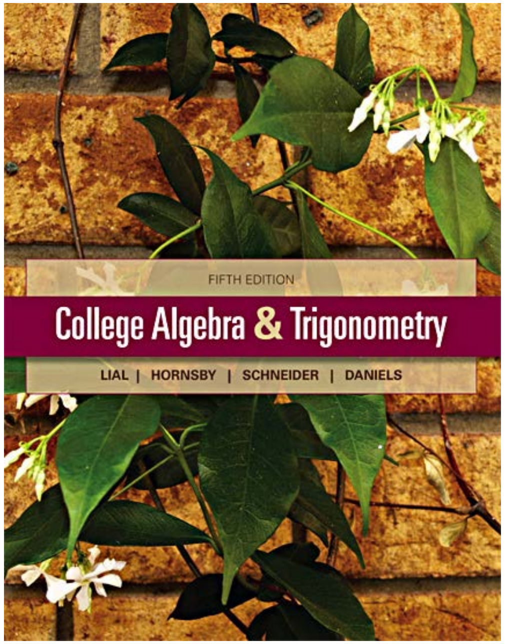
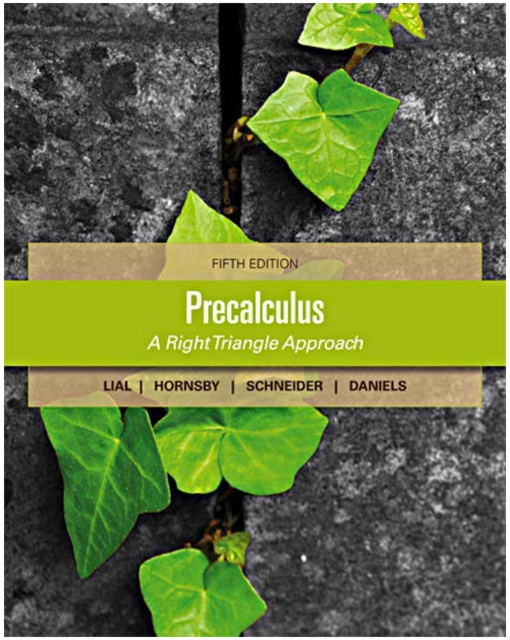
4

Polynomial and Rational Functions





4.1 Quadratic Functions and Models

- Quadratic Functions
- Graphing Techniques
- Completing the Square
- The Vertex Formula
- Quadratic Modelsملغي

Polynomial Function

A **polynomial function** f of degree n, where n is a nonnegative integer, is given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$, and a_0 are real numbers, with $a_n \neq 0$.

Polynomial Functions

Polynomial Function	**Function Name**	**Degree *n***	**Leading Coefficient a_n**
$f(x) = 2$	Constant	0	2
$f(x) = 5x - 1$	Linear	1	5
$f(x) = 4x^2 - x + 1$	Quadratic	2	4
$f(x) = 2x^3 - \frac{1}{2}x + 5$	Cubic	3	2
$f(x) = x^4 + \sqrt{2}x^3 - 3x^2$	Quartic	4	1

Quadratic Function

A function f is a **quadratic function** if

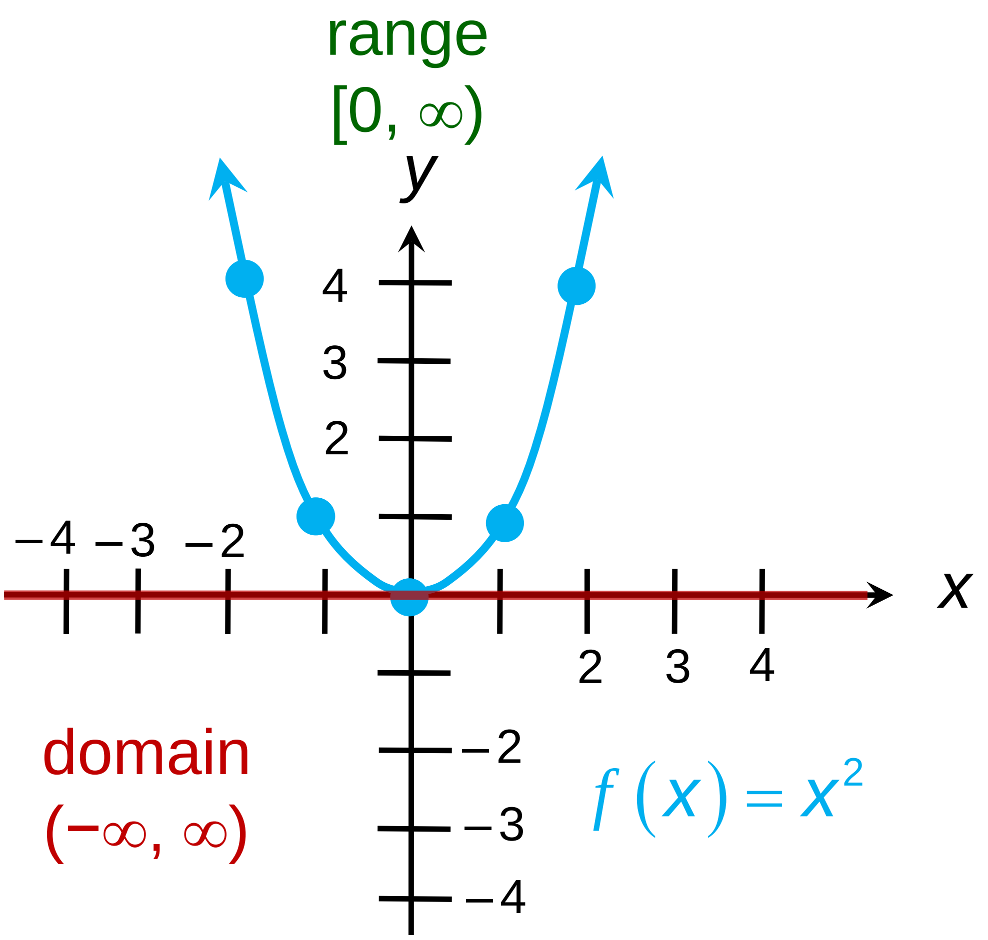
$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers, with $a \neq 0$.

Simplest Quadratic Function

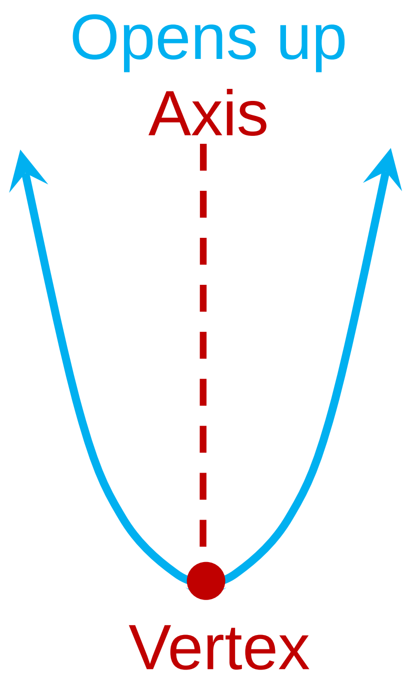
The simplest quadratic function is $f(x) = x^2$.

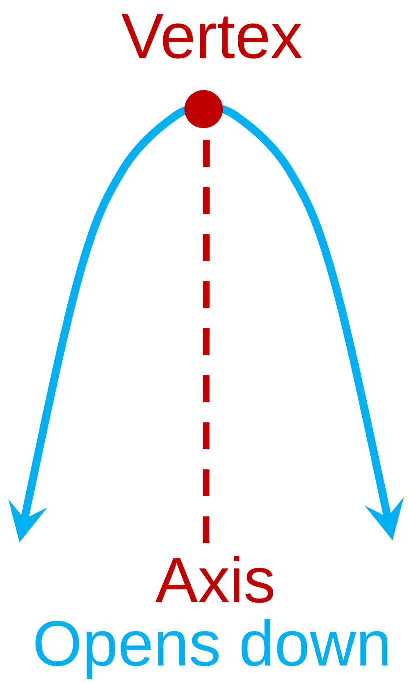
x	$f(x)$
–2	4
–1	1
0	0
1	1
2	4



Parabolas

Parabolas are symmetric with respect to a line. This line is the **axis of symmetry,** or **axis,** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola.





Applying Graphing Techniques to a Quadratic Function

Compared to the basic graph of $f(x) = x^2$, the graph of $F(x) = a(x - h)^2 + k$ has the following characteristics.

$$F(x) = a(x - h)^2 + k$$

- Opens up if $a > 0$
- Opens down if $a < 0$
- Vertically stretched (narrower) if $|a| > 1$
- Vertically shrunk (wider) if $0 < |a| < 1$

Horizontal shift:

- h units right if $h > 0$
- $|h|$ units left if $h < 0$

Vertical shift:

- k units up if $k > 0$
- $|k|$ units down if $k < 0$

Quiz 1

Find the vertex and the axes of each of the following function. And show if its graph opens up or down?

$$a)\ f(x) = 4(x-3)^2 + 5$$

$$b)\ f(x) = -3(x+6)^2 - 7$$

$$c)\ f(x) = (x-8)^2$$

Quiz 1

Solution

$a)\ f(x) = 4(x-3)^2 + 5,$

$vertex = (3,5), axis\ x = 3, a = 4, opens\ \text{up}$

$b)\ f(x) = -3(x+6)^2 - 7$

$vertex = (-6,-7), axis\ x = -6, a = -3, opens\ down$

$c)\ f(x) = (x-8)^2$

$vertex = (8,0), axis\ x = 8, a = 1, opens\ \text{up}$

Example 1 **GRAPHING QUADRATIC FUNCTIONS**

Graph each function. Give the domain and range.

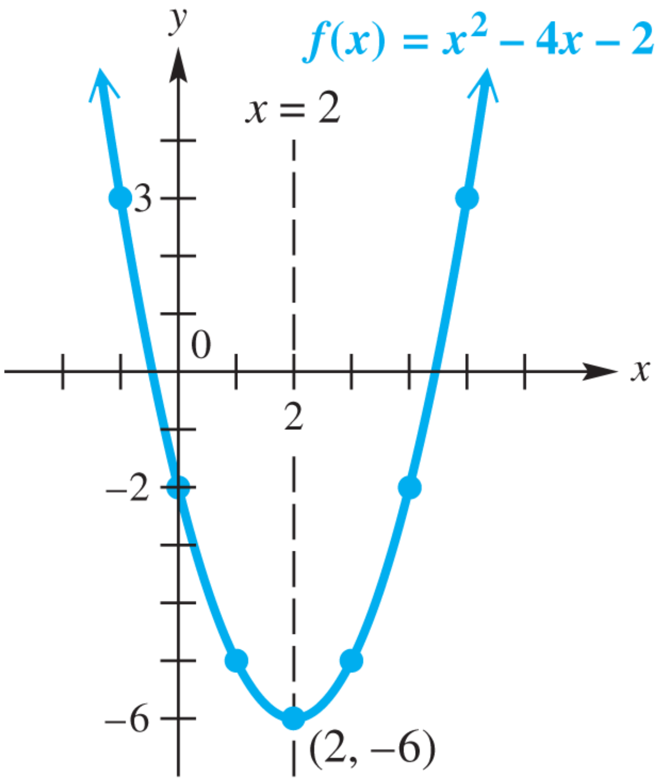
(a) $f(x) = x^2 - 4x - 2$ (by plotting points)

Solution

x	$f(x)$
–1	3
0	–2
1	–5
2	–6
3	–5
4	–2
5	3

Domain $(-\infty, \infty)$

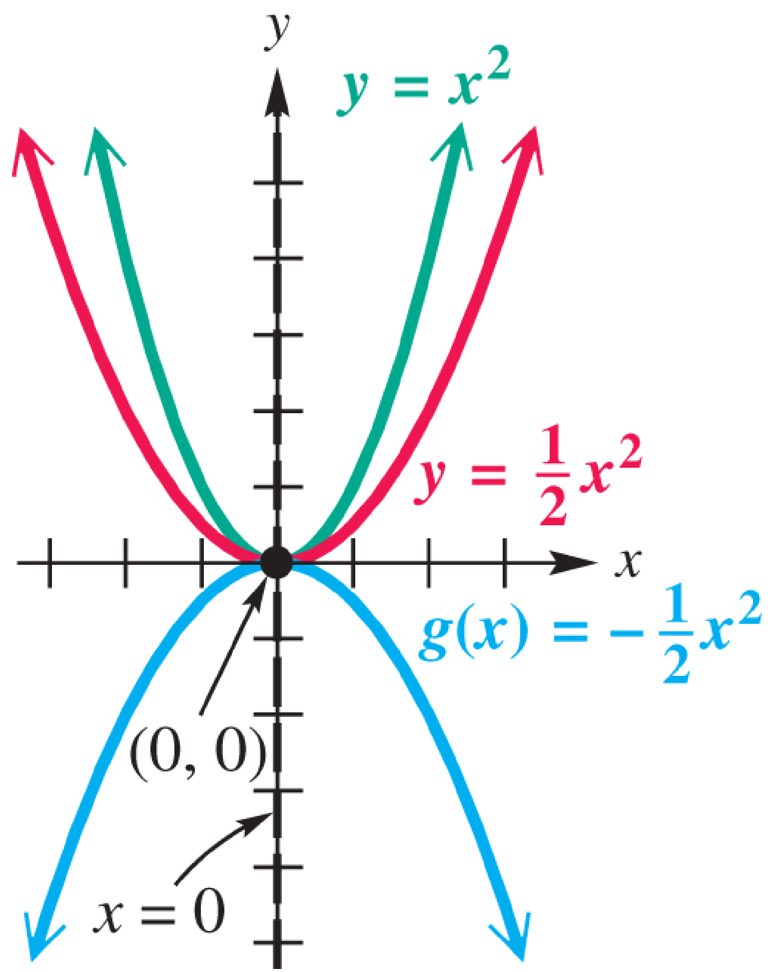
Range $[-6, \infty)$



Graph each function. Give the domain and range.

(b) $g(x) = -\frac{1}{2}x^2$ and compare to $y = x^2$ and $y = \frac{1}{2}x^2$

Solution



Domain
$(-\infty, \infty)$

Range
$(-\infty, 0]$

Graph each function. Give the domain and range.

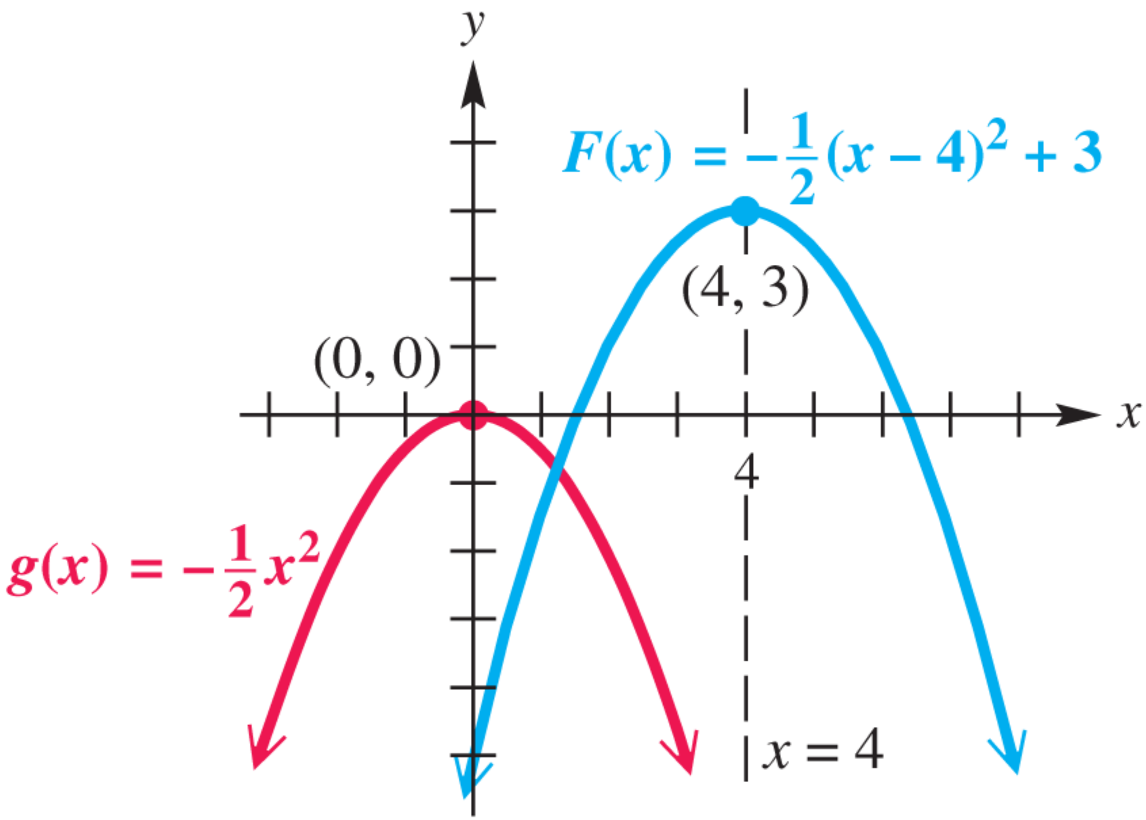
(c) $F(x) = -\frac{1}{2}(x-4)^2 + 3$

and compare to the graph in part (b)

Solution

Domain $(-\infty, \infty)$

Range $(-\infty, 3]$



GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a = 1$)

Graph $f(x) = x^2 - 6x + 7$ by completing the square and locating the vertex. Find the intervals over which the function is increasing or decreasing.

Solution We express $x^2 - 6x + 7$ in the form $(x-h)^2 + k$ by completing the square.

$$f(x) = \left(x^2 - 6x \quad\right) + 7$$

Complete the square.

$$\left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9$$

Homework 1 — GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a = 1$)

Graph $f(x) = x^2 - 6x + 7$ by completing the square and locating the vertex. Find the intervals over which the function is increasing or decreasing.

Solution

$$f(x) = (x^2 - 6x + 9 - 9) + 7$$ Add and subtract 9.

$$f(x) = (x^2 - 6x + 9) - 9 + 7$$ Regroup terms.

$$f(x) = (x - 3)^2 - 2$$ Factor and simplify.

The vertex is (3, –2), and the axis is the line $x = 3$.

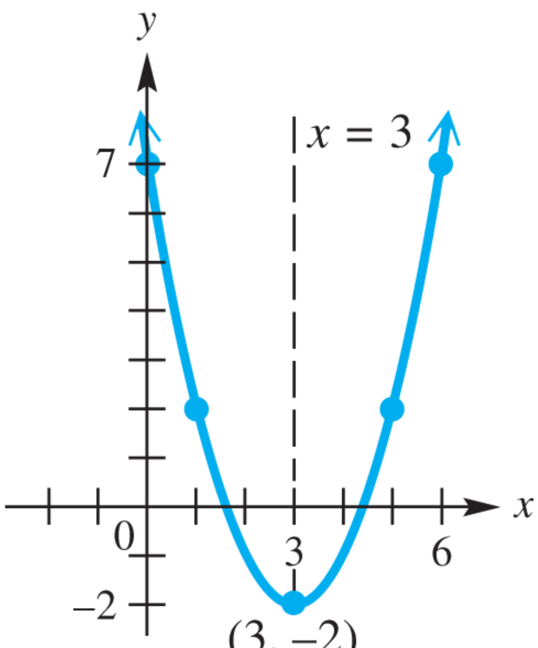
Homework 1

GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a = 1$)

Graph $f(x) = x^2 - 6x + 7$ by completing the square and locating the vertex. Find the intervals over which the function is increasing or decreasing.

Solution

Find additional ordered pairs that satisfy the equation.
Use symmetry about the axis of the parabola to find other ordered pairs.
Connect to obtain the graph.



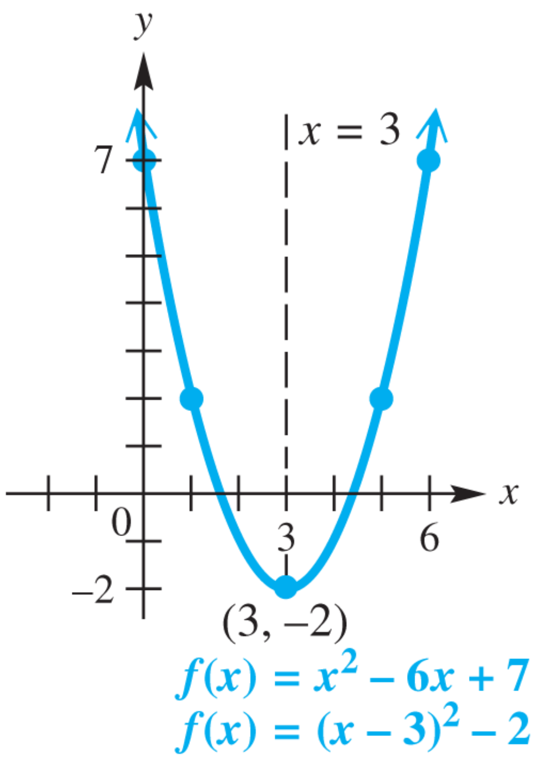
Domain is $(-\infty, \infty)$ Range is $[-2, \infty)$

GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a = 1$)

Graph $f(x) = x^2 - 6x + 7$ by completing the square and locating the vertex. Find the intervals over which the function is increasing or decreasing.

Solution

Since the lowest point on the graph is the vertex (3,–2), the function is decreasing on (–∞,3] and increasing on [3, ∞).



Note In Homework 1 we added and subtracted 9 *on the same side* of the equation to complete the square. This differs from adding the same number to *each side of the equation*, as when we completed the square in **Section 1.3.** Since we want $f(x)$ (or y) alone on one side of the equation, we adjusted that step in the process of completing the square.

Example 2 **GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a \neq 1$)**

Graph $f(x) = -3x^2 - 2x + 1$ by completing the square and locating the vertex. Identify the intercepts of the graph.

Solution To complete the square, the coefficient of x^2 must be 1.

$$f(x) = -3\left(x^2 + \frac{2}{3}x \qquad\right) + 1$$

Factor –3 from the first two terms.

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 1$$

$\left[\frac{1}{2}\left(\frac{2}{3}\right)\right]^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$, so add and subtract $\frac{1}{9}$.

Example 2 GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a \neq 1$)

Graph $f(x) = -3x^2 - 2x + 1$ by completing the square and locating the vertex. Identify the intercepts of the graph.

Solution

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - 3\left(-\frac{1}{9}\right) + 1$$

Distributive property

Be careful here.

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}$$

Factor and simplify.

The vertex is $\left(-\frac{1}{3}, \frac{4}{3}\right)$.

Example 2 GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a \neq 1$)

Graph $f(x) = -3x^2 - 2x + 1$ by completing the square and locating the vertex. Identify the intercepts of the graph.

Solution The intercepts are good additional points to find. The *y*-intercept is found by evaluating $f(0)$.

$$f(0) = -3(0)^2 - 2(0) + 1 = 1$$

The *y*-intercept is 1.

Example 2 GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a \neq 1$)

Graph $f(x) = -3x^2 - 2x + 1$ by completing the square and locating the vertex. Identify the intercepts of the graph.

Solution The *x*-intercepts are found by setting $f(x)$ equal to 0 and solving for *x*.

$0 = -3x^2 - 2x + 1$ Set $f(x) = 0$.

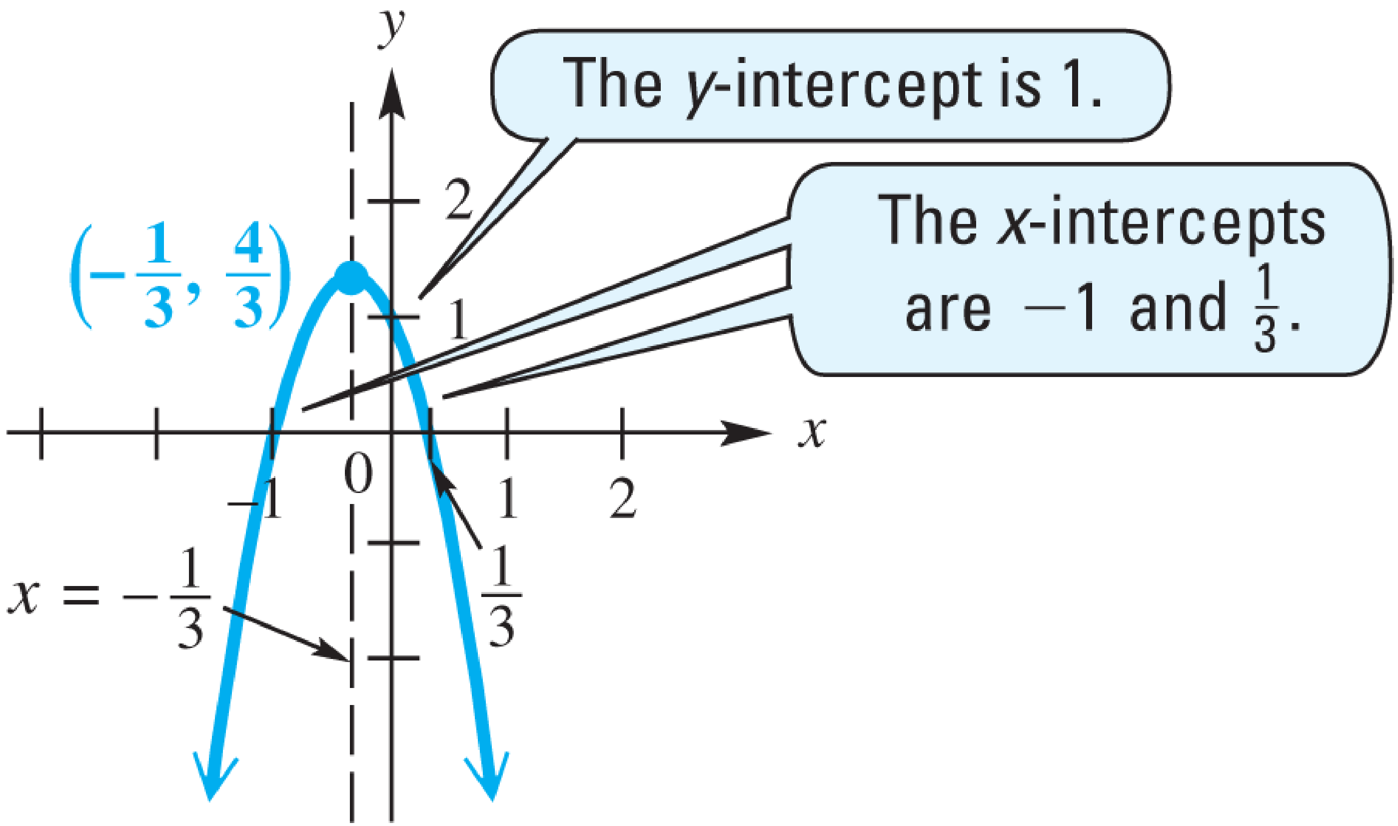
$0 = 3x^2 + 2x - 1$ Multiply by –1.

$0 = (3x - 1)(x + 1)$ Factor.

$x = \frac{1}{3}$ or $x = -1$ Zero-factor property

The *x*-intercepts are $\frac{1}{3}$ and -1.

Example 2 GRAPHING A PARABOLA BY COMPLETING THE SQUARE ($a \neq 1$)



$$f(x) = -3x^2 - 2x + 1$$

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}$$

Note The reverse process

Using the pervious graph to write the quadratic function equation.

Solution.

Since the quadratic function take the fo

$$f(x) = a(x-h)^2+k, \qquad (h,k) = \left(-\frac{1}{3}, \frac{4}{3}\right)$$

$f(x) = a[x - \left(-\frac{1}{3}\right)]^2 + \frac{4}{3} = a[x + \frac{1}{3}]^2 + \frac{4}{3}$, let x=0,y=1

Then a=-3. $f(x) = -3[x + \frac{1}{3}]^2 + \frac{4}{3}$

Graph of a Quadratic Function

The quadratic function defined by $f(x) = ax^2 + bx + c$ can be written as

$$y = f(x) = a(x-h)^2 + k, \quad a \neq 0,$$

where

$$h = -\frac{b}{2a} \text{ and } k = f(h).$$

Graph of a Quadratic Function

The graph of f has the following characteristics.

1. It is a parabola with vertex (h, k) and the vertical line $x = h$ as axis.
2. It opens up if $a > 0$ and down is $a < 0$.
3. It is wider than the graph of $y = x^2$ if $|a| < 1$ and narrower if $|a| > 1$.
4. The y-intercept is $f(0) = c$.
5. The x-intercepts are found by solving the equation $ax^2 + bx + c = 0$.

   If $b^2 - 4ac > 0$, the x-intercepts are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

   If $b^2 - 4ac = 0$, the x-intercept is $-\frac{b}{2a}$.

   If $b^2 - 4ac < 0$, there are no x-intercepts.

Homework 2 USING THE VERTEX FORMULA

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$.

Solution The axis of the parabola is the vertical line

$$x = h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1.$$

The vertex is $(-1, f(-1))$.

Since $f(-1) = 2(-1)^2 + 4(-1) + 5 = 3$, the vertex is $(-1, 3)$.

Quadratic Models ملغى

If air resistance is neglected, the height s (in feet) of an object projected directly upward from an initial height s_0 feet with initial velocity v_0 feet per second is

$$s(t) = -16t^2 + v_0 t + s_0,$$

where t is the number of seconds after the object is projected.

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(a) Give the function that describes the height of the ball in terms of time *t*.

Solution Use the projectile height function with $v_0 = 80$ and $s_0 = 100$.

$$s(t) = -16t^2 + v_0 t + s_0$$

$$s(t) = -16t^2 + 80t + 100$$

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(b) After how many seconds does the projectile reach its maximum height? What is this maximum height?

Solution Find the coordinates of the vertex to determine the maximum height and when it occurs. Let *a* = –16 and *b* = 80 in the vertex formula.

$$x = -\frac{b}{2a} = -\frac{80}{2(-16)} = 2.5$$

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(b) After how many seconds does the projectile reach its maximum height? What is this maximum height?

Solution

$$s(t) = -16t^2 + 80t + 100$$

$$s(2.5) = -16(2.5)^2 + 80(2.5) + 100$$

$$s(2.5) = 200$$

After 2.5 sec the ball reaches its maximum height of 200 ft.

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(c) For what interval of time is the height of the ball greater than 160 ft?

Solution We must solve the quadratic *inequality*

$$-16t^2 + 80t + 100 > 160$$

$$-16t^2 + 80t - 60 > 0 \quad \text{Subtract 160.}$$

$$4t^2 - 20t + 15 < 0 \quad \text{Divide by } -4\text{; reverse the inequality symbol.}$$

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(c) For what interval of time is the height of the ball greater than 160 ft?

Solution By the quadratic formula, the solutions are

$$t = \frac{5 - \sqrt{10}}{2} \approx 0.92 \text{ and } t = \frac{5 + \sqrt{10}}{2} \approx 4.08$$

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(c) For what interval of time is the height of the ball greater than 160 ft?

Solution These numbers divide the number line into three intervals: $(-\infty, 0.92)$, $(0.92, 4.08)$, and $(4.08, \infty)$. Using a test value from each interval shows that $(0.92, 4.08)$ satisfies the *inequality*. The ball is more than 160 ft above the ground between 0.92 sec and 4.08 sec.

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(d) After how many seconds will the ball hit the ground?

Solution The height is 0 when the ball hits the ground. We use the quadratic formula to find the *positive* solution of

$$-16t^2 + 80t + 100 = 0.$$

Example 3 SOLVING A PROBLEM INVOLVING PROJECTILE MOTION

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(d) After how many seconds will the ball hit the ground?

Solution

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(100)}}{2(-16)}$$

$$t \approx \cancel{-1.04} \quad \text{or} \quad t \approx 6.04$$

The ball hits the ground after about 6.04 sec.

Example 4 MODELING THE NUMBER OF HOSPITAL OUTPATIENT VISITS

The number of hospital outpatient visits (in millions) for selected years is shown in the table.

In the table, 95 represents 1995, 100 represents 2000, and so on, and the number of outpatient visits is given in millions.

Year	Visits	Year	Visits
95	483.2	102	640.5
96	505.5	103	648.6
97	520.6	104	662.1
98	545.5	105	673.7
99	573.5	106	690.4
100	592.7	107	693.5
101	612.0	108	710.0

Source: American Hospital Association.

Example 6 MODELING THE NUMBER OF HOSPITAL OUTPATIENT VISITS

(a) Prepare a scatter diagram, and determine a quadratic model for these data.

Solution

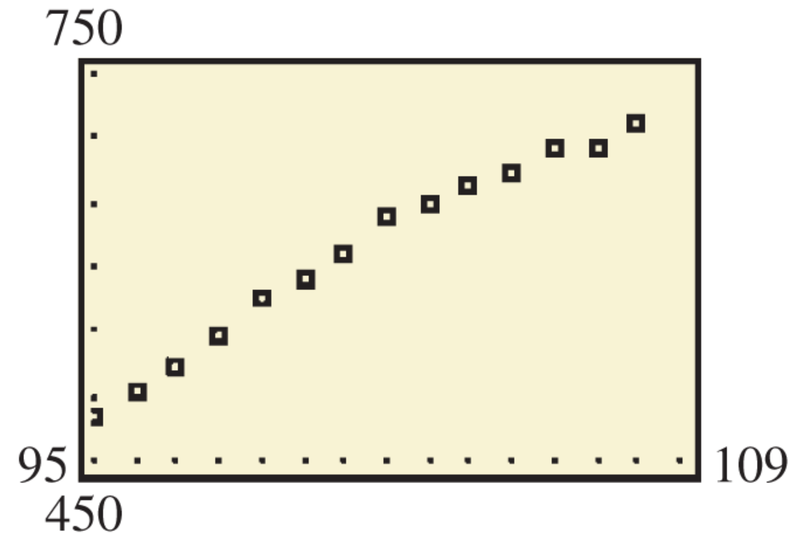
In **Section 2.5** we used linear regression to determine linear equations that modeled data. With a graphing calculator, we can use **quadratic regression** to find quadratic equations that model data.

Example 6 **MODELING THE NUMBER OF HOSPITAL OUTPATIENT VISITS**

(a) Prepare a scatter diagram, and determine a quadratic model for these data.

Solution

The scatter diagram suggests that a quadratic function with a negative value of *a* (so the graph opens down) would be a reasonable model for the data.



Example 6 MODELING THE NUMBER OF HOSPITAL OUTPATIENT VISITS

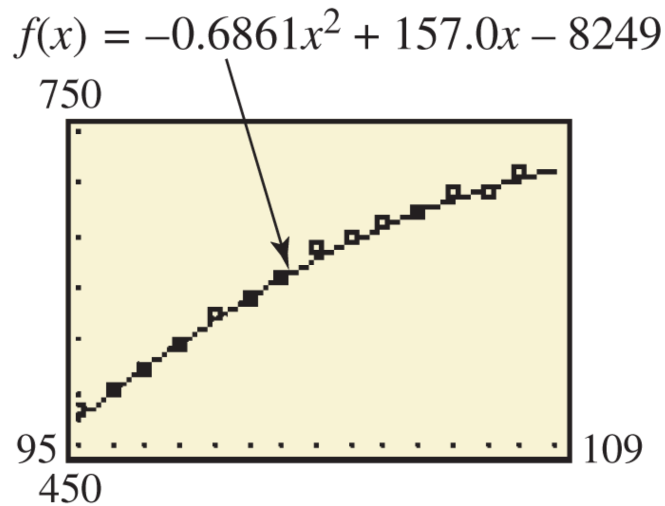
(a) Prepare a scatter diagram, and determine a quadratic model for these data.

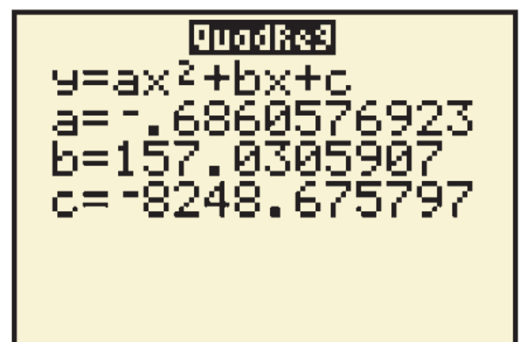
Solution

Using quadratic regression, the quadratic function

$$f(x) = -0.6861x^2 + 157.0x - 8249$$

approximates the data well. The quadratic regression values of *a*, *b*, and *c* are shown.





Example 6 MODELING THE NUMBER OF HOSPITAL OUTPATIENT VISITS

(b) Use the model from part (a) to predict the number of visits in 2012.

Solution The year 2012 corresponds to *x* = 112. The model predicts that there will be 729 million visits in 2012.

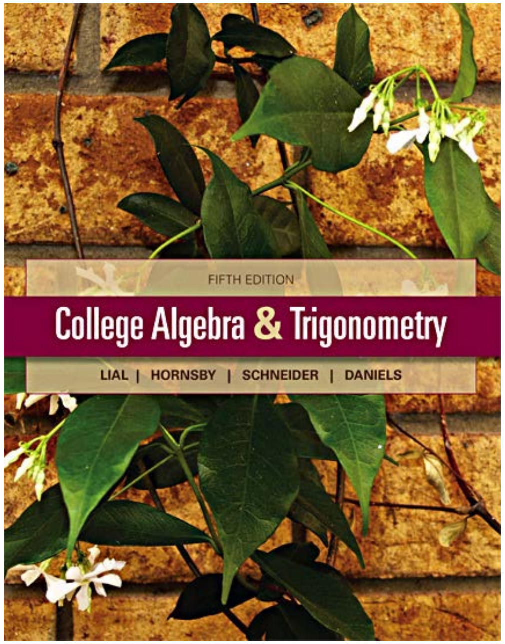
$$f(x) = -0.6861x^2 + 157.0x - 8249$$

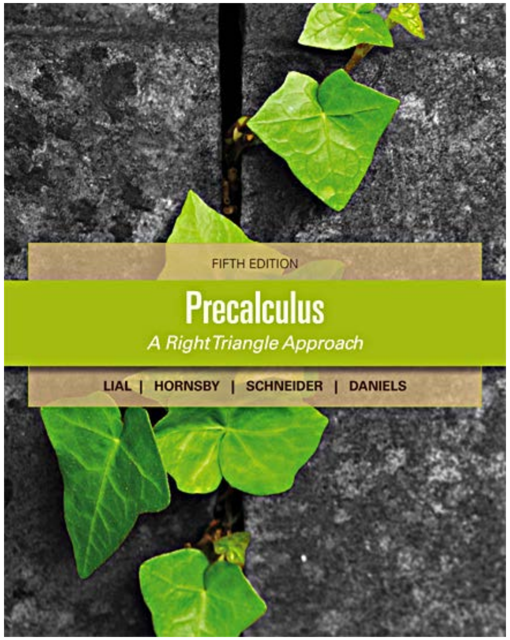
$$f(112) = -0.6861(112)^2 + 157.0(112) - 8249$$

$$\approx 729 \text{ million}$$

4

Polynomial and Rational Functions





4.2 Synthetic Division

- Synthetic Division
- Evaluating Polynomial Functions Using the Remainder Theorem
- Testing Potential Zeros

Division Algorithm

Let $f(x)$ and $g(x)$ be polynomials with $g(x)$ of lesser degree than $f(x)$ and $g(x)$ of degree 1 or more. There exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x) \cdot q(x) + r(x),$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

Synthetic Division

Synthetic division provides an efficient process for dividing a polynomial by a binomial of the form $x - k$.

Synthetic Division

$$\begin{array}{r} 3x^2 + 10x + 40 \\ x-4\,\big)\overline{3x^3 - 2x^2 + 0x - 150} \\ \underline{3x^3 - 12x^2} \\ 10x^2 + 0x \\ \underline{10x^2 - 40x} \\ 40x - 150 \\ \underline{40x - 160} \\ 10 \end{array}$$

Synthetic Division

Here the division process is simplified by omitting all variables and writing only coefficients, with 0 used to represent the coefficient of any missing terms.

$$
\begin{array}{r|rrrrrrr}
 & 3 & & 10 & & 40 & & \\
\hline
-4 & 3 & - & 2 & + & 0 & - & 150 \\
 & 3 & - & 12 & & & & \\
\cline{2-4}
 & & & 10 & + & 0 & & \\
 & & & 10 & - & 40 & & \\
\cline{4-6}
 & & & & & 40 & - & 150 \\
 & & & & & 40 & - & 160 \\
\cline{6-8}
 & & & & & & & 10
\end{array}
$$

Synthetic Division

The numbers in color that are repetitions of the numbers directly above them can be omitted as shown here.

$$
\begin{array}{r|rrrr}
 & 3 & 10 & 40 & \\
\hline
-4 & 3 & -2 & 0 & -150 \\
 & & -12 & & \\
\cline{2-3}
 & & 10 & +\ 0 & \\
 & & & -\ 40 & \\
\cline{3-4}
 & & & 40 & -\ 150 \\
 & & & & -160 \\
\cline{4-5}
 & & & & 10
\end{array}
$$

Synthetic Division

$$
\begin{array}{r|rrrr}
-4 & 3 & -2 & 0 & -150 \\
 & & -12 & -40 & -160 \\
\hline
 & 3 & 10 & 40 & 10
\end{array}
$$

The entire process can now be condensed vertically. The top row of numbers can be omitted since it duplicates the bottom row if the 3 is brought down. The rest of the bottom row is obtained by subtracting –12, –40, and –160 from the corresponding terms above them.

Synthetic Division

Additive inverse →

$$
\begin{array}{r|rrrr}
4 & 3 & -2 & 0 & -150 \\
 & & 12 & 40 & 160 \\
\hline
 & 3 & 10 & 40 & 10
\end{array}
$$

← Signs changed

Quotient → $3x^2 + 10x + 40 + \dfrac{10}{x-4}$ ← Remainder

To simplify the arithmetic, we replace subtraction in the second row by addition and compensate by changing the –4 at the upper left to its additive inverse, 4.

Caution ***To avoid errors, use*** **0** ***as the coefficient for any missing terms, including a missing constant, when setting up the division.***

Example 1 USING SYNTHETIC DIVISION

Use synthetic division to divide.

$$\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}$$

Solution Express $x + 2$ in the form $x - k$ by writing it as $x - (-2)$.

x + 2 leads to –2

$$-2\overline{)\,5 \quad -6 \quad -28 \quad -2}$$ ← Coefficients

Example 1 USING SYNTHETIC DIVISION

Use synthetic division to divide

$$\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}$$

Solution Bring down the 5, and multiply:

$-2(5) = -10$

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & \downarrow & -10 & & \\ \hline & 5 & & & \end{array}$$

Example 1 USING SYNTHETIC DIVISION

Use synthetic division to divide

$$\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}$$

Solution Add –6 and –10 to obtain –16. Multiply –2(–16) = 32.

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & \\ \hline & 5 & -16 & & \end{array}$$

Example 1 **USING SYNTHETIC DIVISION**

Use synthetic division to divide

$$\frac{5x^3-6x^2-28x-2}{x+2}$$

Solution Add –28 and 32, obtaining 4.
Finally, $-2(4) = -8$.

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & -8 \\ \hline & 5 & -16 & 4 & \end{array}$$

Add columns. Be careful with signs.

Example 1 USING SYNTHETIC DIVISION

Use synthetic division to divide

$$\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}$$

Solution Add –2 and –8 to obtain –10.

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & -8 \\ \hline & 5 & -16 & 4 & -10 \end{array}$$

–10 ← Remainder

5 –16 4: Quotient

Example 1 **USING SYNTHETIC DIVISION**

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & -8 \\ \hline & 5 & -16 & 4 & -10 \end{array}$$

Since the divisor *x* – *k* has degree 1, the degree of the quotient will always be written one less than the degree of the polynomial to be divided. Thus,

$$\frac{5x^3 - 6x^2 - 28x - 2}{x+2} = 5x^2 - 16x + 4 + \frac{-10}{x+2}.$$

Remember to add $\frac{\text{remainder}}{\text{divisor}}$.

Special Case of the Division Algorithm

For any polynomial $f(x)$ and any complex number k, there exists a unique polynomial $q(x)$ and number r such that the following holds.

$$f(x) = (x - k)q(x) + r$$

For Example

The mathematical statement

$$\underbrace{5x^3 - 6x^2 - 28x - 2}_{f(x)} = \underbrace{(x+2)}_{=(x-k)}\underbrace{(5x^2 - 16x + 4)}_{q(x)} + \underbrace{(-10)}_{+\ r}.$$

illustrates the special case of the division algorithm. This form of the division algorithm is useful in developing the *remainder theorem.*

Remainder Theorem

If the polynomial $f(x)$ is divided by $x - k$, the remainder is equal to $f(k)$.

Remainder Theorem

In **Example 1,** when $f(x) = 5x^3 - 6x^2 - 28x - 2$ was divided by $x + 2$, or $x - (-2)$, the remainder was –10. Substitute –2 for x in $f(x)$.

$$f(-2) = 5(-2)^3 - 6(-2)^2 - 28(-2) - 2$$

$$= -40 - 24 + 56 - 2$$

$$= -10$$

Use parentheses around substituted values to avoid errors.

Remainder Theorem

An alternative way to find the value of a polynomial is to use synthetic division. By the remainder theorem, instead of replacing x by –2 to find $f(–2)$, divide $f(x)$ by $x + 2$ using synthetic division as in **Example 1.** Then $f(–2)$ is the remainder, –10.

$$
\begin{array}{r|rrrr}
-2 & 5 & -6 & -28 & -2 \\
 & & -10 & 32 & -8 \\
\hline
 & 5 & -16 & 4 & -10
\end{array}
\quad \longleftarrow f(-2)
$$

Homework1 APPLYING THE REMAINDER THEOREM

Let $f(x) = -x^4 + 3x^2 - 4x - 5$. Use the remainder theorem to find $f(-3)$.

Solution Use synthetic division with $k = -3$.

$$\begin{array}{r|rrrrr} -3 & -1 & 0 & 3 & -4 & -5 \\ & & 3 & -9 & 18 & -42 \\ \hline & -1 & 3 & -6 & 14 & -47 \end{array}$$

← Remainder

By this result, $f(-3) = -47$.

Testing Potential Zeros

A **zero** of a polynomial function $f(x)$ is a number k such that $f(k) = 0$. ***The real number zeros are the x-intercepts of the graph of the function.***

The remainder theorem gives a quick way to decide if a number k is a zero of a polynomial function defined by $f(x)$, as follows.

1. Use synthetic division to find $f(k)$.

2. If the remainder is 0, then $f(k) = 0$ and k is a zero of $f(x)$. If the remainder is not 0, then k is not a zero of $f(x)$. A zero of $f(x)$ is a **root,** or **solution,** of the equation $f(x) = 0$.

Example 2 DECIDING WHETHER A NUMBER IS A ZERO

Decide whether the given number k is a zero of $f(x)$.

(a) $f(x) = x^3 - 4x^2 + 9x - 6; \quad k = 1$

Solution Use synthetic division.

Proposed zero →

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 9 & -6 \\ & & 1 & -3 & 6 \\ \hline & 1 & -3 & 6 & 0 \end{array}$$

← $f(x) = x^3 - 4x^2 + 9x - 6$

← Remainder

Since the remainder is 0, $f(1) = 0$, and 1 is a zero of the given polynomial function. An x-intercept of the graph of $f(x)$ is 1, so the graph includes the point (1, 0).

Example 2 DECIDING WHETHER A NUMBER IS A ZERO

Decide whether the given number k is a zero of $f(x)$.

(b) $f(x) = x^4 + x^2 - 3x + 1; \quad k = -1$

Solution Remember to use 0 as coefficient for the missing x^3-term in the synthetic division.

Proposed zero →

$$
\begin{array}{r|rrrrr}
-1 & 1 & 0 & 1 & -3 & 1 \\
 & & -1 & 1 & -2 & 5 \\
\hline
 & 1 & -1 & 2 & -5 & 6
\end{array}
$$

← Remainder

The remainder is not 0, so –1 is not a zero of $f(x) = x^4 + x^2 - 3x + 1$. In fact, $f(-1) = 6$, indicating that (–1, 6) is on the graph of $f(x)$.

Example 2 DECIDING WHETHER A NUMBER IS A ZERO

Decide whether the given number k is a zero of $f(x)$.

(c) $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5; \quad k = 1 + 2i$

Solution Use synthetic division and operations with complex numbers to determine whether $1 + 2i$ is a zero of $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$.

$$
\begin{array}{r|rrrrr}
1+2i & 1 & -2 & 4 & 2 & -5 \\
 & & 1+2i & -5 & -1-2i & 5 \\
\hline
 & 1 & -1+2i & -1 & 1-2i & 0
\end{array}
$$

$(1+2i)(-1+2i)$

$= -1 + 4i^2$

$= -5$

Remainder

Example 2 DECIDING WHETHER A NUMBER IS A ZERO

Decide whether the given number k is a zero of $f(x)$.

(c) $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5; \quad k = 1 + 2i$

Since the remainder is 0, 1 + 2*i* is a zero of the given polynomial function. Notice that 1 + 2*i* is *not* a real number zero. Therefore, it cannot appear as an *x*-intercept on the graph of $f(x)$.

$$
\begin{array}{r|rrrrr}
1+2i & 1 & -2 & 4 & 2 & -5 \\
 & & 1+2i & -5 & -1-2i & 5 \\
\hline
 & 1 & -1+2i & -1 & 1-2i & 0
\end{array}
$$

$(1+2i)(-1+2i)$

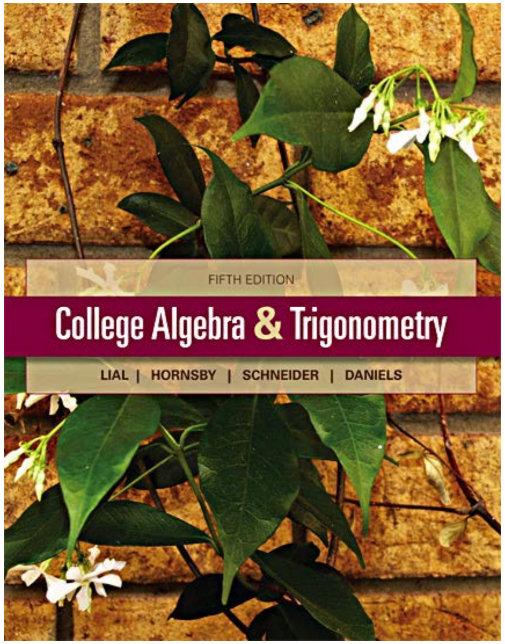
$= -1 + 4i^2$

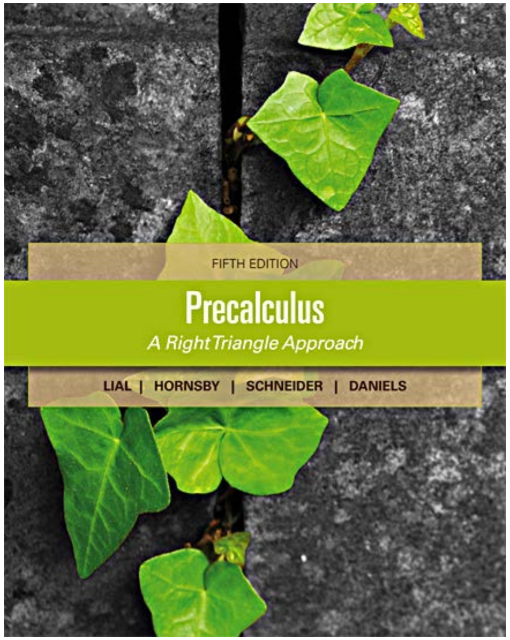
$= -5$

Remainder

4

Polynomial and Rational Functions





4.3 Zeros of Polynomial Functions

- Factor Theorem
- Rational Zeros Theorem ملغي
- Number of Zeros ملغي
- Conjugate Zeros Theorem ملغى
- Finding Zeros of a Polynomial Function ملغى
- Descartes' Rule of Signs ملغي

Factor Theorem

For any polynomial function $f(x)$, $x - k$ is a factor of the polynomial if and only if $f(k) = 0$.

Example 1 DECIDING WHETHER *x* – *k* IS A FACTOR

Determine whether $x - 1$ is a factor of each polynomial.

(a) $f(x) = 2x^4 + 3x^2 - 5x + 7$

Solution By the factor theorem, $x - 1$ will be a factor if $f(1) = 0$. Use synthetic division and the remainder theorem to decide.

1)	2	0	3	–5	7
		2	2	5	0
	2	2	5	0	7

← $f(1) = 7$

Use a zero coefficient for the missing term.

The remainder is 7 and not 0, so $x - 1$ is not a factor of $f(x)$.

Example 1 DECIDING WHETHER $x - k$ IS A FACTOR OF

Determine whether $x - 1$ is a factor of each polynomial.

(b) $f(x) = 3x^5 - 2x^4 + x^3 - 8x^2 + 5x + 1$

Solution

$$
\begin{array}{r|rrrrrr}
1 & 3 & -2 & 1 & -8 & 5 & 1 \\
  &   & 3 & 1 & 2 & -6 & -1 \\
\hline
  & 3 & 1 & 2 & -6 & -1 & 0
\end{array}
\quad \longleftarrow \quad f(1) = 0
$$

Because the remainder is 0, $x - 1$ is a factor. Additionally, we can determine from the coefficients in the bottom row that the other factor is

$$3x^4 + x^3 + 2x^2 - 6x - 1.$$

Example 1 DECIDING WHETHER *x* – *k* IS A FACTOR OF

Determine whether *x* – 1 is a factor of each polynomial.

(b) $f(x) = 3x^5 - 2x^4 + x^3 - 8x^2 + 5x + 1$

Solution

$$
\begin{array}{r|rrrrrr}
1 & 3 & -2 & 1 & -8 & 5 & 1 \\
 & & 3 & 1 & 2 & -6 & -1 \\
\hline
 & 3 & 1 & 2 & -6 & -1 & 0
\end{array}
\quad \longleftarrow \quad f(1) = 0
$$

$$3x^4 + x^3 + 2x^2 - 6x - 1.$$

Thus, $f(x) = (x - 1)(3x^4 + x^3 + 2x^2 - 6x - 1).$

Homework 1 FACTORING A POLYNOMIAL GIVEN A ZERO

Factor $f(x) = 6x^3 + 19x^2 + 2x - 3$ into linear factors if –3 is a zero of f.

Solution Since –3 is a zero of f,

$x - (-3) = x + 3$ is a factor.

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 2 & -3 \\ & & -18 & -3 & 3 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

Use synthetic division to divide $f(x)$ by $x + 3$.

The quotient is $6x^2 + x - 1$, which is the factor that accompanies $x + 3$.

Homework 1 FACTORING A POLYNOMIAL GIVEN A ZERO

Factor the following into linear factors if –3 is a zero of f. $f(x) = 6x^3 + 19x^2 + 2x - 3$

Solution

$$f(x) = (x+3)(6x^2 + x - 1)$$

$$f(x) = (x+3)(2x+1)(3x-1)$$

Factor $6x^2 + x - 1$.

These factors are all linear.

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Rational Zeros Theorem

If $\frac{p}{q}$ is a rational number written in lowest terms, and if $\frac{p}{q}$ is a zero of f, a polynomial function with integer coefficients, then *p* is a factor of the constant term and *q* is a factor of the leading coefficient.

Proof of the Rational Zeros Theorem

$$f\left(\frac{p}{q}\right) = 0 \text{ since } \frac{p}{q} \text{ is a zero of } f(x).$$

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$a_n\left(\frac{p^n}{q^n}\right) + a_{n-1}\left(\frac{p^{n-1}}{q^{n-1}}\right) + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} = -a_0 q^n$$

Multiply by q^n and subtract $a_0\, q^n$.

$$p\left(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}\right) = -a_0 q^n$$

Factor out p.

Proof of the Rational Zeros Theorem

This result shows that $-a_0q^n$ equals the product of the two factors *p* and

$(a_np^{n-1} + \cdots + a_1q^{n-1})$.

For this reason, *p* must be a factor of $-a_0q^n$. Since it was assumed that $\frac{p}{q}$ is written in lowest terms, *p* and *q* have no common factor other than 1, so *p* is not a factor of q^n. Thus, *p* must be a factor of a_0. In a similar way, it can be shown that *q* is a factor of a_n.

Example 3 USING THE RATIONAL ZEROS THEOREM

Consider the polynomial function.

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

(a) List all possible rational zeros.

Solution For a rational number $\frac{p}{q}$ to be a zero, p must be a factor of $a_0 = 2$ and q must be a factor of $a_4 = 6$. Thus, p can be ±1 or ±2, and q can be ±1, ±2, ±3, or ±6. The possible rational zeros, $\frac{p}{q}$, are

$$\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \text{ and } \pm \frac{2}{3}.$$

Example 3 USING THE RATIONAL ZEROS THEOREM

Consider the polynomial function.

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

(b) Find all rational zeros and factor $f(x)$ into linear factors.

Solution Use the remainder theorem to show that 1 is a zero.

Use "trial and error" to find zeros.

$$\begin{array}{r|rrrrr} 1 & 6 & 7 & -12 & -3 & 2 \\ & & 6 & 13 & 1 & -2 \\ \hline & 6 & 13 & 1 & -2 & 0 \end{array}$$

← $f(1) = 0$

The 0 remainder shows that 1 is a zero. The quotient is $6x^3 + 13x^2 + x - 2$, so $f(x) = (x - 1)(6x^3 + 13x^2 + x - 2)$.

Example 3 USING THE RATIONAL ZEROS THEOREM

Consider the polynomial function.

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

(b) Find all rational zeros and factor $f(x)$ into linear equations.

Solution Now, use the quotient polynomial and synthetic division to find that –2 is a zero.

$$\begin{array}{r|rrrr} -2 & 6 & 13 & 1 & -2 \\ & & -12 & -2 & 2 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

← $f(-2) = 0$

The new quotient polynomial is $6x^2 + x - 1$.

Therefore, $f(x)$ can now be completely factored.

Example 3 USING THE RATIONAL ZEROS THEOREM

Consider the polynomial function.

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

(b) Find all rational zeros and factor $f(x)$ into linear equations.

Solution

$$f(x) = (x-1)(x+2)(6x^2 + x - 1)$$

$$f(x) = (x-1)(x+2)(3x-1)(2x+1)$$

Example 3 USING THE RATIONAL ZEROS THEOREM

Consider the polynomial function.

$$f(x)=6x^4+7x^3-12x^2-3x+2$$

(b) Find all rational zeros and factor $f(x)$ into linear equations.

Solution Setting $3x - 1 = 0$ and $2x + 1 = 0$ yields the zeros ⅓ and –½. In summary the rational zeros are 1, –2, ⅓, and –½. The linear factorization of $f(x)$ is

$$\begin{aligned} f(x) &= 6x^4+7x^3-12x^2-3x+2 \\ &= (x-1)(x+2)(3x-1)(2x+1). \end{aligned}$$

Check by multiplying these factors.

Note In **Example 3,** once we obtained the quadratic factor $6x^2 + x - 1$, we were able to complete the work by factoring it directly. Had it not been easily factorable, we could have used the quadratic formula to find the other two zeros (and factors).

Caution ***The rational zeros theorem gives only possible rational zeros. It does not tell us whether these rational numbers are actual zeros.*** We must rely on other methods to determine whether or not they are indeed zeros. Furthermore, the function must have integer coefficients. To apply the rational zeros theorem to a polynomial with fractional coefficients, multiply through by the least common denominator of all the fractions. For example, any rational zeros of $p(x)$ defined below will also be rational zeros of $q(x)$.

$$p(x) = x^4 - \frac{1}{6}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{3}$$

$$q(x) = 6x^4 - x^3 + 4x^2 - x - 2$$

Multiply the terms of $p(x)$ by 6.

Fundamental Theorem of Algebra

Every function defined by a polynomial of degree 1 or more has at least one complex zero.

Fundamental Theorem of Algebra

From the fundamental theorem, if $f(x)$ is of degree 1 or more, then there is some number k_1 such that $f(k_1) = 0$. By the factor theorem,

$$f(x) = (x - k_1)q_1(x)$$

for some polynomial $q_1(x)$.

Fundamental Theorem of Algebra

If $q_1(x)$ is of degree 1 or more, the fundamental theorem and the factor theorem can be used to factor $q_1(x)$ in the same way. There is some number k_2 such that $q_1(k_2) = 0$, so

$$q_1(x) = (x - k_2)q_2(x)$$

and $$f(x) = (x - k_1)(x - k_2)q_2(x).$$

Fundamental Theorem of Algebra

Assuming that $f(x)$ has degree *n* and repeating this process *n* times gives

$$f(x) = a(x - k_1)(x - k_2)\cdots(x - k_n),$$

where *a* is the leading coefficient of $f(x)$. Each of these factors leads to a zero of $f(x)$, so $f(x)$ has the same *n* zeros $k_1, k_2, k_3, \ldots, k_n$. This result suggests the **number of zeros theorem**.

Number of Zeros Theorem

A function defined by a polynomial of degree *n* has *at most n* distinct zeros.

Example 4 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

(a) Zeros of –1, 2, and 4; $f(1) = 3$

Solution These three zeros give $x - (-1) = x + 1$, $x - 2$, and $x - 4$ as factors of $f(x)$. Since $f(x)$ is to be of degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form

$$f(x) = a(x+1)(x-2)(x-4)$$

for some real number a.

Example 4 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

(a) Zeros of –1, 2, and 4; $f(1) = 3$

Solution To find *a*, use the fact that $f(1) = 3$.

$$f(1) = a(1+1)(1-2)(1-4)$$ Let $x = 1$.

$$3 = a(2)(-1)(-3)$$ $f(1) = 3$

$$3 = 6a$$ Multiply.

$$a = \frac{1}{2}$$ Divide by 6.

Example 4 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

(a) Zeros of –1, 2, and 4; $f(1) = 3$

Solution Thus,

$$f(x) = \frac{1}{2}(x+1)(x-2)(x-4),$$

or $$f(x) = \frac{1}{2}x^3 - \frac{5}{2}x^2 + x + 4.$$ Multiply.

Example 4 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

(b) –2 is a zero of multiplicity 3; $f(-1) = 4$

Solution The polynomial function defined by $f(x)$ has the following form.

$$f(x) = a(x+2)(x+2)(x+2)$$

Factor theorem

$$f(x) = a(x+2)^3$$

Example 4 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

(b) –2 is a zero of multiplicity 3; $f(-1) = 4$

Solution To find *a*, use the fact that $f(-1) = 4$.

$$f(-1) = a(-1+2)^3$$

$$4 = a(1)^3$$

$$a = 4$$

Remember: $(x + 2)^3 \neq x^3 + 2^3$

Thus $f(x) = 4(x+2)^3 = 4x^3 + 24x^2 + 48x + 32.$

Note In **Example 4a,** we cannot clear the denominators in $f(x)$ by multiplying each side by 2 because the result would equal $2 \cdot f(x)$, not $f(x)$.

Properties of Conjugates

For any complex numbers *c* and *d*, the following properties hold.

$$\overline{c+d} = \overline{c} + \overline{d}, \quad \overline{c \cdot d} = \overline{c} \cdot \overline{d}, \text{ and } \overline{c^n} = \left(\overline{c}\right)^n$$

Conjugate Zeros Theorem

If $f(x)$ defines a polynomial function ***having only real coefficients*** and if $z = a + bi$ is a zero of $f(x)$, where a and b are real numbers, then

$$\bar{z} = a - bi \text{ is also a zero of } f(x).$$

Proof of the Conjugate Zeros Theorem

Start with the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

where all coefficients are real numbers. If the complex number z is a zero of $f(x)$, then

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Take the conjugate of both sides of this equation.

$$\overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} = \overline{0}.$$

Proof of the Conjugate Zeros Theorem

Using generalizations of the properties

$$\overline{c+d} = \overline{c} + \overline{d} \text{ and } \overline{c \cdot d} = \overline{c} \cdot \overline{d},$$

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \overline{0}$$

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} = \overline{0}.$$

Now use the property $\overline{c^n} = \left(\overline{c}\right)^n$ and the fact that for any real number a, $\overline{a} = a$, to obtain

$$a_n\left(\overline{z}\right)^n + a_{n-1}\left(\overline{z}\right)^{n-1} + \cdots + a_1\left(\overline{z}\right) + a_0 = 0$$

Hence $\overline{z}$ is also a zero of $f(x)$, which completes the proof. $\quad f\left(\overline{z}\right) = 0.$

Caution ***When the conjugate zeros theorem is applied, it is essential that the polynomial have only real coefficients.*** For example,

$$f(x) = x - (1 + i)$$

has $1 + i$ as a zero, but the conjugate $1 - i$ is not a zero.

Example 5 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (COMPLEX ZEROS)

Find a polynomial function of least degree having only real coefficients and zeros 3 and $2 + i$.

Solution The complex number $2 - i$ must also be a zero, so the polynomial has at least three zeros: 3, $2 + i$, and $2 - i$. For the polynomial to be of least degree, these must be the only zeros. By the factor theorem there must be three factors, $x - 3$, $x - (2 + i)$, and $x - (2 - i)$.

Example 5 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (COMPLEX ZEROS)

Find a polynomial function of least degree having only real coefficients and zeros 3 and 2 + *i*.

Solution

$$f(x) = (x-3)[x-(2+i)][x-(2-i)]$$

$$= (x-3)(x-2-i)(x-2+i)$$

$$= (x-3)(x^2-4x+5)$$

$$= x^3 - 7x^2 + 17x - 15$$

Remember: $i^2 = -1$

Example 5 FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (COMPLEX ZEROS)

Find a polynomial function of least degree having only real coefficients and zeros 3 and $2 + i$.

Solution

Any nonzero multiple of

$x^3 - 7x^2 + 17x - 15$ also satisfies the given conditions on zeros. The information on zeros given in the problem is not sufficient to give a specific value for the leading coefficient.

Example 6 **FINDING ALL ZEROS GIVEN ONE ZERO**

Find all zeros of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, given that $1 - i$ is a zero.

Solution Since the polynomial function has only real coefficients and since $1 - i$ is a zero, by the conjugate zeros theorem $1 + i$ is also a zero. To find the remaining zeros, first use synthetic division to divide the original polynomial by $x - (1 - i)$.

Example 6 FINDING ALL ZEROS GIVEN ONE ZERO

Find all zeros of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, given that $1 - i$ is a zero.

Solution

$$
\begin{array}{r|rrrrr}
1-i & 1 & -7 & 18 & -22 & 12 \\
 & & 1-i & -7+5i & 16-6i & -12 \\
\hline
 & 1 & -6-i & 11+5i & -6-6i & 0
\end{array}
$$

Example 6 FINDING ALL ZEROS GIVEN ONE ZERO

Find all zeros of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, given that $1 - i$ is a zero.

Solution By the factor theorem, since $x = 1 - i$ is a zero of $f(x)$, $x - (1 - i)$ is a factor, and $f(x)$ can be written as follows.

$$f(x) = [x - (1 - i)][x^3 + (-6 - i)x^2 + (11 + 5i)x + (-6 - 6i)]$$

We know that $x = 1 + i$ is also a zero of $f(x)$. Continue to use synthetic division and divide the quotient polynomial above by $x - (1 + i)$.

Example 6 FINDING ALL ZEROS GIVEN ONE ZERO

Find all zeros of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, given that $1 - i$ is a zero.

Solution

$$
\begin{array}{r|rrrr}
1+i & 1 & -6-i & 11+5i & -6-6i \\
 & & 1+i & -5-5i & 6+6i \\
\hline
 & 1 & -5 & 6 & 0
\end{array}
$$

Now *f*(*x*) can be written in the following factored form.

$$f(x) = [x-(1-i)][x-(1+i)](x^2-5x+6)$$

$$f(x) = [x-(1-i)][x-(1+i)](x-2)(x-3)$$

Example 6 FINDING ALL ZEROS GIVEN ONE ZERO

Find all zeros of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, given that $1 - i$ is a zero.

Solution

$$f(x) = [x - (1 - i)][x - (1 + i)](x - 2)(x - 3)$$

The remaining zeros are 2 and 3. The four zeros are $1 - i$, $1 + i$, 2, and 3.

Descartes' Rule of Signs

Let $f(x)$ define a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of x.

(a) The number of positive real zeros of f either equals the number of variations in sign occurring in the coefficients of $f(x)$, or is less than the number of variations by a positive even integer.

(b) The number of negative real zeros of f either equals the number of variations in sign occurring in the coefficients of $f(-x)$, or is less than the number of variations by a positive even integer.

Example 7 **APPLYING DESCARTES' RULE OF SIGNS**

Determine the different possibilities for the number of positive, negative, and nonreal complex zeros of

$$f(x) = x^4 - 6x^3 + 8x^2 + 2x - 1.$$

Solution We first consider the possible number of positive zeros by observing that $f(x)$ has three variations in signs:

$$f(x) = +x^4 - 6x^3 + 8x^2 + 2x - 1$$

Example 7 APPLYING DESCARTES' RULE OF SIGNS

Determine the different possibilities for the number of positive, negative, and nonreal complex zeros of

$$f(x) = x^4 - 6x^3 + 8x^2 + 2x - 1.$$

Solution Thus, by Descartes' rule of signs, f has either 3 or 3 – 2 = 1 positive real zeros.

For negative zeros, consider the variations in signs for $f(-x)$:

$$f(-x) = (-x)^4 - 6(-x)^3 + 8(-x)^2 + 2(-x) - 1$$

$$= x^4 + 6x^3 + 8x^2 - 2x - 1.$$

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Example 7 **APPLYING DESCARTES' RULE OF SIGNS**

Determine the different possibilities for the number of positive, negative, and nonreal complex zeros of

$$f(x) = x^4 - 6x^3 + 8x^2 + 2x - 1.$$

Solution

$$= x^4 + 6x^3 + 8x^2 - 2x - 1.$$

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Since there is only one variation in sign, $f(x)$ has exactly one negative real zero.