

# ***Chapter 15***

## ***Markov Analysis***

To accompany  
*Quantitative Analysis for Management, Eleventh Edition,*  
by Render, Stair, and Hanna  
Power Point slides created by Brian Peterson

# ***Learning Objectives***

**After completing this chapter, students will be able to:**

- 1. Determine future states or conditions by using Markov analysis.**
- 2. Compute long-term or steady-state conditions by using only the matrix of transition probabilities.**
- 3. Understand the use of absorbing state analysis in predicting future conditions.**

# ***Chapter Outline***

- 15.1 Introduction**
- 15.2 States and State Probabilities**
- 15.3 Matrix of Transition Probabilities**
- 15.4 Predicting Future Market Shares**
- 15.5 Markov Analysis of Machine Operations**
- 15.6 Equilibrium Conditions**
- 15.7 Absorbing States and the Fundamental Matrix: Accounts Receivable Application**

# ***Introduction***

- ***Markov analysis*** is a technique that deals with the probabilities of future occurrences by analyzing presently known probabilities.
- It has numerous applications in business.
- Markov analysis makes the assumption that the system starts in an initial state or condition.
- The probabilities of changing from one state to another are called a matrix of transition probabilities.
- Solving Markov problems requires basic matrix manipulation.

# ***Introduction***

- **This discussion will be limited to Markov problems that follow four assumptions:**
  - 1. There are a limited or finite number of possible states.**
  - 2. The probability of changing states remains the same over time.**
  - 3. We can predict any future state from the previous state and the matrix of transition probabilities.**
  - 4. The size and makeup of the system do not change during the analysis.**

# ***States and State Probabilities***

- States are used to identify all possible conditions of a process or system.
- It is possible to identify specific states for many processes or systems.
- In Markov analysis we assume that the states are both *collectively exhaustive* and *mutually exclusive*.
- After the states have been identified, the next step is to determine the probability that the system is in this state.

# ***States and State Probabilities***

**The information is placed into a vector of state probabilities:**

$$\begin{aligned}\pi(i) &= \text{vector of state probabilities} \\ &\quad \text{for period } i \\ &= (\pi_1, \pi_2, \pi_3, \dots, \pi_n)\end{aligned}$$

**Where:**

$$\begin{aligned}n &= \text{number of states} \\ \pi_1, \pi_2, \dots, \pi_n &= \text{probability of being in state 1,} \\ &\quad \text{state 2, ..., state } n\end{aligned}$$

# ***States and State Probabilities***

- In some cases it is possible to know with complete certainty in which state an item is located:
- Vector states can then be represented as:

$$\pi(1) = (1, 0)$$

where

$\pi(1)$  = vector of states for the machine  
in period 1

$\pi_1 = 1$  = probability of being in the  
first state

$\pi_2 = 0$  = probability of being in the  
second state



# ***The Vector of State Probabilities for Three Grocery Stores Example***

- **States for people in a small town with three grocery stores.**
- **A total of 100,000 people shop at the three groceries during any given month:**
  - **Forty thousand may be shopping at American Food Store – state 1.**
  - **Thirty thousand may be shopping at Food Mart – state 2.**
  - **Thirty thousand may be shopping at Atlas Foods – state 3.**

# ***The Vector of State Probabilities for Three Grocery Stores Example***

**Probabilities are as follows:**

**State 1 – American Food Store:     $40,000/100,000 = 0.40 = 40\%$**

**State 2 – Food Mart:                     $30,000/100,000 = 0.30 = 30\%$**

**State 3 – Atlas Foods:                     $30,000/100,000 = 0.30 = 30\%$**

**These probabilities can be placed in the following vector of state probabilities**

$$\pi(1) = (0.4, 0.3, 0.3)$$

**where     $\pi(1)$  = vector of state probabilities for the three grocery stores for period 1**

**$\pi_1 = 0.4$  = probability that person will shop at American Food, state 1**

**$\pi_2 = 0.3$  = probability that person will shop at Food Mart, state 2**

**$\pi_3 = 0.3$  = probability that person will shop at Atlas Foods, state 3**

# ***The Vector of State Probabilities for Three Grocery Stores Example***

- The probabilities of the vector states represent the *market shares* for the three groceries.
- Management will be interested in how their market share changes over time.
- Figure 15.1 shows a tree diagram of how the market shares in the next month.

# Tree Diagram for the Three Grocery Stores Example

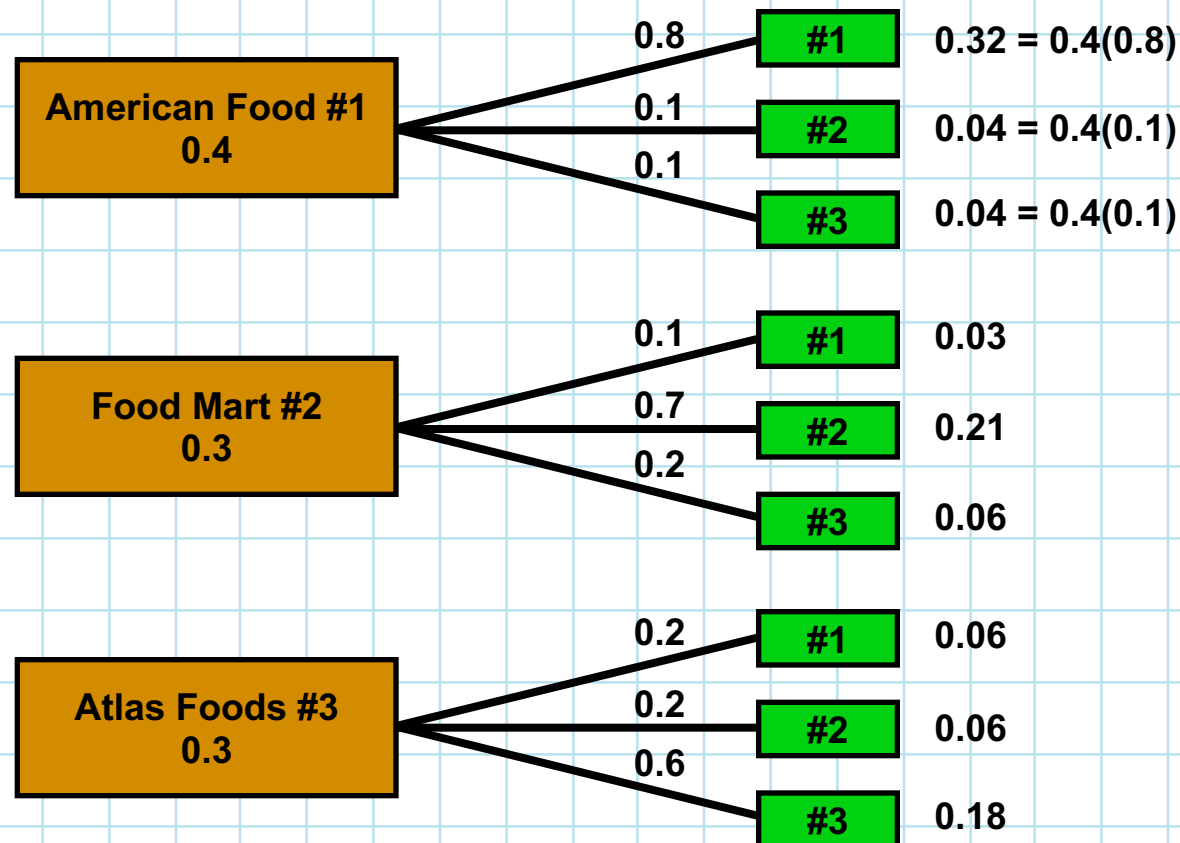


Figure 15.1

# ***Matrix of Transition Probabilities***

**The matrix of transition probabilities allows us to get from a current state to a future state:**

**Let  $P_{ij}$  = conditional probability of being in state  $j$  in the future given the current state of  $i$**

**For example,  $P_{12}$  is the probability of being in state 2 in the future given the event was in state 1 in the period before:**

# ***Matrix of Transition Probabilities***

**Let  $P$  = the matrix of transition probabilities:**

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \dots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \dots & P_{2n} \\ \vdots & & & & \vdots \\ P_{m1} & \dots & & & P_{mn} \end{pmatrix}$$

- Individual  $P_{ij}$  values are determined empirically
- The probabilities in each row will sum to 1

# ***Transition Probabilities for the Three Grocery Stores***

**We used historical data to develop the following matrix:**

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

## ***Row 1***

**$0.8 = P_{11}$  = probability of being in state 1 after being in state 1 in the preceding period**

**$0.1 = P_{12}$  = probability of being in state 2 after being in state 1 in the preceding period**

**$0.1 = P_{13}$  = probability of being in state 3 after being in state 1 in the preceding period**

# ***Transition Probabilities for the Three Grocery Stores***

## **Row 2**

**$0.1 = P_{21}$  = probability of being in state 1 after being in state 2 in the preceding period**

**$0.7 = P_{22}$  = probability of being in state 2 after being in state 2 in the preceding period**

**$0.2 = P_{23}$  = probability of being in state 3 after being in state 2 in the preceding period**

## **Row 3**

**$0.2 = P_{31}$  = probability of being in state 1 after being in state 3 in the preceding period**

**$0.2 = P_{32}$  = probability of being in state 2 after being in state 3 in the preceding period**

**$0.6 = P_{33}$  = probability of being in state 3 after being in state 3 in the preceding period**



# ***Predicting Future Market Shares***

- **One of the purposes of Markov analysis is to predict the future.**
- **Given the vector of state probabilities and the matrix of transitional probabilities, it is not very difficult to determine the state probabilities at a future date.**
- **This type of analysis allows the computation of the probability that a person will be at one of the grocery stores in the future.**
- **Since this probability is equal to market share, it is possible to determine the future market shares of the grocery stores.**

# ***Predicting Future Market Shares***

**When the current period is 0, the state probabilities for the next period 1 are determined as follows:**

$$\pi(1) = \pi(0)P$$

**For any period  $n$  we can compute the state probabilities for period  $n + 1$ :**

$$\pi(n + 1) = \pi(n)P$$

# ***Predicting Future Market Shares***

**The computations for the next period's market share are:**

$$\begin{aligned}\pi(1) &= \pi(0)P \\ &= (0.4, 0.3, 0.3) \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \\ &= [(0.4)(0.8) + (0.3)(0.1) + (0.3)(0.2), \\ &\quad (0.4)(0.1) + (0.3)(0.7) + (0.3)(0.2), \\ &\quad (0.4)(0.1) + (0.3)(0.2) + (0.3)(0.6)] \\ &= (0.41, 0.31, 0.28)\end{aligned}$$

# ***Predicting Future Market Shares***

- **The market share for American Food and Food Mart have increased and the market share for Atlas Foods has decreased.**
- **We can determine if this will continue by looking at the state probabilities will be in the future.**
- **For two time periods from now:**

$$\pi(2) = \pi(1)P$$

# ***Predicting Future Market Shares***

**Since we know that:**

$$\pi(1) = \pi(0)P$$

**We have:**

$$\pi(2) = \pi(1)P = [\pi(0)P]P = \pi(0)PP = \pi(0)P^2$$

**In general:**

$$\pi(n) = \pi(0)P^n$$

**The question of whether American and Food Mart will continue to gain market share and Atlas will continue to loose is best addressed in terms of equilibrium or steady state conditions.**

# ***Markov Analysis of Machine Operations***

- **The owner of Tolsky Works has recorded the operation of his milling machine for several years.**
- **Over the past two years, 80% of the time the milling machine functioned correctly for the current month if it had functioned correctly during the preceding month.**
- **90% of the time the machine remained incorrectly adjusted if it had been incorrectly adjusted in the preceding month.**
- **10% of the time the machine operated correctly in a given month when it had been operating incorrectly the previous month.**

# Tolsky Works

The matrix of transition probabilities for this machine is:

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

where

$P_{11} = 0.8$  = probability that the machine will be *correctly* functioning this month given it was *correctly* functioning last month

$P_{12} = 0.2$  = probability that the machine will *not* be correctly functioning this month given it was *correctly* functioning last month

$P_{21} = 0.1$  = probability that the machine will be *correctly* functioning this month given it was *not* correctly functioning last month

$P_{22} = 0.9$  = probability that the machine will *not* be correctly functioning this month given it was *not* correctly functioning last month

# ***Tolsky Works***

**What is the probability that the machine will be functioning correctly one and two months from now?**

$$\begin{aligned}\pi(1) &= \pi(0)P \\ &= (1, 0) \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \\ &= [(1)(0.8) + (0)(0.1), (1)(0.2) + (0)(0.9)] \\ &= (0.8, 0.2)\end{aligned}$$



# ***Tolsky Works***

**What is the probability that the machine will be functioning correctly one and two months from now?**

$$\begin{aligned}\pi(2) &= \pi(1)P \\ &= (0.8, 0.2) \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \\ &= [(0.8)(0.8) + (0.2)(0.1), (0.8)(0.2) + (0.2)(0.9)] \\ &= (0.66, 0.34)\end{aligned}$$

**State  
Probabilities  
for the  
Machine  
Example for  
15 Periods**

**Table 15.1**

Period	State 1	State 2
1	1.000000	0.000000
2	0.800000	0.200000
3	0.660000	0.340000
4	0.562000	0.438000
5	0.493400	0.506600
6	0.445380	0.554620
7	0.411766	0.588234
8	0.388236	0.611763
9	0.371765	0.628234
10	0.360235	0.639754
11	0.352165	0.647834
12	0.346515	0.653484
13	0.342560	0.657439
14	0.339792	0.660207
15	0.337854	0.662145

# *Equilibrium Conditions*

- It is easy to imagine that all market shares will eventually be 0 or 1.
- But *equilibrium share* of the market values or probabilities generally exist.
- An *equilibrium condition* exists if state probabilities do not change after a large number of periods.
- At equilibrium, state probabilities for the next period equal the state probabilities for current period.
- Equilibrium state probabilities can be computed by repeating Markov analysis for a large number of periods.

# ***Equilibrium Conditions***

- It is always true that

$$\pi (\text{next period}) = \pi (\text{this period})P$$

- Or

$$\pi (n + 1) = \pi (n)P$$

- At equilibrium

$$\pi (n + 1) = \pi (n)$$

- So at equilibrium

$$\pi (n + 1) = \pi (n)P = \pi (n)$$

- Or

$$\pi = \pi P$$

# *Equilibrium Conditions*

- For Tolsky's machine

$$\pi = \pi P$$

$$(\pi_1, \pi_2) = (\pi_1, \pi_2) \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$$

- Using matrix multiplication

$$(\pi_1, \pi_2) = [(\pi_1)(0.8) + (\pi_2)(0.1), (\pi_1)(0.2) + (\pi_2)(0.9)]$$

# ***Equilibrium Conditions***

- The first and second terms on the left side,  $\pi_1$  and  $\pi_2$ , are equal to the first terms on the right side:

$$\pi_1 = 0.8\pi_1 + 0.1\pi_2$$

$$\pi_2 = 0.2\pi_1 + 0.9\pi_2$$

- The state probabilities sum to 1:

$$\pi_1 + \pi_2 + \dots + \pi_n = 1$$

- For Tolsky's machine:

$$\pi_1 + \pi_2 = 1$$

# ***Equilibrium Conditions***

**We arbitrarily decide to solve the following two equations:**

$$\pi_2 = 0.2\pi_1 + 0.9\pi_2$$

$$\pi_1 + \pi_2 = 1$$

**Through rearrangement and substitution we get**

$$0.1\pi_2 = 0.2\pi_1$$

$$\pi_2 = 2\pi_1$$

$$\pi_1 + \pi_2 = 1$$

$$\pi_1 + 2\pi_1 = 1$$

$$3\pi_1 = 1$$

$$\pi_1 = 1/3 = 0.33333333$$

$$\pi_2 = 2/3 = 0.66666667$$

# ***Absorbing States and the Fundamental Matrix***

## ■ **Accounts Receivable example**

- The examples so far assume it is possible to go from one state to another.
- This is not always possible.
- If you must remain in a state it is called an ***absorbing state***.
- An accounts receivable system normally places accounts in three possible states:
  - State 1 ( $\pi_1$ ):*** paid, all bills
  - State 2 ( $\pi_2$ ):*** bad debt, overdue more than three months
  - State 3 ( $\pi_3$ ):*** overdue less than one month
  - State 4 ( $\pi_4$ ):*** overdue between one and three months



# ***Absorbing States and the Fundamental Matrix***

**The matrix of transition probabilities of this problem is:**

<b>THIS MONTH</b>	<b>NEXT MONTH</b>			
	<b>PAID</b>	<b>BAD DEBT</b>	<b>&lt; 1 MONTH</b>	<b>1 TO 3 MONTHS</b>
<b>Paid</b>	1	0	0	0
<b>Bad debt</b>	0	1	0	0
<b>Less than 1 month</b>	0.6	0	0.2	0.2
<b>1 to 3 months</b>	0.4	0.1	0.3	0.2

**Thus:**

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6 & 0 & 0.2 & 0.2 \\ 0.4 & 0.1 & 0.3 & 0.2 \end{bmatrix}$$

# Absorbing States and the Fundamental Matrix

To obtain the fundamental matrix, it is necessary to partition the matrix of transition probabilities as follows:

$$P = \begin{pmatrix} 1 & 0 & | & 0 & 0 \\ 0 & 1 & | & 0 & 0 \\ \hline 0.6 & 0 & | & 0.2 & 0.2 \\ 0.4 & 0.1 & | & 0.3 & 0.2 \end{pmatrix}$$

$\downarrow I$                        $\downarrow 0$

$\uparrow A$                        $\uparrow B$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.6 & 0 \\ 0.4 & 0.1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}$$

where

$I$  = an identity matrix  
 $0$  = a matrix with all 0s

# Absorbing States and the Fundamental Matrix

The fundamental matrix can be computed as:

$$F = (I - B)^{-1}$$

$$F = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{pmatrix} \right)^{-1}$$

$$F = \begin{pmatrix} 0.8 & -0.2 \\ -0.3 & 0.8 \end{pmatrix}^{-1}$$

The inverse of the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{r} & \frac{-b}{r} \\ \frac{-c}{r} & \frac{a}{r} \end{pmatrix}$

where

$$r = ad - bc$$

# Absorbing States and the Fundamental Matrix

To find the matrix  $F$  we compute:

$$r = ad - bc = (0.8)(0.8) - (-0.2)(-0.3) = 0.64 - 0.06 = 0.58$$

With this we have:

$$F = \begin{pmatrix} 0.8 & -0.2 \\ -0.3 & 0.8 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{0.8}{0.58} & \frac{-(-0.2)}{0.58} \\ \frac{-(-0.3)}{0.58} & \frac{0.8}{0.58} \end{pmatrix} = \begin{pmatrix} 1.38 & 0.34 \\ 0.52 & 1.38 \end{pmatrix}$$

# ***Absorbing States and the Fundamental Matrix***

**We can use the matrix  $FA$  to answer questions such as how much of the debt in the less than one month category will be paid back and how much will become bad debt:**

$$**$M = (M_1, M_2, M_3, \dots, M_n)$**$$

**where**

- $n$  = number of nonabsorbing states**
- $M_1$  = amount in the first state or category**
- $M_2$  = amount in the second state or category**
- $M_n$  = amount in the  $n$ th state or category**

# ***Absorbing States and the Fundamental Matrix***

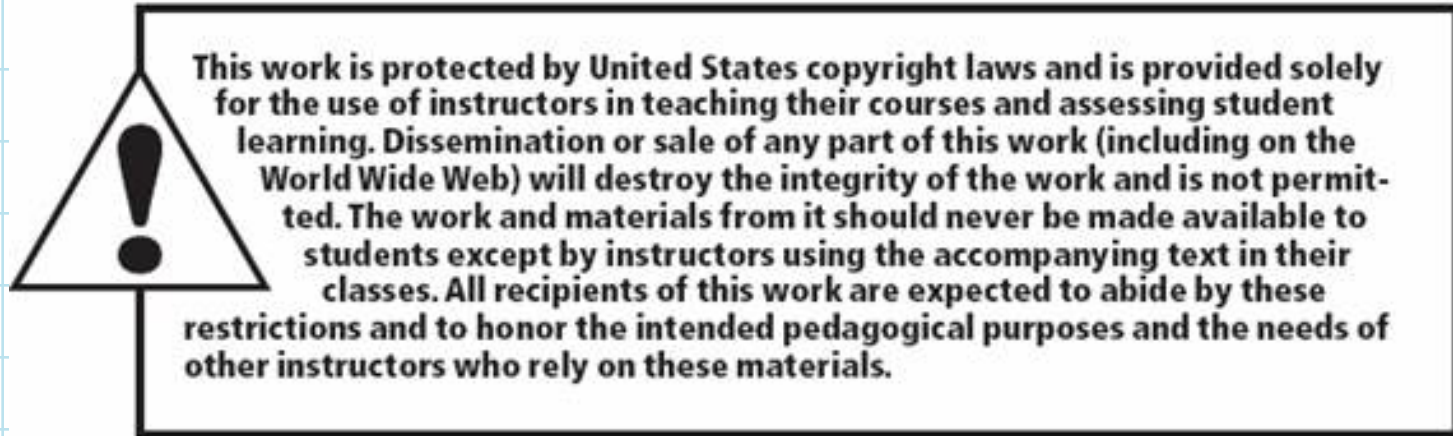
**If we assume there is \$2,000 in the less than one month category and \$5,000 in the one to three month category,  $M$  would be:**

$$**$M = (2,000, 5,000)$**$$

$$\begin{aligned}\text{Amount paid and} & \\ \text{amount in bad debts} &= MFA \\ &= (2,000, 5,000) \begin{bmatrix} 0.97 & 0.03 \\ 0.86 & 0.14 \end{bmatrix} \\ &= (6,240, 760)\end{aligned}$$

**Out of the total of \$7,000, \$6,240 will eventually be paid and \$760 will end up as bad debt.**

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