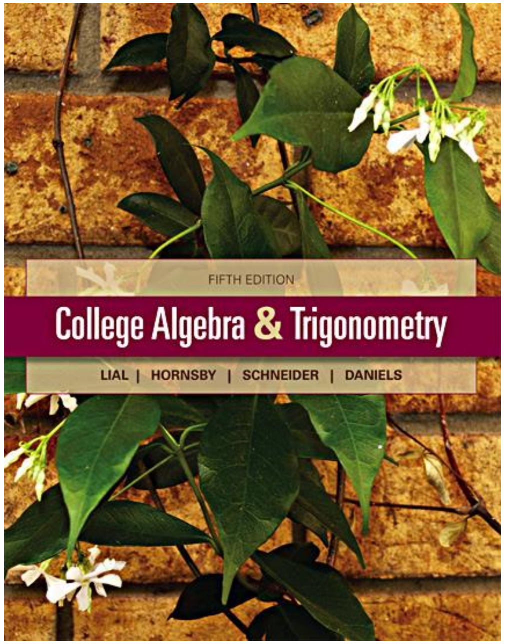
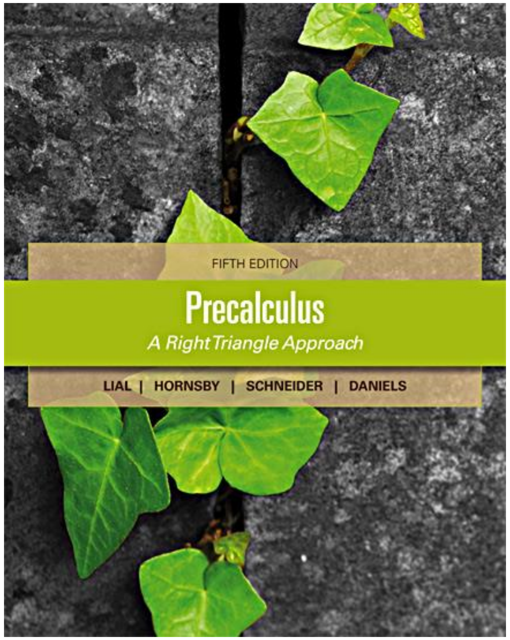
6

Trigonometric Functions





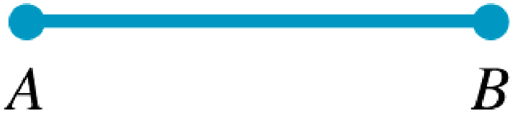
6.1 Angles

Basic Terminology ▪ Degree Measure ▪ Standard Position ▪ Coterminal Angles ملغي

Basic Terminology

Two distinct points determine **line *AB*.**

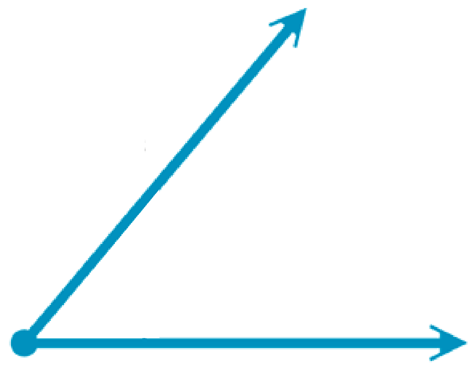
Line segment *AB* is a portion of the line between *A* and *B*, including points *A* and *B.*



Ray *AB* is a portion of line *AB* that starts at *A* and continues through *B*, and on past *B.*

Basic Terminology

An **angle** consists of two rays in a plane with a common endpoint.



The two rays are the **sides** of the angle.

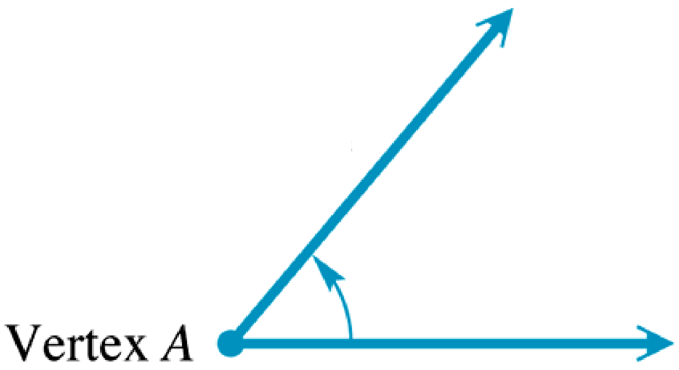
The common endpoint is called the **vertex** of the angle.

Basic Terminology

An angle's **measure** is generated by a rotation about the vertex.

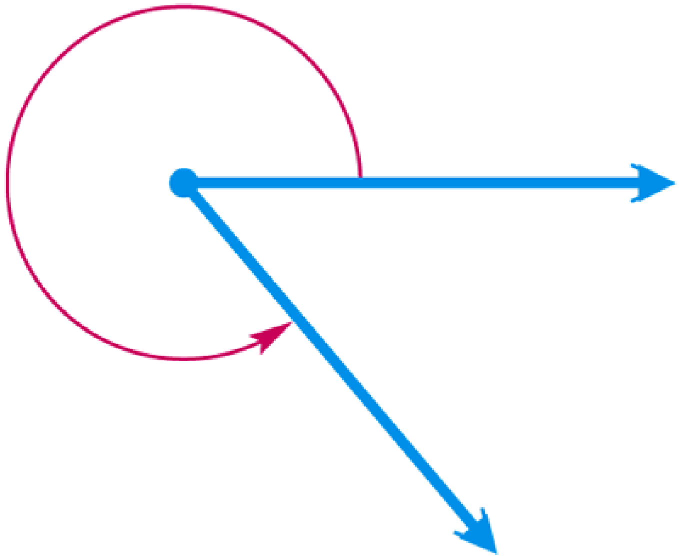
The ray in its initial position is called the **initial side** of the angle.

The ray in its location after the rotation is the **terminal side** of the angle.



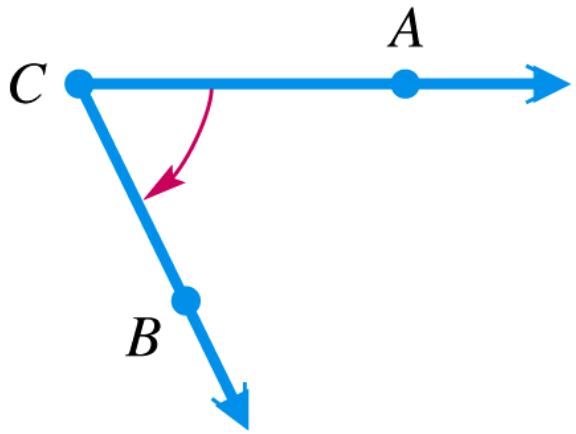
Basic Terminology

Positive angle: The rotation of the terminal side of an angle is counterclockwise.



Positive angle

Negative angle: The rotation of the terminal side is clockwise.

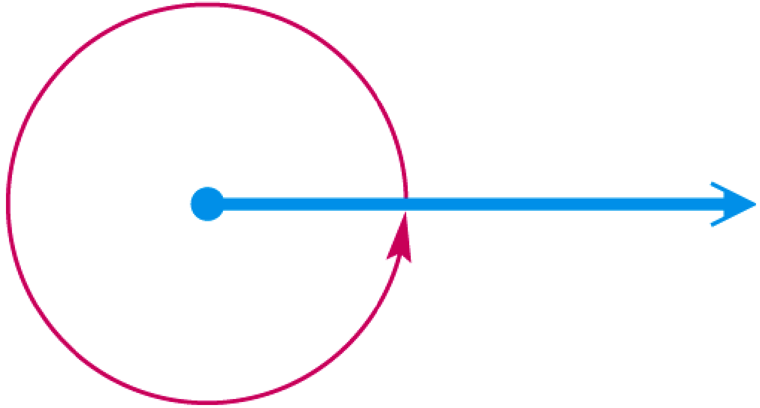


Negative angle

Degree Measure

The most common unit for measuring angles is the **degree**.

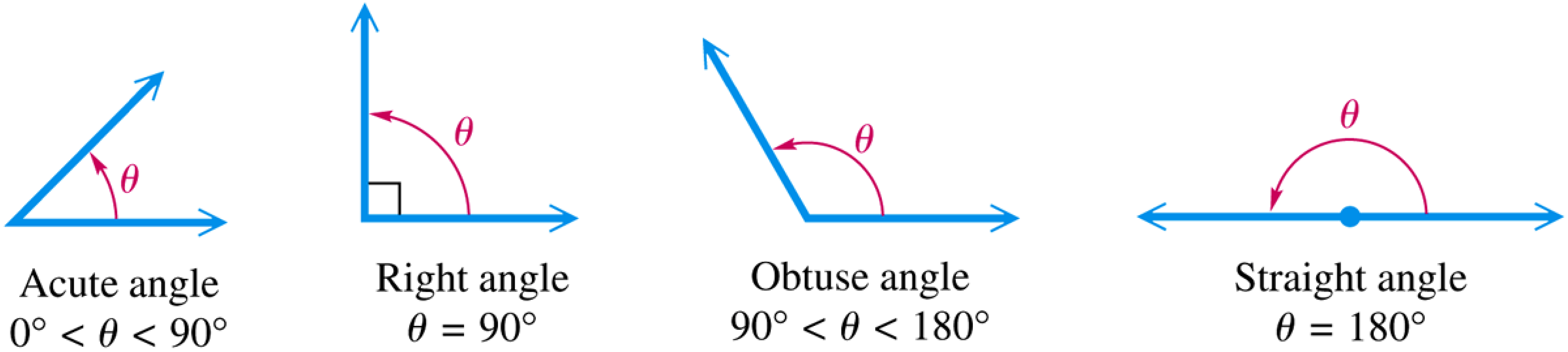
A complete rotation of a ray gives an angle whose measure is 360°.



$\frac{1}{360}$ of complete rotation gives an angle whose measure is 1°.

Degree Measure

Angles are classified by their measures.



FINDING THE COMPLEMENT AND THE SUPPLEMENT OF AN ANGLE

(a) If the sum of the measures of two positive angles is 90°, the angles are complementary and the angles are complements of each other.

$x+y=90$

(b)

If the sum of the measures of two positive angles is 180°, the angles are supplementary and the angles are supplements of each other.

$x+y=180°$

Example 1 FINDING THE COMPLEMENT AND THE SUPPLEMENT OF AN ANGLE

For an angle measuring 40°, find the measure of **(a)** its complement and **(b)** its supplement.

(a) To find the measure of its complement, subtract the measure of the angle from 90°.

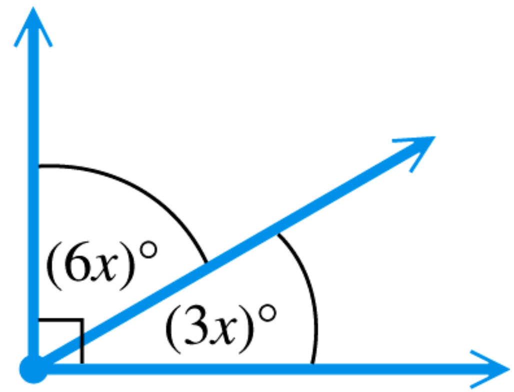
$$90^\circ - 40^\circ = 50^\circ$$ Complement of 40°

(b) To find the measure of its supplement, subtract the measure of the angle from 180°.

$$180^\circ - 40^\circ = 140^\circ$$ Supplement of 40°

FINDING MEASURES OF COMPLEMENTARY AND SUPPLEMENTARY ANGLES

Find the measure of each marked angle.



(a)

Since the two angles form a right angle, they are complementary.

$$6x + 3x = 90$$

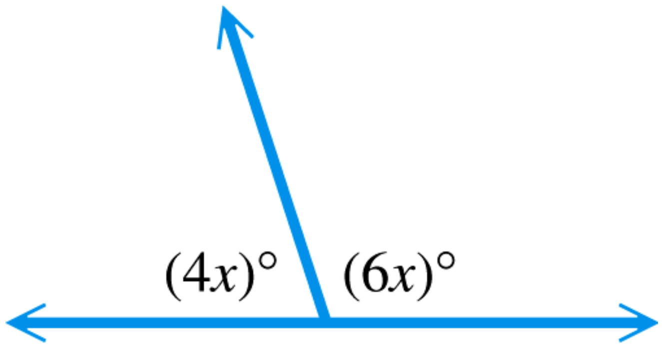
$$9x = 90 \quad \text{Combine like terms.}$$

$$x = 10 \quad \text{Divide by 9.}$$

Determine the measure of each angle by substituting 10 for x: $6(10) = 60°$ $3(10) = 30°$

Homework 1 FINDING MEASURES OF COMPLEMENTARY AND SUPPLEMENTARY ANGLES (continued)

Find the measure of each marked angle.



(b)

Since the two angles form a straight angle, they are supplementary.

$$4x + 6x = 180$$

$$10x = 180$$

$$x = 18$$

The angle measures are $4(18) = 72°$ and $6(18) = 108°$.

Degrees, Minutes, Seconds

One **minute** is 1/60 of a degree.

$$1' = \left(\frac{1}{60}\right)^\circ \quad \text{or} \quad 60' = 1^\circ$$

One **second** is 1/60 of a minute.

$$1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ \quad \text{or} \quad 60'' = 1'$$

Example 2 CALCULATING WITH DEGREES, MINUTES, AND SECONDS

Perform each calculation.

(a) $51°29' + 32°46'$

$$\begin{array}{r} 51°29' \\ +\ 32°46' \\ \hline 83°75' \end{array}$$

Add degrees and minutes separately.

$$\begin{aligned} 83°75' &= 83° + 1°15' \\ &= 84°15' \end{aligned}$$

(b) $90° - 73°12'$

$$\begin{array}{r} 89°60' \\ -\ 73°12' \\ \hline 16°48' \end{array}$$

Write 90° as 89°60′.

CONVERTING BETWEEN DECIMAL DEGREES AND DEGREES, MINUTES, AND SECONDS

(a) Convert 74°08′14″ to decimal degrees to the nearest thousandth.

$$74^\circ 08' 14'' = 74^\circ + \left(\frac{8}{60}\right)^\circ + \left(\frac{14}{3600}\right)^\circ$$

$$\approx 74^\circ + 0.1333^\circ + 0.0039^\circ$$

$$\approx 74.137^\circ$$

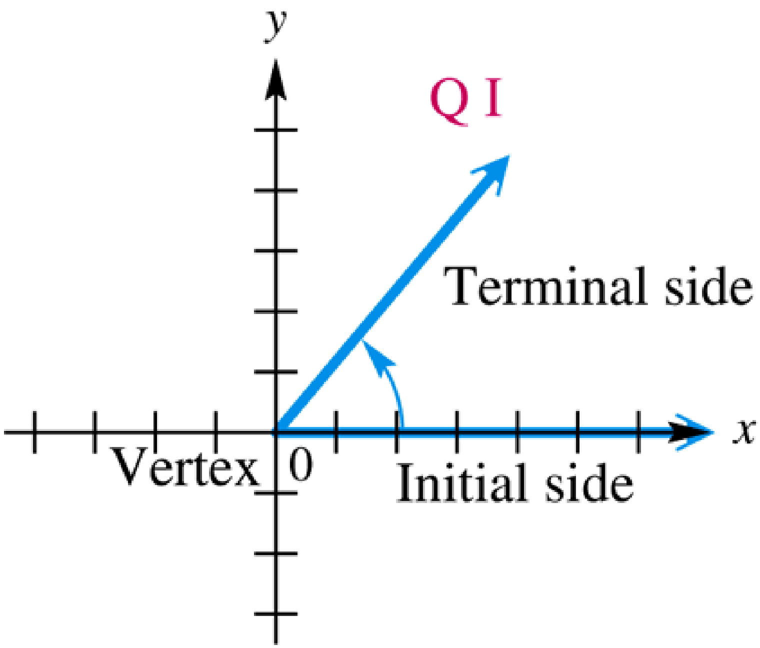
(b) Convert 34.817° to degrees, minutes, and seconds.

$$
\begin{aligned}
34.817^\circ &= 34^\circ + 0.817^\circ \\
&= 34^\circ + 0.817(60') \\
&= 34^\circ + 49.02' \\
&= 34^\circ + 49' + 0.02' \\
&= 34^\circ + 49' + 0.02(60'') \\
&= 34^\circ + 49' + 1.2'' \\
&\approx 34^\circ 49' 01''
\end{aligned}
$$

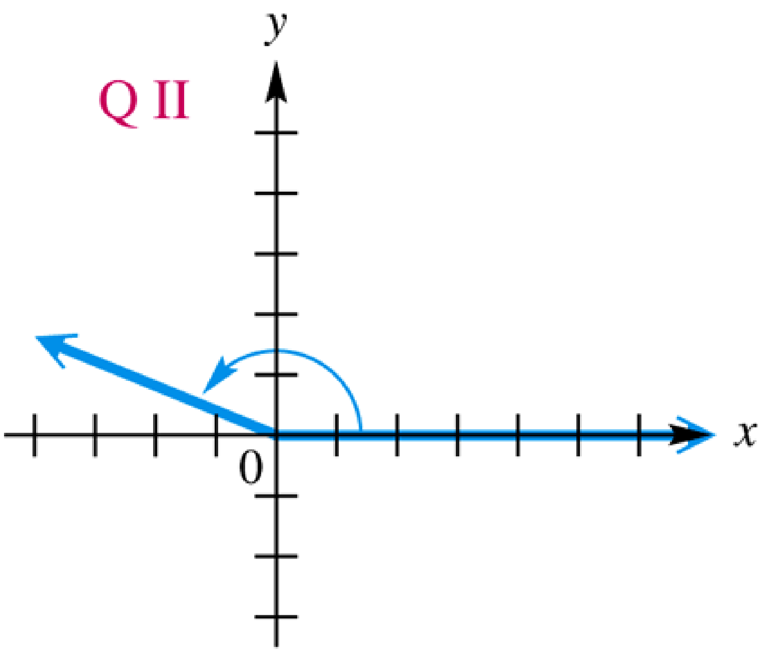
Approximate to the nearest second.

Standard Position

An angle is in **standard position** if its vertex is at the origin and its initial side lies along the positive *x*-axis.



(a)



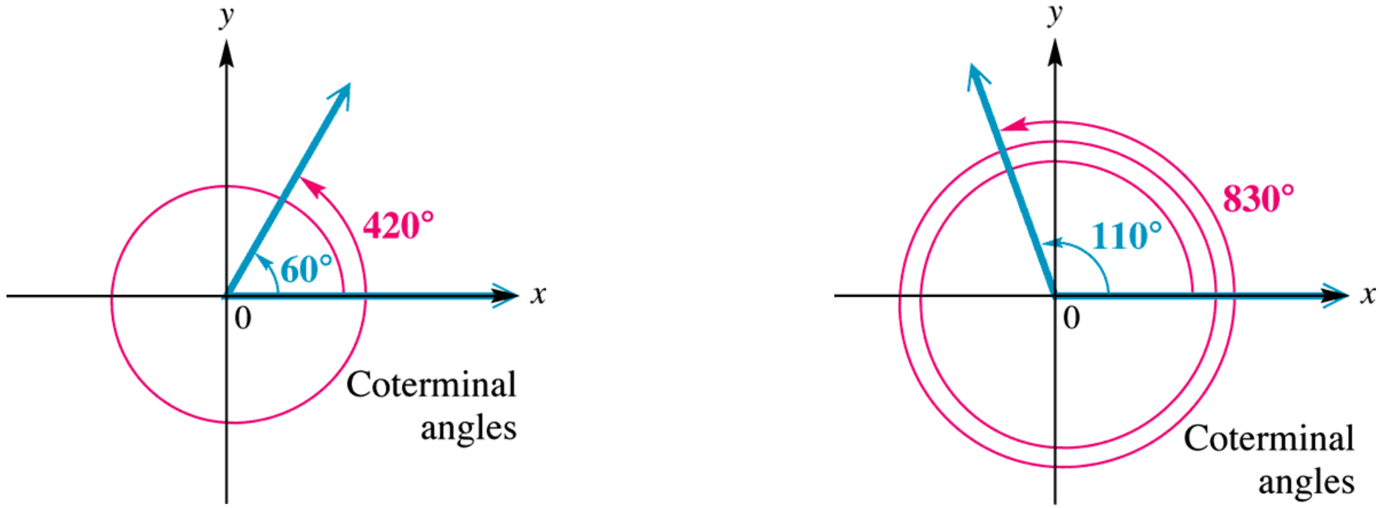
(b)

Quandrantal Angles

Angles in standard position whose terminal sides lie along the *x*-axis or *y*-axis, such as angles with measures 90°, 180°, 270°, and so on, are called **quadrantal angles.**

Coterminal Angles

A complete rotation of a ray results in an angle measuring 360°. By continuing the rotation, angles of measure larger than 360° can be produced. Such angles are called **coterminal angles.**

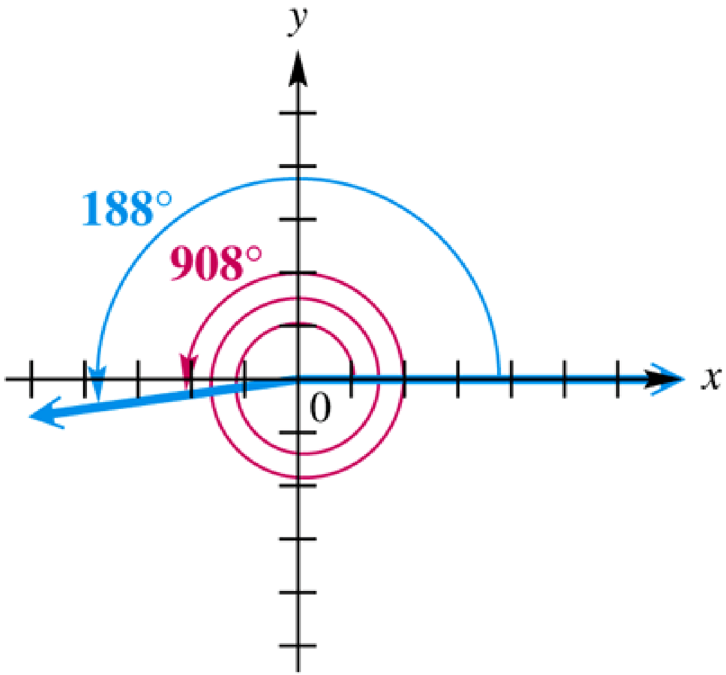


The measures of coterminal angles differ by a multiple of 360°.

Example 3 FINDING MEASURES OF COTERMINAL ANGLES

(a) Find the angle of least positive measure coterminal with an angle of 908°.

Subtract 360° as many times as needed to obtain an angle with measure greater than 0° but less than 360°.



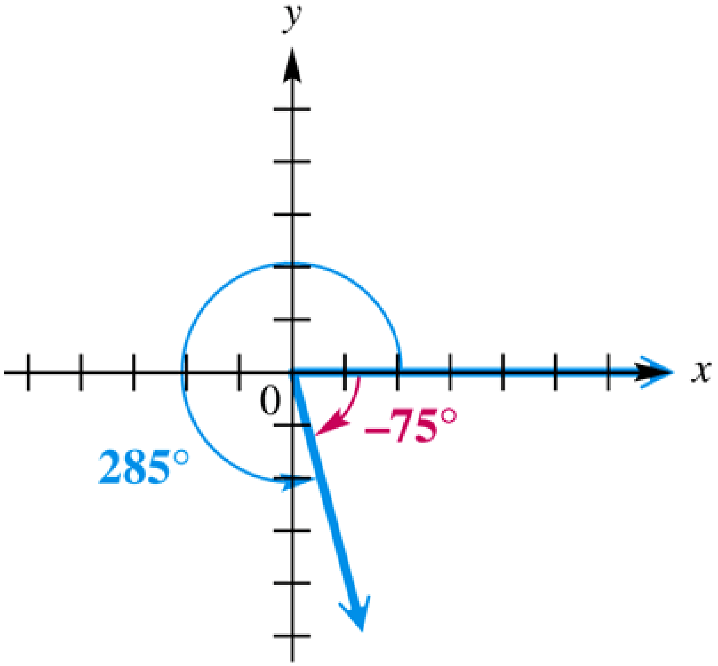
$$908° - 2 \cdot 360° = 188°$$

An angle of 188° is coterminal with an angle of 908°.

▶ Example 3 FINDING MEASURES OF COTERMINAL ANGLES (continued)

(b) Find the angle of least positive measure coterminal with an angle of –75°.

$$360° + (-75°) = 285°$$



An angle of –75 °is coterminal with an angle of 285°.

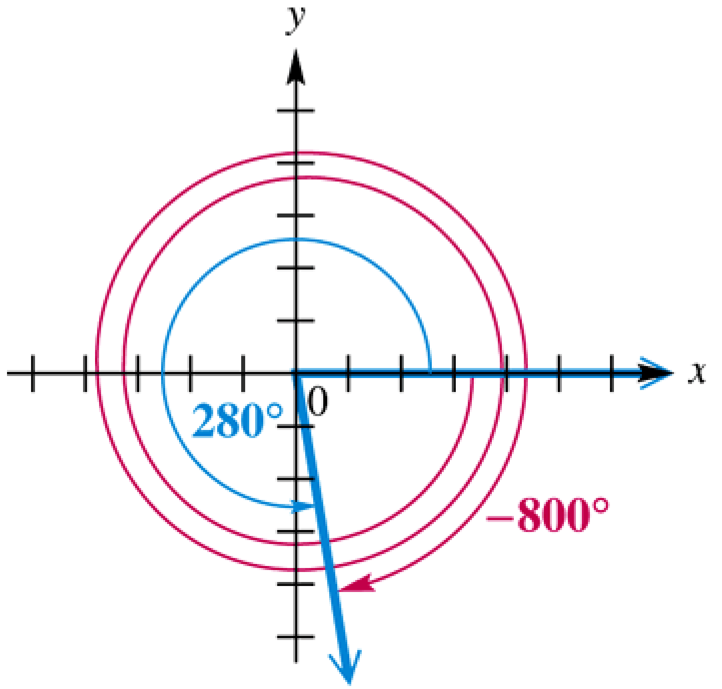
Example 3 FINDING MEASURES OF COTERMINAL ANGLES (continued)

(c) Find the angle of least positive measure coterminal with an angle of –800°.

The least integer multiple of 360° greater than 800° is

$$360^\circ \cdot 3 = 1080^\circ.$$

$$1080^\circ + (-800^\circ) = 280^\circ$$



An angle of –800° is coterminal with an angle of 280°.

Coterminal Angles

To find an expression that will generate all angles coterminal with a given angle, add integer multiples of 360° to the given angle.

For example, the expression for all angles coterminal with 60° is $60° + n \cdot 360°$.

Coterminal Angles

Value of *n*	Angle Coterminal with $60°$
2	$60° + 2 \cdot 360° = 780°$
1	$60° + 1 \cdot 360° = 420°$
0	$60° + 0 \cdot 360° = 60°$ (the angle itself)
-1	$60° + (-1) \cdot 360° = -300°$

Homework 3 ANALYZING THE REVOLUTIONS OF A CD PLAYER ملغي

CD players always spin at the same speed. Suppose a player makes 480 revolutions per min. Through how many degrees will a point on the edge of a CD move in 2 sec?

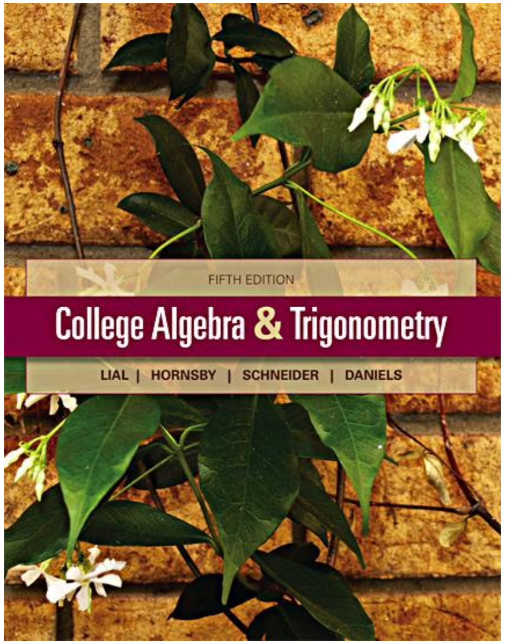
The player revolves 480 times in 1 min or $\frac{480}{60}$ times = 8 times per sec.

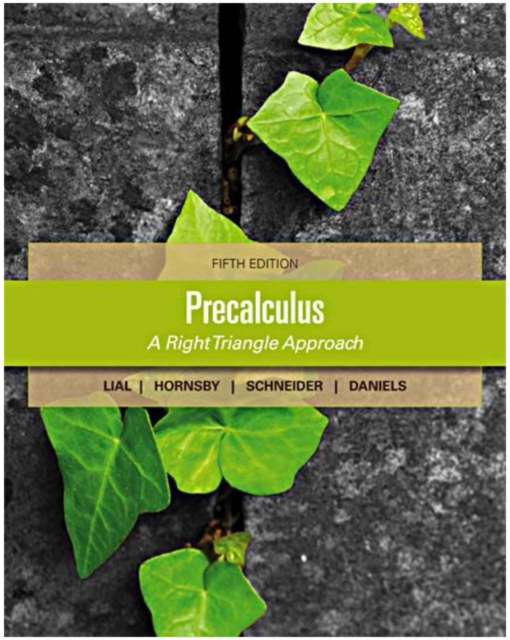
In 2 sec, the player will revolve $2 \cdot 8 = 16$ times.

Each revolution is 360°, so a point on the edge of the CD will revolve $16 \cdot 360° = 5760°$ in 2 sec.

6

Trigonometric Functions





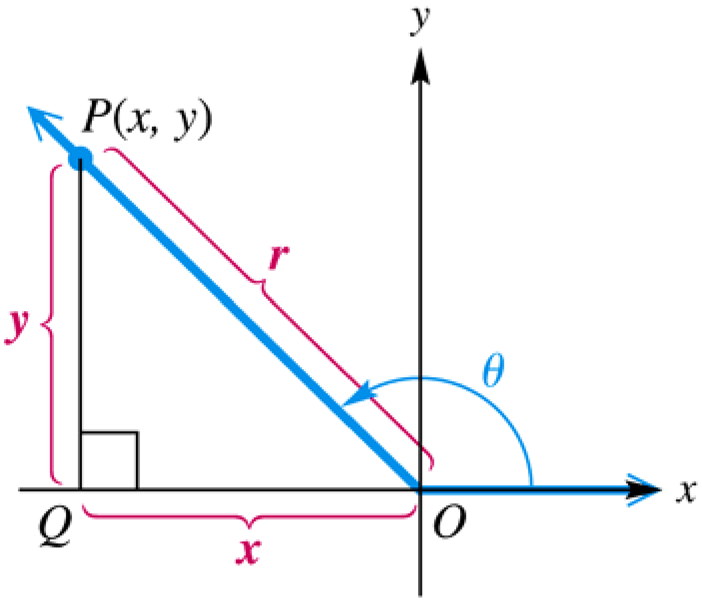
6.2 Trigonometric Functions

Trigonometric Functions ▪ Quadrantal Angles ▪ Reciprocal Identities ▪ Signs and Ranges of Function Values ▪ Pythagorean Identities ▪ Quotient Identities

Trigonometric Functions

Let (x, y) be a point other the origin on the terminal side of an angle θ in standard position. The distance from the point to the origin is

$$r = \sqrt{x^2 + y^2}.$$



Trigonometric Functions

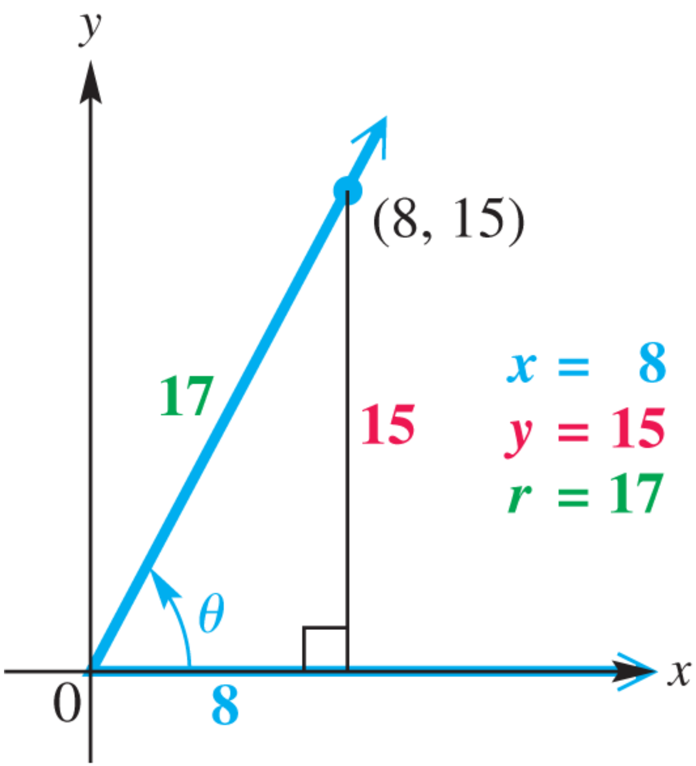
The six trigonometric functions of θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ \ x \neq 0$$

$$\csc\theta = \frac{r}{y},\ \ y \neq 0 \qquad \sec\theta = \frac{r}{x},\ \ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ \ y \neq 0$$

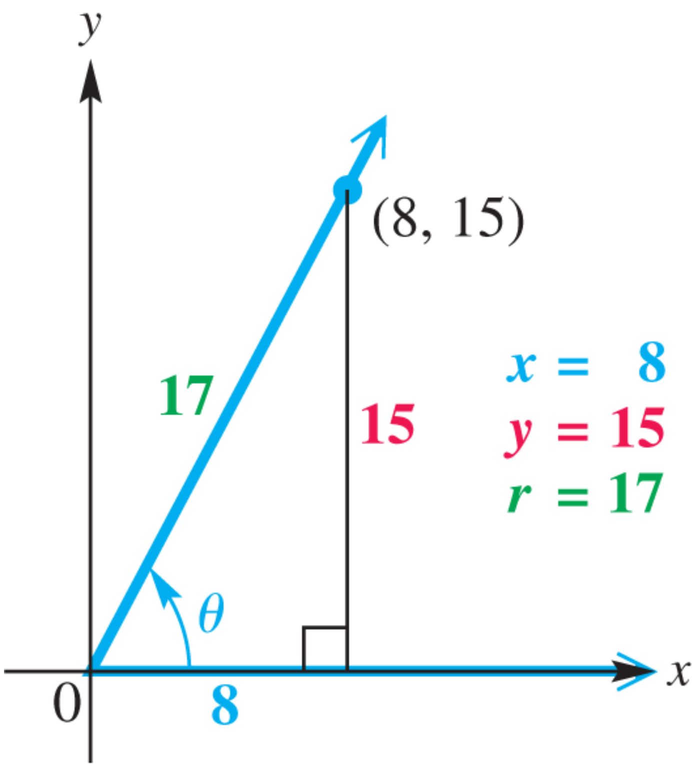
Example 1 FINDING FUNCTION VALUES OF AN ANGLE

The terminal side of angle θ in standard position passes through the point (8, 15). Find the values of the six trigonometric functions of angle θ.



The figure shows angle θ and the triangle formed by dropping a perpendicular from the point (8, 15) to the x-axis. The point (8, 15) is 8 units to the right of the y-axis and 15 units above the x-axis, so $x = 8$ and $y = 15$.

Example 1 FINDING FUNCTION VALUES OF AN ANGLE (continued)



$$r = \sqrt{x^2 + y^2} = \sqrt{8^2 + 15^2}$$

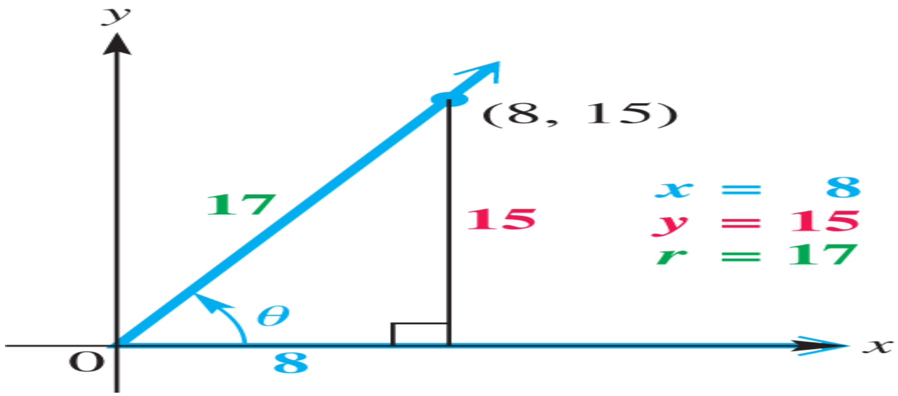
$$= \sqrt{64 + 225} = \sqrt{289} = 17$$

We can now find the values of the six trigonometric functions of angle θ.

Example 1 FINDING FUNCTION VALUES OF AN ANGLE (continued)

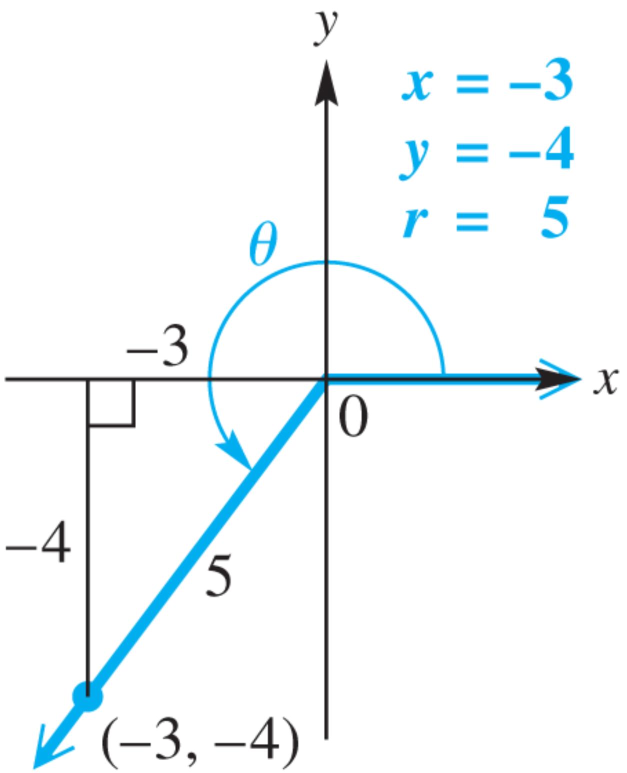
$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$$



HOMEWORK 1 FINDING FUNCTION VALUES OF AN ANGLE

The terminal side of angle θ in standard position passes through the point (–3, –4). Find the values of the six trigonometric functions of angle θ.



$$r = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

FINDING FUNCTION VALUES OF AN ANGLE (continued)

Use the definitions of the trigonometric functions.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

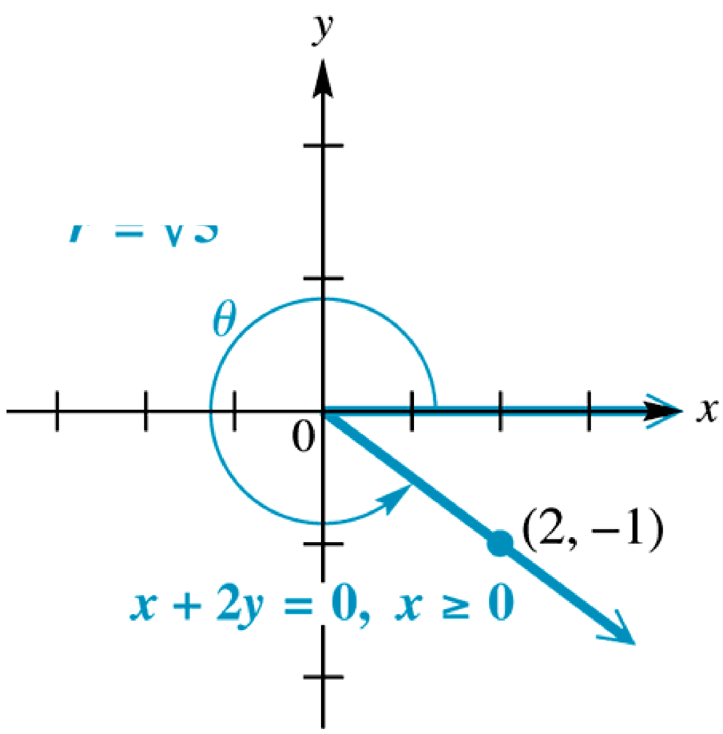
$$\csc\theta = \frac{r}{y},\ y \neq 0 \qquad \sec\theta = \frac{r}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

$$\sin\theta = \frac{-4}{5} = -\frac{4}{5} \qquad \cos\theta = \frac{-3}{5} = -\frac{3}{5} \qquad \tan\theta = \frac{-4}{-3} = \frac{4}{3}$$

$$\csc\theta = \frac{5}{-4} = -\frac{5}{4} \qquad \sec\theta = \frac{5}{-3} = -\frac{5}{3} \qquad \cot\theta = \frac{-3}{-4} = \frac{3}{4}$$

Example 2 FINDING FUNCTION VALUES OF AN ANGLE

Find the six trigonometric function values of the angle θ in standard position, if the terminal side of θ is defined by $x + 2y = 0$, $x \geq 0$.



We can use any point except (0, 0) on the terminal side of θ to find the trigonometric function values.

Choose $x = 2$.

$$x + 2y = 0,\ x \geq 0$$

$$2 + 2y = 0$$

$$2y = -2 \Rightarrow y = -1$$

Example 2 FINDING FUNCTION VALUES OF AN ANGLE (continued)

The point (2, –1) lies on the terminal side, and the corresponding value of *r* is $r = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$

$$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{5}} = \frac{-1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

Multiply by $\frac{\sqrt{5}}{\sqrt{5}}$ to rationalize the denominators.

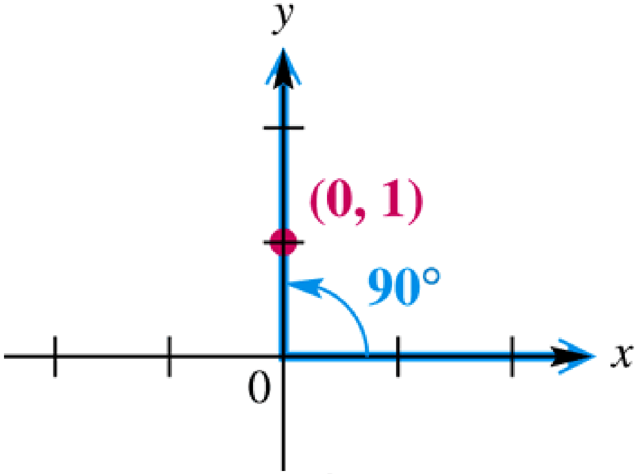
$$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = -\frac{1}{2} \qquad \cot\theta = \frac{x}{y} = -2$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{2} \qquad \csc\theta = \frac{r}{y} = -\sqrt{5}$$

HOMEWORK 2 FINDING FUNCTION VALUES OF QUADRANTAL ANGLES ملغي

(a) Find the values of the six trigonometric functions for an angle of 90°.



The terminal side passes through (0, 1). So $x = 0$, $y = 1$, and $r = 1$.

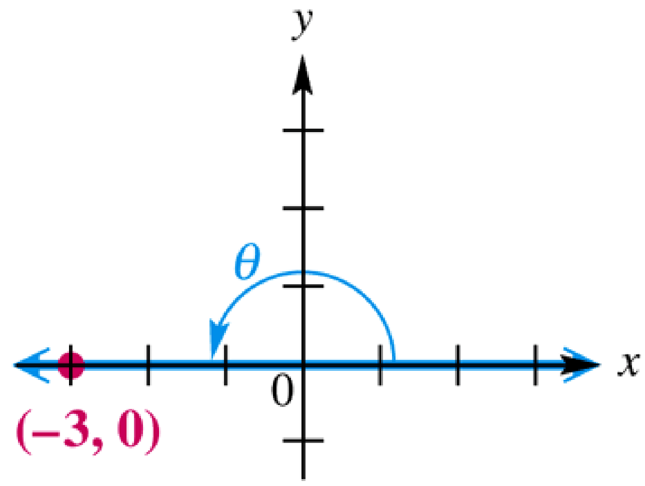
$$\sin 90° = \frac{1}{1} = 1 \qquad \cos 90° = \frac{0}{1} = 0 \qquad \tan 90° = \frac{1}{0}$$

undefined

$$\csc 90° = \frac{1}{1} = 1 \qquad \sec 90° = \frac{1}{0} \qquad \cot 90° = \frac{0}{1} = 0$$

undefined

(b) Find the values of the six trigonometric functions for an angle θ in standard position with terminal side through (–3, 0).



$x = –3$, $y = 0$, and $r = 3$.

$$\sin\theta = \frac{0}{3} = 0 \qquad \cos\theta = \frac{-3}{3} = -1 \qquad \tan\theta = \frac{0}{-3} = 0$$

$$\csc\theta = \frac{3}{0} \qquad \sec\theta = \frac{3}{-3} = -1 \qquad \cot\theta = \frac{-3}{0}$$

undefined   undefined

Function Values

Identify the terminal side of a quadrantal angle.

If the terminal side of the quadrantal angle lies along the y-axis, then the tangent and secant functions are undefined.

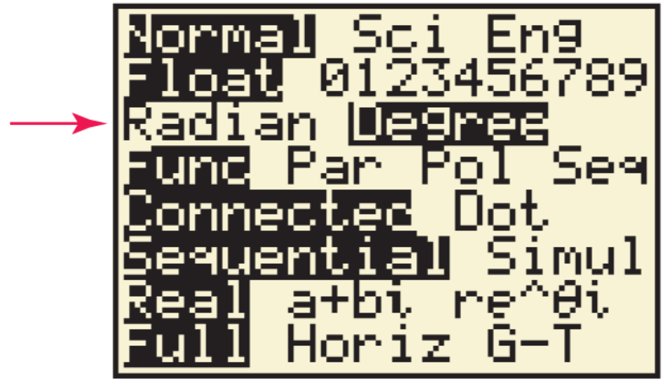
If the terminal side of a quadrantal angle lies along the x-axis, then the cotangent and cosecant functions are undefined.

Function Values of Quadrantal Angles

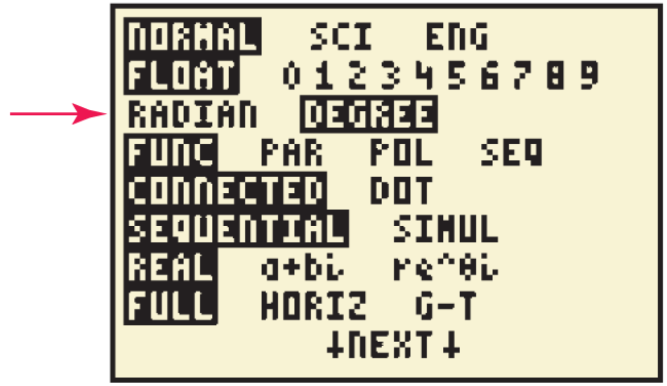
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	1	0	Undefined	1	Undefined
90°	1	0	Undefined	0	Undefined	1
180°	0	-1	0	Undefined	-1	Undefined
270°	-1	0	Undefined	0	Undefined	-1
360°	0	1	0	Undefined	1	Undefined

Using a Calculator

Function values of quadrantal angles can be found with a calculator that has trigonometric function keys. ***Make sure the calculator is set in degree mode.***



TI-83 Plus



TI-84 Plus

Caution

One of the most common errors involving calculators in trigonometry occurs when the calculator is set for radian measure, rather than degree measure. Be sure you know how to set your calculator in degree mode.

Reciprocal Identities

For all angles θ for which both functions are defined,

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Example 3(a) USING THE RECIPROCAL IDENTITIES

Find $\cos\theta$, given that $\sec\theta = \dfrac{5}{3}$.

Since cos *θ* is the reciprocal of sec *θ*,

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$$

Example 3(b) USING THE RECIPROCAL IDENTITIES

Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$.

Since sin *θ* is the reciprocal of csc *θ*,

$$\sin\theta = \frac{1}{\csc\theta} = \frac{1}{-\frac{\sqrt{12}}{2}} = -\frac{2}{\sqrt{12}}$$

$$= -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$\sqrt{12} = \sqrt{4\cdot 3} = 2\sqrt{3}$

$$= -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

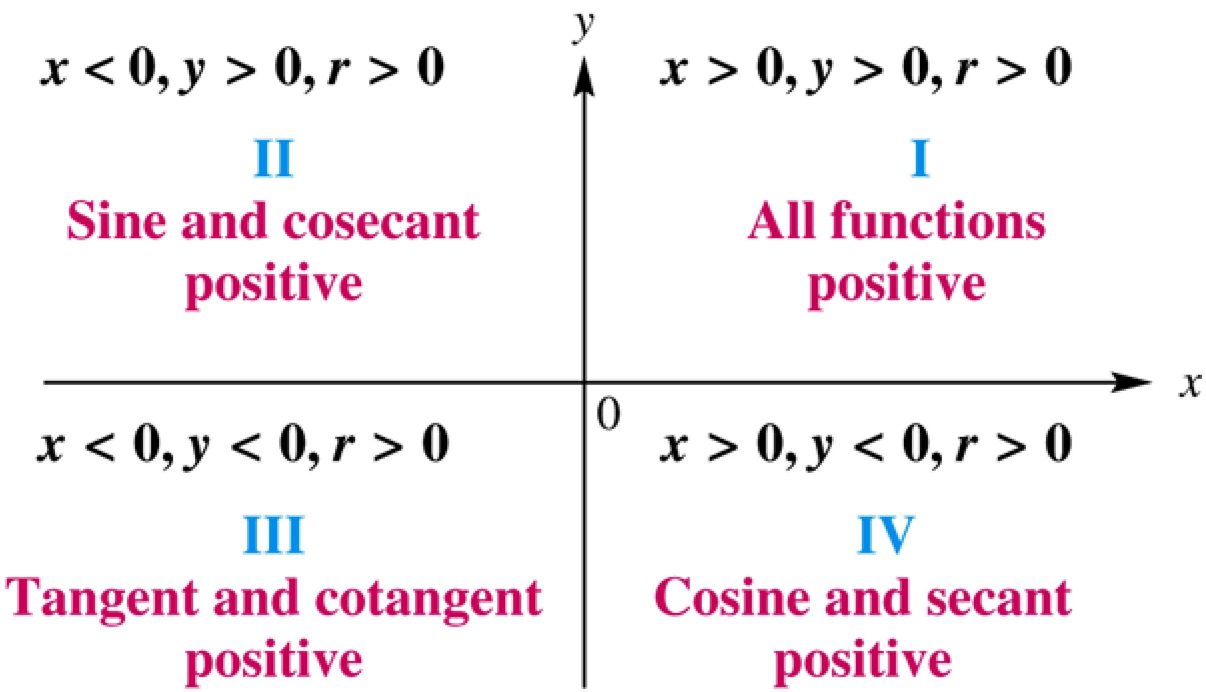
Rationalize the denominator.

$$= -\frac{\sqrt{3}}{3}$$

Signs of Function Values

θ in Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Signs of Function Values



DETERMINING SIGNS OF FUNCTIONS OF NONQUADRANTAL ANGLES

Determine the signs of the trigonometric functions of an angle in standard position with the given measure.

(a) 87°

The angle lies in the first quadrant, so all of its trigonometric function values are positive.

(b) 300°

The angle lies in quadrant IV, so the cosine and secant are positive, while the sine, cosecant, tangent, and cotangent are negative.

DETERMINING SIGNS OF FUNCTIONS OF NONQUADRANTAL ANGLES (cont.)

Determine the signs of the trigonometric functions of an angle in standard position with the given measure.

(c) –200°

The angle lies in quadrant II, so the sine and cosecant are positive, and all other function values are negative.

Example 4 IDENTIFYING THE QUADRANT OF AN ANGLE

Identify the quadrant (or possible quadrants) of any angle θ that satisfies the given conditions.

(a) $\sin \theta > 0$, $\tan \theta < 0$.

Since $\sin \theta > 0$ in quadrants I and II, and $\tan \theta < 0$ in quadrants II and IV, both conditions are met only in quadrant II.

(b) $\cos \theta < 0$, $\sec \theta < 0$

The cosine and secant functions are both negative in quadrants II and III, so θ could be in either of these two quadrants.

Ranges of Trigonometric Functions

Trigonometric Function of θ	Range (Set-Builder Notation)	Range (Interval Notation)
$\sin\theta$, $\cos\theta$	$\{y \mid \lvert y\rvert \le 1\}$	$[-1, 1]$
$\tan\theta$, $\cot\theta$	$\{y \mid y \text{ is a real number}\}$	$(-\infty, \infty)$
$\sec\theta$, $\csc\theta$	$\{y \mid \lvert y\rvert \ge 1\}$	$(-\infty, -1] \cup [1, \infty)$

DECIDING WHETHER A VALUE IS IN THE RANGE OF A TRIGONOMETRIC FUNCTION

Decide whether each statement is *possible* or *impossible*.

(a) $\sin\theta = 2.5$

For any value of θ, we know that $-1 \leq \sin\theta \leq 1$. Since $2.5 > 1$, it is impossible to find a value of θ that satisfies $\sin\theta = 2.5$.

(b) $\tan\theta = 110.47$

The tangent function can take on any real number value. Thus, $\tan\theta = 110.47$ is possible.

(c) $\sec\theta = 0.6$

Since $|\sec\theta| \geq 1$ for all θ for which the secant is defined, the statement $\sec\theta = 0.6$ is impossible.

Example 5 FINDING ALL FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT

Suppose that angle θ is in quadrant II and $\sin\theta = \frac{2}{3}$. Find the values of the other five trigonometric functions.

Choose any point on the terminal side of angle θ.

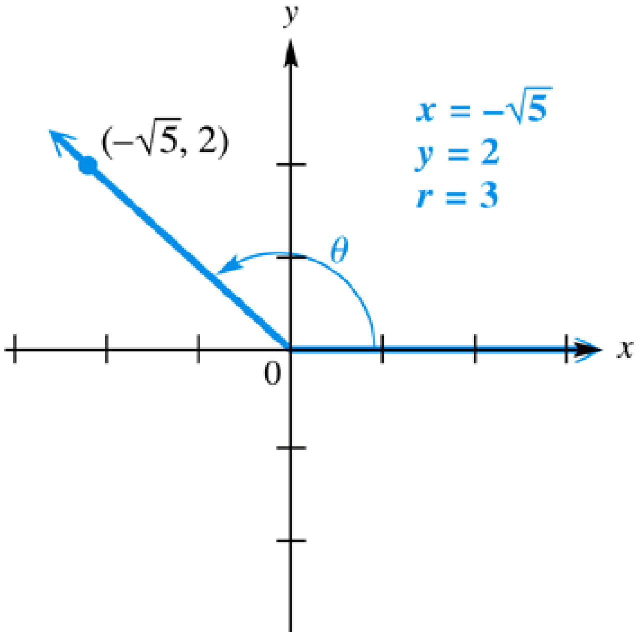
$$\sin\theta = \frac{2}{3} = \frac{y}{r}$$

Let r = 3. Then y = 2.

$$x^2 + y^2 = r^2 \Rightarrow x^2 + 2^2 = 3^2 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$$

Since θ is in quadrant II, $x = -\sqrt{5}$.

Example 5 FINDING ALL FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT (continued)



$$\cos\theta = \frac{x}{r} = -\frac{\sqrt{5}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{3}{-\sqrt{5}}$$

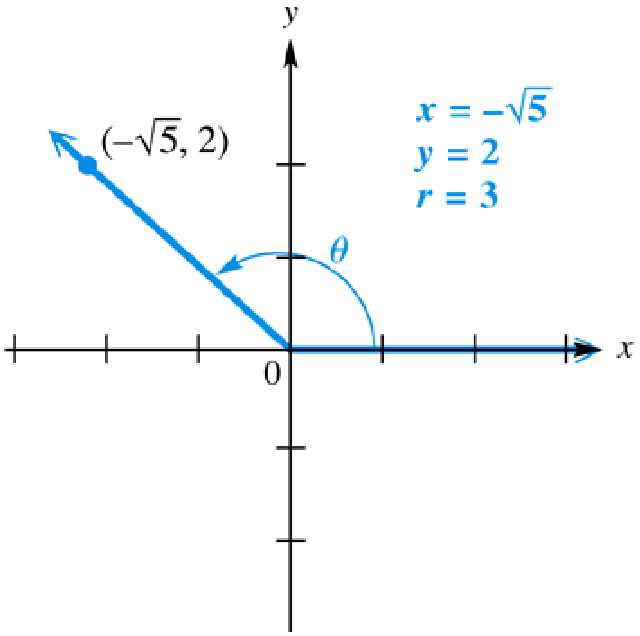
$$= -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

Remember to rationalize the denominator.

$$\tan\theta = \frac{y}{x} = \frac{2}{-\sqrt{5}}$$

$$= -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

Example 5 FINDING ALL FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT (continued)



$$\cot\theta = \frac{x}{y} = -\frac{\sqrt{5}}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{3}{2}$$

Pythagorean Identities
متطابقات فيثاغورث

For all angles θ for which the function values are defined, the following identities hold.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Quotient Identities متطابقات القسمة

For all angles θ for which the denominators are not zero, the following identities hold.

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \qquad \frac{\cos\theta}{\sin\theta} = \cot\theta$$

HOMEWORK 5 USING IDENTITIES TO FIND FUNCTION VALUES

Find $\sin\theta$ and $\tan\theta$, given that $\cos\theta = -\frac{\sqrt{3}}{4}$ and $\sin\theta > 0$.

Start with $\sin^2\theta + \cos^2\theta = 1$.

$$\sin^2\theta + \left(-\frac{\sqrt{3}}{4}\right)^2 = 1$$

$$\sin^2\theta = \frac{13}{16}$$

$$\sin\theta = \pm\sqrt{\frac{13}{16}} = \pm\frac{\sqrt{13}}{4}$$

$$\sin\theta = \frac{\sqrt{13}}{4}$$

Choose the positive square root since sin θ >0.

USING IDENTITIES TO FIND FUNCTION VALUES (continued)

To find tan θ, use the quotient identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{13}}{4}}{-\frac{\sqrt{3}}{4}} = -\frac{\sqrt{13}}{\sqrt{3}}$$

$$= -\frac{\sqrt{13}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{39}}{3}$$

Caution

Be careful to choose the correct sign when square roots are taken.

Example 6 USING IDENTITIES TO FIND FUNCTION VALUES

Find sin θ and cos θ, given that $\tan\theta = \frac{4}{3}$ and θ is in quadrant III.

Solution : Since θ is in quadrant III, sin θ and cos θ will both be negative. It is tempting to say that since

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4}{3} \quad \text{and} \quad \tan\theta = \frac{4}{3}$$

then sin θ = –4 and cos θ = –3. This is *incorrect*, however, since both sin θ and cos θ must be in the interval [–1,1].

Example 6 USING IDENTITIES TO FIND FUNCTION VALUES (continued)

Use the identity $\tan^2\theta + 1 = \sec^2\theta$ to find sec θ. Then use the reciprocal identity to find cos θ.

$$\left(\frac{4}{3}\right)^2 + 1 = \sec^2\theta$$

$$\frac{25}{9} = \sec^2\theta$$

$$-\frac{5}{3} = \sec\theta$$

Choose the negative square root since sec θ <0 when θ is in quadrant III.

$$-\frac{3}{5} = \cos\theta$$

Secant and cosine are reciprocals.

Example 6 USING IDENTITIES TO FIND FUNCTION VALUES (continued)

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$

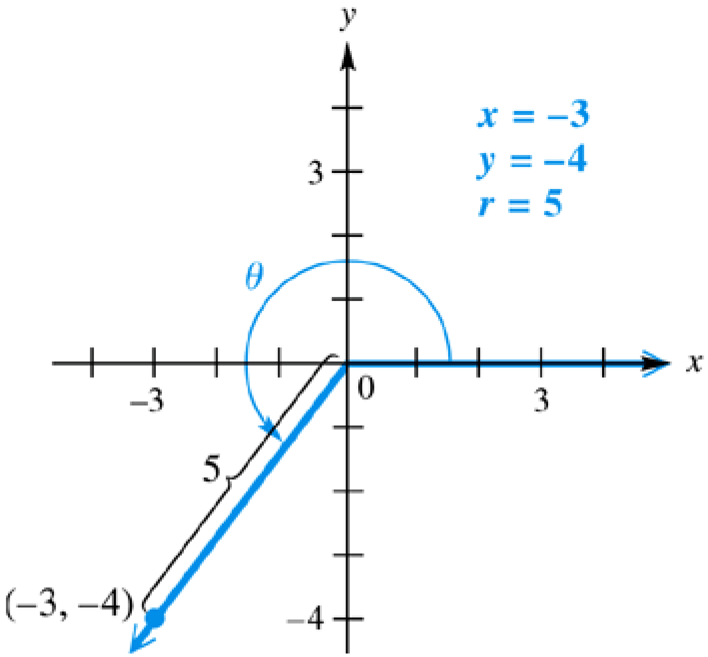
$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = -\frac{4}{5}$$

Choose the negative square root since sin θ <0 for θ in quadrant III.

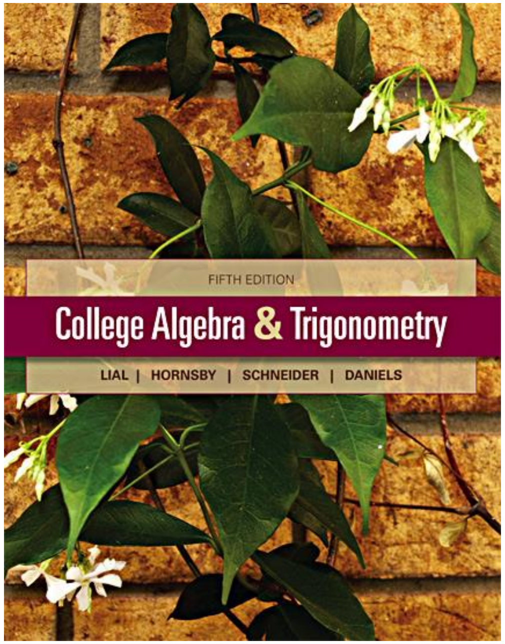
Example 6 USING IDENTITIES TO FIND FUNCTION VALUES (continued)

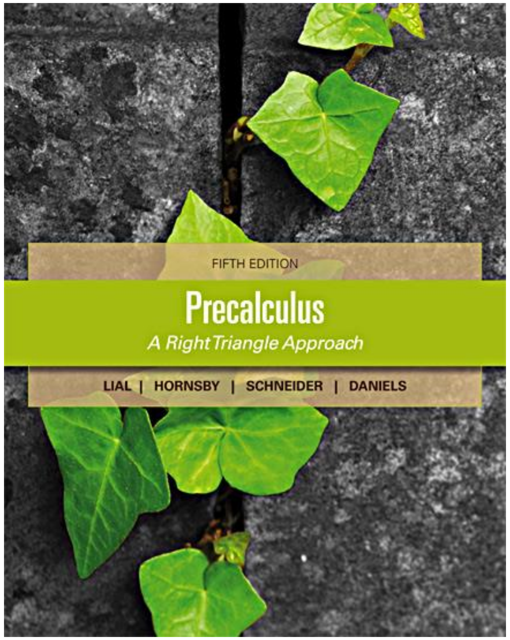
This example can also be worked by sketching θ in standard position in quadrant III, finding r to be 5, and then using the definitions of sin θ and cos θ in terms of x, y, and r.



6

Trigonometric Functions





6.3 Evaluating Trigonometric Functions

Right-Triangle-Based Definitions of the Trigonometric Functions ▪ Cofunctions ▪ Trigonometric Function Values of Special Angles ▪ Reference Angles ▪ Special Angles as Reference Angles ▪ Finding Function Values Using a Calculator ▪ Finding Angle Measures ملغي

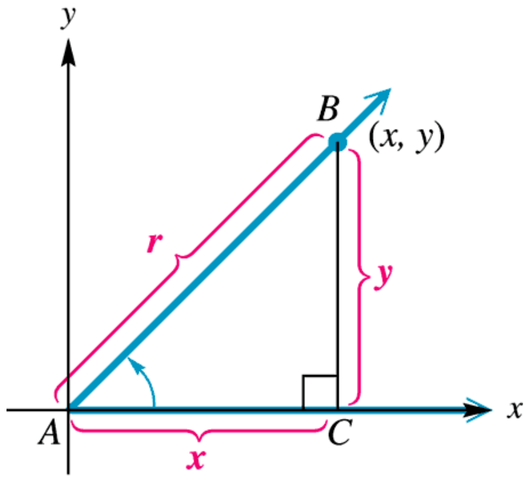
Right-Triangle-Based Definitions of Trigonometric Functions

Let A represent any acute angle in standard position.

$$\sin A = \frac{y}{r} = \frac{\text{side opposite } A}{\text{hypotenuse}} \qquad \csc A = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite } A}$$

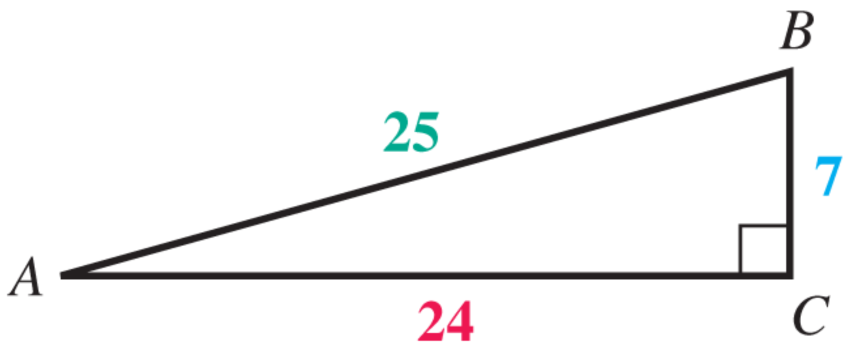
$$\cos A = \frac{x}{r} = \frac{\text{side adjacent to } A}{\text{hypotenuse}} \qquad \sec A = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent to } A}$$

$$\tan A = \frac{y}{x} = \frac{\text{side opposite } A}{\text{side adjacent to } A} \qquad \cot A = \frac{x}{y} = \frac{\text{side adjacent to } A}{\text{side opposite } A}$$



Example 1 **FINDING TRIGONOMETRIC FUNCTION VALUES OF AN ACUTE ANGLE**

Find the sine, cosine, and tangent values for angles *A* and *B.*



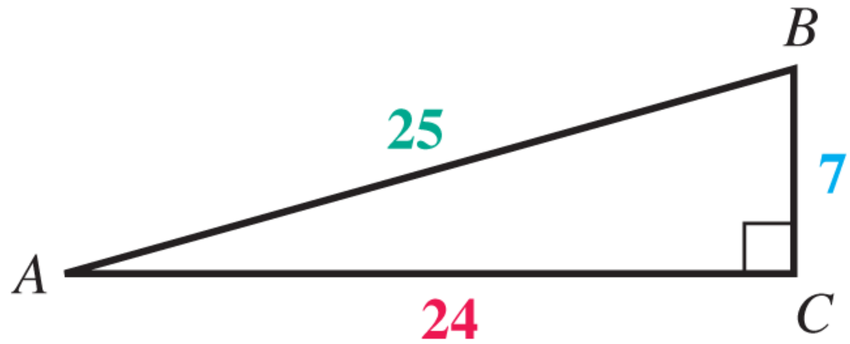
$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{7}{25}$$

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{24}{25}$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{7}{24}$$

Example 1 FINDING TRIGONOMETRIC FUNCTION VALUES OF AN ACUTE ANGLE (cont.)

Find the sine, cosine, and tangent values for angles *A* and *B.*



$$\sin B = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{24}{25}$$

$$\cos B = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{7}{25}$$

$$\tan B = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{24}{7}$$

Cofunction Identities ملغي

For any acute angle A, cofunction values of complementary angles are equal.

$\sin A = \cos(90° - A)$ $\quad$ $\cos A = \sin(90° - A)$

$\tan A = \cot(90° - A)$ $\quad$ $\cot A = \tan(90° - A)$

$\sec A = \csc(90° - A)$ $\quad$ $\csc A = \sec(90° - A)$

WRITING FUNCTIONS IN TERMS OF COFUNCTIONS

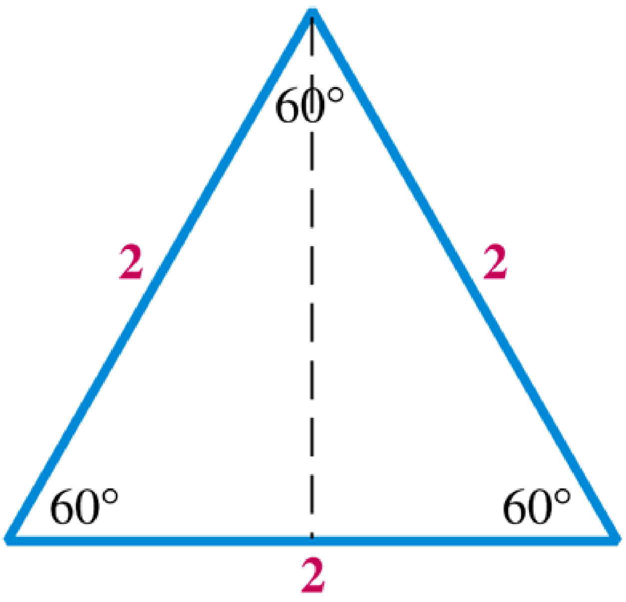
Write each function in terms of its cofunction.

(a) $\cos 52° = \sin (90° - 52°) = \sin 38°$

(b) $\tan 71° = \cot (90° - 71°) = \cot 19°$

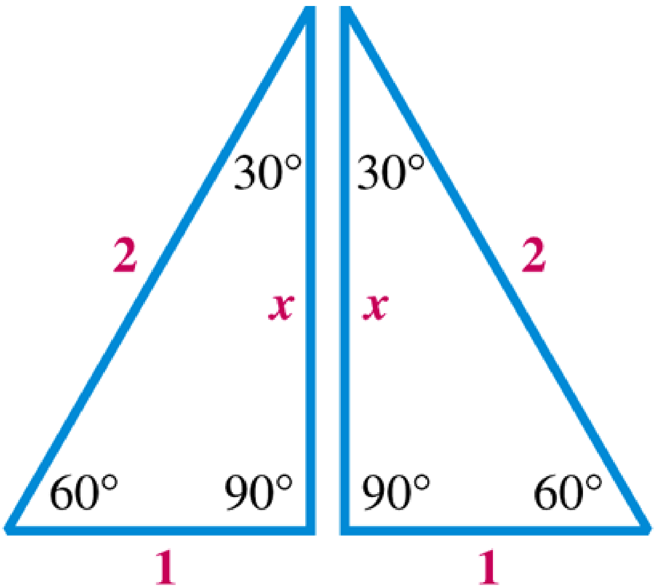
(c) $\sec 24° = \csc (90° - 24°) = \csc 66°$

30°- 60° Triangles



Equilateral triangle

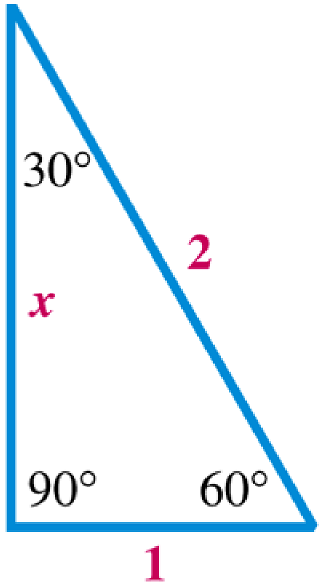
Bisect one angle of an equilateral triangle to create two 30°-60° triangles.



30°–60° right triangle

30°- 60° Triangles

Use the Pythagorean theorem to solve for *x*.



$$2^2 = 1^2 + x^2$$
$$4 = 1 + x^2$$
$$3 = x^2$$
$$\sqrt{3} = x$$

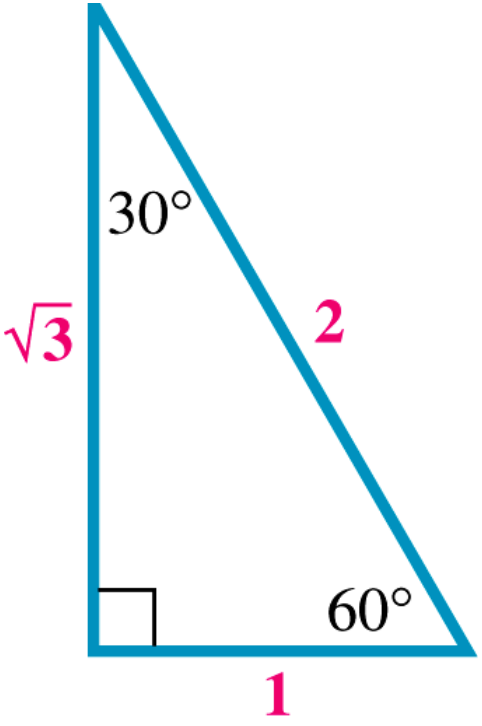
Example 2 FINDING TRIGONOMETRIC FUNCTION VALUES FOR 60°

Find the six trigonometric function values for a 60° angle.

$$\sin 60^\circ = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



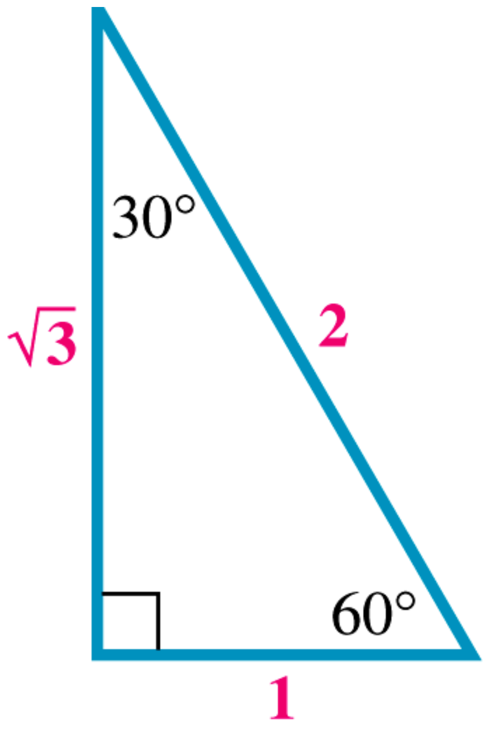
Example 2 FINDING TRIGONOMETRIC FUNCTION VALUES FOR 60° (continued)

Find the six trigonometric function values for a 60° angle.

$$\cot 60^\circ = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec 60^\circ = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{2}{1} = 2$$

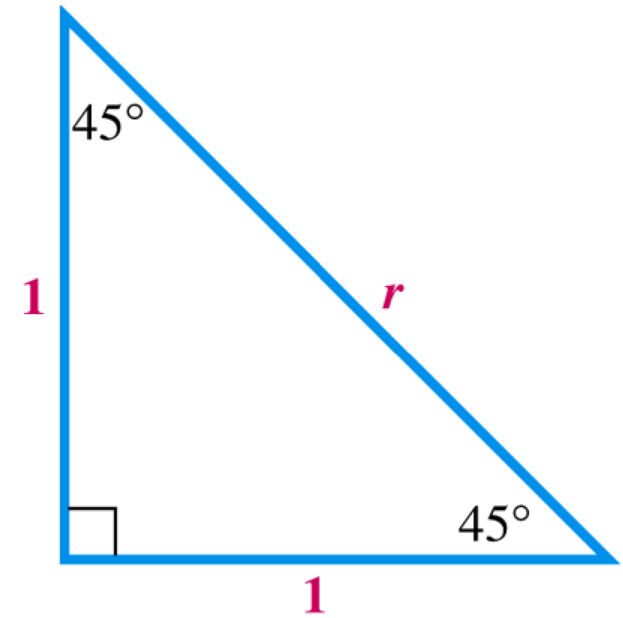
$$\csc 60^\circ = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$



45°- 45° Right Triangles

Use the Pythagorean theorem to solve for *r*.

$$1^2 + 1^2 = r^2$$
$$2 = r^2$$
$$\sqrt{2} = r$$



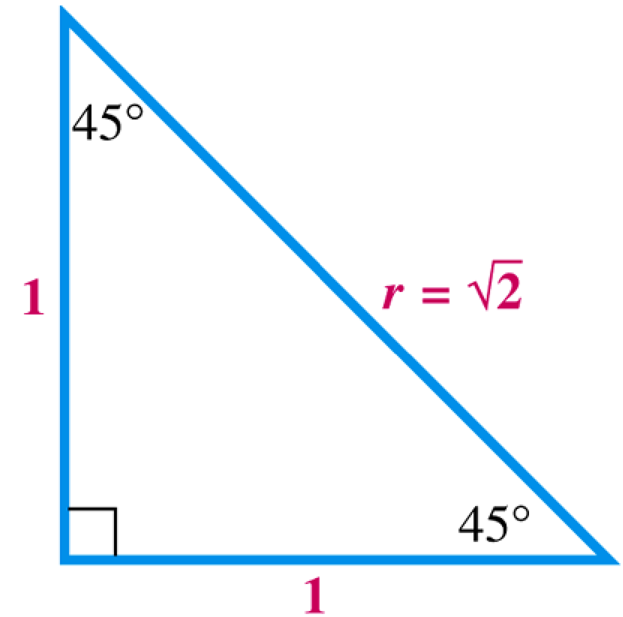
45°– 45° right triangle

45°- 45° Right Triangles

$$\sin 45^\circ = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{1}{1} = 1$$



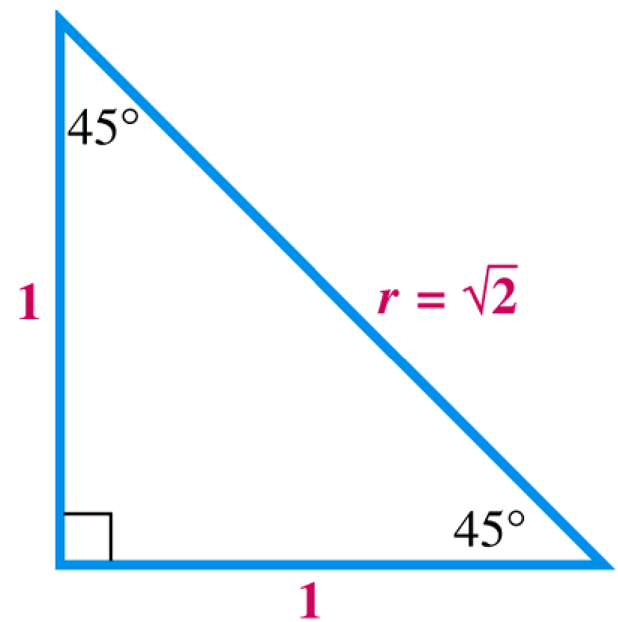
45°– 45° right triangle

45°- 45° Right Triangles

$$\cot 45° = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{1}{1} = 1$$

$$\sec 45° = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\csc 45° = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$



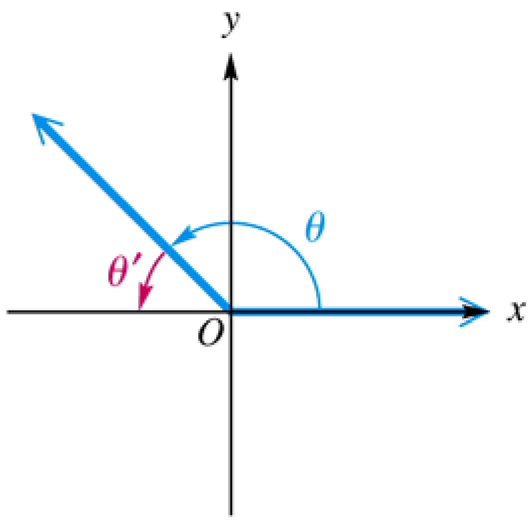
45°– 45° right triangle

Function Values of Special Angles

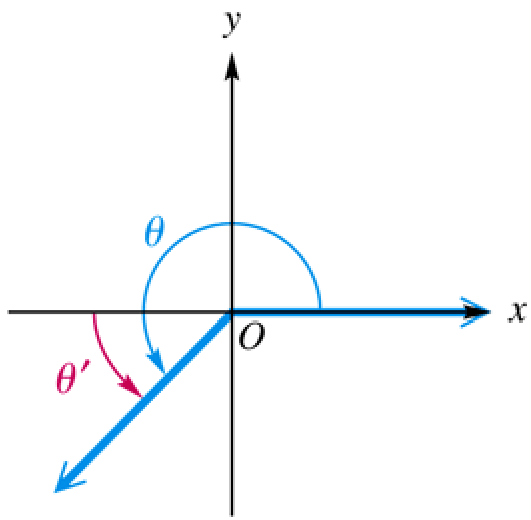
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Reference Angles

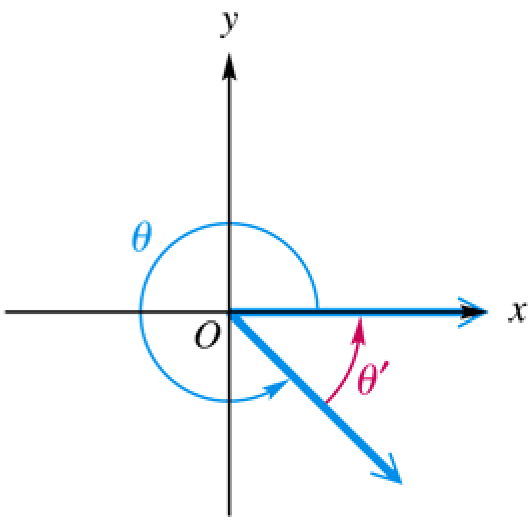
A **reference angle** for an angle θ is the positive acute angle made by the terminal side of angle θ and the *x*-axis.



θ in quadrant II



θ in quadrant III



θ in quadrant IV

Caution

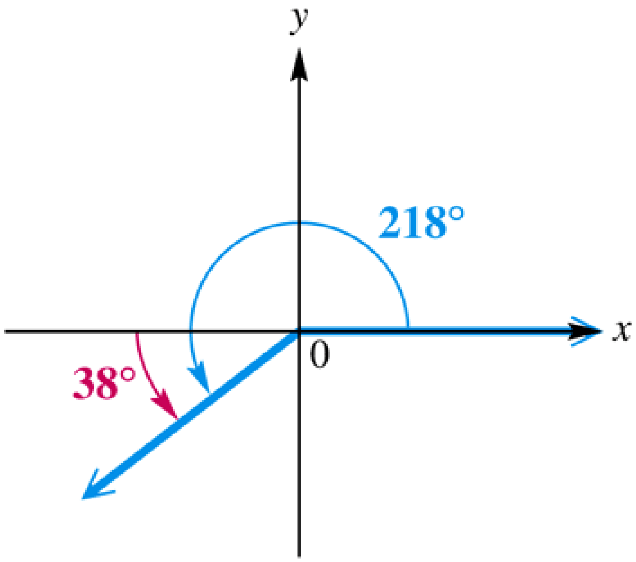
A common error is to find the reference angle by using the terminal side of θ and the *y*-axis.

The reference angle is always found with reference to the x-axis.

FINDING REFERENCE ANGLES

Find the reference angle for an angle of 218°.

The positive acute angle made by the terminal side of this angle and the *x*-axis is 218° – 180° = 38°.



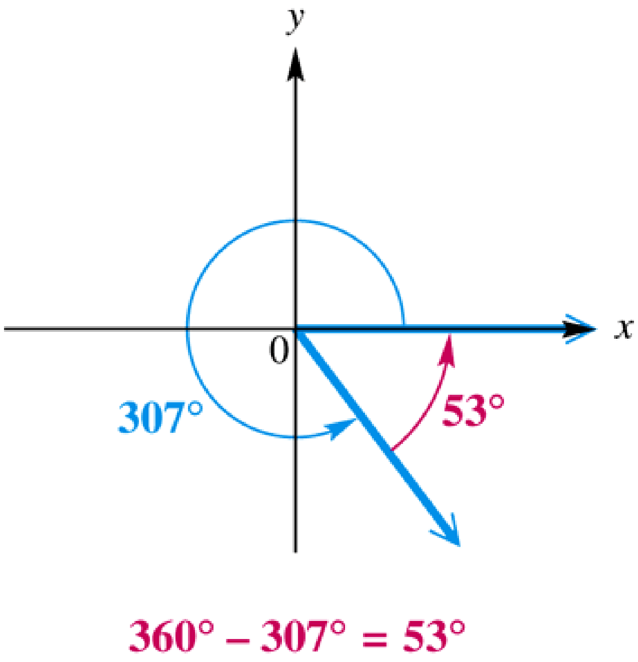
For θ = 218°, the reference angle θ' = 38°.

Example 4(b) FINDING REFERENCE ANGLES

Find the reference angle for an angle of 1387°.

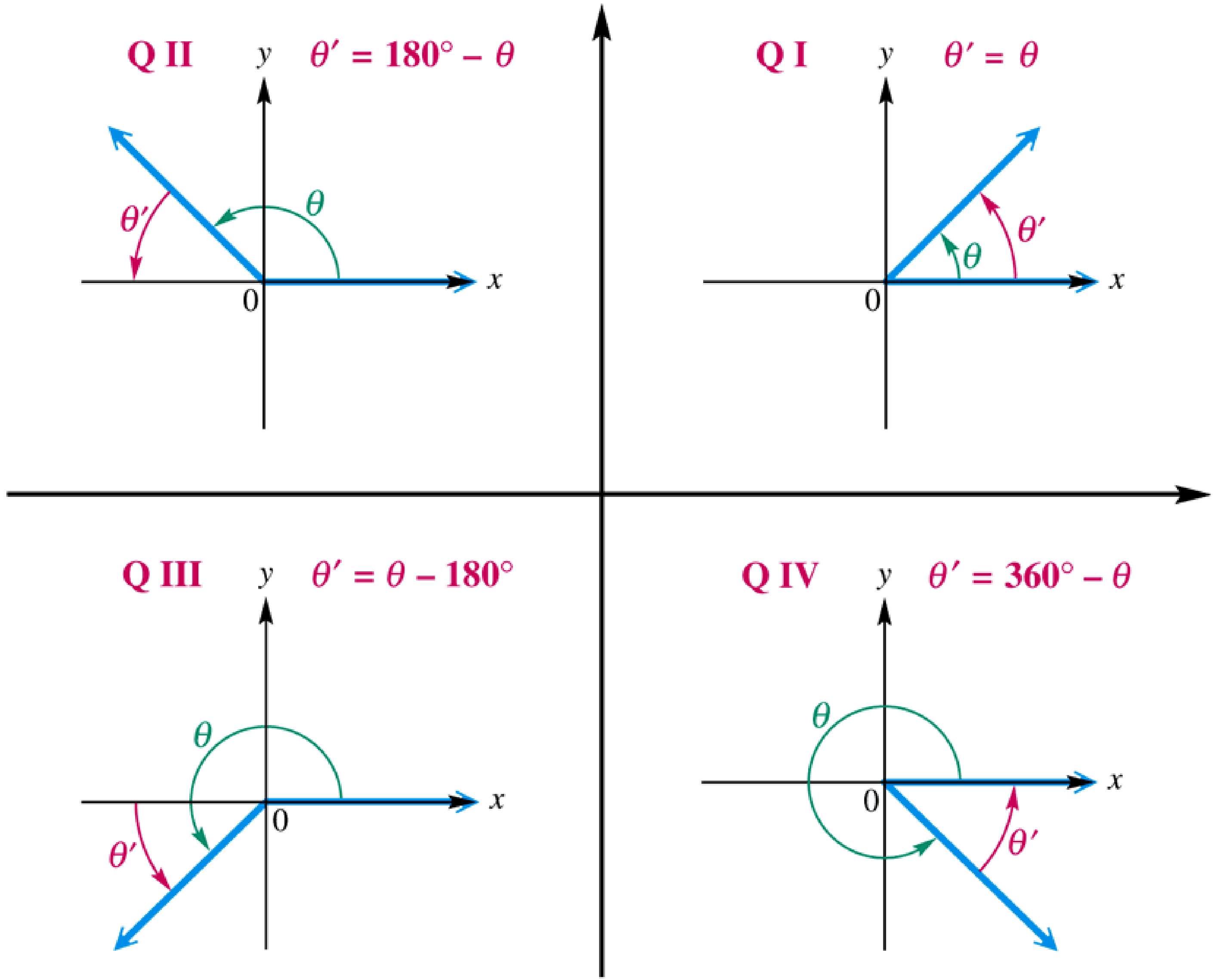
First find a coterminal angle between 0° and 360°.

Divide 1387º by 360 to get a quotient of about 3.9. Begin by subtracting 360° three times.
1387° – 3(360°) = 307°.



The reference angle for 307° (and thus for 1387°) is 360° – 307° = 53°.

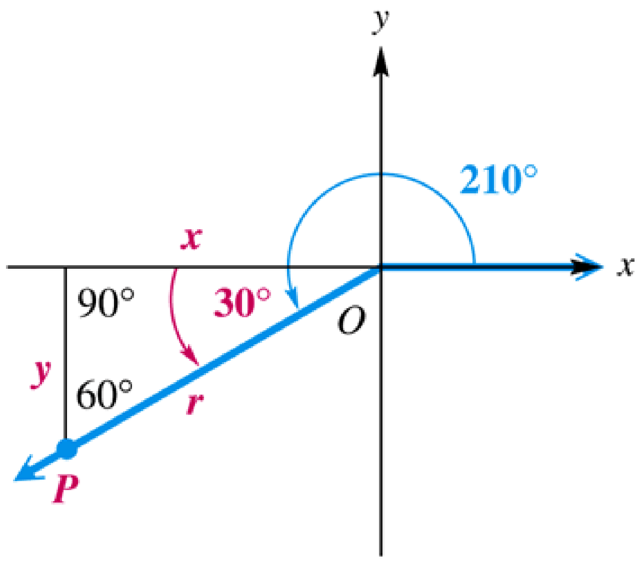
Reference Angle θ' for θ, where $0° < \theta < 360°$



Example 3 FINDING TRIGONOMETRIC FUNCTION VALUES OF A QUADRANT III ANGLE

Find the values of the six trigonometric functions for 210°.

The reference angle for a 210° angle is
210° – 180° = 30°.



Choose point *P* on the terminal side of the angle so the distance from the origin to *P* is 2.

$$r = 2,\ x = -\sqrt{3},\ y = -1$$

Example 3 FINDING TRIGONOMETRIC FUNCTION VALUES OF A QUADRANT III ANGLE (continued)

$$\sin 210° = \frac{-1}{2} = -\frac{1}{2}$$

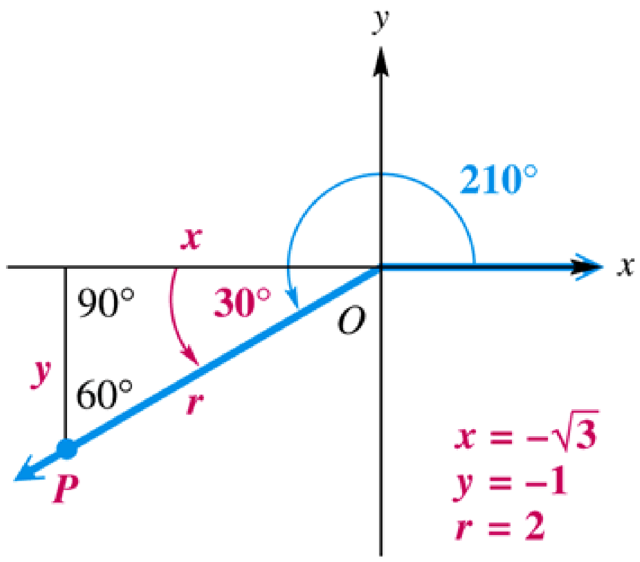
$$\cos 210° = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$\tan 210° = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot 210° = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\sec 210° = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc 210° = \frac{2}{-1} = -2$$



Finding Trigonometric Function Values For Any Nonquadrantal Angle θ

Step 1 If $\theta > 360°$, or if $\theta < 0°$, find a coterminal angle by adding or subtracting 360° as many times as needed to get an angle greater than 0° but less than 360°.

Step 2 Find the reference angle θ'.

Step 3 Find the trigonometric function values for reference angle θ'.

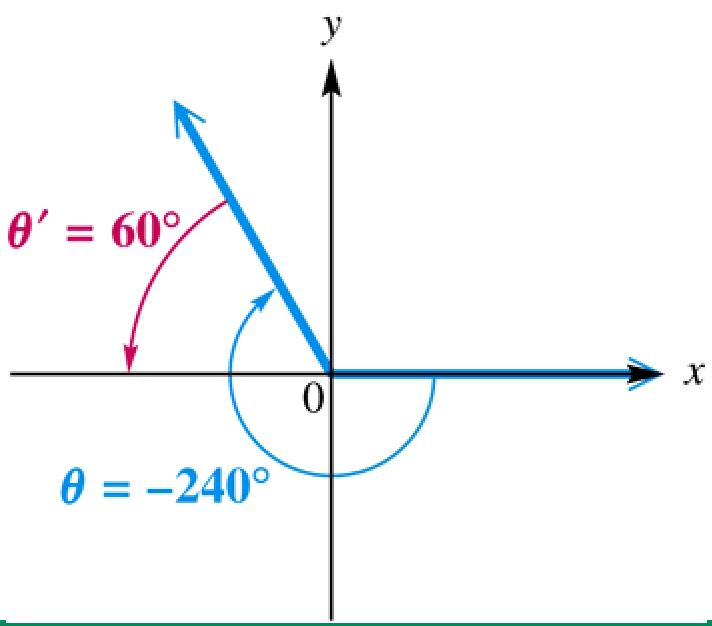
Finding Trigonometric Function Values For Any Nonquadrantal Angle θ (continued)

Step 4 Determine the correct signs for the values found in *Step 3*. This gives the values of the trigonometric functions for angle θ.

FINDING TRIGONOMETRIC FUNCTION VALUES USING REFERENCE ANGLES

Find the exact value of cos (–240°).

Since an angle of –240° is coterminal with an angle of –240° + 360° = 120°, the reference angle is 180° – 120° = 60°.



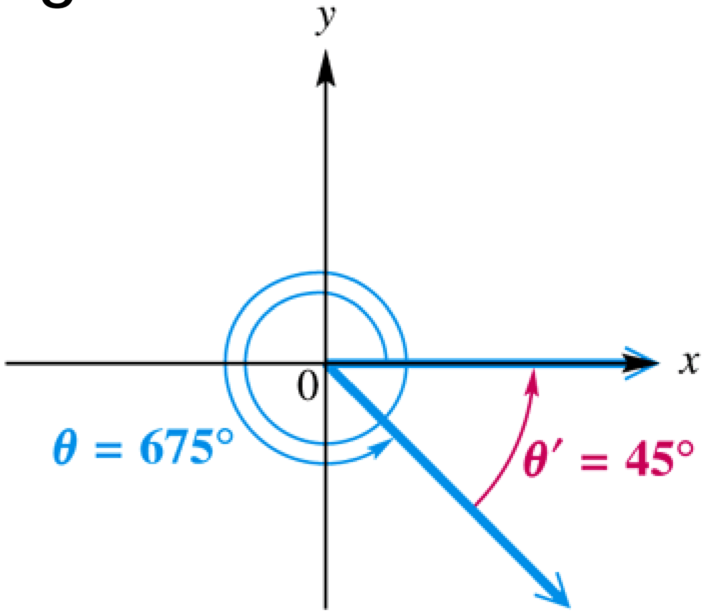
$$\begin{aligned}\cos(-240^\circ) &= \cos 120^\circ \\ &= -\cos 60^\circ \\ &= -\frac{1}{2}\end{aligned}$$

FINDING TRIGONOMETRIC FUNCTION VALUES USING REFERENCE ANGLES

Find the exact value of tan 675°.

Subtract 360° to find a coterminal angle between 0° and 360°: 675° – 360° = 315°.

The reference angle is 360° – 315° = 45º. An angle of 315º is in quadrant IV, so the tangent will be negative.



$$\tan 675^\circ = \tan 315^\circ$$
$$= -\tan 45^\circ$$
$$= -1$$

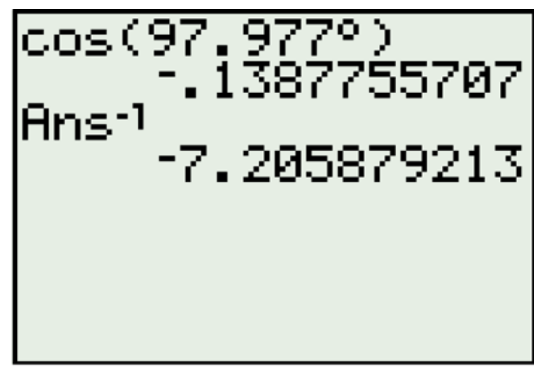
FINDING FUNCTION VALUES WITH A CALCULATOR

Approximate the value of each expression.

(a) sin 49°12′ ≈ 0.75699506

(b) sec 97.977°

Calculators do not have a secant key, so first find cos 97.977° and then take the reciprocal.

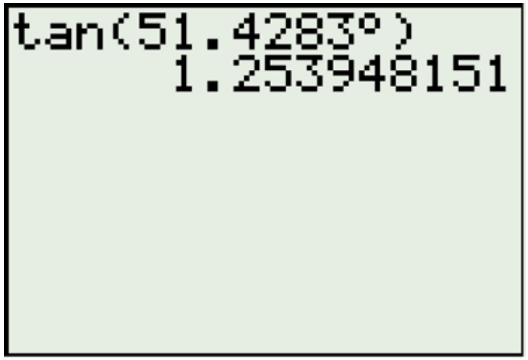


sec 97.977° ≈ –7.20587921

FINDING FUNCTION VALUES WITH A CALCULATOR (continued)

Approximate the value of each expression.

(c) $\dfrac{1}{\cot 51.4283^\circ}$

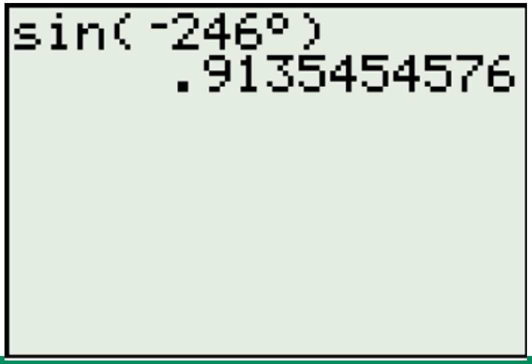


Use the reciprocal identity

$$\tan\theta = \frac{1}{\cot\theta}.$$

$$\frac{1}{\cot 51.4283^\circ} = \tan 51.4283^\circ \approx 1.25394815$$

(d) sin (–246°) ≈ –0.91354546



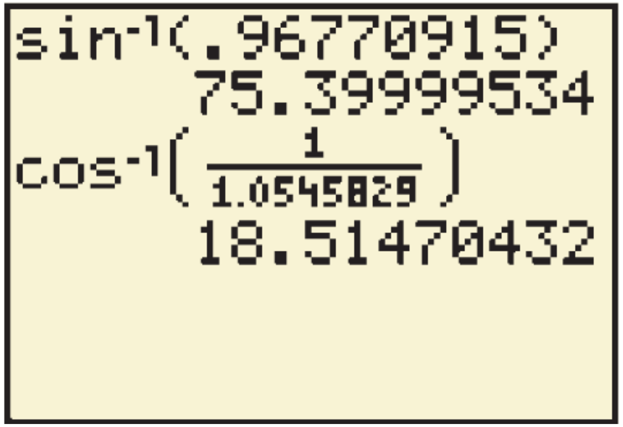
Example 8 USING INVERSE TRIGONOMETRIC FUNCTIONS TO FIND ANGLES

Use a calculator to find an angle θ in the interval [0°, 90°] that satisfies each condition.

(a) $\sin\theta \approx 0.96770915$

Use degree mode and the inverse sine function.

$$\theta = \sin^{-1} 0.96770915 \approx 75.399995°$$



(b) $\sec\theta \approx 1.0545829$

Use the identity $\cos\theta = \dfrac{1}{\sec\theta}$.

$$\theta = \cos^{-1}\left(\frac{1}{1.0545829}\right) \approx 18.514704°$$

Caution

To determine the secant of an angle, we find the *reciprocal of the cosine* of the angle. To determine an angle with a given secant value, we find the *inverse cosine of the reciprocal* of the value.

▶ Example 9 FINDING ANGLE MEASURES GIVEN AN INTERVAL AND A FUNCTION VALUE

Find all values of θ, if θ is in the interval [0°, 360°) and

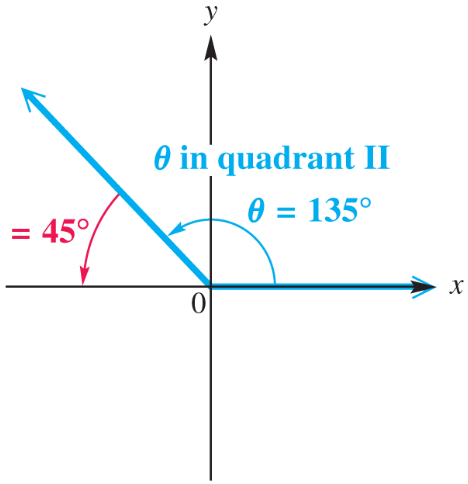
$$\cos\theta = -\frac{\sqrt{2}}{2}.$$

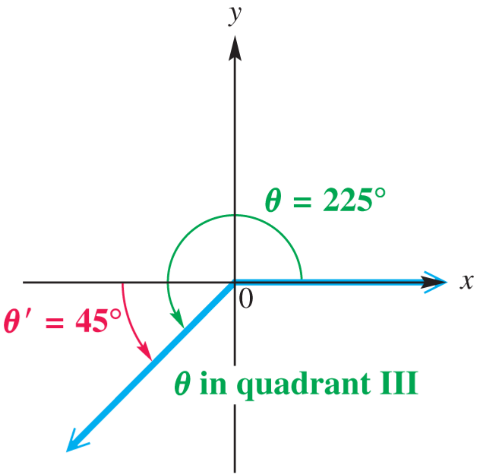
Since cos θ is negative, θ must lie in quadrant II or III.

The absolute value of cos θ is $\frac{\sqrt{2}}{2}$, so the reference angle is 45°.

The angle in quadrant II is 180° – 45° = 135°.

The angle in quadrant III is 180° + 45° = 225°.

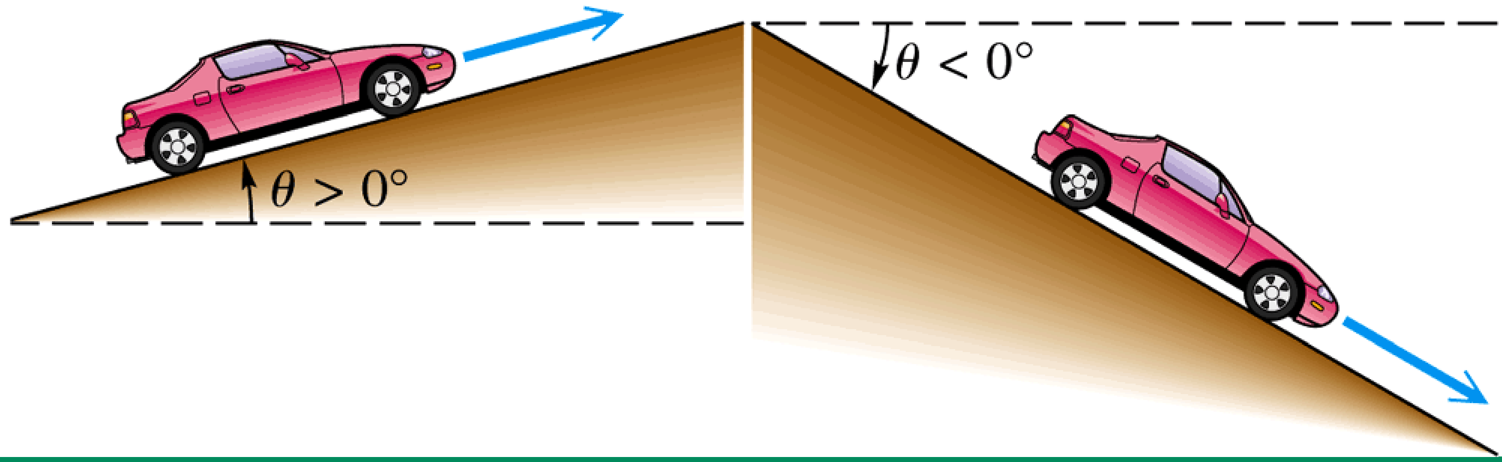




Example 10 FINDING GRADE RESISTANCE

When an automobile travels uphill or downhill on a highway, it experiences a force due to gravity. This force F in pounds is the **grade resistance** and is modeled by the equation $F = W \sin \theta$, where θ is the grade and W is the weight of the automobile.

If the automobile is moving uphill, then $\theta > 0°$; if downhill, then $\theta < 0°$.



Example 10 FINDING GRADE RESISTANCE (cont.)

(a) Calculate F to the nearest 10 lb for a 2500-lb car traveling an uphill grade with $\theta = 2.5°$.

$$F = W \sin\theta = 2500 \sin 2.5° \approx 110 \text{ lb}$$

(b) Calculate F to the nearest 10 lb for a 5000-lb truck traveling a downhill grade with $\theta = -6.1°$.

$$F = W \sin\theta = 5000 \sin(-6.1°) \approx -530 \text{ lb}$$

F is negative because the truck is moving downhill.

Example 10 FINDING GRADE RESISTANCE (cont.)

(c) Calculate F for $\theta = 0°$ and $\theta = 90°$. Do these answers agree with your intuition?

$$F = W\sin\theta = W\sin 0° = W(0) = 0 \text{ lb}$$

$$F = W\sin\theta = W\sin 90° = W(1) = W \text{ lb}$$

If $\theta = 0°$, then there is level ground and gravity does not cause the vehicle to roll.

If $\theta = 90°$, then the road is vertical and the full weight of the vehicle would be pulled downward by gravity, so $F = W$.