3.6 The Mean Value Theorem for Derivatives (MVT)

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1 : Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt{x}$ on $[1 , 4]$

Solution

- **اولا نوجد $f'(c)$:**

$$f'(x) = 2 \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$$

$$f'(c) = \frac{1}{\sqrt{c}}$$

- **ثانيا نوجد الجزء الايمن من القانون:**

$f(4) = 2\sqrt{4} = 2 \cdot 2 = \mathbf{4}$

$f(1) = 2\sqrt{1} = 2 \cdot 1 = \mathbf{2}$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 2}{3} = \frac{2}{3}$$

- **الان نوجد c:**

$$\frac{1}{\sqrt{c}} = \frac{2}{3}$$

$$\sqrt{c} = \frac{3}{2}$$

$$c = \frac{9}{4}$$

Example 2 : Let $f(x) = x^3 - x^2 - x + 1 \; on \; [-1, 2]$. Find all numbers c satisfying the conclusion to the Mean Value Theorem.

Solution

- **اولا نوجد $f'(c)$:**

$$f'(x) = 3x^2 - 2x - 1$$
$$f'(c) = \mathbf{3c^2 - 2c - 1}$$

- **ثانيا نوجد الجزء الايمن من القانون:**

$$f(2) = (2)^3 - (2)^2 - (2) + 1 = 8 - 4 - 2 + 1 = \mathbf{3}$$
$$f(-1) = (-1)^3 - (-1)^2 - (-1) + 1 = -1 - 1 + 1 + 1 = \mathbf{0}$$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{3 - 0}{2 + 1} = \frac{3}{3} = \mathbf{1}$$

- **الان نوجد c:**

$$3c^2 - 2c - 1 = 1$$
$$3c^2 - 2c - 2 = 0$$
$$a = 3, \quad b = -2, \quad c = -2$$

$$c_{1,2} = \frac{2 \pm \sqrt{4 + 24}}{6} = \frac{2 \pm \sqrt{28}}{6}$$

$$\mathbf{c_1 = \frac{2 - \sqrt{28}}{6} \approx -0.55}$$
$$\mathbf{c_2 = \frac{2 + \sqrt{28}}{6} \approx 1.22}$$

Example 3 : Let $f(x) = x^{2/3}$ on $[-8\,, 27]$. Show that the conclusion to the Mean Value Theorem fails .

Solution

- **اولا نوجد $f'(c)$:**

$$f'(x) = \frac{2}{3}\, x^{-1/3}$$

$$f'(c) = \frac{2}{3}\, c^{-1/3}$$

- **ثانيا نوجد الجزء الايمن من القانون:**

$$f(27) = (27)^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = \mathbf{9}$$

$$f(-8) = (-8)^{\frac{2}{3}} = (8)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = \mathbf{4}$$

$$\frac{f(27) - f(-8)}{27 - (-8)} = \frac{9 - 4}{27 + 8} = \frac{5}{35} = \frac{1}{7}$$

- **الان نوجد c:**

$$\frac{2}{3}\, c^{-\frac{1}{3}} = \frac{1}{7}$$

$$c^{-\frac{1}{3}} = \frac{1}{7} \cdot \frac{3}{2}$$

$$c^{-\frac{1}{3}} = \frac{3}{14}$$

$$c = \left(\frac{3}{14}\right)^{-3}$$

$$c = \left(\frac{14}{3}\right)^{3} \approx 102$$