1. We consider a fair coin tossing two times, and let X be a random variable on the probability space of this experiment defined by:

$$X(\omega) = \begin{cases} 1 & \omega \in \{HH, TH, HT\} \\ 0 & \omega \in \{TT\} \end{cases}$$

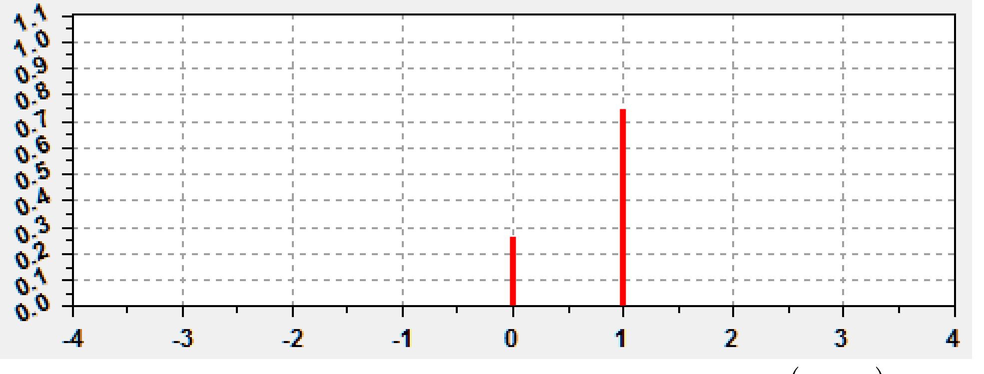
This random variable is called **Bernoulli random variable** with parameter $p = 0.75$ (in this case one say that $p = 0.75$ is the possibility of success). Required:

a. Determine the probability mass function for this random variable and draw its representation.

Answer: The probability mass function for the random variable X is:

$$P(X = k) = P(\{\omega \in \Omega ; X(\omega) = k\}) = \begin{cases} P(\{HH, TH, HT\}) = 0.75 & \textit{for } k = 1 \\ P(\{TT\}) = 0.25 & \textit{for } k = 0 \end{cases}$$

The representation of the random variable X as follow:



The representation of the probability mass function $P\left(X = \bullet\right)$

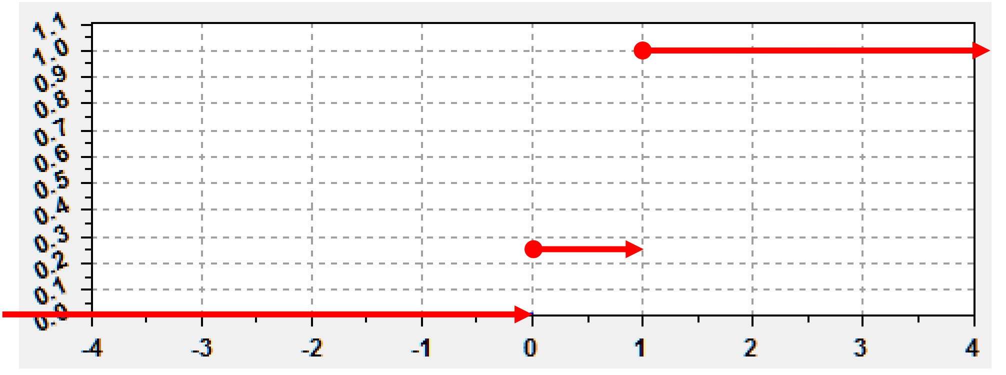
b. Determine the distribution function for this random variable and draw its graph.

Answer: To determine the distribution function for this random variable we must determine the following event:

$$\{\omega \in \Omega \; ; X(\omega) \le x\} = \begin{cases} \{\} = \varnothing & \textit{for } x < 0 \\ \{TT\} & \textit{for } 0 \le x < 1 \\ \{HT, TH, HH\} \cup \{TT\} = \Omega & \textit{for } x \ge 1 \end{cases}$$

Therefore, we get:

$$F_X(x) = P(\{\omega \in \Omega \; ; X(\omega) \le x\}) = \begin{cases} 0 & \textit{for } x < 0 \\ 0.25 & \textit{for } 0 \le x < 1 \\ 0.75 + 0.25 = 1 & \textit{for } x \ge 1 \end{cases}$$



The graph of the distribution function F_X

c. Then calculate the mean, variance and standard deviation of this random variable

Answer: The mean for the random variable X given by:

$$\mathbf{E}(X) = \sum_{i \in I} x_i \, P(X = x_i) = \sum_{k=0}^{1} k \, P(X = k) = 0 \cdot (1-p) + 1 \cdot p = p$$

The second moment of the random variable X is:

$$\mathbf{E}(X^2) = \sum_{i \in I} x_i^2 \, P(X = x_i) = \sum_{k=0}^{1} k^2 \, P(X = k) = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$

Therefore, the variance equal to:

$$\mathbf{var}(X) = \mathbf{E}(X^2) - \left[\mathbf{E}(X)\right]^2 = p - p^2 = p(1-p)$$

$$\Rightarrow \sigma = \sqrt{p(1-p)}$$

.

2. Assume that, the probability that a baby born is a girl in a maternity hospital, is 0.51, and let X be a random variable observe the number births up to a boy is born. Then:

a. Derive the probability mass function and the distribution function of X.

Answer: Assuming that the probability of the birth of a boy is p, then we have the probability that born a boy at the first time after k birth equal to:

$$P(X = k) = \underbrace{(1-p) \cdot (1-p) \cdot \ldots \cdot (1-p)}_{k-1 \text{ factors}} \cdot p = (1-p)^{k-1} p$$

The distribution function of this random variable X is called **geometric distribution** with parameter $p = 0.49$.

The distribution function of X given by

$$F_X(x) = \sum_{\substack{k \in I \\ k \le x}} P(X = k) = \sum_{1 \le k \le x} (1 - 0.49)^{k-1} 0.49 \qquad ; \, x \in \mathbb{R}$$

b. What is the probability that third born is the first boy in the maternity hospital?

Answer: The probability that third born is the first boy in the maternity hospital equal to:

$$P(X=3)=(1-p)\cdot(1-p)\cdot p=(1-0.49)^{3-1}\times 0.49=0.13$$

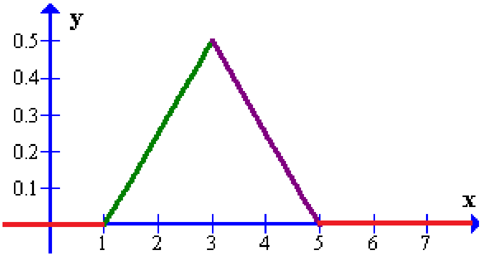
3. Let X be a continuous random variable with probability density function.

$$f_X(x)=\begin{cases}\frac{1}{2}+\frac{1}{4}(x-3) & for\ 1\le x<3\\ \frac{1}{2}-\frac{1}{4}(x-3) & for\ 3\le x<5\\ 0 & otherwise\end{cases}$$

Then:

a. Draw the graph of this probability density function.

Answer: The graph of this probability density function as follow:



b. Determine the distribution function of X.

Answer: The distribution function of X is:

For $x<1$ we have: $F_X(x)=P\left(X\le x\right)=\int_{-\infty}^{1}\underbrace{f_X(t)}_{0}\,dt+=0$

For $1\le x<3$ we have:

$$F_X(x)=P\left(X\le x\right)=\int_{-\infty}^{1}\underbrace{f_X(t)}_{0}\,dt+\int_{1}^{x}f_X(t)\,dt=\frac{1}{2}t\Big|_1^x+\frac{1}{4}\left(\frac{t^2}{2}-3t\right)\Big|_1^x$$

$$=\frac{1}{2}x+\frac{1}{4}\left(\frac{x^2}{2}-3x\right)+\frac{1}{8}=\frac{1}{8}(x^2-2x+1)$$

For $3\le x<5$ we have:

$$F_X(x)=P\left(X\le x\right)=\int_{-\infty}^{1}\underbrace{f_X(t)}_{0}\,dt+\int_{1}^{3}f_X(t)\,dt+\int_{3}^{x}f_X(t)\,dt=\frac{1}{2}+\left[\frac{1}{2}t\Big|_1^x-\frac{1}{4}\left(\frac{t^2}{2}-3t\right)\Big|_1^x\right]$$

$$=\frac{1}{2}+\left[\frac{1}{2}x-\frac{1}{4}\left(\frac{x^2}{2}-3x\right)-\frac{75}{8}\right]=-\frac{1}{8}(x^2+10x+71)$$

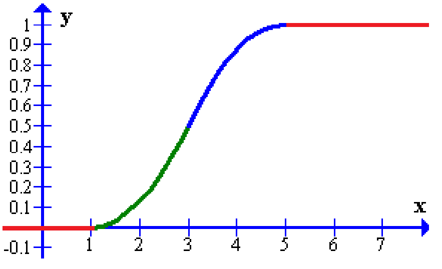
For $x\ge 5$ we have:

$$F_X(x)=P\left(X\le x\right)=\int_{-\infty}^{1}\underbrace{f_X(t)}_{0}\,dt+\int_{1}^{3}f_X(t)\,dt++\int_{3}^{5}f_X(t)\,dt+\int_{5}^{x}f_X(t)\,dt=\frac{1}{2}+\frac{1}{2}=1$$

Therefore, we get:

$$F_X(x) = \begin{cases} 0 & for\ x < 1 \\ \frac{1}{2}x + \frac{1}{4}\left(\frac{x^2}{2} - 3x\right) + \frac{1}{8} & for\ 1 \le x < 3 \\ \frac{1}{2}x - \frac{1}{4}\left(\frac{x^2}{2} - 3x\right) - \frac{71}{8} & for\ 3 \le x < 5 \\ 1 & for\ x \ge 5 \end{cases}$$

The graph of distribution function of X as follow:



4. Let the time for a student to finish the aptitude test of NCAHE (in hours) is a continuous random variable X with:

$$f_X(x) = \begin{cases} k(x-1)(2-x) & for\ 1 \le x < 2 \\ 0 & otherwise \end{cases}$$

Then:

a. Calculate the value of the constant k.

Answer: We have:

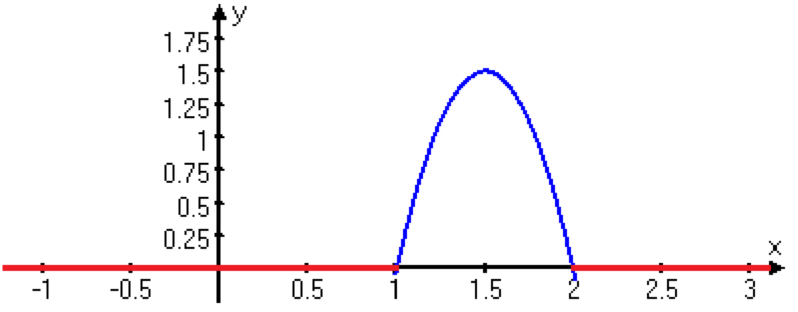
$$1 = \int_{-\infty}^{+\infty} f_X(x)\ dx = \int_{-\infty}^{1} \underset{0}{f_X(x)}\ dx + \int_{1}^{2} f_X(x)\ dx + \int_{2}^{+\infty} \underset{0}{f_X(x)}\ dx$$

$$= \int_1^2 k(x-1)(2-x)\ dx = k\int_1^2 (-x^2 + 3x - 2)\ dx = k\left(-\frac{x^3}{3} + 3\frac{x^2}{2} - 2x\right)\Bigg|_1^2$$

$$= k\left[\left(-\frac{8}{3} + 3\frac{4}{2} - 2\times 2\right) - \left(-\frac{1}{3} + 3\frac{1}{2} - 2\times 1\right)\right] = k\left(\frac{-4}{6} - \frac{-5}{6}\right) = \frac{1}{6}k$$

So we get that $k = 6$.

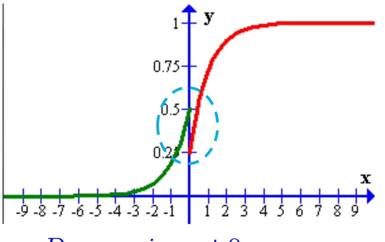
$$f_X(x) = \begin{cases} 6(x-1)(2-x) & for\ 1 \le x < 2 \\ 0 & otherwise \end{cases}$$



5. Determine which of the following is a distribution function:

$$F(x) = \begin{cases} \frac{1}{2}e^{x} & for \ x < 0 \\ 1 - \frac{3}{4}e^{-x} & for \ x \geq 0. \end{cases}$$

$$F(x) = \begin{cases} 0 & for \ x < 0 \\ \frac{x}{1+x} & for \ x \geq 0. \end{cases}$$



Decreasing at 0