CHAPTER 3.

THE DEFINITE INTEGRAL.

One of the great achievements of classical geometry was to obtain formulas for regions, sizes, fields and cones and to find a way to calculate the areas and quantities of general shapes, this method, called integration, is a tool for calculating much more areas and sizes. Integral is of fundamental importance in statistics, science, and engineering.

3.1 INTRODUCTION.

In chapters 1 and 2 we discussed indefinite integrals in which we got different values of an integral by giving different values to the constant of integration. For example $\int x\,dx$ = $\frac{x^2}{2}+C$ and this can be equal to $\frac{x^2}{2}+1$, $\frac{x^2}{2}+2$, $\frac{x^2}{2}-5$ and so on, that is we can get different values for the integral $\int x\,dx$ by giving different values to the constant of integration C, and hence we use the term indefinite integral.

In this chapter we will discuss definite integrals where the integrals will have a definite value, that is, a unique value. A definite integral is denoted by $\int_a^b f(x)\,dx$ (we read it as integral a to b $f(x)$ dx), where a is called the lower limit and b is the upper limit. Usually the definite integral $\int_a^b f(x)\,dx$ is the area of the region bounded by the curve $y=f(x)$, the x- axis and the lines $x=a$ and $x=b$, and we will study Fundamental Theorem and the Mean-Value Theorem of integral calculus, evaluation of definite integral by substitution method, the improper integral and numerical integration.

3.2 THE AREA PROBLEM.

We begin by attempting to solve the area problem : Find the area of the region that lies under the curve *y=f(x)* from a to b. This means that *R*, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $(x) \geq 0$], the vertical lines x=a and x=b and the x -axis.

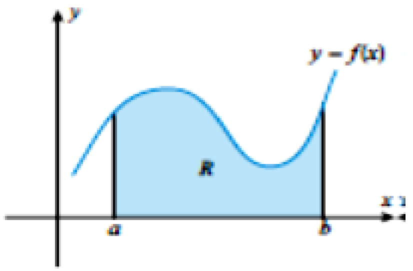

Figure 1

This can be summarized as follows :

Now we develop a method for computing the area beneath the graph of $y = f(x)$ and above the x-axis on an interval $a \leq x \leq b$. You are familiar with the formulas for
computing the area of a rectangle, a circle and a triangle. However, how would you compute the area of a region that's not a rectangle, circle or triangle?
We need a more general description of area, one that can be used to find the area of
almost any two-dimensional region imaginable.
First, assume that $f(x) \geq 0$ and f is continuous
on the interval $[a, b]$, as in Figure 2. We start by dividing the interval $[a, b]$ into n equal pieces. This is called a **regular partition** of $[a, b]$.
The width of each subinterval in the partition is then $\frac{b-a}{n}$, which we denote by Δx (meaning a small change in x).

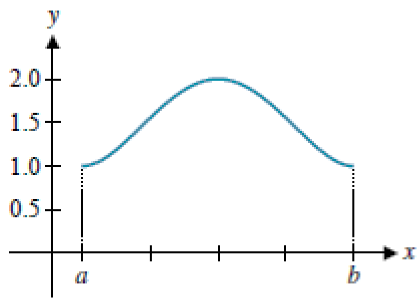

Figure 2.
Area under $y = f(x)$

The points in the partition are denoted by $x_0 = a$, $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$, and so on. In general $x_i = x_0 + i\Delta x$ for $i = 1, 2, \ldots, n$.

See Figure 3 for an illustration of a regular partition for the case where $n = 6$. On each subinterval $[x_{i-1,} x_i]$ (for $i = 1, 2, \ldots, n$), construct a rectangle of height $f(x_i)$ (the value of the function at the right endpoint of the subinterval),

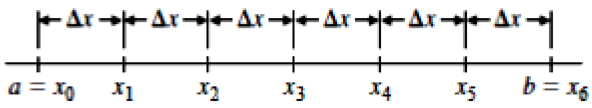

Figure 3.
Regular partition of [a, b]

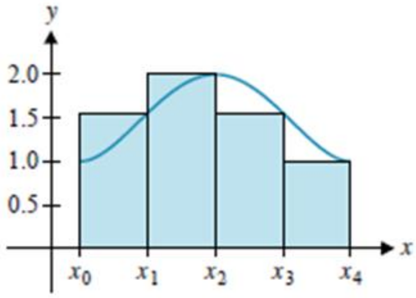

Figure 4.
$A \approx A_4$

as illustrated in Figure 4 for the case where $n = 4$. It should be clear from Figure 4 that the area under the curve A is roughly the same as the sum of the areas of the four rectangles,

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = A_4$$

In particular, notice that although two of these rectangles enclose more area than that under the curve and two enclose less area, on the whole, the sum of the areas of the four rectangles provides an approximation to the total area under the curve. More generally, if we construct n rectangles of equal width on the interval [a, b], we have

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x = A_n$$

One approach to the area problem is to use Archimedes' method of exhaustion in the following way:

• Divide the interval [a, b] into n equal subintervals and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter—it can be above the center, above an endpoint, or above any other point in the subinterval. In (Figure 5) it is above the center.

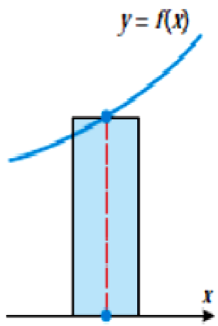

Figure 5.

• For each n, the total area of the rectangles can be viewed as an *approximation* to the exact area under the curve over the interval [a, b].

Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit (Figure 6.)

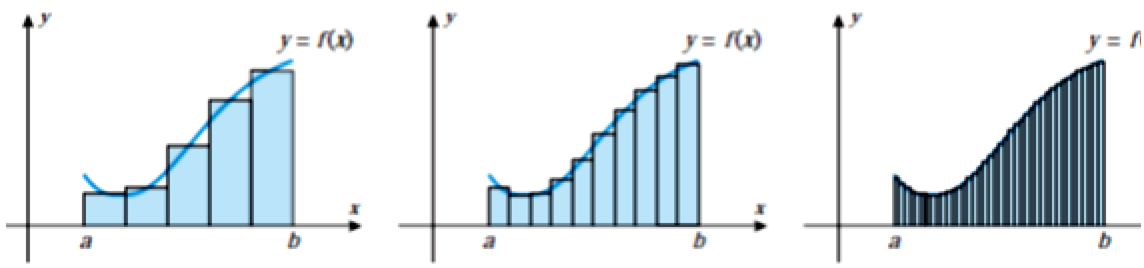

Figure 6.

That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n\to\infty} A_n$$

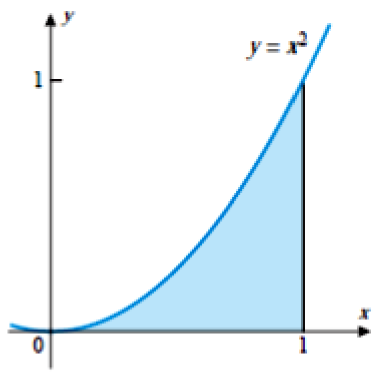

Figure 7.

We will call this the ***rectangle method*** for computing A.
To illustrate this idea, we will use the rectangle
method to approximate the area under the curve
$y = x^2$ over the interval [0, 1] (Figure 7).
We will begin by dividing the interval
[0, 1] into ***n*** equal subintervals, from which it
follows that each subinterval has length $1/n$;
the endpoints of the subintervals occur at
$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1$, (Figure 8).

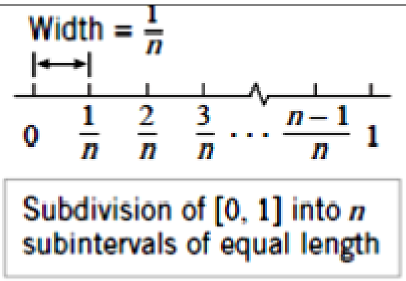

Figure 8.

We want to construct a rectangle over each of these subintervals whose height is the value of the function $f(x) = x^2$ at some point in the subinterval. To be specific, let us use the right endpoints, in which case the heights of our rectangles will be

$$\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, 1^2$$

and since each rectangle has a base of width 1/*n*, the total area A_n of the *n* rectangles will be

$$A_n = \left[\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, 1^2\right]\left(\frac{1}{n}\right) \qquad (*)$$

For example, if *n* = 4, then the total area of the four approximating rectangles would be

$$A_4 = \left[\left(\frac{1}{4}\right)^2, \left(\frac{2}{4}\right)^2, \left(\frac{3}{4}\right)^2, \ldots, 1^2\right]\left(\frac{1}{4}\right) = \frac{15}{32} = 0.46875$$

Table 1 shows the result of evaluating (*) on a computer for some increasingly large values of *n*. These computations suggest that the exact area is close to $\frac{1}{3}$, or in other words

$$\lim_{n\to\infty} A_n = \frac{1}{3}$$

Tabela 1.

n	4	10	100	1000	10,000	100,00
A_n	0.468750	0.385000	0.338350	0.333834	0.333383	0.3333

The area problem.

Given a function *f* that is continuous and nonnegative on an interval [*a*, *b*], find the area between the graph of *f* and the interval [*a*, *b*] on the *x*-axis.(See the Figure 1.)

Example 1. For each of the functions f, find the area $A(x)$ between the graph of *f* and the interval $[a, x] = [-1, x]$, and find the derivative $A'(x)$ of this area function.

$(a)\ f(x) = 2 \qquad (b)\ f(x) = x + 1 \qquad (c)\ f(x) = 2x + 3$

Solution (a): After plotting graph of the function $f(x)=2$, the required area is the shaded region shown in Figure 9. The base of this rectangle is of length $x-(-1)=x+1$ and the height is 2. Thus the required area is

$$A(x) = 2(x - (-1)) = 2(x + 1) = 2x + 2$$

For this area function, $A'(x) = 2 = f(x)$

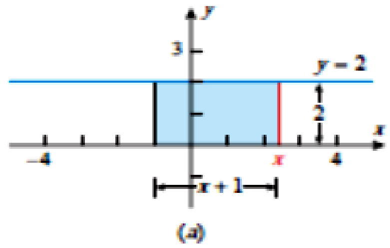

Figure 9.

Solution (b): After plotting graph of the function $f(x)=x+1$, the required area is the shaded region shown in Figure 10. Clearly, the base of this triangle is of length $x-(-1)=x+1$ and the height is the value of $f(x)$ at the end point and it is $x+1$. Thus the required area is

$$A(x) = \frac{1}{2}(x + 1)(x + 1) = \frac{x^2}{2} + x + \frac{1}{2}$$

And $A'(x) = x + 1 = f(x)$

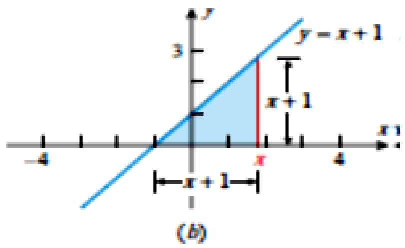

Figure 10.

Solution (c): After plotting graph of the function $f(x)=2x+3$, the required area is the shaded region shown in Figure 11, which is a trapezium.

The lengths of the parallel sides of this trapezium are the value of the function $f(x)$ at the endpoints -1 and x of the interval, which is 1 and $2x+3$.

Also the distance between the parallel sides of this trapezium is $x-(-1)=x+1$.

Recall that the formula for the area of a trapezium is A $= \frac{1}{2}(b_1 + b_2)$h, where b_1 and b_2 denote the lengths of the parallel sides of the trapezoid, and the altitude h denotes the distance between the parallel sides. Thus the required area is given by

$$A(x) = \frac{1}{2}\left((2x + 3) + 1\right)\left(x - (-1)\right) = x^2 + 3x + 2$$

For this area function $A'(x) = 2x + 3 = f(x)$.

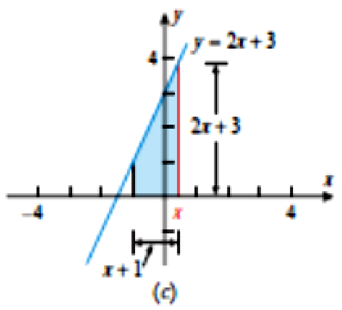

Figure 11.

We summarize the outcome of example 1 above as follows:
Let $f(x)$ be a continuous function defined in the closed interval $[a,b]$ where a and b are any real numbers. Let x be any arbitrary point in the closed interval $[a,b]$. Then the area of the region bounded by the curve $y = f(x)$, the $x-$axis, the line $x = a$ and the line drawn parallel to $x = a$ at the point x, always depends upon the value of x and hence is a function of x. We denote this function by $A(x)$ and call it as the **Area Function**. We also note that derivative of the area function is always equal to the function $f(x)$, that $A'(x) = f(x)$

Example 2 Use the anti-derivative method to find the area under the graph of
$y = x^2$ over the interval $[0, 1]$.

Solution Let x be any point in the interval [0, 1], and let A(x) denote the area under the graph of $f(x) = x^2$ over the interval [0, x]. Then

$$A'(x) = x^2 \qquad (1)$$

By the definition of anti-derivative we get $A(x) = \int A'(x)dx + C$

or
$$A(x) = \frac{1}{3}x^3 + C \qquad (2)$$

for some real constant C. We can determine the specific value for C by considering the case where x = 0. In this case

$$A(0) = C \qquad (3)$$

But if x = 0, then the interval [0, x] reduces to a single point. If we agree that the area above a single point should be taken as zero, then A(0) = 0 and (3) implies that C = 0. Thus, it follows from (2) that $A(x) = \frac{1}{3}x^3$ is the area function we are seeking. This implies that the area under the graph of $y = x^2$ over the interv [0, 1] is $A(1) = \frac{1}{3}(1^3) = \frac{1}{3}$. This is consistent with the result that we previously obtained numerically.

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✔ **QUICK ACTIVITY EXERCISES 3.2 (*See after exercises for answers.*)**

1. Let R denote the region below the graph of $f(x) = \sqrt{1 - x^2}$ and above the interval $[-1, 1]$.

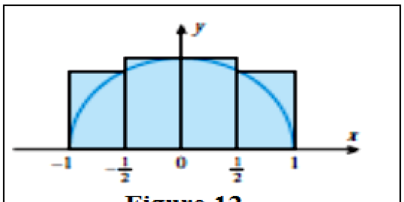

Figure 12.

(a) Use a geometric argument to find the area of R.

(b) What estimate results if the area of R is approximated by the total area within the rectangles of the accompanying figure?

2. Suppose that when the area A between the graph of a function $y = f(x)$ and an interval $[a, b]$ is approximated by the areas of n rectangles, the total area of the rectangles is $An = 2 + \left(\frac{2}{n}\right), n = 1, 2, \ldots$. Then, $A =$ ... ...
3. The area under the graph of $y = x^2$ over the interval $[0, 3]$ is……………………..
4. Find a formula for the area $A(x)$ between the graph of the function $f(x) = x$ and the interval $[0, x]$, and verify that $A'(x) = f(x)$.
5. The area under the graph of $y = f(x)$ over the interval $[0, x]$ is $A(x) = x + e^x - 1$. It follows that $f(x) =$ ... ...

✔ EXERCISES SET 3.2

1. Use a calculating utility with summation capabilities or a CAS to obtain an approximate value for the area between the curve *y* = *f(x)* and the specified interval with *n* = 10, 20, and 50 subintervals using the (a) left endpoint, (b) midpoint, and (c) right endpoint approximations.

a) $f(x) = \sqrt{x}$	; $[a, b] = [0, 4]$
b) $f(x) = \frac{1}{x}$	; $[a, b] = [1, 2]$
c) $f(x) = \sin x$	; $[a, b] = [0, \pi/2]$
d) $f(x) = \frac{1}{x^2}$	; $[a, b] = [1, 3]$

2. Graph each function over the specified interval. Then use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function f and the interval $[a, x]$. Confirm that $A'(x) = f(x)$

$a)\ f(x) = 3 \quad ; [a, x] = [1, x]$
$b)\ f(x) = 5 \quad ; [a, x] = [2, x]$
$c)\ f(x) = 2x + 2 \quad ; [a, x] = [0, x]$
$d)\ f(x) = 3x - 3 \quad ; [a, x] = [1, x]$
e) $f(x) = 2x + 2 \quad ; [a, x] = [1, x]$
$f)\ f(x) = 3x - 3 \quad ; [a, x] = [2, x]$

✔ QUICK ACTIVITY ANSWERS 3.2

1. $(a)\frac{\pi}{2}$, (b) $1 + \frac{\sqrt{3}}{2}$, **2.** 2 , **3.** 9 ,

4. $A(x) = \frac{x^2}{2}$, $A'(x) = x$, **5.** $e^x + 1$.

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3.3 FUNDAMENTAL THEOREM AND THE MEAN-VALUE THEOREM OF INTEGRAL CALCULUS.

Let $f(x)$ be a continuous function defined in the closed interval $[a,b]$ where a and b are any real numbers. Let x be any arbitrary point in the closed interval $[a,b]$. Then the area of the region bounded by the curve $y = f(x)$, the $x-$axis, the line $x = a$ and the line drawn parallel to $x = a$ at the point x, always depends upon the value of x. In other words this area is a function of x and is given by $\int_a^x f(x)\,dx$. We denote this function by $A(x)$ and call it as the **Area Function**.

Theorem1. (First Fundamental Theorem of Integral Calculus, *Part 1*)

Let $f(x)$ be a continuous function defined in the closed interval $[a,b]$ and $A(x)$ be the area function. Then $A'(x) = f(x)$, where $A'(x)$ denotes the derivative of the function $A(x)$ with respect to x.

Theorem2. (Second Fundamental Theorem of Integral Calculus, *Part 1*) Let $f(x)$ be a continuous function defined in the closed interval $[a,b]$ and let $F(x)$ be the anti- derivative of $f(x)$, that is $\int f(x)\,dx = F(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note: 1) The above theorem 2 gives us a direct method to evaluate the definite integral $\int_a^b f(x)\,dx$ without using the limit of a sum.

2) In the definite integral $\int_a^b f(x)\,dx$ it is necessary that the function $f(x)$ is continuous and well defined in the closed interval $[a,b]$.

For example $\int_{-2}^{3} x^2\sqrt{x^2-1}\,dx$ is not meaningful as the function $f(x)=x^2\sqrt{x^2-1}$ becomes imaginary in $-1<x<1$ and hence not defined or not continuous in $-1<x<1$ which is a part of the closed interval $[a,b]$.

Algorithm: To evaluate $\int_a^b f(x)\,dx$ using fundamental theorem of integral calculus.

Step 1: Evaluate the indefinite integral $\int f(x)\,dx$ by using any technique of integration. Let $\int f(x)\,dx = F(x)$. Here there is no need of taking the constant of integration.

Step 2: Find the value of the function $F(x)$ at $x=a$ and $x=b$, that is evaluate $F(a)$ and $F(b)$.

Step 3: Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Example 1. Evaluate

$$\int_1^2 x\,dx$$

Solution:

$$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{4}{2}-\frac{1}{2} = \frac{3}{2}$$

Example 2. Evaluate

$$\int_0^3 (9-x^2)\,dx$$

Solution:

$$\int_0^3 (9-x^2)dx = \left[9x - \frac{x^3}{3}\right]_0^3 = 27 - \frac{27}{3} = 18$$

Example 3. Evaluate

$$\int_0^{\pi} \cos\theta \, d\theta$$

Solution:

$$\int_0^{\pi} \cos\theta \, d\theta = [\sin\theta]_0^{\pi} = \sin\pi - \sin 0 = 0$$

Example 4. Evaluate

$$\int_1^4 x\sqrt{x}\, dx$$

Solution:

$$\int_1^4 x\sqrt{x}\, dx = \int_1^4 x^{\frac{3}{2}}\, dx = \left[\frac{2}{5}\, x^{\frac{5}{2}}\right]_1^4 = \frac{2}{5}\left(4^{\frac{5}{2}} - 1^{\frac{5}{2}}\right) = \frac{62}{5}$$

Example 5. Evaluate

$$(a) \int_0^{\frac{\pi}{2}} 5 \sin x \, dx \qquad (b) \int_0^{\frac{\pi}{3}} \sec^2 x \, dx \qquad (c) \int_0^{\ln 3} 2e^x \, dx$$

$$(d) \int_1^{e^2} \frac{1}{x}\, dx \qquad (e) \int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$$

Solution:

$$(a) \int_0^{\frac{\pi}{2}} 5 \sin x \, dx = -5[\cos x]_0^{\frac{\pi}{2}} = -5\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= -5(0-1) = 5$$

$$(b) \int_0^{\frac{\pi}{3}} \sec^2 x \, dx = [\tan x]_0^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$(c) \int_0^{\ln 3} 2e^x \, dx = 2\,[e^x]_0^{\ln 3} = 2\,(e^{\ln 3} - e^0) = 2(3-1)$$

$$= 4$$

$$(d)\ \int_1^{e^2} \frac{1}{x}\, dx = [\ln|x|]_1^{e^2}$$

$$= \ln|e^2| - \ln 1 = 2 - 0 = 2$$

$$(e)\ \int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = 2\frac{\pi}{6} = \frac{\pi}{3}$$

Example 6. Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx$.

Solution: We will apply the above step by step procedure as follows :

$\int \sqrt{1+\sin 2x}\, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx$

(using the trigonometric identities) = $\int \sqrt{(\sin x + \cos x)^2}\, dx$

= $\int (\sin x + \cos x) dx$ = $-\cos x + \sin x$ = $F(x)$. (Note that we are not taking the constant of integration here).

Now, $F(0) = -\cos 0 + \sin 0$ = -1 and $F\left(\frac{\pi}{4}\right)$ = $-\cos\frac{\pi}{4} + \sin\frac{\pi}{4}$ = $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$ = 0. Then $\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx$ = $F(b)$ - $F(a)$ = $F\left(\frac{\pi}{4}\right)$ - $F(0)$ = 0-(-1) = 1.

Example 7. Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x\, dx$.

Solution: $\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \frac{3\cos x + \cos 3x}{4} dx$ (using the trigonometric identity $\cos 3x = 4\cos^3 x - 3\cos x$ and so $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$)

$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \cos x \, dx + \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos 3x \, dx = \frac{3}{4}[\sin x]_0^{\frac{\pi}{2}} + \frac{1}{4}\left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{2}}$

$= \frac{3}{4}\left(\sin\frac{\pi}{2} - \sin 0\right) + \frac{1}{4}\left(\frac{\sin\frac{3\pi}{2} - \sin 0}{3}\right) = \frac{3}{4}(1-0) + \frac{1}{4}\left(\frac{-1-0}{3}\right) = \frac{8}{12} =$

$\frac{2}{3}$.

Example 8. Evaluate $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}dx$.

Solution: Integrating by parts taking x^2+x as the first function and $\frac{1}{\sqrt{2x+1}}$ as the second function, we get $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}dx = [(x^2+x)\sqrt{2x+1}]_2^4$ - $\int_2^4 (2x+1)\sqrt{2x+1}\,dx$

$= [(x^2+x)\sqrt{2x+1}]_2^4$ - $\int_2^4 (2x+1)^{\frac{3}{2}} dx$ = $[(x^2+x)\sqrt{2x+1}]_2^4$ - $\frac{1}{5}\left[(2x+1)^{\frac{5}{2}}\right]_2^4$

$= (60-6\sqrt{5})$ - $\frac{1}{5}(243-25\sqrt{5})$ = $(60-6\sqrt{5})$ - $\left(\frac{243}{5}-5\sqrt{5}\right) = \left(\frac{57}{5}-\sqrt{5}\right)$

.**Example 9.** Evaluate $\int_1^2 \frac{dx}{x(1+x^2)}$.

Solution: Here we first evaluate $\int \frac{dx}{x(1+x^2)}$ using partial fractions.

Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$ …… (1)

$$\frac{1}{x(1+x^2)} = \frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)} = \frac{(A+B)x^2+Cx+A}{x(1+x^2)}$$

Equating the coefficients of like powers of x in the numerator on both sides, we have $A=1$ (constant terms)

$C=0$ (coefficients of x)

$A+B=0$ (coefficients of x^2)

So $B=-1$. Substituting these values in (1), we get $\frac{1}{x(1+x^2)}$ = $\frac{1}{x}-\frac{x}{1+x^2}$. Therefore $\int\frac{dx}{x(1+x^2)} = \int\frac{1}{x}dx$ - $\int\frac{x}{1+x^2}dx$

$$= \ln|x| - \int\frac{x}{1+x^2}dx \quad \ldots\ldots(2)$$

Now to evaluate $\int\frac{x}{1+x^2}dx$, put $1+x^2=t$, $2x\,dx=dt$.

Then $\int\frac{x}{1+x^2}dx = \frac{1}{2}\int\frac{1}{t}dt = \frac{1}{2}[\ln|t|] = \frac{1}{2}\ln(1+x^2)$. Using this in (2), we get

$$\int\frac{dx}{x(1+x^2)} = \ln|x| - \frac{1}{2}\ln(1+x^2) = \ln|x| - \ln\sqrt{1+x^2} = ln\left|\frac{x}{\sqrt{1+x^2}}\right| = F(x).$$

Therefore $\int_1^2\frac{dx}{x(1+x^2)} = F(2)-F(1) = ln\frac{2}{\sqrt{5}} - ln\frac{1}{\sqrt{2}} = ln\frac{2\sqrt{2}}{\sqrt{5}}$.

Example 10. Evaluate $\int_0^2 x\ (x^2+1)^3\,dx$.

Solution: Put $u = x^2+1$ $so\ that$ $du = 2x\,dx$.
Then we have

$$\int x\ (x^2+1)^3\,dx = \frac{1}{2}\int u^3\ du = \frac{1}{8}\ u^4 = \left[\frac{[(x^2+1)^4]}{8}\right]$$

$$\int_0^2 x\ (x^2+1)^3\ dx = \left[\frac{(x^2+1)^4}{8}\right]_0^2 = \frac{625}{8} - \frac{1}{8} = \frac{624}{8} = 78\ .$$

Example 11. Evaluate

$$(a) \int_0^{\frac{\pi}{8}} sin^5\, 2x\, cos\, 2x\ \ dx \qquad\qquad (b) \int_2^5 (2x-5)(x-3)^9\, dx$$

Solution (a): Put $u = sin\, 2x$ $so\ that$ $du = 2cos\ 2x\ dx$. Then:

$$\int sin^5 2x\ cos2x\, dx = \frac{1}{2}\int u^5\, du = \frac{u^6}{12} = \frac{(sin2x)^6}{12} =$$

$$\int_0^{\frac{\pi}{8}} sin^5\, 2x\, cos\, 2x\ \ dx = \left[\frac{(sin2x)^6}{12}\right]_0^{\frac{\pi}{8}} = \frac{1}{96}$$

Solution (b): Put $u = x - 3$, $so\ that$ $x = u + 3$ and $du = dx$.

Then $\int (2x-5)(x-3)^9\, dx = \int (2u+1)u^9 du = \frac{2u^{11}}{11} + \frac{u^{10}}{10} =$

$\frac{2(x-3)^{11}}{11} + \frac{(x-3)^{10}}{10}$ and

$$\int_2^5 (2x-5)(x-3)^9\, dx = \left[\frac{2(x-3)^{11}}{11} + \frac{(x-3)^{10}}{10}\right]_2^5 = \frac{52233}{110}\ .$$

3.3.1 Part 2 of The Fundamental Theorem Of Calculus.

Theorem 3.
(***The Fundamental Theorem of Calculus, Part 2***) If f is continuous on an interval, then f has an anti- derivative on that interval. In particular, if a is any point in the interval, then the function f defined by

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Example 12.
Using ***The Fundamental Theorem of Calculus, Part 2*** to find the following

$$(a)\ \frac{d}{dx}\left[\int_0^x t^3 dt\right] = x^3 \qquad (b)\ \frac{d}{dx}\left[\int_\pi^x \frac{\sin t}{t} dt\right] = \frac{\sin x}{x}$$

$$(c)\ \frac{d}{dx}\left[\int_1^x \frac{t}{t^2+4} dt\right] = \frac{x}{x^2+4} \qquad (d)\ \frac{d}{dx}\left[\int_0^x \cos t^2\, dt\right] = \cos x^2$$

3.3.2 The Mean-Value Theorem Of Integral Calculus.

Let f be a continuous nonnegative function on $[a, b]$, and let m and M be the minimum and maximum values of $f(x)$ on this interval. Consider the rectangles of heights m and M over the interval $[a, b]$ (Figure 2a.). Then area of the region below the curve $y = f(x)$, above the $x-$ axis and between the lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)dx$$

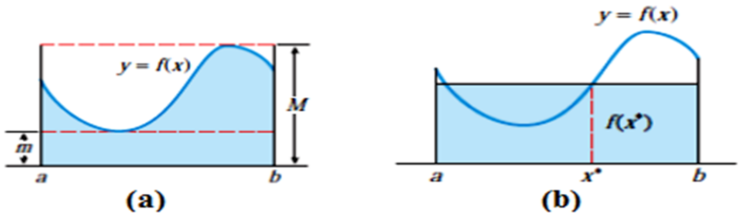

Figure 2.

Geometrically it is clear from this figure that the area under $y = f(x)$ is at least as large as the area of the rectangle of height m and not larger than the area of the rectangle of height M. It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height say $f(x^*)$ between m and M whose area is precisely A (See Figure 2b) , that is

$$\int_a^b f(x)dx = f(x^*)(b-a)\ .$$

This can be summarized as follows :

Theorem 4. (*The Mean-Value Theorem of Integral Calculus*) If f is continuous on a closed interval $[a, b]$, then there is at least one point x^* in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(x^*)[b-a]$$

Example 13. Find x^* that satisfy ***The Mean-Value Theorem of Integral Calculus*** for $y = 2x + 3$ on the interval [1,3].

Solution: By *Mean-Value Theorem of Integral Calculus*

$$\int_a^b f(x)\, dx = f(x^*)[b - a]$$

Here $a = 1$, $b = 3$ and $f(x) = 2x + 3$. Thus

$$\int_1^3 (2x + 3)\, dx = (2x^* + 3)[3 - 1]$$

$$14 = 2(\, 2x^* + 3)$$

$$7 = 2x^* + 3$$

$$x^* = 2$$

Example 14. Find x^* that satisfy ***The Mean-Value Theorem of Integral Calculus*** for $f(x) = x^2$on the interval [1, 4].

Solution:

$$\int_1^4 x^2\ dx = (x^*)^2[4 - 1]$$

$$21 = 3\,(x^*)^2$$

$$\mathrm{x}^* = \pm\sqrt{7}$$

Thus, $\mathrm{x}^* = \sqrt{7} \approx 2.65$
is the point in the interval [1, 4] whose existence is guaranteed by the Mean-Value Theorem of Integral Calculus.

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✔ **QUICK ACTIVITY EXERCISES 3.3 (*See after exercises for answers.*)**

- Evaluate :

1. $\frac{d}{dx}\left[\int_1^x t^3 dt\right] = \ldots\ldots\ldots\ldots,$	**2.** $\frac{d}{dx}\left[\int_0^x \frac{\tan t}{t}\, dt\right] = \ldots\ldots\ldots\ldots$
3. $\int_0^2 (3x^2 - 2x)dx = \ldots\ldots\ldots\ldots,$	**4.** $\int_{-\pi}^{\pi} \cos x\, dx = \ldots\ldots\ldots\ldots$
5. $\int_{\frac{2}{\pi}}^{\frac{2}{\pi}} \sin\theta\cos\theta\, d\theta = \ldots\ldots,$	**6.** $if \int_2^3 f(x)\, dx = 5\ \ then \int_3^2 4\, f(x)dx = \ldots\ldots\ldots$

✔ **EXERCISES SET 3.3**

1. Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.

(1) $\int_{-2}^{1} (x^2 + 6\mathrm{x} + 12)\ \mathrm{dx}$	(6) $\int_0^{\frac{\pi}{4}} sec^2\,\theta\, d\theta$
(2) $\int_{-1}^{2} 4x(1 - x^2)\, dx$	(7) $\int_{\sqrt{e}}^{e} \frac{lnx}{x^2}\, dx$
(3) $\int_1^4 \frac{4}{x^2}\ dx$	(8) $\int_4^9 2x\sqrt{x}\, dx$
(4) $\int_1^2 \frac{dx}{x^6}$	(9) $\int_0^1 (5x + 2)^2 dx$
(5) $\int_1^e x^2 lnx\, dx$	(10) $\int_0^{\frac{\pi}{3}} \theta\ sin\ \theta\, d\theta$

2. Find all values of $x*$ in the stated interval that satisfy the Mean-Value Theorem for Integrals and explain what these numbers represent.

$(a) f(x) = \sqrt{x}$,	$[0, 3]$
$(b) f(x) = x^2 + x;$	$[-12, 0]$
$(c)\ f(x) = \sin x;$	$[-\pi, \pi]$
$(b)\ f(x) = \frac{1}{x^2};$	$[1, 3]$.

3. Evaluate the definite integral two ways: first by a u substitution in the definite integral and then by a *u*-substitution in the corresponding indefinite integral.

$$(a)\int_0^1 (2x+1)^3\,dx \qquad (b)\int_1^2 (4x-2)^5\,dx$$

$$(c)\int_0^8 x\sqrt{1+x}\ dx \qquad (d)\int_{-3}^0 x\sqrt{1-x}\ dx$$

$$(e)\int_{-2}^{-1} \frac{x}{(x^2+2)^3}\,dx \qquad (f)\int_{-2}^{-1} \frac{x}{x^2+2}\ dx$$

4. Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

$$(a)\frac{d}{dx}\left[\int_0^x \frac{\sin y}{1+y^2}\ dy\right] \qquad (b)\frac{d}{dx}\left[\int_1^x \frac{dt}{1+\sqrt{t}}\ dt\right]$$

$$(c)\frac{d}{dx}\left[\int_{ln2}^x e^{\sqrt{t}}\ dt\right] \qquad (d)\frac{d}{dx}\left[\int_0^x \ln y\ dy\right]$$

5. Let

$$F(x) = \int_0^x \sqrt{t^2 + 9}\ dt$$

Find $(a)\ F(4)$ $(b)\ F'(4)$ $(c)\ F''(4)$

✔ QUICK ACTIVITY ANSWERS 3.3

1. x^3 , 2. $\frac{\tan x}{x}$, 3. 4 , 4. 0 , 5. 0 , 6. -20.

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3.4 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION METHOD.

3.4.1 Introduction.

In the definite integral $\int_a^b f(x)\,dx$, we know that a and b are the lower limit and upper limits. These are also known as the limits of integration and these limits are for the variable x of the integrand $f(x)$. To evaluate the definite integral $\int_a^b f(x)\,dx$, sometimes proper substitution will reduce the integral $\int f(x)\,dx$ to a standard form in a new variable t, and then finding out the new limits of integration for the new variable t will reduce the definite integral $\int_a^b f(x)\,dx$ to a standard definite integral in the new variable t which can then be evaluated by using second fundamental theorem of integral calculus.

We summarize the above in the following

Algorithm: To evaluate $\int_a^b f(x)\,dx$ by substitution method.

Step 1: Make a proper substitution say $g(x)=t$, to reduce to the integrand $f(x)$ to some standard form say $h(t)$ in the new variable t.

Step 2: Find the new limits of integration for the new variable t as $t=g(a)$ (lower limit of t) and $t=g(b)$ (upper limit of t).

Step 3: Then $\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} h(t)\,dt$.

Step 4: Now evaluate $\int_{g(a)}^{g(b)} h(t)\,dt$ using second fundamental theorem of integral calculus.

Example 1. Evaluate $\int_2^4 \frac{x}{x^2+1}dx$.

Solution: Here $a=2$, $b=4$ and $f(x)=\frac{x}{x^2+1}$. Put $x^2+1=t$, $2x\,dx=dt$.

Now we change the limits of integration as follows.

When $x=2$, $t=2^2+1$ = 5. (Here $g(x)=x^2+1$ and $t=g(a)$)

When $x=4$, $t=4^2+1$ = 17. ($t=g(b)$)

Then, $\int_2^4 \frac{x}{x^2+1}dx = \frac{1}{2}\int_5^{17}\frac{1}{t}dt = \frac{1}{2}[\ln t]_5^{17} = \frac{1}{2}(ln17 - ln5)$

(By second fundamental theorem of integral calculus).

Example 2. $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$.

Solution: Here $a=-1$, $b=1$ and $f(x)=5x^4\sqrt{x^5+1}$. Now put $x^5+1=t$, $5x^4dx=dt$, when $x=-1$, $t=(-1)^5+1$ = 0 and when $x=1$, $t=1^5+1$ = 2.

Then $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt = \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left(\sqrt{8}-0\right) = \frac{2}{3}\sqrt{8}$.

(By second fundamental theorem of integral calculus).

Example 3. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$.

Solution: Put $\sin x = t$, $\cos x\,dx = dt$. When $x=0$, $t=\sin 0$ = 0 and when $x=\frac{\pi}{2}$, $t=\sin\frac{\pi}{2}$ = 1. Then

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)}dx = \int_0^1 \frac{dt}{(1+t)(2+t)} \qquad \ldots\ldots. \quad (1)$$

We will evaluate $\int_0^1 \frac{dt}{(1+t)(2+t)}$ by resolving to partial fraction.

Let $$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t}+\frac{B}{2+t} \qquad \ldots\ldots. \quad (2)$$

$$\frac{1}{(1+t)(2+t)} = \frac{A(2+t)+B(1+t)}{(1+t)(2+t)}$$

Since the denominators are equal in the above equation we can equate the numerators to get $A(2+t)+B(1+t)=1$.Putting $t=0$, we get $2A+B=1$ and putting $t=-1$, we get $A=1$.

(Students should note the technique used above. After writing equation (2) in simplified form to make denominators equal on both sides, we equate the numerators. Then by putting simple values of the variable we form equations in terms of the unknown.)

Solving the two equations $2A+B=1$ and $A=1$, we get $A=1$ and $B=-1$. Substituting these values in (2), we have

$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t}-\frac{1}{2+t}$. Therefore

$$\int_0^1 \frac{dt}{(1+t)(2+t)} = \int_0^1 \frac{1}{1+t}dt - \int_0^1 \frac{1}{2+t}dt = [\ln(1+t)]_0^1 - [\ln(2+t)]_0^1$$

$= ln2 - ln1 - ln3 + ln2 = 2ln2 - ln3 = ln\frac{4}{3}$. (By second fundamental theorem of integral calculus).

Example 4. Evaluate $\int_0^1 \frac{x\tan^{-1}x}{\left(1+x^2\right)^{\frac{3}{2}}}dx$.

Solution: Put $\tan^{-1}x=t$, that is $x=\tan t$, $dx=\sec^2 t\,dt$ and $1+x^2=1+\tan^2 t=\sec^2 t$.

When $x=0$, $t=\tan^{-1}0=0$ and when $x=1$, $t=\tan^{-1}1=\frac{\pi}{4}$. Then

$$\int_0^1 \frac{x\tan^{-1}x}{\left(1+x^2\right)^{\frac{3}{2}}}dx = \int_0^{\frac{\pi}{4}} \frac{t\tan t\sec^2 t}{\left(\sec^2 t\right)^{\frac{3}{2}}}dt = \int_0^{\frac{\pi}{4}} \frac{t\tan t}{\sec t}dt = \int_0^{\frac{\pi}{4}} t\sin t\,dt.$$

Now integrating by parts taking t as first function and $\sin t$ as second function, we have

$$\int_0^{\frac{\pi}{4}} t\sin t\,dt = \left[-t\cos t\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos t\,dt = \left[-t\cos t\right]_0^{\frac{\pi}{4}} + \left[\sin t\right]_0^{\frac{\pi}{4}}$$

$$= \left(-\frac{\pi}{4}\cos\frac{\pi}{4}-0\right)+\left(\sin\frac{\pi}{4}-\sin 0\right) = -\frac{\pi}{4}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right).$$

3.4.2 Properties of Definite Integral.

In this section we will discuss various properties of definite integrals which will be very useful in evaluating many definite integrals. We list below some properties of definite integrals without proof.

Property 1: $\int_a^b f(x)dx = \int_a^b f(t)dt$.

The above property says that integration is independent of change of variable.

Property 2: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $a\le c\le b$.

Property 3: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

Property 4: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$.

Property 5: $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if} \quad f(2a-x) = f(x) \\ 0 & \text{if} \quad f(2a-x) = -f(x) \end{cases}$

Property 6:
$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is an even function , i.e. } f(-x) = f(x) \\ 0 & \text{if } f(x) \text{ is an odd function , i.e. } f(-x) = -f(x) \end{cases}$

Now we will apply the above properties to evaluate certain definite integrals.

Example 5. Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$.

Solution: Let $f(x) = \sin^5 x \cos^4 x$, then $f(-x) = \sin^5(-x)\cos^4(-x)$

$= (\sin(-x))^5 (\cos(-x))^4 = (-\sin x)^5 (\cos x)^4 = -\sin^5 x \cos^4 x =$ $-f(x)$. Thus $f(x) = \sin^5 x \cos^4 x$ is an odd function and so by property 6 $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx = 0$.

Example 6. Evaluate $\int_{-1}^{2} |x^3 - x|\,dx$.

Solution: In such problems we will first check where does the given function ($x^3 - x$ in this case) cuts the real line. This can be done by solving the equation $x^3 - x = 0$, because if we put $y = x^3 - x$ then this curve will cut the x-axis at the point where $y = 0$, that is $x^3 - x = 0$. Now on solving $x^3 - x = 0$, we get $x = 0, -1, 1$. Note that $x^3 - x > 0$ for $x \in [-1, 0]$ (this can be checked by putting any value between -1 and 0 in $x^3 - x$), $x^3 - x < 0$ for $x \in [0,1]$ and $x^3 - x > 0$ for $x \in [1, 2]$. Then by definition of modulus function we have $\left|x^3-x\right|=x^3-x$ for $x\in[-1,0]$, $\left|x^3-x\right|=-(x^3-x)=x-x^3$ for $x\in[0,1]$ and $\left|x^3-x\right|=x^3-x$ for $x\in[1,2]$. Thus by using property 2, we have

$$\int_{-1}^{2}\left|x^3-x\right|dx = \int_{-1}^{0}\left|x^3-x\right|dx + \int_{0}^{1}\left|x^3-x\right|dx + \int_{1}^{2}\left|x^3-x\right|dx$$

$$= \int_{-1}^{0}(x^3-x)\,dx + \int_{0}^{1}(x-x^3)\,dx + \int_{1}^{2}(x^3-x)\,dx = \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left[\frac{1}{4}-\frac{1}{2}\right] + \left[\frac{1}{2}-\frac{1}{4}\right] + (4-2) - \left[\frac{1}{4}-\frac{1}{2}\right] = \frac{11}{4}.$$

Example 7. Evaluate $\displaystyle\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$.

Solution: Let $\mathrm{I} = \displaystyle\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$. Then by property 3 we have

$$\mathrm{I}=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)(\sin x)}{1+\cos^2 x}dx$$

(since $\sin(\pi-x)=\sin x$ and $\cos(\pi-x)=-\cos x$)

$$\mathrm{I} = \int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx - \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx - \mathrm{I}$$

$$2\mathrm{I} = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx \quad \ldots\ldots (1)$$

Now, put $\cos x=t$, $-\sin x dx=dt$. When $x=0$, $t=1$ and when $x=\pi$, $t=-1$. Then $\displaystyle\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx = \int_1^{-1}\frac{-dt}{1+t^2} = \int_{-1}^{1}\frac{dt}{1+t^2} = 2\int_0^{1}\frac{dt}{1+t^2}$ (by property 6, since $\dfrac{1}{1+t^2}$ is an even function) $= 2\left[\tan^{-1}t\right]_0^1$ =

$2\left[\tan^{-1}1-\tan^{-1}0\right]$ = $2\left(\frac{\pi}{4}-0\right)$ = $\frac{\pi}{2}$ Then from (1) we get 2I = $\pi.\frac{\pi}{2}$ = $\frac{\pi^2}{2}$. Therefore I = $\frac{\pi^2}{4}$.

Example 8. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$.

Solution: Let I = $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$ ….. (1) . Then by property 3, we have

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \quad \text{or} \quad I= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx \qquad \text{…..} \quad (2)$$

Adding (1) and (2), we have

2I = $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$ + $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx$ = $\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\cos x+\sin x}dx$ = $\int_0^{\frac{\pi}{2}} 1dx$ = $[x]_0^{\pi/2}$ = $\frac{\pi}{2}$ .Therefore, I = $\frac{\pi}{4}$.

Example 9. Evaluate

$$\int_0^{\pi/2} \ln\sin x\, dx$$

Solution: Let

$$I = \int_0^{\pi/2} \ln\sin x\, dx \qquad \dots\dots (1)$$

Then by property 3 we find that

$$I = \int_0^{\pi/2} \ln\sin\left(\frac{\pi}{2}-x\right)\, dx \quad \text{or} \quad I = \int_0^{\pi/2} \ln\cos x\, dx \quad \dots\dots\dots \ (2)$$

Adding (1) and (2)

$$2I = \int_0^{\pi/2} (\ln \sin x + \ln \cos x)\, dx$$

$$= \int_0^{\pi/2} (\ln \sin x + \ln \cos x + \ln 2 - ln2)\, dx$$

(Adding and subtracting log2 in the integrand. Students are advised to kindly make note of the need of this addition and subtraction in the next steps)

$$2I = \int_0^{\pi/2} (\ln 2\sin x \cos x - \ln 2\,) dx$$

$$= \int_0^{\pi/2} (\ln 2\sin x \cos x - \int_0^{\pi/2} \ln 2\, dx$$

$$\int_0^{\pi/2} (\ln \sin 2x\, dx - \int_0^{\pi/2} \ln 2\;\, dx = \int_0^{\pi/2} (\ln \sin 2x\, dx - \frac{\pi}{2} ln2$$

(by evaluating the second integral), that is

$$2I = \int_0^{\pi/2} (\ln \sin 2x\, dx - \frac{\pi}{2} ln2 \ldots\ldots \qquad (3).$$

Now put $2x = t$, $dx = \frac{dt}{2}$, When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$. Then

$$\int_0^{\pi/2} (\ln \sin 2x\, dx = \frac{1}{2}\int_0^{\pi} \ln \sin t\, dt = \frac{1}{2}\int_0^{2\pi/2} \ln \sin t\, dt \quad ; \quad (note)$$

$= \frac{1}{2}. 2 \int_0^{\pi/2} \ln \sin t\, dt$ (By property 5, since $ln\ \sin\left(2\frac{\pi}{2} - t\right) = \ln \sin t$) =

$$\int_0^{\pi/2} \ln \sin t\, dt = I \;\; \Longrightarrow \int_0^{\pi/2} \ln \sin t\, dt = \int_0^{\pi/2} \ln \sin x\, dx = I$$

Hence from (3), we get 2I = I - $\frac{\pi}{2}$ln2 or I = - $\frac{\pi}{2}$ln2.

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✔ **QUICK ACTIVITY EXERCISES 3.4** (***See after exercises for answers.***)

- Evaluate:

$$a) \int_0^{\frac{3}{4}} \frac{dx}{1-x}\ dx \qquad\qquad b) \int_0^{ln3} e^x\,(1+e^x)^{\frac{1}{2}}\,dx$$

✔ **EXERCISES SET 3.4**

- **Evaluate the following integrals by using suitable properties of definite integral:**

1). $\int_0^{\frac{\pi}{2}} \frac{\cos x}{3(\cos x+\sin x)} dx$ (Hint : Use property 3).

2) $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x+\cos^5 x} dx$ (Hint : Use property 3).

3). $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$ (Hint: Multiply numerator and denominator in the radical by $a-x$. The given integral becomes sum of two integrals. Integrand in the first integrand is an even function and in the second integral is an odd function. Now apply property 6.).

4). $\int_0^{\pi} \frac{x}{1+\sin x} dx$ (Hint: Use property 3 and some trigonometric identities).

5) $\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x} dx$ (Hint : Use property 3).

6). $\int_0^4 |x-1|\,dx$ (Hint : Use property 2).

7) $\int_{-5}^{5} |x+2|\,dx$ (Hint : Use property 2).

8) $\int_2^8 |x-5|\,dx$ (Hint : Use property 2).

9). $\int_0^{\pi/4} ln\,(1+\tan x)\,dx$ (Hint : Use property 3).

10). $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$ (Hint : Use property 6).

11) $\int_0^{\frac{\pi}{2}} \ln(1+\cos x)\,dx$ (Hint : Use property 3).

12) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$ (Hint : Use property 6 and property 3).

✔ QUICK ACTIVITY ANSWERS 3.4

- **(a):** Put $u = 1 - x$ *so that* $du = -dx$.

When $x=0$, $u=1$ and when $x=\frac{3}{4}$, $u=\frac{1}{4}$. Then

$$\int_0^{\frac{3}{4}} \frac{1}{1-x}\,dx = -\int_1^{\frac{1}{4}} \frac{1}{u}\,du = [-\ln u]_1^{\frac{1}{4}} = \ln 4 .$$

(b): Put $u = 1+e^x$, *so that* $du = e^x dx$.

$when\ \ x = 0, \qquad u = 1+ e^0 = 2$

$when\ \ x = \ln 3,\ u = 1+ e^{ln3} = 4$. Then

$\int_0^{ln3} e^x (1+e^x)^{\frac{1}{2}} dx = \int_2^4 u^{\frac{1}{2}}\ du = \left[\frac{2}{3}\ u^{\frac{3}{2}}\right]_2^4 = \frac{16-4\sqrt{2}}{3}.$

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3.5 IMPROPER INTEGRALS.

3.5.1 Improper Integrals Of The First Kind.

Definition1. We will consider the improper integral whose integrands are bound on an infinite interval.

Let's $f(x)$ be a function defined on the infinite interval $[a,+\infty)$, and integrable over any bounded closed interval $[a,b] \subset [a,+\infty)$.

The improper integral of $f(x)$ over $[a,+\infty)$ is defined.

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to converge and is integral is said to be proper integral of first kind and the limit is defined the value of the integral. Otherwise the improper integral is said to diverge.

Example1. Evaluate:

$$a)\quad \int_o^{+\infty} \frac{dx}{1+x^2} \quad , \qquad b)\int_1^{+\infty} \frac{dx}{x} \quad , \qquad c)\int_o^{+\infty} \sin x\,dx$$

Solution:

a. Following the definition, let's evaluate the finite integral, then take the limit. This yield.

$$\int_{o}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to+\infty} [tan^{-1}x]_0^b$$

$$= \lim_{b\to+\infty} [tan^{-1}b - tan^{-1}0]$$

$$= \lim_{b\to+\infty} [tan^{-1}b] = \frac{\pi}{2},$$ the integral converges.

b. Following the definition:

$$\int_{1}^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} [\ln x]_1^b$$

$$= \lim_{b\to+\infty} [\ln|b| - \ln 1] = \lim_{b\to+\infty} [\ln|b|] = +\infty,$$ the integral diverges.

c. Following the definition, we have:

$$\int_{0}^{+\infty} \sin x dx = \lim_{b\to+\infty} \int_0^b \sin x dx = -\lim_{b\to+\infty} [\cos x]_0^b$$

$$= -\lim_{b\to+\infty} [\cos b - 1] = -\infty$$

We note that the limit doesn't exist and the integral diverges.

Theorem1. $\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & if \quad p > 1 \\ diferges & if \quad p \le 1 \end{cases}$

Example2. Evaluate

$$\int_{0}^{+\infty} (1-x)e^{-x}\, \mathrm{d}x$$

Solution. We begin by evaluating the indefinite integral using integration by parts. setting $= (1-x)$ and $dv = e^{-x}dx$.

$$\int_{0}^{+\infty} (1-x)e^{-x}\,\mathrm{d}x = - e^{-x}\ (1-x) - \int e^{-x}\,\mathrm{d}x$$

$$= - e^{-x} + xe^{-x} + e^{-x} + \mathrm{c}$$

$$= xe^{-x} + c$$

$$\int_{0}^{b} (1-x)e^{-x}\,\mathrm{d}x = \lim_{b\to+\infty} \int_{0}^{b} (1-x)e^{-x}\,\mathrm{d}x = \lim_{b\to+\infty} [xe^{-x}]_0^b = \lim_{b\to+\infty} \frac{b}{e^b}$$

$$= 0\,;$$

Definition2. In the same manner, we can define the improper integral of the first kind of the function f over the interval $(-\infty, \mathrm{b}]$ or $(-\infty, \infty)$.

Let $f(x)$ be a function defined of the interval $(-\infty, \mathrm{b}]$, and interval $[\mathrm{a}, \mathrm{b}] \subset (-\infty, \mathrm{a}]$ the improper integral of $f(x)$ over $(-\infty, \mathrm{b}]$ is defined.

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)dx$$

The integral is to converge if the limit exists and diverge if it doesn't.

The improper integral of $f(x)$ over $(-\infty, +\infty)$ is defined as:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$

Where c is any real number. The improper integral is said to converge if both terms on the right side converge and diverges if either term diverges.

Remark: The choice of c doesn't matter. In deed:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$

$$= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{c} f(x)\,dx \int_{c}^{+\infty} f(x)\,dx$$

Example 3. Evaluate:

$$a. \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} \quad , \quad b. \int_{-\infty}^{o} \cos x \; d\,x \quad , \quad c. \int_{2}^{+\infty} \frac{dx}{x\,(\ln x)^2}$$

Solution:

Following the definition:

$$\text{a)} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{o} \frac{dx}{1+x^2} + \int_{o}^{+\infty} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{o} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{o}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^o + \lim_{b\to\infty} \left[\tan^{-1} x\right]_o^b = \lim_{a\to-\infty} \left[-\tan^{-1} a\right] + \lim_{b\to\infty} \left[\tan^{-1} b\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

The improper integral converges.

$$b) \int_{-\infty}^{o} \cos x \; d\,x = \lim_{a\to-\infty} \int_{a}^{0} \cos x\,dx = \lim_{a\to-\infty} [\sin x]_a^0 = \lim_{a\to-\infty} [-\sin a]$$

$$= +\infty$$

The limit doesn't exist and the improper integral diverges.

$$\text{c)} \int_{2}^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_2^b == \lim_{b\to\infty} \left[-\frac{1}{\ln|b|} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

The improper integral converges.

3.5.2 Improper Integrals of The Second Kind

Definition3. If f is continuous on the interval $[a, b]$, except for an infinite

discontinuity at b, then the ***improper integral of f over the interval*** $[a, b]$ is defined as

$$\int_a^b f(x)dx = \lim_{k\to b^-} \int_a^k f(x)dx$$

In the case where the indicated limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Example 4.

Evaluate: a) $\int_o^1 \frac{dx}{\sqrt{1-x^2}}$ b) $\int_o^1 \frac{dx}{1-x}$

Solution: We note that the integrand $f(x)=\frac{1}{\sqrt{1-x^2}}$ has $b=1$ a discontinuous point (singular point) in the interval $[0,1]$. Thus:

$$\int_o^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{x\to 1^-} \int_o^k \frac{dx}{\sqrt{1-x^2}} = \lim_{k\to 1^-} [\sin^{-1}x]_o^k = \lim_{k\to 1^-} \sin^{-1}k = \frac{\pi}{2}$$

Then the improper integral converges

b) Following the definition:

$$\int_o^1 \frac{dx}{1-x} = \lim_{k\to 1^-} \int_o^k \frac{dx}{1-x} = -\lim_{k\to 1^-} [\ln|1-x|]_o^k$$

$$= -\lim_{k\to 1^-} [\ln|1-k| - \ln 1] = -\lim_{k\to 1^-} \ln|1-k| = +\infty$$

The limit doesn't exist and the improper integral diverges.

Definition4.

Let $f(x)$ be a continuous function over the interval [a, b], except for a singular point (discontinuity) a, then the improper integral over the interval [a, b] is defined as:

$$\int_a^b f(x)dx = \lim_{k\to a^+}\int_k^b f(x)dx$$

The integral is said to converge if the limit exists and diverge if it doesn't.

Remark: If $f(x)$ is a continuous function over [a, b] except on in (a, b), then the improper integral of f over interval [a, b] is defined as:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The improper integral is said to converge is both terms on the right side converge and diverge if either term on the right side diverges.

Example 5. Evaluate

$$\text{a. } \int_{-1}^{o} \frac{dx}{\sqrt{1-x^2}} \quad , \quad \text{b. } \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \quad , \quad \text{c. } \int_{1}^{4} \frac{dx}{(x-2)^{2/3}}$$

Solution:

Following definition:

$$\text{a) } \int_{-1}^{o} \frac{dx}{\sqrt{1-x^2}} = \lim_{k\to -1^+}\int_{k}^{o} \frac{dx}{\sqrt{1-x^2}} = \lim_{k\to -1^+}\left[\sin^{-1}x\right]_k^o = \lim_{k\to -1^+}\left[-\sin^{-1}k\right] = \frac{\pi}{2}$$

$$\text{b) } \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{o} \frac{dx}{\sqrt{1-x^2}} + \int_{o}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

c) the integral is improper, and the integrand function has a discontinuity point $x = 2$:

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\int_1^k \frac{dx}{(x-2)^{2/3}} + \lim_{k\to 2^+}\int_k^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{k\to 2^-}\left[3(k-2)^{1/3} - 3(1-2)^{1/3}\right] + \lim_{k\to 2^+}\left[3(4-2)^{1/3} - 3(k-2)^{1/3}\right]$$

$$= 3 + 3\sqrt[3]{2}$$

the proper integral converges.

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✔ QUICK ACTIVITY EXERCISES 3.5 (*See after exercises for answers.*)

- In each part, determine whether the integral is Improper and if so explain why. Do not evaluate integrals:

1) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x \, dx$	,	2) $\int_{\frac{\pi}{4}}^{\pi} \cot x \, dx$	,	3) $\int_0^{\infty} \frac{\text{dx}}{1+x^2}$

✔ EXERCISES SET 3.5

1. Evaluate the following integrals:

1) $\int_o^1 \frac{dx}{\sqrt{x}}$, 2) $\int_{-\infty}^{o} xe^x \, dx$, 3) $\int_{-\infty}^{6} \frac{dx}{(4-x^2)^2}$, 4) $\int_o^{\infty} x^3 e$

5) $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2}$, 6) $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx$, 7) $\int_{-\infty}^{o} \frac{dx}{(4-x)^2}$, 8) $\int_o^4 \frac{dx}{4-x}$, 10)

2. Evaluate the following integrals :

1) $\int_{\infty}^{-\infty} \frac{\mathrm{dx}}{1+x^2}$	5) $\int_{1}^{2} \frac{\mathrm{dx}}{1-x}$
2) $\int_{1}^{\infty} \frac{\mathrm{dx}}{x^3}$	6) $\int_{0}^{1} \frac{\mathrm{dx}}{\sqrt{x}(x+1)}$
3) $\int_{0}^{1} \frac{\mathrm{dx}}{\sqrt{1-x}}$	7) $\int_{0}^{\infty} \frac{\mathrm{dx}}{\sqrt{x}(x+1)}$
4) $\int_{1}^{4} \frac{\mathrm{dx}}{(x-2)^{3/2}}$	8) $\int_{0}^{\infty} \mathrm{e}^{-3x} dx$

✔ QUICK ACTIVITY ANSWERS 3.5

- 1) proper 2) improper, since cot x has an infinite discontinuity at $x = \pi$
  3) improper, since there is an infinite interval of integration.

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3.6 NUMERICAL INTEGRATION.

To evaluate the definite integral

$$\int_a^b F(x)\, dx \ ,$$

we must find the fundamental function $F(x)$ of the integrand function $f(x)$, where $f(x)$ is a continuous function on the closed interval $[a, b]$, then we write:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

But in some cases, it is difficult to find the fundamental function and in some cases the function is given by its numerical values in the interval $[a, b]$. In such cases the value of the definite integral can be found by some numerical methods as discussed below.

3.6.1 Trapezoidal Method:

To evaluate the integral

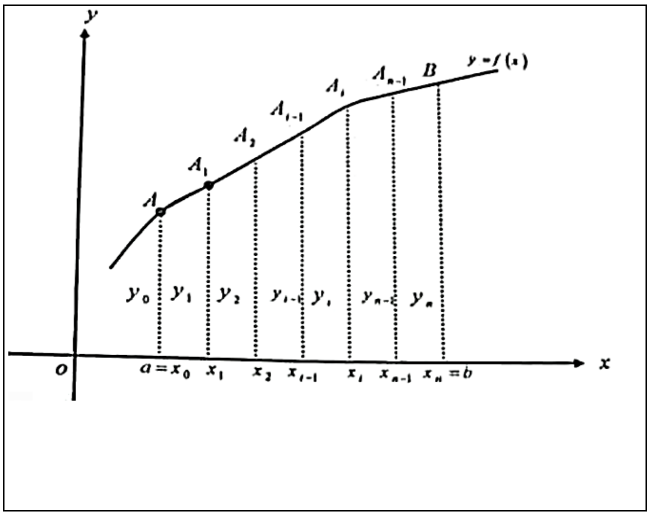
$$\int_a^b F(x)\,dx\,,$$

we divide the interval $[a, b]$ into n equal parts:

$$a = x_o\ ,\ x_1\ ,\ x_2, \cdots, x_{n-1}, x_n = b$$

such that the length of each sub interval is:

$$h = \frac{\mathrm{b} - a}{n} = \Delta x$$



Let $y_o, y_1, \cdots, y_{n-1}, y_n$ be y –coordinate of so that $y_0 = f(x_0)$, $y_1 = f(x_1)$ …. $y_n = f(x_n)$. Then, the area of desired region ($a\ AB\ b$) will be the sum of upright trapezoidal which are bounded from up by arc:

$$AA_1, A_1A_2, \cdots, A_{n-1}A_n$$

Where the area of the first trapezoidal is

$$\left(\frac{y_o + y_1}{2}\right) h,$$

and the area of the second trapezoidal is

$$\left(\frac{y_1 + y_2}{2}\right) h.$$

And the area of the last trapezoidal is

$$\left(\frac{y_{n-1}+y_n}{2}\right)h.$$

Sum of all these areas will be then given by

$$\int_a^b F(x)\,dx = \left(\frac{y_o+y_1}{2}\right)h + \left(\frac{y_1+y_2}{2}\right)h + \cdots + \left(\frac{y_{n-1}+y_n}{2}\right)h$$

$$= \frac{h}{2}(y_o + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

whish's called "trapezoidal rule" to evaluate the definite integral.

Estimation of Error :

We find that the error in the one step is:

$$E = -\frac{1}{12}h^2 f''(x) \quad ; \quad x_o < x < x_1$$

Then the error of the step (subinterval) is:

$$E = -\frac{nh^2}{12} f''(x) \quad ; \quad x \in [x_o, x_n] = [a, b]$$

but$nh = b - a$, then the occurred error by trapezoidal method is given by:

$$E = -\frac{b-a}{12}\mathrm{h}\, f''(x) \quad ; \quad x \in [a, b]$$

Example1. Evaluate the integral

$$\int_o^1 \frac{dx}{1+x^2}$$

by trapezoidal where n = 4, then calculate the occurred error.

Solution:

Divide the interval [0, 1] into four sub intervals of length

$$\mathrm{h} = \frac{\mathrm{b}-\mathrm{a}}{\mathrm{n}} = \frac{1-0}{4} = 0.25$$

then:

x	0	0.25	0.5	0.75	1

y=f(x)	1	0.94	0.8	0.64	0.5

$$\int_{o}^{1} \frac{dx}{1+x^2} = \frac{h}{2}\left[y_o + 2y_1 + 2y_2 + 2y_3 + y_4\right] = \frac{0.25}{2}\left[1+2\left(0.94+0.8+0.64\right)+0.5\right] == 0.7875$$

$$f(x) = \frac{1}{1+x^2}\ ,\ \text{then}: f'(x) = \frac{-2x}{\left(1+x^2\right)^2} \text{ and } f''(x) = \frac{6x^2-2}{\left(1+x^2\right)^3}$$ **, then the** occurred error

$$E = -\frac{b-a}{12} h \max_{0\le x\le 1} f''(x) \le -\frac{(1-0)}{12}(0.25)^5 \max_{o\le x\le 1} f'(x) = -\frac{(0.25)^5}{12}(-2) = 0.5154$$

Example 2. Evaluate the integral $\int_{0.5}^{1} x e^x \, dx$ where $h = 0.1$ and calculate the occurred error.

Solution:

x	0.5	0.6	0.7	0.8	0.9	1
f(x)	0.8244	1.0933	1.4096	1.7804	2.2136	2.7183

$$\int_{0.5}^{1} x e^x \, dx = \frac{h}{2}\left[y_o + 2\left(y_1 + y_2 + y_3 + y_4\right) + y_5\right] = 0.8268$$ then:

$$f(x) = x e^x$$

$$f'(x) = e^x + x e^x$$

$$f''(x) = e^x (2 + x)$$

$$\max f''(x) = f''(1) = e(3) = 8.1548$$

The occurred error:

$$E \le \frac{b-a}{12} h^2 \max_{0.5\le x\le 1} f''(x) = -\frac{(1-0.5)}{12}(0.1)^2 (8.1548) = 0.003398$$

3.6.2 Simpson's Method

We can summarize Simpson's method, by partition the interval $[a,b]$ into n even number of subinterval which are of equal length, then the length of each subinterval:

$$n=\frac{b-a}{h}=\Delta x$$

Substituting the integrand function $f(x)$ with progressive newton's polynomial from second degree which appropriate to $f(x)$, we find:

$$\int_{x_o}^{x_o+2h} f(x)dx=\int_o^2 \left[f_o+\alpha\Delta f_o+\frac{\alpha(\alpha-1)}{2}\Delta^2 f_o\right]h\,d\alpha$$

$$=h\left[f_o\alpha+\frac{\alpha^2}{2}\Delta f_o+\left(\frac{\alpha^3}{6}-\frac{\alpha^2}{4}\right)\Delta^2 f_o\right]_o^2$$

$$=h\left[2f_o+2\Delta f_o+\frac{1}{3}\Delta^2 f_o\right]$$

$$=\frac{h}{3}[f_o+4f_1+f_2]$$

to evaluate the integral $\int_a^b f(x)dx$ on the interval $[a,b]$, we integrate from x_o to $x_2=x_o+2h$, the from x_2 to $x_4,\cdots$, thus from x_{n-2} to x_n, we have:

$$\int_o^b f(x)dx=\int_{a=x_o}^{x_2} f(x)dx+\int_{x_2}^{x_4} f(x)dx+\int_{x_4}^{x_6} f(x)dx+\cdots++\int_{x_{n-2}}^{x_n} f(x)dx=$$

$$\frac{h}{3}[f_o+4f_1+f_2]+\frac{h}{3}[f_2+4f_3+f_4]+\frac{h}{3}[f_4+4f_5+f_6]+\cdots+\frac{h}{3}[f_{n-2}+4f_{n-1}+f_n]$$

$$\Rightarrow\int_a^b f(x)dx=\frac{h}{3}[f_o+4f_1+2f_2+4f_3+2f_4+4f_5+2f_6+\cdots+2f_{n-2}+4f_{n-1}+f_n]$$

Which is called "Simpson's formula" to evaluate the definite integral.

Example 3. Evaluate the integral $\int_o^1 e^x\,dx$ by Simpson's method, which $h = \frac{1}{6}$.

Solution: We have

$$n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

This means that there are even intervals.

So we can apply Simpson's method.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y=f(x)	1	1.1814	1.3956	1.6487	1.9477	2.3010	2.7183

$$\int_o^1 e^x dx = \frac{h}{3}\left[y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right]$$

$$= \frac{1}{18}\left[1 + 4(1.1814 + 1.6487 + 2.3010) + 2(1.3956 + 1.9477) + 2.7183\right] = 1.7183$$

The exact value of the integral is:

$$\int_o^1 e^x dx = \left[e^x\right]_o^1 = e^1 - e^o = 2.7183 - 1 = 1.7183$$

Estimation of error occurred by Simpson's method:

The error occurred in the subinterval $[x_o, x_2]$, where we replace the integrand function by second degree poly no mal, be from the degree of the next term that is:

$$E = h\int_o^2 \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Delta^3 f_o d\alpha$$

$$= \frac{h\Delta^3 f_o}{6}\left[\frac{\alpha^4}{4} - \alpha^3 + \alpha^2\right]_o^2 = \frac{h\Delta^3 f_o}{6}\left[\frac{10}{4} - 8 + 4\right] = 0$$

and therefore the occurred error is from the degree of the next term.

$$E = h\int_{o}^{2} \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\Delta^4 f_o d\alpha = \frac{h\Delta^4 f_o}{24}\int_{o}^{2} \left(\alpha^4 - 6\alpha^3 + 11\alpha^2 - 6\alpha\right)d\alpha$$

$$= \frac{h\Delta^4 f_o}{24}\left[\frac{\alpha^5}{5} - \frac{6\alpha^4}{4} + \frac{11\alpha^3}{3} - 3\alpha^2\right]_o^2 = \frac{h\Delta^4 f_o}{24}\left(-\frac{4}{15}\right)$$

$$\Rightarrow E = \frac{-4h\Delta^4 f_o}{360} = \frac{h\ \Delta^4\ f_o}{90}, \text{ since } \quad f^{(4)}(x) \cong \frac{\Delta^4 f_o}{h^4}, \text{ we get:}$$

$$E = -\frac{h^2}{90} f^{(4)}(x) \quad ; \quad x \in [x_o, x_2]$$

and the occurred error by Simpson method is:

Example 4. Evaluate the integral $\int_{-1}^{1} \frac{dx}{1+e^x}$ by Simpson's method, where $h = 0.25$, and find the occurred error.

Solution: $n = \frac{b-a}{h} = \frac{1-(-1)}{0.25} = \frac{2}{0.25} = 8$

That is, n is even number:

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y=$f(x)$	0.731	0.679	0.622	0.562	0.5	0.437	0.377	0.321	0.268

$$\int_{-1}^{1} \frac{dx}{1+e^x} = \frac{h}{3}\left[y_o + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_8\right]$$

$$= \frac{0.25}{3}\left[0.731 + 4(0.679 + 0.562 + 0.437 + 0.321)\right.$$
$$\left. +2(0.62 + 0.5 + 0.377 + 0.268)\right]$$

$$\Rightarrow \int_{-1}^{1} \frac{dx}{1+e^x} = 0.999$$

$$f(x) = (1+e^x)^{-1} \quad , \ \text{then}: f'(x) = -(1+e^x)^{-2} e^x \ , \ f''(x) = 2(1+e^x)^{-3} e^{2x} - (1+e^x)^{-2} e^x$$

$$f'''(x) = -6(1+e^x)^{-4} e^{3x} + 6(1+e^x)^{-3} e^{2x} - (1+e^x)^{-2} e^x$$

$$f^{(4)}(x) = 24(1+e^x)^{-5} e^{4x} = 36(1+e^x)^{-4} e^{3x} + 14(1+e^x) e^{2x} - (1+e^x)^{-2} e^x$$

and therefore the biggest value of the derivative in $[-1,1]$ is:

$$\max\left|f^{(1)}(x)\right| = 0.1235068$$

The occurred error is:

$$E \le -\frac{(b-a)}{180}(0.25)^4(0.1235068) = -\frac{2}{180}(0.25)^4(0.1235068)$$
$$= -0.000002196$$

Remark: If the number of interval is odd, then we apply Simpson's method on the bigger even number in subinterval and apply the trapezoidal method on the last interval.

Example 5. Evaluate the integral $\int_o^1 e^{-x^2} dx$ by Simpsons method were $h = 0.2$.

Solution:

$$n = \frac{b-a}{h} = \frac{1-0}{0.2} = 5$$

We note that the number of interval is odd:

x	0	0.2	0.4	0.6	0.8	1
y=f(x)	1.000	0.961	0.852	0.698	0.522	0.368

Since the number of interval is five, we apply Simpson's method on four subinterval, then we apply:

$$\int_o^1 e^{-x^2}dx = \frac{h}{3}\left[f_o + 4f_1 + 2f_2 + 4f_3 + f_4\right] + \frac{h}{2}\left[f_o + f_3\right]$$

$$= \frac{0.2}{3}\left[1 + 4\left(0.961 + 0.698\right) + 2\left(0.852\right) + 0.527\right] + + \frac{0.2}{2}\left[0.527 + 0.368\right]$$

$$= 0.6578 + 0.0895 = 0.7473$$

Example 6. Evaluate the integral $\int_o^1 \sqrt{1+x^2}\,dx$ by Simpsons method so that absolute error doesn't exceed. 10^{-3}.

Solution:

Let's determine h for the supposed error, we know that the error by Simpson's method is given by:

$$E = -\frac{(b-a)}{180}h^4 f^{(4)}(x) \quad ; \quad x \in [0,1]$$

The error must verify:

$$\left| -\frac{(b-a)}{180}h^4 f^{(4)}(x) \right| \le 10^{-3}$$

$$f(x) = \sqrt{1+x^2} \ , \ \text{then}: \ f'(x) = x(1+x)^{-\frac{1}{2}}$$

$$f''(x) = \left(1+x^2\right)^{-\frac{1}{2}} - x^2\left(1+x^2\right)^{-\frac{3}{2}} = \frac{1}{\sqrt{\left(1+x^2\right)^3}}$$

$$f'''(x) = -3x\left(1+x^2\right)^{-\frac{1}{2}} + 3x^2\left(1+x^2\right)x^{-\frac{3}{2}}$$

$$= -3x\left[\left(1+x^2\right)^{-\frac{3}{2}} - x^2\left(1+x^2\right)^{-\frac{3}{2}}\right] = \frac{-3x}{\sqrt{\left(1+x^2\right)^3}}$$

$$f^{(4)}(x) = -3\left(1+x^2\right)^{-\frac{3}{2}} + \frac{9}{2}x\left(1+x^2\right)^{-\frac{3}{2}}(2x) + +9x^2\left(1+x^2\right)^{-\frac{5}{2}} - \frac{15}{2}x^3\left(1+x^2\right)^{-\frac{7}{2}}(2x)$$

$$= \frac{-3\left(1+x^2\right)^2 + 18x^2\left(1+x^2\right) - 15x^4}{\left(1+x^2\right)^{\frac{7}{2}}} = \frac{12x^2 - 3}{\sqrt{\left(1+x^2\right)^{\frac{7}{2}}}}$$

$$\Rightarrow f^{(4)}(o) = -3$$

Substituting in (*), we get:

$$\left| -\frac{(1-0)}{180} h^4 (-3) \right| \le 0.001$$

then: $h^4 \le 0.06$

$$h = \sqrt[4]{0.06} \cong 0.4949 \cong 0.5$$

diving the interval $[0,1]$ in two equal interval:

x	0	0.5	1
y = f(x)	1.000	1.1180	1.4142

$$\int_o^1 \sqrt{1+x^2}\, dx = \frac{h}{3}[f_o + 4f_1 + f_2]$$

$$= \frac{0.5}{3}[1 + 4(1.1180) + 1.4142] = 1.1477$$

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✔ EXERCISE SET 3.6

- Evaluate the integral:

1. Evaluate the integral $\int_o^1 e^{-x^2} dx$ by trapezoidal method, where $h = 0.1$.
2. Evaluate the integral $\int_1^2 \frac{dx}{x}$ by trapezoidal method, where $h = 0.1$.

3. Evaluate the integral $\int_{o}^{1} xe^{x}dx$ by trapezoidal method, where $h = 0.2$.
4. Evaluate the integral $\int_{o}^{0.2} \frac{dx}{1+x}$ by Simpson's method, where $h = 0.04$.
5. Evaluate the integral $\int_{o}^{1} e^{\sin x} dx$ by Simpson's method, where $n = 4$.
6. Evaluate the integral $\int_{o}^{2} \frac{dx}{1+x^2}$ by trapezoidal, where $n = 4$.
7. Evaluate the integral $\int_{1}^{2} \frac{dx}{x}$ by Simpson's method, where $h = 0.1$.
8. Evaluate the integral $\int_{o}^{1} xe^{x}dx$ by Simpson's method, where $h = 0.2$.

9. Use Simpson's Rule with to approximate the area of the surface obtained by rotating the curve about the -axis. Compare your answer with the value of the integral produced by your calculator.

$$a)\ y = \ln x\ ;\quad 1 \le x \le 3\ \ ,b)\ \ y = x + \sqrt{x}\ ;\ 1 \le x \le 2\ ,$$
$$c)\ y = xe^{x}\ \ 0 \le x \le 1$$

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