** A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or two spaces horizontally (in either direction) and one space vertically (in either direction). Suppose that we have an infinite chessboard, made up of all squares (m, n) where m and n are nonnegative integers that denote the row number and the column number of the square, respectively. Use mathematical induction to show that a knight starting at (0, 0) can visit every square using a finite sequence of moves. [*Hint:* Use induction on the variable $s = m + n$.]

56. Suppose that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a and b are real numbers. Show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.

57. (*Requires calculus*) Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer. (For the inductive step, use the product rule for derivatives.)

58. Suppose that **A** and **B** are square matrices with the property **AB** = **BA**. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

59. Suppose that m is a positive integer. Use mathematical induction to prove that if a and b are integers with $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ whenever k is a nonnegative integer.

60. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

***61.** Show that

$$[(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_{n-1} \to p_n)] \\ \to [(p_1 \wedge p_2 \wedge \cdots \wedge p_{n-1}) \to p_n]$$

is a tautology whenever $p_1, p_2, \ldots, p_n$ are propositions, where $n \geq 2$.

***62.** Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

****63.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The **arithmetic mean** of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and the **geometric mean** of these numbers is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

64. Use mathematical induction to prove Lemma 3 of Section 4.3, which states that if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

65. Show that if n is a positive integer, then

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is over all nonempty subsets of the set of the n smallest positive integers.)

***66.** Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ and $P(2)$ are true.

Inductive Step: For each positive integer k, if $P(k)$ and $P(k + 1)$ are both true, then $P(k + 2)$ is true.

67. Show that if $A_1, A_2, \ldots, A_n$ are sets where $n \geq 2$, and for all pairs of integers i and j with $1 \leq i < j \leq n$ either A_i is a subset of A_j or A_j is a subset of A_i, then there is an integer i, $1 \leq i \leq n$ such that A_i is a subset of A_j for all integers j with $1 \leq j \leq n$.

***68.** A guest at a party is a **celebrity** if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. Your assignment is to find the celebrity, if one exists, at a party, by asking only one type of question—asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with $3(n - 1)$ questions. [*Hint:* First ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.]

Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals. Exercises 69–71 deal with the gossip problem.

69. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

70. Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$. [*Hint:* In the inductive step, have a new person call a particular person at the start and at the end.]

****71.** Prove that $G(n) = 2n - 4$ for $n \geq 4$.

***72.** Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

***73.** Show that if $I_1, I_2, \ldots, I_n$ is a collection of open intervals on the real number line, $n \geq 2$, and every pair of these intervals has a nonempty intersection, that is, $I_i \cap I_j \neq \emptyset$ whenever $1 \leq i \leq n$ and $1 \leq j \leq n$, then the intersection of all these sets is nonempty, that is, $I_1 \cap I_2 \cap \cdots \cap I_n \neq \emptyset$. (Recall that an **open interval** is the set of real numbers x with $a < x < b$, where a and b are real numbers with $a < b$.)

Sometimes we cannot use mathematical induction to prove a result we believe to be true, but we can use mathematical induction to prove a stronger result. Because the inductive hypothesis of the stronger result provides more to work with, this process is called **inductive loading**. We use inductive loading in Exercise 74.

74. Suppose that we want to prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for all integers greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

75. Let n be an even positive integer. Show that when n people stand in a yard at mutually distinct distances and each person throws a pie at their nearest neighbor, it is possible that everyone is hit by a pie.

76. Construct a tiling using right triominoes of the 4×4 checkerboard with the square in the upper left corner removed.

77. Construct a tiling using right triominoes of the 8×8 checkerboard with the square in the upper left corner removed.

78. Prove or disprove that all checkerboards of these shapes can be completely covered using right triominoes whenever n is a positive integer.

a) 3×2^n **b)** 6×2^n

c) $3^n \times 3^n$ **d)** $6^n \times 6^n$

***79.** Show that a three-dimensional $2^n \times 2^n \times 2^n$ checkerboard with one $1 \times 1 \times 1$ cube missing can be completely covered by $2 \times 2 \times 2$ cubes with one $1 \times 1 \times 1$ cube removed.

***80.** Show that an $n \times n$ checkerboard with one square removed can be completely covered using right triominoes if $n > 5$, n is odd, and $3 \nmid n$.

81. Show that a 5×5 checkerboard with a corner square removed can be tiled using right triominoes.

***82.** Find a 5×5 checkerboard with a square removed that cannot be tiled using right triominoes. Prove that such a tiling does not exist for this board.

83. Use the principle of mathematical induction to show that $P(n)$ is true for $n = b, b+1, b+2, \ldots,$ where b is an integer, if $P(b)$ is true and the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers k with $k \geq b$.

5.2 Strong Induction and Well-Ordering

Introduction

In Section 5.1 we introduced mathematical induction and we showed how to use it to prove a variety of theorems. In this section we will introduce another form of mathematical induction, called **strong induction**, which can often be used when we cannot easily prove a result using mathematical induction. The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. That is, in a strong induction proof that $P(n)$ is true for all positive integers n, the basis step shows that $P(1)$ is true. However, the inductive steps in these two proof methods are different. In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

The validity of both mathematical induction and strong induction follow from the well-ordering property in Appendix 1. In fact, mathematical induction, strong induction, and well-ordering are all equivalent principles (as shown in Exercises 41, 42, and 43). That is, the validity of each can be proved from either of the other two. This means that a proof using one of these two principles can be rewritten as a proof using either of the other two principles. Just as it is sometimes the case that it is much easier to see how to prove a result using strong induction rather than mathematical induction, it is sometimes easier to use well-ordering than one of the two forms of mathematical induction. In this section we will give some examples of how the well-ordering property can be used to prove theorems.

Strong Induction

Before we illustrate how to use strong induction, we state this principle again.

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Note that when we use strong induction to prove that $P(n)$ is true for all positive integers n, our inductive hypothesis is the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$. That is, the inductive hypothesis includes all k statements $P(1), P(2), \ldots, P(k)$. Because we can use all k statements $P(1), P(2), \ldots, P(k)$ to prove $P(k+1)$, rather than just the statement $P(k)$ as in a proof by mathematical induction, strong induction is a more flexible proof technique. Because of this, some mathematicians prefer to always use strong induction instead of mathematical induction, even when a proof by mathematical induction is easy to find.

You may be surprised that mathematical induction and strong induction are equivalent. That is, each can be shown to be a valid proof technique assuming that the other is valid. In particular, any proof using mathematical induction can also be considered to be a proof by strong induction because the inductive hypothesis of a proof by mathematical induction is part of the inductive hypothesis in a proof by strong induction. That is, if we can complete the inductive step of a proof using mathematical induction by showing that $P(k+1)$ follows from $P(k)$ for every positive integer k, then it also follows that $P(k+1)$ follows from all the statements $P(1)$, $P(2), \ldots, P(k)$, because we are assuming that not only $P(k)$ is true, but also more, namely, that the $k-1$ statements $P(1), P(2), \ldots, P(k-1)$ are true. However, it is much more awkward to convert a proof by strong induction into a proof using the principle of mathematical induction. (See Exercise 42.)

Strong induction is sometimes called the **second principle of mathematical induction** or **complete induction**. When the terminology "complete induction" is used, the principle of mathematical induction is called **incomplete induction**, a technical term that is a somewhat unfortunate choice because there is nothing incomplete about the principle of mathematical induction; after all, it is a valid proof technique.

STRONG INDUCTION AND THE INFINITE LADDER To better understand strong induction, consider the infinite ladder in Section 5.1. Strong induction tells us that we can reach all rungs if

1. we can reach the first rung, and
2. for every integer k, if we can reach all the first k rungs, then we can reach the $(k+1)$st rung.

That is, if $P(n)$ is the statement that we can reach the nth rung of the ladder, by strong induction we know that $P(n)$ is true for all positive integers n, because (1) tells us $P(1)$ is true, completing the basis step and (2) tells us that $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ implies $P(k+1)$, completing the inductive step.

Example 1 illustrates how strong induction can help us prove a result that cannot easily be proved using the principle of mathematical induction.

EXAMPLE 1 Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Can we prove that we can reach every rung using the principle of mathematical induction? Can we prove that we can reach every rung using strong induction?

Solution: We first try to prove this result using the principle of mathematical induction.

BASIS STEP: The basis step of such a proof holds; here it simply verifies that we can reach the first rung.

ATTEMPTED INDUCTIVE STEP: The inductive hypothesis is the statement that we can reach the kth rung of the ladder. To complete the inductive step, we need to show that if we assume the inductive hypothesis for the positive integer k, namely, if we assume that we can reach the kth rung of the ladder, then we can show that we can reach the $(k + 1)$st rung of the ladder. However, there is no obvious way to complete this inductive step because we do not know from the given information that we can reach the $(k + 1)$st rung from the kth rung. After all, we only know that if we can reach a rung we can reach the rung two higher.

Now consider a proof using strong induction.

BASIS STEP: The basis step is the same as before; it simply verifies that we can reach the first rung.

INDUCTIVE STEP: The inductive hypothesis states that we can reach each of the first k rungs. To complete the inductive step, we need to show that if we assume that the inductive hypothesis is true, that is, if we can reach each of the first k rungs, then we can reach the $(k + 1)$st rung. We already know that we can reach the second rung. We can complete the inductive step by noting that as long as $k \geq 2$, we can reach the $(k + 1)$st rung from the $(k - 1)$st rung because we know we can climb two rungs from a rung we can already reach, and because $k - 1 \leq k$, by the inductive hypothesis we can reach the $(k - 1)$st rung. This completes the inductive step and finishes the proof by strong induction.

We have proved that if we can reach the first two rungs of an infinite ladder and for every positive integer k if we can reach all the first k rungs then we can reach the $(k + 1)$st rung, then we can reach all rungs of the ladder. ◀

Examples of Proofs Using Strong Induction

Now that we have both mathematical induction and strong induction, how do we decide which method to apply in a particular situation? Although there is no cut-and-dried answer, we can supply some useful pointers. In practice, you should use mathematical induction when it is straightforward to prove that $P(k) \to P(k + 1)$ is true for all positive integers k. This is the case for all the proofs in the examples in Section 5.1. In general, you should restrict your use of the principle of mathematical induction to such scenarios. Unless you can clearly see that the inductive step of a proof by mathematical induction goes through, you should attempt a proof by strong induction. That is, use strong induction and not mathematical induction when you see how to prove that $P(k + 1)$ is true from the assumption that $P(j)$ is true for all positive integers j not exceeding k, but you cannot see how to prove that $P(k + 1)$ follows from just $P(k)$. Keep this in mind as you examine the proofs in this section. For each of these proofs, consider why strong induction works better than mathematical induction.

We will illustrate how strong induction is employed in Examples 2–4. In these examples, we will prove a diverse collection of results. Pay particular attention to the inductive step in each of these examples, where we show that a result $P(k + 1)$ follows under the assumption that $P(j)$ holds for all positive integers j not exceeding k, where $P(n)$ is a propositional function.

We begin with one of the most prominent uses of strong induction, the part of the fundamental theorem of arithmetic that tells us that every positive integer can be written as the product of primes.

EXAMPLE 2 Show that if n is an integer greater than 1, then n can be written as the product of primes.

Extra Examples

Solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

BASIS STEP: $P(2)$ is true, because 2 can be written as the product of one prime, itself. (Note that $P(2)$ is the first case we need to establish.)

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption, that is, that $k+1$ is the product of primes.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b. ◀

Remark: Because 1 can be thought of as the *empty* product of no primes, we could have started the proof in Example 2 with $P(1)$ as the basis step. We chose not to do so because many people find this confusing.

Example 2 completes the proof of the fundamental theorem of arithmetic, which asserts that every nonnegative integer can be written uniquely as the product of primes in nondecreasing order. We showed in Section 4.3 that an integer has at most one such factorization into primes. Example 2 shows there is at least one such factorization.

Next, we show how strong induction can be used to prove that a player has a winning strategy in a game.

EXAMPLE 3 Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Solution: Let n be the number of matches in each pile. We will use strong induction to prove $P(n)$, the statement that the second player can win when there are initially n matches in each pile.

BASIS STEP: When $n = 1$, the first player has only one choice, removing one match from one of the piles, leaving a single pile with a single match, which the second player can remove to win the game.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for all j with $1 \leq j \leq k$, that is, the assumption that the second player can always win whenever there are j matches, where $1 \leq j \leq k$ in each of the two piles at the start of the game. We need to show that $P(k+1)$ is true, that is, that the second player can win when there are initially $k+1$ matches in each pile, under the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$. So suppose that there are $k+1$ matches in each of the two piles at the start of the game and suppose that the first player removes r matches $(1 \leq r \leq k)$ from one of the piles, leaving $k+1-r$ matches in this pile. By removing the same number of matches from the other pile, the second player creates the situation where there are two piles each with $k + 1 - r$ matches. Because $1 \leq k + 1 - r \leq k$, we can now use the inductive hypothesis to conclude that the second player can always win. We complete the proof by noting that if the first player removes all $k + 1$ matches from one of the piles, the second player can win by removing all the remaining matches. ◀

Using the principle of mathematical induction, instead of strong induction, to prove the results in Examples 2 and 3 is difficult. However, as Example 4 shows, some results can be readily proved using either the principle of mathematical induction or strong induction.

Before we present Example 4, note that we can slightly modify strong induction to handle a wider variety of situations. In particular, we can adapt strong induction to handle cases where the inductive step is valid only for integers greater than a particular integer. Let b be a fixed integer and j a fixed positive integer. The form of strong induction we need tells us that $P(n)$ is true for all integers n with $n \geq b$ if we can complete these two steps:

BASIS STEP: We verify that the propositions $P(b), P(b+1), \ldots, P(b+j)$ are true.

INDUCTIVE STEP: We show that $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b + j$.

We will use this alternative form in the strong induction proof in Example 4. That this alternative form is equivalent to strong induction is left as Exercise 28.

EXAMPLE 4 Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: We will prove this result using the principle of mathematical induction. Then we will present a proof using strong induction. Let $P(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

We begin by using the principle of mathematical induction.

BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true. That is, under this hypothesis, postage of k cents can be formed using 4-cent and 5-cent stamps. To complete the inductive step, we need to show that when we assume $P(k)$ is true, then $P(k+1)$ is also true where $k \geq 12$. That is, we need to show that if we can form postage of k cents, then we can form postage of $k + 1$ cents. So, assume the inductive hypothesis is true; that is, assume that we can form postage of k cents using 4-cent and 5-cent stamps. We consider two cases, when at least one 4-cent stamp has been used and when no 4-cent stamps have been used. First, suppose that at least one 4-cent stamp was used to form postage of k cents. Then we can replace this stamp with a 5-cent stamp to form postage of $k + 1$ cents. But if no 4-cent stamps were used, we can form postage of k cents using only 5-cent stamps. Moreover, because $k \geq 12$, we needed at least three 5-cent stamps to form postage of k cents. So, we can replace three 5-cent stamps with four 4-cent stamps to form postage of $k + 1$ cents. This completes the inductive step.

Because we have completed the basis step and the inductive step, we know that $P(n)$ is true for all $n \geq 12$. That is, we can form postage of n cents, where $n \geq 12$ using just 4-cent and 5-cent stamps. This completes the proof by mathematical induction.

Next, we will use strong induction to prove the same result. In this proof, in the basis step we show that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true, that is, that postage of 12, 13, 14, or 15 cents can be formed using just 4-cent and 5-cent stamps. In the inductive step we show how to get postage of $k + 1$ cents for $k \geq 15$ from postage of $k - 3$ cents.

BASIS STEP: We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively. This shows that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true. This completes the basis step.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$. To complete the inductive step, we assume that we can form postage of j cents, where $12 \leq j \leq k$. We need to show that under the assumption that $P(k+1)$ is true, we can also form postage of $k+1$ cents. Using the inductive hypothesis, we can assume that $P(k-3)$ is true because $k-3 \geq 12$, that is, we can form postage of $k-3$ cents using just 4-cent and 5-cent stamps. To form postage of $k+1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k-3$ cents. That is, we have shown that if the inductive hypothesis is true, then $P(k+1)$ is also true. This completes the inductive step.

Because we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that $P(n)$ is true for all integers n with $n \geq 12$. That is, we know that every postage of n cents, where n is at least 12, can be formed using 4-cent and 5-cent stamps. This finishes the proof by strong induction.

(There are other ways to approach this problem besides those described here. Can you find a solution that does not use mathematical induction?) ◀

Using Strong Induction in Computational Geometry

Our next example of strong induction will come from **computational geometry**, the part of discrete mathematics that studies computational problems involving geometric objects. Computational geometry is used extensively in computer graphics, computer games, robotics, scientific calculations, and a vast array of other areas. Before we can present this result, we introduce some terminology, possibly familiar from earlier studies in geometry.

A **polygon** is a closed geometric figure consisting of a sequence of line segments $s_1, s_2, \ldots, s_n$, called **sides**. Each pair of consecutive sides, s_i and s_{i+1}, $i = 1, 2, \ldots, n-1$, as well as the last side s_n and the first side s_1, of the polygon meet at a common endpoint, called a **vertex**. A polygon is called **simple** if no two nonconsecutive sides intersect. Every simple polygon divides the plane into two regions: its **interior**, consisting of the points inside the curve, and its **exterior**, consisting of the points outside the curve. This last fact is surprisingly complicated to prove. It is a special case of the famous Jordan curve theorem, which tells us that every simple curve divides the plane into two regions; see [Or00], for example.

A polygon is called **convex** if every line segment connecting two points in the interior of the polygon lies entirely inside the polygon. (A polygon that is not convex is said to be **nonconvex**.) Figure 1 displays some polygons; polygons (a) and (b) are convex, but polygons (c) and (d) are not. A **diagonal** of a simple polygon is a line segment connecting two nonconsecutive vertices of the polygon, and a diagonal is called an **interior diagonal** if it lies entirely inside the polygon, except for its endpoints. For example, in polygon (d), the line segment connecting a and f is an interior diagonal, but the line segment connecting a and d is a diagonal that is not an interior diagonal.

One of the most basic operations of computational geometry involves dividing a simple polygon into triangles by adding nonintersecting diagonals. This process is called **triangulation**. Note that a simple polygon can have many different triangulations, as shown in Figure 2. Perhaps the most basic fact in computational geometry is that it is possible to triangulate every simple

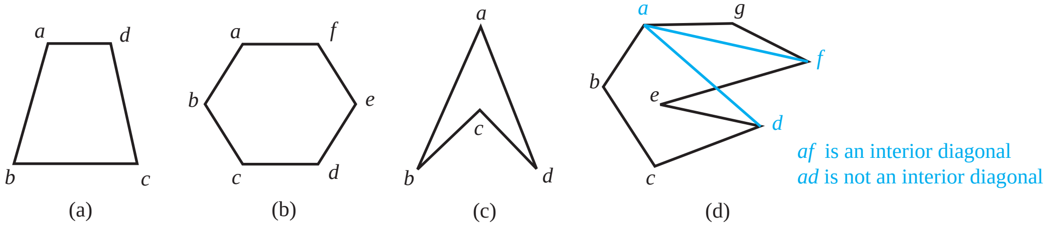
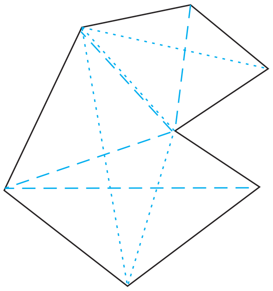


FIGURE 1 **Convex and Nonconvex Polygons.**



Two different triangulations of a simple polygon with seven sides into five triangles, shown with dotted lines and with dashed lines, respectively

FIGURE 2 **Triangulations of a Polygon.**

polygon, as we state in Theorem 1. Furthermore, this theorem tells us that every triangulation of a simple polygon with n sides includes $n - 2$ triangles.

THEOREM 1 A simple polygon with n sides, where n is an integer with $n \geq 3$, can be triangulated into $n - 2$ triangles.

It seems obvious that we should be able to triangulate a simple polygon by successively adding interior diagonals. Consequently, a proof by strong induction seems promising. However, such a proof requires this crucial lemma.

LEMMA 1 Every simple polygon with at least four sides has an interior diagonal.

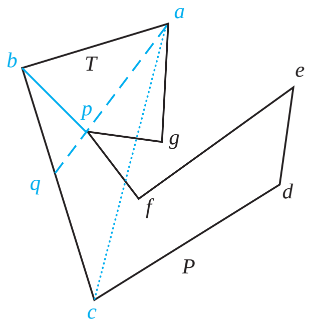
Although Lemma 1 seems particularly simple, it is surprisingly tricky to prove. In fact, as recently as 30 years ago, a variety of incorrect proofs thought to be correct were commonly seen in books and articles. We defer the proof of Lemma 1 until after we prove Theorem 1. It is not uncommon to prove a theorem pending the later proof of an important lemma.

Proof (of Theorem 1): We will prove this result using strong induction. Let $T(n)$ be the statement that every simple polygon with n sides can be triangulated into $n - 2$ triangles.

BASIS STEP: $T(3)$ is true because a simple polygon with three sides is a triangle. We do not need to add any diagonals to triangulate a triangle; it is already triangulated into one triangle, itself. Consequently, every simple polygon with $n = 3$ has can be triangulated into $n - 2 = 3 - 2 = 1$ triangle.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $T(j)$ is true for all integers j with $3 \leq j \leq k$. That is, we assume that we can triangulate a simple polygon with j sides into $j - 2$ triangles whenever $3 \leq j \leq k$. To complete the inductive step, we must show that when we assume the inductive hypothesis, $P(k + 1)$ is true, that is, that every simple polygon with $k + 1$ sides can be triangulated into $(k + 1) - 2 = k - 1$ triangles.

So, suppose that we have a simple polygon P with $k + 1$ sides. Because $k + 1 \geq 4$, Lemma 1 tells us that P has an interior diagonal ab. Now, ab splits P into two simple polygons Q, with s sides, and R, with t sides. The sides of Q and R are the sides of P, together with the side ab, which is a side of both Q and R. Note that $3 \leq s \leq k$ and $3 \leq t \leq k$ because both Q and R have at least one fewer side than P does (after all, each of these is formed from P by deleting at least two sides and replacing these sides by the diagonal ab). Furthermore, the number of sides of P is two less than the sum of the numbers of sides of Q and the number of



T is the triangle *abc*

p is the vertex of *P* inside *T* such that the ∠*bap* is smallest

bp must be an interior diagonal of *P*

FIGURE 3 Constructing an Interior Diagonal of a Simple Polygon.

sides of R, because each side of P is a side of either Q or of R, but not both, and the diagonal ab is a side of both Q and R, but not P. That is, $k + 1 = s + t - 2$.

We now use the inductive hypothesis. Because both $3 \leq s \leq k$ and $3 \leq t \leq k$, by the inductive hypothesis we can triangulate Q and R into $s - 2$ and $t - 2$ triangles, respectively. Next, note that these triangulations together produce a triangulation of P. (Each diagonal added to triangulate one of these smaller polygons is also a diagonal of P.) Consequently, we can triangulate P into a total of $(s - 2) + (t - 2) = s + t - 4 = (k + 1) - 2$ triangles. This completes the proof by strong induction. That is, we have shown that every simple polygon with n sides, where $n \geq 3$, can be triangulated into $n - 2$ triangles. ◁

We now return to our proof of Lemma 1. We present a proof published by Chung-Wu Ho [Ho75]. Note that although this proof may be omitted without loss of continuity, it does provide a correct proof of a result proved incorrectly by many mathematicians.

Proof: Suppose that P is a simple polygon drawn in the plane. Furthermore, suppose that b is the point of P or in the interior of P with the least y-coordinate among the vertices with the smallest x-coordinate. Then b must be a vertex of P, for if it is an interior point, there would have to be a vertex of P with a smaller x-coordinate. Two other vertices each share an edge with b, say a and c. It follows that the angle in the interior of P formed by ab and bc must be less than 180 degrees (otherwise, there would be points of P with smaller x-coordinates than b).

Now let T be the triangle $\triangle abc$. If there are no vertices of P on or inside T, we can connect a and c to obtain an interior diagonal. On the other hand, if there are vertices of P inside T, we will find a vertex p of P on or inside T such that bp is an interior diagonal. (This is the tricky part. Ho noted that in many published proofs of this lemma a vertex p was found such that bp was not necessarily an interior diagonal of P. See Exercise 21.) The key is to select a vertex p such that the angle $\angle bap$ is smallest. To see this, note that the ray starting at a and passing through p hits the line segment bc at a point, say q. It then follows that the triangle $\triangle baq$ cannot contain any vertices of P in its interior. Hence, we can connect b and p to produce an interior diagonal of P. Locating this vertex p is illustrated in Figure 3. ◁

Proofs Using the Well-Ordering Property

The validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of the set of integers, the **well-ordering property** (see Appendix 1). The well-ordering property states that every nonempty set of nonnegative integers has a least element. We will show how the well-ordering property can be used directly in proofs. Furthermore, it can be shown (see Exercises 41, 42, and 43) that the well-ordering property, the principle of mathematical induction, and strong induction are all equivalent. That is, the validity of each of these three proof techniques implies the validity of the other two techniques. In Section 5.1 we showed that the principle of mathematical induction follows from the well-ordering property. The other parts of this equivalence are left as Exercises 31, 42, and 43.

THE WELL-ORDERING PROPERTY Every nonempty set of nonnegative integers has a least element.

The well-ordering property can often be used directly in proofs.

EXAMPLE 5 Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

Extra Examples

Solution: Let S be the set of nonnegative integers of the form $a - dq$, where q is an integer. This set is nonempty because $-dq$ can be made as large as desired (taking q to be a negative integer with large absolute value). By the well-ordering property, S has a least element $r = a - dq_0$.

The integer r is nonnegative. It is also the case that $r < d$. If it were not, then there would be a smaller nonnegative element in S, namely, $a - d(q_0 + 1)$. To see this, suppose that $r \geq d$. Because $a = dq_0 + r$, it follows that $a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0$. Consequently, there are integers q and r with $0 \leq r < d$. The proof that q and r are unique is left as Exercise 37. ◀

EXAMPLE 6 In a round-robin tournament every player plays every other player exactly once and each match has a winner and a loser. We say that the players $p_1, p_2, \ldots, p_m$ form a *cycle* if p_1 beats p_2, p_2 beats $p_3, \ldots, p_{m-1}$ beats p_m, and p_m beats p_1. Use the well-ordering principle to show that if there is a cycle of length m $(m \geq 3)$ among the players in a round-robin tournament, there must be a cycle of three of these players.

Solution: We assume that there is no cycle of three players. Because there is at least one cycle in the round-robin tournament, the set of all positive integers n for which there is a cycle of length n is nonempty. By the well-ordering property, this set of positive integers has a least element k, which by assumption must be greater than three. Consequently, there exists a cycle of players $p_1, p_2, p_3, \ldots, p_k$ and no shorter cycle exists.

Because there is no cycle of three players, we know that $k > 3$. Consider the first three elements of this cycle, p_1, p_2, and p_3. There are two possible outcomes of the match between p_1 and p_3. If p_3 beats p_1, it follows that p_1, p_2, p_3 is a cycle of length three, contradicting our assumption that there is no cycle of three players. Consequently, it must be the case that p_1 beats p_3. This means that we can omit p_2 from the cycle $p_1, p_2, p_3, \ldots, p_k$ to obtain the cycle $p_1, p_3, p_4, \ldots, p_k$ of length $k - 1$, contradicting the assumption that the smallest cycle has length k. We conclude that there must be a cycle of length three. ◀

Exercises

1. Use strong induction to show that if you can run one mile or two miles, and if you can always run two more miles once you have run a specified number of miles, then you can run any number of miles.

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

3. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 8$.

a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 10$.

e) Explain why these steps show that this statement is true whenever $n \geq 8$.

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

5. a) Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

6. a) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

8. Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

***9.** Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.]

10. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

11. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j.

12. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

***13.** A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n - 1$ moves are required to assemble a puzzle with n pieces.

14. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n - 1)/2$.

15. Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is square. [*Hint:* Use strong induction to show that this strategy works. For the first move, the first player chomps all cookies except those in the left and top edges. On subsequent moves, after the second player has chomped cookies on either the top or left edge, the first player chomps cookies in the same relative positions in the left or top edge, respectively.]

***16.** Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is two squares wide, that is, a $2 \times n$ board. [*Hint:* Use strong induction. The first move of the first player should be to chomp the cookie in the bottom row at the far right.]

17. Use strong induction to show that if a simple polygon with at least four sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon.

***18.** Use strong induction to show that when a simple polygon P with consecutive vertices $v_1, v_2, \ldots, v_n$ is triangulated into $n - 2$ triangles, the $n - 2$ triangles can be numbered $1, 2, \ldots, n - 2$ so that v_i is a vertex of triangle i for $i = 1, 2, \ldots, n - 2$.

***19.** **Pick's theorem** says that the area of a simple polygon P in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P) + B(P)/2 - 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of P and on the boundary of P, respectively. Use strong induction on the number of vertices of P to prove Pick's theorem. [*Hint:* For the basis step, first prove the theorem for rectangles, then for right triangles, and finally for all triangles by noting that the area of a triangle is the area of a larger rectangle containing it with the areas of at most three triangles subtracted. For the inductive step, take advantage of Lemma 1.]

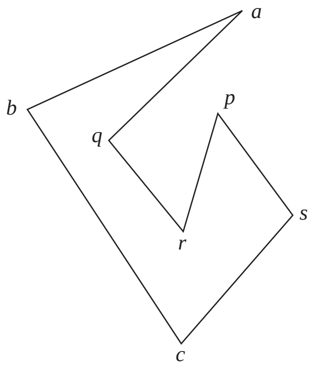
20. Suppose that P is a simple polygon with vertices $v_1, v_2, \ldots, v_n$ listed so that consecutive vertices are connected by an edge, and v_1 and v_n are connected by an edge. A vertex v_i is called an **ear if the line segment connecting the two vertices adjacent to v_i is an interior diagonal of the simple polygon. Two ears v_i and v_j are called **nonoverlapping** if the interiors of the triangles with vertices v_i and its two adjacent vertices and v_j and its two adjacent vertices do not intersect. Prove that every simple polygon with at least four vertices has at least two nonoverlapping ears.

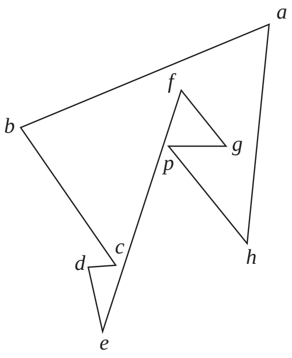
21. In the proof of Lemma 1 we mentioned that many incorrect methods for finding a vertex p such that the line segment bp is an interior diagonal of P have been published. This exercise presents some of the incorrect ways p has been chosen in these proofs. Show, by considering one of the polygons drawn here, that for each of these choices of p, the line segment bp is not necessarily an interior diagonal of P.

a) p is the vertex of P such that the angle $\angle abp$ is smallest.

b) p is the vertex of P with the least x-coordinate (other than b).

c) p is the vertex of P that is closest to b.





Exercises 22 and 23 present examples that show inductive loading can be used to prove results in computational geometry.

***22.** Let $P(n)$ be the statement that when nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two vertices of the polygon are not endpoints of any of these diagonals.

a) Show that when we attempt to prove $P(n)$ for all integers n with $n \geq 3$ using strong induction, the inductive step does not go through.

b) Show that we can prove that $P(n)$ is true for all integers n with $n \geq 3$ by proving by strong induction the stronger assertion $Q(n)$, for $n \geq 4$, where $Q(n)$ states that whenever nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two *nonadjacent* vertices are not endpoints of any of these diagonals.

23. Let $E(n)$ be the statement that in a triangulation of a simple polygon with n sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon.

a) Explain where a proof using strong induction that $E(n)$ is true for all integers $n \geq 4$ runs into difficulties.

b) Show that we can prove that $E(n)$ is true for all integers $n \geq 4$ by proving by strong induction the stronger statement $T(n)$ for all integers $n \geq 4$, which states that in every triangulation of a simple polygon, at least two of the triangles in the triangulation have two sides bordering the exterior of the polygon.

***24.** A stable assignment, defined in the preamble to Exercise 60 in Section 3.1, is called **optimal for suitors** if no stable assignment exists in which a suitor is paired with a suitee whom this suitor prefers to the person to whom this suitor is paired in this stable assignment. Use strong induction to show that the deferred acceptance algorithm produces a stable assignment that is optimal for suitors.

25. Suppose that $P(n)$ is a propositional function. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+2)$ is true.

b) $P(1)$ and $P(2)$ are true; for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

26. Suppose that $P(n)$ is a propositional function. Determine for which nonnegative integers n the statement $P(n)$ must be true if

a) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ is true.

b) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+3)$ is true.

c) $P(0)$ and $P(1)$ are true; for all nonnegative integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

d) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ and $P(n+3)$ are true.

27. Show that if the statement $P(n)$ is true for infinitely many positive integers n and $P(n+1) \rightarrow P(n)$ is true for all positive integers n, then $P(n)$ is true for all positive integers n.

28. Let b be a fixed integer and j a fixed positive integer. Show that if $P(b), P(b+1), \ldots, P(b+j)$ are true and $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b+j$, then $P(n)$ is true for all integers n with $n \geq b$.

29. What is wrong with this "proof" by strong induction?

"Theorem" For every nonnegative integer n, $5n = 0$.

Basis Step: $5 \cdot 0 = 0$.

Inductive Step: Suppose that $5j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. Write $k+1 = i + j$, where i and j are natural numbers less than $k+1$. By the inductive hypothesis, $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$.

***30.** Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the definition of a^0.

Inductive Step: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

***31.** Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

32. Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k + 1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

33. Show that we can prove that $P(n, k)$ is true for all pairs of positive integers n and k if we show

a) $P(1, 1)$ is true and $P(n, k) \rightarrow [P(n + 1, k) \wedge P(n, k + 1)]$ is true for all positive integers n and k.

b) $P(1, k)$ is true for all positive integers k, and $P(n, k) \rightarrow P(n + 1, k)$ is true for all positive integers n and k.

c) $P(n, 1)$ is true for all positive integers n, and $P(n, k) \rightarrow P(n, k + 1)$ is true for all positive integers n and k.

34. Prove that $\sum_{j=1}^{n} j(j + 1)(j + 2) \cdots (j + k - 1) = n(n + 1)(n + 2) \cdots (n + k)/(k + 1)$ for all positive integers k and n. [*Hint*: Use a technique from Exercise 33.]

***35.** Show that if $a_1, a_2, \ldots, a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted into their product. [*Hint:* Use strong induction and consider the last multiplication.]

***36.** The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let S be the set of positive integers of the form $as + bt$, where s and t are integers.

a) Show that S is nonempty.

b) Use the well-ordering property to show that S has a smallest element c.

c) Show that if d is a common divisor of a and b, then d is a divisor of c.

d) Show that $c \mid a$ and $c \mid b$. [*Hint:* First, assume that $c \nmid a$. Then $a = qc + r$, where $0 < r < c$. Show that $r \in S$, contradicting the choice of c.]

e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing that this greatest common divisor is unique.

37. Let a be an integer and d be a positive integer. Show that the integers q and r with $a = dq + r$ and $0 \leq r < d$, which were shown to exist in Example 5, are unique.

38. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

****39.** Can you use the well-ordering property to prove the statement: "Every positive integer can be described using no more than fifteen English words"? Assume the words come from a particular dictionary of English. [*Hint:* Suppose that there are positive integers that cannot be described using no more than fifteen English words. By well ordering, *the smallest positive integer that cannot be described using no more than fifteen English words* would then exist.]

40. Use the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [*Hint:* Use the Archimedean property, given in Appendix 1, to find a positive integer A with $A > 1/(y - x)$. Then show that there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

***41.** Show that the well-ordering property can be proved when the principle of mathematical induction is taken as an axiom.

***42.** Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

***43.** Show that we can prove the well-ordering property when we take strong induction as an axiom instead of taking the well-ordering property as an axiom.

5.3 Recursive Definitions and Structural Induction

Introduction

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion**. For instance, the picture shown in Figure 1 is produced recursively. First, an original picture is given. Then a process of successively superimposing centered smaller pictures on top of the previous pictures is carried out.

CHAPTER

9 Relations

Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on. In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, a real number x and the value $f(x)$ where f is a function, and so on. Relationships such as that between a program and a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science.

Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.

In some computer languages, only the first 31 characters of the name of a variable matter. The relation consisting of ordered pairs of strings where the first string has the same initial 31 characters as the second string is an example of a special type of relation, known as an equivalence relation. Equivalence relations arise throughout mathematics and computer science. We will study equivalence relations, and other special types of relations, in this chapter.

9.1 Relations and Their Properties

Introduction

Links

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. In this section we introduce the basic terminology used to describe binary relations. Later in this chapter we will use relations to solve problems involving communications networks, project scheduling, and identifying elements in sets with common properties.

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a \, R \, b$ to denote that $(a, b) \in R$ and $a \not R \, b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related to** b by R.

Binary relations represent relationships between the elements of two sets. We will introduce n-ary relations, which express relationships among elements of more than two sets, later in this chapter. We will omit the word *binary* when there is no danger of confusion.

Examples 1–3 illustrate the notion of a relation.

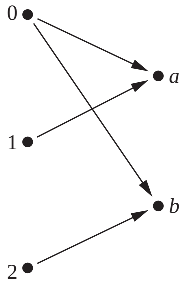
EXAMPLE 1 Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b), where a is a student enrolled in course b. For instance, if Jason Goodfriend and Deborah Sherman are enrolled in CS518, the pairs

(Jason Goodfriend, CS518) and (Deborah Sherman, CS518) belong to R. If Jason Goodfriend is also enrolled in CS510, then the pair (Jason Goodfriend, CS510) is also in R. However, if Deborah Sherman is not enrolled in CS510, then the pair (Deborah Sherman, CS510) is not in R.

Note that if a student is not currently enrolled in any courses there will be no pairs in R that have this student as the first element. Similarly, if a course is not currently being offered there will be no pairs in R that have this course as their second element. ◀

EXAMPLE 2 Let A be the set of cities in the U.S.A., and let B be the set of the 50 states in the U.S.A. Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R. ◀

EXAMPLE 3 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0\, R\, a$, but that $1 \not R\, b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1. We will discuss representations of relations in more detail in Section 9.3. ◀



R	a	b
0	×	×
1	×	
2		×

FIGURE 1 **Displaying the Ordered Pairs in the Relation R from Example 3.**

Functions as Relations

Recall that a function f from a set A to a set B (as defined in Section 2.3) assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B. Moreover, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph.

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph. This can be done by assigning to an element a of A the unique element $b \in B$ such that $(a, b) \in R$. (Note that the relation R in Example 2 is not the graph of a function because Middletown occurs more than once as the first element of an ordered pair in R.)

A relation can be used to express a one-to-many relationship between the elements of the sets A and B (as in Example 2), where an element of A may be related to more than one element of B. A function represents a relation where exactly one element of B is related to each element of A.

Relations are a generalization of graphs of functions; they can be used to express a much wider class of relationships between sets. (Recall that the graph of the function f from A to B is the set of ordered pairs $(a, f(a))$ for $a \in A$.)

Relations on a Set

Relations from a set A to itself are of special interest.

DEFINITION 2 A *relation on a set* A is a relation from A to A.

In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE 4 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

The pairs in this relation are displayed both graphically and in tabular form in Figure 2.

Next, some examples of relations on the set of integers will be given in Example 5.

EXAMPLE 5 Consider these relations on the set of integers:

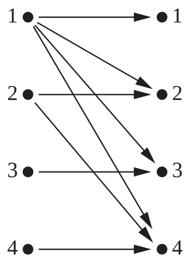
$$\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\},\\
R_2 &= \{(a, b) \mid a > b\},\\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\},\\
R_4 &= \{(a, b) \mid a = b\},\\
R_5 &= \{(a, b) \mid a = b + 1\},\\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}$$

Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Remark: Unlike the relations in Examples 1–4, these are relations on an infinite set.

Solution: The pair $(1, 1)$ is in R_1, R_3, R_4, and R_6; $(1, 2)$ is in R_1 and R_6; $(2, 1)$ is in R_2, R_5, and R_6; $(1, -1)$ is in R_2, R_3, and R_6; and finally, $(2, 2)$ is in R_1, R_3, and R_4.

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a subset of $A \times A$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

FIGURE 2 Displaying the Ordered Pairs in the Relation R from Example 4.

EXAMPLE 6 How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$.

Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these here.

In some relations an element is always related to itself. For instance, let R be the relation on the set of all people consisting of pairs (x, y) where x and y have the same mother and the same father. Then xRx for every person x.

DEFINITION 3 A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Remark: Using quantifiers we see that the relation R on the set A is reflexive if $\forall a((a, a) \in R)$, where the universe of discourse is the set of all elements in A.

We see that a relation on A is reflexive if every element of A is related to itself. Examples 7–9 illustrate the concept of a reflexive relation.

EXAMPLE 7 Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations.

EXAMPLE 8 Which of the relations from Example 5 are reflexive?

Solution: The reflexive relations from Example 5 are R_1 (because $a \leq a$ for every integer a), R_3, and R_4. For each of the other relations in this example it is easy to find a pair of the form (a, a) that is not in the relation. (This is left as an exercise for the reader.)

EXAMPLE 9 Is the "divides" relation on the set of positive integers reflexive?

Solution: Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.)

In some relations an element is related to a second element if and only if the second element is also related to the first element. The relation consisting of pairs (x, y), where x and y are students at your school with at least one common class has this property. Other relations have the property that if an element is related to a second element, then this second element is not related to the first. The relation consisting of the pairs (x, y), where x and y are students at your school, where x has a higher grade point average than y has this property.

DEFINITION 4

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Remark: Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$. Similarly, the relation R on the set A is antisymmetric if $\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$.

That is, a relation is symmetric if and only if a is related to b implies that b is related to a. A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a. That is, the only way to have a related to b and b related to a is for a and b to be the same element. The terms *symmetric* and *antisymmetric* are not opposites, because a relation can have both of these properties or may lack both of them (see Exercise 10). A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b), where $a \neq b$.

Remark: Although relatively few of the 2^{n^2} relations on a set with n elements are symmetric or antisymmetric, as counting arguments can show, many important relations have one of these properties. (See Exercise 47.)

EXAMPLE 10 Which of the relations from Example 7 are symmetric and which are antisymmetric?

Extra Examples

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2, the only thing to check is that both $(2, 1)$ and $(1, 2)$ are in the relation. For R_3, it is necessary to check that both $(1, 2)$ and $(2, 1)$ belong to the relation, and $(1, 4)$ and $(4, 1)$ belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

R_4, R_5, and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with $a \neq b$ such that (a, b) and (b, a) are both in the relation. ◀

EXAMPLE 11 Which of the relations from Example 5 are symmetric and which are antisymmetric?

Solution: The relations R_3, R_4, and R_6 are symmetric. R_3 is symmetric, for if $a = b$ or $a = -b$, then $b = a$ or $b = -a$. R_4 is symmetric because $a = b$ implies that $b = a$. R_6 is symmetric because $a + b \leq 3$ implies that $b + a \leq 3$. The reader should verify that none of the other relations is symmetric.

The relations R_1, R_2, R_4, and R_5 are antisymmetric. R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b$. R_2 is antisymmetric because it is impossible that $a > b$ and $b > a$. R_4 is antisymmetric, because two elements are related with respect to R_4 if and only if they are equal. R_5 is antisymmetric because it is impossible that $a = b + 1$ and $b = a + 1$. The reader should verify that none of the other relations is antisymmetric. ◀

EXAMPLE 12 Is the “divides” relation on the set of positive integers symmetric? Is it antisymmetric?

Solution: This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$. It is antisymmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$ (the verification of this is left as an exercise for the reader). ◀

Let R be the relation consisting of all pairs (x, y) of students at your school, where x has taken more credits than y. Suppose that x is related to y and y is related to z. This means that x has taken more credits than y and y has taken more credits than z. We can conclude that x has taken more credits than z, so that x is related to z. What we have shown is that R has the transitive property, which is defined as follows.

DEFINITION 5 A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark: Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.

EXAMPLE 13 Which of the relations in Example 7 are transitive?

Extra Examples

Solution: R_4, R_5, and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, R_4 is transitive, because $(3, 2)$ and $(2, 1)$, $(4, 2)$ and $(2, 1)$, $(4, 3)$ and $(3, 1)$, and $(4, 3)$ and $(3, 2)$ are the only such sets of pairs, and $(3, 1)$, $(4, 1)$, and $(4, 2)$ belong to R_4. The reader should verify that R_5 and R_6 are transitive.

R_1 is not transitive because $(3, 4)$ and $(4, 1)$ belong to R_1, but $(3, 1)$ does not. R_2 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_2, but $(2, 2)$ does not. R_3 is not transitive because $(4, 1)$ and $(1, 2)$ belong to R_3, but $(4, 2)$ does not. ◀

EXAMPLE 14 Which of the relations in Example 5 are transitive?

Solution: The relations R_1, R_2, R_3, and R_4 are transitive. R_1 is transitive because $a \leq b$ and $b \leq c$ imply that $a \leq c$. R_2 is transitive because $a > b$ and $b > c$ imply that $a > c$. R_3 is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$. R_4 is clearly transitive, as the reader should verify. R_5 is not transitive because $(2, 1)$ and $(1, 0)$ belong to R_5, but $(2, 0)$ does not. R_6 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_6, but $(2, 2)$ does not. ◀

EXAMPLE 15 Is the “divides” relation on the set of positive integers transitive?

Solution: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. It follows that this relation is transitive. ◀

We can use counting techniques to determine the number of relations with specific properties. Finding the number of relations with a particular property provides information about how common this property is in the set of all relations on a set with n elements.

EXAMPLE 16 How many reflexive relations are there on a set with n elements?

Solution: A relation R on a set A is a subset of $A \times A$. Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R. However, if R is reflexive, each of the n ordered pairs (a, a) for $a \in A$ must be in R. Each of the other $n(n - 1)$ ordered pairs of the form (a, b), where $a \neq b$, may or may not be in R. Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations [this is the number of ways to choose whether each element (a, b), with $a \neq b$, belongs to R]. ◀

Formulas for the number of symmetric relations and the number of antisymmetric relations on a set with n elements can be found using reasoning similar to that in Example 16 (see Exercise 47). However, no general formula is known that counts the transitive relations on a set with n elements. Currently, $T(n)$, the number of transitive relations on a set with n elements, is known only for $n \leq 17$. For example, $T(4) = 3{,}994$, $T(5) = 154{,}303$, and $T(6) = 9{,}415{,}189$.

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined. Consider Examples 17–19.

EXAMPLE 17 Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$$\begin{aligned}
R_1 \cup R_2 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\},\\
R_1 \cap R_2 &= \{(1, 1)\},\\
R_1 - R_2 &= \{(2, 2), (3, 3)\},\\
R_2 - R_1 &= \{(1, 2), (1, 3), (1, 4)\}.
\end{aligned}$$

◀

EXAMPLE 18 Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that R_1 consists of all ordered pairs (a, b), where a is a student who has taken course b, and R_2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate. What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, and $R_2 - R_1$?

Solution: The relation $R_1 \cup R_2$ consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate, and $R_1 \cap R_2$ is the set of all ordered pairs (a, b), where a is a student who has taken course b and needs this course to graduate. Also, $R_1 \oplus R_2$ consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it. $R_1 - R_2$ is the set of ordered pairs (a, b), where a has taken course b but does not need it to graduate; that is, b is an elective course that a has taken. $R_2 - R_1$ is the set of all ordered pairs (a, b), where b is a course that a needs to graduate but has not taken. ◀

EXAMPLE 19 Let R_1 be the “less than” relation on the set of real numbers and let R_2 be the “greater than” relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence, $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ or $x > y$. Because the condition $x < y$ or $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$. In other words, the union of the “less than” relation and the “greater than” relation is the “not equals” relation.

Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 because it is impossible that $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x, y) \mid x \neq y\}$. ◀

There is another way that relations are combined that is analogous to the composition of functions.

DEFINITION 6 Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation, as Examples 20 and 21 illustrate.

EXAMPLE 20 What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$ ◀

EXAMPLE 21 **Composing the Parent Relation with Itself** Let R be the relation on the set of all people such that $(a, b) \in R$ if person a is a parent of person b. Then $(a, c) \in R \circ R$ if and only if there is a person b such that $(a, b) \in R$ and $(b, c) \in R$, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c. In other words, $(a, c) \in R \circ R$ if and only if a is a grandparent of c. ◀

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

DEFINITION 7 Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots,$ are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

EXAMPLE 22 Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots.$

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Additional computation shows that R^4 is the same as R^3, so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots.$ The reader should verify this. ◀

The following theorem shows that the powers of a transitive relation are subsets of this relation. It will be used in Section 9.4.

THEOREM 1 The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots.$

Proof: We first prove the "if" part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R. To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. ◁

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$. **b)** $a + b = 4$.
c) $a > b$. **d)** $a \mid b$.
e) $\gcd(a, b) = 1$. **f)** $\operatorname{lcm}(a, b) = 2$.

2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.

b) Display this relation graphically, as was done in Example 4.

c) Display this relation in tabular form, as was done in Example 4.

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.
b) a and b were born on the same day.
c) a has the same first name as b.
d) a and b have a common grandparent.

5. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) everyone who has visited Web page a has also visited Web page b.

b) there are no common links found on both Web page a and Web page b.

c) there is at least one common link on Web page a and Web page b.

d) there is a Web page that includes links to both Web page a and Web page b.

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x + y = 0$. **b)** $x = \pm y$.
c) $x - y$ is a rational number.
d) $x = 2y$. **e)** $xy \geq 0$.
f) $xy = 0$. **g)** $x = 1$.
h) $x = 1$ or $y = 1$.

7. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$. **b)** $xy \geq 1$.
c) $x = y + 1$ or $x = y - 1$.
d) $x \equiv y \pmod{7}$. **e)** x is a multiple of y.
f) x and y are both negative or both nonnegative.
g) $x = y^2$. **h)** $x \geq y^2$.

8. Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.

9. Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.

10. Give an example of a relation on a set that is

a) both symmetric and antisymmetric.
b) neither symmetric nor antisymmetric.

A relation R on the set A is **irreflexive** if for every $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

11. Which relations in Exercise 3 are irreflexive?

12. Which relations in Exercise 4 are irreflexive?

13. Which relations in Exercise 5 are irreflexive?

14. Which relations in Exercise 6 are irreflexive?

15. Can a relation on a set be neither reflexive nor irreflexive?

16. Use quantifiers to express what it means for a relation to be irreflexive.

17. Give an example of an irreflexive relation on the set of all people.

A relation R is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$. Exercises 18–24 explore the notion of an asymmetric relation. Exercise 22 focuses on the difference between asymmetry and antisymmetry.

18. Which relations in Exercise 3 are asymmetric?

19. Which relations in Exercise 4 are asymmetric?

20. Which relations in Exercise 5 are asymmetric?

21. Which relations in Exercise 6 are asymmetric?

22. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.

23. Use quantifiers to express what it means for a relation to be asymmetric.

24. Give an example of an asymmetric relation on the set of all people.

25. How many different relations are there from a set with m elements to a set with n elements?

Let R be a relation from a set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

a) R^{-1}. **b)** $\overline{R}$.

27. Let R be the relation $R = \{(a, b) \mid a$ divides $b\}$ on the set of positive integers. Find

a) R^{-1}. **b)** $\overline{R}$.

28. Let R be the relation on the set of all states in the United States consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1}. **b)** $\overline{R}$.

29. Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f. That is, $R = \{(a, f(a)) \mid a \in A\}$. What is the inverse relation R^{-1}?

30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.

31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations.

a) $R_1 \cup R_2$ **b)** $R_1 \cap R_2$
c) $R_1 \oplus R_2$ **d)** $R_1 - R_2$
e) $R_2 - R_1$

32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

33. Let R be the relation on the set of people consisting of pairs (a, b), where a is a parent of b. Let S be the relation on the set of people consisting of pairs (a, b), where a and b are siblings (brothers or sisters). What are $S \circ R$ and $R \circ S$?

Exercises 34–37 deal with these relations on the set of real numbers:

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the "greater than" relation,
$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the "greater than or equal to" relation,
$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the "less than" relation,
$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the "less than or equal to" relation,
$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the "equal to" relation,
$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the "unequal to" relation.

34. Find

a) $R_1 \cup R_3$. **b)** $R_1 \cup R_5$.
c) $R_2 \cap R_4$. **d)** $R_3 \cap R_5$.
e) $R_1 - R_2$. **f)** $R_2 - R_1$.
g) $R_1 \oplus R_3$. **h)** $R_2 \oplus R_4$.

35. Find

a) $R_2 \cup R_4$. **b)** $R_3 \cup R_6$.
c) $R_3 \cap R_6$. **d)** $R_4 \cap R_6$.
e) $R_3 - R_6$. **f)** $R_6 - R_3$.
g) $R_2 \oplus R_6$. **h)** $R_3 \oplus R_5$.

36. Find

a) $R_1 \circ R_1$. **b)** $R_1 \circ R_2$.
c) $R_1 \circ R_3$. **d)** $R_1 \circ R_4$.
e) $R_1 \circ R_5$. **f)** $R_1 \circ R_6$.
g) $R_2 \circ R_3$. **h)** $R_3 \circ R_3$.

37. Find

a) $R_2 \circ R_1$. **b)** $R_2 \circ R_2$.
c) $R_3 \circ R_5$. **d)** $R_4 \circ R_1$.
e) $R_5 \circ R_3$. **f)** $R_3 \circ R_6$.
g) $R_4 \circ R_6$. **h)** $R_6 \circ R_6$.

38. Let R be the parent relation on the set of all people (see Example 21). When is an ordered pair in the relation R^3?

39. Let R be the relation on the set of people with doctorates such that $(a, b) \in R$ if and only if a was the thesis advisor of b. When is an ordered pair (a, b) in R^2? When is an ordered pair (a, b) in R^n, when n is a positive integer? (Assume that every person with a doctorate has a thesis advisor.)

40. Let R_1 and R_2 be the "divides" and "is a multiple of" relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a$ divides $b\}$ and $R_2 = \{(a, b) \mid a$ is a multiple of $b\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.
e) $R_1 \oplus R_2$.

41. Let R_1 and R_2 be the "congruent modulo 3" and the "congruent modulo 4" relations, respectively, on the set of integers. That is, $R_1 = \{(a, b) \mid a \equiv b \pmod{3}\}$ and $R_2 = \{(a, b) \mid a \equiv b \pmod{4}\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.
e) $R_1 \oplus R_2$.

42. List the 16 different relations on the set $\{0, 1\}$.

43. How many of the 16 different relations on $\{0, 1\}$ contain the pair $(0, 1)$?

44. Which of the 16 relations on $\{0, 1\}$, which you listed in Exercise 42, are

a) reflexive? **b)** irreflexive?
c) symmetric? **d)** antisymmetric?
e) asymmetric? **f)** transitive?

45. a) How many relations are there on the set $\{a, b, c, d\}$?
b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a)?

46. Let S be a set with n elements and let a and b be distinct elements of S. How many relations R are there on S such that

a) $(a, b) \in R$? **b)** $(a, b) \notin R$?
c) no ordered pair in R has a as its first element?
d) at least one ordered pair in R has a as its first element?
e) no ordered pair in R has a as its first element or b as its second element?
f) at least one ordered pair in R either has a as its first element or has b as its second element?

***47.** How many relations are there on a set with n elements that are

a) symmetric? **b)** antisymmetric?
c) asymmetric? **d)** irreflexive?
e) reflexive and symmetric?
f) neither reflexive nor irreflexive?

***48.** How many transitive relations are there on a set with n elements if

a) $n = 1$? **b)** $n = 2$? **c)** $n = 3$?

49. Find the error in the "proof" of the following "theorem."

"*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

50. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

a) $R \cup S$ is reflexive.
b) $R \cap S$ is reflexive.
c) $R \oplus S$ is irreflexive.
d) $R - S$ is irreflexive.
e) $S \circ R$ is reflexive.

51. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.

52. Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.

53. Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.

54. Show that the relation R on a set A is reflexive if and only if the complementary relation $\overline{R}$ is irreflexive.

55. Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

56. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 4)$, $(3, 5)$, $(4, 2)$, $(4, 5)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4. **d)** R^5.

57. Let R be a reflexive relation on a set A. Show that R^n is reflexive for all positive integers n.

***58.** Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n.

59. Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

9.2 *n*-ary Relations and Their Applications

Introduction

Relationships among elements of more than two sets often arise. For instance, there is a relationship involving the name of a student, the student's major, and the student's grade point average. Similarly, there is a relationship involving the airline, flight number, starting point, destination, departure time, and arrival time of a flight. An example of such a relationship in mathematics involves three integers, where the first integer is larger than the second integer, which is larger than the third. Another example is the betweenness relationship involving points on a line, such that three points are related when the second point is between the first and the third.

We will study relationships among elements from more than two sets in this section. These relationships are called ***n*-ary relations**. These relations are used to represent computer databases. These representations help us answer queries about the information stored in databases, such as: Which flights land at O'Hare Airport between 3 A.M. and 4 A.M.? Which students at your

9.3 Representing Relations

Introduction

In this section, and in the remainder of this chapter, all relations we study will be binary relations. Because of this, in this section and in the rest of this chapter, the word relation will always refer to a binary relation. There are many ways to represent a relation between finite sets. As we have seen in Section 9.1, one way is to list its ordered pairs. Another way to represent a relation is to use a table, as we did in Example 3 in Section 9.1. In this section we will discuss two alternative methods for representing relations. One method uses zero–one matrices. The other method uses pictorial representations called directed graphs, which we will discuss later in this section.

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the properties of these relations.

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B.) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)

The use of matrices to represent relations is illustrated in Examples 1–6.

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

EXAMPLE 2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$ ◀

The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$. Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements on the main diagonal of $\mathbf{M}_R$ are equal to 1, as shown in Figure 1. Note that the elements off the main diagonal can be either 0 or 1.

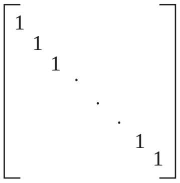


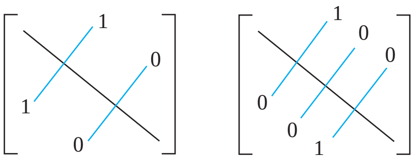
FIGURE 1 The Zero–One Matrix for a Reflexive Relation. (Off Diagonal Elements Can Be 0 or 1.)

The relation R is symmetric if $(a, b) \in R$ implies that $(b, a) \in R$. Consequently, the relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$. In terms of the entries of $\mathbf{M}_R$, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Recalling the definition of the transpose of a matrix from Section 2.6, we see that R is symmetric if and only if

$$\mathbf{M}_R = (\mathbf{M}_R)^t,$$

that is, if $\mathbf{M}_R$ is a symmetric matrix. The form of the matrix for a symmetric relation is illustrated in Figure 2(a).

The relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. The form of the matrix for an antisymmetric relation is illustrated in Figure 2(b).



(a) Symmetric (b) Antisymmetric

FIGURE 2 The Zero–One Matrices for Symmetric and Antisymmetric Relations.

EXAMPLE 3 Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because $\mathbf{M}_R$ is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric. ◀

The Boolean operations join and meet (discussed in Section 2.6) can be used to find the matrices representing the union and the intersection of two relations. Suppose that R_1 and R_2 are relations on a set A represented by the matrices $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$, respectively. The matrix representing the union of these relations has a 1 in the positions where either $\mathbf{M}_{R_1}$ or $\mathbf{M}_{R_2}$ has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$ have a 1. Thus, the matrices representing the union and intersection of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \qquad \text{and} \qquad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

EXAMPLE 4 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

◀

We now turn our attention to determining the matrix for the composite of relations. This matrix can be found using the Boolean product of the matrices (discussed in Section 2.6) for these relations. In particular, suppose that R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements, respectively. Let the zero–one matrices for $S \circ R$, R, and S be $\mathbf{M}_{S \circ R} = [t_{ij}]$, $\mathbf{M}_R = [r_{ij}]$, and $\mathbf{M}_S = [s_{ij}]$, respectively (these matrices have sizes $m \times p$, $m \times n$, and $n \times p$, respectively). The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k. From the definition of the Boolean product, this means that

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

EXAMPLE 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

◀

The matrix representing the composite of two relations can be used to find the matrix for $\mathbf{M}_{R^n}$. In particular,

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]},$$

from the definition of Boolean powers. Exercise 35 asks for a proof of this formula.

EXAMPLE 6 Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

◀

Representing Relations Using Digraphs

We have shown that a relation can be represented by listing all of its ordered pairs or by using a zero–one matrix. There is another important way of representing a relation using a pictorial representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as **directed graphs**, or **digraphs**.

DEFINITION 1 A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

EXAMPLE 7 The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is displayed in Figure 3. ◀

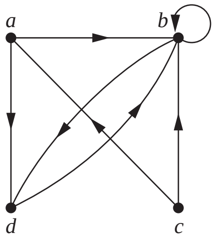


FIGURE 3
A Directed Graph.

The relation R on a set A is represented by the directed graph that has the elements of A as its vertices and the ordered pairs (a, b), where $(a, b) \in R$, as edges. This assignment sets up a one-to-one correspondence between the relations on a set A and the directed graphs with A as their set of vertices. Thus, every statement about relations corresponds to a statement about directed graphs, and vice versa. Directed graphs give a visual display of information about relations. As such, they are often used to study relations and their properties. (Note that relations from a set A to a set B can be represented by a directed graph where there is a vertex for each element of A and a vertex for each element of B, as shown in Section 9.1. However, when $A = B$, such representation provides much less insight than the digraph representations described here.) The use of directed graphs to represent relations on a set is illustrated in Examples 8–10.

EXAMPLE 8 The directed graph of the relation

$$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

on the set $\{1, 2, 3, 4\}$ is shown in Figure 4. ◀

EXAMPLE 9 What are the ordered pairs in the relation R represented by the directed graph shown in Figure 5?

Solution: The ordered pairs (x, y) in the relation are

$$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$$

Each of these pairs corresponds to an edge of the directed graph, with $(2, 2)$ and $(3, 3)$ corresponding to loops. ◀

We will study directed graphs extensively in Chapter 10.

The directed graph representing a relation can be used to determine whether the relation has various properties. For instance, a relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation. A relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (y, x) is in the relation whenever (x, y) is in the relation. Similarly, a relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices. Finally, a relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z (completing a triangle where each side is a directed edge with the correct direction).

Remark: Note that a symmetric relation can be represented by an undirected graph, which is a graph where edges do not have directions. We will study undirected graphs in Chapter 10.

EXAMPLE 10 Determine whether the relations for the directed graphs shown in Figure 6 are reflexive, symmetric, antisymmetric, and/or transitive.

Solution: Because there are loops at every vertex of the directed graph of R, it is reflexive. R is neither symmetric nor antisymmetric because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c. Finally, R is not transitive because there is an edge from a to b and an edge from b to c, but no edge from a to c.

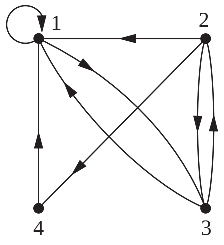


FIGURE 4 **The Directed Graph of the Relation *R*.**

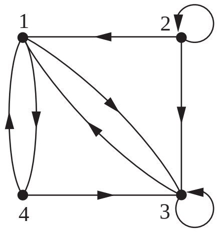
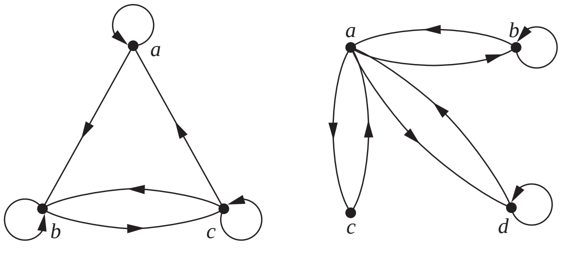


FIGURE 5 **The Directed Graph of the Relation *R*.**



(a) Directed graph of R

(b) Directed graph of S

FIGURE 6 **The Directed Graphs of the Relations *R* and *S*.**

Because loops are not present at all the vertices of the directed graph of S, this relation is not reflexive. It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S, but (c, b) does not belong to S.

Exercises

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$
b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
d) $\{(1, 3), (3, 1)\}$

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

5. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

6. How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?

7. Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

8. Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ consisting of the first 100 positive integers have if R is

a) $\{(a, b) \mid a > b\}$? **b)** $\{(a, b) \mid a \neq b\}$?
c) $\{(a, b) \mid a = b + 1\}$? **d)** $\{(a, b) \mid a = 1\}$?
e) $\{(a, b) \mid ab = 1\}$?

10. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 1000\}$ consisting of the first 1000 positive integers have if R is

a) $\{(a, b) \mid a \leq b\}$?
b) $\{(a, b) \mid a = b \pm 1\}$?
c) $\{(a, b) \mid a + b = 1000\}$?
d) $\{(a, b) \mid a + b \leq 1001\}$?
e) $\{(a, b) \mid a \neq 0\}$?

11. How can the matrix for $\overline{R}$, the complement of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

12. How can the matrix for R^{-1}, the inverse of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

13. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

a) R^{-1}. **b)** $\overline{R}$. **c)** R^2.

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$.
d) $R_1 \circ R_1$. **e)** $R_1 \oplus R_2$.

15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrices that represent

a) R^2. **b)** R^3. **c)** R^4.

16. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{R^{-1}}$, the matrix representing R^{-1}, the inverse of R?

17. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{\overline{R}}$, the matrix representing $\overline{R}$, the complement of R?

18. Draw the directed graphs representing each of the relations from Exercise 1.

19. Draw the directed graphs representing each of the relations from Exercise 2.

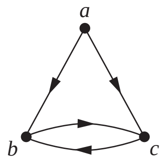
20. Draw the directed graph representing each of the relations from Exercise 3.

21. Draw the directed graph representing each of the relations from Exercise 4.

22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

23.



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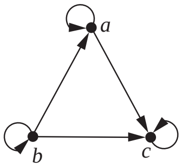

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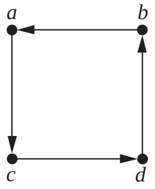

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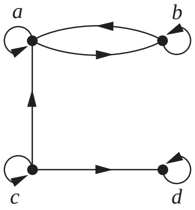

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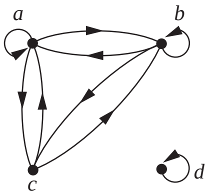

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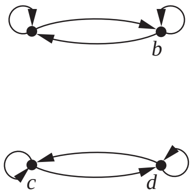

29. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?

30. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is irreflexive?

31. Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

32. Determine whether the relations represented by the directed graphs shown in Exercises 26–28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

33. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1}.

34. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the complementary relation $\overline{R}$.

35. Show that if $\mathbf{M}_R$ is the matrix representing the relation R, then $\mathbf{M}_R^{[n]}$ is the matrix representing the relation R^n.

36. Given the directed graphs representing two relations, how can the directed graph of the union, intersection, symmetric difference, difference, and composition of these relations be found?

9.4 Closures of Relations

Introduction

A computer network has data centers in Boston, Chicago, Denver, Detroit, New York, and San Diego. There are direct, one-way telephone lines from Boston to Chicago, from Boston to Detroit, from Chicago to Detroit, from Detroit to Denver, and from New York to San Diego. Let R be the relation containing (a, b) if there is a telephone line from the data center in a to that in b. How can we determine if there is some (possibly indirect) link composed of one or more telephone lines from one center to another? Because not all links are direct, such as the link from Boston to Denver that goes through Detroit, R cannot be used directly to answer this. In the language of relations, R is not transitive, so it does not contain all the pairs that can be linked. As we will show in this section, we can find all pairs of data centers that have a link by constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R. Here, S is the smallest transitive relation that contains R. This relation is called the **transitive closure** of R.

In general, let R be a relation on a set A. R may or may not have some property **P**, such as reflexivity, symmetry, or transitivity. If there is a relation S with property **P** containing R such that S is a subset of every relation with property **P** containing R, then S is called the **closure** of R with respect to **P**. (Note that the closure of a relation with respect to a property may not exist; see Exercises 15 and 35.) We will show how reflexive, symmetric, and transitive closures of relations can be found.

Closures

The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive. How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding $(2, 2)$ and $(3, 3)$ to R, because these are the only pairs of the form (a, a) that are not in R. Clearly, this new relation contains R. Furthermore, *any* reflexive relation that contains R must also contain $(2, 2)$ and $(3, 3)$. Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the **reflexive closure** of R.

As this example illustrates, given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R. The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R. We see that the reflexive closure of R equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$ is the **diagonal relation** on A. (The reader should verify this.)

EXAMPLE 1 What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?

Solution: The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \le b\}.$$

◀

The relation $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only add $(2, 1)$ and $(1, 3)$, because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R. This new relation is symmetric and contains R. Furthermore, *any* symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain $(2, 1)$ and $(1, 3)$. Consequently, this new relation is called the **symmetric closure** of R.

As this example illustrates, the symmetric closure of a relation R can be constructed by adding all ordered pairs of the form (b, a), where (a, b) is in the relation, that are not already present in R. Adding these pairs produces a relation that is symmetric, that contains R, and that is contained in any symmetric relation that contains R. The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse (defined in the preamble of Exercise 26 in Section 9.1); that is, $R \cup R^{-1}$ is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$. The reader should verify this statement.

EXAMPLE 2 What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?

Extra Examples

Solution: The symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \ne b\}.$$

This last equality follows because R contains all ordered pairs of positive integers where the first element is greater than the second element and R^{-1} contains all ordered pairs of positive integers where the first element is less than the second. ◀

Suppose that a relation R is not transitive. How can we produce a transitive relation that contains R such that this new relation is contained within any transitive relation that contains R? Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation? Consider the relation

$R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $\{1, 2, 3, 4\}$. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R. The pairs of this form not in R are $(1, 2)$, $(2, 3)$, $(2, 4)$, and $(3, 1)$. Adding these pairs does *not* produce a transitive relation, because the resulting relation contains $(3, 1)$ and $(1, 4)$ but does not contain $(3, 4)$. This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure. The rest of this section develops algorithms for constructing transitive closures. As will be shown later in this section, the transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Paths in Directed Graphs

We will see that representing relations by directed graphs helps in the construction of transitive closures. We now introduce some terminology that we will use for this purpose.

A path in a directed graph is obtained by traversing along edges (in the same direction as indicated by the arrow on the edge).

DEFINITION 1

A *path* from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), $(x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length n*. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

EXAMPLE 3 Which of the following are paths in the directed graph shown in Figure 1: a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e? What are the lengths of those that are paths? Which of the paths in this list are circuits?

Solution: Because each of (a, b), (b, e), and (e, d) is an edge, a, b, e, d is a path of length three. Because (c, d) is not an edge, a, e, c, d, b is not a path. Also, b, a, c, b, a, a, b is a path of length six because (b, a), (a, c), (c, b), (b, a), (a, a), and (a, b) are all edges. We see that d, c is a path of length one, because (d, c) is an edge. Also c, b, a is a path of length two, because (c, b) and (b, a) are edges. All of (e, b), (b, a), (a, b), (b, a), (a, b), and (b, e) are edges, so e, b, a, b, a, b, e is a path of length six.

The two paths b, a, c, b, a, a, b and e, b, a, b, a, b, e are circuits because they begin and end at the same vertex. The paths a, b, e, d; c, b, a; and d, c are not circuits. ◀

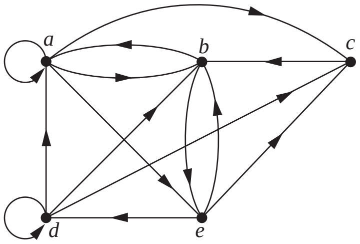

FIGURE 1 **A Directed Graph.**

The term *path* also applies to relations. Carrying over the definition from directed graphs to relations, there is a **path** from a to b in R if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R$, $(x_1, x_2) \in R, \ldots,$ and $(x_{n-1}, b) \in R$. Theorem 1 can be obtained from the definition of a path in a relation.

THEOREM 1 Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof: We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem is true when $n = 1$.

Assume that the theorem is true for the positive integer n. This is the inductive hypothesis. There is a path of length $n + 1$ from a to b if and only if there is an element $c \in A$ such that there is a path of length one from a to c, so $(a, c) \in R$, and a path of length n from c to b, that is, $(c, b) \in R^n$. Consequently, by the inductive hypothesis, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element if and only if $(a, b) \in R^{n+1}$. Therefore, there is a path of length $n + 1$ from a to b if and only if $(a, b) \in R^{n+1}$. This completes the proof. ◁

Transitive Closures

We now show that finding the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path. With this in mind, we define a new relation.

DEFINITION 2 Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b, it follows that R^* is the union of all the sets R^n. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

The connectivity relation is useful in many models.

EXAMPLE 4 Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

Solution: The relation R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ and $(c, b) \in R$, that is, if there is a person c such that a has met c and c has met b. Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1, x_1 has met $x_2, \ldots,$ and x_{n-1} has met b.

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence. (There are many interesting conjectures about R^*. Do you think that this connectivity relation includes the pair with you as the first element and the president of Mongolia as the second element? We will use graphs to model this application in Chapter 10.) ◀

EXAMPLE 5 Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^*?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most $n - 1$ changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary. (The reader should verify these statements.) ◀

EXAMPLE 6 Let R be the relation on the set of all states in the United States that contains (a, b) if state a and state b have a common border. What is R^n, where n is a positive integer? What is R^*?

Solution: The relation R^n consists of the pairs (a, b), where it is possible to go from state a to state b by crossing exactly n state borders. R^* consists of the ordered pairs (a, b), where it is possible to go from state a to state b crossing as many borders as necessary. (The reader should verify these statements.) The only ordered pairs not in R^* are those containing states that are not connected to the continental United States (i.e., those pairs containing Alaska or Hawaii). ◀

Theorem 2 shows that the transitive closure of a relation and the associated connectivity relation are the same.

THEOREM 2 The transitive closure of a relation R equals the connectivity relation R^*.

Proof: Note that R^* contains R by definition. To show that R^* is the transitive closure of R we must also show that R^* is transitive and that $R^* \subseteq S$ whenever S is a transitive relation that contains R.

First, we show that R^* is transitive. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. We obtain a path from a to c by starting with the path from a to b and following it with the path from b to c. Hence, $(a, c) \in R^*$. It follows that R^* is transitive.

Now suppose that S is a transitive relation containing R. Because S is transitive, S^n also is transitive (the reader should verify this) and $S^n \subseteq S$ (by Theorem 1 of Section 9.1). Furthermore, because

$$S^* = \bigcup_{k=1}^{\infty} S^k$$

and $S^k \subseteq S$, it follows that $S^* \subseteq S$. Now note that if $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S. Consequently, $R^* \subseteq S^* \subseteq S$. Thus, any transitive relation that contains R must also contain R^*. Therefore, R^* is the transitive closure of R. ◁

Now that we know that the transitive closure equals the connectivity relation, we turn our attention to the problem of computing this relation. We do not need to examine arbitrarily long paths to determine whether there is a path between two vertices in a finite directed graph. As Lemma 1 shows, it is sufficient to examine paths containing no more than n edges, where n is the number of elements in the set.

LEMMA 1 Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.

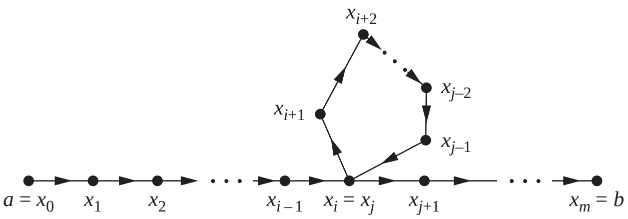

FIGURE 2 Producing a Path with Length Not Exceeding *n*.

Proof: Suppose there is a path from a to b in R. Let m be the length of the shortest such path. Suppose that $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$, is such a path.

Suppose that $a = b$ and that $m > n$, so that $m \geq n + 1$. By the pigeonhole principle, because there are n vertices in A, among the m vertices $x_0, x_1, \ldots, x_{m-1}$, at least two are equal (see Figure 2).

Suppose that $x_i = x_j$ with $0 \leq i < j \leq m - 1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from a to b, leaving a path, namely, $x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_{m-1}, x_m$, from a to b of shorter length. Hence, the path of shortest length must have length less than or equal to n.

The case where $a \neq b$ is left as an exercise for the reader. ◁

From Lemma 1, we see that the transitive closure of R is the union of R, R^2, $R^3, \ldots,$ and R^n. This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i, for some positive integer i with $i \leq n$. Because

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

and the zero–one matrix representing a union of relations is the join of the zero–one matrices of these relations, the zero–one matrix for the transitive closure is the join of the zero–one matrices of the first n powers of the zero–one matrix of R.

THEOREM 3 Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

EXAMPLE 7 Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: By Theorem 3, it follows that the zero–one matrix of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Theorem 3 can be used as a basis for an algorithm for computing the matrix of the relation R^*. To find this matrix, the successive Boolean powers of $\mathbf{M}_R$, up to the nth power, are computed. As each power is calculated, its join with the join of all smaller powers is formed. When this is done with the nth power, the matrix for R^* has been found. This procedure is displayed as Algorithm 1.

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

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**procedure** *transitive closure* ($\mathbf{M}_R$ : zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
**for** $i := 2$ **to** $n$
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
**return** $\mathbf{B}$ {$\mathbf{B}$ is the zero–one matrix for $R^*$}
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We can easily find the number of bit operations used by Algorithm 1 to determine the transitive closure of a relation. Computing the Boolean powers $\mathbf{M}_R, \mathbf{M}_R^{[2]}, \ldots, \mathbf{M}_R^{[n]}$ requires that $n-1$ Boolean products of $n \times n$ zero–one matrices be found. Each of these Boolean products can be found using $n^2(2n-1)$ bit operations. Hence, these products can be computed using $n^2(2n-1)(n-1)$ bit operations.

To find $\mathbf{M}_{R^*}$ from the n Boolean powers of $\mathbf{M}_R$, $n-1$ joins of zero–one matrices need to be found. Computing each of these joins uses n^2 bit operations. Hence, $(n-1)n^2$ bit operations are used in this part of the computation. Therefore, when Algorithm 1 is used, the matrix of the transitive closure of a relation on a set with n elements can be found using $n^2(2n-1)(n-1) + (n-1)n^2 = 2n^3(n-1)$, which is $O(n^4)$ bit operations. The remainder of this section describes a more efficient algorithm for finding transitive closures.

Warshall's Algorithm

Warshall's algorithm, named after Stephen Warshall, who described this algorithm in 1960, is an efficient method for computing the transitive closure of a relation. Algorithm 1 can find the transitive closure of a relation on a set with n elements using $2n^3(n-1)$ bit operations. However, the transitive closure can be found by Warshall's algorithm using only $2n^3$ bit operations.

Remark: Warshall's algorithm is sometimes called the Roy–Warshall algorithm, because Bernard Roy described this algorithm in 1959.

Suppose that R is a relation on a set with n elements. Let $v_1, v_2, \ldots, v_n$ be an arbitrary listing of these n elements. The concept of the **interior vertices** of a path is used in Warshall's algorithm. If $a, x_1, x_2, \ldots, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, \ldots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. For instance, the interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, and b. The interior vertices of a, c, d, a, f, b are c, d, a, and f. (Note that the first vertex in the path is not an interior vertex unless it is visited again by the path, except as the last vertex. Similarly, the last vertex in the path is not an interior vertex unless it was visited previously by the path, except as the first vertex.)

Warshall's algorithm is based on the construction of a sequence of zero–one matrices. These matrices are $\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_n$, where $\mathbf{W}_0 = \mathbf{M}_R$ is the zero–one matrix of this relation, and $\mathbf{W}_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_k\}$ (the first k vertices in the list) and is 0 otherwise. (The first and last vertices in the path may be outside the set of the first k vertices in the list.) Note that $\mathbf{W}_n = \mathbf{M}_{R^*}$, because the (i, j)th entry of $\mathbf{M}_{R^*}$ is 1 if and only if there is a path from v_i to v_j, with all interior vertices in the set $\{v_1, v_2, \ldots, v_n\}$ (but these are the only vertices in the directed graph). Example 8 illustrates what the matrix $\mathbf{W}_k$ represents.

EXAMPLE 8 Let R be the relation with directed graph shown in Figure 3. Let a, b, c, d be a listing of the elements of the set. Find the matrices $\mathbf{W}_0$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, and $\mathbf{W}_4$. The matrix $\mathbf{W}_4$ is the transitive closure of R.

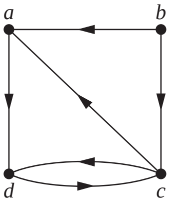


FIGURE 3 The Directed Graph of the Relation *R*.

Solution: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, and $v_4 = d$. $\mathbf{W}_0$ is the matrix of the relation. Hence,

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\mathbf{W}_1$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex. Note that all paths of length one can still be used because they have no interior vertices. Also, there is now an allowable path from b to d, namely, b, a, d. Hence,

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\mathbf{W}_2$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$ as its interior vertices, if any. Because there are no edges that have b as a terminal vertex, no new paths are obtained when we permit b to be an interior vertex. Hence, $\mathbf{W}_2 = \mathbf{W}_1$.

STEPHEN WARSHALL (1935–2006) Stephen Warshall, born in New York City, went to public school in Brooklyn. He attended Harvard University, receiving his degree in mathematics in 1956. He never received an advanced degree, because at that time no programs were available in his areas of interest. However, he took graduate courses at several different universities and contributed to the development of computer science and software engineering.

After graduating from Harvard, Warshall worked at ORO (Operation Research Office), which was set up by Johns Hopkins to do research and development for the U.S. Army. In 1958 he left ORO to take a position at a company called Technical Operations, where he helped build a research and development laboratory for military software projects. In 1961 he left Technical Operations to found Massachusetts Computer Associates. Later, this company became part of Applied Data Research (ADR). After the merger, Warshall sat on the board of directors of ADR and managed a variety of projects and organizations. He retired from ADR in 1982.

During his career Warshall carried out research and development in operating systems, compiler design, language design, and operations research. In the 1971–1972 academic year he presented lectures on software engineering at French universities. There is an interesting anecdote about his proof that the transitive closure algorithm, now known as Warshall's algorithm, is correct. He and a colleague at Technical Operations bet a bottle of rum on who first could determine whether this algorithm always works. Warshall came up with his proof overnight, winning the bet and the rum, which he shared with the loser of the bet. Because Warshall did not like sitting at a desk, he did much of his creative work in unconventional places, such as on a sailboat in the Indian Ocean or in a Greek lemon orchard.

$\mathbf{W}_3$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$, and/or $v_3 = c$ as its interior vertices, if any. We now have paths from d to a, namely, d, c, a, and from d to d, namely, d, c, d. Hence,

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, $\mathbf{W}_4$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has $v_1 = a$, $v_2 = b$, $v_3 = c$, and/or $v_4 = d$ as interior vertices, if any. Because these are all the vertices of the graph, this entry is 1 if and only if there is a path from v_i to v_j. Hence,

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

This last matrix, $\mathbf{W}_4$, is the matrix of the transitive closure. ◀

Warshall's algorithm computes $\mathbf{M}_{R^*}$ by efficiently computing $\mathbf{W}_0 = \mathbf{M}_R, \mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n = \mathbf{M}_{R^*}$. This observation shows that we can compute $\mathbf{W}_k$ directly from $\mathbf{W}_{k-1}$: There is a path from v_i to v_j with no vertices other than $v_1, v_2, \ldots, v_k$ as interior vertices if and only if either there is a path from v_i to v_j with its interior vertices among the first $k - 1$ vertices in the list, or there are paths from v_i to v_k and from v_k to v_j that have interior vertices only among the first $k - 1$ vertices in the list. That is, either a path from v_i to v_j already existed before v_k was permitted as an interior vertex, or allowing v_k as an interior vertex produces a path that goes from v_i to v_k and then from v_k to v_j. These two cases are shown in Figure 4.

The first type of path exists if and only if $w_{ij}^{[k-1]} = 1$, and the second type of path exists if and only if both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. Hence, $w_{ij}^{[k]}$ is 1 if and only if either $w_{ij}^{[k-1]}$ is 1 or both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. This gives us Lemma 2.

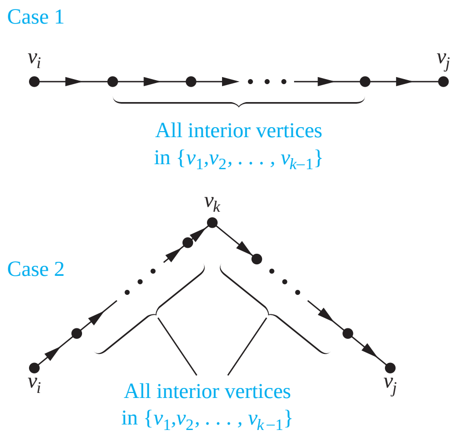


FIGURE 4 Adding v_k to the Set of Allowable Interior Vertices.

LEMMA 2 Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

Lemma 2 gives us the means to compute efficiently the matrices $\mathbf{W}_k$, $k = 1, 2, \ldots, n$. We display the pseudocode for Warshall's algorithm, using Lemma 2, as Algorithm 2.

ALGORITHM 2 Warshall Algorithm.

```
procedure Warshall ($\mathbf{M}_R$ : $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ to $n$
    for $i := 1$ to $n$
        for $j := 1$ to $n$
        $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$
```

The computational complexity of Warshall's algorithm can easily be computed in terms of bit operations. To find the entry $w_{ij}^{[k]}$ from the entries $w_{ij}^{[k-1]}$, $w_{ik}^{[k-1]}$, and $w_{kj}^{[k-1]}$ using Lemma 2 requires two bit operations. To find all n^2 entries of $\mathbf{W}_k$ from those of $\mathbf{W}_{k-1}$ requires $2n^2$ bit operations. Because Warshall's algorithm begins with $\mathbf{W}_0 = \mathbf{M}_R$ and computes the sequence of n zero–one matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n = \mathbf{M}_{R^*}$, the total number of bit operations used is $n \cdot 2n^2 = 2n^3$.

Exercises

1. Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 2)$, and $(3, 0)$. Find the

a) reflexive closure of R. **b)** symmetric closure of R.

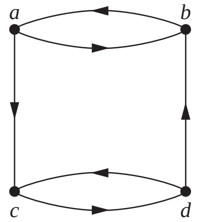
2. Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

3. Let R be the relation $\{(a, b) \mid a$ divides $b\}$ on the set of integers. What is the symmetric closure of R?

4. How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

In Exercises 5–7 draw the directed graph of the reflexive closure of the relations with the directed graph shown.

5.



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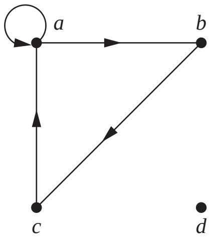

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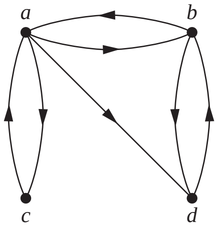


8. How can the directed graph representing the symmetric closure of a relation on a finite set be constructed from the directed graph for this relation?

9. Find the directed graphs of the symmetric closures of the relations with directed graphs shown in Exercises 5–7.

10. Find the smallest relation containing the relation in Example 2 that is both reflexive and symmetric.

11. Find the directed graph of the smallest relation that is both reflexive and symmetric that contains each of the relations with directed graphs shown in Exercises 5–7.

12. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the reflexive closure of R is $\mathbf{M}_R \vee \mathbf{I}_n$.

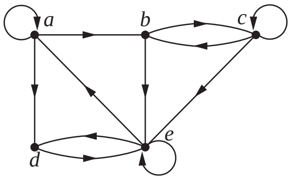
13. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^t$.

14. Show that the closure of a relation R with respect to a property **P**, if it exists, is the intersection of all the relations with property **P** that contain R.

15. When is it possible to define the "irreflexive closure" of a relation R, that is, a relation that contains R, is irreflexive, and is contained in every irreflexive relation that contains R?

16. Determine whether these sequences of vertices are paths in this directed graph.

a) a, b, c, e
b) b, e, c, b, e
c) a, a, b, e, d, e
d) b, c, e, d, a, a, b
e) b, c, c, b, e, d, e, d
f) $a, a, b, b, c, c, b, e, d$



17. Find all circuits of length three in the directed graph in Exercise 16.

18. Determine whether there is a path in the directed graph in Exercise 16 beginning at the first vertex given and ending at the second vertex given.

a) a, b **b)** b, a **c)** b, b
d) a, e **e)** b, d **f)** c, d
g) d, d **h)** e, a **i)** e, c

19. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3)$, $(2, 4)$, $(3, 1)$, $(3, 5)$, $(4, 3)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4.
d) R^5. **e)** R^6. **f)** R^*.

20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

21. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

22. Suppose that the relation R is reflexive. Show that R^* is reflexive.

23. Suppose that the relation R is symmetric. Show that R^* is symmetric.

24. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

25. Use Algorithm 1 to find the transitive closures of these relations on $\{1, 2, 3, 4\}$.

a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
d) $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$

26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.

a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$
b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$
c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$
d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$

27. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 25.

28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.

29. Find the smallest relation containing the relation $\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is

a) reflexive and transitive.
b) symmetric and transitive.
c) reflexive, symmetric, and transitive.

30. Finish the proof of the case when $a \neq b$ in Lemma 1.

31. Algorithms have been devised that use $O(n^{2.8})$ bit operations to compute the Boolean product of two $n \times n$ zero–one matrices. Assuming that these algorithms can be used, give big-O estimates for the number of bit operations using Algorithm 1 and using Warshall's algorithm to find the transitive closure of a relation on a set with n elements.

***32.** Devise an algorithm using the concept of interior vertices in a path to find the length of the shortest path between two vertices in a directed graph, if such a path exists.

33. Adapt Algorithm 1 to find the reflexive closure of the transitive closure of a relation on a set with n elements.

34. Adapt Warshall's algorithm to find the reflexive closure of the transitive closure of a relation on a set with n elements.

35. Show that the closure with respect to the property **P** of the relation $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ on the set $\{0, 1, 2\}$ does not exist if **P** is the property

a) "is not reflexive."
b) "has an odd number of elements."

9.5 Equivalence Relations

Introduction

In some programming languages the names of variables can contain an unlimited number of characters. However, there is a limit on the number of characters that are checked when a compiler determines whether two variables are equal. For instance, in traditional C, only the first eight characters of a variable name are checked by the compiler. (These characters are uppercase or lowercase letters, digits, or underscores.) Consequently, the compiler considers strings longer than eight characters that agree in their first eight characters the same. Let R be the relation on the set of strings of characters such that sRt, where s and t are two strings, if s and t are at least eight characters long and the first eight characters of s and t agree, or $s = t$. It is easy to see that R is reflexive, symmetric, and transitive. Moreover, R divides the set of all strings into classes, where all strings in a particular class are considered the same by a compiler for traditional C.

The integers a and b are related by the "congruence modulo 4" relation when 4 divides $a - b$. We will show later that this relation is reflexive, symmetric, and transitive. It is not hard to see that a is related to b if and only if a and b have the same remainder when divided by 4. It follows that this relation splits the set of integers into four different classes. When we care only what remainder an integer leaves when it is divided by 4, we need only know which class it is in, not its particular value.

These two relations, R and congruence modulo 4, are examples of equivalence relations, namely, relations that are reflexive, symmetric, and transitive. In this section we will show that such relations split sets into disjoint classes of equivalent elements. Equivalence relations arise whenever we care only whether an element of a set is in a certain class of elements, instead of caring about its particular identity.

Equivalence Relations

Links

In this section we will study relations with a particular combination of properties that allows them to be used to relate objects that are similar in some way.

DEFINITION 1

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence relations are important in every branch of mathematics!

Equivalence relations are important throughout mathematics and computer science. One reason for this is that in an equivalence relation, when two elements are related it makes sense to say they are equivalent.

DEFINITION 2

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

For the notion of equivalent elements to make sense, every element should be equivalent to itself, as the reflexive property guarantees for an equivalence relation. It makes sense to say that a and b are related (not just that a is related to b) by an equivalence relation, because when a is related to b, by the symmetric property, b is related to a. Furthermore, because an equivalence relation is transitive, if a and b are equivalent and b and c are equivalent, it follows that a and c are equivalent.

Examples 1–5 illustrate the notion of an equivalence relation.

EXAMPLE 1 Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. In Section 9.1 we showed that R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation. ◀

EXAMPLE 2 Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

Extra Examples

Solution: Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation. ◀

One of the most widely used equivalence relations is congruence modulo m, where m is an integer greater than 1.

EXAMPLE 3 **Congruence Modulo m** Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall from Section 4.1 that $a \equiv b \pmod{m}$ if and only if m divides $a - b$. Note that $a - a = 0$ is divisible by m, because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo m is reflexive. Now suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$. Hence, congruence modulo m is symmetric. Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - b = km$ and $b - c = lm$. Adding these two equations shows that $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c \pmod{m}$. Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an equivalence relation. ◀

EXAMPLE 4 Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Because $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive. Next, suppose that aRb, so that $l(a) = l(b)$. Then bRa, because $l(b) = l(a)$. Hence, R is symmetric. Finally, suppose that aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(a) = l(c)$, so aRc. Consequently, R is transitive. Because R is reflexive, symmetric, and transitive, it is an equivalence relation. ◀

EXAMPLE 5 Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s. For example, let $n = 3$ and let S be the set of all bit strings. Then sR_3t either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_301$ and $00111R_300101$, but $01\not R_3010$ and $01011\not R_301110$.

Show that for every set S of strings and every positive integer n, R_n is an equivalence relation on S.

Solution: The relation R_n is reflexive because $s = s$, so that sR_ns whenever s is a string in S. If sR_nt, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that tR_ns. We conclude that R_n is symmetric.

Now suppose that sR_nt and tR_nu. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters (because in this case we know that s, t, and u are all at least n characters long and both s and u begin with the same n characters as t does). Consequently, R_n is transitive. It follows that R_n is an equivalence relation. ◀

In Examples 6 and 7 we look at two relations that are not equivalence relations.

EXAMPLE 6 Show that the "divides" relation is the set of positive integers in not an equivalence relation.

Solution: By Examples 9 and 15 in Section 9.1, we know that the "divides" relation is reflexive and transitive. However, by Example 12 in Section 9.1, we know that this relation is not symmetric (for instance, $2 \mid 4$ but $4 \nmid 2$). We conclude that the "divides" relation on the set of positive integers is not an equivalence relation. ◀

EXAMPLE 7 Let R be the relation on the set of real numbers such that $x R y$ if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbf{R}$. R is symmetric, for if $x R y$, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that $y R x$. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8$, $y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1$, $|y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8 \, R \, 1.9$, $1.9 \, R \, 1.1$, but $2.8 \not R \, 1.1$. ◀

Equivalence Classes

Let A be the set of all students in your school who graduated from high school. Consider the relation R on A that consists of all pairs (x, y), where x and y graduated from the same high school. Given a student x, we can form the set of all students equivalent to x with respect to R. This set consists of all students who graduated from the same high school as x did. This subset of A is called an equivalence class of the relation.

DEFINITION 3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

EXAMPLE 8 What is the equivalence class of an integer for the equivalence relation of Example 1?

Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$. For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0] = \{0\}$. ◀

EXAMPLE 9 What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution: The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod 4$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}.$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}.$$ ◀

In Example 9 the equivalence classes of 0 and 1 with respect to congruence modulo 4 were found. Example 9 can easily be generalized, replacing 4 with any positive integer m. The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo** m. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a - 2m, a - m, a, a + m, a + 2m, \ldots\}$. For instance, from Example 9 it follows that $[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ and $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.

EXAMPLE 10 What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 from Example 5 on the set of all bit strings? (Recall that $s R_3 t$ if and only if s and t are bit strings with $s = t$ or s and t are strings of at least three bits that start with the same three bits.)

Solution: The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on. Consequently,

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\}.$$ ◀

EXAMPLE 11 **Identifiers in the C Programming Language** In the C programming language, an **identifier** is the name of a variable, a function, or another type of entity. Each identifier is a nonempty string of characters where each character is a lowercase or an uppercase English letter, a digit, or an underscore, and the first character is a lowercase or an uppercase English letter. Identifiers can be any length. This allows developers to use as many characters as they want to name an entity, such as a variable. However, for compilers for some versions of C, there is a limit on the number of characters checked when two names are compared to see whether they refer to the same thing. For example, Standard C compilers consider two identifiers the same when they agree in their first 31 characters. Consequently, developers must be careful not to use identifiers with the same initial 31 characters for different things. We see that two identifiers are considered the same when they are related by the relation R_{31} in Example 5. Using Example 5, we know that R_{31}, on the set of all identifiers in Standard C, is an equivalence relation.

What are the equivalence classes of each of the identifiers Number_of_tropical_storms, Number_of_named_tropical_storms, and Number_of_named_tropical_storms_in_the_Atlantic_in_2005?

Solution: Note that when an identifier is less than 31 characters long, by the definition of R_{31}, its equivalence class contains only itself. Because the identifier Number_of_tropical_storms is 25 characters long, its equivalence class contains exactly one element, namely, itself.

The identifier Number_of_named_tropical_storms is exactly 31 characters long. An identifier is equivalent to it when it starts with these same 31 characters. Consequently, every identifier at least 31 characters long that starts with Number_of_named_tropical_storms is equivalent to this identifier. It follows that the equivalence class of Number_of_named_tropical_storms is the set of all identifiers that begin with the 31 characters Number_of_named_tropical_storms.

An identifier is equivalent to the Number_of_named_tropical_storms_in_the_Atlantic_in_2005 if and only if it begins with its first 31 characters. Because these characters are Number_of_named_tropical_storms, we see that an identifier is equivalent to Number_of_named_tropical_storms_in_the_Atlantic_in_2005 if and only if it is equivalent to Number_of_named_tropical_storms. It follows that these last two identifiers have the same equivalence class. ◀

Equivalence Classes and Partitions

Let A be the set of students at your school who are majoring in exactly one subject, and let R be the relation on A consisting of pairs (x, y), where x and y are students with the same major. Then R is an equivalence relation, as the reader should verify. We can see that R splits all students in A into a collection of disjoint subsets, where each subset contains students with a specified major. For instance, one subset contains all students majoring (just) in computer science, and a second subset contains all students majoring in history. Furthermore, these subsets are equivalence classes of R. This example illustrates how the equivalence classes of an equivalence relation partition a set into disjoint, nonempty subsets. We will make these notions more precise in the following discussion.

Let R be a relation on the set A. Theorem 1 shows that the equivalence classes of two elements of A are either identical or disjoint.

THEOREM 1

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Proof: We first show that (i) implies (ii). Assume that aRb. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc. Because aRb and R is symmetric, we know that bRa. Furthermore, because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

Second, we will show that (ii) implies (iii). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

Next, we will show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent. ◁

We are now in a position to show how an equivalence relation *partitions* a set. Let R be an equivalence relation on a set A. The union of the equivalence classes of R is all of A, because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

In addition, from Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset,$$

when $[a]_R \neq [b]_R$.

Recall that an *index set* is a set whose members label, or index, the elements of a set.

These two observations show that the equivalence classes form a partition of A, because they split A into disjoint subsets. More precisely, a **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if

$$A_i \neq \emptyset \text{ for } i \in I,$$

$$A_i \cap A_j = \emptyset \text{ when } i \neq j,$$

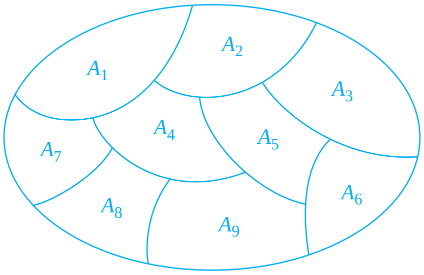


FIGURE 1 **A Partition of a Set.**

and

$$\bigcup_{i \in I} A_i = S.$$

(Here the notation $\bigcup_{i \in I} A_i$ represents the union of the sets A_i for all $i \in I$.) Figure 1 illustrates the concept of a partition of a set.

EXAMPLE 12 Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S. ◀

We have seen that the equivalence classes of an equivalence relation on a set form a partition of the set. The subsets in this partition are the equivalence classes. Conversely, every partition of a set can be used to form an equivalence relation. Two elements are equivalent with respect to this relation if and only if they are in the same subset of the partition.

To see this, assume that $\{A_i \mid i \in I\}$ is a partition on S. Let R be the relation on S consisting of the pairs (x, y), where x and y belong to the same subset A_i in the partition. To show that R is an equivalence relation we must show that R is reflexive, symmetric, and transitive.

We see that $(a, a) \in R$ for every $a \in S$, because a is in the same subset as itself. Hence, R is reflexive. If $(a, b) \in R$, then b and a are in the same subset of the partition, so that $(b, a) \in R$ as well. Hence, R is symmetric. If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset X in the partition, and b and c are in the same subset Y of the partition. Because the subsets of the partition are disjoint and b belongs to X and Y, it follows that $X = Y$. Consequently, a and c belong to the same subset of the partition, so $(a, c) \in R$. Thus, R is transitive.

It follows that R is an equivalence relation. The equivalence classes of R consist of subsets of S containing related elements, and by the definition of R, these are the subsets of the partition. Theorem 2 summarizes the connections we have established between equivalence relations and partitions.

THEOREM 2 Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example 13 shows how to construct an equivalence relation from a partition.

EXAMPLE 13 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 12.

Solution: The subsets in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition. The pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair (6, 6) belongs to R because $\{6\}$ is an equivalence class. No pair other than those listed belongs to R. ◀

The congruence classes modulo m provide a useful illustration of Theorem 2. There are m different congruence classes modulo m, corresponding to the m different remainders possible when an integer is divided by m. These m congruence classes are denoted by $[0]_m, [1]_m, \ldots, [m-1]_m$. They form a partition of the set of integers.

EXAMPLE 14 What are the sets in the partition of the integers arising from congruence modulo 4?

Solution: There are four congruence classes, corresponding to $[0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$. They are the sets

$$\begin{aligned}
[0]_4 &= \{\ldots, -8, -4, 0, 4, 8, \ldots\},\\
[1]_4 &= \{\ldots, -7, -3, 1, 5, 9, \ldots\},\\
[2]_4 &= \{\ldots, -6, -2, 2, 6, 10, \ldots\},\\
[3]_4 &= \{\ldots, -5, -1, 3, 7, 11, \ldots\}.
\end{aligned}$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these congruence classes form a partition. ◀

We now provide an example of a partition of the set of all strings arising from an equivalence relation on this set.

EXAMPLE 15 Let R_3 be the relation from Example 5. What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings? (Recall that $s R_3 t$, where s and t are bit strings, if $s = t$ or s and t are bit strings with at least three bits that agree in their first three bits.)

Solution: Note that every bit string of length less than three is equivalent only to itself. Hence $[\lambda]_{R_3} = \{\lambda\}$, $[0]_{R_3} = \{0\}$, $[1]_{R_3} = \{1\}$, $[00]_{R_3} = \{00\}$, $[01]_{R_3} = \{01\}$, $[10]_{R_3} = \{10\}$, and $[11]_{R_3} = \{11\}$. Note that every bit string of length three or more is equivalent to one of the eight bit strings 000, 001, 010, 011, 100, 101, 110, and 111. We have

$$\begin{aligned}
[000]_{R_3} &= \{000, 0000, 0001, 00000, 00001, 00010, 00011, \ldots\},\\
[001]_{R_3} &= \{001, 0010, 0011, 00100, 00101, 00110, 00111, \ldots\},\\
[010]_{R_3} &= \{010, 0100, 0101, 01000, 01001, 01010, 01011, \ldots\},\\
[011]_{R_3} &= \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\},\\
[100]_{R_3} &= \{100, 1000, 1001, 10000, 10001, 10010, 10011, \ldots\},\\
[101]_{R_3} &= \{101, 1010, 1011, 10100, 10101, 10110, 10111, \ldots\},\\
[110]_{R_3} &= \{110, 1100, 1101, 11000, 11001, 11010, 11011, \ldots\},\\
[111]_{R_3} &= \{111, 1110, 1111, 11100, 11101, 11110, 11111, \ldots\}.
\end{aligned}$$

These 15 equivalence classes are disjoint and every bit string is in exactly one of them. As Theorem 2 tells us, these equivalence classes partition the set of all bit strings. ◀

Exercises

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$

b) $\{(a, b) \mid a$ and b have the same parents$\}$

c) $\{(a, b) \mid a$ and b share a common parent$\}$

d) $\{(a, b) \mid a$ and b have met$\}$

e) $\{(a, b) \mid a$ and b speak a common language$\}$

3. Which of these relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$

b) $\{(f, g) \mid f(0) = g(0)$ or $f(1) = g(1)\}$

c) $\{(f, g) \mid f(x) - g(x) = 1$ for all $x \in \mathbf{Z}\}$

d) $\{(f, g) \mid$ for some $C \in \mathbf{Z}$, for all $x \in \mathbf{Z}$, $f(x) - g(x) = C\}$

e) $\{(f, g) \mid f(0) = g(1)$ and $f(1) = g(0)\}$

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

6. Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.

7. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of **F** and of **T**?

8. Let R be the relation on the set of all sets of real numbers such that $S\,R\,T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\mathbf{Z}$?

9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.

a) Show that R is an equivalence relation on A.

b) What are the equivalence classes of R?

10. Suppose that A is a nonempty set and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.

11. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

12. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.

14. Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n, the nth characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.

15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

17. (*Requires calculus*)

a) Show that the relation R on the set of all differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.

b) Which functions are in the same equivalence class as the function $f(x) = x^2$?

18. (*Requires calculus*)

a) Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the nth derivative of $f(x)$.]

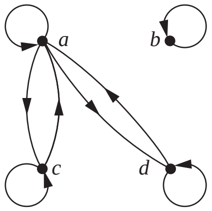
b) Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?

19. Let R be the relation on the set of all URLs (or Web addresses) such that $x\,R\,y$ if and only if the Web page at x is the same as the Web page at y. Show that R is an equivalence relation.

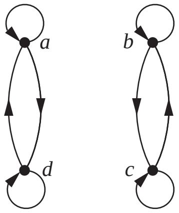
20. Let R be the relation on the set of all people who have visited a particular Web page such that $x\,R\,y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

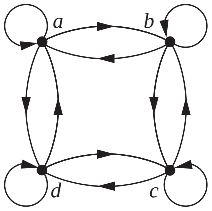
21.



22.



23.



24. Determine whether the relations represented by these zero–one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. Show that the relation R on the set of all bit strings such that $s\ R\ t$ if and only if s and t contain the same number of 1s is an equivalence relation.

26. What are the equivalence classes of the equivalence relations in Exercise 1?

27. What are the equivalence classes of the equivalence relations in Exercise 2?

28. What are the equivalence classes of the equivalence relations in Exercise 3?

29. What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010 **b)** 1011 **c)** 11111 **d)** 01010101

31. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

32. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

33. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_4 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_4 if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

34. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_5 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_5 if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2? **b)** 3? **c)** 6? **d)** -3?

36. What is the congruence class $[4]_m$ when m is

a) 2? **b)** 3? **c)** 6? **d)** 8?

37. Give a description of each of the congruence classes modulo 6.

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

a) *No* **b)** *Yes* **c)** *Help*

39. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [*Hint:* Look at the difference $a - b$ corresponding to (a, b).]

40. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 16?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 16. [*Hint:* Look at the ratio a/b corresponding to (a, b).]

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ **b)** $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

e) the set of bit strings that contain $3k$ ones for some nonnegative integer k; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k.

44. Which of these collections of subsets are partitions of the set of integers?

a) the set of even integers and the set of odd integers

b) the set of positive integers and the set of negative integers

c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

d) the set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6

45. Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

a) the set of pairs (x, y), where x or y is odd; the set of pairs (x, y), where x is even; and the set of pairs (x, y), where y is even

b) the set of pairs (x, y), where both x and y are odd; the set of pairs (x, y), where exactly one of x and y is odd; and the set of pairs (x, y), where both x and y are even

c) the set of pairs (x, y), where x is positive; the set of pairs (x, y), where y is positive; and the set of pairs (x, y), where both x and y are negative

d) the set of pairs (x, y), where $3 \mid x$ and $3 \mid y$; the set of pairs (x, y), where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y), where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y), where $3 \nmid x$ and $3 \nmid y$

e) the set of pairs (x, y), where $x > 0$ and $y > 0$; the set of pairs (x, y), where $x > 0$ and $y \leq 0$; the set of pairs (x, y), where $x \leq 0$ and $y > 0$; and the set of pairs (x, y), where $x \leq 0$ and $y \leq 0$

f) the set of pairs (x, y), where $x \neq 0$ and $y \neq 0$; the set of pairs (x, y), where $x = 0$ and $y \neq 0$; and the set of pairs (x, y), where $x \neq 0$ and $y = 0$

46. Which of these are partitions of the set of real numbers?

a) the negative real numbers, $\{0\}$, the positive real numbers

b) the set of irrational numbers, the set of rational numbers

c) the set of intervals $[k, k+1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

d) the set of intervals $(k, k+1)$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

e) the set of intervals $(k, k+1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

f) the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1)$

47. List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$

c) $\{0, 1, 2\}, \{3, 4, 5\}$

d) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

48. List the ordered pairs in the equivalence relations produced by these partitions of $\{a, b, c, d, e, f, g\}$.

a) $\{a, b\}, \{c, d\}, \{e, f, g\}$

b) $\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}$

c) $\{a, b, c, d\}, \{e, f, g\}$

d) $\{a, c, e, g\}, \{b, d\}, \{f\}$

A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

49. Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3.

50. Show that the partition of the set of people living in the United States consisting of subsets of people living in the same county (or parish) and same state is a refinement of the partition consisting of subsets of people living in the same state.

51. Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.

In Exercises 52 and 53, R_n refers to the family of equivalence relations defined in Example 5. Recall that $s \, R_n \, t$, where s and t are two strings if $s = t$ or s and t are strings with at least n characters that agree in their first n characters.

52. Show that the partition of the set of all bit strings formed by equivalence classes of bit strings with respect to the equivalence relation R_4 is a refinement of the partition formed by equivalence classes of bit strings with respect to the equivalence relation R_3.

53. Show that the partition of the set of all identifiers in C formed by the equivalence classes of identifiers with respect to the equivalence relation R_{31} is a refinement of the partition formed by equivalence classes of identifiers with respect to the equivalence relation R_8. (Compilers for "old" C consider identifiers the same when their names agree in their first eight characters, while compilers in standard C consider identifiers the same when their names agree in their first 31 characters.)

54. Suppose that R_1 and R_2 are equivalence relations on a set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

55. Find the smallest equivalence relation on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (a, c), (d, e)\}$.

56. Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.

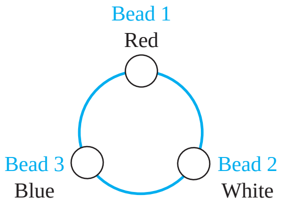
a) $R_1 \cup R_2$ **b)** $R_1 \cap R_2$ **c)** $R_1 \oplus R_2$

57. Consider the equivalence relation from Example 2, namely, $R = \{(x, y) \mid x - y \text{ is an integer}\}$.

a) What is the equivalence class of 1 for this equivalence relation?

b) What is the equivalence class of 1/2 for this equivalence relation?

***58.** Each bead on a bracelet with three beads is either red, white, or blue, as illustrated in the figure shown.



Define the relation R between bracelets as: (B_1, B_2), where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

***59.** Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1, C_2), where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

60. a) Let R be the relation on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that (f, g) belongs to R if and only if f is $\Theta(g)$ (see Section 3.2). Show that R is an equivalence relation.

b) Describe the equivalence class containing $f(n) = n^2$ for the equivalence relation of part (a).

61. Determine the number of different equivalence relations on a set with three elements by listing them.

62. Determine the number of different equivalence relations on a set with four elements by listing them.

***63.** Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?

***64.** Do we necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation?

65. Suppose we use Theorem 2 to form a partition P from an equivalence relation R. What is the equivalence relation R' that results if we use Theorem 2 again to form an equivalence relation from P?

66. Suppose we use Theorem 2 to form an equivalence relation R from a partition P. What is the partition P' that results if we use Theorem 2 again to form a partition from R?

67. Devise an algorithm to find the smallest equivalence relation containing a given relation.

***68.** Let $p(n)$ denote the number of different equivalence relations on a set with n elements (and by Theorem 2 the number of partitions of a set with n elements). Show that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$ and the initial condition $p(0) = 1$. (*Note:* The numbers $p(n)$ are called **Bell numbers** after the American mathematician E. T. Bell.)

69. Use Exercise 68 to find the number of different equivalence relations on a set with n elements, where n is a positive integer not exceeding 10.

9.6 Partial Orderings

Introduction

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

DEFINITION 1 A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

We give examples of posets in Examples 1–3.

EXAMPLE 1 Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers.

Extra Examples

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset. ◀

EXAMPLE 2 The divisibility relation $\mid$ is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 9.1. We see that $(\mathbf{Z}^+, \mid)$ is a poset. Recall that ($\mathbf{Z}^+$ denotes the set of positive integers.) ◀

EXAMPLE 3 Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

Example 4 illustrates a relation that is not a partial ordering.

EXAMPLE 4 Let R be the relation on the set of people such that $x\,R\,y$ if x and y are people and x is older than y. Show that R is not a partial ordering.

Extra Examples

Solution: Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if $x\,R\,y$, then $y \not R\, x$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$. However, R is not reflexive, because no person is older than himself or herself. That is, $x \not R\, x$ for all people x. It follows that R is not a partial ordering. ◀

In different posets different symbols such as $\leq$, $\subseteq$, and $\mid$, are used for a partial ordering. However, we need a symbol that we can use when we discuss the ordering relation in an arbitrary poset. Customarily, the notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R). This notation is used because the "less than or equal to" relation on the set of real numbers is the most familiar example of a partial ordering and the symbol $\preccurlyeq$ is similar to the $\leq$ symbol. (Note that the symbol $\preccurlyeq$ is used to denote the relation in *any* poset, not just the "less than or equals" relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also, we say "a is less than b" or "b is greater than a" if $a \prec b$.

When a and b are elements of the poset $(S, \preccurlyeq)$, it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$. For instance, in $(P(\mathbf{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, because neither set is contained within the other. Similarly, in $(\mathbf{Z}^+, \mid)$, 2 is not related to 3 and 3 is not related to 2, because $2 \nmid 3$ and $3 \nmid 2$. This leads to Definition 2.

DEFINITION 2 The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

EXAMPLE 5 In the poset $(\mathbf{Z}^+, \mid)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀

The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

DEFINITION 3 If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

EXAMPLE 6 The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers. ◀

EXAMPLE 7 The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7. ◀

In Chapter 6 we noted that $(\mathbf{Z}^+, \leq)$ is well-ordered, where $\leq$ is the usual "less than or equal to" relation. We now define well-ordered sets.

DEFINITION 4 $(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

EXAMPLE 8 The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set. The verification of this is left as Exercise 53. The set $\mathbf{Z}$, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $\mathbf{Z}$, has no least element. ◀

At the end of Section 5.3 we showed how to use the principle of well-ordered induction (there called generalized induction) to prove results about a well-ordered set. We now state and prove that this proof technique is valid.

THEOREM 1 **THE PRINCIPLE OF WELL-ORDERED INDUCTION** Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

INDUCTIVE STEP: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false. Consequently, the set $A = \{x \in S \mid P(x) \text{ is false}\}$ is nonempty. Because S is well ordered, A has a least element a. By the choice of a as a least element of A, we know that $P(x)$ is true for all $x \in S$ with $x \prec a$. This implies by the inductive step $P(a)$ is true. This contradiction shows that $P(x)$ must be true for all $x \in S$. ◁

Remark: We do not need a basis step in a proof using the principle of well-ordered induction because if x_0 is the least element of a well ordered set, the inductive step tells us that $P(x_0)$ is true. This follows because there are no elements $x \in S$ with $x \prec x_0$, so we know (using a vacuous proof) that $P(x)$ is true for all $x \in S$ with $x \prec x_0$.

The principle of well-ordered induction is a versatile technique for proving results about well-ordered sets. Even when it is possible to use mathematical induction for the set of positive integers to prove a theorem, it may be simpler to use the principle of well-ordered induction, as we saw in Examples 5 and 6 in Section 6.2, where we proved a result about the well-ordered set $(\mathbf{N} \times \mathbf{N}, \preccurlyeq)$ where $\preccurlyeq$ is lexicographic ordering on $\mathbf{N} \times \mathbf{N}$.

Lexicographic Order

The words in a dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet. This is a special case of an ordering of strings on a set constructed from a partial ordering on the set. We will show how this construction works in any poset.

First, we will show how to construct a partial ordering on the Cartesian product of two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$. The **lexicographic ordering** $\preccurlyeq$ on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in A_1) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in A_2) the second entry of the second pair. In other words, (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

We obtain a partial ordering $\preccurlyeq$ by adding equality to the ordering $\prec$ on $A_1 \times A_2$. The verification of this is left as an exercise.

EXAMPLE 9 Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbf{Z} \times \mathbf{Z}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbf{Z}$.

Solution: Because $3 < 4$, it follows that $(3, 5) \prec (4, 8)$ and that $(3, 8) \prec (4, 5)$. We have $(4, 9) \prec (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$. ◀

In Figure 1 the ordered pairs in $\mathbf{Z}^+ \times \mathbf{Z}^+$ that are less than $(3, 4)$ are highlighted. A lexicographic ordering can be defined on the Cartesian product of n posets $(A_1, \preccurlyeq_1)$, $(A_2, \preccurlyeq_2), \ldots, (A_n, \preccurlyeq_n)$. Define the partial ordering $\preccurlyeq$ on $A_1 \times A_2 \times \cdots \times A_n$ by

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

if $a_1 \prec_1 b_1$, or if there is an integer $i > 0$ such that $a_1 = b_1, \ldots, a_i = b_i$, and $a_{i+1} \prec_{i+1} b_{i+1}$. In other words, one n-tuple is less than a second n-tuple if the entry of the first n-tuple in the first position where the two n-tuples disagree is less than the entry in that position in the second n-tuple.

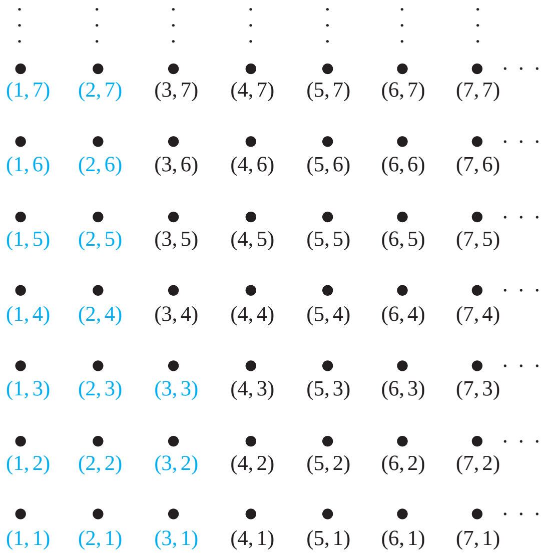


FIGURE 1 **The Ordered Pairs Less Than (3, 4) in Lexicographic Order.**

EXAMPLE 10 Note that $(1, 2, 3, 5) \prec (1, 2, 4, 3)$, because the entries in the first two positions of these 4-tuples agree, but in the third position the entry in the first 4-tuple, 3, is less than that in the second 4-tuple, 4. (Here the ordering on 4-tuples is the lexicographic ordering that comes from the usual "less than or equals" relation on the set of integers.) ◀

We can now define lexicographic ordering of strings. Consider the strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a partially ordered set S. Suppose these strings are not equal. Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if

$$(a_1,\ a_2, \ldots,\ a_t) \prec (b_1,\ b_2, \ldots,\ b_t),\ \text{or}$$
$$(a_1,\ a_2, \ldots,\ a_t) = (b_1,\ b_2, \ldots,\ b_t)\ \text{and}\ m < n,$$

where $\prec$ in this inequality represents the lexicographic ordering of S^t. In other words, to determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms. Then the t-tuples made up of the first t terms of each string are compared using the lexicographic ordering on S^t. One string is less than another string if the t-tuple corresponding to the first string is less than the t-tuple of the second string, or if these two t-tuples are the same, but the second string is longer. The verification that this is a partial ordering is left as Exercise 38 for the reader.

EXAMPLE 11 Consider the set of strings of lowercase English letters. Using the ordering of letters in the alphabet, a lexicographic ordering on the set of strings can be constructed. A string is less than a second string if the letter in the first string in the first position where the strings differ comes before the letter in the second string in this position, or if the first string and the second string agree in all positions, but the second string has more letters. This ordering is the same as that used in dictionaries. For example,

discreet $\prec$ *discrete*,

because these strings differ first in the seventh position, and $e \prec t$. Also,

discreet $\prec$ *discreetness*,

because the first eight letters agree, but the second string is longer. Furthermore,

discrete $\prec$ *discretion*,

because

discrete $\prec$ *discreti*.

◀

Hasse Diagrams

Many edges in the directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure 2(a). Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2(b) loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure 2(c) the edges $(1, 3)$, $(1, 4)$, and $(2, 4)$ are not shown because they must be present. If we assume that all edges are pointed "upward" (as they are drawn in the figure), we do not have to show the directions of the edges; Figure 2(c) does not show directions.

In general, we can represent a finite poset $(S, \preccurlyeq)$ using this procedure: Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops. Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec x$. Finally, arrange each edge so that

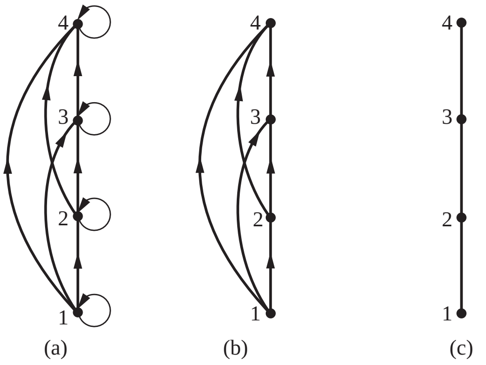


FIGURE 2 Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$.

its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

These steps are well defined, and only a finite number of steps need to be carried out for a finite poset. When all the steps have been taken, the resulting diagram contains sufficient information to find the partial ordering, as we will explain later. The resulting diagram is called the **Hasse diagram** of $(S, \preccurlyeq)$, named after the twentieth-century German mathematician Helmut Hasse who made extensive use of them.

Let $(S, \preccurlyeq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the **covering relation** of $(S, \preccurlyeq)$. From the description of the Hasse diagram of a poset, we see that the edges in the Hasse diagram of $(S, \preccurlyeq)$ are upwardly pointing edges corresponding to the pairs in the covering relation of $(S, \preccurlyeq)$. Furthermore, we can recover a poset from its covering relation, because it is the reflexive transitive closure of its covering relation. (Exercise 31 asks for a proof of this fact.) This tells us that we can construct a partial ordering from its Hasse diagram.

EXAMPLE 12 Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These are (1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12), and (3, 12). Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c). ◀

EXAMPLE 13 Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

HELMUT HASSE (1898–1979) Helmut Hasse was born in Kassel, Germany. He served in the German navy after high school. He began his university studies at Göttingen University in 1918, moving in 1920 to Marburg University to study under the number theorist Kurt Hensel. During this time, Hasse made fundamental contributions to algebraic number theory. He became Hensel's successor at Marburg, later becoming director of the famous mathematical institute at Göttingen in 1934, and took a position at Hamburg University in 1950. Hasse served for 50 years as an editor of *Crelle's Journal*, a famous German mathematics periodical, taking over the job of chief editor in 1936 when the Nazis forced Hensel to resign. During World War II Hasse worked on applied mathematics research for the German navy. He was noted for the clarity and personal style of his lectures and was devoted both to number theory and to his students. (Hasse has been controversial for connections with the Nazi party. Investigations have shown he was a strong German nationalist but not an ardent Nazi.)

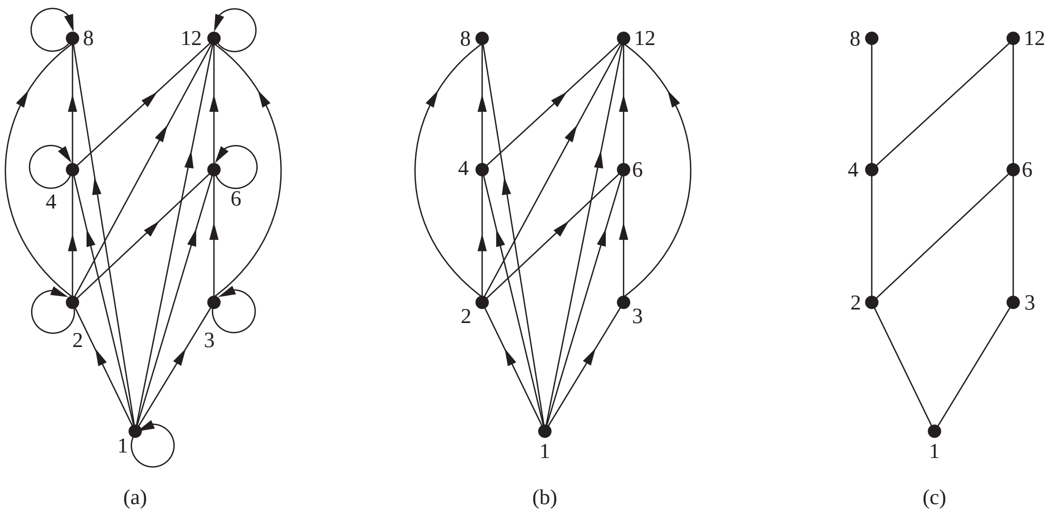


FIGURE 3 Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a, b\})$, $(\emptyset, \{a, c\})$, $(\emptyset, \{b, c\})$, $(\emptyset, \{a, b, c\})$, $(\{a\}, \{a, b, c\})$, $(\{b\}, \{a, b, c\})$, and $(\{c\}, \{a, b, c\})$. Finally all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 4. ◀

Maximal and Minimal Elements

Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

EXAMPLE 14 Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) are maximal, and which are minimal?

Solution: The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element. ◀

Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset $(S, \preccurlyeq)$

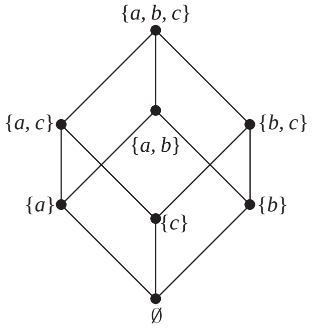


FIGURE 4 The Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$.

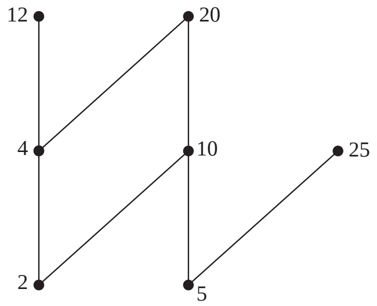
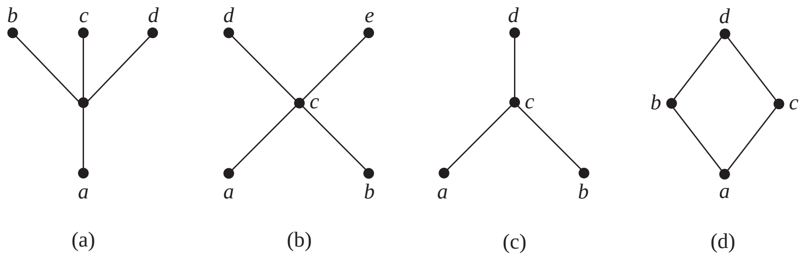


FIGURE 5 The Hasse Diagram of a Poset.



(a) (b) (c) (d)

FIGURE 6 Hasse Diagrams of Four Posets.

if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists [see Exercise 40(a)]. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists [see Exercise 40(b)].

EXAMPLE 15 Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

Solution: The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d. ◀

EXAMPLE 16 Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S. ◀

EXAMPLE 17 Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

Solution: The integer 1 is the least element because $1|n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element. ◀

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

EXAMPLE 18 Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a. ◀

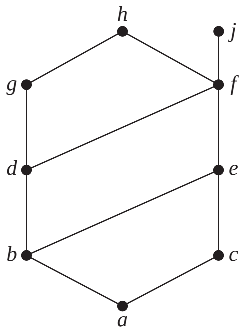


FIGURE 7 The Hasse Diagram of a Poset.

The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A. Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound [see Exercise 42(a)]. That is, x is the least upper bound of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A. Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A. The greatest lower bound of A is unique if it exists [see Exercise 42(b)]. The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively.

EXAMPLE 19 Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

Solution: The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound. ◀

EXAMPLE 20 Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Extra Examples

Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

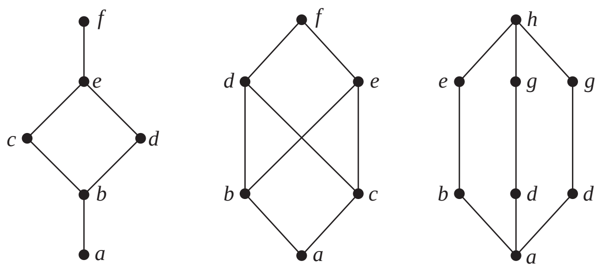
An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

EXAMPLE 21 Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset. ◀



(a) (b) (c)

FIGURE 8 **Hasse Diagrams of Three Posets.**

EXAMPLE 22 Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice. ◀

EXAMPLE 23 Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice. ◀

EXAMPLE 24 Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice. ◀

EXAMPLE 25 **The Lattice Model of Information Flow** In many settings the flow of information from one person or computer program to another is restricted via security clearances. We can use a lattice model to represent different information flow policies. For example, one common information flow policy is the *multilevel security policy* used in government and military systems. Each piece of information is assigned to a security class, and each security class is represented by a pair (A, C) where A is an *authority level* and C is a *category*. People and computer programs are then allowed access to information from a specific restricted set of security classes.

Links

There are billions of pages of classified U.S. government documents.

The typical authority levels used in the U.S. government are unclassified (0), confidential (1), secret (2), and top secret (3). (Information is said to be classified if it is confidential, secret, or top secret.) Categories used in security classes are the subsets of a set of all *compartments* relevant to a particular area of interest. Each compartment represents a particular subject area. For example, if the set of compartments is {*spies, moles, double agents*}, then there are eight different categories, one for each of the eight subsets of the set of compartments, such as {*spies, moles*}.

We can order security classes by specifying that $(A_1, C_1) \preccurlyeq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. Information is permitted to flow from security class (A_1, C_1) into security class (A_2, C_2) if and only if $(A_1, C_1) \preccurlyeq (A_2, C_2)$. For example, information is permitted to flow from the security class (*secret*, {*spies, moles*}) into the security class (*top secret*, {*spies, moles, double agents*}), whereas information is not allowed to flow from the security class (*top secret*, {*spies, moles*}) into either of the security classes (*secret*, {*spies, moles, double agents*}) or (*top secret*, {*spies*}).

We leave it to the reader (see Exercise 48) to show that the set of all security classes with the ordering defined in this example forms a lattice. ◀

Topological Sorting

Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks? To model this problem we set up a partial order on the set of tasks so that $a \prec b$ if and only if a and b are tasks where b

cannot be started until a has been completed. To produce a schedule for the project, we need to produce an order for all 20 tasks that is compatible with this partial order. We will show how this can be done.

We begin with a definition. A total ordering $\preccurlyeq$ is said to be **compatible** with the partial ordering R if $a \preccurlyeq b$ whenever aRb. Constructing a compatible total ordering from a partial ordering is called **topological sorting**.* We will need to use Lemma 1.

LEMMA 1 Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Proof: Choose an element a_0 of S. If a_0 is not minimal, then there is an element a_1 with $a_1 \prec a_0$. If a_1 is not minimal, there is an element a_2 with $a_2 \prec a_1$. Continue this process, so that if a_n is not minimal, there is an element a_{n+1} with $a_{n+1} \prec a_n$. Because there are only a finite number of elements in the poset, this process must end with a minimal element a_n. ◁

The topological sorting algorithm we will describe works for any finite nonempty poset. To define a total ordering on the poset $(A, \preccurlyeq)$, first choose a minimal element a_1; such an element exists by Lemma 1. Next, note that $(A - \{a_1\}, \preccurlyeq)$ is also a poset, as the reader should verify. (Here by $\preccurlyeq$ we mean the restriction of the original relation $\preccurlyeq$ on A to $A - \{a_1\}$.) If it is nonempty, choose a minimal element a_2 of this poset. Then remove a_2 as well, and if there are additional elements left, choose a minimal element a_3 in $A - \{a_1, a_2\}$. Continue this process by choosing a_{k+1} to be a minimal element in $A - \{a_1, a_2, \ldots, a_k\}$, as long as elements remain.

Because A is a finite set, this process must terminate. The end product is a sequence of elements $a_1, a_2, \ldots, a_n$. The desired total ordering $\preccurlyeq_t$ is defined by

$$a_1 \prec_t a_2 \prec_t \cdots \prec_t a_n.$$

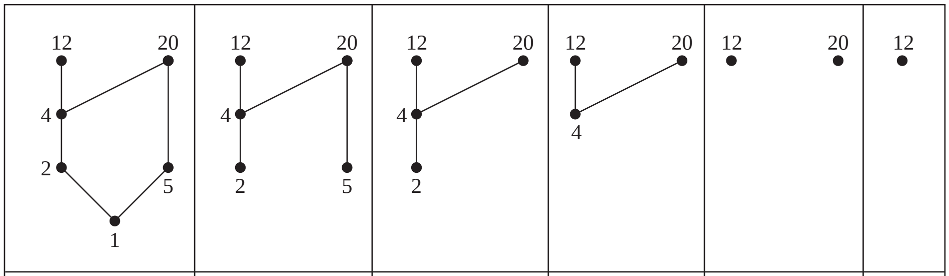
This total ordering is compatible with the original partial ordering. To see this, note that if $b \prec c$ in the original partial ordering, c is chosen as the minimal element at a phase of the algorithm where b has already been removed, for otherwise c would not be a minimal element. Pseudocode for this topological sorting algorithm is shown in Algorithm 1.

ALGORITHM 1 Topological Sorting.

```
procedure topological sort (($S, \preccurlyeq$): finite poset)
$k := 1$
while $S \neq \emptyset$
    $a_k$ := a minimal element of $S$ {such an element exists by Lemma 1}
    $S := S - \{a_k\}$
    $k := k + 1$
return $a_1, a_2, \ldots, a_n$ {$a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S$}
```

EXAMPLE 26 Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

*“Topological sorting” is terminology used by computer scientists; mathematicians use the terminology “linearization of a partial ordering” for the same thing. In mathematics, topology is the branch of geometry dealing with properties of geometric figures that hold for all figures that can be transformed into one another by continuous bijections. In computer science, a topology is any arrangement of objects that can be connected with edges.



Minimal element chosen 1	5	2	4	20	12

FIGURE 9 A Topological Sort of ({1, 2, 4, 5, 12, 20}, |).

Solution: The first step is to choose a minimal element. This must be 1, because it is the only minimal element. Next, select a minimal element of ({2, 4, 5, 12, 20}, |). There are two minimal elements in this poset, namely, 2 and 5. We select 5. The remaining elements are {2, 4, 12, 20}. The only minimal element at this stage is 2. Next, 4 is chosen because it is the only minimal element of ({4, 12, 20}, |). Because both 12 and 20 are minimal elements of ({12, 20}, |), either can be chosen next. We select 20, which leaves 12 as the last element left. This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

The steps used by this sorting algorithm are displayed in Figure 9. ◀

Topological sorting has an application to the scheduling of projects.

EXAMPLE 27 A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task $X \prec$ task Y if task Y cannot be started until task X has been completed. The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in Figure 10. Find an order in which these tasks can be carried out to complete the project.

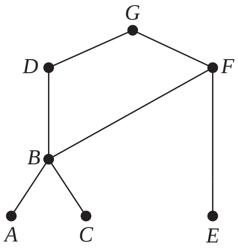
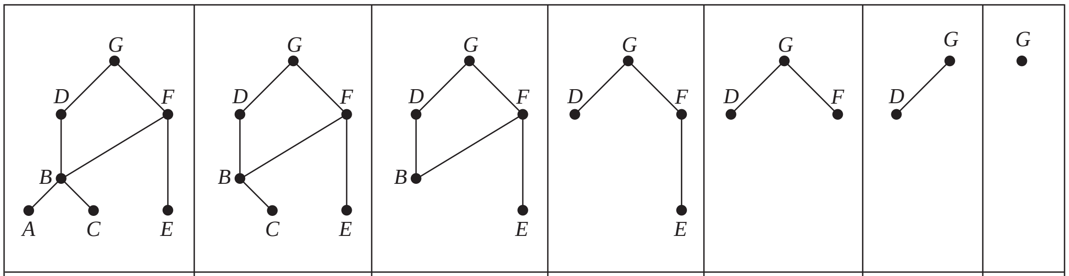


FIGURE 10 The Hasse Diagram for Seven Tasks.

Solution: An ordering of the seven tasks can be obtained by performing a topological sort. The steps of a sort are illustrated in Figure 11. The result of this sort, $A \prec C \prec B \prec E \prec F \prec D \prec G$, gives one possible order for the tasks. ◀



Minimal element chosen *A*	*C*	*B*	*E*	*F*	*D*	*G*

FIGURE 11 A Topological Sort of the Tasks.

Exercises

1. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

3. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

b) a is not taller than b?

c) $a = b$ or a is an ancestor of b?

d) a and b have a common friend?

4. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is no shorter than b?

b) a weighs more than b?

c) $a = b$ or a is a descendant of b?

d) a and b do not have a common friend?

5. Which of these are posets?

a) $(\mathbf{Z}, =)$ **b)** $(\mathbf{Z}, \neq)$ **c)** $(\mathbf{Z}, \geq)$ **d)** $(\mathbf{Z}, \nmid)$

6. Which of these are posets?

a) $(\mathbf{R}, =)$ **b)** $(\mathbf{R}, <)$ **c)** $(\mathbf{R}, \leq)$ **d)** $(\mathbf{R}, \neq)$

7. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

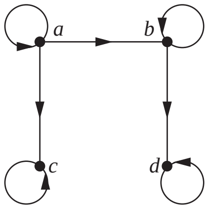
8. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

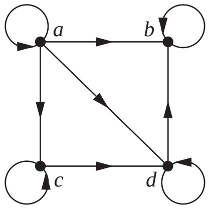
c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

In Exercises 9–11 determine whether the relation with the directed graph shown is a partial order.

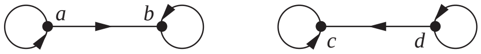
9.



10.



11.



12. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R. The poset (S, R^{-1}) is called the **dual** of (S, R).

13. Find the duals of these posets.

a) $(\{0, 1, 2\}, \leq)$ **b)** $(\mathbf{Z}, \geq)$

c) $(P(\mathbf{Z}), \supseteq)$ **d)** $(\mathbf{Z}^+, |)$

14. Which of these pairs of elements are comparable in the poset $(\mathbf{Z}^+, |)$?

a) 5, 15 **b)** 6, 9 **c)** 8, 16 **d)** 7, 7

15. Find two incomparable elements in these posets.

a) $(P(\{0, 1, 2\}), \subseteq)$ **b)** $(\{1, 2, 4, 6, 8\}, |)$

16. Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual "less than" relation,

a) find all pairs in $S \times S$ less than $(2, 3)$.

b) find all pairs in $S \times S$ greater than $(3, 1)$.

c) draw the Hasse diagram of the poset $(S \times S, \preccurlyeq)$.

17. Find the lexicographic ordering of these n-tuples:

a) $(1, 1, 2), (1, 2, 1)$ **b)** $(0, 1, 2, 3), (0, 1, 3, 2)$

c) $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$

18. Find the lexicographic ordering of these strings of lowercase English letters:

a) *quack, quick, quicksilver, quicksand, quacking*

b) *open, opener, opera, operand, opened*

c) *zoo, zero, zoom, zoology, zoological*

19. Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$.

20. Draw the Hasse diagram for the "greater than or equal to" relation on $\{0, 1, 2, 3, 4, 5\}$.

21. Draw the Hasse diagram for the "less than or equal to" relation on $\{0, 2, 5, 10, 11, 15\}$.

22. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6\}$. **b)** $\{3, 5, 7, 11, 13, 16, 17\}$.

c) $\{2, 3, 5, 10, 11, 15, 25\}$. **d)** $\{1, 3, 9, 27, 81, 243\}$.

23. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. **b)** $\{1, 2, 3, 5, 7, 11, 13\}$.

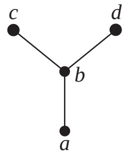
c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$.

d) $\{1, 2, 4, 8, 16, 32, 64\}$.

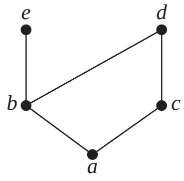
24. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$.

In Exercises 25–27 list all ordered pairs in the partial ordering with the accompanying Hasse diagram.

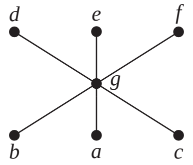
25.



26.



27.



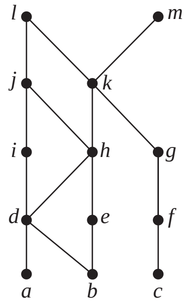
28. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 12\}$?

29. What is the covering relation of the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set of S, where $S = \{a, b, c\}$?

30. What is the covering relation of the partial ordering for the poset of security classes defined in Example 25?

31. Show that a finite poset can be reconstructed from its covering relation. [*Hint:* Show that the poset is the reflexive transitive closure of its covering relation.]

32. Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{a, b, c\}$.

f) Find the least upper bound of $\{a, b, c\}$, if it exists.

g) Find all lower bounds of $\{f, g, h\}$.

h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

33. Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{3, 5\}$.

f) Find the least upper bound of $\{3, 5\}$, if it exists.

g) Find all lower bounds of $\{15, 45\}$.

h) Find the greatest lower bound of $\{15, 45\}$, if it exists.

34. Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{2, 9\}$.

f) Find the least upper bound of $\{2, 9\}$, if it exists.

g) Find all lower bounds of $\{60, 72\}$.

h) Find the greatest lower bound of $\{60, 72\}$, if it exists.

35. Answer these questions for the poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{\{2\}, \{4\}\}$.

f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.

g) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.

h) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists.

36. Give a poset that has

a) a minimal element but no maximal element.

b) a maximal element but no minimal element.

c) neither a maximal nor a minimal element.

37. Show that lexicographic order is a partial ordering on the Cartesian product of two posets.

38. Show that lexicographic order is a partial ordering on the set of strings from a poset.

39. Suppose that $(S, \preccurlyeq_1)$ and $(T, \preccurlyeq_2)$ are posets. Show that $(S \times T, \preccurlyeq)$ is a poset where $(s, t) \preccurlyeq (u, v)$ if and only if $s \preccurlyeq_1 u$ and $t \preccurlyeq_2 v$.

40. **a)** Show that there is exactly one greatest element of a poset, if such an element exists.

b) Show that there is exactly one least element of a poset, if such an element exists.

41. **a)** Show that there is exactly one maximal element in a poset with a greatest element.

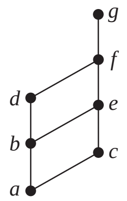
b) Show that there is exactly one minimal element in a poset with a least element.

42. **a)** Show that the least upper bound of a set in a poset is unique if it exists.

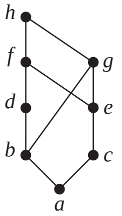
b) Show that the greatest lower bound of a set in a poset is unique if it exists.

43. Determine whether the posets with these Hasse diagrams are lattices.

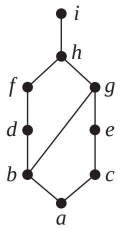
a)



b)



c)



44. Determine whether these posets are lattices.

a) $(\{1, 3, 6, 9, 12\}, |)$ **b)** $(\{1, 5, 25, 125\}, |)$

c) $(\mathbf{Z}, \geq)$

d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

45. Show that every nonempty finite subset of a lattice has a least upper bound and a greatest lower bound.

46. Show that if the poset (S, R) is a lattice then the dual poset (S, R^{-1}) is also a lattice.

47. In a company, the lattice model of information flow is used to control sensitive information with security classes represented by ordered pairs (A, C). Here A is an authority level, which may be nonproprietary (0), proprietary (1), restricted (2), or registered (3). A category C is a subset of the set of all projects {*Cheetah, Impala, Puma*}. (Names of animals are often used as code names for projects in companies.)

a) Is information permitted to flow from (*Proprietary*, {*Cheetah, Puma*}) into (*Restricted*, {*Puma*})?

b) Is information permitted to flow from (*Restricted*, {*Cheetah*}) into (*Registered*, {*Cheetah, Impala*})?

c) Into which classes is information from (*Proprietary*, {*Cheetah*, *Puma*}) permitted to flow?

d) From which classes is information permitted to flow into the security class (*Restricted*, {*Impala*, *Puma*})?

48. Show that the set S of security classes (A, C) is a lattice, where A is a positive integer representing an authority class and C is a subset of a finite set of compartments, with $(A_1, C_1) \preccurlyeq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. [*Hint:* First show that $(S, \preccurlyeq)$ is a poset and then show that the least upper bound and greatest lower bound of (A_1, C_1) and (A_2, C_2) are $(\max(A_1, A_2), C_1 \cup C_2)$ and $(\min(A_1, A_2), C_1 \cap C_2)$, respectively.]

***49.** Show that the set of all partitions of a set S with the relation $P_1 \preccurlyeq P_2$ if the partition P_1 is a refinement of the partition P_2 is a lattice. (See the preamble to Exercise 49 of Section 9.5.)

50. Show that every totally ordered set is a lattice.

51. Show that every finite lattice has a least element and a greatest element.

52. Give an example of an infinite lattice with

a) neither a least nor a greatest element.

b) a least but not a greatest element.

c) a greatest but not a least element.

d) both a least and a greatest element.

53. Verify that $(\mathbf{Z}^+ \times \mathbf{Z}^+, \preccurlyeq)$ is a well-ordered set, where $\preccurlyeq$ is lexicographic order, as claimed in Example 8.

54. Determine whether each of these posets is well-ordered.

a) $(S, \leq)$, where $S = \{10, 11, 12, \ldots\}$

b) $(\mathbf{Q} \cap [0, 1], \leq)$ (the set of rational numbers between 0 and 1 inclusive)

c) $(S, \leq)$, where S is the set of positive rational numbers with denominators not exceeding 3

d) $(\mathbf{Z}^-, \geq)$, where $\mathbf{Z}^-$ is the set of negative integers

A poset $(R, \preccurlyeq)$ is **well-founded** if there is no infinite decreasing sequence of elements in the poset, that is, elements $x_1, x_2, \ldots, x_n$ such that $\cdots \prec x_n \prec \cdots \prec x_2 \prec x_1$. A poset $(R, \preccurlyeq)$ is **dense** if for all $x \in S$ and $y \in S$ with $x \prec y$, there is an element $z \in R$ such that $x \prec z \prec y$.

55. Show that the poset $(\mathbf{Z}, \preccurlyeq)$, where $x \prec y$ if and only if $|x| < |y|$ is well-founded but is not a totally ordered set.

56. Show that a dense poset with at least two elements that are comparable is not well-founded.

57. Show that the poset of rational numbers with the usual "less than or equal to" relation, $(\mathbf{Q}, \leq)$, is a dense poset.

***58.** Show that the set of strings of lowercase English letters with lexicographic order is neither well-founded nor dense.

59. Show that a poset is well-ordered if and only if it is totally ordered and well-founded.

60. Show that a finite nonempty poset has a maximal element.

61. Find a compatible total order for the poset with the Hasse diagram shown in Exercise 32.

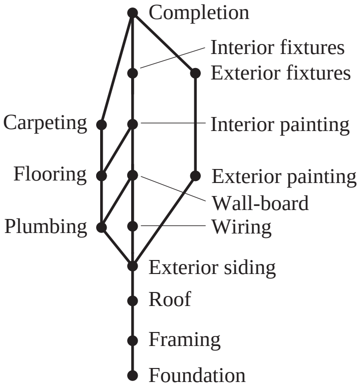
62. Find a compatible total order for the divisibility relation on the set $\{1, 2, 3, 6, 8, 12, 24, 36\}$.

63. Find all compatible total orderings for the poset $(\{1, 2, 4, 5, 12, 20\}, |\}$ from Example 26.

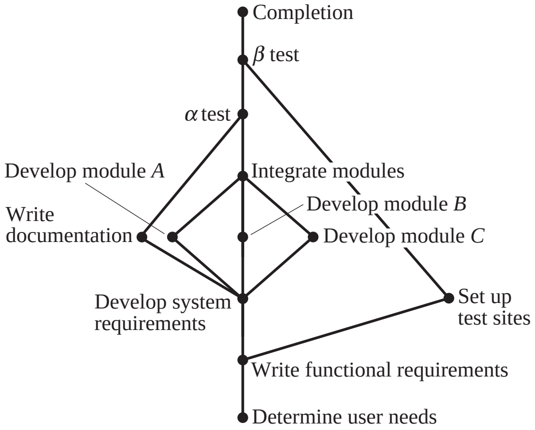
64. Find all compatible total orderings for the poset with the Hasse diagram in Exercise 27.

65. Find all possible orders for completing the tasks in the development project in Example 27.

66. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the figure.



67. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.



Key Terms and Results

TERMS

binary relation from A to B: a subset of $A \times B$

relation on A: a binary relation from A to itself (i.e., a subset of $A \times A$)

$S \circ R$: composite of R and S

R^{-1}: inverse relation of R

R^n: nth power of R

reflexive: a relation R on A is reflexive if $(a, a) \in R$ for all $a \in A$

symmetric: a relation R on A is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$

antisymmetric: a relation R on A is antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$

transitive: a relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

n-ary relation on $A_1, A_2, \ldots, A_n$: a subset of $A_1 \times A_2 \times \cdots \times A_n$

relational data model: a model for representing databases using n-ary relations

primary key: a domain of an n-ary relation such that an n-tuple is uniquely determined by its value for this domain

composite key: the Cartesian product of domains of an n-ary relation such that an n-tuple is uniquely determined by its values in these domains

selection operator: a function that selects the n-tuples in an n-ary relation that satisfy a specified condition

projection: a function that produces relations of smaller degree from an n-ary relation by deleting fields

join: a function that combines n-ary relations that agree on certain fields

directed graph or digraph: a set of elements called vertices and ordered pairs of these elements, called edges

loop: an edge of the form (a, a)

closure of a relation R with respect to a property P: the relation S (if it exists) that contains R, has property **P**, and is contained within any relation that contains R and has property **P**

path in a digraph: a sequence of edges $(a, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1}), (x_{n-1}, b)$ such that the terminal vertex of each edge is the initial vertex of the succeeding edge in the sequence

circuit (or cycle) in a digraph: a path that begins and ends at the same vertex

R^* (connectivity relation): the relation consisting of those ordered pairs (a, b) such that there is a path from a to b

equivalence relation: a reflexive, symmetric, and transitive relation

equivalent: if R is an equivalence relation, a is equivalent to b if aRb

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$[a]_m$ (congruence class modulo m): the set of integers congruent to a modulo m

partition of a set S: a collection of pairwise disjoint nonempty subsets that have S as their union

partial ordering: a relation that is reflexive, antisymmetric, and transitive

poset (S, R): a set S and a partial ordering R on this set

comparable: the elements a and b in the poset $(A, \preccurlyeq)$ are comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$

incomparable: elements in a poset that are not comparable

total (or linear) ordering: a partial ordering for which every pair of elements are comparable

totally (or linearly) ordered set: a poset with a total (or linear) ordering

well-ordered set: a poset $(S, \preccurlyeq)$, where $\preccurlyeq$ is a total order and every nonempty subset of S has a least element

lexicographic order: a partial ordering of Cartesian products or strings

Hasse diagram: a graphical representation of a poset where loops and all edges resulting from the transitive property are not shown, and the direction of the edges is indicated by the position of the vertices

maximal element: an element of a poset that is not less than any other element of the poset

minimal element: an element of a poset that is not greater than any other element of the poset

greatest element: an element of a poset greater than all other elements in this set

least element: an element of a poset less than all other elements in this set

upper bound of a set: an element in a poset greater than all other elements in the set

lower bound of a set: an element in a poset less than all other elements in the set

least upper bound of a set: an upper bound of the set that is less than all other upper bounds

greatest lower bound of a set: a lower bound of the set that is greater than all other lower bounds

lattice: a partially ordered set in which every two elements have a greatest lower bound and a least upper bound

compatible total ordering for a partial ordering: a total ordering that contains the given partial ordering

topological sort: the construction of a total ordering compatible with a given partial ordering

RESULTS

The reflexive closure of a relation R on the set A equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$.

The symmetric closure of a relation R on the set A equals $R \cup R^{-1}$, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

The transitive closure of a relation equals the connectivity relation formed from this relation.

Warshall's algorithm for finding the transitive closure of a relation

Let R be an equivalence relation. Then the following three statements are equivalent: (1) $a\ R\ b$; (2) $[a]_R \cap [b]_R \neq \emptyset$; (3) $[a]_R = [b]_R$.

The equivalence classes of an equivalence relation on a set A form a partition of A. Conversely, an equivalence relation can be constructed from any partition so that the equivalence classes are the subsets in the partition.

The principle of well-ordered induction

The topological sorting algorithm

Review Questions

1. a) What is a relation on a set?
   b) How many relations are there on a set with n elements?

2. a) What is a reflexive relation?
   b) What is a symmetric relation?
   c) What is an antisymmetric relation?
   d) What is a transitive relation?

3. Give an example of a relation on the set $\{1, 2, 3, 4\}$ that is
   a) reflexive, symmetric, and not transitive.
   b) not reflexive, symmetric, and transitive.
   c) reflexive, antisymmetric, and not transitive.
   d) reflexive, symmetric, and transitive.
   e) reflexive, antisymmetric, and transitive.

4. a) How many reflexive relations are there on a set with n elements?
   b) How many symmetric relations are there on a set with n elements?
   c) How many antisymmetric relations are there on a set with n elements?

5. a) Explain how an n-ary relation can be used to represent information about students at a university.
   b) How can the 5-ary relation containing names of students, their addresses, telephone numbers, majors, and grade point averages be used to form a 3-ary relation containing the names of students, their majors, and their grade point averages?
   c) How can the 4-ary relation containing names of students, their addresses, telephone numbers, and majors and the 4-ary relation containing names of students, their student numbers, majors, and numbers of credit hours be combined into a single n-ary relation?

6. a) Explain how to use a zero–one matrix to represent a relation on a finite set.
   b) Explain how to use the zero–one matrix representing a relation to determine whether the relation is reflexive, symmetric, and/or antisymmetric.

7. a) Explain how to use a directed graph to represent a relation on a finite set.
   b) Explain how to use the directed graph representing a relation to determine whether a relation is reflexive, symmetric, and/or antisymmetric.

8. a) Define the reflexive closure and the symmetric closure of a relation.
   b) How can you construct the reflexive closure of a relation?
   c) How can you construct the symmetric closure of a relation?
   d) Find the reflexive closure and the symmetric closure of the relation $\{(1, 2), (2, 3), (2, 4), (3, 1)\}$ on the set $\{1, 2, 3, 4\}$.

9. a) Define the transitive closure of a relation.
   b) Can the transitive closure of a relation be obtained by including all pairs (a, c) such that (a, b) and (b, c) belong to the relation?

c) Describe two algorithms for finding the transitive closure of a relation.

d) Find the transitive closure of the relation $\{(1,1), (1,3), (2,1), (2,3), (2,4), (3,2), (3,4), (4,1)\}$.

10. a) Define an equivalence relation.

b) Which relations on the set $\{a, b, c, d\}$ are equivalence relations and contain (a, b) and (b, d)?

11. a) Show that congruence modulo m is an equivalence relation whenever m is a positive integer.

b) Show that the relation $\{(a, b) \mid a \equiv \pm b \pmod{7}\}$ is an equivalence relation on the set of integers.

12. a) What are the equivalence classes of an equivalence relation?

b) What are the equivalence classes of the "congruent modulo 5" relation?

c) What are the equivalence classes of the equivalence relation in Question 11(b)?

13. Explain the relationship between equivalence relations on a set and partitions of this set.

14. a) Define a partial ordering.

b) Show that the divisibility relation on the set of positive integers is a partial order.

15. Explain how partial orderings on the sets A_1 and A_2 can be used to define a partial ordering on the set $A_1 \times A_2$.

16. a) Explain how to construct the Hasse diagram of a partial order on a finite set.

b) Draw the Hasse diagram of the divisibility relation on the set $\{2, 3, 5, 9, 12, 15, 18\}$.

17. a) Define a maximal element of a poset and the greatest element of a poset.

b) Give an example of a poset that has three maximal elements.

c) Give an example of a poset with a greatest element.

18. a) Define a lattice.

b) Give an example of a poset with five elements that is a lattice and an example of a poset with five elements that is not a lattice.

19. a) Show that every finite subset of a lattice has a greatest lower bound and a least upper bound.

b) Show that every lattice with a finite number of elements has a least element and a greatest element.

20. a) Define a well-ordered set.

b) Describe an algorithm for producing a totally ordered set compatible with a given partially ordered set.

c) Explain how the algorithm from (b) can be used to order the tasks in a project if tasks are done one at a time and each task can be done only after one or more of the other tasks have been completed.

Supplementary Exercises

1. Let S be the set of all strings of English letters. Determine whether these relations are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

a) $R_1 = \{(a, b) \mid a$ and b have no letters in common$\}$

b) $R_2 = \{(a, b) \mid a$ and b are not the same length$\}$

c) $R_3 = \{(a, b) \mid a$ is longer than $b\}$

2. Construct a relation on the set $\{a, b, c, d\}$ that is

a) reflexive, symmetric, but not transitive.

b) irreflexive, symmetric, and transitive.

c) irreflexive, antisymmetric, and not transitive.

d) reflexive, neither symmetric nor antisymmetric, and transitive.

e) neither reflexive, irreflexive, symmetric, antisymmetric, nor transitive.

3. Show that the relation R on $\mathbf{Z} \times \mathbf{Z}$ defined by $(a, b)\ R\ (c, d)$ if and only if $a + d = b + c$ is an equivalence relation.

4. Show that a subset of an antisymmetric relation is also antisymmetric.

5. Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

6. Suppose that R_1 and R_2 are reflexive relations on a set A. Show that $R_1 \oplus R_2$ is irreflexive.

7. Suppose that R_1 and R_2 are reflexive relations on a set A. Is $R_1 \cap R_2$ also reflexive? Is $R_1 \cup R_2$ also reflexive?

8. Suppose that R is a symmetric relation on a set A. Is $\overline{R}$ also symmetric?

9. Let R_1 and R_2 be symmetric relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also symmetric?

10. A relation R is called **circular** if aRb and bRc imply that cRa. Show that R is reflexive and circular if and only if it is an equivalence relation.

11. Show that a primary key in an n-ary relation is a primary key in any projection of this relation that contains this key as one of its fields.

12. Is the primary key in an n-ary relation also a primary key in a larger relation obtained by taking the join of this relation with a second relation?

13. Show that the reflexive closure of the symmetric closure of a relation is the same as the symmetric closure of its reflexive closure.

14. Let R be the relation on the set of all mathematicians that contains the ordered pair (a, b) if and only if a and b have written a published mathematical paper together.

a) Describe the relation R^2.

b) Describe the relation R^*.

c) The **Erdős number** of a mathematician is 1 if this mathematician wrote a paper with the prolific Hungarian mathematician Paul Erdős, it is 2 if this mathematician did not write a joint paper with Erdős but wrote a joint paper with someone who wrote a joint paper with Erdős, and so on (except that the Erdős number of Erdős himself is 0). Give a definition of the Erdős number in terms of paths in R.

15. a) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

b) Show, however, that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

16. a) Let S be the set of subroutines of a computer program. Define the relation R by $\mathbf{P}\,R\,\mathbf{Q}$ if subroutine $\mathbf{P}$ calls subroutine $\mathbf{Q}$ during its execution. Describe the transitive closure of R.

b) For which subroutines $\mathbf{P}$ does $(\mathbf{P}, \mathbf{P})$ belong to the transitive closure of R?

c) Describe the reflexive closure of the transitive closure of R.

17. Suppose that R and S are relations on a set A with $R \subseteq S$ such that the closures of R and S with respect to a property $\mathbf{P}$ both exist. Show that the closure of R with respect to $\mathbf{P}$ is a subset of the closure of S with respect to $\mathbf{P}$.

18. Show that the symmetric closure of the union of two relations is the union of their symmetric closures.

***19.** Devise an algorithm, based on the concept of interior vertices, that finds the length of the longest path between two vertices in a directed graph, or determines that there are arbitrarily long paths between these vertices.

20. Which of these are equivalence relations on the set of all people?

a) $\{(x, y) \mid x$ and y have the same sign of the zodiac$\}$

b) $\{(x, y) \mid x$ and y were born in the same year$\}$

c) $\{(x, y) \mid x$ and y have been in the same city$\}$

***21.** How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?

22. Show that $\{(x, y) \mid x - y \in \mathbf{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbf{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

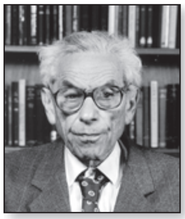
23. Suppose that $P_1 = \{A_1, A_2, \ldots, A_m\}$ and $P_2 = \{B_1, B_2, \ldots, B_n\}$ are both partitions of the set S. Show that the collection of nonempty subsets of the form $A_i \cap B_j$ is a partition of S that is a refinement of both P_1 and P_2 (see the preamble to Exercise 49 of Section 9.5).

***24.** Show that the transitive closure of the symmetric closure of the reflexive closure of a relation R is the smallest equivalence relation that contains R.

25. Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preccurlyeq$ on $\mathbf{R}(S)$ by $R_1 \preccurlyeq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $(\mathbf{R}(S), \preccurlyeq)$ is a poset.

26. Let $\mathbf{P}(S)$ be the set of all partitions of the set S. Define the relation $\preccurlyeq$ on $\mathbf{P}(S)$ by $P_1 \preccurlyeq P_2$ if P_1 is a refinement of P_2 (see Exercise 49 of Section 9.5). Show that $(\mathbf{P}(S), \preccurlyeq)$ is a poset.

Links



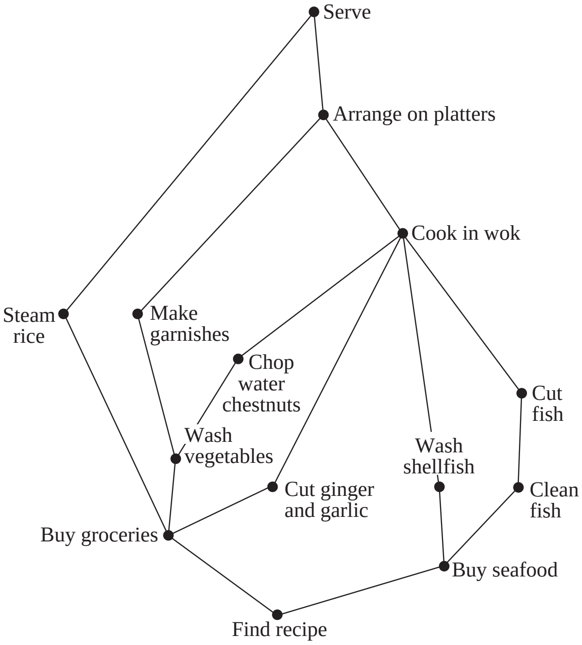
PAUL ERDŐS (1913–1996) Paul Erdős, born in Budapest, Hungary, was the son of two high school mathematics teachers. He was a child prodigy; at age 3 he could multiply three-digit numbers in his head, and at 4 he discovered negative numbers on his own. Because his mother did not want to expose him to contagious diseases, he was mostly home-schooled. At 17 Erdős entered Eőtvős University, graduating four years later with a Ph.D. in mathematics. After graduating he spent four years at Manchester, England, on a postdoctoral fellowship. In 1938 he went to the United States because of the difficult political situation in Hungary, especially for Jews. He spent much of his time in the United States, except for 1954 to 1962, when he was banned as part of the paranoia of the McCarthy era. He also spent considerable time in Israel.

Erdős made many significant contributions to combinatorics and to number theory. One of the discoveries of which he was most proud is his elementary proof (in the sense that it does not use any complex analysis) of the prime number theorem, which provides an estimate for the number of primes not exceeding a fixed positive integer. He also participated in the modern development of the Ramsey theory.

Erdős traveled extensively throughout the world to work with other mathematicians, visiting conferences, universities, and research laboratories. He had no permanent home. He devoted himself almost entirely to mathematics, traveling from one mathematician to the next, proclaiming "My brain is open." Erdős was the author or coauthor of more than 1500 papers and had more than 500 coauthors. Copies of his articles are kept by Ron Graham, a famous discrete mathematician with whom he collaborated extensively and who took care of many of his worldly needs.

Erdős offered rewards, ranging from $10 to $10,000, for the solution of problems that he found particularly interesting, with the size of the reward depending on the difficulty of the problem. He paid out close to $4000. Erdős had his own special language, using such terms as "epsilon" (child), "boss" (woman), "slave" (man), "captured" (married), "liberated" (divorced), "Supreme Fascist" (God), "Sam" (United States), and "Joe" (Soviet Union). Although he was curious about many things, he concentrated almost all his energy on mathematical research. He had no hobbies and no full-time job. He never married and apparently remained celibate. Erdős was extremely generous, donating much of the money he collected from prizes, awards, and stipends for scholarships and to worthwhile causes. He traveled extremely lightly and did not like having many material possessions.

27. Schedule the tasks needed to cook a Chinese meal by specifying their order, if the Hasse diagram representing these tasks is as shown here.



A subset of a poset such that every two elements of this subset are comparable is called a **chain**. A subset of a poset is called an **antichain** if every two elements of this subset are incomparable.

28. Find all chains in the posets with the Hasse diagrams shown in Exercises 25–27 in Section 9.6.

29. Find all antichains in the posets with the Hasse diagrams shown in Exercises 25–27 in Section 9.6.

30. Find an antichain with the greatest number of elements in the poset with the Hasse diagram of Exercise 32 in Section 9.6.

31. Show that every maximal chain in a finite poset $(S, \preccurlyeq)$ contains a minimal element of S. (A maximal chain is a chain that is not a subset of a larger chain.)

****32.** Show that every finite poset can be partitioned into k chains, where k is the largest number of elements in an antichain in this poset.

***33.** Show that in any group of $mn + 1$ people there is either a list of $m + 1$ people where a person in the list (except for the first person listed) is a descendant of the previous person on the list, or there are $n + 1$ people such that none of these people is a descendant of any of the other n people. [*Hint:* Use Exercise 32.]

Suppose that $(S, \preccurlyeq)$ is a well-founded partially ordered set. The *principle of well-founded induction* states that $P(x)$ is true for all $x \in S$ if $\forall x(\forall y(y \prec x \rightarrow P(y)) \rightarrow P(x))$.

34. Show that no separate basis case is needed for the principle of well-founded induction. That is, $P(u)$ is true for all minimal elements u in S if $\forall x(\forall y(y \prec x \rightarrow P(y)) \rightarrow P(x))$.

***35.** Show that the principle of well-founded induction is valid.

A relation R on a set A is a **quasi-ordering** on A if R is reflexive and transitive.

36. Let R be the relation on the set of all functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that (f, g) belongs to R if and only if f is $O(g)$. Show that R is a quasi-ordering.

37. Let R be a quasi-ordering on a set A. Show that $R \cap R^{-1}$ is an equivalence relation.

***38.** Let R be a quasi-ordering and let S be the relation on the set of equivalence classes of $R \cap R^{-1}$ such that (C, D) belongs to S, where C and D are equivalence classes of R, if and only if there are elements c of C and d of D such that (c, d) belongs to R. Show that S is a partial ordering.

Let L be a lattice. Define the **meet** $(\wedge)$ and **join** $(\vee)$ operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements x, y, and z of a lattice L.

a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ **(commutative laws)**

b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ **(associative laws)**

c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ **(absorption laws)**

d) $x \wedge x = x$ and $x \vee x = x$ **(idempotent laws)**

40. Show that if x and y are elements of a lattice L, then $x \vee y = y$ if and only if $x \wedge y = x$.

A lattice L is **bounded** if it has both an **upper bound**, denoted by 1, such that $x \preccurlyeq 1$ for all $x \in L$ and a **lower bound**, denoted by 0, such that $0 \preccurlyeq x$ for all $x \in L$.

41. Show that if L is a bounded lattice with upper bound 1 and lower bound 0 then these properties hold for all elements $x \in L$.

a) $x \vee 1 = 1$ **b)** $x \wedge 1 = x$

c) $x \vee 0 = x$ **d)** $x \wedge 0 = 0$

42. Show that every finite lattice is bounded.

A lattice is called **distributive** if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, and z in L.

***43.** Give an example of a lattice that is not distributive.

44. Show that the lattice $(P(S), \subseteq)$ where $P(S)$ is the power set of a finite set S is distributive.

45. Is the lattice $(\mathbf{Z}^+, |)$ distributive?

The **complement** of an element a of a bounded lattice L with upper bound 1 and lower bound 0 is an element b such that $a \vee b = 1$ and $a \wedge b = 0$. Such a lattice is **complemented** if every element of the lattice has a complement.

46. Give an example of a finite lattice where at least one element has more than one complement and at least one element has no complement.

47. Show that the lattice $(P(S), \subseteq)$ where $P(S)$ is the power set of a finite set S is complemented.

***48.** Show that if L is a finite distributive lattice, then an element of L has at most one complement.

The game of Chomp, introduced in Example 12 in Section 1.8, can be generalized for play on any finite partially ordered set $(S, \preceq)$ with a least element a. In this game, a move consists of selecting an element x in S and removing x and all elements larger than it from S. The loser is the player who is forced to select the least element a.

49. Show that the game of Chomp with cookies arranged in an $m \times n$ rectangular grid, described in Example 12 in Section 1.8, is the same as the game of Chomp on the poset $(S, |)$, where S is the set of all positive integers that divide $p^{m-1}q^{n-1}$, where p and q are distinct primes.

50. Show that if $(S, \preceq)$ has a greatest element b, then a winning strategy for Chomp on this poset exists. [*Hint:* Generalize the argument in Example 12 in Section 1.8.]

Computer Projects

Write programs with these input and output.

1. Given the matrix representing a relation on a finite set, determine whether the relation is reflexive and/or irreflexive.

2. Given the matrix representing a relation on a finite set, determine whether the relation is symmetric and/or antisymmetric.

3. Given the matrix representing a relation on a finite set, determine whether the relation is transitive.

4. Given a positive integer n, display all the relations on a set with n elements.

***5.** Given a positive integer n, determine the number of transitive relations on a set with n elements.

***6.** Given a positive integer n, determine the number of equivalence relations on a set with n elements.

***7.** Given a positive integer n, display all the equivalence relations on the set of the n smallest positive integers.

8. Given an n-ary relation, find the projection of this relation when specified fields are deleted.

9. Given an m-ary relation and an n-ary relation, and a set of common fields, find the join of these relations with respect to these common fields.

10. Given the matrix representing a relation on a finite set, find the matrix representing the reflexive closure of this relation.

11. Given the matrix representing a relation on a finite set, find the matrix representing the symmetric closure of this relation.

12. Given the matrix representing a relation on a finite set, find the matrix representing the transitive closure of this relation by computing the join of the Boolean powers of the matrix representing the relation.

13. Given the matrix representing a relation on a finite set, find the matrix representing the transitive closure of this relation using Warshall's algorithm.

14. Given the matrix representing a relation on a finite set, find the matrix representing the smallest equivalence relation containing this relation.

15. Given a partial ordering on a finite set, find a total ordering compatible with it using topological sorting.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Display all the different relations on a set with four elements.

2. Display all the different reflexive and symmetric relations on a set with six elements.

3. Display all the reflexive and transitive relations on a set with five elements.

***4.** Determine how many transitive relations there are on a set with n elements for all positive integers n with $n \leq 7$.

5. Find the transitive closure of a relation of your choice on a set with at least 20 elements. Either use a relation that corresponds to direct links in a particular transportation or communications network or use a randomly generated relation.

6. Compute the number of different equivalence relations on a set with n elements for all positive integers n not exceeding 20.

7. Display all the equivalence relations on a set with seven elements.

***8.** Display all the partial orders on a set with five elements.

***9.** Display all the lattices on a set with five elements.

Writing Projects

Respond to these with essays using outside sources.

1. Discuss the concept of a fuzzy relation. How are fuzzy relations used?
2. Describe the basic principles of relational databases, going beyond what was covered in Section 9.2. How widely used are relational databases as compared with other types of databases?
3. Look up the original papers by Warshall and by Roy (in French) in which they develop algorithms for finding transitive closures. Discuss their approaches. Why do you suppose that what we call Warshall's algorithm was discovered independently by more than one person?
4. Describe how equivalence classes can be used to define the rational numbers as classes of pairs of integers and how the basic arithmetic operations on rational numbers can be defined following this approach. (See Exercise 40 in Section 9.5.)
5. Explain how Helmut Hasse used what we now call Hasse diagrams.
6. Describe some of the mechanisms used to enforce information flow policies in computer operating systems.
7. Discuss the use of the Program Evaluation and Review Technique (PERT) to schedule the tasks of a large complicated project. How widely is PERT used?
8. Discuss the use of the Critical Path Method (CPM) to find the shortest time for the completion of a project. How widely is CPM used?
9. Discuss the concept of *duality* in a lattice. Explain how duality can be used to establish new results.
10. Explain what is meant by a *modular lattice*. Describe some of the properties of modular lattices and describe how modular lattices arise in the study of projective geometry.

CHAPTER

10 Graphs

Graphs are discrete structures consisting of vertices and edges that connect these vertices. There are different kinds of graphs, depending on whether edges have directions, whether multiple edges can connect the same pair of vertices, and whether loops are allowed. Problems in almost every conceivable discipline can be solved using graph models. We will give examples to illustrate how graphs are used as models in a variety of areas. For instance, we will show how graphs are used to represent the competition of different species in an ecological niche, how graphs are used to represent who influences whom in an organization, and how graphs are used to represent the outcomes of round-robin tournaments. We will describe how graphs can be used to model acquaintanceships between people, collaboration between researchers, telephone calls between telephone numbers, and links between websites. We will show how graphs can be used to model roadmaps and the assignment of jobs to employees of an organization.

Using graph models, we can determine whether it is possible to walk down all the streets in a city without going down a street twice, and we can find the number of colors needed to color the regions of a map. Graphs can be used to determine whether a circuit can be implemented on a planar circuit board. We can distinguish between two chemical compounds with the same molecular formula but different structures using graphs. We can determine whether two computers are connected by a communications link using graph models of computer networks. Graphs with weights assigned to their edges can be used to solve problems such as finding the shortest path between two cities in a transportation network. We can also use graphs to schedule exams and assign channels to television stations. This chapter will introduce the basic concepts of graph theory and present many different graph models. To solve the wide variety of problems that can be studied using graphs, we will introduce many different graph algorithms. We will also study the complexity of these algorithms.

10.1 Graphs and Graph Models

We begin with the definition of a graph.

DEFINITION 1

A *graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

Remark: The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an **infinite graph**, and in comparison, a graph with a finite vertex set and a finite edge set is called a **finite graph**. In this book we will usually consider only finite graphs.

Now suppose that a network is made up of data centers and communication links between computers. We can represent the location of each data center by a point and each communications link by a line segment, as shown in Figure 1.

This computer network can be modeled using a graph in which the vertices of the graph represent the data centers and the edges represent communication links. In general, we visualize

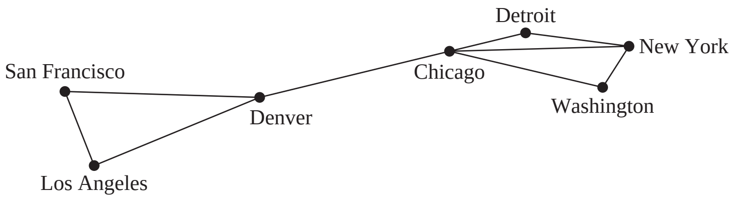


FIGURE 1 A Computer Network.

graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where the endpoints of a line segment representing an edge are the points representing the endpoints of the edge. When we draw a graph, we generally try to draw edges so that they do not cross. However, this is not necessary because any depiction using points to represent vertices and any form of connection between vertices can be used. Indeed, there are some graphs that cannot be drawn in the plane without edges crossing (see Section 10.7). The key point is that the way we draw a graph is arbitrary, as long as the correct connections between vertices are depicted.

Note that each edge of the graph representing this computer network connects two different vertices. That is, no edge connects a vertex to itself. Furthermore, no two different edges connect the same pair of vertices. A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a **simple graph**. Note that in a simple graph, each edge is associated to an unordered pair of vertices, and no other edge is associated to this same edge. Consequently, when there is an edge of a simple graph associated to $\{u, v\}$, we can also say, without possible confusion, that $\{u, v\}$ is an edge of the graph.

A computer network may contain multiple links between data centers, as shown in Figure 2. To model such networks we need graphs that have more than one edge connecting the same pair of vertices. Graphs that may have **multiple edges** connecting the same vertices are called **multigraphs**. When there are m different edges associated to the same unordered pair of vertices $\{u, v\}$, we also say that $\{u, v\}$ is an edge of multiplicity m. That is, we can think of this set of edges as m different copies of an edge $\{u, v\}$.

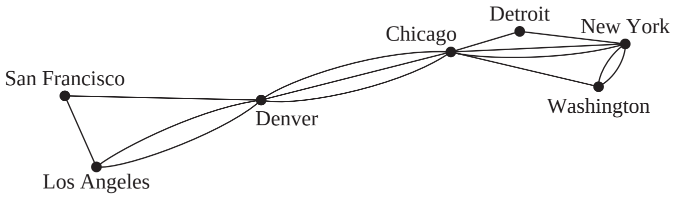


FIGURE 2 A Computer Network with Multiple Links between Data Centers.

Sometimes a communications link connects a data center with itself, perhaps a feedback loop for diagnostic purposes. Such a network is illustrated in Figure 3. To model this network we

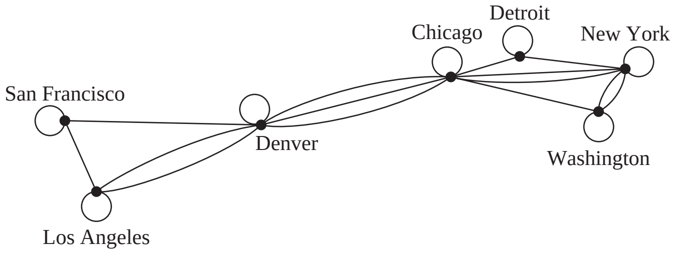


FIGURE 3 A Computer Network with Diagnostic Links.

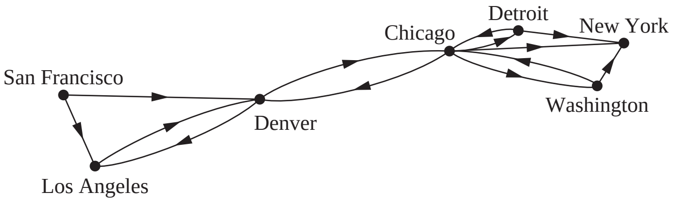


FIGURE 4 **A Communications Network with One-Way Communications Links.**

need to include edges that connect a vertex to itself. Such edges are called **loops**, and sometimes we may even have more than one loop at a vertex. Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called **pseudographs**.

So far the graphs we have introduced are **undirected graphs**. Their edges are also said to be **undirected**. However, to construct a graph model, we may find it necessary to assign directions to the edges of a graph. For example, in a computer network, some links may operate in only one direction (such links are called single duplex lines). This may be the case if there is a large amount of traffic sent to some data centers, with little or no traffic going in the opposite direction. Such a network is shown in Figure 4.

To model such a computer network we use a directed graph. Each edge of a directed graph is associated to an ordered pair. The definition of directed graph we give here is more general than the one we used in Chapter 9, where we used directed graphs to represent relations.

DEFINITION 2

A *directed graph* (or *digraph*) (V, E) consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

When we depict a directed graph with a line drawing, we use an arrow pointing from u to v to indicate the direction of an edge that starts at u and ends at v. A directed graph may contain loops and it may contain multiple directed edges that start and end at the same vertices. A directed graph may also contain directed edges that connect vertices u and v in both directions; that is, when a digraph contains an edge from u to v, it may also contain one or more edges from v to u. Note that we obtain a directed graph when we assign a direction to each edge in an undirected graph. When a directed graph has no loops and has no multiple directed edges, it is called a **simple directed graph**. Because a simple directed graph has at most one edge associated to each ordered pair of vertices (u, v), we call (u, v) an edge if there is an edge associated to it in the graph.

In some computer networks, multiple communication links between two data centers may be present, as illustrated in Figure 5. Directed graphs that may have **multiple directed edges** from a vertex to a second (possibly the same) vertex are used to model such networks. We called such graphs **directed multigraphs**. When there are m directed edges, each associated to an ordered pair of vertices (u, v), we say that (u, v) is an edge of **multiplicity** m.

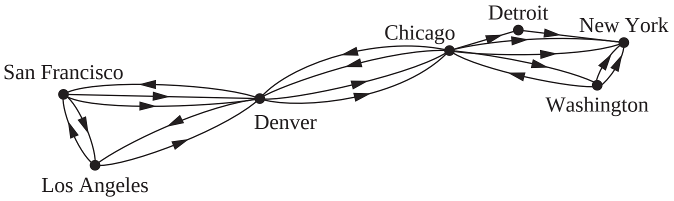


FIGURE 5 **A Computer Network with Multiple One-Way Links.**

TABLE 1 **Graph Terminology.**

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

For some models we may need a graph where some edges are undirected, while others are directed. A graph with both directed and undirected edges is called a **mixed graph**. For example, a mixed graph might be used to model a computer network containing links that operate in both directions and other links that operate only in one direction.

This terminology for the various types of graphs is summarized in Table 1. We will sometimes use the term **graph** as a general term to describe graphs with directed or undirected edges (or both), with or without loops, and with or without multiple edges. At other times, when the context is clear, we will use the term graph to refer only to undirected graphs.

Links

Because of the relatively modern interest in graph theory, and because it has applications to a wide variety of disciplines, many different terminologies of graph theory have been introduced. The reader should determine how such terms are being used whenever they are encountered. The terminology used by mathematicians to describe graphs has been increasingly standardized, but the terminology used to discuss graphs when they are used in other disciplines is still quite varied. Although the terminology used to describe graphs may vary, three key questions can help us understand the structure of a graph:

- Are the edges of the graph undirected or directed (or both)?
- If the graph is undirected, are multiple edges present that connect the same pair of vertices? If the graph is directed, are multiple directed edges present?
- Are loops present?

Answering such questions helps us understand graphs. It is less important to remember the particular terminology used.

Graph Models

Links

Graphs are used in a wide variety of models. We began this section by describing how to construct graph models of communications networks linking data centers. We will complete this section by describing some diverse graph models for some interesting applications. We will return to many of these applications later in this chapter and in Chapter 11. We will introduce additional graph models in subsequent sections of this and later chapters. Also, recall that directed graph models for some applications were introduced in Chapter 9. When we build a graph model, we need to make sure that we have correctly answered the three key questions we posed about the structure of a graph.

Can you find a subject to which graph theory has not been applied?

SOCIAL NETWORKS Graphs are extensively used to model social structures based on different kinds of relationships between people or groups of people. These social structures, and the graphs that represent them, are known as **social networks**. In these graph models, individuals or organizations are represented by vertices; relationships between individuals or organizations are represented by edges. The study of social networks is an extremely active multidisciplinary area, and many different types of relationships between people have been studied using them.

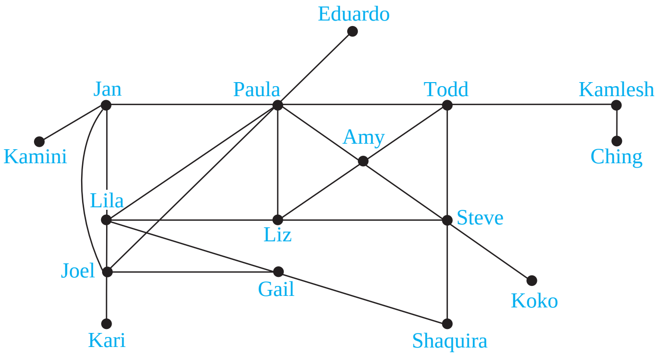


FIGURE 6 **An Acquaintanceship Graph.**

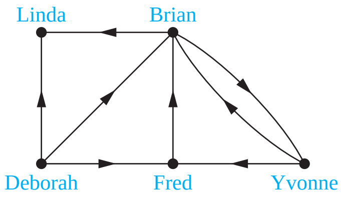


FIGURE 7 **An Influence Graph.**

We will introduce some of the most commonly studied social networks here. More information about social networks can be found in [Ne10] and [EaKl10].

EXAMPLE 1 **Acquaintanceship and Friendship Graphs** We can use a simple graph to represent whether two people know each other, that is, whether they are acquainted, or whether they are friends (either in the real world in the virtual world via a social networking site such as Facebook). Each person in a particular group of people is represented by a vertex. An undirected edge is used to connect two people when these people know each other, when we are concerned only with acquaintanceship, or whether they are friends. No multiple edges and usually no loops are used. (If we want to include the notion of self-knowledge, we would include loops.) A small acquaintanceship graph is shown in Figure 6. The acquaintanceship graph of all people in the world has more than six billion vertices and probably more than one trillion edges! We will discuss this graph further in Section 10.4. ◀

Links

EXAMPLE 2 **Influence Graphs** In studies of group behavior it is observed that certain people can influence the thinking of others. A directed graph called an **influence graph** can be used to model this behavior. Each person of the group is represented by a vertex. There is a directed edge from vertex *a* to vertex *b* when the person represented by vertex *a* can influence the person represented by vertex *b*. This graph does not contain loops and it does not contain multiple directed edges. An example of an influence graph for members of a group is shown in Figure 7. In the group modeled by this influence graph, Deborah cannot be influenced, but she can influence Brian, Fred, and Linda. Also, Yvonne and Brian can influence each other. ◀

EXAMPLE 3 **Collaboration Graphs** A **collaboration graph** is used to model social networks where two people are related by working together in a particular way. Collaboration graphs are simple graphs, as edges in these graphs are undirected and there are no multiple edges or loops. Vertices in these graphs represent people; two people are connected by an undirected edge when the people have collaborated. There are no loops nor multiple edges in these graphs. The **Hollywood graph** is a collaborator graph that represents actors by vertices and connects two actors with an edge if they have worked together on a movie or television show. The Hollywood graph is a huge graph with more than 1.5 million vertices (as of early 2011). We will discuss some aspects of the Hollywood graph later in Section 10.4.

Links

In an **academic collaboration graph**, vertices represent people (perhaps restricted to members of a certain academic community), and edges link two people if they have jointly published a paper. The collaboration graph for people who have published research papers in mathematics was found in 2004 to have more than 400,000 vertices and 675,000 edges, and these numbers have grown considerably since then. We will have more to say about this graph in Section 10.4. Collaboration graphs have also been used in sports, where two professional athletes are considered to have collaborated if they have ever played on the same team during a regular season of their sport. ◀

COMMUNICATION NETWORKS We can model different communications networks using vertices to represent devices and edges to represent the particular type of communications links of interest. We have already modeled a data network in the first part of this section.

EXAMPLE 4 **Call Graphs** Graphs can be used to model telephone calls made in a network, such as a long-distance telephone network. In particular, a directed multigraph can be used to model calls where each telephone number is represented by a vertex and each telephone call is represented by a directed edge. The edge representing a call starts at the telephone number from which the call was made and ends at the telephone number to which the call was made. We need directed edges because the direction in which the call is made matters. We need multiple directed edges because we want to represent each call made from a particular telephone number to a second number.

Links

A small telephone call graph is displayed in Figure 8(a), representing seven telephone numbers. This graph shows, for instance, that three calls have been made from 732-555-1234 to 732-555-9876 and two in the other direction, but no calls have been made from 732-555-4444 to any of the other six numbers except 732-555-0011. When we care only whether there has been a call connecting two telephone numbers, we use an undirected graph with an edge connecting telephone numbers when there has been a call between these numbers. This version of the call graph is displayed in Figure 8(b).

Call graphs that model actual calling activities can be huge. For example, one call graph studied at AT&T, which models calls during 20 days, has about 290 million vertices and 4 billion edges. We will discuss call graphs further in Section 10.4. ◀

INFORMATION NETWORKS Graphs can be used to model various networks that link particular types of information. Here, we will describe how to model the World Wide Web using a graph. We will also describe how to use a graph to model the citations in different types of documents.

EXAMPLE 5 **The Web Graph** The World Wide Web can be modeled as a directed graph where each Web page is represented by a vertex and where an edge starts at the Web page a and ends at the Web page b if there is a link on a pointing to b. Because new Web pages are created and others removed somewhere on the Web almost every second, the Web graph changes on an almost continual basis. Many people are studying the properties of the Web graph to better understand the nature of the Web. We will return to Web graphs in Section 10.4, and in Chapter 11 we will explain how the Web graph is used by the Web crawlers that search engines use to create indexes of Web pages. ◀

Links

EXAMPLE 6 **Citation Graphs** Graphs can be used to represent citations in different types of documents, including academic papers, patents, and legal opinions. In such graphs, each document is represented by a vertex, and there is an edge from one document to a second document if the

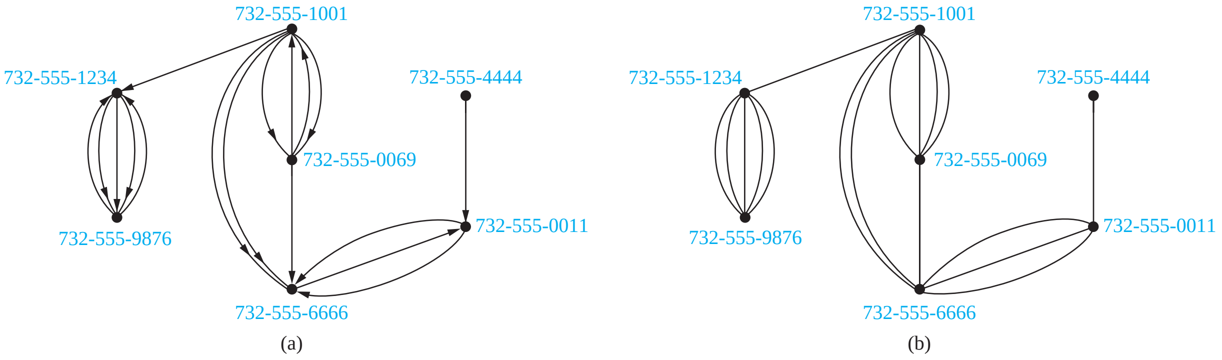


FIGURE 8 **A Call Graph.**

first document cites the second in its citation list. (In an academic paper, the citation list is the bibliography, or list of references; in a patent it is the list of previous patents that are cited; and in a legal opinion it is the list of previous opinions cited.) A citation graph is a directed graph without loops or multiple edges. ◀

SOFTWARE DESIGN APPLICATIONS Graph models are useful tools in the design of software. We will briefly describe two of these models here.

EXAMPLE 7 **Module Dependency Graphs** One of the most important tasks in designing software is how to structure a program into different parts, or modules. Understanding how the different modules of a program interact is essential not only for program design, but also for testing and maintenance of the resulting software. A **module dependency graph** provides a useful tool for understanding how different modules of a program interact. In a program dependency graph, each module is represented by a vertex. There is a directed edge from a module to a second module if the second module depends on the first. An example of a program dependency graph for a web browser is shown in Figure 9. ◀

EXAMPLE 8 **Precedence Graphs and Concurrent Processing** Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed. The dependence of statements on previous statements can be represented by a directed graph. Each statement is represented by a vertex, and there is an edge from one statement to a second statement if the second statement cannot be executed before the first statement. This resulting graph is called a **precedence graph**. A computer program and its graph are displayed in Figure 10. For instance, the graph shows that statement S_5 cannot be executed before statements S_1, S_2, and S_4 are executed. ◀

TRANSPORTATION NETWORKS We can use graphs to model many different types of transportation networks, including road, air, and rail networks, as well shipping networks.

EXAMPLE 9 **Airline Routes** We can model airline networks by representing each airport by a vertex. In particular, we can model all the flights by a particular airline each day using a directed edge to represent each flight, going from the vertex representing the departure airport to the vertex representing the destination airport. The resulting graph will generally be a directed multigraph, as there may be multiple flights from one airport to some other airport during the same day. ◀

EXAMPLE 10 **Road Networks** Graphs can be used to model road networks. In such models, vertices represent intersections and edges represent roads. When all roads are two-way and there is at most one road connecting two intersections, we can use a simple undirected graph to model the road network. However, we will often want to model road networks when some roads are one-way and when there may be more than one road between two intersections. To build such models, we use undirected edges to represent two-way roads and we use directed edges to represent

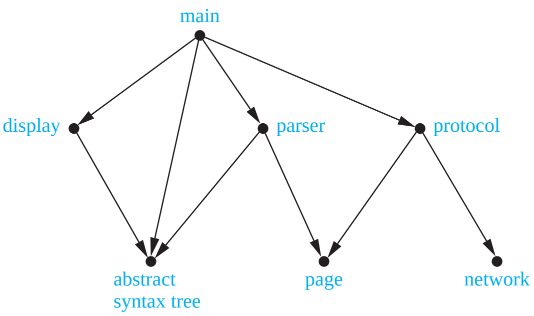
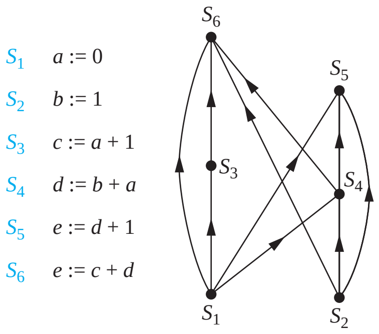


FIGURE 9 **A Module Dependency Graph.**



S_1	$a := 0$
S_2	$b := 1$
S_3	$c := a + 1$
S_4	$d := b + a$
S_5	$e := d + 1$
S_6	$e := c + d$

FIGURE 10 **A Precedence Graph.**

one-way roads. Multiple undirected edges represent multiple two-way roads connecting the same two intersections. Multiple directed edges represent multiple one-way roads that start at one intersection and end at a second intersection. Loops represent loop roads. Mixed graphs are needed to model road networks that include both one-way and two-way roads.

BIOLOGICAL NETWORKS Many aspects of the biological sciences can be modeled using graphs.

EXAMPLE 11 **Niche Overlap Graphs in Ecology** Graphs are used in many models involving the interaction of different species of animals. For instance, the competition between species in an ecosystem can be modeled using a **niche overlap graph**. Each species is represented by a vertex. An undirected edge connects two vertices if the two species represented by these vertices compete (that is, some of the food resources they use are the same). A niche overlap graph is a simple graph because no loops or multiple edges are needed in this model. The graph in Figure 11 models the ecosystem of a forest. We see from this graph that squirrels and raccoons compete but that crows and shrews do not.

Links

Extra Examples

EXAMPLE 12 **Protein Interaction Graphs** A protein interaction in a living cell occurs when two or more proteins in that cell bind to perform a biological function. Because protein interactions are crucial for most biological functions, many scientists work on discovering new proteins and understanding interactions between proteins. Protein interactions within a cell can be modeled using a **protein interaction graph** (also called a **protein–protein interaction network**), an undirected graph in which each protein is represented by a vertex, with an edge connecting the vertices representing each pair of proteins that interact. It is a challenging problem to determine genuine protein interactions in a cell, as experiments often produce false positives, which conclude that two proteins interact when they really do not. Protein interaction graphs can be used to deduce important biological information, such as by identifying the most important proteins for various functions and the functionality of newly discovered proteins.

Because there are thousands of different proteins in a typical cell, the protein interaction graph of a cell is extremely large and complex. For example, yeast cells have more than 6,000 proteins, and more than 80,000 interactions between them are known, and human cells have more than 100,000 proteins, with perhaps as many as 1,000,000 interactions between them. Additional vertices and edges are added to a protein interaction graph when new proteins and interactions between proteins are discovered. Because of the complexity of protein interaction graphs, they are often split into smaller graphs called modules that represent groups of proteins that are involved in a particular function of a cell. Figure 12 illustrates a module of the protein interaction graph described in [Bo04], comprising the complex of proteins that degrade RNA in human cells. To learn more about protein interaction graphs, see [Bo04], [Ne10], and [Hu07].

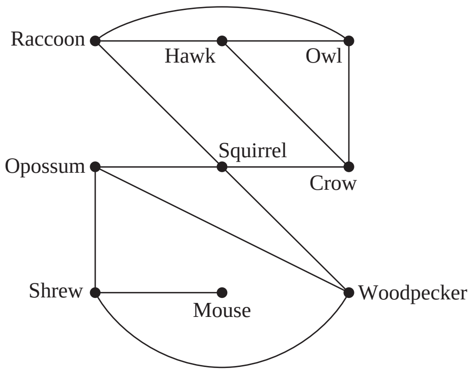


FIGURE 11 **A Niche Overlap Graph.**

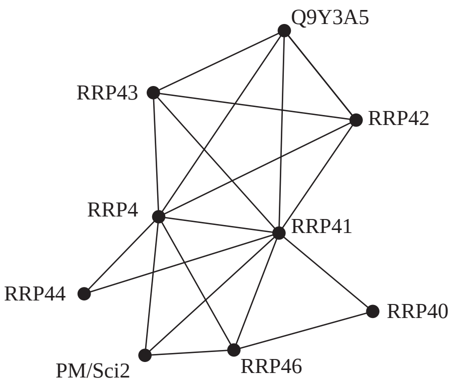


FIGURE 12 **A Module of a Protein Interaction Graph.**

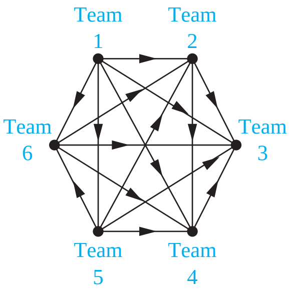


FIGURE 13 **A Graph Model of a Round-Robin Tournament.**

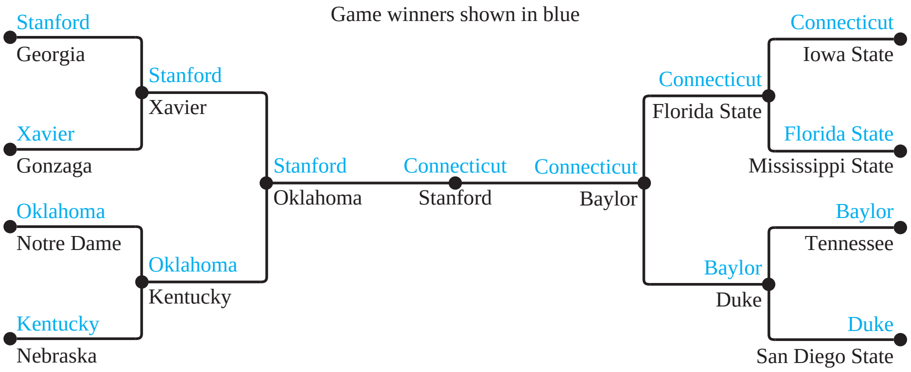


FIGURE 14 **A Single-Elimination Tournament.**

TOURNAMENTS We now give some examples that show how graphs can also be used to model different kinds of tournaments.

EXAMPLE 13 **Round-Robin Tournaments** A tournament where each team plays every other team exactly once and no ties are allowed is called a **round-robin tournament**. Such tournaments can be modeled using directed graphs where each team is represented by a vertex. Note that (a, b) is an edge if team a beats team b. This graph is a simple directed graph, containing no loops or multiple directed edges (because no two teams play each other more than once). Such a directed graph model is presented in Figure 13. We see that Team 1 is undefeated in this tournament, and Team 3 is winless. ◀

EXAMPLE 14 **Single-Elimination Tournaments** A tournament where each contestant is eliminated after one loss is called a **single-elimination tournament**. Single-elimination tournaments are often used in sports, including tennis championships and the yearly NCAA basketball championship. We can model such a tournament using a vertex to represent each game and a directed edge to connect a game to the next game the winner of this game played in. The graph in Figure 14 represents the games played by the final 16 teams in the 2010 NCAA women's basketball tournament. ◀

Exercises

1. Draw graph models, stating the type of graph (from Table 1) used, to represent airline routes where every day there are four flights from Boston to Newark, two flights from Newark to Boston, three flights from Newark to Miami, two flights from Miami to Newark, one flight from Newark to Detroit, two flights from Detroit to Newark, three flights from Newark to Washington, two flights from Washington to Newark, and one flight from Washington to Miami, with

a) an edge between vertices representing cities that have a flight between them (in either direction).

b) an edge between vertices representing cities for each flight that operates between them (in either direction).

c) an edge between vertices representing cities for each flight that operates between them (in either direction), plus a loop for a special sightseeing trip that takes off and lands in Miami.

d) an edge from a vertex representing a city where a flight starts to the vertex representing the city where it ends.

e) an edge for each flight from a vertex representing a city where the flight begins to the vertex representing the city where the flight ends.

2. What kind of graph (from Table 1) can be used to model a highway system between major cities where

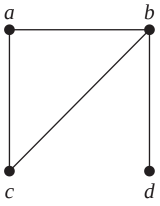
a) there is an edge between the vertices representing cities if there is an interstate highway between them?

b) there is an edge between the vertices representing cities for each interstate highway between them?

c) there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city?

For Exercises 3–9, determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops. Use your answers to determine the type of graph in Table 1 this graph is.

3.



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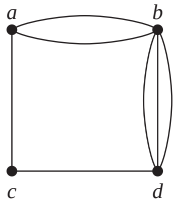

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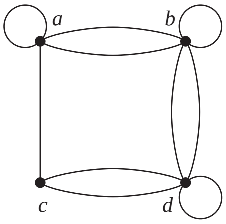

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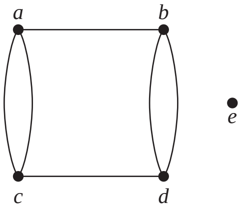

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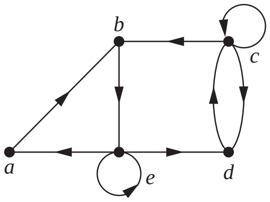

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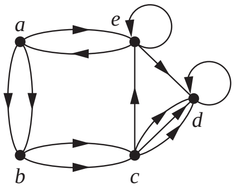
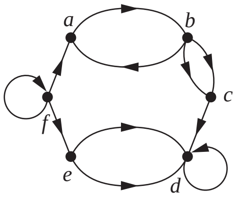


9.



10. For each undirected graph in Exercises 3–9 that is not simple, find a set of edges to remove to make it simple.

11. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

12. Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G.

13. The **intersection graph** of a collection of sets $A_1, A_2, \ldots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

a) $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{0, 1, 2, 3, 4\}$,
$A_3 = \{1, 3, 5, 7, 9\}$, $A_4 = \{5, 6, 7, 8, 9\}$,
$A_5 = \{0, 1, 8, 9\}$

b) $A_1 = \{\ldots, -4, -3, -2, -1, 0\}$,
$A_2 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
$A_3 = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$,
$A_4 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$,
$A_5 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

c) $A_1 = \{x \mid x < 0\}$,
$A_2 = \{x \mid -1 < x < 0\}$,
$A_3 = \{x \mid 0 < x < 1\}$,
$A_4 = \{x \mid -1 < x < 1\}$,
$A_5 = \{x \mid x > -1\}$,
$A_6 = \mathbf{R}$

14. Use the niche overlap graph in Figure 11 to determine the species that compete with hawks.

15. Construct a niche overlap graph for six species of birds, where the hermit thrush competes with the robin and with the blue jay, the robin also competes with the mockingbird, the mockingbird also competes with the blue jay, and the nuthatch competes with the hairy woodpecker.

16. Draw the acquaintanceship graph that represents that Tom and Patricia, Tom and Hope, Tom and Sandy, Tom and Amy, Tom and Marika, Jeff and Patricia, Jeff and Mary, Patricia and Hope, Amy and Hope, and Amy and Marika know each other, but none of the other pairs of people listed know each other.

17. We can use a graph to represent whether two people were alive at the same time. Draw such a graph to represent whether each pair of the mathematicians and computer scientists with biographies in the first five chapters of this book who died before 1900 were contemporaneous. (Assume two people lived at the same time if they were alive during the same year.)

18. Who can influence Fred and whom can Fred influence in the influence graph in Example 2?

19. Construct an influence graph for the board members of a company if the President can influence the Director of Research and Development, the Director of Marketing, and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

20. Which other teams did Team 4 beat and which teams beat Team 4 in the round-robin tournament represented by the graph in Figure 13?

21. In a round-robin tournament the Tigers beat the Blue Jays, the Tigers beat the Cardinals, the Tigers beat the Orioles, the Blue Jays beat the Cardinals, the Blue Jays beat the Orioles, and the Cardinals beat the Orioles. Model this outcome with a directed graph.

22. Construct the call graph for a set of seven telephone numbers 555-0011, 555-1221, 555-1333, 555-8888, 555-2222, 555-0091, and 555-1200 if there were three calls from 555-0011 to 555-8888 and two calls from 555-8888 to 555-0011, two calls from 555-2222 to 555-0091, two calls from 555-1221 to each of the other numbers, and one call from 555-1333 to each of 555-0011, 555-1221, and 555-1200.

23. Explain how the two telephone call graphs for calls made during the month of January and calls made during the month of February can be used to determine the new telephone numbers of people who have changed their telephone numbers.

24. **a)** Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

b) Describe a graph that models the electronic mail sent in a network in a particular week.

25. How can a graph that models e-mail messages sent in a network be used to find people who have recently changed their primary e-mail address?

26. How can a graph that models e-mail messages sent in a network be used to find electronic mail mailing lists used to send the same message to many different e-mail addresses?

27. Describe a graph model that represents whether each person at a party knows the name of each other person at the party. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

28. Describe a graph model that represents a subway system in a large city. Should edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

29. For each course at a university, there may be one or more other courses that are its prerequisites. How can a graph be used to model these courses and which courses are prerequisites for which courses? Should edges be directed or undirected? Looking at the graph model, how can we find courses that do not have any prerequisites and how can we find courses that are not the prerequisite for any other courses?

30. Describe a graph model that represents the positive recommendations of movie critics, using vertices to represent both these critics and all movies that are currently being shown.

31. Describe a graph model that represents traditional marriages between men and women. Does this graph have any special properties?

32. Which statements must be executed before S_6 is executed in the program in Example 8? (Use the precedence graph in Figure 10.)

33. Construct a precedence graph for the following program:

$$\begin{aligned}
S_1&: x := 0\\
S_2&: x := x + 1\\
S_3&: y := 2\\
S_4&: z := y\\
S_5&: x := x + 2\\
S_6&: y := x + z\\
S_7&: z := 4
\end{aligned}$$

34. Describe a discrete structure based on a graph that can be used to model airline routes and their flight times. [*Hint:* Add structure to a directed graph.]

35. Describe a discrete structure based on a graph that can be used to model relationships between pairs of individuals in a group, where each individual may either like, dislike, or be neutral about another individual, and the reverse relationship may be different. [*Hint:* Add structure to a directed graph. Treat separately the edges in opposite directions between vertices representing two individuals.]

36. Describe a graph model that can be used to represent all forms of electronic communication between two people in a single graph. What kind of graph is needed?

10.2 Graph Terminology and Special Types of Graphs

Introduction

Links

We introduce some of the basic vocabulary of graph theory in this section. We will use this vocabulary later in this chapter when we solve many different types of problems. One such problem involves determining whether a graph can be drawn in the plane so that no two of its edges cross. Another example is deciding whether there is a one-to-one correspondence between the vertices of two graphs that produces a one-to-one correspondence between the edges of the graphs. We will also introduce several important families of graphs often used as examples and in models. Several important applications will be described where these special types of graphs arise.

Basic Terminology

First, we give some terminology that describes the vertices and edges of undirected graphs.

DEFINITION 1

Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.

We will also find useful terminology describing the set of vertices adjacent to a particular vertex of a graph.

DEFINITION 2

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

To keep track of how many edges are incident to a vertex, we make the following definition.

DEFINITION 3

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

EXAMPLE 1 What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, and $\deg(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$. In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$. ◀

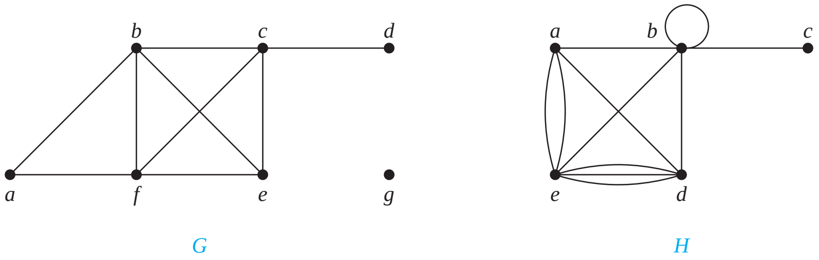


FIGURE 1 **The Undirected Graphs G and H.**

A vertex of degree zero is called **isolated**. It follows that an isolated vertex is not adjacent to any vertex. Vertex g in graph G in Example 1 is isolated. A vertex is **pendant** if and only if it has degree one. Consequently, a pendant vertex is adjacent to exactly one other vertex. Vertex d in graph G in Example 1 is pendant.

Examining the degrees of vertices in a graph model can provide useful information about the model, as Example 2 shows.

EXAMPLE 2 What does the degree of a vertex in a niche overlap graph (introduced in Example 11 in Section 10.1) represent? Which vertices in this graph are pendant and which are isolated? Use the niche overlap graph shown in Figure 11 of Section 10.1 to interpret your answers.

Solution: There is an edge between two vertices in a niche overlap graph if and only if the two species represented by these vertices compete. Hence, the degree of a vertex in a niche overlap graph is the number of species in the ecosystem that compete with the species represented by this vertex. A vertex is pendant if the species competes with exactly one other species in the ecosystem. Finally, the vertex representing a species is isolated if this species does not compete with any other species in the ecosystem.

For instance, the degree of the vertex representing the squirrel in the niche overlap graph in Figure 11 in Section 10.1 is four, because the squirrel competes with four other species: the crow, the opossum, the raccoon, and the woodpecker. In this niche overlap graph, the mouse is the only species represented by a pendant vertex, because the mouse competes only with the shrew and all other species compete with at least two other species. There are no isolated vertices in the graph in this niche overlap graph because every species in this ecosystem competes with at least one other species. ◀

What do we get when we add the degrees of all the vertices of a graph $G = (V, E)$? Each edge contributes two to the sum of the degrees of the vertices because an edge is incident with exactly two (possibly equal) vertices. This means that the sum of the degrees of the vertices is twice the number of edges. We have the result in Theorem 1, which is sometimes called the handshaking theorem (and is also often known as the handshaking lemma), because of the analogy between an edge having two endpoints and a handshake involving two hands.

THEOREM 1 **THE HANDSHAKING THEOREM** Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

EXAMPLE 3 How many edges are there in a graph with 10 vertices each of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$. ◀

Theorem 1 shows that the sum of the degrees of the vertices of an undirected graph is even. This simple fact has many consequences, one of which is given as Theorem 2.

THEOREM 2 An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

Because $\deg(v)$ is even for $v \in V_1$, the first term in the right-hand side of the last equality is even. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is $2m$. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are an even number of vertices of odd degree. ◁

Terminology for graphs with directed edges reflects the fact that edges in directed graphs have directions.

DEFINITION 4 When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to* v and v is said to be *adjacent from* u. The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v). The initial vertex and terminal vertex of a loop are the same.

Because the edges in graphs with directed edges are ordered pairs, the definition of the degree of a vertex can be refined to reflect the number of edges with this vertex as the initial vertex and as the terminal vertex.

DEFINITION 5 In a graph with directed edges the *in-degree of a vertex v*, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of v*, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

EXAMPLE 4 Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.

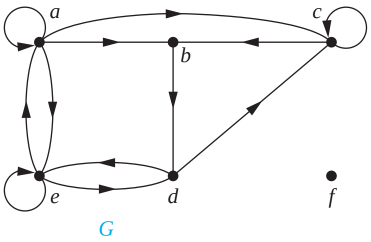


FIGURE 2 **The Directed Graph *G*.**

Solution: The in-degrees in G are $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, and $\deg^-(f) = 0$. The out-degrees are $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, and $\deg^+(f) = 0$. ◀

Because each edge has an initial vertex and a terminal vertex, the sum of the in-degrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same. Both of these sums are the number of edges in the graph. This result is stated as Theorem 3.

THEOREM 3 Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

There are many properties of a graph with directed edges that do not depend on the direction of its edges. Consequently, it is often useful to ignore these directions. The undirected graph that results from ignoring directions of edges is called the **underlying undirected graph**. A graph with directed edges and its underlying undirected graph have the same number of edges.

Some Special Simple Graphs

We will now introduce several classes of simple graphs. These graphs are often used as examples and arise in many applications.

EXAMPLE 5 **Complete Graphs** A **complete graph on n vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**.

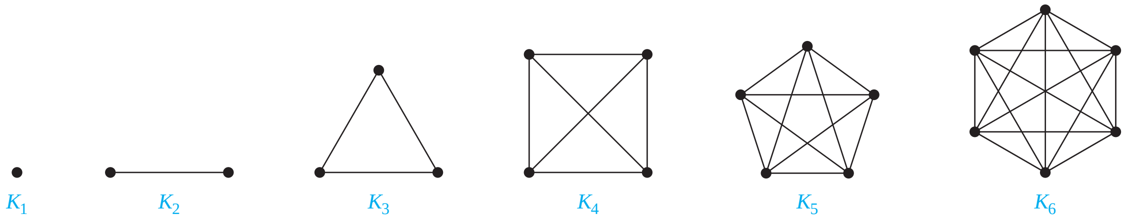


FIGURE 3 **The Graphs K_n for $1 \leq n \leq 6$.**

EXAMPLE 6 **Cycles** A **cycle C_n**, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4.

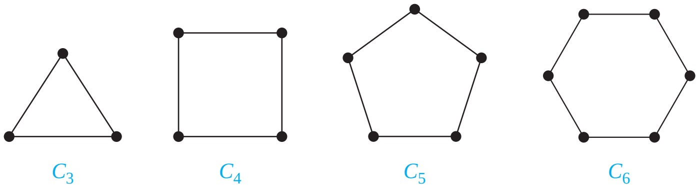


FIGURE 4 **The Cycles C_3, C_4, C_5, and C_6.**

EXAMPLE 7 **Wheels** We obtain a **wheel W_n** when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5, and W_6 are displayed in Figure 5.

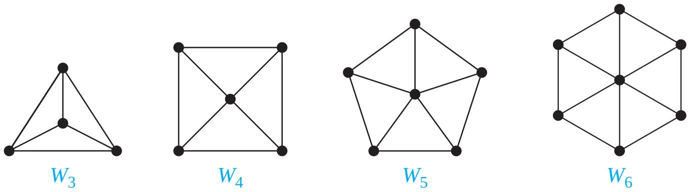


FIGURE 5 **The Wheels W_3, W_4, W_5, and W_6.**

EXAMPLE 8 ***n*-Cubes** An ***n*-dimensional hypercube**, or ***n*-cube**, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. We display Q_1, Q_2, and Q_3 in Figure 6.

Note that you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. In Figure 6, Q_3 is constructed from Q_2 by drawing two copies of Q_2 as the top and bottom faces of Q_3, adding 0 at the beginning of the label of each vertex in the bottom face and 1 at the beginning of the label of each vertex in the top face. (Here, by *face* we mean a face of a cube in three-dimensional space. Think of drawing the graph Q_3 in three-dimensional space with copies of Q_2 as the top and bottom faces of a cube and then drawing the projection of the resulting depiction in the plane.)

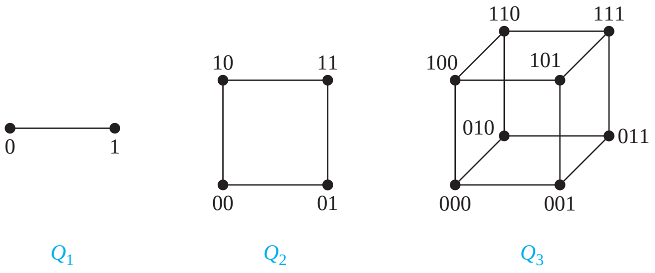


FIGURE 6 **The n-cube Q_n, $n = 1, 2, 3$.**

Bipartite Graphs

Links

Sometimes a graph has the property that its vertex set can be divided into two disjoint subsets such that each edge connects a vertex in one of these subsets to a vertex in the other subset. For example, consider the graph representing marriages between men and women in a village, where each person is represented by a vertex and a marriage is represented by an edge. In this graph, each edge connects a vertex in the subset of vertices representing males and a vertex in the subset of vertices representing females. This leads us to Definition 5.

DEFINITION 6 A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

In Example 9 we will show that C_6 is bipartite, and in Example 10 we will show that K_3 is not bipartite.

EXAMPLE 9 C_6 is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2. ◀

EXAMPLE 10 K_3 is not bipartite. To verify this, note that if we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge. ◀

EXAMPLE 11 Are the graphs G and H displayed in Figure 8 bipartite?

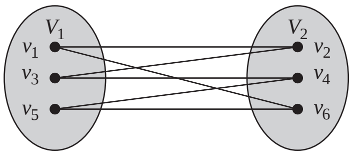


FIGURE 7 **Showing That C_6 Is Bipartite.**

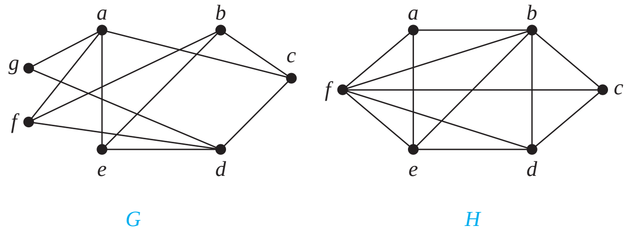


FIGURE 8 **The Undirected Graphs G and H.**

Solution: Graph G is bipartite because its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset. (Note that for G to be bipartite it is not necessary that every vertex in $\{a, b, d\}$ be adjacent to every vertex in $\{c, e, f, g\}$. For instance, b and g are not adjacent.)

Graph H is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. (The reader should verify this by considering the vertices a, b, and f.) ◀

Theorem 4 provides a useful criterion for determining whether a graph is bipartite.

THEOREM 4 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: First, suppose that $G = (V, E)$ is a bipartite simple graph. Then $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2. If we assign one color to each vertex in V_1 and a second color to each vertex in V_2, then no two adjacent vertices are assigned the same color.

Now suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color. Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$. Furthermore, every edge connects a vertex in V_1 and a vertex in V_2 because no two adjacent vertices are either both in V_1 or both in V_2. Consequently, G is bipartite. ◁

We illustrate how Theorem 4 can be used to determine whether a graph is bipartite in Example 12.

EXAMPLE 12 Use Theorem 4 to determine whether the graphs in Example 11 are bipartite.

Solution: We first consider the graph G. We will try to assign one of two colors, say red and blue, to each vertex in G so that no edge in G connects a red vertex and a blue vertex. Without loss of generality we begin by arbitrarily assigning red to a. Then, we must assign blue to c, e, f, and g, because each of these vertices is adjacent to a. To avoid having an edge with two blue endpoints, we must assign red to all the vertices adjacent to either c, e, f, or g. This means that we must assign red to both b and d (and means that a must be assigned red, which it already has been). We have now assigned colors to all vertices, with a, b, and d red and c, e, f, and g blue. Checking all edges, we see that every edge connects a red vertex and a blue vertex. Hence, by Theorem 4 the graph G is bipartite.

Next, we will try to assign either red or blue to each vertex in H so that no edge in H connects a red vertex and a blue vertex. Without loss of generality we arbitrarily assign red to a. Then, we must assign blue to b, e, and f, because each is adjacent to a. But this is not possible because e and f are adjacent, so both cannot be assigned blue. This argument shows that we cannot assign one of two colors to each of the vertices of H so that no adjacent vertices are assigned the same color. It follows by Theorem 4 that H is not bipartite. ◀

Theorem 4 is an example of a result in the part of graph theory known as graph colorings. Graph colorings is an important part of graph theory with important applications. We will study graph colorings further in Section 10.8.

Another useful criterion for determining whether a graph is bipartite is based on the notion of a path, a topic we study in Section 10.4. A graph is bipartite if and only if it is not possible to start at a vertex and return to this vertex by traversing an odd number of distinct edges. We will make this notion more precise when we discuss paths and circuits in graphs in Section 10.4 (see Exercise 63 in that section).

EXAMPLE 13 **Complete Bipartite Graphs** A **complete bipartite graph $K_{m,n}$** is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$ are displayed in Figure 9. ◀

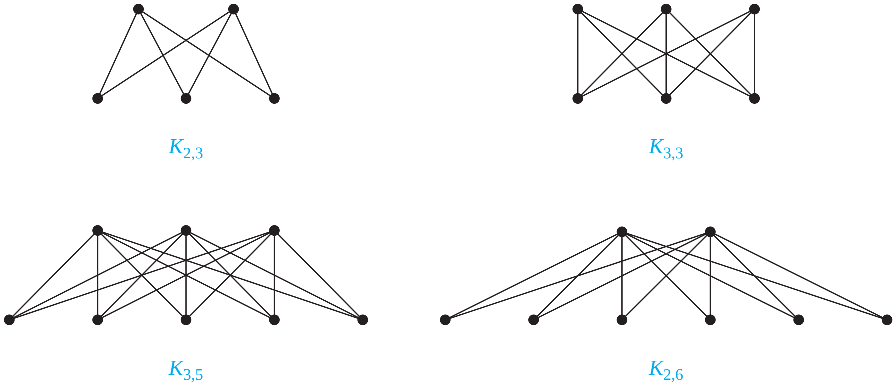


FIGURE 9 **Some Complete Bipartite Graphs.**

Bipartite Graphs and Matchings

Bipartite graphs can be used to model many types of applications that involve matching the elements of one set to elements of another, as Example 14 illustrates.

EXAMPLE 14 **Job Assignments** Suppose that there are m employees in a group and n different jobs that need to be done, where $m \geq n$. Each employee is trained to do one or more of these n jobs. We would like to assign an employee to each job. To help with this task, we can use a graph to model employee capabilities. We represent each employee by a vertex and each job by a vertex. For each employee, we include an edge from that employee to all jobs that the employee has been trained to do. Note that the vertex set of this graph can be partitioned into two disjoint sets, the set of employees and the set of jobs, and each edge connects an employee to a job. Consequently, this graph is bipartite, where the bipartition is (E, J) where E is the set of employees and J is the set of jobs. We now consider two different scenarios.

First, suppose that a group has four employees: Alvarez, Berkowitz, Chen, and Davis; and suppose that four jobs need to be done to complete Project 1: requirements, architecture, implementation, and testing. Suppose that Alvarez has been trained to do requirements and testing; Berkowitz has been trained to do architecture, implementation, and testing; Chen has been trained to do requirements, architecture, and implementation; and Davis has only been trained to do requirements. We model these employee capabilities using the bipartite graph in Figure 10(a).

Second, suppose that a group has second group also has four employees: Washington, Xuan, Ybarra, and Ziegler; and suppose that the same four jobs need to be done to complete Project 2 as are needed to complete Project 1. Suppose that Washington has been trained to do architecture; Xuan has been trained to do requirements, implementation, and testing; Ybarra has been trained to do architecture; and Ziegler has been trained to do requirements, architecture and testing. We model these employee capabilities using the bipartite graph in Figure 10(b).

To complete Project 1, we must assign an employee to each job so that every job has an employee assigned to it, and so that no employee is assigned more than one job. We can do this by assigning Alvarez to testing, Berkowitz to implementation, Chen to architecture, and Davis to requirements, as shown in Figure 10(a) (where blue lines show this assignment of jobs).

To complete Project 2, we must also assign an employee to each job so that every job has an employee assigned to it and no employee is assigned more than one job. However, this is

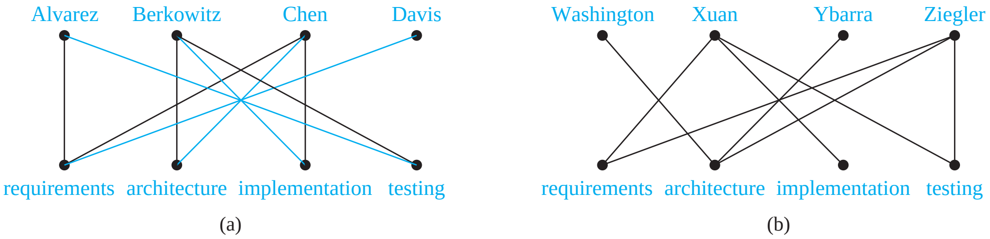


FIGURE 10 **Modeling the Jobs for Which Employees Have Been Trained.**

impossible because there are only two employees, Xuan and Ziegler, who have been trained for at least one of the three jobs of requirements, implementation, and testing. Consequently, there is no way to assign three different employees to these three job so that each job is assigned an employee with the appropriate training. ◀

Finding an assignment of jobs to employees can be thought of as finding a matching in the graph model, where a **matching** M in a simple graph $G = (V, E)$ is a subset of the set E of edges of the graph such that no two edges are incident with the same vertex. In other words, a matching is a subset of edges such that if $\{s, t\}$ and $\{u, v\}$ are distinct edges of the matching, then s, t, u, and v are distinct. A vertex that is the endpoint of an edge of a matching M is said to be **matched** in M; otherwise it is said to be **unmatched**. A **maximum matching** is a matching with the largest number of edges. We say that a matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching from** V_1 **to** V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$. For example, to assign jobs to employees so that the largest number of jobs are assigned employees, we seek a maximum matching in the graph that models employee capabilities. To assign employees to all jobs we seek a complete matching from the set of jobs to the set of employees. In Example 14, we found a complete matching from the set of jobs to the set of employees for Project 1, and this matching is a maximun matching, and we showed that no complete matching exists from the set of jobs to the employees for Project 2.

We now give an example of how matchings can be used to model marriages.

EXAMPLE 15 **Marriages on an Island** Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse. A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair. A maximum matching is a largest possible set of married couples, and a complete matching of V_1 is a set of married couples where every man is married, but possibly not all women. ◀

NECESSARY AND SUFFICIENT CONDITIONS FOR COMPLETE MATCHINGS We now turn our attention to the question of determining whether a complete matching from V_1 to V_2 exists when (V_1, V_2) is a bipartition of a bipartite graph $G = (V, E)$. We will introduce a theorem that provides a set of necessary and sufficient conditions for the existence of a complete matching. This theorem was proved by Philip Hall in 1935.

Hall's marriage theorem is an example of a theorem where obvious necessary conditions are sufficient too.

THEOREM 5 HALL'S MARRIAGE THEOREM The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Proof: We first prove the *only if* part of the theorem. To do so, suppose that there is a complete matching M from V_1 to V_2. Then, if $A \subseteq V_1$, for every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2. Consequently, there are at least as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1. It follows that $|N(A)| \geq |A|$.

To prove the *if* part of the theorem, the more difficult part, we need to show that if $|N(A)| \geq |A|$ for all $A \subseteq V_1$, then there is a complete matching M from V_1 to V_2. We will use strong induction on $|V_1|$ to prove this.

Basis step: If $|V_1| = 1$, then V_1 contains a single vertex v_0. Because $|N(\{v_0\})| \geq |\{v_0\}| = 1$, there is at least one edge connecting v_0 and a vertex $w_0 \in V_2$. Any such edge forms a complete matching from V_1 to V_2.

Inductive step: We first state the inductive hypothesis.

Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2), and $|V_1| = j \leq k$, then there is a complete matching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met.

Now suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$. We will prove that the inductive holds using a proof by cases, using two case. Case *(i)* applies when for all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2. Case *(ii)* applies when for some j with $1 \leq j \leq k$ there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2. Because either Case *(i)* or Case *(ii)* holds, we need only consider these cases to complete the inductive step.

Case (i): Suppose that for all integers j with $1 \leq j \leq k$, the vertices in every subset of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2. Then, we select a vertex $v \in W_1$ and an element $w \in N(\{v\})$, which must exist by our assumption that $|N(\{v\}| \geq |\{v\}| = 1$. We delete v and w and all edges incident to them from H. This produces a bipartite graph H' with bipartition $(W_1 - \{v\}, W_2 - \{w\})$. Because $|W_1 - \{v\}| = k$, the inductive hypothesis tells us there is a complete matching from $W_1 - \{v\}$ to $W_2 - \{w\}$. Adding the edge from v to w to this complete matching produces a complete matching from W_1 to W_2.

Case (ii): Suppose that for some j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2. Let W_2' be the set of these neighbors. Then, by the inductive hypothesis there is a complete matching from W_1' to W_2'. Remove these $2j$ vertices from W_1 and W_2 and all incident edges to produce a bipartite graph K with bipartition $(W_1 - W_1', W_2 - W_2')$.

We will show that the graph K satisfies the condition $|N(A)| \geq |A|$ for all subsets A of $W_1 - W_1'$. If not, there would be a subset of t vertices of $W_1 - W_1'$ where $1 \leq t \leq k + 1 - j$ such that the vertices in this subset have fewer than t vertices of $W_2 - W_2'$ as neighbors. Then, the set of $j + t$ vertices of W_1 consisting of these t vertices together with the j vertices we removed from W_1 has fewer than $j + t$ neighbors in W_2, contradicting the hypothesis that $|N(A)| \geq |A|$ for all $A \subseteq W_1$.

PHILIP HALL (1904–1982) Philip Hall grew up in London, where his mother was a dressmaker. He won a scholarship for board school reserved for needy children, and later a scholarship to King's College of Cambridge University. He received his bachelors degree in 1925. In 1926, unsure of his career goals, he took a civil service exam, but decided to continue his studies at Cambridge after failing.

In 1927 Hall was elected to a fellowship at King's College; soon after, he made his first important discovery in group theory. The results he proved are now known as Hall's theorems. In 1933 he was appointed as a Lecturer at Cambridge, where he remained until 1941. During World War II he worked as a cryptographer at Bletchley Park breaking Italian and Japanese codes. At the end of the war, Hall returned to King's College, and was soon promoted. In 1953 he was appointed to the Sadleirian Chair. His work during the 1950s proved to be extremely influential to the rapid development of group theory during the 1960s.

Hall loved poetry and recited it beautifully in Italian and Japanese, as well as English. He was interested in art, music, and botany. He was quite shy and disliked large groups of people. Hall had an incredibly broad and varied knowledge, and was respected for his integrity, intellectual standards, and judgement. He was beloved by his students.

Hence, by the inductive hypothesis, the graph K has a complete matching. Combining this complete matching with the complete matching from W_1' to W_2', we obtain a complete matching from W_1 to W_2.

We have shown that in both cases there is a complete matching from W_1 to W_2. This completes the inductive step and completes the proof. ◁

We have used strong induction to prove Hall's marriage theorem. Although our proof is elegant, it does have some drawbacks. In particular, we cannot construct an algorithm based on this proof that finds a complete matching in a bipartite graph. For a constructive proof that can be used as the basis of an algorithm, see [Gi85].

Some Applications of Special Types of Graphs

We conclude this section by introducing some additional graph models that involve the special types of graph we have discussed in this section.

EXAMPLE 16 Links

Local Area Networks The various computers in a building, such as minicomputers and personal computers, as well as peripheral devices such as printers and plotters, can be connected using a *local area network*. Some of these networks are based on a *star topology*, where all devices are connected to a central control device. A local area network can be represented using a complete bipartite graph $K_{1,n}$, as shown in Figure 11(a). Messages are sent from device to device through the central control device.

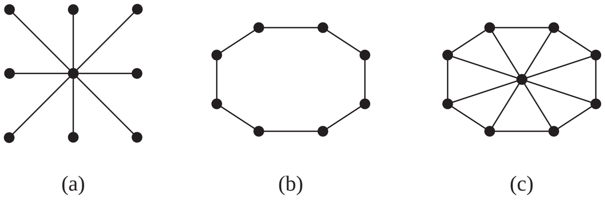


FIGURE 11 **Star, Ring, and Hybrid Topologies for Local Area Networks.**

Other local area networks are based on a *ring topology*, where each device is connected to exactly two others. Local area networks with a ring topology are modeled using n-cycles, C_n, as shown in Figure 11(b). Messages are sent from device to device around the cycle until the intended recipient of a message is reached.

Finally, some local area networks use a hybrid of these two topologies. Messages may be sent around the ring, or through a central device. This redundancy makes the network more reliable. Local area networks with this redundancy can be modeled using wheels W_n, as shown in Figure 11(c). ◀

EXAMPLE 17 **Interconnection Networks for Parallel Computation** For many years, computers executed programs one operation at a time. Consequently, the algorithms written to solve problems were designed to perform one step at a time; such algorithms are called **serial**. (Almost all algorithms described in this book are serial.) However, many computationally intense problems, such as weather simulations, medical imaging, and cryptanalysis, cannot be solved in a reasonable amount of time using serial operations, even on a supercomputer. Furthermore, there is a physical limit to how fast a computer can carry out basic operations, so there will always be problems that cannot be solved in a reasonable length of time using serial operations.

Parallel processing, which uses computers made up of many separate processors, each with its own memory, helps overcome the limitations of computers with a single processor. **Parallel algorithms**, which break a problem into a number of subproblems that can be solved

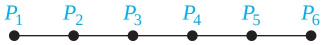


FIGURE 12 A Linear Array for Six Processors.

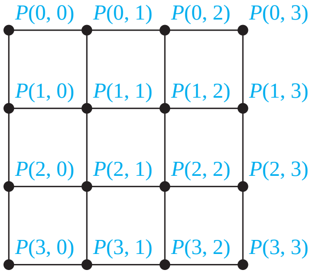


FIGURE 13 A Mesh Network for 16 Processors.

concurrently, can then be devised to rapidly solve problems using a computer with multiple processors. In a parallel algorithm, a single instruction stream controls the execution of the algorithm, sending subproblems to different processors, and directs the input and output of these subproblems to the appropriate processors.

When parallel processing is used, one processor may need output generated by another processor. Consequently, these processors need to be interconnected. We can use the appropriate type of graph to represent the interconnection network of the processors in a computer with multiple processors. In the following discussion, we will describe the most commonly used types of interconnection networks for parallel processors. The type of interconnection network used to implement a particular parallel algorithm depends on the requirements for exchange of data between processors, the desired speed, and, of course, the available hardware.

The simplest, but most expensive, network-interconnecting processors include a two-way link between each pair of processors. This network can be represented by K_n, the complete graph on n vertices, when there are n processors. However, there are serious problems with this type of interconnection network because the required number of connections is so large. In reality, the number of direct connections to a processor is limited, so when there are a large number of processors, a processor cannot be linked directly to all others. For example, when there are 64 processors, $C(64, 2) = 2016$ connections would be required, and each processor would have to be directly connected to 63 others.

On the other hand, perhaps the simplest way to interconnect n processors is to use an arrangement known as a **linear array**. Each processor P_i, other than P_1 and P_n, is connected to its neighbors P_{i-1} and P_{i+1} via a two-way link. P_1 is connected only to P_2, and P_n is connected only to P_{n-1}. The linear array for six processors is shown in Figure 12. The advantage of a linear array is that each processor has at most two direct connections to other processors. The disadvantage is that it is sometimes necessary to use a large number of intermediate links, called **hops**, for processors to share information.

The **mesh network** (or **two-dimensional array**) is a commonly used interconnection network. In such a network, the number of processors is a perfect square, say $n = m^2$. The n processors are labeled $P(i, j)$, $0 \leq i \leq m - 1$, $0 \leq j \leq m - 1$. Two-way links connect processor $P(i, j)$ with its four neighbors, processors $P(i \pm 1, j)$ and $P(i, j \pm 1)$, as long as these are processors in the mesh. (Note that four processors, on the corners of the mesh, have only two adjacent processors, and other processors on the boundaries have only three neighbors. Sometimes a variant of a mesh network in which every processor has exactly four connections is used; see Exercise 72.) The mesh network limits the number of links for each processor. Communication between some pairs of processors requires $O(\sqrt{n}) = O(m)$ intermediate links. (See Exercise 73.) The graph representing the mesh network for 16 processors is shown in Figure 13.

One important type of interconnection network is the hypercube. For such a network, the number of processors is a power of 2, $n = 2^m$. The n processors are labeled $P_0, P_1, \ldots, P_{n-1}$. Each processor has two-way connections to m other processors. Processor P_i is linked to the processors with indices whose binary representations differ from the binary representation of i

in exactly one bit. The hypercube network balances the number of direct connections for each processor and the number of intermediate connections required so that processors can communicate. Many computers have been built using a hypercube network, and many parallel algorithms have been devised that use a hypercube network. The graph Q_m, the m-cube, represents the hypercube network with $n = 2^m$ processors. Figure 14 displays the hypercube network for eight processors. (Figure 14 displays a different way to draw Q_3 than was shown in Figure 6.) ◀

New Graphs from Old

Sometimes we need only part of a graph to solve a problem. For instance, we may care only about the part of a large computer network that involves the computer centers in New York, Denver, Detroit, and Atlanta. Then we can ignore the other computer centers and all telephone lines not linking two of these specific four computer centers. In the graph model for the large network, we can remove the vertices corresponding to the computer centers other than the four of interest, and we can remove all edges incident with a vertex that was removed. When edges and vertices are removed from a graph, without removing endpoints of any remaining edges, a smaller graph is obtained. Such a graph is called a **subgraph** of the original graph.

DEFINITION 7 A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

Given a set of vertices of a graph, we can form a subgraph of this graph with these vertices and the edges of the graph that connect them.

DEFINITION 8 Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.

EXAMPLE 18 The graph G shown in Figure 15 is a subgraph of K_5. If we add the edge connecting c and e to G, we obtain the subgraph induced by $W = \{a, b, c, e\}$. ◀

REMOVING OR ADDING EDGES OF A GRAPH Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by $G - e$, has the same vertex set V as G. Its edge set is $E - e$. Hence,

$$G - e = (V, E - \{e\}).$$

Similarly, if E' is a subset of E, we can produce a subgraph of G by removing the edges in E' from the graph. The resulting subgraph has the same vertex set V as G. Its edge set is $E - E'$.

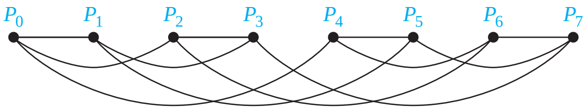


FIGURE 14 **A Hypercube Network for Eight Processors.**

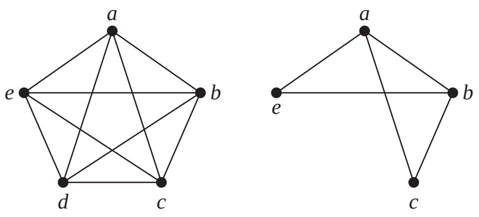


FIGURE 15 **A Subgraph of K_5.**

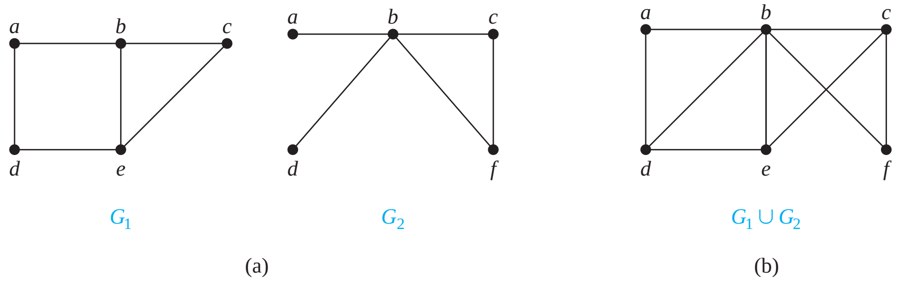


FIGURE 16 **(a) The Simple Graphs G_1 and G_2; (b) Their Union $G_1 \cup G_2$.**

We can also add an edge e to a graph to produce a new larger graph when this edge connects two vertices already in G. We denote by $G + e$ the new graph produced by adding a new edge e, connecting two previously nonincident vertices, to the graph G Hence,

$$G + e = (V, E \cup \{e\}).$$

The vertex set of $G + e$ is the same as the vertex set of G and the edge set is the union of the edge set of G and the set $\{e\}$.

EDGE CONTRACTIONS Sometimes when we remove an edge from a graph, we do not want to retain the endpoints of this edge as separate vertices in the resulting subgraph. In such a case we perform an **edge contraction** which removes an edge e with endpoints u and v and merges u and w into a new single vertex w, and for each edge with u or v as an endpoint replaces the edge with one with w as endpoint in place of u or v and with the same second endpoint. Hence, the contraction of the edge e with endpoints u and v in the graph $G = (V, E)$ produces a new graph $G' = (V', E')$ (which is not a subgraph of G), where $V' = V - \{u, v\} \cup \{w\}$ and E' contains the edges in E which do not have either u or v as endpoints and an edge connecting w to every neighbor of either u or v in V. For example, the contraction of the edge connecting the vertices e and c in the graph G_1 in Figure 16 produces a new graph G_1' with vertices a, b, d, and w. As in G_1, there is an edge in G_1' connecting a and b and an edge connecting a and d. There also is an edge in G_1' that connects b and w that replaces the edges connecting b and c and connecting b and e in G_1 and an edge in G_1' that connects d and w replacing the edge connecting d and e in G_1.

REMOVING VERTICES FROM A GRAPH When we remove a vertex v and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$. Observe that $G - v = (V - v, E')$, where E' is the set of edges of G not incident to v. Similarly, if V' is a subset of V, then the graph $G - V'$ is the subgraph $(V - V', E')$, where E' is the set of edges of G not incident to a vertex in V'.

GRAPH UNIONS Two or more graphs can be combined in various ways. The new graph that contains all the vertices and edges of these graphs is called the **union** of the graphs. We will give a more formal definition for the union of two simple graphs.

DEFINITION 9 The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

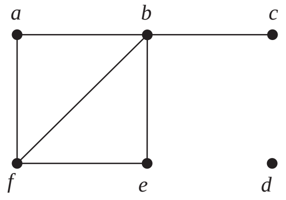
EXAMPLE 19 Find the union of the graphs G_1 and G_2 shown in Figure 16(a). ◀

Solution: The vertex set of the union $G_1 \cup G_2$ is the union of the two vertex sets, namely, $\{a, b, c, d, e, f\}$. The edge set of the union is the union of the two edge sets. The union is displayed in Figure 16(b).

Exercises

In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

1.



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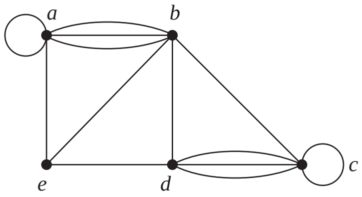

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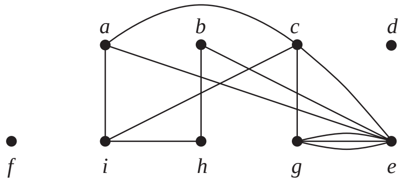


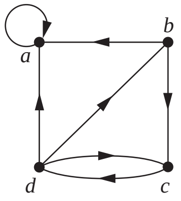
4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph.

5. Can a simple graph exist with 15 vertices each of degree five?

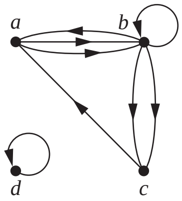
6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

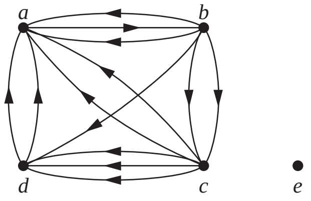
7.



8.



9.



10. For each of the graphs in Exercises 7–9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.

11. Construct the underlying undirected graph for the graph with directed edges in Figure 2.

12. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What does the neighborhood a vertex in this graph represent? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?

13. What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices represent?

14. What does the degree of a vertex in the Hollywood graph represent? What does the neighborhood of a vertex represent? What do the isolated and pendant vertices represent?

15. What do the in-degree and the out-degree of a vertex in a telephone call graph, as described in Example 4 of Section 10.1, represent? What does the degree of a vertex in the undirected version of this graph represent?

16. What do the in-degree and the out-degree of a vertex in the Web graph, as described in Example 5 of Section 10.1, represent?

17. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?

18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

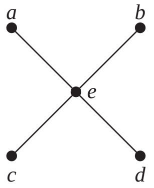
19. Use Exercise 18 to show that in a group of people, there must be two people who are friends with the same number of other people in the group.

20. Draw these graphs.

a) K_7 **b)** $K_{1,8}$ **c)** $K_{4,4}$
d) C_7 **e)** W_7 **f)** Q_4

In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

21.



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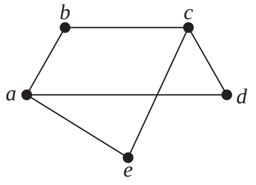

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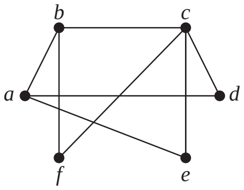

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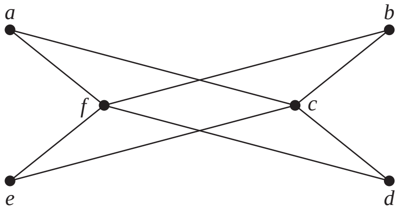

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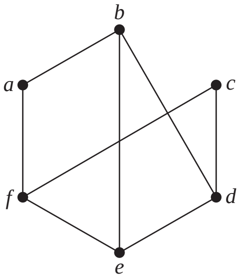


26. For which values of n are these graphs bipartite?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

27. Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

a) Use a bipartite graph to model the four employees and their qualifications.

b) Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.

c) If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

28. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsiblities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

a) Model the capabilities of these employees using a bipartite graph.

b) Find an assignment of responsibilites such that each employee is assigned one responsibility.

c) Is the matching of responsibilities you found in part (b) a complete matching? Is it a maximum matching?

29. Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda. Use Hall's theorem to show there is no matching of the young men and young women on the island such that each young man is matched with a young woman he is willing to marry.

30. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

a) Model the possible marriages on the island using a bipartite graph.

b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

c) Is the matching you found in part (b) a complete matching? Is it a maximum matching?

***31.** Suppose there is an integer k such that every man on a desert island is willing to marry exactly k of the women on the island and every woman on the island is willing to marry exactly k of the men. Also, suppose that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match the men and women on the island so that everyone is matched with someone that they are willing to marry.

***32.** In this exercise we prove a theorem of Øystein Ore. Suppose that $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2) and that $A \subseteq V_1$. Show that the maximum number of vertices of V_1 that are the endpoints of a matching of G equals $|V_1| - \max_{A \subseteq V_1} \text{def}(A)$, where $\text{def}(A) = |A| - |N(A)|$. (Here, $\text{def}(A)$ is called the **deficiency** of A.) [*Hint:* Form a larger graph by adding $\max_{A \subseteq V_1} \text{def}(A)$ new vertices to V_2 and connect all of them to the vertices of V_1.]

33. For the graph G in Exercise 1 find

a) the subgraph induced by the vertices a, b, c, and f.

b) the new graph G_1 obtained from G by contracting the edge connecting b and f.

34. Let n be a positive integer. Show that a subgraph induced by a nonempty subset of the vertex set of K_n is a complete graph.

35. How many vertices and how many edges do these graphs have?

a) K_n **b)** C_n **c)** W_n
d) $K_{m,n}$ **e)** Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 is 4, 4, 4, 3, 2, 1, 0.

36. Find the degree sequences for each of the graphs in Exercises 21–25.

37. Find the degree sequence of each of the following graphs.

a) K_4 **b)** C_4 **c)** W_4
d) $K_{2,3}$ **e)** Q_3

38. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

39. What is the degree sequence of K_n, where n is a positive integer? Explain your answer.

40. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

41. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

42. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 **b)** 6, 5, 4, 3, 2, 1 **c)** 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2 **e)** 3, 3, 2, 2, 2, 2 **f)** 1, 1, 1, 1, 1, 1
g) 5, 3, 3, 3, 3, 3 **h)** 5, 5, 4, 3, 2, 1

43. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3, 3, 3, 3, 2 **b)** 5, 4, 3, 2, 1 **c)** 4, 4, 3, 2, 1
d) 4, 4, 3, 3, 3 **e)** 3, 2, 2, 1, 0 **f)** 1, 1, 1, 1, 1

*44. Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$.

*45. Show that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

*46. Use Exercise 45 to construct a recursive algorithm for determining whether a nonincreasing sequence of positive integers is graphic.

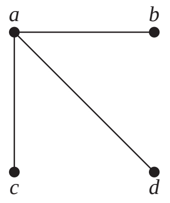
47. Show that every nonincreasing sequence of nonnegative integers with an even sum of its terms is the degree sequence of a pseudograph, that is, an undirected graph where loops are allowed. [*Hint:* Construct such a graph by first adding as many loops as possible at each vertex. Then add additional edges connecting vertices of odd degree. Explain why this construction works.]

48. How many subgraphs with at least one vertex does K_2 have?

49. How many subgraphs with at least one vertex does K_3 have?

50. How many subgraphs with at least one vertex does W_3 have?

51. Draw all subgraphs of this graph.



52. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$. **b)** $2e/v \leq M$.

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called n-**regular** if every vertex in this graph has degree n.

53. For which values of n are these graphs regular?

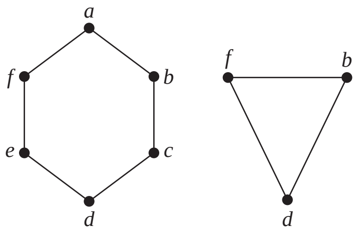
a) K_n **b)** C_n **c)** W_n **d)** Q_n

54. For which values of m and n is $K_{m,n}$ regular?

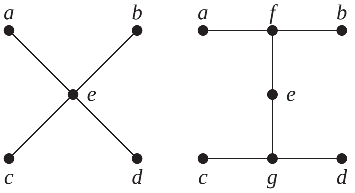
55. How many vertices does a regular graph of degree four with 10 edges have?

In Exercises 56–58 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

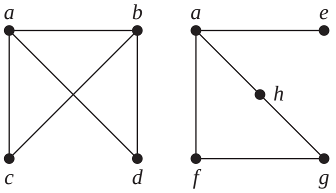
56.



57.



58.



59. The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

a) $\overline{K_n}$ **b)** $\overline{K_{m,n}}$ **c)** $\overline{C_n}$ **d)** $\overline{Q_n}$

60. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

61. If the simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

62. If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of $\overline{G}$?

63. If the degree sequence of the simple graph G is $d_1, d_2, \ldots, d_n$, what is the degree sequence of $\overline{G}$?

***64.** Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

65. Show that if G is a simple graph with n vertices, then the union of G and $\overline{G}$ is K_n.

***66.** Describe an algorithm to decide whether a graph is bipartite based on the fact that a graph is bipartite if and only if it is possible to color its vertices two different colors so that no two vertices of the same color are adjacent.

The **converse** of a directed graph $G = (V, E)$, denoted by G^{conv}, is the directed graph (V, F), where the set F of edges of G^{conv} is obtained by reversing the direction of each edge in E.

67. Draw the converse of each of the graphs in Exercises 7–9 in Section 10.1.

68. Show that $(G^{conv})^{conv} = G$ whenever G is a directed graph.

69. Show that the graph G is its own converse if and only if the relation associated with G (see Section 9.3) is symmetric.

70. Show that if a bipartite graph $G = (V, E)$ is n-regular for some positive integer n (see the preamble to Exercise 53) and (V_1, V_2) is a bipartition of V, then $|V_1| = |V_2|$. That is, show that the two sets in a bipartition of the vertex set of an n-regular graph must contain the same number of vertices.

71. Draw the mesh network for interconnecting nine parallel processors.

72. In a variant of a mesh network for interconnecting $n = m^2$ processors, processor $P(i, j)$ is connected to the four processors $P((i \pm 1) \textbf{ mod } m, j)$ and $P(i, (j \pm 1) \textbf{ mod } m)$, so that connections wrap around the edges of the mesh. Draw this variant of the mesh network for 16 processors.

73. Show that every pair of processors in a mesh network of $n = m^2$ processors can communicate using $O(\sqrt{n}) = O(m)$ hops between directly connected processors.

10.3 Representing Graphs and Graph Isomorphism

Introduction

There are many useful ways to represent graphs. As we will see throughout this chapter, in working with a graph it is helpful to be able to choose its most convenient representation. In this section we will show how to represent graphs in several different ways.

Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs are **isomorphic**. Determining whether two graphs are isomorphic is an important problem of graph theory that we will study in this section.

Representing Graphs

One way to represent a graph without multiple edges is to list all the edges of this graph. Another way to represent a graph with no multiple edges is to use **adjacency lists**, which specify the vertices that are adjacent to each vertex of the graph.

EXAMPLE 1 Use adjacency lists to describe the simple graph given in Figure 1.

Solution: Table 1 lists those vertices adjacent to each of the vertices of the graph. ◀

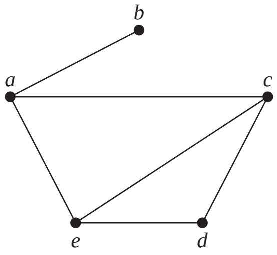


FIGURE 1 **A Simple Graph.**

TABLE 1 **An Adjacency List for a Simple Graph.**

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

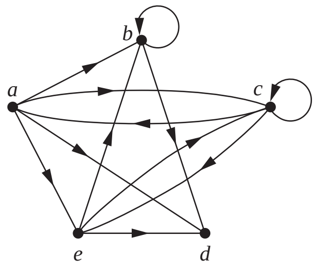


FIGURE 2 **A Directed Graph.**

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

EXAMPLE 2 Represent the directed graph shown in Figure 2 by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

Solution: Table 2 represents the directed graph shown in Figure 2. ◀

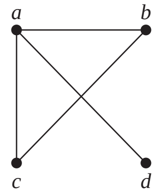


FIGURE 3 **Simple Graph.**

Adjacency Matrices

Carrying out graph algorithms using the representation of graphs by lists of edges, or by adjacency lists, can be cumbersome if there are many edges in the graph. To simplify computation, graphs can be represented using matrices. Two types of matrices commonly used to represent graphs will be presented here. One is based on the adjacency of vertices, and the other is based on incidence of vertices and edges.

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The **adjacency matrix A** (or $\mathbf{A}_G$) of G, with respect to this listing of the vertices, is the n x n zero–one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $\mathbf{A} = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 3 Use an adjacency matrix to represent the graph shown in Figure 3.

Solution: We order the vertices as a, b, c, d. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

◀

EXAMPLE 4 Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

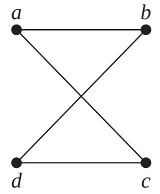


FIGURE 4 **A Graph with the Given Adjacency Matrix.**

Solution: A graph with this adjacency matrix is shown in Figure 4. ◀

Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there may be as many as $n!$ different adjacency matrices for a graph with n vertices, because there are $n!$ different orderings of n vertices.

The adjacency matrix of a simple graph is symmetric, that is, $a_{ij} = a_{ji}$, because both of these entries are 1 when v_i and v_j are adjacent, and both are 0 otherwise. Furthermore, because a simple graph has no loops, each entry a_{ii}, $i = 1, 2, 3, \ldots, n$, is 0.

Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. A loop at the vertex v_i is represented by a 1 at the (i, i)th position of the adjacency matrix. When multiple edges connecting the same pair of vertices v_i and v_j, or multiple loops at the same vertex, are present, the adjacency matrix is no longer a zero–one matrix, because the (i, j)th entry of this matrix equals the number of edges that are associated to $\{v_i, v_j\}$. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

EXAMPLE 5 Use an adjacency matrix to represent the pseudograph shown in Figure 5.

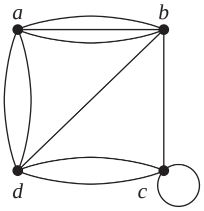


FIGURE 5
A Pseudograph.

Solution: The adjacency matrix using the ordering of vertices a, b, c, d is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

◀

We used zero–one matrices in Chapter 9 to represent directed graphs. The matrix for a directed graph $G = (V, E)$ has a 1 in its (i, j)th position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots, v_n$ is an arbitrary listing of the vertices of the directed graph. In other words, if $\mathbf{A} = [a_{ij}]$ is the adjacency matrix for the directed graph with respect to this listing of the vertices, then

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from v_j to v_i when there is an edge from v_i to v_j.

Adjacency matrices can also be used to represent directed multigraphs. Again, such matrices are not zero–one matrices when there are multiple edges in the same direction connecting two vertices. In the adjacency matrix for a directed multigraph, a_{ij} equals the number of edges that are associated to (v_i, v_j).

TRADE-OFFS BETWEEN ADJACENCY LISTS AND ADJACENCY MATRICES When a simple graph contains relatively few edges, that is, when it is **sparse**, it is usually preferable to use adjacency lists rather than an adjacency matrix to represent the graph. For example, if each vertex has degree not exceeding c, where c is a constant much smaller than n, then each adjacency list contains c or fewer vertices. Hence, there are no more than cn items in all these adjacency lists. On the other hand, the adjacency matrix for the graph has n^2 entries. Note, however, that the adjacency matrix of a sparse graph is a **sparse matrix**, that is, a matrix with few nonzero entries, and there are special techniques for representing, and computing with, sparse matrices.

Now suppose that a simple graph is **dense**, that is, suppose that it contains many edges, such as a graph that contains more than half of all possible edges. In this case, using an adjacency matrix to represent the graph is usually preferable over using adjacency lists. To see why, we compare the complexity of determining whether the possible edge $\{v_i, v_j\}$ is present. Using an adjacency matrix, we can determine whether this edge is present by examining the (i, j)th entry in the matrix. This entry is 1 if the graph contains this edge and is 0 otherwise. Consequently, we need make only one comparison, namely, comparing this entry with 0, to determine whether this edge is present. On the other hand, when we use adjacency lists to represent the graph, we need to search the list of vertices adjacent to either v_i or v_j to determine whether this edge is present. This can require $\Theta(|V|)$ comparisons when many edges are present.

Incidence Matrices

Another common way to represent graphs is to use **incidence matrices**. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 6 Represent the graph shown in Figure 6 with an incidence matrix.

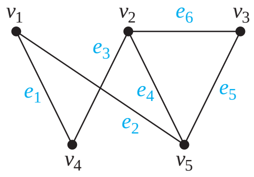


FIGURE 6 An Undirected Graph.

Solution: The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

◀

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries, because these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

EXAMPLE 7 Represent the pseudograph shown in Figure 7 using an incidence matrix.

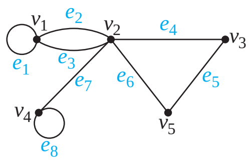


FIGURE 7 A Pseudograph.

Solution: The incidence matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

◀

Isomorphism of Graphs

We often need to know whether it is possible to draw two graphs in the same way. That is, do the graphs have the same structure when we ignore the identities of their vertices? For instance, in chemistry, graphs are used to model chemical compounds (in a way we will describe later). Different compounds can have the same molecular formula but can differ in structure. Such compounds can be represented by graphs that cannot be drawn in the same way. The graphs representing previously known compounds can be used to determine whether a supposedly new compound has been studied before.

There is a useful terminology for graphs with the same structure.

DEFINITION 1 The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*.* Two simple graphs that are not isomorphic are called *nonisomorphic*.

In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship. Isomorphism of simple graphs is an equivalence relation. (We leave the verification of this as Exercise 45.)

EXAMPLE 8 Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure 8, are isomorphic.

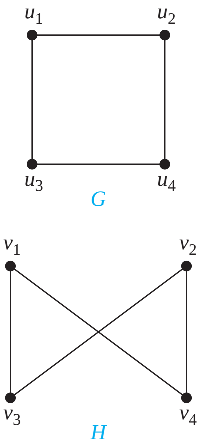


FIGURE 8 **The Graphs G and H.**

Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in H. ◀

Determining whether Two Simple Graphs are Isomorphic

It is often difficult to determine whether two simple graphs are isomorphic. There are $n!$ possible one-to-one correspondences between the vertex sets of two simple graphs with n vertices. Testing each such correspondence to see whether it preserves adjacency and nonadjacency is impractical if n is at all large.

Sometimes it is not hard to show that two graphs are not isomorphic. In particular, we can show that two graphs are not isomorphic if we can find a property only one of the two graphs has, but that is preserved by isomorphism. A property preserved by isomorphism of graphs is called a **graph invariant**. For instance, isomorphic simple graphs must have the same number of vertices, because there is a one-to-one correspondence between the sets of vertices of the graphs.

Links

Isomorphic simple graphs also must have the same number of edges, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges. In addition, the degrees of the vertices in isomorphic simple graphs must be the same. That is, a vertex v of degree d in G must correspond to a vertex $f(v)$ of degree d in H, because a vertex w in G is adjacent to v if and only if $f(v)$ and $f(w)$ are adjacent in H.

EXAMPLE 9 Show that the graphs displayed in Figure 9 are not isomorphic.

Extra Examples

Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic. ◀

The number of vertices, the number of edges, and the number of vertices of each degree are all invariants under isomorphism. If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic. However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic. There are no useful sets of invariants currently known that can be used to determine whether simple graphs are isomorphic.

*The word *isomorphism* comes from the Greek roots *isos* for "equal" and *morphe* for "form."

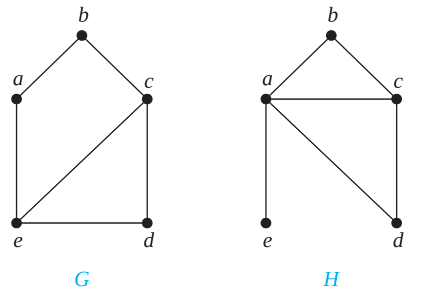


FIGURE 9 **The Graphs G and H.**

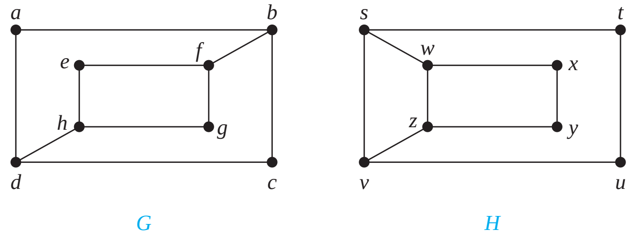


FIGURE 10 **The Graphs G and H.**

EXAMPLE 10 Determine whether the graphs shown in Figure 10 are isomorphic.

Solution: The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic.

However, G and H are not isomorphic. To see this, note that because $\deg(a) = 2$ in G, a must correspond to either t, u, x, or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

Another way to see that G and H are not isomorphic is to note that the subgraphs of G and H made up of vertices of degree three and the edges connecting them must be isomorphic if these two graphs are isomorphic (the reader should verify this). However, these subgraphs, shown in Figure 11, are not isomorphic. ◀

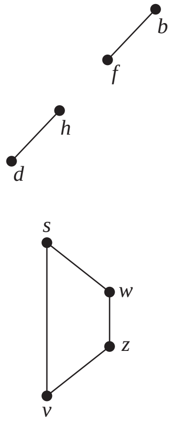


FIGURE 11 **The Subgraphs of G and H Made Up of Vertices of Degree Three and the Edges Connecting Them.**

To show that a function f from the vertex set of a graph G to the vertex set of a graph H is an isomorphism, we need to show that f preserves the presence and absence of edges. One helpful way to do this is to use adjacency matrices. In particular, to show that f is an isomorphism, we can show that the adjacency matrix of G is the same as the adjacency matrix of H, when rows and columns are labeled to correspond to the images under f of the vertices in G that are the labels of these rows and columns in the adjacency matrix of G. We illustrate how this is done in Example 11.

EXAMPLE 11 Determine whether the graphs G and H displayed in Figure 12 are isomorphic.

Solution: Both G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic (as the reader should verify). Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

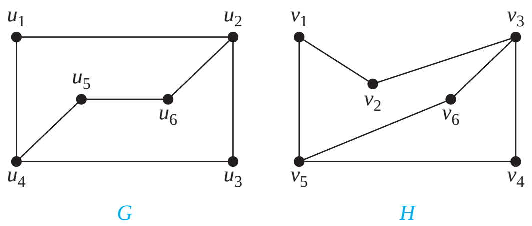


FIGURE 12 **Graphs G and H.**

We now will define a function f and then determine whether it is an isomorphism. Because $\deg(u_1) = 2$ and because u_1 is not adjacent to any other vertex of degree two, the image of u_1 must be either v_4 or v_6, the only vertices of degree two in H not adjacent to a vertex of degree two. We arbitrarily set $f(u_1) = v_6$. [If we found that this choice did not lead to isomorphism, we would then try $f(u_1) = v_4$.] Because u_2 is adjacent to u_1, the possible images of u_2 are v_3 and v_5. We arbitrarily set $f(u_2) = v_3$. Continuing in this way, using adjacency of vertices and degrees as a guide, we set $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, and $f(u_6) = v_2$. We now have a one-to-one correspondence between the vertex set of G and the vertex set of H, namely, $f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, $f(u_6) = v_2$. To see whether f preserves edges, we examine the adjacency matrix of G,

$$\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G,

$$\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Because $\mathbf{A}_G = \mathbf{A}_H$, it follows that f preserves edges. We conclude that f is an isomorphism, so G and H are isomorphic. Note that if f turned out not to be an isomorphism, we would *not* have established that G and H are not isomorphic, because another correspondence of the vertices in G and H may be an isomorphism. ◀

Links

ALGORITHMS FOR GRAPH ISOMORPHISM The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity (in the number of vertices of the graphs). However, linear average-case time complexity algorithms are known that solve this problem, and there is some hope, but also skepticism, that an algorithm with polynomial worst-case time complexity for determining whether two graphs are isomorphic can be found. The best practical general purpose software for isomorphism testing, called NAUTY, can be used to determine whether two graphs with as many as 100 vertices are isomorphic in less than a second on a modern PC. NAUTY software can be downloaded over the Internet and experimented with. Practical algorithms for determining whether two graphs are isomorphic exist for graphs that are restricted in various ways, such as when the maximum degree of vertices is small. The problem of determining whether any two graphs are isomorphic is of special interest because it is one of only a few NP problems (see Exercise 72) not known to be either tractable or NP-complete (see Section 3.3).

APPLICATIONS OF GRAPH ISOMORPHISMS Graph isomorphisms, and functions that are almost graph isomorphisms, arise in applications of graph theory to chemistry and to the design of electronic circuits, and other areas including bioinformatics and computer vision.

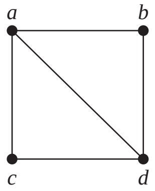
Chemists use multigraphs, known as molecular graphs, to model chemical compounds. In these graphs, vertices represent atoms and edges represent chemical bonds between these atoms. Two structural isomers, molecules with identical molecular formulas but with atoms bonded differently, have nonisomorphic molecular graphs. When a potentially new chemical compound is synthesized, a database of molecular graphs is checked to see whether the molecular graph of the compound is the same as one already known.

Electronic circuits are modeled using graphs in which vertices represent components and edges represent connections between them. Modern integrated circuits, known as chips, are miniaturized electronic circuits, often with millions of transistors and connections between them. Because of the complexity of modern chips, automation tools are used to design them. Graph isomorphism is the basis for the verification that a particular layout of a circuit produced by an automated tool corresponds to the original schematic of the design. Graph isomorphism can also be used to determine whether a chip from one vendor includes intellectual property from a different vendor. This can be done by looking for large isomorphic subgraphs in the graphs modeling these chips.

Exercises

In Exercises 1–4 use an adjacency list to represent the given graph.

1.



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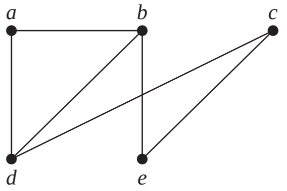

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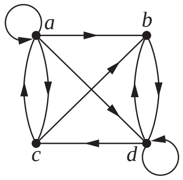

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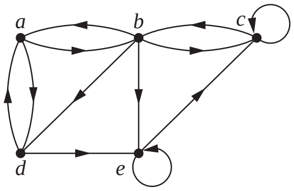


5. Represent the graph in Exercise 1 with an adjacency matrix.

6. Represent the graph in Exercise 2 with an adjacency matrix.

7. Represent the graph in Exercise 3 with an adjacency matrix.

8. Represent the graph in Exercise 4 with an adjacency matrix.

9. Represent each of these graphs with an adjacency matrix.

a) K_4 **b)** $K_{1,4}$ **c)** $K_{2,3}$
d) C_4 **e)** W_4 **f)** Q_3

In Exercises 10–12 draw a graph with the given adjacency matrix.

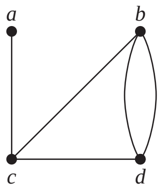
10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

In Exercises 13–15 represent the given graph using an adjacency matrix.

13.



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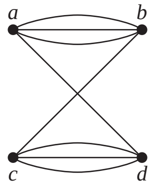
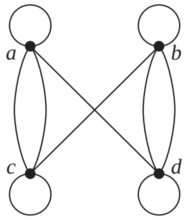


15.



In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

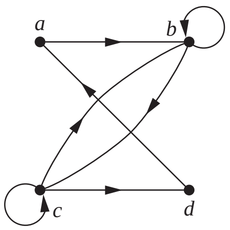
16. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

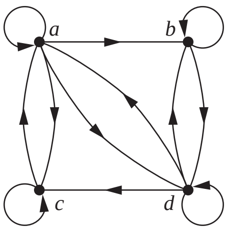
18. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

In Exercises 19–21 find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

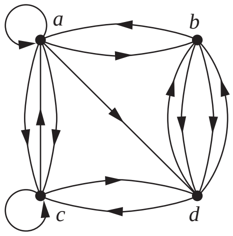
19.



20.



21.



In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **23.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ **24.** $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

25. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

26. Use an incidence matrix to represent the graphs in Exercises 1 and 2.

27. Use an incidence matrix to represent the graphs in Exercises 13–15.

***28.** What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

***29.** What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

30. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

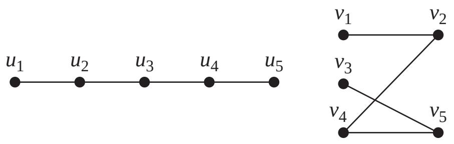
31. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

***32.** Find an adjacency matrix for each of these graphs.
a) K_n **b)** C_n **c)** W_n **d)** $K_{m,n}$ **e)** Q_n

***33.** Find incidence matrices for the graphs in parts (a)–(d) of Exercise 32.

In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

34.



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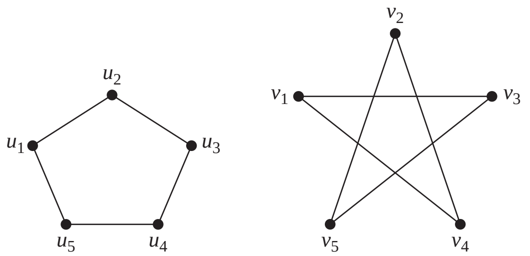
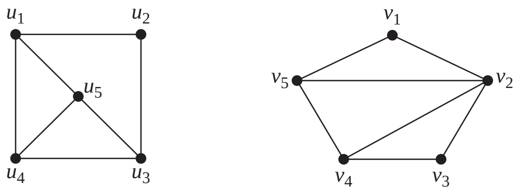


36.



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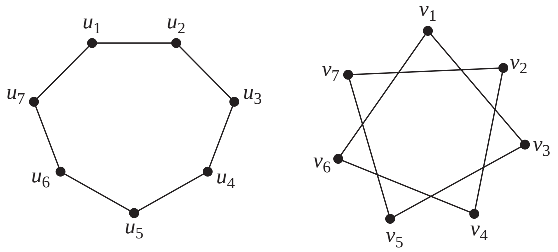

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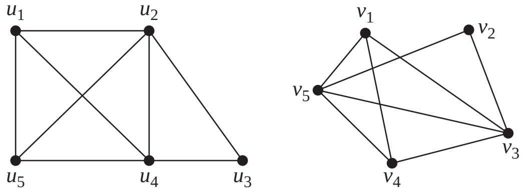

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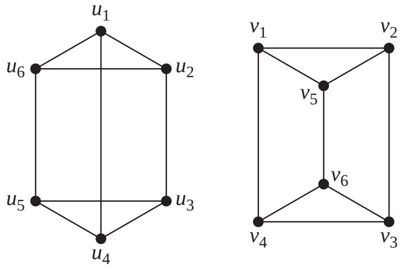

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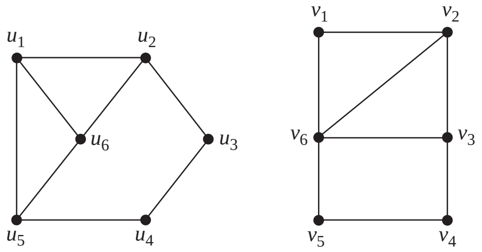

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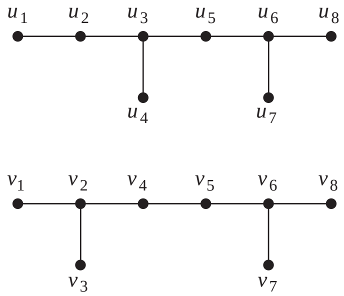
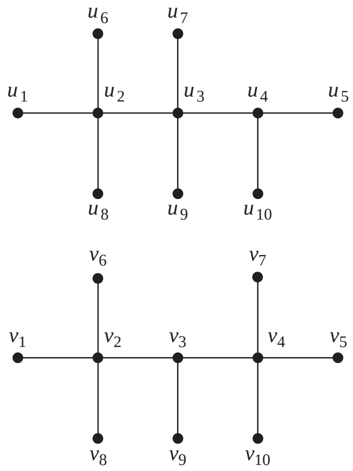
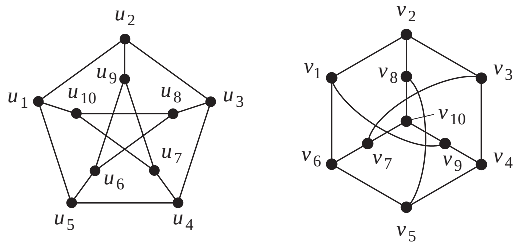


42.



43.



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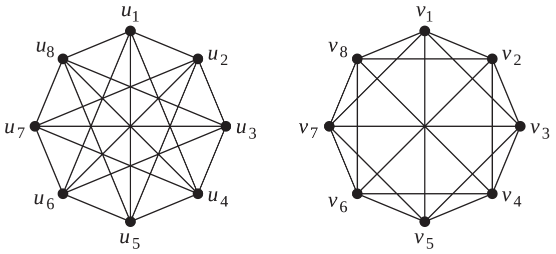


45. Show that isomorphism of simple graphs is an equivalence relation.

46. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

47. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.

48. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.

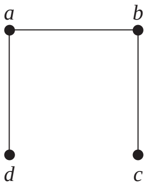
49. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks.

A simple graph G is called **self-complementary** if G and $\overline{G}$ are isomorphic.

50. Show that this graph is self-complementary.



51. Find a self-complementary simple graph with five vertices.

***52.** Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

53. For which integers n is C_n self-complementary?

54. How many nonisomorphic simple graphs are there with n vertices, when n is

a) 2? **b)** 3? **c)** 4?

55. How many nonisomorphic simple graphs are there with five vertices and three edges?

56. How many nonisomorphic simple graphs are there with six vertices and four edges?

57. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

58. Determine whether the graphs without loops with these incidence matrices are isomorphic.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

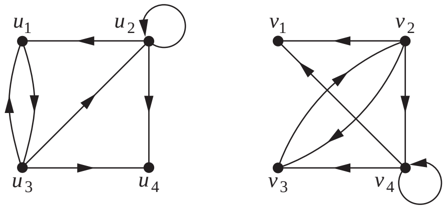
b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

59. Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.

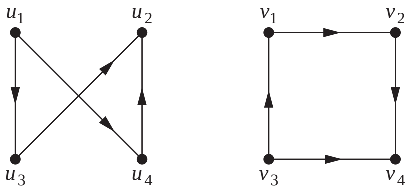
60. Define isomorphism of directed graphs.

In Exercises 61–64 determine whether the given pair of directed graphs are isomorphic. (See Exercise 60.)

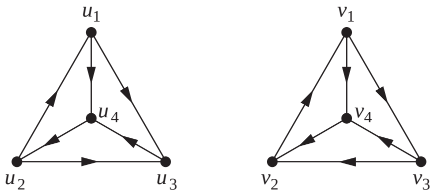
61.



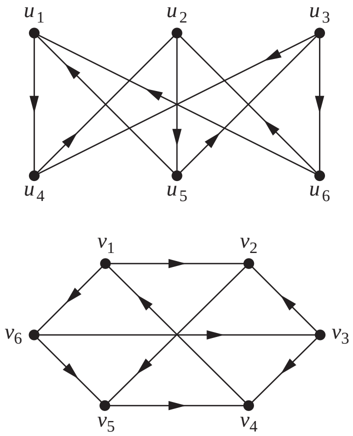
62.



63.



64.



65. Show that if G and H are isomorphic directed graphs, then the converses of G and H (defined in the preamble of Exercise 67 of Section 10.2) are also isomorphic.

66. Show that the property that a graph is bipartite is an isomorphic invariant.

67. Find a pair of nonisomorphic graphs with the same degree sequence (defined in the preamble to Exercise 36 in Section 10.2) such that one graph is bipartite, but the other graph is not bipartite.

***68.** How many nonisomorphic directed simple graphs are there with n vertices, when n is

a) 2? **b)** 3? **c)** 4?

***69.** What is the product of the incidence matrix and its transpose for an undirected graph?

***70.** How much storage is needed to represent a simple graph with n vertices and m edges using

a) adjacency lists?

b) an adjacency matrix?

c) an incidence matrix?

A **devil's pair** for a purported isomorphism test is a pair of nonisomorphic graphs that the test fails to show that they are not isomorphic.

71. Find a devil's pair for the test that checks the degree sequence (defined in the preamble to Exercise 36 in Section 10.2) in two graphs to make sure they agree.

72. Suppose that the function f from V_1 to V_2 is an isomorphism of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Show that it is possible to verify this fact in time polynomial in terms of the number of vertices of the graph, in terms of the number of comparisons needed.

10.4 Connectivity

Introduction

Many problems can be modeled with paths formed by traveling along the edges of graphs. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model. Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

Paths

Informally, a **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.

A formal definition of paths and related terminology is given in Definition 1.

DEFINITION 1

Let n be a nonnegative integer and G an undirected graph. A *path* of *length* n from u to v in G is a sequence of n edges $e_1, \dots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \dots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \dots, n$, the endpoints x_{i-1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$ (because listing these vertices uniquely determines the path). The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to *pass through* the vertices $x_1, x_2, \dots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \dots, e_n$. A path or circuit is *simple* if it does not contain the same edge more than once.

When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2, \dots, e_n$, where e_i is associated with $\{x_{i-1}, x_i\}$ for $i = 1, 2, \dots, n$ by its vertex sequence $x_0, x_1, \dots, x_n$. This notation identifies a path only as far as which vertices it passes through. Consequently, it does not specify a unique path when there is more than one path that passes through this sequence of vertices, which will happen if and only if there are multiple edges between some successive vertices in the list. Note that a path of length zero consists of a single vertex.

Remark: There is considerable variation of terminology concerning the concepts defined in Definition 1. For instance, in some books, the term **walk** is used instead of *path*, where a walk is defined to be an alternating sequence of vertices and edges of a graph, $v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_n$, where v_{i-1} and v_i are the endpoints of e_i for $i = 1, 2, \dots, n$. When this terminology is used, **closed walk** is used instead of *circuit* to indicate a walk that begins and ends at the same vertex, and **trail** is used to denote a walk that has no repeated edge (replacing the term *simple path*). When this terminology is used, the terminology **path** is often used for a trail with no repeated vertices, conflicting with the terminology in Definition 1. Because of this variation in terminology, you will need to make sure which set of definitions are used in a particular book or article when you read about traversing edges of a graph. The text [GrYe06] is a good reference for the alternative terminology described in this remark.

EXAMPLE 1 In the simple graph shown in Figure 1, a, d, c, f, e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge. Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice. ◀

Paths and circuits in directed graphs were introduced in Chapter 9. We now provide more general definitions.

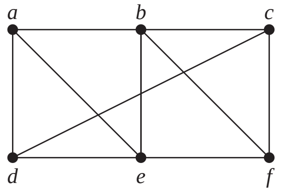


FIGURE 1 **A Simple Graph.**

DEFINITION 2

Let n be a nonnegative integer and G a directed graph. A *path* of length n from u to v in G is a sequence of edges $e_1, e_2, \dots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \dots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a *circuit* or *cycle*. A path or circuit is called *simple* if it does not contain the same edge more than once.

Remark: Terminology other than that given in Definition 2 is often used for the concepts defined there. In particular, the alternative terminology that uses *walk*, *closed walk*, *trail*, and *path* (described in the remarks following Definition 1) may be used for directed graphs. See [GrYe05] for details.

Note that the terminal vertex of an edge in a path is the initial vertex of the next edge in the path. When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2, \dots, e_n$, where e_i is associated with (x_{i-1}, x_i) for $i = 1, 2, \dots, n$, by its vertex sequence $x_0, x_1, \dots, x_n$. The notation identifies a path only as far as which the vertices it passes through. There may be more than one path that passes through this sequence of vertices, which will happen if and only if there are multiple edges between two successive vertices in the list.

Paths represent useful information in many graph models, as Examples 2–4 demonstrate.

EXAMPLE 2

Links

Paths in Acquaintanceship Graphs In an acquaintanceship graph there is a path between two people if there is a chain of people linking these people, where two people adjacent in the chain know one another. For example, in Figure 6 in Section 10.1, there is a chain of six people linking Kamini and Ching. Many social scientists have conjectured that almost every pair of people in the world are linked by a small chain of people, perhaps containing just five or fewer people. This would mean that almost every pair of vertices in the acquaintanceship graph containing all people in the world is linked by a path of length not exceeding four. The play *Six Degrees of Separation* by John Guare is based on this notion. ◀

EXAMPLE 3

Paths in Collaboration Graphs In a collaboration graph, two people a and b are connected by a path when there is a sequence of people starting with a and ending with b such that the endpoints of each edge in the path are people who have collaborated. We will consider two particular collaboration graphs here. First, in the academic collaboration graph of people who have written papers in mathematics, the **Erdős number** of a person m (defined in terms of relations in Supplementary Exercise 14 in Chapter 9) is the length of the shortest path between m and the extremely prolific mathematician Paul Erdős (who died in 1996). That is, the Erdős number of a mathematician is the length of the shortest chain of mathematicians that begins with Paul Erdős and ends with this mathematician, where each adjacent pair of mathematicians have written a joint paper. The number of mathematicians with each Erdős number as of early 2006, according to the Erdős Number Project, is shown in Table 1.

Links

In the Hollywood graph (see Example 3 in Section 10.1) two actors a and b are linked when there is a chain of actors linking a and b, where every two actors adjacent in the chain have acted in the same movie. In the Hollywood graph, the **Bacon number** of an actor c is defined to be the length of the shortest path connecting c and the well-known actor Kevin Bacon. As new movies are made, including new ones with Kevin Bacon, the Bacon number of actors can change. In Table 2 we show the number of actors with each Bacon number as of early 2011 using data from the Oracle of Bacon website. The origins of the Bacon number of an actor dates back to the early 1990s, when Kevin Bacon remarked that he had worked with everyone in Hollywood or someone who worked with them. This lead some people to invent a party

Replace Kevin Bacon by your own favorite actor to invent a new party game

TABLE 1 The Number of Mathematicians with a Given Erdős Number (as of early 2006).

Erdős Number	*Number of People*
0	1
1	504
2	6,593
3	33,605
4	83,642
5	87,760
6	40,014
7	11,591
8	3,146
9	819
10	244
11	68
12	23
13	5

TABLE 2 The Number of Actors with a Given Bacon Number (as of early 2011).

Bacon Number	*Number of People*
0	1
1	2,367
2	242,407
3	785,389
4	200,602
5	14,048
6	1,277
7	114
8	16

game where participants where challenged to find a sequence of movies leading from each actor named to Kevin Bacon. We can find a number similar to a Bacon number using any actor as the center of the acting universe. ◀

Connectedness in Undirected Graphs

When does a computer network have the property that every pair of computers can share information, if messages can be sent through one or more intermediate computers? When a graph is used to represent this computer network, where vertices represent the computers and edges represent the communication links, this question becomes: When is there always a path between two vertices in the graph?

DEFINITION 3 An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called *disconnected*. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Thus, any two computers in the network can communicate if and only if the graph of this network is connected.

EXAMPLE 4 The graph G_1 in Figure 2 is connected, because for every pair of distinct vertices there is a path between them (the reader should verify this). However, the graph G_2 in Figure 2 is not connected. For instance, there is no path in G_2 between vertices a and d. ◀

We will need the following theorem in Chapter 11.

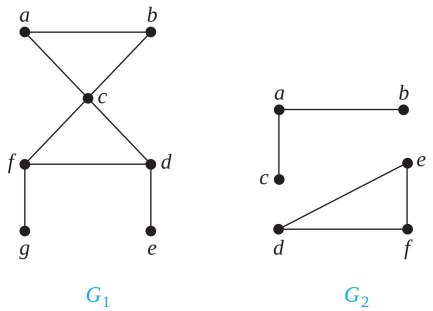


FIGURE 2 **The Graphs G_1 and G_2.**

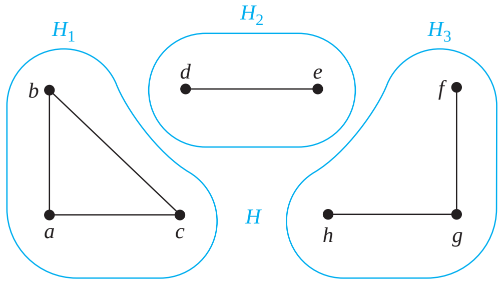


FIGURE 3 **The Graph H and Its Connected Components H_1, H_2, and H_3.**

THEOREM 1 There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof: Let u and v be two distinct vertices of the connected undirected graph $G = (V, E)$. Because G is connected, there is at least one path between u and v. Let $x_0, x_1, \ldots, x_n$, where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length. This path of least length is simple. To see this, suppose it is not simple. Then $x_i = x_j$ for some i and j with $0 \leq i < j$. This means that there is a path from u to v of shorter length with vertex sequence $x_0, x_1, \ldots, x_{i-1}, x_j, \ldots, x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, \ldots, x_{j-1}$. ◁

CONNECTED COMPONENTS A **connected component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

EXAMPLE 5 What are the connected components of the graph H shown in Figure 3?

Solution: The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3, shown in Figure 3. These three subgraphs are the connected components of H. ◀

EXAMPLE 6 **Connected Components of Call Graphs** Two vertices x and y are in the same component of a telephone call graph (see Example 4 in Section 10.1) when there is a sequence of telephone calls beginning at x and ending at y. When a call graph for telephone calls made during a particular day in the AT&T network was analyzed, this graph was found to have 53,767,087 vertices, more than 170 million edges, and more than 3.7 million connected components. Most of these components were small; approximately three-fourths consisted of two vertices representing pairs of telephone numbers that called only each other. This graph has one huge connected component with 44,989,297 vertices comprising more than 80% of the total. Furthermore, every vertex in this component can be linked to any other vertex by a chain of no more than 20 calls. ◀

How Connected is a Graph?

Suppose that a graph represents a computer network. Knowing that this graph is connected tells us that any two computers on the network can communicate. However, we would also like to understand how reliable this network is. For instance, will it still be possible for all computers to communicate after a router or a communications link fails? To answer this and similar questions, we now develop some new concepts.

Sometimes the removal from a graph of a vertex and all incident edges produces a subgraph with more connected components. Such vertices are called **cut vertices** (or **articulation points**). The removal of a cut vertex from a connected graph produces a subgraph that is not connected. Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**. Note that in a graph representing a computer network, a cut vertex and a cut edge represent an essential router and an essential link that cannot fail for all computers to be able to communicate.

EXAMPLE 7 Find the cut vertices and cut edges in the graph G_1 shown in Figure 4.

Solution: The cut vertices of G_1 are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G_1. ◀

VERTEX CONNECTIVITY Not all graphs have cut vertices. For example, the complete graph K_n, where $n \geq 3$, has no cut vertices. When you remove a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph. Connected graphs without cut vertices are called **nonseparable graphs**, and can be thought of as more connected than those with a cut vertex. We can extend this notion by defining a more granulated measure of graph connectivity based on the minimum number of vertices that can be removed to disconnect a graph.

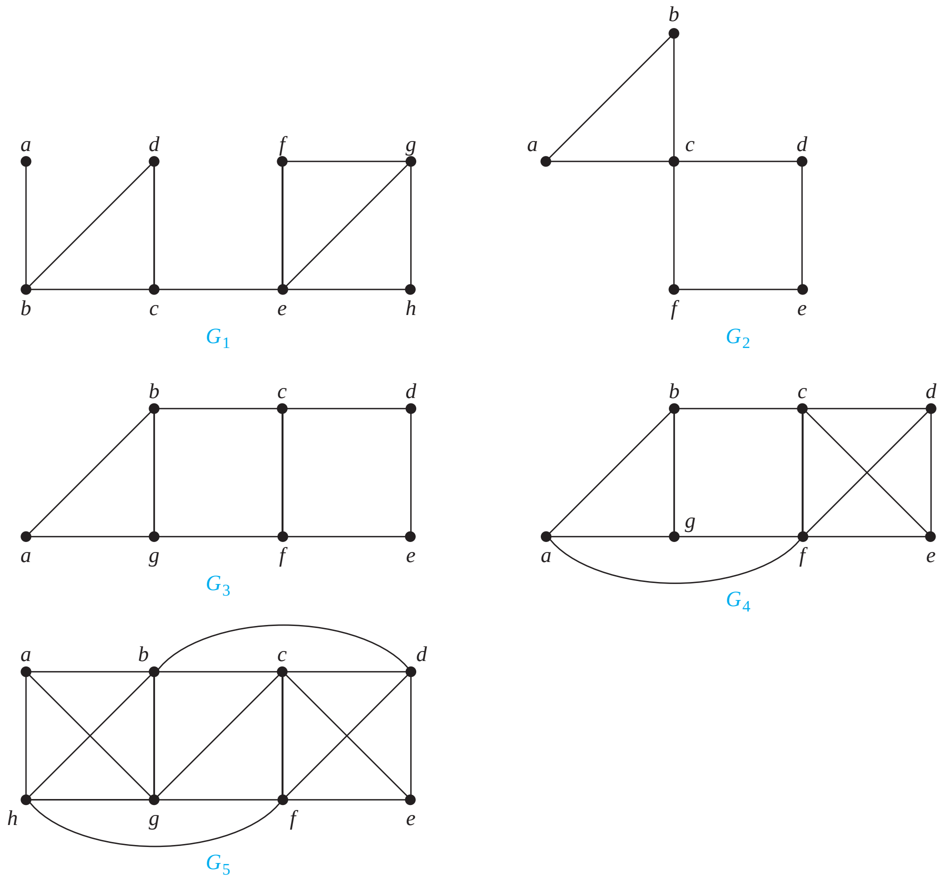


FIGURE 4 **Some Connected Graphs**

A subset V' of the vertex set V of $G = (V, E)$ is a **vertex cut**, or **separating set**, if $G - V'$ is disconnected. For instance, in the graph in Figure 1, the set $\{b, c, e\}$ is a vertex cut with three vertices, as the reader should verify. We leave it to the reader (Exercise 51) to show that every connected graph, except a complete graph, has a vertex cut. We define the **vertex connectivity** of a noncomplete graph G, denoted by $\kappa(G)$, as the minimum number of vertices in a vertex cut.

κ is the lowercase Greek letter kappa.

When G is a complete graph, it has no vertex cuts, because removing any subset of its vertices and all incident edges still leaves a complete graph. Consequently, we cannot define $\kappa(G)$ as the minimum number of vertices in a vertex cut when G is complete. Instead, we set $\kappa(K_n) = n - 1$, the number of vertices needed to be removed to produce a graph with a single vertex.

Consequently, for every graph G, $\kappa(G)$ is minimum number of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex. We have $0 \leq \kappa(G) \leq n - 1$ if G has n vertices, $\kappa(G) = 0$ if and only if G is disconnected or $G = K_1$, and $\kappa(G) = n - 1$ if and only if G is complete [see Exercise 52(a)].

The larger $\kappa(G)$ is, the more connected we consider G to be. Disconnected graphs and K_1 have $\kappa(G) = 0$, connected graphs with cut vertices and K_2 have $\kappa(G) = 1$, graphs without cut vertices that can be disconnected by removing two vertices and K_3 have $\kappa(G) = 2$, and so on. We say that a graph is ***k*-connected** (or ***k*-vertex-connected**), if $\kappa(G) \geq k$. A graph G is 1-connected if it is connected and not a graph containing a single vertex; a graph is 2-connected, or **biconnected**, if it is nonseparable and has at least three vertices. Note that if G is a k-connected graph, then G is a j-connected graph for all j with $0 \leq j \leq k$.

EXAMPLE 8 Find the vertex connectivity for each of the graphs in Figure 4.

Solution: Each of the five graphs in Figure 4 is connected and has more than vertex, so each of these graphs has positive vertex connectivity. Because G_1 is a connected graph with a cut vertex, as shown in Example 7, we know that $\kappa(G_1) = 1$. Similarly, $\kappa(G_2) = 1$, because c is a cut vertex of G_2.

The reader should verify that G_3 has no cut vertices. but that $\{b, g\}$ is a vertex cut. Hence, $\kappa(G_3) = 2$. Similarly, because G_4 has a vertex cut of size two, $\{c, f\}$, but no cut vertices. It follows that $\kappa(G_4) = 2$. The reader can verify that G_5 has no vertex cut of size two, but $\{b, c, f\}$ is a vertex cut of G_5. Hence, $\kappa(G_5) = 3$. ◀

EDGE CONNECTIVITY We can also measure the connectivity of a connected graph $G = (V, E)$ in terms of the minimum number of edges that we can remove to disconnect it. If a graph has a cut edge, then we need only remove it to disconnect G. If G does not have a cut edge, we look for the smallest set of edges that can be removed to disconnect it. A set of edges E' is called an **edge cut** of G if the subgraph $G - E'$ is disconnected. The **edge connectivity** of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G. This defines $\lambda(G)$ for all connected graphs with more than one vertex because it is always possible to disconnect such a graph by removing all edges incident to one of its vertices. Note that $\lambda(G) = 0$ if G is not connected. We also specify that $\lambda(G) = 0$ if G is a graph consisting of a single vertex. It follows that if G is a graph with n vertices, then $0 \leq \lambda(G) \leq n - 1$. We leave it to the reader [Exercise 52(b)] to show that $\lambda(G) = n - 1$ where G is a graph with n vertices if and only if $G = K_n$, which is equivalent to the statement that $\lambda(G) \leq n - 2$ when G is not a complete graph.

λ is the lowercase Greek letter lambda.

EXAMPLE 9 Find the edge connectivity of each of the graphs in Figure 4.

Solution: Each of the five graphs in Figure 4 is connected and has more than one vertex, so we know that all of them have positive edge connectivity. As we saw in Example 7, G_1 has a cut edge, so $\lambda(G_1) = 1$.

The graph G_2 has no cut edges, as the reader should verify, but the removal of the two edges $\{a, b\}$ and $\{a, c\}$ disconnects it. Hence, $\lambda(G_2) = 2$. Similarly, $\lambda(G_3) = 2$, because G_3 has no cut edges, but the removal of the two edges $\{b, c\}$ and $\{f, g\}$ disconnects it.

The reader should verify that the removal of no two edges disconnects G_4, but the removal of the three edges $\{b, c\}$, $\{a, f\}$, and $\{f, g\}$ disconnects it. Hence, $\lambda(G_4) = 3$. Finally, the reader should verify that $\lambda(G_5) = 3$, because the removal of any two of its edges does not disconnect it, but the removal of $\{a, b\}$, $\{a, g\}$, and $\{a, h\}$ does. ◀

AN INEQUALITY FOR VERTEX CONNECTIVITY AND EDGE CONNECTIVITY When $G = (V, E)$ is a noncomplete connected graph with at least three vertices, the minimum degree of a vertex of G is an upper bound for both the vertex connectivity of G and the edge connectivity of G. That is, $\kappa(G) \leq \min_{v \in V} \deg(v)$ and $\lambda(G) \leq \min_{v \in V} \deg(v)$. To see this, observe that deleting all the neighbors of a fixed vertex of minimum degree disconnects G, and deleting all the edges that have a fixed vertex of minimum degree as an endpoint disconnects G.

In Exercise 55, we ask the reader to show that $\kappa(G) \leq \lambda(G)$ when G is a connected noncomplete graph. Note also that $\kappa(K_n) = \lambda(K_n) = \min_{v \in V} \deg(v) = n - 1$ when n is a positive integer and that $\kappa(G) = \lambda(G) = 0$ when G is a disconnected graph. Putting these facts together, establishes that for all graphs G,

$$\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v).$$

APPLICATIONS OF VERTEX AND EDGE CONNECTIVITY Graph connectivity plays an important role in many problems involving the reliability of networks. For instance, as we mentioned in our introduction of cut vertices and cut edges, we can model a data network using vertices to represent routers and edges to represent links between them. The vertex connectivity of the resulting graph equals the minimum number of routers that disconnect the network when they are out of service. If fewer routers are down, data transmission between every pair of routers is still possible. The edge connectivity represents the minimum number of fiber optic links that can be down to disconnect the network. If fewer links are down, it will still be possible for data to be transmitted between every pair of routers.

We can model a highway network, using vertices to represent highway intersections and edges to represent sections of roads running between intersections. The vertex connectivity of the resulting graph represents the minimum number of intersections that can be closed at a particular time that makes it impossible to travel between every two intersections. If fewer intersections are closed, travel between every pair of intersections is still possible. The edge connectivity represents the minimum number of roads that can be closed to disconnect the highway network. If fewer highways are closed, it will still be possible to travel between any two intersections. Clearly, it would be useful for the highway department to take this information into account when planning road repairs.

Connectedness in Directed Graphs

There are two notions of connectedness in directed graphs, depending on whether the directions of the edges are considered.

DEFINITION 4 A directed graph is *strongly connected* if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

For a directed graph to be strongly connected there must be a sequence of directed edges from any vertex in the graph to any other vertex. A directed graph can fail to be strongly connected but still be in “one piece.” Definition 5 makes this notion precise.

DEFINITION 5 A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.

That is, a directed graph is weakly connected if and only if there is always a path between two vertices when the directions of the edges are disregarded. Clearly, any strongly connected directed graph is also weakly connected.

EXAMPLE 10 Are the directed graphs G and H shown in Figure 5 strongly connected? Are they weakly connected?

Solution: G is strongly connected because there is a path between any two vertices in this directed graph (the reader should verify this). Hence, G is also weakly connected. The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H (the reader should verify this). ◀

STRONG COMPONENTS OF A DIRECTED GRAPH The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs, that is, the maximal strongly connected subgraphs, are called the **strongly connected components** or **strong components** of G. Note that if a and b are two vertices in a directed graph, their strong components are either the same or disjoint. (We leave the proof of this last fact as Exercise 17.)

EXAMPLE 11 The graph H in Figure 5 has three strongly connected components, consisting of the vertex a; the vertex e; and the subgraph consisting of the vertices b, c, and d and edges (b, c), (c, d), and (d, b). ◀

EXAMPLE 12 **The Strongly Connected Components of the Web Graph** The Web graph introduced in Example 5 of Section 10.1 represents Web pages with vertices and links with directed edges. A snapshot of the Web in 1999 produced a Web graph with over 200 million vertices and over 1.5 billion edges (numbers that have now grown considerably). (See [Br00] for details.)

Links

In 2010 the Web graph was estimated to have at least 55 billion vertices and one trillion edges. This implies that more than 40 TB of computer memory would have been needed to represent its adjacency matrix.

The underlying undirected graph of this Web graph is not connected, but it has a connected component that includes approximately 90% of the vertices in the graph. The subgraph of the original directed graph corresponding to this connected component of the underlying undirected graph (that is, with the same vertices and all directed edges connecting vertices in this graph) has one very large strongly connected component and many small ones. The former is called the **giant strongly connected component (GSCC)** of the directed graph. A Web page in this component can be reached following links starting at any other page in this component. The GSCC in the Web graph produced by this study was found to have over 53 million vertices. The remaining vertices in the large connected component of the undirected graph represent three different types of Web pages: pages that can be reached from a page in the GSCC, but do not link back to these pages following a series of links; pages that link back to pages in the

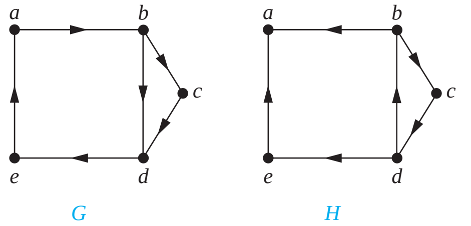


FIGURE 5 **The Directed Graphs G and H.**

GSCC following a series of links, but cannot be reached by following links on pages in the GSCC; and pages that cannot reach pages in the GSCC and cannot be reached from pages in the GSCC following a series of links. In this study, each of these three other sets was found to have approximately 44 million vertices. (It is rather surprising that these three sets are close to the same size.)

Paths and Isomorphism

There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition, paths can be used to construct mappings that may be isomorphisms.

As we mentioned, a useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2. (The proof that this is an invariant is left as Exercise 60.) Example 13 illustrates how this invariant can be used to show that two graphs are not isomorphic.

EXAMPLE 13 Determine whether the graphs G and H shown in Figure 6 are isomorphic.

Solution: Both G and H have six vertices and eight edges. Each has four vertices of degree three, and two vertices of degree two. So, the three invariants—number of vertices, number of edges, and degrees of vertices—all agree for the two graphs. However, H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1, whereas G has no simple circuit of length three, as can be determined by inspection (all simple circuits in G have length at least four). Because the existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

We have shown how the existence of a type of path, namely, a simple circuit of a particular length, can be used to show that two graphs are not isomorphic. We can also use paths to find mappings that are potential isomorphisms.

EXAMPLE 14 Determine whether the graphs G and H shown in Figure 7 are isomorphic.

Solution: Both G and H have five vertices and six edges, both have two vertices of degree three and three vertices of degree two, and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five. Because all these isomorphic invariants agree, G and H may be isomorphic.

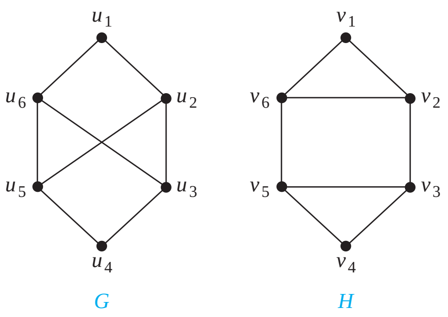


FIGURE 6 **The Graphs G and H.**

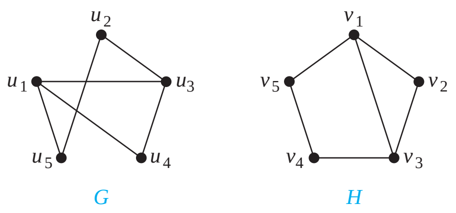


FIGURE 7 **The Graphs G and H.**

To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths u_1, u_4, u_3, u_2, u_5 in G and v_3, v_2, v_1, v_5, v_4 in H both go through every vertex in the graph; start at a vertex of degree three; go through vertices of degrees two, three, and two, respectively; and end at a vertex of degree two. By following these paths through the graphs, we define the mapping f with $f(u_1) = v_3$, $f(u_4) = v_2$, $f(u_3) = v_1$, $f(u_2) = v_5$, and $f(u_5) = v_4$. The reader can show that f is an isomorphism, so G and H are isomorphic, either by showing that f preserves edges or by showing that with the appropriate orderings of vertices the adjacency matrices of G and H are the same. ◀

Counting Paths Between Vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

THEOREM 2 Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

Proof: The theorem will be proved using mathematical induction. Let G be a graph with adjacency matrix $\mathbf{A}$ (assuming an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G). The number of paths from v_i to v_j of length 1 is the (i, j)th entry of $\mathbf{A}$, because this entry is the number of edges from v_i to v_j.

Assume that the (i, j)th entry of $\mathbf{A}^r$ is the number of different paths of length r from v_i to v_j. This is the inductive hypothesis. Because $\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, the (i, j)th entry of $\mathbf{A}^{r+1}$ equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj},$$

where b_{ik} is the (i, k)th entry of $\mathbf{A}^r$. By the inductive hypothesis, b_{ik} is the number of paths of length r from v_i to v_k.

A path of length $r + 1$ from v_i to v_j is made up of a path of length r from v_i to some intermediate vertex v_k, and an edge from v_k to v_j. By the product rule for counting, the number of such paths is the product of the number of paths of length r from v_i to v_k, namely, b_{ik}, and the number of edges from v_k to v_j, namely, a_{kj}. When these products are added for all possible intermediate vertices v_k, the desired result follows by the sum rule for counting. ◁

EXAMPLE 15 How many paths of length four are there from a to d in the simple graph G in Figure 8?

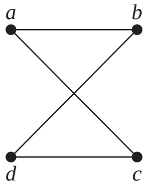


FIGURE 8 **The Graph *G*.**

Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Hence, the number of paths of length four from a to d is the $(1, 4)$th entry of $\mathbf{A}^4$. Because

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

Extra Examples

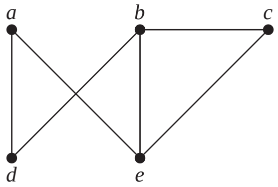
there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d. ◀

Theorem 2 can be used to find the length of the shortest path between two vertices of a graph (see Exercise 56), and it can also be used to determine whether a graph is connected (see Exercises 61 and 62).

Exercises

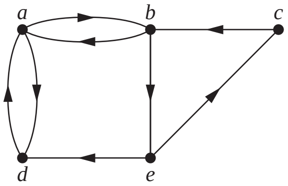
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b **b)** a, e, a, d, b, c, a
c) e, b, a, d, b, e **d)** c, b, d, a, e, c



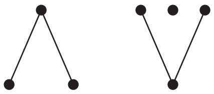
2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b **b)** a, d, a, d, a
c) a, d, b, e, a **d)** a, b, e, c, b, d, a



In Exercises 3–5 determine whether the given graph is connected.

3.



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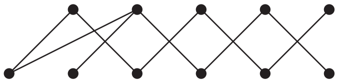

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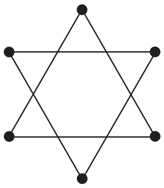


6. How many connected components does each of the graphs in Exercises 3–5 have? For each graph find each of its connected components.

7. What do the connected components of acquaintanceship graphs represent?

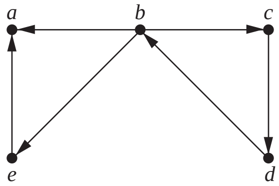
8. What do the connected components of a collaboration graph represent?

9. Explain why in the collaboration graph of mathematicians (see Example 3 in Section 10.1) a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdős if and only if that mathematician has a finite Erdős number.

10. In the Hollywood graph (see Example 3 in Section 10.1), when is the vertex representing an actor in the same connected component as the vertex representing Kevin Bacon?

11. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

a)



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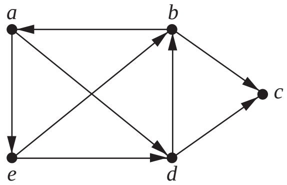

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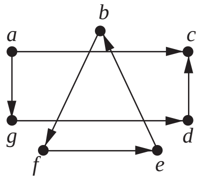
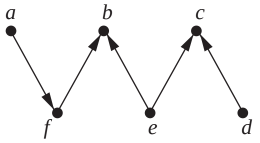


12. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

a)



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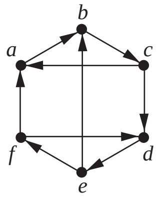

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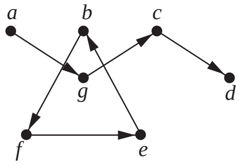
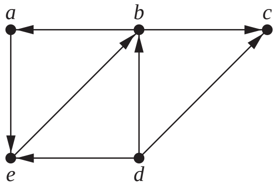


13. What do the strongly connected components of a telephone call graph represent?

14. Find the strongly connected components of each of these graphs.

a)



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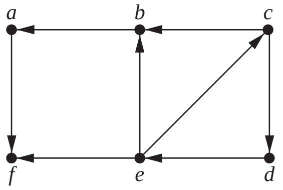

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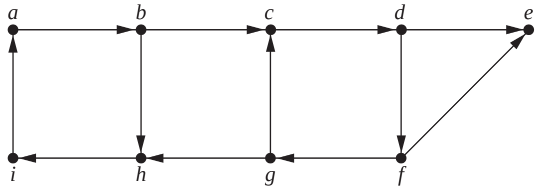
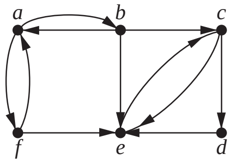


15. Find the strongly connected components of each of these graphs.

a)



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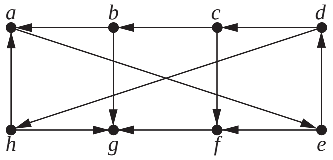

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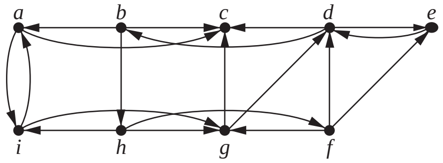


Suppose that $G = (V, E)$ is a directed graph. A vertex $w \in V$ is **reachable** from a vertex $v \in V$ if there is a directed path from v to w. The vertices v and w are **mutually reachable** if there are both a directed path from v to w and a directed path from w to v in G.

16. Show that if $G = (V, E)$ is a directed graph and u, v, and w are vertices in V for which u and v are mutually reachable and v and w are mutually reachable, then u and w are mutually reachable.

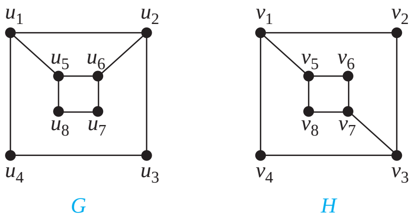
17. Show that if $G = (V, E)$ is a directed graph, then the strong components of two vertices u and v of V are either the same or disjoint. [*Hint:* Use Exercise 16.]

18. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

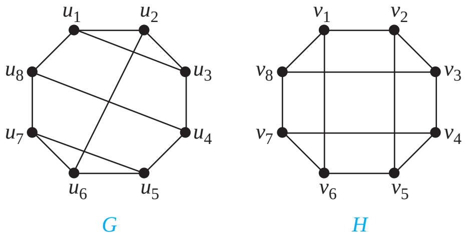
19. Find the number of paths of length n between two different vertices in K_4 if n is

a) 2. **b)** 3. **c)** 4. **d)** 5.

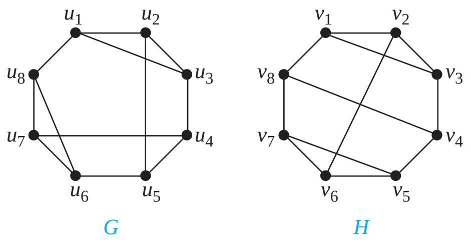
20. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.



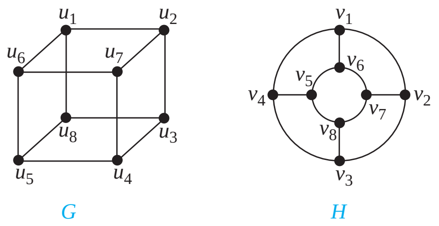
21. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



22. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



23. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



24. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.

25. Find the number of paths of length n between any two nonadjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.

26. Find the number of paths between c and d in the graph in Figure 1 of length

a) 2. **b)** 3. **c)** 4. **d)** 5. **e)** 6. **f)** 7.

27. Find the number of paths from a to e in the directed graph in Exercise 2 of length

a) 2. **b)** 3. **c)** 4. **d)** 5. **e)** 6. **f)** 7.

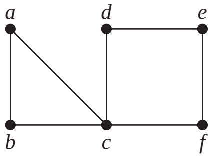
***28.** Show that every connected graph with n vertices has at least $n - 1$ edges.

29. Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$. Show that R is an equivalence relation.

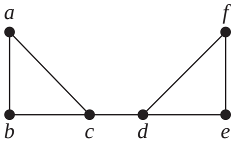
***30.** Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

In Exercises 31–33 find all the cut vertices of the given graph.

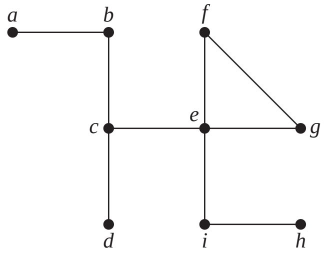
31.



32.



33.



34. Find all the cut edges in the graphs in Exercises 31–33.

***35.** Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

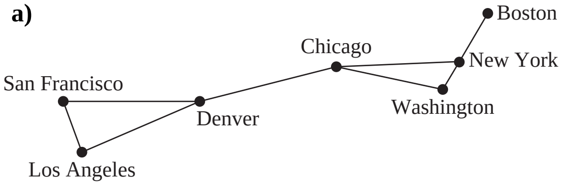
***36.** Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

***37.** Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

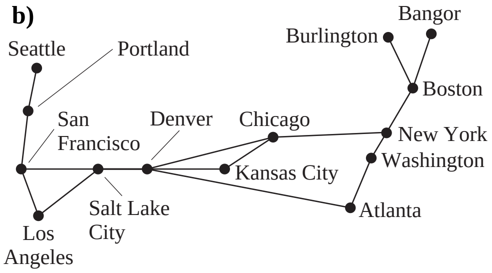
***38.** Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

39. A communications link in a network should be provided with a backup link if its failure makes it impossible for some message to be sent. For each of the communications networks shown here in (a) and (b), determine those links that should be backed up.

a)



b)



A **vertex basis** in a directed graph G is a minimal set B of vertices of G such that for each vertex v of G not in B there is a path to v from some vertex B.

40. Find a vertex basis for each of the directed graphs in Exercises 7–9 of Section 10.2.

41. What is the significance of a vertex basis in an influence graph (described in Example 2 of Section 10.1)? Find a vertex basis in the influence graph in that example.

42. Show that if a connected simple graph G is the union of the graphs G_1 and G_2, then G_1 and G_2 have at least one common vertex.

***43.** Show that if a simple graph G has k connected components and these components have $n_1, n_2, \ldots, n_k$ vertices, respectively, then the number of edges of G does not exceed

$$\sum_{i=1}^{k} C(n_i, 2).$$

***44.** Use Exercise 43 to show that a simple graph with n vertices and k connected components has at most $(n-k)(n-k+1)/2$ edges. [*Hint:* First show that

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k),$$

where n_i is the number of vertices in the ith connected component.]

***45.** Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges.

46. Describe the adjacency matrix of a graph with n connected components when the vertices of the graph are listed so that vertices in each connected component are listed successively.

47. How many nonisomorphic connected simple graphs are there with n vertices when n is

a) 2? **b)** 3? **c)** 4? **d)** 5?

48. Show that each of the following graphs has no cut vertices.

a) C_n where $n \geq 3$

b) W_n where $n \geq 3$

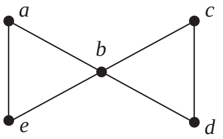
c) $K_{m,n}$ where $m \geq 2$ and $n \geq 2$

d) Q_n where $n \geq 2$

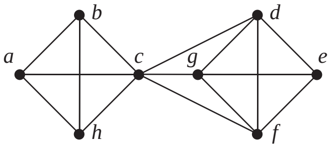
49. Show that each of the graphs in Exercise 48 has no cut edges.

50. For each of these graphs, find $\kappa(G)$, $\lambda(G)$, and $\min_{v \in V} \deg(v)$, and determine which of the two inequalities in $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$ are strict.

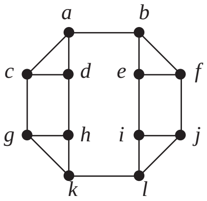
a)



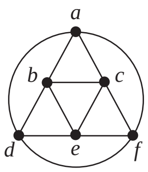
b)



c)



d)



51. Show that if G is a connected graph, then it is possible to remove vertices to disconnect G if and only if G is not a complete graph.

52. Show that if G is a connected graph with n vertices then

a) $\kappa(G) = n - 1$ if and only if $G = K_n$.

b) $\lambda(G) = n - 1$ if and only if $G = K_n$.

53. Find $\kappa(K_{m,n})$ and $\lambda(K_{m,n})$, where m and n are positive integers.

54. Construct a graph G with $\kappa(G) = 1$, $\lambda(G) = 2$, and $\min_{v \in V} \deg(v) = 3$.

***55.** Show that if G is a graph, then $\kappa(G) \leq \lambda(G)$.

56. Explain how Theorem 2 can be used to find the length of the shortest path from a vertex v to a vertex w in a graph.

57. Use Theorem 2 to find the length of the shortest path between a and f in the graph in Figure 1.

58. Use Theorem 2 to find the length of the shortest path from a to c in the directed graph in Exercise 2.

☞ **59.** Let P_1 and P_2 be two simple paths between the vertices u and v in the simple graph G that do not contain the same set of edges. Show that there is a simple circuit in G.

60. Show that the existence of a simple circuit of length k, where k is an integer greater than 2, is an invariant for graph isomorphism.

61. Explain how Theorem 2 can be used to determine whether a graph is connected.

62. Use Exercise 61 to show that the graph G_1 in Figure 2 is connected whereas the graph G_2 in that figure is not connected.

63. Show that a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

64. In an old puzzle attributed to Alcuin of York (735–804), a farmer needs to carry a wolf, a goat, and a cabbage across a river. The farmer only has a small boat, which can carry the farmer and only one object (an animal or a vegetable). He can cross the river repeatedly. However, if the farmer is on the other shore, the wolf will eat the goat, and, similarly, the goat will eat the cabbage. We can describe each state by listing what is on each shore. For example, we can use the pair (*FG*,*WC*) for the state where the farmer and goat are on the first shore and the wolf and cabbage are on the other shore. [The symbol ∅ is used when nothing is on a shore, so that (*FWGC*, ∅) is the initial state.]

a) Find all allowable states of the puzzle, where neither the wolf and the goat nor the goat and the cabbage are left on the same shore without the farmer.

b) Construct a graph such that each vertex of this graph represents an allowable state and the vertices representing two allowable states are connected by an edge if it is possible to move from one state to the other using one trip of the boat.

c) Explain why finding a path from the vertex representing (*FWGC*, ∅) to the vertex representing (∅, *FWGC*) solves the puzzle.

d) Find two different solutions of the puzzle, each using seven crossings.

e) Suppose that the farmer must pay a toll of one dollar whenever he crosses the river with an animal. Which solution of the puzzle should the farmer use to pay the least total toll?

***65.** Use a graph model and a path in your graph, as in Exercise 64, to solve the **jealous husbands problem**. Two married couples, each a husband and a wife, want to cross a river. They can only use a boat that can carry one or two people from one shore to the other shore. Each husband is extremely jealous and is not willing to leave his wife with the other husband, either in the boat or on shore. How can these four people reach the opposite shore?

66. Suppose that you have a three-gallon jug and a five-gallon jug. You may fill either jug with water, you may empty either jug, and you may transfer water from either jug into the other jug. Use a path in a directed graph to show that you can end up with a jug containing exactly one gallon. [*Hint:* Use an ordered pair (a, b) to indicate how much water is in each jug. Represent these ordered pairs by vertices. Add an edge for each allowable operation with the jugs.]

10.5 Euler and Hamilton Paths

Introduction

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once? Although these questions seem to be similar, the first question, which asks whether a graph has an *Euler circuit*, can be easily answered simply by examining the degrees of the vertices of the graph, while the second question, which asks whether a graph has a *Hamilton circuit*, is quite difficult to solve for most graphs. In this section we will study these questions and discuss the difficulty of solving them. Although both questions have many practical applications in many different areas, both arose in old puzzles. We will learn about these old puzzles as well as modern practical applications.

Euler Paths and Circuits

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

Links

Only five bridges connect Kaliningrad today. Of these, just two remain from Euler's day.

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.

The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory. (For a translation of Euler's original paper see [BiLlWi99].) Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges. This multigraph is shown in Figure 2.

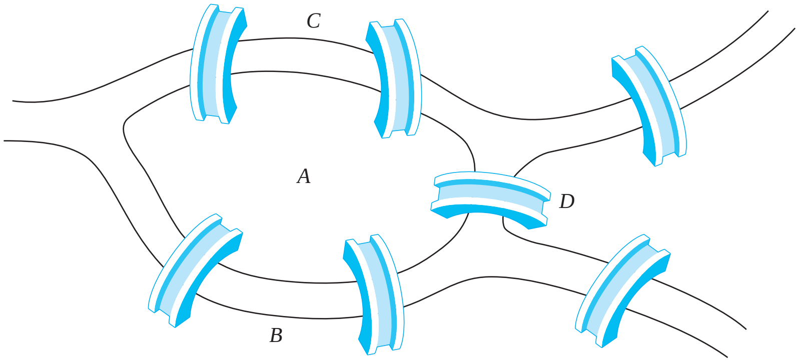


FIGURE 1 **The Seven Bridges of Königsberg.**

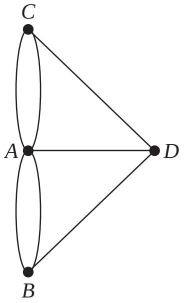
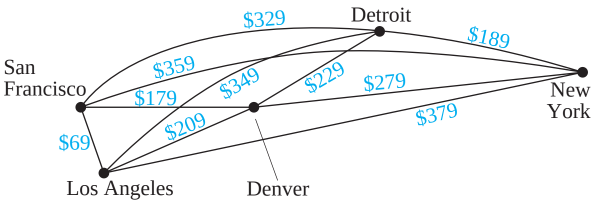
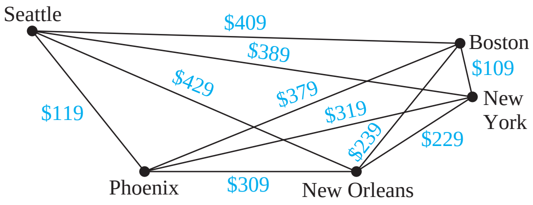


FIGURE 2 **Multigraph Model of the Town of Königsberg.**

27. Find a route with the least total airfare that visits each of the cities in this graph, where the weight on an edge is the least price available for a flight between the two cities.



28. Find a route with the least total airfare that visits each of the cities in this graph, where the weight on an edge is the least price available for a flight between the two cities.



29. Construct a weighted undirected graph such that the total weight of a circuit that visits every vertex at least once is minimized for a circuit that visits some vertices more than once. [*Hint:* There are examples with three vertices.]

30. Show that the problem of finding a circuit of minimum total weight that visits every vertex of a weighted graph at least once can be reduced to the problem of finding a circuit of minimum total weight that visits each vertex of a weighted graph exactly once. Do so by constructing a new weighted graph with the same vertices and edges as the original graph but whose weight of the edge connecting the vertices u and v is equal to the minimum total weight of a path from u to v in the original graph.

***31.** The **longest path problem** in a weighted directed graph with no simple circuits asks for a path in this graph such that the sum of its edge weights is a maximum. Devise an algorithm for solving the longest path problem. [*Hint*: First find a topological ordering of the vertices of the graph.]

10.7 Planar Graphs

Introduction

Consider the problem of joining three houses to each of three separate utilities, as shown in Figure 1. Is it possible to join these houses and utilities so that none of the connections cross? This problem can be modeled using the complete bipartite graph $K_{3,3}$. The original question can be rephrased as: Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?

In this section we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the houses-and-utilities problem.

There are always many ways to represent a graph. When is it possible to find at least one way to represent this graph in a plane without any edges crossing?

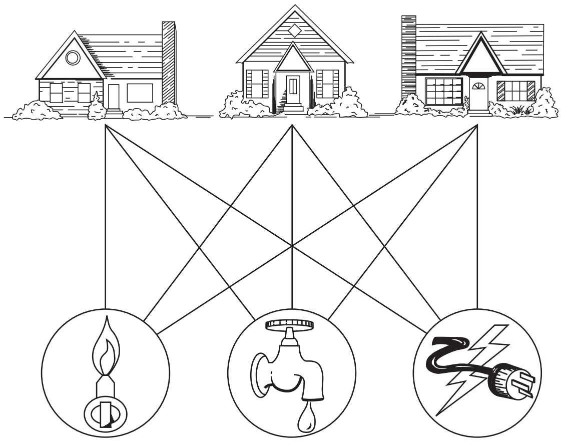


FIGURE 1 **Three Houses and Three Utilities.**

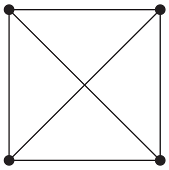


FIGURE 2 The Graph K_4.

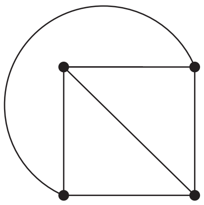


FIGURE 3 K_4 Drawn with No Crossings.

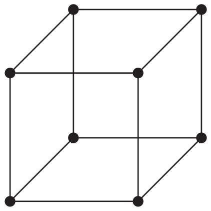


FIGURE 4 The Graph Q_3.

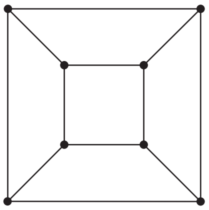


FIGURE 5 A Planar Representation of Q_3.

DEFINITION 1 A graph is called *planar* if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a *planar representation* of the graph.

A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

EXAMPLE 1 Is K_4 (shown in Figure 2 with two edges crossing) planar?

Solution: K_4 is planar because it can be drawn without crossings, as shown in Figure 3. ◀

EXAMPLE 2 Is Q_3, shown in Figure 4, planar?

Solution: Q_3 is planar, because it can be drawn without any edges crossing, as shown in Figure 5. ◀

We can show that a graph is planar by displaying a planar representation. It is harder to show that a graph is nonplanar. We will give an example to show how this can be done in an ad hoc fashion. Later we will develop some general results that can be used to do this.

EXAMPLE 3 Is $K_{3,3}$, shown in Figure 6, planar?

Solution: Any attempt to draw $K_{3,3}$ in the plane with no edges crossing is doomed. We now show why. In any planar representation of $K_{3,3}$, the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2, as shown in Figure 7(a). The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two subregions, R_{21} and R_{22}, as shown in Figure 7(b).

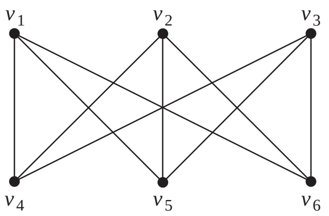


FIGURE 6 The Graph $K_{3,3}$.

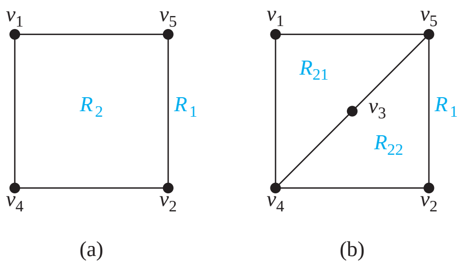


FIGURE 7 Showing that $K_{3,3}$ Is Nonplanar.

Next, note that there is no way to place the final vertex v_6 without forcing a crossing. For if v_6 is in R_1, then the edge between v_6 and v_3 cannot be drawn without a crossing. If v_6 is in R_{21}, then the edge between v_2 and v_6 cannot be drawn without a crossing. If v_6 is in R_{22}, then the edge between v_1 and v_6 cannot be drawn without a crossing.

A similar argument can be used when v_3 is in R_1. The completion of this argument is left for the reader (see Exercise 10). It follows that $K_{3,3}$ is not planar. ◀

Example 3 solves the utilities-and-houses problem that was described at the beginning of this section. The three houses and three utilities cannot be connected in the plane without a crossing. A similar argument can be used to show that K_5 is nonplanar. (See Exercise 11.)

APPLICATIONS OF PLANAR GRAPHS Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar. When this graph is not planar, we must turn to more expensive options. For example, we can partition the vertices in the graph representing the circuit into planar subgraphs. We then construct the circuit using multiple layers. (See the preamble to Exercise 30 to learn about the thickness of a graph.) We can construct the circuit using insulated wires whenever connections cross. In this case, drawing the graph with the fewest possible crossings is important. (See the preamble to Exercise 26 to learn about the crossing number of a graph.)

The planarity of graphs is also useful in the design of road networks. Suppose we want to connect a group of cities by roads. We can model a road network connecting these cities using a simple graph with vertices representing the cities and edges representing the highways connecting them. We can built this road network without using underpasses or overpasses if the resulting graph is planar.

Euler's Formula

A planar representation of a graph splits the plane into **regions**, including an unbounded region. For instance, the planar representation of the graph shown in Figure 8 splits the plane into six regions. These are labeled in the figure. Euler showed that all planar representations of a graph split the plane into the same number of regions. He accomplished this by finding a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

THEOREM 1 **EULER'S FORMULA** Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Proof: First, we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs $G_1, G_2, \ldots, G_e = G$, successively adding an edge at each stage. This is done using the following inductive definition. Arbitrarily pick one edge of G to obtain G_1. Obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already in G_{n-1},

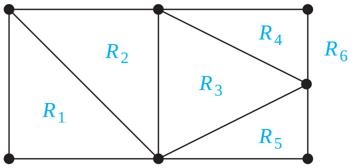


FIGURE 8 The Regions of the Planar Representation of a Graph.

adding the other vertex incident with this edge if it is not already in G_{n-1}. This construction is possible because G is connected. G is obtained after e edges are added. Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of G, respectively.

The proof will now proceed by induction. The relationship $r_1 = e_1 - v_1 + 2$ is true for G_1, because $e_1 = 1$, $v_1 = 2$, and $r_1 = 1$. This is shown in Figure 9.

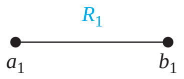


FIGURE 9 The Basis Case of the Proof of Euler's Formula.

Now assume that $r_k = e_k - v_k + 2$. Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}. There are two possibilities to consider. In the first case, both a_{k+1} and b_{k+1} are already in G_k. These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{k+1}, b_{k+1}\}$ to G_k without two edges crossing (and G_{k+1} is planar). The addition of this new edge splits R into two regions. Consequently, in this case, $r_{k+1} = r_k + 1$, $e_{k+1} = e_k + 1$, and $v_{k+1} = v_k$. Thus, each side of the formula relating the number of regions, edges, and vertices increases by exactly one, so this formula is still true. In other words, $r_{k+1} = e_{k+1} - v_{k+1} + 2$. This case is illustrated in Figure 10(a).

In the second case, one of the two vertices of the new edge is not already in G_k. Suppose that a_{k+1} is in G_k but that b_{k+1} is not. Adding this new edge does not produce any new regions, because b_{k+1} must be in a region that has a_{k+1} on its boundary. Consequently, $r_{k+1} = r_k$. Moreover, $e_{k+1} = e_k + 1$ and $v_{k+1} = v_k + 1$. Each side of the formula relating the number of regions, edges, and vertices remains the same, so the formula is still true. In other words, $r_{k+1} = e_{k+1} - v_{k+1} + 2$. This case is illustrated in Figure 10(b).

We have completed the induction argument. Hence, $r_n = e_n - v_n + 2$ for all n. Because the original graph is the graph G_e, obtained after e edges have been added, the theorem is true. ◁

Euler's formula is illustrated in Example 4.

EXAMPLE 4 Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution: This graph has 20 vertices, each of degree 3, so $v = 20$. Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60$, or $e = 30$. Consequently, from Euler's formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12. \quad ◀$$

Euler's formula can be used to establish some inequalities that must be satisfied by planar graphs. One such inequality is given in Corollary 1.

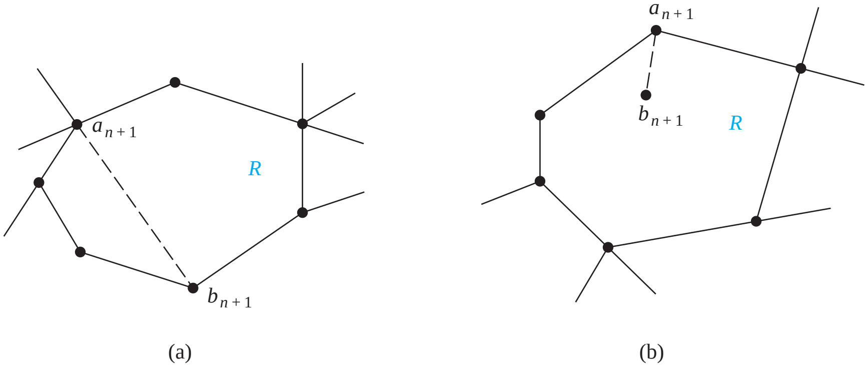


FIGURE 10 Adding an Edge to G_n to Produce G_{n+1}.

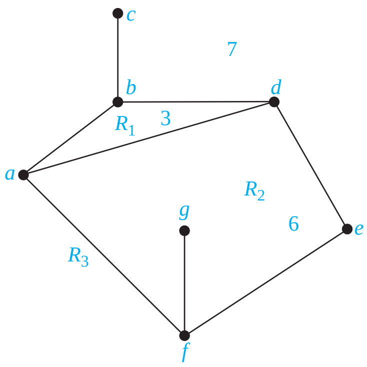


FIGURE 11 **The Degrees of Regions.**

COROLLARY 1 If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Before we prove Corollary 1 we will use it to prove the following useful result.

COROLLARY 2 If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof: If G has one or two vertices, the result is true. If G has at least three vertices, by Corollary 1 we know that $e \leq 3v - 6$, so $2e \leq 6v - 12$. If the degree of every vertex were at least six, then because $2e = \sum_{v \in V} \deg(v)$ (by the handshaking theorem), we would have $2e \geq 6v$. But this contradicts the inequality $2e \leq 6v - 12$. It follows that there must be a vertex with degree no greater than five. ◁

The proof of Corollary 1 is based on the concept of the **degree** of a region, which is defined to be the number of edges on the boundary of this region. When an edge occurs twice on the boundary (so that it is traced out twice when the boundary is traced out), it contributes two to the degree. We denote the degree of a region R by $\deg(R)$. The degrees of the regions of the graph shown in Figure 11 are displayed in the figure.

The proof of Corollary 1 can now be given.

Proof: A connected planar simple graph drawn in the plane divides the plane into regions, say r of them. The degree of each region is at least three. (Because the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree two, or loops that could produce regions of degree one, are permitted.) In particular, note that the degree of the unbounded region is at least three because there are at least three vertices in the graph.

Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph, because each edge occurs on the boundary of a region exactly twice (either in two different regions, or twice in the same region). Because each region has degree greater than or equal to three, it follows that

$$2e = \sum_{\text{all regions } R} \deg(R) \geq 3r.$$

Hence,

$$(2/3)e \geq r.$$

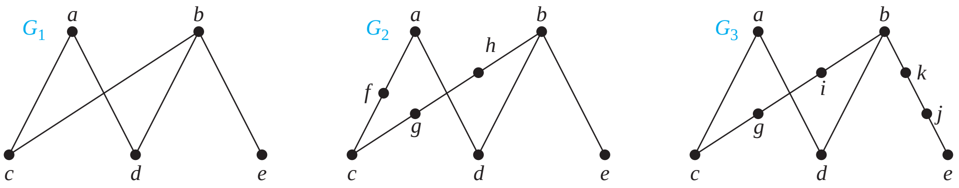


FIGURE 12 **Homeomorphic Graphs.**

Using $r = e - v + 2$ (Euler's formula), we obtain

$$e - v + 2 \leq (2/3)e.$$

It follows that $e/3 \leq v - 2$. This shows that $e \leq 3v - 6$. ◁

This corollary can be used to demonstrate that K_5 is nonplanar.

EXAMPLE 5 Show that K_5 is nonplanar using Corollary 1.

Solution: The graph K_5 has five vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$ and $3v - 6 = 9$. Therefore, K_5 is not planar. ◀

It was previously shown that $K_{3,3}$ is not planar. Note, however, that this graph has six vertices and nine edges. This means that the inequality $e = 9 \leq 12 = 3 \cdot 6 - 6$ is satisfied. Consequently, the fact that the inequality $e \leq 3v - 6$ is satisfied does *not* imply that a graph is planar. However, the following corollary of Theorem 1 can be used to show that $K_{3,3}$ is nonplanar.

COROLLARY 3 If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

The proof of Corollary 3 is similar to that of Corollary 1, except that in this case the fact that there are no circuits of length three implies that the degree of a region must be at least four. The details of this proof are left for the reader (see Exercise 15).

EXAMPLE 6 Use Corollary 3 to show that $K_{3,3}$ is nonplanar.

Solution: Because $K_{3,3}$ has no circuits of length three (this is easy to see because it is bipartite), Corollary 3 can be used. $K_{3,3}$ has six vertices and nine edges. Because $e = 9$ and $2v - 4 = 8$, Corollary 3 shows that $K_{3,3}$ is nonplanar. ◀

Links

KAZIMIERZ KURATOWSKI (1896–1980) Kazimierz Kuratowski, the son of a famous Warsaw lawyer, attended secondary school in Warsaw. He studied in Glasgow, Scotland, from 1913 to 1914 but could not return there after the outbreak of World War I. In 1915 he entered Warsaw University, where he was active in the Polish patriotic student movement. He published his first paper in 1919 and received his Ph.D. in 1921. He was an active member of the group known as the Warsaw School of Mathematics, working in the areas of the foundations of set theory and topology. He was appointed associate professor at the Lwów Polytechnical University, where he stayed for seven years, collaborating with the important Polish mathematicians Banach and Ulam. In 1930, while at Lwów, Kuratowski completed his work characterizing planar graphs.

In 1934 he returned to Warsaw University as a full professor. Until the start of World War II, he was active in research and teaching. During the war, because of the persecution of educated Poles, Kuratowski went into hiding under an assumed name and taught at the clandestine Warsaw University. After the war he helped revive Polish mathematics, serving as director of the Polish National Mathematics Institute. He wrote over 180 papers and three widely used textbooks.

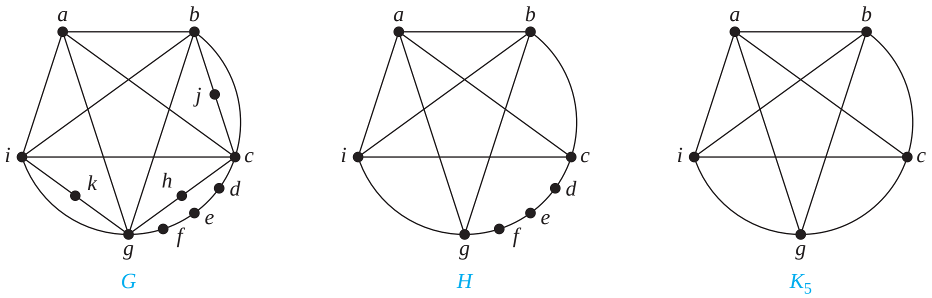


FIGURE 13 **The Undirected Graph G, a Subgraph H Homeomorphic to K_5, and K_5.**

Kuratowski's Theorem

We have seen that $K_{3,3}$ and K_5 are not planar. Clearly, a graph is not planar if it contains either of these two graphs as a subgraph. Surprisingly, all nonplanar graphs must contain a subgraph that can be obtained from $K_{3,3}$ or K_5 using certain permitted operations.

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an **elementary subdivision**. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

EXAMPLE 7 Show that the graphs G_1, G_2, and G_3 displayed in Figure 12 are all homeomorphic.

Solution: These three graphs are homeomorphic because all three can be obtained from G_1 by elementary subdivisions. G_1 can be obtained from itself by an empty sequence of elementary subdivisions. To obtain G_2 from G_1 we can use this sequence of elementary subdivisions: (*i*) remove the edge $\{a, c\}$, add the vertex f, and add the edges $\{a, f\}$ and $\{f, c\}$; (*ii*) remove the edge $\{b, c\}$, add the vertex g, and add the edges $\{b, g\}$ and $\{g, c\}$; and (*iii*) remove the edge $\{b, g\}$, add the vertex h, and add the edges $\{g, h\}$ and $\{b, h\}$. We leave it to the reader to determine the sequence of elementary subdivisions needed to obtain G_3 from G_1. ◀

The Polish mathematician Kazimierz Kuratowski established Theorem 2 in 1930, which characterizes planar graphs using the concept of graph homeomorphism.

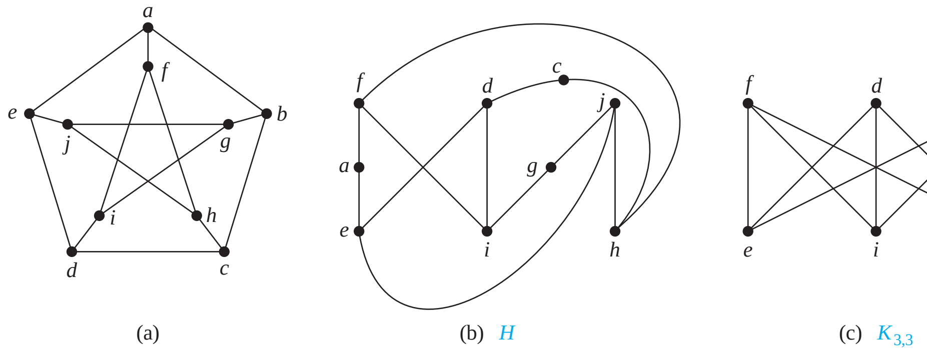
THEOREM 2 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

It is clear that a graph containing a subgraph homeomorphic to $K_{3,3}$ or K_5 is nonplanar. However, the proof of the converse, namely that every nonplanar graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5, is complicated and will not be given here. Examples 8 and 9 illustrate how Kuratowski's theorem is used.

EXAMPLE 8 Determine whether the graph G shown in Figure 13 is planar.

Extra Examples

Solution: G has a subgraph H homeomorphic to K_5. H is obtained by deleting h, j, and k and all edges incident with these vertices. H is homeomorphic to K_5 because it can be obtained from K_5 (with vertices a, b, c, g, and i) by a sequence of elementary subdivisions, adding the vertices d, e, and f. (The reader should construct such a sequence of elementary subdivisions.) Hence, G is nonplanar. ◀



(a)

(b) H

(c) $K_{3,3}$

FIGURE 14 (a) The Petersen Graph, (b) a Subgraph H Homeomorphic to $K_{3,3}$, and (c) $K_{3,3}$.

EXAMPLE 9 Is the Petersen graph, shown in Figure 14(a), planar? (The Danish mathematician Julius Petersen studied this graph in 1891; it is often used to illustrate various theoretical properties of graphs.)

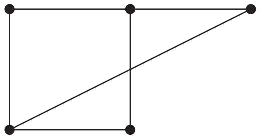
Solution: The subgraph H of the Petersen graph obtained by deleting b and the three edges that have b as an endpoint, shown in Figure 14(b), is homeomorphic to $K_{3,3}$, with vertex sets $\{f, d, j\}$ and $\{e, i, h\}$, because it can be obtained by a sequence of elementary subdivisions, deleting $\{d, h\}$ and adding $\{c, h\}$ and $\{c, d\}$, deleting $\{e, f\}$ and adding $\{a, e\}$ and $\{a, f\}$, and deleting $\{i, j\}$ and adding $\{g, i\}$ and $\{g, j\}$. Hence, the Petersen graph is not planar. ◀

Exercises

1. Can five houses be connected to two utilities without connections crossing?

In Exercises 2–4 draw the given planar graph without any crossings.

2.



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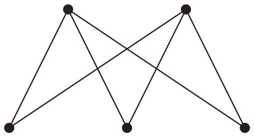
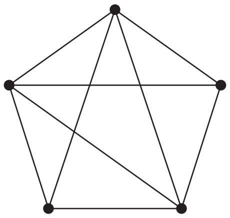
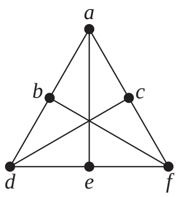


4.

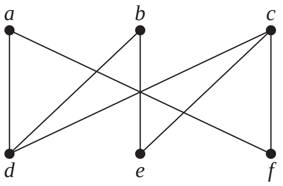


In Exercises 5–9 determine whether the given graph is planar. If so, draw it so that no edges cross.

5.



6.



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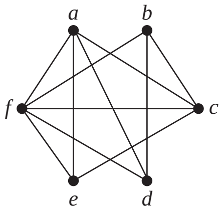

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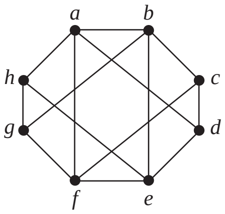

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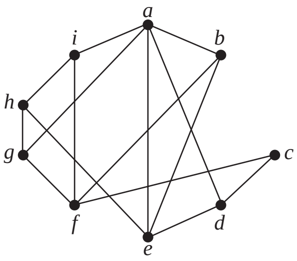


10. Complete the argument in Example 3.

11. Show that K_5 is nonplanar using an argument similar to that given in Example 3.

12. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?

13. Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

14. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

15. Prove Corollary 3.

16. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$ if $v \geq 3$.

***17.** Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

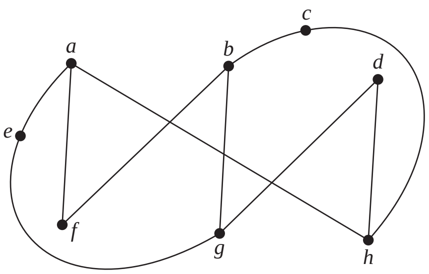
18. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e, v, and k.

19. Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?

a) K_5 **b)** K_6 **c)** $K_{3,3}$ **d)** $K_{3,4}$

In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

20.



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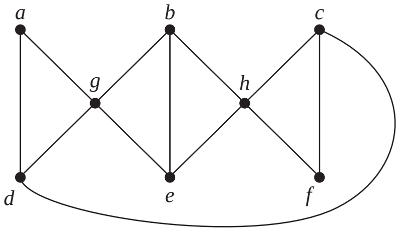

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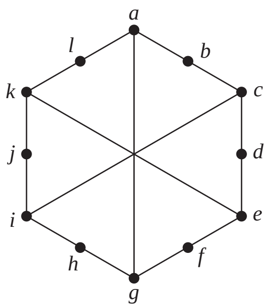
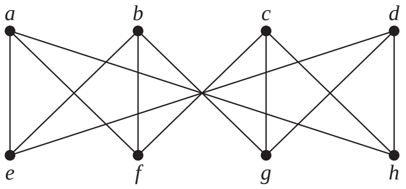


In Exercises 23–25 use Kuratowski's theorem to determine whether the given graph is planar.

23.



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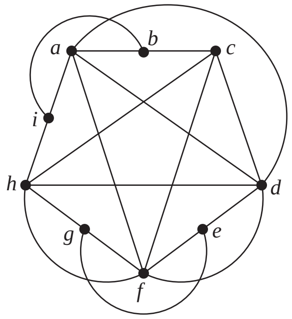

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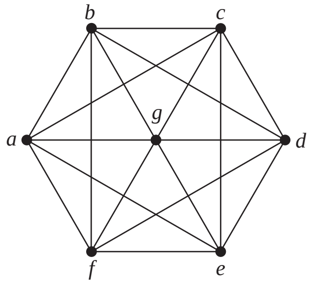


The **crossing number** of a simple graph is the minimum number of crossings that can occur when this graph is drawn in the plane where no three arcs representing edges are permitted to cross at the same point.

26. Show that $K_{3,3}$ has 1 as its crossing number.

****27.** Find the crossing numbers of each of these nonplanar graphs.

a) K_5 **b)** K_6 **c)** K_7
d) $K_{3,4}$ **e)** $K_{4,4}$ **f)** $K_{5,5}$

***28.** Find the crossing number of the Petersen graph.

****29.** Show that if m and n are even positive integers, the crossing number of $K_{m,n}$ is less than or equal to $mn(m-2)(n-2)/16$. [*Hint:* Place m vertices along the x-axis so that they are equally spaced and symmetric about the origin and place n vertices along the y-axis so that they are equally spaced and symmetric about the origin. Now connect each of the m vertices on the x-axis to each of the vertices on the y-axis and count the crossings.]

The **thickness** of a simple graph G is the smallest number of planar subgraphs of G that have G as their union.

30. Show that $K_{3,3}$ has 2 as its thickness.

***31.** Find the thickness of the graphs in Exercise 27.

32. Show that if G is a connected simple graph with v vertices and e edges, where $v \geq 3$, then the thickness of G is at least $\lceil e/(3v-6) \rceil$.

***33.** Use Exercise 32 to show that the thickness of K_n is at least $\lfloor (n+7)/6 \rfloor$ whenever n is a positive integer.

34. Show that if G is a connected simple graph with v vertices and e edges, where $v \geq 3$, and no circuits of length three, then the thickness of G is at least $\lceil e/(2v-4) \rceil$.

35. Use Exercise 34 to show that the thickness of $K_{m,n}$, where m and n are not both 1, is at least $\lceil mn/(2m+2n-4) \rceil$ whenever m and n are positive integers.

***36.** Draw K_5 on the surface of a torus (a doughnut-shaped solid) so that no edges cross.

***37.** Draw $K_{3,3}$ on the surface of a torus so that no edges cross.

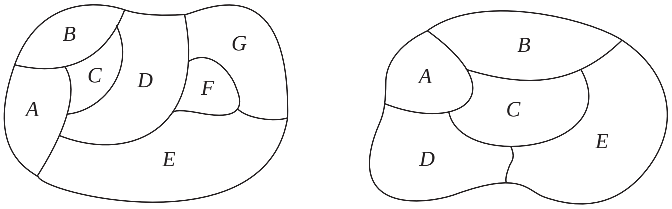


FIGURE 1 **Two Maps.**

10.8 Graph Coloring

Introduction

Links

Problems related to the coloring of maps of regions, such as maps of parts of the world, have generated many results in graph theory. When a map* is colored, two regions with a common border are customarily assigned different colors. One way to ensure that two adjacent regions never have the same color is to use a different color for each region. However, this is inefficient, and on maps with many regions it would be hard to distinguish similar colors. Instead, a small number of colors should be used whenever possible. Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color. For instance, for the map shown on the left in Figure 1, four colors suffice, but three colors are not enough. (The reader should check this.) In the map on the right in Figure 1, three colors are sufficient (but two are not).

Each map in the plane can be represented by a graph. To set up this correspondence, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent. The resulting graph is called the **dual graph** of the map. By the way in which dual graphs of maps are constructed, it is clear that any map in the plane has a planar dual graph. Figure 2 displays the dual graphs that correspond to the maps shown in Figure 1.

The problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color. We now define a graph coloring.

DEFINITION 1

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

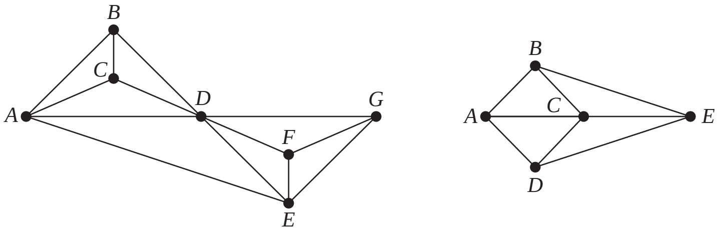


FIGURE 2 **Dual Graphs of the Maps in Figure 1.**

*We will assume that all regions in a map are connected. This eliminates any problems presented by such geographical entities as Michigan.

CHAPTER

11 Trees

A connected graph that contains no simple circuits is called a tree. Trees were used as long ago as 1857, when the English mathematician Arthur Cayley used them to count certain types of chemical compounds. Since that time, trees have been employed to solve problems in a wide variety of disciplines, as the examples in this chapter will show.

Trees are particularly useful in computer science, where they are employed in a wide range of algorithms. For instance, trees are used to construct efficient algorithms for locating items in a list. They can be used in algorithms, such as Huffman coding, that construct efficient codes saving costs in data transmission and storage. Trees can be used to study games such as checkers and chess and can help determine winning strategies for playing these games. Trees can be used to model procedures carried out using a sequence of decisions. Constructing these models can help determine the computational complexity of algorithms based on a sequence of decisions, such as sorting algorithms.

Procedures for building trees containing every vertex of a graph, including depth-first search and breadth-first search, can be used to systematically explore the vertices of a graph. Exploring the vertices of a graph via depth-first search, also known as backtracking, allows for the systematic search for solutions to a wide variety of problems, such as determining how eight queens can be placed on a chessboard so that no queen can attack another.

We can assign weights to the edges of a tree to model many problems. For example, using weighted trees we can develop algorithms to construct networks containing the least expensive set of telephone lines linking different network nodes.

11.1 Introduction to Trees

Links

In Chapter 10 we showed how graphs can be used to model and solve many problems. In this chapter we will focus on a particular type of graph called a **tree**, so named because such graphs resemble trees. For example, *family trees* are graphs that represent genealogical charts. Family trees use vertices to represent the members of a family and edges to represent parent–child relationships. The family tree of the male members of the Bernoulli family of Swiss mathematicians is shown in Figure 1. The undirected graph representing a family tree (restricted to people of just one gender and with no inbreeding) is an example of a tree.

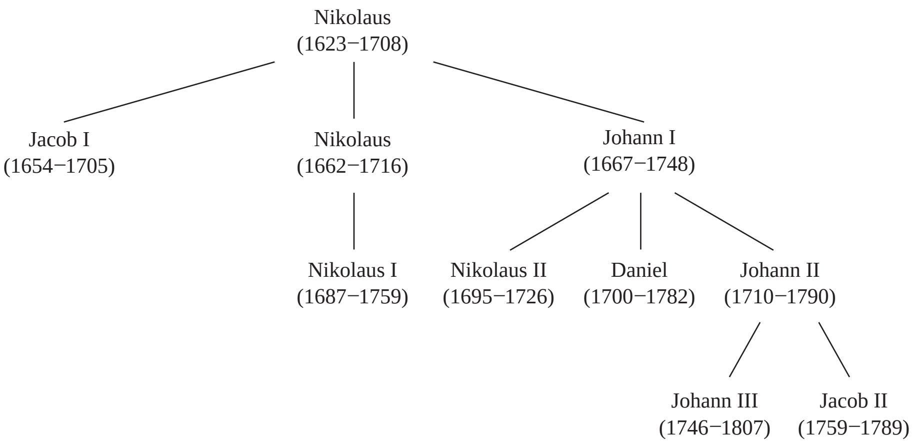


FIGURE 1 **The Bernoulli Family of Mathematicians.**

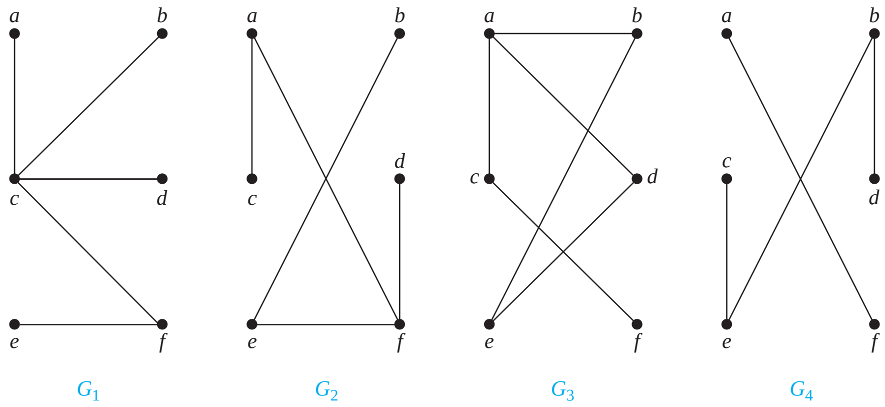


FIGURE 2 **Examples of Trees and Graphs That Are Not Trees.**

DEFINITION 1

A *tree* is a connected undirected graph with no simple circuits.

Because a tree cannot have a simple circuit, a tree cannot contain multiple edges or loops. Therefore any tree must be a simple graph.

EXAMPLE 1 Which of the graphs shown in Figure 2 are trees?

Solution: G_1 and G_2 are trees, because both are connected graphs with no simple circuits. G_3 is not a tree because e, b, a, d, e is a simple circuit in this graph. Finally, G_4 is not a tree because it is not connected. ◀

Any connected graph that contains no simple circuits is a tree. What about graphs containing no simple circuits that are not necessarily connected? These graphs are called **forests** and have the property that each of their connected components is a tree. Figure 3 displays a forest.

Trees are often defined as undirected graphs with the property that there is a unique simple path between every pair of vertices. Theorem 1 shows that this alternative definition is equivalent to our definition.

THEOREM 1

An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

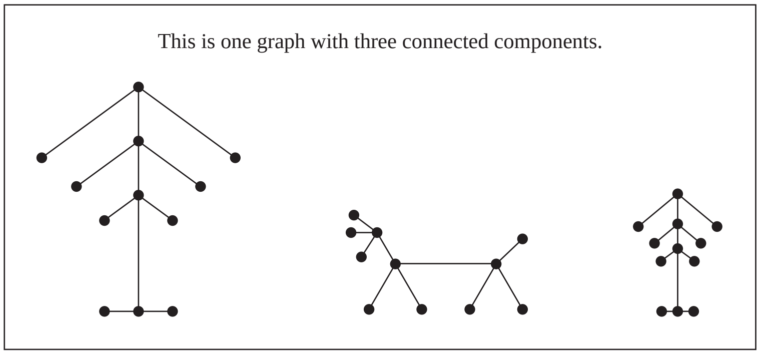


FIGURE 3 **Example of a Forest.**

Proof: First assume that T is a tree. Then T is a connected graph with no simple circuits. Let x and y be two vertices of T. Because T is connected, by Theorem 1 of Section 10.4 there is a simple path between x and y. Moreover, this path must be unique, for if there were a second such path, the path formed by combining the first path from x to y followed by the path from y to x obtained by reversing the order of the second path from x to y would form a circuit. This implies, using Exercise 59 of Section 10.4, that there is a simple circuit in T. Hence, there is a unique simple path between any two vertices of a tree.

Now assume that there is a unique simple path between any two vertices of a graph T. Then T is connected, because there is a path between any two of its vertices. Furthermore, T can have no simple circuits. To see that this is true, suppose T had a simple circuit that contained the vertices x and y. Then there would be two simple paths between x and y, because the simple circuit is made up of a simple path from x to y and a second simple path from y to x. Hence, a graph with a unique simple path between any two vertices is a tree. ◁

Rooted Trees

In many applications of trees, a particular vertex of a tree is designated as the **root**. Once we specify a root, we can assign a direction to each edge as follows. Because there is a unique path from the root to each vertex of the graph (by Theorem 1), we direct each edge away from the root. Thus, a tree together with its root produces a directed graph called a **rooted tree**.

DEFINITION 2 A *rooted tree* is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Rooted trees can also be defined recursively. Refer to Section 5.3 to see how this can be done. We can change an unrooted tree into a rooted tree by choosing any vertex as the root. Note that different choices of the root produce different rooted trees. For instance, Figure 4 displays the rooted trees formed by designating a to be the root and c to be the root, respectively, in the tree T. We usually draw a rooted tree with its root at the top of the graph. The arrows indicating the directions of the edges in a rooted tree can be omitted, because the choice of root determines the directions of the edges.

The terminology for trees has botanical and genealogical origins. Suppose that T is a rooted tree. If v is a vertex in T other than the root, the **parent** of v is the unique vertex u such that there is a directed edge from u to v (the reader should show that such a vertex is unique). When u is the parent of v, v is called a **child** of u. Vertices with the same parent are called **siblings**. The **ancestors** of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root (that is, its parent, its parent's parent, and so on, until the root is reached). The **descendants** of a vertex v are those vertices that have v as

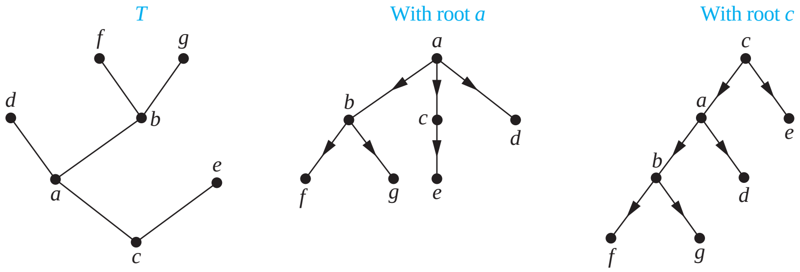


FIGURE 4 **A Tree and Rooted Trees Formed by Designating Two Different Roots.**

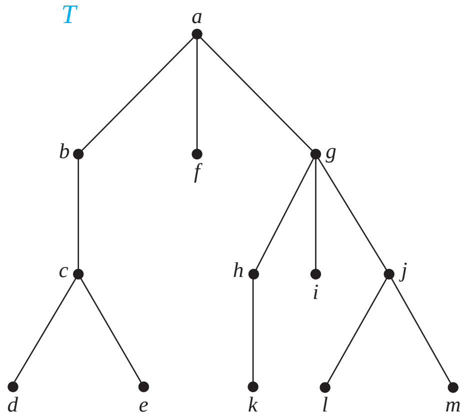


FIGURE 5 **A Rooted Tree T.**

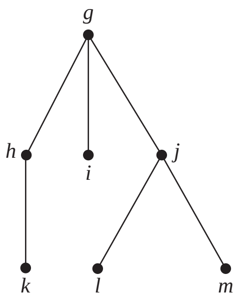


FIGURE 6 **The Subtree Rooted at g.**

an ancestor. A vertex of a rooted tree is called a **leaf** if it has no children. Vertices that have children are called **internal vertices**. The root is an internal vertex unless it is the only vertex in the graph, in which case it is a leaf.

If a is a vertex in a tree, the **subtree** with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

EXAMPLE 2 In the rooted tree T (with root a) shown in Figure 5, find the parent of c, the children of g, the siblings of h, all ancestors of e, all descendants of b, all internal vertices, and all leaves. What is the subtree rooted at g?

Extra Examples

Solution: The parent of c is b. The children of g are h, i, and j. The siblings of h are i and j. The ancestors of e are c, b, and a. The descendants of b are c, d, and e. The internal vertices are a, b, c, g, h, and j. The leaves are d, e, f, i, k, l, and m. The subtree rooted at g is shown in Figure 6. ◀

Rooted trees with the property that all of their internal vertices have the same number of children are used in many different applications. Later in this chapter we will use such trees to study problems involving searching, sorting, and coding.

DEFINITION 3

Links

A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. An m-ary tree with $m = 2$ is called a *binary tree*.

EXAMPLE 3 Are the rooted trees in Figure 7 full m-ary trees for some positive integer m?

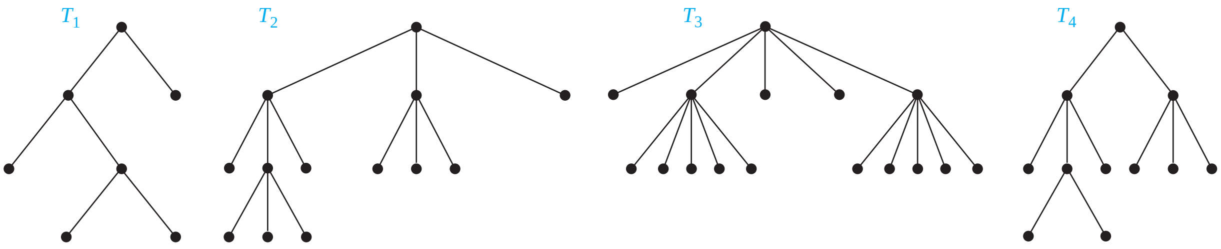


FIGURE 7 **Four Rooted Trees.**

Solution: T_1 is a full binary tree because each of its internal vertices has two children. T_2 is a full 3-ary tree because each of its internal vertices has three children. In T_3 each internal vertex has five children, so T_3 is a full 5-ary tree. T_4 is not a full m-ary tree for any m because some of its internal vertices have two children and others have three children. ◀

ORDERED ROOTED TREES An **ordered rooted tree** is a rooted tree where the children of each internal vertex are ordered. Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right. Note that a representation of a rooted tree in the conventional way determines an ordering for its edges. We will use such orderings of edges in drawings without explicitly mentioning that we are considering a rooted tree to be ordered.

In an ordered binary tree (usually called just a **binary tree**), if an internal vertex has two children, the first child is called the **left child** and the second child is called the **right child**. The tree rooted at the left child of a vertex is called the **left subtree** of this vertex, and the tree rooted at the right child of a vertex is called the **right subtree** of the vertex. The reader should note that for some applications every vertex of a binary tree, other than the root, is designated as a right or a left child of its parent. This is done even when some vertices have only one child. We will make such designations whenever it is necessary, but not otherwise.

Ordered rooted trees can be defined recursively. Binary trees, a type of ordered rooted trees, were defined this way in Section 5.3.

EXAMPLE 4 What are the left and right children of d in the binary tree T shown in Figure 8(a) (where the order is that implied by the drawing)? What are the left and right subtrees of c?

Solution: The left child of d is f and the right child is g. We show the left and right subtrees of c in Figures 8(b) and 8(c), respectively. ◀

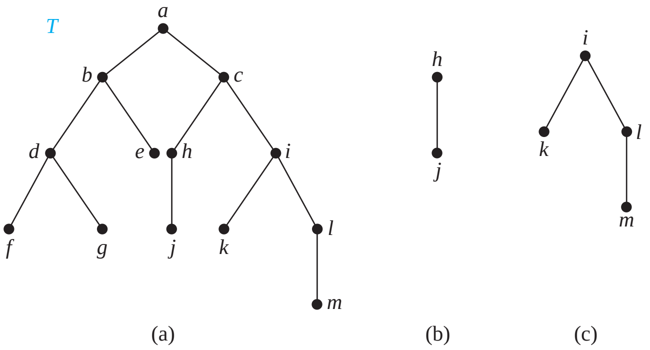
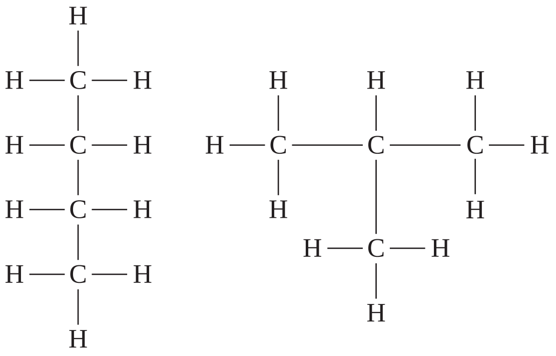


FIGURE 8 **A Binary Tree T and Left and Right Subtrees of the Vertex c.**

Just as in the case of graphs, there is no standard terminology used to describe trees, rooted trees, ordered rooted trees, and binary trees. This nonstandard terminology occurs because trees are used extensively throughout computer science, which is a relatively young field. The reader should carefully check meanings given to terms dealing with trees whenever they occur.

Trees as Models

Trees are used as models in such diverse areas as computer science, chemistry, geology, botany, and psychology. We will describe a variety of such models based on trees.



Butane Isobutane

FIGURE 9 **The Two Isomers of Butane.**

EXAMPLE 5 **Saturated Hydrocarbons and Trees** Graphs can be used to represent molecules, where atoms are represented by vertices and bonds between them by edges. The English mathematician Arthur Cayley discovered trees in 1857 when he was trying to enumerate the isomers of compounds of the form C_nH_{2n+2}, which are called *saturated hydrocarbons*.

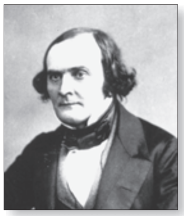
In graph models of saturated hydrocarbons, each carbon atom is represented by a vertex of degree 4, and each hydrogen atom is represented by a vertex of degree 1. There are $3n + 2$ vertices in a graph representing a compound of the form C_nH_{2n+2}. The number of edges in such a graph is half the sum of the degrees of the vertices. Hence, there are $(4n + 2n + 2)/2 = 3n + 1$ edges in this graph. Because the graph is connected and the number of edges is one less than the number of vertices, it must be a tree (see Exercise 15).

The nonisomorphic trees with n vertices of degree 4 and $2n + 2$ of degree 1 represent the different isomers of C_nH_{2n+2}. For instance, when $n = 4$, there are exactly two nonisomorphic trees of this type (the reader should verify this). Hence, there are exactly two different isomers of C_4H_{10}. Their structures are displayed in Figure 9. These two isomers are called butane and isobutane. ◀

EXAMPLE 6 **Representing Organizations** The structure of a large organization can be modeled using a rooted tree. Each vertex in this tree represents a position in the organization. An edge from one vertex to another indicates that the person represented by the initial vertex is the (direct) boss of the person represented by the terminal vertex. The graph shown in Figure 10 displays such a tree. In the organization represented by this tree, the Director of Hardware Development works directly for the Vice President of R&D. ◀

EXAMPLE 7 **Computer File Systems** Files in computer memory can be organized into directories. A directory can contain both files and subdirectories. The root directory contains the entire file

Links



ARTHUR CAYLEY (1821–1895) Arthur Cayley, the son of a merchant, displayed his mathematical talents at an early age with amazing skill in numerical calculations. Cayley entered Trinity College, Cambridge, when he was 17. While in college he developed a passion for reading novels. Cayley excelled at Cambridge and was elected to a 3-year appointment as Fellow of Trinity and assistant tutor. During this time Cayley began his study of n-dimensional geometry and made a variety of contributions to geometry and to analysis. He also developed an interest in mountaineering, which he enjoyed during vacations in Switzerland. Because no position as a mathematician was available to him, Cayley left Cambridge, entering the legal profession and gaining admittance to the bar in 1849. Although Cayley limited his legal work to be able to continue his mathematics research, he developed a reputation as a legal specialist. During his legal career he was able to write more than 300 mathematical papers. In 1863 Cambridge University established a new post in mathematics and offered it to Cayley. He took this job, even though it paid less money than he made as a lawyer.

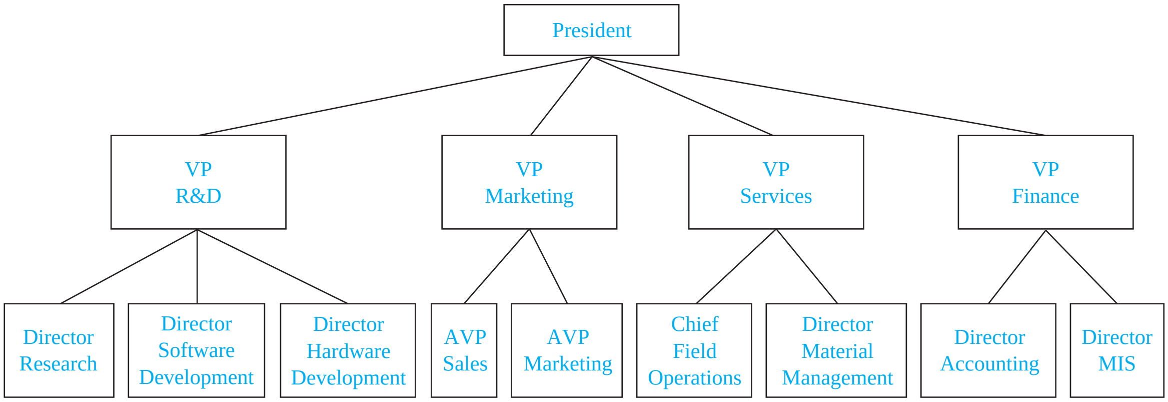


FIGURE 10 **An Organizational Tree for a Computer Company.**

system. Thus, a file system may be represented by a rooted tree, where the root represents the root directory, internal vertices represent subdirectories, and leaves represent ordinary files or empty directories. One such file system is shown in Figure 11. In this system, the file khr is in the directory rje. (Note that links to files where the same file may have more than one pathname can lead to circuits in computer file systems.) ◀

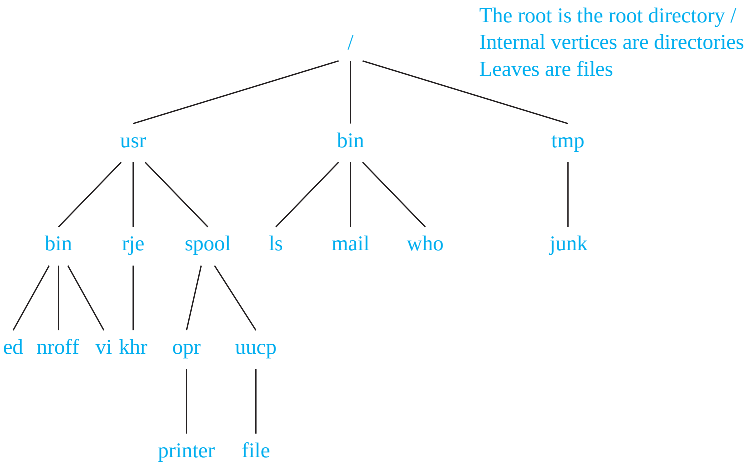


FIGURE 11 **A Computer File System.**

EXAMPLE 8 **Tree-Connected Parallel Processors** In Example 17 of Section 10.2 we described several interconnection networks for parallel processing. A **tree-connected network** is another important way to interconnect processors. The graph representing such a network is a complete binary tree, that is, a full binary tree where every root is at the same level. Such a network interconnects $n = 2^k - 1$ processors, where k is a positive integer. A processor represented by the vertex v that is not a root or a leaf has three two-way connections—one to the processor represented by the parent of v and two to the processors represented by the two children of v. The processor represented by the root has two two-way connections to the processors represented by its two children. A processor represented by a leaf v has a single two-way connection to the parent of v. We display a tree-connected network with seven processors in Figure 12.

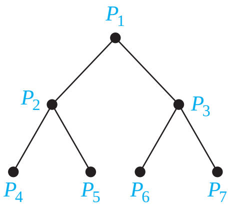


FIGURE 12 **A Tree-Connected Network of Seven Processors.**

We now illustrate how a tree-connected network can be used for parallel computation. In particular, we show how the processors in Figure 12 can be used to add eight numbers, using three steps. In the first step, we add x_1 and x_2 using P_4, x_3 and x_4 using P_5, x_5 and x_6 using P_6, and x_7 and x_8 using P_7. In the second step, we add $x_1 + x_2$ and $x_3 + x_4$ using P_2 and $x_5 + x_6$ and $x_7 + x_8$ using P_3. Finally, in the third step, we add $x_1 + x_2 + x_3 + x_4$ and $x_5 + x_6 + x_7 + x_8$ using P_1. The three steps used to add eight numbers compares favorably to the seven steps required to add eight numbers serially, where the steps are the addition of one number to the sum of the previous numbers in the list. ◀

Properties of Trees

We will often need results relating the numbers of edges and vertices of various types in trees.

THEOREM 2 A tree with n vertices has $n - 1$ edges.

Proof: We will use mathematical induction to prove this theorem. Note that for all the trees here we can choose a root and consider the tree rooted.

BASIS STEP: When $n = 1$, a tree with $n = 1$ vertex has no edges. It follows that the theorem is true for $n = 1$.

INDUCTIVE STEP: The inductive hypothesis states that every tree with k vertices has $k - 1$ edges, where k is a positive integer. Suppose that a tree T has $k + 1$ vertices and that v is a leaf of T (which must exist because the tree is finite), and let w be the parent of v. Removing from T the vertex v and the edge connecting w to v produces a tree T' with k vertices, because the resulting graph is still connected and has no simple circuits. By the inductive hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T', the edge connecting v and w. This completes the inductive step. ◁

Recall that a tree is a connected undirected graph with no simple circuits. So, when G is an undirected graph with n vertices, Theorem 2 tells us that the two conditions *(i)* G is connected and *(ii)* G has no simple circuits, imply *(iii)* G has $n - 1$ edges. Also, when *(i)* and *(iii)* hold, then *(ii)* must also hold, and when *(ii)* and *(iii)* hold, *(i)* must also hold. That is, if G is connected and G has $n - 1$ edges, then G has no simple circuits, so that G is a tree (see Exercise 15(a)), and if G has no simple circuits and G has $n - 1$ edges, then G is connected, and so is a tree (see Exercise 15(b)). Consequently, when two of *(i)*, *(ii)*, and *(iii)* hold, the third condition must also hold, and G must be a tree.

COUNTING VERTICES IN FULL m-ARY TREES The number of vertices in a full m-ary tree with a specified number of internal vertices is determined, as Theorem 3 shows. As in Theorem 2, we will use n to denote the number of vertices in a tree.

THEOREM 3 A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Proof: Every vertex, except the root, is the child of an internal vertex. Because each of the i internal vertices has m children, there are mi vertices in the tree other than the root. Therefore, the tree contains $n = mi + 1$ vertices. ◁

Suppose that T is a full m-ary tree. Let i be the number of internal vertices and l the number of leaves in this tree. Once one of n, i, and l is known, the other two quantities are determined. Theorem 4 explains how to find the other two quantities from the one that is known.

THEOREM 4 A full m-ary tree with

(i) n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n+1]/m$ leaves,

(ii) i internal vertices has $n = mi+1$ vertices and $l = (m-1)i+1$ leaves,

(iii) l leaves has $n = (ml-1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

Proof: Let n represent the number of vertices, i the number of internal vertices, and l the number of leaves. The three parts of the theorem can all be proved using the equality given in Theorem 3, that is, $n = mi+1$, together with the equality $n = l+i$, which is true because each vertex is either a leaf or an internal vertex. We will prove part (i) here. The proofs of parts (ii) and (iii) are left as exercises for the reader.

Solving for i in $n = mi+1$ gives $i = (n-1)/m$. Then inserting this expression for i into the equation $n = l+i$ shows that $l = n-i = n-(n-1)/m = [(m-1)n+1]/m$. ◁

Example 9 illustrates how Theorem 4 can be used.

EXAMPLE 9 Suppose that someone starts a chain letter. Each person who receives the letter is asked to send it on to four other people. Some people do this, but others do not send any letters. How many people have seen the letter, including the first person, if no one receives more than one letter and if the chain letter ends after there have been 100 people who read it but did not send it out? How many people sent out the letter?

Solution: The chain letter can be represented using a 4-ary tree. The internal vertices correspond to people who sent out the letter, and the leaves correspond to people who did not send it out. Because 100 people did not send out the letter, the number of leaves in this rooted tree is $l = 100$. Hence, part (iii) of Theorem 4 shows that the number of people who have seen the letter is $n = (4 \cdot 100 - 1)/(4-1) = 133$. Also, the number of internal vertices is $133 - 100 = 33$, so 33 people sent out the letter. ◀

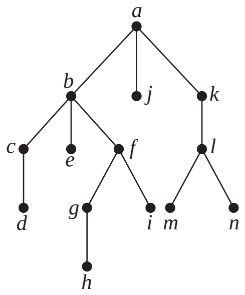


FIGURE 13 A Rooted Tree.

BALANCED m-ARY TREES It is often desirable to use rooted trees that are "balanced" so that the subtrees at each vertex contain paths of approximately the same length. Some definitions will make this concept clear. The **level** of a vertex v in a rooted tree is the length of the unique path from the root to this vertex. The level of the root is defined to be zero. The **height** of a rooted tree is the maximum of the levels of vertices. In other words, the height of a rooted tree is the length of the longest path from the root to any vertex.

EXAMPLE 10 Find the level of each vertex in the rooted tree shown in Figure 13. What is the height of this tree?

Solution: The root a is at level 0. Vertices b, j, and k are at level 1. Vertices c, e, f, and l are at level 2. Vertices d, g, i, m, and n are at level 3. Finally, vertex h is at level 4. Because the largest level of any vertex is 4, this tree has height 4. ◀

A rooted m-ary tree of height h is **balanced** if all leaves are at levels h or $h-1$.

EXAMPLE 11 Which of the rooted trees shown in Figure 14 are balanced?

Solution: T_1 is balanced, because all its leaves are at levels 3 and 4. However, T_2 is not balanced, because it has leaves at levels 2, 3, and 4. Finally, T_3 is balanced, because all its leaves are at level 3. ◀

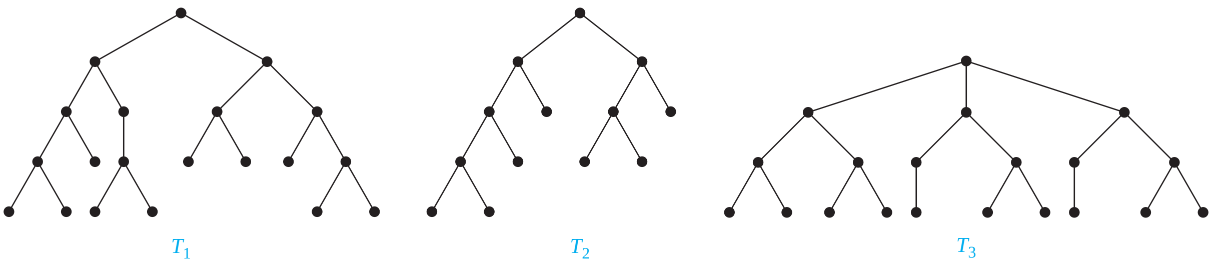


FIGURE 14 **Some Rooted Trees.**

A BOUND FOR THE NUMBER OF LEAVES IN AN m-ARY TREE It is often useful to have an upper bound for the number of leaves in an m-ary tree. Theorem 5 provides such a bound in terms of the height of the m-ary tree.

THEOREM 5 There are at most m^h leaves in an m-ary tree of height h.

Proof: The proof uses mathematical induction on the height. First, consider m-ary trees of height 1. These trees consist of a root with no more than m children, each of which is a leaf. Hence, there are no more than $m^1 = m$ leaves in an m-ary tree of height 1. This is the basis step of the inductive argument.

Now assume that the result is true for all m-ary trees of height less than h; this is the inductive hypothesis. Let T be an m-ary tree of height h. The leaves of T are the leaves of the subtrees of T obtained by deleting the edges from the root to each of the vertices at level 1, as shown in Figure 15.

Each of these subtrees has height less than or equal to $h - 1$. So by the inductive hypothesis, each of these rooted trees has at most m^{h-1} leaves. Because there are at most m such subtrees, each with a maximum of m^{h-1} leaves, there are at most $m \cdot m^{h-1} = m^h$ leaves in the rooted tree. This finishes the inductive argument. ◁

COROLLARY 1 If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. If the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$. (We are using the ceiling function here. Recall that $\lceil x \rceil$ is the smallest integer greater than or equal to x.)

Proof: We know that $l \leq m^h$ from Theorem 5. Taking logarithms to the base m shows that $\log_m l \leq h$. Because h is an integer, we have $h \geq \lceil \log_m l \rceil$. Now suppose that the tree is balanced.

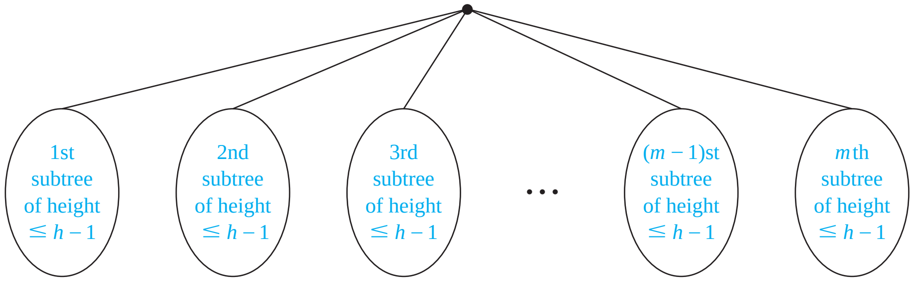

1st subtree of height $\leq h - 1$
2nd subtree of height $\leq h - 1$
3rd subtree of height $\leq h - 1$
. . .
$(m - 1)$st subtree of height $\leq h - 1$
mth subtree of height $\leq h - 1$

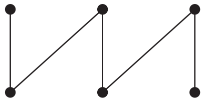
FIGURE 15 **The Inductive Step of the Proof.**

Then each leaf is at level h or $h - 1$, and because the height is h, there is at least one leaf at level h. It follows that there must be more than m^{h-1} leaves (see Exercise 30). Because $l \leq m^h$, we have $m^{h-1} < l \leq m^h$. Taking logarithms to the base m in this inequality gives $h - 1 < \log_m l \leq h$. Hence, $h = \lceil \log_m l \rceil$. ◁

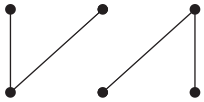
Exercises

1. Which of these graphs are trees?

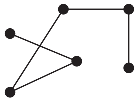
a)



b)



c)



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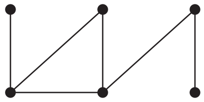

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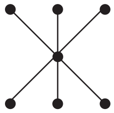

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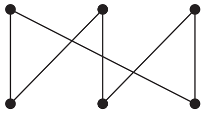
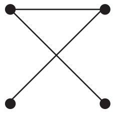


2. Which of these graphs are trees?

a)



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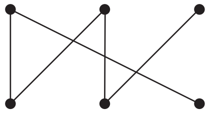

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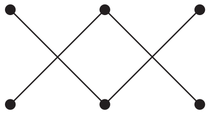

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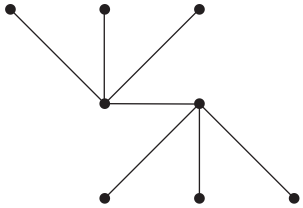

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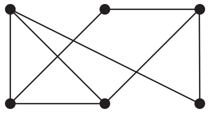

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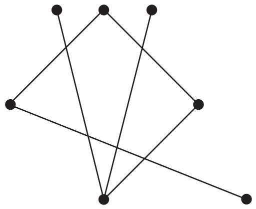
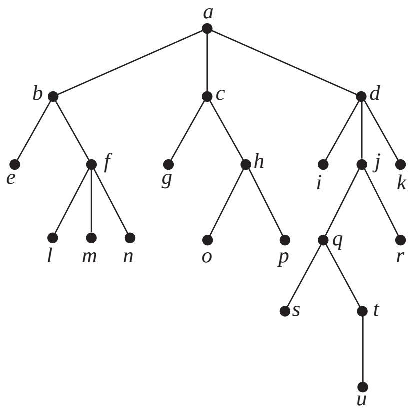
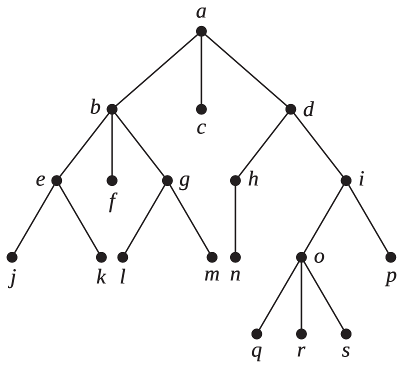


3. Answer these questions about the rooted tree illustrated.



a) Which vertex is the root?
b) Which vertices are internal?
c) Which vertices are leaves?
d) Which vertices are children of j?
e) Which vertex is the parent of h?
f) Which vertices are siblings of o?
g) Which vertices are ancestors of m?
h) Which vertices are descendants of b?

4. Answer the same questions as listed in Exercise 3 for the rooted tree illustrated.



5. Is the rooted tree in Exercise 3 a full m-ary tree for some positive integer m?

6. Is the rooted tree in Exercise 4 a full m-ary tree for some positive integer m?

7. What is the level of each vertex of the rooted tree in Exercise 3?

8. What is the level of each vertex of the rooted tree in Exercise 4?

9. Draw the subtree of the tree in Exercise 3 that is rooted at

a) a. **b)** c. **c)** e.

10. Draw the subtree of the tree in Exercise 4 that is rooted at

a) a. **b)** c. **c)** e.

11. a) How many nonisomorphic unrooted trees are there with three vertices?

b) How many nonisomorphic rooted trees are there with three vertices (using isomorphism for directed graphs)?

***12. a)** How many nonisomorphic unrooted trees are there with four vertices?

b) How many nonisomorphic rooted trees are there with four vertices (using isomorphism for directed graphs)?

***13. a)** How many nonisomorphic unrooted trees are there with five vertices?

b) How many nonisomorphic rooted trees are there with five vertices (using isomorphism for directed graphs)?

***14.** Show that a simple graph is a tree if and only if it is connected but the deletion of any of its edges produces a graph that is not connected.

***15.** Let G be a simple graph with n vertices. Show that

a) G is a tree if and only if it is connected and has $n - 1$ edges.

b) G is a tree if and only if G has no simple circuits and has $n - 1$ edges. [*Hint:* To show that G is connected if it has no simple circuits and $n - 1$ edges, show that G cannot have more than one connected component.]

16. Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

17. How many edges does a tree with 10,000 vertices have?

18. How many vertices does a full 5-ary tree with 100 internal vertices have?

19. How many edges does a full binary tree with 1000 internal vertices have?

20. How many leaves does a full 3-ary tree with 100 vertices have?

21. Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties.)

22. A chain letter starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it or does not send it to anyone. Suppose that 10,000 people send out the letter before the chain ends and that no one receives more than one letter. How many people receive the letter, and how many do not send it out?

23. A chain letter starts with a person sending a letter out to 10 others. Each person is asked to send the letter out to 10 others, and each letter contains a list of the previous six people in the chain. Unless there are fewer than six names in the list, each person sends one dollar to the first person in this list, removes the name of this person from the list, moves up each of the other five names one position, and inserts his or her name at the end of this list. If no person breaks the chain and no one receives more than one letter, how much money will a person in the chain ultimately receive?

***24.** Either draw a full m-ary tree with 76 leaves and height 3, where m is a positive integer, or show that no such tree exists.

***25.** Either draw a full m-ary tree with 84 leaves and height 3, where m is a positive integer, or show that no such tree exists.

***26.** A full m-ary tree T has 81 leaves and height 4.

a) Give the upper and lower bounds for m.

b) What is m if T is also balanced?

A **complete *m*-ary tree** is a full m-ary tree in which every leaf is at the same level.

27. Construct a complete binary tree of height 4 and a complete 3-ary tree of height 3.

28. How many vertices and how many leaves does a complete m-ary tree of height h have?

29. Prove

a) part (*ii*) of Theorem 4.

b) part (*iii*) of Theorem 4.

30. Show that a full m-ary balanced tree of height h has more than m^{h-1} leaves.

31. How many edges are there in a forest of t trees containing a total of n vertices?

32. Explain how a tree can be used to represent the table of contents of a book organized into chapters, where each chapter is organized into sections, and each section is organized into subsections.

33. How many different isomers do these saturated hydrocarbons have?

a) C_3H_8 **b)** C_5H_{12} **c)** C_6H_{14}

34. What does each of these represent in an organizational tree?

a) the parent of a vertex

b) a child of a vertex

c) a sibling of a vertex

d) the ancestors of a vertex

e) the descendants of a vertex

f) the level of a vertex

g) the height of the tree

35. Answer the same questions as those given in Exercise 34 for a rooted tree representing a computer file system.

36. a) Draw the complete binary tree with 15 vertices that represents a tree-connected network of 15 processors.

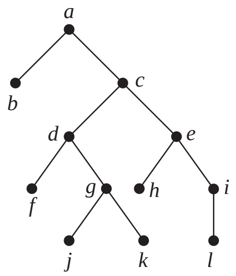
b) Show how 16 numbers can be added using the 15 processors in part (a) using four steps.

37. Let n be a power of 2. Show that n numbers can be added in $\log n$ steps using a tree-connected network of $n - 1$ processors.

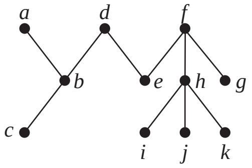
***38.** A **labeled tree** is a tree where each vertex is assigned a label. Two labeled trees are considered isomorphic when there is an isomorphism between them that preserves the labels of vertices. How many nonisomorphic trees are there with three vertices labeled with different integers from the set $\{1, 2, 3\}$? How many nonisomorphic trees are there with four vertices labeled with different integers from the set $\{1, 2, 3, 4\}$?

The **eccentricity** of a vertex in an unrooted tree is the length of the longest simple path beginning at this vertex. A vertex is called a **center** if no vertex in the tree has smaller eccentricity than this vertex. In Exercises 39–41 find every vertex that is a center in the given tree.

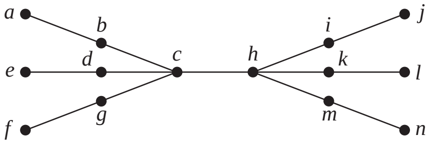
39.



40.



41.



42. Show that a center should be chosen as the root to produce a rooted tree of minimal height from an unrooted tree.

***43.** Show that a tree has either one center or two centers that are adjacent.

44. Show that every tree can be colored using two colors.

The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots,$ the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

45. Draw the first seven rooted Fibonacci trees.

***46.** How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

47. What is wrong with the following "proof" using mathematical induction of the statement that every tree with n vertices has a path of length $n - 1$. *Basis step:* Every tree with one vertex clearly has a path of length 0. *Inductive step:* Assume that a tree with n vertices has a path of length $n - 1$, which has u as its terminal vertex. Add a vertex v and the edge from u to v. The resulting tree has $n + 1$ vertices and has a path of length n. This completes the inductive step.

***48.** Show that the average depth of a leaf in a binary tree with n vertices is $\Omega(\log n)$.

11.2 Applications of Trees

Introduction

We will discuss three problems that can be studied using trees. The first problem is: How should items in a list be stored so that an item can be easily located? The second problem is: What series of decisions should be made to find an object with a certain property in a collection of objects of a certain type? The third problem is: How should a set of characters be efficiently coded by bit strings?

Binary Search Trees

Links

Searching for items in a list is one of the most important tasks that arises in computer science. Our primary goal is to implement a searching algorithm that finds items efficiently when the items are totally ordered. This can be accomplished through the use of a **binary search tree**, which is a binary tree in which each child of a vertex is designated as a right or left child, no vertex has more than one right child or left child, and each vertex is labeled with a key, which is one of the items. Furthermore, vertices are assigned keys so that the key of a vertex is both larger than the keys of all vertices in its left subtree and smaller than the keys of all vertices in its right subtree.

This recursive procedure is used to form the binary search tree for a list of items. Start with a tree containing just one vertex, namely, the root. The first item in the list is assigned as the key of the root. To add a new item, first compare it with the keys of vertices already in the tree, starting at the root and moving to the left if the item is less than the key of the respective vertex if this vertex has a left child, or moving to the right if the item is greater than the key of the respective vertex if this vertex has a right child. When the item is less than the respective vertex and this vertex has no left child, then a new vertex with this item as its key is inserted as a new left child. Similarly, when the item is greater than the respective vertex and this vertex has no right child, then a new vertex with this item as its key is inserted as a new right child. We illustrate this procedure with Example 1.

EXAMPLE 1 Form a binary search tree for the words *mathematics*, *physics*, *geography*, *zoology*, *meteorology*, *geology*, *psychology*, and *chemistry* (using alphabetical order).

Solution: Figure 1 displays the steps used to construct this binary search tree. The word *mathematics* is the key of the root. Because *physics* comes after *mathematics* (in alphabetical order), add a right child of the root with key *physics*. Because *geography* comes before *mathematics*, add a left child of the root with key *geography*. Next, add a right child of the vertex with key *physics*, and assign it the key *zoology*, because *zoology* comes after *mathematics* and after *physics*. Similarly, add a left child of the vertex with key *physics* and assign this new vertex the key *meteorology*. Add a right child of the vertex with key *geography* and assign this new vertex the key *geology*. Add a left child of the vertex with key *zoology* and assign it the key *psychology*. Add a left child of the vertex with key *geography* and assign it the key *chemistry*. (The reader should work through all the comparisons needed at each step.) ◀

Once we have a binary search tree, we need a way to locate items in the binary search tree, as well as a way to add new items. Algorithm 1, an insertion algorithm, actually does both of these tasks, even though it may appear that it is only designed to add vertices to a binary search tree. That is, Algorithm 1 is a procedure that locates an item x in a binary search tree if it is present, and adds a new vertex with x as its key if x is not present. In the pseudocode, v is the vertex currently under examination and *label*(v) represents the key of this vertex. The algorithm begins by examining the root. If x equals the key of v, then the algorithm has found the location of x and terminates; if x is less than the key of v, we move to the left child of v and repeat the procedure; and if x is greater than the key of v, we move to the right child of v and repeat the procedure. If at any step we attempt to move to a child that is not present, we know that x is not present in the tree, and we add a new vertex as this child with x as its key.

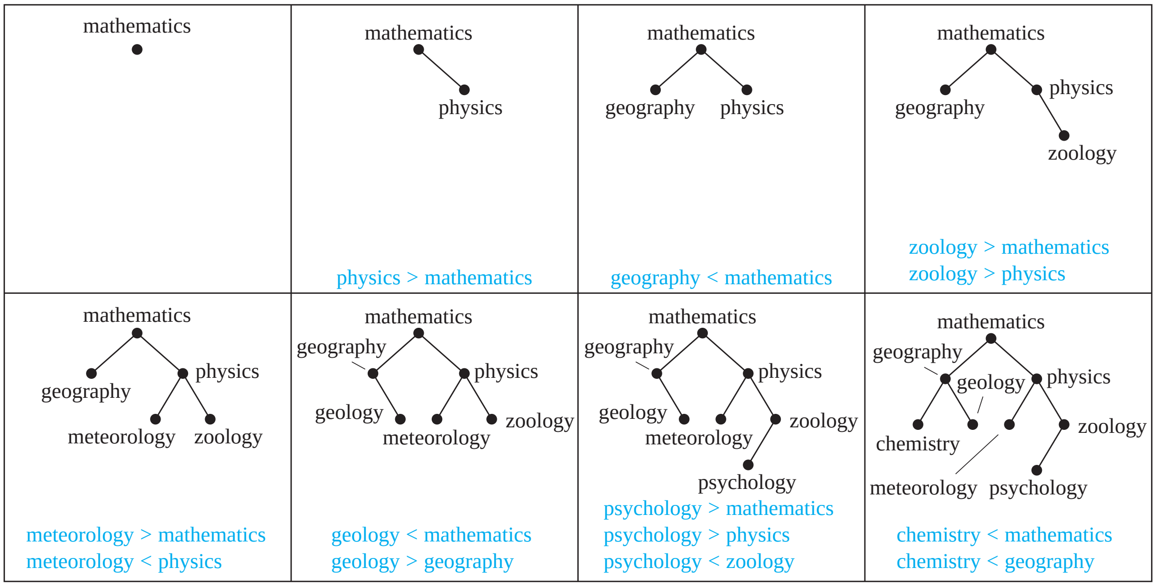


FIGURE 1 **Constructing a Binary Search Tree.**

ALGORITHM 1 Locating an Item in or Adding an Item to a Binary Search Tree.

```
procedure insertion($T$: binary search tree, $x$: item)
$v$ := root of $T$
{a vertex not present in $T$ has the value $null$}
while $v \neq null$ and $label(v) \neq x$
  if $x < label(v)$ then
    if left child of $v \neq null$ then $v$ := left child of $v$
    else add new vertex as a left child of $v$ and set $v := null$
  else
    if right child of $v \neq null$ then $v$ := right child of $v$
    else add new vertex as a right child of $v$ and set $v := null$
if root of $T = null$ then add a vertex $v$ to the tree and label it with $x$
else if $v$ is null or $label(v) \neq x$ then label new vertex with $x$ and let $v$ be this new vertex
return $v$ {$v$ = location of $x$}
```

Example 2 illustrates the use of Algorithm 1 to insert a new item into a binary search tree.

EXAMPLE 2 Use Algorithm 1 to insert the word *oceanography* into the binary search tree in Example 1.

Solution: Algorithm 1 begins with v, the vertex under examination, equal to the root of T, so $label(v) = mathematics$. Because $v \neq null$ and $label(v) = mathematics < oceanography$, we next examine the right child of the root. This right child exists, so we set v, the vertex under examination, to be this right child. At this step we have $v \neq null$ and $label(v) = physics > oceanography$, so we examine the left child of v. This left child exists, so we set v, the vertex under examination, to this left child. At this step, we also have $v \neq null$ and $label(v) = metereology < oceanography$, so we try to examine the right child of v. However, this right child does not exist, so we add a new vertex as the right child of v (which at this point is the vertex with the key *metereology*) and we set $v := null$. We now exit the **while** loop because $v = null$. Because the root of T is not $null$ and $v = null$, we use the **else if** statement at the end of the algorithm to label our new vertex with the key *oceanography*. ◀

We will now determine the computational complexity of this procedure. Suppose we have a binary search tree T for a list of n items. We can form a full binary tree U from T by adding unlabeled vertices whenever necessary so that every vertex with a key has two children. This is illustrated in Figure 2. Once we have done this, we can easily locate or add a new item as a key without adding a vertex.

The most comparisons needed to add a new item is the length of the longest path in U from the root to a leaf. The internal vertices of U are the vertices of T. It follows that U has n internal vertices. We can now use part (ii) of Theorem 4 in Section 11.1 to conclude that U has $n+1$ leaves. Using Corollary 1 of Section 11.1, we see that the height of U is greater than or equal to $h = \lceil \log(n+1) \rceil$. Consequently, it is necessary to perform at least $\lceil \log(n+1) \rceil$ comparisons to add some item. Note that if U is balanced, its height is $\lceil \log(n+1) \rceil$ (by Corollary 1 of Section 11.1). Thus, if a binary search tree is balanced, locating or adding an item requires no more than $\lceil \log(n+1) \rceil$ comparisons. A binary search tree can become unbalanced as items are added to it. Because balanced binary search trees give optimal worst-case complexity for binary searching, algorithms have been devised that rebalance binary search trees as items are added. The interested reader can consult references on data structures for the description of such algorithms.

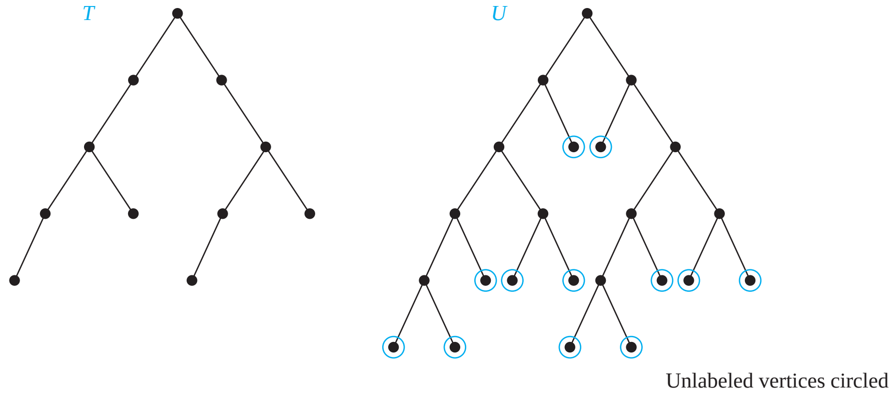


FIGURE 2 **Adding Unlabeled Vertices to Make a Binary Search Tree Full.**

Decision Trees

Links

Rooted trees can be used to model problems in which a series of decisions leads to a solution. For instance, a binary search tree can be used to locate items based on a series of comparisons, where each comparison tells us whether we have located the item, or whether we should go right or left in a subtree. A rooted tree in which each internal vertex corresponds to a decision, with a subtree at these vertices for each possible outcome of the decision, is called a **decision tree**. The possible solutions of the problem correspond to the paths to the leaves of this rooted tree. Example 3 illustrates an application of decision trees.

EXAMPLE 3 Suppose there are seven coins, all with the same weight, and a counterfeit coin that weighs less than the others. How many weighings are necessary using a balance scale to determine which of the eight coins is the counterfeit one? Give an algorithm for finding this counterfeit coin.

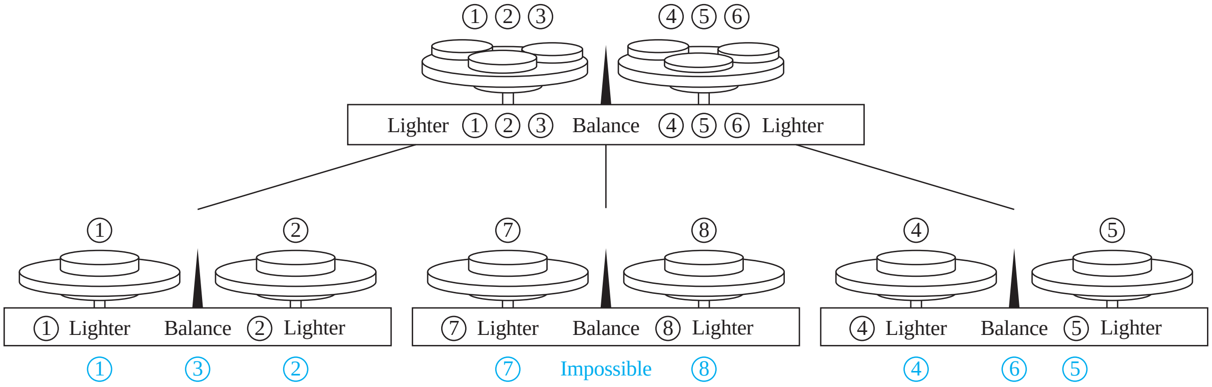
Extra Examples

Solution: There are three possibilities for each weighing on a balance scale. The two pans can have equal weight, the first pan can be heavier, or the second pan can be heavier. Consequently, the decision tree for the sequence of weighings is a 3-ary tree. There are at least eight leaves in the decision tree because there are eight possible outcomes (because each of the eight coins can be the counterfeit lighter coin), and each possible outcome must be represented by at least one leaf. The largest number of weighings needed to determine the counterfeit coin is the height of the decision tree. From Corollary 1 of Section 11.1 it follows that the height of the decision tree is at least $\lceil \log_3 8 \rceil = 2$. Hence, at least two weighings are needed.

It is possible to determine the counterfeit coin using two weighings. The decision tree that illustrates how this is done is shown in Figure 3. ◀

THE COMPLEXITY OF COMPARISON-BASED SORTING ALGORITHMS Many different sorting algorithms have been developed. To decide whether a particular sorting algorithm is efficient, its complexity is determined. Using decision trees as models, a lower bound for the worst-case complexity of sorting algorithms that are based on binary comparisons can be found.

We can use decision trees to model sorting algorithms and to determine an estimate for the worst-case complexity of these algorithms. Note that given n elements, there are $n!$ possible orderings of these elements, because each of the $n!$ permutations of these elements can be the correct order. The sorting algorithms studied in this book, and most commonly used sorting algorithms, are based on binary comparisons, that is, the comparison of two elements at a time. The result of each such comparison narrows down the set of possible orderings. Thus, a sorting algorithm based on binary comparisons can be represented by a binary decision tree in which each internal vertex represents a comparison of two elements. Each leaf represents one of the $n!$ permutations of n elements.


① ② ③
④ ⑤ ⑥
Lighter ① ② ③ Balance ④ ⑤ ⑥ Lighter
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④
⑤
① Lighter Balance ② Lighter
⑦ Lighter Balance ⑧ Lighter
④ Lighter Balance ⑤ Lighter
① ③ ②
⑦ Impossible ⑧
④ ⑥ ⑤

FIGURE 3 A Decision Tree for Locating a Counterfeit Coin. The counterfeit coin is shown in color below each final weighing.

EXAMPLE 4 We display in Figure 4 a decision tree that orders the elements of the list a, b, c.

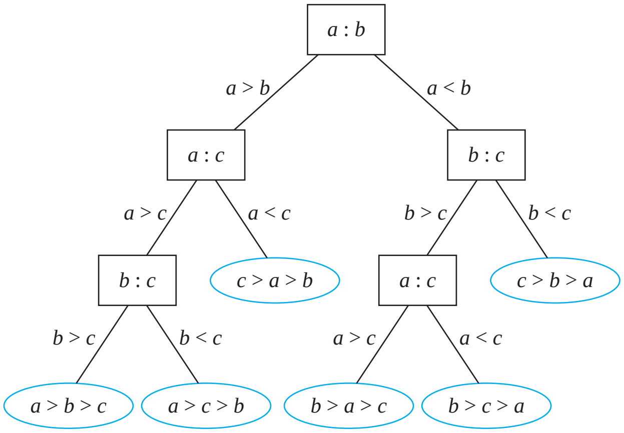


FIGURE 4 A Decision Tree for Sorting Three Distinct Elements.

The complexity of a sort based on binary comparisons is measured in terms of the number of such comparisons used. The largest number of binary comparisons ever needed to sort a list with n elements gives the worst-case performance of the algorithm. The most comparisons used equals the longest path length in the decision tree representing the sorting procedure. In other words, the largest number of comparisons ever needed is equal to the height of the decision tree. Because the height of a binary tree with $n!$ leaves is at least $\lceil \log n! \rceil$ (using Corollary 1 in Section 11.1), at least $\lceil \log n! \rceil$ comparisons are needed, as stated in Theorem 1.

THEOREM 1 A sorting algorithm based on binary comparisons requires at least $\lceil \log n! \rceil$ comparisons.

We can use Theorem 1 to provide a big-Omega estimate for the number of comparisons used by a sorting algorithm based on binary comparison. We need only note that by Exercise 72 in Section 3.2 we know that $\lceil \log n! \rceil$ is $\Theta(n \log n)$, one of the commonly used reference functions for the computational complexity of algorithms. Corollary 1 is a consequence of this estimate.

COROLLARY 1 The number of comparisons used by a sorting algorithm to sort n elements based on binary comparisons is $\Omega(n \log n)$.

A consequence of Corollary 1 is that a sorting algorithm based on binary comparisons that uses $\Theta(n \log n)$ comparisons, in the worst case, to sort n elements is optimal, in the sense that no other such algorithm has better worst-case complexity. Note that by Theorem 1 in Section 5.4 we see that the merge sort algorithm is optimal in this sense.

We can also establish a similar result for the average-case complexity of sorting algorithms. The average number of comparisons used by a sorting algorithm based on binary comparisons is the average depth of a leaf in the decision tree representing the sorting algorithm. By Exercise 48 in Section 11.1 we know that the average depth of a leaf in a binary tree with N vertices is $\Omega(\log N)$. We obtain the following estimate when we let $N = n!$ and note that a function that is $\Omega(\log n!)$ is also $\Omega(n \log n)$ because $\log n!$ is $\Theta(n \log n)$.

THEOREM 2 The average number of comparisons used by a sorting algorithm to sort n elements based on binary comparisons is $\Omega(n \log n)$.

Prefix Codes

Consider the problem of using bit strings to encode the letters of the English alphabet (where no distinction is made between lowercase and uppercase letters). We can represent each letter with a bit string of length five, because there are only 26 letters and there are 32 bit strings of length five. The total number of bits used to encode data is five times the number of characters in the text when each character is encoded with five bits. Is it possible to find a coding scheme of these letters such that, when data are coded, fewer bits are used? We can save memory and reduce transmittal time if this can be done.

Consider using bit strings of different lengths to encode letters. Letters that occur more frequently should be encoded using short bit strings, and longer bit strings should be used to encode rarely occurring letters. When letters are encoded using varying numbers of bits, some method must be used to determine where the bits for each character start and end. For instance, if e were encoded with 0, a with 1, and t with 01, then the bit string 0101 could correspond to *eat*, *tea*, *eaea*, or *tt*.

One way to ensure that no bit string corresponds to more than one sequence of letters is to encode letters so that the bit string for a letter never occurs as the first part of the bit string for another letter. Codes with this property are called **prefix codes**. For instance, the encoding of e as 0, a as 10, and t as 11 is a prefix code. A word can be recovered from the unique bit string that encodes its letters. For example, the string 10110 is the encoding of *ate*. To see this, note that the initial 1 does not represent a character, but 10 does represent a (and could not be the first part of the bit string of another letter). Then, the next 1 does not represent a character, but 11 does represent t. The final bit, 0, represents e.

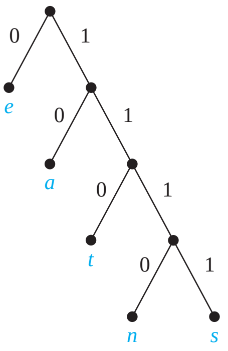


FIGURE 5 **A Binary Tree with a Prefix Code.**

A prefix code can be represented using a binary tree, where the characters are the labels of the leaves in the tree. The edges of the tree are labeled so that an edge leading to a left child is assigned a 0 and an edge leading to a right child is assigned a 1. The bit string used to encode a character is the sequence of labels of the edges in the unique path from the root to the leaf that has this character as its label. For instance, the tree in Figure 5 represents the encoding of e by 0, a by 10, t by 110, n by 1110, and s by 1111.

The tree representing a code can be used to decode a bit string. For instance, consider the word encoded by 11111011100 using the code in Figure 5. This bit string can be decoded by starting at the root, using the sequence of bits to form a path that stops when a leaf is reached.

Each 0 bit takes the path down the edge leading to the left child of the last vertex in the path, and each 1 bit corresponds to the right child of this vertex. Consequently, the initial 1111 corresponds to the path starting at the root, going right four times, leading to a leaf in the graph that has s as its label, because the string 1111 is the code for s. Continuing with the fifth bit, we reach a leaf next after going right then left, when the vertex labeled with a, which is encoded by 10, is visited. Starting with the seventh bit, we reach a leaf next after going right three times and then left, when the vertex labeled with n, which is encoded by 1110, is visited. Finally, the last bit, 0, leads to the leaf that is labeled with e. Therefore, the original word is *sane*.

We can construct a prefix code from any binary tree where the left edge at each internal vertex is labeled by 0 and the right edge by a 1 and where the leaves are labeled by characters. Characters are encoded with the bit string constructed using the labels of the edges in the unique path from the root to the leaves.

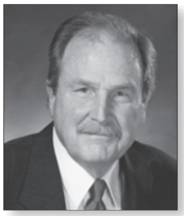
Links

HUFFMAN CODING We now introduce an algorithm that takes as input the frequencies (which are the probabilities of occurrences) of symbols in a string and produces as output a prefix code that encodes the string using the fewest possible bits, among all possible binary prefix codes for these symbols. This algorithm, known as **Huffman coding**, was developed by David Huffman in a term paper he wrote in 1951 while a graduate student at MIT. (Note that this algorithm assumes that we already know how many times each symbol occurs in the string, so we can compute the frequency of each symbol by dividing the number of times this symbol occurs by the length of the string.) Huffman coding is a fundamental algorithm in *data compression*, the subject devoted to reducing the number of bits required to represent information. Huffman coding is extensively used to compress bit strings representing text and it also plays an important role in compressing audio and image files.

Algorithm 2 presents the Huffman coding algorithm. Given symbols and their frequencies, our goal is to construct a rooted binary tree where the symbols are the labels of the leaves. The algorithm begins with a forest of trees each consisting of one vertex, where each vertex has a symbol as its label and where the weight of this vertex equals the frequency of the symbol that is its label. At each step, we combine two trees having the least total weight into a single tree by introducing a new root and placing the tree with larger weight as its left subtree and the tree with smaller weight as its right subtree. Furthermore, we assign the sum of the weights of the two subtrees of this tree as the total weight of the tree. (Although procedures for breaking ties by choosing between trees with equal weights can be specified, we will not specify such procedures here.) The algorithm is finished when it has constructed a tree, that is, when the forest is reduced to a single tree.

Demo

Links



DAVID A. HUFFMAN (1925–1999) David Huffman grew up in Ohio. At the age of 18 he received his B.S. in electrical engineering from The Ohio State University. Afterward he served in the U.S. Navy as a radar maintenance officer on a destroyer that had the mission of clearing mines in Asian waters after World War II. Later, he earned his M.S. from Ohio State and his Ph.D. in electrical engineering from MIT. Huffman joined the MIT faculty in 1953, where he remained until 1967 when he became the founding member of the computer science department at the University of California at Santa Cruz. He played an important role in developing this department and spent the remainder of his career there, retiring in 1994.

Huffman is noted for his contributions to information theory and coding, signal designs for radar and for communications, and design procedures for asynchronous logical circuits. His work on surfaces with zero curvature led him to develop original techniques for folding paper and vinyl into unusual shapes considered works of art by many and publicly displayed in several exhibits. However, Huffman is best known for his development of what is now called Huffman coding, a result of a term paper he wrote during his graduate work at MIT.

Huffman enjoyed exploring the outdoors, hiking, and traveling extensively. He became certified as a scuba diver when he was in his late 60s. He kept poisonous snakes as pets.

ALGORITHM 2 **Huffman Coding.**

```
**procedure** *Huffman*($C$: symbols $a_i$ with frequencies $w_i$, $i = 1, \dots, n$)
$F :=$ forest of $n$ rooted trees, each consisting of the single vertex $a_i$ and assigned weight $w_i$
**while** $F$ is not a tree
    Replace the rooted trees $T$ and $T'$ of least weights from $F$ with $w(T) \geq w(T')$ with a tree
    having a new root that has $T$ as its left subtree and $T'$ as its right subtree. Label the new
    edge to $T$ with 0 and the new edge to $T'$ with 1.
    Assign $w(T) + w(T')$ as the weight of the new tree.
{the Huffman coding for the symbol $a_i$ is the concatenation of the labels of the edges in the
unique path from the root to the vertex $a_i$}
```

Example 5 illustrates how Algorithm 2 is used to encode a set of five symbols.

EXAMPLE 5

Extra Examples

Use Huffman coding to encode the following symbols with the frequencies listed: A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35. What is the average number of bits used to encode a character?

Solution: Figure 6 displays the steps used to encode these symbols. The encoding produced encodes A by 111, B by 110, C by 011, D by 010, E by 10, and F by 00. The average number of bits used to encode a symbol using this encoding is

$$3 \cdot 0.08 + 3 \cdot 0.10 + 3 \cdot 0.12 + 3 \cdot 0.15 + 2 \cdot 0.20 + 2 \cdot 0.35 = 2.45.$$ ◀

Note that Huffman coding is a greedy algorithm. Replacing the two subtrees with the smallest weight at each step leads to an optimal code in the sense that no binary prefix code for these symbols can encode these symbols using fewer bits. We leave the proof that Huffman codes are optimal as Exercise 32.

Huffman coding is used in JPEG image coding

There are many variations of Huffman coding. For example, instead of encoding single symbols, we can encode blocks of symbols of a specified length, such as blocks of two symbols. Doing so may reduce the number of bits required to encode the string (see Exercise 30). We can also use more than two symbols to encode the original symbols in the string (see the preamble to Exercise 28). Furthermore, a variation known as adaptive Huffman coding (see [Sa00]) can be used when the frequency of each symbol in a string is not known in advance, so that encoding is done at the same time the string is being read.

Game Trees

Links

Trees can be used to analyze certain types of games such as tic-tac-toe, nim, checkers, and chess. In each of these games, two players take turns making moves. Each player knows the moves made by the other player and no element of chance enters into the game. We model such games using **game trees**; the vertices of these trees represent the positions that a game can be in as it progresses; the edges represent legal moves between these positions. Because game trees are usually large, we simplify game trees by representing all symmetric positions of a game by the same vertex. However, the same position of a game may be represented by different vertices

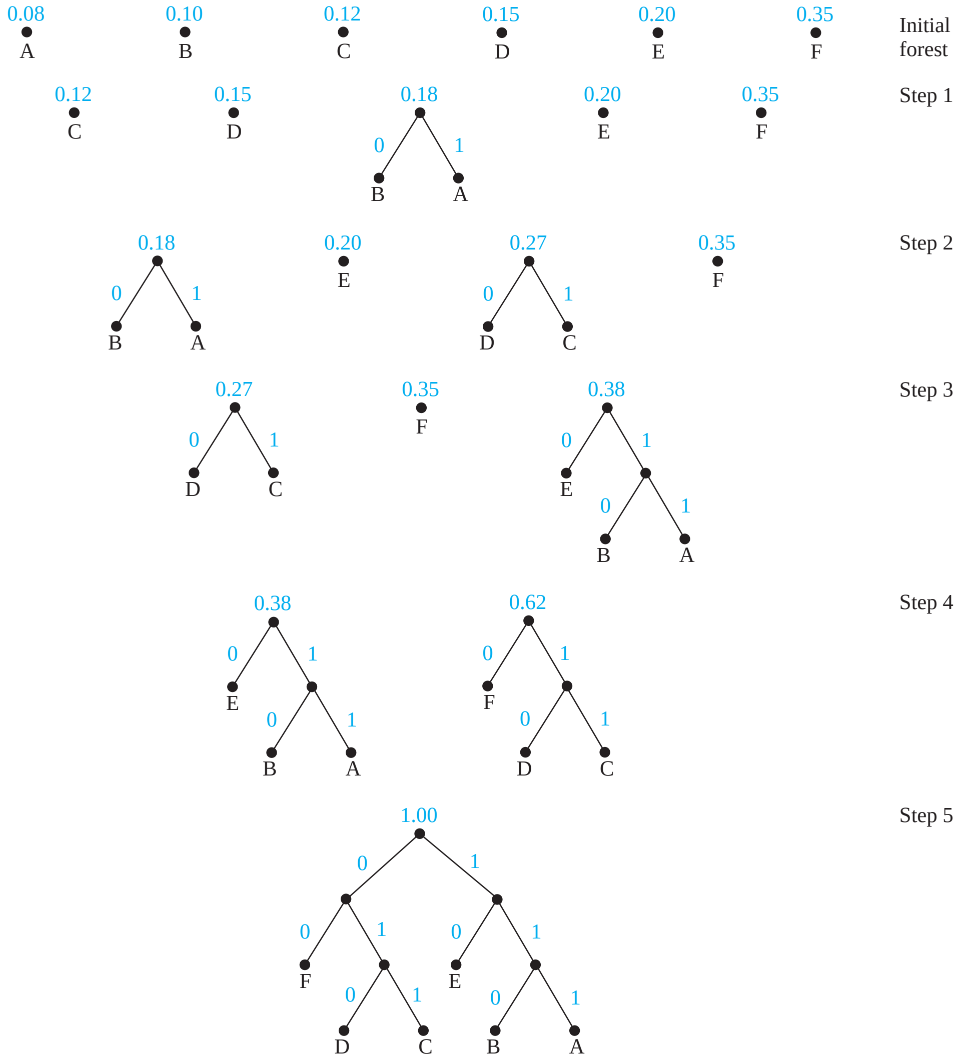


FIGURE 6 **Huffman Coding of Symbols in Example 4.**

if different sequences of moves lead to this position. The root represents the starting position. The usual convention is to represent vertices at even levels by boxes and vertices at odd levels by circles. When the game is in a position represented by a vertex at an even level, it is the first player's move; when the game is in a position represented by a vertex at an odd level, it is the second player's move. Game trees may be infinite when the games they represent never end, such as games that can enter infinite loops, but for most games there are rules that lead to finite game trees.

The leaves of a game tree represent the final positions of a game. We assign a value to each leaf indicating the payoff to the first player if the game terminates in the position represented by this leaf. For games that are win–lose, we label a terminal vertex represented by a circle with a 1 to indicate a win by the first player and we label a terminal vertex represented by a box with a -1 to indicate a win by the second player. For games where draws are allowed, we label a terminal vertex corresponding to a draw position with a 0. Note that for win–lose games, we have assigned values to terminal vertices so that the larger the value, the better the outcome for the first player.

In Example 6 we display a game tree for a well-known and well-studied game.

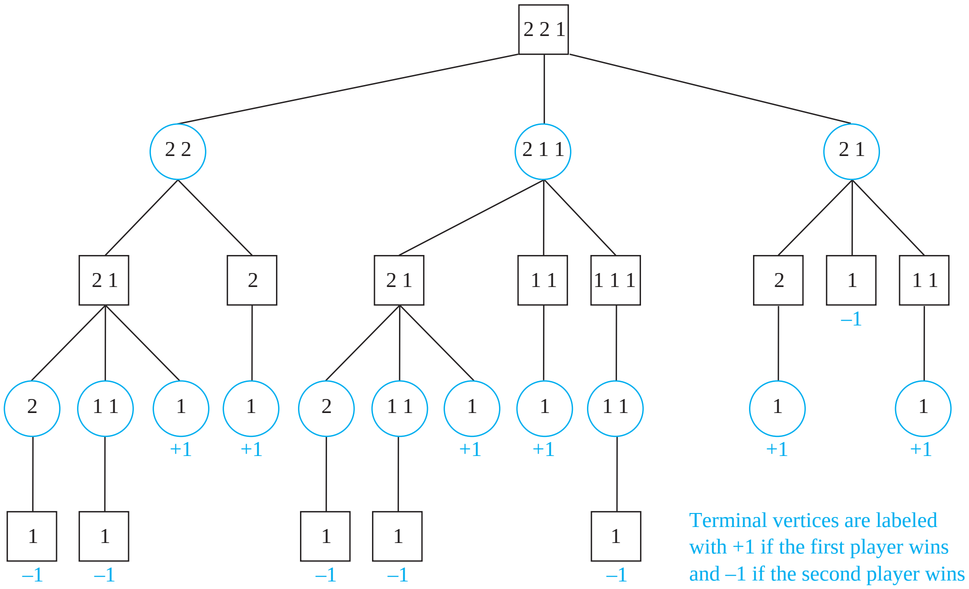

2 2 1
2 2
2 1 1
2 1
2 1
2
2 1
1 1
1 1 1
2
1
−1
1 1
2
1 1
1
+1
1
+1
2
1 1
1
+1
1
+1
1 1
1
+1
1
+1
1
−1
1
−1
1
−1
1
−1
1
−1
Terminal vertices are labeled with +1 if the first player wins and –1 if the second player wins

FIGURE 7 **The Game Tree for a Game of Nim.**

EXAMPLE 6

Although nim is an ancient game, Charles Bouton coined its modern name in 1901 after an archaic English word meaning “to steal.”

Nim In a version of the game of **nim**, at the start of a game there are a number of piles of stones. Two players take turns making moves; a legal move consists of removing one or more stones from one of the piles, without removing all the stones left. A player without a legal move loses. (Another way to look at this is that the player removing the last stone loses because the position with no piles of stones is not allowed.) The game tree shown in Figure 7 represents this version of nim given the starting position where there are three piles of stones containing two, two, and one stone each, respectively. We represent each position with an unordered list of the number of stones in the different piles (the order of the piles does not matter). The initial move by the first player can lead to three possible positions because this player can remove one stone from a pile with two stones (leaving three piles containing one, one, and two stones); two stones from a pile containing two stones (leaving two piles containing two stones and one stone); or one stone from the pile containing one stone (leaving two piles of two stones). When only one pile with one stone is left, no legal moves are possible, so such positions are terminal positions. Because nim is a win–lose game, we label the terminal vertices with $+1$ when they represent wins for the first player and -1 when they represent wins for the second player. ◀

EXAMPLE 7

Tic-tac-toe The game tree for tic-tac-toe is extremely large and cannot be drawn here, although a computer could easily build such a tree. We show a portion of the game tic-tac-toe in Figure 8(a). Note that by considering symmetric positions equivalent, we need only consider three possible initial moves, as shown in Figure 8(a). We also show a subtree of this game tree leading to terminal positions in Figure 8(b), where a player who can win makes a winning move. ◀

We can recursively define the values of all vertices in a game tree in a way that enables us to determine the outcome of this game when both players follow optimal strategies. By a **strategy** we mean a set of rules that tells a player how to select moves to win the game. An optimal strategy for the first player is a strategy that maximizes the payoff to this player and for the second player is a strategy that minimizes this payoff. We now recursively define the value of a vertex.

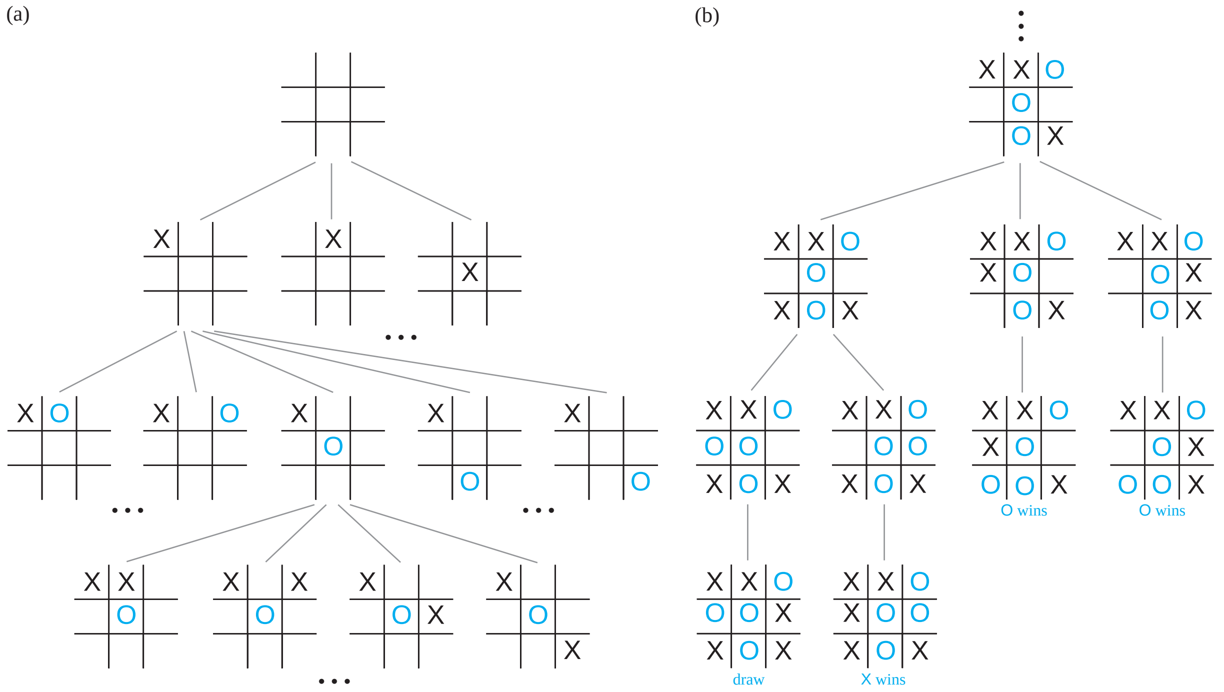


FIGURE 8 **Some of the Game Tree for Tic-Tac-Toe.**

DEFINITION 1

The *value of a vertex in a game tree* is defined recursively as:

(*i*) the value of a leaf is the payoff to the first player when the game terminates in the position represented by this leaf.

(*ii*) the value of an internal vertex at an even level is the maximum of the values of its children, and the value of an internal vertex at an odd level is the minimum of the values of its children.

The strategy where the first player moves to a position represented by a child with maximum value and the second player moves to a position of a child with minimum value is called the **minmax strategy**. We can determine who will win the game when both players follow the minmax strategy by calculating the value of the root of the tree; this value is called the **value** of the tree. This is a consequence of Theorem 3.

THEOREM 3

The value of a vertex of a game tree tells us the payoff to the first player if both players follow the minmax strategy and play starts from the position represented by this vertex.

Proof: We will use induction to prove this theorem.

BASIS STEP: If the vertex is a leaf, by definition the value assigned to this vertex is the payoff to the first player.

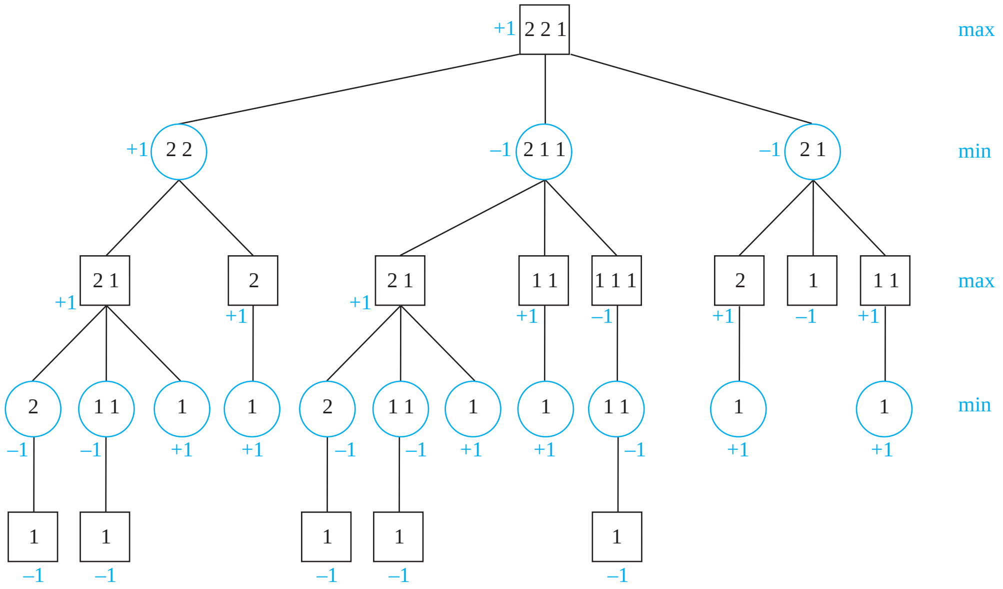


FIGURE 9 **Showing the Values of Vertices in the Game of Nim.**

INDUCTIVE STEP: The inductive hypothesis is the assumption that the values of the children of a vertex are the payoffs to the first player, assuming that play starts at each of the positions represented by these vertices. We need to consider two cases, when it is the first player's turn and when it is the second player's turn.

When it is the first player's turn, this player follows the minmax strategy and moves to the position represented by the child with the largest value. By the inductive hypothesis, this value is the payoff to the first player when play starts at the position represented by this child and follows the minmax strategy. By the recursive step in the definition of the value of an internal vertex at an even level (as the maximum value of its children), the value of this vertex is the payoff when play begins at the position represented by this vertex.

When it is the second player's turn, this player follows the minmax strategy and moves to the position represented by the child with the least value. By the inductive hypothesis, this value is the payoff to the first player when play starts at the position represented by this child and both players follow the minmax strategy. By the recursive definition of the value of an internal vertex at an odd level as the minimum value of its children, the value of this vertex is the payoff when play begins at the position represented by this vertex. ◁

Remark: By extending the proof of Theorem 3, it can be shown that the minmax strategy is the optimal strategy for both players.

Example 8 illustrates how the minmax procedure works. It displays the values assigned to the internal vertices in the game tree from Example 6. Note that we can shorten the computation required by noting that for win–lose games, once a child of a square vertex with value $+1$ is found, the value of the square vertex is also $+1$ because $+1$ is the largest possible payoff. Similarly, once a child of a circle vertex with value -1 is found, this is the value of the circle vertex also.

EXAMPLE 8 In Example 6 we constructed the game tree for nim with a starting position where there are three piles containing two, two, and one stones. In Figure 9 we show the values of the vertices of this game tree. The values of the vertices are computed using the values of the leaves and working one level up at a time. In the right margin of this figure we indicate whether we use the maximum or minimum of the values of the children to find the value of an internal vertex at each level. For example, once we have found the values of the three children of the root, which are 1, -1, and -1, we find the value of the root by computing $\max(1, -1, -1) = 1$. Because the value of the root is 1, it follows that the first player wins when both players follow a minmax strategy. ◀

Chess programs on smartphones can now play at the grandmaster level.

Links

Game trees for some well-known games can be extraordinarily large, because these games have many different possible moves. For example, the game tree for chess has been estimated to have as many as 10^{100} vertices! It may be impossible to use Theorem 3 directly to study a game because of the size of the game tree. Therefore, various approaches have been devised to help determine good strategies and to determine the outcome of such games. One useful technique, called *alpha-beta pruning*, eliminates much computation by pruning portions of the game tree that cannot affect the values of ancestor vertices. (For information about alpha-beta pruning, consult [Gr90].) Another useful approach is to use *evaluation functions*, which estimate the value of internal vertices in the game tree when it is not feasible to compute these values exactly. For example, in the game of tic-tac-toe, as an evaluation function for a position, we may use the number of files (rows, columns, and diagonals) containing no Os (used to indicate moves of the second player) minus the number of files containing no Xs (used to indicate moves of the first player). This evaluation function provides some indication of which player has the advantage in the game. Once the values of an evaluation function are inserted, the value of the game can be computed following the rules used for the minmax strategy. Computer programs created to play chess, such as the famous Deep Blue program, are based on sophisticated evaluation functions. For more information about how computers play chess see [Le91].

Exercises

1. Build a binary search tree for the words *banana, peach, apple, pear, coconut, mango*, and *papaya* using alphabetical order.

2. Build a binary search tree for the words *oenology, phrenology, campanology, ornithology, ichthyology, limnology, alchemy*, and *astrology* using alphabetical order.

3. How many comparisons are needed to locate or to add each of these words in the search tree for Exercise 1, starting fresh each time?

a) *pear* **b)** *banana*
c) *kumquat* **d)** *orange*

4. How many comparisons are needed to locate or to add each of the words in the search tree for Exercise 2, starting fresh each time?

a) *palmistry* **b)** *etymology*
c) *paleontology* **d)** *glaciology*

5. Using alphabetical order, construct a binary search tree for the words in the sentence "*The quick brown fox jumps over the lazy dog.*"

6. How many weighings of a balance scale are needed to find a lighter counterfeit coin among four coins? Describe an algorithm to find the lighter coin using this number of weighings.

7. How many weighings of a balance scale are needed to find a counterfeit coin among four coins if the counterfeit coin may be either heavier or lighter than the others? Describe an algorithm to find the counterfeit coin using this number of weighings.

∗8. How many weighings of a balance scale are needed to find a counterfeit coin among eight coins if the counterfeit coin is either heavier or lighter than the others? Describe an algorithm to find the counterfeit coin using this number of weighings.

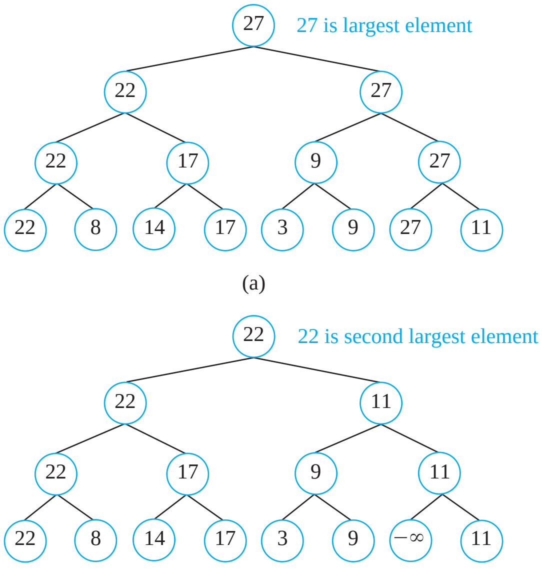
∗9. How many weighings of a balance scale are needed to find a counterfeit coin among 12 coins if the counterfeit coin is lighter than the others? Describe an algorithm to find the lighter coin using this number of weighings.

∗10. One of four coins may be counterfeit. If it is counterfeit, it may be lighter or heavier than the others. How many weighings are needed, using a balance scale, to determine whether there is a counterfeit coin, and if there is, whether it is lighter or heavier than the others? Describe an algorithm to find the counterfeit coin and determine whether it is lighter or heavier using this number of weighings.

11. Find the least number of comparisons needed to sort four elements and devise an algorithm that sorts these elements using this number of comparisons.

∗12. Find the least number of comparisons needed to sort five elements and devise an algorithm that sorts these elements using this number of comparisons.

The **tournament sort** is a sorting algorithm that works by building an ordered binary tree. We represent the elements to be sorted by vertices that will become the leaves. We build up the tree one level at a time as we would construct the tree representing the winners of matches in a tournament. Working left to right, we compare pairs of consecutive elements, adding a parent vertex labeled with the larger of the two elements under comparison. We make similar comparisons between labels of vertices at each level until we reach the root of the tree that is labeled with the largest element. The tree constructed by the tournament sort of 22, 8, 14, 17, 3, 9, 27, 11 is illustrated in part (a) of the figure. Once the largest element has been determined, the leaf with this label is relabeled by $-\infty$, which is defined to be less than every element. The labels of all vertices on the path from this vertex up to the root of the tree are recalculated, as shown in part (b) of the figure. This produces the second largest element. This process continues until the entire list has been sorted.



27 is largest element

(a)

22 is second largest element

(b)

13. Complete the tournament sort of the list 22, 8, 14, 17, 3, 9, 27, 11. Show the labels of the vertices at each step.

14. Use the tournament sort to sort the list 17, 4, 1, 5, 13, 10, 14, 6.

15. Describe the tournament sort using pseudocode.

16. Assuming that n, the number of elements to be sorted, equals 2^k for some positive integer k, determine the number of comparisons used by the tournament sort to find the largest element of the list using the tournament sort.

17. How many comparisons does the tournament sort use to find the second largest, the third largest, and so on, up to the $(n-1)$st largest (or second smallest) element?

18. Show that the tournament sort requires $\Theta(n \log n)$ comparisons to sort a list of n elements. [*Hint:* By inserting the appropriate number of dummy elements defined to be smaller than all integers, such as $-\infty$, assume that $n = 2^k$ for some positive integer k.]

19. Which of these codes are prefix codes?

a) a: 11, e: 00, t: 10, s: 01

b) a: 0, e: 1, t: 01, s: 001

c) a: 101, e: 11, t: 001, s: 011, n: 010

d) a: 010, e: 11, t: 011, s: 1011, n: 1001, i: 10101

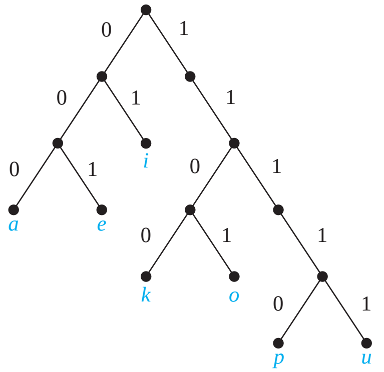
20. Construct the binary tree with prefix codes representing these coding schemes.

a) a: 11, e: 0, t: 101, s: 100

b) a: 1, e: 01, t: 001, s: 0001, n: 00001

c) a: 1010, e: 0, t: 11, s: 1011, n: 1001, i: 10001

21. What are the codes for a, e, i, k, o, p, and u if the coding scheme is represented by this tree?



22. Given the coding scheme a: 001, b: 0001, e: 1, r: 0000, s: 0100, t: 011, x: 01010, find the word represented by

a) 01110100011.

b) 0001110000.

c) 0100101010.

d) 01100101010.

23. Use Huffman coding to encode these symbols with given frequencies: a: 0.20, b: 0.10, c: 0.15, d: 0.25, e: 0.30. What is the average number of bits required to encode a character?

24. Use Huffman coding to encode these symbols with given frequencies: A: 0.10, B: 0.25, C: 0.05, D: 0.15, E: 0.30, F: 0.07, G: 0.08. What is the average number of bits required to encode a symbol?

25. Construct two different Huffman codes for these symbols and frequencies: t: 0.2, u: 0.3, v: 0.2, w: 0.3.

26. a) Use Huffman coding to encode these symbols with frequencies a: 0.4, b: 0.2, c: 0.2, d: 0.1, e: 0.1 in two different ways by breaking ties in the algorithm differently. First, among the trees of minimum weight select two trees with the largest number of vertices to combine at each stage of the algorithm. Second, among the trees of minimum weight select two trees with the smallest number of vertices at each stage.

b) Compute the average number of bits required to encode a symbol with each code and compute the variances of this number of bits for each code. Which tie-breaking procedure produced the smaller variance in the number of bits required to encode a symbol?

27. Construct a Huffman code for the letters of the English alphabet where the frequencies of letters in typical English text are as shown in this table.

Letter	*Frequency*	*Letter*	*Frequency*
A	0.0817	N	0.0662
B	0.0145	O	0.0781
C	0.0248	P	0.0156
D	0.0431	Q	0.0009
E	0.1232	R	0.0572
F	0.0209	S	0.0628
G	0.0182	T	0.0905
H	0.0668	U	0.0304
I	0.0689	V	0.0102
J	0.0010	W	0.0264
K	0.0080	X	0.0015
L	0.0397	Y	0.0211
M	0.0277	Z	0.0005

Suppose that m is a positive integer with $m \geq 2$. An m-ary Huffman code for a set of N symbols can be constructed analogously to the construction of a binary Huffman code. At the initial step, $((N - 1) \textbf{ mod } (m - 1)) + 1$ trees consisting of a single vertex with least weights are combined into a rooted tree with these vertices as leaves. At each subsequent step, the m trees of least weight are combined into an m-ary tree.

28. Describe the m-ary Huffman coding algorithm in pseudocode.

29. Using the symbols 0, 1, and 2 use ternary $(m = 3)$ Huffman coding to encode these letters with the given frequencies: A: 0.25, E: 0.30, N: 0.10, R: 0.05, T: 0.12, Z: 0.18.

30. Consider the three symbols A, B, and C with frequencies A: 0.80, B: 0.19, C: 0.01.

a) Construct a Huffman code for these three symbols.

b) Form a new set of nine symbols by grouping together blocks of two symbols, AA, AB, AC, BA, BB, BC, CA, CB, and CC. Construct a Huffman code for these nine symbols, assuming that the occurrences of symbols in the original text are independent.

c) Compare the average number of bits required to encode text using the Huffman code for the three symbols in part (a) and the Huffman code for the nine blocks of two symbols constructed in part (b). Which is more efficient?

31. Given $n + 1$ symbols $x_1, x_2, \ldots, x_n, x_{n+1}$ appearing $1, f_1, f_2, \ldots, f_n$ times in a symbol string, respectively, where f_j is the jth Fibonacci number, what is the maximum number of bits used to encode a symbol when all possible tie-breaking selections are considered at each stage of the Huffman coding algorithm?

***32.** Show that Huffman codes are optimal in the sense that they represent a string of symbols using the fewest bits among all binary prefix codes.

33. Draw a game tree for nim if the starting position consists of two piles with two and three stones, respectively. When drawing the tree represent by the same vertex symmetric positions that result from the same move. Find the value of each vertex of the game tree. Who wins the game if both players follow an optimal strategy?

34. Draw a game tree for nim if the starting position consists of three piles with one, two, and three stones, respectively. When drawing the tree represent by the same vertex symmetric positions that result from the same move. Find the value of each vertex of the game tree. Who wins the game if both players follow an optimal strategy?

35. Suppose that we vary the payoff to the winning player in the game of nim so that the payoff is n dollars when n is the number of legal moves made before a terminal position is reached. Find the payoff to the first player if the initial position consists of

a) two piles with one and three stones, respectively.

b) two piles with two and four stones, respectively.

c) three piles with one, two, and three stones, respectively.

36. Suppose that in a variation of the game of nim we allow a player to either remove one or more stones from a pile or merge the stones from two piles into one pile as long as at least one stone remains. Draw the game tree for this variation of nim if the starting position consists of three piles containing two, two, and one stone, respectively. Find the values of each vertex in the game tree and determine the winner if both players follow an optimal strategy.

37. Draw the subtree of the game tree for tic-tac-toe beginning at each of these positions. Determine the value of each of these subtrees.

a)

O	X	X
X	O	O
		X

b)

X	O	X
O	X	X
		O

c)

X		O
O	O	
X		X

d)

	O	X
	X	O
X	O	

38. Suppose that the first four moves of a tic-tac-toe game are as shown. Does the first player (whose moves are marked by Xs) have a strategy that will always win?

a)

X	O	X
	O	

b)

X	O	X
	O	

c)

	O	X
	X	
O		

d)

X		
X	O	
O		

39. Show that if a game of nim begins with two piles containing the same number of stones, as long as this number is at least two, then the second player wins when both players follow optimal strategies.

40. Show that if a game of nim begins with two piles containing different numbers of stones, the first player wins when both players follow optimal strategies.

41. How many children does the root of the game tree for checkers have? How many grandchildren does it have?

42. How many children does the root of the game tree for nim have and how many grandchildren does it have if the starting position is

a) piles with four and five stones, respectively.

b) piles with two, three, and four stones, respectively.

c) piles with one, two, three, and four stones, respectively.

d) piles with two, two, three, three, and five stones, respectively.

43. Draw the game tree for the game of tic-tac-toe for the levels corresponding to the first two moves. Assign the value of the evaluation function mentioned in the text that assigns to a position the number of files containing no Os minus the number of files containing no Xs as the value of each vertex at this level and compute the value of the tree for vertices as if the evaluation function gave the correct values for these vertices.

44. Use pseudocode to describe an algorithm for determining the value of a game tree when both players follow a minmax strategy.

11.3 Tree Traversal

Introduction

Links

Ordered rooted trees are often used to store information. We need procedures for visiting each vertex of an ordered rooted tree to access data. We will describe several important algorithms for visiting all the vertices of an ordered rooted tree. Ordered rooted trees can also be used to represent various types of expressions, such as arithmetic expressions involving numbers, variables, and operations. The different listings of the vertices of ordered rooted trees used to represent expressions are useful in the evaluation of these expressions.

Universal Address Systems

Procedures for traversing all vertices of an ordered rooted tree rely on the orderings of children. In ordered rooted trees, the children of an internal vertex are shown from left to right in the drawings representing these directed graphs.

We will describe one way we can totally order the vertices of an ordered rooted tree. To produce this ordering, we must first label all the vertices. We do this recursively:

1. Label the root with the integer 0. Then label its k children (at level 1) from left to right with $1, 2, 3, \ldots, k$.
2. For each vertex v at level n with label A, label its k_v children, as they are drawn from left to right, with $A.1, A.2, \ldots, A.k_v$.

Following this procedure, a vertex v at level n, for $n \geq 1$, is labeled $x_1.x_2.\ldots.x_n$, where the unique path from the root to v goes through the x_1st vertex at level 1, the x_2nd vertex at level 2, and so on. This labeling is called the **universal address system** of the ordered rooted tree.

We can totally order the vertices using the lexicographic ordering of their labels in the universal address system. The vertex labeled $x_1.x_2.\ldots.x_n$ is less than the vertex labeled $y_1.y_2.\ldots.y_m$ if there is an i, $0 \leq i \leq n$, with $x_1 = y_1$, $x_2 = y_2, \ldots, x_{i-1} = y_{i-1}$, and $x_i < y_i$; or if $n < m$ and $x_i = y_i$ for $i = 1, 2, \ldots, n$.

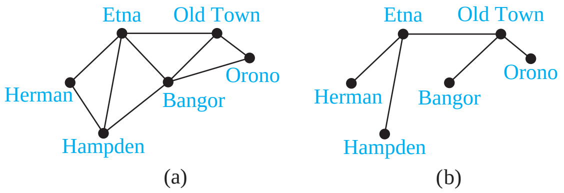


FIGURE 1 **(a) A Road System and (b) a Set of Roads to Plow.**

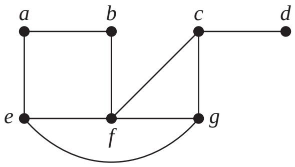


FIGURE 2 **The Simple Graph *G*.**

11.4 Spanning Trees

Introduction

Consider the system of roads in Maine represented by the simple graph shown in Figure 1(a). The only way the roads can be kept open in the winter is by frequently plowing them. The highway department wants to plow the fewest roads so that there will always be cleared roads connecting any two towns. How can this be done?

At least five roads must be plowed to ensure that there is a path between any two towns. Figure 1(b) shows one such set of roads. Note that the subgraph representing these roads is a tree, because it is connected and contains six vertices and five edges.

This problem was solved with a connected subgraph with the minimum number of edges containing all vertices of the original simple graph. Such a graph must be a tree.

DEFINITION 1 Let G be a simple graph. A *spanning tree* of G is a subgraph of G that is a tree containing every vertex of G.

A simple graph with a spanning tree must be connected, because there is a path in the spanning tree between any two vertices. The converse is also true; that is, every connected simple graph has a spanning tree. We will give an example before proving this result.

EXAMPLE 1 Find a spanning tree of the simple graph G shown in Figure 2.

Solution: The graph G is connected, but it is not a tree because it contains simple circuits. Remove the edge $\{a, e\}$. This eliminates one simple circuit, and the resulting subgraph is still connected and still contains every vertex of G. Next remove the edge $\{e, f\}$ to eliminate a second simple circuit. Finally, remove edge $\{c, g\}$ to produce a simple graph with no simple circuits. This subgraph is a spanning tree, because it is a tree that contains every vertex of G. The sequence of edge removals used to produce the spanning tree is illustrated in Figure 3.

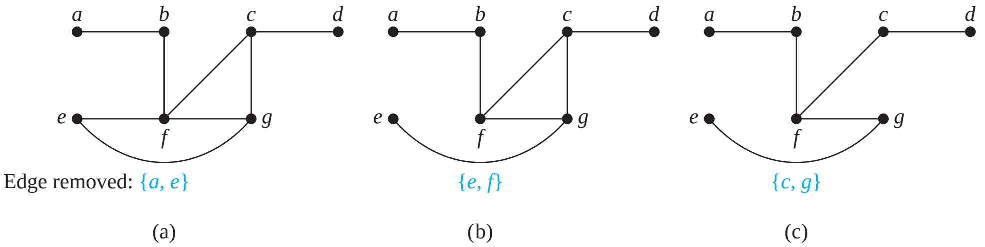


FIGURE 3 **Producing a Spanning Tree for *G* by Removing Edges That Form Simple Circuits.**

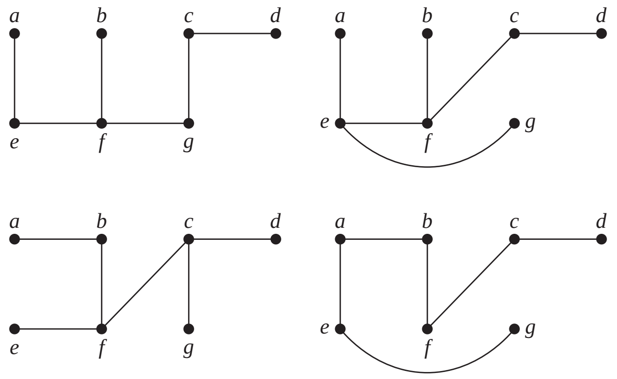


FIGURE 4 Spanning Trees of G.

The tree shown in Figure 3 is not the only spanning tree of G. For instance, each of the trees shown in Figure 4 is a spanning tree of G. ◀

THEOREM 1 A simple graph is connected if and only if it has a spanning tree.

Proof: First, suppose that a simple graph G has a spanning tree T. T contains every vertex of G. Furthermore, there is a path in T between any two of its vertices. Because T is a subgraph of G, there is a path in G between any two of its vertices. Hence, G is connected.

Now suppose that G is connected. If G is not a tree, it must contain a simple circuit. Remove an edge from one of these simple circuits. The resulting subgraph has one fewer edge but still contains all the vertices of G and is connected. This subgraph is still connected because when two vertices are connected by a path containing the removed edge, they are connected by a path not containing this edge. We can construct such a path by inserting into the original path, at the point where the removed edge once was, the simple circuit with this edge removed. If this subgraph is not a tree, it has a simple circuit; so as before, remove an edge that is in a simple circuit. Repeat this process until no simple circuits remain. This is possible because there are only a finite number of edges in the graph. The process terminates when no simple circuits remain. A tree is produced because the graph stays connected as edges are removed. This tree is a spanning tree because it contains every vertex of G. ◁

Spanning trees are important in data networking, as Example 2 shows.

EXAMPLE 2 Links

IP Multicasting Spanning trees play an important role in multicasting over Internet Protocol (IP) networks. To send data from a source computer to multiple receiving computers, each of which is a subnetwork, data could be sent separately to each computer. This type of networking, called unicasting, is inefficient, because many copies of the same data are transmitted over the network. To make the transmission of data to multiple receiving computers more efficient, IP multicasting is used. With IP multicasting, a computer sends a single copy of data over the network, and as data reaches intermediate routers, the data are forwarded to one or more other routers so that ultimately all receiving computers in their various subnetworks receive these data. (Routers are computers that are dedicated to forwarding IP datagrams between subnetworks in a network. In multicasting, routers use Class D addresses, each representing a session that receiving computers may join; see Example 17 in Section 6.1.)

For data to reach receiving computers as quickly as possible, there should be no loops (which in graph theory terminology are circuits or cycles) in the path that data take through the network. That is, once data have reached a particular router, data should never return to this

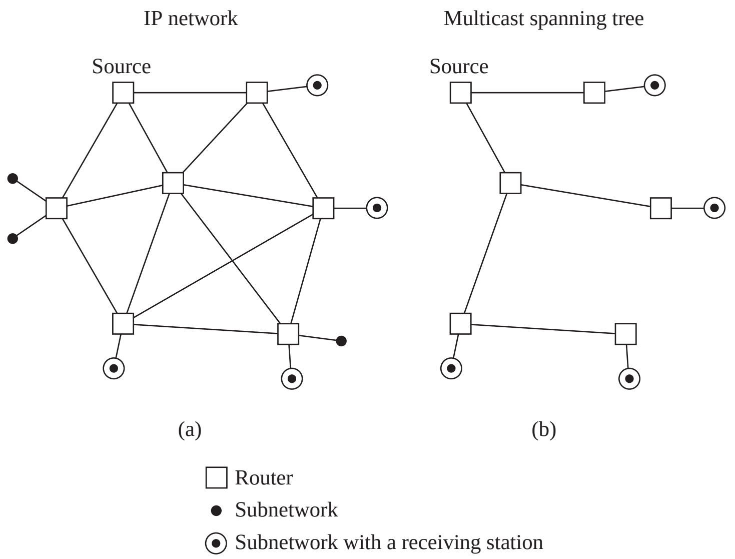


FIGURE 5 A Multicast Spanning Tree.

router. To avoid loops, the multicast routers use network algorithms to construct a spanning tree in the graph that has the multicast source, the routers, and the subnetworks containing receiving computers as vertices, with edges representing the links between computers and/or routers. The root of this spanning tree is the multicast source. The subnetworks containing receiving computers are leaves of the tree. (Note that subnetworks not containing receiving stations are not included in the graph.) This is illustrated in Figure 5. ◀

Depth-First Search

Links

The proof of Theorem 1 gives an algorithm for finding spanning trees by removing edges from simple circuits. This algorithm is inefficient, because it requires that simple circuits be identified. Instead of constructing spanning trees by removing edges, spanning trees can be built up by successively adding edges. Two algorithms based on this principle will be presented here.

Demo

We can build a spanning tree for a connected simple graph using **depth-first search**. We will form a rooted tree, and the spanning tree will be the underlying undirected graph of this rooted tree. Arbitrarily choose a vertex of the graph as the root. Form a path starting at this vertex by successively adding vertices and edges, where each new edge is incident with the last vertex in the path and a vertex not already in the path. Continue adding vertices and edges to this path as long as possible. If the path goes through all vertices of the graph, the tree consisting of this path is a spanning tree. However, if the path does not go through all vertices, more vertices and edges must be added. Move back to the next to last vertex in the path, and, if possible, form a new path starting at this vertex passing through vertices that were not already visited. If this cannot be done, move back another vertex in the path, that is, two vertices back in the path, and try again.

Repeat this procedure, beginning at the last vertex visited, moving back up the path one vertex at a time, forming new paths that are as long as possible until no more edges can be added. Because the graph has a finite number of edges and is connected, this process ends with the production of a spanning tree. Each vertex that ends a path at a stage of the algorithm will be a leaf in the rooted tree, and each vertex where a path is constructed starting at this vertex will be an internal vertex.

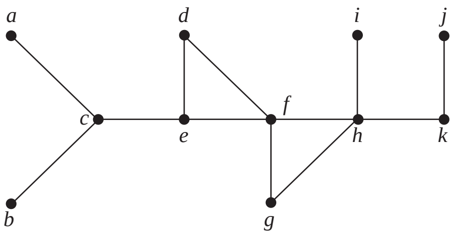
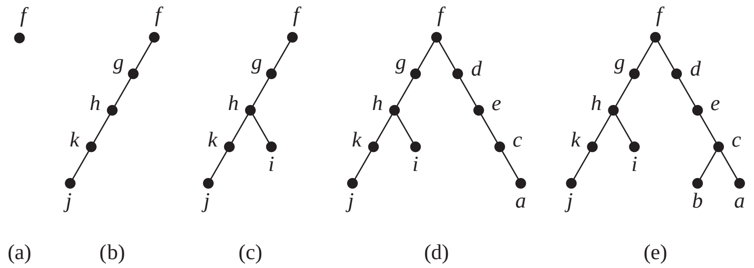


FIGURE 6 **The Graph *G*.**



(a) (b) (c) (d) (e)

FIGURE 7 **Depth-First Search of *G*.**

The reader should note the recursive nature of this procedure. Also, note that if the vertices in the graph are ordered, the choices of edges at each stage of the procedure are all determined when we always choose the first vertex in the ordering that is available. However, we will not always explicitly order the vertices of a graph.

Depth-first search is also called **backtracking**, because the algorithm returns to vertices previously visited to add paths. Example 3 illustrates backtracking.

EXAMPLE 3 Use depth-first search to find a spanning tree for the graph G shown in Figure 6.

Solution: The steps used by depth-first search to produce a spanning tree of G are shown in Figure 7. We arbitrarily start with the vertex f. A path is built by successively adding edges incident with vertices not already in the path, as long as this is possible. This produces a path f, g, h, k, j (note that other paths could have been built). Next, backtrack to k. There is no path beginning at k containing vertices not already visited. So we backtrack to h. Form the path h, i. Then backtrack to h, and then to f. From f build the path f, d, e, c, a. Then backtrack to c and form the path c, b. This produces the spanning tree. ◀

The edges selected by depth-first search of a graph are called **tree edges**. All other edges of the graph must connect a vertex to an ancestor or descendant of this vertex in the tree. These edges are called **back edges**. (Exercise 43 asks for a proof of this fact.)

EXAMPLE 4 In Figure 8 we highlight the tree edges found by depth-first search starting at vertex f by showing them with heavy colored lines. The back edges (e, f) and (f, h) are shown with thinner black lines. ◀

We have explained how to find a spanning tree of a graph using depth-first search. However, our discussion so far has not brought out the recursive nature of depth-first search. To help make the recursive nature of the algorithm clear, we need a little terminology. We say that we

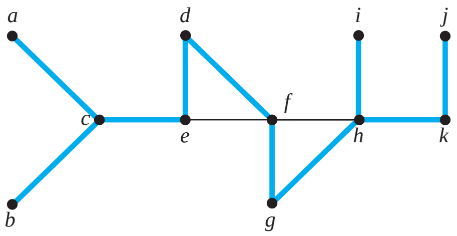


FIGURE 8 **The Tree Edges and Back Edges of the Depth-First Search in Example 4.**

explore from a vertex v when we carry out the steps of depth-first search beginning when v is added to the tree and ending when we have backtracked back to v for the last time. The key observation needed to understand the recursive nature of the algorithm is that when we add an edge connecting a vertex v to a vertex w, we finish exploring from w before we return to v to complete exploring from v.

In Algorithm 1 we construct the spanning tree of a graph G with vertices $v_1, \ldots, v_n$ by first selecting the vertex v_1 to be the root. We initially set T to be the tree with just this one vertex. At each step we add a new vertex to the tree T together with an edge from a vertex already in T to this new vertex and we explore from this new vertex. Note that at the completion of the algorithm, T contains no simple circuits because no edge is ever added that connects two vertices in the tree. Moreover, T remains connected as it is built. (These last two observations can be easily proved via mathematical induction.) Because G is connected, every vertex in G is visited by the algorithm and is added to the tree (as the reader should verify). It follows that T is a spanning tree of G.

ALGORITHM 1 Depth-First Search.

```
procedure DFS($G$: connected graph with vertices $v_1, v_2, \ldots, v_n$)
$T$ := tree consisting only of the vertex $v_1$
visit($v_1$)


procedure visit($v$: vertex of $G$)
for each vertex $w$ adjacent to $v$ and not yet in $T$
    add vertex $w$ and edge $\{v, w\}$ to $T$
    visit($w$)
```

We now analyze the computational complexity of the depth-first search algorithm. The key observation is that for each vertex v, the procedure *visit*(v) is called when the vertex v is first encountered in the search and it is not called again. Assuming that the adjacency lists for G are available (see Section 10.3), no computations are required to find the vertices adjacent to v. As we follow the steps of the algorithm, we examine each edge at most twice to determine whether to add this edge and one of its endpoints to the tree. Consequently, the procedure *DFS* constructs a spanning tree using $O(e)$, or $O(n^2)$, steps where e and n are the number of edges and vertices in G, respectively. [Note that a step involves examining a vertex to see whether it is already in the spanning tree as it is being built and adding this vertex and the corresponding edge if the vertex is not already in the tree. We have also made use of the inequality $e \leq n(n-1)/2$, which holds for any simple graph.]

Depth-first search can be used as the basis for algorithms that solve many different problems. For example, it can be used to find paths and circuits in a graph, it can be used to determine the connected components of a graph, and it can be used to find the cut vertices of a connected graph. As we will see, depth-first search is the basis of backtracking techniques used to search for solutions of computationally difficult problems. (See [GrYe05], [Ma89], and [CoLeRiSt09] for a discussion of algorithms based on depth-first search.)

Breadth-First Search

Demo

We can also produce a spanning tree of a simple graph by the use of **breadth-first search**. Again, a rooted tree will be constructed, and the underlying undirected graph of this rooted tree forms the spanning tree. Arbitrarily choose a root from the vertices of the graph. Then add all

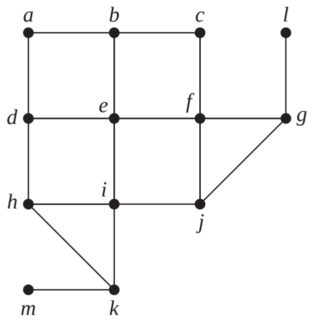


FIGURE 9 **A Graph G.**

Links

edges incident to this vertex. The new vertices added at this stage become the vertices at level 1 in the spanning tree. Arbitrarily order them. Next, for each vertex at level 1, visited in order, add each edge incident to this vertex to the tree as long as it does not produce a simple circuit. Arbitrarily order the children of each vertex at level 1. This produces the vertices at level 2 in the tree. Follow the same procedure until all the vertices in the tree have been added. The procedure ends because there are only a finite number of edges in the graph. A spanning tree is produced because we have produced a tree containing every vertex of the graph. An example of breadth-first search is given in Example 5.

EXAMPLE 5 Use breadth-first search to find a spanning tree for the graph shown in Figure 9.

Extra Examples

Solution: The steps of the breadth-first search procedure are shown in Figure 10. We choose the vertex e to be the root. Then we add edges incident with all vertices adjacent to e, so edges from e to b, d, f, and i are added. These four vertices are at level 1 in the tree. Next, add the edges from these vertices at level 1 to adjacent vertices not already in the tree. Hence, the edges from b to a and c are added, as are edges from d to h, from f to j and g, and from i to k. The new vertices a, c, h, j, g, and k are at level 2. Next, add edges from these vertices to adjacent vertices not already in the graph. This adds edges from g to l and from k to m. ◀

We describe breadth-first search in pseudocode as Algorithm 2. In this algorithm, we assume the vertices of the connected graph G are ordered as $v_1, v_2, \ldots, v_n$. In the algorithm we use the term "process" to describe the procedure of adding new vertices, and corresponding edges, to the tree adjacent to the current vertex being processed as long as a simple circuit is not produced.

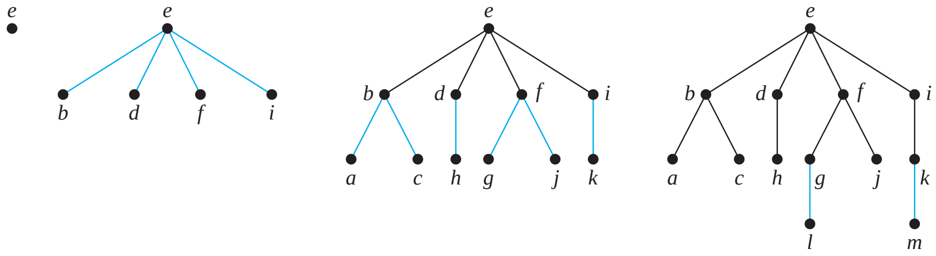


FIGURE 10 **Breadth-First Search of G.**

ALGORITHM 2 **Breadth-First Search.**

```
procedure BFS ($G$: connected graph with vertices $v_1, v_2, \ldots, v_n$)
$T$ := tree consisting only of vertex $v_1$
$L$ := empty list
put $v_1$ in the list $L$ of unprocessed vertices
while $L$ is not empty
    remove the first vertex, $v$, from $L$
    for each neighbor $w$ of $v$
        if $w$ is not in $L$ and not in $T$ then
            add $w$ to the end of the list $L$
            add $w$ and edge $\{v, w\}$ to $T$
```

We now analyze the computational complexity of breadth-first search. For each vertex v in the graph we examine all vertices adjacent to v and we add each vertex not yet visited to the tree T. Assuming we have the adjacency lists for the graph available, no computation is required to determine which vertices are adjacent to a given vertex. As in the analysis of the depth-first search algorithm, we see that we examine each edge at most twice to determine whether we should add this edge and its endpoint not already in the tree. It follows that the breadth-first search algorithm uses $O(e)$ or $O(n^2)$ steps.

Breadth-first search is one of the most useful algorithms in graph theory. In particular, it can serve as the basis for algorithms that solve a wide variety of problems. For example, algorithms that find the connected components of a graph, that determine whether a graph is bipartite, and that find the path with the fewest edges between two vertices in a graph can all be built using breadth-first search.

Backtracking Applications

There are problems that can be solved only by performing an exhaustive search of all possible solutions. One way to search systematically for a solution is to use a decision tree, where each internal vertex represents a decision and each leaf a possible solution. To find a solution via backtracking, first make a sequence of decisions in an attempt to reach a solution as long as this is possible. The sequence of decisions can be represented by a path in the decision tree. Once it is known that no solution can result from any further sequence of decisions, backtrack to the parent of the current vertex and work toward a solution with another series of decisions, if this is possible. The procedure continues until a solution is found, or it is established that no solution exists. Examples 6 to 8 illustrate the usefulness of backtracking.

EXAMPLE 6 **Graph Colorings** How can backtracking be used to decide whether a graph can be colored using n colors?

Solution: We can solve this problem using backtracking in the following way. First pick some vertex a and assign it color 1. Then pick a second vertex b, and if b is not adjacent to a, assign it color 1. Otherwise, assign color 2 to b. Then go on to a third vertex c. Use color 1, if possible, for c. Otherwise use color 2, if this is possible. Only if neither color 1 nor color 2 can be used should color 3 be used. Continue this process as long as it is possible to assign one of the n colors to each additional vertex, always using the first allowable color in the list. If a vertex is reached that cannot be colored by any of the n colors, backtrack to the last assignment made and change the coloring of the last vertex colored, if possible, using the next allowable color in the list. If it is not possible to change this coloring, backtrack farther to previous assignments, one step back at a time, until it is possible to change a coloring of a vertex. Then continue assigning

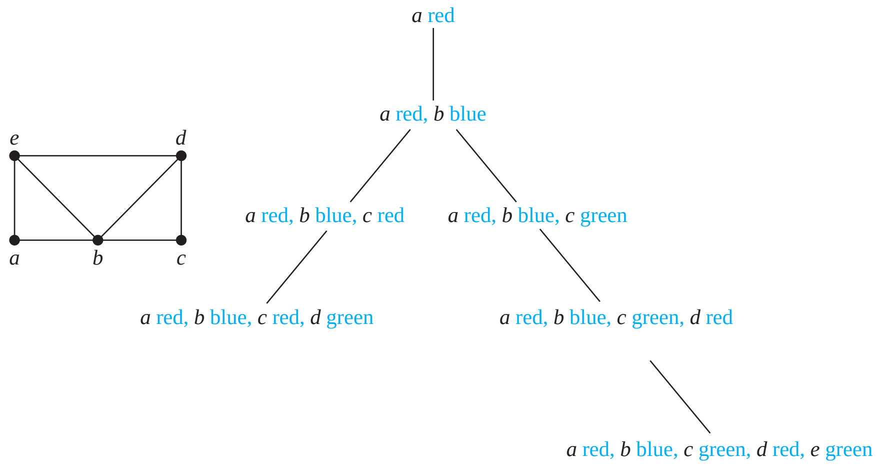

e
d
a
b
c
a red
a red, b blue
a red, b blue, c red
a red, b blue, c green
a red, b blue, c red, d green
a red, b blue, c green, d red
a red, b blue, c green, d red, e green

FIGURE 11 **Coloring a Graph Using Backtracking.**

colors of additional vertices as long as possible. If a coloring using n colors exists, backtracking will produce it. (Unfortunately this procedure can be extremely inefficient.)

In particular, consider the problem of coloring the graph shown in Figure 11 with three colors. The tree shown in Figure 11 illustrates how backtracking can be used to construct a 3-coloring. In this procedure, red is used first, then blue, and finally green. This simple example can obviously be done without backtracking, but it is a good illustration of the technique.

In this tree, the initial path from the root, which represents the assignment of red to a, leads to a coloring with a red, b blue, c red, and d green. It is impossible to color e using any of the three colors when a, b, c, and d are colored in this way. So, backtrack to the parent of the vertex representing this coloring. Because no other color can be used for d, backtrack one more level. Then change the color of c to green. We obtain a coloring of the graph by then assigning red to d and green to e.

EXAMPLE 7 **The n-Queens Problem** The n-queens problem asks how n queens can be placed on an $n \times n$ chessboard so that no two queens can attack one another. How can backtracking be used to solve the n-queens problem?

Links

Solution: To solve this problem we must find n positions on an $n \times n$ chessboard so that no two of these positions are in the same row, same column, or in the same diagonal [a diagonal consists of all positions (i, j) with $i + j = m$ for some m, or $i - j = m$ for some m]. We will use backtracking to solve the n-queens problem. We start with an empty chessboard. At stage $k + 1$ we attempt putting an additional queen on the board in the $(k + 1)$st column, where there are already queens in the first k columns. We examine squares in the $(k + 1)$st column starting with the square in the first row, looking for a position to place this queen so that it is not in the same row or on the same diagonal as a queen already on the board. (We already know it is not in the same column.) If it is impossible to find a position to place the queen in the $(k + 1)$st column, backtrack to the placement of the queen in the kth column, and place this queen in the next allowable row in this column, if such a row exists. If no such row exists, backtrack further.

In particular, Figure 12 displays a backtracking solution to the four-queens problem. In this solution, we place a queen in the first row and column. Then we put a queen in the third row of the second column. However, this makes it impossible to place a queen in the third column. So we backtrack and put a queen in the fourth row of the second column. When we do this, we can place a queen in the second row of the third column. But there is no way to add a queen to the fourth column. This shows that no solution results when a queen is placed in the first row and column. We backtrack to the empty chessboard, and place a queen in the second row of the first column. This leads to a solution as shown in Figure 12.

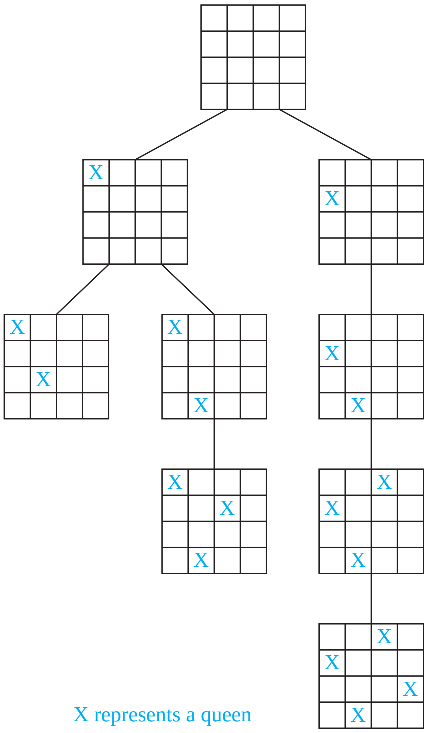


FIGURE 12 **A Backtracking Solution of the Four-Queens Problem.**

EXAMPLE 8 **Sums of Subsets** Consider this problem. Given a set of positive integers $x_1, x_2, \ldots, x_n$, find a subset of this set of integers that has M as its sum. How can backtracking be used to solve this problem?

Solution: We start with a sum with no terms. We build up the sum by successively adding terms. An integer in the sequence is included if the sum remains less than M when this integer is added to the sum. If a sum is reached such that the addition of any term is greater than M, backtrack by dropping the last term of the sum.

Figure 13 displays a backtracking solution to the problem of finding a subset of {31, 27, 15, 11, 7, 5} with the sum equal to 39. ◀

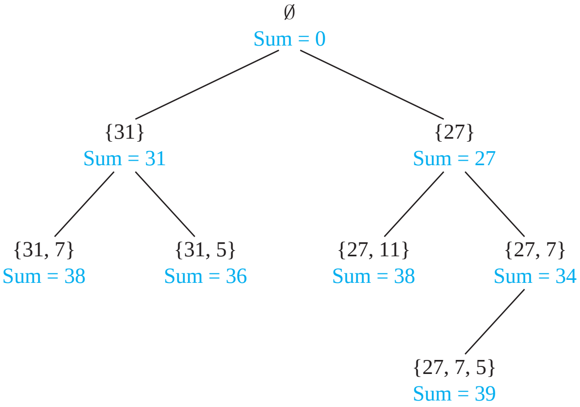


FIGURE 13 **Find a Sum Equal to 39 Using Backtracking.**

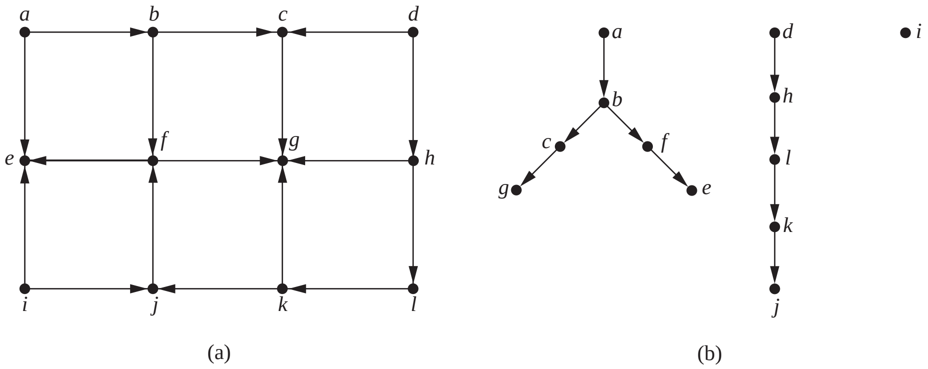


FIGURE 14 **Depth-First Search of a Directed Graph.**

Depth-First Search in Directed Graphs

We can easily modify both depth-first search and breadth-first search so that they can run given a directed graph as input. However, the output will not necessarily be a spanning tree, but rather a spanning forest. In both algorithms we can add an edge only when it is directed away from the vertex that is being visited and to a vertex not yet added. If at a stage of either algorithm we find that no edge exists starting at a vertex already added to one not yet added, the next vertex added by the algorithm becomes the root of a new tree in the spanning forest. This is illustrated in Example 9.

EXAMPLE 9 What is the output of depth-first search given the graph G shown in Figure 14(a) as input?

Solution: We begin the depth-first search at vertex a and add vertices b, c, and g and the corresponding edges where we are blocked. We backtrack to c but we are still blocked, and then backtrack to b, where we add vertices f and e and the corresponding edges. Backtracking takes us all the way back to a. We then start a new tree at d and add vertices h, l, k, and j and the corresponding edges. We backtrack to k, then l, then h, and back to d. Finally, we start a new tree at i, completing the depth-first search. The output is shown in Figure 14(b). ◀

Depth-first search in directed graphs is the basis of many algorithms (see [GrYe05], [Ma89], and [CoLeRiSt09]). It can be used to determine whether a directed graph has a circuit, it can be used to carry out a topological sort of a graph, and it can also be used to find the strongly connected components of a directed graph.

We conclude this section with an application of depth-first search and breadth-first search to search engines on the Web.

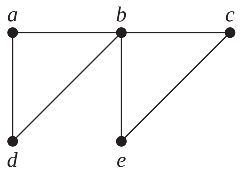
EXAMPLE 10 **Web Spiders** To index websites, search engines such as Google and Yahoo systematically explore the Web starting at known sites. These search engines use programs called Web spiders (or crawlers or bots) to visit websites and analyze their contents. Web spiders use both depth-first searching and breadth-first searching to create indices. As described in Example 5 in Section 10.1, Web pages and links between them can be modeled by a directed graph called the Web graph. Web pages are represented by vertices and links are represented by directed edges. Using depth-first search, an initial Web page is selected, a link is followed to a second Web page (if there is such a link), a link on the second Web page is followed to a third Web page, if there is such a link, and so on, until a page with no new links is found. Backtracking is then used to examine links at the previous level to look for new links, and so on. (Because of practical limitations, Web spiders have limits to the depth they search in depth-first search.) Using breadth-first search, an initial Web page is selected and a link on this page is followed to a second Web page, then a second link on the initial page is followed (if it exists), and so on, until all links of the initial page have been followed. Then links on the pages one level down are followed, page by page, and so on. ◀

Exercises

1. How many edges must be removed from a connected graph with n vertices and m edges to produce a spanning tree?

In Exercises 2–6 find a spanning tree for the graph shown by removing edges in simple circuits.

2.



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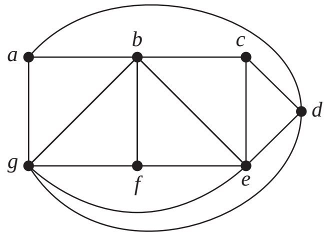

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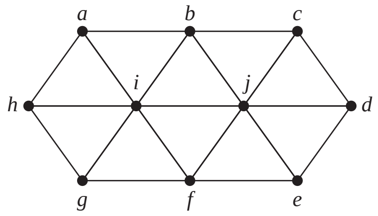

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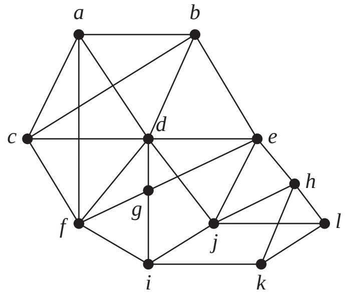

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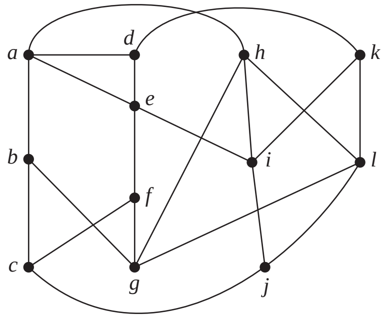
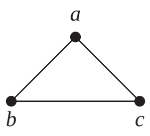


7. Find a spanning tree for each of these graphs.

a) K_5 **b)** $K_{4,4}$ **c)** $K_{1,6}$
d) Q_3 **e)** C_5 **f)** W_5

In Exercises 8–10 draw all the spanning trees of the given simple graphs.

8.



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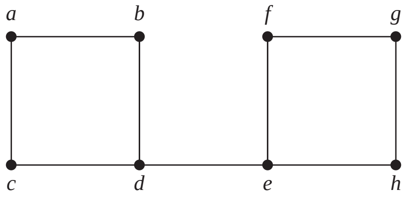

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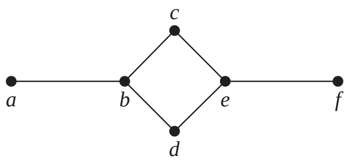


***11.** How many different spanning trees does each of these simple graphs have?

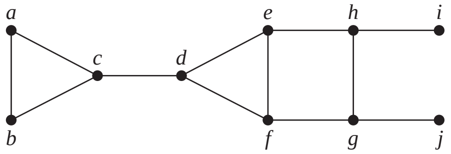
a) K_3 **b)** K_4 **c)** $K_{2,2}$ **d)** C_5

***12.** How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 **b)** K_4 **c)** K_5

In Exercises 13–15 use depth-first search to produce a spanning tree for the given simple graph. Choose a as the root of this spanning tree and assume that the vertices are ordered alphabetically.

13.



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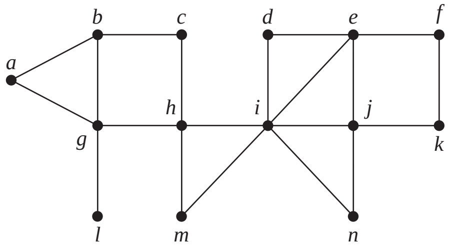

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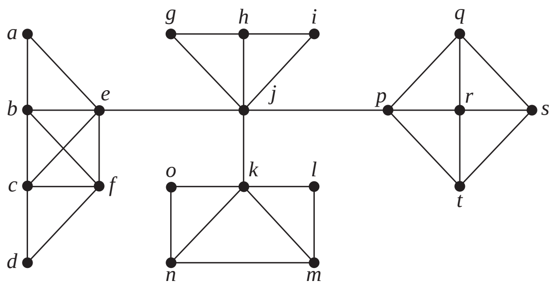


16. Use breadth-first search to produce a spanning tree for each of the simple graphs in Exercises 13–15. Choose a as the root of each spanning tree.

17. Use depth-first search to find a spanning tree of each of these graphs.

a) W_6 (see Example 7 of Section 10.2), starting at the vertex of degree 6

b) K_5

c) $K_{3,4}$, starting at a vertex of degree 3

d) Q_3

18. Use breadth-first search to find a spanning tree of each of the graphs in Exercise 17.

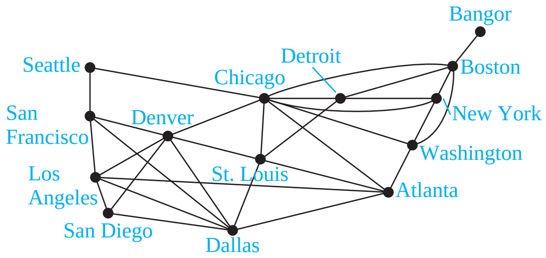
19. Describe the trees produced by breadth-first search and depth-first search of the wheel graph W_n, starting at the vertex of degree n, where n is an integer with $n \geq 3$. (See Example 7 of Section 10.2.) Justify your answers.

20. Describe the trees produced by breadth-first search and depth-first search of the complete graph K_n, where n is a positive integer. Justify your answers.

21. Describe the trees produced by breadth-first search and depth-first search of the complete bipartite graph $K_{m,n}$, starting at a vertex of degree m, where m and n are positive integers. Justify your answers.

22. Describe the tree produced by breadth-first search and depth-first search for the n-cube graph Q_n, where n is a positive integer.

23. Suppose that an airline must reduce its flight schedule to save money. If its original routes are as illustrated here, which flights can be discontinued to retain service between all pairs of cities (where it may be necessary to combine flights to fly from one city to another)?



24. Explain how breadth-first search or depth-first search can be used to order the vertices of a connected graph.

*25. Show that the length of the shortest path between vertices v and u in a connected simple graph equals the level number of u in the breadth-first spanning tree of G with root v.

26. Use backtracking to try to find a coloring of each of the graphs in Exercises 7–9 of Section 10.8 using three colors.

27. Use backtracking to solve the n-queens problem for these values of n.

a) $n = 3$ b) $n = 5$ c) $n = 6$

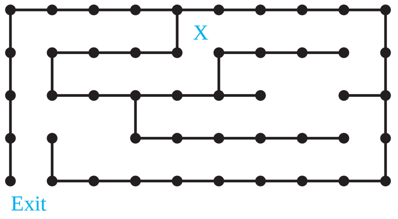
28. Use backtracking to find a subset, if it exists, of the set $\{27, 24, 19, 14, 11, 8\}$ with sum

a) 20. b) 41. c) 60.

29. Explain how backtracking can be used to find a Hamilton path or circuit in a graph.

30. a) Explain how backtracking can be used to find the way out of a maze, given a starting position and the exit position. Consider the maze divided into positions, where at each position the set of available moves includes one to four possibilities (up, down, right, left).

b) Find a path from the starting position marked by X to the exit in this maze.



A **spanning forest** of a graph G is a forest that contains every vertex of G such that two vertices are in the same tree of the forest when there is a path in G between these two vertices.

31. Show that every finite simple graph has a spanning forest.

32. How many trees are in the spanning forest of a graph?

33. How many edges must be removed to produce the spanning forest of a graph with n vertices, m edges, and c connected components?

34. Let G be a connected graph. Show that if T is a spanning tree of G constructed using breadth-first search, then an edge of G not in T must connect vertices at the same level or at levels that differ by 1 in this spanning tree.

35. Explain how to use breadth-first search to find the length of a shortest path between two vertices in an undirected graph.

36. Devise an algorithm based on breadth-first search that determines whether a graph has a simple circuit, and if so, finds one.

37. Devise an algorithm based on breadth-first search for finding the connected components of a graph.

38. Explain how breadth-first search and how depth-first search can be used to determine whether a graph is bipartite.

39. Which connected simple graphs have exactly one spanning tree?

40. Devise an algorithm for constructing the spanning forest of a graph based on deleting edges that form simple circuits.

41. Devise an algorithm for constructing the spanning forest of a graph based on depth-first searching.

42. Devise an algorithm for constructing the spanning forest of a graph based on breadth-first searching.

43. Let G be a connected graph. Show that if T is a spanning tree of G constructed using depth-first search, then an edge of G not in T must be a back edge, that is, it must connect a vertex to one of its ancestors or one of its descendants in T.

44. When must an edge of a connected simple graph be in every spanning tree for this graph?

45. For which graphs do depth-first search and breadth-first search produce identical spanning trees no matter which vertex is selected as the root of the tree? Justify your answer.

46. Use Exercise 43 to prove that if G is a connected, simple graph with n vertices and G does not contain a simple path of length k then it contains at most $(k-1)n$ edges.

47. Use mathematical induction to prove that breadth-first search visits vertices in order of their level in the resulting spanning tree.

48. Use pseudocode to describe a variation of depth-first search that assigns the integer n to the nth vertex visited in the search. Show that this numbering corresponds to the numbering of the vertices created by a preorder traversal of the spanning tree.

49. Use pseudocode to describe a variation of breadth-first search that assigns the integer m to the mth vertex visited in the search.

***50.** Suppose that G is a directed graph and T is a spanning tree constructed using breadth-first search. Show that every edge of G has endpoints that are at the same level or one level higher or lower.

51. Show that if G is a directed graph and T is a spanning tree constructed using depth-first search, then every edge not in the spanning tree is a **forward edge** connecting an ancestor to a descendant, a **back edge** connecting a descendant to an ancestor, or a **cross edge** connecting a vertex to a vertex in a previously visited subtree.

***52.** Describe a variation of depth-first search that assigns the smallest available positive integer to a vertex when the algorithm is totally finished with this vertex. Show that in this numbering, each vertex has a larger number than its children and that the children have increasing numbers from left to right.

Let T_1 and T_2 be spanning trees of a graph. The **distance** between T_1 and T_2 is the number of edges in T_1 and T_2 that are not common to T_1 and T_2.

53. Find the distance between each pair of spanning trees shown in Figures 3(c) and 4 of the graph G shown in Figure 2.

***54.** Suppose that T_1, T_2, and T_3 are spanning trees of the simple graph G. Show that the distance between T_1 and T_3 does not exceed the sum of the distance between T_1 and T_2 and the distance between T_2 and T_3.

****55.** Suppose that T_1 and T_2 are spanning trees of a simple graph G. Moreover, suppose that e_1 is an edge in T_1 that is not in T_2. Show that there is an edge e_2 in T_2 that is not in T_1 such that T_1 remains a spanning tree if e_1 is removed from it and e_2 is added to it, and T_2 remains a spanning tree if e_2 is removed from it and e_1 is added to it.

***56.** Show that it is possible to find a sequence of spanning trees leading from any spanning tree to any other by successively removing one edge and adding another.

A **rooted spanning tree** of a directed graph is a rooted tree containing edges of the graph such that every vertex of the graph is an endpoint of one of the edges in the tree.

57. For each of the directed graphs in Exercises 18–23 of Section 10.5 either find a rooted spanning tree of the graph or determine that no such tree exists.

***58.** Show that a connected directed graph in which each vertex has the same in-degree and out-degree has a rooted spanning tree. [*Hint:* Use an Euler circuit.]

***59.** Give an algorithm to build a rooted spanning tree for connected directed graphs in which each vertex has the same in-degree and out-degree.

***60.** Show that if G is a directed graph and T is a spanning tree constructed using depth-first search, then G contains a circuit if and only if G contains a back edge (see Exercise 51) relative to the spanning tree T.

***61.** Use Exercise 60 to construct an algorithm for determining whether a directed graph contains a circuit.

11.5 Minimum Spanning Trees

Introduction

Links

A company plans to build a communications network connecting its five computer centers. Any pair of these centers can be linked with a leased telephone line. Which links should be made to ensure that there is a path between any two computer centers so that the total cost of the network is minimized? We can model this problem using the weighted graph shown in Figure 1, where vertices represent computer centers, edges represent possible leased lines, and the weights on edges are the monthly lease rates of the lines represented by the edges. We can solve this problem

CHAPTER

12 Boolean Algebra

The circuits in computers and other electronic devices have inputs, each of which is either a 0 or a 1, and produce outputs that are also 0s and 1s. Circuits can be constructed using any basic element that has two different states. Such elements include switches that can be in either the on or the off position and optical devices that can be either lit or unlit. In 1938 Claude Shannon showed how the basic rules of logic, first given by George Boole in 1854 in his *The Laws of Thought*, could be used to design circuits. These rules form the basis for Boolean algebra. In this chapter we develop the basic properties of Boolean algebra. The operation of a circuit is defined by a Boolean function that specifies the value of an output for each set of inputs. The first step in constructing a circuit is to represent its Boolean function by an expression built up using the basic operations of Boolean algebra. We will provide an algorithm for producing such expressions. The expression that we obtain may contain many more operations than are necessary to represent the function. Later in the chapter we will describe methods for finding an expression with the minimum number of sums and products that represents a Boolean function. The procedures that we will develop, Karnaugh maps and the Quine–McCluskey method, are important in the design of efficient circuits.

12.1 Boolean Functions

Introduction

Boolean algebra provides the operations and the rules for working with the set {0, 1}. Electronic and optical switches can be studied using this set and the rules of Boolean algebra. The three operations in Boolean algebra that we will use most are complementation, the Boolean sum, and the Boolean product. The **complement** of an element, denoted with a bar, is defined by $\overline{0} = 1$ and $\overline{1} = 0$. The Boolean sum, denoted by $+$ or by *OR*, has the following values:

$$1 + 1 = 1, \qquad 1 + 0 = 1, \qquad 0 + 1 = 1, \qquad 0 + 0 = 0.$$

The Boolean product, denoted by $\cdot$ or by *AND*, has the following values:

$$1 \cdot 1 = 1, \qquad 1 \cdot 0 = 0, \qquad 0 \cdot 1 = 0, \qquad 0 \cdot 0 = 0.$$

When there is no danger of confusion, the symbol $\cdot$ can be deleted, just as in writing algebraic products. Unless parentheses are used, the rules of precedence for Boolean operators are: first, all complements are computed, followed by all Boolean products, followed by all Boolean sums. This is illustrated in Example 1.

EXAMPLE 1 Find the value of $1 \cdot 0 + \overline{(0 + 1)}$.

Solution: Using the definitions of complementation, the Boolean sum, and the Boolean product, it follows that

$$\begin{aligned} 1 \cdot 0 + \overline{(0 + 1)} &= 0 + \overline{1} \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

◀

The complement, Boolean sum, and Boolean product correspond to the logical operators, $\neg$, $\vee$, and $\wedge$, respectively, where 0 corresponds to **F** (false) and 1 corresponds to **T** (true). Equalities in Boolean algebra can be directly translated into equivalences of compound propositions. Conversely, equivalences of compound propositions can be translated into equalities in Boolean algebra. We will see later in this section why these translations yield valid logical equivalences and identities in Boolean algebra. Example 2 illustrates the translation from Boolean algebra to propositional logic.

EXAMPLE 2 Translate $1 \cdot 0 + \overline{(0 + 1)} = 0$, the equality found in Example 1, into a logical equivalence.

Solution: We obtain a logical equivalence when we translate each 1 into a **T**, each 0 into an **F**, each Boolean sum into a disjunction, each Boolean product into a conjunction, and each complementation into a negation. We obtain

$$(\mathbf{T} \wedge \mathbf{F}) \vee \neg(\mathbf{T} \vee \mathbf{F}) \equiv \mathbf{F}.$$

◀

Example 3 illustrates the translation from propositional logic to Boolean algebra.

EXAMPLE 3 Translate the logical equivalence $(\mathbf{T} \wedge \mathbf{T}) \vee \neg\mathbf{F} \equiv \mathbf{T}$ into an identity in Boolean algebra.

Solution: We obtain an identity in Boolean algebra when we translate each **T** into a 1, each **F** into a 0, each disjunction into a Boolean sum, each conjunction into a Boolean product, and each negation into a complementation. We obtain

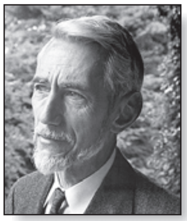
$$(1 \cdot 1) + \overline{0} = 1.$$

◀

Boolean Expressions and Boolean Functions

Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s. The variable x is called a **Boolean variable** if it assumes values only from B, that is, if its only possible values are 0 and 1. A function from B^n to B is called a **Boolean function of degree n**.

Links



CLAUDE ELWOOD SHANNON (1916–2001) Claude Shannon was born in Petoskey, Michigan, and grew up in Gaylord, Michigan. His father was a businessman and a probate judge, and his mother was a language teacher and a high school principal. Shannon attended the University of Michigan, graduating in 1936. He continued his studies at M.I.T., where he took the job of maintaining the differential analyzer, a mechanical computing device consisting of shafts and gears built by his professor, Vannevar Bush. Shannon's master's thesis, written in 1936, studied the logical aspects of the differential analyzer. This master's thesis presents the first application of Boolean algebra to the design of switching circuits; it is perhaps the most famous master's thesis of the twentieth century. He received his Ph.D. from M.I.T. in 1940. Shannon joined Bell Laboratories in 1940, where he worked on transmitting data efficiently. He was one of the first people to use bits to represent information. At Bell Laboratories he worked on determining the amount of traffic that telephone lines can carry. Shannon made many fundamental contributions to information theory. In the early 1950s he was one of the founders of the study of artificial intelligence. He joined the M.I.T. faculty in 1956, where he continued his study of information theory.

Shannon had an unconventional side. He is credited with inventing the rocket-powered Frisbee. He is also famous for riding a unicycle down the hallways of Bell Laboratories while juggling four balls. Shannon retired when he was 50 years old, publishing papers sporadically over the following 10 years. In his later years he concentrated on some pet projects, such as building a motorized pogo stick. One interesting quote from Shannon, published in *Omni Magazine* in 1987, is "I visualize a time when we will be to robots what dogs are to humans. And I am rooting for the machines."

EXAMPLE 4 The function $F(x, y) = x\overline{y}$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$ is a Boolean function of degree 2 with $F(1, 1) = 0$, $F(1, 0) = 1$, $F(0, 1) = 0$, and $F(0, 0) = 0$. We display these values of F in Table 1. ◀

TABLE 1

x	y	$F(x, y)$
1	1	0
1	0	1
0	1	0
0	0	0

Boolean functions can be represented using expressions made up from variables and Boolean operations. The **Boolean expressions** in the variables $x_1, x_2, \ldots, x_n$ are defined recursively as

> $0, 1, x_1, x_2, \ldots, x_n$ are Boolean expressions;
> if E_1 and E_2 are Boolean expressions, then $\overline{E_1}$, (E_1E_2), and $(E_1 + E_2)$ are Boolean expressions.

Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression. In Section 12.2 we will show that every Boolean function can be represented by a Boolean expression.

EXAMPLE 5 Find the values of the Boolean function represented by $F(x, y, z) = xy + \overline{z}$.

Solution: The values of this function are displayed in Table 2. ◀

TABLE 2

x	y	z	xy	$\overline{z}$	$F(x, y, z) = xy + \overline{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Note that we can represent a Boolean function graphically by distinguishing the vertices of the n-cube that correspond to the n-tuples of bits where the function has value 1.

EXAMPLE 6 The function $F(x, y, z) = xy + \overline{z}$ from B^3 to B from Example 5 can be represented by distinguishing the vertices that correspond to the five 3-tuples $(1, 1, 1)$, $(1, 1, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 0)$, where $F(x, y, z) = 1$, as shown in Figure 1. These vertices are displayed using solid black circles. ◀

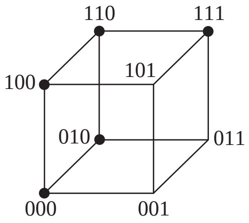


FIGURE 1

Boolean functions F and G of n variables are equal if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B. Two different Boolean expressions that represent the same function are called **equivalent**. For instance, the Boolean expressions xy, $xy + 0$, and $xy \cdot 1$ are equivalent. The **complement** of the Boolean function F is the function $\overline{F}$, where $\overline{F}(x_1, \ldots, x_n) = \overline{F(x_1, \ldots, x_n)}$. Let F and G be Boolean functions of degree n. The **Boolean sum** $F + G$ and the **Boolean product** FG are defined by

$$(F + G)(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) + G(x_1, \ldots, x_n),$$
$$(FG)(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)G(x_1, \ldots, x_n).$$

A Boolean function of degree two is a function from a set with four elements, namely, pairs of elements from $B = \{0, 1\}$, to B, a set with two elements. Hence, there are 16 different Boolean functions of degree two. In Table 3 we display the values of the 16 different Boolean functions of degree two, labeled $F_1, F_2, \ldots, F_{16}$.

TABLE 3 **The 16 Boolean Functions of Degree Two.**

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

EXAMPLE 7 How many different Boolean functions of degree n are there?

Solution: From the product rule for counting, it follows that there are 2^n different n-tuples of 0s and 1s. Because a Boolean function is an assignment of 0 or 1 to each of these 2^n different n-tuples, the product rule shows that there are 2^{2^n} different Boolean functions of degree n. ◀

Table 4 displays the number of different Boolean functions of degrees one through six. The number of such functions grows extremely rapidly.

TABLE 4 **The Number of Boolean Functions of Degree *n*.**

Degree	*Number*
1	4
2	16
3	256
4	65,536
5	4,294,967,296
6	18,446,744,073,709,551,616

Identities of Boolean Algebra

There are many identities in Boolean algebra. The most important of these are displayed in Table 5. These identities are particularly useful in simplifying the design of circuits. Each of the identities in Table 5 can be proved using a table. We will prove one of the distributive laws in this way in Example 8. The proofs of the remaining properties are left as exercises for the reader.

EXAMPLE 8 Show that the distributive law $x(y + z) = xy + xz$ is valid.

Solution: The verification of this identity is shown in Table 6. The identity holds because the last two columns of the table agree. ◀

The reader should compare the Boolean identities in Table 5 to the logical equivalences in Table 6 of Section 1.3 and the set identities in Table 1 in Section 2.2. All are special cases of the same set of identities in a more abstract structure. Each collection of identities can be obtained by making the appropriate translations. For example, we can transform each of the identities in Table 5 into a logical equivalence by changing each Boolean variable into a propositional variable, each 0 into a **F**, each 1 into a **T**, each Boolean sum into a disjunction, each Boolean product into a conjunction, and each complementation into a negation, as we illustrate in Example 9.

TABLE 5 Boolean Identities.

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\ \overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Compare these Boolean identities with the logical equivalences in Section 1.3 and the set identities in Section 2.2.

EXAMPLE 9 Translate the distributive law $x + yz = (x + y)(x + z)$ in Table 5 into a logical equivalence.

Solution: To translate a Boolean identity into a logical equivalence, we change each Boolean variable into a propositional variable. Here we will change the Boolean variables x, y, and z into the propositional variables p, q, and r. Next, we change each Boolean sum into a disjunction and each Boolean product into a conjunction. (Note that 0 and 1 do not appear in this identity and

TABLE 6 Verifying One of the Distributive Laws.

x	y	z	$y + z$	xy	xz	$x(y + z)$	$xy + xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

complementation also does not appear.) This transforms the Boolean identity into the logical equivalence

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

This logical equivalence is one of the distributive laws for propositional logic in Table 6 in Section 1.3. ◀

Identities in Boolean algebra can be used to prove further identities. We demonstrate this in Example 10.

EXAMPLE 10 Prove the **absorption law** $x(x + y) = x$ using the other identities of Boolean algebra shown in Table 5. (This is called an absorption law because absorbing $x + y$ into x leaves x unchanged.)

Extra Examples

Solution: We display steps used to derive this identity and the law used in each step:

$$\begin{aligned} x(x + y) &= (x + 0)(x + y) && \text{Identity law for the Boolean sum} \\ &= x + 0 \cdot y && \text{Distributive law of the Boolean sum over the Boolean product} \\ &= x + y \cdot 0 && \text{Commutative law for the Boolean product} \\ &= x + 0 && \text{Domination law for the Boolean product} \\ &= x && \text{Identity law for the Boolean sum.} \end{aligned}$$

◀

Duality

The identities in Table 5 come in pairs (except for the law of the double complement and the unit and zero properties). To explain the relationship between the two identities in each pair we use the concept of a dual. The **dual** of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

EXAMPLE 11 Find the duals of $x(y + 0)$ and $\overline{x} \cdot 1 + (\overline{y} + z)$.

Solution: Interchanging $\cdot$ signs and $+$ signs and interchanging 0s and 1s in these expressions produces their duals. The duals are $x + (y \cdot 1)$ and $(\overline{x} + 0)(\overline{y}z)$, respectively. ◀

The dual of a Boolean function F represented by a Boolean expression is the function represented by the dual of this expression. This dual function, denoted by F^d, does not depend on the particular Boolean expression used to represent F. An identity between functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken. (See Exercise 30 for the reason why this is true.) This result, called the **duality principle**, is useful for obtaining new identities.

EXAMPLE 12 Construct an identity from the absorption law $x(x + y) = x$ by taking duals.

Solution: Taking the duals of both sides of this identity produces the identity $x + xy = x$, which is also called an absorption law and is shown in Table 5. ◀

The Abstract Definition of a Boolean Algebra

In this section we have focused on Boolean functions and expressions. However, the results we have established can be translated into results about propositions or results about sets. Because of this, it is useful to define Boolean algebras abstractly. Once it is shown that a particular structure is a Boolean algebra, then all results established about Boolean algebras in general apply to this particular structure.

Boolean algebras can be defined in several ways. The most common way is to specify the properties that operations must satisfy, as is done in Definition 1.

DEFINITION 1

A *Boolean algebra* is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $^{-}$ such that these properties hold for all x, y, and z in B:

$$\left.\begin{aligned} x \vee 0 &= x \\ x \wedge 1 &= x \end{aligned}\right\} \quad \text{Identity laws}$$

$$\left.\begin{aligned} x \vee \overline{x} &= 1 \\ x \wedge \overline{x} &= 0 \end{aligned}\right\} \quad \text{Complement laws}$$

$$\left.\begin{aligned} (x \vee y) \vee z &= x \vee (y \vee z) \\ (x \wedge y) \wedge z &= x \wedge (y \wedge z) \end{aligned}\right\} \quad \text{Associative laws}$$

$$\left.\begin{aligned} x \vee y &= y \vee x \\ x \wedge y &= y \wedge x \end{aligned}\right\} \quad \text{Commutative laws}$$

$$\left.\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \end{aligned}\right\} \quad \text{Distributive laws}$$

Using the laws given in Definition 1, it is possible to prove many other laws that hold for every Boolean algebra, such as idempotent and domination laws. (See Exercises 35–42.)

From our previous discussion, $B = \{0, 1\}$ with the *OR* and *AND* operations and the complement operator, satisfies all these properties. The set of propositions in n variables, with the $\vee$ and $\wedge$ operators, **F** and **T**, and the negation operator, also satisfies all the properties of a Boolean algebra, as can be seen from Table 6 in Section 1.3. Similarly, the set of subsets of a universal set U with the union and intersection operations, the empty set and the universal set, and the set complementation operator, is a Boolean algebra as can be seen by consulting Table 1 in Section 2.2. So, to establish results about each of Boolean expressions, propositions, and sets, we need only prove results about abstract Boolean algebras.

Boolean algebras may also be defined using the notion of a lattice, discussed in Chapter 9. Recall that a lattice L is a partially ordered set in which every pair of elements x, y has a least upper bound, denoted by $\text{lub}(x, y)$ and a greatest lower bound denoted by $\text{glb}(x, y)$. Given two elements x and y of L, we can define two operations $\vee$ and $\wedge$ on pairs of elements of L by $x \vee y = \text{lub}(x, y)$ and $x \wedge y = \text{glb}(x, y)$.

For a lattice L to be a Boolean algebra as specified in Definition 1, it must have two properties. First, it must be **complemented**. For a lattice to be complemented it must have a least element 0 and a greatest element 1 and for every element x of the lattice there must exist an element $\overline{x}$ such that $x \vee \overline{x} = 1$ and $x \wedge \overline{x} = 0$. Second, it must be **distributive**. This means that for every x, y, and z in L, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Showing that a complemented, distributive lattice is a Boolean algebra has been left as Supplementary Exercise 39 in Chapter 9.

Exercises

1. Find the values of these expressions.

a) $1 \cdot \overline{0}$ **b)** $1 + \overline{1}$ **c)** $\overline{0} \cdot 0$ **d)** $\overline{(1 + 0)}$

2. Find the values, if any, of the Boolean variable x that satisfy these equations.

a) $x \cdot 1 = 0$ **b)** $x + x = 0$

c) $x \cdot 1 = x$ **d)** $x \cdot \overline{x} = 1$

3. a) Show that $(1 \cdot 1) + (\overline{0 \cdot 1} + 0) = 1$.

b) Translate the equation in part (a) into a propositional equivalence by changing each 0 into an **F**, each 1 into a **T**, each Boolean sum into a disjunction, each Boolean product into a conjunction, each complementation into a negation, and the equals sign into a propositional equivalence sign.

4. a) Show that $(\overline{1} \cdot \overline{0}) + (1 \cdot \overline{0}) = 1$.

b) Translate the equation in part (a) into a propositional equivalence by changing each 0 into an **F**, each 1 into a **T**, each Boolean sum into a disjunction, each Boolean product into a conjunction, each complementation into a negation, and the equals sign into a propositional equivalence sign.

5. Use a table to express the values of each of these Boolean functions.

a) $F(x, y, z) = \overline{x}y$

b) $F(x, y, z) = x + yz$

c) $F(x, y, z) = x\overline{y} + \overline{(xyz)}$

d) $F(x, y, z) = x(yz + \overline{y}\,\overline{z})$

6. Use a table to express the values of each of these Boolean functions.

a) $F(x, y, z) = \overline{z}$

b) $F(x, y, z) = \overline{x}y + \overline{y}z$

c) $F(x, y, z) = x\overline{y}z + \overline{(xyz)}$

d) $F(x, y, z) = \overline{y}(xz + \overline{x}\,\overline{z})$

7. Use a 3-cube Q_3 to represent each of the Boolean functions in Exercise 5 by displaying a black circle at each vertex that corresponds to a 3-tuple where this function has the value 1.

8. Use a 3-cube Q_3 to represent each of the Boolean functions in Exercise 6 by displaying a black circle at each vertex that corresponds to a 3-tuple where this function has the value 1.

9. What values of the Boolean variables x and y satisfy $xy = x + y$?

10. How many different Boolean functions are there of degree 7?

11. Prove the absorption law $x + xy = x$ using the other laws in Table 5.

☞ **12.** Show that $F(x, y, z) = xy + xz + yz$ has the value 1 if and only if at least two of the variables x, y, and z have the value 1.

13. Show that $x\overline{y} + y\overline{z} + \overline{x}z = \overline{x}y + \overline{y}z + x\overline{z}$.

Exercises 14–23 deal with the Boolean algebra $\{0, 1\}$ with addition, multiplication, and complement defined at the beginning of this section. In each case, use a table as in Example 8.

14. Verify the law of the double complement.

15. Verify the idempotent laws.

16. Verify the identity laws.

17. Verify the domination laws.

18. Verify the commutative laws.

19. Verify the associative laws.

20. Verify the first distributive law in Table 5.

21. Verify De Morgan's laws.

22. Verify the unit property.

23. Verify the zero property.

The Boolean operator $\oplus$, called the *XOR* operator, is defined by $1 \oplus 1 = 0$, $1 \oplus 0 = 1$, $0 \oplus 1 = 1$, and $0 \oplus 0 = 0$.

24. Simplify these expressions.

a) $x \oplus 0$ **b)** $x \oplus 1$

c) $x \oplus x$ **d)** $x \oplus \overline{x}$

25. Show that these identities hold.

a) $x \oplus y = (x + y)\overline{(xy)}$

b) $x \oplus y = (x\overline{y}) + (\overline{x}y)$

26. Show that $x \oplus y = y \oplus x$.

27. Prove or disprove these equalities.

a) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

b) $x + (y \oplus z) = (x + y) \oplus (x + z)$

c) $x \oplus (y + z) = (x \oplus y) + (x \oplus z)$

28. Find the duals of these Boolean expressions.

a) $x + y$ **b)** $\overline{x}\,\overline{y}$

c) $xyz + \overline{x}\,\overline{y}\,\overline{z}$ **d)** $x\overline{z} + x \cdot 0 + \overline{x} \cdot 1$

***29.** Suppose that F is a Boolean function represented by a Boolean expression in the variables $x_1, \ldots, x_n$. Show that $F^d(x_1, \ldots, x_n) = \overline{F(\overline{x}_1, \ldots, \overline{x}_n)}$.

***30.** Show that if F and G are Boolean functions represented by Boolean expressions in n variables and $F = G$, then $F^d = G^d$, where F^d and G^d are the Boolean functions represented by the duals of the Boolean expressions representing F and G, respectively. [*Hint:* Use the result of Exercise 29.]

***31.** How many different Boolean functions $F(x, y, z)$ are there such that $F(\overline{x}, \overline{y}, \overline{z}) = F(x, y, z)$ for all values of the Boolean variables x, y, and z?

***32.** How many different Boolean functions $F(x, y, z)$ are there such that $F(\overline{x}, y, z) = F(x, \overline{y}, z) = F(x, y, \overline{z})$ for all values of the Boolean variables x, y, and z?

33. Show that you obtain De Morgan's laws for propositions (in Table 6 in Section 1.3) when you transform De Morgan's laws for Boolean algebra in Table 6 into logical equivalences.

34. Show that you obtain the absorption laws for propositions (in Table 6 in Section 1.3) when you transform the absorption laws for Boolean algebra in Table 6 into logical equivalences.

In Exercises 35–42, use the laws in Definition 1 to show that the stated properties hold in every Boolean algebra.

35. Show that in a Boolean algebra, the **idempotent laws** $x \vee x = x$ and $x \wedge x = x$ hold for every element x.

36. Show that in a Boolean algebra, every element x has a unique complement $\overline{x}$ such that $x \vee \overline{x} = 1$ and $x \wedge \overline{x} = 0$.

37. Show that in a Boolean algebra, the complement of the element 0 is the element 1 and vice versa.

38. Prove that in a Boolean algebra, the **law of the double complement** holds; that is, $\overline{\overline{x}} = x$ for every element x.

39. Show that **De Morgan's laws** hold in a Boolean algebra. That is, show that for all x and y, $\overline{(x \vee y)} = \overline{x} \wedge \overline{y}$ and $\overline{(x \wedge y)} = \overline{x} \vee \overline{y}$.

40. Show that in a Boolean algebra, the **modular properties** hold. That is, show that $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$.

41. Show that in a Boolean algebra, if $x \vee y = 0$, then $x = 0$ and $y = 0$, and that if $x \wedge y = 1$, then $x = 1$ and $y = 1$.

42. Show that in a Boolean algebra, the **dual** of an identity, obtained by interchanging the $\vee$ and $\wedge$ operators and interchanging the elements 0 and 1, is also a valid identity.

43. Show that a complemented, distributive lattice is a Boolean algebra.

12.2 Representing Boolean Functions

Two important problems of Boolean algebra will be studied in this section. The first problem is: Given the values of a Boolean function, how can a Boolean expression that represents this function be found? This problem will be solved by showing that any Boolean function can be represented by a Boolean sum of Boolean products of the variables and their complements. The solution of this problem shows that every Boolean function can be represented using the three Boolean operators $\cdot$, $+$, and $\overline{}$. The second problem is: Is there a smaller set of operators that can be used to represent all Boolean functions? We will answer this question by showing that all Boolean functions can be represented using only one operator. Both of these problems have practical importance in circuit design.

Sum-of-Products Expansions

We will use examples to illustrate one important way to find a Boolean expression that represents a Boolean function.

EXAMPLE 1 Find Boolean expressions that represent the functions $F(x, y, z)$ and $G(x, y, z)$, which are given in Table 1.

Solution: An expression that has the value 1 when $x = z = 1$ and $y = 0$, and the value 0 otherwise, is needed to represent F. Such an expression can be formed by taking the Boolean product of x, $\overline{y}$, and z. This product, $x\overline{y}z$, has the value 1 if and only if $x = \overline{y} = z = 1$, which holds if and only if $x = z = 1$ and $y = 0$.

To represent G, we need an expression that equals 1 when $x = y = 1$ and $z = 0$, or $x = z = 0$ and $y = 1$. We can form an expression with these values by taking the Boolean sum of two different Boolean products. The Boolean product $xy\overline{z}$ has the value 1 if and only if $x = y = 1$ and $z = 0$. Similarly, the product $\overline{x}y\overline{z}$ has the value 1 if and only if $x = z = 0$ and $y = 1$. The Boolean sum of these two products, $xy\overline{z} + \overline{x}y\overline{z}$, represents G, because it has the value 1 if and only if $x = y = 1$ and $z = 0$, or $x = z = 0$ and $y = 1$. ◀

TABLE 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Example 1 illustrates a procedure for constructing a Boolean expression representing a function with given values. Each combination of values of the variables for which the function has the value 1 leads to a Boolean product of the variables or their complements.

DEFINITION 1 A *literal* is a Boolean variable or its complement. A *minterm* of the Boolean variables $x_1, x_2, \dots, x_n$ is a Boolean product $y_1 y_2 \cdots y_n$, where $y_i = x_i$ or $y_i = \overline{x}_i$. Hence, a minterm is a product of n literals, with one literal for each variable.

A minterm has the value 1 for one and only one combination of values of its variables. More precisely, the minterm $y_1 y_2 \dots y_n$ is 1 if and only if each y_i is 1, and this occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = \overline{x}_i$.

EXAMPLE 2 Find a minterm that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$, and equals 0 otherwise.

Solution: The minterm $\overline{x}_1 x_2 \overline{x}_3 x_4 x_5$ has the correct set of values. ◀

By taking Boolean sums of distinct minterms we can build up a Boolean expression with a specified set of values. In particular, a Boolean sum of minterms has the value 1 when exactly one of the minterms in the sum has the value 1. It has the value 0 for all other combinations of values of the variables. Consequently, given a Boolean function, a Boolean sum of minterms can be formed that has the value 1 when this Boolean function has the value 1, and has the value 0 when the function has the value 0. The minterms in this Boolean sum correspond to those combinations of values for which the function has the value 1. The sum of minterms that represents the function is called the **sum-of-products expansion** or the **disjunctive normal form** of the Boolean function.

Links

(See Exercise 42 in Section 1.3 for the development of disjunctive normal form in propositional calculus.)

EXAMPLE 3 Find the sum-of-products expansion for the function $F(x, y, z) = (x + y)\overline{z}$.

Extra Examples

Solution: We will find the sum-of-products expansion of $F(x, y, z)$ in two ways. First, we will use Boolean identities to expand the product and simplify. We find that

$$\begin{aligned}
F(x, y, z) &= (x + y)\overline{z} \\
&= x\overline{z} + y\overline{z} && \text{Distributive law} \\
&= x1\overline{z} + 1y\overline{z} && \text{Identity law} \\
&= x(y + \overline{y})\overline{z} + (x + \overline{x})y\overline{z} && \text{Unit property} \\
&= xy\overline{z} + x\overline{y}\,\overline{z} + xy\overline{z} + \overline{x}y\overline{z} && \text{Distributive law} \\
&= xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}y\,\overline{z}. && \text{Idempotent law}
\end{aligned}$$

Second, we can construct the sum-of-products expansion by determining the values of F for all possible values of the variables x, y, and z. These values are found in Table 2. The sum-of-products expansion of F is the Boolean sum of three minterms corresponding to the three rows of this table that give the value 1 for the function. This gives

$$F(x, y, z) = xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}y\overline{z}.$$

◀

It is also possible to find a Boolean expression that represents a Boolean function by taking a Boolean product of Boolean sums. The resulting expansion is called the **conjunctive normal form** or **product-of-sums expansion** of the function. These expansions can be found from sum-of-products expansions by taking duals. How to find such expansions directly is described in Exercise 10.

TABLE 2

x	y	z	$x + y$	$\overline{z}$	$(x + y)\overline{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Functional Completeness

Every Boolean function can be expressed as a Boolean sum of minterms. Each minterm is the Boolean product of Boolean variables or their complements. This shows that every Boolean function can be represented using the Boolean operators $\cdot$, $+$, and $^{-}$. Because every Boolean function can be represented using these operators we say that the set $\{\cdot, +, ^{-}\}$ is **functionally complete**. Can we find a smaller set of functionally complete operators? We can do so if one of the three operators of this set can be expressed in terms of the other two. This can be done using one of De Morgan's laws. We can eliminate all Boolean sums using the identity

$$x + y = \overline{\overline{x}\,\overline{y}},$$

which is obtained by taking complements of both sides in the second De Morgan law, given in Table 5 in Section 12.1, and then applying the double complementation law. This means that the set $\{\cdot, ^{-}\}$ is functionally complete. Similarly, we could eliminate all Boolean products using the identity

$$xy = \overline{\overline{x} + \overline{y}},$$

which is obtained by taking complements of both sides in the first De Morgan law, given in Table 5 in Section 12.1, and then applying the double complementation law. Consequently $\{+, ^{-}\}$ is functionally complete. Note that the set $\{+, \cdot\}$ is not functionally complete, because it is impossible to express the Boolean function $F(x) = \overline{x}$ using these operators (see Exercise 19).

We have found sets containing two operators that are functionally complete. Can we find a smaller set of functionally complete operators, namely, a set containing just one operator? Such sets exist. Define two operators, the $\mid$ or ***NAND*** operator, defined by $1 \mid 1 = 0$ and $1 \mid 0 = 0 \mid 1 = 0 \mid 0 = 1$; and the $\downarrow$ or ***NOR*** operator, defined by $1 \downarrow 1 = 1 \downarrow 0 = 0 \downarrow 1 = 0$ and $0 \downarrow 0 = 1$. Both of the sets $\{\mid\}$ and $\{\downarrow\}$ are functionally complete. To see that $\{\mid\}$ is functionally complete, because $\{\cdot, ^{-}\}$ is functionally complete, all that we have to do is show that both of the operators $\cdot$ and $^{-}$ can be expressed using just the $\mid$ operator. This can be done as

$$\overline{x} = x \mid x,$$
$$xy = (x \mid y) \mid (x \mid y).$$

The reader should verify these identities (see Exercise 14). We leave the demonstration that $\{\downarrow\}$ is functionally complete for the reader (see Exercises 15 and 16).

Exercises

1. Find a Boolean product of the Boolean variables x, y, and z, or their complements, that has the value 1 if and only if

a) $x = y = 0, z = 1$. **b)** $x = 0, y = 1, z = 0$.
c) $x = 0, y = z = 1$. **d)** $x = y = z = 0$.

2. Find the sum-of-products expansions of these Boolean functions.

a) $F(x, y) = \overline{x} + y$ **b)** $F(x, y) = x\,\overline{y}$
c) $F(x, y) = 1$ **d)** $F(x, y) = \overline{y}$

3. Find the sum-of-products expansions of these Boolean functions.

a) $F(x, y, z) = x + y + z$
b) $F(x, y, z) = (x + z)y$
c) $F(x, y, z) = x$
d) $F(x, y, z) = x\,\overline{y}$

4. Find the sum-of-products expansions of the Boolean function $F(x, y, z)$ that equals 1 if and only if

a) $x = 0$. **b)** $xy = 0$.
c) $x + y = 0$. **d)** $xyz = 0$.

5. Find the sum-of-products expansion of the Boolean function $F(w, x, y, z)$ that has the value 1 if and only if an odd number of w, x, y, and z have the value 1.

6. Find the sum-of-products expansion of the Boolean function $F(x_1, x_2, x_3, x_4, x_5)$ that has the value 1 if and only if three or more of the variables x_1, x_2, x_3, x_4, and x_5 have the value 1.

Another way to find a Boolean expression that represents a Boolean function is to form a Boolean product of Boolean sums of literals. Exercises 7–11 are concerned with representations of this kind.

7. Find a Boolean sum containing either x or $\overline{x}$, either y or $\overline{y}$, and either z or $\overline{z}$ that has the value 0 if and only if

a) $x = y = 1, z = 0$. **b)** $x = y = z = 0$.
c) $x = z = 0, y = 1$.

8. Find a Boolean product of Boolean sums of literals that has the value 0 if and only if $x = y = 1$ and $z = 0$, $x = z = 0$ and $y = 1$, or $x = y = z = 0$. [*Hint:* Take the Boolean product of the Boolean sums found in parts (a), (b), and (c) in Exercise 7.]

9. Show that the Boolean sum $y_1 + y_2 + \cdots + y_n$, where $y_i = x_i$ or $y_i = \overline{x}_i$, has the value 0 for exactly one combination of the values of the variables, namely, when $x_i = 0$ if $y_i = x_i$ and $x_i = 1$ if $y_i = \overline{x}_i$. This Boolean sum is called a **maxterm**.

10. Show that a Boolean function can be represented as a Boolean product of maxterms. This representation is called the **product-of-sums expansion** or **conjunctive normal form** of the function. [*Hint:* Include one maxterm in this product for each combination of the variables where the function has the value 0.]

11. Find the product-of-sums expansion of each of the Boolean functions in Exercise 3.

12. Express each of these Boolean functions using the operators $\cdot$ and $^{-}$.

a) $x + y + z$ **b)** $x + \overline{y}(\overline{x} + z)$
c) $\overline{x + \overline{y}}$ **d)** $\overline{x}(x + \overline{y} + \overline{z})$

13. Express each of the Boolean functions in Exercise 12 using the operators $+$ and $^{-}$.

14. Show that

a) $\overline{x} = x \mid x$. **b)** $xy = (x \mid y) \mid (x \mid y)$.
c) $x + y = (x \mid x) \mid (y \mid y)$.

15. Show that

a) $\overline{x} = x \downarrow x$.
b) $xy = (x \downarrow x) \downarrow (y \downarrow y)$.
c) $x + y = (x \downarrow y) \downarrow (x \downarrow y)$.

16. Show that $\{\downarrow\}$ is functionally complete using Exercise 15.

17. Express each of the Boolean functions in Exercise 3 using the operator $\mid$.

18. Express each of the Boolean functions in Exercise 3 using the operator $\downarrow$.

19. Show that the set of operators $\{+, \cdot\}$ is not functionally complete.

20. Are these sets of operators functionally complete?

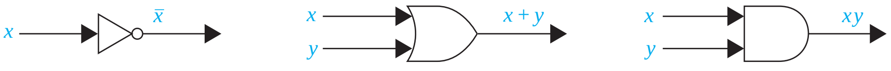
a) $\{+, \oplus\}$ **b)** $\{^{-}, \oplus\}$ **c)** $\{\cdot, \oplus\}$

12.3 Logic Gates

Introduction

Links

Boolean algebra is used to model the circuitry of electronic devices. Each input and each output of such a device can be thought of as a member of the set $\{0, 1\}$. A computer, or other electronic device, is made up of a number of circuits. Each circuit can be designed using the rules of Boolean algebra that were studied in Sections 12.1 and 12.2. The basic elements of circuits



(a) Inverter (b) OR gate (c) AND gate

FIGURE 1 Basic Types of Gates.

are called **gates**, and were introduced in Section 1.2. Each type of gate implements a Boolean operation. In this section we define several types of gates. Using these gates, we will apply the rules of Boolean algebra to design circuits that perform a variety of tasks. The circuits that we will study in this chapter give output that depends only on the input, and not on the current state of the circuit. In other words, these circuits have no memory capabilities. Such circuits are called **combinational circuits** or **gating networks**.

We will construct combinational circuits using three types of elements. The first is an **inverter**, which accepts the value of one Boolean variable as input and produces the complement of this value as its output. The symbol used for an inverter is shown in Figure 1(a). The input to the inverter is shown on the left side entering the element, and the output is shown on the right side leaving the element.

The next type of element we will use is the **OR gate**. The inputs to this gate are the values of two or more Boolean variables. The output is the Boolean sum of their values. The symbol used for an OR gate is shown in Figure 1(b). The inputs to the OR gate are shown on the left side entering the element, and the output is shown on the right side leaving the element.

The third type of element we will use is the **AND gate**. The inputs to this gate are the values of two or more Boolean variables. The output is the Boolean product of their values. The symbol used for an AND gate is shown in Figure 1(c). The inputs to the AND gate are shown on the left side entering the element, and the output is shown on the right side leaving the element.

We will permit multiple inputs to AND and OR gates. The inputs to each of these gates are shown on the left side entering the element, and the output is shown on the right side. Examples of AND and OR gates with n inputs are shown in Figure 2.

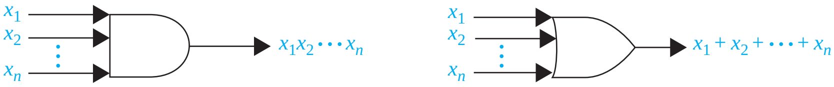


FIGURE 2 Gates with *n* Inputs.

Combinations of Gates

Combinational circuits can be constructed using a combination of inverters, OR gates, and AND gates. When combinations of circuits are formed, some gates may share inputs. This is shown in one of two ways in depictions of circuits. One method is to use branchings that indicate all the gates that use a given input. The other method is to indicate this input separately for each gate. Figure 3 illustrates the two ways of showing gates with the same input values. Note also that output from a gate may be used as input by one or more other elements, as shown in Figure 3. Both drawings in Figure 3 depict the circuit that produces the output $xy + \overline{x}y$.

EXAMPLE 1 Construct circuits that produce the following outputs: (a) $(x + y)\overline{x}$, (b) $\overline{x}\,\overline{(y + \overline{z})}$, and (c) $(x + y + z)(\overline{x}\,\overline{y}\,\overline{z})$.

Solution: Circuits that produce these outputs are shown in Figure 4. ◀

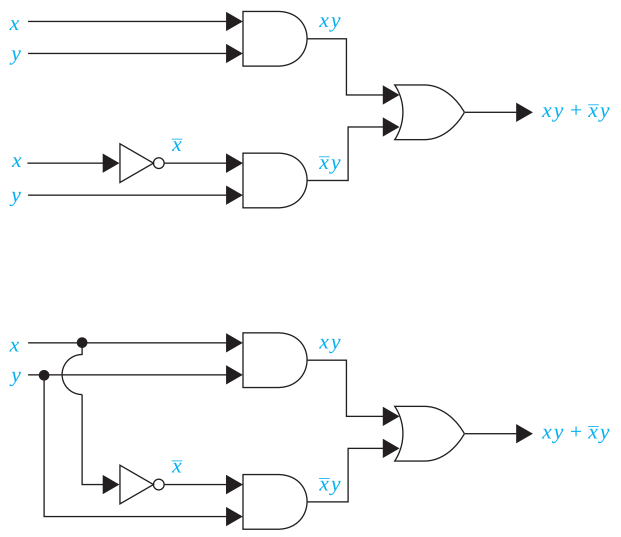


FIGURE 3 **Two Ways to Draw the Same Circuit.**

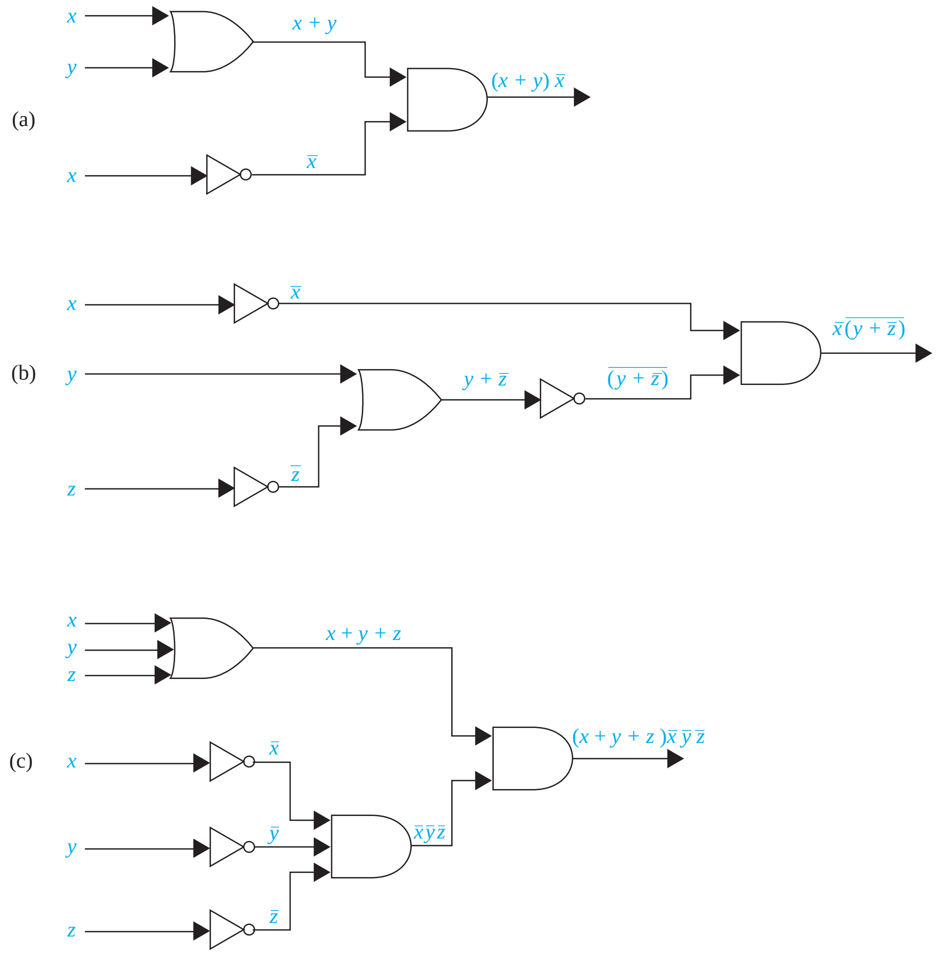


FIGURE 4 **Circuits that Produce the Outputs Specified in Example 1.**

Examples of Circuits

We will give some examples of circuits that perform some useful functions.

EXAMPLE 2 A committee of three individuals decides issues for an organization. Each individual votes either yes or no for each proposal that arises. A proposal is passed if it receives at least two yes votes. Design a circuit that determines whether a proposal passes.

Extra Examples

Solution: Let $x = 1$ if the first individual votes yes, and $x = 0$ if this individual votes no; let $y = 1$ if the second individual votes yes, and $y = 0$ if this individual votes no; let $z = 1$ if the third individual votes yes, and $z = 0$ if this individual votes no. Then a circuit must be designed that produces the output 1 from the inputs x, y, and z when two or more of x, y, and z are 1. One representation of the Boolean function that has these output values is $xy + xz + yz$ (see Exercise 12 in Section 12.1). The circuit that implements this function is shown in Figure 5. ◀

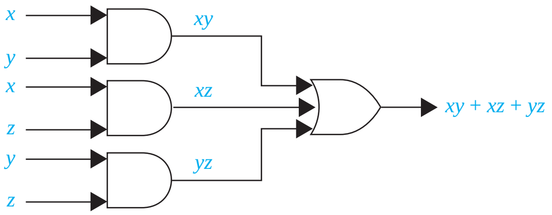


FIGURE 5 **A Circuit for Majority Voting.**

EXAMPLE 3 Sometimes light fixtures are controlled by more than one switch. Circuits need to be designed so that flipping any one of the switches for the fixture turns the light on when it is off and turns the light off when it is on. Design circuits that accomplish this when there are two switches and when there are three switches.

TABLE 1

x	y	$F(x, y)$
1	1	1
1	0	0
0	1	0
0	0	1

Solution: We will begin by designing the circuit that controls the light fixture when two different switches are used. Let $x = 1$ when the first switch is closed and $x = 0$ when it is open, and let $y = 1$ when the second switch is closed and $y = 0$ when it is open. Let $F(x, y) = 1$ when the light is on and $F(x, y) = 0$ when it is off. We can arbitrarily decide that the light will be on when both switches are closed, so that $F(1, 1) = 1$. This determines all the other values of F. When one of the two switches is opened, the light goes off, so $F(1, 0) = F(0, 1) = 0$. When the other switch is also opened, the light goes on, so $F(0, 0) = 1$. Table 1 displays these values. Note that $F(x, y) = xy + \overline{x}\,\overline{y}$. This function is implemented by the circuit shown in Figure 6.

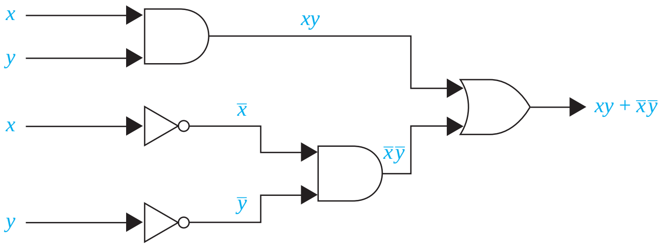


FIGURE 6 **A Circuit for a Light Controlled by Two Switches.**

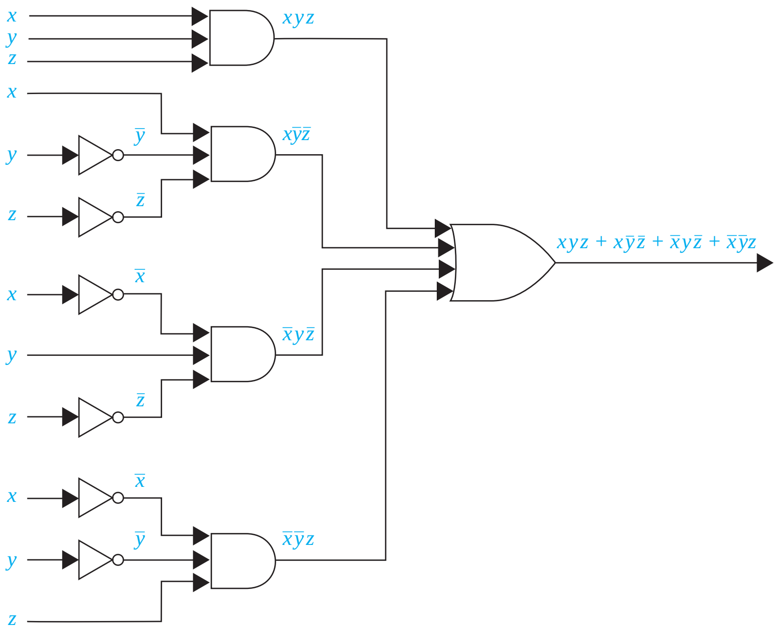

x
y
z
xyz
x
y
$\overline{y}$
$x\overline{y}\overline{z}$
z
$\overline{z}$
$xyz + x\overline{y}\overline{z} + \overline{x}y\overline{z} + \overline{x}\overline{y}z$
x
$\overline{x}$
y
$\overline{x}y\overline{z}$
z
$\overline{z}$
x
$\overline{x}$
y
$\overline{y}$
$\overline{x}\overline{y}z$
z

FIGURE 7 **A Circuit for a Fixture Controlled by Three Switches.**

TABLE 2

x	y	z	$F(x, y, z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

We will now design a circuit for three switches. Let x, y, and z be the Boolean variables that indicate whether each of the three switches is closed. We let $x = 1$ when the first switch is closed, and $x = 0$ when it is open; $y = 1$ when the second switch is closed, and $y = 0$ when it is open; and $z = 1$ when the third switch is closed, and $z = 0$ when it is open. Let $F(x, y, z) = 1$ when the light is on and $F(x, y, z) = 0$ when the light is off. We can arbitrarily specify that the light be on when all three switches are closed, so that $F(1, 1, 1) = 1$. This determines all other values of F. When one switch is opened, the light goes off, so $F(1, 1, 0) = F(1, 0, 1) = F(0, 1, 1) = 0$. When a second switch is opened, the light goes on, so $F(1, 0, 0) = F(0, 1, 0) = F(0, 0, 1) = 1$. Finally, when the third switch is opened, the light goes off again, so $F(0, 0, 0) = 0$. Table 2 shows the values of this function.

The function F can be represented by its sum-of-products expansion as $F(x, y, z) = xyz + x\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z$. The circuit shown in Figure 7 implements this function.

Adders

TABLE 3 **Input and Output for the Half Adder.**

Input		*Output*	
x	y	s	c
1	1	0	1
1	0	1	0
0	1	1	0
0	0	0	0

We will illustrate how logic circuits can be used to carry out addition of two positive integers from their binary expansions. We will build up the circuitry to do this addition from some component circuits. First, we will build a circuit that can be used to find $x + y$, where x and y are two bits. The input to our circuit will be x and y, because these each have the value 0 or the value 1. The output will consist of two bits, namely, s and c, where s is the sum bit and c is the carry bit. This circuit is called a **multiple output circuit** because it has more than one output. The circuit that we are designing is called the **half adder**, because it adds two bits, without considering a carry from a previous addition. We show the input and output for the half adder in Table 3. From Table 3 we see that $c = xy$ and that $s = x\overline{y} + \overline{x}y = (x + y)\overline{(xy)}$. Hence, the circuit shown in Figure 8 computes the sum bit s and the carry bit c from the bits x and y.

We use the **full adder** to compute the sum bit and the carry bit when two bits and a carry are added. The inputs to the full adder are the bits x and y and the carry c_i. The outputs are the sum bit s and the new carry c_{i+1}. The inputs and outputs for the full adder are shown in Table 4.

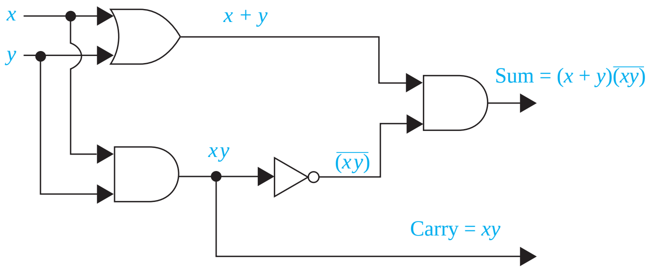


FIGURE 8 The Half Adder.

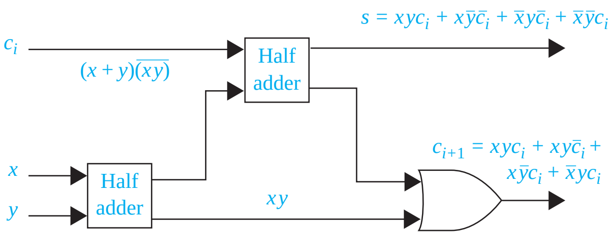

$s = xyc_i + x\overline{y}\overline{c}_i + \overline{x}y\overline{c}_i + \overline{x}\overline{y}c_i$
c_i
$(x + y)\overline{(xy)}$
Half adder
$c_{i+1} = xyc_i + xy\overline{c}_i + x\overline{y}c_i + \overline{x}yc_i$
x
y
Half adder
xy

FIGURE 9 A Full Adder.

TABLE 4 Input and Output for the Full Adder.

Input			*Output*	
x	y	c_i	s	c_{i+1}
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	1	0
0	1	1	0	1
0	1	0	1	0
0	0	1	1	0
0	0	0	0	0

The two outputs of the full adder, the sum bit s and the carry c_{i+1}, are given by the sum-of-products expansions $xyc_i + x\overline{y}\,\overline{c}_i + \overline{x}y\overline{c}_i + \overline{x}\,\overline{y}c_i$ and $xyc_i + xy\overline{c}_i + x\overline{y}c_i + \overline{x}yc_i$, respectively. However, instead of designing the full adder from scratch, we will use half adders to produce the desired output. A full adder circuit using half adders is shown in Figure 9.

Finally, in Figure 10 we show how full and half adders can be used to add the two three-bit integers $(x_2x_1x_0)_2$ and $(y_2y_1y_0)_2$ to produce the sum $(s_3s_2s_1s_0)_2$. Note that s_3, the highest-order bit in the sum, is given by the carry c_2.

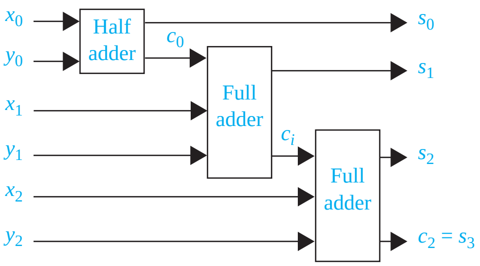
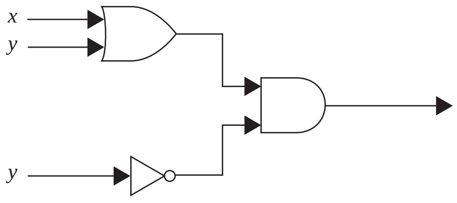


FIGURE 10 Adding Two Three-Bit Integers with Full and Half Adders.

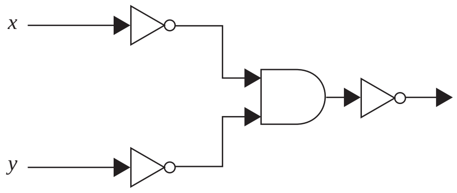
Exercises

In Exercises 1–5 find the output of the given circuit.

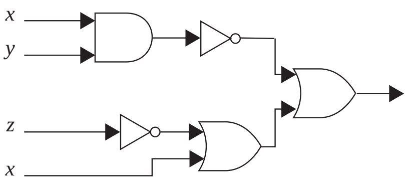
1.



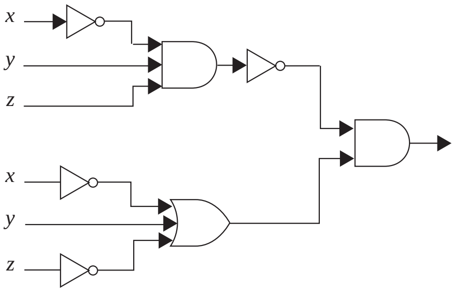
2.



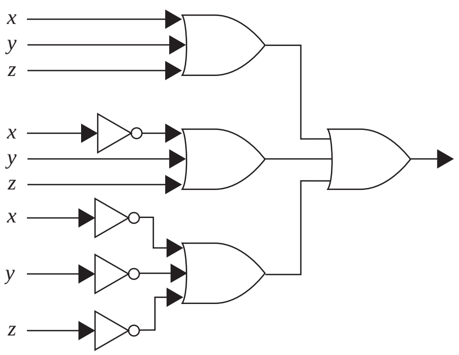
3.



4.



5.



6. Construct circuits from inverters, AND gates, and OR gates to produce these outputs.

a) $\overline{x} + y$ **b)** $\overline{(x + y)x}$

c) $xyz + \overline{x}\,\overline{y}\,\overline{z}$ **d)** $\overline{(\overline{x} + z)(y + \overline{z})}$

7. Design a circuit that implements majority voting for five individuals.

8. Design a circuit for a light fixture controlled by four switches, where flipping one of the switches turns the light on when it is off and turns it off when it is on.

9. Show how the sum of two five-bit integers can be found using full and half adders.

10. Construct a circuit for a half subtractor using AND gates, OR gates, and inverters. A **half subtractor** has two bits as input and produces as output a difference bit and a borrow.

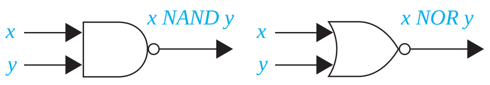
11. Construct a circuit for a full subtractor using AND gates, OR gates, and inverters. A **full subtractor** has two bits and a borrow as input, and produces as output a difference bit and a borrow.

12. Use the circuits from Exercises 10 and 11 to find the difference of two four-bit integers, where the first integer is greater than the second integer.

***13.** Construct a circuit that compares the two-bit integers $(x_1x_0)_2$ and $(y_1y_0)_2$, returning an output of 1 when the first of these numbers is larger and an output of 0 otherwise.

***14.** Construct a circuit that computes the product of the two-bit integers $(x_1x_0)_2$ and $(y_1y_0)_2$. The circuit should have four output bits for the bits in the product.

Two gates that are often used in circuits are NAND and NOR gates. When NAND or NOR gates are used to represent circuits, no other types of gates are needed. The notation for these gates is as follows:



***15.** Use NAND gates to construct circuits with these outputs.

a) $\overline{x}$ **b)** $x + y$

c) xy **d)** $x \oplus y$

***16.** Use NOR gates to construct circuits for the outputs given in Exercise 15.

***17.** Construct a half adder using NAND gates.

***18.** Construct a half adder using NOR gates.

A **multiplexer** is a switching circuit that produces as output one of a set of input bits based on the value of control bits.

19. Construct a multiplexer using AND gates, OR gates, and inverters that has as input the four bits x_0, x_1, x_2, and x_3 and the two control bits c_0 and c_1. Set up the circuit so that x_i is the output, where i is the value of the two-bit integer $(c_1c_0)_2$.

The **depth** of a combinatorial circuit can be defined by specifying that the depth of the initial input is 0 and if a gate has n different inputs at depths $d_1, d_2, \ldots, d_n$, respectively, then its outputs have depth equal to $\max(d_1, d_2, \ldots, d_n) + 1$; this value is also defined to be the depth of the gate. The depth of a combinatorial circuit is the maximum depth of the gates in the circuit.

20. Find the depth of

a) the circuit constructed in Example 2 for majority voting among three people.

b) the circuit constructed in Example 3 for a light controlled by two switches.

c) the half adder shown in Figure 8.

d) the full adder shown in Figure 9.

12.4 Minimization of Circuits

Introduction

The efficiency of a combinational circuit depends on the number and arrangement of its gates. The process of designing a combinational circuit begins with the table specifying the output for each combination of input values. We can always use the sum-of-products expansion of a circuit to find a set of logic gates that will implement this circuit. However, the sum-of-products expansion

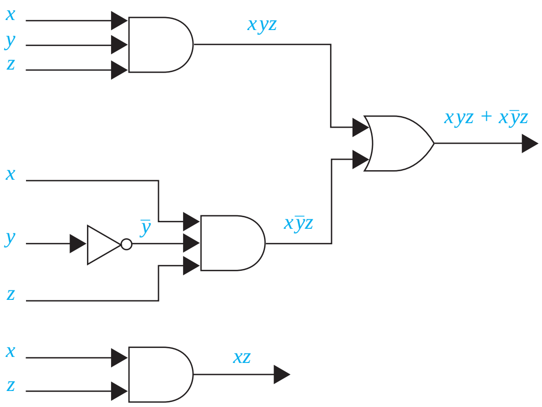


FIGURE 1 **Two Circuits with the Same Output.**

may contain many more terms than are necessary. Terms in a sum-of-products expansion that differ in just one variable, so that in one term this variable occurs and in the other term the complement of this variable occurs, can be combined. For instance, consider the circuit that has output 1 if and only if $x = y = z = 1$ or $x = z = 1$ and $y = 0$. The sum-of-products expansion of this circuit is $xyz + x\overline{y}z$. The two products in this expansion differ in exactly one variable, namely, y. They can be combined as

$$\begin{aligned} xyz + x\overline{y}z &= (y + \overline{y})(xz) \\ &= 1 \cdot (xz) \\ &= xz. \end{aligned}$$

Hence, xz is a Boolean expression with fewer operators that represents the circuit. We show two different implementations of this circuit in Figure 1. The second circuit uses only one gate, whereas the first circuit uses three gates and an inverter.

This example shows that combining terms in the sum-of-products expansion of a circuit leads to a simpler expression for the circuit. We will describe two procedures that simplify sum-of-products expansions.

The goal of both procedures is to produce Boolean sums of Boolean products that represent a Boolean function with the fewest products of literals such that these products contain the fewest literals possible among all sums of products that represent a Boolean function. Finding such a sum of products is called **minimization of the Boolean function**. Minimizing a Boolean function makes it possible to construct a circuit for this function that uses the fewest gates and fewest inputs to the *AND* gates and *OR* gates in the circuit, among all circuits for the Boolean expression we are minimizing.

Until the early 1960s logic gates were individual components. To reduce costs it was important to use the fewest gates to produce a desired output. However, in the mid-1960s, integrated circuit technology was developed that made it possible to combine gates on a single chip. Even though it is now possible to build increasingly complex integrated circuits on chips at low cost, minimization of Boolean functions remains important.

Reducing the number of gates on a chip can lead to a more reliable circuit and can reduce the cost to produce the chip. Also, minimization makes it possible to fit more circuits on the same chip. Furthermore, minimization reduces the number of inputs to gates in a circuit. This reduces the time used by a circuit to compute its output. Moreover, the number of inputs to a gate may be limited because of the particular technology used to build logic gates.

The first procedure we will introduce, known as Karnaugh maps (or K-maps), was designed in the 1950s to help minimize circuits by hand. K-maps are useful in minimizing circuits with up to six variables, although they become rather complex even for five or six variables. The second procedure we will describe, the Quine–McCluskey method, was invented in the 1960s. It automates the process of minimizing combinatorial circuits and can be implemented as a computer program.

COMPLEXITY OF BOOLEAN FUNCTION MINIMIZATION Unfortunately, minimizing Boolean functions with many variables is a computationally intensive problem. It has been shown that this problem is an NP-complete problem (see Section 3.3 and [Ka93]), so the existence of a polynomial-time algorithm for minimizing Boolean circuits is unlikely. The Quine–McCluskey method has exponential complexity. In practice, it can be used only when the number of literals does not exceed ten. Since the 1970s a number of newer algorithms have been developed for minimizing combinatorial circuits (see [Ha93] and [KaBe04]). However, with the best algorithms yet devised, only circuits with no more than 25 variables can be minimized. Also, heuristic (or rule-of-thumb) methods can be used to substantially simplify, but not necessarily minimize, Boolean expressions with a larger number of literals.

Karnaugh Maps

Links

To reduce the number of terms in a Boolean expression representing a circuit, it is necessary to find terms to combine. There is a graphical method, called a **Karnaugh map** or **K-map**, for finding terms to combine for Boolean functions involving a relatively small number of variables. The method we will describe was introduced by Maurice Karnaugh in 1953. His method is based on earlier work by E. W. Veitch. (This method is usually applied only when the function involves six or fewer variables.) K-maps give us a visual method for simplifying sum-of-products expansions; they are not suited for mechanizing this process. We will first illustrate how K-maps are used to simplify expansions of Boolean functions in two variables. We will continue by showing how K-maps can be used to minimize Boolean functions in three variables and then in four variables. Then we will describe the concepts that can be used to extend K-maps to minimize Boolean functions in more than four variables.

There are four possible minterms in the sum-of-products expansion of a Boolean function in the two variables x and y. A K-map for a Boolean function in these two variables consists of four cells, where a 1 is placed in the cell representing a minterm if this minterm is present in the expansion. Cells are said to be **adjacent** if the minterms that they represent differ in exactly one literal. For instance, the cell representing $\overline{x}y$ is adjacent to the cells representing xy and $\overline{x}\,\overline{y}$. The four cells and the terms that they represent are shown in Figure 2.

	y	$\overline{y}$
x	xy	$x\overline{y}$
$\overline{x}$	$\overline{x}y$	$\overline{x}\,\overline{y}$

FIGURE 2 K-maps in Two Variables.

EXAMPLE 1 Find the K-maps for (a) $xy + \overline{x}y$, (b) $x\overline{y} + \overline{x}y$, and (c) $x\overline{y} + \overline{x}y + \overline{x}\,\overline{y}$.

Solution: We include a 1 in a cell when the minterm represented by this cell is present in the sum-of-products expansion. The three K-maps are shown in Figure 3. ◀

We can identify minterms that can be combined from the K-map. Whenever there are 1s in two adjacent cells in the K-map, the minterms represented by these cells can be combined into a product involving just one of the variables. For instance, $x\overline{y}$ and $\overline{x}\,\overline{y}$ are represented by adjacent cells and can be combined into $\overline{y}$, because $x\overline{y} + \overline{x}\,\overline{y} = (x + \overline{x})\overline{y} = \overline{y}$. Moreover, if 1s

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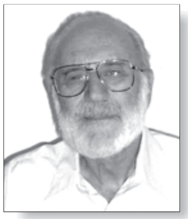
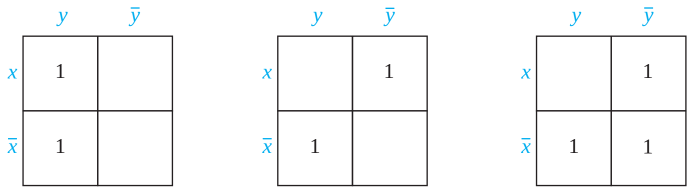


MAURICE KARNAUGH (BORN 1924) Maurice Karnaugh, born in New York City, received his B.S. from the City College of New York and his M.S. and Ph.D. from Yale University. He was a member of the technical staff at Bell Laboratories from 1952 until 1966 and Manager of Research and Development at the Federal Systems Division of AT&T from 1966 to 1970. In 1970 he joined IBM as a member of the research staff. Karnaugh has made fundamental contributions to the application of digital techniques in both computing and telecommunications. His current interests include knowledge-based systems in computers and heuristic search methods.



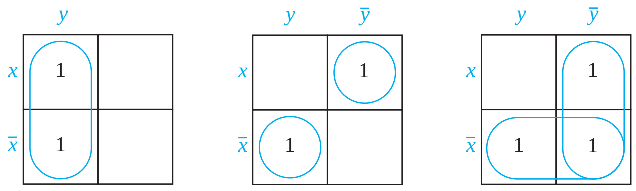
(a) (b) (c)

FIGURE 3 K-maps for the Sum-of-Products Expansions in Example 1.

are in all four cells, the four minterms can be combined into one term, namely, the Boolean expression 1 that involves none of the variables. We circle blocks of cells in the K-map that represent minterms that can be combined and then find the corresponding sum of products. The goal is to identify the largest possible blocks, and to cover all the 1s with the fewest blocks using the largest blocks first and always using the largest possible blocks.

EXAMPLE 2 Simplify the sum-of-products expansions given in Example 1.

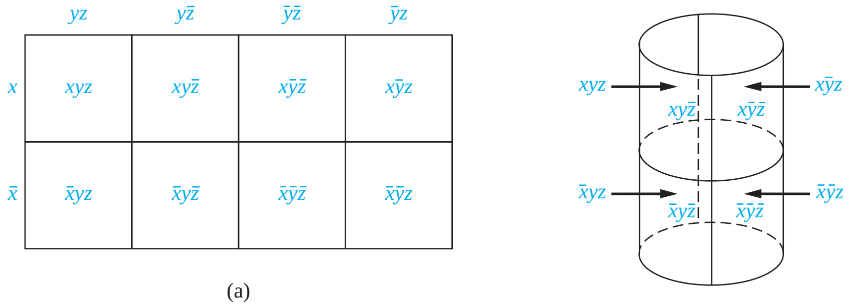
Solution: The grouping of minterms is shown in Figure 4 using the K-maps for these expansions. Minimal expansions for these sums-of-products are (a) y, (b) $x\bar{y} + \bar{x}y$, and (c) $\bar{x} + \bar{y}$. ◀



(a) (b) (c)

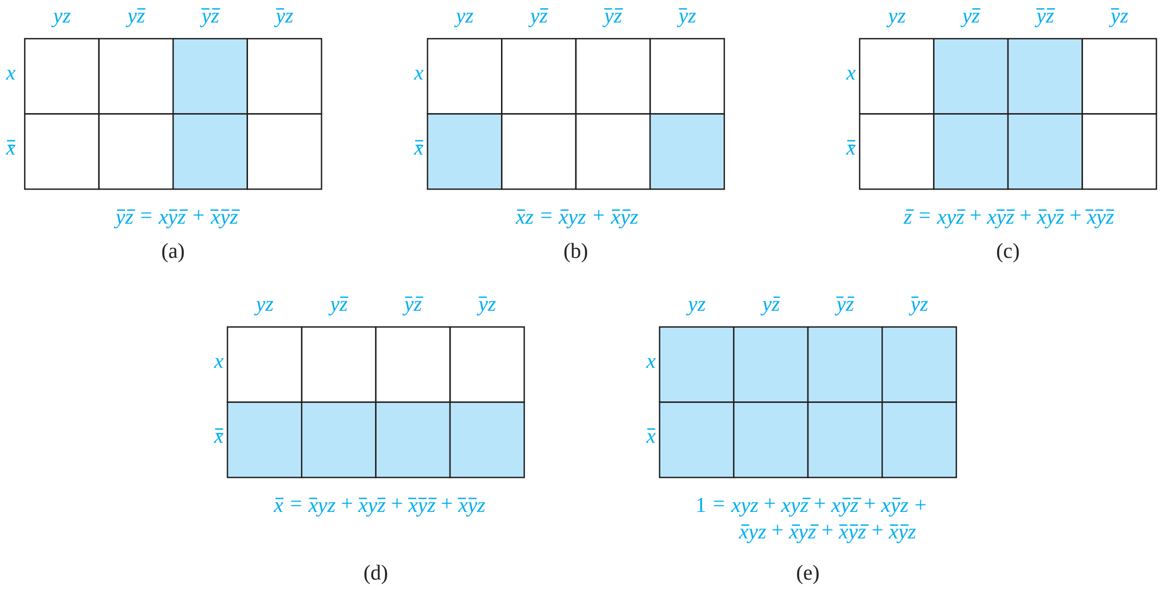
FIGURE 4 Simplifying the Sum-of-Products Expansions from Example 2.

A K-map in three variables is a rectangle divided into eight cells. The cells represent the eight possible minterms in three variables. Two cells are said to be adjacent if the minterms that they represent differ in exactly one literal. One of the ways to form a K-map in three variables is shown in Figure 5(a). This K-map can be thought of as lying on a cylinder, as shown in Figure 5(b). On the cylinder, two cells have a common border if and only if they are adjacent.



(a) (b)

FIGURE 5 K-maps in Three Variables.



$\bar{y}\bar{z} = x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}$

(a)

$\bar{x}z = \bar{x}yz + \bar{x}\bar{y}z$

(b)

$\bar{z} = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$

(c)

$\bar{x} = \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$

(d)

$1 = xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z +$
$\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$

(e)

FIGURE 6 Blocks in K-maps in Three Variables.

To simplify a sum-of-products expansion in three variables, we use the K-map to identify blocks of minterms that can be combined. Blocks of two adjacent cells represent pairs of minterms that can be combined into a product of two literals; 2 × 2 and 4 × 1 blocks of cells represent minterms that can be combined into a single literal; and the block of all eight cells represents a product of no literals, namely, the function 1. In Figure 6, 1 × 2, 2 × 1, 2 × 2, 4×1, and 4 × 2 blocks and the products they represent are shown.

The product of literals corresponding to a block of all 1s in the K-map is called an **implicant** of the function being minimized. It is called a **prime implicant** if this block of 1s is not contained in a larger block of 1s representing the product of fewer literals than in this product.

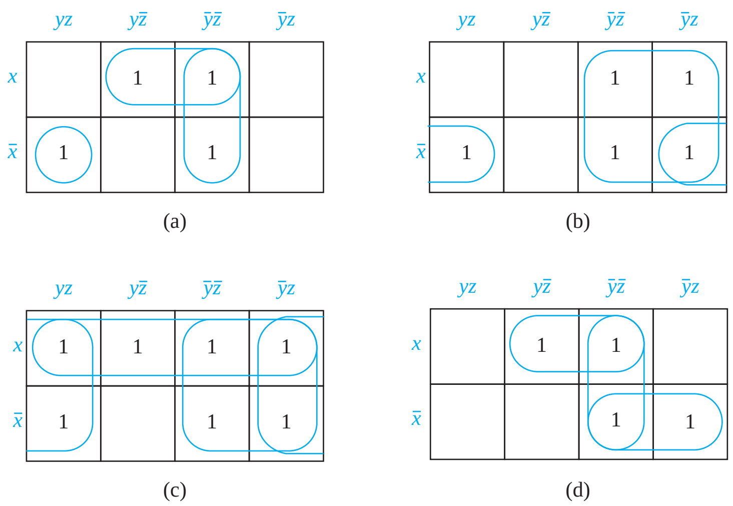
The goal is to identify the largest possible blocks in the map and cover all the 1s in the map with the least number of blocks, using the largest blocks first. The largest possible blocks are always chosen, but we must always choose a block if it is the only block of 1s covering a 1 in the K-map. Such a block represents an **essential prime implicant**. By covering all the 1s in the map with blocks corresponding to prime implicants we can express the sum of products as a sum of prime implicants. Note that there may be more than one way to cover all the 1s using the least number of blocks.

Example 3 illustrates how K-maps in three variables are used.

EXAMPLE 3 Use K-maps to minimize these sum-of-products expansions.

(a) $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$
(b) $x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$
(c) $xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$
(d) $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

Solution: The K-maps for these sum-of-products expansions are shown in Figure 7. The grouping of blocks shows that minimal expansions into Boolean sums of Boolean products are (a) $x\bar{z} + \bar{y}\bar{z} + \bar{x}yz$, (b) $\bar{y} + \bar{x}z$, (c) $x + \bar{y} + z$, and (d) $x\bar{z} + \bar{x}\bar{y}$. In part (d) note that the prime implicants $x\bar{z}$ and $\bar{x}\bar{y}$ are essential prime implicants, but the prime implicant $\bar{y}\bar{z}$ is a prime implicant that is not essential, because the cells it covers are covered by the other two prime implicants. ◀



yz $y\bar{z}$ $\bar{y}\bar{z}$ $\bar{y}z$
x $\bar{x}$
1 1
1 1
(a)
yz $y\bar{z}$ $\bar{y}\bar{z}$ $\bar{y}z$
x $\bar{x}$
1 1
1 1 1
(b)
yz $y\bar{z}$ $\bar{y}\bar{z}$ $\bar{y}z$
x $\bar{x}$
1 1 1 1
1 1 1
(c)
yz $y\bar{z}$ $\bar{y}\bar{z}$ $\bar{y}z$
x $\bar{x}$
1 1
1 1
(d)

FIGURE 7 **Using K-maps in Three Variables.**

A K-map in four variables is a square that is divided into 16 cells. The cells represent the 16 possible minterms in four variables. One of the ways to form a K-map in four variables is shown in Figure 8.

Two cells are adjacent if and only if the minterms they represent differ in one literal. Consequently, each cell is adjacent to four other cells. The K-map of a sum-of-products expansion in four variables can be thought of as lying on a torus, so that adjacent cells have a common boundary (see Exercise 28). The simplification of a sum-of-products expansion in four variables is carried out by identifying those blocks of 2, 4, 8, or 16 cells that represent minterms that can be combined. Each cell representing a minterm must either be used to form a product using fewer literals, or be included in the expansion. In Figure 9 some examples of blocks that represent products of three literals, products of two literals, and a single literal are illustrated.

As is the case in K-maps in two and three variables, the goal is to identify the largest blocks of 1s in the map that correspond to the prime implicants and to cover all the 1s using the fewest blocks needed, using the largest blocks first. The largest possible blocks are always used. Example 4 illustrates how K-maps in four variables are used.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	$wxyz$	$wxy\bar{z}$	$wx\bar{y}\bar{z}$	$wx\bar{y}z$
$w\bar{x}$	$w\bar{x}yz$	$w\bar{x}y\bar{z}$	$w\bar{x}\bar{y}\bar{z}$	$w\bar{x}\bar{y}z$
$\bar{w}\bar{x}$	$\bar{w}\bar{x}yz$	$\bar{w}\bar{x}y\bar{z}$	$\bar{w}\bar{x}\bar{y}\bar{z}$	$\bar{w}\bar{x}\bar{y}z$
$\bar{w}x$	$\bar{w}xyz$	$\bar{w}xy\bar{z}$	$\bar{w}x\bar{y}\bar{z}$	$\bar{w}x\bar{y}z$

FIGURE 8 **K-maps in Four Variables.**

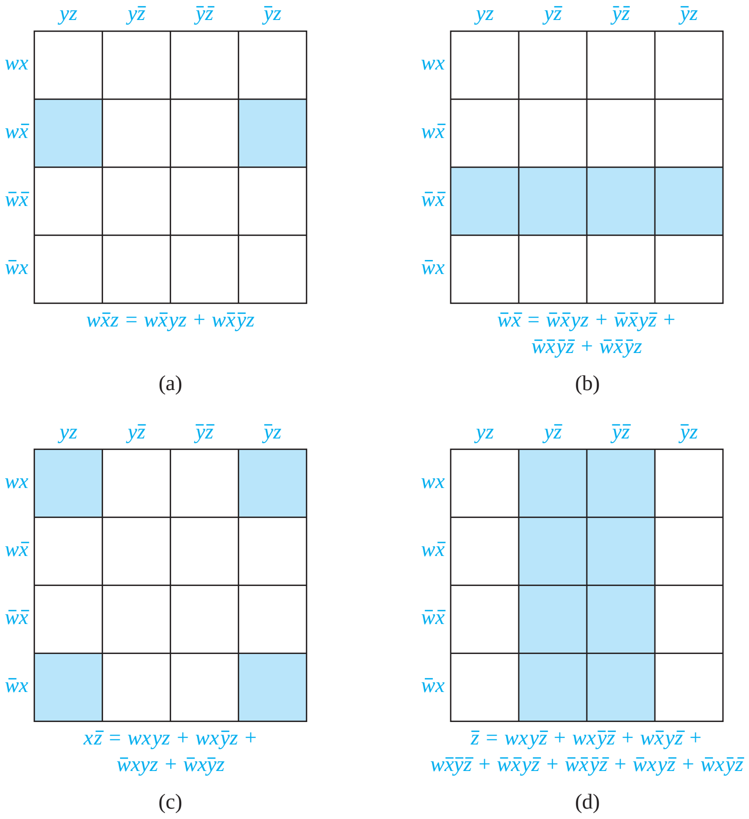

yz
$y\bar{z}$
$\bar{y}\bar{z}$
$\bar{y}z$
wx
$w\bar{x}$
$\bar{w}\bar{x}$
$\bar{w}x$
$w\bar{x}z = w\bar{x}yz + w\bar{x}\bar{y}z$
(a)
$\bar{w}\bar{x} = \bar{w}\bar{x}yz + \bar{w}\bar{x}y\bar{z} +$
$\bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}z$
(b)
$x\bar{z} = wxyz + wx\bar{y}z +$
$\bar{w}xyz + \bar{w}x\bar{y}z$
(c)
$\bar{z} = wxy\bar{z} + wx\bar{y}\bar{z} + w\bar{x}y\bar{z} +$
$w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}xy\bar{z} + \bar{w}x\bar{y}\bar{z}$
(d)

FIGURE 9 **Blocks in K-maps in Four Variables.**

EXAMPLE 4 Use K-maps to simplify these sum-of-products expansions.

(a) $wxyz + wxy\bar{z} + wx\bar{y}\,\bar{z} + w\bar{x}yz + w\bar{x}\,\bar{y}z + w\bar{x}\,\bar{y}\,\bar{z} + \bar{w}x\bar{y}z +$
$\bar{w}\,\bar{x}yz + \bar{w}\,\bar{x}y\bar{z}$

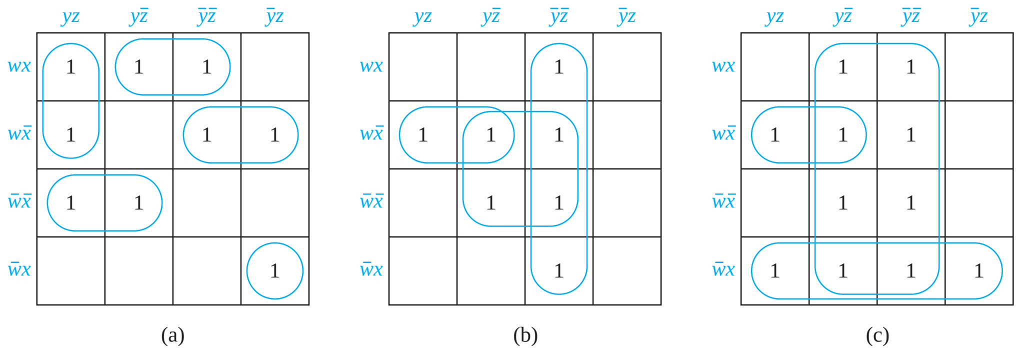
(b) $wx\bar{y}\,\bar{z} + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\,\bar{y}\,\bar{z} + \bar{w}x\bar{y}\,\bar{z} + \bar{w}\,\bar{x}y\bar{z} + \bar{w}\,\bar{x}\,\bar{y}\,\bar{z}$

(c) $wxy\bar{z} + wx\bar{y}\,\bar{z} + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\,\bar{y}\,\bar{z} + \bar{w}xyz + \bar{w}xy\bar{z} + \bar{w}x\bar{y}\,\bar{z} +$
$\bar{w}x\bar{y}z + \bar{w}\,\bar{x}y\bar{z} + \bar{w}\,\bar{x}\,\bar{y}\,\bar{z}$

Solution: The K-maps for these expansions are shown in Figure 10. Using the blocks shown leads to the sum of products (a) $wyz + wx\bar{z} + w\bar{x}\,\bar{y} + \bar{w}\,\bar{x}y + \bar{w}x\bar{y}z$, (b) $\bar{y}\,\bar{z} + w\bar{x}y + \bar{x}\,\bar{z}$, and (c) $\bar{z} + \bar{w}x + w\bar{x}y$. The reader should determine whether there are other choices of blocks in each part that lead to different sums of products representing these Boolean functions. ◀

K-maps can realistically be used to minimize Boolean functions with five or six variables, but beyond that, they are rarely used because they become extremely complicated. However, the concepts used in K-maps play an important role in newer algorithms. Furthermore, mastering these concepts helps you understand these newer algorithms and the computer-aided design (CAD) programs that implement them. As we develop these concepts, we will be able to illustrate them by referring back to our discussion of minimization of Boolean functions in three and in four variables.

The K-maps we used to minimize Boolean functions in two, three, and four variables are built using 2 × 2, 2 × 4, and 4 × 4 rectangles, respectively. Furthermore, corresponding cells in the top row and bottom row and in the leftmost column and rightmost column in each of these



(a)

(b)

(c)

FIGURE 10 Using K-maps in Four Variables.

cases are considered adjacent because they represent minterms differing in only one literal. We can build K-maps for minimizing Boolean functions in more than four variables in a similar way. We use a rectangle containing $2^{\lfloor n/2 \rfloor}$ rows and $2^{\lceil n/2 \rceil}$ columns. (These K-maps contain 2^n cells because $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.) The rows and columns need to be positioned so that the cells representing minterms differing in just one literal are adjacent or are considered adjacent by specifying additional adjacencies of rows and columns. To help (but not entirely) achieve this, the rows and columns of a K-map are arranged using Gray codes (see Section 10.5), where we associate bit strings and products by specifying that a 1 corresponds to the appearance of a variable and a 0 with the appearance of its complement. For example, in a 10-dimensional K-map, the Gray code 01110 used to label a row corresponds to the product $\overline{x}_1 x_2 x_3 x_4 \overline{x}_5$.

EXAMPLE 5 The K-maps we used to minimize Boolean functions with four variables have four rows and four columns. Both the rows and the columns are arranged using the Gray code 11,10,00,01. The rows represent products wx, $w\overline{x}$, $\overline{w}\,\overline{x}$, and $\overline{w}x$, respectively, and the columns correspond to the products yz, $y\overline{z}$, $\overline{y}\,\overline{z}$, and $\overline{y}z$, respectively. Using Gray codes and considering cells adjacent in the first and last rows and in the first and last columns, we ensured that minterms that differ in only one variable are always adjacent.

EXAMPLE 6 To minimize Boolean functions in five variables we use K-maps with $2^3 = 8$ columns and $2^2 = 4$ rows. We label the four rows using the Gray code 11,10,00,01, corresponding to x_1x_2, $x_1\overline{x}_2$, $\overline{x}_1\overline{x}_2$, and $\overline{x}_1x_2$, respectively. We label the eight columns using the Gray code 111,110,100,101,001,000,010,011 corresponding to the terms $x_3x_4x_5$, $x_3x_4\overline{x}_5$, $x_3\overline{x}_4\overline{x}_5$, $x_3\overline{x}_4x_5$, $\overline{x}_3\overline{x}_4x_5$, $\overline{x}_3\overline{x}_4\overline{x}_5$, $\overline{x}_3x_4\overline{x}_5$, and $\overline{x}_3x_4x_5$, respectively. Using Gray codes to label columns and rows ensures that the minterms represented by adjacent cells differ in only one variable. However, to make sure all cells representing products that differ in only one variable are considered adjacent, we consider cells in the top and bottom rows to be adjacent, as well as cells in the first and eighth columns, the first and fourth columns, the second and seventh columns, the third and sixth columns, and the fifth and eighth columns (as the reader should verify).

To use a K-map to minimize a Boolean function in n variables, we first draw a K-map of the appropriate size. We place 1s in all cells corresponding to minterms in the sum-of-products expansion of this function. We then identify all prime implicants of the Boolean function. To do this we look for the blocks consisting of 2^k clustered cells all containing a 1, where $1 \leq k \leq n$. These blocks correspond to the product of $n - k$ literals. (Exercise 33 asks the reader to verify this.) Furthermore, a block of 2^k cells each containing a 1 not contained in a block of 2^{k+1} cells each containing a 1 represents a prime implicant. The reason that this implicant is a prime implicant is that no product obtained by deleting a literal is also represented by a block of cells all containing 1s.

EXAMPLE 7 A block of eight cells representing a product of two literals in a K-map for minimizing Boolean functions in five variables all containing 1s is a prime implicant if it is not contained in a larger block of 16 cells all containing 1s representing a single literal. ◀

Once all prime implicants have been identified, the goal is to find the smallest possible subset of these prime implicants with the property that the cells representing these prime implicants cover all the cells containing a 1 in the K-map. We begin by selecting the essential prime implicants because each of these is represented by a block that covers a cell containing a 1 that is not covered by any other prime implicant. We add additional prime implicants to ensure that all 1s in the K-map are covered. When the number of variables is large, this last step can become exceedingly complicated.

Don’t Care Conditions

Links

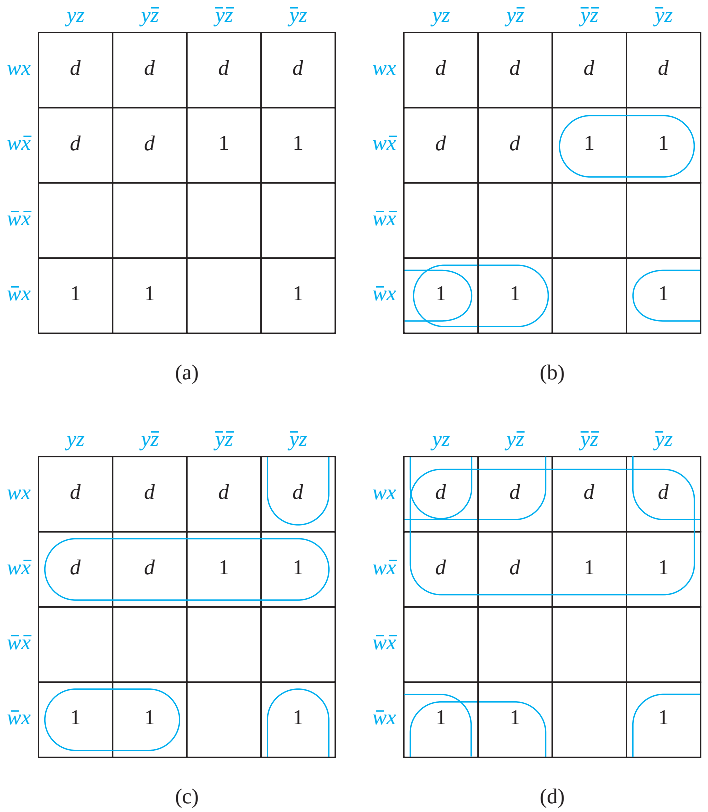
In some circuits we care only about the output for some combinations of input values, because other combinations of input values are not possible or never occur. This gives us freedom in producing a simple circuit with the desired output because the output values for all those combinations that never occur can be arbitrarily chosen. The values of the function for these combinations are called ***don’t care* conditions**. A d is used in a K-map to mark those combinations of values of the variables for which the function can be arbitrarily assigned. In the minimization process we can assign 1s as values to those combinations of the input values that lead to the largest blocks in the K-map. This is illustrated in Example 8.

EXAMPLE 8 One way to code decimal expansions using bits is to use the four bits of the binary expansion of each digit in the decimal expansion. For instance, 873 is encoded as 100001110011. This encoding of a decimal expansion is called a **binary coded decimal expansion**. Because there are 16 blocks of four bits and only 10 decimal digits, there are six combinations of four bits that are not used to encode digits. Suppose that a circuit is to be built that produces an output of 1 if the decimal digit is 5 or greater and an output of 0 if the decimal digit is less than 5. How can this circuit be simply built using OR gates, AND gates, and inverters?

Solution: Let $F(w, x, y, z)$ denote the output of the circuit, where $wxyz$ is a binary expansion of a decimal digit. The values of F are shown in Table 1. The K-map for F, with ds in the *don’t care* positions, is shown in Figure 11(a). We can either include or exclude squares with ds from blocks. This gives us many possible choices for the blocks. For example, excluding all squares with ds and forming blocks, as shown in Figure 11(b), produces the expression $w\overline{x}\,\overline{y} + \overline{w}xy + \overline{w}xz$. Including some of the ds and excluding others and forming blocks, as

TABLE 1

Digit	*w*	*x*	*y*	*z*	*F*
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	0
4	0	1	0	0	0
5	0	1	0	1	1
6	0	1	1	0	1
7	0	1	1	1	1
8	1	0	0	0	1
9	1	0	0	1	1



	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	d	d	d	d
$w\bar{x}$	d	d	1	1
$\bar{w}\bar{x}$				
$\bar{w}x$	1	1		1

(a)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	d	d	d	d
$w\bar{x}$	d	d	1	1
$\bar{w}\bar{x}$				
$\bar{w}x$	1	1		1

(b)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	d	d	d	d
$w\bar{x}$	d	d	1	1
$\bar{w}\bar{x}$				
$\bar{w}x$	1	1		1

(c)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	d	d	d	d
$w\bar{x}$	d	d	1	1
$\bar{w}\bar{x}$				
$\bar{w}x$	1	1		1

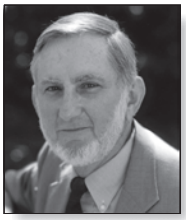
(d)

FIGURE 11 The K-map for *F* Showing Its *Don't Care* Positions.

shown in Figure 11(c), produces the expression $w\bar{x} + \bar{w}xy + x\bar{y}z$. Finally, including all the ds and using the blocks shown in Figure 11(d) produces the simplest sum-of-products expansion possible, namely, $F(x, y, z) = w + xy + xz$. ◀

The Quine–McCluskey Method

We have seen that K-maps can be used to produce minimal expansions of Boolean functions as Boolean sums of Boolean products. However, K-maps are awkward to use when there are more than four variables. Furthermore, the use of K-maps relies on visual inspection to identify terms to group. For these reasons there is a need for a procedure for simplifying sum-of-products expansions that can be mechanized. The Quine–McCluskey method is such a procedure. It can be used for Boolean functions in any number of variables. It was developed in the 1950s by W. V. Quine and E. J. McCluskey, Jr. Basically, the Quine–McCluskey method consists of two



EDWARD J. McCLUSKEY (BORN 1929) Edward McCluskey attended Bowdoin College and M.I.T., where he received his doctorate in electrical engineering in 1956. He joined Bell Telephone Laboratories in 1955, remaining there until 1959. McCluskey was professor of electrical engineering at Princeton University from 1959 until 1966, also serving as director of the Computer Center at Princeton from 1961 to 1966. In 1967 he took a position as professor of computer science and electrical engineering at Stanford University, where he also served as director of the Digital Systems Laboratory from 1969 to 1978. McCluskey has worked in a variety of areas in computer science, including fault-tolerant computing, computer architecture, testing, and logic design. He is the director of the Center for Reliable Computing at Stanford University where he is now an emeritus professor. McCluskey is also an ACM Fellow.

TABLE 2

Minterm	*Bit String*	*Number of 1s*
xyz	111	3
$x\overline{y}z$	101	2
$\overline{x}yz$	011	2
$\overline{x}\,\overline{y}z$	001	1
$\overline{x}\,\overline{y}\,\overline{z}$	000	0

parts. The first part finds those terms that are candidates for inclusion in a minimal expansion as a Boolean sum of Boolean products. The second part determines which of these terms to actually use. We will use Example 9 to illustrate how, by successively combining implicants into implicants with one fewer literal, this procedure works.

EXAMPLE 9 We will show how the Quine–McCluskey method can be used to find a minimal expansion equivalent to

$$xyz + x\overline{y}z + \overline{x}yz + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}.$$

We will represent the minterms in this expansion by bit strings. The first bit will be 1 if x occurs and 0 if $\overline{x}$ occurs. The second bit will be 1 if y occurs and 0 if $\overline{y}$ occurs. The third bit will be 1 if z occurs and 0 if $\overline{z}$ occurs. We then group these terms according to the number of 1s in the corresponding bit strings. This information is shown in Table 2.

Minterms that can be combined are those that differ in exactly one literal. Hence, two terms that can be combined differ by exactly one in the number of 1s in the bit strings that represent them. When two minterms are combined into a product, this product contains two literals. A product in two literals is represented using a dash to denote the variable that does not occur. For instance, the minterms $x\overline{y}z$ and $\overline{x}\,\overline{y}z$, represented by bit strings 101 and 001, can be combined into $\overline{y}z$, represented by the string –01. All pairs of minterms that can be combined and the product formed from these combinations are shown in Table 3.

Next, all pairs of products of two literals that can be combined are combined into one literal. Two such products can be combined if they contain literals for the same two variables, and literals for only one of the two variables differ. In terms of the strings representing the products, these strings must have a dash in the same position and must differ in exactly one of the other two slots. We can combine the products yz and $\overline{y}z$, represented by the strings –11 and –01, into z, represented by the string – –1. We show all the combinations of terms that can be formed in this way in Table 3.

TABLE 3

			Step 1			*Step 2*		
	Term	*Bit String*		*Term*	*String*		*Term*	*String*
1	xyz	111	(1,2)	xz	1–1	(1,2,3,4)	z	– –1
2	$x\overline{y}z$	101	(1,3)	yz	–11			
3	$\overline{x}yz$	011	(2,4)	$\overline{y}z$	–01			
4	$\overline{x}\,\overline{y}z$	001	(3,4)	$\overline{x}z$	0–1			
5	$\overline{x}\,\overline{y}\,\overline{z}$	000	(4,5)	$\overline{x}\,\overline{y}$	00–			

TABLE 4

	xyz	$x\overline{y}z$	$\overline{x}yz$	$\overline{x}\,\overline{y}z$	$\overline{x}\,\overline{y}\,\overline{z}$
z	X	X	X	X	
$\overline{x}\,\overline{y}$				X	X

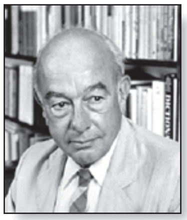
In Table 3 we also indicate which terms have been used to form products with fewer literals; these terms will not be needed in a minimal expansion. The next step is to identify a minimal set of products needed to represent the Boolean function. We begin with all those products that were not used to construct products with fewer literals. Next, we form Table 4, which has a row for each candidate product formed by combining original terms, and a column for each original term; and we put an X in a position if the original term in the sum-of-products expansion was used to form this candidate product. In this case, we say that the candidate product **covers** the original minterm. We need to include at least one product that covers each of the original minterms. Consequently, whenever there is only one X in a column in the table, the product corresponding to the row this X is in must be used. From Table 4 we see that both z and $\overline{x}\,\overline{y}$ are needed. Hence, the final answer is $z + \overline{x}\,\overline{y}$. ◀

As was illustrated in Example 9, the Quine–McCluskey method uses this sequence of steps to simplify a sum-of-products expression.

1. Express each minterm in n variables by a bit string of length n with a 1 in the ith position if x_i occurs and a 0 in this position if $\overline{x}_i$ occurs.

2. Group the bit strings according to the number of 1s in them.

3. Determine all products in $n - 1$ variables that can be formed by taking the Boolean sum of minterms in the expansion. Minterms that can be combined are represented by bit strings that differ in exactly one position. Represent these products in $n - 1$ variables with strings that have a 1 in the ith position if x_i occurs in the product, a 0 in this position if $\overline{x}_i$ occurs, and a dash in this position if there is no literal involving x_i in the product.



WILLARD VAN ORMAN QUINE (1908–2000) Willard Quine, born in Akron, Ohio, attended Oberlin College and later Harvard University, where he received his Ph.D. in philosophy in 1932. He became a Junior Fellow at Harvard in 1933 and was appointed to a position on the faculty there in 1936. He remained at Harvard his entire professional life, except for World War II, when he worked for the U.S. Navy decrypting messages from German submarines. Quine was always interested in algorithms, but not in hardware. He arrived at his discovery of what is now called the Quine–McCluskey method as a device for teaching mathematical logic, rather than as a method for simplifying switching circuits. Quine was one of the most famous philosophers of the twentieth century. He made fundamental contributions to the theory of knowledge, mathematical logic and set theory, and the philosophies of logic and language. His books, including *New Foundations of Mathematical Logic* published in 1937 and *Word and Object* published in 1960, have had a profound impact. Quine retired from Harvard in 1978 but continued to commute from his home in Beacon Hill to his office there. He used the 1927 Remington typewriter on which he prepared his doctoral thesis for his entire life. He even had an operation performed on this machine to add a few special symbols, removing the second period, the second comma, and the question mark. When asked whether he missed the question mark, he replied, "Well, you see, I deal in certainties." There is even a word *quine*, defined in the *New Hacker's Dictionary* as a program that generates a copy of its own source code as its complete output. Producing the shortest possible quine in a given programming language is a popular puzzle for hackers.

4. Determine all products in $n - 2$ variables that can be formed by taking the Boolean sum of the products in $n - 1$ variables found in the previous step. Products in $n - 1$ variables that can be combined are represented by bit strings that have a dash in the same position and differ in exactly one position.
5. Continue combining Boolean products into products in fewer variables as long as possible.
6. Find all the Boolean products that arose that were not used to form a Boolean product in one fewer literal.
7. Find the smallest set of these Boolean products such that the sum of these products represents the Boolean function. This is done by forming a table showing which minterms are covered by which products. Every minterm must be covered by at least one product. The first step in using this table is to find all essential prime implicants. Each essential prime implicant must be included because it is the only prime implicant that covers one of the minterms. Once we have found essential prime implicants, we can simplify the table by eliminating the rows for minterms covered by these prime implicants. Furthermore, we can eliminate any prime implicants that cover a subset of minterms covered by another prime implicant (as the reader should verify). Moreover, we can eliminate from the table the row for a minterm if there is another minterm that is covered by a subset of the prime implicants that cover this minterm. This process of identifying essential prime implicants that must be included, followed by eliminating redundant prime implicants and identifying minterms that can be ignored, is iterated until the table does not change. At this point we use a backtracking procedure to find the optimal solution where we add prime implicants to the cover to find possible solutions, which we compare to the best solution found so far at each step.

A final example will illustrate how this procedure is used to simplify a sum-of-products expansion in four variables.

EXAMPLE 10 Use the Quine–McCluskey method to simplify the sum-of-products expansion $wxy\overline{z} + w\overline{x}yz + w\overline{x}y\overline{z} + \overline{w}xyz + \overline{w}x\overline{y}z + \overline{w}\,\overline{x}yz + \overline{w}\,\overline{x}\,\overline{y}z$.

Solution: We first represent the minterms by bit strings and then group these terms together according to the number of 1s in the bit strings. This is shown in Table 5. All the Boolean products that can be formed by taking Boolean sums of these products are shown in Table 6.

The only products that were not used to form products in fewer variables are $\overline{w}z$, $wy\overline{z}$, $w\overline{x}y$, and $\overline{x}yz$. In Table 7 we show the minterms covered by each of these products. To cover these minterms we must include $\overline{w}z$ and $wy\overline{z}$, because these products are the only products that cover $\overline{w}xyz$ and $wxy\overline{z}$, respectively. Once these two products are included, we see that only one of the two products left is needed. Consequently, we can take either $\overline{w}z + wy\overline{z} + w\overline{x}y$ or $\overline{w}z + wy\overline{z} + \overline{x}yz$ as the final answer. ◀

TABLE 5

Term	*Bit String*	*Number of 1s*
$wxy\overline{z}$	1110	3
$w\overline{x}yz$	1011	3
$\overline{w}xyz$	0111	3
$w\overline{x}y\overline{z}$	1010	2
$\overline{w}x\overline{y}z$	0101	2
$\overline{w}\,\overline{x}yz$	0011	2
$\overline{w}\,\overline{x}\,\overline{y}z$	0001	1

TABLE 6

			Step 1			Step 2		
	Term	*Bit String*		*Term*	*String*		*Term*	*String*
1	$wxy\overline{z}$	1110	(1,4)	$wy\overline{z}$	1–10	(3,5,6,7)	$\overline{w}z$	0 – –1
2	$w\overline{x}yz$	1011	(2,4)	$w\overline{x}y$	101–			
3	$\overline{w}xyz$	0111	(2,6)	$\overline{x}yz$	–011			
4	$w\overline{x}y\overline{z}$	1010	(3,5)	$\overline{w}xz$	01–1			
5	$\overline{w}x\overline{y}z$	0101	(3,6)	$\overline{w}yz$	0–11			
6	$\overline{w}\,\overline{x}yz$	0011	(5,7)	$\overline{w}\,\overline{y}z$	0–01			
7	$\overline{w}\,\overline{x}\,\overline{y}z$	0001	(6,7)	$\overline{w}\,\overline{x}z$	00–1			

TABLE 7

	$wxy\overline{z}$	$w\overline{x}yz$	$\overline{w}xyz$	$w\overline{x}y\overline{z}$	$\overline{w}x\overline{y}z$	$\overline{w}\,\overline{x}yz$	$\overline{w}\,\overline{x}\,\overline{y}z$
$\overline{w}z$			X		X	X	X
$wy\overline{z}$	X			X			
$w\overline{x}y$		X		X			
$\overline{x}yz$		X				X	

Exercises

1. a) Draw a K-map for a function in two variables and put a 1 in the cell representing $\overline{x}y$.

b) What are the minterms represented by cells adjacent to this cell?

2. Find the sum-of-products expansions represented by each of these K-maps.

a)

	y	$\overline{y}$
x	1	
$\overline{x}$	1	1

b)

	y	$\overline{y}$
x	1	1
$\overline{x}$		

c)

	y	$\overline{y}$
x	1	1
$\overline{x}$	1	1

3. Draw the K-maps of these sum-of-products expansions in two variables.

a) $x\overline{y}$ **b)** $xy + \overline{x}\,\overline{y}$

c) $xy + x\overline{y} + \overline{x}y + \overline{x}\,\overline{y}$

4. Use a K-map to find a minimal expansion as a Boolean sum of Boolean products of each of these functions of the Boolean variables x and y.

a) $\overline{x}y + \overline{x}\,\overline{y}$

b) $xy + x\overline{y}$

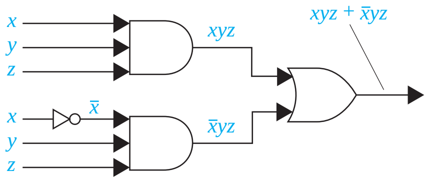
c) $xy + x\overline{y} + \overline{x}y + \overline{x}\,\overline{y}$

5. a) Draw a K-map for a function in three variables. Put a 1 in the cell that represents $\overline{x}y\overline{z}$.

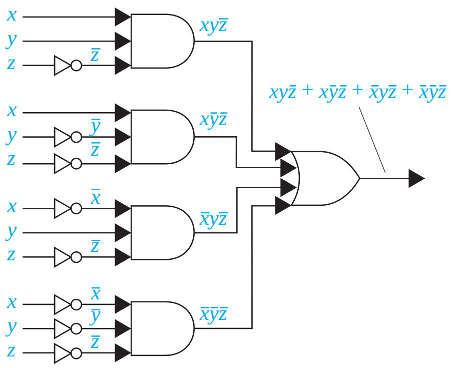
b) Which minterms are represented by cells adjacent to this cell?

6. Use K-maps to find simpler circuits with the same output as each of the circuits shown.

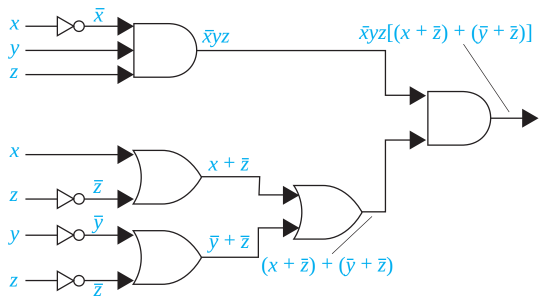
a)



b)



c)



x
y
z
$\overline{x}$
$\overline{x}yz$
$\overline{x}yz[(x + \overline{z}) + (\overline{y} + \overline{z})]$
x
z
$\overline{z}$
y
$\overline{y}$
z
$\overline{z}$
$x + \overline{z}$
$\overline{y} + \overline{z}$
$(x + \overline{z}) + (\overline{y} + \overline{z})$

7. Draw the K-maps of these sum-of-products expansions in three variables.

a) $x\overline{y}\,\overline{z}$ **b)** $\overline{x}yz + \overline{x}\,\overline{y}\,\overline{z}$

c) $xyz + xy\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z$

8. Construct a K-map for $F(x, y, z) = xz + yz + xy\overline{z}$. Use this K-map to find the implicants, prime implicants, and essential prime implicants of $F(x, y, z)$.

9. Construct a K-map for $F(x, y, z) = x\overline{z} + xyz + y\overline{z}$. Use this K-map to find the implicants, prime implicants, and essential prime implicants of $F(x, y, z)$.

10. Draw the 3-cube Q_3 and label each vertex with the minterm in the Boolean variables x, y, and z associated with the bit string represented by this vertex. For each literal in these variables indicate the 2-cube Q_2 that is a subgraph of Q_3 and represents this literal.

11. Draw the 4-cube Q_4 and label each vertex with the minterm in the Boolean variables w, x, y, and z associated with the bit string represented by this vertex. For each literal in these variables, indicate which 3-cube Q_3 that is a subgraph of Q_4 represents this literal. Indicate which 2-cube Q_2 that is a subgraph of Q_4 represents the products wz, $\overline{x}y$, and $\overline{y}\,\overline{z}$.

12. Use a K-map to find a minimal expansion as a Boolean sum of Boolean products of each of these functions in the variables x, y, and z.

a) $\overline{x}yz + \overline{x}\,\overline{y}z$

b) $xyz + xy\overline{z} + \overline{x}yz + \overline{x}y\overline{z}$

c) $xy\overline{z} + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}\,\overline{y}z$

d) $xyz + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z}$

13. a) Draw a K-map for a function in four variables. Put a 1 in the cell that represents $\overline{w}xy\overline{z}$.

b) Which minterms are represented by cells adjacent to this cell?

14. Use a K-map to find a minimal expansion as a Boolean sum of Boolean products of each of these functions in the variables w, x, y, and z.

a) $wxyz + wx\overline{y}z + wx\overline{y}\,\overline{z} + w\overline{x}y\overline{z} + w\overline{x}\,\overline{y}z$

b) $wxy\overline{z} + wx\overline{y}z + w\overline{x}yz + \overline{w}x\overline{y}z + \overline{w}\,\overline{x}y\overline{z} + \overline{w}\,\overline{x}\,\overline{y}z$

c) $wxyz + wxy\overline{z} + wx\overline{y}z + w\overline{x}\,\overline{y}z + w\overline{x}\,\overline{y}\,\overline{z} +$
$\overline{w}x\overline{y}z + \overline{w}\,\overline{x}y\overline{z} + \overline{w}\,\overline{x}\,\overline{y}z$

d) $wxyz + wxy\overline{z} + wx\overline{y}z + w\overline{x}yz + w\overline{x}y\overline{z} +$
$\overline{w}xyz + \overline{w}\,\overline{x}yz + \overline{w}\,\overline{x}y\overline{z} + \overline{w}\,\overline{x}\,\overline{y}z$

15. Find the cells in a K-map for Boolean functions with five variables that correspond to each of these products.

a) $x_1x_2x_3x_4$ **b)** $\overline{x}_1x_3x_5$ **c)** x_2x_4

d) $\overline{x}_3\overline{x}_4$ **e)** x_3 **f)** $\overline{x}_5$

16. How many cells in a K-map for Boolean functions with six variables are needed to represent x_1, $\overline{x}_1x_6$, $\overline{x}_1x_2\overline{x}_6$, $x_2x_3x_4x_5$, and $x_1\overline{x}_2x_4\overline{x}_5$, respectively?

17. a) How many cells does a K-map in six variables have?

b) How many cells are adjacent to a given cell in a K-map in six variables?

18. Show that cells in a K-map for Boolean functions in five variables represent minterms that differ in exactly one literal if and only if they are adjacent or are in cells that become adjacent when the top and bottom rows and cells in the first and eighth columns, the first and fourth columns, the second and seventh columns, the third and sixth columns, and the fifth and eighth columns are considered adjacent.

19. Which rows and which columns of a 4 × 16 map for Boolean functions in six variables using the Gray codes 1111, 1110, 1010, 1011, 1001, 1000, 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101 to label the columns and 11, 10, 00, 01 to label the rows need to be considered adjacent so that cells that represent minterms that differ in exactly one literal are considered adjacent?

***20.** Use K-maps to find a minimal expansion as a Boolean sum of Boolean products of Boolean functions that have as input the binary code for each decimal digit and produce as output a 1 if and only if the digit corresponding to the input is

a) odd. **b)** not divisible by 3.

c) not 4, 5, or 6.

***21.** Suppose that there are five members on a committee, but that Smith and Jones always vote the opposite of Marcus. Design a circuit that implements majority voting of the committee using this relationship between votes.

22. Use the Quine–McCluskey method to simplify the sum-of-products expansions in Example 3.

23. Use the Quine–McCluskey method to simplify the sum-of-products expansions in Exercise 12.

24. Use the Quine–McCluskey method to simplify the sum-of-products expansions in Example 4.

25. Use the Quine–McCluskey method to simplify the sum-of-products expansions in Exercise 14.

***26.** Explain how K-maps can be used to simplify product-of-sums expansions in three variables. [*Hint:* Mark with a 0 all the maxterms in an expansion and combine blocks of maxterms.]

27. Use the method from Exercise 26 to simplify the product-of-sums expansion $(x + y + z)(x + y + \overline{z})(x + \overline{y} + \overline{z})(x + \overline{y} + z)(\overline{x} + y + z)$.

***28.** Draw a K-map for the 16 minterms in four Boolean variables on the surface of a torus.

29. Build a circuit using OR gates, AND gates, and inverters that produces an output of 1 if a decimal digit, encoded using a binary coded decimal expansion, is divisible by 3, and an output of 0 otherwise.

In Exercises 30–32 find a minimal sum-of-products expansion, given the K-map shown with *don't care* conditions indicated with ds.

30.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	d	1	d	1
$w\bar{x}$		d	d	
$\bar{w}\bar{x}$		d	1	
$\bar{w}x$		1	d	

31.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	1			1
$w\bar{x}$		d	1	
$\bar{w}\bar{x}$		1	d	
$\bar{w}x$	d			d

32.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx		d	d	1
$w\bar{x}$	d	d	1	d
$\bar{w}\bar{x}$				
$\bar{w}x$	1	1	1	d

33. Show that products of k literals correspond to 2^{n-k}-dimensional subcubes of the n-cube Q_n, where the vertices of the cube correspond to the minterms represented by the bit strings labeling the vertices, as described in Example 8 of Section 10.2.

Key Terms and Results

TERMS

Boolean variable: a variable that assumes only the values 0 and 1

$\bar{x}$ (complement of x): an expression with the value 1 when x has the value 0 and the value 0 when x has the value 1

$x \cdot y$ (or **xy**) **(Boolean product** or **conjunction of x and y):** an expression with the value 1 when both x and y have the value 1 and the value 0 otherwise

$x + y$ (Boolean sum or **disjunction of x and y):** an expression with the value 1 when either x or y, or both, has the value 1, and 0 otherwise

Boolean expressions: the expressions obtained recursively by specifying that $0, 1, x_1, \ldots, x_n$ are Boolean expressions and $\overline{E}_1$, $(E_1 + E_2)$, and (E_1E_2) are Boolean expressions if E_1 and E_2 are

dual of a Boolean expression: the expression obtained by interchanging + signs and · signs and interchanging 0s and 1s

Boolean function of degree n: a function from B^n to B where $B = \{0, 1\}$

Boolean algebra: a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a complementation operator $^{-}$ that satisfies the identity, complement, associative, commutative, and distributive laws

literal of the Boolean variable x: either x or $\bar{x}$

minterm of $x_1, x_2, \ldots, x_n$: a Boolean product $y_1y_2 \cdots y_n$, where each y_i is either x_i or $\bar{x}_i$

sum-of-products expansion (or **disjunctive normal form):** the representation of a Boolean function as a disjunction of minterms

functionally complete: a set of Boolean operators is called functionally complete if every Boolean function can be represented using these operators

$x \mid y$ (or **x *NAND* y):** the expression that has the value 0 when both x and y have the value 1 and the value 1 otherwise

$x \downarrow y$ (or **x *NOR* y):** the expression that has the value 0 when either x or y or both have the value 1 and the value 0 otherwise

inverter: a device that accepts the value of a Boolean variable as input and produces the complement of the input

OR gate: a device that accepts the values of two or more Boolean variables as input and produces their Boolean sum as output

AND gate: a device that accepts the values of two or more Boolean variables as input and produces their Boolean product as output

half adder: a circuit that adds two bits, producing a sum bit and a carry bit

full adder: a circuit that adds two bits and a carry, producing a sum bit and a carry bit

K-map for n variables: a rectangle divided into 2^n cells where each cell represents a minterm in the variables

minimization of a Boolean function: representing a Boolean function as the sum of the fewest products of literals such that these products contain the fewest literals possible among all sums of products that represent this Boolean function

implicant of a Boolean function: a product of literals with the property that if this product has the value 1, then the value of this Boolean function is 1

prime implicant of a Boolean function: a product of literals that is an implicant of the Boolean function and no product obtained by deleting a literal is also an implicant of this function

essential prime implicant of a Boolean function: a prime implicant of the Boolean function that must be included in a minimization of this function

don't care condition: a combination of input values for a circuit that is not possible or never occurs

RESULTS

The identities for Boolean algebra (see Table 5 in Section 12.1).

An identity between Boolean functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken.

Every Boolean function can be represented by a sum-of-products expansion.

Each of the sets $\{+, ^{-}\}$ and $\{\cdot, ^{-}\}$ is functionally complete.

Each of the sets $\{\downarrow\}$ and $\{\mid\}$ is functionally complete.

The use of K-maps to minimize Boolean expressions.

The Quine–McCluskey method for minimizing Boolean expressions.

Review Questions

1. Define a Boolean function of degree n.

2. How many Boolean functions of degree two are there?

3. Give a recursive definition of the set of Boolean expressions.

4. a) What is the dual of a Boolean expression?

b) What is the duality principle? How can it be used to find new identities involving Boolean expressions?

5. Explain how to construct the sum-of-products expansion of a Boolean function.

6. a) What does it mean for a set of operators to be functionally complete?

b) Is the set $\{+, \cdot\}$ functionally complete?

c) Are there sets of a single operator that are functionally complete?

7. Explain how to build a circuit for a light controlled by two switches using OR gates, AND gates, and inverters.

8. Construct a half adder using OR gates, AND gates, and inverters.

9. Is there a single type of logic gate that can be used to build all circuits that can be built using OR gates, AND gates, and inverters?

10. a) Explain how K-maps can be used to simplify sum-of-products expansions in three Boolean variables.

b) Use a K-map to simplify the sum-of-products expansion $xyz + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}\,\overline{y}\,\overline{z}$.

11. a) Explain how K-maps can be used to simplify sum-of-products expansions in four Boolean variables.

b) Use a K-map to simplify the sum-of-products expansion $wxyz + wxy\overline{z} + wx\overline{y}z + wx\overline{y}\,\overline{z} + w\overline{x}yz + w\overline{x}\,\overline{y}z + \overline{w}xyz + \overline{w}\,\overline{x}yz + \overline{w}\,\overline{x}y\overline{z}$.

12. a) What is a *don't care* condition?

b) Explain how *don't care* conditions can be used to build a circuit using OR gates, AND gates, and inverters that produces an output of 1 if a decimal digit is 6 or greater, and an output of 0 if this digit is less than 6.

13. a) Explain how to use the Quine–McCluskey method to simplify sum-of-products expansions.

b) Use this method to simplify $xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}$.

Supplementary Exercises

1. For which values of the Boolean variables x, y, and z does

a) $x + y + z = xyz$?

b) $x(y + z) = x + yz$?

c) $\overline{x}\,\overline{y}\,\overline{z} = x + y + z$?

2. Let x and y belong to $\{0, 1\}$. Does it necessarily follow that $x = y$ if there exists a value z in $\{0, 1\}$ such that

a) $xz = yz$? **b)** $x + z = y + z$?

c) $x \oplus z = y \oplus z$? **d)** $x \downarrow z = y \downarrow z$?

e) $x \mid z = y \mid z$?

A Boolean function F is called **self-dual** if and only if $F(x_1, \ldots, x_n) = \overline{F(\overline{x}_1, \ldots, \overline{x}_n)}$.

3. Which of these functions are self-dual?

a) $F(x, y) = x$ **b)** $F(x, y) = xy + \overline{x}\,\overline{y}$

c) $F(x, y) = x + y$ **d)** $F(x, y) = xy + \overline{x}y$

4. Give an example of a self-dual Boolean function of three variables.

***5.** How many Boolean functions of degree n are self-dual?

We define the relation $\leq$ on the set of Boolean functions of degree n so that $F \leq G$, where F and G are Boolean functions if and only if $G(x_1, x_2, \ldots, x_n) = 1$ whenever $F(x_1, x_2, \ldots, x_n) = 1$.

6. Determine whether $F \leq G$ or $G \leq F$ for the following pairs of functions.

a) $F(x, y) = x$, $G(x, y) = x + y$

b) $F(x, y) = x + y$, $G(x, y) = xy$

c) $F(x, y) = \overline{x}$, $G(x, y) = x + y$

7. Show that if F and G are Boolean functions of degree n, then

a) $F \leq F + G$. **b)** $FG \leq F$.

8. Show that if F, G, and H are Boolean functions of degree n, then $F + G \leq H$ if and only if $F \leq H$ and $G \leq H$.

***9.** Show that the relation $\leq$ is a partial ordering on the set of Boolean functions of degree n.

***10.** Draw the Hasse diagram for the poset consisting of the set of the 16 Boolean functions of degree two (shown in Table 3 of Section 12.1) with the partial ordering $\leq$.

***11.** For each of these equalities either prove it is an identity or find a set of values of the variables for which it does not hold.

a) $x \mid (y \mid z) = (x \mid y) \mid z$

b) $x \downarrow (y \downarrow z) = (x \downarrow y) \downarrow (x \downarrow z)$

c) $x \downarrow (y \mid z) = (x \downarrow y) \mid (x \downarrow z)$

Define the Boolean operator $\odot$ as follows: $1 \odot 1 = 1$, $1 \odot 0 = 0$, $0 \odot 1 = 0$, and $0 \odot 0 = 1$.

12. Show that $x \odot y = xy + \overline{x}\,\overline{y}$.

13. Show that $x \odot y = \overline{(x \oplus y)}$.

14. Show that each of these identities holds.

a) $x \odot x = 1$ **b)** $x \odot \overline{x} = 0$

c) $x \odot y = y \odot x$

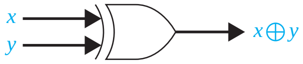
15. Is it always true that $(x \odot y) \odot z = x \odot (y \odot z)$?

***16.** Determine whether the set $\{\odot\}$ is functionally complete.

***17.** How many of the 16 Boolean functions in two variables x and y can be represented using only the given set of operators, variables x and y, and values 0 and 1?

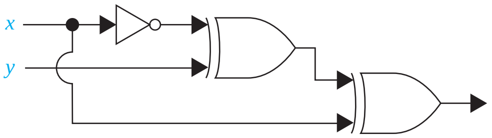
a) $\{\overline{\ }\}$ **b)** $\{\cdot\}$ **c)** $\{+\}$ **d)** $\{\cdot, +\}$

The notation for an **XOR gate**, which produces the output $x \oplus y$ from x and y, is as follows:

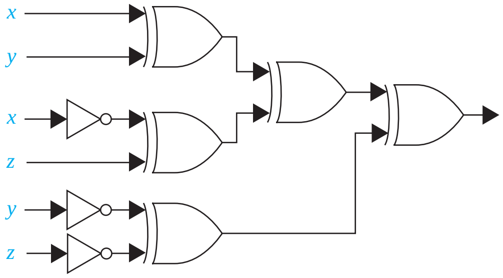


18. Determine the output of each of these circuits.

a)



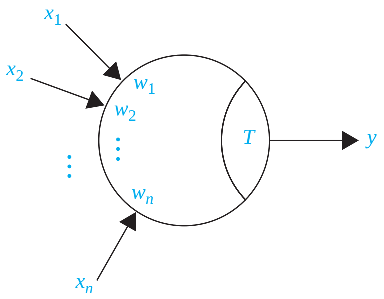
b)



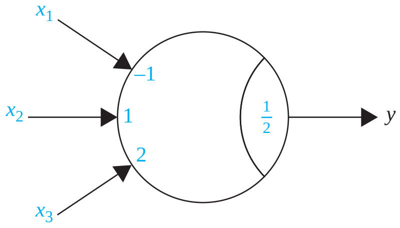
19. Show how a half adder can be constructed using fewer gates than are used in Figure 8 of Section 12.3 when XOR gates can be used in addition to OR gates, AND gates, and inverters.

20. Design a circuit that determines whether three or more of four individuals on a committee vote yes on an issue, where each individual uses a switch for the voting.

A **threshold gate** produces an output y that is either 0 or 1 given a set of input values for the Boolean variables $x_1, x_2, \ldots, x_n$. A threshold gate has a **threshold value** T, which is a real number, and **weights** $w_1, w_2, \ldots, w_n$, each of which is a real number. The output y of the threshold gate is 1 if and only if $w_1x_1 + w_2x_2 + \cdots + w_nx_n \geq T$. The threshold gate with threshold value T and weights $w_1, w_2, \ldots, w_n$ is represented by the following diagram. Threshold gates are useful in modeling in neurophysiology and in artificial intelligence.



21. A threshold gate represents a Boolean function. Find a Boolean expression for the Boolean function represented by this threshold gate.



22. A Boolean function that can be represented by a threshold gate is called a **threshold function**. Show that each of these functions is a threshold function.

a) $F(x) = \overline{x}$ **b)** $F(x, y) = x + y$

c) $F(x, y) = xy$ **d)** $F(x, y) = x \mid y$

e) $F(x, y) = x \downarrow y$ **f)** $F(x, y, z) = x + yz$

g) $F(w, x, y, z) = w + xy + z$

h) $F(w, x, y, z) = wxz + x\overline{y}z$

***23.** Show that $F(x, y) = x \oplus y$ is not a threshold function.

***24.** Show that $F(w, x, y, z) = wx + yz$ is not a threshold function.

Computer Projects

Write programs with these input and output.

1. Given the values of two Boolean variables x and y, find the values of $x + y$, xy, $x \oplus y$, $x \mid y$, and $x \downarrow y$.

2. Construct a table listing the set of values of all 256 Boolean functions of degree three.

3. Given the values of a Boolean function in n variables, where n is a positive integer, construct the sum-of-products expansion of this function.

4. Given the table of values of a Boolean function, express this function using only the operators $\cdot$ and $\overline{\ }$.

5. Given the table of values of a Boolean function, express this function using only the operators $+$ and $\overline{\ }$.

***6.** Given the table of values of a Boolean function, express this function using only the operator $\mid$.

***7.** Given the table of values of a Boolean function, express this function using only the operator ↓.

8. Given the table of values of a Boolean function of degree three, construct its K-map.

9. Given the table of values of a Boolean function of degree four, construct its K-map.

****10.** Given the table of values of a Boolean function, use the Quine–McCluskey method to find a minimal sum-of-products representation of this function.

11. Given a threshold value and a set of weights for a threshold gate and the values of the n Boolean variables in the input, determine the output of this gate.

12. Given a positive integer n, construct a random Boolean expression in n variables in disjunctive normal form.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Compute the number of Boolean functions of degrees seven, eight, nine, and ten.

2. Construct a table of the Boolean functions of degree three.

3. Construct a table of the Boolean functions of degree four.

4. Express each of the different Boolean expressions in three variables in disjunctive normal form with just the *NAND* operator, using as few *NAND* operators as possible. What is the largest number of *NAND* operators required?

5. Express each of the different Boolean expressions in disjunctive normal form in four variables using just the *NOR* operator, with as few *NOR* operators as possible. What is the largest number of *NOR* operators required?

6. Randomly generate 10 different Boolean expressions in four variables and determine the average number of steps required to minimize them using the Quine–McCluskey method.

7. Randomly generate 10 different Boolean expressions in five variables and determine the average number of steps required to minimize them using the Quine–McCluskey method.

Writing Projects

Respond to these with essays using outside sources.

1. Describe some of the early machines devised to solve problems in logic, such as the Stanhope Demonstrator, Jevons's Logic Machine, and the Marquand Machine.

2. Explain the difference between combinational circuits and sequential circuits. Then explain how *flip-flops* are used to build sequential circuits.

3. Define a *shift register* and discuss how shift registers are used. Show how to build shift registers using flip-flops and logic gates.

4. Show how *multipliers* can be built using logic gates.

5. Find out how logic gates are physically constructed. Discuss whether NAND and NOR gates are used in building circuits.

6. Explain how *dependency notation* can be used to describe complicated switching circuits.

7. Describe how multiplexers are used to build switching circuits.

8. Explain the advantages of using threshold gates to construct switching circuits. Illustrate this by using threshold gates to construct half and full adders.

9. Describe the concept of *hazard-free switching circuits* and give some of the principles used in designing such circuits.

10. Explain how to use K-maps to minimize functions of six variables.

11. Discuss the ideas used by newer methods for minimizing Boolean functions, such as Espresso. Explain how these methods can help solve minimization problems in as many as 25 variables.

12. Describe what is meant by the *functional decomposition* of a Boolean function of n variables and discuss procedures for decomposing Boolean functions into a composition of Boolean functions with fewer variables.