CHAPTER 6

Discrete Probability

INTRODUCTION

- *MANY DECISIONS IN BUSINESS, INSURANCE, AND OTHER REAL-LIFE SITUATIONS ARE MADE BY ASSIGNING PROBABILITIES TO ALL POSSIBLE OUTCOMES PERTAINING TO THE SITUATION AND THEN EVALUATING THE RESULTS.*

- *THIS CHAPTER EXPLAINS THE CONCEPTS AND APPLICATIONS OF PROBABILITY DISTRIBUTIONS. IN ADDITION, A SPECIAL PROBABILITY DISTRIBUTION, **BINOMIAL DISTRIBUTION**, IS EXPLAINED.*

DISCRETE PROBABILITY DISTRIBUTION

- *A RANDOM VARIABLE IS A VARIABLE WHOSE VALUES ARE DETERMINED BY CHANCE.*
- *A* ***DISCRETE PROBABILITY DISTRIBUTION*** *CONSISTS OF THE VALUES A RANDOM VARIABLE CAN ASSUME AND THE CORRESPONDING PROBABILITIES OF THE VALUES. THE PROBABILITIES ARE DETERMINED THEORETICALLY OR BY OBSERVATION.*

- ***EX:*** ***CONSTRUCT A PROBABILITY DISTRIBUTION FOR ROLLING A SINGLE DIE.***

SOLUTION:

SINCE THE SAMPLE SPACE IS ***S={1,2,3,4,5,6}*** *AND EACH OUTCOME HAS A PROBABILITY* ***1/6*** *, THE DISTRIBUTION WILL BE*

Outcome x	1	2	3	4	5	6
Probability $P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- ***EX:*** *CONSTRUCT A PROBABILITY DISTRIBUTION FOR THE SAMPLE SPACE FOR TOSSING THREE COINS.*

Number of heads x	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$
Probability $P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

- *EX: DURING THE SUMMER MONTHS, A RENTAL AGENCY KEEPS TRACK OF THE NUMBER OF CHAIN SAWS IT RENTS EACH DAY DURING A PERIOD OF 90 DAYS. THE NUMBER OF SAWS RENTED PER DAY IS REPRESENTED BY THE VARIABLE X. THE RESULTS ARE SHOWN HERE. CONSTRUCT A PROBABILITY DISTRIBUTION.*

x	0	1	2	Total
# of days	**45**	**30**	**15**	**90**

x	0	1	2
$P(x)$	$\frac{45}{90} = 0.5$	$\frac{30}{90} = 0.333$	$\frac{15}{90} = 0.167$

REQUIREMENTS FOR A PROBABILITY DISTRIBUTION

- *THE SUM OF THE PROBABILITIES OF ALL THE EVENTS IN THE SAMPLE SPACE MUST EQUAL 1;*

$$\sum \boldsymbol{P(x) = 1}$$

- *THE PROBABILITY OF EACH EVENT IN THE SAMPLE SPACE MUST BE BETWEEN OR EQUAL TO 0 AND 1;*

$$\boldsymbol{0 \leq P(x) \leq 1}$$

- ***EXAMPLE:***
  - ***DETERMINE WHETHER EACH DISTRIBUTION IS A PROBABILITY DISTRIBUTION.***
    - *A-*

x	0	5	10	15	20
$P(x)$	1/5	1/5	1/5	1/5	1/5

YES, IT IS A PROBABILITY DISTRIBUTION.

- *B-*

x	0	2	4	6
$P(x)$	-1.0	1.5	0.3	0.2

NO, IT IS NOT A PROBABILITY DISTRIBUTION, SINCE P(X) CANNOT BE 1.5 OR -1.0

REQUIREMENTS FOR A PROBABILITY DISTRIBUTION

- *C-*

x	1	2	3	4
$P(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{9}{16}$

YES, IT IS A PROBABILITY DISTRIBUTION.

- *D-*

x	2	3	7
$P(x)$	0.5	0.3	0.4

NO, IT IS NOT, SINCE ***P(X)=1.2***

MEAN OF A PROBABILITY DISTRIBUTION

- *IN ORDER TO FIND THE MEAN FOR A PROBABILITY DISTRIBUTION, ONE MUST MULTIPLY EACH POSSIBLE OUTCOME BY ITS CORRESPONDING PROBABILITY AND FIND THE SUM OF THE PRODUCTS.*

$$\mu = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n) = \sum xp(x)$$

EX: IN A FAMILY WITH TWO CHILDREN, FIND THE MEAN OF THE NUMBER OF CHILDREN WHO WILL BE GIRLS.

THE PROBABILITY DISTRIBUTION IS

# of girls x	0	1	2	$\sum$
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1
$xP(x)$	0	$\frac{1}{2}$	$\frac{2}{4}$	1

- ***EX: IF THREE COINS ARE TOSSED, FIND THE MEAN OF THE NUMBER OF HEADS THAT OCCUR.***

SOLUTION:

THE PROBABILITY DISTRIBUTION IS

# of heads $\boldsymbol{x}$	0	1	2	3	$\sum$
$\boldsymbol{P(x)}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1
$\boldsymbol{xP(x)}$	0	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{3}{8}$	$\frac{12}{8} = \frac{3}{2}$

VARIANCE OF A PROBABILITY DISTRIBUTION

- *THE VARIANCE OF A PROBABILITY DISTRIBUTION IS FOUND BY MULTIPLYING THE SQUARE OF EACH OUTCOME BY ITS CORRESPONDING PROBABILITY, SUMMING THOSE PRODUCTS, AND SUBTRACTING THE SQUARE OF THE MEAN.*
    - *THE FORMULA FOR CALCULATING THE VARIANCE IS:*

$$\boldsymbol{\sigma}^2 = \sum \boldsymbol{x}^2 \boldsymbol{p}(\boldsymbol{x}) - \boldsymbol{\mu}^2$$

    - *THE FORMULA FOR THE STANDARD DEVIATION IS:*

$$\boldsymbol{\sigma} = \sqrt{\boldsymbol{\sigma}^2}$$

- *EX: THE PROBABILITY DISTRIBUTION FOR THE NUMBER OF SPOTS THAT APPEAR WHEN A DIE IS TOSSED*

Outcome x	1	2	3	4	5	6
Probability $\boldsymbol{P}(\boldsymbol{x})$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

FIND THE VARIANCE AND STANDARD DEVIATION OF THE NUMBER OF SPOTS.

Outcome x	1	2	3	4	5	6	$\sum$
Probability $\boldsymbol{P(x)}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$xP(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{21}{6}$
$\boldsymbol{x^2P(x)}$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{25}{6}$	$\frac{36}{6}$	$\frac{91}{6}$

$$\boldsymbol{\mu = \sum xP(x) = \frac{21}{6}}$$

$$\boldsymbol{\sigma^2 = \sum x^2P(x) - \mu^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{441}{36} = \frac{546 - 441}{36}}$$

$$\boldsymbol{= \frac{105}{36} = 2.92}$$

$$\boldsymbol{\sigma = \sqrt{\sigma^2} = \sqrt{2.92} = 1.71}$$

- ***EX: FIVE BALLS NUMBERED 0, 2, 4, 6 AND 8 ARE PLACED IN A BAG. AFTER THE BALLS ARE MIXED, ONE IS SELECTED, ITS NUMBER IS NOTED AND THEN IT IS REPLACED. IF THIS EXPERIMENT IS REPEATED MANY TIMES, AND THE PROBABILITY DISTRIBUTION IS***

# on ball x	0	2	4	6	8
Probability $\boldsymbol{P}(\boldsymbol{x})$	k	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

FIND THE MISSING VALUE (K) , MEAN, VARIANCE AND STANDARD DEVIATION OF THE NUMBERS ON THE BALLS.

$$k+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=1 \rightarrow k+\frac{4}{5}=1 \rightarrow k=1-\frac{4}{5} \rightarrow k=\frac{1}{5}$$

# on ball x	0	2	4	6	8	$\sum$
Probability $P(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
$xP(x)$	0	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$\frac{20}{5}=4$
$x^2P(x)$	0	$\frac{4}{5}$	$\frac{16}{5}$	$\frac{36}{5}$	$\frac{64}{5}$	$\frac{120}{5}=24$

$$\mu=\sum xP(x)=4$$

$$\sigma^2=\sum x^2P(x)-\mu^2=24-(4)^2=24-16=8$$

$$\sigma=\sqrt{\sigma^2}=\sqrt{8}=2.83$$

THE BINOMIAL DISTRIBUTION

- *MANY TYPES OF PROBABILITY PROBLEMS HAVE ONLY TWO POSSIBLE OUTCOMES OR THEY CAN BE REDUCED TO TWO OUTCOMES.*
- ***EXAMPLES:***
  - *WHEN A COIN IS TOSSED IT CAN LAND ON HEADS OR TAILS.*
  - *WHEN A BABY IS BORN IT IS EITHER A BOY OR GIRL.*
  - *A MULTIPLE-CHOICE QUESTION CAN BE CLASSIFIED AS CORRECT OR INCORRECT.*

- *THE BINOMIAL EXPERIMENT IS A PROBABILITY EXPERIMENT THAT SATISFIES THESE REQUIREMENTS:*
  - *EACH TRIAL CAN HAVE ONLY TWO POSSIBLE OUTCOMES - SUCCESS OR FAILURE.*
  - *THERE MUST BE A FIXED NUMBER OF TRIALS.*
  - *THE OUTCOMES OF EACH TRIAL MUST BE INDEPENDENT OF EACH OTHER.*
  - *THE PROBABILITY OF A SUCCESS MUST REMAIN THE SAME FOR EACH TRIAL.*

- *THE OUTCOMES OF A BINOMIAL EXPERIMENT AND THE CORRESPONDING PROBABILITIES OF THESE OUTCOMES ARE CALLED A* ***BINOMIAL DISTRIBUTION*** *WHICH IS THE PROBABILITY OF EXACTLY X SUCCESSES IN N TRIALS*

$$\boldsymbol{P(x) = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}}$$

WHERE

- ***P*** *THE NUMERICAL PROBABILITY OF SUCCESS*
- ***Q*** *THE NUMERICAL PROBABILITY OF FAILURE*

$$\boldsymbol{p + q = 1}$$

- ***N*** *THE NUMBER OF TRIALS*
- ***X*** *THE NUMBER OF SUCCESSES* ***X=0, 1, 2, …, N***

- *EX: A COIN IS TOSSED 3 TIMES. FIND THE PROBABILITY OF GETTING EXACTLY TWO HEADS.*

THIS CAN SOLVED USING THE SAMPLE SPACE

HHH,HHT,HTH,THH,HTT,THT,TTH,TTT

THERE ARE THREE WAYS OF GETTING 2 HEADS.

$$\boldsymbol{P(getting\ 2\ heads) = \frac{n(getting\ 2\ heads)}{n(S)} = \frac{3}{8} = 0.375}$$

- OR USING THE BINOMIAL DISTRIBUTION AS FOLLOWING
  - WE HAVE FIXED NUMBER OF TRIALS (THREE), SO ***N=3***
  - THERE ARE TWO OUTCOMES FOR EACH TRIAL, ***H*** OR ***T***
  - THE OUTCOMES ARE INDEPENDENT OF ONE ANOTHER
  - THE PROBABILITY OF SUCCESS ($\frac{1}{2}$, so $p = \frac{1}{2} \Rightarrow q = 1 - \frac{1}{2} = \frac{1}{2}$

HERE ***X=2*** SINCE WE NEED TO FIND THE PROBABILITY OF

GETTING 2 HEADS,

$$P(x=2) = \frac{n!}{x!\,(n-x)!} p^x q^{n-x} = \frac{3!}{(2!)(1!)} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1$$

$$= \frac{(3)(2)(1)}{(2)(1)(1)} \left(\frac{1}{4}\right) \left(\frac{1}{2}\right) = \frac{6}{16} = \frac{3}{8} = 0.375$$

BINOMIAL DISTRIBUTION PROPERTIES

- *THE MEAN, VARIANCE, AND STANDARD DEVIATION OF A VARIABLE THAT HAS THE BINOMIAL DISTRIBUTION CAN BE FOUND BY USING THE FOLLOWING FORMULAS.*
  - *MEAN*

$$\boldsymbol{\mu = np}$$

  - *VARIANCE*

$$\boldsymbol{\sigma^2 = npq}$$

  - *STANDARD DEVIATION*

$$\boldsymbol{\sigma = \sqrt{\sigma^2}}$$

- *EX: A COIN IS TOSSED 4 TIMES. FIND THE MEAN, VARIANCE AND STANDARD DEVIATION OF THE NUMBER OF HEADS THAT WILL BE OBTAINED.*

In this case $\boldsymbol{n = 4, p = \frac{1}{2} \Rightarrow q = 1 - \frac{1}{2} = \frac{1}{2}}$

$$\boldsymbol{\mu = np = (4)\left(\frac{1}{2}\right) = 2}$$

$$\boldsymbol{\sigma^2 = npq = (4)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1}$$

$$\boldsymbol{\sigma = \sqrt{\sigma^2} = \sqrt{1} = 1}$$