التكامل

تكامل غير محدود

$\int f(x)\,dx = F(x)$

تابع أصلي.

تكامل محدود

$\int_a^b f(x)\,dx$

... التابع الأصلي ...

$f$ : تابع

$F'$ مشتق التابع

$F$ تابع أصلي.

$F'(x) = f(x)$

نقول أن $F$ تابع أصلي.

تذكرة بعض التوابع الأصلية:

| $f(x)$ | $F(x)$ |
|---|---|
| $0$ | $c$ |
| $a$ | $ax$ |
| $x^n$ | $\frac{x^{n+1}}{n+1}$ |
| $\sin x$ | $-\cos x$ |
| $\sin(ax+b)$ | $\frac{-1}{a}\cos(ax+b)$ |
| $\sin(g(x))$ $\underset{\text{تكامل}}{\overset{\text{مشتق}}{\rightleftarrows}}$ | $g'\cos g(x)$ |
| $\cos x$ | $\sin x$ |
| $\cos(ax+b)$ | $\frac{1}{a}\sin(ax+b)$ |
| $\cos(g(x))$ $\underset{\text{تكامل}}{\overset{\text{مشتق}}{\rightleftarrows}}$ | $g'\sin(g(x))$ |

مثلثية

$\tan x \underset{\text{تكامل}}{\overset{\text{مشتق}}{\rightleftarrows}} \frac{1}{\cos^2 x}$

$1+\tan^2 x$

$\cot x \overset{\text{مشتق}}{\rightleftarrows} \frac{-1}{\sin^2 x}$

$-(1+\cot^2 x)$

| | |
|---|---|
| $e^x$ | $e^x$ |
| $e^{ax+b}$ | $\frac{1}{a}e^{ax+b}$ |
| $e^{g(x)}$ $\underset{\text{تكامل}}{\overset{\text{مشتق}}{\rightleftarrows}}$ | $g'\cdot e^{g(x)}$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert$ |
| $\frac{g'}{g}$ | $\ln\lvert g(x)\rvert$ |
| $g'\cdot g^r$ | $\frac{g^{r+1}}{r+1}$ $r \neq -1$ |

$f = (2x+3)(x^2+3x-9)^3$

$F(x) = \frac{(x^2+3x-9)^4}{4}$

$f = \frac{2x+1}{\sqrt{x^2+x+4}}$

$f = (2x+1)(x^2+x+4)^{-\frac{1}{2}}$

$F = \frac{(x^2+x+4)^{\frac{1}{2}}}{\frac{1}{2}}$

$= 2\sqrt{x^2+x+4}$

$f(x) = \frac{2x+5}{x^2+5x+3}$

$F(x) = \ln\lvert x^2+5x+3\rvert$

$f = \tan^2 x$

$= -1 + \underbrace{1+\tan^2 x}$

$F = -x + \tan x$

$f = e^{1-3x}$

$F(x) = \frac{1}{-3} e^{1-3x}$

$f = \tan(-x)$

$f = \frac{\sin(-x)}{\cos(-x)}$

$F = \ln|\cos(-x)|$

$f = \frac{4x+6}{\sqrt{x^2+3x+5}}$

$f(x) = (4x+6)(x^2+3x+5)^{-\frac{1}{2}}$

$= 2\underbrace{(2x+3)}_{g'}\underbrace{(x^2+3x+5)}_{g}^{-\frac{1}{2}}$

$F = 2\frac{(x^2+3x+5)^{\frac{1}{2}}}{\frac{1}{2}}$

$= 4\sqrt{x^2+3x+5}$

$f(x) = 3x^2+4x-5$

$F(x) = 3\frac{x^3}{3} + 4\frac{x^2}{2} - 5x$

$= x^3 + 2x^2 - 5x$

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$F(x) = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} = \frac{x^{\frac{4}{3}}}{\frac{4}{3}}$

$= \frac{3}{4}\sqrt[3]{x^4}$

$x^{\frac{m}{n}} = \sqrt[n]{x^m}$

$f = \frac{x+1}{x^2+2x+1}$

$= \frac{1}{2}\frac{2x+2}{x^2+2x+1}$

$F = \frac{1}{2}\ln|x^2 + 2x+1|$

. دساتير [illegible]

$\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$\cos x \cdot \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$

$\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

$\sin x \cdot \sin y = \frac{-1}{2}[\cos(x+y) - \cos(x-y)]$

$f(x) = \cos 4x \cdot \sin x$

$= \frac{1}{2}[\sin 5x - \sin 3x]$

$F(x) = \frac{1}{2}\left[\frac{-1}{5}\cos 5x + \frac{1}{3}\cos 3x\right]$

دساتير [illegible] :

$\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{1}{2} + \frac{1}{2}\cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2} - \frac{1}{2}\cos 2x$

$G = \frac{1}{\cos^2 x}$ $\quad F = \tan^2 x$

$G' = \frac{-2\cancel{\cos x}(-\sin x)}{\cos^{4} x}$

$= \frac{2\sin x}{\cos^3 x}$

$F'(x) = 2\tan x \frac{1}{\cos^2 x}$

$= 2\frac{\sin x}{\cos x}\,\frac{1}{\cos^2 x}$

$= \frac{2\sin x}{\cos^3 x}$ F ، G أصليتان لـ f.

$1 - \cos^2 x$

$= \sin^2 x$

$G'(x) = -2\cos x(-\sin x)$

$= 2\sin x \cdot \cos x$

$F' = 2\sin x \cdot \cos x$

F ، G أصليتان لـ

نفس ...

$f(x) = \left(x + \frac{1}{x}\right)^2 \quad P(x) = \frac{2(x^4 - 1)}{x^3}$

$f'(x) = 2\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right)$

$= 2\left(\frac{x^2+1}{x}\right)\left(\frac{x^2-1}{x^2}\right)$

$= 2\,\frac{x^4 - 1}{x^3} = P(x)$

F أصلية لـ P على I

$f(x) = \frac{-1}{x(x-1)} \quad P = \frac{2x-1}{x^2(x-1)^2}$

$f'(x) = \frac{-(x-1+x)(-1)}{x^2(x-1)^2}$

$= \frac{2x-1}{x^2(x-1)^2}$

F أصلية لـ P ...

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$$f = \frac{1}{1-2x+x^2}$$

$$= \frac{1}{(x-1)^2}$$

$$= (x-1)^{-2}$$

$$F(x) = \frac{(x-1)^{-1}}{-1}$$

$$= \frac{-1}{x-1}$$

$$f = \frac{2x+1}{(x^2+x)^2}$$

$$= \underbrace{(2x+1)}_{g_1}\underbrace{(x^2+x)^{-2}}_{g} = n$$

$$F = \frac{(x^2+x)^{-1}}{-1} = \frac{-1}{x^2+x}$$

$$f = 8x^3+6x^2-2x+3 \qquad \frac{1}{ccx}$$

$$F = 8\frac{x^4}{4}+6\frac{x^3}{3}-2\frac{x^2}{2}+3x$$

$$= 2x^4+2x^3-x^2+3x$$

$$f = \frac{1}{x^4} = x^{-4}$$

$$F(x) = \frac{x^{-3}}{-3} = \frac{-1}{3}\frac{1}{x^3}$$

$$f = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}} - \frac{3}{x^2}$$

$$= x^{\frac{1}{3}} + x^{-\frac{1}{3}} - 3x^{-2}$$

$$F = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^{\frac{2}{3}}}{\frac{2}{3}} - 3\frac{x^{-1}}{-1}$$

$$= \frac{3}{4}\sqrt[3]{x^4} + \frac{3}{2}\sqrt[3]{x^2} + \frac{3}{x}$$

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$f = \frac{3x+1}{2x}$

$= \frac{3x}{2x} + \frac{1}{2x}$

$= \frac{3}{2} + \frac{1}{2}\,\frac{1}{x}$

$F(x) = \frac{3}{2}x + \frac{1}{2}\ln|x|$

$f = \frac{x+1}{x-2}$

$$\begin{array}{r|l} & 1 \\ x-2 & x+1 \\ & -x\mp 2 \\ \hline & 3 \end{array}$$

$f = 1 + \frac{3}{x-2}$

$= 1 + 3\,\frac{1}{x-2}$

$F = x + 3\ln|x-2|$

$4x)$

$4x)$

$\sin 4x)$

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$f = \frac{4x-2}{\sqrt{x^2-x}}$

$= (4x-2)(x^2-x)^{-\frac{1}{2}}$

$= 2\underbrace{(2x-1)}_{g'}\underbrace{(x^2-x)}_{g}^{-\frac{1}{2}+1}$

$F = 2\,\frac{(x^2-x)^{\frac{1}{2}}}{\frac{1}{2}}$

$= 4\sqrt{x^2-x}$

$f = \frac{5}{4x-3}$

$= 5\,\frac{1}{4x-3}$

$= \frac{5}{4}\,\frac{4}{4x-3}$

$F = \frac{5}{4}\ln|4x-3|$

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$f = \cos 3x \cos x$

$= \frac{1}{2}[\cos 4x + \cos 2x]$

$F = \frac{1}{2}\left[\frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x\right]$

$f = \cot^2 x$

$\cot x \rightarrow \frac{-1}{\sin^2 x} = -(1+\cot^2 x)$

$f = -1 + \underbrace{1 + \cot^2 x}$

$F = -x - \cot x$

$f = \cot x$

$= \frac{\cos x}{\sin x}$

$F = \ln|\sin x|$

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$f = \cos^2 3x$

$f = \frac{1+\cos 6x}{2}$

$= \frac{1}{2} + \frac{1}{2}\cos 6x$

$F(x) = \frac{1}{2}x + \frac{1}{12}\sin 6x$

$f = \cos^4 x$

$= (\cos^2 x)^2$

$= \left(\frac{1+\cos 2x}{2}\right)^2$

$= \frac{1}{4}(1+\cos 2x)^2$

$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)$

$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)$

$F = \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right.$

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$= \frac{3x}{2x}$

$= \frac{3}{2}$

$F(x) = \frac{3}{2}$

$f =$

$x -$

$f = 1$

$= 1$

$F = x$

ii) $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$

iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iv) $\int_a^c f(x) + \int_c^b f = \int_a^b$

5) $\int_a^b f_1 + f_2 = \int_a^c f_1 + \int_a^b f_2$

التكامل المحدد :

$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$

$= F(b) - F(a)$

$\int_0^1 (3x^2+4x-5)\,dx$

$= \left[3\frac{x^3}{3} + 4\frac{x^2}{2} - 5x\right]_0^1$

$= \left[x^3 + 2x^2 - 5x\right]_0^1$

$= 1+2-5-(0+0-0)$

$= -2$

خواص التكامل المحدد :

i) $\int_a^a f(x)\,dx = 0$

$f = x \cdot \sqrt[3]{(x^2+1)^2}$

$= x(x^2+1)^{\frac{2}{3}}$

$= \frac{1}{2} 2x (x^2+1)^{\frac{2}{3}}$

$F = \frac{1}{2} \frac{(x^2+1)^{\frac{5}{3}}}{\frac{5}{3}}$

$= \frac{3}{10} \sqrt[3]{(x^2+1)^5}$

$f = \frac{x}{\sqrt{3-x^2}}$

$= x(3-x^2)^{-\frac{1}{2}}$

$= \frac{1}{-2}(-2x)(3-x^2)^{-\frac{1}{2}}$

$F = -\frac{1}{\cancel{2}} \frac{(3-x^2)^{\frac{1}{2}}}{\frac{1}{\cancel{2}}}$

$= -\sqrt{3-x^2}$

$f = \sqrt{(2x-1)^3}$

$= (2x-1)^{\frac{3}{2}}$

$= \frac{1}{2} 2 (2x-1)^{\frac{3}{2}}$

$F(x) = \frac{1}{2} \frac{(2x-1)^{\frac{5}{2}}}{\frac{5}{2}}$

$= \frac{1}{5}\sqrt{(2x-1)^5}$

$f = \frac{1}{\sqrt{3-2x}}$

$= (3-2x)^{-\frac{1}{2}}$

$= \frac{1}{-2}(-2)(3-2x)^{-\frac{1}{2}}$

$F = -\frac{1}{\cancel{2}} \frac{(3-2x)^{\frac{1}{2}}}{\frac{1}{\cancel{2}}}$

$= -\sqrt{3-2x}$

2) $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) $\int_a^c f(x) + \int_c^b f = \int_a^b$

5) $\int_a^b f_1 \mp f_2 = \int_a^b f_1 \mp \int_a^b f_2$

التكامل المحدد :

$\int_a^b f(x)\,dx = [F(x)]_a^b$

$= F(b) - F(a)$

$\int_0^1 (3x^2 + 4x - 5)\,dx$

$= \left[3\frac{x^3}{3} + 4\frac{x^2}{2} - 5x\right]_0^1$

$= [x^3 + 2x^2 - 5x]_0^1$

$= 1 + 2 - 5 - (0 + 0 - 0)$

$= -2$

خواص التكامل المحدد :

1) $\int_a^a f(x)\,dx = 0$

$f = x \cdot \sqrt[3]{(x^2+1)^2}$

$= x(x^2+1)^{\frac{2}{3}}$

$= \frac{1}{2} 2x (x^2+1)^{\frac{2}{3}}$

$F = \frac{1}{2} \frac{(x^2+1)^{\frac{5}{3}}}{\frac{5}{3}}$

$= \frac{3}{10} \sqrt[3]{(x^2+1)5}$

$f = \frac{x}{\sqrt{3 - x_2}}$

$= x(3 - x^2)^{-\frac{1}{2}}$

$= \frac{1}{-2}(-2x)(3 - x^2)^{-\frac{1}{2}}$

$F = \frac{-1}{\cancel{2}} \frac{(3 - x^2)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}$

$= -\sqrt{3 - x2}$

$f = \sqrt{(2x-1)^3}$

$= (2x-1)^{\frac{3}{2}}$

$= \frac{1}{2} 2 (2x-1)^{\frac{3}{2}}$

$F(x) = \frac{1}{2} \frac{(2x-1)^{\frac{5}{2}}}{\frac{5}{2}}$

$= \frac{1}{5} \sqrt{(2x-1)5}$

$f = \frac{1}{\sqrt{3 - 2x}}$

$= (3 - 2x)^{-\frac{1}{2}}$

$= \frac{1}{-2}(-2)(3 - 2x)^{-\frac{1}{2}}$

$F = \frac{-1}{\cancel{2}} \frac{(3 - 2x)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}$

$= -\sqrt{3 - 2x}$

تكامل القيمة المطلقة:

نجد تكامل القيمة المطلقة عند القيمة التي تعدم ما داخل القيمة المطلقة اذا وجدت ضمن حدود التكامل

$$\int_0^2 |x-1|\,dx$$

$$= \int_0^1 |x-1|\,dx + \int_1^2 |x-1|\,dx$$

$$= \int_0^1 (-x+1) + \int_1^2 (x-1)\,dx$$

$$= \left[-\frac{x^2}{2}+x\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2$$

$$\int_{-2}^{2} |x^2-4x|\,dx$$

$$\int_{-2}^{0}|x^2-4x| + \int_0^2 |x^2-4x|$$

$$\int_{-2}^{0}(x^2-4x) + \int_0^2(-x^2+4x)$$

$$= \left[\frac{x^3}{3}-2x^2\right]_{-2}^{0} + \left[-\frac{x^3}{3}+2x^2\right]_0^2$$

$$\int_0^{\pi} |\cos x|\,dx$$

$$\int_0^{\pi/2}|\cos x| + \int_{\pi/2}^{\pi}|\cos x|$$

$$\int_0^{\pi/2}\cos x + \int_{\pi/2}^{\pi} -\cos x$$

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تكامل القيمة المطلقة:

نجزئ تكامل القيمة المطلقة عند القيمة التي تعدم ما داخل القيمة المطلقة اذا وجدت ضمن حدود التكامل

$$\int_0^2 |x-1|\, dx$$

$$= \int_0^1 |x-1|\,dx + \int_1^2 |x-1|\, dx$$

$$= \int_0^1 (-x+1) + \int_1^2 (x-1)\, dx$$

$$= \left[-\frac{x^2}{2}+x\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2$$

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$$\int_{-2}^{2} |x^2-4x|\, dx$$

$$\int_{-2}^{0} |x^2-4x| + \int_0^2 |x^2-4x|$$

$$\int_{-2}^{0} (x^2-4x) + \int_0^2 (-x^2+4x)$$

$$= \left[\frac{x^3}{3}-2x^2\right]_{-2}^{0} + \left[-\frac{x^3}{3}+2x^2\right]_0^2$$

$$\int_0^{\pi} |\cos x|\, dx$$

$$\int_0^{\frac{\pi}{2}} |\cos x| + \int_{\frac{\pi}{2}}^{\pi} |\cos x|$$

$$\int_0^{\frac{\pi}{2}} \cos x + \int_{\frac{\pi}{2}}^{\pi} -\cos x$$

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$\frac{\pi}{2} - [\sin x]_{\frac{\pi}{2}}^{\pi}$

$-[\sin \pi - \sin \frac{\pi}{2}]$

$) = 1$

التكامل

نستخدم ابجـ

لو كان حدين تابع

مختلفين ( $x$

لـ

طرح

$u =$ جا

$u' =$ جتا

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$$\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b v u'$$

$$\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx$$

$u = x \qquad u' = 1$

$v' = \sin x \qquad v = -\cos x$

$$I = [uv]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} v u'$$

$$= [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x$$

$$= 0 - 0 + [\sin x]_0^{\frac{\pi}{2}} = 1$$

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$$= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}$$

$$= \sin\frac{\pi}{2} - \sin 0 - [\sin \pi - \sin\frac{\pi}{2}]$$

$$= 1 - (0 - 1) = 2$$

التكامل بالتجزئة :

نستخدم التجزئة اذا كان لدينا جداء تابعين من نوعين مختلفين (اي لم نتمكن من استخدام $g' \cdot g^r$)

ملخص

$\int$ صح $\cdot$ (cos / sin / e)

$u =$ صح

$v' =$ cos / sin / e

$\int$ صح $\cdot \ln$

$u = \ln$

$v' =$ صح

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