Chapter 2

Probability Concepts and Applications

To accompany
Quantitative Analysis for Management, Eleventh Edition, by Render, Stair, and Hanna
Power Point slides created by Brian Peterson

Learning Objectives

After completing this chapter, students will be able to:

1. Understand the basic foundations of probability analysis.
2. Describe statistically dependent and independent events.
3. Use Bayes' theorem to establish posterior probabilities.
4. Describe and provide examples of both discrete and continuous random variables.
5. Explain the difference between discrete and continuous probability distributions.
6. Calculate expected values and variances and use the normal table.

Chapter Outline

Chapter Outline

Introduction

- Life is uncertain; we are not sure what the future will bring.
- ***Probability*** is a numerical statement about the likelihood that an event will occur.

Fundamental Concepts

1. The probability, P, of any event or state of nature occurring is greater than or equal to 0 and less than or equal to 1. That is:

$$0 \leq P\,(\text{event}) \leq 1$$

2. The sum of the simple probabilities for all possible outcomes of an activity must equal 1.

Chapters in This Book That Use Probability

CHAPTER	TITLE
3	Decision Analysis
4	Regression Models
5	Forecasting
6	Inventory Control Models
12	Project Management
13	Waiting Lines and Queuing Theory Models
14	Simulation Modeling
15	Markov Analysis
16	Statistical Quality Control
Module 3	Decision Theory and the Normal Distribution
Module 4	Game Theory

Table 2.1

Diversey Paint Example

- Demand for white latex paint at Diversey Paint and Supply has always been either 0, 1, 2, 3, or 4 gallons per day.
- Over the past 200 days, the owner has observed the following frequencies of demand:

QUANTITY DEMANDED	NUMBER OF DAYS	PROBABILITY
0	40	0.20 (= 40/200)
1	80	0.40 (= 80/200)
2	50	0.25 (= 50/200)
3	20	0.10 (= 20/200)
4	10	0.05 (= 10/200)
	Total 200	Total 1.00 (= 200/200)

Diversey Paint Example

Notice the individual probabilities are all between 0 and 1

$$0 \leq P\,(\text{event}) \leq 1$$

And the total of all event probabilities equals 1

$$\sum P\,(\text{event}) = 1.00$$

versey Paint
r 0, 1, 2, 3, or 4

has observed
nd

			PROBABILITY
			0.20 (= 40/200)
			0.40 (= 80/200)
2	50		0.25 (= 50/200)
3	20		0.10 (= 20/200)
4	10		0.05 (= 10/200)
	Total 200	Total	1.00 (= 200/200)

Types of Probability

Determining *objective probability* :

- Relative frequency
  - Typically based on historical data

$$P\text{ (event)} = \frac{\text{Number of occurrences of the event}}{\text{Total number of trials or outcomes}}$$

- Classical or logical method
  - Logically determine probabilities without trials

$$P\text{ (head)} = \frac{1}{2}$$

1 ← Number of ways of getting a head

2 ← Number of possible outcomes (head or tail)

Types of Probability

Subjective probability **is based on the experience and judgment of the person making the estimate.**

- **Opinion polls**
- **Judgment of experts**
- **Delphi method**

Mutually Exclusive Events

Events are said to be ***mutually exclusive*** if only one of the events can occur on any one trial.

- Tossing a coin will result in either a head or a tail.

- Rolling a die will result in only one of six possible outcomes.

Collectively Exhaustive Events

Events are said to be **collectively exhaustive** if the list of outcomes includes every possible outcome.

- Both heads and tails as possible outcomes of coin flips.
- All six possible outcomes of the roll of a die.

OUTCOME OF ROLL	PROBABILITY
1	$^1/_6$
2	$^1/_6$
3	$^1/_6$
4	$^1/_6$
5	$^1/_6$
6	$^1/_6$
	Total 1

Drawing a Card

Draw one card from a deck of 52 playing cards

$$P \text{ (drawing a 7)} = {}^{4}/_{52} = {}^{1}/_{13}$$

$$P \text{ (drawing a heart)} = {}^{13}/_{52} = {}^{1}/_{4}$$

- **These two events are not mutually exclusive since a 7 of hearts can be drawn**
- **These two events are not collectively exhaustive since there are other cards in the deck besides 7s and hearts**

Table of Differences

DRAWS	MUTUALLY EXCLUSIVE	COLLECTIVELY EXHAUSTIVE
1. Draws a spade and a club	Yes	No
2. Draw a face card and a number card	Yes	Yes
3. Draw an ace and a 3	Yes	No
4. Draw a club and a nonclub	Yes	Yes
5. Draw a 5 and a diamond	No	No
6. Draw a red card and a diamond	No	No

Adding Mutually Exclusive Events

We often want to know whether one *or* a second event will occur.

- **When two events are mutually exclusive, the law of addition is:**

$$P\text{ (event } A \text{ or event } B) = P\text{ (event } A) + P\text{ (event } B)$$

$$\begin{aligned} P\text{ (spade or club)} &= P\text{ (spade)} + P\text{ (club)} \\ &= {}^{13}/_{52} + {}^{13}/_{52} \\ &= {}^{26}/_{52} = {}^{1}/_{2} = 0.50 \end{aligned}$$

Adding Not Mutually Exclusive Events

The equation must be modified to account for double counting.

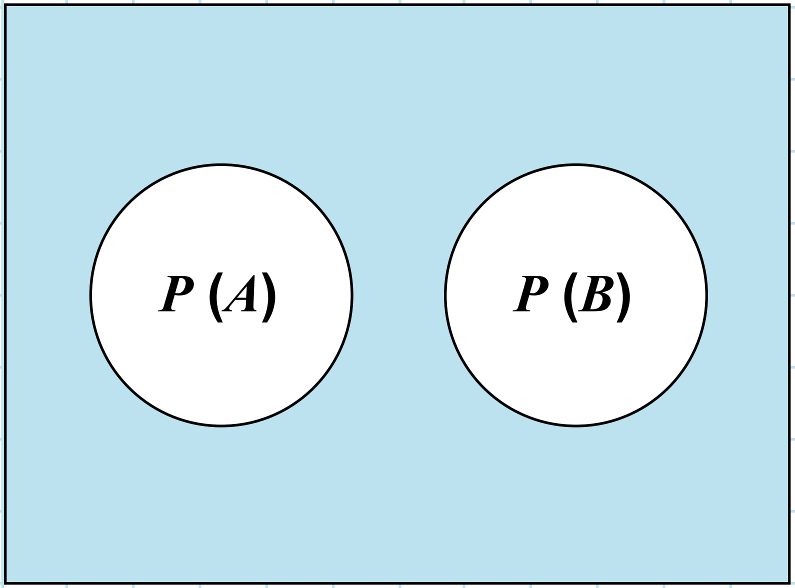
- **The probability is reduced by subtracting the chance of both events occurring together.**

P (event A *or* event B) = P (event A) + P (event B) – P (event A *and* event B both occurring)

$$P\,(A \text{ or } B) = P\,(A) + P\,(B) - P\,(A \text{ and } B)$$

$$\begin{aligned} P(\text{five } \textit{or} \text{ diamond}) &= P(\text{five}) + P(\text{diamond}) - P(\text{five } \textit{and} \text{ diamond}) \\ &= {}^{4}/_{52} + {}^{13}/_{52} - {}^{1}/_{52} \\ &= {}^{16}/_{52} = {}^{4}/_{13} \end{aligned}$$

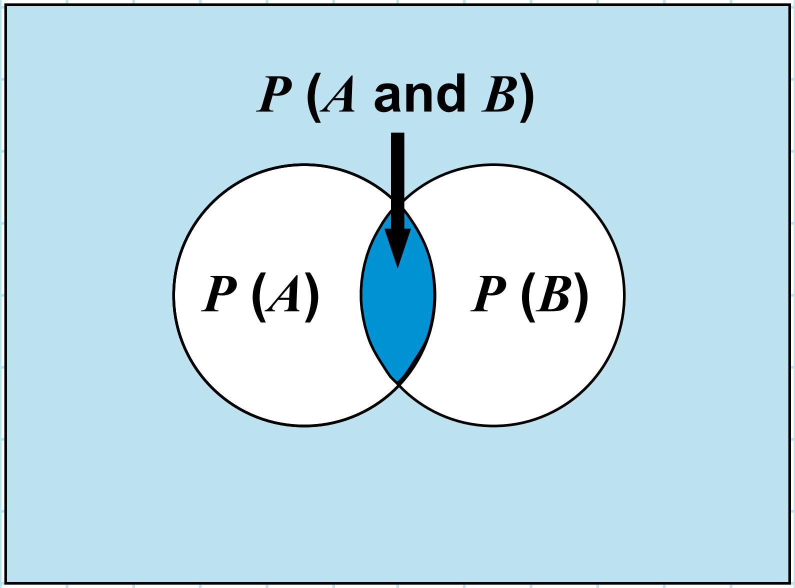
Venn Diagrams



Events that are mutually exclusive.

$$P(A \text{ or } B) = P(A) + P(B)$$

Figure 2.1



Events that are not mutually exclusive.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Figure 2.2

Statistically Independent Events

Events may be either independent or dependent.

- For independent events, the occurrence of one event has no effect on the probability of occurrence of the second event.

Which Sets of Events Are Independent?

1. (a) Your education
   (b) Your income level
   Dependent events

2. (a) Draw a jack of hearts from a full 52-card deck
   (b) Draw a jack of clubs from a full 52-card deck
   Independent events

3. (a) Chicago Cubs win the National League pennant
   (b) Chicago Cubs win the World Series
   Dependent events

4. (a) Snow in Santiago, Chile
   (b) Rain in Tel Aviv, Israel
   Independent events

Three Types of Probabilities

- ***Marginal*** (or ***simple***) probability is just the probability of a single event occurring.

$$P(A)$$

- ***Joint*** probability is the probability of two or more events occurring and is equal to the product of their marginal probabilities for independent events.

$$P(AB) = P(A) \times P(B)$$

- ***Conditional*** probability is the probability of event B given that event A has occurred.

$$P(B \mid A) = \mathrm{P}(B)$$

- Or the probability of event A given that event B has occurred

$$P(A \mid B) = P(A)$$

Joint Probability Example

The probability of tossing a 6 on the first roll of the die and a 2 on the second roll:

P (6 on first and 2 on second)

= P (tossing a 6) x P (tossing a 2)

= $^1/_6$ x $^1/_6$ = $^1/_{36}$ = 0.028

Independent Events

A bucket contains 3 black balls and 7 green balls.

- Draw a ball from the bucket, replace it, and draw a second ball.

1. The probability of a black ball drawn on first draw is:

   $P(B) = 0.30$ (*a marginal probability*)

2. The probability of two green balls drawn is:

   $P(GG) = P(G) \times P(G) = 0.7 \times 0.7 = 0.49$

   (*a joint probability for two independent events*)

Independent Events

A bucket contains 3 black balls and 7 green balls.

- Draw a ball from the bucket, replace it, and draw a second ball.

3. The probability of a black ball drawn on the second draw if the first draw is green is:

$$P\,(B \mid G) = P\,(B) = 0.30$$

(*a conditional probability but equal to the marginal because the two draws are independent events*)

4. The probability of a green ball drawn on the second draw if the first draw is green is:

$$P\,(G \mid G) = P\,(G) = 0.70$$

(*a conditional probability as in event 3*)

Statistically Dependent Events

The *marginal* probability of an event occurring is computed in the same way:

$$P\,(A)$$

Calculating *conditional* probabilities is slightly more complicated. The probability of event A given that event B has occurred is:

$$P\,(A \mid B) = \frac{P\,(AB)}{P\,(B)}$$

The formula for the *joint* probability of two events is:

$$P\,(AB) = P\,(B \mid A)\,P\,(A)$$

When Events Are Dependent

Assume that we have an urn containing 10 balls of the following descriptions:

- 4 are white (W) and lettered (L)
- 2 are white (W) and numbered (N)
- 3 are yellow (Y) and lettered (L)
- 1 is yellow (Y) and numbered (N)

$P(WL) = {}^4/_{10} = 0.4$

$P(WN) = {}^2/_{10} = 0.2$

$P(W) = {}^6/_{10} = 0.6$

$P(Y) = {}^4/_{10} = 0.4$

$P(YL) = {}^3/_{10} = 0.3$

$P(YN) = {}^1/_{10} = 0.1$

$P(L) = {}^7/_{10} = 0.7$

$P(N) = {}^3/_{10} = 0.3$

When Events Are Dependent

The urn contains 10 balls:

Balls	Probability
4 balls White (W) and Lettered (L)	Probability (WL) = $\frac{4}{10}$
2 balls White (W) and Numbered (N)	Probability (WN) = $\frac{2}{10}$
3 balls Yellow (Y) and Lettered (L)	Probability (YL) = $\frac{3}{10}$
1 ball Yellow (Y) and Numbered (N)	Probability (YN) = $\frac{1}{10}$

Figure 2.3

When Events Are Dependent

The conditional probability that the ball drawn is lettered, given that it is yellow, is:

$$P(L \mid Y) = \frac{P(YL)}{P(Y)} = \frac{0.3}{0.4} = 0.75$$

We can verify $P(YL)$ using the joint probability formula

$$P(YL) = P(L \mid Y) \times P(Y) = (0.75)(0.4) = 0.3$$

Joint Probabilities for Dependent Events

If the stock market reaches 12,500 point by January, there is a 70% probability that Tubeless Electronics will go up.

- You believe that there is only a 40% chance the stock market will reach 12,500.
- Let M represent the event of the stock market reaching 12,500 and let T be the event that Tubeless goes up in value.

$$P\,(MT) = P\,(T \mid M) \times P\,(M) = (0.70)(0.40) = 0.28$$

Revising Probabilities with Bayes' Theorem

Bayes' theorem is used to incorporate additional information and help create *posterior probabilities.*

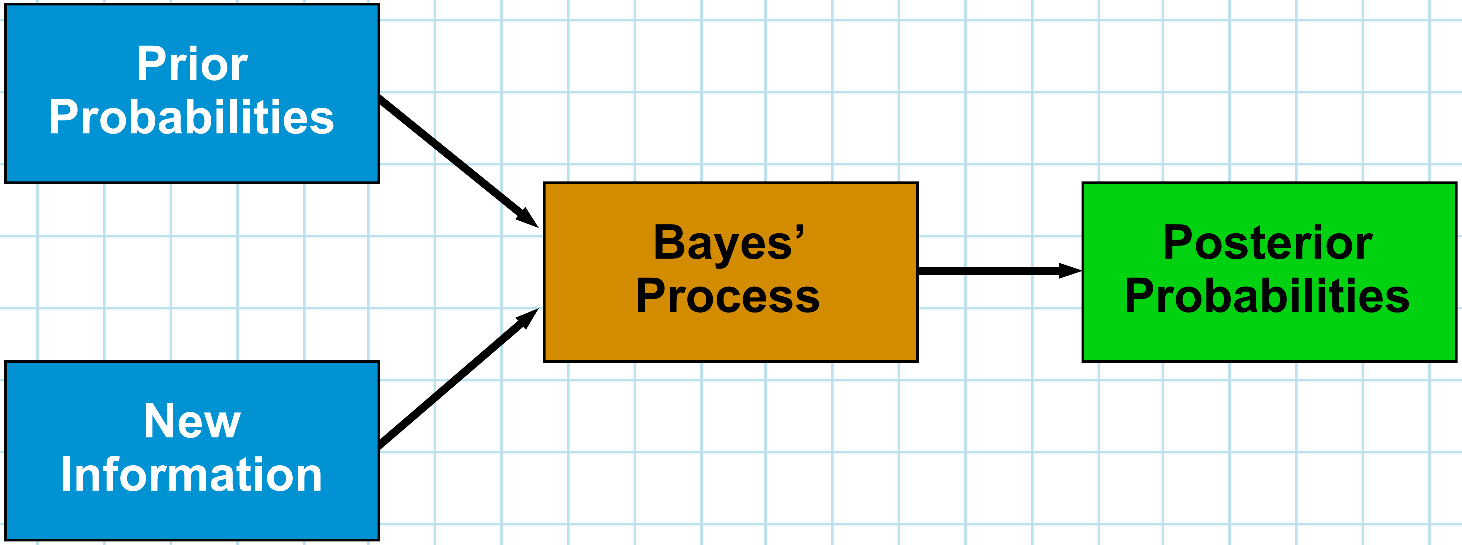


Figure 2.4

Posterior Probabilities

A cup contains two dice identical in appearance but one is fair (unbiased), the other is loaded (biased).

- **The probability of rolling a 3 on the fair die is $^1/_6$ or 0.166.**
- **The probability of tossing the same number on the loaded die is 0.60.**
- **We select one by chance, toss it, and get a 3.**
- **What is the probability that the die rolled was fair?**
- **What is the probability that the loaded die was rolled?**

Posterior Probabilities

We know the probability of the die being fair or loaded is:

$$P\,(\text{fair}) = 0.50 \qquad P\,(\text{loaded}) = 0.50$$

And that

$$P\,(3 \mid \text{fair}) = 0.166 \qquad P\,(3 \mid \text{loaded}) = 0.60$$

We compute the probabilities of P (3 and fair) and P (3 and loaded):

$$P\,(3 \text{ and fair}) = P\,(3 \mid \text{fair}) \times P\,(\text{fair})$$
$$= (0.166)(0.50) = 0.083$$

$$P\,(3 \text{ and loaded}) = P\,(3 \mid \text{loaded}) \times P\,(\text{loaded})$$
$$= (0.60)(0.50) = 0.300$$

Posterior Probabilities

We know the pr
loaded is

P (fair) =

And that

P (3 | fair) =

The sum of these probabilities gives us the unconditional probability of tossing a 3:

$$P(3) = 0.083 + 0.300 = 0.383$$

We compute the probabilities of P (3 and fair) and P (3 and loaded)

$$P(3 \text{ and fair}) = P(3 \mid \text{fair}) \times P(\text{fair})$$
$$= (0.166)(0.50) = 0.083$$

$$P(3 \text{ and loaded}) = P(3 \mid \text{loaded}) \times P(\text{loaded})$$
$$= (0.60)(0.50) = 0.300$$

Posterior Probabilities

If a 3 does occur, the probability that the die rolled was the fair one is:

$$P\,(\text{fair} \mid 3) = \frac{P\,(\text{fair and } 3)}{P\,(3)} = \frac{0.083}{0.383} = 0.22$$

The probability that the die was loaded is:

$$P\,(\text{loaded} \mid 3) = \frac{P\,(\text{loaded and } 3)}{P\,(3)} = \frac{0.300}{0.383} = 0.78$$

- These are the ***revised*** or ***posterior probabilities*** for the next roll of the die.
- We use these to revise our ***prior probability*** estimates.

Bayes' Calculations

Given event B has occurred:

STATE OF NATURE	P (B | STATE OF NATURE)	PRIOR PROBABILITY	JOINT PROBABILITY	POSTERIOR PROBABILITY
A	$P(B \mid A)$	x $P(A)$	= $P(B$ and $A)$	$P(B$ and $A)/P(B) = P(A \mid B)$
A'	$P(B \mid A')$	x $P(A')$	= $P(B$ and $A')$	$P(B$ and $A')/P(B) = P(A' \mid B)$
			$P(B)$	

Table 2.2

Given a 3 was rolled:

STATE OF NATURE	P (B | STATE OF NATURE)	PRIOR PROBABILITY	JOINT PROBABILITY	POSTERIOR PROBABILITY
Fair die	0.166	x 0.5	= 0.083	0.083 / 0.383 = 0.22
Loaded die	0.600	x 0.5	= 0.300	0.300 / 0.383 = 0.78
			$P(3)$ = 0.383	

Table 2.3

General Form of Bayes' Theorem

We can compute revised probabilities more directly by using:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}$$

where

$A' =$ the complement of the event A;
for example, if A is the event "fair die",
then A' is "loaded die".

General Form of Bayes' Theorem

This is basically what we did in the previous example:

Replace A with "fair die"
Replace A' with "loaded die
Replace B with "3 rolled"

We get

$$P(\text{fair die} \mid 3 \text{ rolled})$$

$$= \frac{P(3 \mid \text{fair})P(\text{fair})}{P(3 \mid \text{fair})P(\text{fair}) + P(3 \mid \text{loaded})P(\text{loaded})}$$

$$\frac{(0.166)(0.50)}{(0.166)(0.50) + (0.60)(0.50)} = \frac{0.083}{0.383} = 0.22$$

Further Probability Revisions

We can obtain additional information by performing the experiment a second time

- If you can afford it, perform experiments several times.

We roll the die again and again get a 3.

$$P(\text{fair}) = 0.50 \text{ and } P(\text{loaded}) = 0.50$$

$$P(3,3 \mid \text{fair}) = (0.166)(0.166) = 0.027$$

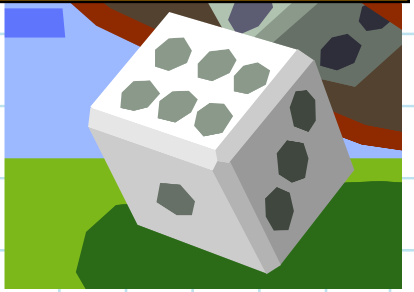
$$P(3,3 \mid \text{loaded}) = (0.6)(0.6) = 0.36$$

Further Probability Revisions

$$P(3,3 \text{ and fair}) = P(3,3 \mid \text{fair}) \times P(\text{fair})$$
$$= (0.027)(0.5) = 0.013$$

$$P(3,3 \text{ and loaded}) = P(3,3 \mid \text{loaded}) \times P(\text{loaded})$$
$$= (0.36)(0.5) = 0.18$$

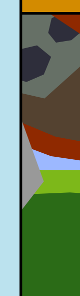
$$P(\text{fair}) = 0.50 \text{ and } P(\text{loaded}) = 0.50$$
$$P(3,3 \mid \text{fair}) = (0.166)(0.166) = 0.027$$
$$P(3,3 \mid \text{loaded}) = (0.6)(0.6) = 0.36$$



Further Probability Revisions

$$P(3,3 \text{ and fair}) = P(3,3 \mid \text{fair}) \times P(\text{fair})$$
$$= (0.027)(0.5) = 0.013$$

$$P(3,3 \text{ and loaded}) = P(3,3 \mid \text{loaded}) \times P(\text{loaded})$$
$$= (0.36)(0.5) = 0.18$$



$$P(\text{fair} \mid 3,3) = \frac{P(3,3 \text{ and fair})}{P(3,3)} = \frac{0.013}{0.193} = 0.067$$

$$P(\text{loaded} \mid 3,3) = \frac{P(3,3 \text{ and loaded})}{P(3,3)} = \frac{0.18}{0.193} = 0.933$$

Further Probability Revisions

After the first roll of the die:

probability the die is fair = 0.22
probability the die is loaded = 0.78

After the second roll of the die:

probability the die is fair = 0.067
probability the die is loaded = 0.933

Random Variables

A ***random variable*** assigns a real number to every possible outcome or event in an experiment.

X = number of refrigerators sold during the day

Discrete random variables can assume only a finite or limited set of values.

Continuous random variables can assume any one of an infinite set of values.

Random Variables – Numbers

EXPERIMENT	OUTCOME	RANDOM VARIABLES	RANGE OF RANDOM VARIABLES
Stock 50 Christmas trees	Number of Christmas trees sold	X	0, 1, 2,…, 50
Inspect 600 items	Number of acceptable items	Y	0, 1, 2,…, 600
Send out 5,000 sales letters	Number of people responding to the letters	Z	0, 1, 2,…, 5,000
Build an apartment building	Percent of building completed after 4 months	R	$0 \leq R \leq 100$
Test the lifetime of a lightbulb (minutes)	Length of time the bulb lasts up to 80,000 minutes	S	$0 \leq S \leq 80{,}000$

Table 2.4

Random Variables – Not Numbers

EXPERIMENT	OUTCOME	RANDOM VARIABLES	RANGE OF RANDOM VARIABLES
Students respond to a questionnaire	Strongly agree (SA) Agree (A) Neutral (N) Disagree (D) Strongly disagree (SD)	$X = \begin{cases} 5 \text{ if SA} \\ 4 \text{ if A} \\ 3 \text{ if N} \\ 2 \text{ if D} \\ 1 \text{ if SD} \end{cases}$	1, 2, 3, 4, 5
One machine is inspected	Defective Not defective	$Y = \begin{cases} 0 \text{ if defective} \\ 1 \text{ if not defective} \end{cases}$	0, 1
Consumers respond to how they like a product	Good Average Poor	$Z = \begin{cases} 3 \text{ if good} \\ 2 \text{ if average} \\ 1 \text{ if poor} \end{cases}$	1, 2, 3

Table 2.5

Probability Distribution of a Discrete Random Variable

For ***discrete random variables*** a probability is assigned to each event.

The students in Pat Shannon's statistics class have just completed a quiz of five algebra problems. The distribution of correct scores is given in the following table:

Probability Distribution of a Discrete Random Variable

RANDOM VARIABLE (X – *Score*)	NUMBER RESPONDING	PROBABILITY $P(X)$
5	10	0.1 = 10/100
4	20	0.2 = 20/100
3	30	0.3 = 30/100
2	30	0.3 = 30/100
1	10	0.1 = 10/100
Total	100	1.0 = 100/100

Table 2.6

The Probability Distribution follows all three rules:

1. Events are mutually exclusive and collectively exhaustive.
2. Individual probability values are between 0 and 1.
3. Total of all probability values equals 1.

Probability Distribution for Dr. Shannon's Class

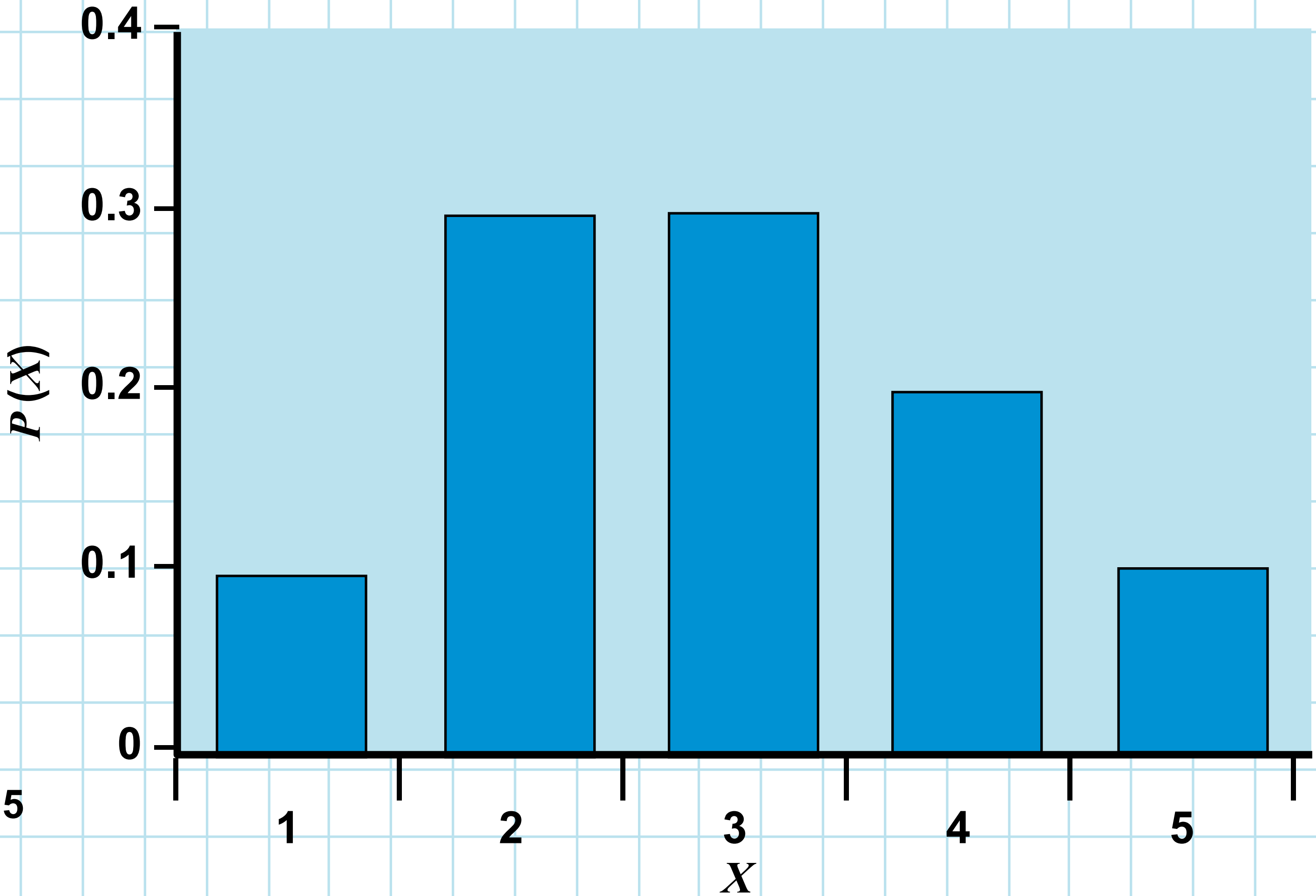


Figure 2.5

Probability Distribution for Dr. Shannon's Class

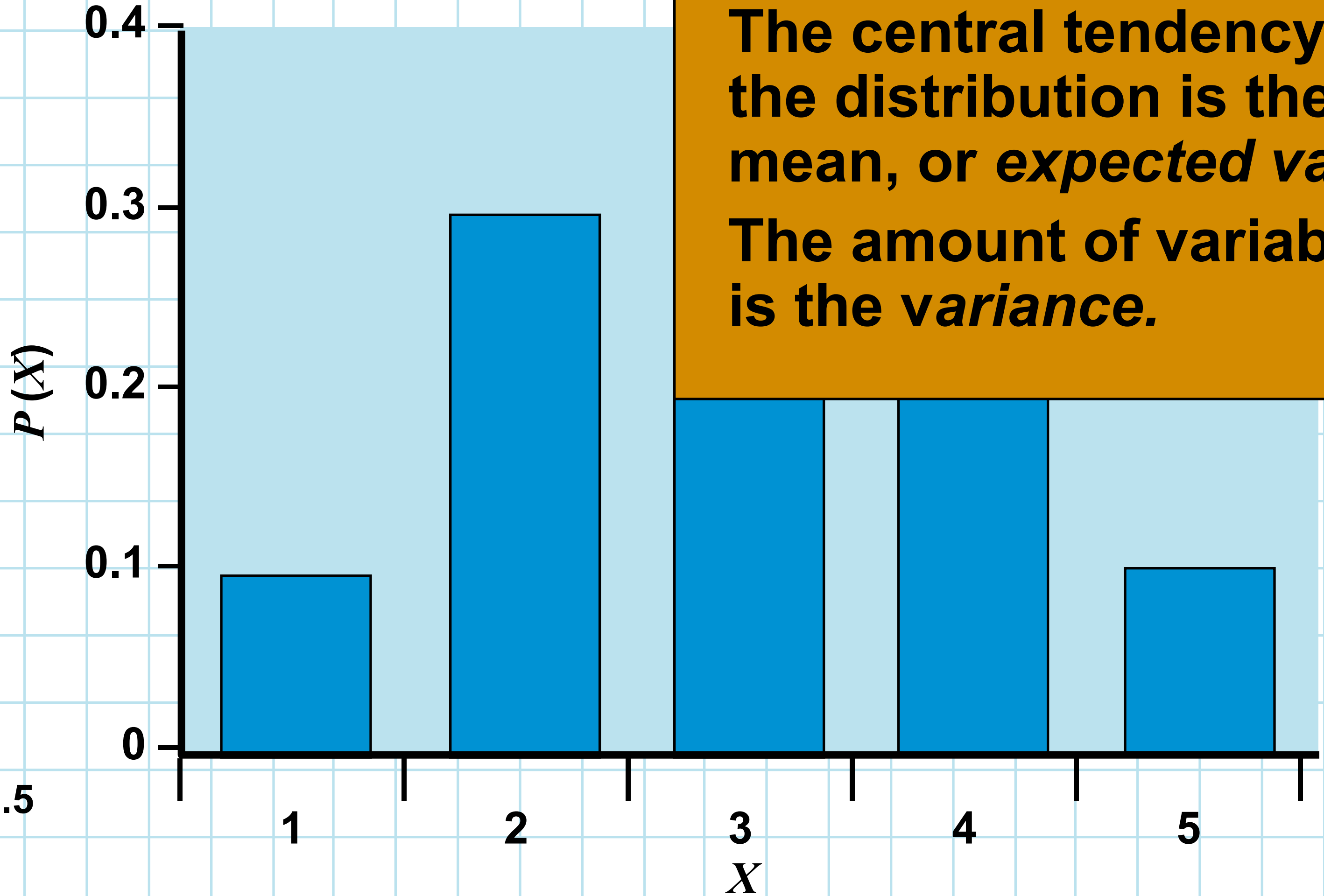


Figure 2.5

The central tendency of the distribution is the mean, or *expected value.*

The amount of variability is the v*ariance.*

Expected Value of a Discrete Probability Distribution

The expected value is a measure of the ***central tendency*** of the distribution and is a weighted average of the values of the random variable.

$$E(X) = \sum_{i=1}^{n} X_i P(X_i)$$

$$= X_1 P(X_1) + X_2 P(X_2) + \ldots + X_n P(X_n)$$

where

X_i = random variable's possible values

$P(X_i)$ = probability of each of the random variable's possible values

$\sum_{i=1}^{n}$ = summation sign indicating we are adding all n possible values

$E(X)$ = expected value or mean of the random sample

Expected Value of a Discrete Probability Distribution

For Dr. Shannon's class:

$$E(X) = \sum_{i=1}^{n} X_i P(X_i)$$

$$= 5(0.1) + 4(0.2) + 3(0.3) + 2(0.3) + 1(0.1)$$

$$= .5 + .8 + .9 + .6 + .1$$

$$= 2.9$$

Variance of a Discrete Probability Distribution

For a discrete probability distribution the variance can be computed by

$$\sigma^2 = \text{Variance} = \sum_{i=1}^{n} [X_i - E(X)]^2 P(X_i)$$

where

X_i = random variable's possible values

$E(X)$ = expected value of the random variable

$[X_i - E(X)]$ = difference between each value of the random variable and the expected mean

$P(X_i)$ = probability of each possible value of the random variable

Variance of a Discrete Probability Distribution

For Dr. Shannon's class:

$$\text{variance} = \sum_{i=1}^{5} [X_i - E(X)]^2 P(X_i)$$

$$\begin{aligned}\text{variance} &= (5-2.9)^2(0.1) + (4-2.9)^2(0.2) + \\ &\quad (3-2.9)^2(0.3) + (2-2.9)^2(0.3) + \\ &\quad (1-2.9)^2(0.1) \\ &= 0.441 + 0.242 + 0.003 + 0.243 + 0.361 \\ &= 1.29\end{aligned}$$

Variance of a Discrete Probability Distribution

A related measure of dispersion is the standard deviation.

$$\sigma = \sqrt{\text{Variance}} = \sqrt{\sigma^2}$$

where

$\sqrt{\ }$ = square root

σ = standard deviation

Variance of a Discrete Probability Distribution

A related measure of dispersion is the standard deviation.

$$\sigma = \sqrt{\text{Variance}} = \sqrt{\sigma^2}$$

where

$\sqrt{\ }$ = s

σ = s

For Dr. Shannon's class:

$$\sigma = \sqrt{\text{Variance}}$$

$$= \sqrt{1.29} = 1.14$$

Using Excel

Formulas in an Excel Spreadsheet for the Dr. Shannon Example

	A	B	C	D
1	**X**	**P(X)**	**XP(X)**	**$(X - E(X))^2P(X)$**
2	5	0.1	=A2*B2	=(A2-C7)^2*B2
3	4	0.2	=A3*B3	=(A3-C7)^2*B3
4	3	0.3	=A4*B4	=(A4-C7)^2*B4
5	2	0.3	=A5*B5	=(A5-C7)^2*B5
6	1	0.1	=A6*B6	=(A6-C7)^2*B6
7		**E(X) = ΣXP(X) =**	=SUM(C2:C6)	=SUM(D2:D6)
8				=SQRT(D7)

Program 2.1A

Using Excel

Excel Output for the Dr. Shannon Example

	A	B	C	D	E	F
1	X	P(X)	XP(X)	$(X - E(X))^2P(X)$		
2	5	0.1	0.5	0.441		
3	4	0.2	0.8	0.242		
4	3	0.3	0.9	0.003		
5	2	0.3	0.6	0.243		
6	1	0.1	0.1	0.361		
7		E(X) = ΣXP(X) =	2.9	1.290	= Variance	
8				1.136	= Standard deviation	

Program 2.1B

Probability Distribution of a Continuous Random Variable

Since random variables can take on an infinite number of values, the fundamental rules for continuous random variables must be modified.

- The sum of the probability values must still equal 1.
- The probability of each individual value of the random variable occurring must equal 0 or the sum would be infinitely large.

The probability distribution is defined by a continuous mathematical function called the probability density function or just the probability function.

- This is represented by $f(X)$.

Probability Distribution of a Continuous Random Variable

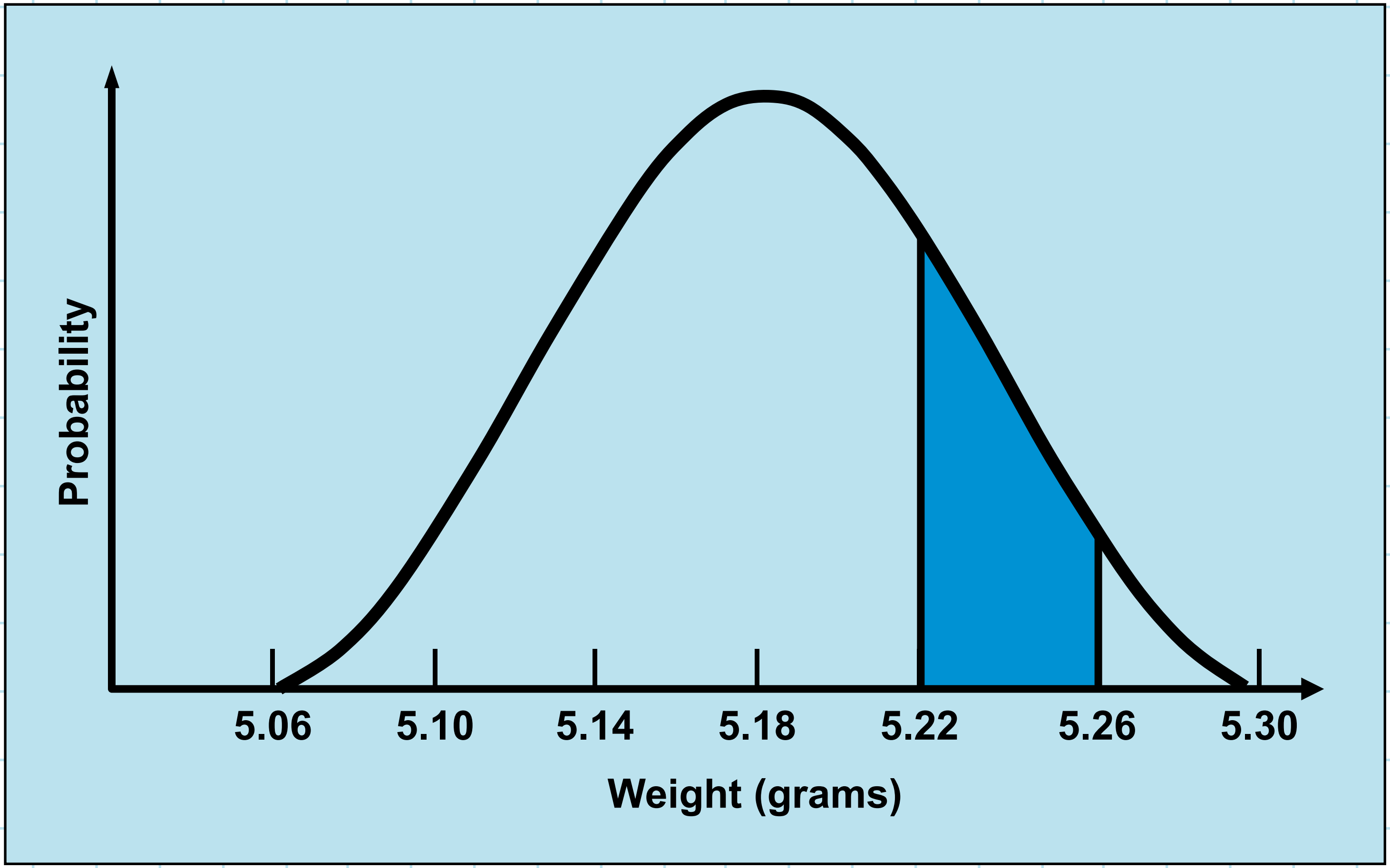


Figure 2.6

The Binomial Distribution

- Many business experiments can be characterized by the Bernoulli process.
- The Bernoulli process is described by the binomial probability distribution.
  1. Each trial has only two possible outcomes.
  2. The probability of each outcome stays the same from one trial to the next.
  3. The trials are statistically independent.
  4. The number of trials is a positive integer.

The Binomial Distribution

The binomial distribution is used to find the probability of a specific number of successes in n trials.

We need to know:

- n = **number of trials**
- p = **the probability of success on any single trial**

We let

- r = **number of successes**
- $q = 1 - p$ = **the probability of a failure**

The Binomial Distribution

The binomial formula is:

Probability of r successes in n trials $= \dfrac{n!}{r!(n-r)!}p^r q^{n-r}$

The symbol ! means factorial, and

$$n! = n(n-1)(n-2)\ldots(1)$$

For example

$$4! = (4)(3)(2)(1) = 24$$

By definition

$$1! = 1 \text{ and } 0! = 1$$

The Binomial Distribution

Binomial Distribution for n = 5 and p = 0.50.

NUMBER OF HEADS (r)	Probability = $\frac{5!}{r!(5-r)!}(0.5)^r(0.5)^{5-r}$
0	$0.03125 = \frac{5!}{0!(5-0)!}(0.5)^0(0.5)^{5-0}$
1	$0.15625 = \frac{5!}{1!(5-1)!}(0.5)^1(0.5)^{5-1}$
2	$0.31250 = \frac{5!}{2!(5-2)!}(0.5)^2(0.5)^{5-2}$
3	$0.31250 = \frac{5!}{3!(5-3)!}(0.5)^3(0.5)^{5-3}$
4	$0.15625 = \frac{5!}{4!(5-4)!}(0.5)^4(0.5)^{5-4}$
5	$0.03125 = \frac{5!}{5!(5-5)!}(0.5)^5(0.5)^{5-5}$

Table 2.7

Solving Problems with the Binomial Formula

We want to find the probability of 4 heads in 5 tosses.

$n = 5$, $r = 4$, $p = 0.5$, and $q = 1 - 0.5 = 0.5$

Thus

$$P = (4 \text{ successes in } 5 \text{ trials}) = \frac{5!}{4!(5-4)!} 0.5^4 0.5^{5-4}$$

$$= \frac{5(4)(3)(2)(1)}{4(3)(2)(1)(1!)} (0.0625)(0.5) = 0.15625$$

Solving Problems with the Binomial Formula

Binomial Probability Distribution for *n* = 5 and *p* = 0.50.

Figure 2.7

Solving Problems with Binomial Tables

MSA Electronics is experimenting with the manufacture of a new transistor.

- **Every hour a random sample of 5 transistors is taken.**
- **The probability of one transistor being defective is 0.15.**

What is the probability of finding 3, 4, or 5 defective?

So n **= 5,** p **= 0.15, and** r **= 3, 4, or 5**

We could use the formula to solve this problem, but using the table is easier.

Solving Problems with Binomial Tables

		P		
n	*r*	0.05	0.10	0.15
5	0	0.7738	0.5905	0.4437
	1	0.2036	0.3281	0.3915
	2	0.0214	0.0729	0.1382
	3	0.0011	0.0081	0.0244
	4	0.0000	0.0005	0.0022
	5	0.0000	0.0000	0.0001

Table 2.8 (partial)

We find the three probabilities in the table for n = 5, p = 0.15, and r = 3, 4, and 5 and add them together.

Solving Problems with Binomial Tables

$$P(\text{3 or more defects}) = P(3) + P(4) + P(5)$$
$$= 0.0244 + 0.0022 + 0.0001 = 0.0267$$

2	0.0214	0.0729	0.1382
3	0.0011	0.0081	$\boxed{0.0244}$
4	0.0000	0.0005	$\boxed{0.0022}$
5	0.0000	0.0000	$\boxed{0.0001}$

Table 2.8 (partial)

We find the three probabilities in the table for n = 5, p = 0.15, and r = 3, 4, and 5 and add them together

Solving Problems with Binomial Tables

It is easy to find the expected value (or mean) and variance of a binomial distribution.

$$\text{Expected value (mean)} = np$$

$$\text{Variance} = np(1 - p)$$

For the MSA example:

$$\text{Expected value} = np = 5(0.15) = 0.75$$

$$\text{Variance} = np(1 - p) = 5(0.15)(0.85) = 0.6375$$

Using Excel

Function in an Excel 2010 Spreadsheet for Binomial Probabilities

Using the cell references eliminates the need to retype the formula if you change a parameter such as *p* or *r*.

The function BINOM.DIST (*r*,*n*,*p*,TRUE) returns the cumulative probability.

	A		
1	**The Binomial D**		
2	X = random variable fo		
3	*n*=	5	number of tr
4	*p*=	0.5	probability of a succes
5	*r*=	4	specific number of successes
6			
7	Cumulative probabilit	P(X ≤ r) =	=BINOM.DIST(B5,B3,B4,TRUE)
8	Probability of exactly	P(X = r) =	=BINOM.DIST(B5,B3,B4,FALSE)

Program 2.2A

Using Excel

Excel Output for the Binomial Example

	A	B	C
1	**The Binomial Distribution**		
2	X = random variable for number of successes		
3	*n*=	5	number of trials
4	*p*=	0.5	probability of a succes
5	*r*=	4	specific number of successes
6			
7	Cumulative probability	P(X ≤ r) =	0.96875
8	Probability of exactly *r* successes	P(X = r) =	0.15625

Program 2.2B

The Normal Distribution

The ***normal distribution*** is the one of the most popular and useful continuous probability distributions.

- The formula for the probability density function is rather complex:

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

- The normal distribution is specified completely when we know the mean, μ, and the standard deviation, σ .

The Normal Distribution

- The normal distribution is symmetrical, with the midpoint representing the mean.
- Shifting the mean does not change the shape of the distribution.
- Values on the X axis are measured in the number of standard deviations away from the mean.
- As the standard deviation becomes larger, the curve flattens.
- As the standard deviation becomes smaller, the curve becomes steeper.

The Normal Distribution

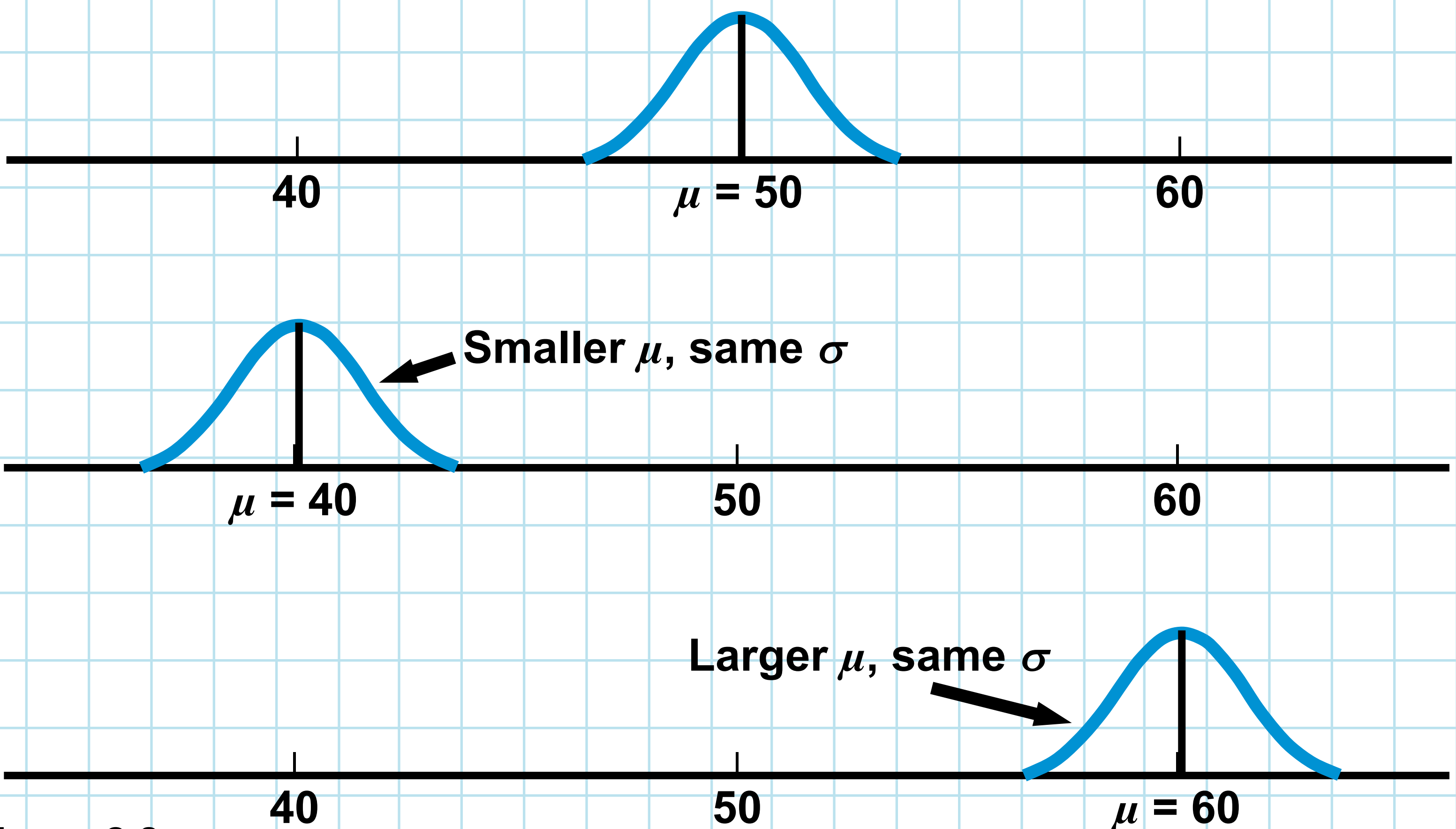


Figure 2.8

The Normal Distribution

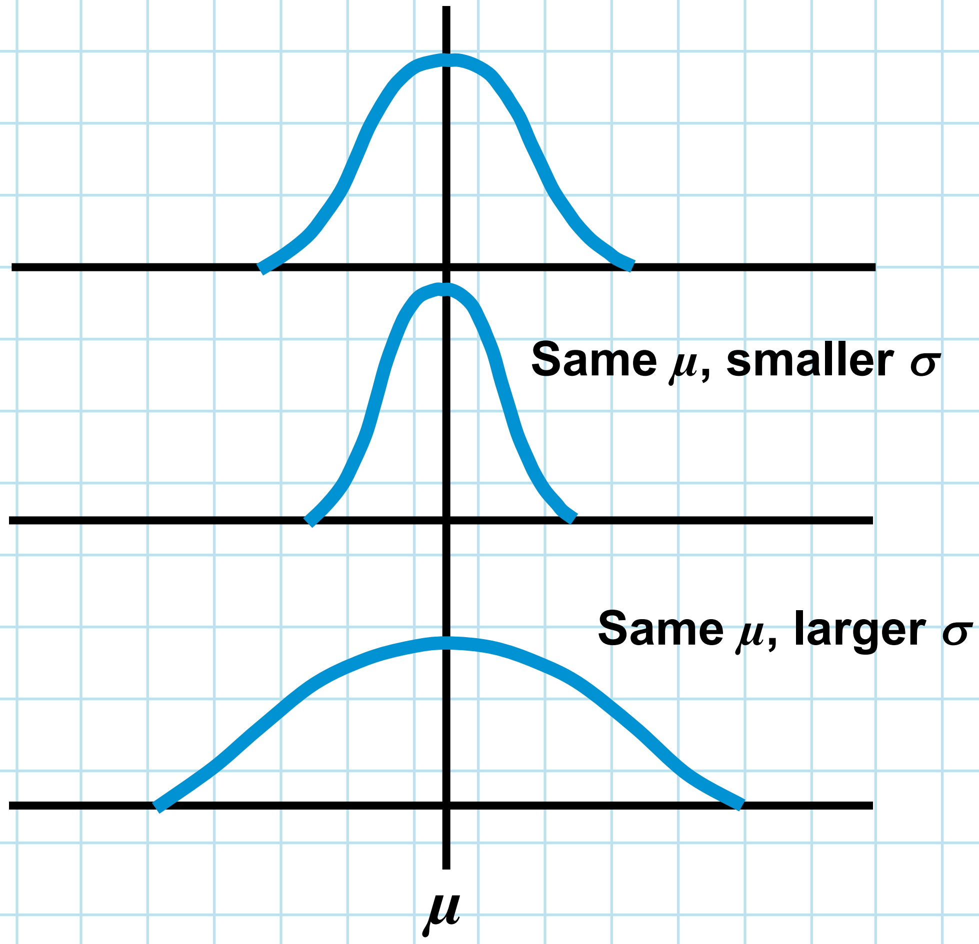


Figure 2.9

Using the Standard Normal Table

Step 1

Convert the normal distribution into a ***standard normal distribution.***

- A standard normal distribution has a mean of 0 and a standard deviation of 1
- The new standard random variable is Z

$$Z = \frac{X - \mu}{\sigma}$$

where

X = value of the random variable we want to measure

μ = mean of the distribution

σ = standard deviation of the distribution

Z = number of standard deviations from X to the mean, μ

Using the Standard Normal Table

For example, $\mu = 100$, $\sigma = 15$, and we want to find the probability that X is less than 130.

$$Z = \frac{X - \mu}{\sigma} = \frac{130 - 100}{15}$$

$$= \frac{30}{15} = 2 \text{ std dev}$$

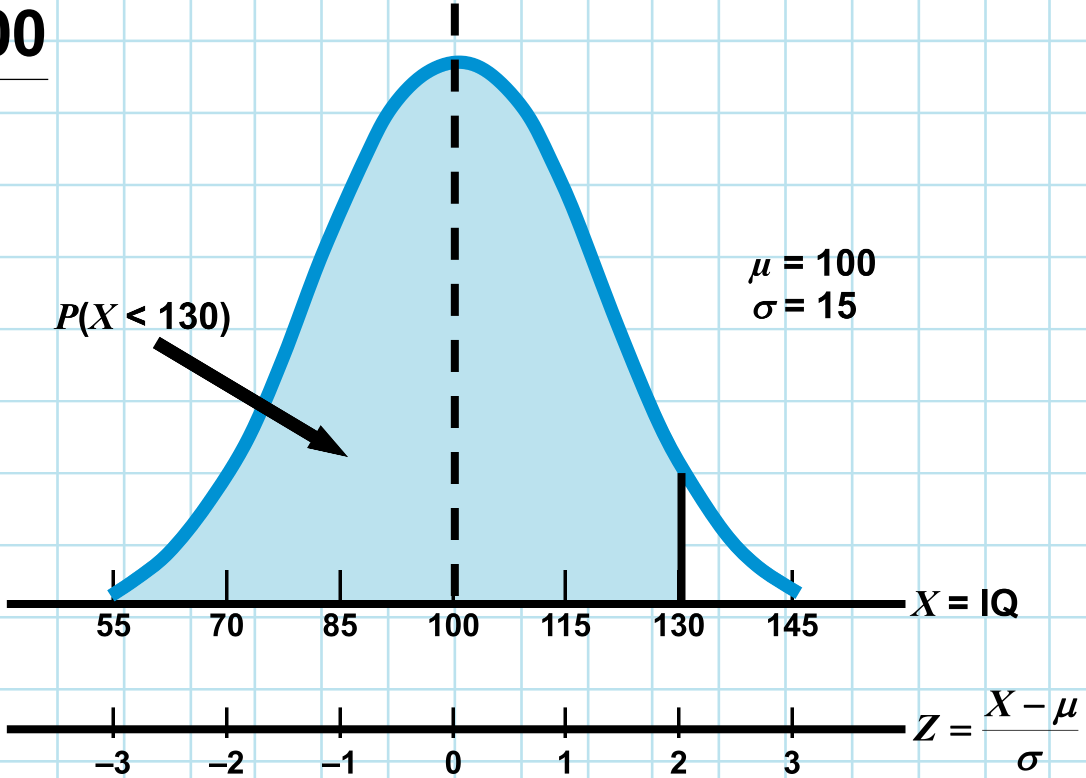


Figure 2.10

Using the Standard Normal Table

Step 2

Look up the probability from a table of normal curve areas.

- **Use Appendix A or Table 2.9 (portion below).**
- **The column on the left has Z values.**
- **The row at the top has second decimal places for the Z values.**

AREA UNDER THE NORMAL CURVE				
Z	0.00	0.01	0.02	0.03
1.8	0.96407	0.96485	0.96562	0.96638
1.9	0.97128	0.97193	0.97257	0.97320
2.0	$\boxed{0.97725}$	0.97784	0.97831	0.97882
2.1	0.98214	0.98257	0.98300	0.98341
2.2	0.98610	0.98645	0.98679	0.98713

Table 2.9 (partial)

$$P(X < 130) = P(Z < 2.00) = 0.97725$$

Haynes Construction Company

Haynes builds three- and four-unit apartment buildings (called triplexes and quadraplexes, respectively).

- Total construction time follows a normal distribution.
- For triplexes, μ = 100 days and σ = 20 days.
- Contract calls for completion in 125 days, and late completion will incur a severe penalty fee.
- What is the probability of completing in 125 days?

Haynes Construction Company

$$Z = \frac{X - \mu}{\sigma} = \frac{125 - 100}{20}$$

$$= \frac{25}{20} = 1.25$$

From Appendix A, for $Z = 1.25$ the area is 0.89435.

- The probability is about 0.89 that Haynes will not violate the contract.

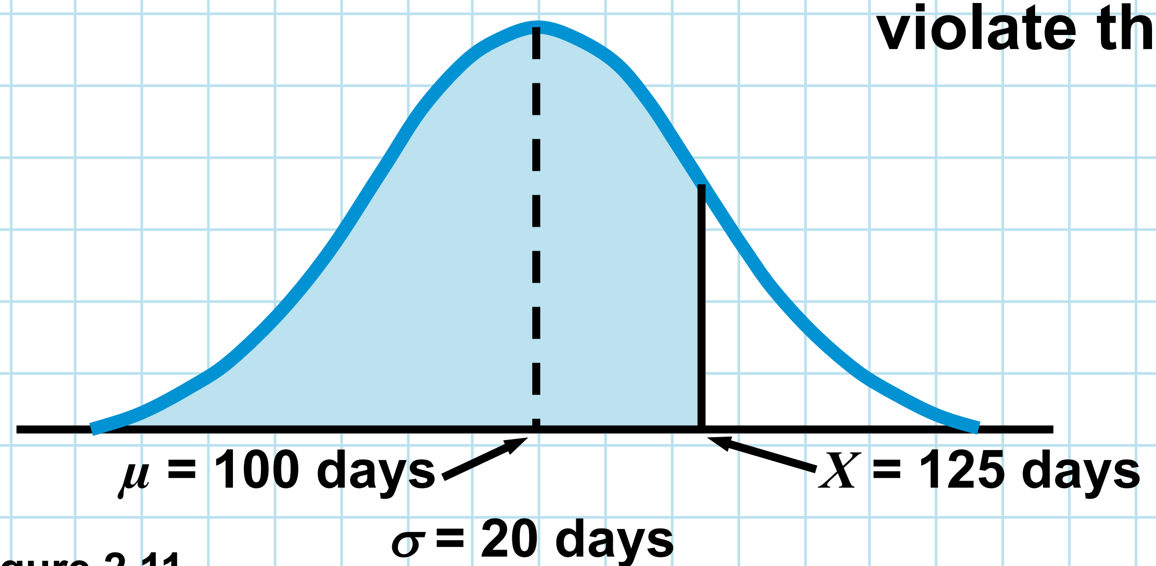


Figure 2.11

Haynes Construction Company

Suppose that completion of a triplex in 75 days or less will earn a bonus of $5,000.

What is the probability that Haynes will get the bonus?

Haynes Construction Company

$$Z = \frac{X - \mu}{\sigma} = \frac{75 - 100}{20}$$

$$= \frac{-25}{20} = -1.25$$

But Appendix A has only positive Z values, and the probability we are looking for is in the negative tail.

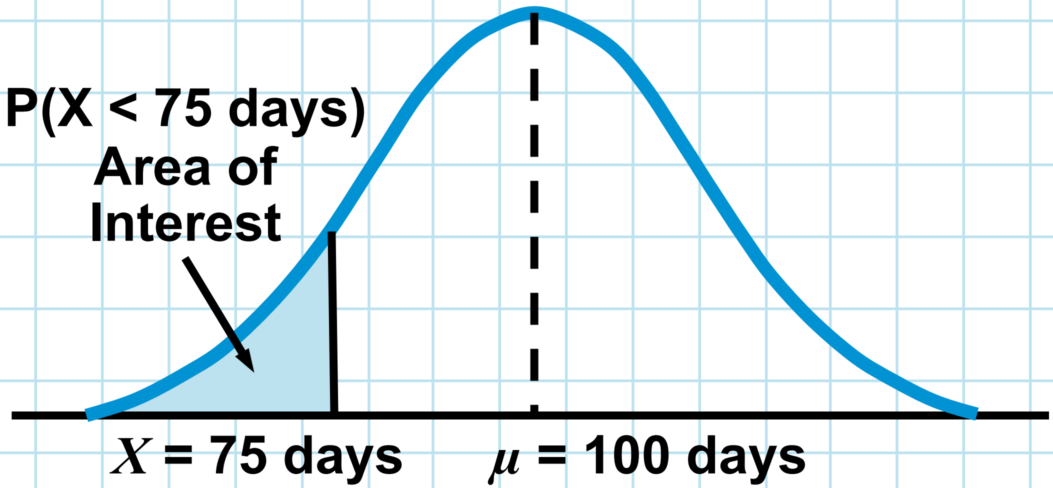


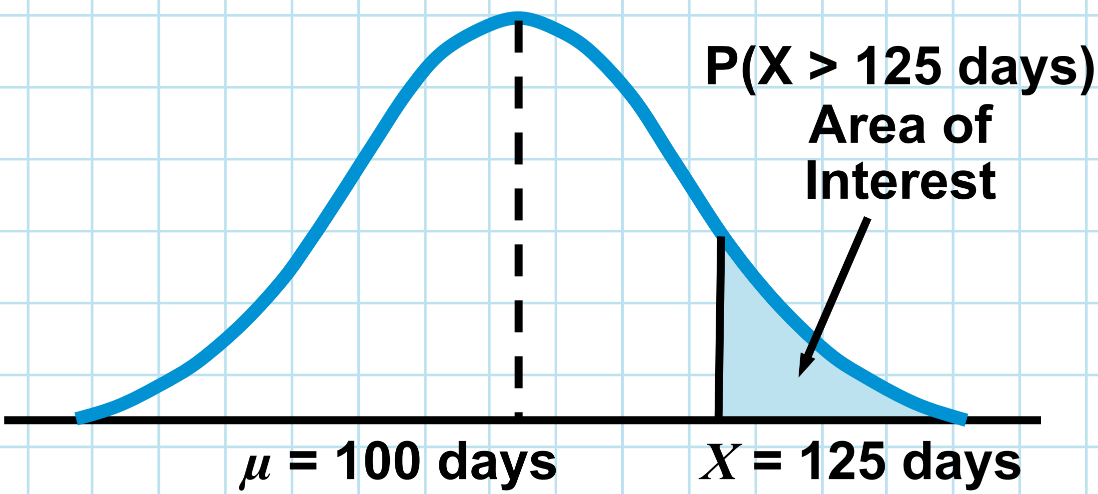
Figure 2.12

Haynes Construction Company

$$Z = \frac{X - \mu}{\sigma} = \frac{75 - 100}{20}$$

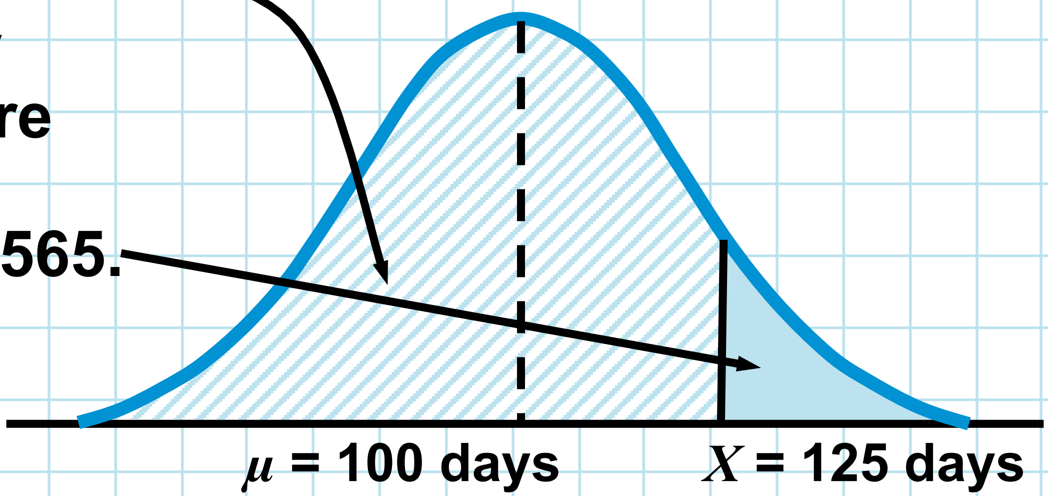
$$= \frac{-25}{20} = -1.25$$

Because the curve is symmetrical, we can look at the probability in the positive tail for the same distance away from the mean.



Haynes Construction Company

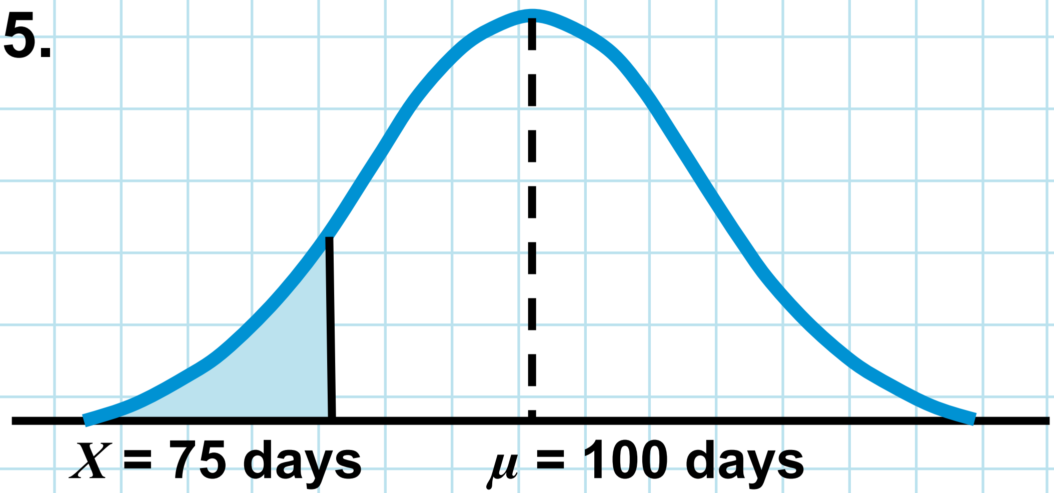
- We know the probability completing in 125 days is 0.89435.
- So the probability completing in more than 125 days is 1 – 0.89435 = 0.10565.



Haynes Construction Company

The probability completing in more than 125 days is
1 – 0.89435 = 0.10565.

Going back to the left tail of the distribution:



The probability of completing in less than 75 days is 0.10565.

Haynes Construction Company

What is the probability of completing a triplex within 110 and 125 days?

We know the probability of completing in 125 days, $P(X < 125) = 0.89435$.

We have to complete the probability of completing in 110 days and find the area between those two events.

Haynes Construction Company

$$Z = \frac{X - \mu}{\sigma} = \frac{110 - 100}{20}$$

$$= \frac{10}{20} = 0.5$$

From Appendix A, for $Z = 0.5$ the area is 0.69146.

$P(110 < X < 125) = 0.89435 - 0.69146 = 0.20289.$

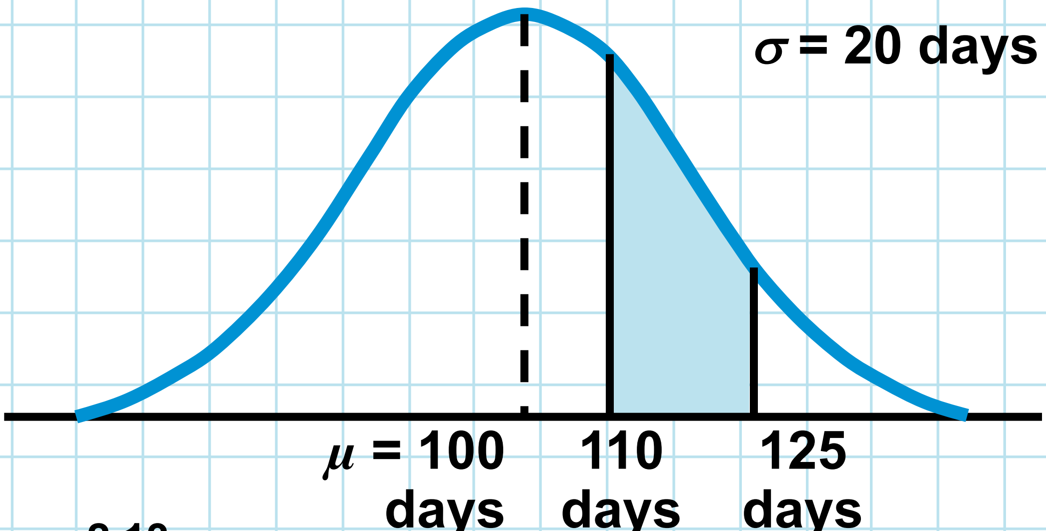


Figure 2.13

Using Excel

Function in an Excel 2010 Spreadsheet for the Normal Distribution Example

	A	B
1	**Normal distribution - X is a**	
2	with mean, µ, and standard	
3	µ =	100
4	σ =	20
5	x =	75
6	P(X ≤ x) =	=NORM.DIST(B5,B3,B4,TRUE)
7	P(X > x) =	=1-B6

Program 2.3A

Using Excel

Excel Output for the Normal Distribution Example

	A	B	C	D
1	**Normal distribution - X is a normal random variable**			
2	with mean, μ, and standard deviation, σ.			
3	μ =	100		
4	σ =	20		
5	x =	75		
6	$P(X \leq x) =$	0.10565		
7	$P(X > x) =$	0.89435		

Program 2.3B

The Empirical Rule

For a normally distributed random variable with mean μ and standard deviation σ, then

1. About 68% of values will be within $\pm 1\sigma$ of the mean.

2. About 95.4% of values will be within $\pm 2\sigma$ of the mean.

3. About 99.7% of values will be within $\pm 3\sigma$ of the mean.

The Empirical Rule

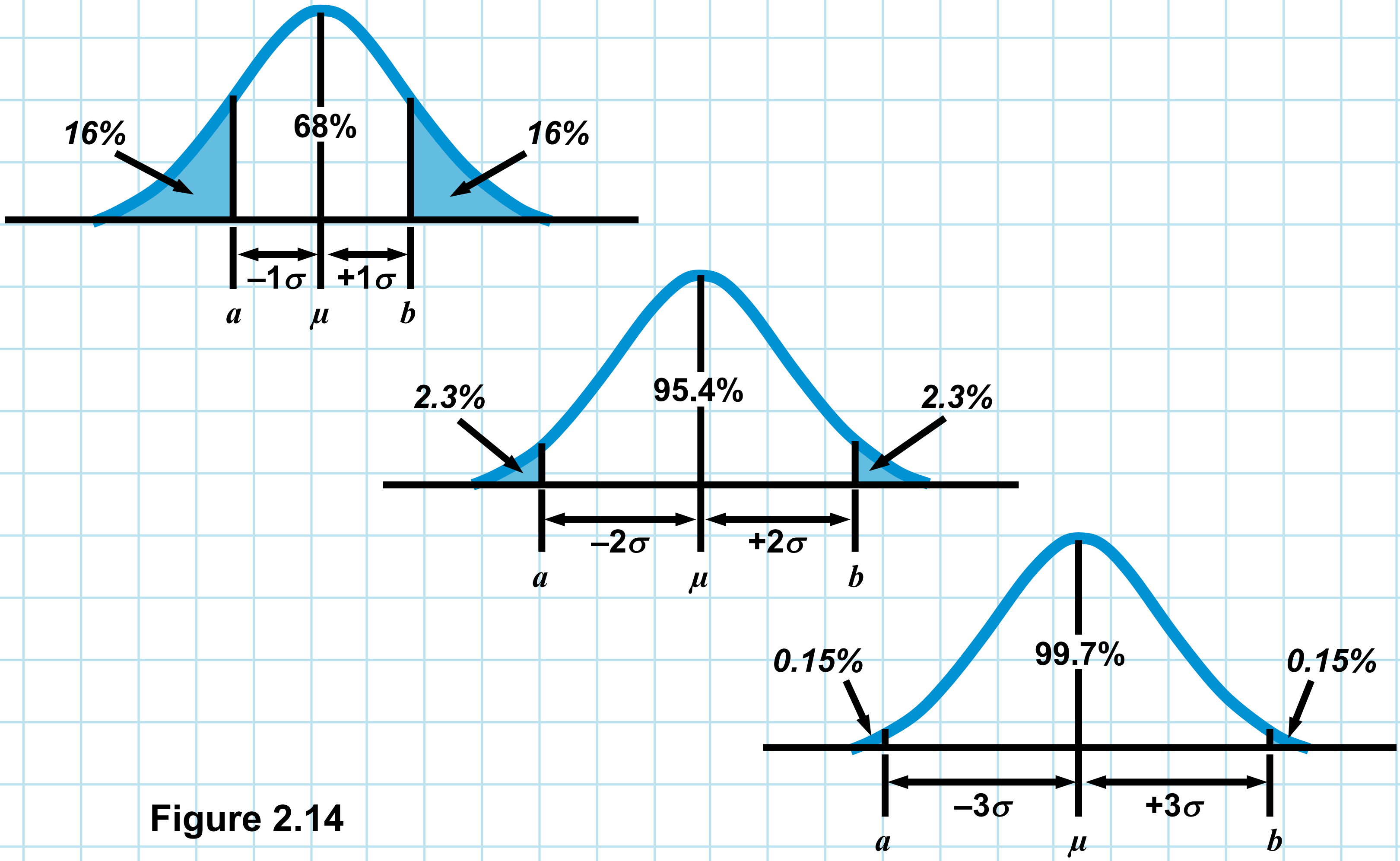


Figure 2.14

The *F* Distribution

- It is a continuous probability distribution.
- The F statistic is the ratio of two sample variances.
- F distributions have two sets of degrees of freedom.
- Degrees of freedom are based on sample size and used to calculate the numerator and denominator of the ratio.

  df_1 = degrees of freedom for the numerator

  df_2 = degrees of freedom for the denominator

- The probabilities of large values of F are very small.

The *F* Distribution

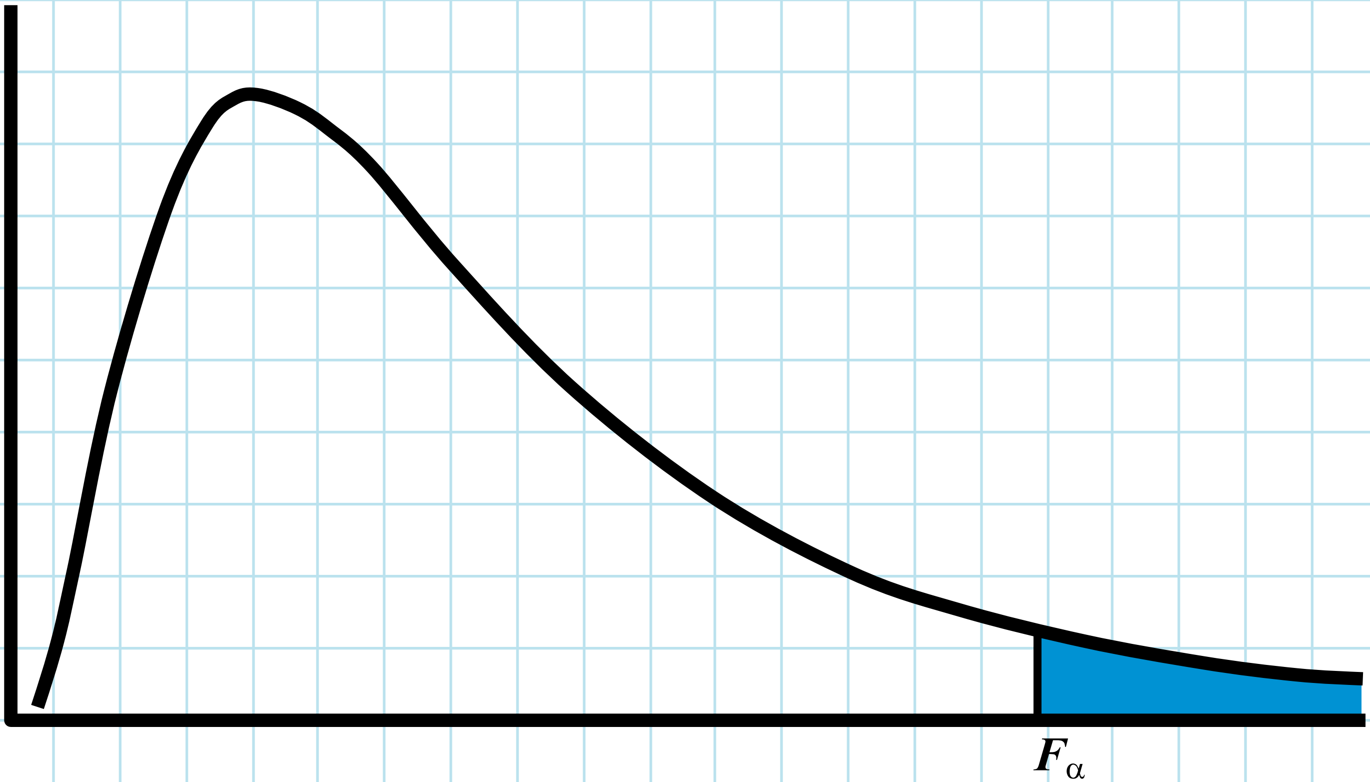


Figure 2.15

The F Distribution

Consider the example:

$df_1 = 5$

$df_2 = 6$

$\alpha = 0.05$

From Appendix D, we get

$$F_{\alpha,\, df_1,\, df_2} = F_{0.05,\, 5,\, 6} = 4.39$$

This means

$$P(F > 4.39) = 0.05$$

The probability is only 0.05 F will exceed 4.39.

The *F* Distribution

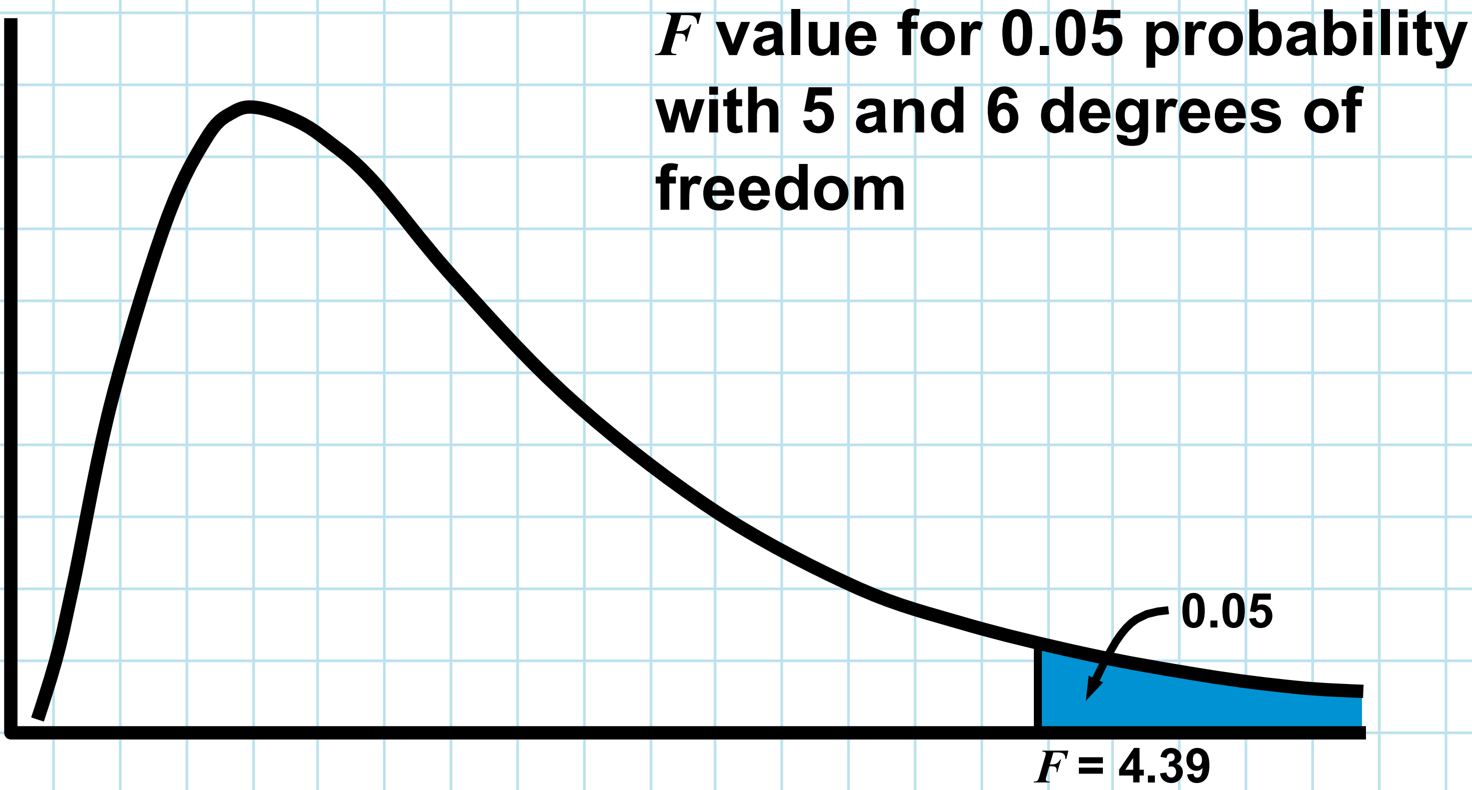


Figure 2.16

Using Excel

	A	B
1	**F Distribution with df**	
2	To find F given α	
3	df1 =	5
4	df2 =	6
5	α =	0.05
6	F-value =	=F.INV.RT(B5,B3,B4)
7		
8	To find the probability	
9	df1 =	5
10	df2 =	6
11	*f* =	4.2
12	P(F > f) =	=F.DIST.RT(B11,B9,B10)

Given the degrees of freedom and the probability $\alpha = 0.05$, this returns the *F*-value corresponding to the right 5% of the area.

This gives the probability to the right of the *F*-value that is specified.

Program 2.4A

Using Excel

Excel Output for the F Distribution

	A	B	C	D	E
1	**F Distribution with df1 and df2 degrees of freedom**				
2	To find F given α				
3	df1 =	5			
4	df2 =	6			
5	α =	0.05			
6	F-value =	4.39			
7					
8	To find the probability to the right of a calculated value, *f*				
9	df1 =	5			
10	df2 =	6			
11	*f* =	4.2			
12	P(F > f) =	0.0548			

Program 2.4B

The Exponential Distribution

- The ***exponential distribution*** (also called the ***negative exponential distribution***) is a continuous distribution often used in queuing models to describe the time required to service a customer. Its probability function is given by:

$$f(X) = \mu e^{-\mu x}$$

where

X = random variable (service times)

μ = average number of units the service facility can handle in a specific period of time

e = 2.718 (the base of natural logarithms)

The Exponential Distribution

$$\text{Expected value} = \frac{1}{\mu} = \text{Average service time}$$

$$\text{Variance} = \frac{1}{\mu^2}$$

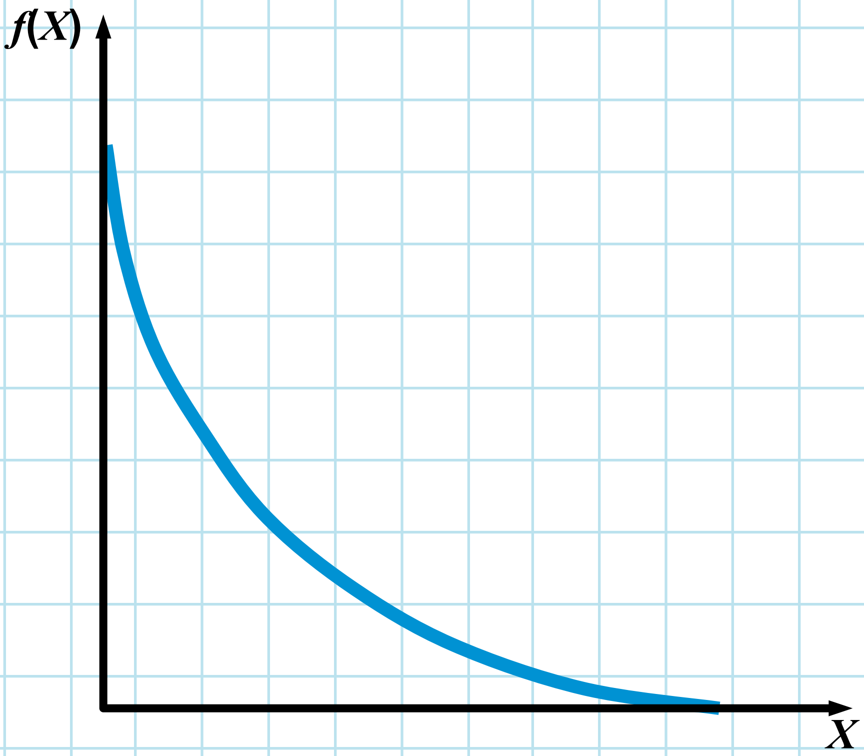


Figure 2.17

Arnold's Muffler Shop

- Arnold's Muffler Shop installs new mufflers on automobiles and small trucks.
- The mechanic can install 3 new mufflers per hour.
- Service time is exponentially distributed.

What is the probability that the time to install a new muffler would be ½ hour or less?

Arnold's Muffler Shop

Here:

X = Exponentially distributed service time

μ = average number of units the served per time period = 3 per hour

t = ½ hour = 0.5hour

$$P(X \leq 0.5) = 1 - e^{-3(0.5)} = 1 - e^{-1.5} = 1 = 0.2231 = 0.7769$$

Arnold's Muffler Shop

Note also that if:

$$P(X \leq 0.5) = 1 - e^{-3(0.5)} = 1 - e^{-1.5} = 1 = 0.2231 = 0.7769$$

Then it must be the case that:

$$P(X > 0.5) = 1 - 0.7769 = 0.2231$$

Arnold's Muffler Shop

Probability That the Mechanic Will Install a Muffler in 0.5 Hour

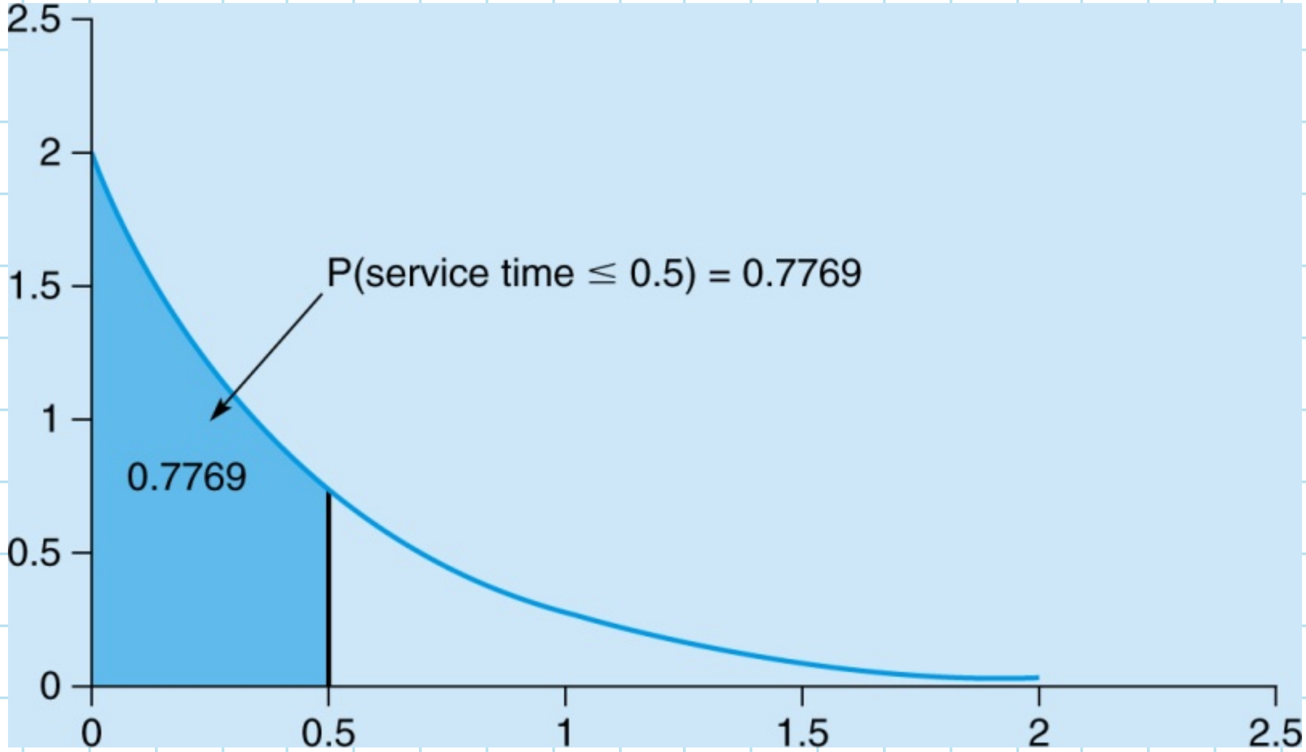


Figure 2.18

Using Excel

Function in an Excel Spreadsheet for the Exponential Distribution

	A	B	C
1	**Exponential distrib**		
2	Average number	3	per hour
3	*t* =	0.5	hours
4	P(X ≤ t) =	=EXPON.DIST(B3,B2,TRUE)	
5	P(X > t) =	=1-B4	
6			
7			

Program 2.5A

Using Excel

Excel Output for the Exponential Distribution

	A	B	C
1	**Exponential distribution - the random variable (X) is time**		
2	Average number per time period = μ =	3	per hour
3	*t* =	0.5000	hours
4	P(X ≤ t) =	0.7769	
5	P(X > t) =	0.2231	

Program 2.5B

The Poisson Distribution

- The ***Poisson distribution*** is a ***discrete*** distribution that is often used in queuing models to describe arrival rates over time. Its probability function is given by:

$$P(X) = \frac{\lambda^x e^{-\lambda}}{X!}$$

where

$P(X)$ = probability of exactly X arrivals or occurrences

λ = average number of arrivals per unit of time (the mean arrival rate)

e = 2.718, the base of natural logarithms

X = specific value (0, 1, 2, 3, …) of the random variable

The Poisson Distribution

The mean and variance of the distribution are both λ.

$$\text{Expected value} = \lambda$$
$$\text{Variance} = \lambda$$

Poisson Distribution

We can use Appendix C to find Poisson probabilities. Suppose that λ = 2. Some probability calculations are:

$$P(X) = \frac{\lambda^x e^{-\lambda}}{X!}$$

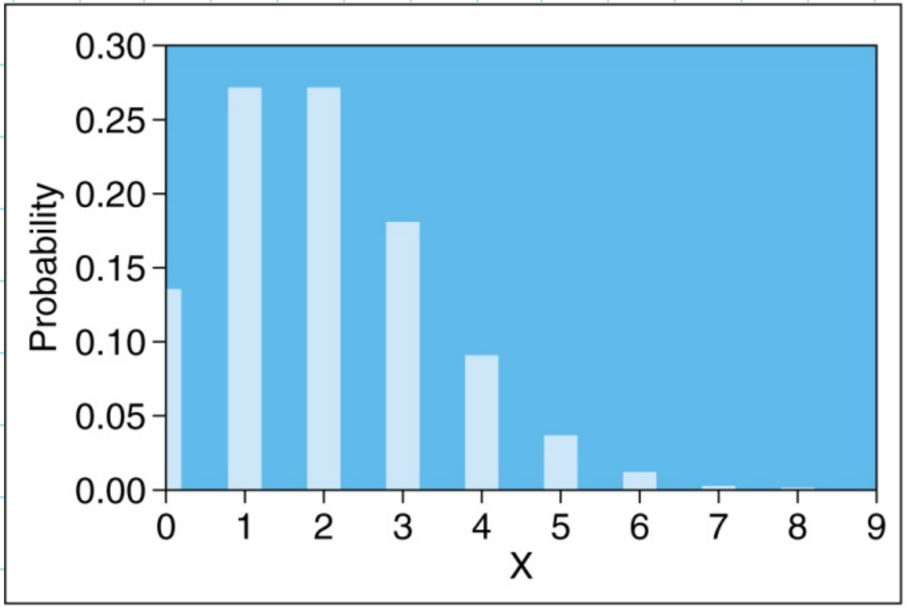
$$P(0) = \frac{2^0 e^{-2}}{0!} = \frac{1(0.1353)}{1} = 0.1353$$

$$P(1) = \frac{2^1 e^{-2}}{1!} = \frac{2(0.1353)}{1} = 0.2706$$

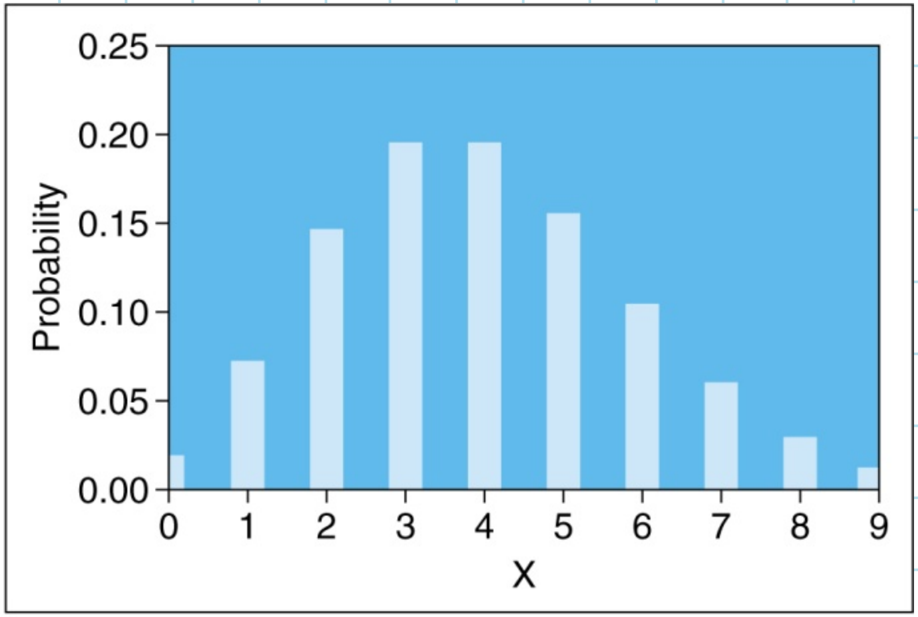
$$P(2) = \frac{2^2 e^{-2}}{2!} = \frac{4(0.1353)}{2} = 0.2706$$

Poisson Distribution

Sample Poisson Distributions with λ = 2 and λ = 4



λ = 2 Distribution



λ = 4 Distribution

Figure 2.19

Using Excel

Functions in an Excel 2010 Spreadsheet for the Poisson Distribution

	A	B	C
1	**Poisson d**		
2	λ =	2	per hour
3	*X*	P(X)	P(X ≤ *x*)
4	0	=POISSON.DIST(A4,B2,FALSE)	=POISSON.DIST(A4,B2,TRUE)
5	1	=POISSON.DIST(A5,B2,FALSE)	=POISSON.DIST(A5,B2,TRUE)
6	2	=POISSON.DIST(A6,B2,FALSE)	=POISSON.DIST(A6,B2,TRUE)

Program 2.6A

Using Excel

Excel Output for the Poisson Distribution

	A	B	C	D	E	F
1	**Poisson distribution - X is the number of occurrences per time period**					
2	λ =	2	per hour			
3	*X*	P(X)	P(X ≤ *x*)			
4	0	0.1353	0.1353			
5	1	0.2707	0.4060			
6	2	0.2707	0.6767			

Program 2.6B

Exponential and Poisson Together

- If the number of occurrences per time period follows a Poisson distribution, then the time between occurrences follows an exponential distribution:
  - Suppose the number of phone calls at a service center followed a Poisson distribution with a mean of 10 calls per hour.
  - Then the time between each phone call would be exponentially distributed with a mean time between calls of 6 minutes (1/10 hour).

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