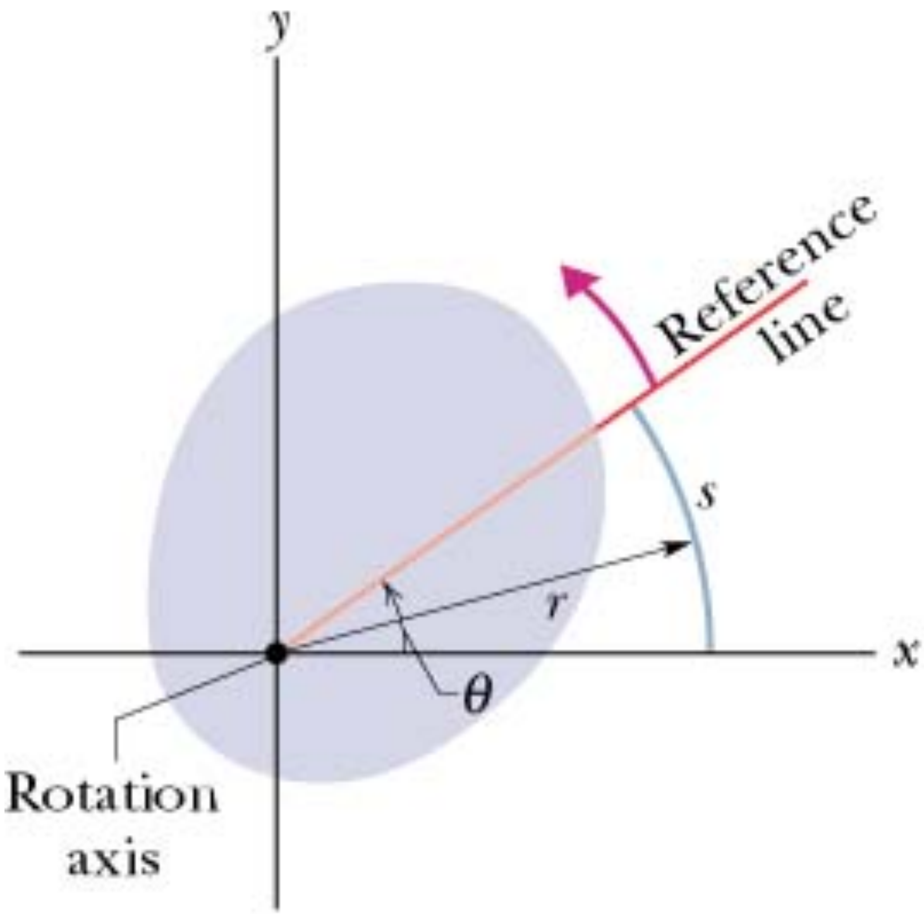
# Chapter 10

## Rotation

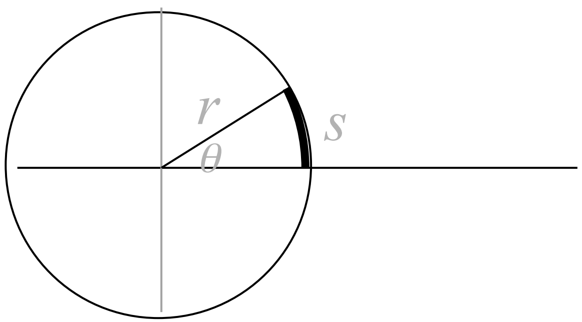
In this chapter we will study the rotational motion of rigid bodies about a fixed axis:

-Angular displacement
-Average and instantaneous angular velocity (symbol: $\boldsymbol{\omega}$ )
-Average and instantaneous angular acceleration (symbol: $\boldsymbol{\alpha}$ )

-Kinetic energy of rotation.



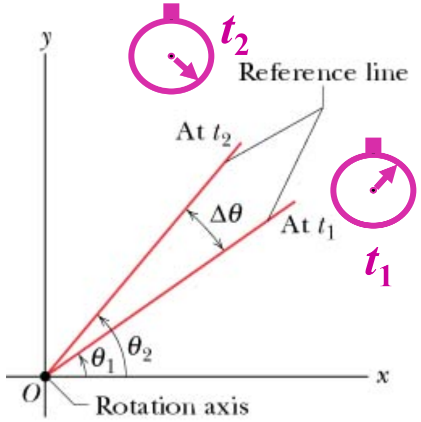

The angle $\theta(t)$ is the angular position of the axis of rotation at any time $t$. $\theta$ is related to the arc length $s$ traveled by a point at a distance $r$ from the axis




**Note:** The angle $\theta$ is measured in **radians.**

$$\theta = \frac{s}{r}.$$

## Angular Displacement :




In the picture we show the reference line at a time $t_1$ and at a later time $t_2$. Between $t_1$ and $t_2$ the body undergoes an angular displacement $\Delta\theta = \theta_2 - \theta_1$. All the points of the rigid body have the same angular displacement because they rotate locked together.

## Angular Velocity :

We define as average angular velocity for the time interval $(t_1, t_2)$ the ratio

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}.$$

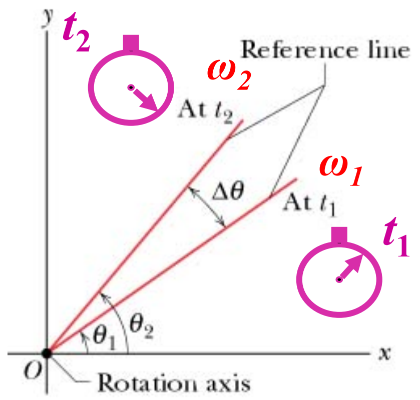
The SI unit for angular velocity is radians/second.

We define as the instantaneous angular velocity the limit of $\dfrac{\Delta\theta}{\Delta t}$ as $\Delta t \to 0$,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}.$$

This is the definition of the first derivative with $t$ : $\boxed{\omega = \dfrac{d\theta}{dt}}$

Algebraic sign of angular frequency: If a rigid body rotates counterclockwise (CCW), $\omega$ has a positive sign. If on the other hand the rotation is clockwise (CW), $\omega$ has a negative sign.




## Angular Acceleration :

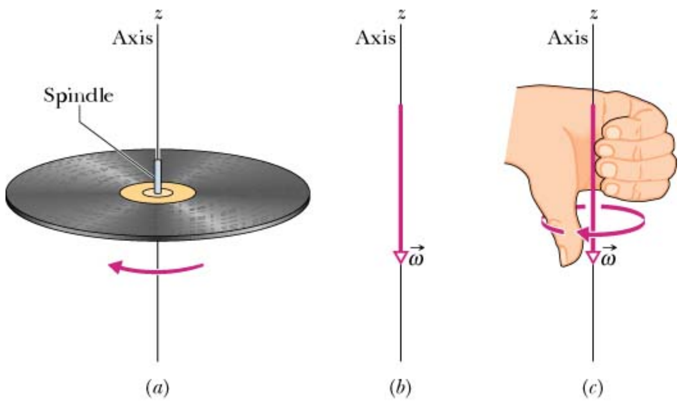
If the angular velocity of a rotating rigid object changes with time we can describe the time rate of change of $\omega$ by defining the angular aceleration.

In the figure we show the reference line at a time $t_1$ and at a later time $t_2$.
The angular velocity of the rotating body is equal to $\omega_1$ at $t_1$ and $\omega_2$ at $t_2$.
We define as average angular acceleration for the time interval $(t_1, t_2)$ the ratio

$\alpha_{\text{avg}} = \dfrac{\omega_2 - \omega_1}{t_2 - t_1} = \dfrac{\Delta\omega}{\Delta t}$. The SI unit for angular velocity is radians/second$^2$.

We define as the instantaneous angular acceleration the limit of $\dfrac{\Delta\omega}{\Delta t}$ as $\Delta t \to 0$,

$\alpha = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\omega}{\Delta t}$. This is the definition of the first derivative with $t$ : $\boxed{\alpha = \dfrac{d\omega}{dt}}$




**Angular Velocity Vector :**

For rotations of rigid bodies about a fixed axis we can describe accurately the angular velocity by assigning an algebraic sign: positive for counterclockwise rotation and negative for clockwise rotation.

We can actually use the vector notation to describe rotational motion, which is more complicated. The angular velocity vector is defined as follows:

The direction of $\vec{\omega}$ is along the rotation axis.

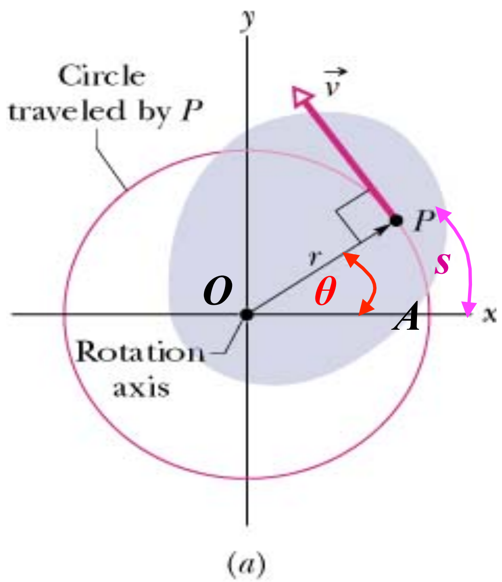
The sense of $\vec{\omega}$ is defined by the right-hand rule (RHL).

Right-hand rule: Curl the right hand so that the fingers point in the direction of the rotation. The thumb of the right hand gives the sense of $\vec{\omega}$.

## Rotation with Constant Angular Acceleration :

When the angular acceleration $\alpha$ is constant we can derive simple expressions that give us the angular velocity $\omega$ and the angular position $\theta$ as a function of time. We could derive these equations in the same way we did in Chapter 2. Instead we will simply write the solutions by exploiting the analogy between translational and rotational motion using the following correspondence between the two motions.

| Translational Motion | | Rotational Motion | |
|---|---|---|---|
| $x$ | $\leftrightarrow$ | $\theta$ | |
| $v$ | $\leftrightarrow$ | $\omega$ | |
| $a$ | $\leftrightarrow$ | $\alpha$ | |
| $v = v_0 + at$ | $\leftrightarrow$ | $\omega = \omega_0 + \alpha t$ | (eq. 1) |
| $x = x_o + v_0 t + \dfrac{at^2}{2}$ | $\leftrightarrow$ | $\theta = \theta + \omega_0 t + \dfrac{\alpha t^2}{2}$ | (eq. 2) |
| $v^2 - v_0^2 = 2a\left(x - x_o\right)$ | $\leftrightarrow$ | $\omega^2 - \omega_0^2 = 2\alpha\left(\theta - \theta_0\right)$ | (eq. 3) |




**Relating the Linear and Angular Variables :**

Consider a point $P$ on a rigid body rotating about a fixed axis. At $t = 0$ the reference line that connects the origin $O$ with point $P$ is on the $x$-axis (point $A$).

During the time interval $t$, point $P$ moves along arc $\overset{\frown}{AP}$ and covers a distance $s$. At the same time, the reference line $OP$ rotates by an angle $\theta$.

**Relation between Angular Velocity and Speed :**

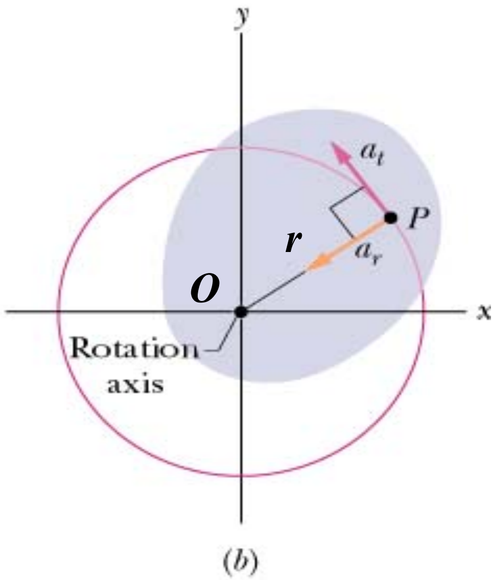
The arc length $s$ and the angle $\theta$ are connected by the equation

$s = r\theta$, where $r$ is the distance $OP$. The speed of point $P$ is $v = \dfrac{ds}{dt} = \dfrac{d\left(r\theta\right)}{dt} = r\dfrac{d\theta}{dt}$.

$$\boxed{v = r\omega}$$

The period $T$ of revolution is given by $T = \dfrac{\text{circumference}}{\text{speed}} = \dfrac{2\pi r}{v} = \dfrac{2\pi r}{\omega r} = \dfrac{2\pi}{\omega}$.

$$\boxed{T = \frac{2\pi}{\omega}} \qquad \boxed{T = \frac{1}{f}} \qquad \boxed{\omega = 2\pi f}$$




## The Acceleration :

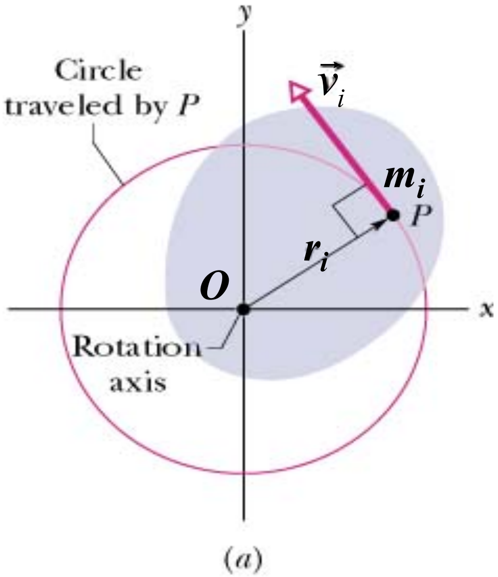
The acceleration of point $P$ is a vector that has two components. The first is a "radial" component along the radius and pointing toward point $O$. We have enountered this component in Chapter 4 where we called it "centripetal" acceleration. Its magnitude is

$$\boxed{a_r = \frac{v^2}{r} = \omega^2 r}$$

The second component is along the tangent to the circular path of $P$ and is thus known as the "tangential" component. Its magnitude is

$$a_t = \frac{dv}{dt} = \frac{d(\omega r)}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad \boxed{a_t = r\alpha}$$

The magnitude of the acceleration vector is $a = \sqrt{a_t^2 + a_r^2}$.




**Kinetic Energy of Rotation :**

Consider the rotating rigid body shown in the figure.

We divide the body into parts of masses $m_1, m_2, m_3, ..., m_i, ....$

The part (or "element") at $P$ has an index $i$ and mass $m_i$.

The kinetic energy of rotation is the sum of the kinetic

energies of the parts: $K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + ....$

$K = \sum_i \frac{1}{2}m_iv_i^2$ The speed of the $i$th element $v_i = \omega r_i \rightarrow K = \sum_i \frac{1}{2}m_i\left(\omega r_i\right)^2$.

$K = \frac{1}{2}\left(\sum_i m_ir_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$ The term $I = \sum_i m_ir_i^2$ is known as

rotational inertia or moment of inertia about the axis of rotation. The axis of rotation must be specified because the value of $I$ for a rigid body depends on its mass, its shape, as well as on the position of the rotation axis. The rotational inertia of an object describes how the mass is distributed about the rotation axis.

$$\boxed{I = \sum_i m_ir_i^2}$$

$$\boxed{I = \int r^2 dm}$$

$$\boxed{K = \frac{1}{2}I\omega^2}$$

**Sample 1:**

The angular position of a point on a rotating wheel is given by

$$\theta = 2.0 + 4.0\, t^2 + 2.0\, t^3,$$

where $\theta$ is in radians and t is in seconds. At t = 0, what are (a) the point's angular position and (b) its angular velocity? (c) What is its angular velocity at t = 4.0 s? (d) Calculate its angular acceleration at t = 2.0 s. (e) Is its angular acceleration constant?

solution (a)

At t =0,

$$\theta = 2.0 + 4.0\, t^2 + 2.0\, t^3 \quad = 2.0 + 4.0\,(0)^2 + 2.0\,(0)^3$$

$$\boxed{\theta = 2.0 \text{ rad}}$$

solution (b)

$$\omega = \frac{d\theta}{dt} \quad = 8.\, t + 6\, t^2 \quad = 8.\,(0) + 6\,(0)^2 = 0$$

$$\boxed{\boldsymbol{\omega = 0}}$$

solution (c)

$$\omega = \frac{d\theta}{dt} = 8.\,t + 6\,t^2 = 8.\,(4) + 6\,(4)^2$$

$$\omega = 128 \text{ rad/s}$$

solution (d)

$$\omega = 8.\ t + 6\ t^2$$

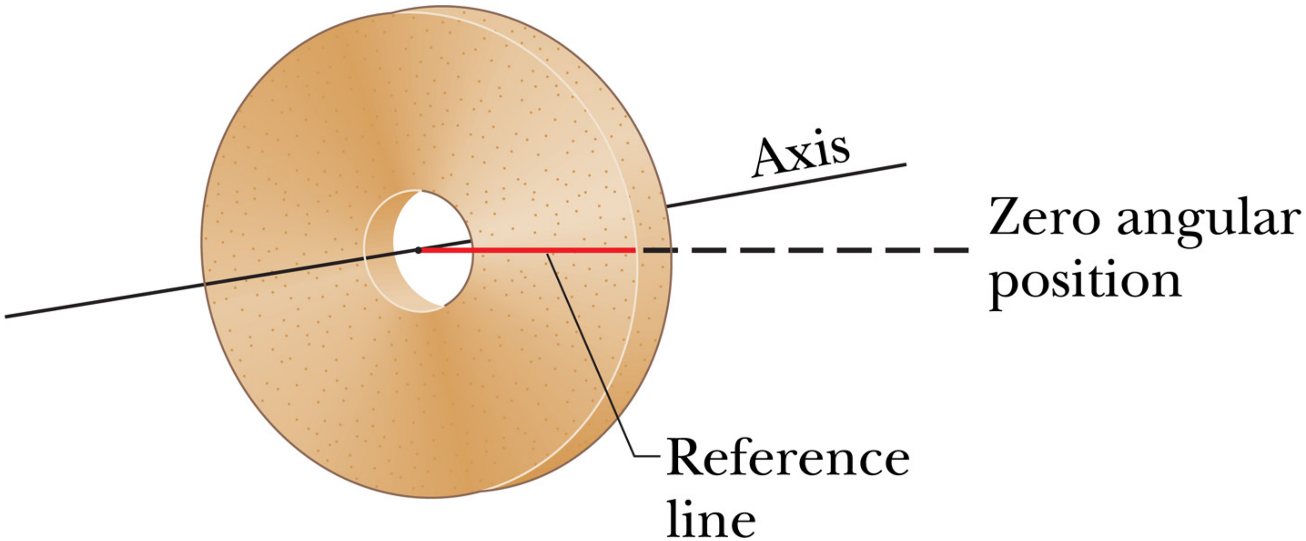
$$\alpha = \frac{d\omega}{dt} = 8.0 + 12\ t$$

at t =2 s $\alpha = 8. + 12 \times (2) = 0$ $\boxed{\alpha = 32 \text{ rad/s}^2}$

answer (e) $\alpha = 8.0 + 12\ t$ $\boxed{\alpha \text{ contains } t\text{, thus } \alpha \text{ is not constant}}$

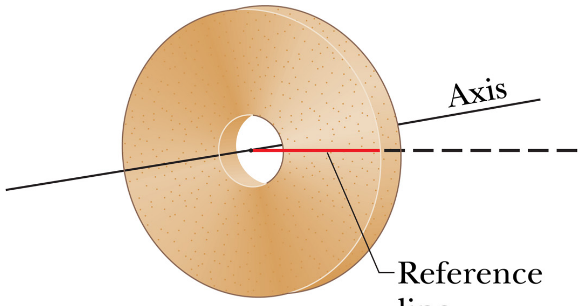
**Sample 2**

A grindstone (Fig. 10-8) rotates at constant angular acceleration $\alpha = 0.35\ \text{rad/s}^2$. At time $t = 0$, it has an angular velocity of $\omega_0 = -4.6\ \text{rad/s}$ and a reference line on it is horizontal, at the angular position $\theta_0 = 0$.



$\alpha = +0.35 \text{ rad/s}^2$

$\omega_0 = -4.6 \text{ rad/s}$




(a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0$ rev?

$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2,$$

$\theta_0 = 0$ and $\theta = 5.0 \text{ rev} = 10\pi$ rad give us

$$10\pi \text{ rad} = (-4.6 \text{ rad/s})t + \tfrac{1}{2}(0.35 \text{ rad/s}^2)t^2.$$

(We converted 5.0 rev to $10\pi$ rad to keep the units consistent.) Solving this quadratic equation for $t$, we find

$$t = 32 \text{ s.} \qquad \text{(Answer)}$$

(b) Describe the grindstone's rotation between t= 0 and t = 32 s.

**Description**: The wheel is initially rotating in the negative (clockwise) direction with angular velocity $\omega_0$ = - 4.6 rad/s, but its angular acceleration a is positive. This initial opposition of the signs of angular velocity and angular acceleration means that the wheel slows in its rotation in the negative direction, stops, and then reverses to rotate in the positive direction. After the reference line comes back through its initial orientation of $\theta = 0$, the wheel turns an additional 5.0 rev by time t = 32 s.

(c) At what time $t$ does the grindstone momentarily stop?

$$\omega = \omega_0 + \alpha t$$

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - (-4.6 \text{ rad/s})}{0.35 \text{ rad/s}^2} = 13 \text{ s.} \qquad \text{(Answer}$$

Sample 3

A flywheel with a diameter of 1.20 m is rotating at an angular speed of 200 rev/min.

(a) What is the angular speed of the flywheel in radians per second?

(b) What is the linear speed of a point on the rim of the flywheel?

(c) What constant angular acceleration (in revolutions per minute- squared) will increase the wheel's angular speed to 1000 rev/min in 60.0 s?

(d) How many revolutions does the wheel make during that 60.0 s?

$$\omega_0 = 200\,\frac{\text{rev}}{\text{min}} \times \frac{2\ \pi\ \text{rad}}{\text{rev}} \times \frac{\text{min}}{60\ \text{s}}$$

$$\boxed{\omega_0 = 20.944\ \text{rad/ s}}$$

$$v = \omega r$$

$$v = 20.944\ (\mathrm{s}^{-1}) \times (1.2\ \mathrm{m}/2)$$

$$\boxed{v = 12.57\ \mathrm{m/s}}$$

Sample

A flywheel with a diameter of 1.20 m is rotating at an angular speed of 200 rev/min.

(a) What is the angular speed of the flywheel in radians per second?

(b) What is the linear speed of a point on the rim of the flywheel?

(c) What constant angular acceleration (in revolutions per minute- squared) will increase the wheel's angular speed to 1000 rev/min in 60.0 s?

(d) How many revolutions does the wheel make during that 60.0 s?

$$\omega = \omega_0 + \alpha t$$

$\omega_0 = 20.944$ rad/ s

$v = 12.57$ m/s

$$\alpha = \frac{\omega - \omega_0}{t}$$

$$\alpha = \frac{1000 - 200 (rev/\min)}{60s} = \frac{1000 - 200 (rev/\min)}{1.0\min}$$

$$\omega = \omega_0 + \alpha t$$

$$\alpha = \frac{\omega - \omega_0}{t}$$

$$\alpha = \frac{800(rev/\min)}{1.0\min} = 800 rev/\min^2$$

$\omega_0 = 20.944$ rad/ s

$v = 12.57$ m/s

Sample 4

A flywheel with a diameter of 1.20 m is rotating at an angular speed of 200 rev/ min.

(a) What is the angular speed of the flywheel in radians per second?

(b) What is the linear speed of a point on the rim of the flywheel?

(c) What constant angular acceleration (in revolutions per minute- squared) will increase the wheel's angular speed to 1000 rev/min in 60.0 s?

**(d) How many revolutions does the wheel make during that 60.0 s?**

$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2} \alpha t^2$$

$\omega_0 = 20.944$ rad/ s

$v = 12.57$ m/s

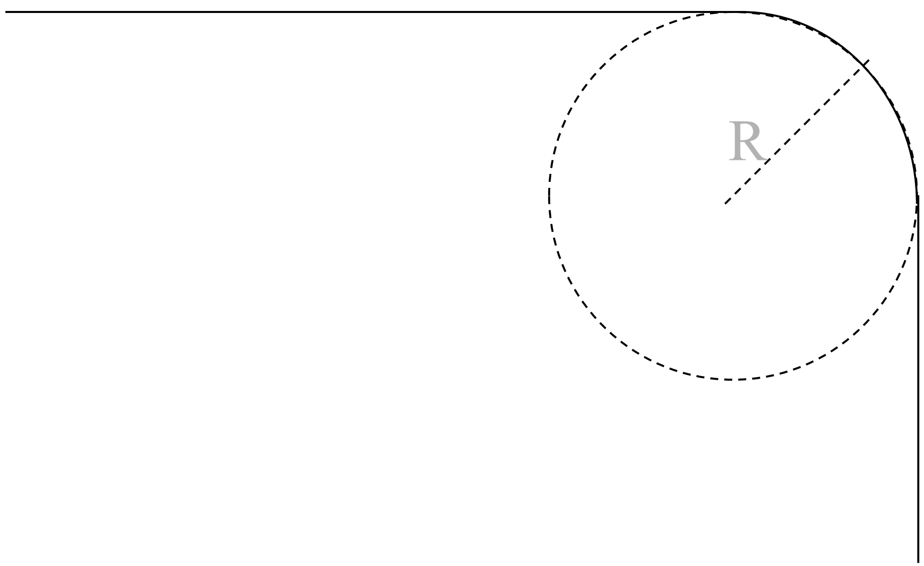
$\alpha = 800$ rev/min$^2$

$$\theta - \theta_0 = 200 \text{ rev/min} \times 1.0 \text{ min} + 0.5 \times 800 \text{ rev/min}^2 \times (1.0 \text{ min})^2$$

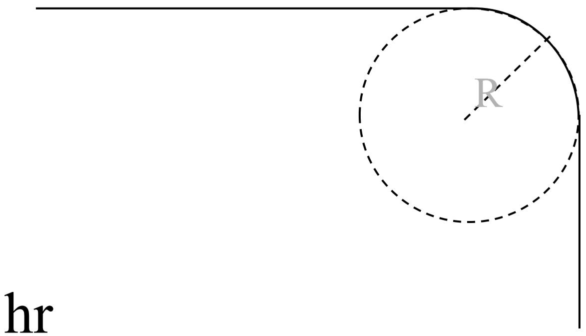
$$\theta - \theta_0 = 600 \text{ rev}$$

Sample 5

•**23** What are the magnitudes of (a) the angular velocity, (b) the radial acceleration, and (c) the tangential acceleration of a spaceship taking a circular turn of radius 3220 km at a speed of 29 000 km/h?



solution of (a)




$$v = \omega r \qquad \omega = v / r$$

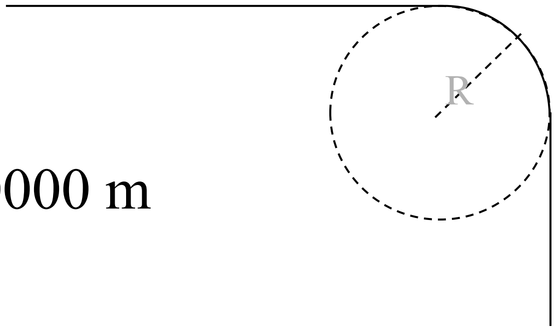
$$v = 290000 \frac{\text{km}}{\text{hr}} \times \frac{1000 \text{ m}}{\text{km}} \times \frac{\text{hr}}{60 \times 60 \text{ s}}$$

$$v = 8055.556 \text{ m/s}$$

$$\omega = (8055.556 \text{ m/s}) \,/\, (3\,220\,000 \text{ m})$$

$$\boxed{\omega = 0.002502 \text{ rad/s}}$$

## solution of (b)




The radial (centripetal) acceleration =

$a_r = v^2 / r$ $a_r = (8055.556 \text{ m/s})^2 / 3220000 \text{ m}$

$$\boxed{a_r = 20.1528 \text{ m/s}^2}$$

## solution of (c)

The tangential (orbital) acceleration =

$a_t = \alpha r$ $v$ is constant $\rightarrow$ $\omega$ is constant $\rightarrow$ $\alpha = 0$

$$\boxed{a_t = 0}$$