Chapter 1

EQUATIONS AND INEQUALITIES

Section 1.1: Linear Equations

1. Solve the equation $2x+7=x-1$

$$\begin{aligned} 2x+7&=x-1\\ 2x+7-x&=x-1-x\\ x+7&=-1\\ x+7-7&=-1-7\\ x&=-8 \end{aligned}$$

Moreover, replacing x with -8 in $2x+7=x-1$ yields a true statement. Therefore, the given statement is true.

2. The left side can be written as

$$5(x-10)=5[x+(-10)]=5x+5(-10)$$
$$=5x+(-50)=5x-50,$$

which is the same as the right side. Therefore, the statement is true.

3. The equations $x^2=4$ and $x+2=4$ are not equivalent. The first has a solution set $\{-2,2\}$ while the solution set of the second is $\{2\}$. Since the equations do not have the same solution set, they are not equivalent. The given statement is false.

4. A linear equation can be a contradiction, an identity, or conditional. If it is a contradiction, it has no solution; if it is an identity, it has more than two solutions; and if it is conditional, it has exactly one solution. Therefore, the given statement is false.

5. Answers will vary.

6. Answers will vary.

7. B cannot be written in the form $ax+b=0$. A can be written as $15x-7=0$ or $15x+(-7)=0$, C can be written as $2x=0$ or $2x+0=0$, and D can be written as $-.04x-.4=0$ or $-.04x+(-.4)=0$.

8. The student's answer is not correct. Additional answers will vary.

9.
$$\begin{aligned} 5x+4&=3x-4\\ 2x+4&=-4\\ 2x&=-8\Rightarrow x=-4 \end{aligned}$$

Solution set: $\{-4\}$

10.
$$\begin{aligned} 9x+11&=7x+1\\ 2x+11&=1\\ 2x&=-10\Rightarrow x=-5 \end{aligned}$$

Solution set: $\{-5\}$

11.
$$\begin{aligned} 6(3x-1)&=8-(10x-14)\\ 18x-6&=8-10x+14\\ 18x-6&=22-10x\\ 28x-6&=22\\ 28x&=28\Rightarrow x=1 \end{aligned}$$

Solution set: $\{1\}$

12.
$$\begin{aligned} 4(-2x+1)&=6-(2x-4)\\ -8x+4&=6-2x+4\\ -8x+4&=10-2x\\ 4&=10+6x\\ -6&=6x\Rightarrow -1=x \end{aligned}$$

Solution set: $\{-1\}$

13.
$$\begin{aligned} \frac{5}{6}x-2x+\frac{4}{3}&=\frac{5}{3}\\ 6\cdot\left[\frac{5}{6}x-2x+\frac{4}{3}\right]&=6\cdot\frac{5}{3}\\ 5x-12x+8&=10\\ -7x+8&=10\\ -7x&=2\Rightarrow x=-\frac{2}{7} \end{aligned}$$

Solution set: $\{-\frac{2}{7}\}$

14.
$$\begin{aligned} \frac{7}{4}+\frac{1}{5}x-\frac{3}{2}&=\frac{4}{5}x\\ 20\cdot\left[\frac{7}{4}+\frac{1}{5}x-\frac{3}{2}\right]&=20\cdot\frac{4}{5}x\\ 35+4x-30&=16x\\ 4x+5&=16x\\ 5&=12x\Rightarrow\frac{5}{12}=x \end{aligned}$$

Solution set: $\{\frac{5}{12}\}$

15.
$$\begin{aligned} 3x+5-5(x+1)&=6x+7\\ 3x+5-5x-5&=6x+7\\ -2x&=6x+7\\ -8x&=7\Rightarrow x=\frac{7}{-8}=-\frac{7}{8} \end{aligned}$$

Solution set: $\{-\frac{7}{8}\}$

16. $5(x+3)+4x-3=-(2x-4)+2$
$5x+15+4x-3=-2x+4+2$
$9x+12=-2x+6$
$11x+12=6$
$11x=-6 \Rightarrow x=\frac{-6}{11}=-\frac{6}{11}$
Solution set: $\left\{-\frac{6}{11}\right\}$

17. $2[x-(4+2x)+3]=2x+2$
$2(x-4-2x+3)=2x+2$
$2(-x-1)=2x+2$
$-2x-2=2x+2$
$-2=4x+2$
$-4=4x \Rightarrow -1=x$
Solution set: $\{-1\}$

18. $4[2x-(3-x)+5]=-7x-2$
$4(2x-3+x+5)=-7x-2$
$4(3x+2)=-7x-2$
$12x+8=-7x-2$
$19x+8=-2$
$19x=-10 \Rightarrow x=\frac{-10}{19}=-\frac{10}{19}$
Solution set: $\left\{-\frac{10}{19}\right\}$

19. $\frac{1}{14}(3x-2)=\frac{x+10}{10}$
$70\cdot\left[\frac{1}{14}(3x-2)\right]=70\cdot\left[\frac{x+10}{10}\right]$
$5(3x-2)=7(x+10)$
$15x-10=7x+70$
$8x-10=70$
$8x=80 \Rightarrow x=10$
Solution set: $\{10\}$

20. $\frac{1}{15}(2x+5)=\frac{x+2}{9}$
$45\cdot\left[\frac{1}{15}(2x+5)\right]=45\cdot\left[\frac{x+2}{9}\right]$
$3(2x+5)=5(x+2)$
$6x+15=5x+10$
$x+15=10 \Rightarrow x=-5$
Solution set: $\{-5\}$

21. $.2x-.5=.1x+7$
$10(.2x-.5)=10(.1x+7)$
$2x-5=x+70$
$x-5=70 \Rightarrow x=75$
Solution set: $\{75\}$

22. $.01x+3.1=2.03x-2.96$
$100(.01x+3.1)=100(2.03x-2.96)$
$x+310=203x-296$
$310=202x-296$
$606=202x \Rightarrow 3=x$
Solution set: $\{3\}$

23. $-4(2x-6)+8x=5x+24+x$
$-8x+24+8x=6x+24$
$24=6x+24$
$0=6x \Rightarrow 0=x$
Solution set: $\{0\}$

24. $-8(3x+4)+6x=4(x-8)+4x$
$-24x-32+6x=4x-32+4x$
$-18x-32=8x-32$
$-32=26x-32$
$0=26x \Rightarrow 0=x$
Solution set: $\{0\}$

25. $.5x+\frac{4}{3}x=x+10$
$\frac{1}{2}x+\frac{4}{3}x=x+10$
$6\left(\frac{1}{2}x+\frac{4}{3}x\right)=6(x+10)$
$3x+8x=6x+60$
$11x=6x+60$
$5x=60 \Rightarrow x=12$
Solution set: $\{12\}$

26. $\frac{2}{3}x+.25x=x+2$
$\frac{2}{3}x+\frac{1}{4}x=x+2$
$12\left(\frac{2}{3}x+\frac{1}{4}x\right)=12(x+2)$
$8x+3x=12x+24$
$11x=12x+24$
$-x=24 \Rightarrow x=-24$
Solution set: $\{-24\}$

27. $.08x+.06(x+12)=7.72$
$100[.08x+.06(x+12)]=100\cdot 7.72$
$8x+6(x+12)=772$
$8x+6x+72=772$
$14x+72=772$
$14x=700 \Rightarrow x=50$
Solution set: $\{50\}$

28.
$$\begin{aligned} .04(x-12)+.06x &= 1.52 \\ 100[.04(x-12)+.06x] &= 100\cdot 1.52 \\ 4(x-12)+6x &= 152 \\ 4x-48+6x &= 152 \\ 10x-48 &= 152 \\ 10x &= 200 \Rightarrow x = 20 \end{aligned}$$
Solution set: $\{20\}$

29.
$$\begin{aligned} 4(2x+7) &= 2x+22+3(2x+2) \\ 8x+28 &= 2x+22+6x+6 \\ 8x+28 &= 8x+28 \\ 28 &= 28 \Rightarrow 0=0 \end{aligned}$$
identity; $\{\text{all real numbers}\}$

30.
$$\begin{aligned} \frac{1}{2}(6x+20) &= x+4+2(x+3) \\ 3x+10 &= x+4+2x+6 \\ 3x+10 &= 3x+10 \\ 10 &= 10 \Rightarrow 0=0 \end{aligned}$$
identity; $\{\text{all real numbers}\}$

31.
$$\begin{aligned} 2(x-8) &= 3x-16 \\ 2x-16 &= 3x-16 \\ -16 &= x-16 \Rightarrow 0 = x \end{aligned}$$
conditional equation; $\{0\}$

32.
$$\begin{aligned} -8(x+3) &= -8x-5(x+1) \\ -8x-24 &= -8x-5x-5 \\ -8x-24 &= -13x-5 \\ 5x-24 &= -5 \\ 5x &= 19 \Rightarrow x = \frac{19}{5} \end{aligned}$$
conditional equation; $\left\{\frac{19}{5}\right\}$

33.
$$\begin{aligned} .3(x+2)-.5(x+2) &= -.2x-.4 \\ 10[.3(x+2)-.5(x+2)] &= 10[-.2x-.4] \\ 3(x+2)-5(x+2) &= -2x-4 \\ 3x+6-5x-10 &= -2x-4 \\ -2x-4 &= -2x-4 \\ 0 &= 0 \end{aligned}$$
identity; $\{\text{all real numbers}\}$

34.
$$\begin{aligned} -.6(x-5)+.8(x-6) &= .2x-1.8 \\ 10[-.6(x-5)+.8(x-6)] &= 10[.2x-1.8] \\ -6(x-5)+8(x-6) &= 2x-18 \\ -6x+30+8x-48 &= 2x-18 \\ 2x-18 &= 2x-18 \\ 0 &= 0 \end{aligned}$$
identity; $\{\text{all real numbers}\}$

35.
$$\begin{aligned} 4(x+7) &= 2(x+12)+2(x+1) \\ 4x+28 &= 2x+24+2x+2 \\ 4x+28 &= 4x+26 \\ 28 &= 26 \end{aligned}$$
contradiction; $\varnothing$

36.
$$\begin{aligned} -6(2x+1)-3(x-4) &= -15x+1 \\ -12x-6-3x+12 &= -15x+1 \\ -15x+6 &= -15x+1 \\ 6 &= 1 \end{aligned}$$
contradiction; $\varnothing$

37. Answers will vary. In solving an equation, you cannot multiply (or divide) both sides of an equation by zero. This is essentially what happened when the student divided both sides of the equation by x. To solve the equation, the student should isolate the variable term, which leads to the solution set $\{0\}$.
$$\begin{aligned} 5x &= 4x \\ 5x-4x &= 4x-4x \\ x &= 0 \end{aligned}$$

38. Answers will vary. If $k \neq 0$, then the equation $x+k=x$ would be a contradiction.

39.
$$\begin{aligned} V &= lwh \\ \frac{V}{wh} &= \frac{lwh}{wh} \\ l &= \frac{V}{wh} \end{aligned}$$

40.
$$\begin{aligned} I &= Prt \\ \frac{I}{rt} &= \frac{Prt}{rt} \\ P &= \frac{I}{rt} \end{aligned}$$

41.
$$\begin{aligned} P &= a+b+c \\ P-a-b &= c \\ c &= P-a-b \end{aligned}$$

42.
$$\begin{aligned} P &= 2l+2w \\ P-2l &= 2w \\ \frac{P-2l}{2} &= \frac{2w}{2} \\ w &= \frac{P-2l}{2} = \frac{P}{2}-l \end{aligned}$$

43.
$$\begin{aligned} A &= \frac{1}{2}h(B+b) \\ 2A &= 2\left[\frac{1}{2}h(B+b)\right] \\ 2A &= h(B+b) \\ 2A &= Bh+bh \\ 2A-bh &= Bh \\ \frac{2A-bh}{h} &= \frac{Bh}{h} \\ B &= \frac{2A-bh}{h} = \frac{2A}{h}-b \end{aligned}$$

44.
$$A=\frac{1}{2}h(B+b)$$
$$2A=2\left[\frac{1}{2}h(B+b)\right]$$
$$2A=h(B+b)$$
$$\frac{2A}{B+b}=\frac{h(B+b)}{B+b}$$
$$h=\frac{2A}{B+b}$$

45.
$$S=2\pi rh+2\pi r^2$$
$$S-2\pi r^2=2\pi rh$$
$$\frac{S-2\pi r^2}{2\pi r}=\frac{2\pi rh}{2\pi r}$$
$$h=\frac{S-2\pi r^2}{2\pi r}=\frac{S}{2\pi r}-r$$

46.
$$s=\frac{1}{2}gt^2$$
$$2s=2\left[\frac{1}{2}gt^2\right]$$
$$2s=gt^2$$
$$\frac{2s}{t^2}=\frac{gt^2}{t^2}$$
$$g=\frac{2s}{t^2}$$

47.
$$S=2lw+2wh+2hl$$
$$S-2lw=2wh+2hl$$
$$S-2lw=(2w+2l)h$$
$$\frac{S-2lw}{2w+2l}=\frac{(2w+2l)h}{2w+2l}$$
$$h=\frac{S-2lw}{2w+2l}$$

48. Answers will vary. This section pertains to solving linear equations. In terms of r, $S=2\pi rh+2\pi r^2$ is not a linear equation.

49.
$$2(x-a)+b=3x+a$$
$$2x-2a+b=3x+a$$
$$-3a+b=x$$
$$x=-3a+b$$

50.
$$5x-(2a+c)=a(x+1)$$
$$5x-2a-c=ax+a$$
$$5x-ax=3a+c$$
$$(5-a)x=3a+c$$
$$x=\frac{3a+c}{5-a}$$

51.
$$ax+b=3(x-a)$$
$$ax+b=3x-3a$$
$$3a+b=3x-ax$$
$$3a+b=(3-a)x$$
$$\frac{3a+b}{3-a}=x$$
$$x=\frac{3a+b}{3-a}$$

52.
$$4a-ax=3b+bx$$
$$4a-3b=bx+ax$$
$$4a-3b=(b+a)x$$
$$\frac{4a-3b}{b+a}=x$$
$$x=\frac{4a-3b}{b+a}$$

53.
$$\frac{x}{a-1}=ax+3$$
$$(a-1)\left[\frac{x}{a-1}\right]=(a-1)(ax+3)$$
$$x=a^2x+3a-ax-3$$
$$3-3a=a^2x-ax-x$$
$$3-3a=(a^2-a-1)x$$
$$\frac{3-3a}{a^2-a-1}=x$$
$$x=\frac{3-3a}{a^2-a-1}$$

54.
$$\frac{x-1}{2a}=\frac{1}{a-b}$$
$$2a(a-b)\left[\frac{x-1}{2a}\right]=2a(a-b)\left(\frac{1}{a-b}\right)$$
$$(a-b)(x-1)=2a$$
$$x-1=\frac{2a}{a-b}$$
$$x=\frac{2a}{a-b}+1$$
$$x=\frac{2a}{a-b}+\frac{a-b}{a-b}$$
$$x=\frac{2a+a-b}{a-b}=\frac{3a-b}{a-b}$$

55.
$$a^2x+3x=2a^2$$
$$(a^2+3)x=2a^2$$
$$x=\frac{2a^2}{a^2+3}$$

56.
$$ax + b^2 = bx - a^2$$
$$a^2 + b^2 = bx - ax$$
$$a^2 + b^2 = (b - a)x$$
$$\frac{a^2 + b^2}{b - a} = x$$
$$x = \frac{a^2 + b^2}{b - a}$$

57.
$$3x = (2x - 1)(m + 4)$$
$$3x = 2xm + 8x - m - 4$$
$$m + 4 = 2xm + 5x$$
$$m + 4 = (2m + 5)x$$
$$\frac{m + 4}{2m + 5} = x$$
$$x = \frac{m + 4}{2m + 5}$$

58.
$$-x = (5x + 3)(3k + 1)$$
$$-x = 15xk + 5x + 9k + 3$$
$$-6x - 15xk = 9k + 3$$
$$(-6 - 15k)x = 9k + 3$$
$$x = \frac{9k + 3}{-6 - 15k}$$

59. **(a)** Here, $r = .08$, $P = 3150$, and $t = \frac{6}{12} = \frac{1}{2}$ (year).

$$I = Prt = 3150(.08)\left(\frac{1}{2}\right) = \$126$$

The interest is $126.

(b) The amount Miguel must pay Julio at the end of the six months is $3150 + $126 = $3276.

60. **(a)** Here, $r = .104$, $P = 20{,}900$, and $t = \frac{18}{12} = \frac{3}{2}$ (year).

$$I = Prt = 20{,}900(.104)\left(\frac{3}{2}\right) = \$3260.40$$

She must pay the bank $20,900 + $3260.40 = $24,160.40.

(b) The interest is $3260.40.

61.
$$F = \frac{9}{5}C + 32$$
$$F = \frac{9}{5} \cdot 40 + 32$$
$$F = 72 + 32$$
$$F = 104$$
Therefore, 40°C = 104°F.

62.
$$F = \frac{9}{5}C + 32$$
$$F = \frac{9}{5} \cdot 200 + 32$$
$$F = 360 + 32$$
$$F = 392$$
Therefore, 200°C = 392°F.

63.
$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(59 - 32)$$
$$C = \frac{5}{9} \cdot 27$$
$$C = 15$$
Therefore, 59°F = 15°C.

64.
$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(86 - 32)$$
$$C = \frac{5}{9} \cdot 54$$
$$C = 30$$
Therefore, 86°F = 30°C.

65.
$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(100 - 32)$$
$$C = \frac{5}{9} \cdot 68$$
$$C \approx 37.8$$
Therefore, 100°F ≈ 37.8°C.

66.
$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(350 - 32)$$
$$C = \frac{5}{9} \cdot 318$$
$$C \approx 176.7$$
Therefore, 350°F ≈ 176.7°C.

67.
$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(867 - 32)$$
$$C = \frac{5}{9} \cdot 835$$
$$C \approx 463.9$$
Therefore, 865°F ≈ 463.9°C.

68. $F = \frac{9}{5}C + 32$
$F = \frac{9}{5} \cdot (-89.4) + 32$
$F = -160.92 + 32$
$F \approx -128.9$
Therefore, –89.4°C ≈ –128.9°F.

69. $C = \frac{5}{9}(F - 32)$
$C = \frac{5}{9}(7 - 32)$
$C = \frac{5}{9} \cdot (-25)$
$C \approx -13.9$
Therefore, 7°F ≈ –14°C.

70. $F = \frac{9}{5}C + 32$
$F = \frac{9}{5} \cdot (26.7) + 32$
$F = 48.06 + 32$
$F \approx 80.0$
Therefore, 26.7°C ≈ 80.0°F.

Section 1.2: Applications and Modeling with Linear Equations

Connections (page 96)

Step 1 compares to Polya's first step, Steps 2 and 3 compare to his second step, Step 4 compares to his third step, and Step 6 compares to his fourth step.

Exercises

1. 15 minutes is $\frac{1}{4}$ of an hour, so multiply 100 mph by $\frac{1}{4}$ to get a distance of 25 mi.

2. 75% is $\frac{3}{4}$, so multiply 120 L by $\frac{3}{4}$, to get 90 L acid.

3. 4% is .04, so multiply $500 by .04 and by 2 yrs to get interest of $40

4. Multiply 60 half-dollars by $.50 to get $30; multiply 200 quarters by $.25 to get $50. Together, the monetary value is $80.

5. Concentration A, 36%, cannot possibly be the concentration of the mixture because it exceeds both the concentrations.

6. Expression D, $x - .60$, does not represent the sales price. $x - .60$ represents x dollars discounted by 60 cents, not x dollars discounted by 60%. All of the other choices are equivalent and represent the sales price.

7. D **8.** A

9. In the formula $P = 2l + 2w$, let $P = 294$ and $w = 57$.
$294 = 2l + 2 \cdot 57$
$294 = 2l + 114$
$180 = 2l \Rightarrow 90 = l$
The length is 90 cm.

10. Let w = width of the rectangular storage shed. Then $w + 6$ = the length of the storage shed. Use the formula for the perimeter of a rectangle.
$P = 2l + 2w$
$44 = 2(w + 6) + 2w$
$44 = 2w + 12 + 2w$
$44 = 4w + 12$
$32 = 4w \Rightarrow 8 = w$
The width is 8 ft and the length is 8 + 6 = 14 ft.

11. Let $x =$ length of shortest side. Then $2x$ = length of each of the longer sides. The perimeter of a triangle is the sum of the measures of the three sides.
$x + 2x + 2x = 30 \Rightarrow 5x = 30 \Rightarrow x = 6$
The length of the shortest side is 6 cm.

12. Let $w =$ width of rectangle. Then $2w - 2.5$ = length of rectangle. Use the formula for the perimeter of a rectangle.
$P = 2l + 2w$
$40.6 = 2(2w - 2.5) + 2w$
$40.6 = 4w - 5 + 2w$
$40.6 = 6w - 5 \Rightarrow 45.6 = 6w \Rightarrow 7.6 = w$
The width is 7.6 cm.

13. Let $x =$ length of shortest side. Then $2x - 200$ = length of longest side and the length of the middle side is $(2x - 200) - 200 = 2x - 400$.
The perimeter of a triangle is the sum of the measures of the three sides.
$$x + (2x - 200) + (2x - 400) = 2400$$
$$x + 2x - 200 + 2x - 400 = 2400$$
$$5x - 600 = 2400$$
$$5x = 3000 \Rightarrow x = 600$$
The length of the shortest side is 600 ft. The middle side is $2 \cdot 600 - 400 = 1200 - 400 = 800$ ft. The longest side is $2 \cdot 600 - 200 = 1200 - 200 = 1000$ ft.

14. Let w = the width of the cake.
Then $w + 10$ = the length of the cake.
Use the formula for the perimeter of a rectangle.

$$P = 2l + 2w$$
$$56 = 2(w+10) + 2w$$
$$56 = 2w + 20 + 2w$$
$$56 = 4w + 20$$
$$36 = 4w \Rightarrow 9 = w$$

The width of the cake was 9 ft and the length was 9 + 10 = 19 ft.

15. Let l = the length of the book.
Then $l - .42$ = the width of the book
Use the formula for the perimeter of a rectangle.

$$P = 2l + 2w$$
$$5.96 = 2l + 2(l - .42)$$
$$5.96 = 2l + 2l - .84$$
$$5.96 = 4l - .84$$
$$6.8 = 4l \Rightarrow 1.7 = l$$

The length of the book is 1.7 cm, and the width of the book is 1.7 − .42 = 1.28 cm.

16. The volume of a right circular cylinder is $V = \pi r^2 h$

$$V = \pi r^2 h$$
$$144\pi = \pi 6^2 h$$
$$144\pi = 36\pi h$$
$$\frac{144\pi}{36\pi} = \frac{36\pi h}{36\pi} \Rightarrow 4 = h$$

The height of the cylinder is 4 in.

17. Let h = the height of box.
Use the formula for the surface area of a rectangular box.

$$S = 2lw + 2wh + 2hl$$
$$496 = 2 \cdot 18 \cdot 8 + 2 \cdot 8h + 2h \cdot 18$$
$$496 = 288 + 16h + 36h$$
$$496 = 288 + 52h$$
$$208 = 52h \Rightarrow 4 = h$$

The height of the box is 4 ft.

18. B and C cannot be correct equations.
In B, $-2x + 7(5 - x) = 52$

$$-2x + 35 - 7x = 52$$
$$-9x + 35 = 52$$
$$-9x = 17 \Rightarrow x = -\tfrac{17}{9}$$

but the length of a rectangle cannot be negative.
In C, $5(x+2) + 5x = 10$

$$5x + 10 + 5x = 10$$
$$10x + 10 = 10 \Rightarrow 10x = 0 \Rightarrow x = 0$$

but the length of a rectangle cannot be zero.

19. Let x = the time (in hours) spent on the way to the business appointment.

	r	t	d
Morning	50	x	$50x$
Afternoon	40	$x + \frac{1}{4}$	$40\left(x + \frac{1}{4}\right)$

The distance on the way to the business appointment is the same as the return trip, so

$$50x = 40\left(x + \tfrac{1}{4}\right)$$
$$50x = 40x + 10$$
$$10x = 10 \Rightarrow x = 1$$

Since she drove 1 hr, her distance traveled would be $50 \cdot 1 = 50$ mi.

20. Let x = time (in hours) on trip from Denver to Minneapolis.

	r	t	d
Going	50	x	$50x$
Returning	55	$32 - x$	$55(32 - x)$

The distance going and returning are the same, so we have

$$50x = 55(32 - x)$$
$$50x = 1760 - 55x$$
$$105x = 1760 \Rightarrow x \approx 16.76$$

Since he traveled approximately 16.8 hr to Minneapolis, the distance would be about $50 \cdot 16.8 = 840$ mi.

21. Let x = David's speed (in mph) on bike.
Then $x + 4.5$ = David's speed (in mph) driving.

	r	t	d
Car	$x + 4.5$	20 min = $\frac{1}{3}$ hr	$\frac{1}{3}(x + 4.5)$
Bike	x	45 min = $\frac{3}{4}$ hr	$\frac{3}{4}x$

The distance by bike and car are the same, so

$$\tfrac{1}{3}(x + 4.5) = \tfrac{3}{4}x$$
$$12\left[\tfrac{1}{3}(x + 4.5)\right] = 12\left[\tfrac{3}{4}x\right]$$
$$4(x + 4.5) = 9x$$
$$4x + 18 = 9x$$
$$18 = 5x \Rightarrow \frac{18}{5} = x$$

Since his rate is $\frac{18}{5}$ (or 3.6) mph, David travels $\frac{3}{4}\left(\frac{18}{5}\right) = \frac{27}{10} = 2.7$ mi to work.

22. Let x = rate (in mph) the San Diego bound plane travels. Then $x + 50$ = rate (in mph) the San Francisco bound plane travels.

	r	t	d
San Diego	x	$\frac{1}{2}$	$\frac{1}{2}x$
San Francisco	$x+50$	$\frac{1}{2}$	$\frac{1}{2}(x+50)$

The distance traveled by the two planes is 275 miles. The rate of the San Diego bound plane can be found by solving $\frac{1}{2}x+\frac{1}{2}(x+50)=275$.

$$\frac{1}{2}x+\frac{1}{2}(x+50)=275$$
$$2\left[\frac{1}{2}x+\frac{1}{2}(x+50)\right]=2[275]$$
$$x+(x+50)=550 \Rightarrow 2x+50=550$$
$$2x=500 \Rightarrow x=250$$

The San Diego bound plane travels at 250 mph, and the San Francisco bound plane travels at $250+50=300$ mph.

23. Let x = time (in hours) it takes for Russ and Janet to be 1.5 mi apart.

	r	t	d
Russ	7	x	$7x$
Janet	5	x	$5x$

Since Russ's rate is faster than Janet's, he travels farther than Janet in the same amount of time. To have the difference between Russ and Janet to be 1.5 mi, solve the following equation.

$7x-5x=1.5 \Rightarrow 2x=1.5 \Rightarrow x=.75$

It will take .75 hr = 45 min for Russ and Janet to be 1.5 mi apart.

24. Let x = time (in hours) Russ runs. Since Janet has a ten-minute start and 10 minutes is $\frac{1}{6}$ hr, Janet's time running is $x+\frac{1}{6}$ hr.

	r	t	d
Russ	7	x	$7x$
Janet	5	$x+\frac{1}{6}$	$5\left(x+\frac{1}{6}\right)$

Since Russ must travel the same distance as Janet, we must solve the following equation.

$$7x=5\left(x+\frac{1}{6}\right)$$
$$7x=5x+\frac{5}{6}$$
$$2x=\frac{5}{6}$$
$$x=\frac{5}{12}$$

It will take $\frac{5}{12}$ hr $=\frac{5}{12}\cdot 60$ min $=$ 25 min for Russ to catch up with Janet.

25. We need to determine how many meters are in 26 miles.

$$26 \text{ mi}\cdot\frac{5,280 \text{ ft}}{1 \text{ mi}}\cdot\frac{1 \text{ m}}{3.281 \text{ ft}}\approx 41,840.9 \text{ m}$$

Tim Montgomery's rate in the 100-m dash would be $r=\frac{d}{t}=\frac{100}{9.78}$ meters per second.

Thus, the time it would take for Tim to run the 26-mi marathon would be

$$t=\frac{d}{r}=\frac{41,840.9}{\frac{100}{9.78}}=41,840.9\cdot\frac{9.78}{100}\approx 4,092 \text{ sec.}$$

Since there is 60 seconds in one minute and $60\cdot 60=3,600$ seconds in one hour,

$4,092 \text{ sec}=1\cdot 3600+8\cdot 60+12 \text{ sec}$

or 1 hr, 8 min, 12 sec. This is about $\frac{1}{2}$ the world record time.

26. We know $26 \text{ mi}\approx 41,840.9 \text{ m}$ from exercise 25. Donovan Bailey's rate in the 100-m dash would be $r=\frac{d}{t}=\frac{100}{9.84}$ meters per second.

Thus, the time it would take for Donovan to run the 26-mi marathon would be

$$t=\frac{d}{r}=\frac{41,840.9}{\frac{100}{9.84}}=41,840.9\cdot\frac{9.84}{100}\approx 4,117.1 \text{ sec}$$

$4,117.1 \text{ sec}=1\cdot 3600+8\cdot 60+37.1 \text{ sec}$

or 1 hr, 8 min, 37.1 sec. This is about $\frac{1}{2}$ the world record time.

27. Let x = speed (in km/hr) of Joann's boat. When Joann is traveling upstream, the current slows her down, so we subtract the speed of the current from the speed of the boat. When she is traveling downstream, the current speeds her up, so we add the speed of the current to the speed of the boat.

	r	t	d
Upstream	$x-5$	20 min = $\frac{1}{3}$ hr	$\frac{1}{3}(x-5)$
Downstream	$x+5$	15 min = $\frac{1}{4}$ hr	$\frac{1}{4}(x+5)$

Since the distance upstream and downstream are the same, we must solve the following equation.

$$\frac{1}{3}(x-5)=\frac{1}{4}(x+5)$$
$$12\left[\frac{1}{3}(x-5)\right]=12\left[\frac{1}{4}(x+5)\right]$$
$$4(x-5)=3(x+5)$$
$$4x-20=3x+15$$
$$x-20=15$$
$$x=35$$

The speed of Joann's boat is 35 km per hour.

28. Let x = speed (in mph) of the wind. When Joe is traveling against the wind, the wind slows him down, so we subtract the speed of the wind from the speed of the plane. When he is traveling with the wind, the wind speeds him up, so we add the speed of the wind to the speed of the plane.

	r	t	d
Against wind	$180-x$	3	$3(180-x)$
With wind	$180+x$	2.8	$2.8(180+x)$

Since the distance going and coming are the same, we must solve the following equation.

$$2.8(180+x)=3(180-x)$$
$$504+2.8x=540-3x$$
$$504+5.8x=540$$
$$5.8x=36 \Rightarrow x\approx 6.2$$

The speed of the wind is about 6.2 mph.

29. Let x = the amount of 5% acid solution (in gallons).

Strength	Gallons of Solution	Gallons of Pure Acid
5%	x	$.05x$
10%	5	$.10\cdot 5=.5$
7%	$x+5$	$.07(x+5)$

The number of gallons of pure acid in the 5% solution plus the number of gallons of pure acid in the 10% solution must equal the number of gallons of pure acid in the 7% solution.

$$.05x+.5=.07(x+5)$$
$$.05x+.5=.07x+.35$$
$$.5=.02x+.35$$
$$.15=.02x$$
$$\frac{.15}{.02}=x\Rightarrow x=7.5=7\tfrac{1}{2}\text{ gal}$$

$7\frac{1}{2}$ gallons of the 5% solution should be added.

30. Let x = the amount of 100% alcohol solution (in gallons).

Strength	Gallons of Solution	Gallons of Pure Alcohol
100%	x	$1x=x$
15%	20	$.15\cdot 20=3$
25%	$x+20$	$.25(x+20)$

The number of gallons of pure alcohol in the 100% solution plus the number of gallons of pure alcohol in the 15% solution must equal the number of gallons of pure alcohol in the 25% solution.

$$x+3=.25(x+20)$$
$$x+3=.25x+5$$
$$.75x+3=5$$
$$.75x=2$$
$$x=\frac{2}{.75}=\frac{200}{75}=\frac{8}{3}=2\tfrac{2}{3}\text{ gal}$$

$2\frac{2}{3}$ gallons of the 100% solution should be added.

31. Let x = the amount of 100% alcohol solution (in liters).

Strength	Liters of Solution	Liters of Pure Alcohol
100%	x	$1x=x$
10%	7	$.10\cdot 7=.7$
30%	$x+7$	$.30(x+7)$

The number of liters of pure alcohol in the 100% solution plus the number of liters of pure alcohol in the 10% solution must equal the number of liters of pure alcohol in the 30% solution.

$$x+.7=.30(x+7)$$
$$x+.7=.30x+2.1$$
$$.7x+.7=2.1$$
$$.7x=1.4$$
$$x=\frac{1.4}{.7}=\frac{14}{7}=2\text{ L}$$

2 L of the 100% solution should be added.

32. Let x = the amount of 5% hydrochloric acid solution (in mL).

Strength	Milliliters of Solution	Milliliters of Hydrochloric Acid
5%	x	$.05x$
20%	60	$.20\cdot 60=12$
10%	$x+60$	$.10(x+60)$

The number of milliliters of hydrochloric acid in the 5% solution plus the number of milliliters of hydrochloric acid in the 20% solution must equal the number of milliliters of hydrochloric acid in the 10% solution.

(*continued on next page*)

(*continued from page 57*)

$$.05x+12=.10(x+60)$$
$$.05x+12=.10x+6$$
$$12=.05x+6$$
$$6=.05x$$
$$\frac{6}{.05}=\frac{600}{5}=120\text{ mL}$$

120 mL of 5% hydrochloric acid solution should be added.

33. Let x = the amount of water (in mL).

Strength	Milliliters of Solution	Milliliters of Salt
6%	8	$.06(8)=.48$
0%	x	$0(x)=0$
4%	$8+x$	$.04(8+x)$

The number of milliliters of salt in the 6% solution plus the number of milliliters of salt in the water (0% solution) must equal the number of milliliters in the 4% solution.

$$.48+0=.04(8+x)$$
$$.48=.32+.04x$$
$$.16=.04x$$
$$\frac{.16}{.04}=x\Rightarrow x=\frac{16}{4}=4\text{ mL}$$

To reduce the saline concentration to 4%, 4 mL of water should be added.

34. Let x = the amount of 100% acid (in liters).

Strength	Liters of Solution	Liters of Pure Acid
100%	x	$1x=x$
30%	18	$.30\cdot 18=5.4$
50%	$x+18$	$.50(x+18)$

The number of liters of acid in the pure acid (100%) plus the number of liters of acid in the 30% solution must equal the number of liters of acid in the 50% solution.

$$x+5.4=.50(x+18)$$
$$x+5.4=.50x+9$$
$$.5x+5.4=9$$
$$.5x=3.6$$
$$x=\frac{3.6}{.5}=\frac{36}{5}=7.2\text{ L}$$

7.2 L pure acid should be added.

35. Let x = amount of the short-term note. Then 240,000 – x = amount of the long-term note.

Amount of Note	Interest Rate	Interest
x	6%	$.06x$
$240,000-x$	5%	$.05(240,000-x)$
$240,000$		$13,000$

The amount of interest from the 6% note plus the amount of interest from the 5% note must equal the total amount of interest.

$$.06x+.05(240,000-x)=13,000$$
$$.06x+12,000-.05x=13,000$$
$$.01x+12,000=13,000$$
$$.01x=1,000$$
$$x=100,000$$

The amount of the short-term note is \$100,000 and the amount of the long-term note is \$240,000 – \$100,000 = \$140,000.

36. Let x = amount paid for the first plot. Then 120,000 – x = amount paid for the second plot.

Amount Paid	Rate of Profit/Loss	Profit/Loss
x	15%	$.15x$
$120,000-x$	–10%	$-.10(120,000-x)$
120,000		5,500

$$.15x-.10(120,000-x)=5500$$
$$.15x-12,000+.10x=5500$$
$$.25x-12,000=5500$$
$$.25x=17,500$$
$$x=\$70,000$$

Carl paid \$70,000 for the first plot and $120,000-70,000=\$50,000$ for the second plot.

37. Let x = amount invested at 2.5%. Then $2x$ = amount invested at 3%.

Amount in Account	Interest Rate	Interest
x	2.5%	$.025x$
$2x$	3%	$.03(2x)=.06x$
		850

The amount of interest from the 2.5% account plus the amount of interest from the 3% account must equal the total amount of interest.

$$.025x+.06x=850$$
$$.085x=850\Rightarrow x=\$10,000$$

Karen deposited \$10,000 at 2.5% and 2(\$10,000) = \$20,000 at 3%.

38. Let x = amount invested at 4%.
Then $4x$ = amount invested at 3.5%.

Amount in Account	Interest Rate	Interest
x	4%	$.04x$
$4x$	3.5%	$.035(4x) = .14x$
		3,600

The amount of interest from the 4% account plus the amount of interest from the 3.5% account must equal the total amount of interest.

$$.04x + .14x = 3600$$
$$.18x = 3600$$
$$x = \$20,000$$

The church invested \$20,000 at 4% and $4 \cdot 20,000 = \$80,000$ at 3.5%.

39. 30% of \$200,000 is \$60,000, so after paying her income tax, Linda had \$140,000 left to invest. Let x = amount invested at 1.5%.
Then $140,000 - x$ = amount invested at 4%.

Amount Invested	Interest Rate	Interest
x	1.5%	$.015x$
$140,000 - x$	4%	$.04(140,000 - x)$
140,000		4350

$$.015x + .04(140,000 - x) = 4350$$
$$.015x + 5600 - .04x = 4350$$
$$-.025x + 5600 = 4350$$
$$-.025x = -1250$$
$$x = \$50,000$$

Linda invested \$50,000 at 1.5% and \$140,000 – \$50,000 = \$90,000 at 4%.

40. 28% of \$48,000 is \$13,440, so after paying her income tax, Maliki had \$34,560 left to invest.
Let x = amount invested at 3.25%.
Then $34,560 - x$ = amount invested at 1.75%.

Amount Invested	Interest Rate	Interest
x	3.25%	$.0325x$
$34,560 - x$	1.75%	$.0175(34,560 - x)$
34,560		904.80

$$.0325x + .0175(34,560 - x) = 904.80$$
$$.0325x + 604.80 - .0175x = 904.80$$
$$.015x + 604.80 = 904.80$$
$$.015x = 300$$
$$x = \$20,000$$

Maliki invested \$20,000 at 3.25% and \$34,560 – \$20,000 = \$14,560 at 1.75%.

41. **(a)** $k = \frac{.132B}{W} = \frac{.132 \cdot 20}{75} = .0352$

(b) $R = (.0352)(.42) \approx .015$
An individual's increased lifetime cancer risk would be 1.5%.

(c) Using an average life expectancy of 72 years, $\frac{.015}{72} \cdot 5000 \approx 1$ case of cancer each year.

42. **(a)** The risk for one year would be $\frac{R}{72} = \frac{1.5 \times 10^{-3}}{72} \approx .000021$ for each individual.

(b) $C = .000021x$

(c) $C = .000021(100,000) = 2.1$
There are approximately 2.1 cancer cases in every 100,000 passive smokers.

(d) $C = \frac{.44(310,000,000)(.26)}{72} \approx 493,000$
There are approximately 493,000 excess deaths caused by smoking each year.

43. **(a)** The volume would be $10 \times 10 \times 8 = 800 \text{ ft}^3$.

(b) Since the paneling has an area of $4 \times 8 = 32$ sq ft, it emits $32 \cdot 3365 = 107,680\ \mu g$ of formaldehyde.

(c) $F = 107,680x$

(d) Since $33\ \mu g / ft^3$ causes irritation, the room would need $33 \cdot 800 = 26,400\ \mu g$ to cause irritation.

$$F = 107,680x$$
$$26,400 = 107,680x$$
$$\frac{26,400}{107,680} = x$$
$$x \approx .25 \text{ day}$$

or approximately 6 hours.

44. **(a)** Since each student needs 15 ft^3 each minute and there are 60 minutes in an hour, the ventilation required by x students per hour would be $V = 60(15x) = 900x$.

(b) The number of air exchanges per hour would be $A = \frac{900x}{15,000} = .06x$.

(c) If $x = 40$, then $A = .06 \cdot 40 = 2.4$ ach.

(d) The ventilation should be increased by $\frac{50}{15} = \frac{10}{3} = 3\frac{1}{3}$ times. (Smoking areas require more than triple the ventilation.)

45. (a) In 2008, $x = 5$.
$$y = .2145x + 15.69$$
$$y = .2145 \cdot 5 + 15.69$$
$$y = 1.0725 + 15.69$$
$$y = 16.7625$$
The projected enrollment for Fall 2008 is approximately 16.8 million

(b)
$$y = .2145x + 15.69$$
$$17 = .2145x + 15.69$$
$$1.31 = .2145x$$
$$\frac{1.31}{.2145} = x$$
$$x \approx 6.1$$
Enrollment is projected to reach 17 million in the year 2009

(c) They are quite close.

(d)
$$y = .2145(-10) + 15.69$$
$$y = -2.145 + 15.69$$
$$y \approx 13.5$$
The enrollment would be approximately 13.5 million

(e) Answers will vary.

46. (a) The 1960s represent $x = 1$.
$$y = -.93x + 20.45$$
$$y = -.93(1) + 20.45$$
$$y = -.93 + 20.45$$
$$y = 19.52$$
The approximate percent of Americans moving in the 1960s is 19.52%. It is .18% less than the 19.7% given in the graph.

(b) The 1980s represent $x = 3$.
$$y = -.93x + 20.45$$
$$y = -.93(3) + 20.45$$
$$y = -2.79 + 20.45$$
$$y = 17.66$$
The approximate percent of Americans moving in the 1980s is 17.66%. It is .24% less than the 17.9% given in the graph.

(c) According to the graph, 15.8% of Americans moved. So, in 2006, 301,000,000(.158) $\approx$ 47,558,000 Americans moved.

Section 1.3: Complex Numbers

1. true

2. true

3. true

4. true

5. false (Every real number is a complex number.)

6. true

7. –4 is real and complex.

8. 0 is real and complex.

9. $13i$ is complex, pure imaginary and nonreal complex.

10. $-7i$ is complex, pure imaginary and nonreal complex.

11. $5 + i$ is complex and nonreal complex.

12. $-6 - 2i$ is complex and nonreal complex.

13. π is real and complex.

14. $\sqrt{24}$ is real and complex.

15. $\sqrt{-25} = 5i$ is complex, pure imaginary and nonreal complex.

16. $\sqrt{-36} = 6i$ is complex, pure imaginary and nonreal complex.

17. $\sqrt{-25} = i\sqrt{25} = 5i$

18. $\sqrt{-36} = i\sqrt{36} = 6i$

19. $\sqrt{-10} = i\sqrt{10}$

20. $\sqrt{-15} = i\sqrt{15}$

21. $\sqrt{-288} = i\sqrt{288} = i\sqrt{144 \cdot 2} = 12i\sqrt{2}$

22. $\sqrt{-500} = i\sqrt{500} = i\sqrt{100 \cdot 5} = 10i\sqrt{5}$

23. $-\sqrt{-18} = -i\sqrt{18} = -i\sqrt{9 \cdot 2} = -3i\sqrt{2}$

24. $-\sqrt{-80} = -i\sqrt{80} = -i\sqrt{16 \cdot 5} = -4i\sqrt{5}$

25.
$$\sqrt{-13} \cdot \sqrt{-13} = i\sqrt{13} \cdot i\sqrt{13}$$
$$= i^2\left(\sqrt{13}\right)^2 = -1 \cdot 13 = -13$$

26.
$$\sqrt{-17} \cdot \sqrt{-17} = i\sqrt{17} \cdot i\sqrt{17}$$
$$= i^2\left(\sqrt{17}\right)^2 = -1 \cdot 17 = -17$$

27. $\sqrt{-3}\cdot\sqrt{-8} = i\sqrt{3}\cdot i\sqrt{8} = i^2\sqrt{3\cdot 8}$
$= -1\cdot\sqrt{24} = -\sqrt{4\cdot 6} = -2\sqrt{6}$

28. $\sqrt{-5}\cdot\sqrt{-15} = i\sqrt{5}\cdot i\sqrt{15} = i^2\sqrt{5\cdot 15}$
$= -1\cdot\sqrt{75} = -\sqrt{25\cdot 3} = -5\sqrt{3}$

29. $\dfrac{\sqrt{-30}}{\sqrt{-10}} = \dfrac{i\sqrt{30}}{i\sqrt{10}} = \sqrt{\dfrac{30}{10}} = \sqrt{3}$

30. $\dfrac{\sqrt{-70}}{\sqrt{-7}} = \dfrac{i\sqrt{70}}{i\sqrt{7}} = \sqrt{\dfrac{70}{7}} = \sqrt{10}$

31. $\dfrac{\sqrt{-24}}{\sqrt{8}} = \dfrac{i\sqrt{24}}{\sqrt{8}} = i\sqrt{\dfrac{24}{8}} = i\sqrt{3}$

32. $\dfrac{\sqrt{-54}}{\sqrt{27}} = \dfrac{i\sqrt{54}}{\sqrt{27}} = i\sqrt{\dfrac{54}{27}} = i\sqrt{2}$

33. $\dfrac{\sqrt{-10}}{\sqrt{-40}} = \dfrac{i\sqrt{10}}{i\sqrt{40}} = \sqrt{\dfrac{10}{40}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$

34. $\dfrac{\sqrt{-40}}{\sqrt{20}} = \dfrac{i\sqrt{40}}{\sqrt{20}} = i\sqrt{\dfrac{40}{20}} = i\sqrt{2}$

35. $\dfrac{\sqrt{-6}\cdot\sqrt{-2}}{\sqrt{3}} = \dfrac{i\sqrt{6}\cdot i\sqrt{2}}{\sqrt{3}} = i^2\sqrt{\dfrac{6\cdot 2}{3}}$
$= -1\cdot\sqrt{\dfrac{12}{3}} = -\sqrt{4} = -2$

36. $\dfrac{\sqrt{-12}\cdot\sqrt{-6}}{\sqrt{8}} = \dfrac{i\sqrt{12}\cdot i\sqrt{6}}{\sqrt{8}} = i^2\sqrt{\dfrac{12\cdot 6}{8}}$
$= -1\cdot\sqrt{\dfrac{72}{8}} = -\sqrt{9} = -3$

37. $\dfrac{-6-\sqrt{-24}}{2} = \dfrac{-6-\sqrt{-4\cdot 6}}{2} = \dfrac{-6-2i\sqrt{6}}{2}$
$= \dfrac{2\left(-3-i\sqrt{6}\right)}{2} = -3-i\sqrt{6}$

38. $\dfrac{-9-\sqrt{-18}}{3} = \dfrac{-9-\sqrt{-9\cdot 2}}{3} = \dfrac{-9-3i\sqrt{2}}{3}$
$= \dfrac{3\left(-3-i\sqrt{2}\right)}{3} = -3-i\sqrt{2}$

39. $\dfrac{10+\sqrt{-200}}{5} = \dfrac{10+\sqrt{-100\cdot 2}}{5}$
$= \dfrac{10+10i\sqrt{2}}{5} = \dfrac{5\left(2+2i\sqrt{2}\right)}{5}$
$= 2+2i\sqrt{2}$

40. $\dfrac{20+\sqrt{-8}}{2} = \dfrac{20+\sqrt{-4\cdot 2}}{2} = \dfrac{20+2i\sqrt{2}}{2}$
$= \dfrac{2\left(10+i\sqrt{2}\right)}{2} = 10+i\sqrt{2}$

41. $\dfrac{-3+\sqrt{-18}}{24} = \dfrac{-3+\sqrt{-9\cdot 2}}{24} = \dfrac{-3+3i\sqrt{2}}{24}$
$= \dfrac{3\left(-1+i\sqrt{2}\right)}{24} = \dfrac{-1+i\sqrt{2}}{8}$
$= -\dfrac{1}{8}+\dfrac{\sqrt{2}}{8}i$

42. $\dfrac{-5+\sqrt{-50}}{10} = \dfrac{-5+\sqrt{-25\cdot 2}}{10} = \dfrac{-5+5i\sqrt{2}}{10}$
$= \dfrac{5\left(-1+i\sqrt{2}\right)}{10} = \dfrac{-1+i\sqrt{2}}{2}$
$= -\dfrac{1}{2}+\dfrac{\sqrt{2}}{2}i$

43. $(3+2i)+(9-3i) = (3+9)+\left[2+(-3)\right]i$
$= 12+(-1)i = 12-i$

44. $(4-i)+(8+5i) = (4+8)+(-1+5)i$
$= 12+4i$

45. $(-2+4i)-(-4+4i)$
$= \left[-2-(-4)\right]+(4-4)i$
$= 2+0i = 2$

46. $(-3+2i)-(-4+2i)$
$= \left[-3-(-4)\right]+(2-2)i = 1+0i = 1$

47. $(2-5i)-(3+4i)-(-2+i)$
$= \left[2-3-(-2)\right]+(-5-4-1)i$
$= 1+(-10)i = 1-10i$

48. $(-4-i)-(2+3i)+(-4+5i)$
$= \left[-4-2+(-4)\right]+(-1-3+5)i = -10+i$

49. $-i-2-(6-4i)-(5-2i)$
$= (-2-6-5)+\left[-1-(-4)-(-2)\right]i$
$= -13+5i$

50. $3-(4-i)-4i+(-2+5i)$
$= \left[3-4+(-2)\right]+\left[-(-1)-4+5\right]i$
$= -3+2i$

51. $(2+i)(3-2i)$
$= 2(3)+2(-2i)+i(3)+i(-2i)$
$= 6-4i+3i-2i^2 = 6-i-2(-1)$
$= 6-i+2 = 8-i$

52. $(-2+3i)(4-2i)$
$= -2(4)-2(-2i)+3i(4)+3i(-2i)$
$= -8+4i+12i-6i^2 = -8+16i-6(-1)$
$= -8+16i+6 = -2+16i$

53. $(2+4i)(-1+3i)$
$= 2(-1)+2(3i)+4i(-1)+4i(3i)$
$= -2+6i-4i+12i^2 = -2+2i+12(-1)$
$= -2+2i-12 = -14+2i$

54. $(1+3i)(2-5i)$
$= 1(2)+1(-5i)+3i(2)+3i(-5i)$
$= 2-5i+6i-15i^2 = 2+i-15(-1)$
$= 2+i+15 = 17+i$

55. $(3-2i)^2 = 3^2-2(3)(2i)+(2i)^2$
$= 9-12i-4 = 5-12i$

56. $(2+i)^2 = 2^2+2(2)(i)+i^2 = 4+4i+i^2$
$= 4+4i+(-1) = 3+4i$

57. $(3+i)(3-i) = 3^2-i^2 = 9-(-1) = 10$

58. $(5+i)(5-i) = 5^2-i^2 = 25-(-1) = 26$

59. $(-2-3i)(-2+3i) = (-2)^2-(3i)^2 = 4-9i^2$
$= 4-9(-1) = 13$

60. $(6-4i)(6+4i) = 6^2-(4i)^2$
$= 36-16i^2 = 36-16(-1)$
$= 36+16 = 52$

61. $(\sqrt{6}+i)(\sqrt{6}-i) = (\sqrt{6})^2-i^2$
$= 6-(-1) = 6+1 = 7$

62. $(\sqrt{2}-4i)(\sqrt{2}+4i) = (\sqrt{2})^2-(4i)^2 = 2-16i^2$
$= 2-16(-1) = 2+16 = 18$

63. $i(3-4i)(3+4i) = i[(3-4i)(3+4i)]$
$= i[3^2-(4i)^2]$
$= i[9-16i^2]$
$= i[9-16(-1)]$
$= i(9+16) = 25i$

64. $i(2+7i)(2-7i) = i[(2+7i)(2-7i)]$
$= i[2^2-(7i)^2]$
$= i[4-49i^2]$
$= i[4-49(-1)]$
$= i(4+49) = 53i$

65. $3i(2-i)^2 = 3i(2^2-2(2i)+i^2)$
$= 3i(4-4i-1) = 3i(3-4i)$
$= 9i-12i^2 = 9i-12(-1)$
$= 12+9i$

66. $-5i(4-3i)^2 = -5i[4^2-2(4)(3i)+(3i)^2]$
$= -5i[16-24i+9i^2]$
$= -5i[16-24i+9(-1)]$
$= -5i(16-24i-9)$
$= -5i(7-24i)$
$= -35i+120i^2 = -35i+120(-1)$
$= -35i-120 = -120-35i$

67. $(2+i)(2-i)(4+3i) = [(2+i)(2-i)](4+3i)$
$= [2^2-i^2](4+3i)$
$= [4-(-1)](4+3i)$
$= 5(4+3i) = 20+15i$

68. $(3-i)(3+i)(2-6i) = [(3-i)(3+i)](2-6i)$
$= [3^2-i^2](2-6i)$
$= [9-(-1)](2-6i)$
$= 10(2-6i) = 20-60i$

69. $i^{25} = i^{24}\cdot i = (i^4)^6\cdot i = 1^6\cdot i = i$

70. $i^{29} = i^{28}\cdot i = (i^4)^7\cdot i = 1^7\cdot i = i$

71. $i^{22} = i^{20}\cdot i^2 = (i^4)^5\cdot(-1) = 1^5\cdot(-1) = -1$

72. $i^{26} = i^{24}\cdot i^2 = (i^4)^6\cdot(-1) = 1^6\cdot(-1) = -1$

73. $i^{23} = i^{20}\cdot i^3 = (i^4)^5\cdot i^3 = 1^5\cdot(-i) = -i$

74. $i^{27} = i^{24}\cdot i^3 = (i^4)^6\cdot i^3 = 1^6\cdot(-i) = -i$

75. $i^{32} = (i^4)^8 = 1^8 = 1$

76. $i^{40} = \left(i^4\right)^{10} = 1^{10} = 1$

77. $i^{-13} = i^{-16} \cdot i^3 = \left(i^4\right)^{-4} \cdot i^3 = 1^{-4} \cdot (-i) = -i$

78. $i^{-14} = i^{-16} \cdot i^2 = \left(i^4\right)^{-4} \cdot i^2 = 1^{-4} \cdot (-1) = -1$

79. $\dfrac{1}{i^{-11}} = i^{11} = i^8 \cdot i^3 = \left(i^4\right)^2 \cdot i^3 = 1^2 \cdot (-i) = -i$

80. $\dfrac{1}{i^{-12}} = i^{12} = \left(i^4\right)^3 = 1^3 = 1$

81. Answers will vary.

82. Answers will vary.

83. $$\frac{6+2i}{1+2i} = \frac{(6+2i)(1-2i)}{(1+2i)(1-2i)} = \frac{6-12i+2i-4i^2}{1^2-(2i)^2} = \frac{6-10i-4(-1)}{1-4i^2} = \frac{6-10i+4}{1-4(-1)} = \frac{10-10i}{1+4} = \frac{10-10i}{5} = \frac{10}{5} - \frac{10}{5}i = 2-2i$$

84. $$\frac{14+5i}{3+2i} = \frac{(14+5i)(3-2i)}{(3+2i)(3-2i)} = \frac{42-28i+15i-10i^2}{3^2-(2i)^2} = \frac{42-13i-10(-1)}{9-4i^2} = \frac{42-13i+10}{9-4(-1)} = \frac{52-13i}{9+4} = \frac{52-13i}{13} = \frac{52}{13} - \frac{13}{13}i = 4-i$$

85. $$\frac{2-i}{2+i} = \frac{(2-i)(2-i)}{(2+i)(2-i)} = \frac{2^2-2(2i)+i^2}{2^2-i^2} = \frac{4-4i+(-1)}{4-(-1)} = \frac{3-4i}{5} = \frac{3}{5} - \frac{4}{5}i$$

86. $$\frac{4-3i}{4+3i} = \frac{(4-3i)(4-3i)}{(4+3i)(4-3i)} = \frac{4^2-2(4)(3i)+(3i)^2}{4^2-(3i)^2} = \frac{16-24i+9i^2}{16-9i^2} = \frac{16-24i+9(-1)}{16-9(-1)} = \frac{16-24i-9}{16+9} = \frac{7-24i}{25} = \frac{7}{25} - \frac{24}{25}i$$

87. $$\frac{1-3i}{1+i} = \frac{(1-3i)(1-i)}{(1+i)(1-i)} = \frac{1-i-3i+3i^2}{1^2-i^2} = \frac{1-4i+3(-1)}{1-(-1)} = \frac{1-4i-3}{2} = \frac{-2-4i}{2} = \frac{-2}{2} - \frac{4}{2}i = -1-2i$$

88. $$\frac{-3+4i}{2-i} = \frac{(-3+4i)(2+i)}{(2-i)(2+i)} = \frac{-6-3i+8i+4i^2}{2^2-i^2} = \frac{-6+5i+4(-1)}{4-(-1)} = \frac{-6+5i-4}{5} = \frac{-10+5i}{5} = \frac{-10}{5} + \frac{5}{5}i = -2+i$$

89. $$\frac{-5}{i} = \frac{-5(-i)}{i(-i)} = \frac{5i}{-i^2} = \frac{5i}{-(-1)} = \frac{5i}{1} = 5i \text{ or } 0+5i$$

90. $$\frac{-6}{i} = \frac{-6(-i)}{i(-i)} = \frac{6i}{-i^2} = \frac{6i}{-(-1)} = \frac{6i}{1} = 6i \text{ or } 0+6i$$

91. $$\frac{8}{-i} = \frac{8 \cdot i}{-i \cdot i} = \frac{8i}{-i^2} = \frac{8i}{-(-1)} = \frac{8i}{1} = 8i \text{ or } 0+8i$$

92. $$\frac{12}{-i} = \frac{12 \cdot i}{-i \cdot i} = \frac{12i}{-i^2} = \frac{12i}{-(-1)} = \frac{12i}{1} = 12i \text{ or } 0+12i$$

93. $$\frac{2}{3i} = \frac{2(-3i)}{3i \cdot (-3i)} = \frac{-6i}{-9i^2} = \frac{-6i}{-9(-1)} = \frac{-6i}{9} = -\frac{2}{3}i \text{ or } 0 - \frac{2}{3}i$$

Note: In the above solution, we multiplied the numerator and denominator by the complex conjugate of $3i$, namely $-3i$. Since there is a reduction in the end, the same results can be achieved by multiplying the numerator and denominator by $-i$.

94. $$\frac{5}{9i} = \frac{5(-9i)}{9i \cdot (-9i)} = \frac{-45i}{-81i^2} = \frac{-45i}{-81(-1)} = \frac{-45i}{81} = -\frac{5}{9}i \text{ or } 0 - \frac{5}{9}i$$

95. We need to show that $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2 = i.$

$$\begin{aligned}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2 &= \left(\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}i+\left(\frac{\sqrt{2}}{2}i\right)^2\\ &= \frac{2}{4}+2\cdot\frac{2}{4}i+\frac{2}{4}i^2 = \frac{1}{2}+i+\frac{1}{2}i^2\\ &= \frac{1}{2}+i+\frac{1}{2}(-1)=\frac{1}{2}+i-\frac{1}{2}=i\end{aligned}$$

96. We need to show that $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^3 = i.$

$$\begin{aligned}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^3 &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^2\\ &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left[\left(\frac{\sqrt{3}}{2}\right)^2+2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}i+\left(\frac{1}{2}i\right)^2\right]\\ &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left[\frac{3}{4}+\frac{\sqrt{3}}{2}i+\frac{1}{4}i^2\right]\\ &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left[\frac{3}{4}+\frac{\sqrt{3}}{2}i+\frac{1}{4}(-1)\right]\\ &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left[\frac{3}{4}+\frac{\sqrt{3}}{2}i-\frac{1}{4}\right]\\ &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(\frac{2}{4}+\frac{\sqrt{3}}{2}i\right)\\ &= \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}i+\frac{1}{2}\cdot\frac{1}{2}i+\frac{1}{2}\frac{\sqrt{3}}{2}i^2\\ &= \frac{\sqrt{3}}{4}+\frac{3}{4}i+\frac{1}{4}i+\frac{\sqrt{3}}{4}i^2\\ &= \frac{\sqrt{3}}{4}+\frac{4}{4}i+\frac{\sqrt{3}}{4}(-1)=\frac{\sqrt{3}}{4}+i+\left(-\frac{\sqrt{3}}{4}\right)=i\end{aligned}$$

97. Let $z = 3-2i.$

$$\begin{aligned}3z-z^2 &= 3(3-2i)-(3-2i)^2\\ &= 9-6i-\left[3^2-2(6i)+(2i)^2\right]\\ &= 9-6i-\left(9-12i+4i^2\right)\\ &= 9-6i-\left[9-12i+4(-1)\right]\\ &= 9-6i-\left[9-12i+(-4)\right]\\ &= 9-6i-(5-12i)\\ &= 9-6i-5+12i = 4+6i\end{aligned}$$

98. Let $z = -6i.$

$$\begin{aligned}-2z+z^3 &= -2(-6i)+(-6i)^3 = 12i+(-6)^3\,i^3\\ &= 12i+(-216)(-i) = 12i+216i = 228i\end{aligned}$$

Section 1.4: Quadratic Equations

1. $x^2 = 25$
$x = \pm\sqrt{25} = \pm 5;$ G

2. $x^2 = -25$
$x = \pm\sqrt{-25} = \pm 5i;$ A

3. $x^2 + 5 = 0$
$x^2 = -5$
$x = \pm\sqrt{-5} = \pm i\sqrt{5};$ C

4. $x^2 - 5 = 0$
$x^2 = 5$
$x = \pm\sqrt{5};$ E

5. $x^2 = -20$
$x = \pm\sqrt{-20} = \pm 2i\sqrt{5};$ H

6. $x^2 = 20$
$x = \pm\sqrt{20} = \pm 2\sqrt{5};$ B

7. $x - 5 = 0$
$x = 5;$ D

8. $x + 5 = 0$
$x = -5;$ F

9. D is the only one set up for direct use of the zero-factor property.
$(3x-1)(x-7) = 0$
$3x - 1 = 0$ or $x - 7 = 0$
$x = \frac{1}{3}$ or $x = 7$
Solution set: $\left\{\frac{1}{3}, 7\right\}$

10. B is the only one set up for direct use of the square root property.

$$(2x+5)^2 = 7$$
$$2x+5 = \pm\sqrt{7}$$
$$2x = -5 \pm \sqrt{7} \Rightarrow x = \frac{-5 \pm \sqrt{7}}{2}$$

Solution set: $\left\{\frac{-5\pm\sqrt{7}}{2}\right\}$

11. C is the only one that does not require Step 1 of the method of completing the square.

$$x^2 + x = 12$$
$$x^2 + x + \frac{1}{4} = 12 + \frac{1}{4}$$

Note: $\left[\frac{1}{2}\cdot 1\right]^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$$\left(x+\frac{1}{2}\right)^2 = \frac{49}{4}$$
$$x + \frac{1}{2} = \pm\sqrt{\frac{49}{4}}$$
$$x + \frac{1}{2} = \pm\frac{7}{2} \Rightarrow x = -\frac{1}{2} \pm \frac{7}{2}$$
$$-\frac{1}{2} - \frac{7}{2} = \frac{-8}{2} = -4 \text{ and } -\frac{1}{2} + \frac{7}{2} = \frac{6}{2} = 3$$

Solution set: $\{-4, 3\}$

12. A is the only one set up so that the values of a, b, and c can be determined immediately.

$3x^2 - 17x - 6 = 0$ yields $a = 3$, $b = -17$, and $c = -6$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-17) \pm \sqrt{(-17)^2 - 4(3)(-6)}}{2(3)}$$
$$= \frac{17 \pm \sqrt{289 - (-72)}}{6} = \frac{17 \pm \sqrt{361}}{6}$$
$$= \frac{17 \pm 19}{6}$$
$$\frac{17+19}{6} = \frac{36}{6} = 6 \text{ and } \frac{17-19}{6} = \frac{-2}{6} = -\frac{1}{3}$$

Solution set: $\left\{-\frac{1}{3}, 6\right\}$

13. $x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x - 2 = 0 \Rightarrow x = 2$ or $x - 3 = 0 \Rightarrow x = 3$

Solution set: $\{2, 3\}$

14. $x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x + 4 = 0 \Rightarrow x = -4$ or $x - 2 = 0 \Rightarrow x = 2$

Solution set: $\{-4, 2\}$

15. $5x^2 - 3x - 2 = 0$

$(5x+2)(x-1) = 0$

$5x + 2 = 0 \Rightarrow x = -\frac{2}{5}$ or $x - 1 = 0 \Rightarrow x = 1$

Solution set: $\left\{-\frac{2}{5}, 1\right\}$

16. $2x^2 - x - 15 = 0$

$(2x+5)(x-3) = 0$

$2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$ or $x - 3 = 0 \Rightarrow x = 3$

Solution set: $\left\{-\frac{5}{2}, 3\right\}$

17. $-4x^2 + x = -3$

$0 = 4x^2 - x - 3$

$0 = (4x+3)(x-1)$

$4x + 3 = 0 \Rightarrow x = -\frac{3}{4}$ or $x - 1 = 0 \Rightarrow x = 1$

Solution set: $\left\{-\frac{3}{4}, 1\right\}$

18. $-6x^2 + 7x = -10$

$0 = 6x^2 - 7x - 10 = 0$

$0 = (6x+5)(x-2) = 0$

$6x + 5 = 0 \Rightarrow x = -\frac{5}{6}$ or $x - 2 = 0 \Rightarrow x = 2$

Solution set: $\left\{-\frac{5}{6}, 2\right\}$

19. $x^2 = 16$

$x = \pm\sqrt{16} = \pm 4$

Solution set: $\{\pm 4\}$

20. $x^2 = 25$

$x = \pm\sqrt{25} = \pm 5$

Solution set: $\{\pm 5\}$

21. $27 - x^2 = 0$

$27 = x^2$

$x = \pm\sqrt{27} = \pm 3\sqrt{3}$

Solution set: $\{\pm 3\sqrt{3}\}$

22. $48 - x^2 = 0$

$48 = x^2$

$x = \pm\sqrt{48} = \pm 4\sqrt{3}$

Solution set: $\{\pm 4\sqrt{3}\}$

23. $x^2 = -81$

$x = \pm\sqrt{-81} = \pm 9i$

Solution set: $\{\pm 9i\}$

24. $x^2 = -400$
$x = \pm\sqrt{-400} = \pm 20i$
Solution set: $\{\pm 20i\}$

25. $(3x-1)^2 = 12$
$3x - 1 = \pm\sqrt{12}$
$3x = 1 \pm 2\sqrt{3}$
$x = \dfrac{1 \pm 2\sqrt{3}}{3}$
Solution set: $\left\{\frac{1\pm 2\sqrt{3}}{3}\right\}$

26. $(4x+1)^2 = 20$
$4x + 1 = \pm\sqrt{20}$
$4x = -1 \pm 2\sqrt{5}$
$x = \dfrac{-1 \pm 2\sqrt{5}}{4}$
Solution set: $\left\{\frac{-1\pm 2\sqrt{5}}{4}\right\}$

27. $(x+5)^2 = -3$
$x + 5 = \pm\sqrt{-3}$
$x + 5 = \pm i\sqrt{3}$
$x = -5 \pm i\sqrt{3}$
Solution set: $\{-5 \pm i\sqrt{3}\}$

28. $(x-4)^2 = -5$
$x - 4 = \pm\sqrt{-5}$
$x - 4 = \pm i\sqrt{5}$
$x = 4 \pm i\sqrt{5}$
Solution set: $\{4 \pm i\sqrt{5}\}$

29. $(5x-3)^2 = -3$
$5x - 3 = \pm\sqrt{-3}$
$5x - 3 = \pm i\sqrt{3}$
$5x = 3 \pm i\sqrt{3}$
$x = \dfrac{3 \pm i\sqrt{3}}{5} = \dfrac{3}{5} \pm \dfrac{\sqrt{3}}{5}i$
Solution set: $\left\{\frac{3}{5} \pm \frac{\sqrt{3}}{5}i\right\}$

30. $(-2x+5)^2 = -8$
$-2x + 5 = \pm\sqrt{-8}$
$-2x + 5 = \pm 2i\sqrt{2}$
$-2x = -5 \pm 2i\sqrt{2}$
$x = \dfrac{-5 \pm 2i\sqrt{2}}{-2} = \dfrac{5}{2} \pm i\sqrt{2}$
Solution set: $\left\{\frac{5}{2} \pm i\sqrt{2}\right\}$

31. $x^2 - 4x + 3 = 0$
$x^2 - 4x + 4 = -3 + 4$ Note: $\left[\frac{1}{2}\cdot 4\right]^2 = 2^2 = 4$
$(x-2)^2 = 1$
$x - 2 = \pm\sqrt{1}$
$x - 2 = \pm 1$
$x = 2 \pm 1$
$2 - 1 = 1$ and $2 + 1 = 3$
Solution set: $\{1, 3\}$

32. $x^2 - 7x + 12 = 0$
$x^2 - 7x + \frac{49}{4} = -12 + \frac{49}{4}$ Note: $\left[\frac{1}{2}\cdot 7\right]^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$
$\left(x - \frac{7}{2}\right)^2 = \frac{1}{4}$
$x - \frac{7}{2} = \pm\sqrt{\frac{1}{4}}$
$x - \frac{7}{2} = \pm\frac{1}{2}$
$x = \frac{7}{2} \pm \frac{1}{2}$
$\frac{7}{2} - \frac{1}{2} = \frac{6}{2} = 3$ and $\frac{7}{2} + \frac{1}{2} = \frac{8}{2} = 4$
Solution set: $\{3, 4\}$

33. $2x^2 - x - 28 = 0$
$x^2 - \frac{1}{2}x - 14 = 0$ Multiply by $\frac{1}{2}$.
$x^2 - \frac{1}{2}x + \frac{1}{16} = 14 + \frac{1}{16}$
Note: $\left[\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\right]^2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}$
$\left(x - \frac{1}{4}\right)^2 = \frac{225}{16}$
$x - \frac{1}{4} = \pm\sqrt{\frac{225}{16}}$
$x - \frac{1}{4} = \pm\frac{15}{4}$
$x = \frac{1}{4} \pm \frac{15}{4}$
$\frac{1}{4} - \frac{15}{4} = \frac{-14}{4} = -\frac{7}{2}$ and $\frac{1}{4} + \frac{15}{4} = \frac{16}{4} = 4$
Solution set: $\left\{-\frac{7}{2}, 4\right\}$

34. $4x^2 - 3x - 10 = 0$

$x^2 - \frac{3}{4}x - \frac{10}{4} = 0$

$x^2 - \frac{3}{4}x - \frac{5}{2} = 0$

$x^2 - \frac{3}{4}x + \frac{9}{64} = \frac{5}{2} + \frac{9}{64}$

Note: $\left[\frac{1}{2} \cdot \left(-\frac{3}{4}\right)\right]^2 = \left(-\frac{3}{8}\right)^2 = \frac{9}{64}$

$\left(x - \frac{3}{8}\right)^2 = \frac{169}{64}$

$x - \frac{3}{8} = \pm\sqrt{\frac{169}{64}} = \pm\frac{13}{8}$

$x = \frac{3}{8} \pm \frac{13}{8}$

$\frac{3}{8} - \frac{13}{8} = \frac{-10}{8} = -\frac{5}{4}$ and $\frac{3}{8} + \frac{13}{8} = \frac{16}{8} = 2$

Solution set: $\left\{-\frac{5}{4}, 2\right\}$

35. $x^2 - 2x - 2 = 0$

$x^2 - 2x + 1 = 2 + 1$

Note: $\left[\frac{1}{2} \cdot (-2)\right]^2 = (-1)^2 = 1$

$(x-1)^2 = 3$

$x - 1 = \pm\sqrt{3}$

$x = 1 \pm \sqrt{3}$

Solution set: $\left\{1 \pm \sqrt{3}\right\}$

36. $x^2 - 3x - 6 = 0$

$x^2 - 3x + \frac{9}{4} = 6 + \frac{9}{4}$

Note: $\left[\frac{1}{2} \cdot (-3)\right]^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$

$\left(x - \frac{3}{2}\right)^2 = \frac{33}{4}$

$x - \frac{3}{2} = \pm\sqrt{\frac{33}{4}}$

$x - \frac{3}{2} = \pm\frac{\sqrt{33}}{2}$

$x = \frac{3}{2} \pm \frac{\sqrt{33}}{2} = \frac{3 \pm \sqrt{33}}{2}$

Solution set: $\left\{\frac{3 \pm \sqrt{33}}{2}\right\}$

37. $2x^2 + x = 10$

$x^2 + \frac{1}{2}x = 5$

$x^2 + \frac{1}{2}x + \frac{1}{16} = 5 + \frac{1}{16}$

Note: $\left[\frac{1}{2} \cdot \frac{1}{2}\right]^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$\left(x + \frac{1}{4}\right)^2 = \frac{81}{16}$

$x + \frac{1}{4} = \pm\sqrt{\frac{81}{16}}$

$x + \frac{1}{4} = \pm\frac{9}{4}$

$x = -\frac{1}{4} \pm \frac{9}{4}$

$-\frac{1}{4} - \frac{9}{4} = \frac{-10}{4} = -\frac{5}{2}$ and $-\frac{1}{4} + \frac{9}{4} = \frac{8}{4} = 2$

Solution set: $\left\{-\frac{5}{2}, 2\right\}$

38. $3x^2 + 2x = 5$

$x^2 + \frac{2}{3}x = \frac{5}{3}$

$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{5}{3} + \frac{1}{9}$

Note: $\left[\frac{1}{2} \cdot \left(-\frac{2}{3}\right)\right]^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$

$\left(x + \frac{1}{3}\right)^2 = \frac{16}{9}$

$x + \frac{1}{3} = \pm\sqrt{\frac{16}{9}}$

$x + \frac{1}{3} = \pm\frac{4}{3}$

$x = -\frac{1}{3} \pm \frac{4}{3}$

$-\frac{1}{3} - \frac{4}{3} = -\frac{5}{3}$ and $-\frac{1}{3} + \frac{4}{3} = \frac{3}{3} = 1$

Solution set: $\left\{-\frac{5}{3}, 1\right\}$

39. $2x^2 - 4x - 3 = 0$

$x^2 - 2x - \frac{3}{2} = 0$

$x^2 - 2x + 1 = \frac{3}{2} + 1$

Note: $\left[\frac{1}{2} \cdot (-2)\right]^2 = (-1)^2 = 1$

$(x-1)^2 = \frac{5}{2}$

$x - 1 = \pm\sqrt{\frac{5}{2}} = \pm\frac{\sqrt{10}}{2}$

$x = 1 \pm \frac{\sqrt{10}}{2} = \frac{2 \pm \sqrt{10}}{2}$

Solution set: $\left\{\frac{2 \pm \sqrt{10}}{2}\right\}$

40. $-3x^2 + 6x + 5 = 0$

$x^2 - 2x - \frac{5}{3} = 0$

$x^2 - 2x + 1 = \frac{5}{3} + 1$

Note: $\left[\frac{1}{2} \cdot (-2)\right]^2 = (-1)^2 = 1$

$(x-1)^2 = \frac{8}{3}$

$x - 1 = \pm\sqrt{\frac{8}{3}} = \pm\frac{\sqrt{24}}{3} = \pm\frac{2\sqrt{6}}{3}$

$x = 1 \pm \frac{2\sqrt{6}}{3} = \frac{3 \pm 2\sqrt{6}}{3}$

Solution set: $\left\{\frac{3 \pm 2\sqrt{6}}{3}\right\}$

41. $-4x^2 + 8x = 7$

$x^2 - 2x = -\frac{7}{4}$

$x^2 - 2x + 1 = -\frac{7}{4} + 1$

Note: $\left[\frac{1}{2} \cdot (-2)\right]^2 = (-1)^2 = 1$

$(x-1)^2 = \frac{-3}{4}$

$x - 1 = \pm\sqrt{\frac{-3}{4}} = \pm\frac{i\sqrt{3}}{2}$

$x = 1 \pm \frac{\sqrt{3}}{2}i$

Solution set: $\left\{1 \pm \frac{\sqrt{3}}{2}i\right\}$

42. $3x^2-9x=-7$

$x^2-3x=-\frac{7}{3}$

$x^2-3x+\frac{9}{4}=-\frac{7}{3}+\frac{9}{4}=\frac{-28}{12}+\frac{27}{12}$

Note: $\left[\frac{1}{2}\cdot(-3)\right]^2=\left(-\frac{3}{2}\right)^2=\frac{9}{4}$

$\left(x-\frac{3}{2}\right)^2=\frac{-1}{12}$

$x-\frac{3}{2}=\pm\sqrt{\frac{-1}{12}}=\pm\frac{i\sqrt{12}}{12}=\pm\frac{2\sqrt{3}}{12}i=\pm\frac{\sqrt{3}}{6}i$

$x=\frac{3}{2}\pm\frac{\sqrt{3}}{6}i$

Solution set: $\left\{\frac{3}{2}\pm\frac{\sqrt{3}}{6}i\right\}$

43. Francisco is incorrect because $c = 0$ and the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be evaluated with $a=1, b=-8,$ and $c=0.$

44. Francisca is incorrect because $b = 0$ and the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be evaluated with $a=1, b=0,$ and $c=-19.$

45. $x^2-x-1=0$

Let $a=1, b=-1,$ and $c=-1.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$$

Solution set: $\left\{\frac{1\pm\sqrt{5}}{2}\right\}$

46. $x^2-3x-2=0$

Let $a=1, b=-3,$ and $c=-2.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-2)}}{2(1)}=\frac{3\pm\sqrt{9+8}}{2}=\frac{3\pm\sqrt{17}}{2}$$

Solution set: $\left\{\frac{3\pm\sqrt{17}}{2}\right\}$

47. $x^2-6x=-7$

$x^2-6x+7=0$

Let $a=1, b=-6,$ and $c=7.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(7)}}{2(1)}=\frac{6\pm\sqrt{36-28}}{2}=\frac{6\pm\sqrt{8}}{2}=\frac{6\pm2\sqrt{2}}{2}=3\pm\sqrt{2}$$

Solution set: $\left\{3\pm\sqrt{2}\right\}$

48. $x^2-4x=-1$

$x^2-4x+1=0$

Let $a=1, b=-4,$ and $c=1.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

Solution set: $\left\{2\pm\sqrt{3}\right\}$

49. $x^2=2x-5$

$x^2-2x+5=0$

Let $a=1, b=-2,$ and $c=5.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)}=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm\sqrt{-16}}{2}=\frac{2\pm4i}{2}=1\pm2i$$

Solution set: $\{1\pm2i\}$

50. $x^2=2x-10$

$x^2-2x+10=0$

Let $a=1, b=-2,$ and $c=10.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(10)}}{2(1)}=\frac{2\pm\sqrt{4-40}}{2}=\frac{2\pm\sqrt{-36}}{2}=\frac{2\pm6i}{2}=1\pm3i$$

Solution set: $\{1\pm3i\}$

51. $-4x^2 = -12x + 11$

$0 = 4x^2 - 12x + 11$

Let $a = 4, b = -12,$ and $c = 11.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(11)}}{2(4)} = \frac{12 \pm \sqrt{144-176}}{8} = \frac{12 \pm \sqrt{-32}}{8} = \frac{12 \pm 4i\sqrt{2}}{8} = \frac{12}{8} \pm \frac{4\sqrt{2}}{8}i = \frac{3}{2} \pm \frac{\sqrt{2}}{2}i$$

Solution set: $\left\{\frac{3}{2} \pm \frac{\sqrt{2}}{2}i\right\}$

52. $-6x^2 = 3x + 2$

$0 = 6x^2 + 3x + 2$

Let $a = 6, b = 3,$ and $c = 2.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(6)(2)}}{2(6)} = \frac{-3 \pm \sqrt{9-48}}{12} = \frac{-3 \pm \sqrt{-39}}{12} = \frac{-3 \pm i\sqrt{39}}{12} = -\frac{3}{12} \pm \frac{\sqrt{39}}{12}i = -\frac{1}{4} \pm \frac{\sqrt{39}}{12}i$$

Solution set: $\left\{-\frac{1}{4} \pm \frac{\sqrt{39}}{12}i\right\}$

53. $\frac{1}{2}x^2 + \frac{1}{4}x - 3 = 0$

$4\left(\frac{1}{2}x^2 + \frac{1}{4}x - 3\right) = 4 \cdot 0$

$2x^2 + x - 12 = 0$

Let $a = 2, b = 1,$ and $c = -12.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(2)(-12)}}{2(2)} = \frac{-1 \pm \sqrt{1+96}}{4} = \frac{-1 \pm \sqrt{97}}{4}$$

Solution set: $\left\{\frac{-1 \pm \sqrt{97}}{4}\right\}$

54. $\frac{2}{3}x^2 + \frac{1}{4}x = 3$

$12\left(\frac{2}{3}x^2 + \frac{1}{4}x\right) = 12 \cdot 3$

$8x^2 + 3x = 36$

$8x^2 + 3x - 36 = 0$

Let $a = 8, b = 3,$ and $c = -36.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(8)(-36)}}{2(8)} = \frac{-3 \pm \sqrt{9+1152}}{16} = \frac{-3 \pm \sqrt{1161}}{16} = \frac{-3 \pm 3\sqrt{129}}{16}$$

Solution set: $\left\{\frac{-3 \pm 3\sqrt{129}}{16}\right\}$

55. $.2x^2 + .4x - .3 = 0$

$10\left(.2x^2 + .4x - .3\right) = 10 \cdot 0$

$2x^2 + 4x - 3 = 0$

Let $a = 2, b = 4,$ and $c = -3.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} = \frac{-4 \pm \sqrt{16+24}}{4} = \frac{-4 \pm \sqrt{40}}{4} = \frac{-4 \pm 2\sqrt{10}}{4} = \frac{-2 \pm \sqrt{10}}{2}$$

Solution set: $\left\{\frac{-2 \pm \sqrt{10}}{2}\right\}$

56. $.1x^2 - .1x = .3$

$10\left(.1x^2 - .1x\right) = 10 \cdot .3$

$x^2 - x = 3$

$x^2 - x - 3 = 0$

Let $a = 1, b = -1,$ and $c = -3.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

Solution set: $\left\{\frac{1 \pm \sqrt{13}}{2}\right\}$

57. $(4x-1)(x+2) = 4x$

$4x^2 + 7x - 2 = 4x \Rightarrow 4x^2 + 3x - 2 = 0$

Let $a = 4, b = 3,$ and $c = -2.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)} = \frac{-3 \pm \sqrt{9+32}}{8} = \frac{-3 \pm \sqrt{41}}{8}$$

Solution set: $\left\{\frac{-3 \pm \sqrt{41}}{8}\right\}$

58. $(3x+2)(x-1)=3x$

$3x^2-x-2=3x$

$3x^2-4x-2=0$

Let $a=3, b=-4,$ and $c=-2$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-2)}}{2(3)}=\frac{4\pm\sqrt{16+24}}{6}=\frac{4\pm\sqrt{40}}{6}=\frac{4\pm2\sqrt{10}}{6}=\frac{2\pm\sqrt{10}}{3}$$

Solution set: $\left\{\frac{2\pm\sqrt{10}}{3}\right\}$

59. $(x-9)(x-1)=-16$

$x^2-10x+9=-16$

$x^2-10x+25=0$

Let $a=1$, $b=-10$, and $c=25$.

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-(4)(1)(25)}}{2}=\frac{10\pm\sqrt{100-100}}{2}=\frac{10\pm0}{2}=5$$

Solution set: $\{5\}$

60. Answers will vary.

$-2x^2+3x-6=0\Rightarrow-1\left(2x^2-3x+6\right)=0\Rightarrow$

$2x^2-3x+6=0$

Therefore, the two equations have the same solution set.

61. $x^3-8=0$

$x^3-2^3=0$

$(x-2)\left(x^2+2x+4\right)=0$

$x-2=0\Rightarrow x=2$ or

$x^2+2x+4=0$

$a=1, b=2,$ and $c=4$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}=\frac{-2\pm\sqrt{4-16}}{2}=\frac{-2\pm\sqrt{-12}}{2}=\frac{-2\pm2i\sqrt{3}}{2}=-1\pm\sqrt{3}i$$

Solution set: $\left\{2,-1\pm\sqrt{3}i\right\}$

62. $x^3-27=0$

$x^3-3^3=0$

$(x-3)\left(x^2+3x+9\right)=0$

$x-3=0\Rightarrow x=3$ or

$x^2+3x+9=0$

$a=1, b=3,$ and $c=9$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-3\pm\sqrt{3^2-4(1)(9)}}{2(1)}=\frac{-3\pm\sqrt{9-36}}{2}=\frac{-3\pm\sqrt{-27}}{2}=\frac{-3\pm3i\sqrt{3}}{2}=-\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i$$

Solution set: $\left\{3,-\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i\right\}$

63. $x^3+27=0$

$x^3+3^3=0$

$(x+3)\left(x^2-3x+9\right)=0$

$x+3=0\Rightarrow x=-3$ or

$x^2-3x+9=0$

$a=1, b=-3,$ and $c=9$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}=\frac{3\pm\sqrt{9-36}}{2}=\frac{3\pm\sqrt{-27}}{2}=\frac{3\pm3i\sqrt{3}}{2}=\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i$$

Solution set: $\left\{-3,\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i\right\}$

64. $x^3+64=0$

$x^3+4^3=0$

$(x+4)\left(x^2-4x+16\right)=0$

$x+4=0\Rightarrow x=-4$

$x^2-4x+16=0$

$a=1, b=-4,$ and $c=16$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(16)}}{2(1)}=\frac{4\pm\sqrt{16-64}}{2}=\frac{4\pm\sqrt{-48}}{2}=\frac{4\pm4i\sqrt{3}}{2}=2\pm2i\sqrt{3}$$

Solution set: $\left\{-4,2\pm2i\sqrt{3}\right\}$

65. $s = \frac{1}{2}gt^2$

$2s = 2\left[\frac{1}{2}gt^2\right] \Rightarrow 2s = gt^2 \Rightarrow \frac{2s}{g} = \frac{gt^2}{g} \Rightarrow$

$t^2 = \frac{2s}{g} \Rightarrow t = \pm\sqrt{\frac{2s}{g}} = \frac{\pm\sqrt{2s}}{\sqrt{g}} \cdot \frac{\sqrt{g}}{\sqrt{g}} = \frac{\pm\sqrt{2sg}}{g}$

66. $A = \pi r^2$

$\frac{A}{\pi} = \frac{\pi r^2}{\pi} \Rightarrow r^2 = \frac{A}{\pi} \Rightarrow r = \pm\sqrt{\frac{A}{\pi}} \Rightarrow$

$r = \frac{\pm\sqrt{A}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} = \frac{\pm\sqrt{A\pi}}{\pi}$

67. $F = \frac{kMv^2}{r}$

$rF = r\left[\frac{kMv^2}{r}\right] \Rightarrow Fr = kMv^2 \Rightarrow$

$\frac{Fr}{kM} = \frac{kMv^2}{kM} \Rightarrow v^2 = \frac{Fr}{kM} \Rightarrow v = \pm\sqrt{\frac{Fr}{kM}} \Rightarrow$

$v = \frac{\pm\sqrt{Fr}}{\sqrt{kM}} \cdot \frac{\sqrt{kM}}{\sqrt{kM}} = \frac{\pm\sqrt{FrkM}}{kM}$

68. $s = s_0 + gt^2 + k$

$s - s_0 - k = gt^2$

$\frac{s - s_0 - k}{g} = \frac{gt^2}{g}$

$t^2 = \frac{s - s_0 - k}{g}$

$t = \pm\sqrt{\frac{s - s_0 - k}{g}} = \frac{\pm\sqrt{s - s_0 - k}}{\sqrt{g}} \cdot \frac{\sqrt{g}}{\sqrt{g}}$

$t = \frac{\pm\sqrt{(s - s_0 - k)g}}{g}$

69. $h = -16t^2 + v_0t + s_0$

$16t^2 - v_0t + h - s_0 = 0$

$16t^2 - v_0t + (h - s_0) = 0$ $\quad a = 16, b = -v_0, c = h - s_0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-v_0) \pm \sqrt{(-v_0)^2 - 4(16)(h - s_0)}}{2(16)}$

$= \frac{v_0 \pm \sqrt{v_0^2 - 64(h - s_0)}}{32}$

$= \frac{v_0 \pm \sqrt{v_0^2 - 64h + 64s_0}}{32}$

70. $S = 2\pi rh + 2\pi r^2$

$0 = 2\pi r^2 + 2\pi rh - S$

$0 = (2\pi)r^2 + (2\pi h)r - S$ $\quad a = 2\pi, b = 2\pi h, c = -S$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2\pi h \pm \sqrt{(2\pi h)^2 - 4(2\pi)(-S)}}{2(2\pi)}$

$= \frac{-2\pi h \pm \sqrt{4\pi^2h^2 + 8\pi S}}{4\pi}$

$= \frac{-2\pi h \pm 2\sqrt{\pi^2h^2 + 2\pi S}}{4\pi}$

$= \frac{-\pi h \pm \sqrt{\pi^2h^2 + 2\pi S}}{2\pi}$

71. $4x^2 - 2xy + 3y^2 = 2$

$4x^2 - 2xy + 3y^2 - 2 = 0$

(a) Solve for *x* in terms of *y*.

$4x^2 - (2y)x + (3y^2 - 2) = 0$

$a = 4$, $b = -2y$, and $c = 3y^2 - 2$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4(4)(3y^2 - 2)}}{2(4)}$

$= \frac{2y \pm \sqrt{4y^2 - 16(3y^2 - 2)}}{8}$

$= \frac{2y \pm \sqrt{4y^2 - 48y^2 + 32}}{8}$

$= \frac{2y \pm \sqrt{32 - 44y^2}}{8} = \frac{2y \pm \sqrt{4(8 - 11y^2)}}{8}$

$= \frac{2y \pm 2\sqrt{8 - 11y^2}}{8} = \frac{y \pm \sqrt{8 - 11y^2}}{4}$

(b) Solve for y in terms of x.

$3y^2 - (2x)y + (4x^2 - 2) = 0$

$a = 3$, $b = -2x$, and $c = 4x^2 - 2$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(3)(4x^2 - 2)}}{2(3)} = \frac{2x \pm \sqrt{4x^2 - 12(4x^2 - 2)}}{6} = \frac{2x \pm \sqrt{4x^2 - 48x^2 + 24}}{6}$$

$$= \frac{2x \pm \sqrt{24 - 44x^2}}{6} = \frac{2x \pm \sqrt{4(6 - 11y^2)}}{6} = \frac{2x \pm 2\sqrt{6 - 11y^2}}{6} = \frac{x \pm \sqrt{6 - 11y^2}}{3}$$

72. $3y^2 + 4xy - 9x^2 = -1$

$-9x^2 + 4xy + 3y^2 + 1 = 0$

(a) Solve for x in terms of y.

$-9x^2 + (4y)x + (3y^2 + 1) = 0$

$a = -9$, $b = 4y$, and $c = 3y^2 + 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4y \pm \sqrt{(4y)^2 - 4(-9)(3y^2 + 1)}}{2(-9)} = \frac{-4y \pm \sqrt{16y^2 + 36(3y^2 + 1)}}{-18} = \frac{-4y \pm \sqrt{16y^2 + 108y^2 + 36}}{-18}$$

$$= \frac{-4y \pm \sqrt{124y^2 + 36}}{-18} = \frac{-4y \pm \sqrt{4(31y^2 + 9)}}{-18} = \frac{-4y \pm 2\sqrt{31y^2 + 9}}{-18} = \frac{-2y \pm \sqrt{31y^2 + 9}}{-9} = \frac{2y \pm \sqrt{31y^2 + 9}}{9}$$

(b) Solve for y in terms of x.

$3y^2 + (4x)y + (1 - 9x^2) = 0$

$a = 3$, $b = 4x$, and $c = 1 - 9x^2$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4x \pm \sqrt{(4x)^2 - 4(3)(1 - 9x^2)}}{2(3)} = \frac{-4x \pm \sqrt{16x^2 - 12(1 - 9x^2)}}{6} = \frac{-4x \pm \sqrt{16x^2 - 12 + 108x^2}}{6}$$

$$= \frac{-4x \pm \sqrt{124x^2 - 12}}{6} = \frac{-4x \pm \sqrt{4(31x^2 - 3)}}{6} = \frac{-4x \pm 2\sqrt{31x^2 - 3}}{6} = \frac{-2x \pm \sqrt{31x^2 - 3}}{3}$$

73. $x^2 - 8x + 16 = 0$

$a = 1$, $b = -8$, and $c = 16$

$b^2 - 4ac = (-8)^2 - 4(1)(16) = 64 - 64 = 0$

one rational solution (a double solution)

74. $x^2 + 4x + 4 = 0$

$a = 1$, $b = 4$, and $c = 4$

$b^2 - 4ac = 4^2 - 4(1)(4) = 16 - 16 = 0$

one rational solution (a double solution)

75. $3x^2 + 5x + 2 = 0$

$a = 3$, $b = 5$, and $c = 2$

$b^2 - 4ac = 5^2 - 4(3)(2) = 25 - 24 = 1 = 1^2$

two distinct rational solutions

76. $8x^2 = -14x - 3$

$8x^2 + 14x + 3 = 0$

$a = 8$, $b = 14$, and $c = 3$

$b^2 - 4ac = 14^2 - 4(8)(3)$

$= 196 - 96 = 100 = 10^2$

two distinct rational solutions

77. $4x^2 = -6x + 3$
$4x^2 + 6x - 3 = 0$
$a = 4, b = 6,$ and $c = -3$
$b^2 - 4ac = 6^2 - 4(4)(-3) = 36 + 48 = 84$
two distinct irrational solutions

78. $2x^2 + 4x + 1 = 0$
$a = 2, b = 4,$ and $c = 1$
$b^2 - 4ac = 4^2 - 4(2)(1) = 16 - 8 = 8$
two distinct irrational solutions

79. $9x^2 + 11x + 4 = 0$
$a = 9, b = 11,$ and $c = 4$
$b^2 - 4ac = 11^2 - 4(9)(4) = 121 - 144 = -23$
two distinct nonreal complex solutions

80. $3x^2 = 4x - 5$
$3x^2 - 4x + 5 = 0$
$a = 3, b = -4,$ and $c = 5$
$b^2 - 4ac = (-4)^2 - 4(3)(5) = 16 - 60 = -44$
two distinct nonreal complex solutions

81. $8x^2 - 72 = 0$
$a = 8, b = 0,$ and $c = -72$
$b^2 - 4ac = 0^2 - 4(8)(-72) = 2304 = 48^2$
two distinct rational solutions

82. Answers will vary.
$\sqrt{2}x^2 + 5x - 3\sqrt{2} = 0$
$a = \sqrt{2}, b = 5,$ and $c = -3\sqrt{2}$
$b^2 - 4ac = 5^2 - 4(\sqrt{2})(-3\sqrt{2})$
$= 25 + 12 \cdot 2 = 25 + 24 = 49$
This does not contradict the discussion in this section because a condition that is placed on the quadratic equation is that it has integer coefficients in order to investigate the discriminant.

83. It is not possible for the solution set of a quadratic equation with integer coefficients to consist of a single irrational number. Additional responses will vary.

84. It is not possible for the solution set of a quadratic equation with real coefficients to consist of one real number and one nonreal complex number. Answers will vary.

In exercises 85−88, there are other possible answers.

85. $x = 4$ or $x = 5$
$x - 4 = 0$ or $x - 5 = 0$
$(x-4)(x-5) = 0$
$x^2 - 5x - 4x + 20 = 0$
$x^2 - 9x + 20 = 0$
$a = 1, b = -9,$ and $c = 20$

86. $x = -3$ or $x = 2$
$x + 3 = 0$ or $x - 2 = 0$
$(x+3)(x-2) = 0$
$x^2 - 2x + 3x - 6 = 0$
$x^2 + x - 6 = 0$
$a = 1, b = 1,$ and $c = -6$

87. $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$
$x - (1 + \sqrt{2}) = 0$ or $x - (1 - \sqrt{2}) = 0$
$[x - (1+\sqrt{2})][x - (1-\sqrt{2})] = 0$
$x^2 - x(1-\sqrt{2}) - x(1+\sqrt{2})$
$+(1+\sqrt{2})(1-\sqrt{2}) = 0$
$x^2 - x + x\sqrt{2} - x - x\sqrt{2} + \left[1^2 - (\sqrt{2})^2\right] = 0$
$x^2 - 2x + (1-2) = 0$
$x^2 - 2x - 1 = 0$
$a = 1, b = -2,$ and $c = -1$

88. $x = i$ or $x = -i$
$x - i = 0$ or $x + i = 0$
$(x-i)(x+i) = 0$
$x^2 - i^2 = 0$
$x^2 - (-1) = 0 \Rightarrow x^2 + 1 = 0$
$a = 1, b = 0,$ and $c = 1$

Chapter 1 Quiz
(Sections 1.1–1.4)

1. $3(x-5) + 2 = 1 - (4 + 2x)$
$3x - 15 + 2 = 1 - 4 - 2x$
$3x - 13 = -3 - 2x$
$5x - 13 = -3$
$5x = 10 \Rightarrow x = 2$
Solution set $\{2\}$

2. **(a)** $4x - 5 = -2(3 - 2x) + 3$
$4x - 5 = -6 + 4x + 3$
$4x - 5 = 4x - 3$
$-5 = -3$
contradiction; solution set: $\varnothing$

(b) $5x-9=5(-2+x)+1$
$5x-9=-10+5x+1$
$5x-9=5x-9$
identity; solution set: {all real numbers} or $(-\infty,\infty)$

(c) $5x-4=3(6-x)$
$5x-4=18-3x$
$8x-4=18$
$8x=22 \Rightarrow x=\frac{22}{8}=\frac{11}{4}$
conditional equation; solution set: $\left\{\frac{11}{4}\right\}$

3. $ay+2x=y+5x$
$ay-3x=y$
$-3x=y-ay=y(1-a)$
$3x=y(a-1)$
$\frac{3x}{a-1}=y$

4. Let x = the amount deposited at 2.5% interest. Then $2x$ = the amount depositied at 3.0% interest. The interest earned on x dollars at 2.5% is $0.025x$, and the interest earned on $2x$ at 3.0% is $(2x)(0.03)=0.06x$. The total earned is $850, so we have
$.025x+.06x=850$
$.085x=850 \Rightarrow x=10{,}000$
$10,000 was invested at 2.5%, and $20,000 was invested at 3.0%.

5. Substitute 1999 for x in the equation:
$y=.12(1999)-234.42=5.46$
So, the model predicts that the minimum hourly wage for 1999 was $5.46. The difference between the actual minimum wage and the predicted wage is
$5.46 – $5.15 = $0.31.

6. $$\frac{-4+\sqrt{-24}}{8}=\frac{-4+\sqrt{-4\cdot 6}}{8}=\frac{-4}{8}+\frac{2i\sqrt{6}}{8}=-\frac{1}{2}+\frac{\sqrt{6}}{4}i$$

7. $$\frac{7-2i}{2+4i}=\frac{7-2i}{2+4i}\cdot\frac{2-4i}{2-4i}=\frac{14-28i-4i+(-8)}{4-(-16)}=\frac{6-32i}{20}=\frac{6}{20}-\frac{32}{20}i=\frac{3}{10}-\frac{8}{5}i$$

8. $3x^2-x=-1 \Rightarrow 3x^2-x+1=0$
Use the quadratic formula. $a=3$, $b=-1$, $c=1$
$$\frac{-(-1)\pm\sqrt{(-1)^2-4(3)(1)}}{2(3)}=\frac{1\pm\sqrt{-11}}{6}=\frac{1}{6}\pm\frac{\sqrt{11}}{6}i$$
Solution set: $\left\{\frac{1}{6}\pm\frac{\sqrt{11}}{6}i\right\}$

9. $x^2-29=0 \Rightarrow x^2=29 \Rightarrow x=\pm\sqrt{29}$
Solution set: $\{\pm 29\}$

10. $A=\pi r^2$
$\frac{A}{\pi}=r^2$
$\pm\sqrt{\frac{A}{\pi}}=\pm\frac{\sqrt{A\pi}}{\pi}=r$
Note that this is the formula for the area of a circle. If it is used in that context, then r must be greater than or equal to zero.

Section 1.5: Applications and Modeling with Quadratic Equations

Connections (page 125)

1. **(a)** $d=16t^2=16(5)^2=400$
It will fall 400 feet in 5 seconds.

(b) $d=16t^2=16(10)^2=1600$
It will fall 1600 feet in 10 seconds.
No, the second answer is $2^2=4$ times the first because the number of seconds is squared in the formula.

2. Both formulas involve the number 16 times the square of time. However, in the formula for the distance an object falls, 16 is positive, while in the formula for a projected object, it is preceded by a negative sign. Also in the formula for a projected object, the initial velocity and height affect the distance.

Exercises

1. The length of the parking area is $2x+200$, while the width is x, so the area is $(2x+200)x$. Set the area equal to 40,000.
$(2x+200)x=40{,}000$, so choice A is the correct choice.

2. The diagonal of this rectangle is the hypotenuse of a right triangle with legs r feet and s feet. By the Pythagorean theorem, the length of the diagonal is $\sqrt{r^2+s^2}$, so the correct choice is C.

3. Use the Pythagorean theorem with $a = x$, $b = 2x - 2$, and $c = x + 4$.

$$x^2 + (2x-2)^2 = (x+4)^2$$

The correct choice is D.

4. The length of the picture is $34 - 2x$, while the width is $21 - 2x$, giving an area of $(34 - 2x)(21 - 2x)$. Use the formula for the area of a rectangle, $A = lw$, and set the area equal to 600. $(34-2x)(21-2x) = 600$

The correct choice is B.

5. Let x = the first integer. Then $x + 1$ = the next consecutive integer.

$$x(x+1) = 56 \Rightarrow x^2 + x = 56$$
$$x^2 + x - 56 = 0 \Rightarrow (x+8)(x-7) = 0$$
$$x+8 = 0 \Rightarrow x = -8 \text{ or } x - 7 = 0 \Rightarrow x = 7$$

If $x = -8$, then $x + 1 = -7$. If $x = 7$, then $x + 1 = 8$. So the two integers are -8 and -7, or 7 and 8.

6. Let x = the first integer. Then $x + 1$ = the next consecutive integer.

$$x(x+1) = 110 \Rightarrow x^2 + x = 110 \Rightarrow$$
$$x^2 + x - 110 = 0 \Rightarrow (x+11)(x-10) = 0$$
$$x+11 = 0 \Rightarrow x = -11 \text{ or}$$
$$x-10 = 0 \Rightarrow x = 10$$

If $x = -11$, then $x + 1 = -10$. If $x = 10$, then $x + 1 = 11$. So the two integers are -11 and -10, or 10 and 11.

7. Let x = the first even integer. Then $x + 2$ = the next consecutive even integer.

$$x(x+2) = 168 \Rightarrow x^2 + 2x = 168$$
$$x^2 + 2x - 168 = 0 \Rightarrow (x+14)(x-12) = 0$$
$$x+14 = 0 \Rightarrow x = -14 \text{ or}$$
$$x-12 = 0 \Rightarrow x = 12$$

If $x = -14$, then $x + 2 = -12$. If $x = 12$, then $x + 2 = 14$. so, the two even integers are -14 and -12, or 12 and 14.

8. Let x = the first even integer. Then $x + 2$ = the next consecutive even integer.

$$x(x+2) = 224 \Rightarrow x^2 + 2x = 224 \Rightarrow$$
$$x^2 + 2x - 224 = 0 \Rightarrow (x+16)(x-14) = 0$$
$$x+16 = 0 \Rightarrow x = -16 \text{ or}$$
$$x-14 = 0 \Rightarrow x = 14$$

If $x = -16$, then $x + 2 = -14$. If $x = 14$, then $x + 2 = 16$. So, the two even integers are -16 and -14, or 14 and 16.

9. Let x = the first odd integer. Then $x + 2$ = the next consecutive odd integer.

$$x(x+2) = 63 \Rightarrow x^2 + 2x = 63 \Rightarrow$$
$$x^2 + 2x - 63 = 0 \Rightarrow (x+9)(x-7) = 0$$
$$x+9 = 0 \Rightarrow x = -9 \text{ or}$$
$$x-7 = 0 \Rightarrow x = 7$$

If $x = -9$, then $x + 2 = -7$. If $x = 7$, then $x + 2 = 9$. so, the two odd integers are -9 and -7, or 7 and 9.

10. Let x = the first odd integer. Then $x + 2$ = the next consecutive odd integer.

$$x(x+2) = 143 \Rightarrow x^2 + 2x = 143 \Rightarrow$$
$$x^2 + 2x - 143 = 0 \Rightarrow (x+13)(x-11) = 0$$
$$x+13 = 0 \Rightarrow x = -13 \text{ or}$$
$$x-11 = 0 \Rightarrow x = 11$$

If $x = -13$, then $x + 2 = -11$. If $x = 11$, then $x + 2 = 13$. so, the two odd integers are -13 and -11, or 11 and 13.

11. Let x = the first integer. Then $x + 1$ = the next consecutive integer.

$$x^2 + (x+1)^2 = 61$$
$$x^2 + x^2 + 2x + 1 = 61 \Rightarrow 2x^2 + 2x + 1 = 61$$
$$2x^2 + 2x - 60 = 0 \Rightarrow 2\left(x^2 + x - 30\right) = 0$$
$$x^2 + x - 30 = 0 \Rightarrow (x+6)(x-5) = 0$$
$$x+6 = 0 \Rightarrow x = -6 \text{ or}$$
$$x-5 = 0 \Rightarrow x = 5$$

If $x = -6$, then $x + 1 = -5$. If $x = 5$, then $x + 1 = 6$. So the two integers are -6 and -5, or 5 and 6.

12. Let x = the first even integer. Then $x + 2$ = the next consecutive even integer.

$$x^2 + (x+2)^2 = 52$$
$$x^2 + x^2 + 4x + 4 = 52 \Rightarrow 2x^2 + 4x + 4 = 52 \Rightarrow$$
$$2x^2 + 4x - 48 = 0 \Rightarrow 2\left(x^2 + 2x - 24\right) = 0 \Rightarrow$$
$$x^2 + 2x - 24 = 0 \Rightarrow (x+6)(x-4) = 0$$
$$x+6 = 0 \Rightarrow x = -6 \text{ or}$$
$$x-4 = 0 \Rightarrow x = 4$$

If $x = -6$, then $x + 2 = -4$. If $x = 4$, then $x + 2 = 6$. So the two even integers are -6 and -4, or 4 and 6.

13. Let x = the first odd integer. Then $x + 2$ = the next consecutive odd integer.

$$x^2 + (x+2)^2 = 202$$
$$x^2 + x^2 + 4x + 4 = 202$$
$$2x^2 + 4x + 4 = 202 \Rightarrow 2x^2 + 4x - 198 = 0$$
$$2(x^2 + 2x - 99) = 0 \Rightarrow x^2 + 2x - 99 = 0$$
$$(x+11)(x-9) = 0$$
$$x + 11 = 0 \Rightarrow x = -11 \text{ or}$$
$$x - 9 = 0 \Rightarrow x = 9$$

If $x = -11$, then $x + 2 = -9$. If $x = 9$, then $x + 2 = 11$. So the two integers are -11 and -9, or 9 and 11.

14. Let x = the first odd integer. Then $x + 2$ = the next consecutive odd integer.

$$(x+2)^2 - x^2 = 32$$
$$x^2 + 4x + 4 - x^2 = 32 \Rightarrow 4x + 4 = 32$$
$$4x = 28 \Rightarrow x = 7$$

If $x = 7$, then $x + 2 = 9$. So the two integers are 7 and 9.

15. Let x = the length of one leg, $x + 2$ = the length of the other leg, and $x + 4$ = the length of the hypotenuse. (Remember that the hypotenuse is the longest side in a right triangle.) The Pythagorean theorem gives

$$x^2 + (x+2)^2 = (x+4)^2$$
$$x^2 + x^2 + 4x + 4 = x^2 + 8x + 16$$
$$x^2 - 4x - 12 = 0 \Rightarrow (x-6)(x+2) = 0$$
$$x - 6 = 0 \Rightarrow x = 6 \text{ or}$$
$$x + 2 = 0 \Rightarrow x = -2$$

Length cannot be negative, so reject that solution. If $x = 6$, then $x + 2 = 8$ and $x + 4 = 10$. The sides of the right triangle are 6, 8, and 10.

16. Let x = one of the numbers. Then $x + 4$ is the other number.

$$x^2 + (x+4)^2 = 208$$
$$x^2 + x^2 + 8x + 16 = 208 \Rightarrow 2x^2 + 8x - 192 = 0$$
$$x^2 + 4x - 96 = 0 \Rightarrow (x-8)(x+12) = 0$$
$$x - 8 = 0 \Rightarrow x = 8 \text{ or}$$
$$x + 12 = 0 \Rightarrow x = -12$$

The problem asks for a positive number, so reject $x = -12$. If $x = 8$, then $x + 4 = 12$. The two numbers are 8 and 12.

17. Let x = the length of the side of the smaller square. Then $x + 3$ = the length of the side of the larger square.

$$(x+3)^2 + x^2 = 149$$
$$x^2 + 6x + 9 + x^2 = 149 \Rightarrow 2x^2 + 6x - 140 = 0$$
$$x^2 + 3x - 70 = 0 \Rightarrow (x-7)(x+10) = 0$$
$$x - 7 = 0 \Rightarrow x = 7 \text{ or}$$
$$x + 10 = 0 \Rightarrow x = -10$$

Length cannot be negative, so reject that solution. If $x = 7$, then $x + 3 = 10$. The length of the side of smaller square is 7 in., and the length of the side of the larger square is 10 in.

18. Let x = the length of the side of the smaller square. Then $x + 5$ = the length of the side of the larger square.

$$(x+5)^2 - x^2 = 95$$
$$x^2 + 10x + 25 - x^2 = 95$$
$$10x + 25 = 95 \Rightarrow 10x = 70 \Rightarrow x = 7$$

If $x = 7$, then $x + 5 = 12$. The length of the side of the smaller square is 7 in., and the length of the side of the larger square is 12 in.

19. Use the figure and equation A from Exercise 1.

$$x(2x + 200) = 40{,}000$$
$$2x^2 + 200x = 40{,}000$$
$$2x^2 + 200x - 40{,}000 = 0$$
$$x^2 + 100x - 20{,}000 = 0$$
$$(x-100)(x+200) = 0$$
$$x = 100 \text{ or } x = -200$$

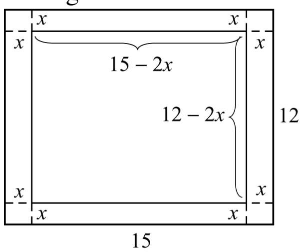
The negative solution is not meaningful. If $x = 100$, then $2x + 200 = 400$. The dimensions of the lot are 100 yd by 400 yd.

20. Use the formula for the area of a rectangle.

$$A = lw$$
$$5000 = (150 - x)x$$
$$5000 = 150x - x^2$$
$$x^2 - 150x + 5000 = 0 \Rightarrow (x-100)(x-50) = 0$$
$$x - 100 = 0 \Rightarrow x = 100 \text{ or}$$
$$x - 50 = 0 \Rightarrow x = 50$$

If $x = 100$, then $150 - x = 50$. If $x = 50$, then $150 - x = 100$. The dimensions of the garden are 50 m by 100 m.

21. Let x = the width of the strip of floor around the rug.

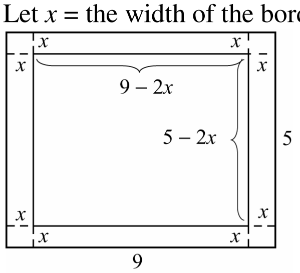


The dimensions of the carpet are $15 - 2x$ by $12 - 2x$. Since $A = lw$, the equation for the carpet area is $(15 - 2x)(12 - 2x) = 108$. Put this equation in standard form and solve by factoring.

$$(15-2x)(12-2x)=108$$
$$180-30x-24x+4x^2=108$$
$$180-54x+4x^2=108$$
$$4x^2-54x+72=0$$
$$2x^2-27x+36=0$$
$$(2x-3)(x-12)=0$$
$$2x-3=0 \Rightarrow x=\frac{3}{2}$$
$$x-12=0 \Rightarrow x=12$$

The solutions of the quadratic equation are $\frac{3}{2}$ and 12. We eliminate 12 as meaningless in this problem. If $x = \frac{3}{2}$, then $15 - 2x = 12$ and $12 - 2x = 9$. The dimensions of the carpet are 9 ft by 12 ft.

22. Let x = the width of the border.

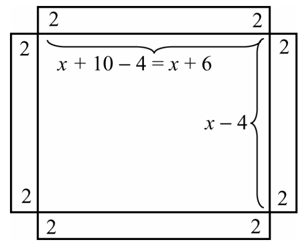


The dimensions of the center plot are $9 - 2x$ by $5 - 2x$. The total area is $5 \cdot 9 = 45$ sq ft. The border area is 24 sq ft, so the area of the center plot is $45 - 24 = 21$ sq ft. Apply the formula for the area of a rectangle to the center plot.

$$A = lw$$
$$(9-2x)(5-2x)=21$$
$$45-18x-10x+4x^2=21$$
$$45-28x+4x^2=21$$
$$4x^2-28x+24=0 \Rightarrow x^2-7x+6=0$$
$$(x-6)(x-1)=0 \Rightarrow x-6=0 \Rightarrow x=6 \text{ or}$$
$$x-1=0 \Rightarrow x=1$$

The solutions are 1 and 6. We eliminate 6 as meaningless in this problem. The border can be 1 ft wide.

23. Let x = the width of the metal. The dimensions of the base of the box are $x-4$ by $x+6$.

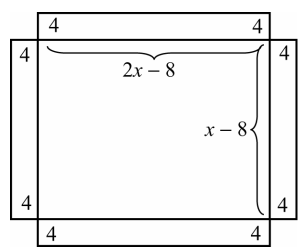


Since the formula for the volume of a box is $V = lwh$, we have

$$(x+6)(x-4)(2)=832$$
$$(x+6)(x-4)=416$$
$$x^2-4x+6x-24=416$$
$$x^2+2x-24=416$$
$$x^2+2x-440=0 \Rightarrow (x+22)(x-20)=0$$
$$x+22=0 \Rightarrow x=-22 \text{ or}$$
$$x-20=0 \Rightarrow x=20$$

The negative solution is not meaningful. If $x = 20$, then $x + 10 = 30$. The dimensions of the sheet of metal are 20 in by 30 in.

24. Let x = the width of the metal. The dimensions of the base of the box are $x-8$ by $2x-8$.



Since the formula for the volume of a box is $V = lwh$, we have

$$(2x-8)(x-8)(4)=1536$$
$$(2x-8)(x-8)=384$$
$$2x^2-16x-8x+64=384$$
$$2x^2-24x+64=384$$
$$2x^2-24x-320=0$$
$$x^2-12x-160=0 \Rightarrow (x-20)(x+8)=0$$
$$x-20=0 \Rightarrow x=20 \text{ or}$$
$$x+8=0 \Rightarrow x=-8$$

The negative solution is not meaningful. If $x = 20$, then $2x = 40$. The dimensions of the sheet of metal are 20 in by 40 in.

25. Let h = height and r = radius.
Area of side = $2\pi rh$ and Area of circle = πr^2
Surface area = area of side + area of top + area of bottom
Surface area = $2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2$
$371 = 2\pi r(12) + 2\pi r^2$
$371 = 24\pi r + 2\pi r^2$
$0 = 2\pi r^2 + 24\pi r - 371$
$a = 2\pi$, $b = 24\pi$, and $c = -371$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} r$$
$$= \frac{-24\pi \pm \sqrt{(24\pi)^2 - 4(2\pi)(-371)}}{2(2\pi)}$$
$$= \frac{-24\pi \pm \sqrt{576\pi^2 + 2968\pi}}{4\pi}$$
$r \approx -15.75$ *or* $r \approx 3.75$
The negative solution is not meaningful. The radius of the circular top is approximately 3.75 cm.

26. Let x = length, then $x - 3.1875$ = width, and 2.3125 = depth. $V = lwh$
$182.742 = x(x - 3.1875)(2.3125)$
$182.742 = 2.3125x^2 - 7.3711x$
$0 = 2.3125x^2 - 7.3711x - 182.742$
$a = 2.3125$, $b = -7.3711$, and $c = -182.742$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-7.3711) \pm \sqrt{(-7.3711)^2 - 4(2.3125)(-182.742)}}{2(2.3125)}$$
$x \approx -7.438$ or $x \approx 10.625$
A box cannot have a negative length, so reject –7.438 as a solution. The length is approximately 10.625 in and the width is approximately 10.625 – 3.1875 = 7.438 in.

27. Let h = height and r = radius.
Surface area = $2\pi rh + 2\pi r^2$
$8\pi = 2\pi r(3) + 2\pi r^2$
$8\pi = 6\pi r + 2\pi r^2$
$0 = 2\pi r^2 + 6\pi r - 8\pi$
$0 = 2\pi\left(r^2 + 3r - 4\right) \Rightarrow 0 = (r + 4)(r - 1)$
$r + 4 = 0 \Rightarrow r = -4$ or $r - 1 = 0 \Rightarrow r = 1$
The r represents the radius of a cylinder, so -4 is not reasonable. The radius of the circular top is approximately 1 ft.

28. Let h = height and r = radius. Volume = $\pi r^2 h$
$\pi r = \pi r^2(3) \Rightarrow r = 3r^2 \Rightarrow 0 = 3r^2 - r \Rightarrow$
$0 = r(3r - 1) \Rightarrow r = 0$ or $3r - 1 = 0 \Rightarrow r = \frac{1}{3}$
A circle must have a radius greater than 0. The radius of the circular top is $\frac{1}{3}$ ft or 4 in.

29. Let x = length of side of square. Area = x^2 and perimeter = $4x$
$x^2 = 4x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x - 4) = 0 \Rightarrow$
$x = 0$ or $x = 4$
We reject 0 since x must be greater than 0. The side of the square measures 4 units.

30. Let x = width of rectangle.
Then $2x$ = length of rectangle.
Area = lw and Perimeter = $2l + 2w$
$(2x)(x) = 2\left[2(2x) + 2x\right]$
$2x^2 = 2(4x + 2x) \Rightarrow 2x^2 = 2(6x)$
$2x^2 = 12x \Rightarrow 2x^2 - 12x = 0$
$2x(x - 6) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$ or
$x - 6 = 0 \Rightarrow x = 6$
We reject 0 since x must be greater than 0. The width of the rectangle measures 6 units. The length of the rectangle measures 12 units.

31. Let h = the height of the dock.
Then $2h + 3$ = the length of the rope from the boat to the top of the dock.
Apply the Pythagorean theorem to the triangle shown in the text.
$h^2 + 12^2 = (2h + 3)^2$
$h^2 + 144 = (2h)^2 + 2(6h) + 3^2$
$h^2 + 144 = 4h^2 + 12h + 9$
$0 = 3h^2 + 12h - 135$
$0 = h^2 + 4h - 45 \Rightarrow 0 = (h + 9)(h - 5)$
$h + 9 = 0 \Rightarrow h = -9$ or $h - 5 = 0 \Rightarrow h = 5$
The negative solution is not meaningful. The height of the dock is 5 ft.

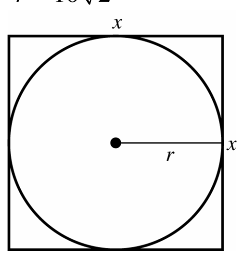
32. Let x = the horizontal distance
Apply the Pythagorean theorem to the right triangle shown in the text.

$$a^2 + b^2 = c^2$$
$$x^2 + (x+10)^2 = 50^2$$
$$x^2 + x^2 + 2(10x) + 10^2 = 2500$$
$$x^2 + x^2 + 20x + 100 = 2500$$
$$2x^2 + 20x - 2400 = 0$$
$$x^2 + 10x - 1200 = 0$$
$$(x+40)(x-30) = 0$$
$$x + 40 = 0 \Rightarrow x = -40 \text{ or}$$
$$x - 30 = 0 \Rightarrow x = 30$$

The negative solution is not meaningful. The kite's horizontal distance is 30 ft and the vertical distance is 40 ft.

33. Let r = radius of circle and x = length of side of square. The radius is $\frac{1}{2}$ the length of the side of the square. Area = x^2

$800 = x^2 \Rightarrow x = \sqrt{800} = 20\sqrt{2} \Rightarrow$
$r = 10\sqrt{2}$



The radius is $10\sqrt{2}$ feet.

34. Let x = length of short leg.
Then $2x$ = length of long leg.
Apply the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$26^2 = x^2 + (2x)^2$$
$$676 = x^2 + 4x^2$$
$$676 = 5x^2$$
$$135.2 = x^2$$
$$\pm\sqrt{135.2} = x$$

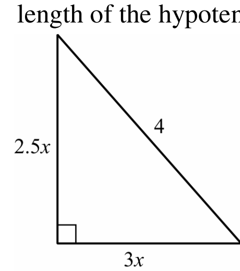
The negative solution is not meaningful. The short leg should be $\sqrt{135.2} \approx 11.6$ in. and the long leg should be $2\sqrt{135.2} \approx 23.3$ in.

35. Let x = length of ladder
Distance from building to ladder = 8 + 2 = 10.
Distance from ground to window = 13
Apply the Pythagorean theorem.

$$a^2 + b^2 = c^2$$
$$10^2 + 13^2 = x^2 \Rightarrow 100 + 169 = x^2 \Rightarrow$$
$$269 = x^2 \Rightarrow \pm\sqrt{269} = x$$

$x \approx -16.4$ or $x \approx 16.4$
The negative solution is not meaningful. The worker will need a 16.4-ft ladder.

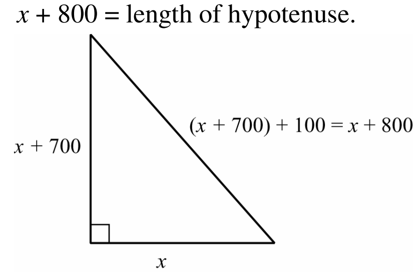
36. Let x = the number of hours they can talk to each other on the walkie-talkies.
Use $d = rt$ to determine how far each boy walks in x hours. Then $2.5x$ = the number of miles Tanner walks north and $3x$ = the number of miles Sheldon walks east. This forms a right triangle with legs of length $2.5x$ and $3x$, and length of the hypotenuse is the distance between the boys. We want to find x when the length of the hypotenuse is 4 mi.



$$a^2 + b^2 = c^2$$
$$(2.5x)^2 + (3x)^2 = 4^2 \Rightarrow 6.25x^2 + 9x^2 = 16 \Rightarrow$$
$$15.25x^2 = 16 \Rightarrow x^2 \approx 1.049 \Rightarrow x \approx \pm 1.02$$

The negative solution is not meaningful.
$1.02 \text{ hr} = 1.02\,(60 \text{ min}) \approx 61 \text{ min}$
They will be able to talk for about 61 min.

37. Let x = length of short leg, x + 700 = length of long leg, and x + 700 + 100 or x + 800 = length of hypotenuse.



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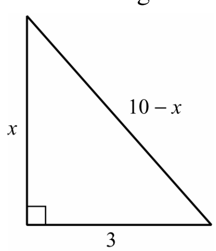
Apply the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$(x+800)^2 = x^2 + (x+700)^2$$
$$x^2 + 1600x + 640,000 = x^2 + x^2 + 1400x + 490,000$$
$$0 = x^2 - 200x - 150,000$$
$$0 = (x+300)(x-500)$$
$$x + 300 = 0 \Rightarrow x = -300 \text{ or}$$
$$x - 500 = 0 \Rightarrow x = 500$$

The negative solution is not meaningful.
500 = length of short leg
500 + 700 = 1200 = length of long leg
1200 + 100 = 1300 = length of hypotenuse
500 + 1200 + 1300 = 3000 = length of walkway. The total length is 3000 yd.

38. Let x = height of the break
$10 - x$ = length of hypotenuse



Apply the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$(10 - x)^2 = x^2 + 3^2$$
$$100 - 20x + x^2 = x^2 + 9$$
$$100 - 20x = 9 \Rightarrow -20x = -91 \Rightarrow x = 4.55$$

The height of the break is 4.55 ft.

39. (a)
$$s = -16t^2 + v_0 t$$
$$s = -16t^2 + 96t$$
$$80 = -16t^2 + 96t$$
$$16t^2 - 96t + 80 = 0$$

$a = 16$, $b = -96$ and $c = 80$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-96) \pm \sqrt{(-96)^2 - 4(16)(80)}}{2(16)} = \frac{96 \pm \sqrt{9216 - 5120}}{32} = \frac{96 \pm \sqrt{4096}}{32} = \frac{96 \pm 64}{32}$$
$$t = \frac{96 - 64}{32} = 1 \text{ or } t = \frac{96 + 64}{32} = 5$$

The projectile will reach 80 ft at 1 sec and 5 sec.

(b)
$$s = -16t^2 + 96t$$
$$0 = -16t^2 + 96t$$
$$0 = -16t(t - 6)$$
$$-16t = 0 \Rightarrow t = 0 \quad \text{or} \quad t - 6 = 0 \Rightarrow t = 6$$

The projectile will return to the ground after 6 sec.

40. (a)
$$s = -16t^2 + v_0 t$$
$$s = -16t^2 + 128t$$
$$80 = -16t^2 + 128t$$
$$16t^2 - 128t + 80 = 0$$

$a = 16$, $b = -128$ and $c = 80$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-128) \pm \sqrt{(-128)^2 - 4(16)(80)}}{2(16)} = \frac{128 \pm \sqrt{16384 - 5120}}{32} = \frac{128 \pm \sqrt{11264}}{32}$$
$$t = \frac{128 - \sqrt{11264}}{32} \approx .68 \text{ or}$$
$$t = \frac{128 + \sqrt{11264}}{32} \approx 7.32$$

The projectile will reach 80 ft at .68 sec and 7.32 sec.

(b)
$$s = -16t^2 + 128t$$
$$0 = -16t^2 + 128t$$
$$0 = -16t(t - 8)$$
$$-16t = 0 \Rightarrow t = 0 \quad \text{or} \quad t - 8 = 0 \Rightarrow t = 8$$

The projectile will return to the ground after 8 sec.

41. (a)
$$s = -16t^2 + v_0 t$$
$$s = -16t^2 + 32t$$
$$80 = -16t^2 + 32t$$
$$16t^2 - 32t + 80 = 0$$

$a = 16$, $b = -32$ and $c = 80$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-32) \pm \sqrt{(-32)^2 - 4(16)(80)}}{2(16)} = \frac{32 \pm \sqrt{1024 - 5120}}{32} = \frac{32 \pm \sqrt{-4096}}{32} = \frac{32 \pm 64i}{32}$$

The projectile will not reach 80 ft.

(b) $s = -16t^2 + 32t$

$0 = -16t^2 + 32t \Rightarrow 0 = -16t(t-2) \Rightarrow$

$-16t = 0 \Rightarrow t = 0$ or $t - 2 = 0 \Rightarrow t = 2$

The projectile will return to the ground after 2 sec.

42. **(a)** $s = -16t^2 + v_0 t \Rightarrow s = -16t^2 + 16t$

$80 = -16t^2 + 16t \Rightarrow 16t^2 - 16t + 80 = 0$

$a = 16$, $b = -16$ and $c = 80$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(16)(80)}}{2(16)} = \frac{16 \pm \sqrt{256 - 5120}}{32} = \frac{16 \pm \sqrt{-4864}}{32} = \frac{16 \pm 16i\sqrt{19}}{32}$$

The projectile will not reach 80 ft.

(b) $s = -16t^2 + 16t$

$0 = -16t^2 + 16t \Rightarrow 0 = -16t(t-1) \Rightarrow$

$-16t = 0 \Rightarrow t = 0$ or $t - 1 = 0 \Rightarrow t = 1$

The projectile will return to the ground after 1 sec.

43. The height of the ball is given by

$h = -2.7t^2 + 30t + 6.5.$

(a) When the ball is 12 ft above the moon's surface, $h = 12$. Set $h = 12$ and solve for t.

$12 = -2.7t^2 + 30t + 6.5$

$2.7t^2 - 30t + 5.5 = 0$

Use the quadratic formula with $a = 2.7$, $b = –30$, and $c = 5.5$.

$$t = \frac{30 \pm \sqrt{900 - 4(2.7)(5.5)}}{2(2.7)} = \frac{30 \pm \sqrt{840.6}}{5.4}$$

$$\frac{30 + \sqrt{840.6}}{5.4} \approx 10.92 \text{ or } \frac{30 - \sqrt{840.6}}{5.4} \approx .19$$

Therefore, the ball reaches 12 ft first after .19 sec (on the way up), then again after 10.92 sec (on the way down).

(b) When the ball returns to the surface, $h = 0$.

$0 = -2.7t^2 + 30t + 6.5$

Use the quadratic formula with $a = –2.7$, $b = 30$, and $c = 6.5$.

$$t = \frac{-30 \pm \sqrt{900 - 4(-2.7)(6.5)}}{2(-2.7)} = \frac{-30 \pm \sqrt{970.2}}{-5.4}$$

$$\frac{-30 + \sqrt{970.2}}{-5.4} \approx -.21 \text{ or}$$

$$\frac{-30 - \sqrt{970.2}}{-5.4} \approx 11.32$$

The negative solution is not meaningful. Therefore, the ball returns to the surface after 11.32 sec.

44. When the quadratic formula is applied to the equation $-2.7t^2 + 30t + 6.5 = 100 \Rightarrow$ $-2.7t^2 + 30t - 93.5 = 0$, the discriminant,

$$b^2 - 4ac = 30^2 - 4(-2.7)(-93.5) = 900 - 1009.8 = -109.8$$

is negative. Since this equation has no real solution, the ball will never reach a height of 100 ft.

45. **(a)** Let $x = 50$.

$T = .00787(50)^2 - 1.528(50) + 75.89 \approx 19.2$

The exposure time when $x = 50$ ppm is approximately 19.2 hr.

(b) Let $T = 3$ and solve for x.

$3 = .00787x^2 - 1.528x + 75.89$

$.00787x^2 - 1.528x + 72.89 = 0$

Use the quadratic formula with $a = .00787$, $b = –1.528$, and $c = 72.89$.

$$x = \frac{-(-1.528) \pm \sqrt{(-1.528)^2 - 4(.00787)(72.89)}}{2(.00787)} = \frac{1.528 \pm \sqrt{2.334784 - 2.2945772}}{.01574} = \frac{1.528 \pm \sqrt{.0402068}}{.01574}$$

$$\frac{1.528 + \sqrt{.0402068}}{.01574} \approx 109.8 \text{ or}$$

$$\frac{1.528 - \sqrt{.0402068}}{.01574} \approx 84.3$$

We reject the potential solution 109.8 because it is not in the interval [50, 100]. So, 84.3 ppm carbon monoxide concentration is necessary for a person to reach the 4% to 6% CoHb level in 3 hr.

46. **(a)** Let $C = 1700$ and solve for x.

$1700 = -6.57x^2 + 50.89x + 1631 \Rightarrow$

$6.57x^2 - 50.89x + 69 = 0$

Use the quadratic formula with $a = 6.57$, $b = -50.89$, and $c = 69$.

$$x = \frac{-(-50.89) \pm \sqrt{(-50.89)^2 - 4(6.57)(69)}}{2(6.57)} = \frac{50.89 \pm \sqrt{2589.7921 - 1831.32}}{13.14} = \frac{50.89 \pm \sqrt{776.4721}}{13.14}$$

$$\frac{50.89 + \sqrt{776.4721}}{13.14} \approx 5.994$$

$$\frac{50.89 - \sqrt{776.4721}}{13.14} \approx 1.752$$

We reject the potential solution 5.994 because it is not in the interval [0, 4]. So, the model predicts that the emissions reached 1700 million tons about 1.752 years after 1998, which is late 1999.

(b) 2003 is represented by $x = 5$. Substitute $x = 5$ into the equation to find C:

$$C = -6.57(5)^2 + 50.89(5) + 1631 \approx 1721 \text{ million tons}$$

This would indicate that emissions began to decrease, which is inconsistent with the trend during the period 1998–2002.

47. **(a)** Let $x = 600$ and solve for T.

$$T = .0002x^2 - .316x + 127.9 = .0002(600)^2 - .316(600) + 127.9 = 10.3$$

The exposure time when $x = 600$ ppm is 10.3 hr.

(b) Let $T = 4$ and solve for x.

$4 = .0002x^2 - .316x + 127.9$

$.0002x^2 - .316x + 123.9 = 0$

Use the quadratic formula with $a = .0002$, $b = -.316$, and $c = 123.9$.

$$x = \frac{-(-.316) \pm \sqrt{(-.316)^2 - 4(.0002)(123.9)}}{2(.0002)} = \frac{.316 \pm \sqrt{.099856 - .09912}}{.0004} = \frac{.316 \pm \sqrt{.000736}}{.0004}$$

$$\frac{.316 + \sqrt{.000736}}{.0004} \approx 857.8 \text{ or}$$

$$\frac{.316 - \sqrt{.000736}}{.0004} \approx 722.2$$

857.8 is not in the interval [500, 800]. A concentration of 722.2 ppm is required.

48. Let $y = 12{,}400$ and solve for x.

$12,400 = -6.31x^2 + 494.6x + 8438 \Rightarrow$

$6.31x^2 - 494.6x + 3962 = 0$

Use the quadratic formula with $a = 6.31$, $b = -494.6$, and $c = 3962$.

$$x = \frac{-(-494.6) \pm \sqrt{(-494.6)^2 - 4(6.31)(3962)}}{2(6.31)} = \frac{494.6 \pm \sqrt{244,629.16 - 100,000.88}}{12.62} = \frac{494.6 \pm \sqrt{144,628.28}}{12.62}$$

$$\frac{494.6 + \sqrt{144,628.28}}{12.62} \approx 69.3265$$

$$\frac{494.6 - \sqrt{144,628.28}}{12.62} \approx 9.0570$$

Reject 69.3265 because it is not in the interval [0, 9]. Based on this model, the cost was \$12,400 about 9.06 years after 1997 or in 2006.

49. Let $x = 4$ and solve for y.

$$y = .808x^2 + 2.625x + .502 = .808(4)^2 + 2.625(4) + .502 = .808(16) + 10.5 + .502 = 12.928 + 11.002 = 23.93$$

Approximately 23.93 million households are expected to pay at least one bill online each month in 2004.

50. Let $x = 7$ and solve for y.

$$y = 1.318x^2 - 3.526x + 2.189 = 1.318(7)^2 - 3.526(7) + 2.189 = 1.318(49) - 24.682 + 2.189 = 64.582 - 22.493 = 42.089$$

Approximately 42.1 million households are expected to have high-definition television in 2007.

51. For each \$20 increase in rent over \$300, one unit will remain vacant. Therefore, for x \$20 increases, x units will remain vacant. Therefore, the number of rented units will be $80 - x$.

52. x is the number of \$20 increases in rent. Therefore, the rent will be $300 + 20x$ dollars.

53. 300 + 20x is the rent for each apartment, and 80 – x is the number of apartments that will be rented at that cost. The revenue generated will then be the product of 80 – x and 300 + 20x, so the correct expression is

$$(80-x)(300+20x) = 24,000+1600x-300x-20x^2 = 24,000+1300x-20x^2.$$

54. Set the revenue equal to \$35,000. This gives the equation $35,000 = 24,000+1300x-20x^2$. Rewrite this equation in standard form: $20x^2-1300x+11,000=0$.

55.

$$20x^2-1300x+11,000=0$$
$$x^2-65x+550=0$$
$$(x-10)(x-55)=0$$
$$x-10=0 \quad\text{or}\quad x-55=0$$
$$x=10 \quad\text{or}\quad x=55$$

If x = 10, 80 – x = 70. If x = 55, 80 – x = 25. Because of the restriction that at least 30 units must be rented, only x = 10 is valid here, and the number of units rented is 70.

56. Let x = number of weeks the manager should wait. Then $100+5x$ = number of pounds and $.40-.02x$ = cost per pound

(Cost per pound)(Number of pounds) = Revenue

$$(.40-.02x)(100+5x)=38.40$$
$$40+2x-2x-.1x^2=38.40$$
$$40-.1x^2=38.40$$
$$-.1x^2=-1.6$$
$$-10(-.1x^2)=-10(-1.6)$$
$$x^2=16 \Rightarrow x=\pm 4$$

The negative solution is not meaningful. The farmer should wait 4 weeks to get an average revenue of \$38.40 per tree.

57. Let x = number of passengers in excess of 75. Then 225 – 5x = the cost per passenger (in dollars) and 75 + x = the number of passengers.

(Cost per passenger)(Number of passengers) = Revenue

$$(225-5x)(75+x)=16,000$$
$$16,875+225x-375x-5x^2=16,000$$
$$16,875-150x-5x^2=16,000$$
$$0=5x^2+150x-875$$
$$0=x^2+30x-175 \Rightarrow 0=(x+35)(x-5)$$
$$x+35=0 \Rightarrow x=-35 \text{ or } x-5=0 \Rightarrow x=5$$

The negative solution is not meaningful. Since there are 5 passengers in excess of 75, the total number of passengers is 80.

58. Let x = number of days the scouts should wait. Then $4-.1x$ = the price the scouts will receive per hundred pounds, and $120+4x$ = the number of hundreds of pounds of cans the scouts can collect.

(price per hundred pounds)(Number of pounds) = Revenue

$$(4-.1x)(120+4x)=490$$
$$480+16x-12x-.4x^2=490$$
$$480+4x-.4x^2=490$$
$$0=.4x^2-4x+10$$
$$10\cdot 0=10(.4x^2-4x+10)$$
$$0=4x^2-40x+100$$
$$0=x^2-10x+25$$
$$0=(x-5)^2$$
$$0=x-5 \Rightarrow 5=x$$

The scouts should wait 5 days in order to get \$490 for their cans.

Section 1.6: Other Types of Equations and Applications

1. $\dfrac{5}{2x+3}-\dfrac{1}{x-6}=0$

$2x+3\neq 0 \Rightarrow x\neq -\frac{3}{2}$ and $x-6\neq 0 \Rightarrow x\neq 6$.

2. $\dfrac{2}{x+1}+\dfrac{3}{5x+5}=0$ or $\dfrac{2}{x+1}+\dfrac{3}{5(x+1)}=0$

$5(x+1)\neq 0$ and $x+1\neq 0 \Rightarrow x\neq -1$

3. $\dfrac{3}{x-2}+\dfrac{1}{x+1}=\dfrac{3}{x^2-x-2}$

or $\dfrac{3}{x-2}+\dfrac{1}{x+1}=\dfrac{3}{(x-2)(x+1)}$

$x-2\neq 0 \Rightarrow x\neq 2$ and $x+1\neq 0 \Rightarrow x\neq -1$

4. $\dfrac{2}{x+3}-\dfrac{5}{x-1}=\dfrac{-5}{x^2+2x-3}$ or

$\dfrac{2}{x+3}-\dfrac{5}{x-1}=\dfrac{-5}{(x+3)(x-1)}$

$x+3\neq 0 \Rightarrow x\neq -3$ and $x-1\neq 0 \Rightarrow x\neq 1$

5. $\dfrac{1}{4x}-\dfrac{2}{x}=3$

$4x\neq 0 \Rightarrow x\neq 0$

6. $\frac{5}{2x}+\frac{2}{x}=6$

$2x \neq 0 \Rightarrow x \neq 0$

7. $\frac{2x+5}{2}-\frac{3x}{x-2}=x$

The least common denominator is $2(x-2)$, which is equal to 0 if $x=2$. Therefore, 2 cannot possibly be a solution of this equation.

$$2(x-2)\left[\frac{2x+5}{2}-\frac{3x}{x-2}\right]=2(x-2)(x)$$
$$(x-2)(2x+5)-2(3x)=2x(x-2)$$
$$2x^2+5x-4x-10-6x=2x^2-4x$$
$$-5x-10=-4x \Rightarrow -10=x$$

The restriction $x \neq 2$ does not affect the result. Therefore, the solution set is $\{-10\}$.

8. $\frac{4x+3}{4}-\frac{2x}{x+1}=x$

The least common denominator is $4(x+1)$, which is equal to 0 if $x=-1$. Therefore, -1 cannot possibly be a solution of this equation.

$$4(x+1)\left[\frac{4x+3}{4}-\frac{2x}{x+1}\right]=4(x+1)(x)$$
$$(x+1)(4x+3)-4(2x)=4x(x+1)$$
$$4x^2+3x+4x+3-8x=4x^2+4x$$
$$-x+3=4x$$
$$3=5x \Rightarrow \frac{3}{5}=x$$

The restriction $x \neq -1$ does not affect the result. Therefore, the solution set is $\left\{\frac{3}{5}\right\}$.

9. $\frac{x}{x-3}=\frac{3}{x-3}+3$

The least common denominator is $x-3$, which is equal to 0 if $x=3$. Therefore, 3 cannot possibly be a solution of this equation.

$$(x-3)\left(\frac{x}{x-3}\right)=(x-3)\left[\frac{3}{x-3}+3\right]$$
$$x=3+3(x-3)$$
$$x=3+3x-9$$
$$x=3x-6 \Rightarrow -2x=-6 \Rightarrow x=3$$

The only possible solution is 3. However, the variable is restricted to real numbers except 3. Therefore, the solution set is: $\varnothing$.

10. $\frac{x}{x-4}=\frac{4}{x-4}+4$

The least common denominator is $x-4$, which is equal to 0 if $x=4$. Therefore, 4 cannot possibly be a solution of this equation.

$$(x-4)\left(\frac{x}{x-4}\right)=(x-4)\left[\frac{4}{x-4}+4\right]$$
$$x=4+4(x-4)$$
$$x=4+4x-16$$
$$x=4x-12$$
$$-3x=-12 \Rightarrow x=4$$

The only possible solution is 4. However, the variable is restricted to real numbers except 4. Therefore, the solution set is: $\varnothing$.

11. $\frac{-2}{x-3}+\frac{3}{x+3}=\frac{-12}{x^2-9}$ or $\frac{-2}{x-3}+\frac{3}{x+3}=\frac{-12}{(x+3)(x-3)}$

The least common denominator is $(x+3)(x-3)$, which is equal to 0 if $x=-3$ or $x=3$. Therefore, -3 and 3 cannot possibly be solutions of this equation.

$$(x+3)(x-3)\left[\frac{-2}{x-3}+\frac{3}{x+3}\right]$$
$$=(x+3)(x-3)\left(\frac{-12}{(x+3)(x-3)}\right)$$
$$-2(x+3)+3(x-3)=-12$$
$$-2x-6+3x-9=-12$$
$$-15+x=-12 \Rightarrow x=3$$

The only possible solution is 3. However, the variable is restricted to real numbers except -3 and 3. Therefore, the solution set is: $\varnothing$.

12. $\frac{3}{x-2}+\frac{1}{x+2}=\frac{12}{x^2-4}$ or $\frac{3}{x-2}+\frac{1}{x+2}=\frac{12}{(x+2)(x-2)}$

The least common denominator is $(x+2)(x-2)$, which is equal to 0 if $x=-2$ or $x=2$. Therefore, -2 and 2 cannot possibly be solutions of this equation.

$$(x+2)(x-2)\left[\frac{3}{x-2}+\frac{1}{x+2}\right]$$
$$=(x+2)(x-2)\left(\frac{12}{(x+2)(x-2)}\right)$$

$$3(x+2)+(x-2)=12$$
$$3x+6+x-2=12$$
$$4x+4=12$$
$$4x=8 \Rightarrow x=2$$

The only possible solution is 2. However, the variable is restricted to real numbers except -2 and 2. Therefore, the solution set is: $\varnothing$.

13. $\dfrac{4}{x^2+x-6}-\dfrac{1}{x^2-4}=\dfrac{2}{x^2+5x+6}$ or

$$\frac{4}{(x+3)(x-2)}-\frac{1}{(x+2)(x-2)}=\frac{2}{(x+2)(x+3)}$$

The least common denominator is $(x+3)(x-2)(x+2)$, which is equal to 0 if $x=-3$ or $x=2$ or $x=-2$. Therefore, -3 and 2 and -2 cannot possibly be solutions of this equation.

$$(x+3)(x-2)(x+2)\cdot\left[\frac{4}{(x+3)(x-2)}-\frac{1}{(x+2)(x-2)}\right]=(x+3)(x-2)(x+2)\left(\frac{2}{(x+2)(x+3)}\right)$$
$$4(x+2)-1(x+3)=2(x-2)$$
$$4x+8-x-3=2x-4$$
$$3x+5=2x-4 \Rightarrow x+5=-4 \Rightarrow x=-9$$

The restrictions $x \neq -3$, $x \neq 2$, and $x \neq -2$ do not affect the result. Therefore, the solution set is $\{-9\}$.

14. $$\frac{3}{x^2+x-2}-\frac{1}{x^2-1}=\frac{7}{2x^2+6x+4}$$
$$\frac{3}{(x+2)(x-1)}-\frac{1}{(x+1)(x-1)}=\frac{7}{2(x^2+3x+2)}$$
$$\frac{3}{(x+2)(x-1)}-\frac{1}{(x+1)(x-1)}=\frac{7}{2(x+2)(x+1)}$$

The least common denominator is $2(x+1)(x-1)(x+2)$, which is equal to 0 if $x=-1$ or $x=1$ or $x=-2$. Therefore, -1 and 1 and -2 cannot possibly be solutions of this equation.

$$2(x+1)(x-1)(x+2)\cdot\left[\frac{3}{(x+2)(x-1)}-\frac{1}{(x+1)(x-1)}\right]=2(x+1)(x-1)(x+2)\left(\frac{7}{2(x+2)(x+1)}\right)$$
$$2(3)(x+1)-2(x+2)=7(x-1)$$
$$6(x+1)-2(x+2)=7(x-1)$$
$$6x+6-2x-4=7x-7$$
$$4x+2=7x-7$$
$$2=3x-7$$
$$9=3x \Rightarrow 3=x$$

The restrictions $x \neq -1$, $x \neq 1$, and $x \neq -2$ do not affect the result. Therefore, the solution set is $\{3\}$.

15. $\dfrac{2x+1}{x-2}+\dfrac{3}{x}=\dfrac{-6}{x^2-2x}$ or

$$\frac{2x+1}{x-2}+\frac{3}{x}=\frac{-6}{x(x-2)}$$

Multiply each term in the equation by the least common denominator, $x(x-2)$, assuming $x \neq 0, 2$.

$$x(x-2)\left[\frac{2x+1}{x-2}+\frac{3}{x}\right]=x(x-2)\left(\frac{-6}{x(x-2)}\right)$$
$$x(2x+1)+3(x-2)=-6$$
$$2x^2+x+3x-6=-6$$
$$2x^2+4x-6=-6$$
$$2x^2+4x=0 \Rightarrow 2x(x+2)=0$$
$$2x=0 \Rightarrow x=0 \quad \text{or} \quad x+2=0 \Rightarrow x=-2$$

Because of the restriction $x \neq 0$, the only valid solution is -2. The solution set is $\{-2\}$.

16. $\dfrac{4x+3}{x+1}+\dfrac{2}{x}=\dfrac{1}{x^2+x}$ or $\dfrac{4x+3}{x+1}+\dfrac{2}{x}=\dfrac{1}{x(x+1)}$

Multiply each term in the equation by the least common denominator, $x(x+1)$, assuming $x \neq 0, -1$.

$$x(x+1)\left[\frac{4x+3}{x+1}+\frac{2}{x}\right]=x(x+1)\left(\frac{1}{x(x+1)}\right)$$
$$x(4x+3)+2(x+1)=1$$
$$4x^2+3x+2x+2=1$$
$$4x^2+5x+2=1 \Rightarrow 4x^2+5x+1=0$$
$$(4x+1)(x+1)=0$$
$$4x+1=0 \Rightarrow x=-\tfrac{1}{4} \quad \text{or} \quad x+1=0 \Rightarrow x=-1$$

Because of the restriction $x \neq -1$, the only valid solution is $-\frac{1}{4}$. The solution set is $\left\{-\frac{1}{4}\right\}$.

17. $\frac{x}{x-1}-\frac{1}{x+1}=\frac{2}{x^2-1}$ or
$\frac{x}{x-1}-\frac{1}{x+1}=\frac{2}{(x+1)(x-1)}$

Multiply each term in the equation by the least common denominator, $(x+1)(x-1)$, assuming $x\neq\pm1$.

$$(x+1)(x-1)\left[\frac{x}{x-1}-\frac{1}{x+1}\right]=(x+1)(x-1)\left(\frac{2}{(x+1)(x-1)}\right)$$

$$x(x+1)-(x-1)=2\Rightarrow x^2+x-x+1=2$$
$$x^2+1=2\Rightarrow x^2-1=0$$
$$(x+1)(x-1)=0\Rightarrow x+1=0\Rightarrow x=-1 \text{ or}$$
$$x-1=0\Rightarrow x=1$$

Because of the restriction $x\neq\pm1$, the solution set is $\varnothing$.

18. $\frac{-x}{x+1}-\frac{1}{x-1}=\frac{-2}{x^2-1}$ or
$\frac{-x}{x+1}-\frac{1}{x-1}=\frac{-2}{(x+1)(x-1)}$

Multiply each term in the equation by the least common denominator, $(x+1)(x-1)$, assuming $x\neq\pm1$.

$$(x+1)(x-1)\left[\frac{-x}{x+1}-\frac{1}{x-1}\right]=(x+1)(x-1)\left(\frac{-2}{(x+1)(x-1)}\right)$$

$$-x(x-1)-(x+1)=-2$$
$$-x^2+x-x-1=-2\Rightarrow -x^2-1=-2\Rightarrow$$
$$-x^2+1=0\Rightarrow x^2-1=0\Rightarrow$$
$$(x+1)(x-1)=0\Rightarrow x+1=0\Rightarrow x=-1 \text{ or}$$
$$x-1=0\Rightarrow x=1$$

Because of the restriction $x\neq\pm1$, the solution set is: $\varnothing$.

19. $\frac{5}{x^2}-\frac{43}{x}=18$

Multiply each term in the equation by the least common denominator, x^2, assuming $x\neq0$.

$$x^2\left[\frac{5}{x^2}-\frac{43}{x}\right]=x^2(18)$$
$$5-43x=18x^2\Rightarrow 0=18x^2+43x-5$$
$$0=(2x+5)(9x-1)$$
$$2x+5=0\Rightarrow x=-\tfrac{5}{2} \text{ or}$$
$$9x-1=0\Rightarrow x=\tfrac{1}{9}$$

The restriction $x\neq0$ does not affect the result. Therefore, the solution set is $\left\{-\frac{5}{2},\frac{1}{9}\right\}$.

20. $\frac{7}{x^2}+\frac{19}{x}=6$

Multiply each term in the equation by the least common denominator, x^2, assuming $x\neq0$.

$$x^2\left[\frac{7}{x^2}+\frac{19}{x}\right]=x^2(6)$$
$$7+19x=6x^2$$
$$0=6x^2-19x-7$$
$$0=(3x+1)(2x-7)$$
$$3x+1=0\Rightarrow x=-\tfrac{1}{3} \quad\text{or}\quad 2x-7=0\Rightarrow x=\tfrac{7}{2}$$

The restriction $x\neq0$ does not affect the result. Therefore, the solution set is $\left\{-\frac{1}{3},\frac{7}{2}\right\}$.

21. $2=\frac{3}{2x-1}+\frac{-1}{(2x-1)^2}$

Multiply each term in the equation by the least common denominator, $(2x-1)^2$, assuming $x\neq\frac{1}{2}$.

$$(2x-1)^2(2)=(2x-1)^2\left[\frac{3}{2x-1}+\frac{-1}{(2x-1)^2}\right]$$
$$2(4x^2-4x+1)=3(2x-1)-1$$
$$8x^2-8x+2=6x-3-1$$
$$8x^2-8x+2=6x-4\Rightarrow 8x^2-14x+6=0$$
$$2(4x^2-7x+3)=0\Rightarrow 2(4x-3)(x-1)=0$$
$$4x-3=0\Rightarrow x=\tfrac{3}{4} \text{ or } x-1=0\Rightarrow x=1$$

The restriction $x\neq\frac{1}{2}$ does not affect the result. Therefore the solution set is $\left\{\frac{3}{4},1\right\}$.

22. $6=\frac{7}{2x-3}+\frac{3}{(2x-3)^2}$

Multiply each term in the equation by the least common denominator, $(2x-3)^2$, assuming $x\neq\frac{3}{2}$.

$$(2x-3)^2(6)=(2x-3)^2\left[\frac{7}{2x-3}+\frac{3}{(2x-3)^2}\right]$$
$$6(4x^2-12x+9)=7(2x-3)+3$$
$$24x^2-72x+54=14x-21+3$$
$$24x^2-72x+54=14x-18$$
$$24x^2-86x+72=0$$
$$2(12x^2-43x+36)=0\Rightarrow 2(4x-9)(3x-4)=0$$

$4x-9=0\Rightarrow x=\frac{9}{4}$ or $3x-4=0\Rightarrow x=\frac{4}{3}$

The restriction $x\neq\frac{3}{2}$ does not affect the result.

Therefore, the solution set is $\left\{\frac{9}{4},\frac{4}{3}\right\}$.

23. $\frac{2x-5}{x}=\frac{x-2}{3}$

Multiply each term in the equation by the least common denominator, $3x$, assuming $x\neq 0$.

$$3x\left(\frac{2x-5}{x}\right)=3x\left(\frac{x-2}{3}\right)$$
$$3(2x-5)=x(x-2)\Rightarrow 6x-15=x^2-2x\Rightarrow$$
$$0=x^2-8x+15=(x-3)(x-5)$$

$x-3=0\Rightarrow x=3$ or $x-5=0\Rightarrow x=5$

The restriction $x\neq 0$ does not affect the result. Therefore, the solution set is $\{3, 5\}$.

24. $\frac{x+4}{2x}=\frac{x-1}{3}$

Multiply each term in the equation by the least common denominator, $6x$, assuming $x\neq 0$.

$$6x\left(\frac{x+4}{2x}\right)=6x\left(\frac{x-1}{3}\right)$$
$$3(x+4)=2x(x-1)$$
$$3x+12=2x^2-2x$$
$$0=2x^2-5x-12$$
$$0=(2x+3)(x-4)$$

$2x+3=0\Rightarrow x=-\frac{3}{2}$ or $x-4=0\Rightarrow x=4$

The restriction $x\neq 0$ does not affect the result.

Therefore the solution set is $\left\{-\frac{3}{2},4\right\}$.

25. $\frac{2x}{x-2}=5+\frac{4x^2}{x-2}$

Multiply each term in the equation by the least common denominator, $x-2$, assuming $x\neq 2$.

$$(x-2)\left(\frac{2x}{x-2}\right)=(x-2)\left[5+\frac{4x^2}{x-2}\right]$$
$$2x=5(x-2)+4x^2$$
$$2x=5x-10+4x^2$$
$$0=4x^2+3x-10$$
$$0=(x+2)(4x-5)$$

$x+2=0\Rightarrow x=-2$ or $4x-5=0\Rightarrow x=\frac{5}{4}$

The restriction $x\neq 2$ does not affect the result.

Therefore the solution set is $\left\{-2,\ \frac{5}{4}\right\}$.

26. $\frac{-3x}{2}+\frac{9x-5}{3}=\frac{11x+8}{6x}$

Multiply each term in the equation by the least common denominator, $6x$, assuming $x\neq 0$.

$$6x\left[\frac{-3x}{2}+\frac{9x-5}{3}\right]=6x\left(\frac{11x+8}{6x}\right)$$
$$3x(-3x)+2x(9x-5)=11x+8$$
$$-9x^2+18x^2-10x=11x+8$$
$$9x^2-10x=11x+8$$
$$9x^2-21x-8=0$$
$$(3x+1)(3x-8)=0$$

$3x+1=0\Rightarrow x=-\frac{1}{3}$ or $3x-8=0\Rightarrow x=\frac{8}{3}$

The restriction $x\neq 0$ does not affect the result.

Therefore, the solution set is $\left\{-\frac{1}{3},\frac{8}{3}\right\}$.

27. Let x = the amount of time (in hours) it takes Joe and Sam to paint the house.

	r	t	Part of the Job Accomplished
Joe	$\frac{1}{3}$	x	$\frac{1}{3}x$
Sam	$\frac{1}{5}$	x	$\frac{1}{5}x$

Since Joe and Sam must accomplish 1 job (painting a house), we must solve the following equation.

$$\frac{1}{3}x+\frac{1}{5}x=1$$
$$15\left[\frac{1}{3}x+\frac{1}{5}x\right]=15\cdot 1$$
$$5x+3x=15\Rightarrow 8x=15\Rightarrow x=\frac{15}{8}=1\tfrac{7}{8}$$

It takes Joe and Sam $1\frac{7}{8}$ hr working together to paint the house.

28. Let x = the amount of time (in hours) it takes Joe and Sam to paint the house.

	Rate	Time	Part of the Job Accomplished
Joe	$\frac{1}{6}$	x	$\frac{1}{6}x$
Sam	$\frac{1}{8}$	x	$\frac{1}{8}x$

Since Joe and Sam must accomplish 1 job (painting a house), we must solve the following equation.

$$\frac{1}{6}x+\frac{1}{8}x=1$$
$$24\left[\frac{1}{6}x+\frac{1}{8}x\right]=24\cdot 1$$
$$4x+3x=24$$
$$7x=24 \Rightarrow x=\frac{24}{7}=3\tfrac{3}{7}$$

It takes Joe and Sam $3\frac{3}{7}$ hr working together to paint the house.

29. Let x = the amount of time (in hours) it takes plant A to produce the pollutant. Then $2x$ = the amount of time (in hours) it takes plant B to produce the pollutant.

	Rate	Time	Part of the Job Accomplished
Pollution from A	$\frac{1}{x}$	26	$\frac{1}{x}(26)$
Pollution from B	$\frac{1}{2x}$	26	$\frac{1}{2x}(26)$

Since plant A and B accomplish 1 job (producing the pollutant), we must solve the following equation.

$$\frac{1}{x}(26)+\frac{1}{2x}(26)=1$$
$$\frac{26}{x}+\frac{13}{x}=1$$
$$x\left[\frac{39}{x}\right]=x\cdot 1$$
$$39=x$$

Plant B will take $2\cdot 39=78$ hr to produce the pollutant.

30. Let x = the amount of time (in hours) the second pipe operates.

	Rate	Time	Part of the Job Accomplished
First pipe	$\frac{1}{10}$	$5+x$	$\frac{1}{10}(5+x)$
Second pipe	$\frac{1}{12}$	x	$\frac{1}{12}x$

Since the two pipes are working together, we must solve the following equation.

$$\frac{1}{10}(5+x)+\frac{1}{12}x=1$$
$$60\left[\frac{1}{10}(5+x)+\frac{1}{12}x\right]=60\cdot 1$$
$$6(5+x)+5x=60$$
$$30+6x+5x=60 \Rightarrow 30+11x=60 \Rightarrow$$
$$11x=30 \Rightarrow x=\frac{30}{11}=2\tfrac{8}{11}\text{ hr}$$

It will take $2\frac{8}{11}$ hr after the second pipe is opened to fill the pond.

31. Let x = the amount of time (in hours) to fill the pool with both pipes open.

	Rate	Time	Part of the Job Accomplished
Inlet pipe	$\frac{1}{5}$	x	$\frac{1}{5}x$
Outlet pipe	$\frac{1}{8}$	x	$\frac{1}{8}x$

Filling the pool is 1 whole job, but because the outlet pipe empties the pool, its contribution should be subtracted from the contribution of the inlet pipe.

$$\frac{1}{5}x-\frac{1}{8}x=1$$
$$40\left[\frac{1}{5}x-\frac{1}{8}x\right]=40\cdot 1 \Rightarrow 8x-5x=40 \Rightarrow$$
$$3x=40 \Rightarrow x=\frac{40}{3}=13\tfrac{1}{3}\text{ hr}$$

It took $13\frac{1}{3}$ hr to fill the pool.

32. We need to determine how much of the pool was filled after 1 hour. To do this, we evaluate $\frac{1}{5}x-\frac{1}{8}x$ when $x=1$. After 1 hour, $\frac{1}{5}\cdot 1-\frac{1}{8}\cdot 1=\frac{1}{5}-\frac{1}{8}=\frac{8}{40}-\frac{5}{40}=\frac{3}{40}$ of the pool has been filled. What remains to be filled is $1-\frac{3}{40}=\frac{40}{40}-\frac{3}{40}=\frac{37}{40}$. If we now let x be the amount of time it takes to complete filling the pool, we must solve the following.

$$\frac{1}{5}x=\frac{37}{40}$$
$$5\left(\frac{1}{5}x\right)=5\left(\frac{37}{40}\right)$$
$$x=\frac{37}{8}=4\tfrac{5}{8}\text{ hr}$$

It will take $4\frac{5}{8}$ hr more to fill the pool.

33. Let x = the amount of time (in minutes) to fill the sink with both pipes open.

	Rate	Time	Part of the Job Accomplished
Tap	$\frac{1}{5}$	x	$\frac{1}{5}x$
Drain	$\frac{1}{10}$	x	$\frac{1}{10}x$

Filling the sink is 1 whole job, but because the sink is draining, its contribution should be subtracted from the contribution of the taps.

$$\frac{1}{5}x - \frac{1}{10}x = 1$$

$$10\left[\frac{1}{5}x - \frac{1}{10}x\right] = 10 \cdot 1 \Rightarrow 2x - x = 10 \Rightarrow x = 10$$

It will take 10 minutes to fill the sink if Mark forgets to put in the stopper.

34. We need to determine how much of the sink was filled after 1 minute. To do this, we evaluate $\frac{1}{5}x - \frac{1}{10}x$ when $x = 1$. After 1 minute, $\frac{1}{5} \cdot 1 - \frac{1}{10} \cdot 1 = \frac{1}{5} - \frac{1}{10} = \frac{2}{10} - \frac{1}{10} = \frac{1}{10}$ of the sink has been filled. What remains to be filled is $1 - \frac{1}{10} = \frac{10}{10} - \frac{1}{10} = \frac{9}{10}$. If we now let x be the amount of time it takes to complete filling the sink, we must solve the following.

$$\frac{1}{5}x = \frac{9}{10}$$

$$5\left(\frac{1}{5}x\right) = 5\left(\frac{9}{10}\right) \Rightarrow x = \frac{45}{10} = 4\tfrac{1}{2} \text{ min}$$

It will take $4\frac{1}{2}$ min more to fill the sink.

35. $x - \sqrt{2x+3} = 0$

$$x = \sqrt{2x+3} \Rightarrow x^2 = \left(\sqrt{2x+3}\right)^2$$

$$x^2 = 2x+3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow$$

$$(x+1)(x-3) = 0 \Rightarrow x = -1 \text{ or } x = 3$$

Check $x = -1$.

$$x - \sqrt{2x+3} = 0$$

$$-1 - \sqrt{2(-1)+3} \stackrel{?}{=} 0$$

$$-1 - \sqrt{-2+3} = 0$$

$$-1 - \sqrt{1} = 0 \Rightarrow -1 - 1 = 0 \Rightarrow -2 = 0$$

This is a false statement. -1 is not a solution.

Check $x = 3$.

$$x - \sqrt{2x+3} = 0$$

$$3 - \sqrt{2(3)+3} \stackrel{?}{=} 0$$

$$3 - \sqrt{6+3} = 0$$

$$3 - \sqrt{9} = 0 \Rightarrow 3 - 3 = 0 \Rightarrow 0 = 0$$

This is a true statement. 3 is a solution.

Solution set: $\{3\}$

36. $x - \sqrt{3x+18} = 0$

$$x = \sqrt{3x+18} \Rightarrow x^2 = \left(\sqrt{3x+18}\right)^2$$

$$x^2 = 3x+18 \Rightarrow x^2 - 3x - 18 = 0$$

$$(x+3)(x-6) = 0 \Rightarrow x = -3 \text{ or } x = 6$$

Check $x = -3$.

$$x - \sqrt{3x+18} = 0$$

$$-3 - \sqrt{3(-3)+18} \stackrel{?}{=} 0$$

$$-3 - \sqrt{-9+18} = 0$$

$$-3 - \sqrt{9} = 0 \Rightarrow -3 - 3 = 0 \Rightarrow -6 = 0$$

This is a false statement. -3 is not a solution.

Check $x = 6$.

$$x - \sqrt{3x+18} = 0$$

$$6 - \sqrt{3(6)+18} \stackrel{?}{=} 0$$

$$6 - \sqrt{18+18} = 0$$

$$6 - \sqrt{36} = 0 \Rightarrow 6 - 6 = 0 \Rightarrow 0 = 0$$

This is a true statement. 6 is a solution

Solution set: $\{6\}$

37. $\sqrt{3x+7} = 3x+5$

$$\left(\sqrt{3x+7}\right)^2 = (3x+5)^2$$

$$3x+7 = 9x^2 + 30x + 25$$

$$0 = 9x^2 + 27x + 18$$

$$0 = 9\left(x^2 + 3x + 2\right) = 9(x+2)(x+1)$$

$x = -2$ or $x = -1$

Check $x = -2$.

$$\sqrt{3x+7} = 3x+5$$

$$\sqrt{3(-2)+7} \stackrel{?}{=} 3(-2)+5$$

$$\sqrt{-6+7} = -6+5$$

$$\sqrt{1} = -1 \Rightarrow 1 = -1$$

This is a false statement. -2 is not a solution.

Check $x = -1$

$$\sqrt{3x+7} = 3x+5$$

$$\sqrt{3(-1)+7} \stackrel{?}{=} 3(-1)+5$$

$$\sqrt{-3+7} = -3+5$$

$$\sqrt{4} = 2 \Rightarrow 2 = 2$$

This is a true statement. -1 is a solution.

Solution set: $\{-1\}$

38. $\sqrt{4x+13}=2x-1$
$$\left(\sqrt{4x+13}\right)^2=(2x-1)^2$$
$$4x+13=4x^2-4x+1$$
$$0=4x^2-8x-12$$
$$0=4\left(x^2-2x-3\right)$$
$$0=4(x+1)(x-3)$$
$x=-1$ or $x=3$

Check $x=-1$.
$$\sqrt{4x+13}=2x-1$$
$$\sqrt{4(-1)+13}\stackrel{?}{=}2(-1)-1$$
$$\sqrt{-4+13}=-2-1$$
$$\sqrt{9}=-3\Rightarrow 3=-3$$
This is a false statement. -1 is not a solution.

Check $x=3$.
$$\sqrt{4x+13}=2x-1$$
$$\sqrt{4(3)+13}\stackrel{?}{=}2(3)-1$$
$$\sqrt{12+13}=6-1$$
$$\sqrt{25}=5\Rightarrow 5=5$$
This is a true statement. 3 is a solution.

Solution set: $\{3\}$

39. $\sqrt{4x+5}-6=2x-11$
$$\sqrt{4x+5}=2x-5$$
$$\left(\sqrt{4x+5}\right)^2=(2x-5)^2$$
$$4x+5=4x^2-20x+25$$
$$0=4x^2-24x+20$$
$$0=4\left(x^2-6x+5\right)=4(x-1)(x-5)$$
$x=1$ or $x=5$

Check $x=1$.
$$\sqrt{4x+5}-6=2x-11$$
$$\sqrt{4(1)+5}-6\stackrel{?}{=}2(1)-11$$
$$\sqrt{4+5}-6=2-11$$
$$\sqrt{9}-6=-9$$
$$3-6=-9\Rightarrow -3=-9$$
This is a false statement. 1 is not a solution.

Check $x=5$.
$$\sqrt{4x+5}-6=2x-11$$
$$\sqrt{4(5)+5}-6\stackrel{?}{=}2(5)-11$$
$$\sqrt{20+5}-6=10-11$$
$$\sqrt{25}-6=-1\Rightarrow 5-6=-1\Rightarrow -1=-1$$
This is a true statement. 5 is a solution.

Solution set: $\{5\}$

40. $\sqrt{6x+7}-9=x-7$
$$\sqrt{6x+7}=x+2$$
$$\left(\sqrt{6x+7}\right)^2=(x+2)^2$$
$$6x+7=x^2+4x+4$$
$$0=x^2-2x-3=(x+1)(x-3)$$
$x=-1$ or $x=3$

Check $x=-1$.
$$\sqrt{6x+7}-9=x-7$$
$$\sqrt{6(-1)+7}-9\stackrel{?}{=}-1-7$$
$$\sqrt{-6+7}-1=0$$
$$\sqrt{1}-1=0\Rightarrow 1-1=0\Rightarrow 0=0$$
This is a true statement. -1 is a solution.

Check $x=3$.
$$\sqrt{6x+7}-9=x-7$$
$$\sqrt{6(3)+7}-9\stackrel{?}{=}3-7$$
$$\sqrt{18+7}-9=-4$$
$$\sqrt{25}-9=-4\Rightarrow 5-9=-4\Rightarrow -4=-4$$
This is a true statement. 3 is a solution.

Solution set: $\{-1,3\}$

41. $\sqrt{4x}-x+3=0$
$$\sqrt{4x}=x-3$$
$$\left(\sqrt{4x}\right)^2=(x-3)^2$$
$$4x=x^2-6x+9$$
$$0=x^2-10x+9=(x-1)(x-9)$$
$$x=1 \text{ or } x=9$$

Check $x=1$.
$$\sqrt{4x}-x+3=0$$
$$\sqrt{4(1)}-1+3\stackrel{?}{=}0$$
$$\sqrt{4}-1+3=0$$
$$2-1+3=0\Rightarrow 4=0$$
This is a false statement. 1 is not a solution.

Check $x=9$.
$$\sqrt{4x}-x+3=0$$
$$\sqrt{4(9)}-9+3\stackrel{?}{=}0$$
$$\sqrt{36}-9+3=0$$
$$6-9+3=0\Rightarrow 0=0$$
This is a true statement. 9 is a solution.

Solution set: $\{9\}$

42. $\sqrt{2x}-x+4=0$

$$\sqrt{2x}=x-4$$
$$\left(\sqrt{2x}\right)^2=(x-4)^2$$
$$2x=x^2-8x+16$$
$$0=x^2-10x+16=(x-2)(x-8)$$
$$x=2 \text{ or } x=8$$

Check $x=2$.

$$\sqrt{2x}-x+4=0$$
$$\sqrt{2(2)}-2+4\overset{?}{=}0$$
$$\sqrt{4}-2+4=0$$
$$2-2+4=0\Rightarrow 4=0$$

This is a false statement. 2 is not a solution.

Check $x=8$.

$$\sqrt{2x}-x+4=0$$
$$\sqrt{2(8)}-8+4\overset{?}{=}0$$
$$\sqrt{16}-8+4=0$$
$$4-8+4=0\Rightarrow 0=0$$

This is a true statement. 8 is a solution.

Solution set: $\{8\}$

43. $\sqrt{x}-\sqrt{x-5}=1$

$$\sqrt{x}=1+\sqrt{x-5}\Rightarrow\left(\sqrt{x}\right)^2=\left(1+\sqrt{x-5}\right)^2$$
$$x=1+2\sqrt{x-5}+(x-5)$$
$$x=x+2\sqrt{x-5}-4\Rightarrow 4=2\sqrt{x-5}$$
$$2=\sqrt{x-5}\Rightarrow 2^2=\left(\sqrt{x-5}\right)^2$$
$$4=x-5\Rightarrow 9=x$$

Check $x=9$.

$$\sqrt{x}-\sqrt{x-5}=1$$
$$\sqrt{9}-\sqrt{9-5}\overset{?}{=}1$$
$$3-\sqrt{4}=1$$
$$3-2=1\Rightarrow 1=1$$

This is a true statement.

Solution set is: $\{9\}$

44. $\sqrt{x}-\sqrt{x-12}=2$

$$\sqrt{x}=2+\sqrt{x-12}\Rightarrow\left(\sqrt{x}\right)^2=\left(2+\sqrt{x-12}\right)^2$$
$$x=4+4\sqrt{x-12}+(x-12)$$
$$x=x+4\sqrt{x-12}-8\Rightarrow 8=4\sqrt{x-12}$$
$$2=\sqrt{x-12}\Rightarrow 2^2=\left(\sqrt{x-12}\right)^2$$
$$4=x-12\Rightarrow 16=x$$

Check $x=16$.

$$\sqrt{x}-\sqrt{x-12}=2$$
$$\sqrt{16}-\sqrt{16-12}\overset{?}{=}2$$
$$4-\sqrt{4}=2\Rightarrow 4-2=2\Rightarrow 2=2$$

This is a true statement.

Solution set: $\{16\}$

45. $\sqrt{x+7}+3=\sqrt{x-4}$

$$\left(\sqrt{x+7}+3\right)^2=\left(\sqrt{x-4}\right)^2$$
$$(x+7)+6\sqrt{x+7}+9=x-4$$
$$x+6\sqrt{x+7}+16=x-4\Rightarrow 6\sqrt{x+7}=-20$$
$$3\sqrt{x+7}=-10\Rightarrow\left(3\sqrt{x+7}\right)^2=(-10)^2$$
$$9(x+7)=100\Rightarrow 9x+63=100$$
$$9x=37\Rightarrow x=\frac{37}{9}$$

Check $x=\frac{37}{9}$.

$$\sqrt{x+7}+3=\sqrt{x-4}$$
$$\sqrt{\tfrac{37}{9}+7}+3\overset{?}{=}\sqrt{\tfrac{37}{9}-4}$$
$$\sqrt{\tfrac{37}{9}+\tfrac{63}{9}}+3=\sqrt{\tfrac{37}{9}-\tfrac{36}{9}}$$
$$\sqrt{\tfrac{100}{9}}+3=\sqrt{\tfrac{1}{9}}$$
$$\tfrac{10}{3}+3=\tfrac{1}{3}\Rightarrow\tfrac{10}{3}+\tfrac{9}{3}=\tfrac{1}{3}\Rightarrow\tfrac{19}{3}=\tfrac{1}{3}$$

This is a false statement.

Solution set: $\varnothing$

46. $\sqrt{x+5}-2=\sqrt{x-1}$

$$\left(\sqrt{x+5}-2\right)^2=\left(\sqrt{x-1}\right)^2$$
$$(x+5)-4\sqrt{x+5}+4=x-1$$
$$x+9-4\sqrt{x+5}=x-1\Rightarrow -4\sqrt{x+5}=-10$$
$$2\sqrt{x+5}=5\Rightarrow\left(2\sqrt{x+5}\right)^2=5^2$$
$$4(x+5)=25\Rightarrow 4x+20=25$$
$$4x=5\Rightarrow x=\frac{5}{4}$$

Check $x=\frac{5}{4}$.

$$\sqrt{x+5}-2=\sqrt{x-1}$$
$$\sqrt{\tfrac{5}{4}+5}-2\overset{?}{=}\sqrt{\tfrac{5}{4}-1}$$
$$\sqrt{\tfrac{5}{4}+\tfrac{20}{4}}-2=\sqrt{\tfrac{5}{4}-\tfrac{4}{4}}$$
$$\sqrt{\tfrac{25}{4}}-2=\sqrt{\tfrac{1}{4}}$$
$$\tfrac{5}{2}-2=\tfrac{1}{2}\Rightarrow\tfrac{5}{2}-\tfrac{4}{2}=\tfrac{1}{2}\Rightarrow\tfrac{1}{2}=\tfrac{1}{2}$$

This is a true statement.

Solution set: $\left\{\frac{5}{4}\right\}$

47. $\sqrt{x+2}-\sqrt{2x+5}=1$

$$\sqrt{x+2}=\sqrt{2x+5}-1$$
$$\left(\sqrt{x+2}\right)^2=\left(\sqrt{2x+5}-1\right)^2$$
$$x+2=(2x+5)-2\sqrt{2x+5}+1$$
$$x+2=2x+6-2\sqrt{2x+5}$$
$$2\sqrt{2x+5}=x+4$$
$$\left(2\sqrt{2x+5}\right)^2=(x+4)^2$$
$$4(2x+5)=x^2+8x+16$$
$$8x+20=x^2+8x+16$$
$$0=x^2-4\Rightarrow 0=(x+2)(x-2)$$
$$x=\pm 2$$

Check $x = 2$.

$$\sqrt{x+2}=\sqrt{2x+5}-1$$
$$\sqrt{2+2}\stackrel{?}{=}\sqrt{2(2)+5}-1$$
$$\sqrt{4}=\sqrt{4+5}-1\Rightarrow 2=\sqrt{9}-1$$
$$2=3-1\Rightarrow 2=2$$

This is a true statement. 2 is a solution.

Check $x = -2$.

$$\sqrt{x+2}=\sqrt{2x+5}-1$$
$$\sqrt{-2+2}\stackrel{?}{=}\sqrt{2(-2)+5}-1$$
$$\sqrt{0}=\sqrt{-4+5}-1\Rightarrow 0=\sqrt{1}-1$$
$$0=1-1\Rightarrow 0=0$$

This is a true statement. -2 is a solution.

Solution set: $\{\pm 2\}$

48. $\sqrt{4x+1}-\sqrt{x-1}=2$

$$\sqrt{4x+1}=\sqrt{x-1}+2$$
$$\left(\sqrt{4x+1}\right)^2=\left(\sqrt{x-1}+2\right)^2$$
$$4x+1=(x-1)+4\sqrt{x-1}+4$$
$$4x+1=x+3+4\sqrt{x-1}$$
$$3x-2=4\sqrt{x-1}$$
$$(3x-2)^2=\left(4\sqrt{x-1}\right)^2$$
$$9x^2-12x+4=16(x-1)$$
$$9x^2-12x+4=16x-16$$
$$9x^2-28x+20=0\Rightarrow(9x-10)(x-2)=0$$
$$x=\tfrac{10}{9}\text{ or }x=2$$

Check $x=\frac{10}{9}$.

$$\sqrt{4x+1}=\sqrt{x-1}+2$$
$$\sqrt{4\left(\tfrac{10}{9}\right)+1}\stackrel{?}{=}\sqrt{\tfrac{10}{9}-1}+2$$
$$\sqrt{\tfrac{40}{9}+1}=\sqrt{\tfrac{10}{9}-\tfrac{9}{9}}+2$$
$$\sqrt{\tfrac{40}{9}+\tfrac{9}{9}}=\sqrt{\tfrac{1}{9}}+2\Rightarrow\sqrt{\tfrac{49}{9}}=\tfrac{1}{3}+2$$
$$\tfrac{7}{3}=\tfrac{1}{3}+\tfrac{6}{3}\Rightarrow\tfrac{7}{3}=\tfrac{7}{3}$$

This is a true statement. $\frac{10}{9}$ is a solution.

Check $x = 2$.

$$\sqrt{4x+1}=\sqrt{x-1}+2$$
$$\sqrt{4(2)+1}\stackrel{?}{=}\sqrt{2-1}+2$$
$$\sqrt{8+1}=\sqrt{1}+2$$
$$\sqrt{9}=1+2\Rightarrow 3=3$$

This is a true statement. 2 is a solution.

Solution set: $\left\{\frac{10}{9},2\right\}$

49. $\sqrt{3x}=\sqrt{5x+1}-1$

$$\left(\sqrt{3x}\right)^2=\left(\sqrt{5x+1}-1\right)^2$$
$$3x=(5x+1)-2\sqrt{5x+1}+1$$
$$3x=5x+2-2\sqrt{5x+1}$$
$$2\sqrt{5x+1}=2+2x\Rightarrow\sqrt{5x+1}=1+x$$
$$\left(\sqrt{5x+1}\right)^2=(1+x)^2\Rightarrow 5x+1=1+2x+x^2$$
$$0=x^2-3x\Rightarrow 0=x(x-3)\Rightarrow$$
$$x=0\text{ or }x=3$$

Check $x = 0$.

$$\sqrt{3x}=\sqrt{5x+1}-1$$
$$\sqrt{3(0)}\stackrel{?}{=}\sqrt{5(0)+1}-1$$
$$\sqrt{0}=\sqrt{0+1}-1\Rightarrow 0=\sqrt{1}-1$$
$$0=1-1\Rightarrow 0=0$$

This is a true statement. 0 is a solution.

Check $x = 3$.

$$\sqrt{3x}=\sqrt{5x+1}-1$$
$$\sqrt{3(3)}\stackrel{?}{=}\sqrt{5(3)+1}-1$$
$$\sqrt{9}=\sqrt{15+1}-1\Rightarrow 3=\sqrt{16}-1$$
$$3=4-1\Rightarrow 3=3$$

This is a true statement. 3 is a solution.

Solution set: $\{0, 3\}$

50. $\sqrt{2x}=\sqrt{3x+12}-2$

$$\left(\sqrt{2x}\right)^2=\left(\sqrt{3x+12}-2\right)^2$$
$$2x=3x+12-4\sqrt{3x+12}+4$$
$$2x=3x+16-4\sqrt{3x+12}$$
$$4\sqrt{3x+12}=x+16$$
$$\left(4\sqrt{3x+12}\right)^2=(x+16)^2$$
$$16(3x+12)=x^2+32x+256$$
$$48x+192=x^2+32x+256$$
$$0=x^2-16x+64$$
$$0=(x-8)^2\Rightarrow x=8$$

Check $x = 8$.

$$\sqrt{2x} = \sqrt{3x+12} - 2$$
$$\sqrt{2(8)} \stackrel{?}{=} \sqrt{3(8)+12} - 2$$
$$\sqrt{16} = \sqrt{24+12} - 2$$
$$4 = \sqrt{36} - 2 \Rightarrow 4 = 6 - 2 \Rightarrow 4 = 4$$

This is a true statement.
Solution set: $\{8\}$

51. $$\sqrt{x+2} = 1 - \sqrt{3x+7}$$
$$\left(\sqrt{x+2}\right)^2 = (1 - \sqrt{3x+7})^2$$
$$x+2 = 1 - 2\sqrt{3x+7} + (3x+7)$$
$$x+2 = 3x+8 - 2\sqrt{3x+7}$$
$$2\sqrt{3x+7} = 2x+6$$
$$2\sqrt{3x+7} = 2(x+3)$$
$$\sqrt{3x+7} = x+3 \Rightarrow \left(\sqrt{3x+7}\right)^2 = (x+3)^2$$
$$3x+7 = x^2+6x+9 \Rightarrow 0 = x^2+3x+2$$
$$0 = (x+2)(x+1)$$
$$x = -2 \text{ or } x = -1$$

Check $x = -2$.

$$\sqrt{x+2} = 1 - \sqrt{3x+7}$$
$$\sqrt{-2+2} \stackrel{?}{=} 1 - \sqrt{3(-2)+7}$$
$$\sqrt{0} = 1 - \sqrt{-6+7}$$
$$0 = 1 - \sqrt{1}$$
$$0 = 1 - 1 \Rightarrow 0 = 0$$

This is a true statement. -2 is a solution.

Check $x = -1$.

$$\sqrt{x+2} = 1 - \sqrt{3x+7}$$
$$\sqrt{-1+2} \stackrel{?}{=} 1 - \sqrt{3(-1)+7}$$
$$\sqrt{1} = 1 - \sqrt{-3+7}$$
$$1 = 1 - \sqrt{4}$$
$$1 = 1 - 2 \Rightarrow 1 = -1$$

This is a false statement.
-1 is not a solution.
Solution set: $\{-2\}$

52. $$\sqrt{2x-5} = 2 + \sqrt{x-2}$$
$$\left(\sqrt{2x-5}\right)^2 = \left(2 + \sqrt{x-2}\right)^2$$
$$2x-5 = 4 + 4\sqrt{x-2} + (x-2)$$
$$2x-5 = x+2+4\sqrt{x-2}$$
$$x-7 = 4\sqrt{x-2}$$
$$(x-7)^2 = \left(4\sqrt{x-2}\right)^2$$
$$x^2-14x+49 = 16(x-2)$$
$$x^2-14x+49 = 16x-32$$
$$x^2-30x+81 = 0 \Rightarrow (x-3)(x-27) = 0 \Rightarrow$$
$$x = 3 \text{ or } x = 27$$

Check $x = 3$.

$$\sqrt{2x-5} = 2 + \sqrt{x-2}$$
$$\sqrt{2(3)-5} \stackrel{?}{=} 2 + \sqrt{3-2}$$
$$\sqrt{6-5} = 2 + \sqrt{1}$$
$$\sqrt{1} = 2 + 1 \Rightarrow 1 = 3$$

This is a false statement. 3 is not a solution.

Check $x = 27$.

$$\sqrt{2x-5} = 2 + \sqrt{27-2}$$
$$\sqrt{2(27)-5} \stackrel{?}{=} 2 + \sqrt{27-2}$$
$$\sqrt{54-5} = 2 + \sqrt{25}$$
$$\sqrt{49} = 2 + 5 \Rightarrow 7 = 7$$

This is a true statement. 27 is a solution.
Solution set: $\{27\}$

53. $$\sqrt{2\sqrt{7x+2}} = \sqrt{3x+2}$$
$$\left(\sqrt{2\sqrt{7x+2}}\right)^2 = \left(\sqrt{3x+2}\right)^2$$
$$2\sqrt{7x+2} = 3x+2$$
$$\left(2\sqrt{7x+2}\right)^2 = (3x+2)^2$$
$$4(7x+2) = 9x^2+12x+4$$
$$28x+8 = 9x^2+12x+4$$
$$0 = 9x^2-16x-4$$
$$0 = (9x+2)(x-2)$$
$$x = -\tfrac{2}{9} \text{ or } x = 2$$

Check $x = -\frac{2}{9}$.

$$\sqrt{2\sqrt{7x+2}} = \sqrt{3x+2}$$
$$\sqrt{2\sqrt{7\left(-\tfrac{2}{9}\right)+2}} \stackrel{?}{=} \sqrt{3\left(-\tfrac{2}{9}\right)+2}$$
$$\sqrt{2\sqrt{-\tfrac{14}{9}+2}} = \sqrt{-\tfrac{2}{3}+2}$$
$$\sqrt{2\sqrt{-\tfrac{14}{9}+\tfrac{18}{9}}} = \sqrt{-\tfrac{2}{3}+\tfrac{6}{3}}$$
$$\sqrt{2\sqrt{\tfrac{4}{9}}} = \sqrt{\tfrac{4}{3}} \Rightarrow \sqrt{2\left(\tfrac{2}{3}\right)} = \frac{\sqrt{4}}{\sqrt{3}}$$
$$\sqrt{\tfrac{4}{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \Rightarrow \frac{\sqrt{4}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \Rightarrow \frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

This is a true statement.
$-\frac{2}{9}$ is a solution.

(*continued on next page*)

(*continued from page 93*)

Check $x = 2$.

$$\sqrt{2\sqrt{7x+2}} = \sqrt{3x+2}$$
$$\sqrt{2\sqrt{7(2)+2}} \stackrel{?}{=} \sqrt{3(2)+2}$$
$$\sqrt{2\sqrt{14+2}} = \sqrt{6+2}$$
$$\sqrt{2\sqrt{16}} = \sqrt{8}$$
$$\sqrt{2(4)} = 2\sqrt{2}$$
$$\sqrt{8} = 2\sqrt{2} \Rightarrow 2\sqrt{2} = 2\sqrt{2}$$

This is a true statement. 2 is a solution.

Solution set: $\left\{-\frac{2}{9}, 2\right\}$

54.

$$\sqrt{3\sqrt{2x+3}} = \sqrt{5x-6}$$
$$\left(\sqrt{3\sqrt{2x+3}}\right)^2 = \left(\sqrt{5x-6}\right)^2$$
$$3\sqrt{2x+3} = 5x-6$$
$$\left(3\sqrt{2x+3}\right)^2 = (5x-6)^2$$
$$9(2x+3) = 25x^2 - 60x + 36$$
$$18x + 27 = 25x^2 - 60x + 36$$
$$0 = 25x^2 - 78x + 9$$
$$0 = (25x-3)(x-3)$$
$$x = \frac{3}{25} \text{ or } x = 3$$

Check $x = \frac{3}{25}$.

$$\sqrt{3\sqrt{2x+3}} = \sqrt{5x-6}$$
$$\sqrt{3\sqrt{2\left(\frac{3}{25}\right)+3}} \stackrel{?}{=} \sqrt{5\left(\frac{3}{25}\right)-6}$$
$$\sqrt{3\sqrt{\frac{6}{25}+3}} = \sqrt{\frac{3}{5}-6}$$
$$\sqrt{3\sqrt{\frac{6}{25}+\frac{75}{25}}} = \sqrt{\frac{3}{5}-\frac{30}{5}}$$
$$\sqrt{3\sqrt{\frac{81}{25}}} = \sqrt{-\frac{27}{5}} \Rightarrow \sqrt{3\left(\frac{9}{5}\right)} = \frac{\sqrt{-27}}{\sqrt{5}}$$
$$\sqrt{\frac{27}{5}} = \frac{3i\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$
$$\frac{\sqrt{27}}{\sqrt{5}} = \frac{3i\sqrt{15}}{5}$$
$$\frac{3\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3i\sqrt{15}}{5} \Rightarrow \frac{3\sqrt{15}}{5} = \frac{3i\sqrt{15}}{5}$$

This is a false statement. $\frac{3}{25}$ is not a solution.

Check $x = 3$.

$$\sqrt{3\sqrt{2x+3}} = \sqrt{5x-6}$$
$$\sqrt{3\sqrt{2(3)+3}} \stackrel{?}{=} \sqrt{5(3)-6}$$
$$\sqrt{3\sqrt{6+3}} = \sqrt{15-6}$$
$$\sqrt{3\sqrt{9}} = \sqrt{9}$$
$$\sqrt{3(3)} = 3 \Rightarrow \sqrt{9} = 3 \Rightarrow 3 = 3$$

This is a true statement. 3 is a solution

Solution set: $\{3\}$

55.

$$3 - \sqrt{x} = \sqrt{2\sqrt{x}-3}$$
$$\left(3-\sqrt{x}\right)^2 = \left(\sqrt{2\sqrt{x}-3}\right)^2$$
$$9 - 6\sqrt{x} + x = 2\sqrt{x} - 3$$
$$12 + x = 8\sqrt{x}$$
$$(12+x)^2 = \left(8\sqrt{x}\right)^2$$
$$144 + 24x + x^2 = 64x$$
$$x^2 - 40x + 144 = 0$$
$$(x-36)(x-4) = 0 \Rightarrow x = 36 \text{ or } x = 4$$

Check $x = 36$.

$$3 - \sqrt{x} = \sqrt{2\sqrt{x}-3}$$
$$3 - \sqrt{36} \stackrel{?}{=} \sqrt{2\sqrt{36}-3}$$
$$3 - 6 = \sqrt{2(6)-3}$$
$$-3 = \sqrt{12-3} \Rightarrow -3 = \sqrt{9} \Rightarrow -3 = 3$$

This is a false statement. 36 is not a solution.

Check $x = 4$.

$$3 - \sqrt{x} = \sqrt{2\sqrt{x}-3}$$
$$3 - \sqrt{4} \stackrel{?}{=} \sqrt{2\sqrt{4}-3}$$
$$3 - 2 = \sqrt{2(2)-3}$$
$$1 = \sqrt{4-3} \Rightarrow 1 = \sqrt{1} \Rightarrow 1 = 1$$

This is a true statement. 4 is a solution.

Solution set: $\{4\}$

56.

$$\sqrt{x} + 2 = \sqrt{4+7\sqrt{x}}$$
$$\left(\sqrt{x}+2\right)^2 = \left(\sqrt{4+7\sqrt{x}}\right)^2$$
$$x + 4\sqrt{x} + 4 = 4 + 7\sqrt{x}$$
$$x = 3\sqrt{x} \Rightarrow x^2 = \left(3\sqrt{x}\right)^2$$
$$x^2 = 9x \Rightarrow x^2 - 9x = 0$$
$$x(x-9) = 0 \Rightarrow x = 0 \text{ or } x = 9$$

Check $x = 0$.

$$\sqrt{x} + 2 = \sqrt{4+7\sqrt{x}}$$
$$\sqrt{0} + 2 \stackrel{?}{=} \sqrt{4+7\sqrt{0}}$$
$$0 + 2 = \sqrt{4+7(0)}$$
$$2 = \sqrt{4+0} \Rightarrow 2 = \sqrt{4} \Rightarrow 2 = 2$$

This is a true statement. 0 is a solution.

Check $x = 9$.

$$\sqrt{x}+2=\sqrt{4+7\sqrt{x}}$$
$$\sqrt{9}+2\stackrel{?}{=}\sqrt{4+7\sqrt{9}}$$
$$3+2=\sqrt{4+7(3)}$$
$$5=\sqrt{4+21}\Rightarrow 5=\sqrt{25}\Rightarrow 5=5$$

This is a true statement. 9 is a solution.

Solution set: $\{0,9\}$

57. $\sqrt[3]{4x+3}=\sqrt[3]{2x-1}$

$$\left(\sqrt[3]{4x+3}\right)^3=\left(\sqrt[3]{2x-1}\right)^3$$
$$4x+3=2x-1\Rightarrow 2x=-4\Rightarrow x=-2$$

Check $x=-2$.

$$\sqrt[3]{4(-2)+3}=\sqrt[3]{2(-2)-1}$$
$$\sqrt[3]{-5}\stackrel{?}{=}\sqrt[3]{-5}\Rightarrow -\sqrt[3]{5}=-\sqrt[3]{5}$$

This is a true statement. -2 is a solution.

Solution set: $\{-2\}$

58. $\sqrt[3]{2x}=\sqrt[3]{5x+2}\Rightarrow\left(\sqrt[3]{2x}\right)^3=\left(\sqrt[3]{5x+2}\right)^3$

$$2x=5x+2\Rightarrow -3x=2\Rightarrow x=-\frac{2}{3}$$

Check $x=-\frac{2}{3}$.

$$\sqrt[3]{2\left(-\tfrac{2}{3}\right)}=\sqrt[3]{5\left(-\tfrac{2}{3}\right)+2}$$
$$\sqrt[3]{-\tfrac{4}{3}}\stackrel{?}{=}\sqrt[3]{-\tfrac{10}{3}+2}\Rightarrow-\sqrt[3]{\tfrac{4}{3}}=\sqrt[3]{-\tfrac{10}{3}+\tfrac{6}{3}}\Rightarrow$$
$$-\frac{\sqrt[3]{4}}{\sqrt[3]{3}}\cdot\frac{\sqrt[3]{9}}{\sqrt[3]{9}}=\sqrt[3]{-\tfrac{4}{3}}\Rightarrow-\frac{\sqrt[3]{36}}{3}=-\sqrt[3]{\tfrac{4}{3}}\Rightarrow$$
$$-\frac{\sqrt[3]{36}}{3}=-\frac{\sqrt[3]{4}}{\sqrt[3]{3}}\cdot\frac{\sqrt[3]{9}}{\sqrt[3]{9}}\Rightarrow-\frac{\sqrt[3]{36}}{3}=-\frac{\sqrt[3]{36}}{3}$$

This is a true statement.

Solution set: $\left\{-\frac{2}{3}\right\}$

59. $\sqrt[3]{5x^2-6x+2}-\sqrt[3]{x}=0$

$$\sqrt[3]{5x^2-6x+2}=\sqrt[3]{x}$$
$$\left(\sqrt[3]{5x^2-6x+2}\right)^3=\left(\sqrt[3]{x}\right)^3$$
$$5x^2-6x+2=x$$
$$5x^2-7x+2=0$$
$$(5x-2)(x-1)=0\Rightarrow x=\tfrac{2}{5}\text{ or }x=1$$

Check $x=\frac{2}{5}$.

$$\sqrt[3]{5x^2-6x+2}-\sqrt[3]{x}=0$$
$$\sqrt[3]{5\left(\tfrac{2}{5}\right)^2-6\left(\tfrac{2}{5}\right)+2}-\sqrt[3]{\tfrac{2}{5}}\stackrel{?}{=}0$$
$$\sqrt[3]{5\left(\tfrac{4}{25}\right)-\tfrac{12}{5}+2}-\sqrt[3]{\tfrac{2}{5}}=0$$
$$\sqrt[3]{\tfrac{4}{5}-\tfrac{12}{5}+\tfrac{10}{5}}-\sqrt[3]{\tfrac{2}{5}}=0$$
$$\sqrt[3]{\tfrac{2}{5}}-\sqrt[3]{\tfrac{2}{5}}=0\Rightarrow 0=0$$

This is a true statement. $\frac{2}{5}$ is a solution.

Check $x=1$.

$$\sqrt[3]{5x^2-6x+2}-\sqrt[3]{x}=0$$
$$\sqrt[3]{5(1)^2-6(1)+2}-\sqrt[3]{1}\stackrel{?}{=}0$$
$$\sqrt[3]{5(1)-6+2}-1=0$$
$$\sqrt[3]{5-6+2}-1=0$$
$$\sqrt[3]{1}-1=0\Rightarrow 1-1=0\Rightarrow 0=0$$

This is a true statement. 1 is a solution.

Solution set: $\left\{\frac{2}{5},1\right\}$

60. $\sqrt[3]{3x^2-9x+8}=\sqrt[3]{x}$

$$\left(\sqrt[3]{3x^2-9x+8}\right)^3=\left(\sqrt[3]{x}\right)^3$$
$$3x^2-9x+8=x\Rightarrow 3x^2-10x+8=0\Rightarrow$$
$$(3x-4)(x-2)=0\Rightarrow x=\tfrac{4}{3}\text{ or }x=2$$

Check $x=\frac{4}{3}$.

$$\sqrt[3]{3x^2-9x+8}=\sqrt[3]{x}$$
$$\sqrt[3]{3\left(\tfrac{4}{3}\right)^2-9\left(\tfrac{4}{3}\right)+8}\stackrel{?}{=}\sqrt[3]{\tfrac{4}{3}}$$
$$\sqrt[3]{3\left(\tfrac{16}{9}\right)-12+8}=\tfrac{\sqrt[3]{4}}{\sqrt[3]{3}}\cdot\tfrac{\sqrt[3]{9}}{\sqrt[3]{9}}\Rightarrow\sqrt[3]{\tfrac{16}{3}-4}=\tfrac{\sqrt[3]{36}}{3}$$
$$\sqrt[3]{\tfrac{16}{3}-\tfrac{12}{3}}=\tfrac{\sqrt[3]{36}}{3}\Rightarrow\sqrt[3]{\tfrac{4}{3}}=\tfrac{\sqrt[3]{36}}{3}$$
$$\tfrac{\sqrt[3]{4}}{\sqrt[3]{3}}\cdot\tfrac{\sqrt[3]{9}}{\sqrt[3]{9}}=\tfrac{\sqrt[3]{36}}{3}\Rightarrow\tfrac{\sqrt[3]{36}}{3}=\tfrac{\sqrt[3]{36}}{3}$$

This is a true statement. $\frac{4}{3}$ is a solution.

Check $x=2$.

$$\sqrt[3]{3x^2-9x+8}=\sqrt[3]{x}$$
$$\sqrt[3]{3(2)^2-9(2)+8}\stackrel{?}{=}\sqrt[3]{2}$$
$$\sqrt[3]{3(4)-18+8}=\sqrt[3]{2}$$
$$\sqrt[3]{12-10}=\sqrt[3]{2}\Rightarrow\sqrt[3]{2}=\sqrt[3]{2}$$

This is a true statement. 2 is a solution.

Solution set: $\left\{\frac{4}{3},2\right\}$

61. $(2x+5)^{1/3}-(6x-1)^{1/3}=0$

$$(2x+5)^{1/3}=(6x-1)^{1/3}$$
$$\left[(2x+5)^{1/3}\right]^3=\left[(6x-1)^{1/3}\right]^3$$
$$2x+5=6x-1$$
$$5=4x-1\Rightarrow 6=4x$$
$$\tfrac{6}{4}=x\Rightarrow x=\tfrac{3}{2}$$

Check $x=\frac{3}{2}$.

$$(2x+5)^{1/3}-(6x-1)^{1/3}=0$$
$$\left[2\left(\tfrac{3}{2}\right)+5\right]^{1/3}-\left[6\left(\tfrac{3}{2}\right)-1\right]^{1/3}\stackrel{?}{=}0$$
$$(3+5)^{1/3}-(9-1)^{1/3}=0$$
$$8^{1/3}-8^{1/3}=0$$
$$2-2=0\Rightarrow 0=0$$

This is a true statement.

Solution set: $\left\{\frac{3}{2}\right\}$

62. $(3x+7)^{1/3}-(4x+2)^{1/3}=0$

$$(3x+7)^{1/3}=(4x+2)^{1/3}$$
$$\left[(3x+7)^{1/3}\right]^3=\left[(4x+2)^{1/3}\right]^3$$
$$3x+7=4x+2\Rightarrow 5=x$$

Check $x=5$.

$$(3x+7)^{1/3}-(4x+2)^{1/3}=0$$
$$\left[3(5)+7\right]^{1/3}-\left[4(5)+2\right]^{1/3}\stackrel{?}{=}0$$
$$(15+7)^{1/3}-(20+2)^{1/3}=0$$
$$22^{1/3}-22^{1/3}=0\Rightarrow 0=0$$

This is a true statement.

Solution set: $\{5\}$

63. $\sqrt[4]{x-15}=2\Rightarrow\left(\sqrt[4]{x-15}\right)^4=2^4\Rightarrow$

$x-15=16\Rightarrow x=31$

Check $x=31$.

$$\sqrt[4]{x-15}=2\Rightarrow\sqrt[4]{31-15}\stackrel{?}{=}2$$
$$\sqrt[4]{16}=2\Rightarrow 2=2$$

This is a true statement.

Solution set: $\{31\}$

64. $\sqrt[4]{3x+1}=1$

$$\left(\sqrt[4]{3x+1}\right)^4=1^4\Rightarrow 3x+1=1$$
$$3x=0\Rightarrow x=0$$

Check $x=0$.

$$\sqrt[4]{3x+1}=1\Rightarrow\sqrt[4]{3(0)+1}\stackrel{?}{=}1$$
$$\sqrt[4]{0+1}=1\Rightarrow\sqrt[4]{1}=1\Rightarrow 1=1$$

This is a true statement.

Solution set: $\{0\}$

65. $\sqrt[4]{x^2+2x}=\sqrt[4]{3}\Rightarrow\left(\sqrt[4]{x^2+2x}\right)^4=\left(\sqrt[4]{3}\right)^4$

$$x^2+2x=3\Rightarrow x^2+2x-3=0$$
$$(x+3)(x-1)=0\Rightarrow x=-3\text{ or }x=1$$

Check $x=-3$.

$$\sqrt[4]{x^2+2x}=\sqrt[4]{3}$$
$$\sqrt[4]{(-3)^2+2(-3)}\stackrel{?}{=}\sqrt[4]{3}$$
$$\sqrt[4]{9-6}=\sqrt[4]{3}\Rightarrow\sqrt[4]{3}=\sqrt[4]{3}$$

This is a true statement. -3 is a solution.

Check $x=1$.

$$\sqrt[4]{x^2+2x}=\sqrt[4]{3}$$
$$\sqrt[4]{1^2+2(1)}\stackrel{?}{=}\sqrt[4]{3}$$
$$\sqrt[4]{1+2}=\sqrt[4]{3}\Rightarrow\sqrt[4]{3}=\sqrt[4]{3}$$

This is a true statement. 1 is a solution.

Solution set: $\{-3,1\}$

66. $\sqrt[4]{x^2+6x}=2\Rightarrow\left(\sqrt[4]{x^2+6x}\right)^4=2^4$

$$x^2+6x=16\Rightarrow x^2+6x-16=0$$
$$(x+8)(x-2)=0\Rightarrow x=-8\text{ or }x=2$$

Check $x=-8$.

$$\sqrt[4]{x^2+6x}=2$$
$$\sqrt[4]{(-8)^2+6(-8)}\stackrel{?}{=}2$$
$$\sqrt[4]{64-48}=2\Rightarrow\sqrt[4]{16}=2\Rightarrow 2=2$$

This is a true statement. -8 is a solution.

Check $x=2$.

$$\sqrt[4]{x^2+6x}=2$$
$$\sqrt[4]{2^2+6(2)}\stackrel{?}{=}2$$
$$\sqrt[4]{4+12}=2\Rightarrow\sqrt[4]{16}=2\Rightarrow 2=2$$

This is a true statement. 2 is a solution.

Solution set: $\{-8,2\}$

67. $(x^2+24x)^{1/4}=3 \Rightarrow \left[\left(x^2+24x\right)^{1/4}\right]^4=3^4 \Rightarrow$

$x^2+24x=81 \Rightarrow x^2+24x-81=0 \Rightarrow$

$(x+27)(x-3)=0 \Rightarrow x+27=0 \Rightarrow x=-27$ or

$x-3=0 \Rightarrow x=3$

Check $x=-27$.

$(x^2+24x)^{1/4}=3$

$\left[(-27)^2+24(-27)\right]^{1/4} \stackrel{?}{=} 3$

$(729-648)^{1/4}=3$

$81^{1/4}=3 \Rightarrow 3=3$

This is a true statement. -27 is a solution.

Check $x=3$.

$(x^2+24x)^{1/4}=3$

$\left[3^2+24(3)\right]^{1/4} \stackrel{?}{=} 3$

$(9+72)^{1/4}=3 \Rightarrow 81^{1/4}=3 \Rightarrow 3=3$

This is a true statement. 3 is a solution.

Solution set: $\{-27, 3\}$

68. $\left(3x^2+52x\right)^{1/4}=4 \Rightarrow \left[\left(3x^2+52x\right)^{1/4}\right]^4=4^4$

$3x^2+52x=256 \Rightarrow 3x^2+52x-256=0$

$(3x+64)(x-4)=0 \Rightarrow x=-\frac{64}{3}$ or $x=4$

Check $x=-\frac{64}{3}$.

$\left(3x^2+52x\right)^{1/4}=4$

$\left[3\left(-\frac{64}{3}\right)^2+52\left(-\frac{64}{3}\right)\right]^{1/4} \stackrel{?}{=} 4$

$\left[3\left(\frac{4096}{9}\right)-\frac{3328}{3}\right]^{1/4}=4$

$\left(\frac{4096}{3}-\frac{3328}{3}\right)^{1/4}=4$

$\left(\frac{768}{3}\right)^{1/4}=4$

$256^{1/4}=4 \Rightarrow 4=4$

This is a true statement.

$-\frac{64}{3}$ is a solution.

Check $x=4$.

$\left(3x^2+52x\right)^{1/4}=4$

$\left[3(4)^2+52(4)\right]^{1/4} \stackrel{?}{=} 4$

$\left[3(16)+208\right]^{1/4}=4$

$(48+208)^{1/4}=4$

$256^{1/4}=4 \Rightarrow 4=4$

This is a true statement. 4 is a solution.

Solution set: $\left\{-\frac{64}{3}, 4\right\}$

69. $2x^4-7x^2+5=0$

Let $u=x^2$; then $u^2=x^4$. With this substitution, the equation becomes

$2u^2-7u+5=0$.

$2u^2-7u+5=0 \Rightarrow (u-1)(2u-5)=0 \Rightarrow$

$u=1$ or $u=\frac{5}{2}$

To find x, replace u with x^2.

$x^2=1 \Rightarrow x=\pm 1$ or

$x^2=\frac{5}{2} \Rightarrow x=\pm\sqrt{\frac{5}{2}}=\pm\frac{\sqrt{5}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\pm\frac{\sqrt{10}}{2}$

Solution set: $\left\{\pm 1, \pm\frac{\sqrt{10}}{2}\right\}$

70. $4x^4-8x^2+3=0$

Let $u=x^2$; then $u^2=x^4$.

$4u^2-8u+3=0 \Rightarrow (2u-1)(2u-3)=0 \Rightarrow$.

$u=\frac{1}{2}$ or $u=\frac{3}{2}$

To find x, replace u with x^2.

$x^2=\frac{3}{2} \Rightarrow x=\pm\sqrt{\frac{3}{2}}=\pm\frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \Rightarrow x=\pm\frac{\sqrt{6}}{2}$ or

$x^2=\frac{1}{2} \Rightarrow x=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \Rightarrow x=\pm\frac{\sqrt{2}}{2}$

Solution set: $\left\{\pm\frac{\sqrt{6}}{2}, \pm\frac{\sqrt{2}}{2}\right\}$

71. $x^4+2x^2-15=0$

Let $u=x^2$; then $u^2=x^4$.

$u^2+2u-15=0 \Rightarrow (u-3)(u+5)=0$.

$u=3$ or $u=-5$

To find x, replace u with x^2.

$x^2=3 \Rightarrow x=\pm\sqrt{3}$ or

$x^2=-5 \Rightarrow x=\pm\sqrt{-5}=\pm i\sqrt{5}$

Solution set: $\left\{\pm\sqrt{3}, \pm i\sqrt{5}\right\}$

72. $3x^4+10x^2-25=0$

Let $u=x^2$; then $u^2=x^4$.

$3u^2+10u-25=0 \Rightarrow (u+5)(3u-5)=0 \Rightarrow$.

$u=-5$ or $u=\frac{5}{3}$

To find x, replace u with x^2.

$x^2=-5 \Rightarrow x=\pm\sqrt{-5}=\pm i\sqrt{5}$ or

$x^2=\frac{5}{3} \Rightarrow x=\pm\sqrt{\frac{5}{3}} \Rightarrow x=\pm\frac{\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\pm\frac{\sqrt{15}}{3}$

Solution set: $\left\{\pm i\sqrt{5}, \pm\frac{\sqrt{15}}{3}\right\}$

73. $(2x-1)^{2/3} = x^{1/3}$

$[(2x-1)^{2/3}]^3 = (x^{1/3})^3$

$(2x-1)^2 = x \Rightarrow 4x^2 - 4x + 1 = x$

$4x^2 - 5x + 1 = 0 \Rightarrow (4x-1)(x-1) = 0 \Rightarrow$

$x = \dfrac{1}{4}$ or $x = 1$

Check $x = \frac{1}{4}$.

$(2x-1)^{2/3} = x^{1/3} \Rightarrow \left[2\left(\frac{1}{4}\right)-1\right]^{2/3} \stackrel{?}{=} \left(\frac{1}{4}\right)^{1/3}$

$\left[\frac{1}{2}-1\right]^{2/3} = \frac{1}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} \Rightarrow \left[-\frac{1}{2}\right]^{2/3} = \frac{\sqrt[3]{2}}{2} \Rightarrow$

$\left[\left(-\frac{1}{2}\right)^2\right]^{1/3} = \frac{\sqrt[3]{2}}{2} \Rightarrow \left(\frac{1}{4}\right)^{1/3} = \frac{\sqrt[3]{2}}{2}$

$\frac{1}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{\sqrt[3]{2}}{2} \Rightarrow \frac{\sqrt[3]{2}}{2} = \frac{\sqrt[3]{2}}{2}$

This is a true statement. $\frac{1}{4}$ is a solution.

Check $x = 1$.

$(2x-1)^{2/3} = x^{1/3} \Rightarrow \left[2(1)-1\right]^{2/3} \stackrel{?}{=} (1)^{1/3}$

$\left[2-1\right]^{2/3} = 1 \Rightarrow 1^{2/3} = 1 \Rightarrow 1 = 1$

This is a true statement. 1 is a solution.

Solution set: $\left\{\frac{1}{4}, 1\right\}$

74. $(x-3)^{2/5} = (4x)^{1/5}$

$\left[(x-3)^{2/5}\right]^5 = \left[(4x)^{1/5}\right]^5$

$(x-3)^2 = 4x$

$x^2 - 6x + 9 = 4x$

$x^2 - 10x + 9 = 0$

$(x-1)(x-9) = 0 \Rightarrow x = 1$ or $x = 9$

Check $x = 1$.

$(x-3)^{2/5} = (4x)^{1/5}$

$(1-3)^{2/5} \stackrel{?}{=} (4 \cdot 1)^{1/5}$

$(-2)^{2/5} = 4^{1/5}$

$\left[(-2)^2\right]^{1/5} = \sqrt[5]{4} \Rightarrow 4^{1/5} = \sqrt[5]{4} \Rightarrow \sqrt[5]{4} = \sqrt[5]{4}$

This is a true statement. 1 is a solution.

Check $x = 9$.

$(x-3)^{2/5} = (4x)^{1/5}$

$(9-3)^{2/5} \stackrel{?}{=} (4 \cdot 9)^{1/5}$

$6^{2/5} = 36^{1/5} \Rightarrow \left[6^2\right]^{1/5} = \sqrt[5]{36}$

$36^{1/5} = \sqrt[5]{36} \Rightarrow \sqrt[5]{36} = \sqrt[5]{36}$

This is a true statement. 9 is a solution.

Solution set: $\{1, 9\}$

75. $x^{2/3} = 2x^{1/3} \Rightarrow \left(x^{2/3}\right)^3 = \left(2x^{1/3}\right)^3 \Rightarrow$

$x^2 = 8x \Rightarrow x^2 - 8x = 0 \Rightarrow x(x-8) = 0 \Rightarrow$

$x = 0$ or $x = 8$

Check $x = 0$.

$x^{2/3} = 2x^{1/3}$

$0^{2/3} \stackrel{?}{=} 2\left(0^{1/3}\right) \Rightarrow 0 = 2 \cdot 0 \Rightarrow 0 = 0$

This is a true statement. 0 is a solution.

Check $x = 8$.

$x^{2/3} = 2x^{1/3} \Rightarrow 8^{2/3} \stackrel{?}{=} 2\left(8^{1/3}\right)$

$\left(8^2\right)^{1/3} = 2 \cdot 2 \Rightarrow 64^{1/3} = 4 \Rightarrow 4 = 4$

This is a true statement. 8 is a solution.

Solution set: $\{0, 8\}$

76. $3x^{3/4} = x^{1/2} \Rightarrow \left(3x^{3/4}\right)^4 = \left(x^{1/2}\right)^4 \Rightarrow$

$81x^3 = x^2 \Rightarrow 81x^3 - x^2 = 0 \Rightarrow$

$x^2(81x-1) = 0 \Rightarrow x = 0$ or $x = \frac{1}{81}$

Check $x = 0$.

$3x^{3/4} = x^{1/2}$

$3(0)^{3/4} \stackrel{?}{=} 0^{1/2} \Rightarrow 3 \cdot 0 = 0 \Rightarrow 0 = 0$

This is a true statement. 0 is a solution.

Check $x = \frac{1}{81}$.

$3x^{3/4} = x^{1/2}$

$3\left(\frac{1}{81}\right)^{3/4} \stackrel{?}{=} \left(\frac{1}{81}\right)^{1/2} \Rightarrow 3 \cdot \left[\left(\frac{1}{81}\right)^{1/4}\right]^3 = \frac{1}{9}$

$3 \cdot \left(\frac{1}{3}\right)^3 = \frac{1}{9} \Rightarrow 3 \cdot \frac{1}{27} = \frac{1}{9} \Rightarrow \frac{1}{9} = \frac{1}{9}$

This is a true statement. $\frac{1}{81}$ is a solution.

Solution set: $\left\{0, \frac{1}{81}\right\}$

77. $(x-1)^{2/3} + (x-1)^{1/3} - 12 = 0$

Let $u = (x-1)^{1/3}$ then,

$u^2 = \left[(x-1)^{1/3}\right]^2 = (x-1)^{2/3}$.

$u^2 + u - 12 = 0 \Rightarrow (u+4)(u-3) = 0 \Rightarrow$

$u = -4$ or $u = 3$

To find x, replace u with $(x-1)^{1/3}$.

$(x-1)^{1/3} = -4 \Rightarrow \left[(x-1)^{1/3}\right]^3 = (-4)^3 \Rightarrow$

$x - 1 = -64 \Rightarrow x = -63$ or

$(x-1)^{1/3} = 3 \Rightarrow \left[(x-1)^{1/3}\right]^3 = 3^3 \Rightarrow$

$x - 1 = 27 \Rightarrow x = 28$

Check $x = -63$.

$$(x-1)^{2/3} + (x-1)^{1/3} - 12 = 0$$
$$(-63-1)^{2/3} + (-63-1)^{1/3} - 12 \stackrel{?}{=} 0$$
$$(-64)^{2/3} + (-64)^{1/3} - 12 = 0$$
$$\left[(-64)^{1/3}\right]^2 - 4 - 12 = 0$$
$$(-4)^2 - 4 - 12 = 0$$
$$16 - 4 - 12 = 0 \Rightarrow 0 = 0$$

This is a true statement. -63 is a solution.

Check $x = 28$.

$$(x-1)^{2/3} + (x-1)^{1/3} - 12 = 0$$
$$(28-1)^{2/3} + (28-1)^{1/3} - 12 \stackrel{?}{=} 0$$
$$27^{2/3} + 27^{1/3} - 12 = 0$$
$$\left[27^{1/3}\right]^2 + 3 - 12 = 0$$
$$3^2 + 3 - 12 = 0$$
$$9 + 3 - 12 = 0 \Rightarrow 0 = 0$$

This is a true statement. 28 is a solution.

Solution set: $\{-63, 28\}$

78. $(2x-1)^{2/3} + 2(2x-1)^{1/3} - 3 = 0$

Let $u = (2x-1)^{1/3}$ then,

$u^2 = \left[(2x-1)^{1/3}\right]^2 = (2x-1)^{2/3}$.

$u^2 + 2u - 3 = 0 \Rightarrow (u+3)(u-1) = 0 \Rightarrow$

$u = -3$ or $u = 1$

To find x, replace u with $(2x-1)^{1/3}$.

$(2x-1)^{1/3} = -3 \Rightarrow \left[(2x-1)^{1/3}\right]^3 = (-3)^3 \Rightarrow$

$2x - 1 = -27 \Rightarrow 2x = -26 \Rightarrow x = -13$ or

$(2x-1)^{1/3} = 1 \Rightarrow \left[(2x-1)^{1/3}\right]^3 = 1^3 \Rightarrow$

$2x - 1 = 1 \Rightarrow 2x = 2 \Rightarrow x = 1$

Check $x = -13$.

$$(2x-1)^{2/3} + 2(2x-1)^{1/3} - 3 = 0$$
$$[2(-13)-1]^{2/3} + 2[2(-13)-1]^{1/3} - 3 \stackrel{?}{=} 0$$
$$(-26-1)^{2/3} + 2(-26-1)^{1/3} - 3 = 0$$
$$(-27)^{2/3} + 2(-27)^{1/3} - 3 = 0$$
$$\left[(-27)^{1/3}\right]^2 + 2(-3) - 3 = 0$$
$$(-3)^2 - 6 - 3 = 0$$
$$9 - 6 - 3 = 0$$
$$0 = 0$$

This is a true statement. -13 is a solution.

Check $x = 1$.

$$(2x-1)^{2/3} + 2(2x-1)^{1/3} - 3 = 0$$
$$[2(1)-1]^{2/3} + 2[2(1)-1]^{1/3} - 3 \stackrel{?}{=} 0$$
$$(2-1)^{2/3} + 2(2-1)^{1/3} - 3 = 0$$
$$1^{2/3} + 2(1)^{1/3} - 3 = 0$$
$$1 + 2(1) - 3 = 0$$
$$1 + 2 - 3 = 0$$
$$0 = 0$$

This is a true statement. 1 is a solution

Solution set: $\{-13, 1\}$.

79. $(x+1)^{2/5} - 3(x+1)^{1/5} + 2 = 0$

Let $u = (x+1)^{1/5}$ then,

$u^2 = \left[(x+1)^{1/5}\right]^2 = (x+1)^{2/5}$.

$u^2 - 3u + 2 = 0 \Rightarrow (u-1)(u-2) = 0 \Rightarrow$

$u = 1$ or $u = 2$

To find x, replace u with $(x+1)^{1/5}$.

$(x+1)^{1/5} = 1 \Rightarrow \left[(x+1)^{1/5}\right]^5 = 1^5 \Rightarrow$ or

$x + 1 = 1 \Rightarrow x = 0$

$(x+1)^{1/5} = 2 \Rightarrow \left[(x+1)^{1/5}\right]^5 = 2^5 \Rightarrow$

$x + 1 = 32 \Rightarrow x = 31$

Check $x = 0$.

$$(x+1)^{2/5} - 3(x+1)^{1/5} + 2 = 0$$
$$(0+1)^{2/5} - 3(0+1)^{1/5} + 2 \stackrel{?}{=} 0$$
$$1^{2/5} - 3(1)^{1/5} + 2 = 0$$
$$1 - 3(1) + 2 = 0 \Rightarrow 1 - 3 + 2 = 0$$

This is a true statement. 0 is a solution.

Check $x = 31$.

$$(x+1)^{2/5} - 3(x+1)^{1/5} + 2 = 0$$
$$(31+1)^{2/5} - 3(31+1)^{1/5} + 2 \stackrel{?}{=} 0$$
$$32^{2/5} - 3(32)^{1/5} + 2 = 0$$
$$\left[(32)^{1/5}\right]^2 - 3(2) + 2 = 0$$
$$2^2 - 6 + 2 = 0$$
$$4 - 6 + 2 = 0 \Rightarrow 0 = 0$$

This is a true statement. 31 is a solution.

Solution set: $\{0, 31\}$

80. $(x+5)^{4/3}+(x+5)^{2/3}-20=0$

Let $u=(x+5)^{2/3}$ then,

$u^2=\left[(x+5)^{2/3}\right]^2=(x+5)^{4/3}$.

$u^2+u-20=0 \Rightarrow (u+5)(u-4)=0 \Rightarrow$
$u=-5$ or $u=4$

To find *x*, replace *u* with $(x+5)^{2/3}$.

$(x+5)^{2/3}=-5 \Rightarrow \left[(x+5)^{2/3}\right]^3=(-5)^3 \Rightarrow$
$(x+5)^2=-125 \Rightarrow x+5=\pm\sqrt{-125} \Rightarrow$
$x+5=\pm 5i\sqrt{5} \Rightarrow x=-5\pm 5i\sqrt{5}$ or

$(x+5)^{2/3}=4 \Rightarrow \left[(x+5)^{2/3}\right]^3=4^3 \Rightarrow$
$(x+5)^2=64 \Rightarrow x+5=\pm\sqrt{64} \Rightarrow$
$x+5=\pm 8 \Rightarrow x=-5\pm 8$
$x=-5-8$ or $x=-5+8 \Rightarrow x=-13$ or $x=3$

Check $x=-5-5i\sqrt{5}$.

$$(x+5)^{4/3}+(x+5)^{2/3}-20=0$$
$$(-5-5i\sqrt{5}+5)^{4/3}+(-5-5i\sqrt{5}+5)^{2/3}-20\stackrel{?}{=}0$$
$$\left(-5i\sqrt{5}\right)^{4/3}+\left(-5i\sqrt{5}\right)^{2/3}-20=0$$
$$\left[\left(-5i\sqrt{5}\right)^4\right]^{1/3}+\left[\left(-5i\sqrt{5}\right)^2\right]^{1/3}-20=0$$
$$\left[(-5)^4 i^4\left(\sqrt{5}\right)^4\right]^{1/3}+\left[(-5)^2 i^2\left(\sqrt{5}\right)^2\right]^{1/3}-20=0$$
$$625(1)(25)+25(-1)(5)-20=0$$
$$15{,}625-125-20=0$$
$$15{,}480=0$$

This is a false statement. $-5-5i\sqrt{5}$ is not a solution.

Check $x=-5+5i\sqrt{5}$.

$$(x+5)^{4/3}+(x+5)^{2/3}-20=0$$
$$(-5+5i\sqrt{5}+5)^{4/3}+(-5+5i\sqrt{5}+5)^{2/3}-20\stackrel{?}{=}0$$
$$\left(5i\sqrt{5}\right)^{4/3}+\left(5i\sqrt{5}\right)^{2/3}-20=0$$
$$\left[\left(5i\sqrt{5}\right)^4\right]^{1/3}+\left[\left(5i\sqrt{5}\right)^2\right]^{1/3}-20=0$$
$$\left[5^4 i^4\left(\sqrt{5}\right)^4\right]^{1/3}+\left[5^2 i^2\left(\sqrt{5}\right)^2\right]^{1/3}-20=0$$
$$625(1)(25)+25(-1)(5)-20=0$$
$$15{,}625-125-20=0$$
$$15{,}480=0$$

This is a false statement. $-5+5i\sqrt{5}$ is not a solution.

Check $x=-13$.

$$(x+5)^{4/3}+(x+5)^{2/3}-20=0$$
$$(-13+5)^{4/3}+(-13+5)^{2/3}-20\stackrel{?}{=}0$$
$$(-8)^{4/3}+(-8)^{2/3}-20=0$$
$$\left[(-8)^{1/3}\right]^4+\left[(-8)^{1/3}\right]^2-20=0$$
$$(-2)^4+(-2)^2-20=0$$
$$16+4-20=0 \Rightarrow 0=0$$

This is a true statement -13 is a solution.

Check $x=3$.

$$(x+5)^{4/3}+(x+5)^{2/3}-20=0$$
$$(3+5)^{4/3}+(3+5)^{2/3}-20\stackrel{?}{=}0$$
$$(8)^{4/3}+(8)^{2/3}-20=0$$
$$\left[(8)^{1/3}\right]^4+\left[(8)^{1/3}\right]^2-20=0$$
$$2^4+2^2-20=0$$
$$16+4-20=0 \Rightarrow 0=0$$

This is a true statement. 3 is a solution.

Solution set: $\{-13, 3\}$

81. $6(x+2)^4-11(x+2)^2=-4$
$6(x+2)^4-11(x+2)^2+4=0$

Let $u=(x+2)^2$ then $u^2=(x+2)^4$.

$6u^2-11u+4=0 \Rightarrow (3u-4)(2u-1)=0 \Rightarrow$
$u=\frac{4}{3}$ or $u=\frac{1}{2}$

To find *x*, replace *u* with $(x+2)^2$.

$(x+2)^2=\frac{4}{3} \Rightarrow x+2=\pm\sqrt{\frac{4}{3}}=\pm\frac{2\sqrt{3}}{3}$ or
$x=-2\pm\frac{2\sqrt{3}}{3}=-\frac{6}{3}\pm\frac{2\sqrt{3}}{3}=\frac{-6\pm 2\sqrt{3}}{3}$

$(x+2)^2=\frac{1}{2} \Rightarrow x+2=\pm\sqrt{\frac{1}{2}}=\pm\frac{\sqrt{2}}{2}$
$x=-2\pm\frac{\sqrt{2}}{2}=-\frac{4}{2}\pm\frac{\sqrt{2}}{2}=\frac{-4\pm\sqrt{2}}{2}$

Solution set: $\left\{\frac{-6\pm 2\sqrt{3}}{3}, \frac{-4\pm\sqrt{2}}{2}\right\}$

82. $8(x-4)^4-10(x-4)^2=-3$
$8(x-4)^4-10(x-4)^2+3=0$

Let $u=(x-4)^2$; then $u^2=(x-4)^4$.

$8u^2-10u+3=0 \Rightarrow (2u-1)(4u-3)=0 \Rightarrow$
$u=\frac{1}{2}$ or $u=\frac{3}{4}$

$(x-4)^2 = \frac{1}{2} \Rightarrow x-4 = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$ or

$x = 4 \pm \frac{\sqrt{2}}{2} = \frac{8}{2} \pm \frac{\sqrt{2}}{2} = \frac{8\pm\sqrt{2}}{2}$

$(x-4)^2 = \frac{3}{4} \Rightarrow x-4 = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

$x = 4 \pm \frac{\sqrt{3}}{2} = \frac{8}{2} \pm \frac{\sqrt{3}}{2} = \frac{8\pm\sqrt{3}}{2}$

Solution set: $\left\{\frac{8\pm\sqrt{2}}{2}, \frac{8\pm\sqrt{3}}{2}\right\}$

83. $10x^{-2} + 33x^{-1} - 7 = 0$

Let $u = x^{-1}$; then $u^2 = x^{-2}$.

$10u^2 + 33u - 7 = 0 \Rightarrow (2u+7)(5u-1) = 0$

$u = -\frac{7}{2}$ or $u = \frac{1}{5}$

To find *x*, replace *u* with x^{-1}.

$x^{-1} = -\frac{7}{2} \Rightarrow x = -\frac{2}{7}$ or $x^{-1} = \frac{1}{5} \Rightarrow x = 5$

Solution set: $\left\{-\frac{2}{7}, 5\right\}$

84. $7x^{-2} - 10x^{-1} - 8 = 0$

Let $u = x^{-1}$; then $u^2 = x^{-2}$.

$7u^2 - 10u - 8 = 0 \Rightarrow (7u+4)(u-2) = 0$

$u = -\frac{4}{7}$ or $u = 2$

To find *x*, replace *u* with x^{-1}.

$x^{-1} = -\frac{4}{7} \Rightarrow x = -\frac{7}{4}$ or $x^{-1} = 2 \Rightarrow x = \frac{1}{2}$

Solution set: $\left\{-\frac{7}{4}, \frac{1}{2}\right\}$

85. $x^{-2/3} + x^{-1/3} - 6 = 0$

Let $u = x^{-1/3}$; then $u^2 = \left(x^{-1/3}\right)^2 = x^{-2/3}$.

$u^2 + u - 6 = 0 \Rightarrow (u+3)(u-2) = 0$

$u = -3$ or $u = 2$

To find *x*, replace *u* with $x^{-1/3}$.

$x^{-1/3} = -3 \Rightarrow \left(x^{-1/3}\right)^{-3} = (-3)^{-3} \Rightarrow$ or

$x = \frac{1}{(-3)^3} \Rightarrow x = -\frac{1}{27}$

$x^{-1/3} = 2 \Rightarrow \left(x^{-1/3}\right)^{-3} = 2^{-3} \Rightarrow$

$x = \frac{1}{2^3} \Rightarrow x = \frac{1}{8}$

Check $x = -\frac{1}{27}$.

$$x^{-2/3} + x^{-1/3} - 6 = 0$$
$$\left(-\tfrac{1}{27}\right)^{-2/3} + \left(-\tfrac{1}{27}\right)^{-1/3} - 6 \stackrel{?}{=} 0$$
$$(-27)^{2/3} + (-27)^{1/3} - 6 = 0$$
$$\left[(-27)^{1/3}\right]^2 - 3 - 6 = 0$$
$$(-3)^2 - 3 - 6 = 0$$
$$9 - 3 - 6 = 0 \Rightarrow 0 = 0$$

This is a true statement.

Check $x = \frac{1}{8}$.

$$x^{-2/3} + x^{-1/3} - 6 = 0$$
$$\left(\tfrac{1}{8}\right)^{-2/3} + \left(\tfrac{1}{8}\right)^{-1/3} - 6 \stackrel{?}{=} 0$$
$$8^{2/3} + 8^{1/3} - 6 = 0$$
$$\left(8^{1/3}\right)^2 + 2 - 6 = 0$$
$$2^2 + 2 - 6 = 0$$
$$4 + 2 - 6 = 0 \Rightarrow 0 = 0$$

This is a true statement.

Solution set: $\left\{-\frac{1}{27}, \frac{1}{8}\right\}$

86. $2x^{-4/3} - x^{-2/3} - 1 = 0$

Let $u = x^{-2/3}$; then $u^2 = \left(x^{-2/3}\right)^2 = x^{-4/3}$.

$2u^2 - u - 1 = 0 \Rightarrow (2u+1)(u-1) = 0$

$u = -\frac{1}{2}$ or $u = 1$

To find *x*, replace *u* with $x^{-2/3}$.

$x^{-2/3} = -\frac{1}{2} \Rightarrow \left(x^{-2/3}\right)^{-3} = \left(-\frac{1}{2}\right)^{-3}$

$x^2 = (-2)^3 \Rightarrow x^2 = -8 \Rightarrow x = \pm\sqrt{-8} = \pm 2i\sqrt{2}$

or $x^{-2/3} = 1 \Rightarrow \left(x^{-2/3}\right)^{-3} = 1^{-3} \Rightarrow x^2 = 1^3 \Rightarrow$

$x^2 = 1 \Rightarrow x = \pm 1$

Check $x = -2i\sqrt{2}$.

$$2x^{-4/3} - x^{-2/3} - 1 = 0$$
$$2\left(-2i\sqrt{2}\right)^{-4/3} - \left(-2i\sqrt{2}\right)^{-2/3} - 1 \stackrel{?}{=} 0$$
$$2\left[\left(-\frac{1}{2i\sqrt{2}}\right)^4\right]^{1/3} - \left[\left(-\frac{1}{2i\sqrt{2}}\right)^2\right]^{1/3} - 1 = 0$$
$$2\left(\frac{1}{2^4 i^4 \left(\sqrt{2}\right)^4}\right)^{1/3} - \left(\frac{1}{2^2 i^2 \left(\sqrt{2}\right)^2}\right)^{1/3} - 1 = 0$$
$$2\left(\frac{1}{16(1)(4)}\right)^{1/3} - \left(\frac{1}{4(-1)(2)}\right)^{1/3} - 1 = 0$$
$$2\left(\frac{1}{64}\right)^{1/3} - \left(\frac{1}{-8}\right)^{1/3} - 1 = 0$$
$$2\left(\frac{1}{4}\right) - \left(\frac{1}{-2}\right) - 1 = 0$$
$$\frac{1}{2} + \frac{1}{2} - 1 = 0$$
$$0 = 0$$

This is a true statement.

(*continued on next page*)

(*continued from page 101*)

$-2i\sqrt{2}$ is a solution.

Check $x = 2i\sqrt{2}$.

$$2x^{-4/3} - x^{-2/3} - 1 = 0$$
$$2\left(2i\sqrt{2}\right)^{-4/3} - \left(2i\sqrt{2}\right)^{-2/3} - 1 \stackrel{?}{=} 0$$
$$2\left[\left(\frac{1}{2i\sqrt{2}}\right)^4\right]^{1/3} - \left[\left(\frac{1}{2i\sqrt{2}}\right)^2\right]^{1/3} - 1 = 0$$
$$2\left(\frac{1}{2^4 i^4 \left(\sqrt{2}\right)^4}\right)^{1/3} - \left(\frac{1}{2^2 i^2 \left(\sqrt{2}\right)^2}\right)^{1/3} - 1 = 0$$
$$2\left(\frac{1}{16(1)(4)}\right)^{1/3} - \left(\frac{1}{4(-1)(2)}\right)^{1/3} - 1 = 0$$
$$2\left(\frac{1}{64}\right)^{1/3} - \left(\frac{1}{-8}\right)^{1/3} - 1 = 0$$
$$2\left(\frac{1}{4}\right) - \left(\frac{1}{-2}\right) - 1 = 0$$
$$\frac{1}{2} + \frac{1}{2} - 1 = 0 \Rightarrow 0 = 0$$

This is a true statement. $2i\sqrt{2}$ is a solution.

Check $x = -1$.

$$2x^{-4/3} - x^{-2/3} - 1 = 0$$
$$2(-1)^{-4/3} - (-1)^{-2/3} - 1 \stackrel{?}{=} 0$$
$$2\left[(-1)^4\right]^{1/3} - \left[(-1)^2\right]^{1/3} - 1 = 0$$
$$2(1)^{1/3} - (1)^{1/3} - 1 = 0$$
$$2(1) - 1 - 1 = 0$$
$$2 - 1 - 1 = 0 \Rightarrow 0 = 0$$

This is a true statement. -1 is a solution.

Check $x = 1$.

$$2x^{-4/3} - x^{-2/3} - 1 = 0$$
$$2(1)^{-4/3} - 1^{-2/3} - 1 \stackrel{?}{=} 0$$
$$2\left[1^4\right]^{1/3} - \left[1^2\right]^{1/3} - 1 = 0$$
$$2(1)^{1/3} - (1)^{1/3} - 1 = 0$$
$$2(1) - 1 - 1 = 0$$
$$2 - 1 - 1 = 0 \Rightarrow 0 = 0$$

This is a true statement. 1 is a solution.

Solution set: $\left\{\pm 1, \pm 2i\sqrt{2}\right\}$

87. $16x^{-4} - 65x^{-2} + 4 = 0$

Let $u = x^{-2}$; then $u^2 = x^{-4}$. Solve the resulting equation by factoring:

$$16u^2 - 65u + 4 = 0 \Rightarrow (u-4)(16u-1) = 0 \Rightarrow$$
$$u = 4 \text{ or } u = \frac{1}{16}$$

Find x by replacing u with x^{-2}:

$$x^{-2} = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$
$$x^{-2} = \frac{1}{16} \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$$
:

Check $x = \frac{1}{2}$

$$16\left(\frac{1}{2}\right)^{-4} - 65\left(\frac{1}{2}\right)^{-2} + 4 = 0$$
$$16(2)^4 - 65(2)^2 + 4 \stackrel{?}{=} 0$$
$$16(16) - 65(4) + 4 = 0$$
$$256 - 260 + 4 = 0$$
$$0 = 0$$

This is a true statement, so $\frac{1}{2}$ is a solution.

Check $x = -\frac{1}{2}$

$$16\left(-\frac{1}{2}\right)^{-4} - 65\left(-\frac{1}{2}\right)^{-2} + 4 = 0$$
$$16(-2)^4 - 65(-2)^2 + 4 \stackrel{?}{=} 0$$
$$16(16) - 65(4) + 4 = 0$$
$$256 - 260 + 4 = 0$$
$$0 = 0$$

This is a true statement, so $-\frac{1}{2}$ is a solution.

Check $x = 4$

$$16(4)^{-4} - 65(4)^{-2} + 4 = 0$$
$$16\left(\frac{1}{4}\right)^4 - 65\left(\frac{1}{4}\right)^2 + 4 \stackrel{?}{=} 0$$
$$16\left(\frac{1}{256}\right) - 65\left(\frac{1}{16}\right) + 4 = 0$$
$$\frac{1}{16} - \frac{65}{16} + 4 = 0$$
$$0 = 0$$

This is a true statement, so 4 is a solution.

Check $x = -4$

$$16(-4)^{-4} - 65(-4)^{-2} + 4 = 0$$
$$16\left(-\frac{1}{4}\right)^4 - 65\left(-\frac{1}{4}\right)^2 + 4 \stackrel{?}{=} 0$$
$$16\left(-\frac{1}{256}\right) - 65\left(-\frac{1}{16}\right) + 4 = 0$$
$$\frac{1}{16} - \frac{65}{16} + 4 = 0$$
$$0 = 0$$

This is a true statement, so -4 is a solution.

Solution set: $\left\{\pm\frac{1}{2}, \pm 4\right\}$

88. $625x^{-4} - 125x^{-2} + 4 = 0$

Let $u = x^{-2}$; then $u^2 = x^{-4}$. Solve the resulting equation by factoring:

$$625u^2 - 125u + 4 = 0 \Rightarrow$$
$$(25u-4)(25u-1) = 0 \Rightarrow u = \frac{4}{25} \text{ or } u = \frac{1}{25}$$

Find x by replacing u with x^{-2}:

$$x^{-2} = \frac{4}{25} \Rightarrow x^2 = \frac{25}{4} \Rightarrow x = \pm\frac{5}{2}$$
$$x^{-2} = \frac{1}{25} \Rightarrow x^2 = 5 \Rightarrow x = \pm 5$$

Check $x = \frac{5}{2}$

$$625\left(\frac{5}{2}\right)^{-4} - 125\left(\frac{5}{2}\right)^{-2} + 4 \stackrel{?}{=} 0$$
$$625\left(\frac{2}{5}\right)^{4} - 125\left(\frac{2}{5}\right)^{2} + 4 = 0$$
$$625\left(\frac{16}{625}\right) - 125\left(\frac{4}{25}\right) + 4 = 0$$
$$16 - 20 + 4 = 0 \Rightarrow 0 = 0$$

This is a true statement, so $\frac{5}{2}$ is a solution.

Check $x = -\frac{5}{2}$

$$625\left(-\frac{5}{2}\right)^{-4} - 125\left(-\frac{5}{2}\right)^{-2} + 4 \stackrel{?}{=} 0$$
$$625\left(-\frac{2}{5}\right)^{4} - 125\left(-\frac{2}{5}\right)^{2} + 4 = 0$$
$$625\left(\frac{16}{625}\right) - 125\left(\frac{4}{25}\right) + 4 = 0$$
$$16 - 20 + 4 = 0 \Rightarrow 0 = 0$$

This is a true statement, so $-\frac{5}{2}$ is a solution.

Check $x = 5$

$$625(5)^{-4} - 125(5)^{-2} + 4 \stackrel{?}{=} 0$$
$$625\left(\frac{1}{5}\right)^{4} - 125\left(\frac{1}{5}\right)^{2} + 4 = 0$$
$$625\left(\frac{1}{625}\right) - 125\left(\frac{1}{25}\right) + 4 = 0$$
$$1 - 5 + 4 = 0 \Rightarrow 0 = 0$$

This is a true statement, so 5 is a solution.

Check $x = -5$

$$625(-5)^{-4} - 125(-5)^{-2} + 4 \stackrel{?}{=} 0$$
$$625\left(-\frac{1}{5}\right)^{4} - 125\left(-\frac{1}{5}\right)^{2} + 4 = 0$$
$$625\left(\frac{1}{625}\right) - 125\left(\frac{1}{25}\right) + 4 = 0$$
$$1 - 5 + 4 = 0 \Rightarrow 0 = 0$$

This is a true statement, so -5 is a solution.

Solution set: $\left\{\pm\frac{5}{2}, \pm 5\right\}$

89. $x - \sqrt{x} - 12 = 0$

Let $u = \sqrt{x}$; then $u^2 = x$. Solve the resulting equation by factoring.

$u^2 - u - 12 = 0 \Rightarrow (u-4)(u+3) = 0$

$u = 4$ or $u = -3$

To find x, replace u with $\sqrt{x}$.

$\sqrt{x} = 4 \Rightarrow \left(\sqrt{x}\right)^2 = 4^2 \Rightarrow x = 16$ or

$\sqrt{x} = -3 \Rightarrow \left(\sqrt{x}\right)^2 = (-3)^2 \Rightarrow x = 9$

But $\sqrt{9} \neq -3$

So when $u = -3$, there is no solution for x.

Solution set: $\{16\}$

90. $x - \sqrt{x} - 12 = 0$

Solve by isolating $\sqrt{x}$, then squaring both sides.

$x - 12 = \sqrt{x}$

$(x-12)^2 = (\sqrt{x})^2 \Rightarrow x^2 - 24x + 144 = x$

$x^2 - 25x + 144 = 0 \Rightarrow (x-16)(x-9) = 0$

$x = 16$ or $x = 9$

Check $x = 16$.

$x - \sqrt{x} - 12 = 0$

$16 - \sqrt{16} - 12 \stackrel{?}{=} 0$

$16 - 4 - 12 = 0 \Rightarrow 0 = 0$

This is a true statement.

Check $x = 9$.

$x - \sqrt{x} - 12 = 0$

$9 - \sqrt{9} - 12 \stackrel{?}{=} 0 \Rightarrow 9 - 3 - 12 = 0 \Rightarrow -6 = 0$

This is a false statement. 9 does not satisfy the equation.

Solution set: $\{16\}$

91. Answers will vary.

92. $3x - 2\sqrt{x} - 8 = 0$

Solve by substitution.

Let $u = \sqrt{x}$; then $u^2 = x$. Solve the resulting equation by factoring.

$3u^2 - 2u - 8 = 0 \Rightarrow (3u+4)(u-2) = 0$

$u = -\frac{4}{3}$ or $u = 2$

To find x, replace u with $\sqrt{x}$.

$\sqrt{x} = -\dfrac{4}{3}$ has no solution, because the result of a square root is never a negative real number.

$\sqrt{x} = 2 \Rightarrow x = 4$

Check $x = 4$.

$3x - 2\sqrt{x} - 8 = 0$

$3(4) - 2\sqrt{4} - 8 \stackrel{?}{=} 0$

$3(4) - 2(2) - 8 = 0$

$12 - 4 - 8 = 0 \Rightarrow 0 = 0$

This is a true statement.

Solution set: $\{4\}$

93. $d = k\sqrt{h}$ for h

$$\frac{d}{k} = \sqrt{h} \Rightarrow \frac{d^2}{k^2} = h$$

So, $h = \dfrac{d^2}{k^2}$.

94. $x^{2/3} + y^{2/3} = a^{2/3}$ for y

$$y^{2/3} = a^{2/3} - x^{2/3}$$
$$\left(y^{2/3}\right)^3 = \left(a^{2/3} - x^{2/3}\right)^3$$
$$y^2 = \left(a^{2/3} - x^{2/3}\right)^3$$
$$y = \pm\sqrt{(a^{2/3} - x^{2/3})^3}$$
$$y = \pm\left(a^{2/3} - x^{2/3}\right)^{3/2}$$

95. $m^{3/4} + n^{3/4} = 1$ for m

$$m^{3/4} = 1 - n^{3/4}$$

Raise both sides to the $\frac{4}{3}$ power.

$$(m^{3/4})^{4/3} = (1 - n^{3/4})^{4/3}$$
$$m = (1 - n^{3/4})^{4/3}$$

96. $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$ for R

$$Rr_1r_2\left(\frac{1}{R}\right) = Rr_1r_2\left(\frac{1}{r_1}\right) + Rr_1r_2\left(\frac{1}{r_2}\right)$$

Multiply both sides by Rr_1r_2.

$$r_1r_2 = Rr_2 + Rr_1$$
$$r_1r_2 = R(r_2 + r_1)$$
$$\frac{r_1r_2}{r_2 + r_1} = R$$

So, $R = \dfrac{r_1r_2}{r_1 + r_2}$.

97. $\dfrac{E}{e} = \dfrac{R + r}{r}$ for e

$$er\left(\frac{E}{e}\right) = er\left(\frac{R + r}{r}\right)$$

Multiply both sides by er.

$$Er = eR + er$$
$$Er = e(R + r)$$
$$\frac{Er}{R + r} = e$$

So, $e = \dfrac{Er}{R + r}$.

98. $a^2 + b^2 = c^2$ for b

$$b^2 = c^2 - a^2$$
$$b = \pm\sqrt{c^2 - a^2}$$

Summary Exercises on Solving Equations

1. $4x - 3 = 2x + 3 \Rightarrow 2x - 3 = 3 \Rightarrow$

$2x = 6 \Rightarrow x = 3$

Solution set: $\{3\}$

2. $5 - (6x + 3) = 2(2 - 2x)$

$5 - 6x - 3 = 4 - 4x$

$2 - 6x = 4 - 4x \Rightarrow 2 = 4 + 2x$

$-2 = 2x \Rightarrow -1 = x$

Solution set: $\{-1\}$.

3. $x(x + 6) = 9 \Rightarrow x^2 + 6x = 9 \Rightarrow x^2 + 6x - 9 = 0$

Solve by completing the square.

$x^2 + 6x + 9 = 9 + 9$

Note: $\left[\frac{1}{2}\cdot 6\right]^2 = 3^2 = 9$

$(x + 3)^2 = 18 \Rightarrow x + 3 = \pm\sqrt{18} \Rightarrow$

$x + 3 = \pm 3\sqrt{2} \Rightarrow x = -3 \pm 3\sqrt{2}$

Solve by the quadratic formula.

Let $a = 1, b = 6,$ and $c = -9.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(1)(-9)}}{2(1)} = \frac{-6 \pm \sqrt{36 + 36}}{2} = \frac{-6 \pm \sqrt{72}}{2} = \frac{-6 \pm 6\sqrt{2}}{2} = -3 \pm 3\sqrt{2}$$

Solution set: $\left\{-3 \pm \sqrt{2}\right\}$

4. $x^2 = 8x - 12 \Rightarrow x^2 - 8x + 12 = 0$

Solve by factoring.

$x^2 - 8x + 12 = 0 \Rightarrow (x - 2)(x - 6) = 0 \Rightarrow$

$x = 2$ or $x = 6$

or solve by completing the square.

$x^2 - 8x + 16 = -12 + 16$

Note: $\left[\frac{1}{2}\cdot(-8)\right]^2 = (-4)^2 = 16$

$(x - 4)^2 = 4 \Rightarrow x - 4 = \pm\sqrt{4} \Rightarrow$

$x - 4 = \pm 2 \Rightarrow x = 4 \pm 2 \Rightarrow$

$x = 4 - 2 = 2$ or $x = 4 + 2 = 6$

or solve by the quadratic formula.

Let $a = 1,\ b = -8,$ and $c = 12.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(12)}}{2(1)} = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm \sqrt{16}}{2} = \frac{8 \pm 4}{2} = 4 \pm 2$$

$x = 4 - 2 = 2$ or $x = 4 + 2 = 6$

Solution set: $\{2, 6\}$

5. $$\sqrt{x+2}+5=\sqrt{x+15}$$
$$\left(\sqrt{x+2}+5\right)^2=\left(\sqrt{x+15}\right)^2$$
$$(x+2)+10\sqrt{x+2}+25=x+15\Rightarrow$$
$$x+27+10\sqrt{x+2}=x+15\Rightarrow$$
$$27+10\sqrt{x+2}=15\Rightarrow$$
$$10\sqrt{x+2}=-12\Rightarrow 5\sqrt{x+2}=-6$$
$$\left(5\sqrt{x+2}\right)^2=(-6)^2\Rightarrow$$
$$25(x+2)=36\Rightarrow 25x+50=36$$
$$25x=-14\Rightarrow x=-\frac{14}{25}$$

Check $x=-\frac{14}{25}$.

$$\sqrt{x+2}+5=\sqrt{x+15}$$
$$\sqrt{-\tfrac{14}{25}+2}+5\stackrel{?}{=}\sqrt{-\tfrac{14}{25}+15}$$
$$\sqrt{-\tfrac{14}{25}+\tfrac{50}{25}}+5=\sqrt{-\tfrac{14}{25}+\tfrac{375}{25}}$$
$$\sqrt{\tfrac{36}{25}}+5=\sqrt{\tfrac{361}{25}}\Rightarrow \tfrac{6}{5}+\tfrac{25}{5}=\tfrac{19}{5}\Rightarrow \tfrac{31}{5}=\tfrac{19}{5}$$

This is a false statement. Solution set: $\varnothing$

6. $\frac{5}{x+3}-\frac{6}{x-2}=\frac{3}{x^2+x-6}$ or
$$\frac{5}{x+3}-\frac{6}{x-2}=\frac{3}{(x+3)(x-2)}$$

The least common denominator is $(x+3)(x-2)$, which is equal to 0 if $x=-3$ or $x=2$. Therefore, -3 and 2 cannot possibly be solutions of this equation.

$$(x+3)(x-2)\left[\frac{5}{x+3}-\frac{6}{x-2}\right]=(x+3)(x-2)\left(\frac{3}{(x+3)(x-2)}\right)$$
$$5(x-2)-6(x+3)=3$$
$$5x-10-6x-18=3$$
$$-x-28=3\Rightarrow -x=31\Rightarrow x=-31$$

The restrictions $x\neq-3$ and $x\neq 2$ do not affect the result. Therefore, the solution set is $\{-31\}$.

7. $\frac{3x+4}{3}-\frac{2x}{x-3}=x$

The least common denominator is $3(x-3)$, which is equal to 0 if $x=3$. Therefore, 3 cannot possibly be a solution of this equation.

$$3(x-3)\left[\frac{3x+4}{3}-\frac{2x}{x-3}\right]=3(x-3)(x)$$
$$(x-3)(3x+4)-3(2x)=3x(x-3)$$
$$3x^2+4x-9x-12-6x=3x^2-9x$$
$$3x^2-11x-12=3x^2-9x$$
$$-11x-12=-9x$$
$$-12=2x\Rightarrow -6=x$$

The restriction $x\neq 3$ does not affect the result. Therefore, the solution set is $\{-6\}$.

8. $\frac{x}{2}+\frac{4}{3}x=x+5\Rightarrow 6\left(\frac{x}{2}+\frac{4}{3}x\right)=6(x+5)$
$$3x+8x=6x+30\Rightarrow 11x=6x+30$$
$$5x=30\Rightarrow x=6$$

Solution set: $\{6\}$

9. $5-\frac{2}{x}+\frac{1}{x^2}=0$

The least common denominator is x^2, which is equal to 0 if $x=0$. Therefore, 0 cannot possibly be a solution of this equation.

$$x^2\left[5-\frac{2}{x}+\frac{1}{x^2}\right]=x^2(0)\Rightarrow 5x^2-2x+1=0$$

Solve by completing the square.

$$x^2-\tfrac{2}{5}x+\tfrac{1}{5}=0\qquad\text{Multiply by }\tfrac{1}{5}.$$
$$x^2-\tfrac{2}{5}x+\tfrac{1}{25}=-\tfrac{1}{5}+\tfrac{1}{25}$$
$$\text{Note:}\left[\tfrac{1}{2}\cdot\left(-\tfrac{2}{5}\right)\right]^2=\left(-\tfrac{1}{5}\right)^2=\tfrac{1}{25}$$
$$\left(x-\tfrac{1}{5}\right)^2=-\tfrac{5}{25}+\tfrac{1}{25}=\tfrac{-4}{25}$$
$$x-\tfrac{1}{5}=\pm\sqrt{\tfrac{-4}{25}}$$
$$x-\tfrac{1}{5}=\pm\tfrac{2}{5}i\Rightarrow x=\tfrac{1}{5}\pm\tfrac{2}{5}i$$

Solve by the quadratic formula.
Let $a=5$, $b=-2$, and $c=1$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4(5)(1)}}{2(5)}$$
$$=\frac{2\pm\sqrt{4-20}}{10}=\frac{2\pm\sqrt{-16}}{10}$$
$$=\frac{2\pm 4i}{10}=\frac{2}{10}\pm\frac{4}{10}i=\frac{1}{5}\pm\frac{2}{5}i$$

The restriction $x\neq 0$ does not affect the result. Therefore, the solution set is $\left\{\frac{1}{5}\pm\frac{2}{5}i\right\}$.

10. $(2x+1)^2=9\Rightarrow 2x+1=\pm\sqrt{9}\Rightarrow 2x+1=\pm3$
$$2x=-1\pm3\Rightarrow x=\frac{-1\pm3}{2}$$
$$x=\frac{-1-3}{2}=\frac{-4}{2}=-2\text{ or }x=\frac{-1+3}{2}=\frac{2}{2}=1$$

Solution set: $\{-2,1\}$

11. $x^{-2/5} - 2x^{-1/5} - 15 = 0$

Let $u = x^{-1/5}$; then $u^2 = \left(x^{-1/5}\right)^2 = x^{-2/5}$.

$u^2 - 2u - 15 = 0 \Rightarrow (u+3)(u-5) = 0 \Rightarrow$
$u = -3$ or $u = 5$

To find x, replace u with $x^{-1/5}$.

$x^{-1/5} = -3 \Rightarrow \left(x^{-1/5}\right)^{-5} = (-3)^{-5}$
$x = \frac{1}{(-3)^5} \Rightarrow x = -\frac{1}{243}$ or

$x^{-1/5} = 5 \Rightarrow \left(x^{-1/5}\right)^{-5} = 5^{-5} \Rightarrow x = \frac{1}{5^5} \Rightarrow$
$x = \frac{1}{3125}$

Check $x = -\frac{1}{243}$.

$x^{-2/5} - 2x^{-1/5} - 15 = 0$
$\left(-\frac{1}{243}\right)^{-2/5} - 2\left(-\frac{1}{243}\right)^{-1/5} - 15 = 0$?
$(-243)^{2/5} - 2(-243)^{1/5} - 15 = 0$
$\left[(-243)^{1/5}\right]^2 - 2(-3) - 15 = 0$
$(-3)^2 + 6 - 15 = 0 \Rightarrow 9 + 6 - 15 = 0 \Rightarrow 0 = 0$

This is a true statement. $-\frac{1}{243}$ is a solution.

Check $x = \frac{1}{3125}$.

$x^{-2/5} - 2x^{-1/5} - 15 = 0$
$\left(\frac{1}{3125}\right)^{-2/5} - 2\left(\frac{1}{3125}\right)^{-1/5} - 15 \stackrel{?}{=} 0$
$(3125)^{2/5} - 2(3125)^{1/5} - 15 = 0$
$\left[(3125)^{1/5}\right]^2 - 2(5) - 15 = 0$
$5^2 - 10 - 15 = 0$
$25 - 10 - 15 = 0 \Rightarrow 0 = 0$

This is a true statement. $\frac{1}{3125}$ is a solution.

Solution set: $\left\{-\frac{1}{243}, \frac{1}{3125}\right\}$

12. $\sqrt{x+2} + 1 = \sqrt{2x+6}$
$\left(\sqrt{x+2} + 1\right)^2 = \left(\sqrt{2x+6}\right)^2$
$x + 2 + 2\sqrt{x+2} + 1 = 2x + 6$
$x + 3 + 2\sqrt{x+2} = 2x + 6$
$2\sqrt{x+2} = x + 3$
$\left(2\sqrt{x+2}\right)^2 = (x+3)^2$
$4(x+2) = x^2 + 6x + 9$
$4x + 8 = x^2 + 6x + 9$
$0 = x^2 + 2x + 1 = (x+1)^2$
$0 = x + 1 \Rightarrow -1 = x$

Check $x = -1$.
$\sqrt{x+2} + 1 = \sqrt{2x+6}$
$\sqrt{-1+2} + 1 \stackrel{?}{=} \sqrt{2(-1)+6}$
$\sqrt{1} + 1 = \sqrt{-2+6} \Rightarrow 2 = \sqrt{4} \Rightarrow 2 = 2$

This is a true statement.
Solution set: $\{-1\}$

13. $x^4 - 3x^2 - 4 = 0$

Let $u = x^2$; then $u^2 = x^4$.

$u^2 - 3u - 4 = 0 \Rightarrow (u+1)(u-4) = 0 \Rightarrow$.
$u = -1$ or $u = 4$

To find x, replace x with x^2.

$x^2 = -1 \Rightarrow x = \pm\sqrt{-1} = \pm i$ or
$x^2 = 4 \Rightarrow x = \pm\sqrt{4} = \pm 2$

Solution set: $\{\pm i, \pm 2\}$

14. $1.2x + .3 = .7x - .9$
$10[1.2x + .3] = 10[.7x - .9]$
$12x + 3 = 7x - 9 \Rightarrow 5x + 3 = -9 \Rightarrow$
$5x = -12 \Rightarrow x = -2.4$

Solution set: $\{-2.4\}$

15. $\sqrt[6]{2x+1} = \sqrt[6]{9} \Rightarrow \left(\sqrt[6]{2x+1}\right)^6 = \left(\sqrt[6]{9}\right)^6$
$2x + 1 = 9 \Rightarrow 2x = 8 \Rightarrow x = 4$

Check $x = 4$.

$\sqrt[6]{2x+1} = \sqrt[6]{9} \Rightarrow \sqrt[6]{2(4)+1} \stackrel{?}{=} \sqrt[6]{9}$
$\sqrt[6]{8+1} = \sqrt[6]{9} \Rightarrow \sqrt[6]{9} = \sqrt[6]{9}$

This is a true statement.
Solution set: {4}

16. $3x^2 - 2x = -1 \Rightarrow 3x^2 - 2x + 1 = 0$

Solve by completing the square.

$3x^2 - 2x = -1$
$x^2 - \frac{2}{3}x = -\frac{1}{3}$ Multiply by $\frac{1}{3}$.
$x^2 - \frac{2}{3}x + \frac{1}{9} = -\frac{1}{3} + \frac{1}{9}$
Note: $\left[\frac{1}{2}\cdot\left(-\frac{2}{3}\right)\right]^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$
$\left(x - \frac{1}{3}\right)^2 = -\frac{3}{9} + \frac{1}{9} = \frac{-2}{9}$
$x - \frac{1}{3} = \pm\sqrt{\frac{-2}{9}}$
$x - \frac{1}{3} = \pm\frac{\sqrt{2}}{3}i \Rightarrow x = \frac{1}{3} \pm \frac{\sqrt{2}}{3}i$

Solve by the quadratic formula.
Let $a = 3$, $b = -2$, and $c = 1$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(1)}}{2(3)}$$
$$=\frac{2\pm\sqrt{4-12}}{6}=\frac{2\pm\sqrt{-8}}{6}$$
$$=\frac{2\pm 2i\sqrt{2}}{6}=\frac{2}{6}\pm\frac{2\sqrt{2}}{6}i=\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$$

Solution set: $\left\{\frac{1}{3}\pm\frac{\sqrt{2}}{3}i\right\}$.

17. $3\left[2x-(6-2x)+1\right]=5x$
$$3(2x-6+2x+1)=5x$$
$$3(4x-5)=5x$$
$$12x-15=5x\Rightarrow -15=-7x\Rightarrow$$
$$\frac{-15}{-7}=x\Rightarrow x=\frac{15}{7}$$

Solution set: $\left\{\frac{15}{7}\right\}$

18. $\sqrt{x}+1=\sqrt{11-\sqrt{x}}$
$$\left(\sqrt{x}+1\right)^2=\left(\sqrt{11-\sqrt{x}}\right)^2$$
$$x+2\sqrt{x}+1=11-\sqrt{x}$$
$$x+3\sqrt{x}+1=11\Rightarrow 3\sqrt{x}=10-x\Rightarrow$$
$$\left(3\sqrt{x}\right)^2=(10-x)^2$$
$$9x=100-20x+x^2$$
$$0=100-29x+x^2$$
$$0=x^2-29x+100$$
$$0=(x-4)(x-25)\Rightarrow$$
$$x=4 \text{ or } x=25$$

Check $x = 4$.
$$\sqrt{x}+1=\sqrt{11-\sqrt{x}}$$
$$\sqrt{4}+1\stackrel{?}{=}\sqrt{11-\sqrt{4}}$$
$$2+1=\sqrt{11-2}\Rightarrow 3=\sqrt{9}\Rightarrow 3=3$$

This is a true statement.

Check $x = 25$.
$$\sqrt{x}+1=\sqrt{11-\sqrt{x}}$$
$$\sqrt{25}+1\stackrel{?}{=}\sqrt{11-\sqrt{25}}$$
$$5+1=\sqrt{11-5}\Rightarrow 6=\sqrt{6}$$

This is a false statement.

Solution set: $\{4\}$

19. $(14-2x)^{2/3}=4$
$$[(14-2x)^{2/3}]^3=4^3$$
$$(14-2x)^2=64$$
$$196-56x+4x^2=64$$
$$4x^2-56x+132=0$$
$$4\left(x^2-14x+33\right)=0$$
$$4(x-3)(x-11)=0\Rightarrow x=3 \text{ or } x=11$$

Check $x = 3$.
$$(14-2x)^{2/3}=4$$
$$\left[14-2(3)\right]^{2/3}\stackrel{?}{=}4$$
$$(14-6)^{2/3}=4\Rightarrow 8^{2/3}=4$$
$$\left(8^{1/3}\right)^2=4\Rightarrow 2^2=4\Rightarrow 4=4$$

This is a true statement.

Check $x = 11$.
$$(14-2x)^{2/3}=4$$
$$\left[14-2(11)\right]^{2/3}\stackrel{?}{=}4$$
$$(14-22)^{2/3}=4\Rightarrow(-8)^{2/3}=4$$
$$\left[(-8)^{1/3}\right]^2=4\Rightarrow(-2)^2=4\Rightarrow 4=4$$

This is a true statement.

Solution set: $\{3,11\}$

20. $2x^{-1}-x^{-2}=1$
$$-x^{-2}+2x^{-1}-1=0$$
$$x^{-2}-2x^{-1}+1=0$$

Let $u=x^{-1}$; then $u^2=x^{-2}$.
$$u^2-2u+1=0\Rightarrow(u-1)^2=0\Rightarrow u=1$$

To find x, replace u with x^{-1}.
$$x^{-1}=1\Rightarrow x=1$$

Solution set: $\{1\}$

21. $\frac{3}{x-3}=\frac{3}{x-3}$

The least common denominator is $(x-3)$ which is equal to 0 if $x=3$. Therefore, 3 cannot possibly be a solution of this equation.

Solution set: $\{x\,|\,x\neq 3\}$.

22. $a^3+b^3=c^3$ for a
$$a^3=c^3-b^3\Rightarrow a=\sqrt[3]{c^3-b^3}$$

Section 1.7: Inequalities

1. $x < -6$
The interval includes all real numbers less than -6 not including -6. The correct interval notation is $(-\infty, -6)$, so the correct choice is F.

2. $x \le 6$
The interval includes all real numbers less than or equal to 6, so it includes 6. The correct interval notation is $(-\infty, 6]$, so the correct choice is J.

3. $-2 < x \le 6$
The interval includes all real numbers from –2 to 6, not including –2, but including 6. The correct interval notation is (–2, 6], so the correct choice is A.

4. $x^2 \le 9$
The interval includes all real numbers between -3 and 3, including -3 and 3. The correct interval notation is [–3, 3], so the correct choice is H.

5. $x \ge -6$
The interval includes all real numbers greater than or equal to –6, so it includes –6. The correct interval notation is $[-6, \infty)$, so the correct choice is I.

6. $6 \le x$
The interval includes all real numbers greater than or equal to 6, so it includes 6. The correct interval notation is $[6, \infty)$, so the correct choice is D.

7. The interval shown on the number line includes all real numbers between –2 and 6, including –2, but not including 6. The correct interval notation is [–2, 6), so the correct choice is B.

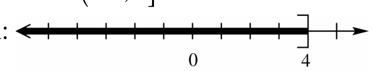
8. The interval shown on the number line includes all real numbers between 0 and 8, not including 0 or 8. The correct interval notation is (0, 8), so the correct choice is G.

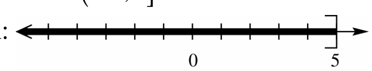
9. The interval shown on the number line includes all real numbers less than -3, not including -3, and greater than 3, not including 3. The correct interval notation is $(-\infty, -3) \cup (3, \infty)$, so the correct choice is E.

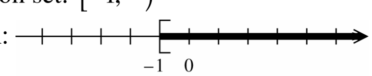
10. The interval includes all real numbers less than or equal to –6, so it includes –6, The correct interval notation is $(-\infty, -6]$, so the correct choice is C.

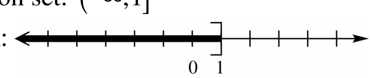
11. Answers wil1 vary.

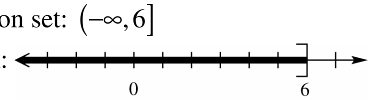
12. D
$-8 < x < -10$ would mean $-8 < x$ and $x < -10$, which is equivalent to $x > -8$ and $x < -10$. There is no real number that is simultaneously to the right of –8 and to the left of –10 on a number line.

13. $2x + 8 \le 16 \Rightarrow 2x + 8 - 8 \le 16 - 8 \Rightarrow$
$$2x \le 8 \Rightarrow \frac{2x}{2} \le \frac{8}{2} \Rightarrow x \le 4$$
Solution set: $(-\infty, 4]$
Graph: 

14. $3x - 8 \le 7 \Rightarrow 3x - 8 + 8 \le 7 + 8 \Rightarrow$
$$3x \le 15 \Rightarrow \frac{3x}{3} \le \frac{15}{3} \Rightarrow x \le 5$$
Solution set: $(-\infty, 5]$
Graph: 

15.
$$-2x - 2 \le 1 + x$$
$$-2x - 2 + 2 \le 1 + x + 2$$
$$-2x \le x + 3 \Rightarrow -2x - x \le 3 \Rightarrow$$
$$-3x \le 3 \Rightarrow \frac{-3x}{-3} \ge \frac{3}{-3} \Rightarrow x \ge -1$$
Solution set: $[-1, \infty)$
Graph: 

16.
$$-4x + 3 \ge -2 + x$$
$$-4x + 3 - 3 \ge -2 - 3 + x$$
$$-4x \ge -5 + x \Rightarrow -4x - x \ge -5 \Rightarrow$$
$$-5x \ge -5 \Rightarrow \frac{-5x}{-5} \le \frac{-5}{-5} \Rightarrow x \le 1$$
Solution set: $(-\infty, 1]$
Graph: 

17. $2(x + 5) + 1 \ge 5 + 3x$
$$2x + 10 + 1 \ge 5 + 3x \Rightarrow 2x + 11 \ge 5 + 3x \Rightarrow$$
$$2x + 11 - 3x \ge 5 + 3x - 3x \Rightarrow -x + 11 \ge 5 \Rightarrow$$
$$-x + 11 - 11 \ge 5 - 11 \Rightarrow \frac{-x}{-1} \le \frac{-6}{-1} \Rightarrow x \le 6$$
Solution set: $(-\infty, 6]$
Graph: 

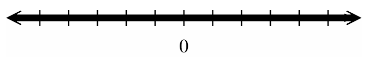
18. $6x-(2x+3)\ge 4x-5$

$6x-2x-3\ge 4x-5 \Rightarrow 4x-3\ge 4x-5$

$4x-4x-3\ge 4x-5-4x \Rightarrow -3\ge -5$

The inequality is true when x is any real number.

Solution set: $(-\infty,\infty)$

Graph: 

19. $8x-3x+2<2(x+7)$

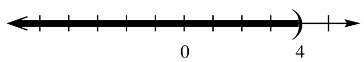
$5x+2<2x+14$

$5x+2-2x<2x+14-2x$

$3x+2<14 \Rightarrow 3x+2-2<14-2 \Rightarrow$

$3x<12 \Rightarrow \dfrac{3x}{3}<\dfrac{12}{3} \Rightarrow x<4$

Solution set: $(-\infty,4)$

Graph: 

20. $2-4x+5(x-1)<-6(x-2)$

$2-4x+5x-5<-6x+12$

$x-3<-6x+12$

$x-3+6x<-6x+12+6x$

$7x-3<12 \Rightarrow 7x-3+3<12+3 \Rightarrow$

$7x<15 \Rightarrow \dfrac{7x}{7}<\dfrac{15}{7} \Rightarrow x<\frac{15}{7}$

Solution set: $\left(-\infty,\frac{15}{7}\right)$

Graph: 

21. $\dfrac{4x+7}{-3}\le 2x+5$

$(-3)\left(\dfrac{4x+7}{-3}\right)\ge(-3)(2x+5)$

$4x+7\ge -6x-15$

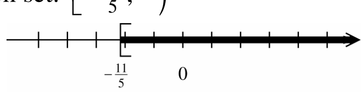
$4x+7+6x\ge -6x-15+6x$

$10x+7\ge -15$

$10x+7-7\ge -15-7 \Rightarrow 10x\ge -22 \Rightarrow$

$\dfrac{10x}{10}\ge\dfrac{-22}{10} \Rightarrow x\ge -\frac{11}{5}$

Solution set: $\left[-\frac{11}{5},\infty\right)$

Graph: 

22. $\dfrac{2x-5}{-8}\le 1-x$

$(-8)\left(\dfrac{2x-5}{-8}\right)\ge(-8)(1-x)$

$2x-5\ge -8+8x$

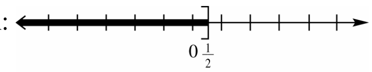
$2x-5-8x\ge -8+8x-8x$

$-6x-5\ge -8$

$-6x-5+5\ge -8+5 \Rightarrow -6x\ge -3$

$\dfrac{-6x}{-6}\le\dfrac{-3}{-6} \Rightarrow x\le\frac{1}{2}$

Solution set: $\left(-\infty,\frac{1}{2}\right]$

Graph: 

23. $\dfrac{1}{3}x+\dfrac{2}{5}x-\dfrac{1}{2}(x+3)\le\dfrac{1}{10}$

$30\left[\dfrac{1}{3}x+\dfrac{2}{5}x-\dfrac{1}{2}(x+3)\right]\le 30\left[\dfrac{1}{10}\right]$

$10x+12x-15(x+3)\le 3$

$10x+12x-15x-45\le 3$

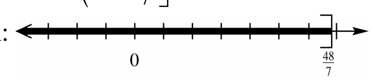
$7x-45\le 3$

$7x-45+45\le 3+45$

$7x\le 48$

$\dfrac{7x}{7}\le\dfrac{48}{7} \Rightarrow x\le\frac{48}{7}$

Solution set: $\left(-\infty,\frac{48}{7}\right]$

Graph: 

24. $-\dfrac{2}{3}x-\dfrac{1}{6}x+\dfrac{2}{3}(x+1)\le\dfrac{4}{3}$

$(-6)\left[-\dfrac{2}{3}x-\dfrac{1}{6}x+\dfrac{2}{3}(x+1)\right]\ge(-6)\left[\dfrac{4}{3}\right]$

$4x+x-4(x+1)\ge -8$

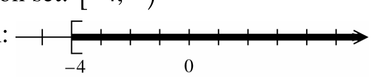
$4x+x-4x-4\ge -8$

$x-4\ge -8$

$x-4+4\ge -8+4$

$x\ge -4$

Solution set: $[-4,\infty)$

Graph: 

25. $-5<5+2x<11$

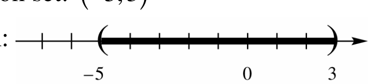
$-5-5<5+2x-5<11-5$

$-10<2x<6$

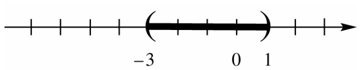
$\dfrac{-10}{2}<\dfrac{2x}{2}<\dfrac{6}{2}$

$-5<x<3$

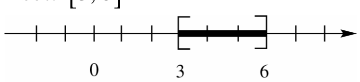
Solution set: $(-5,3)$

Graph: 

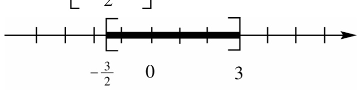
26.
$$-7 < 2 + 3x < 5$$
$$-7 - 2 < 2 + 3x - 2 < 5 - 2$$
$$-9 < 3x < 3$$
$$\frac{-9}{3} < \frac{3x}{3} < \frac{3}{3}$$
$$-3 < x < 1$$
Solution set: $(-3,1)$

Graph:


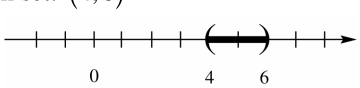
27.
$$10 \le 2x + 4 \le 16$$
$$10 - 4 \le 2x + 4 - 4 \le 16 - 4$$
$$6 \le 2x \le 12$$
$$\frac{6}{2} \le \frac{2x}{2} \le \frac{12}{2}$$
$$3 \le x \le 6$$
Solution set: $[3,6]$

Graph:


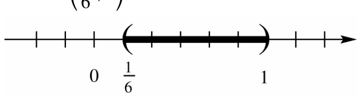
28.
$$-6 \le 6x + 3 \le 21$$
$$-6 - 3 \le 6x + 3 - 3 \le 21 - 3$$
$$-9 \le 6x \le 18$$
$$\frac{-9}{6} \le \frac{6x}{6} \le \frac{18}{6}$$
$$-\tfrac{3}{2} \le x \le 3$$
Solution set: $\left[-\tfrac{3}{2}, 3\right]$

Graph:


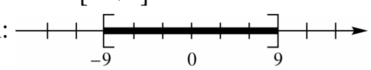
29.
$$-11 > -3x + 1 > -17$$
$$-11 - 1 > -3x + 1 - 1 > -17 - 1$$
$$-12 > -3x > -18$$
$$\frac{-12}{-3} < \frac{-3x}{-3} < \frac{-18}{-3}$$
$$4 < x < 6$$
Solution set: $(4,6)$

Graph:


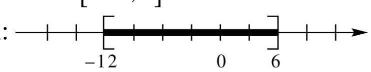
30.
$$2 > -6x + 3 > -3$$
$$2 - 3 > -6x + 3 - 3 > -3 - 3$$
$$-1 > -6x > -6$$
$$\frac{-1}{-6} < \frac{-6x}{-6} < \frac{-6}{-6}$$
$$\tfrac{1}{6} < x < 1$$
Solution set: $\left(\tfrac{1}{6}, 1\right)$

Graph:


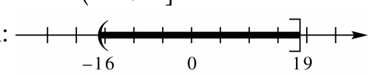
31.
$$-4 \le \frac{x+1}{2} \le 5$$
$$2(-4) \le 2\left(\frac{x+1}{2}\right) \le 2(5)$$
$$-8 \le x + 1 \le 10$$
$$-8 - 1 \le x + 1 - 1 \le 10 - 1 \Rightarrow -9 \le x \le 9$$
Solution set: $[-9,9]$

Graph:


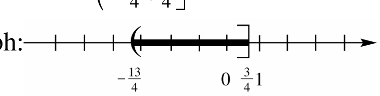
32.
$$-5 \le \frac{x-3}{3} \le 1$$
$$3(-5) \le 3\left(\frac{x-3}{3}\right) \le 3(1)$$
$$-15 \le x - 3 \le 3$$
$$-15 + 3 \le x - 3 + 3 \le 3 + 3 \Rightarrow -12 \le x \le 6$$
Solution set: $[-12,6]$

Graph:


33.
$$-3 \le \frac{x-4}{-5} < 4$$
$$(-5)(-3) \ge (-5)\left(\frac{x-4}{-5}\right) > (-5)(4)$$
$$15 \ge x - 4 > -20$$
$$15 + 4 \ge x - 4 + 4 > -20 + 4$$
$$19 \ge x > -16 \Rightarrow -16 < x \le 19$$
Solution set: $(-16,19]$

Graph:


34.
$$1 \le \frac{4x-5}{-2} < 9$$
$$(-2)(1) \ge (-2)\left(\frac{4x-5}{-2}\right) > (-2)(9)$$
$$-2 \ge 4x - 5 > -18$$
$$-2 + 5 \ge 4x - 5 + 5 > -18 + 5$$
$$3 \ge 4x > -13$$
$$\frac{3}{4} \ge \frac{4x}{4} > \frac{-13}{4} \Rightarrow -\frac{13}{4} < x \le \frac{3}{4}$$
Solution set: $\left(-\tfrac{13}{4}, \tfrac{3}{4}\right]$

Graph:


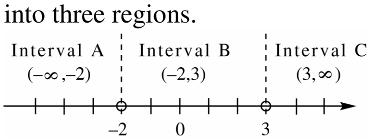
35. $C = 50x + 5000$; $R = 60x$
The product will at least break even when $R \ge C$. Set $R \ge C$ and solve for x.
$60x \ge 50x + 5000 \Rightarrow 10x \ge 5000 \Rightarrow x \ge 500$
The break-even point is at $x = 500$.
This product will at least break even if the number of units of picture frames produced is in interval $[500, \infty)$.

36. $C = 100x + 6000$; $R = 500x$
The product will at least break even when $R \ge C$. Set $R \ge C$ and solve for x.
$500x \ge 100x + 6000 \Rightarrow 400x \ge 6000 \Rightarrow x \ge 15$
The break-even point is $x = 15$.
The product will at least break even when the number of units of baseball caps produced is in the interval $[15, \infty)$.

37. $C = 85x + 900$; $R = 105x$
The product will at least break even when $R \ge C$. Set $R \ge C$ and solve for x.
$105x \ge 85x + 900 \Rightarrow 20x \ge 900 \Rightarrow x \ge 45$
The break-even point is $x = 45$.
The product will at least break even when the number of units of coffee cups produced is in the interval $[45, \infty)$.

38. $C = 70x + 500$; $R = 60x$
The product will at least break even when $R \ge C$. Set $R \ge C$ and solve for x.
$60x \ge 70x + 500 \Rightarrow -10x \ge 500 \Rightarrow x \le -50$
The product will never break even.

39. $x^2 - x - 6 > 0$
Step 1: Find the values of x that satisfy $x^2 - x - 6 = 0$.
$x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0$
$x + 2 = 0 \Rightarrow x = -2$ or $x - 3 = 0 \Rightarrow x = 3$
Step 2: The two numbers divide a number line into three regions.

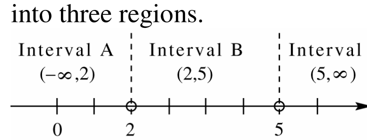


Step 3: Choose a test value to see if it satisfies the inequality, $x^2 - x - 6 > 0$.

Interval	Test Value	Is $x^2 - x - 6 > 0$ True or False?
A: $(-\infty, -2)$	-3	$(-3)^2 - (-3) - 6 \overset{?}{>} 0$ $6 > 0$ True
B: $(-2, 3)$	0	$0^2 - 0 - 6 \overset{?}{>} 0$ $-6 > 0$ False
C: $(3, \infty)$	4	$4^2 - 4 - 6 \overset{?}{>} 0$ $6 > 0$ True

Solution set: $(-\infty, -2) \cup (3, \infty)$

40. $x^2 - 7x + 10 > 0$
Step 1: Find the values of x that satisfy the corresponding equation.
$x^2 - 7x + 10 = 0$
$(x-2)(x-5) = 0$
$x - 2 = 0 \Rightarrow x = 2$ or $x - 5 = 0 \Rightarrow x = 5$
Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2 - 7x + 10 > 0$.

Interval	Test Value	Is $x^2 - 7x + 10 > 0$ True or False?
A: $(-\infty, 2)$	0	$0^2 - 7(0) + 10 \overset{?}{>} 0$ $10 > 0$ True
B: $(2, 5)$	3	$3^2 - 7(3) + 10 \overset{?}{>} 0$ $-2 > 0$ False
C: $(5, \infty)$	6	$6^2 - 7(6) + 10 \overset{?}{>} 0$ $4 > 0$ True

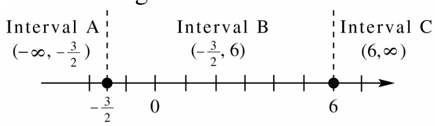
Solution set: $(-\infty, 2) \cup (5, \infty)$

41. $2x^2 - 9x \le 18$
Step 1: Find the values of x that satisfy the corresponding equation.
$2x^2 - 9x = 18$
$2x^2 - 9x - 18 = 0$
$(2x+3)(x-6) = 0$
$2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$ or $x - 6 = 0 \Rightarrow x = 6$

(*continued on next page*)

(*continued from page 111*)

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $2x^2 - 9x \le 18$

Interval	Test Value	Is $2x^2 - 9x \le 18$ True or False?
A: $\left(-\infty, -\frac{3}{2}\right)$	−2	$2(-2)^2 - 9(-2) \overset{?}{\le} 18$ $26 \le 18$ False
B: $\left(-\frac{3}{2}, 6\right)$	0	$2(0)^2 - 9(0) \overset{?}{\le} 18$ $0 \le 18$ True
C: $(6, \infty)$	7	$2(7)^2 - 9(7) \overset{?}{\le} 18$ $35 \le 18$ False

Solution set: $\left[-\frac{3}{2}, 6\right]$

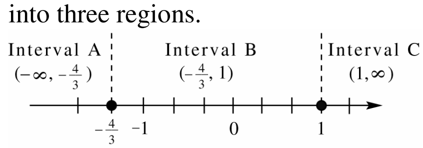
42. $3x^2 + x \le 4$

Step 1: Find the values of x that satisfy the corresponding equation.

$3x^2 + x = 4 \Rightarrow 3x^2 + x - 4 = 0 \Rightarrow$
$(3x + 4)(x - 1) = 0$

$3x + 4 = 0 \Rightarrow x = -\frac{4}{3}$ or $x - 1 = 0 \Rightarrow x = 1$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $3x^2 + x \le 4$

Interval	Test Value	Is $3x^2 + x \le 4$ True or False?
A: $\left(-\infty, -\frac{4}{3}\right)$	−2	$3(-2)^2 + (-2) \overset{?}{\le} 4$ $10 \le 4$ False
B: $\left(-\frac{4}{3}, 1\right)$	0	$3(0)^2 + (0) \overset{?}{\le} 4$ $0 \le 4$ True
C: $(1, \infty)$	2	$3(2)^2 + 2 \overset{?}{\le} 4$ $14 \le 4$ False

Solution set: $\left[-\frac{4}{3}, 1\right]$

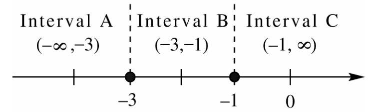
43. $-x^2 - 4x - 6 \le -3$

Step 1: Find the values of x that satisfy the corresponding equation.

$-x^2 - 4x - 6 = -3$
$x^2 + 4x + 3 = 0$
$(x + 3)(x + 1) = 0$

$x + 3 = 0 \Rightarrow x = -3$ or $x + 1 = 0 \Rightarrow x = -1$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $-x^2 - 4x - 6 \le -3$

Interval	Test Value	Is $-x^2 - 4x - 6 \le -3$ True or False?
A: $(-\infty, -3)$	−4	$-(-4)^2 - 4(-4) - 6 \overset{?}{\le} -3$ $-6 \le -3$ True
B: $(-3, -1)$	−2	$-(-2)^2 - 4(-2) - 6 \overset{?}{\le} -3$ $-2 \le -3$ False
C: $(-1, \infty)$	0	$-(0)^2 - 4(0) - 6 \overset{?}{\le} -3$ $-6 \le -3$ True

Solution set: $(-\infty, -3] \cup [-1, \infty)$

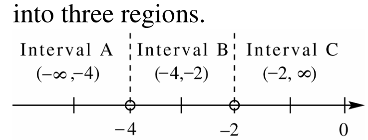
44. $-x^2 - 6x - 16 > -8$

Step 1: Find the values of x that satisfy the corresponding equation.

$-x^2 - 6x - 16 = -8$
$x^2 + 6x + 8 = 0$
$(x + 4)(x + 2) = 0$

$x + 4 = 0 \Rightarrow x = -4$ or $x + 2 = 0 \Rightarrow x = -2$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2 + 6x + 16 < 8$.

Interval	Test Value	Is $-x^2 - 6x - 16 > -8$ True or False?
A: $(-\infty, -4)$	-5	$-(-5)^2 - 6(-5) - 16 > -8$ $-11 > -8$ False
B: $(-4, -2)$	-3	$-(-3)^2 - 6(-3) - 16 > -8$ $-7 > -8$ True
C: $(-2, \infty)$	0	$-(0)^2 - 6(0) - 16 > -8$ $-16 > -8$ False

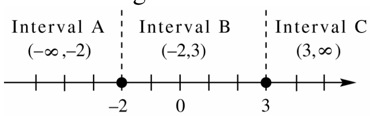
Solution set: $(-4, -2)$

45. $x(x-1) \le 6 \Rightarrow x^2 - x \le 6 \Rightarrow x^2 - x - 6 \le 0$

Step 1: Find the values of x that satisfy $x^2 - x - 6 = 0$.

$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$
$$x + 2 = 0 \Rightarrow x = -2 \quad \text{or} \quad x - 3 = 0 \Rightarrow x = 3$$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x(x-1) \le 6$.

Interval	Test Value	Is $x(x-1) \le 6$ True or False?
A: $(-\infty, -2)$	-3	$-3(-3-1) \overset{?}{\le} 6$ $12 \le 6$ False
B: $(-2, 3)$	0	$0(0-1) \overset{?}{\le} 6$ $0 \le 6$ True
C: $(3, \infty)$	4	$4(4-1) \overset{?}{\le} 6$ $12 \le 6$ False

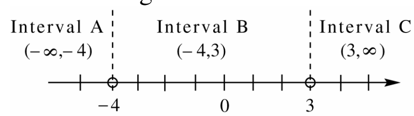
Solution set: $[-2, 3]$

46. $x(x+1) < 12 \Rightarrow x(x+1) < 12 \Rightarrow$
$x^2 + x < 12 \Rightarrow x^2 + x - 12 < 0$

Step 1: Find the values of x that satisfy $x^2 + x - 12 = 0$.

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x + 4 = 0 \Rightarrow x = -4 \quad \text{or} \quad x - 3 = 0 \Rightarrow x = 3$$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x(x+1) < 12$.

Interval	Test Value	Is $x(x+1) < 12$ True or False?
A: $(-\infty, -4)$	-5	$-5(-5+1) \overset{?}{<} 12$ $20 < 12$ False
B: $(-4, 3)$	0	$0(0+1) \overset{?}{<} 12$ $0 < 12$ True
C: $(3, \infty)$	4	$4(4+1) \overset{?}{<} 12$ $20 < 12$ False

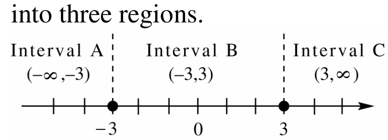
Solution set: $(-4, 3)$

47. $x^2 \le 9$

Step 1: Find the values of x that satisfy $x^2 \le 9$

$$x^2 = 9$$
$$x^2 - 9 = 0$$
$$(x+3)(x-3) = 0$$
$$x + 3 = 0 \Rightarrow x = -3 \quad \text{or} \quad x - 3 = 0 \Rightarrow x = 3$$

Step 2: The two numbers divide a number line into three regions.



(*continued on next page*)

(*continued from page 113*)

Step 3: Choose a test value to see if it satisfies the inequality, $x^2 \le 9$.

Interval	Test Value	Is $x^2 \le 9$ True or False?
A: $(-\infty, -3)$	-4	$(-4)^2 \overset{?}{\le} 9$ $16 \le 9$ False
B: $(-3, 3)$	0	$(0)^2 \overset{?}{\le} 9$ $0 \le 9$ True
C: $(3, \infty)$	4	$(4)^2 \overset{?}{\le} 9$ $16 \le 9$ False

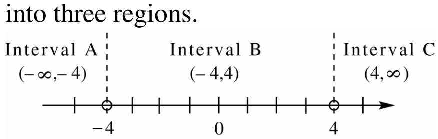
Solution set: [–3, 3]

48. $x^2 > 16 \Rightarrow x^2 - 16 > 0$

Step 1: Find the values of x that satisfy

$$x^2 - 16 = 0$$
$$(x+4)(x-4) = 0$$
$$x + 4 = 0 \Rightarrow x = -4 \quad \text{or} \quad x - 4 = 0 \Rightarrow x = 4$$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2 > 16$.

Interval	Test Value	Is $x^2 > 16$ True or False?
A: $(-\infty, -4)$	-5	$(-5)^2 \overset{?}{>} 16$ $25 > 16$ True
B: $(-4, 4)$	0	$(0)^2 \overset{?}{>} 16$ $0 > 16$ False
C: $(4, \infty)$	5	$(5)^2 \overset{?}{>} 16$ $25 > 16$ True

Solution set: $(-\infty, -4) \cup (4, \infty)$

49. $x^2 + 5x - 2 < 0$

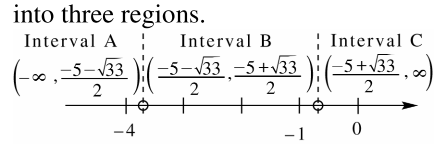
Step 1: Find the values of x that satisfy $x^2 + 5x - 2 = 0$.

Use the quadratic formula to solve the equation.

Let $a = 1$, $b = 5$, and $c = -2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4(1)(-2)}}{2(1)}$$
$$= \frac{-5 \pm \sqrt{25+8}}{2} = \frac{-5 \pm \sqrt{33}}{2}$$
$$x = \frac{-5 - \sqrt{33}}{2} \approx -5.4 \text{ or}$$
$$x = \frac{-5 + \sqrt{33}}{2} \approx .4$$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2 + 5x - 2 < 0$.

Interval	Test Value	Is $x^2 + 5x - 2 < 0$ True or False?
A: $\left(-\infty, \frac{-5-\sqrt{33}}{2}\right)$	-6	$(-6)^2 + 5(-6) - 2 \overset{?}{<} 0$ $4 < 0$ False
B: $\left(\frac{-5-\sqrt{33}}{2}, \frac{-5+\sqrt{33}}{2}\right)$	0	$(0)^2 + 5(0) - 2 \overset{?}{<} 0$ $-2 < 0$ True
C: $\left(\frac{-5+\sqrt{33}}{2}, \infty\right)$	1	$(1)^2 + 5(1) - 2 \overset{?}{<} 0$ $4 < 0$ False

Solution set: $\left(\frac{-5-\sqrt{33}}{2}, \frac{-5+\sqrt{33}}{2}\right)$

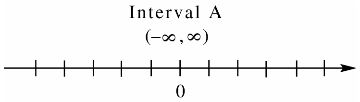
50. $4x^2 + 3x + 1 \le 0$

Step 1: Find the values of x that satisfy $4x^2 + 3x + 1 = 0$.

Use the quadratic formula to solve the equation. Let $a = 4$, $b = 3$, and $c = 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(4)(1)}}{2(4)}$$
$$= \frac{-3 \pm \sqrt{9-16}}{8} = \frac{-3 \pm \sqrt{-7}}{8} = \frac{-3 \pm i\sqrt{7}}{8}$$
$$= \frac{-3}{8} \pm \frac{\sqrt{7}}{8} i$$

Step 2: The number line is one region, $(-\infty, \infty)$.



Step 3: Since there are no real values of x that satisfy $4x^2+3x+1=0$, $4x^2+3x+1$ is either always positive or always negative. By substituting an arbitrary value such as $x=0$, we see that $4x^2+3x+1$ will be positive and thus the solution set is $\varnothing$.

Interval	Test Value	Is $4x^2+3x+1 \le 0$ True or False?
A: $(-\infty, \infty)$	0	$4(0)^2+3(0)+1 \overset{?}{\le} 0$ $1 \le 0$

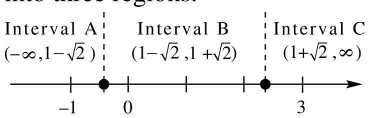
Solution set: $\varnothing$

51. $x^2-2x \le 1 \Rightarrow x^2-2x-1 \le 0$
Step 1: Find the values of x that satisfy $x^2-2x-1=0$.
Use the quadratic formula to solve the equation.
Let $a=1$, $b=-2$, and $c=-1$.

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$
$$=\frac{2 \pm \sqrt{4+4}}{2}=\frac{2 \pm \sqrt{8}}{2}$$
$$=\frac{2 \pm 2\sqrt{2}}{2}=1 \pm \sqrt{2}$$

$1-\sqrt{2} \approx -.4$ or $1+\sqrt{2} \approx 2.4$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2-2x \le 1$.

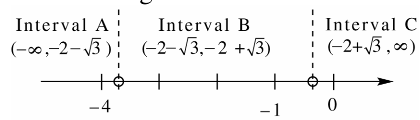
Interval	Test Value	Is $x^2-2x \le 1$ True or False?
A: $(-\infty, 1-\sqrt{2})$	-1	$(-1)^2-2(-1) \overset{?}{\le} 1$ $3 \le 1$ False
B: $(1-\sqrt{2}, 1+\sqrt{2})$	0	$0^2-2(0) \overset{?}{\le} 1 \Rightarrow 0 \le 1$ True
C: $(1+\sqrt{2}, \infty)$	3	$3^2-2(3) \overset{?}{\le} 1 \Rightarrow 3 \le 1$ False

Solution set: $\left[1-\sqrt{2}, 1+\sqrt{2}\right]$

52. $x^2+4x > -1 \Rightarrow x^2+4x+1 > 0$
Step 1: Find the values of x that satisfy $x^2+4x+1=0$.
Use the quadratic formula to solve the equation. Let $a=1$, $b=4$, and $c=1$.

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}=\frac{-4 \pm \sqrt{4^2-4(1)(1)}}{2(1)}$$
$$=\frac{-4 \pm \sqrt{16-4}}{2}=\frac{-4 \pm \sqrt{12}}{2}=\frac{-4 \pm 2\sqrt{3}}{2}$$
$$=-2 \pm \sqrt{3} \Rightarrow x \approx -3.7 \text{ or } x \approx -.3$$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2+4x > -1$.

Interval	Test Value	Is $x^2+4x > -1$ True or False?
A: $(-\infty, -2-\sqrt{3})$	-4	$(-4)^2+4(-4) \overset{?}{>} -1$ $0 > -1$ True
B: $(-2-\sqrt{3}, -2+\sqrt{3})$	-1	$(-1)^2+4(-1) \overset{?}{>} -1$ $-3 > -1$ False
C: $(-2+\sqrt{3}, \infty)$	0	$0^2+4(0) \overset{?}{>} -1$ $0 > -1$ True

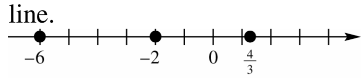
Solution set: $(-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, \infty)$

53. A; $(x+3)^2$ is equal to zero when $x=-3$. For any other real number, $(x+3)^2$ is positive. $(x+3)^2 \geq 0$ has solution set $(-\infty,\infty)$.

54. D; $(8x-7)^2$ is never negative, so $(8x-7)^2 < 0$ has solution set $\varnothing$.

55. $(3x-4)(x+2)(x+6)=0$
Set each factor to zero and solve.
$3x-4=0 \Rightarrow x=\frac{4}{3}$ or $x+2=0 \Rightarrow x=-2$ or $x+6=0 \Rightarrow x=-6$
Solution set: $\left\{\frac{4}{3},-2,-6\right\}$

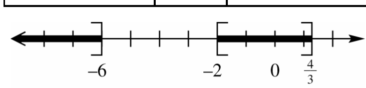
56. Plot the solutions –6, –2, and $\frac{4}{3}$ on a number line.



57.

Interval	Test Value	Is $(3x-4)(x+2)(x+6) \leq 0$ True or False?
A: $(-\infty,-6)$	−10	$[3(-10)-4][-10+2] \cdot[-10+6] \stackrel{?}{\leq} 0$ $-1088 \leq 0$ True
B: $(-6,-2)$	–4	$[3(-4)-4][-4+2] \cdot[-4+6] \stackrel{?}{\leq} 0$ $64 \leq 0$ False
C: $\left(-2,\frac{4}{3}\right)$	0	$[3(0)-4][0+2][0+6] \stackrel{?}{\leq} 0$ $-48 \leq 0$ True
D: $\left(\frac{4}{3},\infty\right)$	4	$[3(4)-4][4+2][4+6] \stackrel{?}{\leq} 0$ $480 \leq 0$ False

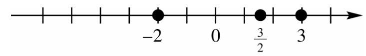
58.



Solution set: $(-\infty,-6] \cup \left[-2,\frac{4}{3}\right]$

59. $(2x-3)(x+2)(x-3) \geq 0$
Step 1: Solve $(2x-3)(x+2)(x-3)=0$.
Set each factor to zero and solve.
$2x-3=0 \Rightarrow x=\frac{3}{2}$ or $x+2=0 \Rightarrow x=-2$ or $x-3=0 \Rightarrow x=3$
Solution set: $\left\{-2,\frac{3}{2},3\right\}$

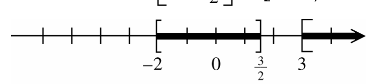
Step 2: Plot the solutions $-2, \frac{3}{2},$ and 3 on a number line.



Step 3: Choose a test value to see if it satisfies the inequality, $(2x-3)(x+2)(x-3) \geq 0$.

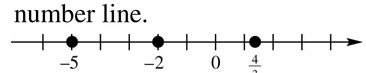
Interval	Test Value	Is $(2x-3)(x+2)(x-3) \geq 0$ True or False?
A: $(-\infty,-2)$	–3	$[2(-3)-3][-3+2] \cdot[-3-3] \stackrel{?}{\geq} 0$ $-54 \geq 0$ False
B: $\left(-2,\frac{3}{2}\right)$	0	$[2(0)-3][0+2] \cdot[0-3] \stackrel{?}{\geq} 0$ $18 \geq 0$ True
C: $\left(\frac{3}{2},3\right)$	2	$[2(2)-3][2+2] \cdot[2-3] \stackrel{?}{\geq} 0$ $-4 \geq 0$ False
D: $(3,\infty)$	4	$[2(4)-3][4+2] \cdot[4-3] \stackrel{?}{\geq} 0$ $30 \geq 0$ True

Solution set: $\left[-2,\frac{3}{2}\right] \cup [3,\infty)$



60. $(x+5)(3x-4)(x+2) \geq 0$
Step 1: Solve $(x+5)(3x-4)(x+2)=0$.
Set each factor to zero and solve.
$x+5=0 \Rightarrow x=-5$ or $3x-4=0 \Rightarrow x=\frac{4}{3}$ or $x+2=0 \Rightarrow x=-2$
Solution set: $\left\{-5,-2,\frac{4}{3}\right\}$

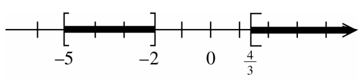
Step 2: Plot the solutions $-5,-2,$ and $\frac{4}{3}$ on a number line.



Step 3: Choose a test value to see if it satisfies the inequality, $(x+5)(3x-4)(x+2) \geq 0$.

Interval	Test Value	Is $(x+5)(3x-4)(x+2) \ge 0$ True or False?
A: $(-\infty,-5)$	-6	$[-6+5][3(-6)-4] \cdot [-6+2] \overset{?}{\ge} 0$ $-88 \ge 0$ False
B: $(-5,-2)$	-3	$[-3+5][3(-3)-4] \cdot [-3+2] \overset{?}{\ge} 0$ $26 \ge 0$ True
C: $\left(-2,\frac{4}{3}\right)$	0	$[0+5][3(0)-4] \cdot [0+2] \overset{?}{\ge} 0$ $-40 \ge 0$ False
D: $\left(\frac{4}{3},\infty\right)$	2	$[2+5][3(2)-4] \cdot [2+2] \ge 0$ $56 \ge 0$ True

Solution set: $[-5,-2] \cup \left[\frac{4}{3},\infty\right)$



61. $4x - x^3 \ge 0$

Step 1: Solve $4x - x^3 = 0$.

$$4x - x^3 = 0 \Rightarrow x\left(4 - x^2\right) = 0 \Rightarrow$$

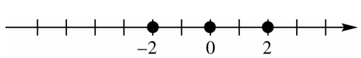
$x(2+x)(2-x) = 0$

Set each factor to zero and solve.

$x = 0$ or $2 + x = 0 \Rightarrow x = -2$ or $2 - x = 0 \Rightarrow x = 2$

Solution set: $\{-2, 0, 2\}$

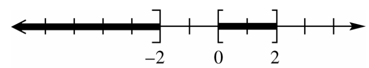
Step 2: The values -2, 0, and 2 divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $4x - x^3 \ge 0$.

Interval	Test Value	Is $4x - x^3 \ge 0$ True or False?
A: $(-\infty,-2)$	-3	$4(-3)-(-3)^3 \overset{?}{\ge} 0$ $15 \ge 0$ True
B: $(-2,0)$	-1	$4(-1)-(-1)^3 \overset{?}{\ge} 0$ $-3 \ge 0$ False
C: $(0,2)$	1	$4(1)-1^3 \overset{?}{\ge} 0$ $3 \ge 0$ True
D: $(2,\infty)$	3	$4(3)-3^3 \overset{?}{\ge} 0$ $-15 \ge 0$ False

Solution set: $(-\infty,-2] \cup [0,2]$



62. $16x - x^3 \ge 0$

Step 1: Solve $16x - x^3 = 0$.

$$16x - x^3 = 0 \Rightarrow x\left(16 - x^2\right) = 0 \Rightarrow$$

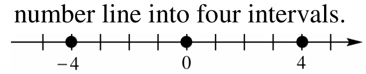
$x(4+x)(4-x) = 0$

Set each factor to zero and solve.

$x = 0$ or $4 + x = 0 \Rightarrow x = -4$ or $4 - x = 0 \Rightarrow x = 4$

Solution set: $\{-4, 0, 4\}$

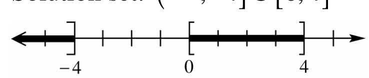
Step 2: The values -4, 0, and 4 divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $16x - x^3 \ge 0$.

Interval	Test Value	Is $16x - x^3 \ge 0$ True or False?
A: $(-\infty,-4)$	-5	$16(-5)-(-5)^3 \overset{?}{\ge} 0$ $45 \ge 0$ True
B: $(-4,0)$	-1	$16(-1)-(-1)^3 \overset{?}{\ge} 0$ $-15 \ge 0$ False
C: $(0,4)$	1	$16(1)-1^3 \overset{?}{\ge} 0$ $15 \ge 0$ True
D: $(4,\infty)$	5	$16(5)-5^3 \overset{?}{\ge} 0$ $-45 \ge 0$ False

Solution set: $(-\infty,-4] \cup [0,4]$



63. $(x+1)^2(x-3)<0$

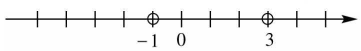
Step 1: Solve $(x+1)^2(x-3)=0$.

Set each distinct factor to zero and solve.

$x+1=0 \Rightarrow x=-1 \quad \text{or} \quad x-3=0 \Rightarrow x=3$

Solution set: $\{-1,3\}$

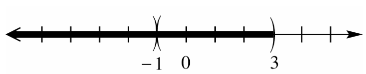
Step 2: The values -1 and 3 divide the number line into three intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $(x+1)^2(x-3)<0$.

Interval	Test Value	Is $(x+1)^2(x-3)<0$ True or False?
A: $(-\infty,-1)$	-2	$(-2+1)^2(-2-3)\overset{?}{<}0$ $-5<0$ True
B: $(-1,3)$	0	$(0+1)^2(0-3)\overset{?}{<}0$ $-3<0$ True
C: $(3,\infty)$	4	$(4+1)^2(4-3)\overset{?}{<}0$ $25<0$ False

Solution set: $(-\infty,-1)\cup(-1,3)$



64. $(x-5)^2(x+1)<0$

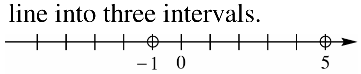
Step 1: Solve $(x-5)^2(x+1)<0$.

Set each distinct factor to zero and solve.

$x-5=0 \Rightarrow x=5 \quad \text{or} \quad x+2=0 \Rightarrow x=-1$

Solution set: $\{-1,5\}$

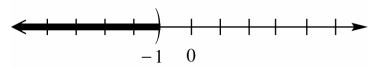
Step 2: The values -1 and 5 divide the number line into three intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $(x-5)^2(x+1)<0$.

Interval	Test Value	Is $(x-5)^2(x+1)<0$ True or False?
A: $(-\infty,-1)$	-2	$(-2-5)^2(-2+1)\overset{?}{<}0$ $-49<0$ True
B: $(-1,5)$	0	$(0-5)^2(0+1)\overset{?}{<}0$ $25<0$ False
C: $(5,\infty)$	6	$(6-5)^2(6+1)\overset{?}{<}0$ $7<0$ False

Solution set: $(-\infty,-1)$



65. $x^3+4x^2-9x\geq 36$

Step 1: Solve $x^3+4x^2-9x\geq 36$

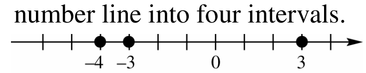
$$x^3+4x^2-9x\geq 36$$
$$x^3+4x^2-9x-36=0$$
$$x^2(x+4)-9(x+4)=0$$
$$(x+4)(x^2-9)=0$$
$$(x+4)(x+3)(x-3)=0$$

Set each factor to zero and solve.

$x+4=0 \Rightarrow x=-4 \quad \text{or} \quad x+3=0 \Rightarrow x=-3 \quad \text{or}$
$x-3=0 \Rightarrow x=3$

Solution set: $\{-4,-3,3\}$

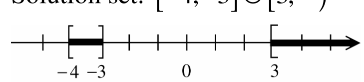
Step 2: The values $-4,-3,$ and 3 divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $x^3+4x^2-9x\geq 36$

Interval	Test Value	Is $x^3+4x^2-9x\geq 36$ True or False?
A: $(-\infty,-4)$	-5	$(-5)^3+4(-5)^2-9(-5)\overset{?}{\geq}36$ $20\geq 0$ False
B: $(-4,-3)$	-3.5	$(-3.5)^3+4(-3.5)^2$ $-9(-3.5)\overset{?}{\geq}36$ $37.625\geq 0$ True
C: $(-3,3)$	0	$0^3+4(0)^2-9(0)\overset{?}{\geq}36$ $0\geq 0$ False
D: $(3,\infty)$	4	$4^3+4(4)^2-9(4)\overset{?}{\geq}36$ $92\geq 36$ True

Solution set: $[-4,-3]\cup[3,\infty)$



66. $x^3 + 3x^2 - 16x \le 48$

Step 1: Solve $x^3 + 3x^2 - 16x = 48$.

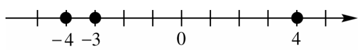
$$x^3 + 3x^2 - 16x = 48$$
$$x^3 + 3x^2 - 16x - 48 = 0$$
$$x^2(x+3) - 16(x+3) = 0$$
$$(x+3)(x^2 - 16) = 0$$
$$(x+3)(x+4)(x-4) = 0$$

Set each factor to zero and solve.

$x+3=0 \Rightarrow x=-3$ or $x+4=0 \Rightarrow x=-4$ or $x-4=0 \Rightarrow x=4$

Solution set: $\{-4, -3, 4\}$

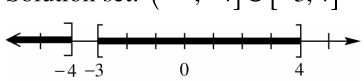
Step 2: The values $-4, -3$, and 4 divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $x^3 + 3x^2 - 16x \le 48$.

Interval	Test Value	Is $x^3 + 3x^2 - 16x \le 48$. True or False?
A: $(-\infty, -4)$	-5	$(-5)^3 + 3(-5)^2 - 16(-5) \overset{?}{\le} 48$ $30 \le 48$ True
B: $(-4, -3)$	-3.5	$(-3.5)^3 + 3(-3.5)^2 - 16(-3.5) \overset{?}{\le} 48$ $49.875 \le 48$ False
C: $(-3, 4)$	0	$0^3 + 3(0)^2 - 16(0) \overset{?}{\le} 48$ $0 \le 48$ True
D: $(4, \infty)$	5	$5^3 + 3(5)^2 - 16(5) \le 48$ $120 \le 48$ False

Solution set: $(-\infty, -4] \cup [-3, 4]$



67. $x^2(x+4)^2 \ge 0$

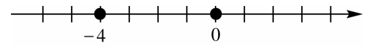
Step 1: Solve $x^2(x+4)^2 \ge 0$.

Set each distinct factor to zero and solve.

$x = 0$ or $x + 4 = 0 \Rightarrow x = -4$

Solution set: $\{-4, 0\}$

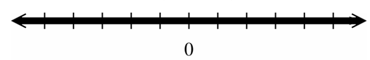
Step 2: The values -4 and 0 divide the number line into three intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2(x+4)^2 \ge 0$.

Interval	Test Value	Is $x^2(x+4)^2 \ge 0$ True or False?
A: $(-\infty, -4)$	-5	$(-5)^2(-5+4)^2 \overset{?}{\ge} 0$ $25 \ge 0$ True
B: $(-4, 0)$	-1	$(-1)^2(-1+4)^2 \overset{?}{\ge} 0$ $9 \ge 0$ True
C: $(0, \infty)$	1	$1^2(1+4)^2 \overset{?}{\ge} 0$ $25 \ge 0$ True

Solution set: $(-\infty, \infty)$



68. $x^2(2x-3)^2 < 0$

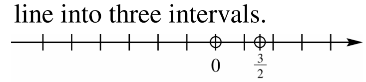
Step 1: Solve $x^2(2x-3)^2 < 0$.

Set each distinct factor to zero and solve.

$x = 0$ or $2x - 3 = 0 \Rightarrow x = \frac{3}{2}$

Solution set: $\{0, \frac{3}{2}\}$

Step 2: The values 0 and $\frac{3}{2}$ divide the number line into three intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2(2x-3)^2 < 0$.

Interval	Test Value	Is $x^2(2x-3)^2 < 0$ True or False?
A: $(-\infty, 0)$	-1	$(-1)^2[2(-1)-3]^2 \overset{?}{<} 0$ $25 < 0$ False
B: $\left(0, \frac{3}{2}\right)$	1	$1^2[2(1)-3]^2 \overset{?}{<} 0$ $1 < 0$ False
C: $\left(\frac{3}{2}, \infty\right)$	2	$2^2[2(2)-3]^2 \overset{?}{<} 0$ $4 < 0$ False

Solution set: $\varnothing$

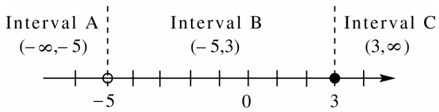
69. $\dfrac{x-3}{x+5} \le 0$

Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$x-3=0 \Rightarrow x=3 \quad \text{or} \quad x+5=0 \Rightarrow x=-5$

The values –5 and 3 divide the number line into three regions. Use an open circle on –5 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{x-3}{x+5} \le 0.$

Interval	Test Value	Is $\frac{x-3}{x+5} \le 0$ True or False?
A: $(-\infty,-5)$	–6	$\frac{-6-3}{-6+5} \overset{?}{\le} 0$ $9 \le 0$ False
B: $(-5,3)$	0	$\frac{0-3}{0+5} \overset{?}{\le} 0$ $-\frac{3}{5} \le 0$ True
C: $(3,\infty)$	4	$\frac{4-3}{4+5} \overset{?}{\le} 0$ $\frac{1}{9} \le 0$ False

Interval B satisfies the inequality. The endpoint −5 is not included because it makes the denominator 0.

Solution set: (–5, 3]

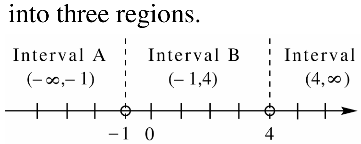
70. $\dfrac{x+1}{x-4} > 0$

Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$x+1=0 \Rightarrow x=-1 \quad \text{or} \quad x-4=0 \Rightarrow x=4$

The values –1 and 4 divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{x+1}{x-4} > 0.$

Interval	Test Value	Is $\frac{x+1}{x-4} > 0$ True or False?
A: $(-\infty,-1)$	–2	$\frac{-2+1}{-2-4} \overset{?}{>} 0$ $\frac{1}{6} > 0$ True
B: $(-1,4)$	0	$\frac{0+1}{0-4} \overset{?}{>} 0$ $-\frac{1}{4} > 0$ False
C: $(4,\infty)$	5	$\frac{5+1}{5-4} \overset{?}{>} 0$ $6 > 0$ True

Solution set: $(-\infty,-1) \cup (4,\infty)$

71. $\dfrac{1-x}{x+2} < -1$

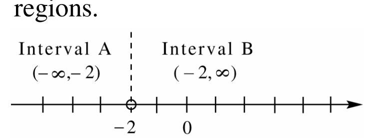
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{1-x}{x+2} < -1 \Rightarrow \frac{x-1}{x+2} > 1$$
$$\frac{x-1}{x+2} - 1 > 0 \Rightarrow \frac{x-1}{x+2} - \frac{x+2}{x+2} > 0$$
$$\frac{x-1-(x+2)}{x+2} > 0 \Rightarrow \frac{x-1-x-2}{x+2} > 0$$
$$\frac{-3}{x+2} > 0$$

Step 2: Since the numerator is a constant, determine the values that will cause denominator to equal 0.

$x+2=0 \Rightarrow x=-2$

The value –2 divides the number line into two regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{1-x}{x+2} < -1$

Interval	Test Value	Is $\frac{1-x}{x+2} < -1$ True or False?
A: $(-\infty,-2)$	–3	$\frac{-1-(-3)}{-3+2} \overset{?}{<} -1$ $-2 < -1$ True
B: $(-2,\infty)$	–1	$\frac{1-(-1)}{-1+2} \overset{?}{<} 1$ $2 < -1$ False

Solution set: $(-\infty,-2)$

72. $\dfrac{6-x}{x+2}>1$

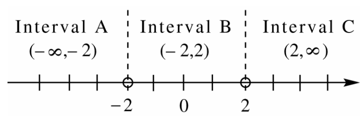
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{6-x}{x+2}>1\Rightarrow\frac{x-6}{x+2}<-1$$
$$\frac{x-6}{x+2}+1<0\Rightarrow\frac{x-6}{x+2}+\frac{x+2}{x+2}<0$$
$$\frac{x-6+x+2}{x+2}<0\Rightarrow\frac{2x-4}{x+2}<0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$2x-4=0\Rightarrow x=2$ or $x+2=0\Rightarrow x=-2$

The values –2 and 2 divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{6-x}{x+2}>1$

Interval	Test Value	Is $\frac{6-x}{x+2}>1$ True or False?
A: $(-\infty,-2)$	-3	$\frac{-6-(-3)}{-3+2}\overset{?}{>}1$ $-9>-1$ False
B: $(-2,2)$	0	$\frac{6-0}{0+2}\overset{?}{>}1$ $3>1$ True
C: $(2,\infty)$	3	$\frac{6-3}{3+2}\overset{?}{>}1$ $\frac{3}{5}>1$ False

Solution set: (–2, 2)

73. $\dfrac{3}{x-6}\le 2$

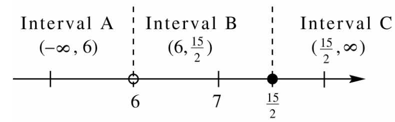
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{3}{x-6}-2\le 0\Rightarrow\frac{3}{x-6}-\frac{2(x-6)}{x-6}\le 0$$
$$\frac{3-2(x-6)}{x-6}\le 0\Rightarrow\frac{3-2x+12}{x-6}\le 0$$
$$\frac{15-2x}{x-6}\le 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$15-2x=0\Rightarrow x=\frac{15}{2}$ or $x-6=0\Rightarrow x=6$

The values 6 and $\frac{15}{2}$ divide the number line into three regions. Use an open circle on 6 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{3}{x-6}\le 2.$

Interval	Test Value	Is $\frac{3}{x-6}\le 2$ True or False?
A: $(-\infty,6)$	0	$\frac{3}{0-6}\overset{?}{\le}2$ $-\frac{1}{2}\le 2$ True
B: $\left(6,\frac{15}{2}\right)$	7	$\frac{3}{7-6}\overset{?}{\le}2$ $3\le 2$ False
C: $\left(\frac{15}{2},\infty\right)$	8	$\frac{3}{8-6}\overset{?}{\le}2$ $\frac{3}{2}\le 2$ True

Intervals A and C satisfiy the inequality. The endpoint 6 is not included because it makes the denominator 0.

Solution set: $(-\infty,6)\cup\left[\frac{15}{2},\infty\right)$

74. $\dfrac{3}{x-2}<1$

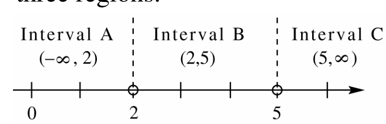
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{3}{x-2}-1<0$$
$$\frac{3}{x-2}-\frac{x-2}{x-2}<0\Rightarrow\frac{3-(x-2)}{x-2}<0$$
$$\frac{3-x+2}{x-2}<0\Rightarrow\frac{5-x}{x-2}<0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$5-x=0\Rightarrow 5=x$ or $x-2=0\Rightarrow x=2$

The values 2 and 5 divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{3}{x-2}<1.$

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Interval	Test Value	Is $\frac{3}{x-2} < 1$ True or False?
A: $(-\infty, 2)$	0	$\frac{3}{0-2} \overset{?}{<} 1$ $-\frac{3}{2} < 1$ True
B: $(2, 5)$	3	$\frac{3}{3-2} \overset{?}{<} 1$ $3 < 1$ False
C: $(5, \infty)$	6	$\frac{3}{6-2} \overset{?}{<} 1$ $\frac{3}{4} < 1$ True

Solution set: $(-\infty, 2) \cup (5, \infty)$

75. $\frac{-4}{1-x} < 5$

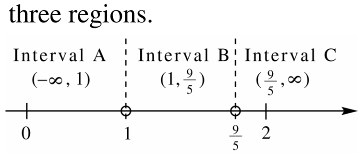
Step 1: Rewrite the inequality to compare a single fraction to 0:

$$\frac{-4}{1-x} < 5 \Rightarrow \frac{-4}{1-x} - 5 < 0$$
$$\frac{-4}{1-x} - \frac{5(1-x)}{1-x} < 0 \Rightarrow \frac{-4-5(1-x)}{1-x} < 0$$
$$\frac{-4-5+5x}{1-x} < 0 \Rightarrow \frac{-9+5x}{1-x} < 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-9+5x = 0 \Rightarrow x = \frac{9}{5}$ or $1 - x = 0 \Rightarrow x = 1$

The values 1 and $\frac{9}{5}$ divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{-4}{1-x} < 5$

Interval	Test Value	Is $\frac{-4}{1-x} < 5$ True or False?
A: $(-\infty, 1)$	0	$\frac{-4}{1-0} \overset{?}{<} 5$ $-4 < 5$ True
B: $(1, \frac{9}{5})$	$\frac{6}{5}$	$\frac{-4}{1-\frac{6}{5}} \overset{?}{<} 5$ $20 < 5$ False
C: $(\frac{9}{5}, \infty)$	2	$\frac{-4}{1-2} \overset{?}{<} 5$ $4 < 5$ True

Solution set: $(-\infty, 1) \cup \left(\frac{9}{5}, \infty\right)$

76. $\frac{-6}{3x-5} \le 2$

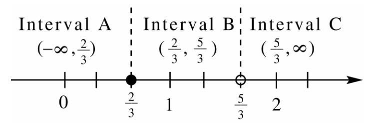
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{-6}{3x-5} \le 2 \Rightarrow \frac{-6}{3x-5} - \frac{2(3x-5)}{3x-5} \le 0$$
$$\frac{-6-2(3x-5)}{3x-5} \le 0 \Rightarrow \frac{-6-6x+10}{3x-5} \le 0$$
$$\frac{-6x+4}{3x-5} \le 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-6x + 4 = 0 \Rightarrow x = \frac{2}{3}$ or $3x - 5 = 0 \Rightarrow x = \frac{5}{3}$

The values $\frac{2}{3}$ and $\frac{5}{3}$ divide the number line into three regions. Use an open circle on $\frac{5}{3}$ because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{6}{5-3x} \le 2$.

Interval	Test Value	Is $\frac{6}{5-3x} \le 2$ True or False?
A: $\left(-\infty, \frac{2}{3}\right)$	0	$\frac{6}{5-3\cdot 0} \overset{?}{\le} 2$ $\frac{6}{5} \le 2$ True
B: $\left(\frac{2}{3}, \frac{5}{3}\right)$	1	$\frac{6}{5-3\cdot 1} \overset{?}{\le} 2$ $3 \le 2$ False
Interval	Test Value	Is $\frac{6}{5-3x} \le 2$ True or False?
C: $\left(\frac{5}{3}, \infty\right)$	2	$\frac{6}{5-3\cdot 2} \overset{?}{\le} 2$ $-6 \le 2$ True

Intervals A and C satisfy the inequality. The endpoint $\frac{5}{3}$ is not included because it makes the denominator 0.

Solution set: $\left(-\infty, \frac{2}{3}\right] \cup \left(\frac{5}{3}, \infty\right)$

77. $\frac{10}{3+2x} \le 5$

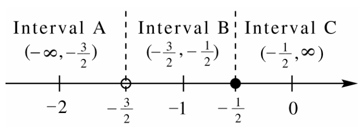
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{10}{3+2x} - 5 \le 0 \Rightarrow \frac{10}{3+2x} - \frac{5(3+2x)}{3+2x} \le 0$$
$$\frac{10-5(3+2x)}{3+2x} \le 0 \Rightarrow \frac{10-15-10x}{3+2x} \le 0$$
$$\frac{-10x-5}{3+2x} \le 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-10x-5=0 \Rightarrow x=-\frac{1}{2}$ or $3+2x=0 \Rightarrow x=-\frac{3}{2}$

The values $-\frac{3}{2}$ and $-\frac{1}{2}$ divide the number line into three regions. Use an open circle on $-\frac{3}{2}$ because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{10}{3+2x} \le 5$.

Interval	Test Value	Is $\frac{10}{3+2x} \le 5$ True or False?
A: $\left(-\infty,-\frac{3}{2}\right)$	-2	$\frac{10}{3+2(-2)} \overset{?}{\le} 5$ $-10 \le 5$ True
B: $\left(-\frac{3}{2},-\frac{1}{2}\right)$	-1	$\frac{10}{3+2(-1)} \overset{?}{\le} 5$ $10 \le 5$ False
C: $\left(-\frac{1}{2},\infty\right)$	0	$\frac{10}{3+2(0)} \overset{?}{\le} 5$ $\frac{10}{3} \le 5$ True

Intervals A and C satisfy the inequality. The endpoint $-\frac{3}{2}$ is not included because it makes the denominator 0.

Solution set: $\left(-\infty,-\frac{3}{2}\right) \cup \left[-\frac{1}{2},\infty\right)$

78. $\frac{1}{x+2} \ge 3$

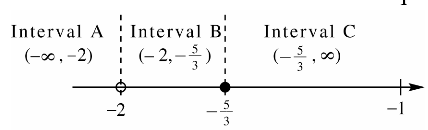
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{1}{x+2} - 3 \ge 0 \Rightarrow \frac{1}{x+2} - \frac{3(x+2)}{x+2} \ge 0$$
$$\frac{1-3(x+2)}{x+2} \ge 0 \Rightarrow \frac{1-3x-6}{x+2} \ge 0$$
$$\frac{-3x-5}{x+2} \ge 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-3x-5=0 \Rightarrow x=-\frac{5}{3}$ or $x+2=0 \Rightarrow x=-2$

The values –2 and $-\frac{5}{3}$ divide the number line into three regions. Use an open circle on –2 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{1}{x+2} \ge 3$.

Interval	Test Value	Is $\frac{1}{x+2} \ge 3$ True or False?
A: $(-\infty,-2)$	-3	$\frac{1}{-3+2} \overset{?}{\ge} 3$ $-1 \ge 3$ False
B: $\left(-2,-\frac{5}{3}\right)$	$-\frac{11}{6}$	$\frac{1}{-\frac{11}{6}+2} \overset{?}{\ge} 3$ $6 \ge 3$ True
C: $\left(-\frac{5}{3},\infty\right)$	0	$\frac{1}{0+2} \overset{?}{\ge} 3$ $\frac{1}{2} \ge 3$ False

Interval B satisfies the inequality. The endpoint -2 is not included because it makes the denominator 0.

Solution set: $\left(-2,-\frac{5}{3}\right]$

79. $\frac{7}{x+2} \ge \frac{1}{x+2}$

Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{7}{x+2} - \frac{1}{x+2} \ge 0 \Rightarrow \frac{6}{x+2} \ge 0$$

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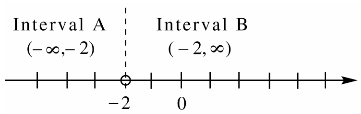
Step 2: Since the numerator is a constant, determine the value that will cause the denominator to equal 0.

$x+2=0 \Rightarrow x=-2$

The value -2 divides the number line into two regions. Use an open circle on –2 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{7}{x+2} \geq \frac{1}{x+2}$.

Interval	Test Value	Is $\frac{7}{x+2} \geq \frac{1}{x+2}$ True or False?
A: $(-\infty,-2)$	-3	$\frac{7}{-3+2} \overset{?}{\geq} \frac{1}{-3+2}$ $-7 \geq -1$ False
B: $(-2,\infty)$	0	$\frac{7}{0+2} \overset{?}{\geq} \frac{1}{0+2}$ $\frac{7}{2} \geq \frac{1}{2}$ True

Interval B satisfies the inequality. The endpoint -2 is not included because it makes the denominator 0.

Solution set: $(-2,\infty)$

80. $\frac{5}{x+1} > \frac{12}{x+1}$

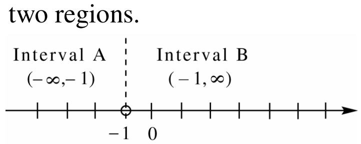
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{5}{x+1} - \frac{12}{x+1} > 0 \Rightarrow \frac{-7}{x+1} > 0$$

Step 2: Since the numerator is a constant, determine the value that will cause the denominator to equal 0.

$x+1=0 \Rightarrow x=-1$

The value -1 divides the number line into two regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{5}{x+1} > \frac{12}{x+1}$.

Interval	Test Value	Is $\frac{5}{x+1} > \frac{12}{x+1}$ True or False?
A: $(-\infty,-1)$	-2	$\frac{5}{-2+1} \overset{?}{>} \frac{12}{-2+1}$ $-5 > -12$ True
B: $(-1,\infty)$	0	$\frac{5}{0+1} \overset{?}{>} \frac{12}{0+1}$ $5 > 12$ False

Solution set: $(-\infty,-1)$

81. $\frac{3}{2x-1} > \frac{-4}{x}$

Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

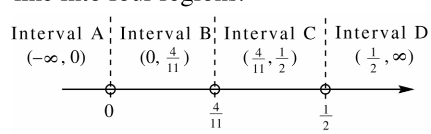
$$\frac{3}{2x-1} + \frac{4}{x} > 0$$
$$\frac{3x}{x(2x-1)} + \frac{4(2x-1)}{x(2x-1)} > 0$$
$$\frac{3x+4(2x-1)}{x(2x-1)} > 0$$
$$\frac{3x+8x-4}{x(2x-1)} > 0 \Rightarrow \frac{11x-4}{x(2x-1)} > 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$11x-4=0 \Rightarrow x=\frac{4}{11}$ or $x=0$ or

$2x-1=0 \Rightarrow x=\frac{1}{2}$

The values 0, $\frac{4}{11}$, and $\frac{1}{2}$ divide the number line into four regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{3}{2x-1} > \frac{-4}{x}$.

Interval	Test Value	Is $\frac{3}{2x-1} > \frac{-4}{x}$ True or False?
A: $(-\infty,0)$	-1	$\frac{3}{2(-1)-1} \overset{?}{>} \frac{-4}{-1}$ $-1 > 4$ False
B: $\left(0,\frac{4}{11}\right)$	$\frac{1}{11}$	$\frac{3}{2\left(\frac{1}{11}\right)-1} \overset{?}{>} \frac{-4}{\frac{1}{11}}$ or $-\frac{11}{3} \overset{?}{>} -44$ $-3\frac{2}{3} > -44$ True

Interval	Test Value	Is $\frac{3}{2x-1} > \frac{-4}{x}$ True or False?
C: $\left(\frac{4}{11}, \frac{1}{2}\right)$	$\frac{9}{22}$	$\frac{3}{2\left(\frac{9}{22}\right)-1} \overset{?}{>} \frac{-4}{\frac{9}{22}}$ or $-\frac{33}{2} \overset{?}{>} -\frac{88}{9}$ $-16\frac{1}{2} > -9\frac{7}{9}$ False
D: $\left(\frac{1}{2}, \infty\right)$	1	$\frac{3}{2(1)-1} \overset{?}{>} \frac{-4}{1}$ $3 > -4$ True

Solution set: $\left(0, \frac{4}{11}\right) \cup \left(\frac{1}{2}, \infty\right)$

82. $\frac{-5}{3x+2} \geq \frac{5}{x}$

Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{-5}{3x+2} - \frac{5}{x} \geq 0$$

$$\frac{-5x}{x(3x+2)} - \frac{5(3x+2)}{x(3x+2)} \geq 0$$

$$\frac{-5x-5(3x+2)}{x(3x+2)} \geq 0$$

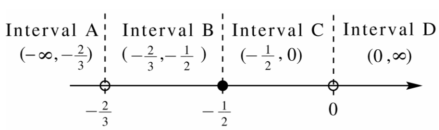
$$\frac{-5x-15x-10}{x(3x+2)} \geq 0 \Rightarrow \frac{-20x-10}{x(3x+2)} \geq 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-20x-10=0$ or $x=0$ or $3x+2=0$

$x=-\frac{1}{2}$ or $x=0$ or $x=-\frac{2}{3}$

The values $-\frac{2}{3}$, $-\frac{1}{2}$, and 0 divide the number line into four regions. Use an open circle on 0 and $-\frac{2}{3}$ because they makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{-5}{3x+2} \geq \frac{5}{x}$.

Interval	Test Value	Is $\frac{-5}{3x+2} \geq \frac{5}{x}$ True or False?
A: $\left(-\infty, -\frac{2}{3}\right)$	-1	$\frac{-5}{3(-1)+2} \overset{?}{\geq} \frac{5}{-1}$ $5 \geq -5$ True
B: $\left(-\frac{2}{3}, -\frac{1}{2}\right)$	$-\frac{7}{12}$	$\frac{-5}{3\left(-\frac{7}{12}\right)+2} \overset{?}{\geq} \frac{5}{-\frac{7}{12}}$ $-20 \geq -\frac{60}{7}$ $-20 \geq -8\frac{4}{7}$ False
C: $\left(-\frac{1}{2}, 0\right)$	$-\frac{1}{4}$	$\frac{-5}{3\left(-\frac{1}{4}\right)+2} \overset{?}{\geq} \frac{5}{-\frac{1}{4}}$ $-4 \geq -20$ True
D: $(0, \infty)$	1	$\frac{-5}{3(1)+2} \overset{?}{\geq} \frac{5}{1}$ $-1 \geq 5$ False

Intervals A and C satisfy the inequality. The endpoints $-\frac{2}{3}$ and 0 are not included because they make the denominator 0.

Solution set: $\left(-\infty, -\frac{2}{3}\right) \cup \left[-\frac{1}{2}, 0\right)$

83. $\frac{4}{2-x} \geq \frac{3}{1-x}$

Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{4}{2-x} \geq \frac{3}{1-x}$$

$$\frac{4}{2-x} - \frac{3}{1-x} \geq 0$$

$$\frac{4(1-x)}{(2-x)(1-x)} - \frac{3(2-x)}{(1-x)(2-x)} \geq 0$$

$$\frac{4(1-x)-3(2-x)}{(x-2)(1-x)} \geq 0$$

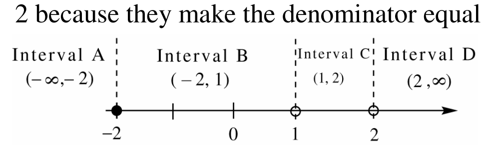
$$\frac{4-4x-6+3x}{(2-x)(1-x)} \geq 0$$

$$\frac{-2-x}{(2-x)(1-x)} \geq 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-2-x=0 \Rightarrow x=-2$ or $2-x=0 \Rightarrow x=2$ or $1-x=0 \Rightarrow x=1$

The values –2, 1, and 2 divide the number line into four regions. Use an open circle on 1 and 2 because they make the denominator equal 0.



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Step 3: Choose a test value to see if it satisfies the inequality, $\frac{4}{2-x} \geq \frac{3}{1-x}$

Interval	Test Value	Is $\frac{4}{2-x} \geq \frac{3}{1-x}$ True or False?
A: $(-\infty,-2)$	-3	$\frac{4}{-2-(-3)} \overset{?}{\geq} \frac{3}{-1-(-3)}$ $4 \geq \frac{3}{2}$ True
B: $(-2,1)$	0	$\frac{4}{2-0} \overset{?}{\geq} \frac{3}{1-0}$ $2 \geq 3$ False
C: $(1,2)$	1.5	$\frac{4}{2-1.5} \overset{?}{\geq} \frac{3}{1-1.5}$ $8 \geq -6$ True
D: $(2,\infty)$	3	$\frac{4}{2-3} \overset{?}{\geq} \frac{3}{1-3}$ $-4 \geq -\frac{3}{2}$ False

Intervals A and C satisfy the inequality. The endpoints 1 and 2 are not included because they make the denominator 0.

Solution set: $(-\infty,-2] \cup (1,2)$

84. $\frac{4}{x+1} < \frac{2}{x+3}$

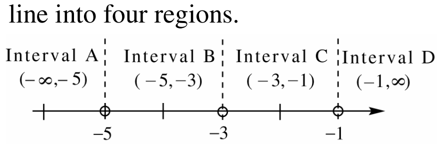
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{4}{x+1} - \frac{2}{x+3} < 0$$
$$\frac{4(x+3)}{(x+1)(x+3)} - \frac{2(x+1)}{(x+3)(x+1)} < 0$$
$$\frac{4(x+3)-2(x+1)}{(x+1)(x+3)} < 0$$
$$\frac{4x+12-2x-2}{(x+1)(x+3)} < 0$$
$$\frac{2x+10}{(x+1)(x+3)} < 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$2x+10=0 \Rightarrow x=-5$ or $x+1=0 \Rightarrow x=-1$ or $x+3=0 \Rightarrow x=-3$

The values -5, -3, and -1 divide the number line into four regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{4}{x+1} < \frac{2}{x+3}$.

Interval	Test Value	Is $\frac{4}{x+1} < \frac{2}{x+3}$ True or False?
A: $(-\infty,-5)$	-6	$\frac{4}{-6+1} \overset{?}{<} \frac{2}{-6+3}$ or $-\frac{4}{5} \overset{?}{<} -\frac{2}{3}$ $-\frac{12}{15} < -\frac{10}{15}$ True
B: $(-5,-3)$	-4	$\frac{4}{-4+1} \overset{?}{<} \frac{2}{-4+3}$ $-\frac{4}{3} < -2$ False
C: $(-3,-1)$	-2	$\frac{4}{-2+1} \overset{?}{<} \frac{2}{-2+3}$ $-4 < 2$ True
D: $(-1,\infty)$	0	$\frac{4}{0+1} \overset{?}{<} \frac{2}{0+3}$ $4 < \frac{2}{3}$ False

Solution set: $(-\infty,-5) \cup (-3,-1)$

85. $\frac{x+3}{x-5} \leq 1$

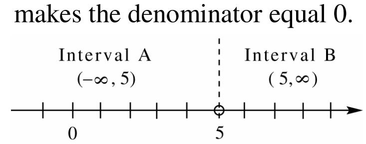
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{x+3}{x-5} - 1 \leq 0 \Rightarrow \frac{x+3}{x-5} - \frac{x-5}{x-5} \leq 0$$
$$\frac{x+3-(x-5)}{x-5} \leq 0 \Rightarrow \frac{x+3-x+5}{x-5} \leq 0$$
$$\frac{8}{x-5} \leq 0$$

Step 2: Since the numerator is a constant, determine the value that will cause the denominator to equal 0.

$x-5=0 \Rightarrow x=5$

The value 5 divides the number line into two regions. Use an open circle on 5 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{x+3}{x-5} \leq 1$.

Interval	Test Value	Is $\frac{x+3}{x-5} \le 1$ True or False?
A: $(-\infty, 5)$	0	$\frac{0+3}{0-5} \overset{?}{\le} 1$ $-\frac{3}{5} \le 1$ True
B: $(5, \infty)$	6	$\frac{6+3}{6-5} \overset{?}{\le} 1$ $9 \le 1$ False

Interval A satisfies the inequality. The endpoint 5 is not included because it makes the denominator 0.

Solution set: $(-\infty, 5)$

86. $\dfrac{x+2}{3+2x} \le 5$

Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{x+2}{3+2x} - 5 \le 0$$

$$\frac{x+2}{3+2x} - \frac{5(3+2x)}{3+2x} \le 0$$

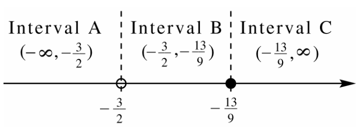
$$\frac{x+2-5(3+2x)}{3+2x} \le 0$$

$$\frac{x+2-15-10x}{3+2x} \le 0 \Rightarrow \frac{-9x-13}{3+2x} \le 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-9x - 13 = 0 \Rightarrow x = -\frac{13}{9}$ or $3 + 2x = 0 \Rightarrow x = -\frac{3}{2}$

The values $-\frac{3}{2}$ and $-\frac{13}{9}$ divide the number line into three regions. Use an open circle on $-\frac{3}{2}$ because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{x+2}{3+2x} \le 5$.

Interval	Test Value	Is $\frac{x+2}{3+2x} \le 5$ True or False?
A: $\left(-\infty, -\frac{3}{2}\right)$	−2	$\frac{-2+2}{3+2(-2)} \overset{?}{\le} 5$ $0 \le 5$ True
B: $\left(-\frac{3}{2}, -\frac{13}{9}\right)$	−1.45	$\frac{-1.45+2}{3+2(-1.45)} \overset{?}{\le} 5$ $5.5 \le 5$ False
C: $\left(-\frac{13}{9}, \infty\right)$	0	$\frac{0+2}{3+2(0)} \overset{?}{\le} 5$ $\frac{2}{3} \le 5$ True

Intervals A and C satisfy the inequality. The endpoint $-\frac{3}{2}$ is not included because it makes the denominator 0.

Solution set: $\left(-\infty, -\frac{3}{2}\right) \cup \left[-\frac{13}{9}, \infty\right)$

87. $\dfrac{2x-3}{x^2+1} \ge 0$

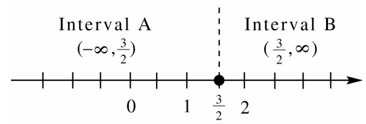
Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$2x - 3 = 0$ or $x^2 + 1 = 0$

$x = \frac{3}{2}$ has no real solutions

$\frac{3}{2}$ divides the number line into two intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{2x-3}{x^2+1} \ge 0$.

Interval	Test Value	Is $\frac{2x-3}{x^2+1} \ge 0$ True or False?
A: $\left(-\infty, \frac{3}{2}\right)$	0	$\frac{2(0)-3}{0^2+1} \overset{?}{\ge} 0$ $-3 \ge 0$ False
B: $\left(\frac{3}{2}, \infty\right)$	2	$\frac{2(2)-3}{2^2+1} \overset{?}{\ge} 0$ $\frac{1}{5} \ge 0$ True

Solution set: $\left[\frac{3}{2}, \infty\right)$

88. $\dfrac{9x-8}{4x^2+25} < 0$

Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$9x - 8 = 0 \Rightarrow x = \frac{8}{9}$ or

$4x^2 + 25 = 0,$ which has no real solutions

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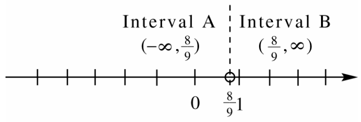
The value $\frac{8}{9}$ divides the number line into two intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{9x-8}{4x^2+25} < 0$.

Interval	Test Value	Is $\frac{9x-8}{4x^2+25} < 0$ True or False?
A: $\left(-\infty, \frac{8}{9}\right)$	0	$\frac{9(0)-8}{4(0)^2+25} \overset{?}{<} 0$ $-\frac{8}{25} < 0$ True
B: $\left(\frac{8}{9}, \infty\right)$	1	$\frac{9(1)-8}{4(1)^2+25} \overset{?}{<} 0$ $\frac{1}{29} < 0$ False

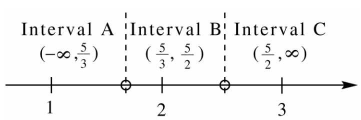
Solution set: $\left(-\infty, \frac{8}{9}\right)$

89. $\frac{(5-3x)^2}{(2x-5)^3} > 0$

Since one side of the inequality is already 0, we start with Step 2.
Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$5-3x=0 \Rightarrow x=\frac{5}{3}$ or $2x-5=0 \Rightarrow x=\frac{5}{2}$

The values $\frac{5}{3}$ and $\frac{5}{2}$ divide the number line into three intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{(5-3x)^2}{(2x-5)^3} > 0$.

Interval	Test Value	Is $\frac{(5-3x)^2}{(2x-5)^3} > 0$ True or False?
A: $\left(-\infty, \frac{5}{3}\right)$	0	$\frac{(5-3\cdot 0)^2}{(2\cdot 0-5)^3} \overset{?}{>} 0$ $-\frac{1}{5} > 0$ False
B: $\left(\frac{5}{3}, \frac{5}{2}\right)$	2	$\frac{(5-3\cdot 2)^2}{(2\cdot 2-5)^3} \overset{?}{>} 0$ $-1 > 0$ False
C: $\left(\frac{5}{2}, \infty\right)$	3	$\frac{(5-3\cdot 3)^2}{(2\cdot 3-5)^3} \overset{?}{>} 0$ $16 > 0$ True

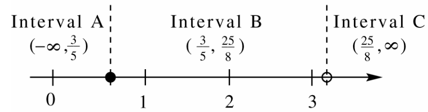
Solution set: $\left(\frac{5}{2}, \infty\right)$

90. $\frac{(5x-3)^3}{(25-8x)^2} \le 0$

Since one side of the inequality is already 0, we start with Step 2.
Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$5x-3=0 \Rightarrow x=\frac{3}{5}$ or $25-8x=0 \Rightarrow x=\frac{25}{8}$

The values $\frac{3}{5}$ and $\frac{25}{8}$ divide the number line into three intervals. Use an open circle on $\frac{25}{8}$ because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{(5x-3)^3}{(25-8x)^2} \le 0$.

Interval	Test Value	Is $\frac{(5x-3)^3}{(25-8x)^2} \le 0$ True or False?
A: $\left(-\infty, \frac{3}{5}\right)$	0	$\frac{(5\cdot 0-3)^3}{(25-8\cdot 0)^2} \overset{?}{\le} 0$ $-\frac{27}{625} \le 0$ True
B: $\left(\frac{3}{5}, \frac{25}{8}\right)$	2	$\frac{(5\cdot 2-3)^3}{(25-8\cdot 2)^2} \overset{?}{\le} 0$ $\frac{343}{81} \le 0$ False
C: $\left(\frac{25}{8}, \infty\right)$	4	$\frac{(5\cdot 4-3)^3}{(25-8\cdot 4)^2} \overset{?}{\le} 0$ $\frac{4913}{49} \le 0$ False

Solution set: $\left(-\infty, \frac{3}{5}\right]$

91. $\dfrac{(2x-3)(3x+8)}{(x-6)^3} \geq 0$

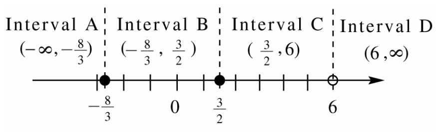
Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$2x-3=0$ or $3x+8=0$ or $x-6=0$

$x=\frac{3}{2}$ or $x=-\frac{8}{3}$ or $x=6$

The values $-\frac{8}{3}$, $\frac{3}{2}$, and 6 divide the number line into four intervals. Use an open circle on 6 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{(2x-3)(3x+8)}{(x-6)^3} \geq 0$.

Interval	Test Value	Is $\frac{(2x-3)(3x+8)}{(x-6)^3} \geq 0$ True or False?
A: $\left(-\infty, -\frac{8}{3}\right)$	–3	$\frac{[2(-3)-3][3(-3)+8]}{(-3-6)^3} \overset{?}{\geq} 0$ $-\frac{1}{81} \geq 0$ False
B: $\left(-\frac{8}{3}, \frac{3}{2}\right)$	0	$\frac{(2\cdot 0-3)(3\cdot 0+8)}{(0-6)^3} \overset{?}{\geq} 0$ $\frac{1}{9} \geq 0$ True
C: $\left(\frac{3}{2}, 6\right)$	2	$\frac{(2\cdot 2-3)(3\cdot 2+8)}{(2-6)^3} \overset{?}{\geq} 0$ $-\frac{7}{32} \geq 0$ False
D: $(6, \infty)$	7	$\frac{(2\cdot 7-3)(3\cdot 7+8)}{(7-6)^3} \overset{?}{\geq} 0$ $319 \geq 0$ True

Solution set: $\left[-\frac{8}{3}, \frac{3}{2}\right] \cup (6, \infty)$

92. $\dfrac{(9x-11)(2x+7)}{(3x-8)^3} > 0$

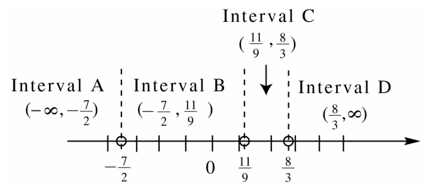
Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$9x-11=0$ or $2x+7=0$ or $3x-8=0$

$x=\frac{11}{9}$ or $x=-\frac{7}{2}$ or $x=\frac{8}{3}$

The values $-\frac{7}{2}$, $\frac{11}{9}$, and $\frac{8}{3}$ divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{(9x-11)(2x+7)}{(3x-8)^3} > 0$.

Interval	Test Value	Is $\frac{(9x-11)(2x+7)}{(3x-8)^3} > 0$ True or False?
A: $\left(-\infty, -\frac{7}{2}\right)$	–4	$\frac{[9(-4)-11][2(-4)+7]}{[3(-4)-8]^3} \overset{?}{>} 0$ $-\frac{47}{8000} > 0$ False
B: $\left(-\frac{7}{2}, \frac{11}{9}\right)$	0	$\frac{(9\cdot 0-11)(2\cdot 0+7)}{(3\cdot 0-8)^3} \overset{?}{>} 0$ $\frac{77}{512} > 0$ True
C: $\left(\frac{11}{9}, \frac{8}{3}\right)$	2	$\frac{(9\cdot 2-11)(2\cdot 2+7)}{(3\cdot 2-8)^3} \overset{?}{>} 0$ $-\frac{77}{8} > 0$ False
D: $\left(\frac{8}{3}, \infty\right)$	3	$\frac{(9\cdot 3-11)(2\cdot 3+7)}{(3\cdot 3-8)^3} \overset{?}{>} 0$ $208 > 0$ True

Solution set: $\left(-\frac{7}{2}, \frac{11}{9}\right) \cup \left(\frac{8}{3}, \infty\right)$

93. **(a)** Let $R = 5.3$ and then solve for x.

$5.3 = 0.28944x + 3.5286$

$1.7714 = 0.28944x$

$6.1 \approx x$

The model predicts that the receipts reach \$5.3 billion about 6.1 years after 1986 which is in 1992.

(b) Let $R = 7$ and then solve for x.

$7 = 0.28944x + 3.5286$

$3.4714 = 0.28944x$

$12.0 \approx x$

The model predicts that the receipts reach \$7 billion about 12.0 years after 1986 which is in 1998.

94. **(a)** Let $W = 25$ and solve for x.

$W = .6286x + 27.662$

$.6286x + 27.662 > 25$

$.6286x > -2.662$

$x > -4.23$ (approximately)

According to the model, the percent of waste recovered first exceeded 25% about 4.23 years before 1998, which is in 1993.

(b) Solve for x for values between 26 and 28.

$$W = .6286x + 27.662$$
$$26 < .6286x + 27.662 < 28$$
$$-1.662 < .6286x < .338$$
$$-2.64 < x < .54 \text{ (approximately)}$$

According to the model, the percent of waste recovered was between 26% and 28% about 2.6 years before 1998, which is in 1995, until about .54 years after 1998, which is in 1998.

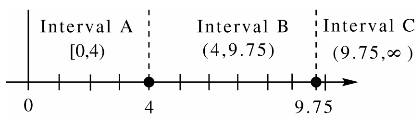
95. $-16t^2 + 220t \geq 624$

Step 1: Find the values of x that satisfy $-16t^2 + 220t = 624$.

$$-16t^2 + 220t = 624 \Rightarrow 0 = 16t^2 - 220t + 624$$
$$0 = 4t^2 - 55t + 156$$
$$0 = (t-4)(4t-39)$$

$t - 4 = 0 \Rightarrow t = 4$ or $4t - 39 = 0 \Rightarrow t = \frac{39}{4} = 9.75$

Step 2: The two numbers divide a number line into three regions, where $t \geq 0$.



Step 3: Choose a test value to see if it satisfies the inequality, $-16t^2 + 220t \geq 624$.

Interval	Test Value	Is $-16t^2 + 220t \geq 624$ True or False?
A: $(0,4)$	1	$-16 \cdot 1^2 + 220 \cdot 1 \overset{?}{\geq} 624$; $204 \geq 624$ False
B: $(4,9.75)$	5	$-16 \cdot 5^2 + 220 \cdot 5 \overset{?}{\geq} 624$; $700 \geq 624$ True
C: $(9.75,\infty)$	10	$-16 \cdot 10^2 + 220 \cdot 10 \overset{?}{\geq} 624$; $600 \geq 624$ False

The projectile will be at least 624 feet above ground between 4 sec and 9.75 sec (inclusive).

96. $2t^2 - 5t - 12 < 0$

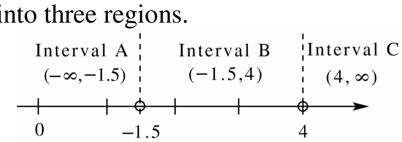
Step 1: Find the values of x that satisfy $2t^2 - 5t - 12 = 0$.

$$2t^2 - 5t - 12 = 0 \Rightarrow (2t+3)(t-4) = 0$$

$2t + 3 = 0 \Rightarrow t = -\frac{3}{2} = -1.5$ or

$t - 4 = 0 \Rightarrow t = 4$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $2t^2 - 5t - 12 < 0$.

Interval	Test Value	Is $2t^2 - 5t - 12 < 0$ True or False?
A: $(-\infty,-1.5)$	-2	$2(-2)^2 - 5(-2) - 12 \overset{?}{<} 0$; $6 < 0$; False
B: $(-1.5,4)$	0	$2 \cdot 0^2 - 5 \cdot 0 - 12 \overset{?}{<} 0$; $-12 < 0$; True
C: $(4,\infty)$	5	$2 \cdot 5^2 - 5 \cdot 5 - 12 \overset{?}{<} 0$; $13 < 0$; False

The velocity will be negative between -1.5 sec and 4 sec.

97. **(a)** $1.5 \times 10^{-3} \leq R \leq 6.0 \times 10^{-3}$

$$2.08 \times 10^{-5} \leq \frac{R}{72} \leq 8.33 \times 10^{-5}$$

(Approximate values are given.)

(b) Let N be the number of additional lung cancer deaths each year. Then N would be determined by taking the annual individual risk times the total number of people. Thus, $N = (310 \times 10^6)\left(\frac{R}{72}\right)$. The range for N would be approximately

$$(310 \times 10^6)(2.08 \times 10^{-5}) \leq N \leq (310 \times 10^6)(8.33 \times 10^{-5})$$
$$6448 \leq N \leq 25{,}823$$

Thus, radon gas exposure is expected by the EPA to cause approximately between 6400 and 25,800 cases of lung cancer each year in the United States.

98. Answers will vary.

The student's answer is not correct. The expression $x + 2$ is a variable expression, not a constant. You lose a critical value if you multiply through by $x + 2$. The correct solution can be found using the methods described in this section.

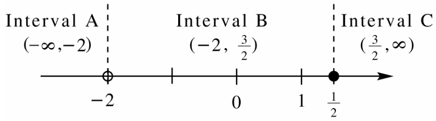
$$\frac{2x-3}{x+2} \leq 0$$

Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$2x-3=0 \Rightarrow x=\frac{3}{2}$ or $x+2=0 \Rightarrow x=-2$

The values -2 and $\frac{3}{2}$ divide the number line into three intervals. Use an open circle on -2 because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{2x-3}{x+2} \le 0.$

Interval	Test Value	Is $\frac{2x-3}{x+2} \le 0$ True or False?
A: $(-\infty,-2)$	-3	$\frac{2(-3)-3}{-3+2} \overset{?}{\le} 0$ $7 \le 0$ False
B: $\left(-2,\frac{3}{2}\right)$	0	$\frac{2\cdot 0-3}{0+2} \overset{?}{\le} 0$ $-\frac{3}{2} \le 0$ True
C: $\left(\frac{3}{2},\infty\right)$	2	$\frac{2\cdot 2-3}{2+2} \overset{?}{\le} 0$ $\frac{1}{4} \le 0$ False

Interval B satisfies the inequality. The endpoint -2 is not included because it makes the denominator 0. Solution set: $\left(-2,\frac{3}{2}\right]$

99. Answers will vary. The student's answer is not correct. Taking the square root of both sides (and including a $\pm$) can be done if one is examining an equation, not an inequality. The correct solution can be found using the methods described in this section.

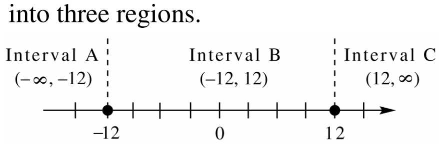
$x^2 \le 144$

Step 1: Find the values of x that satisfy $x^2 = 144.$

$x^2 = 144 \Rightarrow x^2 - 144 = 0 \Rightarrow (x+12)(x-12)=0$

$x+12=0 \Rightarrow x=-12$ or $x-12=0 \Rightarrow x=12$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2 \le 144.$

Interval	Test Value	Is $x^2 \le 144$ True or False?
A: $(-\infty,-12)$	-20	$(-20)^2 \overset{?}{\le} 144$ $400 \le 144$ False
B: $(-12,12)$	0	$0^2 \overset{?}{\le} 144$ $0 \le 16$ True
C: $(12,\infty)$	20	$20^2 \overset{?}{\le} 144$ $400 \le 144$ False

Solution set: $[-12,12]$

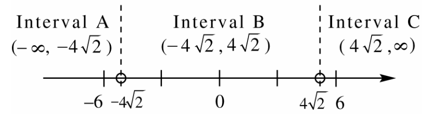
100. When $b^2-4ac>0$ where $a=1, b=-k,$ and $c=8,$ two real solutions will occur (if the discriminant were equal to zero, we would have one real solution).

$(-k)^2 - 4(1)(8) = k^2 - 32 > 0$

Step 1: Find the values of k that satisfy $k^2 - 32 = 0.$

$k^2 = 32 \Rightarrow k = \pm\sqrt{32} \Rightarrow k = \pm 4\sqrt{2}$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $k^2 - 32 > 0.$

Interval	Test Value	Is $k^2 - 32 > 0$ True or False?
A: $(-\infty,-4\sqrt{2})$	-6	$(-6)^2 - 32 \overset{?}{>} 0$ $4 > 0$ True
B: $(-4\sqrt{2},4\sqrt{2})$	0	$0^2 - 32 \overset{?}{>} 0$ $-32 > 0$ False
C: $(4\sqrt{2},\infty)$	6	$6^2 - 32 \overset{?}{>} 0$ $4 > 0$ True

When $k < -4\sqrt{2}$ or $k > 4\sqrt{2}$ we will have two real solutions.

Section 1.8: Absolute Value Equations and Inequalities

1. $|x| = 7$

The solution set includes any value of x whose absolute value is 7; thus $x = 7$or $x = -7$ are both solutions. The correct graph is F.

2. $|x| = -7$

There is no solution, since the absolute value of any real number is never negative. The correct choice is B.

3. $|x| > -7$

The solution set is all real numbers, since the absolute value of any real number is always greater than –7. The correct graph is D, which shows the entire number line.

4. $|x| > 7$

The solution set includes any value of x whose absolute value is greater than 7; thus $x > 7$ or $x < -7$. The correct graph is E.

5. $|x| < 7$

The solution set includes any value of x whose absolute value is less than 7; thus x must be between –7 and 7, not including –7 or 7. The correct graph is G.

6. $|x| \ge 7$

The solution set includes any value of x whose absolute value is greater than or equal to 7; thus $x \ge 7$ or $x \le -7$. The correct graph is A.

7. $|x| \le 7$

The solution set includes any value of x whose absolute value is less than or equal to 7; thus x must be between –7 and 7, including –7 and 7. The correct graph is C.

8. $|x| \ne 7$

The solution set includes any value of x whose absolute value is not equal to 7; thus, x can equal all real numbers except –7 and 7. The correct graph is H.

9. $|3x - 1| = 2$

$3x - 1 = 2 \Rightarrow 3x = 3 \Rightarrow x = 1$ or

$3x - 1 = -2 \Rightarrow 3x = -1 \Rightarrow x = -\frac{1}{3}$

Solution set: $\left\{-\frac{1}{3}, 1\right\}$

10. $|4x + 2| = 5$

$4x + 2 = 5 \Rightarrow 4x = 3 \Rightarrow x = \frac{3}{4}$ or

$4x + 2 = -5 \Rightarrow 4x = -7 \Rightarrow x = -\frac{7}{4}$

Solution set: $\left\{-\frac{7}{4}, \frac{3}{4}\right\}$

11. $|5 - 3x| = 3$

$5 - 3x = 3 \Rightarrow 2 = 3x \Rightarrow \frac{2}{3} = x$ or

$5 - 3x = -3 \Rightarrow 8 = 3x \Rightarrow \frac{8}{3} = x$

Solution set: $\left\{\frac{2}{3}, \frac{8}{3}\right\}$

12. $|7 - 3x| = 3$

$7 - 3x = 3 \Rightarrow -3x = -4 \Rightarrow x = \frac{4}{3}$ or

$7 - 3x = -3 \Rightarrow -3x = -10 \Rightarrow x = \frac{10}{3}$

Solution set: $\left\{\frac{4}{3}, \frac{10}{3}\right\}$

13. $\left|\frac{x-4}{2}\right| = 5$

$\frac{x-4}{2} = 5 \Rightarrow x - 4 = 10 \Rightarrow x = 14$ or

$\frac{x-4}{2} = -5 \Rightarrow x - 4 = -10 \Rightarrow x = -6$

Solution set: $\{-6, 14\}$

14. $\left|\frac{x+2}{2}\right| = 7$

$\frac{x+2}{2} = 7 \Rightarrow x + 2 = 14 \Rightarrow x = 12$ or

$\frac{x+2}{2} = -7 \Rightarrow x + 2 = -14 \Rightarrow x = -16$

Solution set: $\{-16, 12\}$

15. $\left|\frac{5}{x-3}\right| = 10$

$\frac{5}{x-3} = 10 \Rightarrow 5 = 10(x-3) \Rightarrow 5 = 10x - 30 \Rightarrow$

$35 = 10x \Rightarrow x = \frac{35}{10} = \frac{7}{2}$ or

$\frac{5}{x-3} = -10 \Rightarrow 5 = -10(x-3) \Rightarrow$

$5 = -10x + 30 \Rightarrow -25 = -10x \Rightarrow x = \frac{-25}{-10} = \frac{5}{2}$

Solution set: $\left\{\frac{5}{2}, \frac{7}{2}\right\}$

16. $\left|\frac{3}{2x-1}\right| = 4$

$\frac{3}{2x-1} = 4 \Rightarrow 3 = 4(2x-1) \Rightarrow 3 = 8x-4 \Rightarrow$
$7 = 8x \Rightarrow \frac{7}{8} = x$ or
$\frac{3}{2x-1} = -4 \Rightarrow 3 = -4(2x-1) \Rightarrow$
$3 = -8x+4 \Rightarrow -1 = -8x \Rightarrow x = \frac{1}{8}$

Solution set: $\left\{\frac{1}{8}, \frac{7}{8}\right\}$

17. $\left|\frac{6x+1}{x-1}\right| = 3$

$\frac{6x+1}{x-1} = 3 \Rightarrow 6x+1 = 3(x-1) \Rightarrow$
$6x+1 = 3x-3 \Rightarrow 3x = -4 \Rightarrow x = -\frac{4}{3}$ or
$\frac{6x+1}{x-1} = -3 \Rightarrow 6x+1 = -3(x-1) \Rightarrow$
$6x+1 = -3x+3 \Rightarrow 9x = 2 \Rightarrow x = \frac{2}{9}$

Solution set: $\left\{-\frac{4}{3}, \frac{2}{9}\right\}$

18. $\left|\frac{2x+3}{3x-4}\right| = 1$

$\frac{2x+3}{3x-4} = 1 \Rightarrow 2x+3 = 1(3x-4) \Rightarrow$
$2x+3 = 3x-4 \Rightarrow x = 7$ or
$\frac{2x+3}{3x-4} = -1 \Rightarrow 2x+3 = -1(3x-4) \Rightarrow$
$2x+3 = -3x+4 \Rightarrow 5x = 1 \Rightarrow x = \frac{1}{5}$

Solution set: $\left\{\frac{1}{5}, 7\right\}$

19. $|2x-3| = |5x+4|$

$2x-3 = 5x+4 \Rightarrow -7 = 3x \Rightarrow -\frac{7}{3} = x$ or
$2x-3 = -(5x+4) \Rightarrow 2x-3 = -5x-4 \Rightarrow$
$7x = -1 \Rightarrow x = \frac{-1}{7} = -\frac{1}{7}$

Solution set: $\left\{-\frac{7}{3}, -\frac{1}{7}\right\}$

20. $|x+1| = |1-3x|$

$x+1 = 1-3x \Rightarrow 4x = 0 \Rightarrow x = 0$ or
$x+1 = -(1-3x) \Rightarrow x+1 = -1+3x \Rightarrow$
$2 = 2x \Rightarrow 1 = x$
Solution set: $\{1, 0\}$

21. $|4-3x| = |2-3x|$

$4-3x = 2-3x \Rightarrow 4 = 2$ False or
$4-3x = -(2-3x) \Rightarrow 4-3x = -2+3x \Rightarrow$
$6 = 6x \Rightarrow 1 = x$
Solution set: $\{1\}$

22. $|3-2x| = |5-2x|$

$3-2x = 5-2x \Rightarrow 3 = 5$ False or
$3-2x = -(5-2x) \Rightarrow 3-2x = -5+2x \Rightarrow$
$8 = 4x \Rightarrow 2 = x$
Solution set: $\{2\}$

23. $|5x-2| = |2-5x|$

$5x-2 = 2-5x \Rightarrow 10x = 4 \Rightarrow x = \frac{4}{10} = \frac{2}{5}$ or
$5x-2 = -(2-5x) \Rightarrow 5x-2 = -2+5x \Rightarrow$
$0 = 0$ True
Solution set: $(-\infty, \infty)$

24. Answers will vary.
If x is negative, then $3x$ will also be negative. Since the outcome of an absolute value can never be negative, a negative value of x is not possible.

25. Answers will vary.
If x is positive, then $-5x$ will be negative. Since the outcome of an absolute value can never be negative, a positive value of x is not possible.

26. **(a)** $-|x| = |x|$

$|x|$ will equal its own opposite only if $x = 0$.
Solution set: $\{0\}$

(b) $|-x| = |x|$

Any number and its opposite have the same absolute value.
Solution set: $(-\infty, \infty)$

(c) $|x^2| = |x|$

Solution set: $\{-1, 0, 1\}$

(d) $-|x| = 9$

$|x| = -9$ is never true.
Solution set: $\varnothing$

27. $|2x+5| < 3$

$-3 < 2x+5 < 3$
$-8 < 2x < -2$
$-4 < x < -1$

Solution set: $(-4, -1)$

28. $|3x-4|<2$
$-2<3x-4<2$
$2<3x<6$
$\frac{2}{3}<x<2$
Solution set: $\left(\frac{2}{3},2\right)$

29. $|2x+5|\ge 3$
$2x+5\le -3 \Rightarrow 2x\le -8 \Rightarrow x\le -4$ or
$2x+5\ge 3 \Rightarrow 2x\ge -2 \Rightarrow x\ge -1$
Solution set: $(-\infty,-4]\cup[-1,\infty)$

30. $|3x-4|\ge 2$
$3x-4\le -2 \Rightarrow 3x\le 2 \Rightarrow x\le \frac{2}{3}$ or
$3x-4\ge 2 \Rightarrow 3x\ge 6 \Rightarrow x\ge 2$
Solution set: $\left(-\infty,\frac{2}{3}\right]\cup[2,\infty)$

31. $\left|\frac{1}{2}-x\right|<2$
$-2<\frac{1}{2}-x<2$
$2(-2)<2\left(\frac{1}{2}-x\right)<2(2)$
$-4<1-2x<4$
$-5<-2x<3$
$\frac{5}{2}>x>-\frac{3}{2}$
Solution set: $\left(-\frac{3}{2},\frac{5}{2}\right)$

32. $\left|\frac{3}{5}+x\right|<1$
$-1<\frac{3}{5}+x<1$
$5(-1)<5\left(\frac{3}{5}+x\right)<5(1)$
$-5<3+5x<5$
$-8<5x<2$
$-\frac{8}{5}<x<\frac{2}{5}$
Solution set: $\left(-\frac{8}{5},\frac{2}{5}\right)$

33. $4|x-3|>12 \Rightarrow |x-3|>3$
$x-3<-3 \Rightarrow x<0$ or $x-3>3 \Rightarrow x>6$
Solution set: $(-\infty,0)\cup(6,\infty)$

34. $5|x+1|>10 \Rightarrow |x+1|>2$
$x+1<-2 \Rightarrow x<-3$ or $x+1>2 \Rightarrow x>1$
Solution set: $(-\infty,-3)\cup(1,\infty)$

35. $|5-3x|>7$
$5-3x<-7 \Rightarrow -3x<-12 \Rightarrow x>4$ or
$5-3x>7 \Rightarrow -3x>2 \Rightarrow x<-\frac{2}{3}$
Solution set: $\left(-\infty,-\frac{2}{3}\right)\cup(4,\infty)$

36. $|7-3x|>4$
$7-3x<-4 \Rightarrow -3x<-11 \Rightarrow x>\frac{11}{3}$ or
$7-3x>4 \Rightarrow -3x>-3 \Rightarrow x<1$
Solution set: $(-\infty,1)\cup\left(\frac{11}{3},\infty\right)$

37. $|5-3x|\le 7$
$-7\le 5-3x\le 7$
$-12\le -3x\le 2$
$4\ge x\ge -\frac{2}{3}$
$-\frac{2}{3}\le x\le 4$
Solution set: $\left[-\frac{2}{3},4\right]$

38. $|7-3x|\le 4$
$-4\le 7-3x\le 4$
$-11\le -3x\le -3$
$\frac{11}{3}\ge x\ge 1$
$1\le x\le \frac{11}{3}$
Solution set: $\left[1,\frac{11}{3}\right]$

39. $\left|\frac{2}{3}x+\frac{1}{2}\right|\le \frac{1}{6}$
$-\frac{1}{6}\le \frac{2}{3}x+\frac{1}{2}\le \frac{1}{6}$
$6\left(-\frac{1}{6}\right)\le 6\left(\frac{2}{3}x+\frac{1}{2}\right)\le 6\left(\frac{1}{6}\right)$
$-1\le 4x+3\le 1$
$-4\le 4x\le -2$
$-1\le x\le -\frac{1}{2}$
Solution set: $\left[-1,-\frac{1}{2}\right]$

40. $\left|\frac{5}{3}-\frac{1}{2}x\right|>\frac{2}{9}$
$\frac{5}{3}-\frac{1}{2}x<-\frac{2}{9} \Rightarrow 18\left(\frac{5}{3}-\frac{1}{2}x\right)<18\left(-\frac{2}{9}\right) \Rightarrow$
$30-9x<-4 \Rightarrow -9x<-34 \Rightarrow x>\frac{34}{9}$ or
$\frac{5}{3}-\frac{1}{2}x>\frac{2}{9} \Rightarrow 18\left(\frac{5}{3}-\frac{1}{2}x\right)>18\left(\frac{2}{9}\right) \Rightarrow$
$30-9x>4 \Rightarrow -9x>-26 \Rightarrow x<\frac{26}{9}$
Solution set: $\left(-\infty,\frac{26}{9}\right)\cup\left(\frac{34}{9},\infty\right)$

41. $|.01x+1|<.01$
$-.01<.01x+1<.01$
$-1<x+100<1$
$-101<x<-99$
Solution set: $(-101,-99)$

42. The absolute value of any number is the same as the absolute value of the opposite of that number. Therefore, $|x|=|-x|$ for all values of x.

43. $|4x+3|-2=-1 \Rightarrow |4x+3|=1$
$4x+3=1 \Rightarrow 4x=-2 \Rightarrow x=\frac{-2}{4}=-\frac{1}{2}$ or
$4x+3=-1 \Rightarrow 4x=-4 \Rightarrow x=-1$
Solution set: $\left\{-\frac{1}{2}, -1\right\}$

44. $|8-3x|-3=-2 \Rightarrow |8-3x|=1$
$8-3x=1 \Rightarrow -3x=-7 \Rightarrow x=\frac{7}{3}$ or
$8-3x=-1 \Rightarrow -3x=-9 \Rightarrow x=3$
Solution set: $\left\{\frac{7}{3}, 3\right\}$

45. $|6-2x|+1=3 \Rightarrow |6-2x|=2$
$6-2x=2 \Rightarrow -2x=-4 \Rightarrow x=2$ or
$6-2x=-2 \Rightarrow -2x=-8 \Rightarrow x=4$
Solution set: $\{2, 4\}$

46. $|4-4x|+2=4 \Rightarrow |4-4x|=2$
$4-4x=2 \Rightarrow -4x=-2 \Rightarrow x=\frac{-2}{-4}=\frac{1}{2}$ or
$4-4x=-2 \Rightarrow -4x=-6 \Rightarrow x=\frac{-6}{-4}=\frac{3}{2}$
Solution set: $\left\{\frac{1}{2}, \frac{3}{2}\right\}$

47. $|3x+1|-1<2 \Rightarrow |3x+1|<3$
$-3<3x+1<3$
$-4<3x<2$
$-\frac{4}{3}<x<\frac{2}{3}$
Solution set: $\left(-\frac{4}{3}, \frac{2}{3}\right)$

48. $|5x+2|-2<3 \Rightarrow |5x+2|<5$
$-5<5x+2<5$
$-7<5x<3$
$-\frac{7}{5}<x<\frac{3}{5}$
Solution set: $\left(-\frac{7}{5}, \frac{3}{5}\right)$

49. $\left|5x+\frac{1}{2}\right|-2<5 \Rightarrow \left|5x+\frac{1}{2}\right|<7$
$-7<5x+\frac{1}{2}<7$
$2(-7)<2\left(5x+\frac{1}{2}\right)<2(7)$
$-14<10x+1<14$
$-15<10x<13$
$-\frac{15}{10}<x<\frac{13}{10} \Rightarrow -\frac{3}{2}<x<\frac{13}{10}$
Solution set: $\left(-\frac{3}{2}, \frac{13}{10}\right)$

50. $\left|2x+\frac{1}{3}\right|+1<4 \Rightarrow \left|2x+\frac{1}{3}\right|<3$
$-3<2x+\frac{1}{3}<3$
$3(-3)<3\left(2x+\frac{1}{3}\right)<3(3)$
$-9<6x+1<9$
$-10<6x<8$
$\frac{-10}{6}<x<\frac{8}{6}$
$-\frac{5}{3}<x<\frac{4}{3}$
Solution set: $\left(-\frac{5}{3}, \frac{4}{3}\right)$

51. $|10-4x|+1\geq 5 \Rightarrow |10-4x|\geq 4$
$10-4x\leq -4 \Rightarrow -4x\leq -14 \Rightarrow x\geq \frac{-14}{-4} \Rightarrow x\geq \frac{7}{2}$
or
$10-4x\geq 4 \Rightarrow -4x\geq -6 \Rightarrow x\leq \frac{-6}{-4} \Rightarrow x\leq \frac{3}{2}$
Solution set: $\left(-\infty, \frac{3}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$

52. $|12-6x|+3\geq 9 \Rightarrow |12-6x|\geq 6$
$12-6x\leq -6 \Rightarrow -6x\leq -18 \Rightarrow x\geq 3$ or
$12-6x\geq 6 \Rightarrow -6x\geq -6 \Rightarrow x\leq 1$
Solution set: $(-\infty, 1] \cup [3, \infty)$

53. $|3x-7|+1<-2 \Rightarrow |3x-7|<-3$
An absolute value cannot be negative.
Solution set: $\varnothing$

54. $|-5x+7|-4<-6 \Rightarrow |-5x+7|<-2$
An absolute value cannot be negative.
Solution set: $\varnothing$

55. Since the absolute value of a number is always nonnegative, the inequality $|10-4x|\geq -4$ is always true. The solution set is $(-\infty, \infty)$.

56. Since the absolute value of a number is always nonnegative, the inequality $|12-9x|\geq -12$ is always true. The solution set is $(-\infty, \infty)$.

57. There is no number whose absolute value is less than any negative number. The solution set of $|6-3x|<-11$ is $\varnothing$.

58. There is no number whose absolute value is less than any negative number. The solution set of $|18-3x|<-3$ is $\varnothing$.

59. The absolute value of a number will be 0 if that number is 0. Therefore $|8x+5|=0$ is equivalent to $8x+5=0$, which has solution set $\left\{-\frac{5}{8}\right\}$.

60. The absolute value of a number will be 0 if that number is 0. Therefore $|7+2x|=0$ is equivalent to $7+2x=0,$ which has solution set $\left\{-\frac{7}{2}\right\}.$

61. Any number less than zero will be negative. There is no number whose absolute value is a negative number. The solution set of $|4.3x+9.8|<0$ is $\varnothing.$

62. Any number less than zero will be negative. There is no number whose absolute value is a negative number. The solution set of $|1.5x-14|<0$ is $\varnothing.$

63. Since the absolute value of a number is always nonnegative, $|2x+1|<0$ is never true, so $|2x+1|\le 0$ is only true when $|2x+1|=0.$

$|2x+1|=0 \Rightarrow 2x+1=0 \Rightarrow 2x=-1 \Rightarrow x=-\frac{1}{2}$

Solution set: $\left\{-\frac{1}{2}\right\}$

64. Since the absolute value of a number is always nonnegative, $|3x+2|<0$ is never true, so $|3x+2|\le 0$ is only true when $|3x+2|=0.$

$|3x+2|=0 \Rightarrow 3x+2=0 \Rightarrow 3x=-2 \Rightarrow x=-\frac{2}{3}$

Solution set: $\left\{-\frac{2}{3}\right\}$

65. $|3x+2|>0$ will be false only when $3x+2=0,$ which occurs when $x=-\frac{2}{3}.$ So the solution set for $|3x+2|>0$ is $\left(-\infty,-\frac{2}{3}\right)\cup\left(-\frac{2}{3},\infty\right).$

66. $|4x+3|>0$ will be false only when $4x+3=0,$ which occurs when $x=-\frac{3}{4}.$ So the solution set for $|4x+3|>0$ is $\left(-\infty,-\frac{3}{4}\right)\cup\left(-\frac{3}{4},\infty\right).$

67. 6 and the opposite of 6, namely $-6.$

68. $x^2-x=6$

$x^2-x-6=0$

$(x+2)(x-3)=0$

$x+2=0 \Rightarrow x=-2$ or $x-3=0 \Rightarrow x=3$

Solution set: $\{-2,3\}$

69. $x^2-x=-6 \Rightarrow x^2-x+6=0$

The quadratic formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be evaluated with $a=1, b=-1,$ and $c=6.$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(6)}}{2(1)}=\frac{1\pm\sqrt{1-24}}{2}=\frac{1\pm\sqrt{-23}}{2}=\frac{1\pm i\sqrt{23}}{2}=\frac{1}{2}\pm\frac{\sqrt{23}}{2}i$$

Solution set: $\left\{\frac{1}{2}\pm\frac{\sqrt{23}}{2}i\right\}$

70. $\left\{-2,3,\frac{1}{2}\pm\frac{\sqrt{23}}{2}i\right\}$

71. $|4x^2-23x-6|=0$

Because 0 and the opposite of 0 represent the same value, only one equation needs to be solved.

$4x^2-23x-6=0 \Rightarrow (4x+1)(x-6)=0$

$4x+1=0$ or $x-6=0$

$x=-\frac{1}{4}$ or $x=6$

Solution set: $\left\{-\frac{1}{4},6\right\}$

72. $|6x^3+23x^2+7x|=0$

Because 0 and the opposite of 0 represent the same value, only one equation needs to be solved.

$6x^3+23x^2+7x=0$

$x\left(6x^2+23x+7\right)=0$

$x(2x+7)(3x+1)=0$

$x=0$ or $2x+7=0 \Rightarrow x=-\frac{7}{2}$ or

$3x+1=0 \Rightarrow x=-\frac{1}{3}$

Solution set: $\left\{-\frac{7}{2},-\frac{1}{3},0\right\}$

73. $|x^2+1|-|2x|=0$

$|x^2+1|-|2x|=0 \Rightarrow |x^2+1|=|2x|$

$x^2+1=2x \Rightarrow x^2-2x+1=0$

$(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ or

$x^2+1=-2x \Rightarrow x^2+2x+1=0 \Rightarrow (x+1)^2=0$

$x+1=0 \Rightarrow x=-1$

Solution set: $\{-1,1\}$

74. $\left|\frac{x^2+2}{x}\right|-\frac{11}{3}=0$

$\left|\frac{x^2+2}{x}\right|-\frac{11}{3}=0 \Rightarrow \left|\frac{x^2+2}{x}\right|=\frac{11}{3}$

$\frac{x^2+2}{x}=\frac{11}{3} \Rightarrow 3x\left(\frac{x^2+2}{x}\right)=3x\left(\frac{11}{3}\right)$

$3\left(x^2+2\right)=11x \Rightarrow 3x^2+6=11x \Rightarrow$

$3x^2-11x+6=0 \Rightarrow (3x-2)(x-3)=0$

$3x-2=0 \Rightarrow x=\frac{2}{3}$ or $x-3=0 \Rightarrow x=3$ or

$\frac{x^2+2}{x}=-\frac{11}{3}$

$3x\left(\frac{x^2+2}{x}\right)=3x\left(-\frac{11}{3}\right)$

$3\left(x^2+2\right)=-11x \Rightarrow 3x^2+6=-11x \Rightarrow$

$3x^2+11x+6=0 \Rightarrow (3x+2)(x+3)=0$

$3x+2=0 \Rightarrow x=-\frac{2}{3}$ or $x+3=0 \Rightarrow x=-3$

Solution set: $\left\{-3,-\frac{2}{3},\frac{2}{3},3\right\}$

75. Any number less than zero will be negative. There is no number whose absolute value is a negative number. The solution set of $\left|x^4+2x^2+1\right|<0$ is $\varnothing$.

76. $\left|x^4+2x^2+1\right|\geq 0$

Since the outcome of absolute value of a number is always 0 or greater, any real number will satisfy this inequality.

Solution set: $(-\infty,\infty)$

77. $\left|\frac{x-4}{3x+1}\right|\geq 0$

This inequality will be true, except where $\frac{x-4}{3x+1}$ is undefined. This occurs when $3x+1=0$, or $x=-\frac{1}{3}$.

Solution set: $\left(-\infty,-\frac{1}{3}\right)\cup\left(-\frac{1}{3},\infty\right)$

78. $\left|\frac{9-x}{7+8x}\right|\geq 0$

This inequality will be true, except where $\frac{9-x}{7+8x}$ is undefined. This occurs when $7+8x=0$, or $x=-\frac{7}{8}$.

Solution set: $\left(-\infty,-\frac{7}{8}\right)\cup\left(-\frac{7}{8},\infty\right)$

79. $|p-q|=2$, which is equivalent to $|q-p|=2$, indicates that the distance between p and q is 2 units.

80. $|r-s|=6$, which is equivalent to $|s-r|=6$, indicates that the distance between r and s is 6 units.

81. "m is no more than 2 units from 7" means that m is 2 units or less from 7. Thus the distance between m and 7 is less than or equal to 2, or $|m-7|\leq 2$.

82. "z is no less than 8 units from 4" means that z is 8 units or more from 4. Thus, the distance between z and 4 is greater than or equal to 8, or $|z-4|\geq 8$.

83. "p is within .0001 units of 6" means that p is less than .0001 units from 6. Thus the distance between p and 6 is less than .0001, or $|p-6|<.0001$.

84. "k is within .0002 units of 7" means that k is less than .0002 units from 7. Thus the distance between k and 7 is less than .0002, or $|k-7|<.0002$.

85. "r is no less than 1 unit from 29" means that r is 1 unit or more from 29. Thus the distance between r and 29 is greater than or equal to 1, or $|r-29|\geq 1$.

86. "q is no more than 4 units from 22" means that q is 4 units or less from 22. Thus the distance between q and 22 is less than or equal to 4, or $|q-22|\leq 4$.

87. Since we want y to be within .002 unit of 6, we have $|y-6|<.002$ or $|5x+1-6|<.002$.

$|5x-5|<.002$

$-.002<5x-5<.002$

$4.998<5x<5.002$

$.9996<x<1.0004$

Values of x in the interval $(.9996, 1.0004)$ would satisfy the condition.

88. Since we want y to be within .002 unit of 6, we have $|y-6|<.002$ or $|10x+2-6|<.002$.

$|10x-4|<.002 \Rightarrow -.002<10x-4<.002$

$3.998<10x<4.002$

$.3998<x<.4002$

Values of x in the interval $(.3998, .4002)$ would satisfy the condition.

89. $|y-8.2| \le 1.5$
$-1.5 \le y-8.2 \le 1.5$
$6.7 \le y \le 9.7$
The range of weights, in pounds, is [6.5, 9.5].

90. $|C+84| \le 56$
$-56 \le C+84 \le 56$
$-140 \le C \le -28$
In degrees Celsius, the range of temperature is the interval [–140, –28].

91. 780 is 50 more than 730 and 680 is 50 less than 730, so all of the temperatures in the acceptable range are within 50° of 730°. That is $|F-730| \le 50$.

92. Let x = the speed of the kite. 148 is 25 more than 123, and 98 is 25 less than 123, so all the speeds are within 25 ft per sec of 123 ft per sec, that is, $|x-123| \le 25$.

Let x = speed of the wind. 26 is 5 more than 21, and 16 is 5 less than 21, so the wind speeds are within 5 ft per sec of 21 ft per sec, that is, $|x-21| \le 5$.

93. $|R_L-26.75| \le 1.42$
$-1.42 \le R_L-26.75 \le 1.42$
$25.33 \le R_L \le 28.17$
$|R_E-38.75| \le 2.17$
$-2.17 \le R_E-38.75 \le 2.17$
$36.58 \le R_E \le 40.92$

94. Since there are 225 students and R_L and R_E are individual rates, the total amounts of carbon dioxide emitted would be $T_L = 225R_L$ and $T_E = 225R_E$. Thus,
$(225)(25.33) \le T_L \le (225)(28.17)$
$5699.25 \le T_L \le 6338.25$
$(225)(36.58) \le T_E \le (225)(40.92)$
$8230.5 \le T_E \le 9207$

95. Answers will vary.

96. Answers will vary.
Yes, $|a-b|^2$ is equal to $(a-b)^2$, if a and b are real numbers.

Chapter 1: Review Exercises

1. $2x+8=3x+2 \Rightarrow 8=x+2 \Rightarrow 6=x$
Solution set: $\{6\}$

2. $4x-2(x-1)=12 \Rightarrow 4x-2x+2=12 \Rightarrow$
$2x+2=12 \Rightarrow 2x=10 \Rightarrow x=5$
Solution set: $\{5\}$

3. $5x-2(x+4)=3(2x+1)$
$5x-2x-8=6x+3 \Rightarrow 3x-8=6x+3 \Rightarrow$
$-8=3x+3 \Rightarrow -11=3x \Rightarrow -\frac{11}{3}=x$
Solution set: $\left\{-\frac{11}{3}\right\}$

4. $9x-11(k+p)=x(a-1)$
$9x-11k-11p=ax-x$
$10x-ax=11k+11p$
$(10-a)x=11k+11p$
$x=\frac{11k+11p}{10-a}=\frac{11(k+p)}{10-a}$

5. $A=\frac{24f}{B(p+1)}$ for f (approximate annual interest rate)
$B(p+1)A=B(p+1)\left(\frac{24f}{B(p+1)}\right)$
$AB(p+1)=24f$
$\frac{AB(p+1)}{24}=f$
$f=\frac{AB(p+1)}{24}$

6. B and C cannot be equations to solve a geometry problem. The length of a rectangle must be positive.

A. $2x+2(x+2)=20$
$2x+2x+4=20$
$4x+4=20 \Rightarrow 4x=16 \Rightarrow x=4$

B. $2x+2(5+x)=-2$
$2x+10+2x=-2$
$4x+10=-2 \Rightarrow 4x=-12 \Rightarrow x=-3$

C. $8(x+2)+4x=16$
$8x+16+4x=16$
$12x+16=16 \Rightarrow 12x=0 \Rightarrow x=0$

D. $2x+2(x-3)=10$
$2x+2x-6=10$
$4x-6=10 \Rightarrow 4x=16 \Rightarrow x=4$

7. A and B cannot be equations used to find the number of pennies in a jar. The number of pennies must be a whole number.

A. $5x+3=11 \Rightarrow 5x=8 \Rightarrow x=\frac{8}{5}$

B. $12x+6=-4 \Rightarrow 12x=-10 \Rightarrow$
$x=-\frac{10}{12}=-\frac{5}{6}$

C. $100x=50(x+3)$
$100x=50x+150 \Rightarrow 50x=150 \Rightarrow x=3$

D. $6(x+4)=x+24$
$6x+24=x+24$
$5x+24=24 \Rightarrow 5x=0 \Rightarrow x=0$

8. Let l = the length of the carry-on (in inches).
Let w = the width of the carry-on (in inches).
Let h = the height of the carry-on (in inches).
Linear inches = $l+w+h$.

(a) Linear inches $=l+w+h$
$=9+12+21=42$ in
No; all airlines on the list will allow the Samsonite carry-on bag.

(b) Linear inches $=l+w+h$
$=10+14+22=46$ in
On Southwest and USAirways, the carry-on is allowed.

9. Let x = the original length of the square (in inches). Since the perimeter of a square is 4 times the length of one side, we have $4(x-4)=\frac{1}{2}(4x)+10.$ Solve this equation for x to determine the length of each side of the original square.
$4x-16=2x+10$
$2x-16=10 \Rightarrow 2x=26 \Rightarrow x=13$
The original square is 13 in. on each side.

10. Let x = rate of Becky riding her bike to library. Then $x-8$ = rate of Becky riding her bike home.

To	r	t	d
Library	x	20 min = $\frac{1}{3}$ hr	$\frac{1}{3}x$
Home	$x-8$	30 min = $\frac{1}{2}$ hr	$\frac{1}{2}(x-8)$

Since the distance going to the library is the same as going home, we solve the following.
$\frac{1}{3}x=\frac{1}{2}(x-8) \Rightarrow 6\left[\frac{1}{3}x\right]=6\left[\frac{1}{2}(x-8)\right]$
$2x=3(x-8) \Rightarrow 2x=3x-24$
$-x=-24 \Rightarrow x=24$

To find the distance, substitute $x=24$ into $d=\frac{1}{3}x.$ Since $d=\frac{1}{3}(24)=8$, Becky lives 8 mi from the library.

11. Let x = the amount of 100% alcohol solution (in liters).

Strength	Liters of Solution	Liters of Pure Alcohol
100%	x	$1x=x$
10%	12	$.10\cdot 12=1.2$
30%	$x+12$	$.30(x+12)$

The number of liters of pure alcohol in the 100% solution plus the number of liters of pure alcohol in the 10% solution must equal the number of liters of pure alcohol in the 30% solution.
$x+1.2=.30(x+12) \Rightarrow x+1.2=.30x+3.6$
$.7x+1.2=3.6 \Rightarrow .7x=2.4$
$x=\dfrac{2.4}{.7}=\dfrac{24}{7}=3\frac{3}{7}$ L

$3\frac{3}{7}$ L of the 100% solution should be added.

12. Let x = amount borrowed at 11.5%.
Then $90{,}000-x$ = amount borrowed at 12%.

Amount Borrowed	Interest Rate	Interest
x	11.5%	$.115x$
$90{,}000-x$	12%	$.12(90{,}000-x)$
90,000		10,525

The amount of interest borrowed at 11.5% plus the amount of interest borrowed at 12% note must equal the total amount of interest.
$.115x+.12(90{,}000-x)=10{,}525$
$.115x+10{,}800-.12x=10{,}525$
$-.005x+10{,}800=10{,}525$
$-.005x=-275$
$x=55{,}000$
The amount borrowed at 11.5% is $55,000 and the amount borrowed at 12% is $90,000 – $55,000 = $35,000.

13. Let x = average speed upriver.
Then $x+5$ = average speed on return trip.

	r	t	d
Upriver	x	1.2	$1.2x$
Downriver	$x+5$	.9	$.9(x+5)$

Since the distance upriver and downriver are the same, we solve the following.
$1.2x=.9(x+5)$
$1.2x=.9x+4.5 \Rightarrow .3x=4.5 \Rightarrow x=15$
The average speed of the boat upriver is 15 mph.

14. Let x = number of hours for slower plant (Plant II) to release that amount. Then $\frac{1}{2}x$ = number of hours for faster plant (Plant I) to release that amount. (If the plant is twice as fast, it takes half the time.)

	Rate	Time	Part of the Job Accomplished
Plant II	$\frac{1}{x}$	3	$\frac{1}{x}(3)=\frac{3}{x}$
Plant I	$\frac{1}{\frac{1}{2}x}=\frac{2}{x}$	3	$\frac{2}{x}(3)=\frac{6}{x}$

Since Plant I and Plant II accomplish 1 job (releasing toxic waste) we must solve the following equation.

$\frac{3}{x}+\frac{6}{x}=1 \Rightarrow \frac{9}{x}=1 \Rightarrow x\left(\frac{9}{x}\right)=x\cdot 1 \Rightarrow 9=x$

It takes the slower plant (Plant I) 9 hours to release that same amount.

15. (a) In one year, the maximum amount of lead ingested would be

$$.05\ \frac{\text{mg}}{\text{liter}}\cdot 2\ \frac{\text{liters}}{\text{day}}\cdot 365.25\ \frac{\text{days}}{\text{year}} = 36.525\ \frac{\text{mg}}{\text{year}}.$$

The maximum amount A of lead (in milligrams) ingested in x years would be $A = 36.525x$.

(b) If $x = 72$, then $A = 36.525(72) = 2629.8$ mg. The EPA maximum lead intake from water over a lifetime is 2629.8 mg.

16. In 2009, $x = 3$.

$y = 31.86x + 201.82$

$y = 31.86\cdot 3 + 201.82 = 297.4$

Based on the model, retail e-commerce sales will be approximately $297.4 billion in 2009.

17. (a) Using 1955 for $x = 0$, then for 1985, $x = 30$

$y = .118x + .056$

$y = .118(30) + .056 \approx 3.60$

The minimum wage in 1985 was $3.60 according to the model. This is $0.25 more than the actual value of $3.35.

(b) Let y = $4.25 and then solve for x.

$4.25 = .118x + .056$

$4.194 = .118x$

$35.5 \approx x$

The model predicts the minimum wage to be $4.25 about 3.5 years after 1955, which is mid-1990. This is consistent with the minimum wage changing to $4.25 in 1991.

18. (a) 1955: 166 million (.203) ≈ 33.7 million
1965: 194 million (.197) ≈ 38.2 million
1975: 216 million (.182) ≈ 39.3 million
1985: 238 million (.179) ≈ 42.6 million
1995: 263 million (.167) ≈ 43.9 million

(b) Answers will vary.

19. $(6-i)+(7-2i)=(6+7)+[-1+(-2)]i = 13+(-3)i = 13-3i$

20. $(-11+2i)-(8-7i)=(-11-8)+[2-(-7)]i = -19+9i$

21. $15i-(3+2i)-11=(-3-11)+(15-2)i = -14+13i$

22. $-6+4i-(8i-2)=[-6-(-2)]+(4-8)i = -4+(-4)i=-4-4i$

23. $(5-i)(3+4i)=5(3)+5(4i)-i(3)-i(4i)$
$=15+20i-3i-4i^2$
$=15+17i-4(-1)$
$=15+17i+4=19+17i$

24. $(-8+2i)(-1+i)$
$=-8(-1)-8(i)+2i(-1)+2i(i)$
$=8-8i-2i+2i^2=8-10i+2(-1)$
$=8-10i-2=6-10i$

25. $(5-11i)(5+11i)=5^2-(11i)^2$ Product of the sum and difference of two terms
$=25-121i^2$
$=25-121(-1)$
$=25+121=146$

26. $(4-3i)^2=4^2-2(4)(3i)+(3i)^2$ Square of a binomial
$=16-24i+9i^2$
$=16-24i+9(-1)$
$=16-24i-9=7-24i$

27. $-5i(3-i)^2=-5i[3^2-2(3)(i)+i^2]$
$=-5i[9-6i+(-1)]$
$=-5i(8-6i)=-40i+30i^2$
$=-40i+30(-1)=-40i+(-30)$
$=-30-40i$

28. $4i(2+5i)(2-i) = 4i\left[(2+5i)(2-i)\right]$
$$= 4i\left[\begin{array}{l}2(2)+2(-i)\\ \quad +5i(2)+5i(-i)\end{array}\right]$$
$$= 4i\left[4-2i+10i-5i^2\right]$$
$$= 4i\left[4+8i-5(-1)\right]$$
$$= 4i\left[4+8i+5\right]$$
$$= 4i\left[9+8i\right] = 36i+32i^2$$
$$= 36i+32(-1) = 36i+(-32)$$
$$= -32+36i$$

29. $\dfrac{-12-i}{-2-5i} = \dfrac{(-12-i)(-2+5i)}{(-2-5i)(-2+5i)}$
$$= \frac{24-60i+2i-5i^2}{(-2)^2-(5i)^2} = \frac{24-58i-5(-1)}{4-25i^2}$$
$$= \frac{24-58i+5}{4-25(-1)} = \frac{29-58i}{4+25} = \frac{29-58i}{29}$$
$$= \frac{29}{29}-\frac{58}{29}i = 1-2i$$

30. $\dfrac{-7+i}{-1-i} = \dfrac{(-7+i)(-1+i)}{(-1-i)(-1+i)} = \dfrac{7-7i-i+i^2}{(-1)^2-i^2}$
$$= \frac{7-8i+(-1)}{1-(-1)} = \frac{6-8i}{2} = \frac{6}{2}-\frac{8}{2}i$$
$$= 3-4i$$

31. $i^{11} = i^8 \cdot i^3 = 1\cdot(-i) = -i$

32. $i^{60} = \left(i^4\right)^{15} = 1^{15} = 1$

33. $i^{1001} = i^{1000}\cdot i = \left(i^4\right)^{250}\cdot i = 1^{250}\cdot i = i$

34. $i^{110} = i^{108}\cdot i^2 = \left(i^4\right)^{27}\cdot(-1) = 1^{27}\cdot(-1) = -1$

35. $i^{-27} = i^{-28}\cdot i = \left(i^4\right)^{-7}\cdot i = 1^{-7}\cdot i = i$

36. $\frac{1}{i^{17}} = i^{-17} = i^{-20}\cdot i^3$
$$= \left(i^4\right)^{-5}\cdot i^3 = 1^{-5}\cdot(-i) = -i$$

37. $(x+7)^2 = 5 \Rightarrow x+7 = \pm\sqrt{5} \Rightarrow x = -7\pm\sqrt{5}$

Solution set: $\{-7\pm\sqrt{5}\}$

38. $(2-3x)^2 = 8$

$2-3x = \pm\sqrt{8} \Rightarrow 2-3x = \pm 2\sqrt{2} \Rightarrow$

$2\pm 2\sqrt{2} = 3x \Rightarrow \dfrac{2\pm 2\sqrt{2}}{3} = x$

Solution set: $\left\{\frac{2\pm 2\sqrt{2}}{3}\right\}$

39. $2x^2+x-15 = 0$

$(x+3)(2x-5) = 0$

$x+3 = 0 \Rightarrow x = -3$ or $2x-5 = 0 \Rightarrow x = \frac{5}{2}$

Solution set: $\left\{-3, \frac{5}{2}\right\}$

40. $12x^2 = 8x-1 \Rightarrow 12x^2-8x+1 = 0$

$(6x-1)(2x-1) = 0$

$6x-1 = 0 \Rightarrow x = \frac{1}{6}$ or $2x-1 = 0 \Rightarrow x = \frac{1}{2}$

Solution set: $\left\{\frac{1}{6}, \frac{1}{2}\right\}$

41. $-2x^2+11x = -21 \Rightarrow -2x^2+11x+21 = 0$

$2x^2-11x-21 = 0 \Rightarrow (2x+3)(x-7) = 0$

$2x+3 = 0 \Rightarrow x = -\frac{3}{2}$ or $x-7 = 0 \Rightarrow x = 7$

Solution set: $\left\{-\frac{3}{2}, 7\right\}$

42. $-x(3x+2) = 5 \Rightarrow -3x^2-2x = 5$

$-3x^2-2x-5 = 0 \Rightarrow 3x^2+2x+5 = 0$

Solve by completing the square.

$$3x^2+2x+5 = 0$$
$$x^2+\tfrac{2}{3}x+\tfrac{5}{3} = 0$$
$$x^2+\tfrac{2}{3}x+\tfrac{1}{9} = -\tfrac{5}{3}+\tfrac{1}{9}$$

Note: $\left[\frac{1}{2}\cdot\left(-\frac{2}{3}\right)\right]^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$

$$\left(x+\tfrac{1}{3}\right)^2 = -\tfrac{14}{9}$$
$$x+\tfrac{1}{3} = \pm\sqrt{-\tfrac{14}{9}}$$
$$x+\tfrac{1}{3} = \pm\tfrac{\sqrt{14}}{3}i$$
$$x = -\tfrac{1}{3}\pm\tfrac{\sqrt{14}}{3}i$$

Solve by the quadratic formula.

Let $a = 3$, $b = 2$, and $c = 5$.

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-2\pm\sqrt{2^2-4(3)(5)}}{2(3)}$$
$$= \frac{-2\pm\sqrt{4-60}}{6} = \frac{-2\pm\sqrt{-56}}{6}$$
$$= \frac{-2\pm 2i\sqrt{14}}{6} = -\tfrac{2}{6}\pm\tfrac{2\sqrt{14}}{6}i = -\tfrac{1}{3}\pm\tfrac{\sqrt{14}}{3}i$$

Solution set: $\left\{-\frac{1}{3}\pm\frac{\sqrt{14}}{3}i\right\}$

43. $(2x+1)(x-4)=x \Rightarrow 2x^2-8x+x-4=x \Rightarrow$
$2x^2-7x-4=x \Rightarrow 2x^2-8x-4=0 \Rightarrow$
$x^2-4x-2=0$

Solve by completing the square.

$x^2-4x-2=0$
$x^2-4x+4=2+4$

Note: $\left[\frac{1}{2}\cdot(-4)\right]^2=(-2)^2=2$

$(x-2)^2=6 \Rightarrow x-2=\pm\sqrt{6} \Rightarrow x=2\pm\sqrt{6}$

Solve by the quadratic formula.
Let $a=1$, $b=-4$, and $c=-2$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-2)}}{2(1)}=\frac{4\pm\sqrt{16+8}}{2}=\frac{4\pm\sqrt{24}}{2}=\frac{4\pm2\sqrt{6}}{2}=2\pm\sqrt{6}$$

Solution set: $\left\{2\pm\sqrt{6}\right\}$

44. $\sqrt{2}x^2-4x+\sqrt{2}=0$
Using the quadratic formula would be the most direct approach.
$a=\sqrt{2}, b=-4,$ and $c=\sqrt{2}$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4\cdot\sqrt{2}\cdot\sqrt{2}}}{2\cdot\sqrt{2}}=\frac{4\pm\sqrt{16-8}}{2\sqrt{2}}=\frac{4\pm\sqrt{8}}{2\sqrt{2}}=\frac{4\pm2\sqrt{2}}{2\sqrt{2}}=\frac{2\pm\sqrt{2}}{\sqrt{2}}=\frac{(2\pm\sqrt{2})(\sqrt{2})}{\sqrt{2}(\sqrt{2})}=\frac{2\sqrt{2}\pm2}{2}=\sqrt{2}\pm1$$

Solution set: $\{\sqrt{2}\pm1\}$

45. $x^2-\sqrt{5}x-1=0$
Using the quadratic formula would be the most direct approach.
$a=1, b=-\sqrt{5},$ and $c=-1$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-\sqrt{5})\pm\sqrt{(-\sqrt{5})^2-4\cdot1\cdot(-1)}}{2\cdot1}=\frac{\sqrt{5}\pm\sqrt{5+4}}{2}=\frac{\sqrt{5}\pm\sqrt{9}}{2}=\frac{\sqrt{5}\pm3}{2}$$

Solution set: $\left\{\frac{\sqrt{5}\pm3}{2}\right\}$

46. $(x+4)(x+2)=2x \Rightarrow x^2+2x+4x+8=2x$
$x^2+6x+8=2x \Rightarrow x^2+4x+8=0$

Solve by completing the square.

$x^2+4x+8=0$
$x^2+4x+4=-8+4$

Note: $\left[\frac{1}{2}\cdot(-4)\right]^2=(-2)^2=2$

$(x+2)^2=-4 \Rightarrow x+2=\pm\sqrt{-4} \Rightarrow$
$x+2=\pm2i \Rightarrow x=-2\pm2i$

Solve by the quadratic formula.
Let $a=1$, $b=4$, and $c=8$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{4^2-4(1)(8)}}{2(1)}=\frac{-4\pm\sqrt{16-32}}{2}=\frac{-4\pm\sqrt{-16}}{2}=\frac{-4\pm4i}{2}=-2\pm2i$$

Solution set: $\{-2\pm2i\}$

47. D; $(7x+4)^2=11$
This equation has two real, distinct solutions since the positive number 11 has a positive square root and a negative square root.

48. B and C are the equations that have exactly one real solution because the positive and negative square root of 0 represent the same number.

49. A; $(3x-4)^2=-9$
This equation has two imaginary solutions since the negative number –9 has two imaginary square roots.

50. $8x^2=-2x-6 \Rightarrow 8x^2+2x+6=0$
$a=8$, $b=2$, and $c=6$
$b^2-4ac=2^2-4(8)(6)=4-192=-188$
The equation has two distinct nonreal complex solutions since the discriminant is negative.

51. $-6x^2+2x=-3 \Rightarrow -6x^2+2x+3=0$
$a=-6$, $b=2$, and $c=3$
$b^2-4ac=2^2-4(-6)(3)=4+72=76$
The equation has two distinct irrational solutions since the discriminant is positive but not a perfect square.

52. $16x^2 + 3 = -26x \Rightarrow 16x^2 + 26x + 3 = 0$
$a = 16$, $b = 26$, and $c = 3$
$$b^2 - 4ac = 26^2 - 4(16)(3) = 676 - 192 = 484 = 22^2$$
The equation has two distinct rational solutions since the discriminant is a positive perfect square.

53. $-8x^2 + 10x = 7 \Rightarrow 0 = 8x^2 - 10x + 7$
$a = 8$, $b = -10$, and $c = 7$
$$b^2 - 4ac = (-10)^2 - 4(8)(7) = 100 - 224 = -124$$
The equation has two distinct nonreal complex solutions since the discriminant is negative.

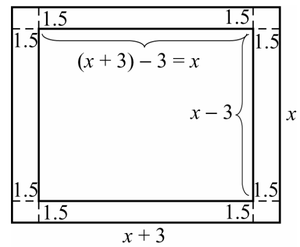
54. $25x^2 + 110x + 121 = 0$
$a = 25$, $b = 110$, and $c = 121$
$$b^2 - 4ac = 110^2 - 4(25)(121) = 12{,}100 - 12{,}100 = 0$$
The equation has one rational solution (a double solution) since the discriminant is equal to zero.

55. $x(9x + 6) = -1 \Rightarrow 9x^2 + 6x = -1 \Rightarrow$
$9x^2 + 6x + 1 = 0$
$a = 9$, $b = 6$, and $c = 1$
$$b^2 - 4ac = 6^2 - 4(9)(1) = 36 - 36 = 0$$
The equation has one rational solution (a double solution) since the discriminant is equal to zero.

56. Answers will vary.

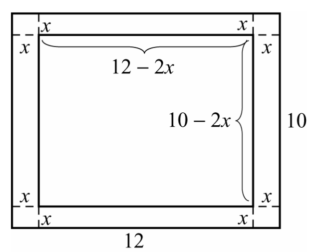
57. The projectile will be 750 ft above the ground whenever $220t - 16t^2 = 750$.
Solve this equation for t.
$$220t - 16t^2 = 750 \Rightarrow 0 = 16t^2 - 220t + 750 \Rightarrow$$
$$0 = 8t^2 - 110t + 375 \Rightarrow 0 = (4t - 25)(2t - 15)$$
$4t - 25 = 0$ or $2t - 15 = 0$
$t = \frac{25}{4} = 6.25$ or $t = \frac{15}{2} = 7.5$
The projectile will be 750 ft high at 6.25 sec and at 7.5 sec.

58. Let x = width of the frame.
Then $x + 3$ = length of the frame.
Set up an equation that represents the area of the unframed picture.
$$x(x - 3) = 70$$
$$x^2 - 3x = 70$$
$$x^2 - 3x - 70 = 0$$
$$(x + 7)(x - 10) = 0$$
$x + 7 = 0 \Rightarrow x = -7$ or $x - 10 = 0 \Rightarrow x = 10$
We disregard the negative solution.



Since x represents the width of the frame, the frame is 10 in wide and $10 + 3 = 13$ in. long.

59. Let x = width of border.
Apply the formula $A = LW$ to both the outside and inside rectangles.
Inside area= Outside area – Border area
$$(12 - 2x)(10 - 2x) = 12 \cdot 10 - 21$$
$$120 - 24x - 20x + 4x^2 = 120 - 21$$
$$120 - 44x + 4x^2 = 99$$
$$4x^2 - 44x + 120 = 99$$
$$4x^2 - 44x + 21 = 0$$
$$(2x - 21)(2x - 1) = 0$$
$2x = 21 \Rightarrow x = \frac{21}{2} = 10\frac{1}{2}$ or
$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$



The border width cannot be $10\frac{1}{2}$ since this exceeds the width of the outside rectangle, so reject this solution. The width of the border is $\frac{1}{2}$ ft.

60. $D = .1s^2 - 3s + 22 \Rightarrow 800 = .1s^2 - 3s + 22$
$$0 = .1s^2 - 3s - 778$$
$$10 \cdot 0 = 10(.1s^2 - 3s - 778)$$
$$0 = s^2 - 30s - 7780$$
Solve by the quadratic formula.
Let $a = 1$, $b = -30$, and $c = -7780$.
$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(-7780)}}{2(1)} = \frac{30 \pm \sqrt{900 + 31{,}120}}{2} = \frac{30 \pm \sqrt{32{,}020}}{2}$$

(*continued on next page*)

(*continued from page 143*)

$s = \frac{30 - \sqrt{32,020}}{2} \approx -74.5$ or

$s = \frac{30 + \sqrt{32,020}}{2} \approx 104.5$

We disregard the negative solution. The appropriate landing speed would be approximately 104.5 ft per sec.

61. In 1980, $x = 10$.

$y = -6.77x^2 + 445.34x + 11,279.82$

$y = -6.77 \cdot 10^2 + 445.34 \cdot 10 + 11,279.82$

$y = -6.77 \cdot 100 + 4453.4 + 11,279.82$

$y = -677 + 4453.4 + 11,279.82 = 15,056.22$

Approximately 15,056 airports

62. Let x = the length of the middle side. Then $x - 7$ = the length of the shorter side and $x + 1$ = the length of the hypotenuse. Use the Pythagorean theorem.

$x^2 + (x-7)^2 = (x+1)^2$

$x^2 + x^2 - 14x + 49 = x^2 + 2x + 1$

$x^2 - 16x + 48 = 0$

$(x-12)(x-4) = 0 \Rightarrow x = 12$ or $x = 4$

If $x = 12$, then $x - 7 = 5$ and $x + 1 = 13$.
If $x = 4$, then $x - 7 = -3$, which is not possible.
The sides are 5 inches, 12 inches, and 13 inches long.

63. $4x^4 + 3x^2 - 1 = 0$

Let $u = x^2$; then $u^2 = x^4$.
With this substitution, the equation becomes $4u^2 + 3u - 1 = 0$.
Solve this equation by factoring.

$(u+1)(4u-1) = 0$

$u + 1 \Rightarrow u = -1$ or $4u - 1 = 0 \Rightarrow u = \frac{1}{4}$

To find x, replace u with x^2.

$x^2 = -1 \Rightarrow x = \pm\sqrt{-1} \Rightarrow x = \pm i$ or

$x^2 = \frac{1}{4} \Rightarrow x = \pm\sqrt{\frac{1}{4}} \Rightarrow x = \pm\frac{1}{2}$

Solution set: $\left\{\pm i, \pm\frac{1}{2}\right\}$

64. $x^2 - 2x^4 = 0 \Rightarrow x^2\left(1 - 2x^2\right) = 0$

$x^2 = 0 \Rightarrow x = \pm\sqrt{0} \Rightarrow x = 0$ or

$1 - 2x^2 = 0 \Rightarrow 1 = 2x^2$

$\frac{1}{2} = x^2 \Rightarrow \pm\sqrt{\frac{1}{2}} = x$

$x = \pm\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \Rightarrow x = \pm\frac{\sqrt{2}}{2}$

Solution set: $\left\{0, \pm\frac{\sqrt{2}}{2}\right\}$

65. $\frac{2}{x} - \frac{4}{3x} = 8 + \frac{3}{x}$

$3x\left(\frac{2}{x} - \frac{4}{3x}\right) = 3x\left(8 + \frac{3}{x}\right)$

$6 - 4 = 24x + 9$

$2 = 24x + 9$

$-7 = 24x$

$-\frac{7}{24} = x$

Solution set: $\left\{-\frac{7}{24}\right\}$

66. $2 - \frac{5}{x} = \frac{3}{x^2}$

$x^2\left(2 - \frac{5}{x}\right) = x^2\left(\frac{3}{x^2}\right)$

$2x^2 - 5x = 3$

$2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0$

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ or $x - 3 = 0 \Rightarrow x = 3$

Solution set: $\left\{-\frac{1}{2}, 3\right\}$

67. $\frac{10}{4x-4} = \frac{1}{1-x} \Rightarrow \frac{10}{4(x-1)} = \frac{1}{1-x} \Rightarrow$

$\frac{10}{4(x-1)} = \frac{(-1)\cdot 1}{(-1)(1-x)} \Rightarrow \frac{10}{4(x-1)} = \frac{-1}{x-1}$

Multiply each term in the equation by the least common denominator, $4(x-1)$, assuming $x \neq 1$.

$4(x-1)\left[\frac{10}{4(x-1)}\right] = 4(x-1)\left(\frac{-1}{x-1}\right)$

$10 = -4 \Rightarrow 14 = 0$

This is a false statement, the solution set is $\varnothing$.
Alternate solution:

$\frac{10}{4x-4} = \frac{1}{1-x}$ or $\frac{10}{4(x-1)} = \frac{1}{1-x}$

Multiply each term in the equation by the least common denominator, $4(x-1)(1-x)$, assuming $x \neq 1$.

$4(x-1)(1-x)\left[\frac{10}{4(x-1)}\right]$

$= 4(x-1)(1-x)\left(\frac{1}{1-x}\right)$

$10(1-x) = 4(x-1) \Rightarrow 10 - 10x = 4x - 4$

$10 = 14x - 4 \Rightarrow 14 = 14x \Rightarrow 1 = x$

Because of the restriction $x \neq 1$, the solution set is $\varnothing$.

68. $\frac{13}{x^2+10}=\frac{2}{x}$

Multiply both sides by the least common denominator, $x\left(x^2+10\right)$, assuming $x \neq 0$.

$$x(x^2+10)\left(\frac{13}{x^2+10}\right)=x\left(x^2+10\right)\left(\frac{2}{x}\right)$$
$$13x=2\left(x^2+10\right)$$
$$13x=2x^2+20$$
$$0=2x^2-13x+20$$
$$0=(2x-5)(x-4)$$

$2x-5=0 \Rightarrow x=\frac{5}{2}$ or $x-4=0 \Rightarrow x=4$

The restriction $x \neq 0$ does not affect the result. Therefore, the solution set is $\left\{\frac{5}{2},4\right\}$.

69. $\frac{x}{x+2}+\frac{1}{x}+3=\frac{2}{x^2+2x} \Rightarrow$

$$\frac{x}{x+2}+\frac{1}{x}+3=\frac{2}{x(x+2)}$$

Multiply each term in the equation by the least common denominator, $x(x+2)$, assuming $x \neq 0,-2$.

$$x(x+2)\left[\frac{x}{x+2}+\frac{1}{x}+3\right]=x(x+2)\left(\frac{2}{x(x+2)}\right)$$
$$x^2+(x+2)+3x(x+2)=2$$
$$x^2+x+2+3x^2+6x=2$$
$$4x^2+7x+2=2$$
$$4x^2+7x=0 \Rightarrow x(4x+7)=0$$

$x=0$ or $4x+7=0 \Rightarrow x=-\frac{7}{4}$

Because of the restriction $x \neq 0$, the only valid solution is $-\frac{7}{4}$. The solution set is $\left\{-\frac{7}{4}\right\}$.

70. $\frac{2}{x+2}+\frac{1}{x+4}=\frac{4}{x^2+6x+8} \Rightarrow$

$$\frac{2}{x+2}+\frac{1}{x+4}=\frac{4}{(x+4)(x+2)}$$

The least common denominator is $(x+4)(x+2)$, which is equal to 0 if $x=-4$ or $x=-2$. Therefore, -4 and -2 cannot possibly be solutions of this equation.

$$(x+4)(x+2)\left[\frac{2}{x+2}+\frac{1}{x+4}\right]$$
$$=(x+4)(x+2)\left(\frac{4}{(x+4)(x+2)}\right)$$
$$2(x+4)+(x+2)=4$$
$$2x+8+x+2=4$$
$$3x+10=4 \Rightarrow 3x=-6 \Rightarrow x=-2$$

The only possible solution is -2. However, the variable is restricted to real numbers except -4 and -2. Therefore, the solution set is: $\varnothing$.

71. $(2x+3)^{2/3}+(2x+3)^{1/3}-6=0$

Let $u=(2x+3)^{1/3}$. Then

$u^2=[(2x+3)^{1/3}]^2=(2x+3)^{2/3}$.

With this substitution, the equation becomes $u^2+u-6=0$. Solve by factoring.

$(u+3)(u-2)=0$

$u+3=0 \Rightarrow u=-3$ or $u-2=0 \Rightarrow u=2$

To find *x*, replace *u* with $(2x+3)^{1/3}$.

$(2x+3)^{1/3}=-3 \Rightarrow \left[(2x+3)^{1/3}\right]^3=(-3)^3 \Rightarrow$

$2x+3=-27 \Rightarrow 2x=-30 \Rightarrow x=-15$

or

$(2x+3)^{1/3}=2 \Rightarrow \left[(2x+3)^{1/3}\right]^3=2^3 \Rightarrow$

$2x+3=8 \Rightarrow 2x=5 \Rightarrow x=\frac{5}{2}$

Check $x=-15$.

$$(2x+3)^{2/3}+(2x+3)^{1/3}=6$$
$$[2(-15)+3]^{2/3}+[2(-15)+3]^{1/3}=6$$
$$(-30+3)^{2/3}+(-30+3)^{1/3}=6$$
$$(-27)^{2/3}+(-27)^{1/3}=6$$
$$\left[(-27)^{1/3}\right]^2+(-3)=6$$
$$(-3)^2-3=6$$
$$9-3=6 \Rightarrow 6=6$$

This is a true statement. -15 is a solution.

Check $x=\frac{5}{2}$.

$$(2x+3)^{2/3}+(2x+3)^{1/3}=6$$
$$\left[2\left(\tfrac{5}{2}\right)+3\right]^{2/3}+\left[2\left(\tfrac{5}{2}\right)+3\right]^{1/3}=6$$
$$(5+3)^{2/3}+(5+3)^{1/3}=6$$
$$8^{2/3}+8^{1/3}=6$$
$$\left[8^{1/3}\right]^2+2=6$$
$$(2)^2+2=6$$
$$4+2=6 \Rightarrow 6=6$$

This is a true statement. $\frac{5}{2}$ is a solution.

Solution set: $\left\{-15,\frac{5}{2}\right\}$

72. $(x+3)^{-2/3} - 2(x+3)^{-1/3} = 3 \Rightarrow$
$(x+3)^{-2/3} - 2(x+3)^{-1/3} - 3 = 0$

Let $u = (x+3)^{-1/3}$; then

$u^2 = \left[(x+3)^{-1/3}\right]^2 = (x+3)^{-2/3}.$

$u^2 - 2u - 3 = 0 \Rightarrow (u+1)(u-3) = 0$
$u = -1$ or $u = 3$

To find x, replace u with $(x+3)^{-1/3}$.

$(x+3)^{-1/3} = -1 \Rightarrow \left[(x+3)^{-1/3}\right]^{-3} = (-1)^{-3} \Rightarrow$
$x+3 = \frac{1}{(-1)^3} \Rightarrow x+3 = \frac{1}{-1} = -1 \Rightarrow x = -4$

or

$(x+3)^{-1/3} = 3 \Rightarrow \left[(x+3)^{-1/3}\right]^{-3} = 3^{-3} \Rightarrow$
$x+3 = \frac{1}{3^3} \Rightarrow x+3 = \frac{1}{27} \Rightarrow$
$x = -3 + \frac{1}{27} = -\frac{81}{27} + \frac{1}{27} = -\frac{80}{27}$

Check $x = -4$.

$$(x+3)^{-2/3} - 2(x+3)^{-1/3} - 3 = 0$$
$$(-4+3)^{-2/3} - 2(-4+3)^{-1/3} - 3 = 0$$
$$(-1)^{-2/3} - 2(-1)^{-1/3} - 3 = 0$$
$$(-1)^{2/3} - 2(-1)^{1/3} - 3 = 0$$
$$\left[(-1)^{1/3}\right]^2 - 2(-1)^{1/3} - 3 = 0$$
$$(-1)^2 - 2(-1) - 3 = 0$$
$$1 + 2 - 3 = 0$$
$$0 = 0$$

This is a true statement. -4 is a solution.

Check $x = -\frac{80}{27}$.

$$(x+3)^{-2/3} - 2(x+3)^{-1/3} - 3 = 0$$
$$\left(-\tfrac{80}{27}+3\right)^{-2/3} - 2\left(-\tfrac{80}{27}+3\right)^{-1/3} - 3 = 0\ ?$$
$$\left(-\tfrac{80}{27}+\tfrac{81}{27}\right)^{-2/3} - 2\left(-\tfrac{80}{27}+\tfrac{81}{27}\right)^{-1/3} - 3 = 0$$
$$\left(\tfrac{1}{27}\right)^{-2/3} - 2\left(\tfrac{1}{27}\right)^{-1/3} - 3 = 0$$
$$(27)^{2/3} - 2(27)^{1/3} - 3 = 0$$
$$\left[27^{1/3}\right]^2 - 2(27)^{1/3} - 3 = 0$$
$$3^2 - 2(3) - 3 = 0$$
$$9 - 6 - 3 = 0$$
$$0 = 0$$

This is a true statement. $-\frac{80}{27}$ is a solution.

Solution set: $\left\{-4, -\frac{80}{27}\right\}$

73. $\sqrt{4x-2} = \sqrt{3x+1}$

$$\left(\sqrt{4x-2}\right)^2 = \left(\sqrt{3x+1}\right)^2$$
$$4x - 2 = 3x + 1$$
$$x - 2 = 1$$
$$x = 3$$

Check $x = 3$.

$$\sqrt{4x-2} = \sqrt{3x+1}$$
$$\sqrt{4(3)-2} = \sqrt{3(3)+1}$$
$$\sqrt{12-2} = \sqrt{9+1}$$
$$\sqrt{10} = \sqrt{10}$$

This is a true statement.
Solution set: $\{3\}$

74. $\sqrt{2x+3} = x+2$

$$\left(\sqrt{2x+3}\right)^2 = (x+2)^2$$
$$2x+3 = x^2+4x+4$$
$$0 = x^2+2x+1$$
$$0 = (x+1)^2$$
$$x+1 = 0 \Rightarrow x = -1$$

Check $x = -1$.

$$\sqrt{2x+3} = x+2$$
$$\sqrt{2(-1)+3} = -1+2$$
$$\sqrt{-2+3} = 1$$
$$\sqrt{1} = 1$$
$$1 = 1$$

This is a true statement.
Solution set: $\{-1\}$

75. $\sqrt{x+2} - x = 2 \Rightarrow \sqrt{x+2} = 2 + x$

$$\left(\sqrt{x+2}\right)^2 = (2+x)^2 \Rightarrow x+2 = 4+4x+x^2$$
$$0 = x^2+3x+2 \Rightarrow 0 = (x+2)(x+1)$$
$$x+2 = 0 \Rightarrow x = -2 \quad \text{or} \quad x+1 = 0 \Rightarrow x = -1$$

Check $x = -2$.

$$\sqrt{x+2} = 2+x$$
$$\sqrt{-2+2} = 2+(-2)$$
$$\sqrt{0} = 0 \Rightarrow 0 = 0$$

This is a true statement. -2 is a solution.

Check $x = -1$.

$$\sqrt{x+2} = 2+x$$
$$\sqrt{-1+2} = 2+(-1)\ ?$$
$$\sqrt{1} = 1 \Rightarrow 1 = 1$$

This is a true statement. -1 is a solution.
Solution set: $\{-2, -1\}$

76. $\sqrt{x}-\sqrt{x+3}=-1 \Rightarrow \sqrt{x}=\sqrt{x+3}-1$

$$\left(\sqrt{x}\right)^2=\left(\sqrt{x+3}-1\right)^2$$
$$x=(x+3)-2\sqrt{x+3}+1$$
$$x=x+4-2\sqrt{x+3}$$
$$0=4-2\sqrt{x+3}$$
$$2\sqrt{x+3}=4 \Rightarrow \sqrt{x+3}=2$$
$$\left(\sqrt{x+3}\right)=2^2 \Rightarrow x+3=4 \Rightarrow x=1$$

Check $x=1$.

$$\sqrt{x}-\sqrt{x+3}=-1$$
$$\sqrt{1}-\sqrt{1+3}=-1\ ?$$
$$1-\sqrt{4}=-1$$
$$1-2=-1$$
$$-1=-1$$

This is a true statement.
Solution set: $\{1\}$

77. $\sqrt{x+3}-\sqrt{3x+10}=1$

$$\sqrt{x+3}=1+\sqrt{3x+10}$$
$$\left(\sqrt{x+3}\right)^2=\left(1+\sqrt{3x+10}\right)^2$$
$$x+3=1+2\sqrt{3x+10}+(3x+10)$$
$$x+3=3x+11+2\sqrt{3x+10}$$
$$-2x-8=2\sqrt{3x+10}$$
$$x+4=-\sqrt{3x+10}$$
$$(x+4)^2=\left(-\sqrt{3x+10}\right)^2$$
$$x^2+8x+16=3x+10$$
$$x^2+5x+6=0 \Rightarrow (x+2)(x+3)=0$$
$$x+3=0 \Rightarrow x=-3 \quad \text{or} \quad x+2=0 \Rightarrow x=-2$$

Check $x=-3$.

$$\sqrt{x+3}-\sqrt{3x+10}=1$$
$$\sqrt{-3+3}-\sqrt{3(-3)+10}=1$$
$$\sqrt{0}-\sqrt{-9+10}=1$$
$$0-\sqrt{1}=1$$
$$0-1=1 \Rightarrow -1=1$$

This is a false statement. -3 is not a solution.
Check $x=-2$.

$$\sqrt{x+3}-\sqrt{3x+10}=1$$
$$\sqrt{-2+3}-\sqrt{3(-2)+10}=1$$
$$\sqrt{1}-\sqrt{-6+10}=1$$
$$1-\sqrt{4}=1$$
$$1-2=1 \Rightarrow -1=1$$

This is a false statement. -2 is not a solution.
Since neither of the proposed solutions satisfies the original equation, the equation has no solution.
Solution set: $\varnothing$

78. $\sqrt{5x-15}-\sqrt{x+1}=2$

$$\sqrt{5x-15}=\sqrt{x+1}+2$$
$$\left(\sqrt{5x-15}\right)^2=\left(\sqrt{x+1}+2\right)^2$$
$$5x-15=(x+1)+4\sqrt{x+1}+4$$
$$5x-15=x+5+4\sqrt{x+1}$$
$$4x-20=4\sqrt{x+1}$$
$$x-5=\sqrt{x+1}$$
$$(x-5)^2=\left(\sqrt{x+1}\right)^2$$
$$x^2-10x+25=x+1$$
$$x^2-11x+24=0 \Rightarrow (x-3)(x-8)=0$$
$$x-3=0 \Rightarrow x=3 \quad \text{or} \quad x-8=0 \Rightarrow x=8$$

Check $x=3$.

$$\sqrt{5x-15}-\sqrt{x+1}=2$$
$$\sqrt{5(3)-15}-\sqrt{3+1}=2$$
$$\sqrt{15-15}-\sqrt{4}=2$$
$$\sqrt{0}-2=2$$
$$0-2=2 \Rightarrow -2=2$$

This is a false statement. 3 is not a solution.
Check $x=8$.

$$\sqrt{5x-15}-\sqrt{x+1}=2$$
$$\sqrt{5(8)-15}-\sqrt{8+1}=2$$
$$\sqrt{40-15}-\sqrt{9}=2$$
$$\sqrt{25}-3=2$$
$$5-3=2 \Rightarrow 2=2$$

This is a true statement. 8 is a solution.
Solution set: $\{8\}$

79. $\sqrt{x^2+3x}-2=0$

$$\sqrt{x^2+3x}=2 \Rightarrow \left(\sqrt{x^2+3x}\right)^2=2^2$$
$$x^2+3x=4 \Rightarrow x^2+3x-4=0$$
$$(x-1)(x+4)=0$$
$$x-1=0 \Rightarrow x=1 \quad \text{or} \quad x+4=0 \Rightarrow x=-4$$

Check $x=-4$.

$$\sqrt{x^2+3x}-2=0$$
$$\sqrt{(-4)^2+3(-4)}-2=0$$
$$\sqrt{16+(-12)}-2=0$$
$$\sqrt{4}-2=0 \Rightarrow 2-2=0 \Rightarrow 0=0$$

This is a true statement. -4 is a solution.

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(*continued from page 147*)

Check $x = 1$.

$\sqrt{x^2 + 3x} - 2 = 0$

$\sqrt{1^2 + 3(1)} - 2 = 0$

$\sqrt{1+3} - 2 = 0$

$\sqrt{4} - 2 = 0 \Rightarrow 2 - 2 = 0 \Rightarrow 0 = 0$

This is a true statement. 1 is a solution.

Solution set: $\{-4, 1\}$

80. $\sqrt[5]{2x} = \sqrt[5]{3x+2}$

$\left(\sqrt[5]{2x}\right)^5 = \left(\sqrt[5]{3x+2}\right)^5$

$2x = 3x + 2 \Rightarrow -x = 2 \Rightarrow x = -2$

Check $x = -2$.

$\sqrt[5]{2x} = \sqrt[5]{3x+2}$

$\sqrt[5]{2(-2)} = \sqrt[5]{3(-2)+2}$

$\sqrt[5]{-4} = \sqrt[5]{-6+2}$

$-\sqrt[5]{4} = \sqrt[5]{-4}$

$-\sqrt[5]{4} = -\sqrt[5]{4}$

This is a true statement.

Solution set: $\{-2\}$

81. $\sqrt[3]{6x+2} - \sqrt[3]{4x} = 0$

$\sqrt[3]{6x+2} = \sqrt[3]{4x}$

$\left(\sqrt[3]{6x+2}\right)^3 = \left(\sqrt[3]{4x}\right)^3$

$6x + 2 = 4x \Rightarrow 2 = -2x \Rightarrow -1 = x$

Check $x = -1$.

$\sqrt[3]{6x+2} - \sqrt[3]{4x} = 0$

$\sqrt[3]{6(-1)+2} - \sqrt[3]{4(-1)} = 0$

$\sqrt[3]{-6+2} - \sqrt[3]{-4} = 0$

$\sqrt[3]{-4} - \left(-\sqrt[3]{4}\right) = 0$

$-\sqrt[3]{4} + \sqrt[3]{4} = 0 \Rightarrow 0 = 0$

This is a true statement.

Solution set: $\{-1\}$

82. $(x-2)^{2/3} = x^{1/3}$

$\left[(x-2)^{2/3}\right]^3 = \left(x^{1/3}\right)^3$

$(x-2)^2 = x$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

$(x-4)(x+1) = 0$

$x - 4 = 0 \Rightarrow x = 4$ or $x - 1 = 0 \Rightarrow x = 1$

Check $x = 1$.

$(x-2)^{2/3} = x^{1/3} \Rightarrow$

$(1-2)^{2/3} = 1^{1/3} \Rightarrow (-1)^{2/3} = 1$

$\left[(-1)^{1/3}\right]^2 = 1 \Rightarrow (-1)^2 = 1 \Rightarrow 1 = 1$

This is a true statement. 1 is a solution.

Check $x = 4$.

$(x-2)^{2/3} = x^{1/3} \Rightarrow$

$(4-2)^{2/3} = 4^{1/3} \Rightarrow (2)^{2/3} = 4^{1/3}$

$\left[2^2\right]^{1/3} = 4^{1/3} \Rightarrow 4^{1/3} = 4^{1/3}$

This is a true statement. 4 is a solution.

Solution set: $\{1, 4\}$

83. $-9x + 3 < 4x + 10$

$-13x < 7$

$x > -\dfrac{7}{13}$

Solution set: $\left(-\frac{7}{13}, \infty\right)$

84. $11x \ge 2(x-4)$

$11x \ge 2x - 8$

$9x \ge -8$

$x \ge -\dfrac{8}{9}$

Solution set: $\left[-\frac{8}{9}, \infty\right)$

85. $-5x - 4 \ge 3(2x-5)$

$-5x - 4 \ge 6x - 15$

$-11x - 4 \ge -15$

$-11x \ge -11$

$x \le 1$

Solution set: $(-\infty, 1]$

86. $7x - 2(x-3) \le 5(2-x)$

$7x - 2x + 6 \le 10 - 5x$

$5x + 6 \le 10 - 5x$

$10x + 6 \le 10$

$10x \le 4$

$x \le \frac{4}{10}$

$x \le \frac{2}{5}$

Solution set: $\left(-\infty, \frac{2}{5}\right]$

87. $5 \le 2x - 3 \le 7$

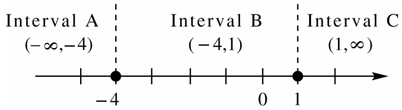
$8 \le 2x \le 10$

$4 \le x \le 5$

Solution set: $[4, 5]$

88. $-8 > 3x - 5 > -12$
$-3 > 3x > -7$
$-1 > x > -\frac{7}{3}$
$-\frac{7}{3} < x < -1$
Solution set: $\left(-\frac{7}{3}, -1\right)$

89. $x^2 + 3x - 4 \le 0$
Step 1: Find the values of x that satisfy $x^2 + 3x - 4 = 0$.
$x^2 + 3x - 4 = 0$
$(x+4)(x-1) = 0$
$x + 4 = 0 \Rightarrow x = -4$ or $x - 1 = 0 \Rightarrow x = 1$
Step 2: The two numbers divide a number line into three regions.

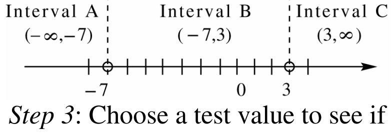


Step 3: Choose a test value to see if it satisfies the inequality, $x^2 + 3x - 4 \le 0$.

Interval	Test Value	Is $x^2 + 3x - 4 \le 0$ True or False?
A: $(-\infty, -4)$	-5	$(-5)^2 + 3(-5) - 4 \overset{?}{\le} 0$ $6 \le 0$ False
B: $(-4, 1)$	0	$0^2 + 3(0) - 4 \overset{?}{\le} 0$ $-4 \le 0$ True
C: $(1, \infty)$	2	$2^2 + 3(2) - 4 \overset{?}{\le} 0$ $6 \le 0$ False

Solution set: $[-4, 1]$

90. $x^2 + 4x - 21 > 0$
Step 1: Find the values of x that satisfy $x^2 + 4x - 21 = 0$.
$x^2 + 4x - 21 = 0$
$(x+7)(x-3) = 0$
$x + 7 = 0 \Rightarrow x = -7$ or $x - 3 = 0 \Rightarrow x = 3$
Step 2: The two numbers divide a number line into three regions.

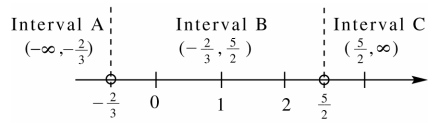


Step 3: Choose a test value to see if it satisfies the inequality, $x^2 + 4x - 21 > 0$

Interval	Test Value	Is $x^2 + 4x - 21 > 0$ True or False?
A: $(-\infty, -7)$	-8	$(-8)^2 + 4(-8) - 21 \overset{?}{>} 0$ $11 > 0$ True
B: $(-7, 3)$	0	$0^2 + 4(0) - 21 \overset{?}{>} 0$ $-21 > 0$ False
C: $(3, \infty)$	4	$4^2 + 4(4) - 21 \overset{?}{>} 0$ $11 > 0$ True

Solution set: $(-\infty, -7) \cup (3, \infty)$

91. $6x^2 - 11x < 10$
Step 1: Find the values of x that satisfy $6x^2 - 11x = 10$.
$6x^2 - 11x = 10$
$6x^2 - 11x - 10 = 0$
$(3x+2)(2x-5) = 0$
$3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$ or $2x - 5 = 0 \Rightarrow x = \frac{5}{2}$
Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $6x^2 - 11x < 10$

Interval	Test Value	Is $6x^2 - 11x < 10$ True or False?
A: $\left(-\infty, -\frac{2}{3}\right)$	-1	$6(-1)^2 - 11(-1) \overset{?}{<} 10$ $17 < 0$ False
B: $\left(-\frac{2}{3}, \frac{5}{2}\right)$	0	$6 \cdot 0^2 - 11 \cdot 0 - 10 \overset{?}{<} 10$ $-10 < 10$ True
C: $\left(\frac{5}{2}, \infty\right)$	3	$6 \cdot 3^2 - 11 \cdot 3 - 10 \overset{?}{<} 10$ $11 < 10$

Solution set: $\left(-\frac{2}{3}, \frac{5}{2}\right)$

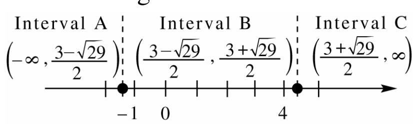
92. $x^2 - 3x \geq 5$

Step 1: Find the values of x that satisfy $x^2 - 3x = 5 \Rightarrow x^2 - 3x - 5 = 0$ Use the quadratic formula Let $a = 1,\ b = -3,$ and $c = -5.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)} = \frac{3 \pm \sqrt{9+20}}{2} = \frac{3 \pm \sqrt{29}}{2}$$

$$x = \frac{3 - \sqrt{29}}{2} \approx -1.2 \text{ or } x = \frac{3 + \sqrt{29}}{2} \approx 4.2$$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $x^2 - 3x \geq 5$

Interval	Test Value	Is $x^2 - 3x \geq 5$ True or False?
A: $\left(-\infty, \frac{3-\sqrt{29}}{2}\right)$	−2	$(-2)^2 - 3(-2) \overset{?}{\geq} 5$ $10 \geq 5$ True
B: $\left(\frac{3-\sqrt{29}}{2}, \frac{3+\sqrt{29}}{2}\right)$	0	$0^2 - 3 \cdot 0 - 5 \overset{?}{\geq} 5$ $-5 \geq 5$ False
C: $\left(\frac{3+\sqrt{29}}{2}, \infty\right)$	5	$5^2 - 3 \cdot 5 \overset{?}{\geq} 5$ $10 \geq 5$ True

Solution set: $\left(-\infty, \frac{3-\sqrt{29}}{2}\right] \cup \left[\frac{3+\sqrt{29}}{2}, \infty\right)$

93. $x^3 - 16x \leq 0$

Step 1: Solve $x^3 - 16x = 0.$

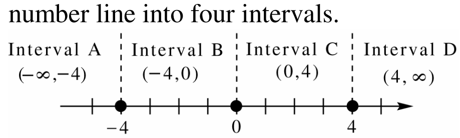
$x^3 - 16x = 0 \Rightarrow x\left(x^2 - 16\right) = 0 \Rightarrow$

$x(x+4)(x-4) = 0$

Set each factor to zero and solve.

$x = 0$ or $x + 4 = 0 \Rightarrow x = -4$ or $x - 4 = 0 \Rightarrow x = 4$

Step 2: The values -4, 0, and, 4 divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $x^3 - 16x \leq 0.$

Interval	Test Value	Is $x^3 - 16x \leq 0$ True or False?
A: $(-\infty, -4)$	−5	$(-5)^3 - 16(-5) \overset{?}{\leq} 0$ $-45 \leq 0$ True
B: $(-4, 0)$	−1	$(-1)^3 - 16(-1) \overset{?}{\leq} 0$ $15 \leq 0$ False
C: $(0, 4)$	1	$1^3 - 16 \cdot 1 \overset{?}{\leq} 0$ $-15 \leq 0$ True
D: $(4, \infty)$	5	$5^3 - 16 \cdot 5 \overset{?}{\leq} 0$ $45 \leq 0$ False

Solution set: $(-\infty, -4] \cup [0, 4]$

94. $2x^3 - 3x^2 - 5x < 0$

Step 1: Solve $2x^3 - 3x^2 - 5x = 0.$

$2x^3 - 3x^2 - 5x = 0 \Rightarrow x\left(2x^2 - 3x - 5\right) = 0 \Rightarrow$

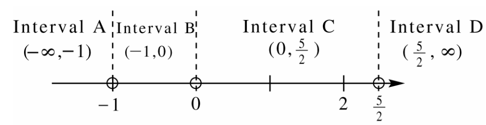
$x(x+1)(2x-5) = 0$

Set each factor to zero and solve.

$x = 0$ or $x + 1 = 0$ or $2x - 5 = 0$

$x = 0$ or $x = -1$ or $x = \frac{5}{2}$

Step 2: The values -1, 0, and, $\frac{5}{2}$ divide the number line into four intervals.



Step 3: Choose a test value to see if it satisfies the inequality, $2x^3 - 3x^2 - 5x < 0.$

Interval	Test Value	Is $2x^3 - 3x^2 - 5x < 0$ True or False?
A: $(-\infty, -1)$	−2	$2(-2)^3 - 3(-2)^2 - 5(-2) \overset{?}{<} 0$ $-18 < 0$ True
B: $(-1, 0)$	−.5	$2(-.5)^3 - 3(-.5)^2 - 5(-.5) \overset{?}{<} 0$ $1.5 < 0$ False

Interval	Test Value	Is $2x^3-3x^2-5x<0$ True or False?
C: $\left(0,\frac{5}{2}\right)$	1	$2\cdot1^3-3\cdot1^2-5\cdot1\overset{?}{<}0$ $-6<0$ True
D: $\left(\frac{5}{2},\infty\right)$	3	$2\cdot3^3-3\cdot3^2-5\cdot3\overset{?}{<}0$ $12<0$ False

Solution set: $(-\infty,-1)\cup\left(0,\frac{5}{2}\right)$

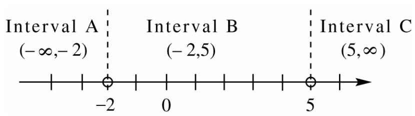
95. $\dfrac{3x+6}{x-5}>0$

Since one side of the inequality is already 0, we start with Step 2.

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$3x+6=0\Rightarrow x=-2 \quad \text{or} \quad x-5=0\Rightarrow x=5$

The values –2 and 5 to divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{3x+6}{x-5}>0$.

Interval	Test Value	Is $\frac{3x+6}{x-5}>0$ True or False?
A: $(-\infty,-2)$	−3	$\frac{3(-3)+6}{-3-5}\overset{?}{>}0$ $\frac{3}{8}>0$ True
B: $(-2,5)$	0	$\frac{3(0)+6}{0-5}\overset{?}{>}0$ $-\frac{6}{5}>0$ False
C: $(5,\infty)$	6	$\frac{3(6)+6}{6-5}\overset{?}{>}0$ $24>0$ True

Solution set: $(-\infty,-2)\cup(5,\infty)$

96. $\dfrac{x+7}{2x+1}-1\le0$

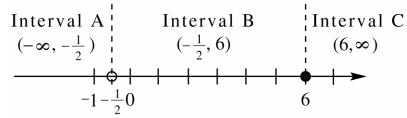
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{x+7}{2x+1}-1\le0\Rightarrow\frac{x+7}{2x+1}-\frac{2x+1}{2x+1}\le0\Rightarrow$$
$$\frac{x+7-(2x+1)}{2x+1}\le0\Rightarrow\frac{x+7-2x-1}{2x+1}\le0\Rightarrow$$
$$\frac{6-x}{2x+1}\le0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$6-x=0\Rightarrow x=6 \quad \text{or} \quad 2x+1=0\Rightarrow x=-\frac{1}{2}$

The values $-\frac{1}{2}$ and 6 divide the number line into three regions. Use an open circle on $-\frac{1}{2}$ because it makes the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{x+7}{2x+1}-1\le0$.

Interval	Test Value	Is $\frac{x+7}{2x+1}-1\le0$ True or False?
A: $\left(-\infty,-\frac{1}{2}\right)$	−1	$\frac{-1+7}{2(-1)+1}-1\overset{?}{\le}0$ $-7\le1$ True
B: $\left(-\frac{1}{2},6\right)$	0	$\frac{0+7}{2\cdot0+1}-1\overset{?}{\le}0$ $6\le1$ False
C: $(6,\infty)$	7	$\frac{7+7}{2\cdot7+1}-1\overset{?}{\le}0$ $-\frac{1}{15}\le0$ True

Intervals A and C satisfy the inequality. The endpoint $-\frac{1}{2}$ is not included because it makes the denominator 0.

Solution set: $\left(-\infty,-\frac{1}{2}\right)\cup[6,\infty)$

97. $\dfrac{3x-2}{x}-4>0$

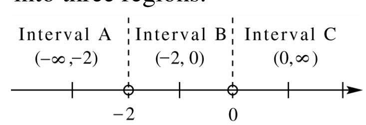
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{3x-2}{x}-4>0\Rightarrow\frac{3x-2}{x}-\frac{4x}{x}>0\Rightarrow$$
$$\frac{3x-2-4x}{x}>0\Rightarrow\frac{-x-2}{x}>0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-x-2=0\Rightarrow x=-2 \quad \text{or} \quad x=0$

The values -2 and 0 divide the number line into three regions.



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Step 3: Choose a test value to see if it satisfies the inequality, $\frac{3x-2}{x}-4>0$.

Interval	Test Value	Is $\frac{3x-2}{x}-4>0$ True or False?
A: $(-\infty,-2)$	-3	$\frac{3(-3)-2}{-3}-4\overset{?}{>}0$ $-\frac{1}{3}>4$ False
B: $(-2,0)$	-1	$\frac{3(-1)-2}{-1}-4\overset{?}{>}0$ $1>4$ True
C: $(0,\infty)$	1	$\frac{3\cdot1-2}{1}-4\overset{?}{>}0$ $-3>0$ False

Solution set: (–2, 0)

98. $\frac{5x+2}{x}<-1$

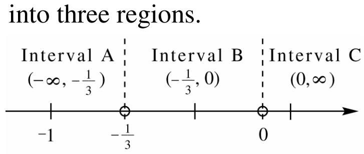
Step 1: Rewrite the inequality to compare a single fraction with 0.

$$\frac{5x+2}{x}+1<0\Rightarrow\frac{5x+2}{x}+\frac{x}{x}<0\Rightarrow$$
$$\frac{5x+2+x}{x}<0\Rightarrow\frac{6x+2}{x}<0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$6x+2=0\Rightarrow x=-\frac{1}{3}$ or $x=0$

The values $-\frac{1}{3}$ and 0 divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{5x+2}{x}<-1$.

Interval	Test Value	Is $\frac{5x+2}{x}<-1$ True or False?
A: $\left(-\infty,-\frac{1}{3}\right)$	-1	$\frac{5(-1)+2}{-1}\overset{?}{<}-1$ $3<-1$ False
B: $\left(-\frac{1}{3},0\right)$	$-.1$	$\frac{5(-.1)+2}{-.1}\overset{?}{<}-1$ $-15<-1$ True
C: $(0,\infty)$	1	$\frac{5\cdot1+2}{1}\overset{?}{<}-1$ $7<-1$ False

Solution set: $\left(-\frac{1}{3},0\right)$

99. $\frac{3}{x-1}\le\frac{5}{x+3}$

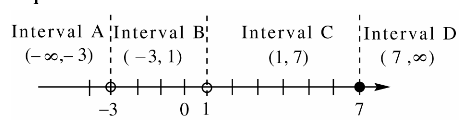
Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{3}{x-1}-\frac{5}{x+3}\le0$$
$$\frac{3(x+3)}{(x-1)(x+3)}-\frac{5(x-1)}{(x+3)(x-1)}\le0$$
$$\frac{3(x+3)-5(x-1)}{(x-1)(x+3)}\le0$$
$$\frac{3x+9-5x+5}{(x-1)(x+3)}\le0$$
$$\frac{-2x+14}{(x-1)(x+3)}\le0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-2x+14=0\Rightarrow x=7$ or $x-1\Rightarrow x=1$ or
$x+3=0\Rightarrow x=-3$

The values –3, 1 and 7 divide the number line into four regions. Use an open circle on -3 and 1 because they make the denominator equal 0.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{3}{x-1}\le\frac{5}{x+3}$.

Interval	Test Value	Is $\frac{3}{x-1}\le\frac{5}{x+3}$ True or False?
A: $(-\infty,-3)$	-4	$\frac{3}{-4-1}\overset{?}{\le}\frac{5}{-4+3}$ $-\frac{3}{5}\le-5$ False
B: $(-3,1)$	0	$\frac{3}{0-1}\overset{?}{\le}\frac{5}{0+3}$ $-3\le\frac{5}{3}$ True
C: $(1,7)$	2	$\frac{3}{2-1}\overset{?}{\le}\frac{5}{2+3}$ $3\le1$ False
D: $(7,\infty)$	8	$\frac{3}{8-1}\overset{?}{\le}\frac{5}{8+3}$ $\frac{3}{7}\overset{?}{\le}\frac{5}{11}$ $\frac{33}{77}\le\frac{35}{77}$ True

Intervals B and D satisfy the inequality. The endpoints -3 and 1 are not included because they make the denominator 0.

Solution set: $(-3,1)\cup[7,\infty)$

100. $\dfrac{3}{x+2} > \dfrac{2}{x-4}$

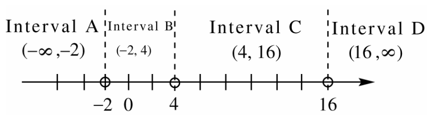
Step 1: Rewrite the inequality to compare a single fraction with 0.

$$\frac{3}{x+2} - \frac{2}{x-4} > 0$$
$$\frac{3(x-4)}{(x+2)(x-4)} - \frac{2(x+2)}{(x-4)(x+2)} > 0$$
$$\frac{3(x-4)-2(x+2)}{(x+2)(x-4)} > 0$$
$$\frac{3x-12-2x-4}{(x+2)(x-4)} > 0$$
$$\frac{x-16}{(x+2)(x-4)} > 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$x-16=0 \Rightarrow x=16$ or $x+2=0 \Rightarrow x=-2$ or $x-4=0 \Rightarrow x=4$

The values –2, 4 and 16 divide the number line into four regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\dfrac{3}{x+2} > \dfrac{2}{x-4}$.

Interval	Test Value	Is $\frac{3}{x+2} > \frac{2}{x-4}$ True or False?
A: $(-\infty,-2)$	–3	$\frac{3}{-3+2} \overset{?}{>} \frac{2}{-3-4}$ $-3 > -\frac{2}{7}$ False
B: $(-2,4)$	0	$\frac{3}{0+2} \overset{?}{>} \frac{2}{0-4}$ $\frac{3}{2} > -\frac{1}{2}$ True
C: $(4,16)$	5	$\frac{3}{5+2} \overset{?}{>} \frac{2}{5-4}$ $\frac{3}{7} > 2$ False
D: $(16,\infty)$	17	$\frac{3}{17+2} \overset{?}{>} \frac{2}{17-4}$ $\frac{3}{19} \overset{?}{>} \frac{2}{13}$ $\frac{39}{247} > \frac{38}{247}$ True

Solution set: $(-2,4)\cup(16,\infty)$

101. **(a)** Answers will vary.

(b) Let x = the maximum initial concentration of ozone.

$x - .43x \le 50 \Rightarrow .57x \le 50$
$x \le 87.7$ (approximately)

The filter will reduce ozone concentrations that don't exceed 87.7 ppb.

102. $C = 3x + 1500$, $R = 8x$

The company will at least break even when $R \ge C$.

$8x \ge 3x + 1500 \Rightarrow 5x > 1500 \Rightarrow x \ge 300$

The break-even point is at $x = 300$. The company will at least break even if the number of units produced is in the interval $[300,\infty)$.

103. $s = 320 - 16t^2$

(a) When $s = 0$, the projectile will be at ground level.

$0 = 320t - 16t^2 \Rightarrow 16t^2 - 320t = 0 \Rightarrow$
$t^2 - 20t = 0 \Rightarrow t(t-20) = 0 \Rightarrow$
$t = 0$ or $t = 20$

The projectile will return to the ground after 20 sec.

(b) Solve $s > 576$ for t.

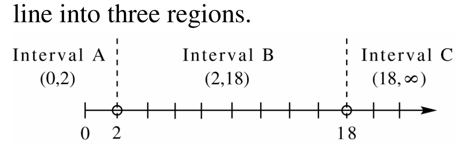
$$320t - 16t^2 > 576$$
$$0 > 16t^2 - 320t + 576$$
$$0 > t^2 - 20t + 36$$

Step 1: Find the values of x that satisfy $t^2 - 20t + 36 = 0$.

$t^2 - 20t + 36 = 0 \Rightarrow (t-2)(t-18) = 0 \Rightarrow$

$t-2=0 \Rightarrow t=2$ or $t-18=0 \Rightarrow t=18$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $320t - 16t^2 > 576$.

Interval	Test Value	Is $320t - 16t^2 > 576$ True or False?
A: $(0,2)$	1	$320(1) - 16(1)^2 \overset{?}{>} 576$ $304 > 576$ False
B: $(2,18)$	3	$320(3) - 16(3)^2 \overset{?}{>} 576$ $816 > 576$ True

(*continued on next page*)

(*continued from page 153*)

Interval	Test Value	Is $320t-16t^2>576$ True or False?
C: $(18,\infty)$	20	$320(20)-16(20)^2\overset{?}{>}576$ $0>576$ False

The projectile will be more than 576 ft above the ground between 2 and 18 sec.

104. $y=18.1x+326.3$
$18.1x+326.3>500$
$18.1x>173.7$
$x>9.6$ (approximately)
Based on the model, the amount paid by the government first exceeds \$500 billion about 9.6 years after 1994, which is in 2003. This is consistent with the graph.

105. Answers will vary. 3 cannot be in the solution set because when 3 is substituted into $\frac{14x+9}{x-3}$, division by zero occurs.

106. Answers will vary. -4 must be in the solution set because when -4 is substituted into $\frac{x+4}{2x+1}$, the result is zero.

107. "at least 65" means that the number is 65 or greater; $W\ge 65$.

108. "up to 35" means 35 or less; $w\le 35$
"up to 45" means 45 or less; $L\le 45$.

109. "as many as 100,000" means 100,000 or less; $a\le 100{,}000$

110. "fewer than 2000" means less than 2000; $p<2000$.

111. $|x+4|=7$
$x+4=7\Rightarrow x=3$ or $x+4=-7\Rightarrow x=-11$
Solution set: $\{-11, 3\}$

112. $|2-x|-3=0\Rightarrow |2-x|=3$
$2-x=3\Rightarrow x=-1$ or $2-x=-3\Rightarrow x=5$
Solution set: $\{-1, 5\}$

113. $\left|\frac{7}{2-3x}\right|-9=0\Rightarrow\left|\frac{7}{2-3x}\right|=9$
$\frac{7}{2-3x}=9\Rightarrow 7=9(2-3x)\Rightarrow 7=18-27x\Rightarrow$
$-11=-27x\Rightarrow \frac{-11}{-27}=x\Rightarrow x=\frac{11}{27}$ or
$\frac{7}{2-3x}=-9\Rightarrow 7=-9(2-3x)\Rightarrow$
$7=-18+27x\Rightarrow 25=27x\Rightarrow \frac{25}{27}=x\Rightarrow x=\frac{25}{27}$
Solution set: $\left\{\frac{11}{27},\frac{25}{27}\right\}$

114. $\left|\frac{8x-1}{3x+2}\right|=7$
$\frac{8x-1}{3x+2}=7\Rightarrow 8x-1=7(3x+2)\Rightarrow$
$8x-1=21x+14\Rightarrow -1=13x+14\Rightarrow$
$-15=13x\Rightarrow x=-\frac{15}{13}$ or
$\frac{8x-1}{3x+2}=-7\Rightarrow 8x-1=-7(3x+2)\Rightarrow$
$8x-1=-21x-14\Rightarrow 29x-1=-14\Rightarrow$
$29x=-13\Rightarrow x=-\frac{13}{29}$
Solution set: $\left\{-\frac{15}{13},-\frac{13}{29}\right\}$

115. $|5x-1|=|2x+3|$
$5x-1=2x+3\Rightarrow 3x-1=3\Rightarrow 3x=4\Rightarrow x=\frac{4}{3}$
or
$5x-1=-(2x+3)\Rightarrow 5x-1=-2x-3\Rightarrow$
$7x-1=-3\Rightarrow 7x=-2\Rightarrow x=-\frac{2}{7}$
Solution set: $\left\{-\frac{2}{7},\frac{4}{3}\right\}$

116. $|x+10|=|x-11|$
$x+10=x-11\Rightarrow 10=-11$ False
or
$x+10=-(x-11)\Rightarrow x+10=-x+11\Rightarrow$
$2x+10=11\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

117. $|2x+9|\le 3$
$-3\le 2x+9\le 3$
$-12\le 2x\le -6$
$-6\le x\le -3$
Solution set: $[-6,-3]$

118. $|8-5x|\ge 2$
$8-5x\ge 2\Rightarrow -5x\ge -6\Rightarrow x\le\frac{6}{5}$ or
$8-5x\le -2\Rightarrow -5x\le -10\Rightarrow x\ge 2$
Solution set: $\left(-\infty,\frac{6}{5}\right]\cup[2,\infty)$

119. $|7x-3|>4$
$7x-3<-4\Rightarrow 7x<-1\Rightarrow x<-\frac{1}{7}$ or
$7x-3>4\Rightarrow 7x>7\Rightarrow x>1$
Solution set: $\left(-\infty,-\frac{1}{7}\right)\cup(1,\infty)$

120. $\left|\frac{1}{2}x+\frac{2}{3}\right|<3$

$$-3<\tfrac{1}{2}x+\tfrac{2}{3}<3$$
$$6(-3)<6\left(\tfrac{1}{2}x+\tfrac{2}{3}\right)<6(3)$$
$$-18<3x+4<18$$
$$-22<3x<14\Rightarrow -\tfrac{22}{3}<x<\tfrac{14}{3}$$

Solution set: $\left(-\frac{22}{3},\frac{14}{3}\right)$

121. $|3x+7|-5=0\Rightarrow|3x+7|=5$

$3x+7=5\Rightarrow 3x=-2\Rightarrow x=-\frac{2}{3}$ or
$3x+7=-5\Rightarrow 3x=-12\Rightarrow x=-4$

Solution set: $\left\{-4,\ -\frac{2}{3}\right\}$

122. $|7x+8|-6>-3\Rightarrow|7x+8|>3$

$7x+8<-3\Rightarrow 7x<-11\Rightarrow x<-\frac{11}{7}$ or
$7x+8>3\Rightarrow 7x>-5\Rightarrow x>-\frac{5}{7}$

Solution set: $\left(-\infty,-\frac{11}{7}\right)\cup\left(-\frac{5}{7},\infty\right)$

123. Since the absolute value of a number is always nonnegative, the inequality $|4x-12|\ge -3$ is always true. The solution set is $(-\infty,\infty)$.

124. There is no number whose absolute value is less than or equal to any negative number. The solution set of $|7-2x|\le -9$ is $\varnothing$.

125. Since the absolute value of a number is always nonnegative, $\left|x^2+4x\right|<0$ is never true, so $\left|x^2+4x\right|\le 0$ is only true when $\left|x^2+4x\right|=0$.

$$\left|x^2+4x\right|=0\Rightarrow x^2+4x=0\Rightarrow x(x+4)=0$$
$$x=0 \quad\text{or}\quad x+4=0$$
$$x-0 \quad\text{or}\quad x=-4$$

Solution set: $\{-4,0\}$

126. $\left|x^2+4x\right|>0$ will be false only when $x^2+4x=0,$ which occurs when $x=-4$ or $x=0$ (see last exercise). So the solution set for $\left|x^2+4x\right|>0$ is $(-\infty,-4)\cup(-4,0)\cup(0,\infty)$.

127. "*k* is 12 units from 6 on the number line" means that the distance between *k* and 6 is 12 units, or $|k-6|=12$ or $|6-k|=12$.

128. "*p* is at least 3 units from 1 on the number line" means that *p* is 3 units or more from 1. Thus, the distance between *p* and 1 is greater than or equal to 3, or $|p-1|\ge 3$ or $|1-p|\ge 3$.

129. "*t* is no less than .01 unit from 5" means that *t* is .01 unit or more from 5. Thus, the distance between *t* and 5 is greater than or equal to .01, or $|t-5|\ge .01$ or $|5-t|\ge .01$

130. "*s* is no more than .001 unit from 100" means that *s* is .001 unit or less from 100. Thus, the distance between *s* and 100 is less than or equal to .001, or $|s-100|\le .001$ or $|100-s|\le .001$

Chapter 1: Test

1. $3(x-4)-5(x+2)=2-(x+24)$

$$3x-12-5x-10=2-x-24$$
$$-2x-22=-x-22$$
$$-22=x-22$$
$$0=x$$

Solution set: $\{0\}$

2. $\frac{2}{3}x+\frac{1}{2}(x-4)=x-4$

$$6\left[\frac{2}{3}x+\frac{1}{2}(x-4)\right]=6(x-4)$$
$$4x+3(x-4)=6x-24$$
$$4x+3x-12=6x-24$$
$$7x-12=6x-24$$
$$x-12=-24$$
$$x=-12$$

Solution set: $\{-12\}$

3. $6x^2-11x-7=0$

$$(2x+1)(3x-7)=0$$

$2x+1=0\Rightarrow x=-\frac{1}{2}$ or $3x-7=0\Rightarrow x=\frac{7}{3}$

Solution set: $\left\{-\frac{1}{2},\frac{7}{3}\right\}$

4. $(3x+1)^2=8$

$$3x+1=\pm\sqrt{8}=\pm 2\sqrt{2}$$
$$3x=-1\pm 2\sqrt{2}\Rightarrow x=\frac{-1\pm 2\sqrt{2}}{3}$$

Solution set: $\left\{\frac{-1\pm 2\sqrt{2}}{3}\right\}$

5. $3x^2 + 2x = -2$
Solve by completing the square.

$$3x^2 + 2x = -2$$
$$3x^2 + 2x + 2 = 0$$
$$x^2 + \tfrac{2}{3}x + \tfrac{2}{3} = 0 \Rightarrow x^2 + \tfrac{2}{3}x + \tfrac{1}{9} = -\tfrac{2}{3} + \tfrac{1}{9}$$
$$\text{Note:}\left[\tfrac{1}{2}\cdot\left(-\tfrac{2}{3}\right)\right]^2 = \left(-\tfrac{1}{3}\right)^2 = \tfrac{1}{9}$$
$$\left(x + \tfrac{1}{3}\right)^2 = -\tfrac{5}{9} \Rightarrow x + \tfrac{1}{3} = \pm\sqrt{-\tfrac{5}{9}} \Rightarrow$$
$$x + \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{3}i \Rightarrow x = -\tfrac{1}{3} \pm \tfrac{\sqrt{5}}{3}i$$

Solve by the quadratic formula.
Let $a = 3$, $b = 2$, and $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(3)(2)}}{2(3)} = \frac{-2 \pm \sqrt{4 - 24}}{6} = \frac{-2 \pm \sqrt{-20}}{6} = \frac{-2 \pm 2i\sqrt{5}}{6} = -\tfrac{2}{6} \pm \tfrac{2\sqrt{5}}{6}i = -\tfrac{1}{3} \pm \tfrac{\sqrt{5}}{3}i$$

Solution set: $\left\{-\tfrac{1}{3} \pm \tfrac{\sqrt{5}}{3}i\right\}$

6.
$$\frac{12}{x^2 - 9} = \frac{2}{x-3} - \frac{3}{x+3}$$
$$\frac{12}{(x+3)(x-3)} + \frac{3}{x+3} = \frac{2}{x-3}$$

Multiply each term in the equation by the least common denominator, $(x+3)(x-3)$ assuming $x \neq -3, 3$.

$$(x+3)(x-3)\left[\frac{12}{(x+3)(x-3)} + \frac{3}{x+3}\right] = (x+3)(x-3)\left(\frac{2}{x-3}\right)$$
$$12 + 3(x-3) = 2(x+3)$$
$$12 + 3x - 9 = 2x + 6$$
$$3x + 3 = 2x + 6$$
$$x + 3 = 6 \Rightarrow x = 3$$

The only possible solution is 3. However, the variable is restricted to real numbers except -3 and 3. Therefore, the solution set is $\varnothing$.

7. $\dfrac{4x}{x-2} + \dfrac{3}{x} = \dfrac{-6}{x^2 - 2x}$ or $\dfrac{4x}{x-2} + \dfrac{3}{x} = \dfrac{-6}{x(x-2)}$
Multiply each term in the equation by the least common denominator, $x(x-2)$, assuming $x \neq 0, 2$.

$$x(x-2)\left[\frac{4x}{x-2} + \frac{3}{x}\right] = x(x-2)\left(\frac{-6}{x(x-2)}\right)$$
$$4x^2 + 3(x-2) = -6 \Rightarrow 4x^2 + 3x - 6 = -6$$
$$4x^2 + 3x = 0 \Rightarrow x(4x+3) = 0$$
$$x = 0 \quad \text{or} \quad 4x + 3 = 0 \Rightarrow x = -\tfrac{3}{4}$$

Because of the restriction $x \neq 0$, the only valid solution is $-\tfrac{3}{4}$. The solution set is $\left\{-\tfrac{3}{4}\right\}$.

8. $\sqrt{3x+4} + 5 = 2x + 1 \Rightarrow \sqrt{3x+4} = 2x - 4$

$$\left(\sqrt{3x+4}\right)^2 = (2x-4)^2$$
$$3x + 4 = 4x^2 - 16x + 16$$
$$0 = 4x^2 - 19x + 12$$
$$0 = (4x-3)(x-4)$$
$$4x - 3 = 0 \Rightarrow x = \tfrac{3}{4} \quad \text{or} \quad x - 4 = 0 \Rightarrow x = 4$$

Check $x = \tfrac{3}{4}$.

$$\sqrt{3x+4} + 4 = 2x$$
$$\sqrt{3\left(\tfrac{3}{4}\right) + 4} + 4 = 2\left(\tfrac{3}{4}\right)$$
$$\sqrt{\tfrac{9}{4} + 4} + 4 = \tfrac{3}{2} \Rightarrow \sqrt{\tfrac{25}{4}} + 4 = \tfrac{3}{2}$$
$$\tfrac{5}{2} + 4 = \tfrac{3}{2} \Rightarrow \tfrac{13}{2} = \tfrac{3}{2}$$

This is a false statement. $\tfrac{3}{4}$ is a not solution.

Check $x = 4$.

$$\sqrt{3x+4} + 4 = 2x$$
$$\sqrt{3(4)+4} + 4 = 2(4)$$
$$\sqrt{12+4} + 4 = 8$$
$$\sqrt{16} + 4 = 8$$
$$4 + 4 = 8 \Rightarrow 8 = 8$$

This is a true statement. 4 is a solution.
Solution set: $\{4\}$

9. $\sqrt{-2x+3} + \sqrt{x+3} = 3$

$$\sqrt{-2x+3} = 3 - \sqrt{x+3}$$
$$\left(\sqrt{-2x+3}\right)^2 = \left(3 - \sqrt{x+3}\right)^2$$
$$-2x + 3 = 9 - 6\sqrt{x+3} + (x+3)$$
$$-2x + 3 = 12 + x - 6\sqrt{x+3}$$
$$-3x - 9 = -6\sqrt{x+3}$$
$$x + 3 = 2\sqrt{x+3}$$
$$(x+3)^2 = \left(2\sqrt{x+3}\right)^2$$
$$x^2 + 6x + 9 = 4(x+3)$$
$$x^2 + 6x + 9 = 4x + 12$$
$$x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0$$
$$x + 3 = 0 \Rightarrow x = -3 \quad \text{or} \quad x - 1 = 0 \Rightarrow x = 1$$

Check $x=-3$.

$$\sqrt{-2x+3}+\sqrt{x+3}=3$$
$$\sqrt{-2(-3)+3}+\sqrt{-3+3}=3$$
$$\sqrt{6+3}+\sqrt{0}=3$$
$$\sqrt{9}+0=3$$
$$3+0=3\Rightarrow 3=3$$

This is a true statement. -3 is a solution.

Check $x=1$.

$$\sqrt{-2x+3}+\sqrt{x+3}=3$$
$$\sqrt{-2(1)+3}+\sqrt{1+3}=3$$
$$\sqrt{-2+3}+\sqrt{4}=3$$
$$\sqrt{1}+2=3$$
$$1+2=3\Rightarrow 3=3$$

This is a true statement. 1 is a solution.

Solution set: $\{-3,1\}$

10. $\sqrt[3]{3x-8}=\sqrt[3]{9x+4}$

$$\left(\sqrt[3]{3x-8}\right)^3=\left(\sqrt[3]{9x+4}\right)^3$$
$$3x-8=9x+4\Rightarrow -8=6x+4\Rightarrow$$
$$-12=6x\Rightarrow -2=x$$

Check $x=-2$.

$$\sqrt[3]{3x-8}=\sqrt[3]{9x+4}$$
$$\sqrt[3]{3(-2)-8}=\sqrt[3]{9(-2)+4}$$
$$\sqrt[3]{-6-8}=\sqrt[3]{-18+4}$$
$$\sqrt[3]{-14}=\sqrt[3]{-14}\Rightarrow -\sqrt[3]{14}=-\sqrt[3]{14}$$

This is a true statement.

Solution set: $\{-2\}$

11. $x^4-17x^2+16=0$

Let $u=x^2$; then $u^2=x^4$.

With this substitution, the equation becomes $u^2-17u+16=0$.

Solve this equation by factoring.

$(u-1)(u-16)=0$

$u-1=0\Rightarrow u=1$ or $u-16=0\Rightarrow u=16$

To find x, replace u with x^2.

$x^2=1\Rightarrow x=\pm\sqrt{1}\Rightarrow x=\pm1$ or

$x^2=16\Rightarrow x=\pm\sqrt{16}\Rightarrow x=\pm4$

Solution set: $\{\pm1,\pm4\}$

12. $(x+3)^{2/3}+(x+3)^{1/3}-6=0$

Let $u=(x+3)^{1/3}$. Then

$u^2=\left[(x+3)^{1/3}\right]^2=(x+3)^{2/3}$.

$u^2+u-6=0\Rightarrow (u+3)(u-2)=0$

$u+3=0\Rightarrow u=-3$ or $u-2=0\Rightarrow u=2$

To find x, replace u with $(x+3)^{1/3}$.

$(x+3)^{1/3}=-3\Rightarrow\left[(x+3)^{1/3}\right]^3=(-3)^3\Rightarrow$

$x+3=27\Rightarrow x=-30$ or

$(x+3)^{1/3}=2\Rightarrow\left[(x+3)^{1/3}\right]^3=2^3\Rightarrow$

$x+3=8\Rightarrow x=5$

Check $x=-30$.

$$(x+3)^{2/3}+(x+3)^{1/3}-6=0$$
$$(-30+3)^{2/3}+(-30+3)^{1/3}-6=0$$
$$(-27)^{2/3}+(-27)^{1/3}-6=0$$
$$\left[(-27)^{1/3}\right]^2+(-3)-6=0$$
$$(-3)^2-3-6=0$$
$$9-3-6=0\Rightarrow 0=0$$

This is a true statement. -30 is a solution.

Check $x=5$.

$$(x+3)^{2/3}+(x+3)^{1/3}-6=0$$
$$(5+3)^{2/3}+(5+3)^{1/3}-6=0$$
$$8^{2/3}+8^{1/3}-6=0$$
$$\left[8^{1/3}\right]^2+2-6=0$$
$$2^2+2-6=0$$
$$4+2-6=0\Rightarrow 0=0$$

This is a true statement. 5 is a solution.

Solution set: $\{-30,5\}$

13. $|4x+3|=7$

$4x+3=7\Rightarrow 4x=4\Rightarrow x=1$ or

$4x+3=-7\Rightarrow 4x=-10\Rightarrow x=-\frac{10}{4}=-\frac{5}{2}$

Solution set: $\left\{-\frac{5}{2},1\right\}$

14. $|2x+1|=|5-x|$

$2x+1=5-x\Rightarrow 3x+1=5\Rightarrow 3x=4\Rightarrow x=\frac{4}{3}$

or

$2x+1=-(5-x)\Rightarrow 2x+1=-5+x\Rightarrow x=-6$

Solution set: $\left\{-6,\frac{4}{3}\right\}$

15.
$$S=2HW+2LW+2LH$$
$$S-2LH=2HW+2LW$$
$$S-2LH=W(2H+2L)$$
$$\frac{S-2LH}{2H+2L}=W$$
$$W=\frac{S-2LH}{2H+2L}$$

16. **(a)** $(9-3i)-(4+5i)=(9-4)+(-3-5)i$
$=5-8i$

(b) $(4+3i)(-5+3i) = -20+12i-15i+9i^2$
$= -20-3i+9(-1)$
$= -20-3i-9 = -29-3i$

(c) $(8+3i)^2 = 8^2 + 2(8)(3i) + (3i)^2$
$= 64+48i+9i^2$
$= 64+48i+9(-1)$
$= 64+48i-9 = 55+48i$

(d) $\frac{3+19i}{1+3i} = \frac{(3+19i)(1-3i)}{(1+3i)(1-3i)}$
$= \frac{3-9i+19i-57i^2}{1-(3i)^2}$
$= \frac{3+10i-57(-1)}{1-9i^2} = \frac{3+10i+57}{1-9(-1)}$
$= \frac{60+10i}{1+9} = \frac{60+10i}{10} = 6+i$

17. **(a)** $i^{42} = i^{40} \cdot i^2 = \left(i^4\right)^{10} \cdot (-1) = 1^{10} \cdot (-1) = -1$

(b) $i^{-31} = i^{-32} \cdot i = \left(i^4\right)^{-8} \cdot i = 1^{-8} \cdot i = i$

(c) $\frac{1}{i^{19}} = i^{-19} = i^{-20} \cdot i = \left(i^4\right)^{-5} \cdot i = 1^{-5} \cdot i = i$

18. **(a)** Minimum:
$1120\frac{\text{gal}}{\text{min}} \cdot 60\frac{\text{min}}{\text{hr}} \cdot 12\frac{\text{hr}}{\text{day}} = 806{,}400\frac{\text{gal}}{\text{day}}$
The equation that will calculate the minimum amount of water pumped after x days would be $A = 806{,}400x$.

(b) $A = 806{,}400x$ when $x = 30$ would be $A = 806{,}400(30) = 24{,}192{,}000$ gal.

(c) Since there would be $806{,}400\frac{\text{gal}}{\text{day}}$ minimum and each pool requires 20,000 gal, there would be a minimum of $\frac{806{,}400}{20{,}000} = 40.32$ pools that could be filled each day. The equation that will calculate the minimum number of pools that could be filled after x days would be $P = 40.32x$. Approximately 40 pools could be filled each day.

(d) Solve $P = 40.32x$ where $P = 1000$.
$1000 = 40.32x \Rightarrow x = \frac{1000}{40.32} \approx 24.8$ days.

19. Let $w =$ width of rectangle. Then
$2w - 20 =$ length of rectangle.
Use the formula for the perimeter of a rectangle.
$P = 2l + 2w$
$620 = 2(2w-20) + 2w$
$620 = 4w - 40 + 2w$
$620 = 6w - 40 \Rightarrow 660 = 6w \Rightarrow 110 = w$
The width is 110 m and the length is $2(110) - 20 = 220 - 20 = 200$ m.

20. Let x = amount of cashews (in pounds). Then $35 - x$ = amount of walnuts (in pounds).

	Cost per Pound	Amount of Nuts	
Cashews	7.00	x	$7.00x$
Walnuts	5.50	$35-x$	$5.50(35-x)$
Mixture	6.50	35	$35 \cdot 6.50$

Solve the following equation.
$7.00x + 5.50(35-x) = 35 \cdot 6.50$
$7x + 192.5 - 5.5x = 227.5$
$1.5x + 192.5 = 227.5$
$1.5x = 35$
$x = \frac{35}{1.5} = \frac{350}{15} = \frac{70}{3} = 23\frac{1}{3}$

The fruit and nut stand owner should mix $23\frac{1}{3}$ lbs of cashews with $35 - 23\frac{1}{3} = 11\frac{2}{3}$ lbs of walnuts.

21. Let x = time (in hours) the mother spent driving to meet plane.
Since Mary Lynn has been in the plane for 15 minutes, and 15 minutes is $\frac{1}{4}$ hr, she has been traveling by plane for $x + \frac{1}{4}$ hr.

	d	r	t
Mary Lynn by plane	420		$x+\frac{1}{4}$
Mother by car	20	40	x

The time driven by Mary Lynn's mother can be found by $20 = 40x \Rightarrow x = \frac{1}{2}$ hr. Mary Lynn, therefore, flew for $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$ hr.
The rate of Mary Lynn's plane can be found by $r = \frac{d}{t} = \frac{420}{\frac{3}{4}} = 420 \cdot \frac{4}{3} = 560$ km per hour.

22. $h = -16t^2 + 96t$

(a) Let $h = 80$ and solve for t.

$80 = -16t^2 + 96t \Rightarrow 16t^2 - 96t + 80 = 0$

$t^2 - 6t + 5 = 0$

$(t-1)(t-5) = 0$

$t - 1 = 0 \Rightarrow t = 1 \quad \text{or} \quad t - 5 = 0 \Rightarrow t = 5$

The projectile will reach a height of 80 ft at 1 sec and 5 sec.

(b) Let $h = 0$ and solve for t.

$0 = -16t^2 + 96t$

$0 = -16t(t-6)$

$t = 0 \quad \text{or} \quad t - 6 = 0 \Rightarrow t = 6$

The projectile will return to the ground at 6 sec.

23. The table shows each equation evaluated at the years 1975, 1994, and 2006. Equation B best models the data.

24. $-2(x-1) - 12 < 2(x+1)$

$-2x + 2 - 12 < 2x + 2$

$-2x - 10 < 2x + 2$

$-4x - 10 < 2$

$-4x < 12$

$x > -3$

Solution set: $(-3, \infty)$

25. $-3 \le \frac{1}{2}x + 2 \le 3$

$2(-3) \le 2\left(\frac{1}{2}x + 2\right) \le 2(3)$

$-6 \le x + 4 \le 6$

$-10 \le x \le 2$

Solution set: $[-10, 2]$

26. $2x^2 - x \ge 3$

Step 1: Find the values of x that satisfy $2x^2 - x = 3$.

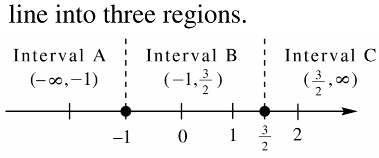
$2x^2 - x = 3$

$2x^2 - x - 3 = 0$

$(x+1)(2x-3) = 0$

$x + 1 = 0 \Rightarrow x = -1 \quad \text{or} \quad 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$

Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $2x^2 - x \ge 3$

Interval	Test Value	Is $2x^2 - x \ge 3$ True or False?
A: $(-\infty, -1)$	-2	$2(-2)^2 - (-2) \overset{?}{\ge} 3$ $10 \ge 3$ True
B: $\left(-1, \frac{3}{2}\right)$	0	$2 \cdot 0^2 - 0 \overset{?}{\ge} 3$ $0 \ge 3$ False
C: $\left(\frac{3}{2}, \infty\right)$	2	$2 \cdot 2^2 - 2 \overset{?}{\ge} 3$ $6 \ge 3$ True

Solution set: $(-\infty, -1] \cup \left[\frac{3}{2}, \infty\right)$

27. $\frac{x+1}{x-3} < 5$

Step 1: Rewrite the inequality so that 0 is on one side and there is a single fraction on the other side.

$$\frac{x+1}{x-3} < 5 \Rightarrow \frac{x+1}{x-3} - 5 < 0$$

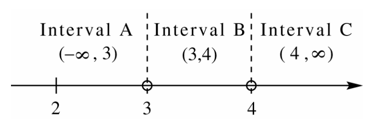
$$\frac{x+1}{x-3} - \frac{5(x-3)}{x-3} < 0 \Rightarrow \frac{x+1-5(x-3)}{x-3} < 0$$

$$\frac{x+1-5x+15}{x-3} < 0 \Rightarrow \frac{-4x+16}{x-3} < 0$$

Step 2: Determine the values that will cause either the numerator or denominator to equal 0.

$-4x + 16 = 0 \Rightarrow x = 4 \quad \text{or} \quad x - 3 = 0 \Rightarrow x = 3$

The values 3 and 4 divide the number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $\frac{x+1}{x-3} < 5$.

Interval	Test Value	Is $\frac{x+1}{x-3} < 5$ True or False?
A: $(-\infty, 3)$	0	$\frac{0+1}{0-3} \overset{?}{<} 5$ $-\frac{1}{3} < 5$ True
B: $(3, 4)$	3.5	$\frac{3.5+1}{3.5-3} \overset{?}{<} 5$ $9 < 5$ False
C: $(4, \infty)$	5	$\frac{5+1}{5-3} \overset{?}{<} 5$ $3 < 5$ True

Solution set: $(-\infty, 3) \cup (4, \infty)$

28. $|2x-5|<9$
$-9<2x-5<9$
$-4<2x<14$
$-2<x<7$
Solution set: (–2, 7)

29. $|2x+1|-11\geq 0 \Rightarrow |2x+1|\geq 11$

$$2x+1\leq -11 \quad \text{or} \quad 2x+1\geq 11$$
$$2x\leq -12 \qquad 2x\geq 10$$
$$x\leq -6 \quad \text{or} \quad x\geq 5$$

Solution set: $(-\infty,-6]\cup[5,\infty)$

30. $|3x+7|\leq 0 \Rightarrow 3x+7\leq 0 \Rightarrow x\leq -\frac{7}{3}$

However, if $x<-\frac{7}{3}$, the expression inside the absolute value bars is negative, so x cannot be less than $-\frac{7}{3}$. The solution set of $|3x+7|\leq 0$ is $\left\{-\frac{7}{3}\right\}$.

Chapter 1: Quantitative Reasoning

Let x = number of new shares of stock issued. Currently, the number of shares the acquaintance holds is $.05(900{,}000)=45{,}000.$ For his number of shares to represent 10% of the total number of shares, we need to find the number of new shares that should be issued. To do this, we solve the following.

$$45{,}000+x=.10(900{,}000+x)$$
$$45{,}000+x=90{,}000+.10x$$
$$45{,}000+.90x=90{,}000$$
$$.90x=45{,}000$$
$$x=50{,}000$$

50,000 new shares should be issued.

Chapter 2

GRAPHS AND FUNCTIONS

Section 2.1: Rectangular Coordinates and Graphs

Connections (page 190)

1. Answers will vary.
2. Answers will vary.
   Latitude and longitude values pinpoint distances north or south of the equator and east or west of the prime meridian. Similarly on a Cartesian coordinate system, x- and y-coordinates give distances and directions from the y-axis and x-axis, respectively.

Exercises

1. False. $(-1, 3)$ lies in Quadrant II.
2. False. The expression should be $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.
3. True. The origin has coordinates $(0,0)$. So, the distance from $(0,0)$ to (a,b) is $d=\sqrt{(a-0)^2+(b-0)^2}=\sqrt{a^2+b^2}$
4. True. The midpoint has coordinates $\left(\frac{a+3a}{2},\frac{b+(-3b)}{2}\right)=\left(\frac{4a}{2},\frac{-2b}{2}\right)=(2a,-b)$.
5. True. When $x = 0$, $y = 2(0) + 4 = 4$, so the y-intercept is 4. When $y = 0$, $0 = 2x + 4 \Rightarrow x = -2$, so the x-intercept is -2.
6. Answers will vary.
7. Any three of the following: $(2,-5),(-1,7),(3,-9),(5,-17),(6,-21)$
8. Any three of the following: $(3,3),(-5,-21),(8,18),(4,6),(0,-6)$
9. Any three of the following: $(1993,31),(1995,35),(1997,37),(1999,35),(2001,28),(2003,25)$
10. Any three of the following: $(1997,87.8),(1998,90.0),(1999,83.7),(2000,88.5),(2001,84.3)$
11. $P(-5,-7)$, $Q(-13,1)$
    (a) $d(P,Q)=\sqrt{[-13-(-5)]^2+[1-(-7)]^2}=\sqrt{(-8)^2+8^2}=\sqrt{128}=8\sqrt{2}$
    (b) The midpoint M of the segment joining points P and Q has coordinates $\left(\frac{-5+(-13)}{2},\frac{-7+1}{2}\right)=\left(\frac{-18}{2},\frac{-6}{2}\right)=(-9,-3)$.
12. $P(-4,3)$, $Q(2,-5)$
    (a) $d(P,Q)=\sqrt{[2-(-4)]^2+(-5-3)^2}=\sqrt{6^2+(-8)^2}=\sqrt{100}=10$
    (b) The midpoint M of the segment joining points P and Q has coordinates $\left(\frac{-4+2}{2},\frac{3+(-5)}{2}\right)=\left(\frac{-2}{2},\frac{-2}{2}\right)=(-1,-1)$.
13. $P(8,2)$, $Q(3,5)$
    (a) $d(P,Q)=\sqrt{(3-8)^2+(5-2)^2}=\sqrt{(-5)^2+3^2}=\sqrt{25+9}=\sqrt{34}$
    (b) The midpoint M of the segment joining points P and Q has coordinates $\left(\frac{8+3}{2},\frac{2+5}{2}\right)=\left(\frac{11}{2},\frac{7}{2}\right)$.
14. $P(-8,4)$, $Q(3,-5)$
    (a) $d(P,Q)=\sqrt{\left[3-(-8)\right]^2+(-5-4)^2}=\sqrt{11^2+(-9)^2}=\sqrt{121+81}=\sqrt{202}$
    (b) The midpoint M of the segment joining points P and Q has coordinates $\left(\frac{-8+3}{2},\frac{4+(-5)}{2}\right)=\left(-\frac{5}{2},-\frac{1}{2}\right)$.

15. $P(-6, -5)$, $Q(6, 10)$

(a) $d(P, Q) = \sqrt{[6-(-6)]^2 + [10-(-5)]^2}$
$= \sqrt{12^2 + 15^2} = \sqrt{144 + 225}$
$= \sqrt{369} = 3\sqrt{41}$

(b) The midpoint M of the segment joining points P and Q has coordinates
$\left(\frac{-6+6}{2}, \frac{-5+10}{2}\right) = \left(\frac{0}{2}, \frac{5}{2}\right) = \left(0, \frac{5}{2}\right).$

16. $P(6, -2)$, $Q(4, 6)$

(a) $d(P, Q) = \sqrt{(4-6)^2 + [6-(-2)]^2}$
$= \sqrt{(-2)^2 + 8^2}$
$= \sqrt{4 + 64} = \sqrt{68} = 2\sqrt{17}$

(b) The midpoint M of the segment joining points P and Q has coordinates
$\left(\frac{6+4}{2}, \frac{-2+6}{2}\right) = \left(\frac{10}{2}, \frac{4}{2}\right) = (5, 2)$

17. $P\left(3\sqrt{2}, 4\sqrt{5}\right)$, $Q\left(\sqrt{2}, -\sqrt{5}\right)$

(a) $d(P, Q)$
$= \sqrt{\left(\sqrt{2} - 3\sqrt{2}\right)^2 + \left(-\sqrt{5} - 4\sqrt{5}\right)^2}$
$= \sqrt{\left(-2\sqrt{2}\right)^2 + \left(-5\sqrt{5}\right)^2}$
$= \sqrt{8 + 125} = \sqrt{133}$

(b) The midpoint M of the segment joining points P and Q has coordinates
$\left(\frac{3\sqrt{2} + \sqrt{2}}{2}, \frac{4\sqrt{5} + (-\sqrt{5})}{2}\right)$
$= \left(\frac{4\sqrt{2}}{2}, \frac{3\sqrt{5}}{2}\right) = \left(2\sqrt{2}, \frac{3\sqrt{5}}{2}\right).$

18. $P\left(-\sqrt{7}, 8\sqrt{3}\right), Q\left(5\sqrt{7}, -\sqrt{3}\right)$

(a) $d(P, Q)$
$= \sqrt{[5\sqrt{7} - (-\sqrt{7})]^2 + (-\sqrt{3} - 8\sqrt{3})^2}$
$= \sqrt{(6\sqrt{7})^2 + (-9\sqrt{3})^2} = \sqrt{252 + 243}$
$= \sqrt{495} = 3\sqrt{55}$

(b) The midpoint M of the segment joining points P and Q has coordinates
$\left(\frac{-\sqrt{7} + 5\sqrt{7}}{2}, \frac{8\sqrt{3} + (-\sqrt{3})}{2}\right)$
$= \left(\frac{4\sqrt{7}}{2}, \frac{7\sqrt{3}}{2}\right) = \left(2\sqrt{7}, \frac{7\sqrt{3}}{2}\right).$

19. Label the points $A(-6, -4)$, $B(0, -2)$, and $C(-10, 8)$. Use the distance formula to find the length of each side of the triangle.

$d(A, B) = \sqrt{[0-(-6)]^2 + [-2-(-4)]^2}$
$= \sqrt{6^2 + 2^2} = \sqrt{36 + 4} = \sqrt{40}$

$d(B, C) = \sqrt{(-10-0)^2 + [8-(-2)]^2}$
$= \sqrt{(-10)^2 + 10^2} = \sqrt{100 + 100}$
$= \sqrt{200}$

$d(A, C) = \sqrt{[-10-(-6)]^2 + [8-(-4)]^2}$
$= \sqrt{(-4)^2 + 12^2} = \sqrt{16 + 144} = \sqrt{160}$

Since $\left(\sqrt{40}\right)^2 + \left(\sqrt{160}\right)^2 = \left(\sqrt{200}\right)^2$, triangle ABC is a right triangle.

20. Label the points $A(-2, -8)$, $B(0, -4)$, and $C(-4, -7)$. Use the distance formula to find the length of each side of the triangle.

$d(A, B) = \sqrt{[0-(-2)]^2 + [-4-(-8)]^2}$
$= \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20}$

$d(B, C) = \sqrt{(-4-0)^2 + [-7-(-4)]^2}$
$= \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9}$
$= \sqrt{25} = 5$

$d(A, C) = \sqrt{[-4-(-2)]^2 + [-7-(-8)]^2}$
$= \sqrt{(-2)^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}$

Since $(\sqrt{5})^2 + (\sqrt{20})^2 = 5 + 20 = 25 = 5^2$, triangle ABC is a right triangle.

21. Label the points $A(-4, 1)$, $B(1, 4)$, and $C(-6, -1)$.

$d(A, B) = \sqrt{[1-(-4)]^2 + (4-1)^2}$
$= \sqrt{5^2 + 3^2} = \sqrt{25 + 9} = \sqrt{34}$

$d(B, C) = \sqrt{(-6-1)^2 + (-1-4)^2}$
$= \sqrt{(-7)^2 + (-5)^2} = \sqrt{49 + 25} = \sqrt{74}$

$d(A, C) = \sqrt{[-6-(-4)]^2 + (-1-1)^2}$
$= \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8}$

Since $(\sqrt{8})^2 + (\sqrt{34})^2 \neq (\sqrt{74})^2$ because $8 + 34 = 42 \neq 74$, triangle ABC is not a right triangle.

22. Label the points $A(-2, -5)$, $B(1, 7)$, and $C(3, 15)$.

$$d(A, B) = \sqrt{[1-(-2)]^2 + [7-(-5)]^2}$$
$$= \sqrt{3^2 + 12^2} = \sqrt{9+144} = \sqrt{153}$$
$$d(B, C) = \sqrt{(3-1)^2 + (15-7)^2}$$
$$= \sqrt{2^2 + 8^2} = \sqrt{4+64} = \sqrt{68}$$
$$d(A, C) = \sqrt{[3-(-2)]^2 + [15-(-5)]^2}$$
$$= \sqrt{5^2 + 20^2} = \sqrt{25+400} = \sqrt{425}$$

Since $(\sqrt{68})^2 + (\sqrt{153})^2 \neq \left(\sqrt{425}\right)^2$ because $68 + 153 = 221 \neq 425$, triangle ABC is not a right triangle.

23. Label the points $A(-4, 3)$, $B(2, 5)$, and $C(-1, -6)$.

$$d(A, B) = \sqrt{\left[2-(-4)\right]^2 + (5-3)^2}$$
$$= \sqrt{6^2 + 2^2} = \sqrt{36+4} = \sqrt{40}$$
$$d(B, C) = \sqrt{(-1-2)^2 + (-6-5)^2}$$
$$= \sqrt{(-3)^2 + (-11)^2}$$
$$= \sqrt{9+121} = \sqrt{130}$$
$$d(A, C) = \sqrt{\left[-1-(-4)\right]^2 + (-6-3)^2}$$
$$= \sqrt{3^2 + (-9)^2} = \sqrt{9+81} = \sqrt{90}$$

Since $\left(\sqrt{40}\right)^2 + \left(\sqrt{90}\right)^2 = \left(\sqrt{130}\right)^2$, triangle ABC is a right triangle.

24. Label the points $A(-7, 4)$, $B(6, -2)$, and $C(0, -15)$.

$$d(A, B) = \sqrt{\left[6-(-7)\right]^2 + (-2-4)^2}$$
$$= \sqrt{13^2 + (-6)^2}$$
$$= \sqrt{169+36} = \sqrt{205}$$
$$d(B, C) = \sqrt{(0-6)^2 + \left[-15-(-2)\right]^2}$$
$$= \sqrt{(-6)^2 + (-13)^2}$$
$$= \sqrt{36+169} = \sqrt{205}$$
$$d(A, C) = \sqrt{\left[0-(-7)\right]^2 + (-15-4)^2}$$
$$= \sqrt{7^2 + (-19)^2}$$
$$= \sqrt{49+361} = \sqrt{410}$$

Since $\left(\sqrt{205}\right)^2 + \left(\sqrt{205}\right)^2 = \left(\sqrt{410}\right)^2$, triangle ABC is a right triangle.

25. Label the given points $A(0, -7)$, $B(-3, 5)$, and $C(2, -15)$. Find the distance between each pair of points.

$$d(A, B) = \sqrt{(-3-0)^2 + \left[5-(-7)\right]^2}$$
$$= \sqrt{(-3)^2 + 12^2} = \sqrt{9+144}$$
$$= \sqrt{153} = 3\sqrt{17}$$
$$d(B, C) = \sqrt{\left[2-(-3)\right]^2 + (-15-5)^2}$$
$$= \sqrt{5^2 + (-20)^2} = \sqrt{25+400}$$
$$= \sqrt{425} = 5\sqrt{17}$$
$$d(A, C) = \sqrt{(2-0)^2 + \left[-15-(-7)\right]^2}$$
$$= \sqrt{2^2 + (-8)^2} = \sqrt{68} = 2\sqrt{17}$$

Since $d(A, B) + d(A, C) = d(B, C)$ or $3\sqrt{17} + 2\sqrt{17} = 5\sqrt{17}$, the points are collinear.

26. Label the points $A(-1, 4)$, $B(-2, -1)$, and $C(1, 14)$. Apply the distance formula to each pair of points.

$$d(A, B) = \sqrt{\left[-2-(-1)\right]^2 + (-1-4)^2}$$
$$= \sqrt{(-1)^2 + (-5)^2} = \sqrt{26}$$
$$d(B, C) = \sqrt{\left[1-(-2)\right]^2 + \left[14-(-1)\right]^2}$$
$$= \sqrt{3^2 + 15^2} = \sqrt{234} = 3\sqrt{26}$$
$$d(A, C) = \sqrt{\left[1-(-1)\right]^2 + (14-4)^2}$$
$$= \sqrt{2^2 + 10^2} = \sqrt{104} = 2\sqrt{26}$$

Because $\sqrt{26} + 2\sqrt{26} = 3\sqrt{26}$, the points are collinear.

27. Label the points $A(0, 9)$, $B(-3, -7)$, and $C(2, 19)$.

$$d(A, B) = \sqrt{(-3-0)^2 + (-7-9)^2}$$
$$= \sqrt{(-3)^2 + (-16)^2} = \sqrt{9+256}$$
$$= \sqrt{265} \approx 16.279$$
$$d(B, C) = \sqrt{\left[2-(-3)\right]^2 + \left[19-(-7)\right]^2}$$
$$= \sqrt{5^2 + 26^2} = \sqrt{25+676}$$
$$= \sqrt{701} \approx 26.476$$
$$d(A, C) = \sqrt{(2-0)^2 + (19-9)^2}$$
$$= \sqrt{2^2 + 10^2} = \sqrt{4+100}$$
$$= \sqrt{104} \approx 10.198$$

(*continued on next page*)

(*continued from page 163*)

Since $d(A, B) + d(A, C) \neq d(B, C)$

or $\sqrt{265} + \sqrt{104} \neq \sqrt{701}$

$16.279 + 10.198 \neq 26.476,$

$26.477 \neq 26.476,$

the three given points are not collinear. (Note, however, that these points are very close to lying on a straight line and may appear to lie on a straight line when graphed.)

28. Label the points $A(-1, -3)$, $B(-5, 12)$, and $C(1, -11)$.

$$d(A, B) = \sqrt{\left[-5-(-1)\right]^2 + \left[12-(-3)\right]^2} = \sqrt{(-4)^2 + 15^2} = \sqrt{16+225} = \sqrt{241} \approx 15.5242$$

$$d(B, C) = \sqrt{\left[1-(-5)\right]^2 + (-11-12)^2} = \sqrt{6^2 + (-23)^2} = \sqrt{36+529} = \sqrt{565} \approx 23.7697$$

$$d(A, C) = \sqrt{\left[1-(-1)\right]^2 + \left[-11-(-3)\right]^2} = \sqrt{2^2 + (-8)^2} = \sqrt{4+64} = \sqrt{68} \approx 8.2462$$

Since $d(A, B) + d(A, C) \neq d(B, C)$

or $\sqrt{241} + \sqrt{68} \neq \sqrt{565}$

$15.5242 + 8.2462 \neq 23.7697$

$23.7704 \neq 23.7697,$

the three given points are not collinear. (Note, however, that these points are very close to lying on a straight line and may appear to lie on a straight line when graphed.)

29. Label the points $A(-7, 4)$, $B(6,-2)$, and $C(-1,1)$.

$$d(A, B) = \sqrt{\left[6-(-7)\right]^2 + (-2-4)^2} = \sqrt{13^2 + (-6)^2} = \sqrt{169+36} = \sqrt{205} \approx 14.3178$$

$$d(B, C) = \sqrt{(-1-6)^2 + \left[1-(-2)\right]^2} = \sqrt{(-7)^2 + 3^2} = \sqrt{49+9} = \sqrt{58} \approx 7.6158$$

$$d(A, C) = \sqrt{\left[-1-(-7)\right]^2 + (1-4)^2} = \sqrt{6^2 + (-3)^2} = \sqrt{36+9} = \sqrt{45} \approx 6.7082$$

Since $d(B, C) + d(A, C) \neq d(A, B)$ or

$\sqrt{58} + \sqrt{45} \neq \sqrt{205}$

$7.6158 + 6.7082 \neq 14.3178$

$14.3240 \neq 14.3178,$

the three given points are not collinear. (Note, however, that these points are very close to lying on a straight line and may appear to lie on a straight line when graphed.)

30. Label the given points $A(-4, 3)$, $B(2, 5)$, and $C(-1,4)$. Find the distance between each pair of points.

$$d(A, B) = \sqrt{\left[2-(-4)\right]^2 + (5-3)^2} = \sqrt{6^2 + 2^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$

$$d(B, C) = \sqrt{(-1-2)^2 + (4-5)^2} = \sqrt{(-3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$$

$$d(A, C) = \sqrt{\left[-1-(-4)\right]^2 + (4-3)^2} = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

Since $d(B, C) + d(A, C) = d(A, B)$ or

$\sqrt{10} + \sqrt{10} = 2\sqrt{10},$ the points are collinear.

31. Midpoint (5, 8), endpoint (13, 10)

$\frac{13+x}{2} = 5$ and $\frac{10+y}{2} = 8$

$13 + x = 10$ and $10 + y = 16$

$x = -3$ and $y = 6.$

The other endpoint has coordinates (–3, 6).

32. Midpoint (–7, 6), endpoint (–9, 9)

$\frac{-9+x}{2} = -7$ and $\frac{9+y}{2} = 6$

$-9 + x = -14$ and $9 + y = 12$

$x = -5$ and $y = 3.$

The other endpoint has coordinates (–5, 3).

33. Midpoint (12, 6), endpoint (19, 16)

$\frac{19+x}{2} = 12$ and $\frac{16+y}{2} = 6$

$19 + x = 24$ and $16 + y = 12$

$x = 5$ and $y = -4.$

The other endpoint has coordinates (5, –4).

34. Midpoint (–9, 8), endpoint (–16, 9)

$$\frac{-16+x}{2}=-9 \quad\text{and}\quad \frac{9+y}{2}=8$$
$$-16+x=-18 \quad\text{and}\quad 9+y=16$$
$$x=-2 \quad\text{and}\quad y=7$$

The other endpoint has coordinates (–2, 7).

35. Midpoint (a, b), endpoint (p, q)

$$\frac{p+x}{2}=a \quad\text{and}\quad \frac{q+y}{2}=b$$
$$p+x=2a \quad\text{and}\quad q+y=2b$$
$$x=2a-p \quad\text{and}\quad y=2b-q$$

The other endpoint has coordinates $(2a-p, 2b-q)$.

36. Midpoint $\left(\frac{a+b}{2},\frac{c+d}{2}\right)$, endpoint (b,d)

$$\frac{b+x}{2}=\frac{a+b}{2} \quad\text{and}\quad \frac{d+y}{2}=\frac{c+d}{2}$$
$$b+x=a+b \quad\text{and}\quad d+y=c+d$$
$$x=a \quad\text{and}\quad y=c$$

The other endpoint has coordinates (a, c).

37. The endpoints of the segment are (1990, 20.3) and (2006, 28.0).

$$M=\left(\frac{1990+2006}{2},\frac{20.3+28.0}{2}\right)=(1998,\ 24.15)$$

The estimate is 24.15%. This is close to the actual figure of 24.4%.

38. The endpoints are (2000, 387) and (2004, 506)

$$M=\left(\frac{2000+2004}{2},\frac{387+506}{2}\right)=(2002,446.5)$$

The average payment to families in 2002 was \$446.50

39. The points to use would be (1970, 3968) and (2004, 19157). Their midpoint is

$$\left(\frac{1970+2004}{2},\frac{3968+19{,}157}{2}\right)=(1987,11562.50).$$

In 1987, the poverty level cutoff was approximately \$11,563.

40. (a) To estimate the enrollment for 1998, use the points (1995, 11092) and (2001, 12233)

$$M=\left(\frac{1995+2001}{2},\frac{11{,}092+12{,}233}{2}\right)=(1998,11662.5)$$

The enrollment for 1998 was about 11,663 thousand.

(b) To estimate the enrollment for 2004, use the points (2001, 12233) and (2007, 13555)

$$M=\left(\frac{2001+2007}{2},\frac{12{,}233+13{,}555}{2}\right)=(2004,12894)$$

The enrollment for 2004 was about 12,894 thousand.

41. The midpoint M has coordinates $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

$$d(P,M)=\sqrt{\left(\frac{x_1+x_2}{2}-x_1\right)^2+\left(\frac{y_1+y_2}{2}-y_1\right)^2}$$
$$=\sqrt{\left(\frac{x_1+x_2}{2}-\frac{2x_1}{2}\right)^2+\left(\frac{y_1+y_2}{2}-\frac{2y_1}{2}\right)^2}$$
$$=\sqrt{\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2-y_1}{2}\right)^2}$$
$$=\sqrt{\frac{(x_2-x_1)^2}{4}+\frac{(y_2-y_1)^2}{4}}$$
$$=\sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{4}}$$
$$=\tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d(M,Q)=\sqrt{\left(x_2-\frac{x_1+x_2}{2}\right)^2+\left(y_2-\frac{y_1+y_2}{2}\right)^2}$$
$$=\sqrt{\left(\frac{2x_2}{2}-\frac{x_1+x_2}{2}\right)^2+\left(\frac{2y_2}{2}-\frac{y_1+y_2}{2}\right)^2}$$
$$=\sqrt{\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2-y_1}{2}\right)^2}$$
$$=\sqrt{\frac{(x_2-x_1)^2}{4}+\frac{(y_2-y_1)^2}{4}}$$
$$=\sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{4}}$$
$$=\tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

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$d(P,Q)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Since $\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$+\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$,

this shows $d(P,M)+d(M,Q)=d(P,Q)$ and $d(P,M)=d(M,Q)$.

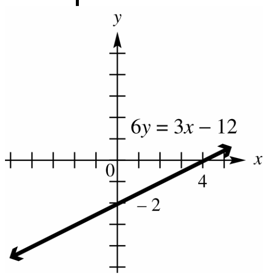
42. The distance formula, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, can be written as $d=[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}$.

In exercises 43−54, other ordered pairs are possible.

43. (a)

x	*y*	
0	−2	*y*-intercept: $x=0\Rightarrow$ $6y=3(0)-12\Rightarrow$ $6y=-12\Rightarrow y=-2$
4	0	*x*-intercept: $y=0\Rightarrow$ $6(0)=3x-12\Rightarrow$ $0=3x-12\Rightarrow$ $12=3x\Rightarrow 4=x$
2	−1	additional point

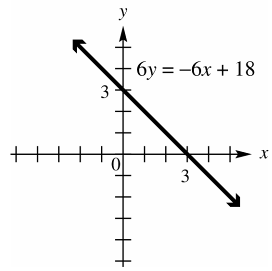
(b)



44. (a)

x	*y*	
0	3	*y*-intercept: $x=0\Rightarrow$ $6y=-6(0)+18\Rightarrow$ $6y=18\Rightarrow y=3$
3	0	*x*-intercept: $y=0\Rightarrow$ $6(0)=-6x+18\Rightarrow$ $0=-6x+18\Rightarrow$ $6x=18\Rightarrow x=3$
1	2	additional point

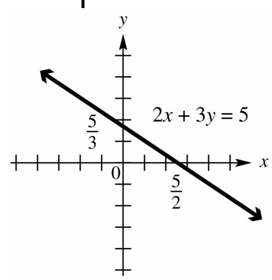
(b)



45. (a)

x	*y*	
0	$\frac{5}{3}$	*y*-intercept: $x=0\Rightarrow$ $2(0)+3y=5\Rightarrow$ $3y=5\Rightarrow y=\frac{5}{3}$
$\frac{5}{2}$	0	*x*-intercept: $y=0\Rightarrow$ $2x+3(0)=5\Rightarrow$ $2x=5\Rightarrow x=\frac{5}{2}$
4	−1	additional point

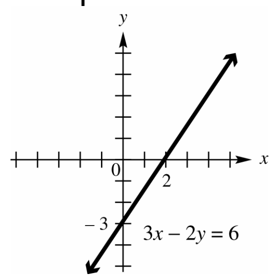
(b)



46. (a)

x	*y*	
0	−3	*y*-intercept: $x=0\Rightarrow$ $3(0)-2y=6\Rightarrow$ $-2y=6\Rightarrow y=-3$
2	0	*x*-intercept: $y=0\Rightarrow$ $3x-2(0)=6\Rightarrow$ $3x=6\Rightarrow x=2$
4	3	additional point

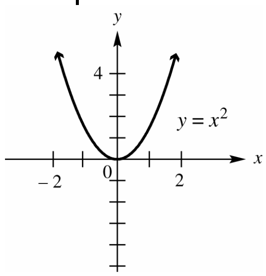
(b)



47. **(a)**

x	y	
0	0	x- and y-intercept: $0 = 0^2$
1	1	additional point
-2	4	additional point

(b)

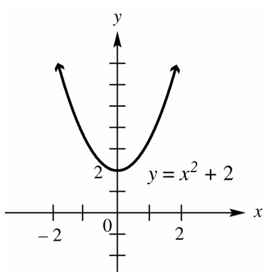


48. **(a)**

x	y	
0	2	y-intercept: $x = 0 \Rightarrow$ $y = 0^2 + 2 \Rightarrow$ $y = 0 + 2 \Rightarrow y = 2$
-1	3	additional point
2	6	additional point

no x-intercept:
$y = 0 \Rightarrow 0 = x^2 + 2 \Rightarrow$
$-2 = x^2 \Rightarrow \pm\sqrt{-2} = x$

(b)

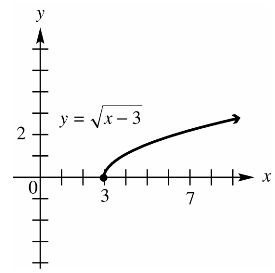


49. **(a)**

x	y	
3	0	x-intercept: $y = 0 \Rightarrow$ $0 = \sqrt{x-3} \Rightarrow$ $0 = x - 3 \Rightarrow 3 = x$
4	1	additional point
7	2	additional point

no y-intercept:
$x = 0 \Rightarrow y = \sqrt{0-3} \Rightarrow y = \sqrt{-3}$

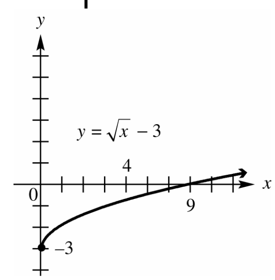
(b)



50. **(a)**

x	y	
0	-3	y-intercept: $x = 0 \Rightarrow$ $y = \sqrt{0} - 3 \Rightarrow$ $y = 0 - 3 \Rightarrow y = -3$
4	-1	additional point
9	0	x-intercept: $y = 0 \Rightarrow$ $0 = \sqrt{x} - 3 \Rightarrow$ $3 = \sqrt{x} \Rightarrow 9 = x$

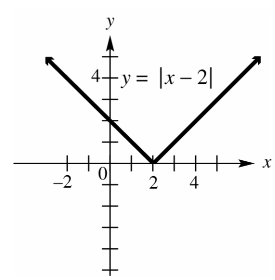
(b)



51. **(a)**

x	y	
0	2	y-intercept: $x = 0 \Rightarrow$ $y = \lvert 0 - 2 \rvert \Rightarrow$ $y = \lvert -2 \rvert \Rightarrow y = 2$
2	0	x-intercept: $y = 0 \Rightarrow$ $0 = \lvert x - 2 \rvert \Rightarrow$ $0 = x - 2 \Rightarrow 2 = x$
-2	4	additional point
4	2	additional point

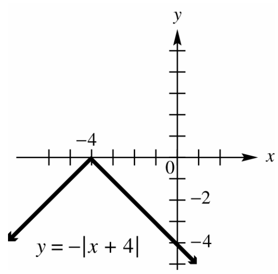
(b)



52. (a)

x	y	
−2	−2	additional point
−4	0	x-intercept: $y = 0 \Rightarrow$ $0 = -\lvert x+4\rvert \Rightarrow$ $0 = \lvert x+4\rvert \Rightarrow$ $0 = x+4 \Rightarrow -4 = x$
0	−4	y-intercept: $x = 0 \Rightarrow$ $y = -\lvert 0+4\rvert \Rightarrow$ $y = -\lvert 4\rvert \Rightarrow y = -4$

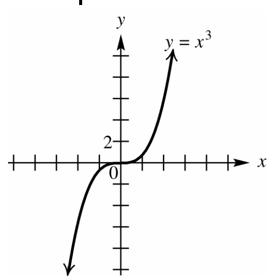
(b)



53. (a)

x	y	
0	0	x- and y-intercept: $0 = 0^3$
−1	−1	additional point
2	8	additional point

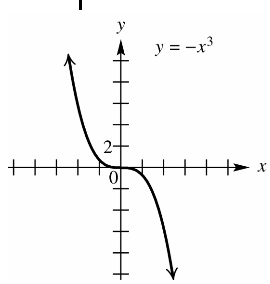
(b)



54. (a)

x	y	
0	0	x- and y-intercept: $0 = -0^3$
1	−1	additional point
2	−8	additional point

(b)

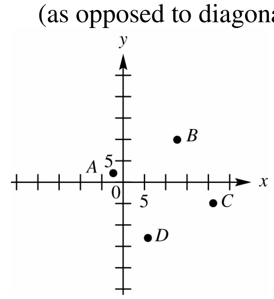


55. Points on the x-axis have y-coordinates equal to 0. The point on the x-axis will have the same x-coordinate as point (4, 3). Therefore, the line will intersect the x-axis at (4, 0).

56. Points on the y-axis have x-coordinates equal to 0. The point on the y-axis will have the same y-coordinate as point (4, 3). Therefore, the line will intersect the y-axis at (0, 3).

57. Since (a, b) is in the second quadrant, a is negative and b is positive. Therefore, $(a, -b)$ will have a negative x -coordinate and a negative y-coordinate and will lie in quadrant III. $(-a, b)$ will have a positive x-coordinate and a positive y-coordinate and will lie in quadrant I. Also, $(-a, -b)$ will have a positive x -coordinate and a negative y-coordinate and will lie in quadrant IV. Finally, (b, a) will have a positive x-coordinate and a negative y-coordinate and will lie in quadrant IV.

58. Label the points $A(-2, 2)$, $B(13, 10)$, $C(21, -5)$, and $D(6, -13)$. To determine which points form sides of the quadrilateral (as opposed to diagonals), plot the points.



Use the distance formula to find the length of each side.

$$d(A, B) = \sqrt{\left[13 - (-2)\right]^2 + (10 - 2)^2} = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

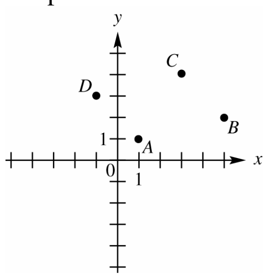
$$d(B, C) = \sqrt{(21 - 13)^2 + (-5 - 10)^2} = \sqrt{8^2 + (-15)^2} = \sqrt{64 + 225} = \sqrt{289} = 17$$

$$d(C, D) = \sqrt{(6 - 21)^2 + \left[-13 - (-5)\right]^2} = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

$$d(D,A)=\sqrt{(-2-6)^2+[2-(-13)]^2}$$
$$=\sqrt{(-8)^2+15^2}$$
$$=\sqrt{64+225}=\sqrt{289}=17$$

Since all sides have equal length, the four points form a rhombus.

59. To determine which points form sides of the quadrilateral (as opposed to diagonals), plot the points.



Use the distance formula to find the length of each side.

$$d(A,B)=\sqrt{(5-1)^2+(2-1)^2}$$
$$=\sqrt{4^2+1^2}=\sqrt{16+1}=\sqrt{17}$$
$$d(B,C)=\sqrt{(3-5)^2+(4-2)^2}$$
$$=\sqrt{(-2)^2+2^2}=\sqrt{4+4}=\sqrt{8}$$
$$d(C,D)=\sqrt{(-1-3)^2+(3-4)^2}$$
$$=\sqrt{(-4)^2+(-1)^2}$$
$$=\sqrt{16+1}=\sqrt{17}$$
$$d(D,A)=\sqrt{[1-(-1)]^2+(1-3)^2}$$
$$=\sqrt{2^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}$$

Since $d(A, B) = d(C, D)$ and $d(B, C) = d(D, A)$, the points are the vertices of a parallelogram. Since $d(A, B) \neq d(B, C)$, the points are not the vertices of a rhombus.

60. For the points $A(4, 5)$ and $D(10, 14)$, the difference of the x-coordinates is $10 - 4 = 6$ and the difference of the y-coordinates is $14 - 5 = 9$. Dividing these differences by 3, we obtain 2 and 3, respectively. Adding 2 and 3 to the x and y coordinates of point A, respectively, we obtain $B(4 + 2, 5 + 3)$ or $B(6, 8)$.
Adding 2 and 3 to the x- and y- coordinates of point B, respectively, we obtain $C(6 + 2, 8 + 3)$ or $C(8, 11)$. The desired points are $B(6, 8)$ and $C(8, 11)$.
We check these by showing that $d(A, B) = d(B, C) = d(C, D)$ and that $d(A, D) = d(A, B) + d(B, C) + d(C, D)$.

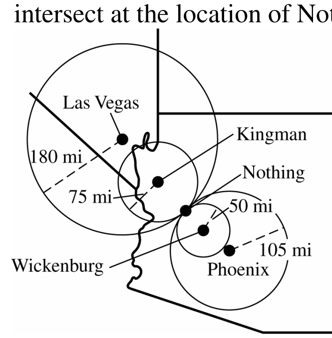
$$d(A,B)=\sqrt{(6-4)^2+(8-5)^2}$$
$$=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}$$
$$d(B,C)=\sqrt{(8-6)^2+(11-8)^2}$$
$$=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}$$
$$d(C,D)=\sqrt{(10-8)^2+(14-11)^2}$$
$$=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}$$
$$d(A,D)=\sqrt{(10-4)^2+(14-5)^2}$$
$$=\sqrt{6^2+9^2}=\sqrt{36+81}$$
$$=\sqrt{117}=\sqrt{9(13)}=3\sqrt{13}$$

$d(A, B)$, $d(B, C)$, and $d(C, D)$ all have the same measure and
$d(A, D) = d(A, B) + d(B, C) + d(C, D)$ since
$3\sqrt{13}=\sqrt{13}+\sqrt{13}+\sqrt{13}$.

Section 2.2: Circles

Connections (page 198)

Using compasses, draw circles centered at Wickenburg, Kingman, Phoenix, and Las Vegas with scaled radii of 50, 75, 105, and 180 miles respectively. The four circles should intersect at the location of Nothing.

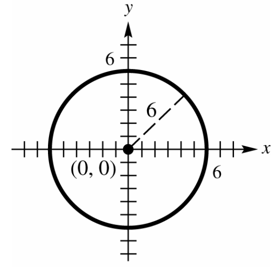


Exercises

1. **(a)** Center (0, 0), radius 6

$$\sqrt{(x-0)^2+(y-0)^2}=6$$
$$(x-0)^2+(y-0)^2=6^2$$
$$x^2+y^2=36$$

(b)

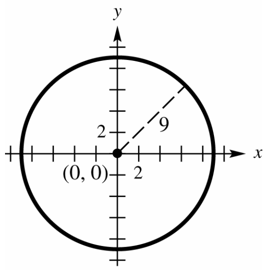


$x^2 + y^2 = 36$

2. **(a)** Center (0, 0), radius 9

$$\sqrt{(x-0)^2+(y-0)^2}=9$$
$$(x-0)^2+(y-0)^2=9^2$$
$$x^2+y^2=81$$

(b)

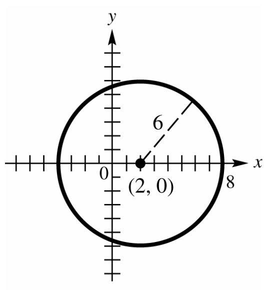


$x^2+y^2=81$

3. **(a)** Center (2, 0), radius 6

$$\sqrt{(x-2)^2+(y-0)^2}=6$$
$$(x-2)^2+(y-0)^2=6^2$$
$$(x-2)^2+y^2=36$$

(b)

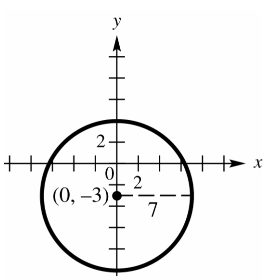


$(x-2)^2+y^2=36$

4. **(a)** Center (0, –3), radius 7

$$\sqrt{(x-0)^2+[y-(-3)]^2}=7$$
$$(x-0)^2+[y-(-3)]^2=7^2$$
$$x^2+(y+3)^2=49$$

(b)

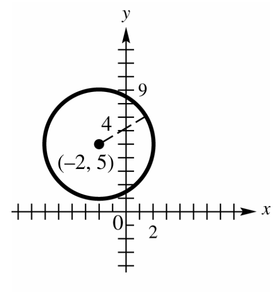


$x^2+(y+3)^2=49$

5. **(a)** Center (–2, 5), radius 4

$$\sqrt{[x-(-2)]^2+(y-5)^2}=4$$
$$[x-(-2)]^2+(y-5)^2=4^2$$
$$(x+2)^2+(y-5)^2=16$$

(b)

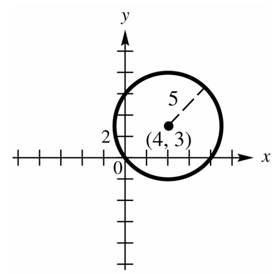


$(x+2)^2+(y-5)^2=16$

6. **(a)** Center (4, 3), radius 5

$$\sqrt{(x-4)^2+(y-3)^2}=5$$
$$(x-4)^2+(y-3)^2=5^2$$
$$(x-4)^2+(y-3)^2=25$$

(b)

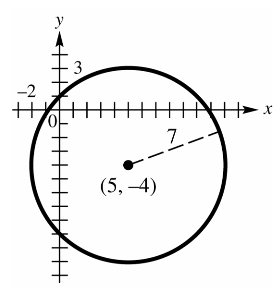


$(x-4)^2+(y-3)^2=25$

7. **(a)** Center (5, –4), radius 7

$$\sqrt{(x-5)^2+[y-(-4)]^2}=7$$
$$(x-5)^2+[y-(-4)]^2=7^2$$
$$(x-5)^2+(y+4)^2=49$$

(b)

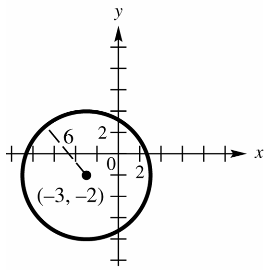


$(x-5)^2+(y+4)^2=49$

8. **(a)** Center (–3, –2), radius 6

$$\sqrt{[x-(-3)]^2+[y-(-2)]^2}=6$$
$$[x-(-3)]^2+[y-(-2)]^2=6^2$$
$$(x+3)^2+(y+2)^2=36$$

(b)

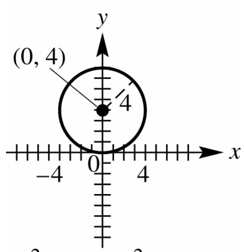


$(x+3)^2+(y+2)^2=36$

9. (a) Center (0, 4), radius 4

$$\sqrt{(x-0)^2+(y-4)^2}=4$$
$$x^2+(y-4)^2=16$$

(b)

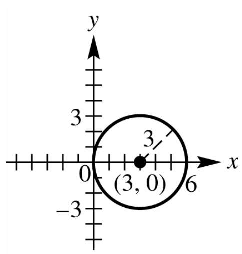


$x^2+(y-4)^2=16$

10. (a) Center (3, 0), radius 3

$$\sqrt{(x-3)^2+(y-0)^2}=3$$
$$(x-3)^2+y^2=9$$

(b)

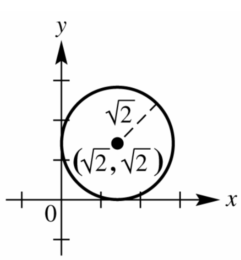


$(x-3)^2+y^2=9$

11. (a) Center $\left(\sqrt{2},\sqrt{2}\right)$, radius $\sqrt{2}$

$$\sqrt{\left(x-\sqrt{2}\right)^2+\left(y-\sqrt{2}\right)^2}=\sqrt{2}$$
$$\left(x-\sqrt{2}\right)^2+\left(y-\sqrt{2}\right)^2=2$$

(b)

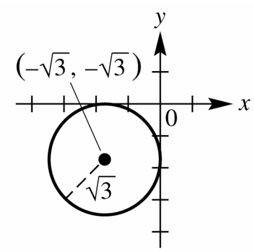


$(x-\sqrt{2})^2+(y-\sqrt{2})^2=2$

12. (a) Center $\left(-\sqrt{3},-\sqrt{3}\right)$, radius $\sqrt{3}$

$$\sqrt{\left[x-\left(-\sqrt{3}\right)\right]^2+\left[y-\left(-\sqrt{3}\right)\right]^2}=\sqrt{3}$$
$$\left[x-\left(-\sqrt{3}\right)\right]^2+\left[y-\left(-\sqrt{3}\right)\right]^2=\left(\sqrt{3}\right)^2$$
$$\left(x+\sqrt{3}\right)^2+\left(y+\sqrt{3}\right)^2=3$$

(b)



$(x+\sqrt{3})^2+(y+\sqrt{3})^2=3$

13. (a) The center of the circle is located at the midpoint of the diameter determined by the points (1, 1) and (5, 1). Using the midpoint formula, we have

$C=\left(\frac{1+5}{2},\frac{1+1}{2}\right)=(3,1)$. The radius is one-half the length of the diameter:

$$r=\frac{1}{2}\sqrt{(5-1)^2+(1-1)^2}=2$$

The equation of the circle is

$$(x-3)^2+(y-1)^2=4$$

(b) Expand $(x-3)^2+(y-1)^2=4$ to find the equation of the circle in general form:

$$(x-3)^2+(y-1)^2=4$$
$$x^2-6x+9+y^2-2y+1=4$$
$$x^2+y^2-6x-2y+6=0$$

14. (a) The center of the circle is located at the midpoint of the diameter determined by the points (−1, 1) and (−1, −5).
Using the midpoint formula, we have

$$C=\left(\frac{-1+(-1)}{2},\frac{1+(-5)}{2}\right)=(-1,-2).$$

The radius is one-half the length of the diameter:

$$r=\frac{1}{2}\sqrt{\left[-1-(-1)\right]^2+(-5-1)^2}=3$$

The equation of the circle is

$$(x+1)^2+(y+2)^2=9$$

(b) Expand $(x+1)^2+(y+2)^2=9$ to find the equation of the circle in general form:

$$\begin{aligned}(x+1)^2+(y+2)^2&=9\\ x^2+2x+1+y^2+4y+4&=9\\ x^2+y^2+2x+4y-4&=0\end{aligned}$$

15. **(a)** The center of the circle is located at the midpoint of the diameter determined by the points (–2, 4) and (–2, 0). Using the midpoint formula, we have

$$C=\left(\frac{-2+(-2)}{2},\frac{4+0}{2}\right)=(-2,2).$$

The radius is one-half the length of the diameter:

$$r=\frac{1}{2}\sqrt{\left[-2-(-2)\right]^2+(4-0)^2}=2$$

The equation of the circle is

$$(x+2)^2+(y-2)^2=4$$

(b) Expand $(x+2)^2+(y-2)^2=4$ to find the equation of the circle in general form:

$$\begin{aligned}(x+2)^2+(y-2)^2&=4\\ x^2+4x+4+y^2-4y+4&=4\\ x^2+y^2+4x-4y+4&=0\end{aligned}$$

16. **(a)** The center of the circle is located at the midpoint of the diameter determined by the points (0, –3) and (6, –3). Using the midpoint formula, we have

$$C=\left(\frac{0+6}{2},\frac{-3+(-3)}{2}\right)=(3,-3).$$

The radius is one-half the length of the diameter:

$$r=\frac{1}{2}\sqrt{(6-0)^2+\left[-3-(-3)\right]^2}=3$$

The equation of the circle is

$$(x-3)^2+(y+3)^2=9$$

(b) Expand $(x-3)^2+(y+3)^2=9$ to find the equation of the circle in general form:

$$\begin{aligned}(x-3)^2+(y+3)^2&=9\\ x^2-6x+9+y^2+6y+9&=9\\ x^2+y^2-6x+6y+9&=0\end{aligned}$$

17. Since the center (–3, 5) is in quadrant II, choice B is the correct graph.

18. Answers will vary. If $m>0$, the graph is a circle. If $m=0$, the graph is a point. If $m<0$, the graph does not exist.

19. $x^2+y^2+6x+8y+9=0$

Complete the square on x and y separately.

$$\begin{aligned}\left(x^2+6x\right)+\left(y^2+8y\right)&=-9\\ \left(x^2+6x+9\right)+\left(y^2+8y+16\right)&=-9+9+16\\ (x+3)^2+(y+4)^2&=16\end{aligned}$$

Yes, it is a circle. The circle has its center at (–3, –4) and radius 4.

20. $x^2+y^2+8x-6y+16=0$

Complete the square on x and y separately.

$$\begin{aligned}\left(x^2+8x\right)+\left(y^2-6y\right)&=-16\\ \left(x^2+8x+16\right)+\left(y^2-6y+9\right)&=-16+16+9\\ (x+4)^2+(y-3)^2&=9\end{aligned}$$

Yes, it is a circle. The circle has its center at (–4, 3) and radius 3.

21. $x^2+y^2-4x+12y=-4$

Complete the square on x and y separately.

$$\begin{aligned}x^2-4x+y^2+12y&=-4\\ \left(x^2-4x\right)+\left(y^2+12y\right)&=-4\\ \left(x^2-4x+4\right)+\left(y^2+12y+36\right)&=-4+4+36\\ (x-2)^2+(y+6)^2&=36\end{aligned}$$

Yes, it is a circle. The circle has its center at (2, –6) and radius 6.

22. $x^2+y^2-12x+10y=-25$

Complete the square on x and y separately.

$$\begin{aligned}\left(x^2-12x\right)+\left(y^2+10y\right)&=-25\\ \left(x^2-12x+36\right)+\left(y^2+10y+25\right)&=\\ &-25+36+25\\ (x-6)^2+(y+5)^2&=36\end{aligned}$$

Yes, it is a circle. The circle has its center at (6, –5) and radius 6.

23. $4x^2+4y^2+4x-16y-19=0$

Complete the square on x and y separately.

$$\begin{aligned}4\left(x^2+x\right)+4\left(y^2-4y\right)&=19\\ 4\left(x^2+x+\tfrac{1}{4}\right)+4\left(y^2-4y+4\right)&=\\ &19+4\left(\tfrac{1}{4}\right)+4(4)\\ 4\left(x+\tfrac{1}{2}\right)^2+4(y-2)^2&=36\\ \left(x+\tfrac{1}{2}\right)^2+(y-2)^2&=9\end{aligned}$$

Yes, it is a circle with center $\left(-\frac{1}{2},2\right)$ and radius 3.

24. $9x^2+9y^2+12x-18y-23=0$

Complete the square on x and y separately.

$$9\left(x^2+\tfrac{4}{3}x\right)+9\left(y^2-2y\right)=23$$
$$9\left(x^2+\tfrac{4}{3}x+\tfrac{4}{9}\right)+9\left(y^2-2y+1\right)=23+9\left(\tfrac{4}{9}\right)+9(1)$$
$$9\left(x+\tfrac{2}{3}\right)^2+9(y-1)^2=36$$
$$\left(x+\tfrac{2}{3}\right)^2+(y-1)^2=4$$

Yes, it is a circle with center $\left(-\frac{2}{3},1\right)$ and radius 2.

25. $x^2+y^2+2x-6y+14=0$

Complete the square on x and y separately.

$$\left(x^2+2x\right)+\left(y^2-6y\right)=-14$$
$$\left(x^2+2x+1\right)+\left(y^2-6y+9\right)=-14+1+9$$
$$(x+1)^2+(y-3)^2=-4$$

The graph is nonexistent.

26. $x^2+y^2+4x-8y+32=0$

Complete the square on x and y separately.

$$\left(x^2+4x\right)+\left(y^2-8y\right)=-32$$
$$\left(x^2+4x+4\right)+\left(y^2-8y+16\right)=-32+4+16$$
$$(x+2)^2+(y-4)^2=-12$$

The graph is nonexistent.

27. $x^2+y^2-6x-6y+18=0$

Complete the square on x and y separately.

$$\left(x^2-6x\right)+\left(y^2-6y\right)=-18$$
$$\left(x^2-6x+9\right)+\left(y^2-6y+9\right)=-18+9+9$$
$$(x-3)^2+(y-3)^2=0$$

The graph is the point (3, 3).

28. $x^2+y^2+4x+4y+8=0$

Complete the square on x and y separately.

$$\left(x^2+4x\right)+\left(y^2+4y\right)=-8$$
$$\left(x^2+4x+4\right)+\left(y^2+4y+4\right)=-8+4+4$$
$$(x+2)^2+(y+2)^2=0$$

The graph is the point (−2, −2).

29. $9x^2+9y^2+36x=-32$

Complete the square on x and y separately.

$$9\left(x^2+4x\right)+9y^2=-32$$
$$9\left(x^2+4x+4\right)+9(y-0)^2=-32+9(4)$$
$$9(x+2)^2+9(y-0)^2=4$$
$$(x+2)^2+(y-0)^2=\tfrac{4}{9}=\left(\tfrac{2}{3}\right)^2$$

Yes, it is a circle with center $(-2,0)$ and radius $\frac{2}{3}$.

30. $4x^2+4y^2+4x-4y-7=0$

Complete the square on x and y separately.

$$4\left(x^2+x\right)+4\left(y^2-y\right)=7$$
$$4\left(x^2+x+\tfrac{1}{4}\right)+4\left(y^2-y+\tfrac{1}{4}\right)=7+4\left(\tfrac{1}{4}\right)+4\left(\tfrac{1}{4}\right)$$
$$4\left(x+\tfrac{1}{2}\right)^2+4\left(y-\tfrac{1}{2}\right)^2=9$$
$$\left(x+\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{9}{4}$$

Yes, it is a circle with center $\left(-\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{3}{2}$.

31. The midpoint M has coordinates

$$\left(\frac{-1+5}{2},\frac{3+(-9)}{2}\right)=\left(\frac{4}{2},\frac{-6}{2}\right)=(2,-3).$$

32. Use points $C(2, -3)$ and $P(-1, 3)$.

$$d(C,P)=\sqrt{(-1-2)^2+\left[3-(-3)\right]^2}=\sqrt{(-3)^2+6^2}=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}$$

The radius is $3\sqrt{5}$.

33. Use points $C(2, -3)$ and $Q(5, -9)$.

$$d(C,Q)=\sqrt{(5-2)^2+\left[-9-(-3)\right]^2}=\sqrt{3^2+(-6)^2}=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}$$

The radius is $3\sqrt{5}$.

34. Use the points $P(-1, 3)$ and $Q(5, -9)$.

Since $d(P, Q) = \sqrt{\left[5-(-1)\right]^2 + (-9-3)^2}$

$= \sqrt{6^2 + (-12)^2} = \sqrt{36+144} = \sqrt{180}$

$= 6\sqrt{5}$,the radius is $\frac{1}{2}d(P, Q)$. Thus

$r = \frac{1}{2}\left(6\sqrt{5}\right) = 3\sqrt{5}.$

35. The center-radius form for this circle is

$(x-2)^2 + (y+3)^2 = (3\sqrt{5})^2 \Rightarrow$

$(x-2)^2 + (y+3)^2 = 45.$

36. Label the endpoints of the diameter $P(3, -5)$ and $Q(-7, 3)$. The midpoint M of the segment joining P and Q has coordinates

$\left(\frac{3+(-7)}{2}, \frac{-5+3}{2}\right) = \left(\frac{-4}{2}, \frac{-2}{2}\right) = (-2, -1).$

The center is $C(-2, -1)$. To find the radius, we can use points $C(-2, -1)$.and $P(3, -5)$

$d(C, P) = \sqrt{\left[3-(-2)\right]^2 + \left[-5-(-1)\right]^2}$ We

$= \sqrt{5^2 + (-4)^2} = \sqrt{25+16} = \sqrt{41}$

could also use points $C(-2, -1)$.and $Q(-7, 3)$.

$d(C, Q) = \sqrt{\left[-7-(-2)\right]^2 + \left[3-(-1)\right]^2}$

$= \sqrt{(-5)^2 + 4^2} = \sqrt{25+16} = \sqrt{41}$

We could also use points $P(3, -5)$ and $Q(-7, 3)$.

$d(C, Q) = \sqrt{(-7-3)^2 + \left[3-(-5)\right]^2}$

$= \sqrt{(-10)^2 + 8^2} = \sqrt{100+64}$

$= \sqrt{164} = 2\sqrt{41}$

$\frac{1}{2}d(P, Q) = \frac{1}{2}\left(2\sqrt{41}\right) = \sqrt{41}$

The center-radius form of the equation of the circle is

$[x-(-2)]^2 + [y-(-1)]^2 = (\sqrt{41})^2$

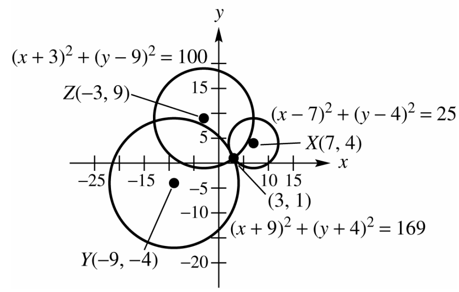
$(x+2)^2 + (y+1)^2 = 41$

37. The equations of the three circles are

$(x-7)^2 + (y-4)^2 = 25$,

$(x+9)^2 + (y+4)^2 = 169$, and

$(x+3)^2 + (y-9)^2 = 100$. From the graph of the three circles, it appears that the epicenter is located at (3, 1).



Check algebraically:

$(x-7)^2 + (y-4)^2 = 25$

$(3-7)^2 + (1-4)^2 = 25$

$4^2 + 3^2 = 25 \Rightarrow 25 = 25$

$(x+9)^2 + (y+4)^2 = 169$

$(3+9)^2 + (1+4)^2 = 169$

$12^2 + 5^2 = 169 \Rightarrow 169 = 169$

$(x+3)^2 + (y-9)^2 = 100$

$(3+3)^2 + (1-9)^2 = 100$

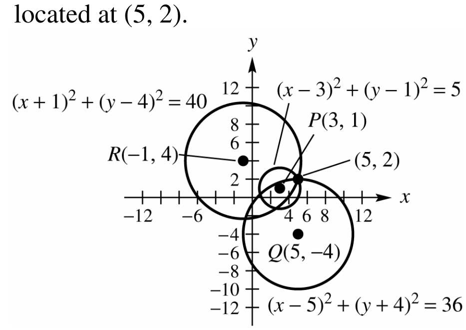
$6^2 + (-8)^2 = 100 \Rightarrow 100 = 100$

(3, 1) satisfies all three equations, so the epicenter is at (3, 1).

38. The three equations are $(x-3)^2 + (y-1)^2 = 5$,

$(x-5)^2 + (y+4)^2 = 36$, and

$(x+1)^2 + (y-4)^2 = 40$. From the graph of the three circles, it appears that the epicenter is located at (5, 2).



Check algebraically:

$(x-3)^2 + (y-1)^2 = 5$

$(5-3)^2 + (2-1)^2 = 5$

$2^2 + 1^2 = 5 \Rightarrow 5 = 5$

$(x-5)^2 + (y+4)^2 = 36$

$(5-5)^2 + (2+4)^2 = 36$

$6^2 = 36 \Rightarrow 36 = 36$

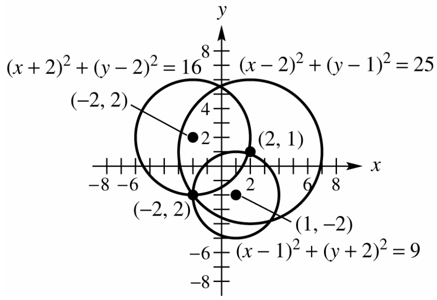
$(x+1)^2 + (y-4)^2 = 40$

$(5+1)^2 + (2-4)^2 = 40$

$6^2 + (-2)^2 = 40 \Rightarrow 40 = 40$

(5, 2) satisfies all three equations, so the epicenter is at (5, 2).

39. From the graph of the three circles, it appears that the epicenter is located at (−2, −2).

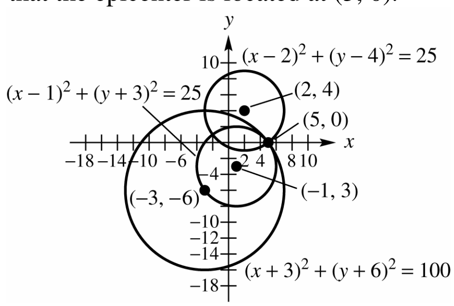


Check algebraically:

$$(x-2)^2+(y-1)^2=25$$
$$(-2-2)^2+(-2-1)^2=25$$
$$(-4)^2+(-3)^2=25$$
$$25=25$$
$$(x+2)^2+(y-2)^2=16$$
$$(-2+2)^2+(-2-2)^2=16$$
$$0^2+(-4)^2=16$$
$$16=16$$
$$(x-1)^2+(y-2)^2=9$$
$$(-2-1)^2+(-2-2)^2=9$$
$$(-3)^2+0^2=9$$
$$9=9$$

(−2, −2) satisfies all three equations, so the epicenter is at (−2, −2).

40. From the graph of the three circles, it appears that the epicenter is located at (5, 0).



Check algebraically:

$$(x-2)^2+(y-4)^2=25$$
$$(5-2)^2+(0-4)^2=25$$
$$3^2+(-4)^2=25$$
$$25=25$$
$$(x-1)^2+(y+3)^2=25$$
$$(5-1)^2+(0+3)^2=25$$
$$4^2+3^2=25$$
$$25=25$$
$$(x+3)^2+(y+6)^2=100$$
$$(5+3)^2+(0+6)^2=100$$
$$8^2+6^2=100$$
$$100=100$$

(5, 0) satisfies all three equations, so the epicenter is at (5, 0).

41. The radius of this circle is the distance from the center $C(3, 2)$ to the x-axis. This distance is 2, so $r = 2$.

$$(x-3)^2+(y-2)^2=2^2 \Rightarrow$$
$$(x-3)^2+(y-2)^2=4$$

42. The radius is the distance from the center $C(-4, 3)$ to the point $P(5, 8)$.

$$r=\sqrt{[5-(-4)]^2+(8-3)^2}$$
$$=\sqrt{9^2+5^2}=\sqrt{106}$$

The equation of the circle is

$$[x-(-4)]^2+(y-3)^2=(\sqrt{106})^2 \Rightarrow$$
$$(x+4)^2+(y-3)^3=106$$

43. Label the points $P(x, y)$ and $Q(1, 3)$.

If $d(P,Q)=4$, $\sqrt{(1-x)^2+(3-y)^2}=4 \Rightarrow$

$(1-x)^2+(3-y)^2=16.$

If $x = y$, then we can either substitute x for y or y for x. Substituting x for y we solve the following:

$$(1-x)^2+(3-x)^2=16$$
$$1-2x+x^2+9-6x+x^2=16$$
$$2x^2-8x+10=16$$
$$2x^2-8x-6=0$$
$$x^2-4x-3=0$$

To solve this equation, we can use the quadratic formula with $a = 1$, $b = -4$, and $c = -3$.

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)}$$
$$=\frac{4\pm\sqrt{16+12}}{2}=\frac{4\pm\sqrt{28}}{2}$$
$$=\frac{4\pm2\sqrt{7}}{2}=2\pm\sqrt{7}$$

Since $x = y$, the points are $\left(2+\sqrt{7}, 2+\sqrt{7}\right)$ and $\left(2-\sqrt{7}, 2-\sqrt{7}\right)$.

44. Let $P(-2, 3)$ be a point which is 8 units from $Q(x, y)$. We have

$$d(P, Q) = \sqrt{(-2-x)^2 + (3-y)^2} = 8 \Rightarrow$$

$$(-2-x)^2 + (3-y)^2 = 64.$$

Since $x + y = 0$, $x = -y$. We can either substitute $-x$ for y or $-y$ for x. Substituting $-x$ for y we solve the following:

$$(-2-x)^2 + [3-(-x)]^2 = 64$$
$$(-2-x)^2 + (3+x)^2 = 64$$
$$4 + 4x + x^2 + 9 + 6x + x^2 = 64$$
$$2x^2 + 10x + 13 = 64$$
$$2x^2 + 10x - 51 = 0$$

To solve this equation, use the quadratic formula with $a = 2$, $b = 10$, and $c = -51$.

$$x = \frac{-10 \pm \sqrt{10^2 - 4(2)(-51)}}{2(2)} = \frac{-10 \pm \sqrt{100+408}}{4} = \frac{-10 \pm \sqrt{508}}{4} = \frac{-10 \pm \sqrt{4(127)}}{4} = \frac{-10 \pm 2\sqrt{127}}{4} = \frac{-5 \pm \sqrt{127}}{2}$$

Since $y = -x$ the points are

$$\left(\frac{-5-\sqrt{127}}{2}, \frac{5+\sqrt{127}}{2}\right) \text{ and } \left(\frac{-5+\sqrt{127}}{2}, \frac{5-\sqrt{127}}{2}\right).$$

45. Let $P(x, y)$ be a point whose distance from $A(1, 0)$ is $\sqrt{10}$ and whose distance from $B(5, 4)$ is $\sqrt{10}$. $d(P, A) = \sqrt{10}$, so

$$\sqrt{(1-x)^2 + (0-y)^2} = \sqrt{10} \Rightarrow$$

$(1-x)^2 + y^2 = 10.$ $d(P, B) = \sqrt{10}$, so

$$\sqrt{(5-x)^2 + (4-y)^2} = \sqrt{10} \Rightarrow$$

$(5-x)^2 + (4-y)^2 = 10.$ Thus,

$$(1-x)^2 + y^2 = (5-x)^2 + (4-y)^2$$
$$1 - 2x + x^2 + y^2 = 25 - 10x + x^2 + 16 - 8y + y^2$$
$$1 - 2x = 41 - 10x - 8y$$
$$8y = 40 - 8x$$
$$y = 5 - x$$

Substitute $5 - x$ for y in the equation $(1-x)^2 + y^2 = 10$ and solve for x.

$$(1-x)^2 + (5-x)^2 = 10 \Rightarrow$$
$$1 - 2x + x^2 + 25 - 10x + x^2 = 10$$
$$2x^2 - 12x + 26 = 10 \Rightarrow 2x^2 - 12x + 16 = 0$$
$$x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0 \Rightarrow$$
$$x - 2 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 2 \quad \text{or} \quad x = 4$$

To find the corresponding values of y use the equation $y = 5 - x$. If $x = 2$, then $y = 5 - 2 = 3$. If $x = 4$, then $y = 5 - 4 = 1$. The points satisfying the conditions are $(2, 3)$ and $(4, 1)$.

46. The circle of smallest radius that contains the points $A(1, 4)$ and $B(-3, 2)$ within or on its boundary will be the circle having points A and B as endpoints of a diameter. The center will be M, the midpoint:

$$\left(\frac{1+(-3)}{2}, \frac{4+2}{2}\right) = \left(\frac{-2}{2}, \frac{6}{2}\right) = (-1, 3).$$

The radius will be the distance from M to either A or B:

$$d(M, A) = \sqrt{[1-(-1)]^2 + (4-3)^2} = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$$

The equation of the circle is

$$[x-(-1)]^2 + (y-3)^2 = \left(\sqrt{5}\right)^2 \Rightarrow$$
$$(x+1)^2 + (y-3)^2 = 5.$$

47. Label the points $A(3, y)$ and $B(-2, 9)$. If $d(A, B) = 12$, then

$$\sqrt{(-2-3)^2 + (9-y)^2} = 12$$
$$\sqrt{(-5)^2 + (9-y)^2} = 12$$
$$(-5)^2 + (9-y)^2 = 12^2$$
$$25 + 81 - 18y + y^2 = 144$$
$$y^2 - 18y - 38 = 0$$

Solve this equation by using the quadratic formula with $a = 1$, $b = -18$, and $c = -38$:

$$y = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(1)(-38)}}{2(1)} = \frac{18 \pm \sqrt{324+152}}{2(1)} = \frac{18 \pm \sqrt{476}}{2} = \frac{18 \pm \sqrt{4(119)}}{2} = \frac{18 \pm 2\sqrt{119}}{2} = 9 \pm \sqrt{119}$$

The values of y are $9 + \sqrt{119}$ and $9 - \sqrt{119}$.

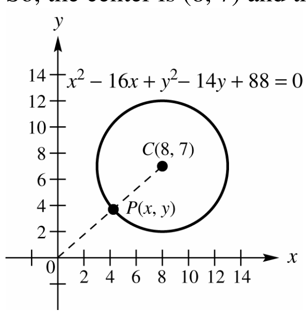
48. Since the center is in the third quadrant, the radius is $\sqrt{2}$, and the circle is tangent to both axes, the center must be at $(-\sqrt{2}, -\sqrt{2})$. Using the center-radius of the equation of a circle, we have

$$\left[x-\left(-\sqrt{2}\right)\right]^2+\left[y-\left(-\sqrt{2}\right)\right]^2=\left(\sqrt{2}\right)^2 \Rightarrow$$
$$\left(x+\sqrt{2}\right)^2+\left(y+\sqrt{2}\right)^2=2.$$

49. Let $P(x, y)$ be the point on the circle whose distance from the origin is the shortest. Complete the square on x and y separately to write the equation in center-radius form:

$$x^2-16x+y^2-14y+88=0$$
$$x^2-16x+64+y^2-14y+49=-88+64+49$$
$$(x-8)^2+(y-7)^2=25$$

So, the center is (8, 7) and the radius is 5.



$d(P, O)=\sqrt{8^2+7^2}=\sqrt{113}$. Since the length of the radius is 5, $d(P, O)=\sqrt{113}-5$.

50. The equation of the circle centered at (3, 0) with radius 2 is $(x-3)^2+y^2=4$. Let $y = 1$ and solve for x:

$$(x-3)^2+1^2=4 \Rightarrow (x-3)^2=3 \Rightarrow$$
$$x-3=\pm\sqrt{3} \Rightarrow x=3+\sqrt{3} \text{ or } x=3-\sqrt{3}$$

So the coordinates of the points of intersection are $\left(3+\sqrt{3}, 1\right)$ and $\left(3-\sqrt{3}, 1\right)$.

Section 2.3: Functions

1. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{5, 3, 4, 7} x-values

{1, 2, 9, 8} y-values

2. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{8, 5, 9, 3} x-values

{0, 7, 3, 8} y-values

3. Two ordered pairs, namely $(2,4)$ and $(2,6)$, have the same x-value paired with different y-values, so the relation is not a function.

4. Two ordered pairs, namely $(9,-2)$ and $(9,1)$, have the same x-value paired with different y-values, so the relation is not a function.

5. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{−3, 4, −2} x-values

{1, 7} y-values

6. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{−12, −10, 8} x-values

{5, 3} y-values

7. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{3, 7, 10} x-values

{−4} y-values

8. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{−4, 0, 4} x-values

{$\sqrt{2}$} y-values

9. Two sets of ordered pairs, namely $(1,1)$ and $(1,-1)$ as well as $(2,4)$ and $(2,-4)$, have the same x-value paired with different y-values, so the relation is not a function.
domain: $\{0,1,2\}$; range: $\{-4,-1,0,1,4\}$

10. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{2, 3, 4, 5} x-values
↓ ↓ ↓ ↓
{5, 7, 9, 11} y-values

domain: $\{2,3,4,5\}$; range: $\{5,7,9,11\}$

11. The relation is a function because for each different x-value there is exactly one y-value.
domain: $\{2,3,5,11,17\}$; range: $\{1,7,20\}$

12. Two ordered pairs, namely $(2,15)$ and $(2,19)$, have the same x-value paired with different y-values, so the relation is not a function. domain: $\{1,2,3,5\}$; range: $\{10,15,19,27\}$

13. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{0, −1, −2} x-values
↓ ↓ ↓
{0, 1, 2} y-values

Domain: {0, −1, −2}; range: {0, 1, 2}

14. The relation is a function because for each different x-value there is exactly one y-value. This correspondence can be shown as follows.

{0, 1, 2} x-values
↓ ↓ ↓
{0, −1, −2} y-values

Domain: {0, 1, 2}; range: {0, −1, −2}

15. The relation is a function because for each different year, there is exactly one number for visitors to the Grand Canyon.
domain: {2001, 2002, 2003, 2004}
range: {4,400,823, 4,339,139, 4,464,400, 4,672,911}

16. The relation is a function because for each basketball season, there is only one number for attendance.
domain: {2002, 2003, 2004, 2005}
range: {10,163,629, 10,016,106, 9,940,466, 9,902,850}

17. This graph represents a function. If you pass a vertical line through the graph, one x-value corresponds to only one y-value.
domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$

18. This graph represents a function. If you pass a vertical line through the graph, one x-value corresponds to only one y-value.
domain: $(-\infty,\infty)$; range: $(-\infty,4]$

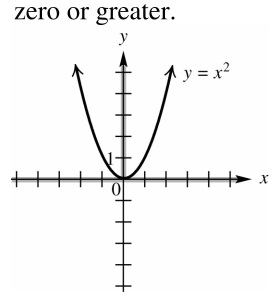
19. This graph does not represent a function. If you pass a vertical line through the graph, there are places where one value of x corresponds to two values of y.
domain: $[3,\infty)$; range: $(-\infty,\infty)$

20. This graph represents a function. If you pass a vertical line through the graph, one x-value corresponds to only one y-value.
domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$

21. This graph does not represent a function. If you pass a vertical line through the graph, there are places where one value of x corresponds to two values of y.
domain: $[-4,4]$; range: $[-3,3]$

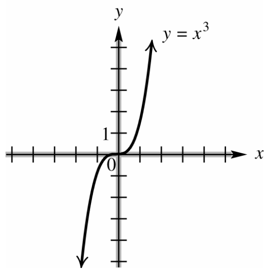
22. This graph represents a function. If you pass a vertical line through the graph, one x-value corresponds to only one y-value.
domain: $[-2,2]$; range: $[0,4]$

23. $y = x^2$ represents a function since y is always found by squaring x. Thus, each value of x corresponds to just one value of y. x can be any real number. Since the square of any real number is not negative, the range would be zero or greater.



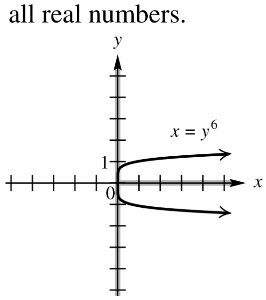
domain: $(-\infty,\infty)$; range: $[0,\infty)$

24. $y = x^3$ represents a function since y is always found by cubing x. Thus, each value of x corresponds to just one value of y. x can be any real number. Since the cube of any real number could be negative, positive, or zero, the range would be any real number.



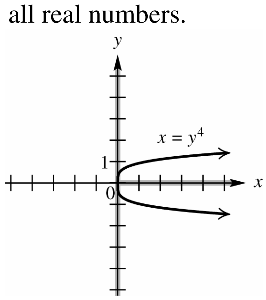
domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

25. The ordered pairs $(1,1)$ and $(1,-1)$ both satisfy $x = y^6$. This equation does not represent a function. Because x is equal to the sixth power of y, the values of x are nonnegative. Any real number can be raised to the sixth power, so the range of the relation is all real numbers.



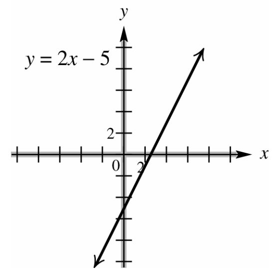
domain: $[0, \infty)$ range: $(-\infty, \infty)$

26. The ordered pairs $(1,1)$ and $(1,-1)$ both satisfy $x = y^4$. This equation does not represent a function. Because x is equal to the fourth power of y, the values of x are nonnegative. Any real number can be raised to the fourth power, so the range of the relation is all real numbers.



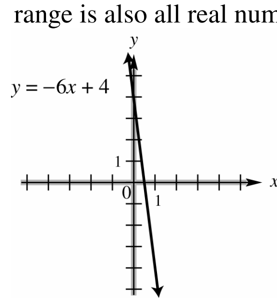
domain: $[0, \infty)$ range: $(-\infty, \infty)$

27. $y = 2x - 5$ represents a function since y is found by multiplying x by 2 and subtracting 5. Each value of x corresponds to just one value of y. x can be any real number, so the domain is all real numbers. Since y is twice x, less 5, y also may be any real number, and so the range is also all real numbers.



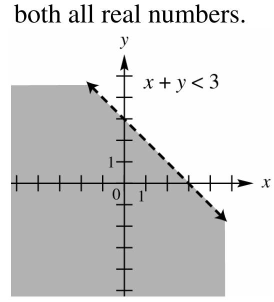
domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

28. $y = -6x + 4$ represents a function since y is found by multiplying x by -6 and adding 4. Each value of x corresponds to just one value of y. x can be any real number, so the domain is all real numbers. Since y is -6 times x, plus 4, y also may be any real number, and so the range is also all real numbers.



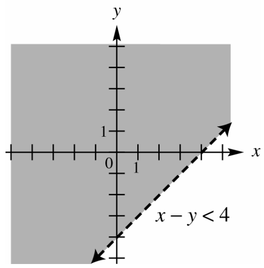
domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

29. By definition, y is a function of x if every value of x leads to exactly one value of y. Substituting a particular value of x, say 1, into $x + y < 3$, corresponds to many values of y. The ordered pairs (0, 2) (1, 1) (1, 0) (1, −1) and so on, all satisfy the inequality. Note that the points on the graphed line do not satisfy the inequality and only indicate the boundary of the solution set. This does not represent a function. Any number can be used for x or for y, so the domain and range of this relation are both all real numbers.



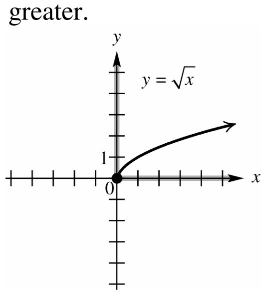
domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

30. By definition, y is a function of x if every value of x leads to exactly one value of y. Substituting a particular value of x, say 1, into $x - y < 4$ corresponds to many values of y. The ordered pairs (1, −1) (1, 0) (1, 1) (1, 2) and so on, all satisfy the inequality. Note that the points on the graphed line do not satisfy the inequality and only indicate the boundary of the solution set. This does not represent a function. Any number can be used for x or for y, so the domain and range of this relation are both all real numbers.



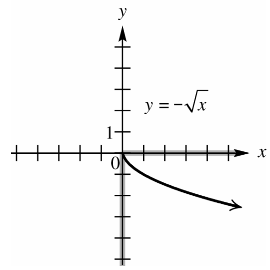
domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

31. For any choice of x in the domain of $y = \sqrt{x}$, there is exactly one corresponding value of y, so this equation defines a function. Since the quantity under the square root cannot be negative, we have $x \geq 0$. Because the radical is nonnegative, the range is also zero or greater.



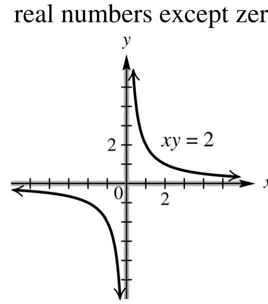
domain: $[0, \infty)$; range: $[0, \infty)$

32. For any choice of x in the domain of $y = -\sqrt{x}$, there is exactly one corresponding value of y, so this equation defines a function. Since the quantity under the square root cannot be negative, we have $x \geq 0$. The outcome of the radical is nonnegative, when you change the sign (by multiplying by −1), the range becomes nonpositive. Thus the range is zero or less.



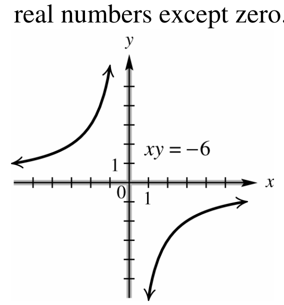
domain: $[0, \infty)$; range: $(-\infty, 0]$

33. Since $xy = 2$ can be rewritten as $y = \frac{2}{x}$, we can see that y can be found by dividing x into 2. This process produces one value of y for each value of x in the domain, so this equation is a function. The domain includes all real numbers except those that make the denominator equal to zero, namely $x = 0$. Values of y can be negative or positive, but never zero. Therefore, the range will be all real numbers except zero.



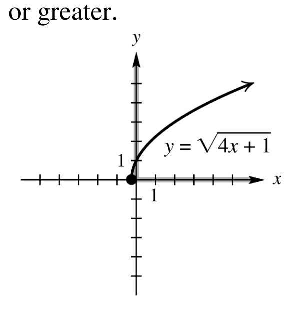
domain: $(-\infty, 0) \cup (0, \infty)$;
range: $(-\infty, 0) \cup (0, \infty)$

34. Since $xy = -6$ can be rewritten as $y = \frac{-6}{x}$, we can see that y can be found by dividing x into −6. This process produces one value of y for each value of x in the domain, so this equation is a function. The domain includes all real numbers except those that make the denominator equal to zero, namely $x = 0$. Values of y can be negative or positive, but never zero. Therefore, the range will be all real numbers except zero.



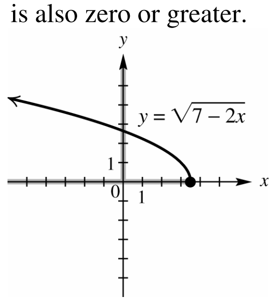
domain: $(-\infty, 0) \cup (0, \infty)$;
range: $(-\infty, 0) \cup (0, \infty)$

35. For any choice of x in the domain of $y=\sqrt{4x+1}$ there is exactly one corresponding value of y, so this equation defines a function. Since the quantity under the square root cannot be negative, we have $4x+1\geq 0 \Rightarrow 4x\geq -1 \Rightarrow x\geq -\frac{1}{4}$. Because the radical is nonnegative, the range is also zero or greater.



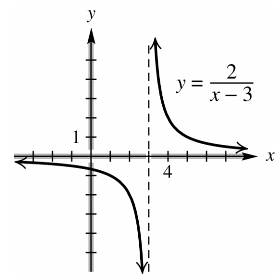
domain: $\left[-\frac{1}{4},\infty\right)$; range: $[0,\infty)$

36. For any choice of x in the domain of $y=\sqrt{7-2x}$ there is exactly one corresponding value of y, so this equation defines a function. Since the quantity under the square root cannot be negative, we have $7-2x\geq 0 \Rightarrow -2x\geq -7 \Rightarrow x\leq \frac{-7}{-2}$ or $x\leq \frac{7}{2}$. Because the radical is nonnegative, the range is also zero or greater.



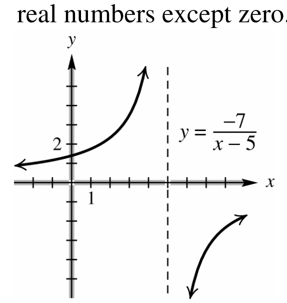
domain: $\left(-\infty,\frac{7}{2}\right]$; range: $[0,\infty)$

37. Given any value in the domain of $y=\frac{2}{x-3}$, we find y by subtracting 3, then dividing into 2. This process produces one value of y for each value of x in the domain, so this equation is a function. The domain includes all real numbers except those that make the denominator equal to zero, namely $x=3$. Values of y can be negative or positive, but never zero. Therefore, the range will be all real numbers except zero.



domain: $(-\infty,3)\cup(3,\infty)$;
range: $(-\infty,0)\cup(0,\infty)$

38. Given any value in the domain of $y=\frac{-7}{x-5}$, we find y by subtracting 5, then dividing into -7. This process produces one value of y for each value of x in the domain, so this equation is a function. The domain includes all real numbers except those that make the denominator equal to zero, namely $x=5$. Values of y can be negative or positive, but never zero. Therefore, the range will be all real numbers except zero.



domain: $(-\infty,5)\cup(5,\infty)$;
range: $(-\infty,0)\cup(0,\infty)$

39. B

40. Answers will vary. An example is: <u>The cost of gasoline</u> depends on <u>the number of gallons used</u>; so <u>cost</u> is a function of <u>number of gallons</u>.

41. $f(x)=-3x+4$
$f(0)=-3\cdot 0+4=0+4=4$

42. $f(x)=-3x+4$
$f(-3)=-3(-3)+4=9+4=13$

43. $g(x)=-x^2+4x+1$
$g(-2)=-(-2)^2+4(-2)+1$
$=-4+(-8)+1=-11$

44. $g(x)=-x^2+4x+1$
$g(10)=-10^2+4\cdot 10+1$
$=-100+40+1=-59$

45. $f(x) = -3x + 4$
$f\left(\frac{1}{3}\right) = -3\left(\frac{1}{3}\right) + 4 = -1 + 4 = 3$

46. $f(x) = -3x + 4$
$f\left(-\frac{7}{3}\right) = -3\left(-\frac{7}{3}\right) + 4 = 7 + 4 = 11$

47. $g(x) = -x^2 + 4x + 1$
$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) + 1$
$= -\frac{1}{4} + 2 + 1 = \frac{11}{4}$

48. $g(x) = -x^2 + 4x + 1$
$g\left(-\frac{1}{4}\right) = -\left(-\frac{1}{4}\right)^2 + 4\left(-\frac{1}{4}\right) + 1$
$= -\frac{1}{16} - 1 + 1 = -\frac{1}{16}$

49. $f(x) = -3x + 4$
$f(p) = -3p + 4$

50. $g(x) = -x^2 + 4x + 1$
$g(k) = -k^2 + 4k + 1$

51. $f(x) = -3x + 4$
$f(-x) = -3(-x) + 4 = 3x + 4$

52. $g(x) = -x^2 + 4x + 1$
$g(-x) = -(-x)^2 + 4(-x) + 1$
$= -x^2 - 4x + 1$

53. $f(x) = -3x + 4$
$f(x+2) = -3(x+2) + 4$
$= -3x - 6 + 4 = -3x - 2$

54. $f(x) = -3x + 4$
$f(a+4) = -3(a+4) + 4$
$= -3a - 12 + 4 = -3a - 8$

55. $f(x) = -3x + 4$
$f(2m-3) = -3(2m-3) + 4$
$= -6m + 9 + 4 = -6m + 13$

56. $f(x) = -3x + 4$
$f(3t-2) = -3(3t-2) + 4$
$= -9t + 6 + 4 = -9t + 10$

57. **(a)** $f(2) = 2$ **(b)** $f(-1) = 3$

58. **(a)** $f(2) = 5$ **(b)** $f(-1) = 11$

59. **(a)** $f(2) = 15$ **(b)** $f(-1) = 10$

60. **(a)** $f(2) = 1$ **(b)** $f(-1) = 7$

61. **(a)** $f(2) = 3$ **(b)** $f(-1) = -3$

62. **(a)** $f(2) = -3$ **(b)** $f(-1) = 2$

63. **(a)** $x + 3y = 12$
$3y = -x + 12$
$y = \frac{-x + 12}{3}$
$y = -\frac{1}{3}x + 4 \Rightarrow f(x) = -\frac{1}{3}x + 4$

(b) $f(3) = -\frac{1}{3}(3) + 4 = -1 + 4 = 3$

64. **(a)** $x - 4y = 8$
$x = 8 + 4y$
$x - 8 = 4y$
$\frac{x - 8}{4} = y$
$y = \frac{1}{4}x - 2 \Rightarrow f(x) = \frac{1}{4}x - 2$

(b) $f(3) = \frac{1}{4}(3) - 2 = \frac{3}{4} - 2 = \frac{3}{4} - \frac{8}{4} = -\frac{5}{4}$

65. **(a)** $y + 2x^2 = 3 - x$
$y = -2x^2 - x + 3$
$f(x) = -2x^2 - x + 3$

(b) $f(3) = -2(3)^2 - 3 + 3$
$= -2 \cdot 9 - 3 + 3 = -18$

66. **(a)** $y - 3x^2 = 2 + x$
$y = 3x^2 + x + 2$
$f(x) = 3x^2 + x + 2$

(b) $f(3) = 3(3)^2 + 3 + 2$
$= 3 \cdot 9 + 3 + 2 = 32$

67. **(a)** $4x - 3y = 8$
$4x = 3y + 8$
$4x - 8 = 3y$
$\frac{4x - 8}{3} = y$
$y = \frac{4}{3}x - \frac{8}{3} \Rightarrow f(x) = \frac{4}{3}x - \frac{8}{3}$

(b) $f(3) = \frac{4}{3}(3) - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$

68. **(a)** $-2x + 5y = 9$
$5y = 2x + 9$
$y = \frac{2x + 9}{5}$
$y = \frac{2}{5}x + \frac{9}{5} \Rightarrow f(x) = \frac{2}{5}x + \frac{9}{5}$

(b) $f(3) = \frac{2}{5}(3) + \frac{9}{5} = \frac{6}{5} + \frac{9}{5} = \frac{15}{5} = 3$

69. $f(3) = 4$

70. Since $f(0.2) = 0.2^2 + 3(0.2) + 1 =$ $0.04 + 0.6 + 1 = 1.64$, the height of the rectangle is 1.64 units. The base measures $0.3 - 0.2 = 0.1$ unit. Since the area of a rectangle is base times height, the area of this rectangle is $0.1(1.64) = 0.164$ square unit.

71. $f(3)$ is the y-component of the coordinate, which is -4.

72. $f(-2)$ is the y-component of the coordinate, which is -3.

73. **(a)** $f(-2) = 0$ **(b)** $f(0) = 4$
(c) $f(1) = 2$ **(d)** $f(4) = 4$

74. **(a)** $f(-2) = 5$ **(b)** $f(0) = 0$
(c) $f(1) = 2$ **(d)** $f(4) = 4$

75. **(a)** $f(-2) = -3$ **(b)** $f(0) = -2$
(c) $f(1) = 0$ **(d)** $f(4) = 2$

76. **(a)** $f(-2) = 3$ **(b)** $f(0) = 3$
(c) $f(1) = 3$ **(d)** $f(4) = 3$

77. **(a)** $[4, \infty)$ **(b)** $(-\infty, -1]$
(c) $[-1, 4]$

78. **(a)** $(-\infty, 1]$ **(b)** $[4, \infty)$
(c) $[1, 4]$

79. **(a)** $(-\infty, 4]$ **(b)** $[4, \infty)$
(c) none

80. **(a)** none **(b)** $(-\infty, \infty)$
(c) none

81. **(a)** none **(b)** $(-\infty, -2]$; $[3, \infty)$
(c) $(-2, 3)$

82. **(a)** $(3, \infty)$ **(b)** $(-\infty, -3)$
(c) $(-3, 3]$

83. **(a)** Yes, it is the graph of a function.
(b) $[0, 24]$
(c) When $t = 8$, $y = 1200$ from the graph. At 8 A.M., approximately 1200 megawatts is being used.
(d) The most electricity was used at 17 hr or 5 P.M. The least electricity was used at 4 A.M.
(e) $f(12) = 2000$; At 12 noon, electricity use is 2000 megawatts.
(f) increasing from 4 A.M. to 5 P.M.; decreasing from midnight to 4 A.M. and from 5 P.M. to midnight

84. **(a)** At $t = 2$, $y = 240$ from the graph. Therefore, at 2 seconds, the ball is 240 feet high.
(b) At $y = 192$, $x = 1$ and $x = 5$ from the graph. Therefore, the height will be 192 feet at 1 second and at 5 seconds.
(c) The ball is going up from 0 to 3 seconds and down from 3 to 7 seconds.
(d) The coordinate of the highest point is (3, 256). Therefore, it reaches a maximum height of 256 feet at 3 seconds.
(e) At $x = 7$, $y = 0$. Therefore, at 7 seconds, the ball hits the ground.

85. **(a)** At $t = 12$ and $t = 20$, $y = 55$ from the graph. Therefore, after about 12 noon until about 8 P.M. the temperature was over 55º.
(b) At $t = 5$ and $t = 22$, $y = 40$ from the graph. Therefore, until about 6 A.M. and after 10 P.M. the temperature was below 40º.
(c) The temperature at noon in Bratenahl, Ohio was 55º. Since the temperature in Greenville is 7º higher, we are looking for the time at which Bratenahl, Ohio was $55° - 7°$ or 48º. This occurred at approximately 10 A.M and 8:30 P.M.

86. **(a)** At $t = 8$, $y = 24$ from the graph. Therefore, there are 24 units of the drug in the bloodstream at 8 hours.
(b) The level increases between 0 and 2 hours after the drug is taken and decreases between 2 and 12 hours after the drug is taken.
(c) The coordinates of the highest point are (2, 64). Therefore, at 2 hours, the level of the drug in the bloodstream reaches its greatest value of 64 units.

(d) After the peak, $y = 16$ at $t = 10$.
10 hours – 2 hours = 8 hours after the peak. 8 additional hours are required for the level to drop to 16 units

Section 2.4: Linear Functions

1. B; $f(x) = 3x + 6$ is a linear function with y-intercept 6.

2. H; $x = 9$ is a vertical line.

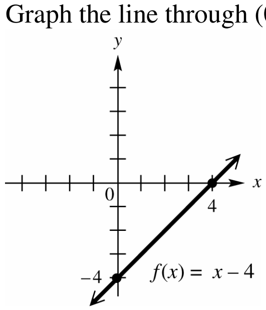
3. C; $f(x) = -8$ is a constant function.

4. G; $2x - y = -4$ or $y = 2x + 4$ is a linear equation with x-intercept –2 and y-intercept 4.

5. A; $f(x) = 5x$ is a linear function whose graph passes through the origin, (0, 0).
$f(0) = 2(0) = 0$.

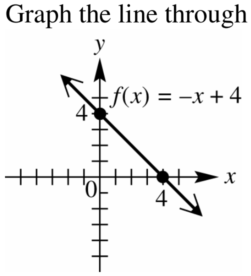
6. D; $f(x) = x^2$ is a function that is not linear.

7. $f(x) = x - 4$; Use the intercepts.
$f(0) = 0 - 4 = -4$: y-intercept
$0 = x - 4 \Rightarrow x = 4$: x-intercept
Graph the line through (0, –4) and (4, 0).



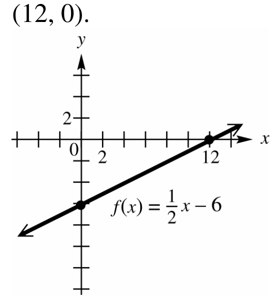
The domain and range are both $(-\infty, \infty)$.

8. $f(x) = -x + 4$; Use the intercepts.
$f(0) = -0 + 4 = 4$: y-intercept
$0 = -x + 4 \Rightarrow x = 4$: x-intercept
Graph the line through (0, 4) and (4, 0).



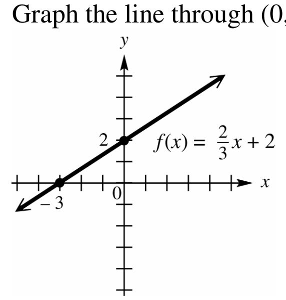
The domain and range are both $(-\infty, \infty)$.

9. $f(x) = \frac{1}{2}x - 6$; Use the intercepts.
$f(0) = \frac{1}{2}(0) - 6 = -6$: y-intercept
$0 = \frac{1}{2}x - 6 \Rightarrow 6 = \frac{1}{2}x \Rightarrow x = 12$: x-intercept
Graph the line through (0, –6) and (12, 0).



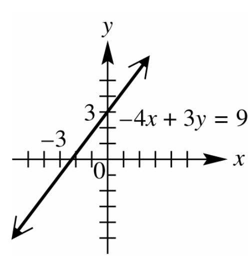
The domain and range are both $(-\infty, \infty)$.

10. $f(x) = \frac{2}{3}x + 2$; Use the intercepts.
$f(0) = \frac{2}{3}(0) + 2 = 2$: y-intercept
$0 = \frac{2}{3}x + 2 \Rightarrow -2 = \frac{2}{3}x \Rightarrow x = -3$: x-intercept
Graph the line through (0, 2) and (–3, 0).



The domain and range are both $(-\infty, \infty)$.

11. $-4x + 3y = 9$; Use the intercepts.
$-4(0) + 3y = 9 \Rightarrow 3y = 9 \Rightarrow$
$y = 3$: y-intercept
$-4x + 3(0) = 9 \Rightarrow -4x = 9 \Rightarrow$
$x = -\frac{9}{4}$: x-intercept
Graph the line through $(0, 3)$ and $\left(-\frac{9}{4}, 0\right)$.



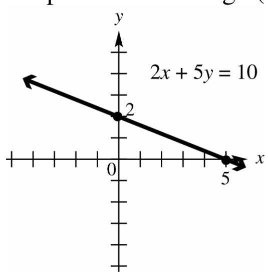
The domain and range are both $(-\infty, \infty)$.

12. $2x+5y=10$; Use the intercepts.

$2(0)+5y=10 \Rightarrow 5y=10 \Rightarrow$
$y=2$: y-intercept

$2x+5(0)=10 \Rightarrow 2x=10 \Rightarrow$
$x=5$: x-intercept

Graph the line through (0, 2) and (5, 0):



The domain and range are both $(-\infty,\infty)$.

13. $3y-4x=0$; Use the intercepts.

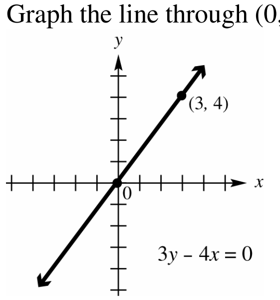
$3y-4(0)=0 \Rightarrow 3y=0 \Rightarrow y=0$: y-intercept

$3(0)-4x=0 \Rightarrow -4x=0 \Rightarrow x=0$: x-intercept

The graph has just one intercept. Choose an additional value, say 3, for x.

$3y-4(3)=0 \Rightarrow 3y-12=0$
$3y=12 \Rightarrow y=4$

Graph the line through (0, 0) and (3, 4):



The domain and range are both $(-\infty,\infty)$.

14. $3x+2y=0$; Use the intercepts.

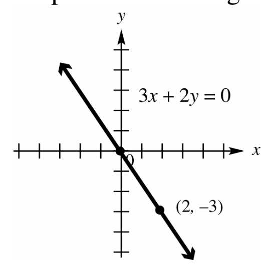
$3(0)+2y=0 \Rightarrow 2y=0 \Rightarrow y=0$: y-intercept

$3x+2(0)=0 \Rightarrow 3x=0 \Rightarrow x=0$: x-intercept

The graph has just one intercept. Choose an additional value, say 2, for x.

$3(2)+2y=0 \Rightarrow 6+2y=0 \Rightarrow$
$2y=-6 \Rightarrow y=-3$

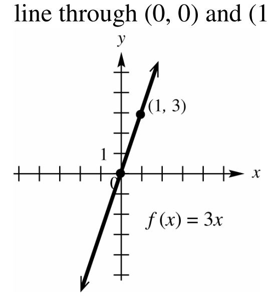
Graph the line through (0, 0) and (2, −3):



The domain and range are both $(-\infty,\infty)$.

15. $f(x)=3x$

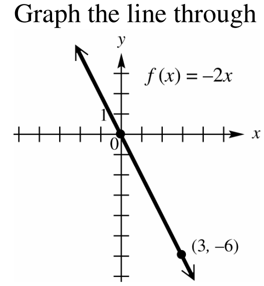
The x-intercept and the y-intercept are both zero. This gives us only one point, (0, 0). If $x=1$, $y=3(1)=3$. Another point is (1, 3). Graph the line through (0, 0) and (1, 3).



The domain and range are both $(-\infty,\infty)$.

16. $y=-2x$

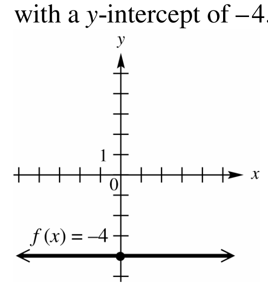
The x-intercept and the y-intercept are both zero. This gives us only one point, (0, 0). If $x=3$, $y=-2(3)=-6$, so another point is (3, −6). Graph the line through (0, 0) and (3, −6).



The domain and range are both $(-\infty,\infty)$.

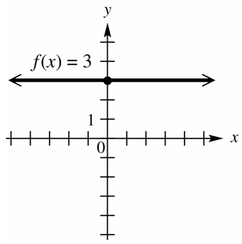
17. $f(x)=-4$ is a constant function.

The graph of $f(x)=-4$ is a horizontal line with a y-intercept of −4.



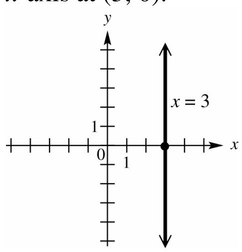
domain: $(-\infty,\infty)$; range: $\{-4\}$

18. $f(x) = 3$ is a constant function. The graph of $f(x) = 3$ is a horizontal line with y-intercept of 3.



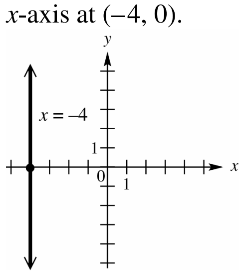
domain: $(-\infty, \infty)$; range: $\{3\}$

19. $x = 3$ is a vertical line, intersecting the x-axis at (3, 0).



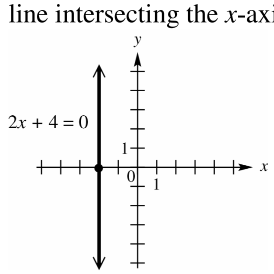
domain: $\{3\}$; range: $(-\infty, \infty)$

20. $x = -4$ is a vertical line intersecting the x-axis at (–4, 0).



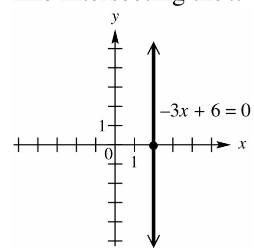
domain: $\{-4\}$; range: $(-\infty, \infty)$

21. $2x + 4 = 0 \Rightarrow 2x = -4 \Rightarrow x = -2$ is a vertical line intersecting the x-axis at (–2, 0).



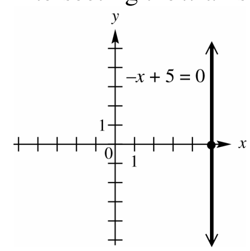
domain: $\{-2\}$; range: $(-\infty, \infty)$

22. $-3x + 6 = 0 \Rightarrow -3x = -6 \Rightarrow x = 2$ is a vertical line intersecting the x-axis at (2, 0).



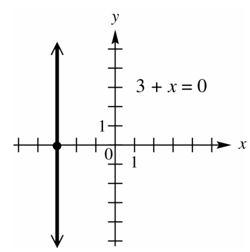
domain: $\{2\}$; range: $(-\infty, \infty)$

23. $-x + 5 = 0 \Rightarrow x = 5$ is a vertical line intersecting the x-axis at (5, 0).



domain: $\{5\}$; range: $(-\infty, \infty)$

24. $3 + x = 0 \Rightarrow x = -3$ is a vertical line intersecting the x-axis at $(-3, 0)$.



domain: $\{-3\}$; range: $(-\infty, \infty)$

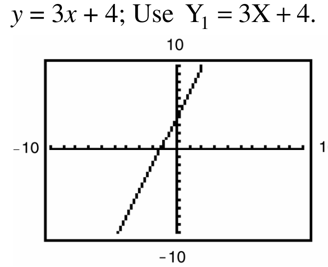
25. $y = 5$ is a horizontal line with y-intercept 5. Choice A resembles this.

26. $y = -5$ is a horizontal line with y-intercept –5. Choice C resembles this.

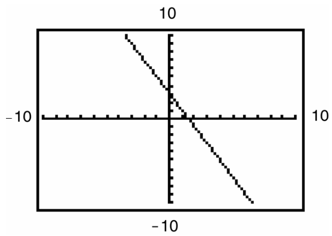
27. $x = 5$ is a vertical line with x-intercept 5. Choice D resembles this.

28. $x = -5$ is a vertical line with x-intercept –5. Choice B resembles this.

29. $y = 3x + 4$; Use $Y_1 = 3X + 4$.



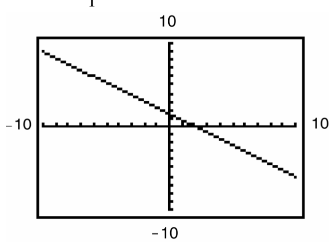
30. $y = -2x + 3$; Use $Y_1 = -2X + 3$



31. $3x + 4y = 6$; Solve for y.

$$3x + 4y = 6$$
$$4y = -3x + 6$$
$$y = -\tfrac{3}{4}x + \tfrac{3}{2}$$

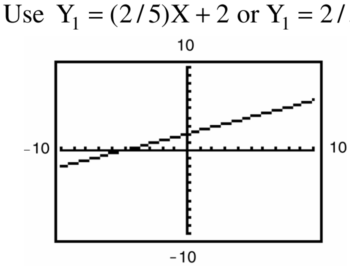
Use $Y_1 = (-3/4)X + (3/2)$
or $Y_1 = -3/4X + 3/2$.



32. $-2x + 5y = 10$; Solve for y.

$$-2x + 5y = 10$$
$$5y = 2x + 10$$
$$y = \tfrac{2}{5}x + 2$$

Use $Y_1 = (2/5)X + 2$ or $Y_1 = 2/5X + 2$



33. The rise is 2.5 feet while the run is 10 feet so the slope is $\frac{2.5}{10} = .25 = 25\% = \frac{1}{4}$. So A = 0.25, C = $\frac{2.5}{10}$, D = 25%, and E = $\frac{1}{4}$ are all expressions of the slope.

34. The pitch or slope is $\frac{1}{4}$. If the rise is 4 feet then $\frac{1}{4} = \frac{\text{rise}}{\text{run}} = \frac{4}{x}$ or $x = 16$ feet. So 16 feet in the horizontal direction corresponds to a rise of 4 feet.

35. Through (2, –1) and (–3, –3)
Let $x_1 = 2$, $y_1 = -1$, $x_2 = -3$, and $y_2 = -3$.
Then rise $= \Delta y = -3 - (-1) = -2$ and run $= \Delta x = -3 - 2 = -5$.
The slope is $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{-2}{-5} = \frac{2}{5}$.

36. Through (5, –3) and (1, –7)
Let $x_1 = 5$, $y_1 = -3$, $x_2 = 1$, and $y_2 = -7$.
Then rise $= \Delta y = -7 - (-3) = -4$ and run $= \Delta x = 1 - 5 = -4$.
The slope is $m = \frac{\Delta y}{\Delta x} = \frac{-4}{-4} = 1$.

37. Through (5, 9) and (–2, 9)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 9}{-2 - 5} = \frac{0}{-7} = 0$$

38. Through (–2, 4) and (6, 4)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 4}{6 - (-2)} = \frac{0}{8} = 0$$

39. Through (5, 1) and (4, 1)
This is a horizontal line. The slope of every horizontal line is zero, so $m = 0$.

40. Through (3, 5) and (4, 5)
This is a horizontal line. The slope of every horizontal line is zero, so $m = 0$.

41. Vertical, through (4, –7)
The slope of every vertical line is undefined; m is undefined.

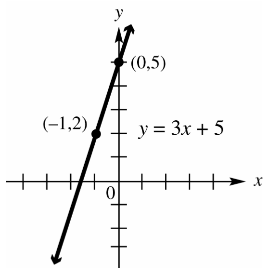
42. Vertical, through (–8, 5)
The slope of every vertical line is undefined; m is undefined.

43. Both B and C can be used to find the slope.
The form $m = \frac{y_2 - y_1}{x_2 - x_1}$ is the form that is standardly used. If you rename points 1 and 2, you will get the formula stated in choice B. Choice D is incorrect because it shows a change in x to a change in y, which is not how slope is defined. Choice A is incorrect because the y-values are subtracted in one way, and the x-values in the opposite way. This will result in the opposite (additive inverse) of the actual value of the slope of the line that passes between the two points.

44. Answers will vary. No, the graph of a linear function cannot have an undefined slope. A line that has an undefined slope is vertical. With a vertical line, more than one y-value is associated with the x-value.

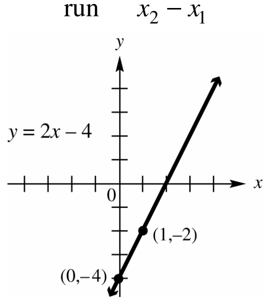
45. $y = 3x + 5$
Find two ordered pairs that are solutions to the equation.
If $x = 0$, then $y = 3(0) + 5 \Rightarrow y = 5$.
If $x = -1$ then
$y = 3(-1) + 5 \Rightarrow y = -3 + 5 \Rightarrow y = 2$. Thus two ordered pairs are $(0,5)$ and $(-1,2)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-5}{-1-0} = \frac{-3}{-1} = 3.$$



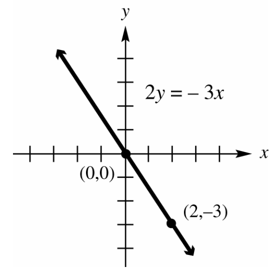
46. $y = 2x - 4$
Find two ordered pairs that are solutions to the equation. If $x = 0$, then $y = 2(0) - 4 \Rightarrow$
$y = -4$. If $x = 1$, then $y = 2(1) - 4 \Rightarrow$
$y = 2 - 4 \Rightarrow y = -2$. Thus two ordered pairs are $(0,-4)$ and $(1,-2)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-4)}{1-0} = \frac{2}{1} = 2.$$



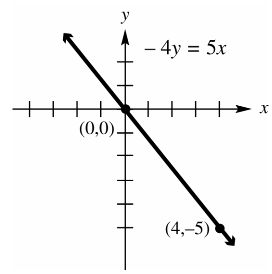
47. $2y = -3x$
Find two ordered pairs that are solutions to the equation. If $x = 0$, then $2y = 0 \Rightarrow y = 0$.
If $y = -3$, then $2(-3) = -3x \Rightarrow -6 = -3x \Rightarrow$
$x = 2$. Thus two ordered pairs are $(0,0)$ and $(2,-3)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-0}{2-0} = -\frac{3}{2}.$$



48. $-4y = 5x$
Find two ordered pairs that are solutions to the equation. If $x = 0$, then $-4y = 0 \Rightarrow y = 0$.
If $x = 4$, then $-4y = 5(4) \Rightarrow -4y = 20$
$\Rightarrow y = -5$. Thus two ordered pairs are $(0,0)$ and $(4,-5)$.

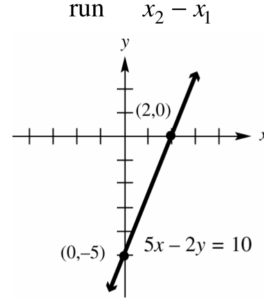
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-0}{4-0} = -\frac{5}{4}.$$



49. $5x - 2y = 10$
Find two ordered pairs that are solutions to the equation. If $x = 0$, then $5(0) - 2y = 10 \Rightarrow$
$\Rightarrow y = -5$. If $y = 0$, then $5x - 2(0) = 10 \Rightarrow$
$5x = 10 \Rightarrow x = 2$.

Thus two ordered pairs are $(0,-5)$ and $(2,0)$.

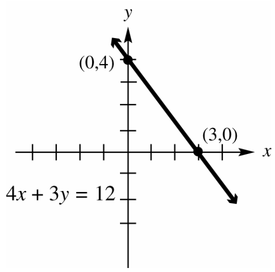
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-5)}{2-0} = \frac{5}{2}.$$



50. $4x + 3y = 12$

Find two ordered pairs that are solutions to the equation. If $x = 0$, then $4(0) + 3y = 12 \Rightarrow$ $3y = 12 \Rightarrow y = 4$. If $y = 0$, then $4x + 3(0) = 12 \Rightarrow 4x = 12 \Rightarrow x = 3$. Thus two ordered pairs are $(0,4)$ and $(3,0)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{3-0} = -\frac{4}{3}.$$

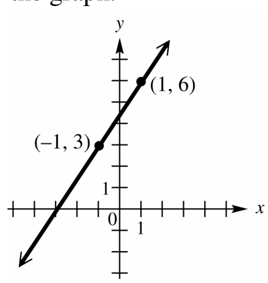


51. Answers will vary.

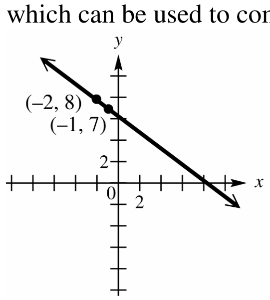
52. Answers will vary.

53. Through (–1, 3), $m = \frac{3}{2}$

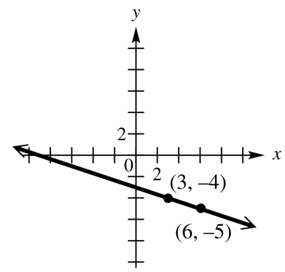
First locate the point (–1, 3). Since the slope is $\frac{3}{2}$, a change of 2 units horizontally (2 units to the right) produces a change of 3 units vertically (3 units up). This gives a second point, (1, 6), which can be used to complete the graph.



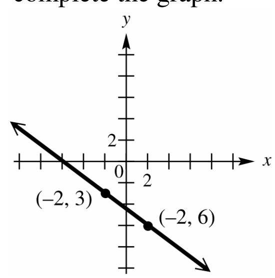
54. Through (–2, 8), $m = -1$. Since the slope is –1, a change of 1 unit horizontally (to the right) produces a change of –1 unit vertically (1 unit down). This gives a second point (–1, 7), which can be used to complete the graph.



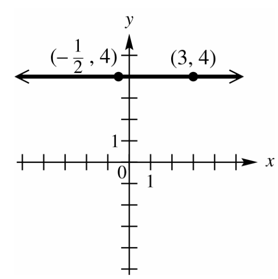
55. Through (3, –4), $m = -\frac{1}{3}$. First locate the point (3, –4). Since the slope is $-\frac{1}{3}$, a change of 3 units horizontally (3 units to the right) produces a change of –1 unit vertically (1 unit down). This gives a second point, (6, –5), which can be used to complete the graph.



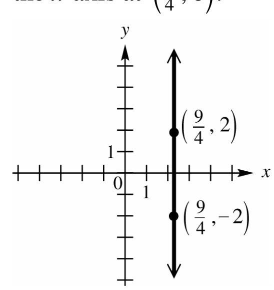
56. Through (–2, –3), $m = -\frac{3}{4}$. Since the slope is $-\frac{3}{4} = \frac{-3}{4}$, a change of 4 units horizontally (4 units to the right) produces a change of –3 units vertically (3 units down). This gives a second point (2, –6), which can be used to complete the graph.



57. Through $\left(-\frac{1}{2}, 4\right)$, $m = 0$. The graph is the horizontal line through $\left(-\frac{1}{2}, 4\right)$.



58. Through $\left(\frac{9}{4}, 2\right)$, undefined slope. The slope is undefined, so the line is vertical, intersecting the x-axis at $\left(\frac{9}{4}, 0\right)$.



59. $m = \frac{1}{3}$ matches graph D because the line rises gradually as x increases.

60. $m = -3$ matches graph C because the line falls rapidly as x increases.

61. $m = 0$ matches graph A because horizontal lines have slopes of 0.

62. $m = -\frac{1}{3}$ matches graph F because the line falls gradually as x increases.

63. $m = 3$ matches graph E because the line rises rapidly as x increases.

64. m is undefined for graph B because vertical lines have undefined slopes.

65. The average rate of change is $m = \frac{\Delta y}{\Delta x}$

$\frac{20-4}{0-4} = \frac{-16}{4} = -\$4$ (thousand) per year. The value of the machine is decreasing \$4000 each year during these years.

66. The average rate of change is $m = \frac{\Delta y}{\Delta x}$

$= \frac{200-0}{4-0} = \frac{200}{4} = \$50$ per month. The amount saved is increasing \$50 each month during these months.

67. The average rate of change is $m = \frac{\Delta y}{\Delta x}$

$\frac{3-3}{4-0} = \frac{0}{4} = 0\%$ per year. The percent of pay raise is not changing - it is 3% each year.

68. The graph is a horizontal line, so the average rate of change (slope) is 0. That means that the number of named hurricanes remained the same, 10, for the four consecutive years shown.

69. For a constant function, the average rate of change is zero.

70. **(a)** The slope of –0.0187 indicates that the average rate of change of the winning time for the 5000 m run is 0.0187 min less (faster). It is negative because the times are generally decreasing as time progresses.

(b) The Olympics were not held during World Wars I (1914–1919) and II (1939–1945).

(c) $y = -.0187(1996) + 50.60 \approx 13.27$ min
The times differ by
$13.27 - 13.13 = .14$ min

71. **(a)** Answers will vary.

(b) $m = \frac{12{,}057 - 2773}{1999 - 1950} = \frac{9284}{49} \approx 189.5$
This means that the average rate of change in the number of radio stations per year is an increase of about 189.5 stations.

72. **(a)** To find the change in subscribers, we need to subtract the number of subscribers in consecutive years.

Years	Change in subscribers (in thousands)
2000–2001	$128{,}375 - 109{,}478 = 18{,}897$
2001–2002	$140{,}766 - 128{,}375 = 12{,}391$
2002–2003	$158{,}722 - 140{,}766 = 17{,}956$
2003–2004	$182{,}140 - 158{,}722 = 23{,}418$
2004–2005	$207{,}896 - 182{,}140 = 25{,}756$

(b) The change in successive years not the same. An approximately straight line could not be drawn through the points if they were plotted.

73. **(a)** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{21.9 - 27.6}{2004 - 1994}$
$= \frac{-5.7}{10} = -0.57$ million recipients per year

(b) The negative slope means the numbers of recipients *decreased* by 0.57 million each year.

74. **(a)** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13.1 - 14.6}{1996 - 1912} = \frac{-1.5}{84}$
≈ -0.0179 min per year. The winning time decreased an average of .0179 min each event year from 1912 to 1996.

(b) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13.6 - 14.6}{2000 - 1912} = \frac{-1}{88}$
≈ -0.0114 min per year. The winning time decreased an average of .0114 min each event year from 1912 to 2000.

75. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1.4 - 3.7}{2004 - 2000} = \dfrac{-2.3}{4}$
$= -0.575$ per year

The percent of freshman listing computer science as their probable field of study decreased an average of 0.575% per year from 2000 to 2004.

76. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{90 - 500}{2007 - 1997} = \dfrac{-410}{10} = -\$41$

The price decreased an average of \$41 each year from 1997 to 2007.

77. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{19.788 - 0.315}{2006 - 1997} = \dfrac{19.473}{9}$

≈ 2.16 million per year. Sales of DVD players increased an average of 2.16 million each year from 1997 to 2006.

78. The first two points are $A(0, -6)$ and $B(1, -3)$.

$m = \dfrac{-3 - (-6)}{1 - 0} = \dfrac{3}{1} = 3$

79. The second and third points are B(1, –3) and C(2, 0).

$m = \dfrac{0 - (-3)}{2 - 1} = \dfrac{3}{1} = 3$

80. If we use any two points on a line to find its slope, we find that the slope is <u>the same</u> in all cases.

81. The first two points are $A(0, -6)$ and $B(1, -3)$.

$d(A, B) = \sqrt{[-3 - (-6)]^2 + (1 - 0)^2}$
$= \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10}$

82. The second and fourth points are $B(1, -3)$ and $D(3, 3)$.

$d(B, D) = \sqrt{[3 - (-3)]^2 + (3 - 1)^2}$
$= \sqrt{6^2 + 2^2} = \sqrt{36 + 4}$
$= \sqrt{40} = 2\sqrt{10}$

83. The first and fourth points are $A(0, -6)$ and $D(3, 3)$.

$d(A, D) = \sqrt{[3 - (-6)]^2 + (3 - 0)^2}$
$= \sqrt{9^2 + 3^2} = \sqrt{81 + 9}$
$= \sqrt{90} = 3\sqrt{10}$

84. $\sqrt{10} + 2\sqrt{10} = 3\sqrt{10}$; The sum is $3\sqrt{10}$, which is equal to the answer in Exercise 83.

85. If points A, B, and C lie on a line in that order, then the distance between A and B added to the distance between <u>B</u> and <u>C</u> is equal to the distance between <u>A</u> and <u>C</u>.

86. The midpoint of the segment joining $A(0, -6)$ and $G(6, 12)$ has coordinates $\left(\frac{0+6}{2}, \frac{-6+12}{2}\right) = \left(\frac{6}{2}, \frac{6}{2}\right) = (3, 3)$. The midpoint is $M(3, 3)$, which is the same as the middle entry in the table.

87. The midpoint of the segment joining $E(4, 6)$ and $F(5, 9)$ has coordinates $\left(\frac{4+5}{2}, \frac{6+9}{2}\right) = \left(\frac{9}{2}, \frac{15}{2}\right) = (4.5, 7.5)$. If the x-value 4.5 were in the table, the corresponding y-value would be 7.5.

88. **(a)** $C(x) = 10x + 500$

(b) $R(x) = 35x$

(c) $P(x) = R(x) - C(x)$
$= 35x - (10x + 500)$
$= 35x - 10x - 500 = 25x - 500$

(d) $C(x) = R(x)$
$10x + 500 = 35x$
$500 = 25x$
$20 = x$
20 units; do not produce

89. **(a)** $C(x) = 11x + 180$

(b) $R(x) = 20x$

(c) $P(x) = R(x) - C(x)$
$= 20x - (11x + 180)$
$= 20x - 11x - 180 = 9x - 180$

(d) $C(x) = R(x)$
$11x + 180 = 20x$
$180 = 9x$
$20 = x$
20 units; produce

90. **(a)** $C(x) = 150x + 2700$

(b) $R(x) = 280x$

(c) $P(x) = R(x) - C(x)$
$= 280x - (150x + 2700)$
$= 280x - 150x - 2700$
$= 130x - 2700$

(d) $C(x) = R(x)$
$150x + 2700 = 280x$
$2700 = 130x$
$20.77 \approx x$ or 21 units
21 units; produce

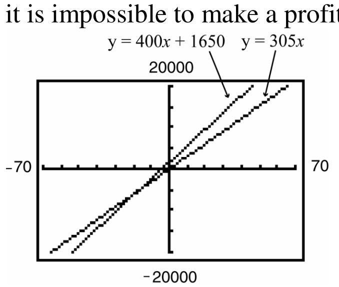
91. **(a)** $C(x) = 400x + 1650$

(b) $R(x) = 305x$

(c) $P(x) = R(x) - C(x)$
$= 305x - (400x + 1650)$
$= 305x - 400x - 1650$
$= -95x - 1650$

(d)
$$C(x) = R(x)$$
$$400x + 1650 = 305x$$
$$95x + 1650 = 0$$
$$95x = -1650$$
$$x \approx -17.37 \text{ units}$$

This result indicates a negative "break-even point," but the number of units produced must be a positive number. A calculator graph of the lines $Y_1 = 400X + 1650$ and $Y_2 = 305X$ on the same screen or solving the inequality $305x < 400x + 1650$ will show that $R(x) < C(x)$ for all positive values of x (in fact whenever x is greater than -17.4). Do not produce the product since it is impossible to make a profit.



92. **(a)** $C(x) = R(x) \Rightarrow 200x + 1000 = 240x \Rightarrow$
$1000 = 40x \Rightarrow 25 = x$
25 units

(b) $C(25) = 200(25) + 1000 = \$6000$ which is the same as $R(25) = 240(25) = \$6000$

(c) $C(x) = R(x) \Rightarrow 220x + 1000 = 240x \Rightarrow$
$1000 = 20x \Rightarrow 50 = x$
The break-even point is 50 units instead of 25 units. The manager is not better off because twice as many units must be sold before beginning to show a profit.

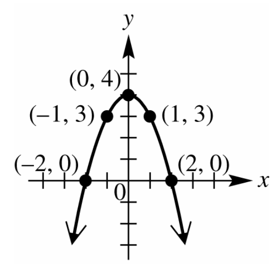
Chapter 2 Quiz

(Sections 2.1–2.4)

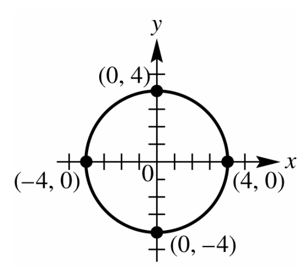
1. $d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(-8 - (-4))^2 + (-3 - 2)^2}$
$= \sqrt{(-4)^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$

2. To find an estimate for 1985, find the midpoint of (1980, 4.50) and (1990, 5.20):
$$M = \left(\frac{1980 + 1990}{2}, \frac{4.50 + 5.20}{2}\right)$$
$= (1985, 4.85)$. The enrollment in 1985 was 4.85 million.
To find an estimate for 1995, find the midpoint of (1990, 5.20) and (2000, 5.80):
$$M = \left(\frac{1990 + 2000}{2}, \frac{5.20 + 5.80}{2}\right)$$
$= (1995, 5.50)$. The enrollment in 1985 was 5.50 million.

3.



4.



5. $x^2 + y^2 - 4x + 8y + 3 = 0$
Complete the square on x and y separately.
$(x^2 - 4x + 4) + (y^2 + 8y + 16) = -3 + 4 + 16 \Rightarrow$
$(x - 2)^2 + (y + 4)^2 = 17$
The radius is $\sqrt{17}$ and the midpoint of the circle is (2, −4).

6. $f(-1) = |-1 + 3| = |2| = 2$

7. Domain: $(-\infty, \infty)$; range: $[0, \infty)$

8. **(a)** The largest interval over which *f* is decreasing is $(-\infty,-3]$.

(b) The largest interval over which *f* is increasing is $[-3,\infty)$.

(c) There is no interval over which the function is constant.

9. **(a)** $m=\dfrac{11-5}{5-1}=\dfrac{6}{4}=\dfrac{3}{2}$

(b) $m=\dfrac{4-4}{-1-(-7)}=\dfrac{0}{6}=0$

(c) $m=\dfrac{-4-12}{6-6}=\dfrac{-16}{0}\Rightarrow$ the slope is undefined.

10. The points to use would be (1982, 79.1) and (2002, 69.3). The average rate of change is
$\dfrac{69.3-79.1}{2002-1982}=\dfrac{-9.8}{20}=-0.49$
The number of college freshmen age 18 or younger on December 31 decreased an average of 0.49% per year from 1982 to 2002.

Section 2.5: Equations of Lines; Curve Fitting

1. $y=\frac{1}{4}x+2$ is graphed in D.
The slope is $\frac{1}{4}$ and the *y*-intercept is 2.

2. $4x + 3y = 12$ or $3y = -4x + 12$ or $y=-\frac{4}{3}x+4$ is graphed in B. The slope is $-\frac{4}{3}$ and the *y*-intercept is 4.

3. $y-(-1)=\frac{3}{2}(x-1)$ is graphed in C. The slope is $\frac{3}{2}$ and a point on the graph is (1, –1).

4. $y = 4$ is graphed in A. $y = 4$ is a horizontal line with *y*-intercept 4.

5. Through (1, 3), $m = -2$.
Write the equation in point-slope form.
$y-y_1=m(x-x_1)\Rightarrow y-3=-2(x-1)$
Then, change to standard form.
$y-3=-2x+2\Rightarrow 2x+y=5$

6. Through (2, 4), m = –1
Write the equation in point-slope form.
$y-y_1=m(x-x_1)\Rightarrow y-4=-1(x-2)$
Then, change to standard form.
$y-4=-x+2\Rightarrow x+y=6$

7. Through (–5, 4), $m=-\frac{3}{2}$
Write the equation in point-slope form.
$y-4=-\frac{3}{2}\left[x-(-5)\right]$
Change to standard form.
$2(y-4)=-3(x+5)$
$2y-8=-3x-15$
$3x+2y=-7$

8. Through $(-4,3)$, $m=\frac{3}{4}$
Write the equation in point-slope form.
$y-3=\frac{3}{4}\left[x-(-4)\right]$
Change to standard form.
$4(y-3)=3(x+4)$
$4y-12=3x+12$
$-3x+4y=24$ or $3x-4y=-24$

9. Through (–8, 4), undefined slope
Since undefined slope indicates a vertical line, the equation will have the form $x = a$. The equation of the line is $x = -8$.

10. Through (5, 1), undefined slope
This is a vertical line through (5, 1), so the equation is $x = 5$.

11. Through (5, −8), $m = 0$
This is a horizontal line through (5, −8), so the equation is $y = -8$.

12. Through (−3, 12), $m = 0$
This is a horizontal line through (−3, 12), so the equation is $y = 12$.

13. Through (–1, 3) and (3, 4)
First find *m*.
$m=\dfrac{4-3}{3-(-1)}=\dfrac{1}{4}$
Use either point and the point-slope form.
$y-4=\frac{1}{4}(x-3)$
Change to slope-intercept form.
$4(y-4)=x-3$
$4y-16=x-3$
$4y=x+13$
$y=\frac{1}{4}x+\frac{13}{4}$

14. Through (8, –1) and (4, 3)
First find *m*.
$m=\dfrac{3-(-1)}{4-8}=\dfrac{4}{-4}=-1$
Use either point and the point-slope form.
$y-3=-1(x-4)$
$y-3=-x+4$
$y=-x+7$

15. x-intercept 3, y-intercept –2
The line passes through (3, 0) and (0, –2). Use these points to find m.
$$m = \frac{-2-0}{0-3} = \frac{2}{3}$$
Using slope-intercept form we have
$y = \frac{2}{3}x - 2.$

16. x-intercept –2, y-intercept 4
The line passes through the points (–2, 0) and (0, 4). Use these points to find m.
$$m = \frac{4-0}{0-(-2)} = 2$$
Using slope-intercept form we have
$y = 2x + 4.$

17. Vertical, through (–6, 4)
The equation of a vertical line has an equation of the form $x = a$. Since the line passes through (–6, 4), the equation is $x = -6$. (Since this slope of a vertical line is undefined, this equation cannot be written in slope-intercept form.)

18. Vertical, through (2, 7)
The equation of a horizontal line has an equation of the form $x = a$. Since the line passes through (2, 7), the equation is $x = 2$. (Since this slope of a vertical line is undefined, this equation cannot be written in slope-intercept form.)

19. Horizontal, through (−7, 4)
The equation of a horizontal line has an equation of the form $y = b$. Since the line passes through (−7, 4), the equation is $y = 4$.

20. Horizontal, through (−8, −2)
The equation of a horizontal line has an equation of the form $y = b$. Since the line passes through (−8, −2), the equation is $y = -2$.

21. $m = 5, b = 15$
Using slope-intercept form, we have
$y = 5x + 15.$

22. $m = -2, b = 12$
Using slope-intercept form, we have
$y = -2x + 12.$

23. $m = -\frac{2}{3}, b = -\frac{4}{5}$
Using slope-intercept form, we have
$y = -\frac{2}{3}x - \frac{4}{5}.$

24. $m = -\frac{5}{8}, b = -\frac{1}{3}$
Using slope-intercept form, we have
$y = -\frac{5}{8}x - \frac{1}{3}.$

25. slope 0, y-intercept $\frac{3}{2}$
These represent $m = 0$ and $b = \frac{3}{2}$. Using slope-intercept form we have
$y = (0)x + \frac{3}{2} \Rightarrow y = \frac{3}{2}.$

26. slope 0, y-intercept $-\frac{5}{4}$
These represent $m = 0$ and $b = -\frac{5}{4}$. Using slope-intercept form we have
$y = (0)x - \frac{5}{4} \Rightarrow y = -\frac{5}{4}.$

27. The line $x + 2 = 0$ has x-intercept –2. It does not have a y-intercept. The slope of this line is undefined.
The line $4y = 2$ has y-intercept $\frac{1}{2}$. It does not have an x-intercept. The slope of this line is 0.

28. **(a)** The graph of $y = 3x + 2$ has a positive slope and a positive y-intercept. These conditions match graph D.

(b) The graph of $y = -3x + 2$ has a negative slope and a positive y-intercept. These conditions match graph B.

(c) The graph of $y = 3x - 2$ has a positive slope and a negative y-intercept. These conditions match graph A.

(d) The graph of $y = -3x - 2$ has a negative slope and a negative y-intercept. These conditions match graph C.

29. **(a)** The graph of $y = 2x + 3$ has a positive slope and a positive y-intercept. These conditions match graph B.

(b) The graph of $y = -2x + 3$ has a negative slope and a positive y-intercept. These conditions match graph D.

(c) The graph of $y = 2x - 3$ has a positive slope and a negative y-intercept. These conditions match graph A.

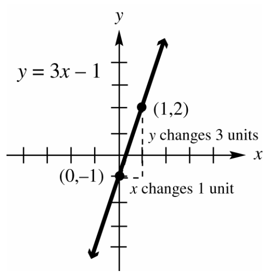
(d) The graph of $y = -2x - 3$ has a negative slope and a negative y-intercept. These conditions match graph C.

30. **(a)** Use the first two points in the table, $A(-2, -11)$ and $B(-1, -8)$.
$$m = \frac{-8-(-11)}{-1-(-2)} = \frac{3}{1} = 3$$

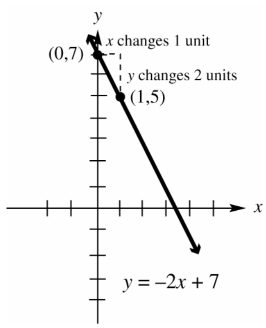
(b) When $x = 0$, $y = -5$. The y-intercept is -5.

(c) Substitute 3 for m and -5 for b in the slope-intercept form.
$y = mx + b \Rightarrow y = 3x - 5$

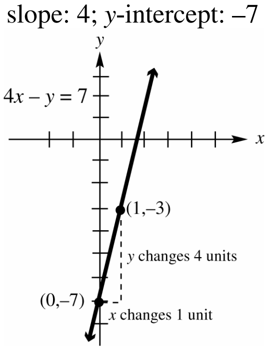
31. $y = 3x - 1$
This equation is in the slope-intercept form, $y = mx + b$.
slope: 3;
y-intercept: -1



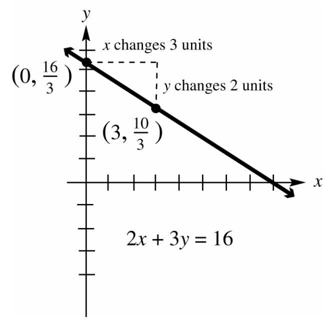
32. $y = -2x + 7$
slope: -2;
y-intercept: 7



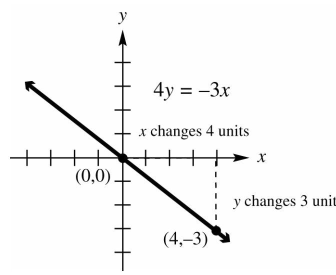
33. $4x - y = 7$
Solve for y to write the equation in slope-intercept form.
$-y = -4x + 7 \Rightarrow y = 4x - 7$
slope: 4; y-intercept: -7



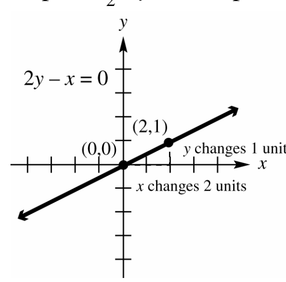
34. $2x + 3y = 16$
Solve the equation for y to write the equation in slope-intercept form.
$3y = -2x + 16 \Rightarrow y = -\frac{2}{3}x + \frac{16}{3}$
slope: $-\frac{2}{3}$; y-intercept: $\frac{16}{3}$



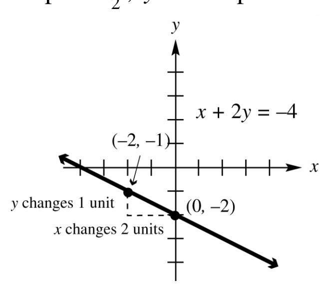
35. $4y = -3x$
$y = -\frac{3}{4}x$ or
$y = -\frac{3}{4}x + 0$
slope: $-\frac{3}{4}$;
y-intercept 0



36. $2y - x = 0$
$2y = x \Rightarrow y = \frac{1}{2}x$ or $y = \frac{1}{2}x + 0$
slope is $\frac{1}{2}$; y-intercept: 0



37. $x + 2y = -4$
Solve the equation for y to write the equation in slope-intercept form.
$2y = -x - 4 \Rightarrow y = -\frac{1}{2}x - 2$
slope: $-\frac{1}{2}$; y-intercept: -2

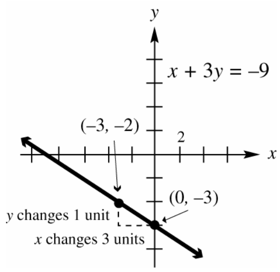


38. $x + 3y = -9$

Solve the equation for y to write the equation in slope-intercept form.

$3y = -x - 9 \Rightarrow y = -\frac{1}{3}x - 3$

slope: $-\frac{1}{3}$; y-intercept: -3

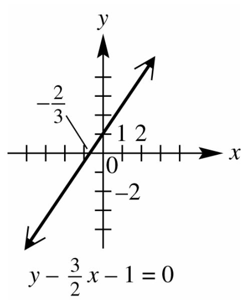


39. $y - \frac{3}{2}x - 1 = 0$

Solve the equation for y to write the equation in slope-intercept form.

$y - \frac{3}{2}x - 1 = 0 \Rightarrow y = \frac{3}{2}x + 1$

slope: $\frac{3}{2}$; y-intercept: 1



40. **(a)** Solve the equation for y to write the equation in slope-intercept form.

$Ax + By = C \Rightarrow By = -Ax + C \Rightarrow$

$y = -\dfrac{A}{B}x + \dfrac{B}{C}$

The slope is $-\frac{A}{B}$

(b) The y-intercept is $\frac{C}{B}$.

41. **(a)** The line falls 2 units each time the x value increases by 1 unit. Therefore the slope is -2. The graph intersects the y-axis at the point (0, 1) and intersects the x-axis at $\left(\frac{1}{2}, 0\right)$, so the y-intercept is 1 and the x-intercept is $\frac{1}{2}$.

(b) The equation defining f is $y = -2x + 1$.

42. **(a)** The line rises 2 units each time the x value increases by 1 unit. Therefore the slope is 2. The graph intersects the y-axis at the point (0, −1) and intersects the x-axis at $\left(\frac{1}{2}, 0\right)$, so the y-intercept is −1 and the x-intercept is $\frac{1}{2}$.

(b) The equation defining f is $y = 2x - 1$.

43. **(a)** The line falls 1 unit each time the x value increases by 3 units. Therefore the slope is $-\frac{1}{3}$. The graph intersects the y-axis at the point (0, 2), so the y-intercept is 2. The graph passes through (3, 1) and will fall 1 unit when the x value increases by 3, so the x-intercept is 6.

(b) The equation defining f is $y = -\frac{1}{3}x + 2$.

44. **(a)** The line rises 3 units each time the x value increases by 4 units. Therefore the slope is $\frac{3}{4}$. The graph intersects the y-axis at the point (0, −3) and intersects the x-axis at (4, 0), so the y-intercept is −3 and the x-intercept is 4.

(b) The equation defining f is $y = \frac{3}{4}x - 3$.

45. **(a)** The line falls 200 units each time the x value increases by 1 unit. Therefore the slope is −200. The graph intersects the y-axis at the point (0, 300) and intersects the x-axis at $\left(\frac{3}{2}, 0\right)$, so the y-intercept is 300 and the x-intercept is $\frac{3}{2}$.

(b) The equation defining f is $y = -200x + 300$.

46. **(a)** The line rises 100 units each time the x value increases by 5 units. Therefore the slope is 20. The graph intersects the y-axis at the point (0, −50) and intersects the x-axis at $\left(\frac{5}{2}, 0\right)$, so the y-intercept is −50 and the x-intercept is $\frac{5}{2}$.

(b) The equation defining f is $y = 20x - 50$.

47. **(a)** through (–1, 4), parallel to $x + 3y = 5$
Find the slope of the line $x + 3y = 5$ by writing this equation in slope-intercept form.
$x+3y=5 \Rightarrow 3y=-x+5 \Rightarrow$
$y=-\frac{1}{3}x+\frac{5}{3}$
The slope is $-\frac{1}{3}$. Since the lines are parallel, $-\frac{1}{3}$ is also the slope of the line whose equation is to be found. Substitute $m=-\frac{1}{3}$, $x_1=-1$, and $y_1=4$ into the point-slope form.
$$y-y_1=m(x-x_1)$$
$$y-4=-\frac{1}{3}\left[x-(-1)\right]$$
$$y-4=-\frac{1}{3}(x+1)$$
$$3y-12=-x-1 \Rightarrow x+3y=11$$

(b) Solve for *y*.
$3y=-x+11 \Rightarrow y=-\frac{1}{3}x+\frac{11}{3}$

48. **(a)** through (3, –2), parallel to $2x - y = 5$
Find the slope of the line $2x - y = 5$ by writing this equation in slope-intercept form.
$2x-y=5 \Rightarrow -y=-2x+5 \Rightarrow$
$y=2x-5$
The slope is 2. Since the lines are parallel, the slope of the line whose equation is to be found is also 2.
Substitute $m=2$, $x_1=3$, and $y_1=-2$ into the point-slope form.
$y-y_1=m(x-x_1) \Rightarrow$
$y+2=2(x-3) \Rightarrow y+2=2x-6 \Rightarrow$
$-2x+y=-8$ or $2x-y=8$

(b) Solve for *y*. $y = 2x – 8$

49. **(a)** through (1, 6), perpendicular to $3x + 5y = 1$
Find the slope of the line $3x + 5y = 1$ by writing this equation in slope-intercept form.
$3x+5y=1 \Rightarrow 5y=-3x+1 \Rightarrow$
$y=-\frac{3}{5}x+\frac{1}{5}$
This line has a slope of $-\frac{3}{5}$. The slope of any line perpendicular to this line is $\frac{5}{3}$, since $-\frac{3}{5}\left(\frac{5}{3}\right)=-1.$ Substitute $m=\frac{5}{3}$, $x_1=1,$ and $y_1=6$ into the point-slope form.
$$y-6=\frac{5}{3}(x-1)$$
$$3(y-6)=5(x-1)$$
$$3y-18=5x-5$$
$$-13=5x-3y \text{ or } 5x-3y=-13$$

(b) Solve for *y*. $3y=5x+13 \Rightarrow y=\frac{5}{3}x+\frac{13}{3}$

50. **(a)** through (–2, 0), perpendicular to $8x – 3y = 7$
Find the slope of the line $8x – 3y = 7$ by writing the equation in slope-intercept form.
$8x-3y=7 \Rightarrow -3y=-8x+7 \Rightarrow$
$y=\frac{8}{3}x-\frac{7}{3}$
This line has a slope of $\frac{8}{3}$. The slope of any line perpendicular to this line is $-\frac{3}{8}$, since $\frac{8}{3}\left(-\frac{3}{8}\right)=-1.$
Substitute $m=-\frac{3}{8}$, $x_1=-2,$ and $y_1=0$ into the point-slope form.
$$y-0=-\frac{3}{8}(x+2)$$
$$8y=-3(x+2)$$
$$8y=-3x-6 \Rightarrow 3x+8y=-6$$

(b) Solve for *y*. $8y=-3x-6 \Rightarrow y=-\frac{3}{8}x-\frac{6}{8} \Rightarrow$
$y=-\frac{3}{8}x-\frac{3}{4}$

51. **(a)** through (4, 1), parallel to $y=-5$
Since $y=-5$ is a horizontal line, any line parallel to this line will be horizontal and have an equation of the form $y=b$. Since the line passes through (4, 1), the equation is $y = 1$.

(b) The slope-intercept form is $y = 1$.

52. **(a)** through $(-2,-2)$, parallel to $y=3$
Since $y=3$ is a horizontal line, any line parallel to this line will be horizontal and have an equation of the form $y=b$.
Since the line passes through $(-2,-2)$, the equation is $y=-2$.

(b) The slope-intercept form is $y=-2$.

53. **(a)** through (–5, 6), perpendicular to $x = –2$.
Since $x = –2$ is a vertical line, any line perpendicular to this line will be horizontal and have an equation of the form $y = b$. Since the line passes through (–5, 6), the equation is $y = 6$.

(b) The slope-intercept form is $y = 6$.

54. **(a)** Through (4, –4), perpendicular to $x = 4$
Since $x = 4$ is a vertical line, any line perpendicular to this line will be horizontal and have an equation of the form $y = b$. Since the line passes through (4, –4), the equation is $y = -4$.

(b) The slope-intercept form is $y = -4$.

55. **(a)** Find the slope of the line $3y + 2x = 6$.
$3y+2x=6 \Rightarrow 3y=-2x+6 \Rightarrow$
$y=-\frac{2}{3}x+2$
Thus, $m=-\frac{2}{3}$. A line parallel to $3y + 2x = 6$ also has slope $-\frac{2}{3}$.
Solve for k using the slope formula.

$$\frac{2-(-1)}{k-4}=-\frac{2}{3}$$
$$\frac{3}{k-4}=-\frac{2}{3}$$
$$3(k-4)\left(\frac{3}{k-4}\right)=3(k-4)\left(-\frac{2}{3}\right)$$
$$9=-2(k-4)$$
$$9=-2k+8$$
$$2k=-1\Rightarrow k=-\frac{1}{2}$$

(b) Find the slope of the line $2y - 5x = 1$.
$2y-5x=1 \Rightarrow 2y=5x+1 \Rightarrow$
$y=\frac{5}{2}x+\frac{1}{2}$
Thus, $m=\frac{5}{2}$. A line perpendicular to $2y - 5x = 1$ will have slope $-\frac{2}{5}$, since $\frac{5}{2}\left(-\frac{2}{5}\right)=-1$.
Solve this equation for k.

$$\frac{3}{k-4}=-\frac{2}{5}$$
$$5(k-4)\left(\frac{3}{k-4}\right)=5(k-4)\left(-\frac{2}{5}\right)$$
$$15=-2(k-4)$$
$$15=-2k+8$$
$$2k=-7\Rightarrow k=-\frac{7}{2}$$

56. **(a)** Find the slope of the line $2x-3y=4$.
$2x-3y=4 \Rightarrow -3y=-2x+4 \Rightarrow$
$y=\frac{2}{3}x-\frac{4}{3}$
Thus, $m=\frac{2}{3}$. A line parallel to $2x - 3y = 4$ also has slope $\frac{2}{3}$.
Solve for r using the slope formula.

$$\frac{r-6}{-4-2}=\frac{2}{3}\Rightarrow\frac{r-6}{-6}=\frac{2}{3}\Rightarrow$$
$$-6\left(\frac{r-6}{-6}\right)=-6\left(\frac{2}{3}\right)\Rightarrow$$
$$r-6=-4\Rightarrow r=2$$

(b) Find the slope of the line $x + 2y = 1$.
$x+2y=1 \Rightarrow 2y=-x+1 \Rightarrow$
$y=-\frac{1}{2}x+\frac{1}{2}$
Thus, $m=-\frac{1}{2}$. A line perpendicular to the line $x + 2y = 1$ has slope 2, since $-\frac{1}{2}(2)=-1$. Solve for r using the slope formula.

$$\frac{r-6}{-4-2}=2\Rightarrow\frac{r-6}{-6}=2\Rightarrow$$
$$r-6=-12\Rightarrow r=-6$$

57. (1970, 43.3), (2005, 59.3)

$$m=\frac{59.3-43.3}{2005-1970}=\frac{16}{35}\approx 0.457$$

Now use either point, say (1970, 43.3), and the point-slope form to find the equation.

$$y-43.3=0.457(x-1970)$$
$$y-43.3=0.457x-900.29$$
$$y=0.457x-856.99$$

Let $x = 2006$
$y = 0.457\ (2006) - 856.99 \approx 59.8$
The percent of women in the civilian labor force is predicted to be 59.8%.
This figure is very close to the actual figure.

58. (1975, 46.3), (2000, 59.9)

$$m=\frac{59.9-46.3}{2000-1975}=\frac{13.6}{25}=.544$$

Now use either point, say (2000, 59.9), and the point-slope form to find the equation.

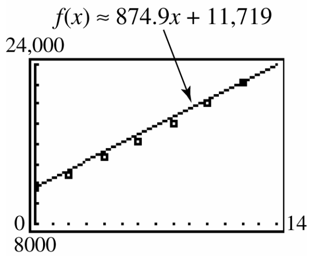
$$y-59.9=0.544(x-2000)$$
$$y-59.9=0.544x-1088$$
$$y=0.544x-1028.1$$

Let $x = 1996$.
$y=0.544(1996)-1028.1\approx 57.7$
The percent of women in the civilian labor force is predicted to be 57.7%. This figure is reasonably close to the actual figure.

59. **(a)** (0, 11719), (12, 22218)

$$m=\frac{22,218-11,719}{12-0}=\frac{10,499}{12}\approx 874.9$$

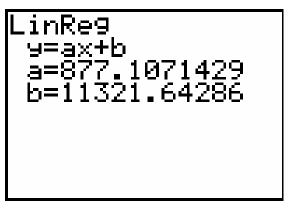
From the point (0, 11719), the value of b is 11,719. Therefore we have
$f(x)\approx 874.9x+11,719$



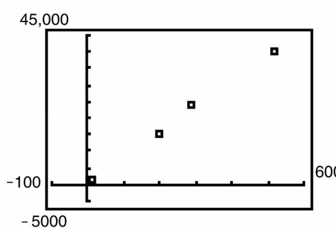
The average tuition increase is about \$875 per year for the period, because this is the slope of the line.

(b) 2005 corresponds to $x = 11$.
$f(11) \approx 874.9(11) + 11{,}719 \approx \$21{,}343$
This is a fairly good approximation.

(c) From the calculator,
$f(x) \approx 877.1x + 11{,}322$



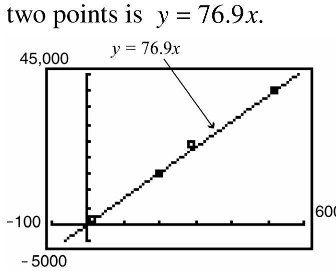
60. (a) There appears to be a linear relationship between the data. The farther the galaxy is from Earth, the faster it is receding.



(b) Using the points (520, 40,000) and (0, 0), we obtain

$$m = \frac{40{,}000 - 0}{520 - 0} = \frac{40{,}000}{520} \approx 76.9.$$

The equation of the line through these two points is $y = 76.9x$.



(c) $76.9x = 60{,}000$

$$x = \frac{60{,}000}{76.9} \Rightarrow x \approx 780$$

The galaxy Hydra is approximately 780 megaparsecs away.

(d) $A = \dfrac{9.5 \times 10^{11}}{m}$

$$A = \frac{9.5 \times 10^{11}}{76.9} \approx 1.235 \times 10^{10} \approx 12.35 \times 10^{9}$$

Using $m = 76.9$, we estimate that the age of the universe is approximately 12.35 billion years.

(e) $A = \dfrac{9.5 \times 10^{11}}{50} = 1.9 \times 10^{10}$ or 19×10^{9}

$$A = \frac{9.5 \times 10^{11}}{100} = 9.5 \times 10^{9}$$

The range for the age of the universe is between 9.5 billion and 19 billion years.

61. (a) The ordered pairs are (0, 32) and (100, 212).

The slope is $m = \dfrac{212 - 32}{100 - 0} = \dfrac{180}{100} = \dfrac{9}{5}$.

Use $(x_1, y_1) = (0, 32)$ and $m = \frac{9}{5}$ in the point-slope form.

$y - y_1 = m(x - x_1)$
$y - 32 = \frac{9}{5}(x - 0)$
$y - 32 = \frac{9}{5}x$
$y = \frac{9}{5}x + 32 \Rightarrow F = \frac{9}{5}C + 32$

(b) $F = \frac{9}{5}C + 32$
$5F = 9(C + 32)$
$5F = 9C + 160 \Rightarrow 9C = 5F - 160 \Rightarrow$
$9C = 5(F - 32) \Rightarrow C = \frac{5}{9}(F - 32)$

(c) $F = C \Rightarrow F = \frac{5}{9}(F - 32) \Rightarrow$
$9F = 5(F - 32) \Rightarrow 9F = 5F - 160 \Rightarrow$
$4F = -160 \Rightarrow F = -40$
$F = C$ when F is $-40°$.

62. (a) The ordered pairs are (0, 1) and (100, 3.92).
The slope is

$$m = \frac{3.92 - 1}{100 - 0} = \frac{2.92}{100} = .0292 \quad \text{and } b = 1.$$

Using slope-intercept form we have
$y = .0292x + 1$ or $p(x) = .0292x + 1$.

(b) Let $x = 60$.
$p(60) = .0292(60) + 1 = 2.752$
The pressure at 60 feet is approximately 2.75 atmospheres.

63. (a) Since we are wanting to find C as a function of I, use the points (8795, 6739) and (10904, 8746), where the first component represents the independent variable, I. First find the slope of the line.

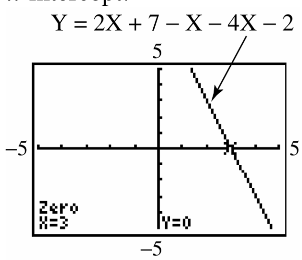
$$m = \frac{8746 - 6739}{10,904 - 8795} = \frac{2007}{2109} \approx 0.952$$

Now use either point, say (8795, 6739), and the point-slope form to find the equation.

$$y - 6739 = 0.952(x - 8795)$$
$$y - 6739 \approx 0.952x - 8373$$
$$y \approx 0.952x - 1634$$
$$\text{or } C = 0.952I - 1634$$

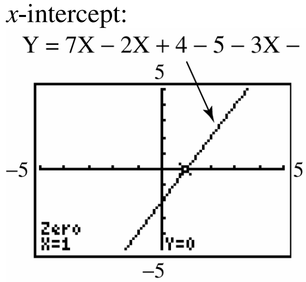
(b) Since the slope is 0.952, the marginal propensity to consume is 0.952.

64. Write the equation as an equivalent equation with 0 on one side: $2x + 7 - x = 4x - 2 \Rightarrow$ $2x + 7 - x - 4x - 2 = 0$. Now graph $\text{Y} = 2\text{X} + 7 - \text{X} - 4\text{X} - 2$ to find the x-intercept:



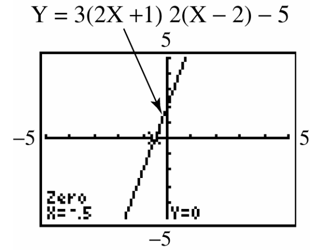
Solution set: $\{3\}$

65. Write the equation as an equivalent equation with 0 on one side: $7x - 2x + 4 - 5 = 3x + 1 \Rightarrow$ $7x - 2x + 4 - 5 - 3x - 1 = 0$. Now graph $\text{Y} = 7\text{X} - 2\text{X} + 4 - 5 - 3\text{X} - 1 = 0$ to find the x-intercept:



Solution set: {1}

66. Write the equation as an equivalent equation with 0 on one side: $3(2x+1) - 2(x-2) = 5 \Rightarrow$ $3(2x+1) - 2(x-2) - 5 = 0$. Now graph $\text{Y} = 3(2\text{X}+1) - 2(\text{X}-2) - 5$ to find the x-intercept:

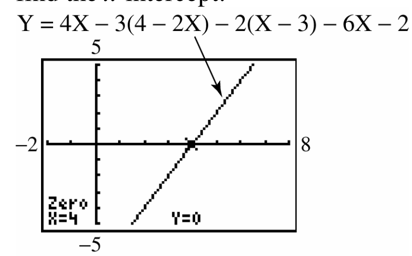


Solution set: $\left\{-\frac{1}{2}\right\}$ or $\{-.5\}$

67. Write the equation as an equivalent equation with 0 on one side:

$$4x - 3(4 - 2x) = 2(x - 3) + 6x + 2 \Rightarrow$$
$$4x - 3(4 - 2x) - 2(x - 3) - 6x - 2 = 0.$$

Now graph $\text{Y} = 4\text{X} - 3(4 - 2\text{X}) - 2(\text{X} - 3) - 6\text{X} - 2$ to find the x-intercept:



Solution set: $\{4\}$

68. D is the only possible answer, since the x-intercept occurs when $y = 0$, we can see from the graph that the value of the x-intercept exceeds 10.

69. (a)

$$-2(x - 5) = -x - 2$$
$$-2x + 10 = -x - 2$$
$$10 = x - 2$$
$$12 = x$$

Solution set: $\{12\}$

(b) Answers will vary. The largest value of x that is displayed in the standard viewing window is 10. As long as 12 is either a minimum or a maximum, or between the minimum and maximum, then the solution will be seen.

70. The Pythagorean Theorem and its converse assure us that in triangle OPQ, angle POQ is a right angle if and only if $[d(O, P)]^2 + [d(O, Q)]^2 = [d(P, Q)]^2$.

71. $d(O, P) = \sqrt{(x_1 - 0)^2 + (m_1 x_1 - 0)^2}$
$= \sqrt{x_1^2 + m_1^2 x_1^2}$

72. $d(O, Q) = \sqrt{(x_2 - 0)^2 + (m_2 x_2 - 0)^2}$
$= \sqrt{x_2^2 + m_2^2 x_2^2}$

73. $d(P, Q) = \sqrt{(x_2 - x_1)^2 + (m_2 x_2 - m_1 x_1)^2}$

74. $[d(O, P)]^2 + [d(O, Q)]^2 = [d(P, Q)]^2$

$$\left[\sqrt{x_1^2 + m_1^2 x_1^2}\right]^2 + \left[\sqrt{x_2^2 + m_2^2 x_2^2}\right]^2 = \left[\sqrt{(x_2 - x_1)^2 + (m_2 x_2 - m_1 x_1)^2}\right]^2$$

$$\left(x_1^2 + m_1^2 x_1^2\right) + \left(x_2^2 + m_2^2 x_2^2\right) = \left(x_2 - x_1\right)^2 + \left(m_2 x_2 - m_1 x_1\right)^2$$

$$x_1^2 + m_1^2 x_1^2 + x_2^2 + m_2^2 x_2^2 = x_2^2 - 2x_2 x_1 + x_1^2 + m_2^2 x_2^2 - 2m_1 m_2 x_1 x_2 + m_1^2 x_1^2$$

$$0 = -2x_2 x_1 - 2m_1 m_2 x_1 x_2 \Rightarrow$$

$$-2m_1 m_2 x_1 x - 2x_2 x_1 = 0$$

75. $-2m_1 m_2 x_1 x_2 - 2x_1 x_2 = 0$
$-2x_1 x_2 (m_1 m_2 + 1) = 0$

76. $-2x_1 x_2 (m_1 m_2 + 1) = 0$
Since $x_1 \neq 0$ and $x_2 \neq 0$, we have $m_1 m_2 + 1 = 0$ implying that $m_1 m_2 = -1$.

77. If two nonvertical lines are perpendicular, then the product of the slopes of these lines is –1.

78. To show that the line $y = x$ is the perpendicular bisector of the segment with endpoints (a, b) and (b, a), we must show that the line bisects the segment and is perpendicular to the segment. To show that it bisects the segment, we find the midpoint of the segment. The midpoint is $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$. Since $\frac{a+b}{2} = \frac{b+a}{2}$, we have a point that lies on the line $y = x$. Thus the line does bisect the segment. In order to show that the line $y = x$ is perpendicular to the segment with endpoints (a, b) and (b, a), we must find the slope of the segment (the line has slope 1). Since $m = \frac{a-b}{b-a} = -1$ represents the slope of the segment, we have that the line $y = x$ and the segment are perpendicular since $1(-1) = -1$. Thus, $y = x$ is the perpendicular bisector of the segment with endpoints (a, b) and (b, a).

79. Label the points as follows: $A(-1, 5)$, $B(2, -4)$, and $C(4, -10)$.

For A and B: $m = \frac{-4-5}{2-(-1)} = \frac{-9}{3} = -3$

For B and C, $m = \frac{-10-(-4)}{4-2} = \frac{-6}{2} = -3$

For A and C, $m = \frac{-10-5}{4-(-1)} = \frac{-15}{5} = -3$

Since all three slopes are the same, the points are collinear.

80. $A(0, -7)$, $B(-3, 5)$, $C(2, -15)$

For A and B, $m = \frac{5-(-7)}{-3-0} = \frac{12}{-3} = -4$

For B and C, $m = \frac{-15-5}{2-(-3)} = \frac{-20}{5} = -4$

For A and C, $m = \frac{-15-(-7)}{2-0} = \frac{-8}{2} = -4$

Since all three slopes are the same, the points are collinear.

81. $A(-1, 4)$, $B(-2, -1)$, $C(1, 14)$

For A and B, $m = \frac{-1-4}{-2-(-1)} = \frac{-5}{-1} = 5$

For B and C, $m = \frac{14-(-1)}{1-(-2)} = \frac{15}{3} = 5$

For A and C, $m = \frac{14-4}{1-(-1)} = \frac{10}{2} = 5$

Since all three slopes are the same, the points are collinear.

82. $A(0, 9)$, $B(-3, -7)$, $C(2, 19)$

For A and B, $m = \frac{-7-9}{-3-0} = \frac{-16}{-3} = \frac{16}{3}$

For B and C, $m = \frac{19-(-7)}{2-(-3)} = \frac{26}{5}$

For A and C, $m = \frac{19-9}{2-0} = \frac{10}{2} = 5$

Since all three slopes are not the same, the points are not collinear.

83. $A(-1, -3)$, $B(-5, 12)$, $C(1, -11)$

For A and B, $m = \frac{12-(-3)}{-5-(-1)} = -\frac{15}{4}$

For B and C, $m = \frac{-11-12}{1-(-5)} = -\frac{23}{6}$

For A and C, $m = \frac{-11-(-3)}{1-(-1)} = -\frac{8}{2} = -4$

Since all three slopes are not the same, the points are not collinear.

Summary Exercises on Graphs, Functions, and Equations

1. $P(3, 5),\ Q(2, -3)$

(a) $d(P, Q) = \sqrt{(2-3)^2 + (-3-5)^2}$
$= \sqrt{(-1)^2 + (-8)^2}$
$= \sqrt{1+64} = \sqrt{65}$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{3+2}{2}, \frac{5+(-3)}{2}\right) = \left(\frac{5}{2}, \frac{2}{2}\right) = \left(\frac{5}{2}, 1\right).$$

(c) First find m: $m = \frac{-3-5}{2-3} = \frac{-8}{-1} = 8$

Use either point and the point-slope form.
$y - 5 = 8(x-3)$
Change to slope-intercept form.
$y - 5 = 8x - 24 \Rightarrow y = 8x - 19$

2. $P(-1, 0),\ Q(4, -2)$

(a) $d(P, Q) = \sqrt{[4-(-1)]^2 + (-2-0)^2}$
$= \sqrt{5^2 + (-2)^2}$
$= \sqrt{25+4} = \sqrt{29}$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{-1+4}{2}, \frac{0+(-2)}{2}\right) = \left(\frac{3}{2}, \frac{-2}{2}\right) = \left(\frac{3}{2}, -1\right).$$

(c) First find m: $m = \frac{-2-0}{4-(-1)} = \frac{-2}{5} = -\frac{2}{5}$

Use either point and the point-slope form.
$y - 0 = -\frac{2}{5}[x-(-1)]$
Change to slope-intercept form.
$5y = -2(x+1)$
$5y = -2x - 2$
$y = -\frac{2}{5}x - \frac{2}{5}$

3. $P(-2, 2),\ Q(3, 2)$

(a) $d(P, Q) = \sqrt{[3-(-2)]^2 + (2-2)^2}$
$= \sqrt{5^2 + 0^2} = \sqrt{25+0} = \sqrt{25} = 5$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{-2+3}{2}, \frac{2+2}{2}\right) = \left(\frac{1}{2}, \frac{4}{2}\right) = \left(\frac{1}{2}, 2\right).$$

(c) First find m: $m = \frac{2-2}{3-(-2)} = \frac{0}{5} = 0$

All lines that have a slope of 0 are horizontal lines. The equation of a horizontal line has an equation of the form $y = b$. Since the line passes through (3, 2), the equation is $y = 2$.

4. $P\left(2\sqrt{2}, \sqrt{2}\right),\ Q\left(\sqrt{2}, 3\sqrt{2}\right)$

(a) $d(P, Q) = \sqrt{\left(\sqrt{2} - 2\sqrt{2}\right)^2 + \left(3\sqrt{2} - \sqrt{2}\right)^2}$
$= \sqrt{\left(-\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}$
$= \sqrt{2+8} = \sqrt{10}$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{2\sqrt{2}+\sqrt{2}}{2}, \frac{\sqrt{2}+3\sqrt{2}}{2}\right) = \left(\frac{3\sqrt{2}}{2}, \frac{4\sqrt{2}}{2}\right) = \left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right).$$

(c) First find m: $m = \frac{3\sqrt{2}-\sqrt{2}}{\sqrt{2}-2\sqrt{2}} = \frac{2\sqrt{2}}{-\sqrt{2}} = -2$

Use either point and the point-slope form.
$y - \sqrt{2} = -2\left(x - 2\sqrt{2}\right)$
Change to slope-intercept form.
$y - \sqrt{2} = -2x + 4\sqrt{2} \Rightarrow y = -2x + 5\sqrt{2}$

5. $P(5, -1),\ Q(5, 1)$

(a) $d(P, Q) = \sqrt{(5-5)^2 + [1-(-1)]^2}$
$= \sqrt{0^2 + 2^2} = \sqrt{0+4} = \sqrt{4} = 2$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{5+5}{2}, \frac{-1+1}{2}\right) = \left(\frac{10}{2}, \frac{0}{2}\right) = (5, 0).$$

(c) First find m.

$m = \frac{1-(-1)}{5-5} = \frac{2}{0} =$ undefined

All lines that have an undefined slope are vertical lines. The equation of a vertical line has an equation of the form $x = a$. Since the line passes through (5, 1), the equation is $x = 5$. (Since this slope of a vertical line is undefined, this equation cannot be written in slope-intercept form.)

6. $P(1, 1),\ Q(-3, -3)$

(a) $d(P, Q) = \sqrt{(-3-1)^2 + (-3-1)^2} = \sqrt{(-4)^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{1+(-3)}{2}, \frac{1+(-3)}{2}\right) = \left(\frac{-2}{2}, \frac{-2}{2}\right) = (-1, -1).$$

(c) First find m: $m = \frac{-3-1}{-3-1} = \frac{-4}{-4} = 1$

Use either point and the point-slope form.

$y - 1 = 1(x-1)$

Change to slope-intercept form.

$y - 1 = x - 1 \Rightarrow y = x$

7. $P\left(2\sqrt{3}, 3\sqrt{5}\right),\ Q\left(6\sqrt{3}, 3\sqrt{5}\right)$

(a) $d(P, Q) = \sqrt{\left(6\sqrt{3} - 2\sqrt{3}\right)^2 + \left(3\sqrt{5} - 3\sqrt{5}\right)^2} = \sqrt{\left(4\sqrt{3}\right)^2 + 0^2} = \sqrt{48} = 4\sqrt{3}$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{2\sqrt{3}+6\sqrt{3}}{2}, \frac{3\sqrt{5}+3\sqrt{5}}{2}\right) = \left(\frac{8\sqrt{3}}{2}, \frac{6\sqrt{5}}{2}\right) = \left(4\sqrt{3}, 3\sqrt{5}\right).$$

(c) First find m: $m = \frac{3\sqrt{5} - 3\sqrt{5}}{6\sqrt{3} - 2\sqrt{3}} = \frac{0}{4\sqrt{3}} = 0$

All lines that have a slope of 0 are horizontal lines. The equation of a horizontal line has an equation of the form $y = b$. Since the line passes through $\left(2\sqrt{3}, 3\sqrt{5}\right)$, the equation is $y = 3\sqrt{5}$.

8. $P(0, -4),\ Q(3, 1)$

(a) $d(P, Q) = \sqrt{(3-0)^2 + [1-(-4)]^2} = \sqrt{3^2 + 5^2} = \sqrt{9+25} = \sqrt{34}$

(b) The midpoint M of the segment joining points P and Q has coordinates

$$\left(\frac{0+3}{2}, \frac{-4+1}{2}\right) = \left(\frac{3}{2}, \frac{-3}{2}\right) = \left(\frac{3}{2}, -\frac{3}{2}\right).$$

(c) First find m: $m = \frac{1-(-4)}{3-0} = \frac{5}{3}$

Using slope-intercept form we have $y = \frac{5}{3}x - 4$.

9. Through $(-2, 1)$ and $(4, -1)$

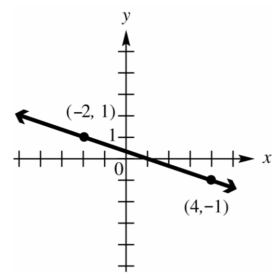
First find m: $m = \frac{-1-1}{4-(-2)} = \frac{-2}{6} = -\frac{1}{3}$

Use either point and the point-slope form.

$y - (-1) = -\frac{1}{3}(x-4)$

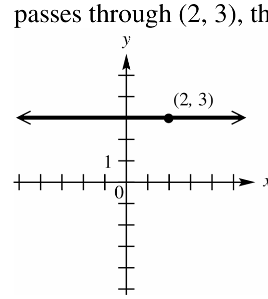
Change to slope-intercept form.

$3(y+1) = -(x-4) \Rightarrow 3y + 3 = -x + 4 \Rightarrow$
$3y = -x + 1 \Rightarrow y = -\frac{1}{3}x + \frac{1}{3}$



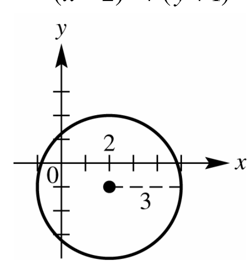
10. the horizontal line through (2, 3)

The equation of a horizontal line has an equation of the form $y = b$. Since the line passes through (2, 3), the equation is $y = 3$.



11. the circle with center $(2, -1)$ and radius 3

$(x-2)^2 + [y-(-1)]^2 = 3^2$
$(x-2)^2 + (y+1)^2 = 9$



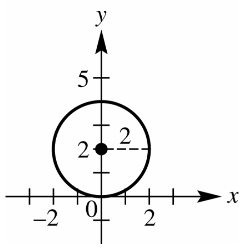
12. the circle with center (0, 2) and tangent to the x-axis

The distance from the center of the circle to the x-axis is 2, so $r = 2$.

(*continued on next page*)

(*continued from page 203*)

$(x-0)^2+(y-2)^2=2^2 \Rightarrow x^2+(y-2)^2=4$



13. the line through $(3,-5)$ with slope $-\frac{5}{6}$

Write the equation in point-slope form.

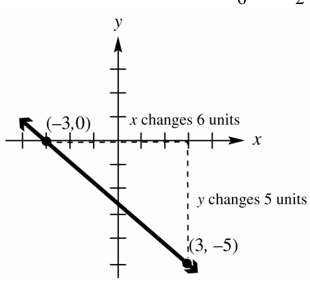
$y-(-5)=-\frac{5}{6}(x-3)$

Change to standard form.

$6(y+5)=-5(x-3) \Rightarrow 6y+30=-5x+15$

$6y=-5x-15 \Rightarrow y=-\frac{5}{6}x-\frac{15}{6}$

$y=-\frac{5}{6}x-\frac{5}{2}$



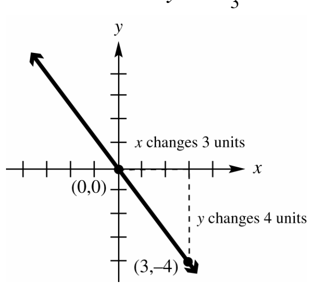
14. a line through the origin and perpendicular to the line $3x-4y=2$

First, find the slope of the line $3x-4y=2$ by writing this equation in slope-intercept form.

$3x-4y=2 \Rightarrow -4y=-3x+2 \Rightarrow$

$y=\frac{3}{4}x-\frac{2}{4} \Rightarrow y=\frac{3}{4}x-\frac{1}{2}$

This line has a slope of $\frac{3}{4}$. The slope of any line perpendicular to this line is $-\frac{4}{3}$, since $-\frac{4}{3}\left(\frac{3}{4}\right)=-1$. Using slope-intercept form we have $y=-\frac{4}{3}x+0$ or $y=-\frac{4}{3}x$.



15. a line through (–3, 2) and parallel to the line $2x+3y=6$

First, find the slope of the line $2x+3y=6$ by writing this equation in slope-intercept form.

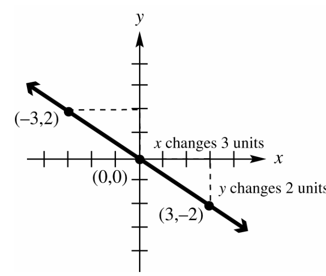
$2x+3y=6 \Rightarrow 3y=-2x+6 \Rightarrow y=-\frac{2}{3}x+2$

The slope is $-\frac{2}{3}$. Since the lines are parallel, $-\frac{2}{3}$ is also the slope of the line whose equation is to be found. Substitute $m=-\frac{2}{3}$, $x_1=-3$, and $y_1=2$ into the point-slope form.

$y-y_1=m(x-x_1) \Rightarrow y-2=-\frac{2}{3}[x-(-3)] \Rightarrow$

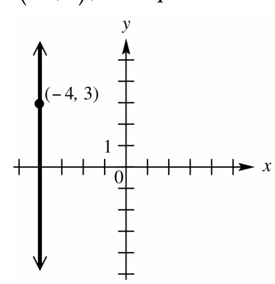
$3(y-2)=-2(x+3) \Rightarrow 3y-6=-2x-6 \Rightarrow$

$3y=-2x \Rightarrow y=-\frac{2}{3}x$



16. the vertical line through $(-4,3)$

The equation of a vertical line has an equation of the form $x = a$. Since the line passes through $(-4,3)$, the equation is $x = -4$.



17. $x^2-4x+y^2+2y=4$

Complete the square on x and y separately.

$(x^2-4x)+(y^2+2y)=4$

$(x^2-4x+4)+(y^2+2y+1)=4+4+1$

$(x-2)^2+(y+1)^2=9$

Yes, it is a circle. The circle has its center at $(2,-1)$ and radius 3.

18. $x^2+6x+y^2+10y+36=0$

Complete the square on x and y separately.

$$\left(x^2+6x\right)+\left(y^2+10y\right)=-36$$
$$\left(x^2+6x+9\right)+\left(y^2+10y+25\right)=-36+9+25$$
$$\left(x+3\right)^2+\left(y+5\right)^2=-2$$

No, it is not a circle.

19. $x^2-12x+y^2+20=0$

Complete the square on x and y separately.

$$\left(x^2-12x\right)+y^2=-20$$
$$\left(x^2-12x+36\right)+y^2=-20+36$$
$$\left(x-6\right)^2+y^2=16$$

Yes, it is a circle. The circle has its center at (6, 0) and radius 4.

20. $x^2+2x+y^2+16y=-61$

Complete the square on x and y separately.

$$\left(x^2+2x\right)+\left(y^2+16y\right)=-61$$
$$\left(x^2+2x+1\right)+\left(y^2+16y+64\right)=-61+1+64$$
$$\left(x+1\right)^2+\left(y+8\right)^2=4$$

Yes, it is a circle. The circle has its center at $\left(-1,-8\right)$ and radius 2.

21. $x^2-2x+y^2+10=0$

Complete the square on x and y separately.

$$\left(x^2-2x\right)+y^2=-10$$
$$\left(x^2-2x+1\right)+y^2=-10+1$$
$$\left(x-1\right)^2+y^2=-9$$

No, it is not a circle.

22. $x^2+y^2-8y-9=0$

Complete the square on x and y separately.

$$x^2+\left(y^2-8y\right)=9$$
$$x^2+\left(y^2-8y+16\right)=9+16$$
$$x^2+\left(y-4\right)^2=25$$

Yes, it is a circle. The circle has its center at $\left(0,4\right)$ and radius 5.

23. The equation of the circle is $(x-4)^2+(y-5)^2=4^2$. Let $y=2$ and solve for x: $(x-4)^2+(2-5)^2=4^2\Rightarrow$

$(x-4)^2+(-3)^2=4^2\Rightarrow(x-4)^2=7\Rightarrow$

$x-4=\pm\sqrt{7}\Rightarrow x=4\pm\sqrt{7}$

The points of intersection are $\left(4+\sqrt{7},2\right)$ and $\left(4-\sqrt{7},2\right)$

24. Write the equation in center-radius form by completing the square on x and y separately:

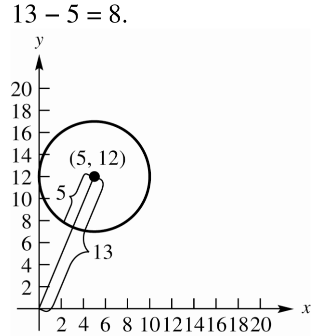
$$x^2+y^2-10x-24y+144=0$$
$$\left(x^2-10x+\quad\right)+\left(y^2-24y+144\right)=0$$
$$(x^2-10x+25)+(y^2-24y+144)=25$$
$$(x-5)^2+(y-12)^2=25$$

The center of the circle is (5, 12) and the radius is 5.

Now use the distance formula to find the distance from the center (5, 12) to the origin:

$$d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$$
$$=\sqrt{(5-0)^2+(12-0)^2}=\sqrt{25+144}$$
$$=\sqrt{169}=13$$

Since the radius is 5, the shortest distance from the origin to the graph of the circle is 13 − 5 = 8.



25. **(a)** The equation can be rewritten as $-4y=-x-6\Rightarrow y=\frac{1}{4}x+\frac{6}{4}\Rightarrow y=\frac{1}{4}x+\frac{3}{2}$.

x can be any real number, so the domain is all real numbers and the range is also all real numbers.

domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$

(b) Each value of x corresponds to just one value of y. $x-4y=-6$ represents a function.

$y=\frac{1}{4}x+\frac{3}{2}\Rightarrow f\left(x\right)=\frac{1}{4}x+\frac{3}{2}$

$f\left(-2\right)=\frac{1}{4}\left(-2\right)+\frac{3}{2}=-\frac{1}{2}+\frac{3}{2}=\frac{2}{2}=1$

26. **(a)** The equation can be rewritten as $y^2 - 5 = x$. y can be any real number. Since the square of any real number is not negative, y^2 is never negative. Taking the constant term into consideration, domain would be $[-5, \infty)$.
domain: $[-5, \infty)$; range: $(-\infty, \infty)$

(b) Since $(-4, 1)$ and $(-4, -1)$ both satisfy the relation, $y^2 - x = 5$ does not represent a function.

27. **(a)** $(x+2)^2 + y^2 = 25$ is a circle centered at $(-2, 0)$ with a radius of 5. The domain will start 5 units to the left of –2 and end 5 units to the right of –2. The domain will be $[-2-5, -2+5] = [-7, 3]$. The range will start 5 units below 0 and end 5 units above 0. The range will be $[0-5, 0+5] = [-5, 5]$.

(b) Since $(-2, 5)$ and $(-2, -5)$ both satisfy the relation, $(x+2)^2 + y^2 = 25$ does not represent a function.

28. **(a)** The equation can be rewritten as $-2y = -x^2 + 3 \Rightarrow y = \frac{1}{2}x^2 - \frac{3}{2}$. x can be any real number. Since the square of any real number is not negative, $\frac{1}{2}x^2$ is never negative. Taking the constant term into consideration, range would be $\left[-\frac{3}{2}, \infty\right)$.
domain: $(-\infty, \infty)$; range: $\left[-\frac{3}{2}, \infty\right)$

(b) Each value of x corresponds to just one value of y. $x^2 - 2y = 3$ represents a function.
$y = \frac{1}{2}x^2 - \frac{3}{2} \Rightarrow f(x) = \frac{1}{2}x^2 - \frac{3}{2}$
$f(-2) = \frac{1}{2}(-2)^2 - \frac{3}{2} = \frac{1}{2}(4) - \frac{3}{2} = \frac{4}{2} - \frac{3}{2} = \frac{1}{2}$

Section 2.6: Graphs of Basic Functions

1. The equation $y = x^2$ matches graph E. The domain is $(-\infty, \infty)$.

2. The equation of y = $|x|$ matches graph G. The function is increasing on $[0, \infty)$.

3. The equation $y = x^3$ matches graph A. The range is $(-\infty, \infty)$.

4. Graph C is not the graph of a function. Its equation is $x = y^2$.

5. Graph F is the graph of the identity function. Its equation is $y = x$.

6. The equation $y = [\![x]\!]$ matches graph B. $y = [\![1.5]\!] = 1$

7. The equation $y = \sqrt[3]{x}$ matches graph H. No, there is no interval over which the function is decreasing.

8. The equation of $y = \sqrt{x}$ matches graph D. The domain is $[0, \infty)$.

9. The graph in B is discontinuous at many points. Assuming the graph continues, the range would be $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

10. The graphs in E and G decrease over part of the domain and increase over part of the domain. They both decrease over $(-\infty, 0]$ and increase over $[0, \infty)$.

11. The function is continuous over the entire domain of real numbers $(-\infty, \infty)$.

12. The function is continuous over the entire domain of real numbers $(-\infty, \infty)$.

13. The function is continuous over the interval $[0, \infty)$.

14. The function is continuous over the interval $(-\infty, 0]$.

15. The function has a point of discontinuity at $x = 1$. It is continuous over the interval $(-\infty, 1)$ and the interval $[1, \infty)$.

16. The function has a point of discontinuity at $x = 1$. It is continuous over the interval $(-\infty, 1)$ and the interval $(1.\infty)$.

17. $f(x) = \begin{cases} 2x & \text{if } x \le -1 \\ x - 1 & \text{if } x > -1 \end{cases}$

(a) $f(-5) = 2(-5) = -10$

(b) $f(-1) = 2(-1) = -2$

(c) $f(0) = 0 - 1 = -1$

(d) $f(3) = 3 - 1 = 2$

18. $f(x) = \begin{cases} x - 2 & \text{if } x < 3 \\ 5 - x & \text{if } x \geq 3 \end{cases}$

(a) $f(-5) = -5 - 2 = -7$

(b) $f(-1) = -1 - 2 = -3$

(c) $f(0) = 0 - 2 = -2$

(d) $f(3) = 5 - 3 = 2$

19. $f(x) = \begin{cases} 2 + x & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$

(a) $f(-5) = 2 + (-5) = -3$

(b) $f(-1) = -(-1) = 1$

(c) $f(0) = -0 = 0$

(d) $f(3) = 3 \cdot 3 = 9$

20. $\begin{cases} -2x & \text{if } x < -3 \\ 3x - 1 & \text{if } -3 \leq x \leq 2 \\ -4x & \text{if } x > 2 \end{cases}$

(a) $f(-5) = -2(-5) = 10$

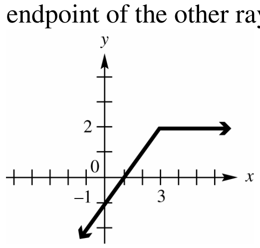
(b) $f(-1) = 3(-1) - 1 = -3 - 1 = -4$

(c) $f(0) = 3(0) - 1 = 0 - 1 = -1$

(d) $f(3) = -4(3) = -12$

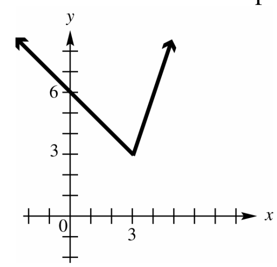
21. $f(x) = \begin{cases} x - 1 & \text{if } x \leq 3 \\ 2 & \text{if } x > 3 \end{cases}$

Draw the graph of $y = x - 1$ to the left of $x = 3$, including the endpoint at $x = 3$. Draw the graph of $y = 2$ to the right of $x = 3$, and note that the endpoint at $x = 3$ coincides with the endpoint of the other ray.



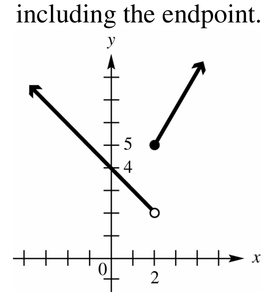
22. $f(x) = \begin{cases} 6 - x & \text{if } x \leq 3 \\ 3x - 6 & \text{if } x > 3 \end{cases}$

Graph the line $y = 6 - x$ to the left of $x = 3$, including the endpoint. Draw $y = 3x - 6$ to the right of $x = 3$. Note that the endpoint at $x = 3$ coincides with the endpoint of the other ray.



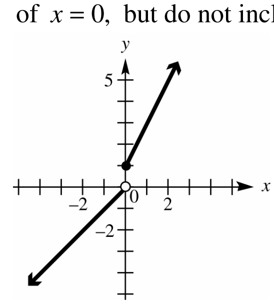
23. $f(x) = \begin{cases} 4 - x & \text{if } x < 2 \\ 1 + 2x & \text{if } x \geq 2 \end{cases}$

Draw the graph of $y = 4 - x$ to the left of $x = 2$, but do not include the endpoint. Draw the graph of $y = 1 + 2x$ to the right of $x = 2$, including the endpoint.



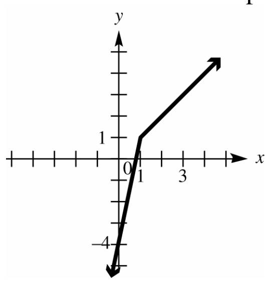
24. $f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

Graph the line $y = 2x + 1$ to the right of $x = 0$, including the endpoint. Draw $y = x$ to the left of $x = 0$, but do not include the endpoint.



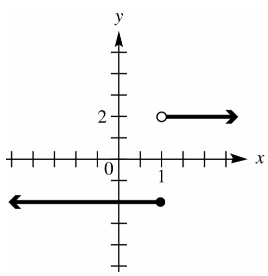
25. $f(x)=\begin{cases}5x-4 & \text{if } x\le 1\\ x & \text{if } x>1\end{cases}$

Graph the line $y=5x-4$ to the left of $x=1$, including the endpoint. Draw $y=x$ to the right of $x=1$; note that the endpoint at $x=1$ coincides with the endpoint of the other ray.



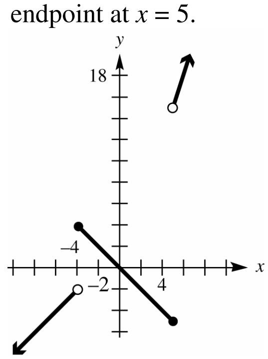
26. $f(x)=\begin{cases}-2 & \text{if } x\le 1\\ 2 & \text{if } x>1\end{cases}$

Graph the line $y=-2$ to the left of $x=1$, including the endpoint. Draw $y=2$ to the right of $x=1$, but do not include the endpoint.



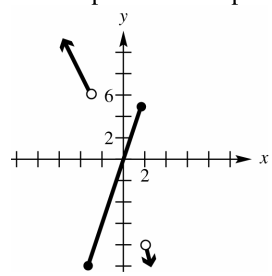
27. $f(x)=\begin{cases}2+x & \text{if } x<-4\\ -x & \text{if } -4\le x\le 5\\ 3x & \text{if } x>5\end{cases}$

Draw the graph of $y=2+x$ to the left of –4, but do not include the endpoint at $x=4$. Draw the graph of $y=-x$ between –4 and 5, including both endpoints. Draw the graph of $y=3x$ to the right of 5, but do not include the endpoint at $x=5$.



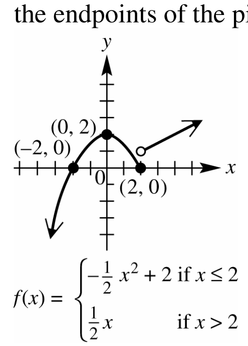
28. $f(x)=\begin{cases}-2x & \text{if } x<-3\\ 3x-1 & \text{if } -3\le x\le 2\\ -4x & \text{if } x>2\end{cases}$

Graph the line $y=-2x$ to the left of $x=-3$, but do not include the endpoint. Draw $y=3x-1$ between $x=-3$ and $x=2$, and include both endpoints. Draw $y=-4x$ to the right of $x=2$, but do not include the endpoint. Notice that the endpoints of the pieces do not coincide.



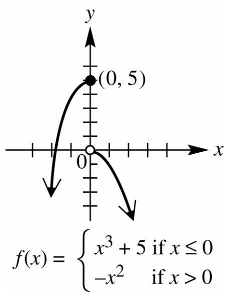
29. $f(x)=\begin{cases}-\frac{1}{2}x^2+2 & \text{if } x\le 2\\ \frac{1}{2}x & \text{if } x>2\end{cases}$

Graph the curve $y=-\frac{1}{2}x^2+2$ to the left of $x=2$, including the end point at (2, 0). Graph the line $y=\frac{1}{2}x$ to the right of $x=2$, but do not include the endpoint at (2, 1). Notice that the endpoints of the pieces do not coincide.



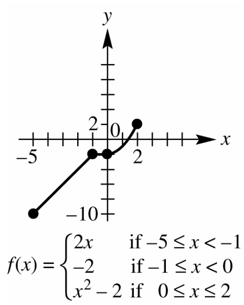
30. $f(x)=\begin{cases}x^3+5 & \text{if } x\le 0\\ -x^2 & \text{if } x>0\end{cases}$

Graph the curve $y=x^3+5$ to the left of $x=0$, including the end point at (0, 5). Graph the line $y=-x^2$ to the right of $x=0$, but do not include the endpoint at (0, 0). Notice that the endpoints of the pieces do not coincide.



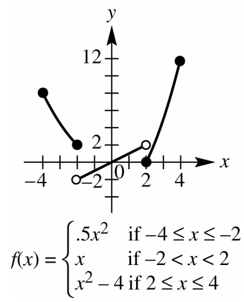
31. $f(x)=\begin{cases}2x & \text{if } -5\le x<-1\\ -2 & \text{if } -1\le x<0\\ x^2-2 & \text{if } 0\le x\le 2\end{cases}$

Graph the line $y = 2x$ between $x = -5$ and $x = -1$, including the left end point at $(-5, -10)$, but not including the right endpoint at $(-1, -2)$. Graph the line $y = -2$ between $x = -1$ and $x = 0$, including the left endpoint at $(-1, -2)$ and not including the right endpoint at $(0, -2)$. Note that $(-1, -2)$ coincides with the first two sections, so it is included. Graph the curve $y = x^2 - 2$ from $x = 0$ to $x = 2$, including the endpoints at $(0, -2)$ and $(2, 2)$. Note that $(0, -2)$ coincides with the second two sections, so it is included. The graph ends at $x = -5$ and $x = 2$.



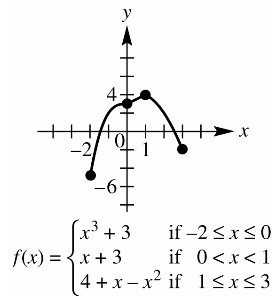
32. $f(x)=\begin{cases}0.5x^2 & \text{if } -4\le x\le -2\\ x & \text{if } -2<x<2\\ x^2-4 & \text{if } 2\le x\le 4\end{cases}$

Graph the curve $y = 0.5x^2$ between $x = -4$ and $x = -2$, including the endpoints at $(-4, 8)$ and $(-2, 2)$. Graph the line $y = x$ between $x = -2$ and $x = 2$, but do not include the endpoints at $(-2, -2)$ and $(2, 2)$. Graph the curve $y = x^2 - 4$ from $x = 2$ to $x = 4$, including the endpoints at $(2, 0)$ and $(4, 12)$. The graph ends at $x = -4$ and $x = 4$.



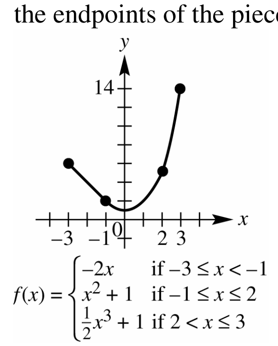
33. $f(x)=\begin{cases}x^3+3 & \text{if } -2\le x\le 0\\ x+3 & \text{if } 0<x<1\\ 4+x-x^2 & \text{if } 1\le x\le 3\end{cases}$

Graph the curve $y = x^3 + 3$ between $x = -2$ and $x = 0$, including the endpoints at $(-2, -5)$ and $(0, 3)$. Graph the line $y = x + 3$ between $x = 0$ and $x = 1$, but do not include the endpoints at $(0, 3)$ and $(1, 4)$. Graph the curve $y = 4 + x - x^2$ from $x = 1$ to $x = 3$, including the endpoints at $(1, 4)$ and $(3, -2)$. The graph ends at $x = -2$ and $x = 3$.



34. $f(x)=\begin{cases}-2x & \text{if } -3\le x<-1\\ x^2+1 & \text{if } -1\le x\le 2\\ \frac{1}{2}x^3+1 & \text{if } 2<x\le 3\end{cases}$

Graph the curve $y = -\frac{1}{2}x^2 + 2$ to the left of $x = 2$, including the end point at $(2, 0)$. Graph the line $y = \frac{1}{2}x$ to the right of $x = 2$, but do not include the endpoint at $(2, 1)$. Notice that the endpoints of the pieces do not coincide.



35. The solid circle on the graph shows that the endpoint (0, –1) is part of the graph, while the open circle shows that the endpoint (0, 1) is not part of the graph. The graph is made up of parts of two horizontal lines. The function which fits this graph is

$$f(x)=\begin{cases}-1 & \text{if } x\le 0\\ 1 & \text{if } x>0.\end{cases}$$

domain: $(-\infty,\infty)$; range: $\{-1, 1\}$

36. We see that $y = 1$ for every value of x except $x = 0$, and that when $x = 0$, $y = 0$. We can write the function as

$$f(x)=\begin{cases}1 & \text{if } x\ne 0\\ 0 & \text{if } x=0.\end{cases}$$

domain: $(-\infty,\infty)$; range: $\{0, 1\}$

37. The graph is made up of parts of two horizontal lines. The solid circle shows that the endpoint (0, 2) of the one on the left belongs to the graph, while the open circle shows that the endpoint (0, –1) of the one on the right does not belong to the graph. The function that fits this graph is

$$f(x)=\begin{cases}2 & \text{if } x\le 0\\ -1 & \text{if } x>1.\end{cases}$$

domain: $(-\infty,0]\cup(1,\infty)$; range: $\{-1, 2\}$

38. We see that $y = 1$ when $x\le -1$ and that $y = -1$ when $x > 2$. We can write the function as

$$f(x)=\begin{cases}1 & \text{if } x\le -1\\ -1 & \text{if } x>2.\end{cases}$$

domain: $(-\infty,-1]\cup(2,\infty)$; range: $\{-1, 1\}$

39. For $x\le 0$, that piece of the graph goes through the points (–1, –1) and (0, 0). The slope is 1, so the equation of this piece is $y = x$. For $x > 0$, that piece of the graph is a horizontal line passing through (2, 2), so its equation is $y = 2$. We can write the function as

$$f(x)=\begin{cases}x & \text{if } x\le 0\\ 2 & \text{if } x>0\end{cases}$$

domain: $(-\infty,\infty)$ range: $(-\infty,0]\cup\{2\}$

40. For $x < 0$, that piece of the graph is a horizontal line passing though (–3, –3), so the equation of this piece is $y = -3$. For $x\ge 0$, the curve passes through (1, 1) and (4, 2), so the equation of this piece is $y=\sqrt{x}$. We can write the function as $f(x)=\begin{cases}-3 & \text{if } x<0\\ \sqrt{x} & \text{if } x\ge 0\end{cases}$

domain: $(-\infty,\infty)$ range: $\{-3\}\cup[0,\infty)$

41. For $x < 1$, that piece of the graph is a curve passes through (–8, –2), (–1, –1) and (1, 1), so the equation of this piece is $y=\sqrt[3]{x}$. The right piece of the graph passes through (1, 2) and (2, 3). $m=\dfrac{2-3}{1-2}=1$, and the equation of the line is $y-2=x-1\Rightarrow y=x+1$. We can write the function as $f(x)=\begin{cases}\sqrt[3]{x} & \text{if } x<1\\ x+1 & \text{if } x\ge 1\end{cases}$

domain: $(-\infty,\infty)$ range: $(-\infty,1)\cup[2,\infty)$

42. For all values except $x = 2$, the graph is a line. It passes through (0, –3) and (1, 1). The slope is 2, so the equation is $y = 2x - 3$. At $x = 2$, the graph is the point (2, 3). We can write the function as $f(x)=\begin{cases}3 & \text{if } x=2\\ 2x-3 & \text{if } x\ne 2\end{cases}$

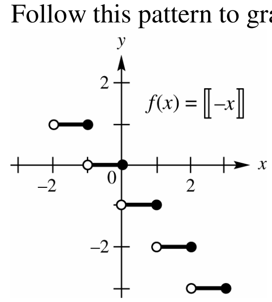
domain: $(-\infty,\infty)$ range: $(-\infty,1)\cup(1,\infty)$

43. $f(x)=[\![-x]\!]$

Plot points.

x	$-x$	$f(x)=[\![-x]\!]$
–2	2	2
–1.5	1.5	1
–1	1	1
–0.5	0.5	0
0	0	0
0.5	–0.5	–1
1	–1	–1
1.5	–1.5	–2
2	–2	–2

More generally, to get $y = 0$, we need $0\le -x<1\Rightarrow 0\ge x>-1\Rightarrow -1<x\le 0$.
To get $y = 1$, we need $1\le -x<2\Rightarrow$ $-1\ge x>-2\ \Rightarrow -2<x\le -1$.
Follow this pattern to graph the step function.



domain: $(-\infty,\infty)$; range: $\{\ldots,-2,-1,0,1,2,\ldots\}$

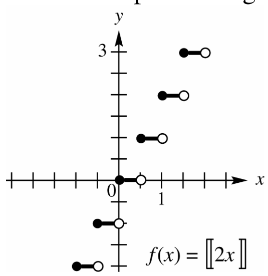
44. $f(x) = [\![2x]\!]$

To get $y = 0$, we need $0 \le 2x < 1 \Rightarrow 0 \le x < \frac{1}{2}$.

To get $y = 1$, we need $1 \le 2x < 2 \Rightarrow \frac{1}{2} \le x < 1$.

To get $y = 2$, we need $2 \le 2x < 3 \Rightarrow 1 \le x < \frac{3}{2}$.

Follow this pattern to graph the step function.



domain: $(-\infty, \infty)$; range: $\{..., -2, -1, 0, 1, 2, ...\}$

45. $g(x) = [\![2x - 1]\!]$

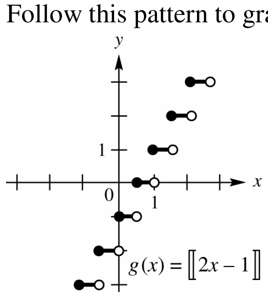
To get $y = 0$, we need

$0 \le 2x - 1 < 1 \Rightarrow 1 \le 2x < 2 \Rightarrow \frac{1}{2} \le x < 1$.

To get $y = 1$, we need

$1 \le 2x - 1 < 2 \Rightarrow 2 \le 2x < 3 \Rightarrow 1 \le x < \frac{3}{2}$.

Follow this pattern to graph the step function.



domain: $(-\infty, \infty)$; range: $\{..., 2, -1, 0, 1, 2, ...\}$

46. The function value is half the integer.

If $x = 2$, then $f(2) = \left[\!\left[\frac{1}{2}(2)\right]\!\right] = [\![1]\!] = 1$, if

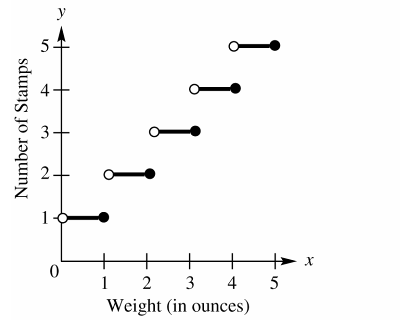
$x = 4$, then $f(4) = \left[\!\left[\frac{1}{2}(4)\right]\!\right] = [\![2]\!] = 2$, etc.

In general, if x is an even integer, it is of the form $2n$, where n is an integer.

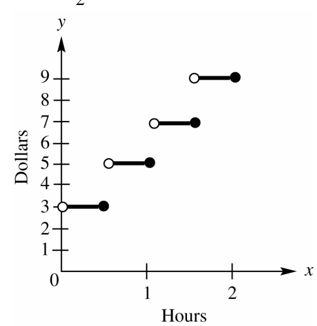
$$f(x) = f(2n) = \left[\!\left[\frac{1}{2}(2n)\right]\!\right] = [\![n]\!] = n$$

Since $x = 2n$, then $n = \frac{1}{2}x$.

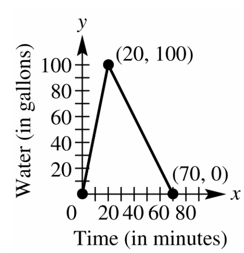
47. The cost of mailing a letter that weights more than 1 ounce and less than 2 ounces is the same as the cost of a 2-ounce letter, and the cost of mailing a letter that weighs more than 2 ounces and less than 3 ounces is the same as the cost of a 3-ounce letter, etc.



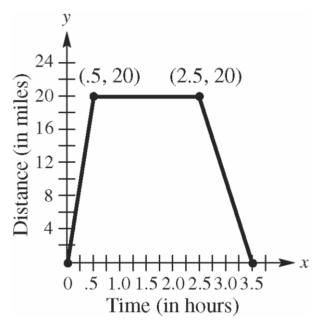
48. The cost is the same for all cars parking between $\frac{1}{2}$ hour and 1-hour, between 1 hour and $1\frac{1}{2}$ hours, etc.



49.



50.



51. **(a)** For $0 \le x \le 4$, $m = \frac{39.2 - 42.8}{4 - 0} = -0.9$, so $y = -0.9x + 42.8$. For $4 < x \le 8$, $m = \frac{32.7 - 39.2}{8 - 4} = -1.625$, so the equation is $y - 32.7 = -1.625(x - 8) \Rightarrow$ $y = -1.625x + 45.7$

(b) $f(x) = \begin{cases} -0.9x + 42.8 & \text{if } 0 \le x \le 4 \\ -1.625x + 45.7 & \text{if } 4 < x \le 8 \end{cases}$

52. When $0 \le x \le 3$, the slope is 5, which means that the inlet pipe is open, and the outlet pipe is closed. When $3 < x \le 5$, the slope is 2, which means that both pipes are open. When $5 < x \le 8$, the slope is 0, which means that both pipes are closed. When $8 < x \le 10$, the slope is −3, which means that the inlet pipe is closed, and the outlet pipe is open.

53. **(a)** The initial amount is 50,000 gallons. The final amount is 30,000 gallons.

(b) The amount of water in the pool remained constant during the first and fourth days.

(c) $f(2) \approx 45{,}000$; $f(4) = 40{,}000$

(d) The slope of the segment between (1, 50000) and (3, 40000) is −5000, so the water was being drained at 5000 gallons per day.

54. **(a)** There were 20 gallons of gas in the tank at $x = 3$.

(b) The slope is steepest between $t = 1$ and $t \approx 2.9$, so that is when the car burned gasoline at the fastest rate.

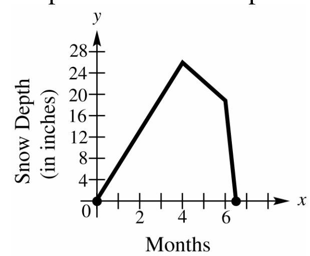
55. **(a)** Since there is no charge for additional length, we use the greatest integer function. The cost is based on multiples of two feet, so $f(x) = 0.8\left[\!\left[\frac{x}{2}\right]\!\right]$ if $6 \le x \le 18$.

(b) $f(8.5) = 0.8\left[\!\left[\frac{8.5}{2}\right]\!\right] = 0.8(4) = \$3.20$

$f(15.2) = 0.8\left[\!\left[\frac{15.2}{2}\right]\!\right] = 0.8(7) = \$5.60$

56. **(a)** $f(x) = \begin{cases} 6.5x & \text{if } 0 \le x \le 4 \\ -5.5x + 48 & \text{if } 4 < x \le 6 \\ -30x + 195 & \text{if } 6 < x \le 6.5 \end{cases}$

Draw a graph of $y = 6.5x$ between 0 and 4, including the endpoints. Draw the graph of $y = -5.5x + 48$ between 4 and 6, including the endpoint at 6 but not the one at 4. Draw the graph of $y = -30x + 195$, including the endpoint at 6.5 but not the one at 6. Notice that the endpoints of the three pieces coincide.



(b) From the graph, observe that the snow depth, y, reaches its deepest level (26 in.) when $x = 4$, $x = 4$ represents 4 months after the beginning of October, which is the beginning of February.

(c) From the graph, the snow depth y is nonzero when x is between 0 and 6.5. Snow begins at the beginning of October and ends 6.5 months later, in the middle of April.

Section 2.7: Graphing Techniques

Connections (page 269)

Answers will vary.

Exercises

1. **(a)** B; $y = (x-7)^2$ is a shift of $y = x^2$, 7 units to the right.

(b) D; $y = x^2 - 7$ is a shift of $y = x^2$, 7 units downward.

(c) E; $y = 7x^2$ is a vertical stretch of $y = x^2$, by a factor of 7.

(d) A; $y = (x+7)^2$ is a shift of $y = x^2$, 7 units to the left.

(e) C; $y = x^2 + 7$ is a shift of $y = x^2$, 7 units upward.

2. **(a)** E; $y = 4\sqrt[3]{x}$ is a vertical stretch of $y = \sqrt[3]{x}$, by a factor of 4.

(b) C; $y=-\sqrt[3]{x}$ is a reflection of $y=\sqrt[3]{x}$, over the x-axis.

(c) D; $y=\sqrt[3]{-x}$ is a reflection of $y=\sqrt[3]{x}$, over the y-axis.

(d) A; $y=\sqrt[3]{x-4}$ is a shift of $y=\sqrt[3]{x}$, 4 units to the right.

(e) B; $y=\sqrt[3]{x}-4$ is a shift of $y=\sqrt[3]{x}$, 4 units down.

3. **(a)** B; $y=x^2+2$ is a shift of $y=x^2$, 2 units upward.

(b) A; $y=x^2-2$ is a shift of $y=x^2$, 2 units downward.

(c) G; $y=(x+2)^2$ is a shift of $y=x^2$, 2 units to the left.

(d) C; $y=(x-2)^2$ is a shift of $y=x^2$, 2 units to the right.

(e) F; $y=2x^2$ is a vertical stretch of $y=x^2$, by a factor of 2.

(f) D; $y=-x^2$ is a reflection of $y=x^2$, across the x-axis.

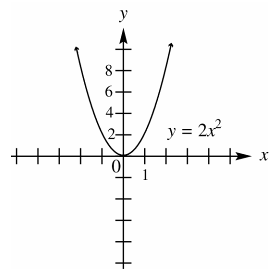
(g) H; $y=(x-2)^2+1$ is a shift of $y=x^2$, 2 units to the right and 1 unit upward.

(h) E; $y=(x+2)^2+1$ is a shift of $y=x^2$, 2 units to the left and 1 unit upward.

(i) I; $y=(x+2)^2-1$ is a shift of $y=x^2$, 2 units to the left and 1 unit down.

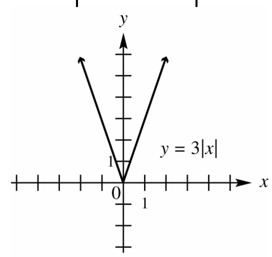
4. $y=2x^2$

x	$y=x^2$	$y=2x^2$
−2	4	8
−1	1	2
0	0	0
1	1	2
2	4	8



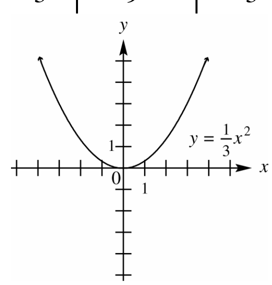
5. $y=3|x|$

x	$y=\|x\|$	$y=3\|x\|$
−2	2	6
−1	1	3
0	0	0
1	1	3
2	2	6



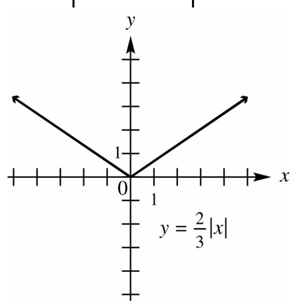
6. $y=\frac{1}{3}x^2$

x	$y=x^2$	$y=\frac{1}{3}x^2$
−3	9	3
−2	4	$\frac{4}{3}$
−1	1	$\frac{1}{3}$
0	0	0
1	1	$\frac{1}{3}$
2	4	$\frac{4}{3}$
3	9	3



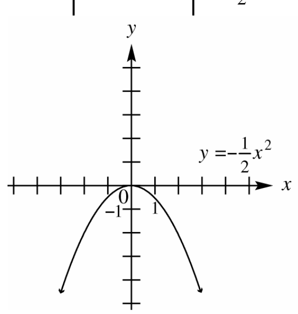
7. $y = \frac{2}{3}|x|$

x	$y = \|x\|$	$y = \frac{2}{3}\|x\|$
−3	3	2
−2	2	$\frac{4}{3}$
−1	1	$\frac{2}{3}$
0	0	0
1	1	$\frac{2}{3}$
2	2	$\frac{4}{3}$
3	3	2



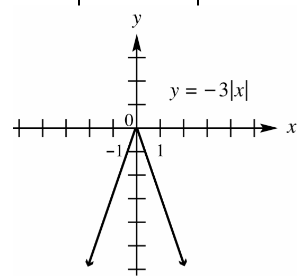
8. $y = -\frac{1}{2}x^2$

x	$y = x^2$	$y = -\frac{1}{2}x^2$
−3	9	$-\frac{9}{2}$
−2	4	−2
−1	1	$-\frac{1}{2}$
0	0	0
1	1	$-\frac{1}{2}$
2	4	−2
3	9	$-\frac{9}{2}$



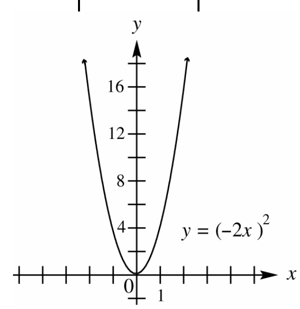
9. $y = -3|x|$

x	$y = \|x\|$	$y = -3\|x\|$
−2	2	−6
−1	1	−3
0	0	0
1	1	−3
2	2	−6



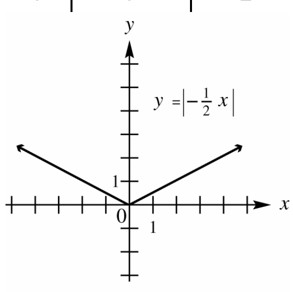
10. $y = (-2x)^2$

x	$y = x^2$	$y = (-2x)^2 = (-2)^2 x^2 = 4x^2$
−2	4	16
−1	1	4
0	0	0
1	1	4
2	4	16



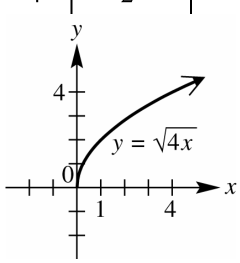
11. $y = \left|-\frac{1}{2}x\right|$

x	$y = \lvert x \rvert$	$y = \left\lvert -\frac{1}{2}x \right\rvert = \left\lvert -\frac{1}{2}\right\rvert \lvert x \rvert = \frac{1}{2}\lvert x \rvert$
−4	4	2
−3	3	$\frac{3}{2}$
−2	2	1
−1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	2	1
3	3	$\frac{3}{2}$
4	4	2



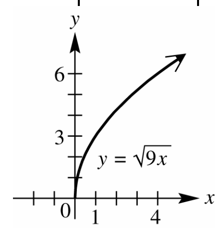
12. $y = \sqrt{4x}$

x	$y = \sqrt{x}$	$y = \sqrt{4x} = 2\sqrt{x}$
0	0	0
1	1	2
2	$\sqrt{2}$	$2\sqrt{2}$
3	$\sqrt{3}$	$2\sqrt{3}$
4	2	4



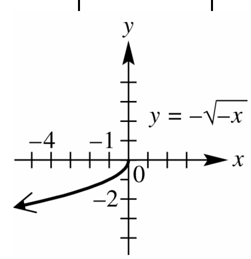
13. $y = \sqrt{9x}$

x	$y = \sqrt{x}$	$y = \sqrt{4x} = 3\sqrt{x}$
0	0	0
1	1	3
2	$\sqrt{2}$	$3\sqrt{2}$
3	$\sqrt{3}$	$3\sqrt{3}$
4	2	6



14. $y = -\sqrt{-x}$

x	$y = \sqrt{-x}$	$y = -\sqrt{-x}$
−4	2	−2
−3	$\sqrt{3}$	$-\sqrt{3}$
−2	$\sqrt{2}$	$-\sqrt{2}$
−1	1	−1
0	0	0



15. (a) $y = f(x+4)$ is a horizontal translation of *f*, 4 units to the left. The point that corresponds to $(8,12)$ on this translated function would be $(8-4,12) = (4,12)$.

(b) $y = f(x)+4$ is a vertical translation of *f*, 4 units up. The point that corresponds to $(8,12)$ on this translated function would be $(8,12+4) = (8,16)$.

16. **(a)** $y = \frac{1}{4} f(x)$ is a vertical shrinking of *f*, by a factor of $\frac{1}{4}$. The point that corresponds to $(8,12)$ on this translated function would be $\left(8, \frac{1}{4} \cdot 12\right) = (8,3)$.

(b) $y = 4f(x)$ is a vertical stretching of *f*, by a factor of 4. The point that corresponds to $(8,12)$ on this translated function would be $(8, 4 \cdot 12) = (8,48)$.

17. **(a)** $y = f(4x)$ is a horizontal shrinking of *f*, by a factor of 4. The point that corresponds to $(8,12)$ on this translated function is $\left(8 \cdot \frac{1}{4}, 12\right) = (2,12)$.

(b) $y = f\left(\frac{1}{4}x\right)$ is a horizontal stretching of *f*, by a factor of 4. The point that corresponds to $(8,12)$ on this translated function is $(8 \cdot 4, 12) = (32,12)$.

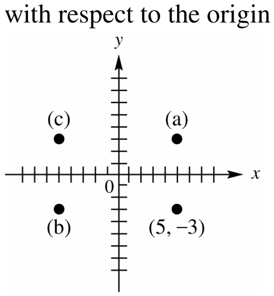
18. **(a)** The point that corresponds to $(8,12)$ when reflected across the x-axis would be $(8,-12)$.

(b) The point that corresponds to $(8,12)$ when reflected across the y-axis would be $(-8,12)$.

19. **(a)** The point that is symmetric to (5, –3) with respect to the *x*-axis is (5, 3).

(b) The point that is symmetric to (5, –3) with respect to the *y*-axis is (–5, –3).

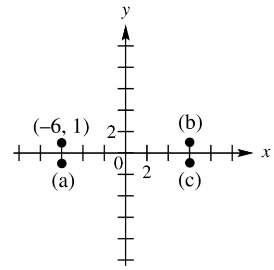
(c) The point that is symmetric to (5, –3) with respect to the origin is (–5, 3).



20. **(a)** The point that is symmetric to (–6, 1) with respect to the *x*-axis is (–6, –1).

(b) The point that is symmetric to (–6, 1) with respect to the *y*-axis is (6, 1).

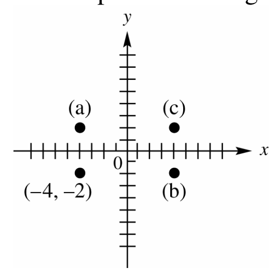
(c) The point that is symmetric to (–6, 1) with respect to the origin is (6, –1).



21. **(a)** The point that is symmetric to (–4, –2) with respect to the *x*-axis is (–4, 2).

(b) The point that is symmetric to (–4, –2) with respect to the *y*-axis is (4, –2).

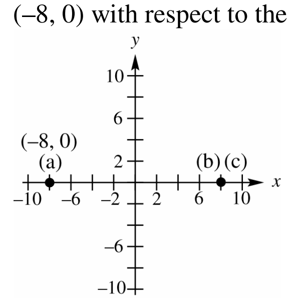
(c) The point that is symmetric to (–4, –2) with respect to the origin is (4, 2).



22. **(a)** The point that is symmetric to (–8, 0) with respect to the x-axis is (–8, 0), since this point lies on the *x*-axis.

(b) The point that is symmetric to the point (–8, 0) with respect to the *y*-axis is (8, 0).

(c) The point that is symmetric to the point (–8, 0) with respect to the origin is (8, 0).



23. $y = x^2 + 5$

Replace *x* with –*x* to obtain $y = (-x)^2 + 5 = x^2 + 5$. The result is the same as the original equation, so the graph is symmetric with respect to the *y*-axis. Since *y* is a function of *x*, the graph cannot be symmetric with respect to the *x*-axis. Replace *x* with –*x* and *y* with –*y* to obtain $-y = (-x)^2 + 2 \Rightarrow -y = x^2 + 2 \Rightarrow y = -x^2 - 2$.

The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the y-axis only.

24. $y = 2x^4 - 3$
Replace x with $-x$ to obtain
$y = 2(-x)^4 - 3 = 2x^4 - 3$
The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Since y is a function of x, the graph cannot be symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $-y = 2(-x)^4 - 3 \Rightarrow -y = 2x^4 - 3 \Rightarrow$ $y = -2x^4 + 3$. The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the y-axis only.

25. $x^2 + y^2 = 12$
Replace x with $-x$ to obtain
$(-x)^2 + y^2 = 12 \Rightarrow x^2 + y^2 = 12$.
The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Replace y with $-y$ to obtain
$x^2 + (-y)^2 = 12 \Rightarrow x^2 + y^2 = 12$
The result is the same as the original equation, so the graph is symmetric with respect to the x-axis. Since the graph is symmetric with respect to the x-axis and y-axis, it is also symmetric with respect to the origin.

26. $y^2 = \dfrac{-6}{x^2}$
Replace x with $-x$ to obtain
$y^2 = \dfrac{-6}{(-x)^2} \Rightarrow y^2 = \dfrac{-6}{x^2}$.
The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Replace y with $-y$ to obtain
$(-y)^2 = \dfrac{-6}{x^2} \Rightarrow y^2 = \dfrac{-6}{x^2}$.
The result is the same as the original equation, so the graph is symmetric with respect to the x-axis. Since the graph is symmetric with respect to the x-axis and y-axis, it is also symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the x-axis, the y-axis, and the origin.

27. $y = -4x^3$
Replace x with $-x$ to obtain
$y = -4(-x)^3 \Rightarrow y = -4(-x^3) \Rightarrow y = 4x^3$.
The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Replace y with $-y$ to obtain $-y = -4x^3 \Rightarrow y = 4x^3$.
The result is not the same as the original equation, so the graph is not symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain
$-y = -4(-x)^3 \Rightarrow -y = -4(-x^3) \Rightarrow$
$-y = 4x^3 \Rightarrow y = -4x^3$.
The result is the same as the original equation, so the graph is symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the origin only.

28. $y = x^3 - x$
Replace x with $-x$ to obtain
$y = (-x)^3 - (-x) \Rightarrow y = -x^3 + x$.
The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Replace y with $-y$ to obtain $-y = x^3 - x \Rightarrow y = -x^3 + x$. The result is not the same as the original equation, so the graph is not symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $-y = (-x)^3 - (-x) \Rightarrow -y = -x^3 + x \Rightarrow$ $y = x^3 - x$. The result is the same as the original equation, so the graph is symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the origin only.

29. $y = x^2 - x + 7$
Replace x with $-x$ to obtain
$y = (-x)^2 - (-x) + 7 \Rightarrow y = x^2 + x + 7$.
The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Since y is a function of x, the graph cannot be symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $-y = (-x)^2 - (-x) + 7 \Rightarrow$
$-y = x^2 + x + 7 \Rightarrow y = -x^2 - x - 7$.
The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph has none of the listed symmetries.

30. $y = x + 15$

Replace x with $-x$ to obtain $y = (-x) + 15 \Rightarrow y = -x + 15.$

The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Since y is a function of x, the graph cannot be symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $-y = (-x) + 15 \Rightarrow y = x - 15.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph has none of the listed symmetries.

31. $f(x) = -x^3 + 2x$

$$f(-x) = -(-x)^3 + 2(-x)$$
$$= x^3 - 2x = -\left(-x^3 + 2x\right) = -f(x)$$

The function is odd.

32. $f(x) = x^5 - 2x^3$

$$f(-x) = (-x)^5 - 2(-x)^3$$
$$= -x^5 + 2x^3 = -\left(x^5 - 2x^3\right) = -f(x)$$

The function is odd.

33. $f(x) = .5x^4 - 2x^2 + 6$

$$f(-x) = .5(-x)^4 - 2(-x)^2 + 6$$
$$= .5x^4 - 2x^2 + 6 = f(x)$$

The function is even.

34. $f(x) = .75x^2 + |x| + 4$

$$f(-x) = .75(-x)^2 + |-x| + 4$$
$$= .75x^2 + |x| + 4 = f(x)$$

The function is even.

35. $f(x) = x^3 - x + 9$

$$f(x) = (-x)^3 - (-x) + 9$$
$$= -x^3 + x + 9 = -\left(x^3 - x - 9\right) \neq -f(x)$$

The function is neither.

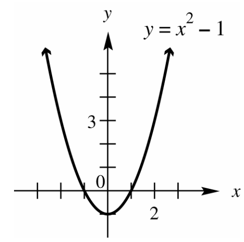
36. $f(x) = x^4 - 5x + 8$

$$f(-x) = (-x)^4 - 5(-x) + 8$$
$$= x^4 + 5x + 8 \neq f(x)$$

The function is neither.

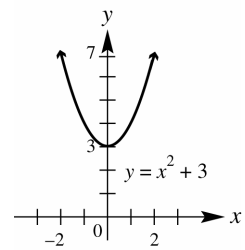
37. $y = x^2 - 1$

This graph may be obtained by translating the graph of $y = x^2$, 1 unit downward.



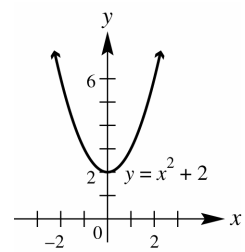
38. $y = x^2 + 3$

This graph may be obtained by translating the graph of $y = x^2$, 3 units upward.



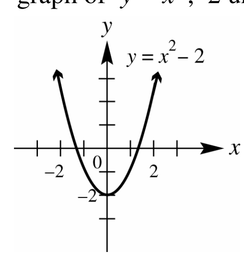
39. $y = x^2 + 2$

This graph may be obtained by translating the graph of $y = x^2$, 2 units upward.



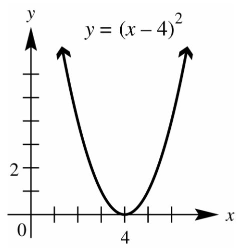
40. $y = x^2 - 2$

This graph may be obtained by translating the graph of $y = x^2$, 2 units downward.

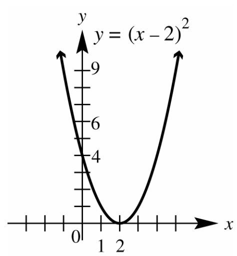


41. $y = (x - 4)^2$

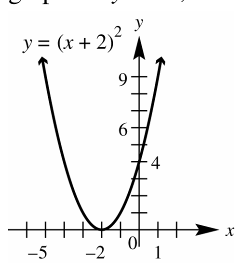
This graph may be obtained by translating the graph of $y = x^2$, 4 units to the right.



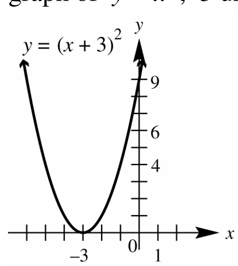
42. $y=(x-2)^2$
This graph may be obtained by translating the graph of $y=x^2$, 2 units to the right.



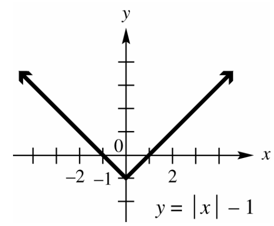
43. $y=(x+2)^2$
This graph may be obtained by translating the graph of $y=x^2$, 2 units to the left.



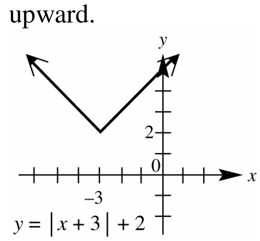
44. $y=(x+3)^2$
This graph may be obtained by translating the graph of $y=x^2$, 3 units to the left.



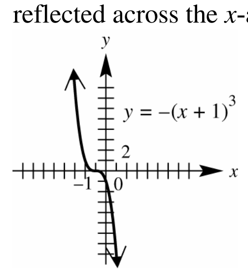
45. $y=|x|-1$
The graph is obtained by translating the graph of $y=|x|$, 1 unit downward.



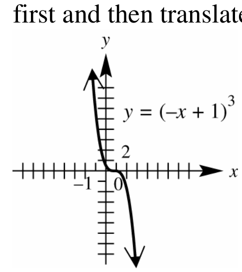
46. $y=|x+3|+2$
This graph may be obtained by translating the graph of $y=|x|$, 3 units to the left and 2 units upward.



47. $y=-(x+1)^3$
This graph may be obtained by translating the graph of $y=x^3$, 1 unit to the left. It is then reflected across the *x*-axis.

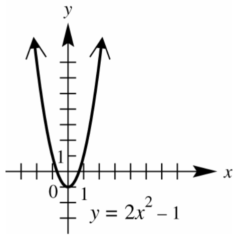


48. $y=(-x+1)^3$
If the given equation is written in the form $y=[-1(x-1)]^3=(-1)^3(x-1)^3=-(x-1)^3$, we see that this graph can be obtained by translating the graph of $y=x^3$, 1 unit to the right. It is then reflected across the *y*-axis. (We may also reflect the graph about the *y*-axis first and then translate it 1 unit to the right.)



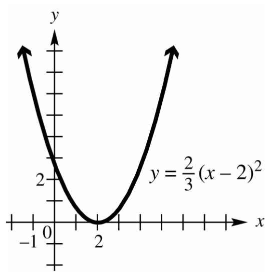
49. $y = 2x^2 - 1$

This graph may be obtained by translating the graph of $y = x^2$, 1 unit down. It is then stretched vertically by a factor of 2.



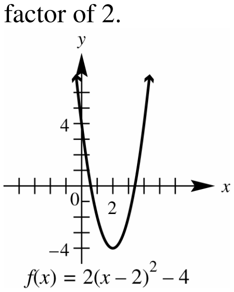
50. $y = \frac{2}{3}(x-2)^2$

This graph may be obtained by translating the graph of $y = x^2$, 2 units to the right. It is then shrunk vertically by a factor of $\frac{2}{3}$.



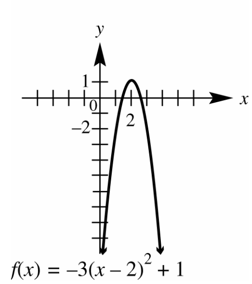
51. $f(x) = 2(x-2)^2 - 4$

This graph may be obtained by translating the graph of $y = x^2$, 2 units to the right and 4 units down. It is then stretched vertically by a factor of 2.



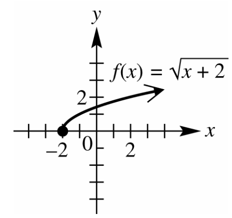
52. $f(x) = -3(x-2)^2 + 1$

This graph may be obtained by translating the graph of $y = x^2$, 2 units to the right and 1 unit up. It is then stretched vertically by a factor of 3 and reflected over the x-axis.



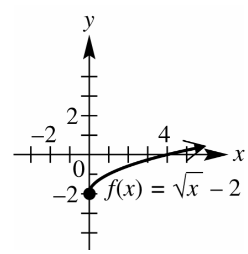
53. $f(x) = \sqrt{x+2}$

This graph may be obtained by translating the graph of $y = \sqrt{x}$ two units to the left.



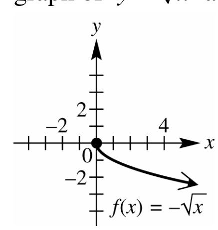
54. $f(x) = \sqrt{x} - 2$

This graph may be obtained by translating the graph of $y = \sqrt{x}$ two units down.



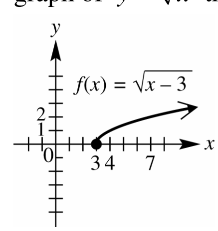
55. $f(x) = -\sqrt{x}$

This graph may be obtained by reflecting the graph of $y = \sqrt{x}$ across the x-axis.



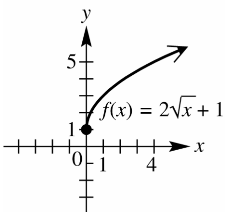
56. $f(x) = \sqrt{x-3}$

This graph may be obtained by translating the graph of $y = \sqrt{x}$ three units to the right.



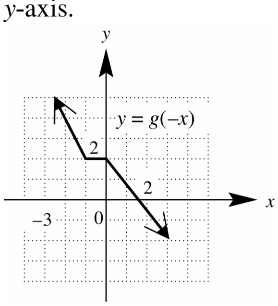
57. $f(x) = 2\sqrt{x} + 1$

This graph may be obtained by stretching the graph of $y = \sqrt{x}$ vertically by a factor of two and then translating the resulting graph one unit up.

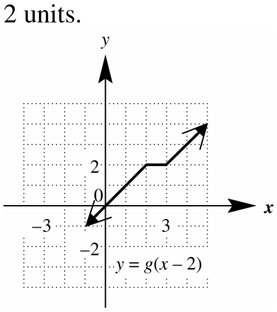


58. Because $g(x) = |-x| = |x| = f(x)$, the graphs are the same.

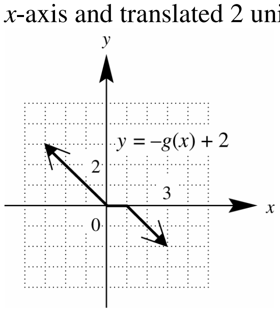
59. (a) $y = g(-x)$
The graph of $g(x)$ is reflected across the y-axis.



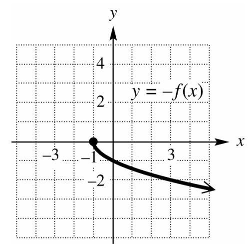
(b) $y = g(x - 2)$
The graph of $g(x)$ is translated to the right 2 units.



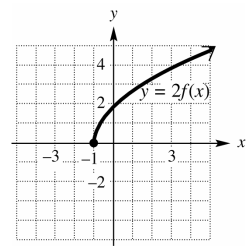
(c) $y = -g(x) + 2$
The graph of $g(x)$ is reflected across the x-axis and translated 2 units up.



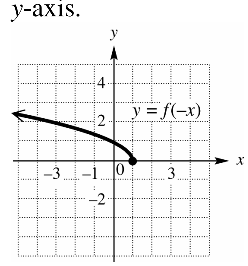
60. (a) $y = -f(x)$
The graph of $f(x)$ is reflected across the x-axis.



(b) $y = 2f(x)$
The graph of $f(x)$ is stretched vertically by a factor of 2.



(c) $y = f(-x)$
The graph of $f(x)$ is reflected across the y-axis.



61. It is the graph of $f(x) = |x|$ translated 1 unit to the left, reflected across the x-axis, and translated 3 units up. The equation is $y = -|x+1| + 3$.

62. It is the graph of $g(x) = \sqrt{x}$ translated 4 units to the left, reflected across the x-axis, and translated 2 units up. The equation is $y = -\sqrt{x+4} + 2$.

63. It is the graph of $f(x) = \sqrt{x}$ translated one unit right and then three units down. The equation is $y = \sqrt{x-1} - 3$.

64. It is the graph of $f(x) = |x|$ translated 2 units to the right, shrunken vertically by a factor of $\frac{1}{2}$, and translated 1 unit down. The equation is $y = \frac{1}{2}|x-2| - 1$.

65. It is the graph of $g(x) = \sqrt{x}$ translated 4 units to the left, stretched vertically by a factor of 2, and translated 4 units down. The equation is $y = 2\sqrt{x+4} - 4$.

66. It is the graph of $f(x) = |x|$ reflected across the x-axis and then shifted down two units. The equation is $y = -|x| - 2$.

67. Since $f(3) = 6$, the point (3, 6) is on the graph. Since the graph is symmetric with respect to the origin, the point (–3, –6) is on the graph. Therefore, $f(-3) = -6$.

68. Since $f(3) = 6$, (3, 6) is a point on the graph. Since the graph is symmetric with respect to the y-axis, (–3, 6) is on the graph. Therefore, $f(-3) = 6$.

69. Since $f(3) = 6$, the point (3, 6) is on the graph. Since the graph is symmetric with respect to the line $x = 6$ and since the point (3, 6) is 3 units to the left of the line $x = 6$, the image point of (3, 6), 3 units to the right of the line $x = 6$, is (9, 6). Therefore, $f(9) = 6$.

70. Since $f(3) = 6$ and since $f(-x) = f(x)$, $f(-3) = f(3)$. Therefore, $f(-3) = 6$.

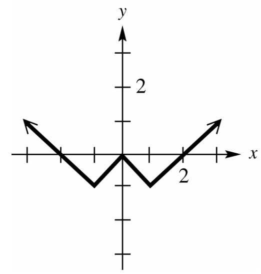
71. An odd function is a function whose graph is symmetric with respect to the origin. Since (3, 6) is on the graph, (–3, –6) must also be on the graph. Therefore, $f(-3) = -6$.

72. If f is an odd function, $f(-x) = -f(x)$. Since $f(3) = 6$ and $f(-x) = -f(x)$, $f(-3) = -f(3)$. Therefore, $f(-3) = -6$.

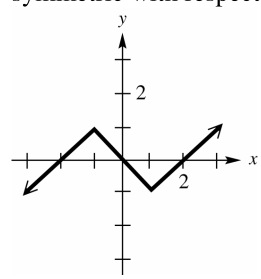
73. $f(x) = 2x + 5$: Translate the graph of $f(x)$ up 2 units to obtain the graph of
$t(x) = (2x+5) + 2 = 2x + 7$.
Now translate the graph of $t(x) = 2x + 7$ left 3 units to obtain the graph of
$g(x) = 2(x+3) + 7 = 2x + 6 + 7 = 2x + 13$.
(Note that if the original graph is first translated to the left 3 units and then up 2 units, the final result will be the same.)

74. $f(x) = 3 - x$: Translate the graph of $f(x)$ down 2 units to obtain the graph of
$t(x) = (3 - x) - 2 = -x + 1$. Now translate the graph of $t(x) = -x + 1$ right 3 units to obtain the graph of
$g(x) = -(x-3) + 1 = -x + 3 + 1 = -x + 4$.
(Note that if the original graph is first translated to the right 3 units and then down 2 units, the final result will be the same.)

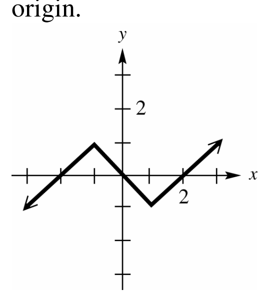
75. **(a)** Since $f(-x) = f(x)$, the graph is symmetric with respect to the y-axis.



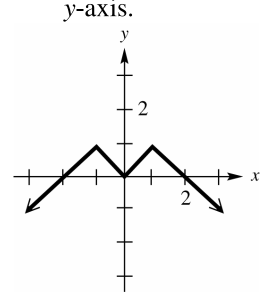
(b) Since $f(-x) = -f(x)$, the graph is symmetric with respect to the origin.



76. **(a)** $f(x)$ is odd. An odd function has a graph symmetric with respect to the origin. Reflect the left half of the graph in the origin.



(b) $f(x)$ is even. An even function has a graph symmetric with respect to the y-axis. Reflect the left half of the graph in the y-axis.



77. Answers will vary.
There are four possibilities for the constant, c.

i) $c > 0$ $|c| > 1$ The graph of $F(x)$ is stretched vertically by a factor of c.

ii) $c > 0$ $|c| < 1$ The graph of $F(x)$ is shrunk vertically by a factor of c.

iii) $c < 0$ $|c| > 1$ The graph of $F(x)$ is stretched vertically by a factor of $-c$ and reflected over the x-axis.

iv) $c < 0$ $|c| < 1$ The graph of $F(x)$ is shrunk vertically by a factor of $-c$ and reflected over the x-axis.

78. The graph of $y = F(x+h)$ represents a horizontal shift of the graph of $y = F(x)$. If $h > 0$, it is a shift to the left h units. If $h < 0$, it is a shift to the left $-h$ units (h is negative). The graph of $y = F(x)+h$ is not the same as the graph of $y = F(x+h)$. The graph of $y = F(x)+h$ represents a vertical shift of the graph of $y = F(x)$.

Chapter 2 Quiz
(Sections 2.5–2.7)

1. (a) First, find the slope: $m = \dfrac{9-5}{-1-(-3)} = 2$
Choose either point, say, (–3, 5), to find the equation of the line:
$y - 5 = 2(x-(-3)) \Rightarrow y = 2(x+3)+5 \Rightarrow$
$y = 2x + 11$.

(b) To find the x-intercept, let $y = 0$ and solve for x: $0 = 2x + 11 \Rightarrow x = -\frac{11}{2}$. The x-intercept is $-\frac{11}{2}$.

2. Write $3x - 2y = 6$ in slope-intercept form to find its slope: $3x - 2y = 6 \Rightarrow y = \frac{3}{2}x - 3$. Then, the slope of the line perpendicular to this graph is $-\frac{2}{3}$. $y - 4 = -\frac{2}{3}(x-(-6)) \Rightarrow$
$y = -\frac{2}{3}(x+6)) + 4 \Rightarrow y = -\frac{2}{3}x$

3. (a) $x = -8$ **(b)** $y = 5$

4. (a) Cubing function; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$; increasing over $(-\infty, \infty)$.

(b) Absolute value function; domain: $(-\infty, \infty)$; range: $[0, \infty)$; decreasing over $(-\infty, 0]$; increasing over $[0, \infty)$

(c) Cube root function: domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$; increasing over $(-\infty, \infty)$.

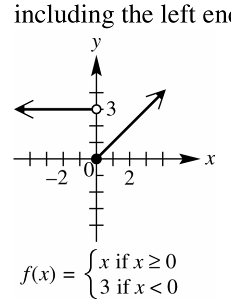
5. (a) The highest speed limit is 55 miles per hour. The lowest speed limit is 30 miles per hour.

(b) There are about 12 miles of highway with a speed limit of 55 miles per hour.

(c) $f(4) = 40; f(12) = 30; f(18) = 55$

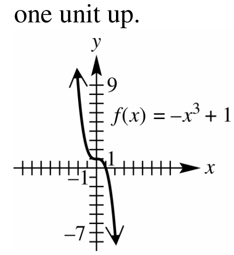
6. $f(x) = \begin{cases} x & \text{if } x \ge 0 \\ 3 & \text{if } x < 0 \end{cases}$

For values of $x < 0$, the graph is the horizontal line $y = 3$. Do not include the right endpoint (0, 3). Graph the line $y = x$ for values of $x \ge 0$, including the left endpoint (0, 0).



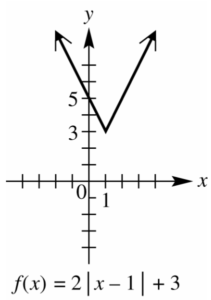
7. $f(x) = -x^3 + 1$

Reflect the graph of $f(x) = x^3$ across the x-axis, and then translate the resulting graph one unit up.



8. $f(x)=2|x-1|+3$

Shift the graph of $f(x)=|x|$ one unit right, stretch the resulting graph vertically by a factor of 2, then shift this graph three units up.



9. This is the graph of $f(x)=\sqrt{x}$, translated four units to the left, reflected across the x-axis, and then translated two units down. The equation is $y=-\sqrt{x+4}-2$.

10. (a) $y=x^2-7$

Replace x with $-x$ to obtain
$y=(-x)^2-7 \Rightarrow y=x^2-7$
The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Replace y with $-y$ to obtain $-y=x^2-7 \Rightarrow y=-x^2+7$. The result is not the same as the original equation, so the graph is not symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain
$-y=(-x)^2-7 \Rightarrow y=x^2+7$. The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the y-axis only.

(b) $x=y^4+y^2-9$

Replace x with $-x$ to obtain
$-x=y^4+y^2-9=-y^4-y^2+9$
The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Replace y with $-y$ to obtain
$x=(-y)^4+(-y)^2-9=y^4+y^2-9$,
which is the same as the original equation, so the graph is symmetric with respect to the x-axis.
Replace x with $-x$ and y with $-y$ to obtain
$-x=(-y)^4+(-y)^2-9=y^4+y^2-9$.
The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the x-axis only.

(c) $x^2=1-y^2$

Replace x with $-x$ to obtain
$(-x)^2=1-y^2 \Rightarrow x^2=1-y^2$
The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Replace y with $-y$ to obtain $x^2=1-(-y)^2 \Rightarrow x^2=1-y^2$, which is the same as the original equation, so the graph is symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain
$(-x)^2=1-(-y)^2 \Rightarrow x^2=1-y^2$. The result is the same as the original equation, so the graph is symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the x-axis, y-axis, and the origin.

Section 2.8: Function Operations and Composition

In Exercises 1–8, $f(x)=x^2+3$ and $g(x)=-2x+6$.

1. $$\begin{aligned}(f+g)(3)&=f(3)+g(3)\\&=\left[(3)^2+3\right]+\left[-2(3)+6\right]\\&=12+0=12\end{aligned}$$

2. $$\begin{aligned}(f+g)(-5)&=f(-5)+g(-5)\\&=[(-5)^2+3]+[-2(-5)+6]\\&=28+16=44\end{aligned}$$

3. $$\begin{aligned}(f-g)(-5)&=f(-1)-g(-1)\\&=[(-1)^2+3]-[-2(-1)+6]\\&=4-8=-4\end{aligned}$$

4. $$\begin{aligned}(f-g)(4)&=f(4)-g(4)\\&=[(4)^2+3]-[-2(4)+6]\\&=19-(-2)=21\end{aligned}$$

5. $$\begin{aligned}(fg)(4)&=f(4)\cdot g(4)\\&=[4^2+3]\cdot[-2(4)+6]\\&=19\cdot(-2)=-38\end{aligned}$$

6. $$\begin{aligned}(fg)(-3)&=f(-3)\cdot g(-3)\\&=[(-3)^2+3]\cdot[-2(-3)+6]\\&=12\cdot 12=144\end{aligned}$$

7. $\left(\frac{f}{g}\right)(-1) = \frac{f(-1)}{g(-1)} = \frac{(-1)^2+3}{-2(-1)+6} = \frac{4}{8} = \frac{1}{2}$

8. $\left(\frac{f}{g}\right)(5) = \frac{f(5)}{g(5)} = \frac{(5)^2+3}{-2(5)+6} = \frac{28}{-4} = -7$

9. $f(x) = 3x+4,\ g(x) = 2x-5$

i) $(f+g)(x) = f(x)+g(x)$
$= (3x+4)+(2x-5) = 5x-1$

ii) $(f-g)(x) = f(x)-g(x)$
$= (3x+4)-(2x-5) = x+9$

iii) $(fg)(x) = f(x)\cdot g(x) = (3x+4)(2x-5)$
$= 6x^2-15x+8x-20$
$= 6x^2-7x-20$

iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x+4}{2x-5}$

The domains of both *f* and *g* are the set of all real numbers, so the domains of $f+g$, $f-g$, and fg are all $(-\infty,\infty)$. The domain of $\frac{f}{g}$ is the set of all real numbers for which $g(x)\neq 0$. This is the set of all real numbers except $\frac{5}{2}$, which is written in interval notation as $\left(-\infty,\frac{5}{2}\right)\cup\left(\frac{5}{2},\infty\right)$.

10. $f(x) = 6-3x,\ g(x) = -4x+1$

i) $(f+g)(x) = f(x)+g(x)$
$= (6-3x)+(-4x+1)$
$= -7x+7$

ii) $(f-g)(x) = f(x)-g(x)$
$= (6-3x)-(-4x+1) = x+5$

iii) $(fg)(x) = f(x)\cdot g(x) = (6-3x)(-4x+1)$
$= -24x+6+12x^2-3x$
$= 12x^2-27x+6$

iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{6-3x}{-4x+1}$

The domains of both *f* and *g* are the set of all real numbers, so the domains of $f+g$, $f-g$, and fg are all $(-\infty,\infty)$. The domain of $\frac{f}{g}$ is the set of all real numbers for which $g(x)\neq 0$. This is the set of all real numbers except $\frac{1}{4}$, which is written in interval notation as $\left(-\infty,\frac{1}{4}\right)\cup\left(\frac{1}{4},\infty\right)$.

11. $f(x) = 2x^2-3x,\ g(x) = x^2-x+3$

i) $(f+g)(x) = f(x)+g(x)$
$= (2x^2-3x)+(x^2-x+3)$
$= 3x^2-4x+3$

ii) $(f-g)(x) = f(x)-g(x)$
$= (2x^2-3x)-(x^2-x+3)$
$= 2x^2-3x-x^2+x-3$
$= x^2-2x-3$

iii) $(fg)(x) = f(x)\cdot g(x)$
$= (2x^2-3x)(x^2-x+3)$
$= 2x^4-2x^3+6x^2-3x^3+3x^2-9x$
$= 2x^4-5x^3+9x^2-9x$

iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2-3x}{x^2-x+3}$

The domains of both *f* and *g* are the set of all real numbers, so the domains of $f+g$, $f-g$, and fg are all $(-\infty,\infty)$. The domain of $\frac{f}{g}$ is the set of all real numbers for which $g(x)\neq 0$. If $x^2-x+3=0$, then by the quadratic formula $x=\frac{1\pm i\sqrt{11}}{2}$. The equation has no real solutions. There are no real numbers which make the denominator zero. Thus, the domain of $\frac{f}{g}$ is also $(-\infty,\infty)$.

12. $f(x) = 4x^2+2x,\ g(x) = x^2-3x+2$

i) $(f+g)(x) = f(x)+g(x)$
$= (4x^2+2x)+(x^2-3x+2)$
$= 5x^2-x+2$

ii) $(f-g)(x) = f(x)-g(x)$
$= (4x^2+2x)-(x^2-3x+2)$
$= 4x^2+2x-x^2+3x-2$
$= 3x^2+5x-2$

iii) $(fg)(x) = f(x)\cdot g(x)$
$= (4x^2+2x)(x^2-3x+2)$
$= 4x^4-12x^3+8x^2+2x^3-6x^2+4x$
$= 4x^4-10x^3+2x^2+4x$

iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4x^2+2x}{x^2-3x+2}$

The domains of both f and g are the set of all real numbers, so the domains of $f+g$, $f-g$, and fg are all $(-\infty,\infty)$. The domain of $\frac{f}{g}$ is the set of all real numbers x such that $x^2-3x+2 \neq 0$. Since $x^2-3x+2=(x-1)(x-2)$, the numbers which give this denominator a value of 0 are $x = 1$ and $x = 2$. Therefore, the domain of $\frac{f}{g}$ is the set of all real numbers except 1 and 2, which is written in interval notation as $(-\infty, 1)\cup(1, 2)\cup(2, \infty)$.

13. $f(x)=\sqrt{4x-1},\ g(x)=\frac{1}{x}$

i) $(f+g)(x) = f(x)+g(x) = \sqrt{4x-1}+\frac{1}{x}$

ii) $(f-g)(x) = f(x)-g(x) = \sqrt{4x-1}-\frac{1}{x}$

iii) $(fg)(x) = f(x)\cdot g(x)$
$= \sqrt{4x-1}\left(\frac{1}{x}\right) = \frac{\sqrt{4x-1}}{x}$

iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{4x-1}}{\frac{1}{x}} = x\sqrt{4x-1}$

Since $4x-1\geq 0 \Rightarrow 4x \geq 1 \Rightarrow x\geq \frac{1}{4}$, the domain of f is $\left[\frac{1}{4}, \infty\right)$. The domain of g is $(-\infty,0)\cup(0,\infty)$. Considering the intersection of the domains of f and g, the domains of $f+g$, $f-g$, and fg are all $\left[\frac{1}{4}, \infty\right)$. Since $\frac{1}{x}\neq 0$ for any value of x, the domain of $\frac{f}{g}$ is also $\left[\frac{1}{4}, \infty\right)$.

14. $f(x)=\sqrt{5x-4},\ g(x)=-\frac{1}{x}$

i) $(f+g)(x) = f(x)+g(x)$
$= \sqrt{5x-4}+\left(-\frac{1}{x}\right)$
$= \sqrt{5x-4}-\frac{1}{x}$

ii) $(f-g)(x) = f(x)-g(x)$
$= \sqrt{5x-4}-\left(-\frac{1}{x}\right)$
$= \sqrt{5x-4}+\frac{1}{x}$

iii) $(fg)(x) = f(x)\cdot g(x)$
$= \left(\sqrt{5x-4}\right)\left(-\frac{1}{x}\right) = -\frac{\sqrt{5x-4}}{x}$

iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{5x-4}}{-\frac{1}{x}} = -x\sqrt{5x-4}$

Since $5x-4\geq 0 \Rightarrow 5x\geq 4 \Rightarrow x\geq \frac{4}{5}$, the domain of f is $\left[\frac{4}{5}, \infty\right)$. The domain of g is $(-\infty,0)\cup(0,\infty)$. Considering the intersection of the domains of f and g, the domains of $f+g$, $f-g$, and fg are all $\left[\frac{4}{5}, \infty\right)$. Since $-\frac{1}{x}\neq 0$ for any value of x, the domain of $\frac{f}{g}$ is also $\left[\frac{4}{5}, \infty\right)$.

In the responses to Exercises 15–16, numerical answers may vary.

15. $G(1996)\approx 7.7$ and $B(1996)\approx 11.8$, thus
$T(1996) = G(1996)+B(1996)$
$= 7.7+11.8 = 19.5.$

16. $G(2006)\approx 18$ and $B(2006)\approx 21$, thus
$T(2006) = G(2006)+B(2006)$
$= 18+21 = 39$

17. Looking at the graphs of the functions, the slopes of the line segments for the period 1996–2006 are much steeper than the slopes of the corresponding line segments for the period 1991–1996. Thus, the number of sodas increased more rapidly during the period 1996–2006.

18. Answers will vary.

In the responses to Exercises 19–20, numerical answers may vary.

19. $(T-S)(2000) = T(2000)-S(2000)$
$= 19-13 = 6$
It represents the dollars in billions spent for general science in 2000.

20. $(T-G)(2005) = T(2005)-G(2005)$
$= 23-8 = 15$
It represents the dollars in billions spent on space and other technologies in 2005.

21. In space and other technologies spending was almost static in the years 1995–2000.

22. In space and other technologies spending increased the most during the years 2000–2005.

23. **(a)** $(f+g)(2) = f(2) + g(2)$
$= 4 + (-2) = 2$

(b) $(f-g)(1) = f(1) - g(1) = 1 - (-3) = 4$

(c) $(fg)(0) = f(0) \cdot g(0) = 0(-4) = 0$

(d) $\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = \frac{1}{-3} = -\frac{1}{3}$

24. **(a)** $(f+g)(0) = f(0) + g(0) = 0 + 2 = 2$

(b) $(f-g)(-1) = f(-1) - g(-1)$
$= -2 - 1 = -3$

(c) $(fg)(1) = f(1) \cdot g(1) = 2 \cdot 1 = 2$

(d) $\left(\frac{f}{g}\right)(2) = \frac{f(2)}{g(2)} = \frac{4}{-2} = -2$

25. **(a)** $(f+g)(-1) = f(-1) + g(-1) = 0 + 3 = 3$

(b) $(f-g)(-2) = f(-2) - g(-2)$
$= -1 - 4 = -5$

(c) $(fg)(0) = f(0) \cdot g(0) = 1 \cdot 2 = 2$

(d) $\left(\frac{f}{g}\right)(2) = \frac{f(2)}{g(2)} = \frac{3}{0} =$ undefined

26. **(a)** $(f+g)(1) = f(1) + g(1) = -3 + 1 = -2$

(b) $(f-g)(0) = f(0) - g(0) = -2 - 0 = -2$

(c) $(fg)(-1) = f(-1) \cdot g(-1) = -3(-1) = 3$

(d) $\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = \frac{-3}{1} = -3$

27. **(a)** $(f+g)(2) = f(2) + g(2) = 7 + (-2) = 5$

(b) $(f-g)(4) = f(4) - g(4) = 10 - 5 = 5$

(c) $(fg)(-2) = f(-2) \cdot g(-2) = 0 \cdot 6 = 0$

(d) $\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{5}{0} =$ undefined

28. **(a)** $(f+g)(2) = f(2) + g(2) = 5 + 4 = 9$

(b) $(f-g)(4) = f(4) - g(4) = 0 - 0 = 0$

(c) $(fg)(-2) = f(-2) \cdot g(-2) = -4 \cdot 2 = -8$

(d) $\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{8}{-1} = -8$

29.

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-2	0	6	$0+6=6$	$0-6=-6$	$0 \cdot 6 = 0$	$\frac{0}{6}=0$
0	5	0	$5+0=5$	$5-0=5$	$5 \cdot 0 = 0$	$\frac{5}{0}=$ undefined
2	7	-2	$7+(-2)=5$	$7-(-2)=9$	$7(-2)=-14$	$\frac{7}{-2}=-3.5$
4	10	5	$10+5=15$	$10-5=5$	$10 \cdot 5 = 50$	$\frac{10}{5}=2$

30.

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-2	-4	2	$-4+2=-2$	$-4-2=-6$	$-4 \cdot 2 = -8$	$\frac{-4}{2}=-2$
0	8	-1	$8+(-1)=7$	$8-(-1)=9$	$8(-1)=-8$	$\frac{8}{-1}=-8$
2	5	4	$5+4=9$	$5-4=1$	$5 \cdot 4 = 20$	$\frac{5}{4}=1.25$
4	0	0	$0+0=0$	$0-0=0$	$0 \cdot 0 = 0$	$\frac{0}{0}=$ undefined

31. Answers will vary.
The difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, represents the slope of the secant line which passes through points $(x, f(x))$ and $(x+h, f(x+h))$. The formula is derived by applying the rule that slope represents a change in y to a change in x.

32. Answers will vary. The secant line PQ represents the line that is formed between points P and Q. This line exists when h is positive. The tangent line at point P is created when the difference in the x values between points P and Q (namely h) becomes zero.

33. $f(x)=2-x$

(a) $f(x+h)=2-(x+h)=2-x-h$

(b) $f(x+h)-f(x)=(2-x-h)-(2-x)$
$=2-x-h-2+x=-h$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{-h}{h}=-1$

34. $f(x)=1-x$

(a) $f(x+h)=1-(x+h)=1-x-h$

(b) $f(x+h)-f(x)=(1-x-h)-(1-x)$
$=1-x-h-1+x=-h$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{-h}{h}=-1$

35. $f(x)=6x+2$

(a) $f(x+h)=6(x+h)+2=6x+6h+2$

(b) $f(x+h)-f(x)$
$=(6x+6h+2)-(6x+2)$
$=6x+6h+2-6x-2=6h$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{6h}{h}=6$

36. $f(x)=4x+11$

(a) $f(x+h)=4(x+h)+11=4x+4h+11$

(b) $f(x+h)-f(x)$
$=(4x+4h+11)-(4x+11)$
$=4x+4h+11-4x-11=4h$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{4h}{h}=4$

37. $f(x)=-2x+5$

(a) $f(x+h)=-2(x+h)+5$
$=-2x-2h+5$

(b) $f(x+h)-f(x)$
$=(-2x-2h+5)-(-2x+5)$
$=-2x-2h+5+2x-5$
$=-2h$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{-2h}{h}=-2$

38. $f(x)=1-x^2$

(a) $f(x+h)=1-(x+h)^2$
$=1-(x^2+2xh+h^2)$
$=1-x^2-2xh-h^2$

(b) $f(x+h)-f(x)$
$=(1-x^2-2xh-h^2)-(1-x^2)$
$=1-x^2-2xh-h^2-1+x^2$
$=-2xh-h^2$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{-2xh-h^2}{h}$
$=\dfrac{h(-2x-h)}{h}$
$=-2x-h$

39. $f(x)=\dfrac{1}{x}$

(a) $f(x+h)=\dfrac{1}{x+h}$

(b) $f(x+h)-f(x)$
$=\dfrac{1}{x+h}-\dfrac{1}{x}=\dfrac{x-(x+h)}{x(x+h)}$
$=\dfrac{-h}{x(x+h)}$

(c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\frac{-h}{x(x+h)}}{h}=\dfrac{-h}{hx(x+h)}$
$=-\dfrac{1}{x(x+h)}$

40. $f(x)=\dfrac{1}{x^2}$

(a) $f(x+h)=\dfrac{1}{(x+h)^2}$

(b) $f(x+h)-f(x)$

$$= \frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2-(x+h)^2}{x^2(x+h)^2}$$
$$= \frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}$$
$$= \frac{-2xh-h^2}{x^2(x+h)^2}$$

(c) $$\frac{f(x+h)-f(x)}{h} = \frac{\frac{-2xh-h^2}{x^2(x+h)^2}}{h}$$
$$= \frac{-2xh-h^2}{hx^2(x+h)^2}$$
$$= \frac{h(-2x-h)}{hx^2(x+h)^2}$$
$$= \frac{-2x-h}{x^2(x+h)^2}$$

41. Since $g(x) = -x+3$, $g(4) = -4+3 = -1$.
Therefore, $(f \circ g)(4) = f[g(4)] = f(-1)$
$= 2(-1)-3 = -2-3 = -5.$

42. Since $g(x) = -x+3$, $g(2) = -2+3 = 1$.
Therefore, $(f \circ g)(2) = f[g(2)] = f(1)$
$= 2(1)-3 = 2-3 = -1.$

43. Since $g(x) = -x+3$, $g(-2) = -(-2)+3 = 5$.
Therefore, $(f \circ g)(-2) = f[g(-2)] = f(5)$
$= 2(5)-3 = 10-3 = 7.$

44. Since $f(x) = 2x-3$,
$f(3) = 2(3)-3 = 6-3 = 3.$
Therefore, $(g \circ f)(3) = g[f(3)]$
$= g(3) = -3+3 = 0.$

45. Since $f(x) = 2x-3$,
$f(0) = 2(0)-3 = 0-3 = -3.$
Therefore, $(g \circ f)(0) = g[f(0)]$
$= g(-3) = -(-3)+3 = 3+3 = 6.$

46. Since $f(x) = 2x-3$,
$f(-2) = 2(-2)-3 = -4-3 = -7.$
Therefore, $(g \circ f)(-2) = g[f(-2)]$
$= g(-7) = -(-7)+3 = 7+3 = 10.$

47. Since $f(x) = 2x-3$,
$f(2) = 2(2)-3 = 4-3 = 1.$
Therefore, $(f \circ f)(2) = f[f(2)]$
$= f(1) = 2(1)-3 = 2-3 = -1.$

48. Since $g(x) = -x+3$, $g(-2) = -(-2)+3 = 5$.
Therefore, $(g \circ g)(-2) = g[g(-2)]$
$= g(5) = -5+3 = -2.$

49. $(f \circ g)(2) = f[g(2)] = f(3) = 1$

50. $(f \circ g)(7) = f[g(7)] = f(6) = 9$

51. $(g \circ f)(3) = g[f(3)] = g(1) = 9$

52. $(g \circ f)(6) = g[f(6)] = g(9) = 12$

53. $(f \circ f)(4) = f[f(4)] = f(3) = 1$

54. $(g \circ g)(1) = g[g(1)] = g(9) = 12$

55. $(f \circ g)(1) = f[g(1)] = f(9)$
However, $f(9)$ cannot be determined from the table given.

56. $(g \circ (f \circ g))(7) = g(f(g(7)))$
$= g(f(6)) = g(9) = 12$

57. **(a)** $(f \circ g)(x) = f(g(x)) = f(5x+7)$
$= -6(5x+7)+9$
$= -30x-42+9 = -30x-33$
The domain and range of both f and g are $(-\infty, \infty)$, so the domain of $f \circ g$ is $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g(-6x+9)$
$= 5(-6x+9)+7$
$= -30x+45+7 = -30x+52$
The domain of $g \circ f$ is $(-\infty, \infty)$.

58. **(a)** $(f \circ g)(x) = f(g(x)) = f(3x-1)$
$= 8(3x-1)+12$
$= 24x-8+12 = 24x+4$
The domain and range of both f and g are $(-\infty, \infty)$, so the domain of $f \circ g$ is $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g(8x+12)$
$= 3(8x+12) - 1$
$= 24x + 36 - 1 = 24x + 35$
The domain of $g \circ f$ is $(-\infty, \infty)$.

59. (a) $(f \circ g)(x) = f(g(x)) = f(x+3) = \sqrt{x+3}$
The domain and range of g are $(-\infty, \infty)$, however, the domain and range of *f* are $[0, \infty)$. So, $x + 3 \ge 0 \Rightarrow x \ge -3$.
Therefore, the domain of $f \circ g$ is $[-3, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{x} + 3$
The domain and range of g are $(-\infty, \infty)$, however, the domain and range of *f* are $[0, \infty)$ .Therefore, the domain of $g \circ f$ is $[0, \infty)$.

60. (a) $(f \circ g)(x) = f(g(x)) = f(x-1) = \sqrt{x-1}$
The domain and range of g are $(-\infty, \infty)$, however, the domain and range of *f* are $[0, \infty)$. So, $x - 1 \ge 0 \Rightarrow x \ge 1$. Therefore, the domain of $f \circ g$ is $[1, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\sqrt{x} = \sqrt{x} - 1$
The domain and range g are $(-\infty, \infty)$, however, the domain and range of *f* are $[0, \infty)$ .Therefore, the domain of $g \circ f$ is $[0, \infty)$.

61. (a) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x - 1)$
$= (x^2 + 3x - 1)^3$
The domain and range of *f* and g are $(-\infty, \infty)$, so the domain of $f \circ g$ is $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^3)$
$= \left(x^3\right)^2 + 3\left(x^3\right) - 1$
$= x^6 + 3x^3 - 1$
The domain and range of *f* and g are $(-\infty, \infty)$, so the domain of $g \circ f$ is $(-\infty, \infty)$.

62. (a) $(f \circ g)(x) = f(g(x)) = f(x^4 + x^2 - 3x - 4)$
$= x^4 + x^2 - 3x - 4 + 2$
$= x^4 + x^2 - 3x - 2$
The domain and range of *f* and g are $(-\infty, \infty)$, so the domain of $f \circ g$ is $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g(x+2)$
$= (x+2)^4 + (x+2)^2 - 3(x+2) - 4$
The domain and range of *f* and g are $(-\infty, \infty)$, so the domain of $g \circ f$ is $(-\infty, \infty)$.

63. (a) $(f \circ g)(x) = f(g(x)) = f(3x) = \sqrt{3x-1}$
The domain and range of g are $(-\infty, \infty)$, however, the domain and range of *f* are $[1, \infty)$. So, $3x - 1 \ge 0 \Rightarrow x \ge \frac{1}{3}$. Therefore, the domain of $f \circ g$ is $\left[\frac{1}{3}, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x-1}\right)$
$= 3\sqrt{x-1}$
The domain and range of g are $(-\infty, \infty)$, however, the range of *f* is $[0, \infty)$. So $x - 1 \ge 0 \Rightarrow x \ge 1$. Therefore, the domain of $g \circ f$ is $[1, \infty)$.

64. (a) $(f \circ g)(x) = f(g(x)) = f(2x) = \sqrt{2x-2}$
The domain and range of g are $(-\infty, \infty)$, however, the domain of *f* is $[2, \infty)$. So, $2x - 2 \ge 0 \Rightarrow x \ge 1$. Therefore, the domain of $f \circ g$ is $\left[1, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x-2}\right)$
$= 2\sqrt{x-2}$
The domain and range of g are $(-\infty, \infty)$, however, the range of *f* is $[0, \infty)$. So $x - 2 \ge 0 \Rightarrow x = 2$. Therefore, the domain of $g \circ f$ is $[2, \infty)$.

65. (a) $(f \circ g)(x) = f(g(x)) = f(x+1) = \frac{2}{x+1}$
The domain and range of g are $(-\infty, \infty)$, however, the domain of *f* is $(-\infty, 0) \cup (0, \infty)$. So, $x + 1 \ne 0 \Rightarrow x \ne -1$.
Therefore, the domain of $f \circ g$ is $(-\infty, -1) \cup (-1, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{2}{x}\right) = \frac{2}{x} + 1$
The domain and range of *f* is $(-\infty, 0) \cup (0, \infty)$, however, the domain and range of *g* are $(-\infty, \infty)$. So $x \neq 0$. Therefore, the domain of $g \circ f$ is $(-\infty, 0) \cup (0, \infty)$.

66. (a) $(f \circ g)(x) = f(g(x)) = f(x+4) = \frac{4}{x+4}$
The domain and range of *g* are $(-\infty, \infty)$, however, the domain of *f* is $(-\infty, 0) \cup (0, \infty)$. So, $x + 4 \neq 0 \Rightarrow x \neq -4$. Therefore, the domain of $f \circ g$ is $(-\infty, -4) \cup (-4, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{4}{x}\right) = \frac{4}{x} + 4$
The domain and range of *f* is $(-\infty, 0) \cup (0, \infty)$, however, the domain and range of *g* are $(-\infty, \infty)$. So $x \neq 0$. Therefore, the domain of $g \circ f$ is $(-\infty, 0) \cup (0, \infty)$.

67. (a) $(f \circ g)(x) = f(g(x)) = f\left(-\frac{1}{x}\right) = \sqrt{-\frac{1}{x} + 2}$
The domain and range of *g* are $(-\infty, 0) \cup (0, \infty)$, however, the domain of *f* is $[-2, \infty)$. So, $-\frac{1}{x} + 2 \geq 0 \Rightarrow$ $x < 0$ or $x \geq \frac{1}{2}$ (using test intervals). Therefore, the domain of $f \circ g$ is $(-\infty, 0) \cup \left[\frac{1}{2}, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+2}\right) = -\frac{1}{\sqrt{x+2}}$
The domain of *f* is $[-2, \infty)$ and its range is $(-\infty, \infty)$. The domain and range of *g* are $(-\infty, 0) \cup (0, \infty)$. So $x + 2 > 0 \Rightarrow x > -2$. Therefore, the domain of $g \circ f$ is $(-2, \infty)$.

68. (a) $(f \circ g)(x) = f(g(x)) = f\left(-\frac{2}{x}\right) = \sqrt{-\frac{2}{x} + 4}$
The domain and range of *g* are $(-\infty, 0) \cup (0, \infty)$, however, the domain of *f* is $[-4, \infty)$. So, $-\frac{2}{x} + 4 \geq 0 \Rightarrow$ $x < 0$ or $x \geq \frac{1}{2}$ (using test intervals). Therefore, the domain of $f \circ g$ is $(-\infty, 0) \cup \left[\frac{1}{2}, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+4}\right) = -\frac{2}{\sqrt{x+4}}$
The domain of *f* is $[-4, \infty)$ and its range is $(-\infty, \infty)$. The domain and range of *g* are $(-\infty, 0) \cup (0, \infty)$. So $x + 4 > 0 \Rightarrow x > -4$. Therefore, the domain of $g \circ f$ is $(-4, \infty)$.

69. (a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x+5}\right) = \sqrt{\frac{1}{x+5}}$
The domain of *g* is $(-\infty, -5) \cup (-5, \infty)$, and the range of *g* is $(-\infty, 0) \cup (0, \infty)$. The domain of *f* is $[0, \infty)$. Therefore, the domain of $f \circ g$ is $(-5, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \frac{1}{\sqrt{x}+5}$
The domain and range of *f* is $[0, \infty)$. The domain of *g* is $(-\infty, -5) \cup (-5, \infty)$. Therefore, the domain of $g \circ f$ is $[0, \infty)$.

70. (a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{3}{x+6}\right) = \sqrt{\frac{3}{x+6}}$
The domain of *g* is $(-\infty, -6) \cup (-6, \infty)$, and the range of *g* is $(-\infty, 0) \cup (0, \infty)$. The domain of *f* is $[0, \infty)$. Therefore, the domain of $f \circ g$ is $(-6, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \frac{3}{\sqrt{x}+6}$
The domain and range of *f* is $[0, \infty)$. The domain of *g* is $(-\infty, -6) \cup (-6, \infty)$. Therefore, the domain of $g \circ f$ is $[0, \infty)$.

71. (a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{1/x - 2} = \frac{x}{1-2x}$
The domain and range of *g* are $(-\infty, 0) \cup (0, \infty)$. The domain of *f* is $(-\infty, -2) \cup (-2, \infty)$, and the range of *f* is $(-\infty, 0) \cup (0, \infty)$. So, $\frac{x}{1-2x} < 0 \Rightarrow x < 0$ or $0 < x < \frac{1}{2}$ or $x > \frac{1}{2}$ (using test intervals). Thus, $x \neq 0$ and $x \neq \frac{1}{2}$. Therefore, the domain of $f \circ g$ is $(-\infty, 0) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

(b) $(g\circ f)(x) = g(f(x)) = g\left(\frac{1}{x-2}\right) = \frac{1}{1/(x-2)}$
$= x-2$
The domain and range of g are $(-\infty,0)\cup(0,\infty)$. The domain of f is $(-\infty,-2)\cup(-2,\infty)$, and the range of f is $(-\infty,0)\cup(0,\infty)$. Therefore, the domain of $g\circ f$ is $(-\infty,-2)\cup(-2,\infty)$.

72. (a) $(f\circ g)(x) = f(g(x)) = f\left(-\frac{1}{x}\right) = \frac{1}{-1/x+4}$
$= \frac{x}{-1+4x}$
The domain and range of g are $(-\infty,0)\cup(0,\infty)$. The domain of f is $(-\infty,-4)\cup(-4,\infty)$, and the range of f is $(-\infty,0)\cup(0,\infty)$. So, $\frac{x}{-1+4x} < 0 \Rightarrow x < 0$ or $0 < x < \frac{1}{4}$ or $-1+4x < 0 \Rightarrow x > \frac{1}{4}$ (using test intervals). Thus, $x \neq 0$ and $x \neq \frac{1}{4}$. Therefore, the domain of $f\circ g$ is $(-\infty,0)\cup\left(0,\frac{1}{4}\right)\cup\left(\frac{1}{4},\infty\right)$.

(b) $(g\circ f)(x) = g(f(x)) = g\left(\frac{1}{x+4}\right) = -\frac{1}{1/(x+4)}$
$= -x-4$
The domain and range of g are $(-\infty,0)\cup(0,\infty)$. The domain of f is $(-\infty,-4)\cup(-4,\infty)$, and the range of f is $(-\infty,0)\cup(0,\infty)$. Therefore, the domain of $g\circ f$ is $(-\infty,-4)\cup(-4,\infty)$.

73. $g[f(2)] = g(1) = 2$ and $g[f(3)] = g(2) = 5$
Since $g[f(1)] = 7$ and $f(1) = 3$, $g(3) = 7$.

x	$f(x)$	$g(x)$	$g[f(x)]$
1	3	2	7
2	1	5	**2**
3	2	**7**	**5**

74. Since $f(x)$ is odd, $f(-1) = -f(1) = -(-2) = 2$. Since $g(x)$ is even, $g(1) = g(-1) = 2$ and $g(2) = g(-2) = 0$. Since $(f\circ g)(-1) = 1$, $f[g(-1)] = 1$ and $f(2) = 1$. Since $f(x)$ is odd, $f(-2) = -f(2) = -1$. Thus, $(f\circ g)(-2) = f[g(-2)] = f(0) = 0$ and $(f\circ g)(1) = f[g(1)] = f(2) = 1$ and $(f\circ g)(2) = f[g(2)] = f(0) = 0$.

x	−2	−1	0	1	2
$f(x)$	**−1**	**2**	0	−2	**1**
$g(x)$	0	2	1	**2**	**0**
$(f\circ g)(x)$	**0**	1	−2	**1**	**0**

75. Answers will vary. In general, composition of functions is not commutative.
$(f\circ g)(x) = f(2x-3) = 3(2x-3)-2$
$= 6x-9-2 = 6x-11$
$(g\circ f)(x) = g(3x-2) = 2(3x-2)-3$
$= 6x-4-3 = 6x-7$
Thus, $(f\circ g)(x) \neq (g\circ f)(x)$.

76. Answers will vary. To find $f\circ g$, the function g must be substituted into the function f.
$(f\circ g)(x) = f[g(x)] = 2(x^2+3)-5$
$= 2x^2+6-5 = 2x^2+1$

77. $(f\circ g)(x) = f[g(x)] = 4\left[\frac{1}{4}(x-2)\right]+2$
$= \left(4\cdot\frac{1}{4}\right)(x-2)+2$
$= (x-2)+2 = x-2+2 = x$
$(g\circ f)(x) = g[f(x)] = \frac{1}{4}[(4x+2)-2]$
$= \frac{1}{4}(4x+2-2) = \frac{1}{4}(4x) = x$

78. $(f\circ g)(x) = f[g(x)] = -3\left(-\frac{1}{3}x\right)$
$= \left[-3\left(-\frac{1}{3}\right)\right]x = x$
$(g\circ f)(x) = g[f(x)] = -\frac{1}{3}(-3x)$
$= \left[-\frac{1}{3}(-3)\right]x = x$

79. $(f\circ g)(x) = f[g(x)] = \sqrt[3]{5\left(\frac{1}{5}x^3-\frac{4}{5}\right)+4}$
$= \sqrt[3]{x^3-4+4} = \sqrt[3]{x^3} = x$
$(g\circ f)(x) = g[f(x)] = \frac{1}{5}\left(\sqrt[3]{5x+4}\right)^3 - \frac{4}{5}$
$= \frac{1}{5}(5x+4) - \frac{4}{5} = \frac{5x}{5} + \frac{4}{5} - \frac{4}{5}$
$= \frac{5x}{5} = x$

80. $(f\circ g)(x) = f[g(x)] = \sqrt[3]{(x^3-1)+1}$
$= \sqrt[3]{x^3-1+1} = \sqrt[3]{x^3} = x$
$(g\circ f)(x) = g[f(x)] = \left(\sqrt[3]{x^3+1}\right)^3 - 1$
$= x^3+1-1 = x$

In Exercises 81–86, we give only one of many possible ways.

81. $h(x) = (6x-2)^2$

Let $g(x) = 6x - 2$ and $f(x) = x^2$.

$(f \circ g)(x) = f(6x-2) = (6x-2)^2 = h(x)$

82. $h(x) = (11x^2 + 12x)^2$

Let $g(x) = 11x^2 + 12x$ and $f(x) = x^2$.

$(f \circ g)(x) = f(11x^2 + 12x)$
$= (11x^2 + 12x)^2 = h(x)$

83. $h(x) = \sqrt{x^2 - 1}$

Let $g(x) = x^2 - 1$ and $f(x) = \sqrt{x}$.

$(f \circ g)(x) = f(x^2 - 1) = \sqrt{x^2 - 1} = h(x)$.

84. $h(x) = (2x-3)^3$

Let $g(x) = 2x - 3$ and $f(x) = x^3$.

$(f \circ g)(x) = f(2x-3) = (2x-3)^3 = h(x)$

85. $h(x) = \sqrt{6x} + 12$

Let $g(x) = 6x$ and $f(x) = \sqrt{x} + 12$.

$(f \circ g)(x) = f(6x) = \sqrt{6x} + 12 = h(x)$

86. $h(x) = \sqrt[3]{2x+3} - 4$

Let $g(x) = 2x + 3$ and $f(x) = \sqrt[3]{x} - 4$.

$(f \circ g)(x) = f(2x+3) = \sqrt[3]{2x+3} - 4 = h(x)$

87. $f(x) = 12x$, $g(x) = 5280x$

$(f \circ g)(x) = f[g(x)] = f(5280x)$
$= 12(5280x) = 63{,}360x$

The function $f \circ g$ computes the number of inches in x miles.

88. **(a)** $x = 4s \Rightarrow \frac{x}{4} = s \Rightarrow s = \frac{x}{4}$

(b) $y = s^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$

(c) $y = \frac{6^2}{16} = \frac{36}{16} = 2.25$ square units

89. $A(x) = \frac{\sqrt{3}}{4}x^2$

(a) $A(2x) = \frac{\sqrt{3}}{4}(2x)^2 = \frac{\sqrt{3}}{4}(4x^2) = \sqrt{3}x^2$

(b) $A(16) = A(2 \cdot 8) = \sqrt{3}(8)^2$
$= 64\sqrt{3}$ square units

90. **(a)** $y_1 = .04x$

(b) $y_2 = .025(x + 500)$

(c) $y_1 + y_2$ represents the total annual interest.

(d) $(y_1 + y_2)(250) = y_1(250) + y_2(250)$
$= .04(250) + .025(250 + 500)$
$= 10 + .025(750) = 10 + 18.75$
$= \$28.75$

91. **(a)** $r(t) = 4t$ and $A(r) = \pi r^2$

$(A \circ r)(t) = A[r(t)]$
$= A(4t) = \pi(4t)^2 = 16\pi t^2$

(b) $(A \circ r)(t)$ defines the area of the leak in terms of the time *t*, in minutes.

(c) $A(3) = 16\pi(3)^2 = 144\pi \text{ ft}^2$

92. **(a)** $(A \circ r)(t) = A[r(t)]$
$= A(2t) = \pi(2t)^2 = 4\pi t^2$

(b) It defines the area of the circular layer in terms of the time *t,* in hours.

(c) $(A \circ r)(4) = 4\pi(4)^2 = 64\pi \text{ mi}^2$

93. Let x = the number of people less than 100 people that attend.

(a) x people fewer than 100 attend, so $100 - x$ people do attend $N(x) = 100 - x$

(b) The cost per person starts at \$20 and increases by \$5 for each of the x people that do not attend. The total increase is \$5x, and the cost per person increases to \$20 + \$5x. Thus, $G(x) = 20 + 5x$.

(c) $C(x) = N(x) \cdot G(x) = (100 - x)(20 + 5x)$

(d) If 80 people attend, $x = 100 - 80 = 20$.

$C(20) = (100 - 20)\left[20 + 5(20)\right]$
$= (80)(20 + 100)$
$= (80)(120) = \$9600$

94. If the area of a square is x^2 square inches, each side must have a length of x inches. If 3 inches is added to one dimension and 1 inch is subtracted from the other, the new dimensions will be $x + 3$ and $x - 1$. Thus, the area of the resulting rectangle is $A(x) = (x + 3)(x - 1)$.

Chapter 2: Review Exercises

1. $P(3, -1), Q(-4, 5)$

$$d(P, Q) = \sqrt{(-4-3)^2 + [5-(-1)]^2}$$
$$= \sqrt{(-7)^2 + 6^2} = \sqrt{49+36} = \sqrt{85}$$

Midpoint:

$$\left(\frac{3+(-4)}{2}, \frac{-1+5}{2}\right) = \left(\frac{-1}{2}, \frac{4}{2}\right) = \left(-\frac{1}{2}, 2\right)$$

2. $M(-8, 2), N(3, -7)$

$$d(M, N) = \sqrt{[3-(-8)]^2 + (-7-2)^2}$$
$$= \sqrt{11^2 + (-9)^2} = \sqrt{121+81} = \sqrt{202}$$

Midpoint: $\left(\frac{-8+3}{2}, \frac{2+(-7)}{2}\right) = \left(-\frac{5}{2}, -\frac{5}{2}\right)$

3. $A(-6, 3), B(-6, 8)$

$$d(A, B) = \sqrt{[-6-(-6)]^2 + (8-3)^2}$$
$$= \sqrt{0+5^2} = \sqrt{25} = 5$$

Midpoint:

$$\left(\frac{-6+(-6)}{2}, \frac{3+8}{2}\right) = \left(\frac{-12}{2}, \frac{11}{2}\right) = \left(-6, \frac{11}{2}\right)$$

4. Label the points $A(5, 7)$, $B(3, 9)$, and $C(6, 8)$.

$$d(A, B) = \sqrt{(3-5)^2 + (9-7)^2}$$
$$= \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8}$$
$$d(A, C) = \sqrt{(6-5)^2 + (8-7)^2}$$
$$= \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$
$$d(B, C) = \sqrt{(6-3)^2 + (8-9)^2}$$
$$= \sqrt{3^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$$

Since $\left(\sqrt{8}\right)^2 + \left(\sqrt{2}\right)^2 = \left(\sqrt{10}\right)^2$, triangle ABC is a right triangle.

5. Label the points $A(-1, 2)$, $B(-10, 5)$, and $C(-4, k)$.

$$d(A, B) = \sqrt{[-1-(-10)]^2 + (2-5)^2}$$
$$= \sqrt{9^2 + (-3)^2} = \sqrt{81+9} = \sqrt{90}$$
$$d(A, C) = \sqrt{[-4-(-1)]^2 + (k-2)^2}$$
$$= \sqrt{9+(k-2)^2}$$
$$d(B, C) = \sqrt{[-10-(-4)]^2 + (5-k)^2}$$
$$= \sqrt{36+(k-5)^2}$$

If segment AB is the hypotenuse, then

$$\left(\sqrt{90}\right)^2 = \left[\sqrt{9+(k-2)^2}\right]^2 + \left[\sqrt{36+(k-5)^2}\right]^2.$$

$$\left(\sqrt{90}\right)^2 = \left[\sqrt{9+(k-2)^2}\right]^2 + \left[\sqrt{36+(k-5)^2}\right]^2$$
$$90 = 9 + (k-2)^2 + 36 + (k-5)^2$$
$$90 = 9 + k^2 - 4k + 4 + 36 + k^2 - 10k + 25$$
$$0 = 2k^2 - 14k - 16 \Rightarrow 0 = k^2 - 7k - 8 \Rightarrow$$
$$0 = (k-8)(k+1) \Rightarrow k = 8 \text{ or } k = -1$$

Another approach is if segment AB is the hypotenuse, the product of the slopes of lines AC and BC is –1 since the product of slopes of perpendicular lines is –1.

$$\left(\frac{k-2}{-4-(-1)}\right)\cdot\left(\frac{k-5}{-4-(-10)}\right) = -1$$
$$\left(\frac{k-2}{-3}\right)\cdot\left(\frac{k-5}{6}\right) = -1$$
$$\frac{(k-2)(k-5)}{-18} = -1$$
$$\frac{k^2-5k-2k+10}{-18} = -1$$
$$k^2 - 7k + 10 = 18 \Rightarrow k^2 - 7k - 8 = 0 \Rightarrow$$
$$(k+8)(k-1) = 0 \Rightarrow k = 8 \text{ or } k = -1$$

We will use the second approach for investigating the other two sides of the triangle. If segment AC is the hypotenuse, the product of the slopes of lines AB and BC is –1 since the product of slopes of perpendicular lines is –1.

$$\left(\frac{5-2}{-10-(-1)}\right)\cdot\left(\frac{k-5}{-4-(-10)}\right) = -1$$
$$\left(\frac{3}{-9}\right)\cdot\left(\frac{k-5}{6}\right) = -1$$
$$\frac{k-5}{-18} = -1$$
$$k - 5 = 18 \Rightarrow k = 23$$

If segment BC is the hypotenuse, the product of the slopes of lines AB and AC is –1.

$$\left(\frac{3}{-9}\right)\cdot\left(\frac{k-2}{-4-(-1)}\right) = -1$$
$$\left(\frac{-1}{3}\right)\cdot\left(\frac{k-2}{-3}\right) = -1$$
$$\frac{k-2}{9} = -1$$
$$k - 2 = -9 \Rightarrow k = -7$$

The possible values of k are –7, 23, 8, and –1.

6. $P(-2, -5)$, $Q(1, 7)$, $R(3, 15)$

$$\begin{aligned} d(P, Q) &= \sqrt{(-2-1)^2 + (-5-7)^2} \\ &= \sqrt{(-3)^2 + (-12)^2} = \sqrt{9+144} \\ &= \sqrt{153} = 3\sqrt{17} \end{aligned}$$

$$\begin{aligned} d(Q, R) &= \sqrt{(3-1)^2 + (15-7)^2} \\ &= \sqrt{2^2 + 8^2} = \sqrt{4+64} \\ &= \sqrt{68} = 2\sqrt{17} \end{aligned}$$

$$\begin{aligned} d(P, R) &= \sqrt{(-2-3)^2 + (-5-15)^2} \\ &= \sqrt{(-5)^2 + (-20)^2} = \sqrt{25+400} \\ &= \sqrt{425} = 5\sqrt{17} \end{aligned}$$

Since $d(P, Q) + d(Q, R) = 3\sqrt{17} + 2\sqrt{17}$ $= 5\sqrt{17} = d(P, R)$, these three points are collinear.

7. Center $(-2, 3)$, radius 15

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ [x-(-2)]^2 + (y-3)^2 &= 15^2 \\ (x+2)^2 + (y-3)^2 &= 225 \end{aligned}$$

8. Center $(\sqrt{5}, -\sqrt{7})$, radius $\sqrt{3}$

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ \left(x-\sqrt{5}\right)^2 + \left[y-\left(-\sqrt{7}\right)\right]^2 &= \left(\sqrt{3}\right)^2 \\ \left(x-\sqrt{5}\right)^2 + \left(y+\sqrt{7}\right)^2 &= 3 \end{aligned}$$

9. Center $(-8, 1)$, passing through $(0, 16)$

The radius is the distance from the center to any point on the circle. The distance between $(-8, 1)$ and $(0, 16)$ is

$$\begin{aligned} r &= \sqrt{(-8-0)^2 + (1-16)^2} = \sqrt{(-8)^2 + (-15)^2} \\ &= \sqrt{64+225} = \sqrt{289} = 17. \end{aligned}$$

The equation of the circle is

$$\begin{aligned} [x-(-8)]^2 + (y-1)^2 &= 17^2 \\ (x+8)^2 + (y-1)^2 &= 289 \end{aligned}$$

10. Center $(3, -6)$, tangent to the x-axis

The point $(3, -6)$ is 6 units directly below the x-axis. Any segment joining a circle's center to a point on the circle must be a radius, so in this case the length of the radius is 6 units.

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ (x-3)^2 + [y-(-6)]^2 &= 6^2 \\ (x-3)^2 + (y+6)^2 &= 36 \end{aligned}$$

11. The center of the circle is $(0, 0)$. Use the distance formula to find the radius:

$$r^2 = (3-0)^2 + (5-0)^2 = 9 + 25 = 34$$

The equation is $x^2 + y^2 = 34$.

12. The center of the circle is $(0, 0)$. Use the distance formula to find the radius:

$$r^2 = (-2-0)^2 + (3-0)^2 = 4 + 9 = 13$$

The equation is $x^2 + y^2 = 13$.

13. The center of the circle is $(0, 3)$. Use the distance formula to find the radius:

$$r^2 = (-2-0)^2 + (6-3)^2 = 4 + 9 = 13$$

The equation is $x^2 + (y-3)^2 = 13$.

14. The center of the circle is $(5, 6)$. Use the distance formula to find the radius:

$$r^2 = (4-5)^2 + (9-6)^2 = 1 + 9 = 10$$

The equation is $(x-5)^2 + (y-6)^2 = 10$.

15. $x^2 - 4x + y^2 + 6y + 12 = 0$

Complete the square on x and y to put the equation in center-radius form.

$$\begin{aligned} \left(x^2 - 4x\right) + \left(y^2 + 6y\right) &= -12 \\ \left(x^2 - 4x + 4\right) + \left(y^2 + 6y + 9\right) &= -12 + 4 + 9 \\ \left(x-2\right)^2 + \left(y+3\right)^2 &= 1 \end{aligned}$$

The circle has center $(2, -3)$ and radius 1.

16. $x^2 - 6x + y^2 - 10y + 30 = 0$

Complete the square on x and y to put the equation in center-radius form.

$$\begin{aligned} (x^2 - 6x + 9) + (y^2 - 10y + 25) &= -30 + 9 + 25 \\ (x-3)^2 + (y-5)^2 &= 4 \end{aligned}$$

The circle has center $(3, 5)$ and radius 2.

17.

$$\begin{aligned} 2x^2 + 14x + 2y^2 + 6y + 2 &= 0 \\ x^2 + 7x + y^2 + 3y + 1 &= 0 \\ \left(x^2 + 7x\right) + \left(y^2 + 3y\right) &= -1 \\ \left(x^2 + 7x + \tfrac{49}{4}\right) + \left(y^2 + 3y + \tfrac{9}{4}\right) &= -1 + \tfrac{49}{4} + \tfrac{9}{4} \\ \left(x + \tfrac{7}{2}\right)^2 + \left(y + \tfrac{3}{2}\right)^2 &= -\tfrac{4}{4} + \tfrac{49}{4} + \tfrac{9}{4} \\ \left(x + \tfrac{7}{2}\right)^2 + \left(y + \tfrac{3}{2}\right)^2 &= \tfrac{54}{4} \end{aligned}$$

The circle has center $\left(-\frac{7}{2}, -\frac{3}{2}\right)$ and radius $\sqrt{\frac{54}{4}} = \frac{\sqrt{54}}{\sqrt{4}} = \frac{\sqrt{9 \cdot 6}}{\sqrt{4}} = \frac{3\sqrt{6}}{2}$.

18.
$$3x^2+33x+3y^2-15y=0$$
$$x^2+11x+y^2-5y=0$$
$$\left(x^2+11x\right)+\left(y^2-5y\right)=0$$
$$\left(x^2+11x+\tfrac{121}{4}\right)+\left(y^2-5y+\tfrac{25}{4}\right)=0+\tfrac{121}{4}+\tfrac{25}{4}$$
$$\left(x+\tfrac{11}{2}\right)^2+\left(y-\tfrac{5}{2}\right)^2=\tfrac{146}{4}$$
The circle has center $\left(-\frac{11}{2},\frac{5}{2}\right)$ and radius $\frac{\sqrt{146}}{2}$.

19. Find all possible values of x so that the distance between $(x, -9)$ and $(3, -5)$ is 6.
$$\sqrt{(3-x)^2+(-5+9)^2}=6$$
$$\sqrt{9-6x+x^2+16}=6$$
$$\sqrt{x^2-6x+25}=6$$
$$x^2-6x+25=36$$
$$x^2-6x-11=0$$
Apply the quadratic formula where $a=1, b=-6,$ and $c=-11$.
$$x=\frac{6\pm\sqrt{36-4(1)(-11)}}{2}=\frac{6\pm\sqrt{36+44}}{2}$$
$$=\frac{6\pm\sqrt{80}}{2}=\frac{6\pm4\sqrt{5}}{2}=\frac{2(3\pm2\sqrt{5})}{2}$$
$x=3+2\sqrt{5}$ or $x=3-2\sqrt{5}$

20. This is not the graph of a function because a vertical line can intersect it in two points.
domain: $(-\infty,\infty)$; range: $[0,\infty)$

21. This is not the graph of a function because a vertical line can intersect it in two points.
domain: $[-6,6]$; range: $[-6,6]$

22. This is the graph of a function. No vertical line will intersect the graph in more than one point.
domain: $(-\infty,-2]\cup[2,\infty)$; range: $[0,\infty)$

23. This is not the graph of a function because a vertical line can intersect it in two points.
domain: $(-\infty,\infty)$; range: $(-\infty,-1]\cup[1,\infty)$

24. This is the graph of a function. No vertical line will intersect the graph in more than one point.
domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$

25. This is not the graph of a function because a vertical line can intersect it in two points.
domain: $[0,\infty)$; range: $(-\infty,\infty)$

26. The equation $x=\frac{1}{3}y^2$ does not define y as a function of x. For some values of x, there will be more than one value of y. For example, ordered pairs (3, 3) and (3, −3) satisfy the relation. Thus, the relation would not be a function.

27. $y=6-x^2$
Each value of x corresponds to exactly one value of y, so this equation defines a function.

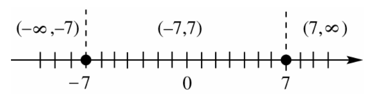
28. The equation $y=-\dfrac{4}{x}$ defines y as a function of x because for every x in the domain, which is $(-\infty,0)\cup(0,\infty)$, there will be exactly one value of y.

29. The equation $y=\pm\sqrt{x-2}$ does not define y as a function of x. For some values of x, there will be more than one value of y. For example, ordered pairs (3, 1) and (3, −1) satisfy the relation.

30. In the function $f(x)=-4+|x|$, we may use any real number for x. The domain is $(-\infty,\infty)$.

31. $f(x)=\dfrac{8+x}{8-x}$
x can be any real number except 8, since this will give a denominator of zero. Thus, the domain is $(-\infty,8)\cup(8,\infty)$.

32. $y=\sqrt{49-x^2}$
In the function $y=\sqrt{49-x^2}$, we must have $49-x^2\ge 0$.
Step 1: Find the values of x that satisfy $49-x^2=0$.
$$49-x^2=0$$
$$(7+x)(7-x)=0\Rightarrow x=-7 \text{ or } x=7$$
Step 2: The two numbers divide a number line into three regions.



Step 3: Choose a test value to see if it satisfies the inequality, $49 - x^2 \geq 0$.

Interval	Test Value	Is $49 - x^2 \geq 0$ true or false?
$(-\infty, -7)$	-8	$49 - (-8)^2 \geq 0$? $-15 \geq 0$ False
$(-7, 7)$	0	$49 - 0^2 \geq 0$? $49 \geq 0$ True
$(7, \infty)$	8	$49 - 8^2 \geq 0$? $-15 \geq 0$ False

Solving this inequality, we obtain a solution interval of [–7, 7], so the domain is [–7, 7].

33. **(a)** As x is getting larger on the interval $[2, \infty)$, the value of y is increasing.

(b) As x is getting larger on the interval $(-\infty, -2]$, the value of y is decreasing.

34. We need to consider the solid dot. Thus, $f(0) = 0$.

35. $f(x) = -2x^2 + 3x - 6$

$$f(3) = -2 \cdot 3^2 + 3 \cdot 3 - 6$$
$$= -2 \cdot 9 + 3 \cdot 3 - 6$$
$$= -18 + 9 - 6 = -15$$

36. $f(x) = -2x^2 + 3x - 6$

$$f(-0.5) = -2(-0.5)^2 + 3(-0.5) - 6$$
$$= -2(0.25) + 3(-0.5) - 6$$
$$= -0.5 - 1.5 - 6 = -8$$

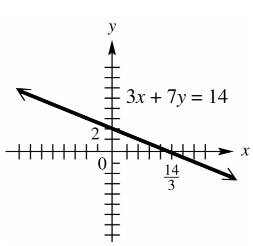
37. $f(x) = -2x^2 + 3x - 6 \Rightarrow f(k) = -2k^2 + 3k - 6$

38. $3x + 7y = 14$

$$7y = -3x + 14$$
$$y = -\frac{3}{7}x + 2$$

The graph is the line with slope of $-\frac{3}{7}$ and y-intercept 2. It may also be graphed using intercepts. To do this, locate the x-intercept by setting $y = 0$:

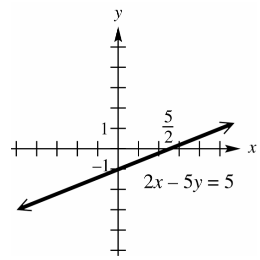
$3x + 7(0) = 14 \Rightarrow 3x = 14 \Rightarrow x = \frac{14}{3}$



39. $2x - 5y = 5 \Rightarrow -5y = -2x + 5 \Rightarrow y = \frac{2}{5}x - 1$

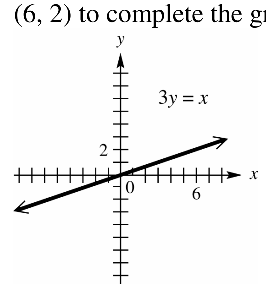
The graph is the line with slope $\frac{2}{5}$ and y-intercept –1. It may also be graphed using intercepts. To do this, locate the x-intercept: x-intercept: $y = 0$

$2x - 5(0) = 5 \Rightarrow 2x = 5 \Rightarrow x = \frac{5}{2}$



40. $3y = x \Rightarrow y = \frac{1}{3}x$

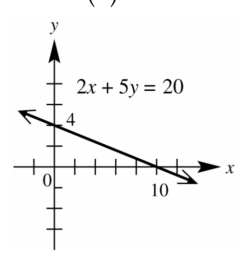
The graph is the line with slope $\frac{1}{3}$ and y-intercept 0, which means that it passes through the origin. Use another point such as (6, 2) to complete the graph.



41. $2x + 5y = 20 \Rightarrow 5y = -2x + 20 \Rightarrow y = -\frac{2}{5}x + 4$

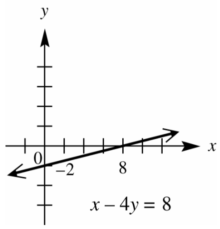
The graph is the line with slope of $-\frac{2}{5}$ and y-intercept 4. It may also be graphed using intercepts. To do this, locate the x-intercept: x-intercept: $y = 0$

$2x + 5(0) = 20 \Rightarrow 2x = 20 \Rightarrow x = 10$

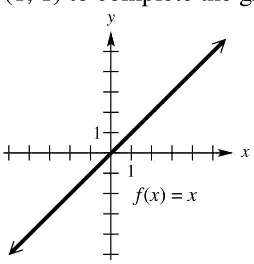


42. $x - 4y = 8$
$-4y = -x + 8$
$y = \frac{1}{4}x - 2$

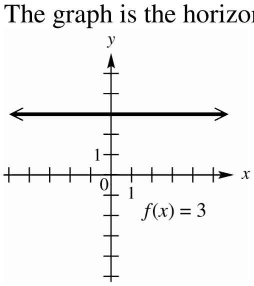
The graph is the line with slope $\frac{1}{4}$ and y-intercept –2. It may also be graphed using intercepts. To do this, locate the x-intercept:
x-intercept: $y = 0 \quad x - 4(0) = 8 \Rightarrow x = 8$



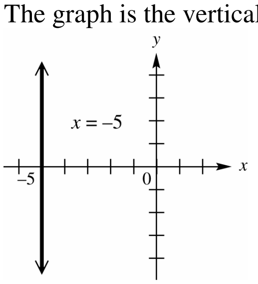
43. $f(x) = x$
The graph is the line with slope 1 and y-intercept 0, which means that it passes through the origin. Use another point such as (1, 1) to complete the graph.



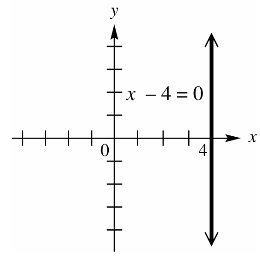
44. $f(x) = 3$
The graph is the horizontal line through (0, 3).



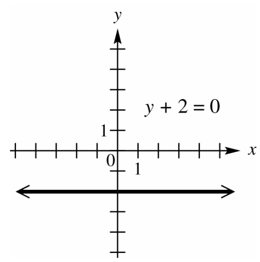
45. $x = -5$
The graph is the vertical line through (–5, 0).



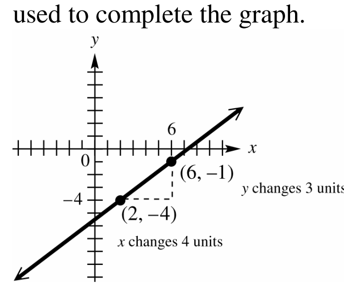
46. $x - 4 = 0 \Rightarrow x = 4$
The graph is the vertical line through (4, 0).



47. $y + 2 = 0 \Rightarrow y = -2$
The graph is the horizontal line through $(0, -2)$.

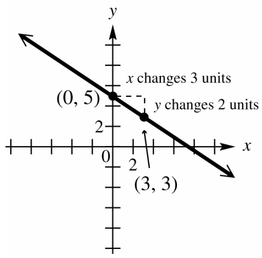


48. Line through (2, –4), $m = \frac{3}{4}$
First locate the point (2, –4).
Since the slope is $\frac{3}{4}$, a change of 4 units horizontally(4 units to the right) produces a change of 3 units vertically (3 units up). This gives a second point, (6, –1), which can be used to complete the graph.



49. Line through (0, 5), $m = -\frac{2}{3}$

Note that $m = -\frac{2}{3} = \frac{-2}{3}$.
Begin by locating the point (0, 5). Since the slope is $\frac{-2}{3}$, a change of 3 units horizontally (3 units to the right) produces a change of –2 units vertically (2 units down). This gives a second point, (3, 3), which can be used to complete the graph.



50. through (8, 7) and $\left(\frac{1}{2}, -2\right)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-7}{\frac{1}{2}-8} = \frac{-9}{-\frac{15}{2}} = -9\left(-\frac{2}{15}\right) = \frac{18}{15} = \frac{6}{5}$$

51. through (2, –2) and (3, –4)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-(-2)}{3-2} = \frac{-2}{1} = -2$$

52. through (5, 6) and (5, –2)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-6}{5-5} = \frac{-8}{0}$$

The slope is undefined.

53. through (0, –7) and (3, –7)

$$m = \frac{-7-(-7)}{3-0} = \frac{0}{3} = 0$$

54. $9x - 4y = 2$.
Solve for y to put the equation in slope-intercept form.

$$-4y = -9x + 2 \Rightarrow y = \frac{9}{4}x - \frac{1}{2}$$

Thus, the slope is $\frac{9}{4}$.

55. $11x + 2y = 3$

Solve for y to put the equation in slope-intercept form.

$$2y = -11x + 3 \Rightarrow y = -\frac{11}{2}x + \frac{3}{2}$$

Thus, the slope is $-\frac{11}{2}$.

56. $x - 5y = 0$.
Solve for y to put the equation in slope-intercept form.

$$-5y = -x \Rightarrow y = \frac{1}{5}x$$

Thus, the slope is $\frac{1}{5}$.

57. $x - 2 = 0 \Rightarrow x = 2$
The graph is a vertical line, through (2, 0). The slope is undefined.

58. **(a)** This is the graph of a function since no vertical line intersects the graph in more than one point.

(b) The lowest point on the graph occurs in December, so the most jobs lost occurred in December. The highest point on the graph occurs in January, so the most jobs gained occurred in January.

(c) The number of jobs lost in December is approximately 6000. The number of jobs gained in January is approximately 2000.

(d) It shows a slight downward trend.

59. Initially, the car is at home. After traveling for 30 mph for 1 hr, the car is 30 mi away from home. During the second hour the car travels 20 mph until it is 50 mi away. During the third hour the car travels toward home at 30 mph until it is 20 mi away. During the fourth hour the car travels away from home at 40 mph until it is 60 mi away from home. During the last hour, the car travels 60 mi at 60 mph until it arrived home.

60. We need to find the slope of a line that passes between points $(1970, 10{,}000)$ and $(2006, 56{,}000)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{56{,}000 - 10{,}000}{2006 - 1970} = \frac{46{,}000}{36} \approx \$1278 \text{ per year}$$

61. **(a)** We need to first find the slope of a line that passes between points (0, 30.7) and (5, 70.7)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{70.7 - 30.7}{5-0} = \frac{40}{5} = 8$$

Now use the point-slope form with $(x_1, y_1) = (0, 30.7)$ and $m = 8$. (The other point, (5, 70.7), could also have been used.)

$$y - 30.7 = 8(x - 0) \Rightarrow y = 8x + 30.7$$

The slope, 8, indicates that the number of e-filing taxpayers increased by 8% each year from 2001 to 2005.

(b) For 2005, we evaluate the function for $x = 4$. $y = 8(4) + 30.7 = 62.7$

62.7% of the tax returns are predicted to have been filed electronically.

62. **(a)** through (–2, 4) and (1, 3)
First find the slope.
$$m = \frac{3-4}{1-(-2)} = \frac{-1}{3}$$
Now use the point-slope form with $(x_1, y_1) = (1, 3)$ and $m = -\frac{1}{3}$.
$$y - y_1 = m(x - x_1)$$
$$y - 3 = -\tfrac{1}{3}(x-1)$$
$$3(y-3) = -1(x-1)$$
$$3y - 9 = -x + 1$$
$$3y = -x + 10 \Rightarrow y = -\tfrac{1}{3}x + \tfrac{10}{3}$$

(b) Standard form:
$$y = -\tfrac{1}{3}x + \tfrac{10}{3} \Rightarrow 3y = -x + 10 \Rightarrow x + 3y = 10$$

63. **(a)** through (3, –5) with slope –2
Use the point-slope form.
$$y - y_1 = m(x - x_1)$$
$$y - (-5) = -2(x-3)$$
$$y + 5 = -2(x-3)$$
$$y + 5 = -2x + 6$$
$$y = -2x + 1$$

(b) Standard form: $y = -2x + 1 \Rightarrow 2x + y = 1$

64. **(a)** x-intercept –3, y-intercept 5
Two points of the line are (–3, 0) and (0, 5). First, find the slope.
$$m = \frac{5-0}{0+3} = \frac{5}{3}$$
The slope is $\frac{5}{3}$ and the y-intercept is 5.
Write the equation in slope-intercept form: $y = \frac{5}{3}x + 5$

(b) Standard form:
$$y = \tfrac{5}{3}x + 5 \Rightarrow 3y = 5x + 15 \Rightarrow -5x + 3y = 15 \Rightarrow 5x - 3y = -15$$

65. **(a)** through (2, –1) parallel to $3x - y = 1$
Find the slope of $3x - y = 1$.
$$3x - y = 1 \Rightarrow -y = -3x + 1 \Rightarrow y = 3x - 1$$
The slope of this line is 3. Since parallel lines have the same slope, 3 is also the slope of the line whose equation is to be found. Now use the point-slope form with $(x_1, y_1) = (2, -1)$ and $m = 3$.
$$y - y_1 = m(x - x_1)$$
$$y - (-1) = 3(x-2)$$
$$y + 1 = 3x - 6 \Rightarrow y = 3x - 7$$

(b) Standard form:
$$y = 3x - 7 \Rightarrow -3x + y = -7 \Rightarrow 3x - y = 7$$

66. **(a)** through (0, 5), perpendicular to $8x + 5y = 3$
Find the slope of $8x + 5y = 3$.
$$8x + 5y = 3 \Rightarrow 5y = -8x + 3 \Rightarrow y = -\tfrac{8}{5}x + \tfrac{3}{5}$$
The slope of this line is $-\frac{8}{5}$. The slope of any line perpendicular to this line is $\frac{5}{8}$, since $-\frac{8}{5}\left(\frac{5}{8}\right) = -1$.
The equation in slope-intercept form with slope $\frac{5}{8}$ and y-intercept 5 is $y = \frac{5}{8}x + 5$.

(b) Standard form:
$$y = \tfrac{5}{8}x + 5 \Rightarrow 8y = 5x + 40 \Rightarrow -5x + 8y = 40 \Rightarrow 5x - 8y = -40$$

67. **(a)** through (2, –10), perpendicular to a line with an undefined slope
A line with an undefined slope is a vertical line. Any line perpendicular to a vertical line is a horizontal line, with an equation of the form $y = b$. Since the line passes through (2, –10), the equation of the line is $y = -10$.

(b) Standard form: $y = -10$

68. **(a)** through (3, –5), parallel to $y = 4$
This will be a horizontal line through (3, –5). Since y has the same value at all points on the line, the equation is $y = -5$.

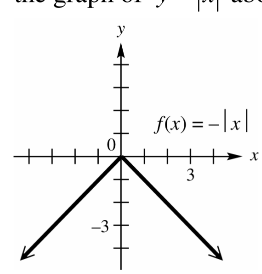
(b) Standard form: $y = -5$

69. **(a)** through (–7, 4), perpendicular to $y = 8$
The line $y = 8$ is a horizontal line, so any line perpendicular to it will be a vertical line. Since x has the same value at all points on the line, the equation is $x = -7$. It is not possible to write this in slope-intercept form.

(b) Standard form: $x = -7$

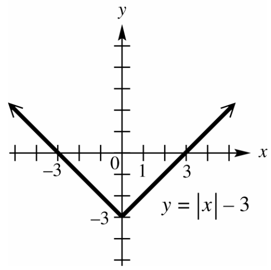
70. $f(x) = -|x|$

The graph of $f(x) = -|x|$ is the reflection of the graph of $y = |x|$ about the x-axis.



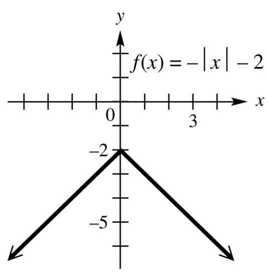
71. $f(x) = |x| - 3$

The graph is the same as that of $y = |x|$, except that it is translated 3 units downward.



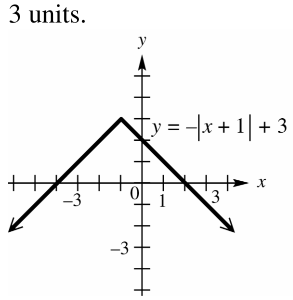
72. $f(x) = -|x| - 2$

The graph of $f(x) = -|x| - 2$ is the reflection of the graph of $y = |x|$ about the x-axis, translated down 2 units.



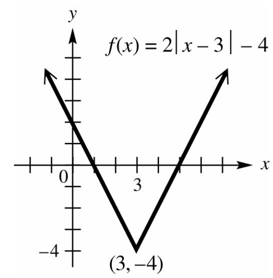
73. $f(x) = -|x+1| + 3 = -|x-(-1)| + 3$

The graph of $f(x) = -|x+1| + 3$ is a translation of the graph of $y = |x|$ to the left 1 unit, reflected over the x-axis and translated up 3 units.



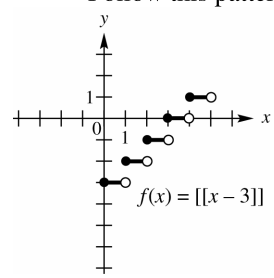
74. $f(x) = 2|x-3| - 4$

The graph of $f(x) = 2|x-3| - 4$ is a translation of the graph of $y = |x|$ to the right 3 units, stretched vertically by a factor of 2, and translated down 4 units.



75. $f(x) = [\![x-3]\!]$

To get $y = 0$, we need $0 \le x - 3 < 1 \Rightarrow$ $3 \le x < 4.$ To get $y = 1$, we need $1 \le x - 3 < 2 \Rightarrow \ 4 \le x < 5.$
Follow this pattern to graph the step function.

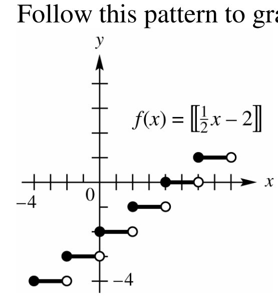


76. $f(x) = [\![\frac{1}{2}x - 2]\!]$

For y to be 0, we need $0 \le \frac{1}{2}x - 2 < 1 \Rightarrow$

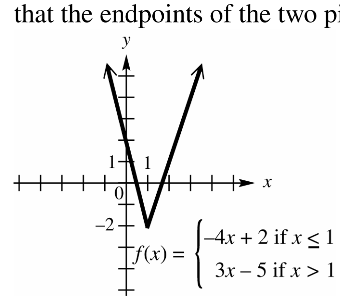
$2 \le \frac{1}{2}x < 3 \Rightarrow 4 \le x < 6.$

Follow this pattern to graph the step function.



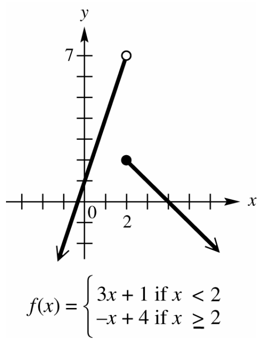
77. $f(x) = \begin{cases} -4x + 2 & \text{if } x \le 1 \\ 3x - 5 & \text{if } x > 1 \end{cases}$

Draw the graph of $y = -4x + 2$ to the left of $x = 1$, including the endpoint at $x = 1$. Draw the graph of $y = 3x - 5$ to the right of $x = 1$, but do not include the endpoint at $x = 1$. Observe that the endpoints of the two pieces coincide.



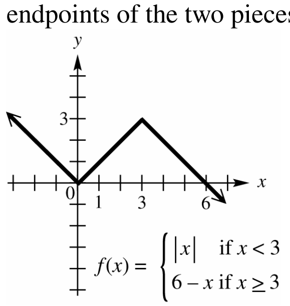
78. $f(x) = \begin{cases} 3x+1 & \text{if } x < 2 \\ -x+4 & \text{if } x \geq 2 \end{cases}$

Graph the line $y = 3x + 1$ to the left of $x = 2$, and graph the line $y = -x + 4$ to the right of $x = 2$. The graph has an open circle at (2, 7) and a closed circle at (2, 2).



79. $f(x) = \begin{cases} |x| & \text{if } x < 3 \\ 6 - x & \text{if } x \geq 3 \end{cases}$

Draw the graph of $y = |x|$ to the left of $x = 3$, but do not include the endpoint. Draw the graph of $y = 6 - x$ to the right of $x = 3$, including the endpoint. Observe that the endpoints of the two pieces coincide.



80. The graph of a nonzero function cannot be symmetric with respect to the *x*-axis. Such a graph would fail the vertical line test, so the statement is true.

81. The graph of an even function is symmetric with respect to the *y*-axis. This statement is true.

82. The graph of an odd function is symmetric with respect to the origin. This statement is true.

83. If (*a*, *b*) is on the graph of an even function, so is (*a*, –*b*). The statement is false. For example, $f(x) = x^2$ is even, and (2, 4) is on the graph but (2, –4) is not.

84. If (*a*, *b*) is on the graph of an odd function, so is (–*a*, *b*). This statement is false. For example, $f(x) = x^3$ is odd, and (2, 8) is on the graph but (–2, 8) is not.

85. The constant function $f(x) = 0$ is both even and odd. Since $f(-x) = 0 = f(x)$, the function is even. Also since $f(-x) = 0 = -0 = -f(x)$, the function is odd. This statement is true.

86. $5y^2 + 5x^2 = 30$

Replace *x* with –*x* to obtain $5y^2 + 5(-x)^2 = 30 \Rightarrow 5y^2 + 5x^2 = 30.$ The result is the same as the original equation, so the graph is symmetric with respect to the *y*-axis. Replace *y* with –*y* to obtain $5(-y)^2 + 5x^2 = 30 \Rightarrow 5y^2 + 5x^2 = 30.$ The result is the same as the original equation, so the graph is symmetric with respect to the *x*-axis. Since the graph is symmetric with respect to the *y*-axis and *x*-axis, it must also be symmetric with respect to the origin. Note that this equation is the same as $y^2 + x^2 = 6$, which is a circle centered at the origin.

87. $x + y^2 = 10$

Replace *x* with –*x* to obtain $(-x) + y^2 = 10.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the *y*-axis. Replace *y* with –*y* to obtain $x + (-y)^2 = 10 \Rightarrow x + y^2 = 10.$ The result is the same as the original equation, so the graph is symmetric with respect to the *x*-axis. Replace *x* with –*x* and *y* with –*y* to obtain $(-x) + (-y)^2 = 10 \Rightarrow (-x) + y^2 = 10.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. The graph is symmetric with respect to the *x*-axis only.

88. $y^3 = x + 4$

Replace *x* with –*x* to obtain $y^3 = -x + 4$. The result is not the same as the original equation, so the graph is not symmetric with respect to the *y*-axis. Replace *y* with –*y* to obtain $(-y)^3 = x + 4 \Rightarrow -y^3 = x + 4 \Rightarrow$ $y^3 = -x - 4$ The result is not the same as the original equation, so the graph is not symmetric with respect to the *x*-axis.

Replace x with $-x$ and y with $-y$ to obtain $(-y)^3 = (-x)+1 \Rightarrow -y^3 = -x+1 \Rightarrow y^3 = x-1.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph has none of the listed symmetries.

89. $x^2 = y^3$
Replace x with $-x$ to obtain $(-x)^2 = y^3 \Rightarrow x^2 = y^3.$ The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Replace y with $-y$ to obtain $x^2 = (-y)^3 \Rightarrow x^2 = -y^3.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $(-x)^2 = (-y)^3 \Rightarrow x^2 = -y^3.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the y-axis only.

90. $|y| = -x$
Replace x with $-x$ to obtain $|y| = -(-x) \Rightarrow |y| = x.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Replace y with $-y$ to obtain $|-y| = -x \Rightarrow |y| = -x.$ The result is the same as the original equation, so the graph is symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $|-y| = -(-x) \Rightarrow |y| = x.$ The result is not the same as the original equation, so the graph is not symmetric with respect to the origin. Therefore, the graph is symmetric with respect to the x-axis only.

91. $6x + y = 4$
Replace x with $-x$ to obtain $6(-x) + y = 4 \Rightarrow -6x + 7 = 4$. The result is not the same as the original equation, so the graph is not symmetric with respect to the y-axis. Replace y with $-y$ to obtain $6x + (-y) = 4 \Rightarrow 6x - y = 4$. The result is not the same as the original equation, so the graph is not symmetric with respect to the x-axis. Replace x with $-x$ and y with $-y$ to obtain $6(-x) + (-y) = 4 \Rightarrow -6x - y = 4$. This equation is not equivalent to the original one, so the graph is not symmetric with respect to the origin. Therefore, the graph has none of the listed symmetries.

92. $|x| = |y|$
Replace x with $-x$ to obtain $|-x| = |y| \Rightarrow |x| = |y|.$ The result is the same as the original equation, so the graph is symmetric with respect to the y-axis. Replace y with $-y$ to obtain $|x| = |-y| \Rightarrow |x| = |y|.$ The result is the same as s the original equation, so the graph is symmetric with respect to the x-axis. Since the graph is symmetric with respect to the x-axis and with respect to the y-axis, it must also by symmetric with respect to the origin.

93. To obtain the graph of $g(x) = -|x|$, reflect the graph of $f(x) = |x|$ across the x-axis.

94. To obtain the graph of $h(x) = |x| - 2$, translate the graph of $f(x) = |x|$ down 2 units.

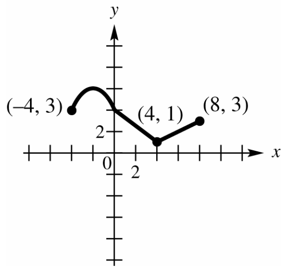
95. To obtain the graph of $k(x) = 2|x-4|$, translate the graph of $f(x) = |x|$ to the right 4 units and stretch vertically by a factor of 2.

96. If the graph of $f(x) = 3x - 4$ is reflected about the x-axis, we obtain a graph whose equation is $y = -(3x-4) = -3x + 4.$

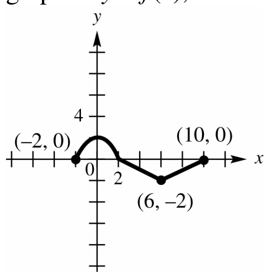
97. If the graph of $f(x) = 3x - 4$ is reflected about the y-axis, we obtain a graph whose equation is $y = f(-x) = 3(-x) - 4 = -3x - 4.$

98. If the graph of $f(x) = 3x - 4$ is reflected about the origin, every point (x, y) will be replaced by the point $(-x, -y)$. The equation for the graph will change from $y = 3x - 4$ to $-y = 3(-x) - 4 \Rightarrow -y = -3x - 4 \Rightarrow y = 3x + 4.$

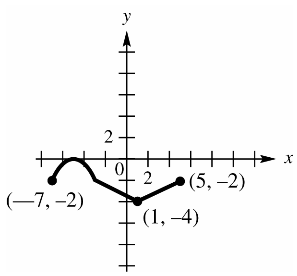
99. **(a)** To graph $y = f(x)+3$, translate the graph of $y = f(x)$, 3 units up.



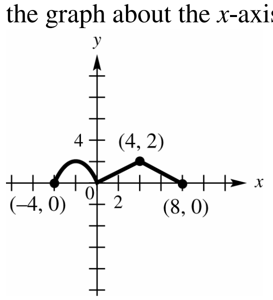
(b) To graph $y = f(x-2)$, translate the graph of $y = f(x)$, 2 units to the right.



(c) To graph $y = f(x+3)-2$, translate the graph of $y = f(x)$, 3 units to the left and 2 units down.



(d) To graph $y = |f(x)|$, keep the graph of $y = f(x)$ as it is where $y \geq 0$ and reflect the graph about the x-axis where $y < 0$.



For Exercises 100–108, $f(x) = 3x^2 - 4$ and $g(x) = x^2 - 3x - 4$.

100. $(f+g)(x) = f(x)+g(x)$
$= (3x^2-4)+(x^2-3x-4)$
$= 4x^2-3x-8$

101. $(fg)(x) = f(x)\cdot g(x)$
$= (3x^2-4)(x^2-3x-4)$
$= 3x^4-9x^3-12x^2-4x^2+12x+16$
$= 3x^4-9x^3-16x^2+12x+16$

102. $(f-g)(4) = f(4)-g(4)$
$= (3\cdot 4^2-4)-(4^2-3\cdot 4-4)$
$= (3\cdot 16-4)-(16-3\cdot 4-4)$
$= (48-4)-(16-12-4)$
$= 44-0 = 44$

103. $(f+g)(-4) = f(-4)+g(-4)$
$= [3(-4)^2-4]+[(-4)^2-3(-4)-4]$
$= [3(16)-4]+[16-3(-4)-4]$
$= [48-4]+[16+12-4]$
$= 44+24 = 68$

104. $(f+g)(2k) = f(2k)+g(2k)$
$= [3(2k)^2-4]+[(2k)^2-3(2k)-4]$
$= [3(4)k^2-4]+[4k^2-3(2k)-4]$
$= (12k^2-4)+(4k^2-6k-4)$
$= 16k^2-6k-8$

105. $\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \frac{3\cdot 3^2-4}{3^2-3\cdot 3-4} = \frac{3\cdot 9-4}{9-3\cdot 3-4}$
$= \frac{27-4}{9-9-4} = \frac{23}{-4} = -\frac{23}{4}$

106. $\left(\frac{f}{g}\right)(-1) = \frac{3(-1)^2-4}{(-1)^2-3(-1)-4} = \frac{3(1)-4}{1-3(-1)-4}$
$= \frac{3-4}{1+3-4} = \frac{-1}{0} = \text{undefined}$

107. The domain of $(fg)(x)$ is the intersection of the domain of $f(x)$ and the domain of $g(x)$. Both have domain $(-\infty, \infty)$, so the domain of $(fg)(x)$ is $(-\infty, \infty)$.

108. $\left(\frac{f}{g}\right)(x) = \frac{3x^2-4}{x^2-3x-4} = \frac{3x^2-4}{(x+1)(x-4)}$

Since both $f(x)$ and $g(x)$ have domain $(-\infty, \infty)$, we are concerned about values of x that make $g(x) = 0$. Thus, the expression is undefined if $(x+1)(x-4) = 0$, that is, if $x = -1$ or $x = 4$. Thus, the domain is the set of all real numbers except $x = -1$ and $x = 4$, or $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$.

109. $f(x) = \frac{1}{x},\ g(x) = x^2 + 1$

Since

$(f \circ g)(x) = f[g(x)]$ and $(f \circ g)(x) = \frac{1}{x^2 + 1}$,

choices (C) and (D) are not equal to $(f \circ g)(x)$.

110. $f(x) = 2x + 9$

$f(x+h) = 2(x+h) + 9 = 2x + 2h + 9$

$f(x+h) - f(x) = (2x + 2h + 9) - (2x + 9)$
$= 2x + 2h + 9 - 2x - 9 = 2h$

Thus, $\frac{f(x+h) - f(x)}{h} = \frac{2h}{h} = 2.$

111. $f(x) = x^2 - 5x + 3$

$f(x+h) = (x+h)^2 - 5(x+h) + 3$
$= x^2 + 2xh + h^2 - 5x - 5h + 3$

$f(x+h) - f(x)$
$= (x^2 + 2xh + h^2 - 5x - 5h + 3) - (x^2 - 5x + 3)$
$= x^2 + 2xh + h^2 - 5x - 5h + 3 - x^2 + 5x - 3$
$= 2xh + h^2 - 5h$

$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 5h}{h}$
$= \frac{h(2x + h - 5)}{h} = 2x + h - 5$

For Exercises 112–116,
$f(x) = \sqrt{x-2}$ and $g(x) = x^2$.

112. $(f \circ g)(x) = f[g(x)] = f(x^2) = \sqrt{x^2 - 2}$

113. $(g \circ f)(x) = g[f(x)] = g\left(\sqrt{x-2}\right)$
$= \left(\sqrt{x-2}\right)^2 = x - 2$

114. Since $g(x) = x^2,\ g(-6) = (-6)^2 = 36.$

Therefore, $(f \circ g)(2) = f\left[g(-6)\right] = f(36)$
$= \sqrt{36 - 2} = \sqrt{34}$.

115. Since $f(x) = \sqrt{x-2},\ f(3) = \sqrt{3-2} = \sqrt{1} = 1.$

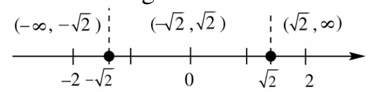
Therefore, $(g \circ f)(3) = g\left[f(3)\right] = g(1)$
$= 1^2 = 1.$

116. To find the domain of $f \circ g$, we must consider the domain of g as well as the composed function, $f \circ g$. Since

$(f \circ g)(x) = f\left[g(x)\right] = \sqrt{x^2 - 2}$ we need to determine when $x^2 - 2 \ge 0.$

Step 1: Find the values of x that satisfy
$x^2 - 2 = 0.$
$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Step 2: The two numbers divide a number line into three regions.



Step 3 Choose a test value to see if it satisfies the inequality, $x^2 - 2 \ge 0.$

Interval	Test Value	Is $x^2 - 2 \ge 0$ true or false?
$\left(-\infty, -\sqrt{2}\right)$	-2	$(-2)^2 - 2 \ge 0$? $2 \ge 0$ True
$\left(-\sqrt{2}, \sqrt{2}\right)$	0	$0^2 - 2 \ge 0$? $-2 \ge 0$ False
$\left(\sqrt{2}, \infty\right)$	2	$2^2 - 2 \ge 0$? $2 \ge 0$ True

The domain of $f \circ g$ is
$\left(-\infty, -\sqrt{2}\right] \cup \left[\sqrt{2}, \infty\right).$

117. $(f + g)(1) = f(1) + g(1) = 7 + 1 = 8$

118. $(f - g)(3) = f(3) - g(3) = 9 - 9 = 0$

119. $(fg)(-1) = f(-1) \cdot g(-1) = 3(-2) = -6$

120. $\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{5}{0} =$ undefined

121. $(g \circ f)(-2) = g[f(-2)] = g(1) = 2$

122. $(f \circ g)(3) = f[g(3)] = f(-2) = 1$

123. $(f \circ g)(2) = f[g(2)] = f(2) = 1$

124. $(g \circ f)(3) = g[f(3)] = g(4) = 8$

125. Let x = number of yards.
$f(x) = 36x$, where $f(x)$ is the number of inches.
$g(x) = 1760x$, where $g(x)$ is the number of miles. Then
$(g \circ f)(x) = g\left[f(x)\right] = 1760(36x) = 63{,}360x.$
There are $63{,}360x$ inches in x miles

.

126. Use the definition for the perimeter of a rectangle.
P = length + width + length + width
$P(x) = 2x + x + 2x + x = 6x$
This is a linear function.

127. If $V(r) = \frac{4}{3}\pi r^3$ and if the radius is increased by 3 inches, then the amount of volume gained is given by
$V_g(r) = V(r+3) - V(r) = \frac{4}{3}\pi(r+3)^3 - \frac{4}{3}\pi r^3.$

128. **(a)** $V = \pi r^2 h$
If d is the diameter of its top, then $h = d$ and $r = \frac{d}{2}$. So,
$V(d) = \pi\left(\frac{d}{2}\right)^2 (d) = \pi\left(\frac{d^2}{4}\right)(d) = \frac{\pi d^3}{4}.$

(b) $S = 2\pi r^2 + 2\pi rh \Rightarrow$
$$S(d) = 2\pi\left(\frac{d}{2}\right)^2 + 2\pi\left(\frac{d}{2}\right)(d) = \frac{\pi d^2}{2} + \pi d^2$$
$$= \frac{\pi d^2}{2} + \frac{2\pi d^2}{2} = \frac{3\pi d^2}{2}$$

Chapter 2: Test

1. **(a)** The domain of $f(x) = \sqrt{x} + 3$ occurs when $x \geq 0$. In interval notation, this correlates to the interval in D, $[0, \infty)$.

(b) The range of $f(x) = \sqrt{x-3}$ is all real numbers greater than or equal to 0. In interval notation, this correlates to the interval in D, $[0, \infty)$.

(c) The domain of $f(x) = x^2 - 3$ is all real numbers. In interval notation, this correlates to the interval in C, $(-\infty, \infty)$.

(d) The range of $f(x) = x^2 + 3$ is all real numbers greater than or equal to 3. In interval notation, this correlates to the interval in B, $[3, \infty)$.

(e) The domain of $f(x) = \sqrt[3]{x-3}$ is all real numbers. In interval notation, this correlates to the interval in C, $(-\infty, \infty)$.

(f) The range of $f(x) = \sqrt[3]{x} + 3$ is all real numbers. In interval notation, this correlates to the interval in C, $(-\infty, \infty)$.

(g) The domain of $f(x) = |x| - 3$ is all real numbers. In interval notation, this correlates to the interval in C, $(-\infty, \infty)$.

(h) The range of $f(x) = |x+3|$ is all real numbers greater than or equal to 0. In interval notation, this correlates to the interval in D, $[0, \infty)$.

(i) The domain of $x = y^2$ is $x \geq 0$ since when you square any value of y, the outcome will be nonnegative. In interval notation, this correlates to the interval in D, $[0, \infty)$.

(j) The range of $x = y^2$ is all real numbers. In interval notation, this correlates to the interval in C, $(-\infty, \infty)$.

2. Consider the points $(-2, 1)$ and $(3, 4)$.
$$m = \frac{4-1}{3-(-2)} = \frac{3}{5}$$

3. We label the points $A(-2, 1)$ and $B(3, 4)$.
$$d(A, B) = \sqrt{[3-(-2)]^2 + (4-1)^2}$$
$$= \sqrt{5^2 + 3^2} = \sqrt{25+9} = \sqrt{34}$$

4. The midpoint has coordinates
$$\left(\frac{-2+3}{2}, \frac{1+4}{2}\right) = \left(\frac{1}{2}, \frac{5}{2}\right).$$

5. Use the point-slope form with $(x_1, y_1) = (-2, 1)$ and $m = \frac{3}{5}$.
$$y - y_1 = m(x - x_1)$$
$$y - 1 = \tfrac{3}{5}[x - (-2)]$$
$$y - 1 = \tfrac{3}{5}(x+2) \Rightarrow 5(y-1) = 3(x+2) \Rightarrow$$
$$5y - 5 = 3x + 6 \Rightarrow 5y = 3x + 11 \Rightarrow$$
$$-3x + 5y = 11 \Rightarrow 3x - 5y = -11$$

6. Solve $3x - 5y = -11$ for y.
$$3x - 5y = -11$$
$$-5y = -3x - 11$$
$$y = \tfrac{3}{5}x + \tfrac{11}{5}$$
Therefore, the linear function is
$f(x) = \frac{3}{5}x + \frac{11}{5}.$

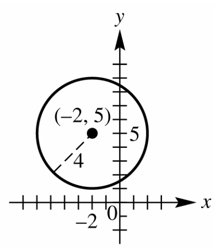
7. **(a)** The center is at (0, 0) and the radius is 2, so the equation of the circle is
$x^2 + y^2 = 4$.

(b) The center is at (1, 4) and the radius is 1, so the equation of the circle is
$(x-1)^2+(y-4)^2=1$

8. $x^2+y^2+4x-10y+13=0$
Complete the square on x and y to write the equation in standard form:

$$x^2+y^2+4x-10y+13=0$$
$$\left(x^2+4x+\quad\right)+\left(y^2-10y+\quad\right)=-13$$
$$\left(x^2+4x+4\right)+\left(y^2-10y+25\right)=-13+4+25$$
$$\left(x+2\right)^2+\left(y-5\right)^2=16$$

The circle has center (−2, 5) and radius 4.



$x^2+y^2+4x-10y+13=0$

9. **(a)** This is not the graph of a function because some vertical lines intersect it in more than one point. The domain of the relation is [0, 4]. The range is [−4, 4].

(b) This is the graph of a function because no vertical line intersects the graph in more than one point. The domain of the function is $(-\infty,-1)\cup(-1,\infty)$. The range is $(-\infty,0)\cup(0,\infty)$. As x is getting larger on the intervals $(-\infty,-1)$ and $(-1,\infty)$, the value of y is decreasing, so the function is decreasing on these intervals. (The function is never increasing or constant.)

10. Point A has coordinates (5, −3).

(a) The equation of a vertical line through A is $x = 5$.

(b) The equation of a horizontal line through A is $y = -3$.

11. The slope of the graph of $y=-3x+2$ is −3.

(a) A line parallel to the graph of $y=-3x+2$ has a slope of −3.
Use the point-slope form with $(x_1,\ y_1)=(2,3)$ and $m=-3$.

$$y-y_1=m(x-x_1)$$
$$y-3=-3(x-2)$$
$$y-3=-3x+6\Rightarrow y=-3x+9$$

(b) A line perpendicular to the graph of $y=-3x+2$ has a slope of $\frac{1}{3}$ since $-3\left(\frac{1}{3}\right)=-1$.

$$y-3=\tfrac{1}{3}(x-2)$$
$$3\left(y-3\right)=x-2\Rightarrow 3y-9=x-2\Rightarrow$$
$$3y=x+7\Rightarrow y=\tfrac{1}{3}x+\tfrac{7}{3}$$

12. **(a)** $\left(-\infty,-3\right)$

(b) $\left(4,\infty\right)$

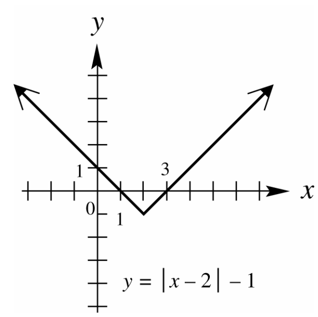
(c) $\left[-3,4\right]$

(d) $\left(-\infty,-3\right)$; $\left[-3,4\right]$; $\left(4,\infty\right)$

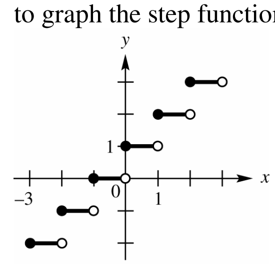
(e) $\left(-\infty,\infty\right)$

(f) $\left(-\infty,2\right)$

13. To graph $y=\left|x-2\right|-1$, we translate the graph of $y=\left|x\right|$, 2 units to the right and 1 unit down.



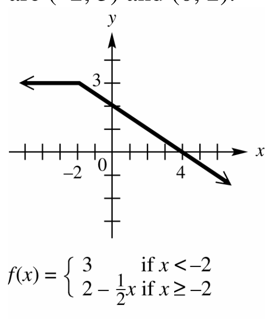
14. $f(x)=[\![x+1]\!]$
To get $y = 0$, we need $0\le x+1<1\Rightarrow -1\le x<0$. To get $y = 1$, we need $1\le x+1<2\Rightarrow 0\le x<1$. Follow this pattern to graph the step function.



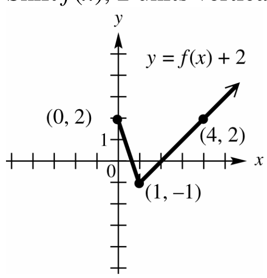
$f(x)=[\![x+1]\!]$

15. $f(x) = \begin{cases} 3 & \text{if } x < -2 \\ 2 - \frac{1}{2}x & \text{if } x \geq -2 \end{cases}$

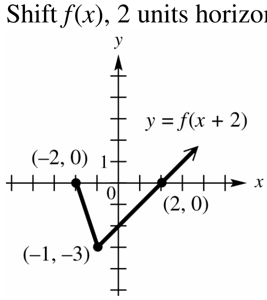
For values of x with $x < -2$, we graph the horizontal line $y = 3$. For values of x with $x \geq -2$, we graph the line with a slope of $-\frac{1}{2}$ and a y-intercept of 2. Two points on this line are (–2, 3) and (0, 2).



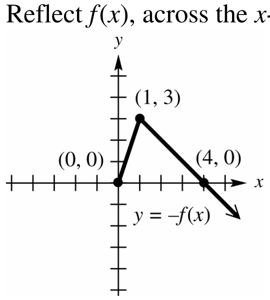
16. **(a)** Shift $f(x)$, 2 units vertically upward.



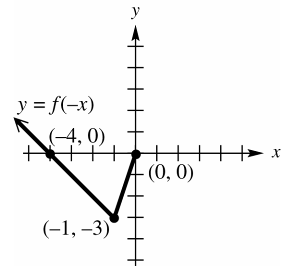
(b) Shift $f(x)$, 2 units horizontally to the left.



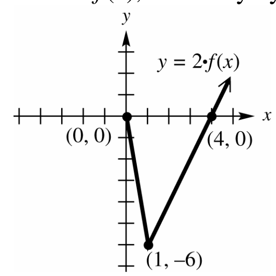
(c) Reflect $f(x)$, across the x-axis.



(d) Reflect $f(x)$, across the y-axis.



(e) Stretch $f(x)$, vertically by a factor of 2.



17. Answers will vary. Starting with $y = \sqrt{x}$, we shift it to the left 2 units and stretch it vertically by a factor of 2. The graph is then reflected over the x-axis and then shifted down 3 units.

18. $3x^2 - y^2 = 3$

(a) Replace y with $-y$ to obtain

$3x^2 - (-y)^2 = 3 \Rightarrow 3x^2 - y^2 = 3.$

The result is the same as the original equation, so the graph is symmetric with respect to the x-axis.

(b) Replace x with $-x$ to obtain

$3(-x)^2 - y^2 = 3 \Rightarrow 3x^2 - y^2 = 3.$

The result is the same as the original equation, so the graph is symmetric with respect to the y-axis.

(c) Since the graph is symmetric with respect to the x-axis and with respect to the y-axis, it must also be symmetric with respect to the origin.

19. $f(x) = 2x^2 - 3x + 2, \; g(x) = -2x + 1$

(a)
$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= \left(2x^2 - 3x + 2\right) - (-2x + 1) \\ &= 2x^2 - 3x + 2 + 2x - 1 \\ &= 2x^2 - x + 1\end{aligned}$$

(b) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x^2 - 3x + 2}{-2x + 1}$

(c) We must determine which values solve the equation $-2x+1=0$.

$-2x+1=0 \Rightarrow -2x=-1 \Rightarrow x=\frac{1}{2}$

Thus, $\frac{1}{2}$ is excluded from the domain, and the domain is $\left(-\infty,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$..

(d) $f(x)=2x^2-3x+2$

$$\begin{aligned} f(x+h) &= 2(x+h)^2-3(x+h)+2 \\ &= 2\left(x^2+2xh+h^2\right)-3x-3h+2 \\ &= 2x^2+4xh+2h^2-3x-3h+2 \end{aligned}$$

$$\begin{aligned} &f(x+h)-f(x) \\ &= (2x^2+4xh+2h^2-3x-3h+2) \\ &\qquad -(2x^2-3x+2) \\ &= 2x^2+4xh+2h^2-3x \\ &\qquad -3h+2-2x^2+3x-2 \\ &= 4xh+2h^2-3h \end{aligned}$$

$$\frac{f(x+h)-f(x)}{h}=\frac{4xh+2h^2-3h}{h}=\frac{h(4x+2h-3)}{h}=4x+2h-3$$

20. **(a)**
$$\begin{aligned} (f+g)(1) &= f(1)+g(1) \\ &= (2\cdot1^2-3\cdot1+2)+(-2\cdot1+1) \\ &= (2\cdot1-3\cdot1+2)+(-2\cdot1+1) \\ &= (2-3+2)+(-2+1) \\ &= 1+(-1)=0 \end{aligned}$$

(b)
$$\begin{aligned} (fg)(2) &= f(2)\cdot g(2) \\ &= (2\cdot2^2-3\cdot2+2)\cdot(-2\cdot2+1) \\ &= (2\cdot4-3\cdot2+2)\cdot(-2\cdot2+1) \\ &= (8-6+2)\cdot(-4+1) \\ &= 4(-3)=-12 \end{aligned}$$

(c) $g(x)=-2x+1 \Rightarrow g(0)=-2(0)+1$ $=0+1=1$. Therefore,

$$\begin{aligned} (f\circ g)(0) &= f\left[g(0)\right] \\ &= f(1)=2\cdot1^2-3\cdot1+2 \\ &= 2\cdot1-3\cdot1+2 \\ &= 2-3+2=1 \end{aligned}$$

21. $(f\circ g)=f(g(x))=f(2x-7)$
$=\sqrt{(2x-7)+1}=\sqrt{2x-6}$

The domain and range of g are $(-\infty,\infty)$, while the domain of *f* is $[0,\infty)$. We need to find the values of x which fit the domain of *f*: $2x-6\ge0 \Rightarrow x\ge3$. So, the domain of $f\circ g$ is $[3,\infty)$.

22. $(g\circ f)=g(f(x))=g\left(\sqrt{x+1}\right)$
$=2\sqrt{x+1}-7$

The domain and range of g are $(-\infty,\infty)$, while the domain of *f* is $[0,\infty)$. We need to find the values of x which fit the domain of *f*: $x+1\ge0 \Rightarrow x\ge-1$. So, the domain of $g\circ f$ is $[-1,\infty)$.

23.
$$\begin{aligned} f(x) &= .4[\![x]\!]+.75 \\ f(5.5) &= .4[\![5.5]\!]+.75=.4(5)+.75 \\ &= 2+.75=\$2.75 \end{aligned}$$

24. **(a)** $C(x) = 3300 + 4.50x$

(b) $R(x) = 10.50x$

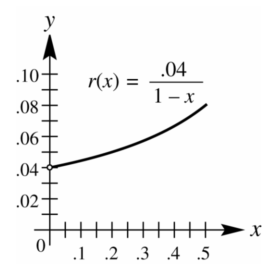
(c)
$$\begin{aligned} P(x) &= R(x)-C(x) \\ &= 10.50x-(3300+4.50x) \\ &= 6.00x-3300 \end{aligned}$$

(d)
$$\begin{aligned} P(x) &> 0 \\ 6.00x-3300 &> 0 \\ 6.00x &> 3300 \\ x &> 550 \end{aligned}$$

He must produce and sell 551 items before he earns a profit.

Chapter 2: Quantitative Reasoning

1.

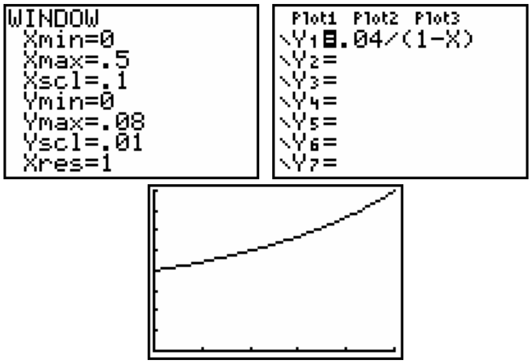


x	$r(x)=\frac{.04}{1-x}$
.1	$r(.1)=\frac{.04}{1-.1}=\frac{.04}{.9}\approx.044$
.2	$r(.2)=\frac{.04}{1-.2}=\frac{.04}{.8}=.05$
.3	$r(.3)=\frac{.04}{1-.3}=\frac{.04}{.7}\approx.057$
.4	$r(.4)=\frac{.04}{1-.4}=\frac{.04}{.6}\approx.067$
.5	$r(.5)=\frac{.04}{1-.5}=\frac{.04}{.5}=.08$

(*continued on next page*)

(*continued from page 249*)

Using the graphing calculator, we have the following screens.



This is not a linear function because it cannot be written in the form $y = ax + b$. The $1 - x$ in the denominator prevents this. Also, when you look at the graph, it doesn't appear to form a line.

2. Evaluate $r(x) = \frac{.04}{1-x}$, where $x = .31$.

$$r(.31) = \frac{.04}{1-.31} = \frac{.04}{.69} \approx .058 \text{ or } 5.8\%$$

3. Solve $r(x) = \frac{.04}{1-x}$, where $r(x) = .0626$.

$$\begin{aligned} .0626 &= \frac{.04}{1-x} \\ .0626(1-x) &= .04 \\ .0626 - .0626x &= .04 \\ -.0626x &= -.0226 \\ x &= \tfrac{-.0226}{-.0626} \approx .36 \text{ or } 36\% \end{aligned}$$

Chapter 3

POLYNOMIAL AND RATIONAL FUNCTIONS

Section 3.1: Quadratic Functions and Models

1. $f(x) = (x+3)^2 - 4$

(a) domain: $(-\infty, \infty)$; range: $[-4, \infty)$

(b) vertex: $(h, k) = (-3, -4)$

(c) axis: $x = -3$

(d) To find the y-intercept, let $x = 0$.
$y = (0+3)^2 - 4 = 3^2 - 4 = 9 - 4 = 5$
y-intercept: 5

(e) To find the x-intercepts, let $f(x) = 0$.

$$\begin{aligned} 0 &= (x+3)^2 - 4 \\ (x+3)^2 &= 4 \\ x+3 &= \pm\sqrt{4} = \pm 2 \\ x &= -3 \pm 2 \end{aligned}$$

$x = -3 - 2 = -5$ or $x = -3 + 2 = -1$
x-intercepts: -5 and -1

2. $f(x) = (x-5)^2 - 4$

(a) domain: $(-\infty, \infty)$; range: $[-4, \infty)$

(b) vertex: $(h, k) = (5, -4)$

(c) axis: $x = 5$

(d) To find the y-intercept, let $x = 0$.
$y = (0-5)^2 - 4 = (-5)^2 - 4 = 25 - 4 = 21$
y-intercept: 21

(e) To find the x-intercepts, let $f(x) = 0$.

$$\begin{aligned} 0 &= (x-5)^2 - 4 \\ (x-5)^2 &= 4 \\ x-5 &= \pm\sqrt{4} = \pm 2 \\ x &= 5 \pm 2 \end{aligned}$$

$x = 5 - 2 = 3$ or $x = 5 + 2 = 7$
x-intercepts: 3 and 7

3. $f(x) = -2(x+3)^2 + 2$

(a) domain: $(-\infty, \infty)$; range: $(-\infty, 2]$

(b) vertex: $(h, k) = (-3, 2)$

(c) axis: $x = -3$

(d) To find the y-intercept, let $x = 0$.

$$\begin{aligned} y &= -2(0+3)^2 + 2 = -2 \cdot 3^2 + 2 \\ &= -2 \cdot 9 + 2 = -18 + 2 = -16 \end{aligned}$$

y-intercept: -16

(e) To find the x-intercepts, let $f(x) = 0$.

$$\begin{aligned} 0 &= -2(x+3)^2 + 2 \\ (x+3)^2 &= 1 \\ x+3 &= \pm\sqrt{1} = \pm 1 \\ x &= -3 \pm 1 \end{aligned}$$

$x = -3 - 1 = -4$ or $x = -3 + 1 = -2$
x-intercepts: -4 and -2

4. $f(x) = -3(x-2)^2 + 1$

(a) domain: $(-\infty, \infty)$; range: $(-\infty, 1]$

(b) vertex: $(h, k) = (2, 1)$

(c) axis: $x = 2$

(d) To find the y-intercept, let $x = 0$.

$$\begin{aligned} y &= -3(0-2)^2 + 1 = -3(-2)^2 + 1 \\ &= -3 \cdot 4 + 1 = -12 + 1 = -11 \end{aligned}$$

y-intercept: -11

(e) To find the x-intercepts, let $f(x) = 0$.

$$\begin{aligned} 0 &= -3(x-2)^2 + 1 \\ (x-2)^2 &= \frac{1}{3} \\ x-2 &= \pm\frac{\sqrt{3}}{3} \\ x &= 2 \pm \frac{\sqrt{3}}{3} = \frac{6 \pm \sqrt{3}}{3} \end{aligned}$$

x-intercepts: $\frac{6-\sqrt{3}}{3}$ and $\frac{6+\sqrt{3}}{3}$

5. $f(x) = (x-4)^2 - 3$
Since a > 0, the parabola opens upward. The vertex is at (4, –3). The correct graph, therefore, is B.

6. $f(x) = -(x-4)^2 + 3$
Since a < 0, the parabola opens downward. The vertex is (4, 3). The correct graph, therefore, is A.

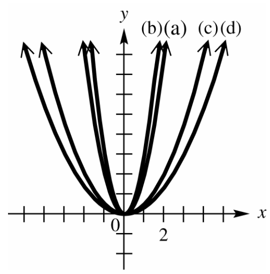
7. $f(x)=(x+4)^2-3$

Since a > 0, the parabola opens upward. The vertex is at (–4, –3). The correct graph, therefore, is D.

8. $f(x)=-(x+4)^2+3$

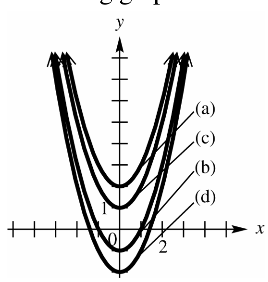
Since a < 0, the parabola opens downward. The vertex is (–4, 3). The correct graph, therefore, is C.

9. For parts (a), (b), (c), and (d), see the following graph.



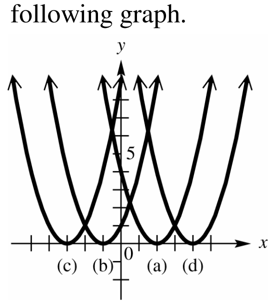
(e) If the absolute value of the coefficient is greater than 1, it causes the graph to be stretched vertically, so it is narrower. If the absolute value of the coefficient is between 0 and 1, it causes the graph to shrink vertically, so it is broader.

10. For parts (a), (b), (c), and (d), see the following graph:



(e) The graph of x^2+k is translated k units upward if k is positive and $|k|$ units downward if k is negative.

11. For parts (a), (b), (c), and (d), see the following graph.



(e) The graph of $(x-h)^2$ is translated h units to the right if h is positive and $|h|$ units to the left if h is negative.

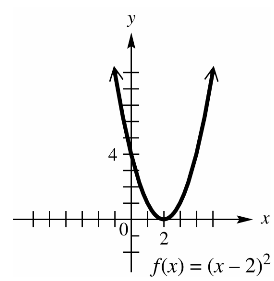
12. **(a)** $y=(x+4)^2+2$ has a vertex (–4, 2) and opens up since a = 1; this is choice C.

(b) $y=(x+2)^2+4$ has vertex (–2, 4) and opens up since a = 1; this is choice A.

(c) $y=-(x+4)^2+2$ has vertex (–4, 2) and opens down since a = –1; this is choice D.

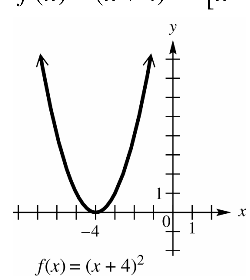
(d) $y=-(x+2)^2+4$ has vertex (–2, –4) and opens down since a = –1; this is choice B.

13. $f(x)=(x-2)^2$



This equation is of the form $y=(x-h)^2$, with h = 2. The graph opens upward and has the same shape as that of $y=x^2$. It is a horizontal translation of the graph of $y=x^2$, 2 units to the right. The vertex is (2, 0) and the axis is the vertical line x = 2. Additional points on the graph are $(1,1)$ and $(3,1)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is 0 and the graph opens upward, the range is $[0, \infty)$.

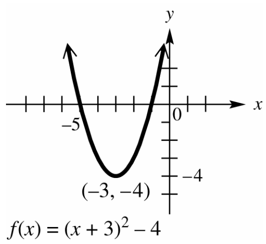
14. $f(x)=(x+4)^2=[x-(-4)]^2$



This equation is of the form $y=(x-h)^2$, with $h=-4$. The graph opens upward and has the same shape as that of $y=x^2$. It is a horizontal translation of the graph of $y=x^2$, 4 units to the left.

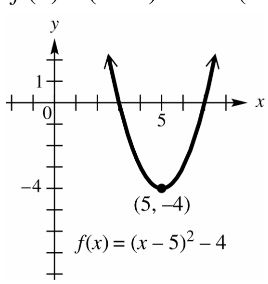
The vertex is (–4, 0). The axis is the vertical line $x = -4$. Additional points on the graph are $(-5, 1)$ and $(-3, 1)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is 0 and the graph opens upward, the range is $[0, \infty)$.

15. $f(x) = (x+3)^2 - 4 = [x-(-3)]^2 + (-4)$



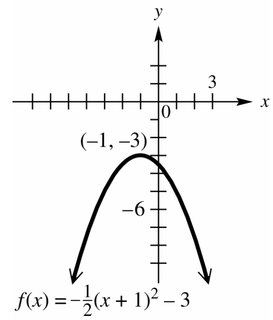
This equation is of the form $y = (x-h)^2 + k$, with $h = -3$ and $k = -4$. The graph opens upward and has the same shape as $y = x^2$. The vertex is (–3, –4). It is a translation of $y = x^2$, 3 units to the left and 4 units down. The axis is the vertical line $x = -3$. Additional points on the graph are $(-4, -3)$ and $(-2, -3)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is – 4 and the graph opens upward, the range is $[-4, \infty)$.

16. $f(x) = (x-5)^2 - 4 = (x-5)^2 + (-4)$



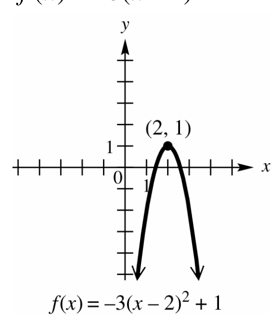
This equation is of the form $y = (x-h)^2 + k$, with $h = 5$ and $k = -4$. The graph opens upward and has the same shape as $y = x^2$. The vertex is (5, –4). It is a translation of $y = x^2$, 5 units to the right and 4 units down. The axis is the vertical line $x = 5$. Additional points on the graph are $(4, -3)$ and $(6, -3)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is – 4 and the graph opens upward, the range is $[-4, \infty)$.

17. $f(x) = -\frac{1}{2}(x+1)^2 - 3 = -\frac{1}{2}[x-(-1)]^2 + (-3)$



This equation is of the form $y = a(x-h)^2 + k$, with $h = -1$, $k = -3$, and $a = -\frac{1}{2}$. The graph opens downward and is wider than $y = x^2$. The vertex is (–1, –3). It is a translation of the graph $y = -\frac{1}{2}x^2$, 1 unit to the left and 3 units down. The axis is the vertical line $x = -1$. Additional points on the graph are $\left(-2, -3\frac{1}{2}\right)$ and $\left(0, -3\frac{1}{2}\right)$. The domain is $(-\infty, \infty)$. Since the largest value of y is –3 and the graph opens downward, the range is $(-\infty, -3]$.

18. $f(x) = -3(x-2)^2 + 1$



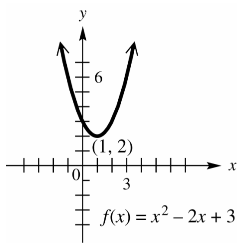
This equation is of the form $y = a(x-h)^2 + k$, with $h = 2$, $k = 1$, and $a = -3$. The graph opens downward and is narrower than $y = x^2$. The vertex is (2, 1). It is a translation of the graph $y = -3x^2$, 2 units to the right and 1 unit up. The axis is the vertical line $x = 2$. Additional points on the graph are $(1, -2)$ and $(3, -2)$. The domain is $(-\infty, \infty)$. Since the largest value of y is 1 and the graph opens downward, the range is $(-\infty, 1]$.

19. $f(x) = x^2 - 2x + 3$; Rewrite by completing the square on x.

$f(x) = x^2 - 2x + 3 = (x^2 - 2x + 1 - 1) + 3$

Note: $\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1$

$= (x^2 - 2x + 1) - 1 + 3 = (x-1)^2 + 2$



This equation is of the form $y = (x-h)^2 + k$, with $h = 1$ and $k = 2$. The graph opens upward and has the same shape as $y = x^2$. The vertex is (1, 2). It is a translation of the graph $y = x^2$, 1 unit to the right and 2 units up. The axis is the vertical line $x = 1$. Additional points on the graph are $(0,3)$ and $(2,3)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is 2 and the graph opens upward, the range is $[2, \infty)$.

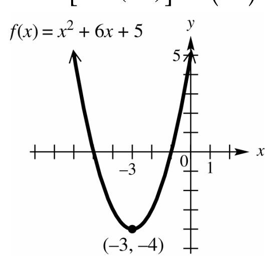
20. $f(x) = x^2 + 6x + 5$; Rewrite by completing the square on x.

$f(x) = x^2 + 6x + 5 = (x^2 + 6x + 9 - 9) + 5$

Note: $\left[\frac{1}{2}(6)\right]^2 = 3^2 = 9$

$= (x^2 + 6x + 9) - 9 + 5 = (x+3)^2 - 4$

$= [x - (-3)]^2 + (-4)$



This equation is of the form $y = (x-h)^2 + k$, with $h = -3$ and $k = -4$. The graph opens upward and has the same shape as $y = x^2$. The vertex is $(-3, -4)$. It is a translation of the graph $y = x^2$, 3 units to the left and 4 units down. The axis is the vertical line $x = -3$. Additional points on the graph are $(-4,-3)$ and $(-2,-3)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is -4 and the graph opens upward, the range is $[-4, \infty)$.

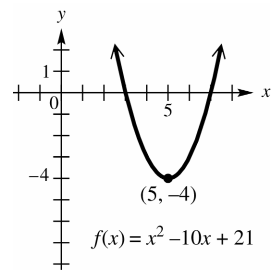
21. $f(x) = x^2 - 10x + 21$; Rewrite by completing the square on x.

$f(x) = x^2 - 10x + 21 = (x^2 - 10x + 25 - 25) + 21$

Note: $\left[\frac{1}{2}(-10)\right]^2 = (-5)^2 = 25$

$= (x^2 - 10x + 25) - 25 + 21 = (x-5)^2 - 4$

$= (x-5)^2 + (-4)$



This equation is of the form $y = (x-h)^2 + k$, with $h = 5$ and $k = -4$. The graph opens upward and has the same shape as $y = x^2$. The vertex is $(5, -4)$. It is a translation of the graph $y = x^2$, 5 units to the right and 4 units down. The axis is the vertical line $x = 5$. Additional points on the graph are $(4,-3)$ and $(6,-3)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is -4 and the graph opens upward, the range is $[-4, \infty)$.

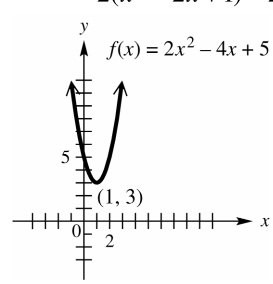
22. $f(x) = 2x^2 - 4x + 5$; Rewrite by completing the square on x.

$f(x) = 2x^2 - 4x + 5 = 2(x^2 - 2x) + 5$

$= 2(x^2 - 2x + 1 - 1) + 5$

Note: $\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1$

$= 2(x^2 - 2x + 1) - 2 + 5 = 2(x-1)^2 + 3$

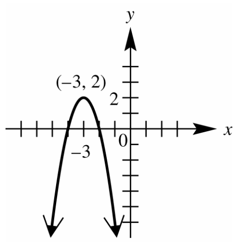


This equation is of the form $y = a(x-h)^2 + k$, with $h = 1$, $k = 3$, and $a = 2$. The graph opens upward and is narrower than $y = x^2$. The vertex is (1, 3).

It is a translation of the graph $y = 2x^2$, 1 unit to the right and 3 units up. The axis is the vertical line $x = 1$. Additional points on the graph are $(0,5)$ and $(2,5)$. The domain is $(-\infty, \infty)$. Since the smallest value of y is 3 and the graph opens upward, the range is $[3, \infty)$.

23. $f(x) = -2x^2 - 12x - 16$; Rewrite by completing the square on x.

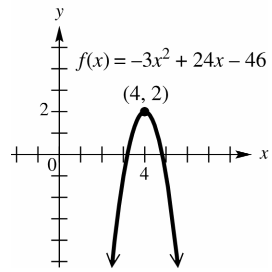
$$f(x) = -2x^2 - 12x - 16 = -2(x^2 + 6x) - 16$$
$$= -2(x^2 + 6x + 9 - 9) - 16$$
$$\text{Note: } \left[\tfrac{1}{2}(6)\right]^2 = 3^2 = 9$$
$$= -2(x^2 + 6x + 9) + 18 - 16$$
$$= -2(x+3)^2 + 2 = -2[x - (-3)]^2 + 2$$



This equation is of the form $y = a(x-h)^2 + k$, with $h = -3$, $k = 2$, and $a = -2$. The graph opens downward and is narrower than $y = x^2$. The vertex is $(-3,2)$. It is a translation of the graph $y = -2x^2$, 3 units to the left and 2 units up. The axis is the vertical line $x = -3$. Additional points on the graph are $(-4,0)$ and $(-2,0)$. The domain is $(-\infty, \infty)$. Since the largest value of y is 2 and the graph opens downward, the range is $(-\infty, 2]$.

24. $f(x) = -3x^2 + 24x - 46$; Rewrite by completing the square on x.

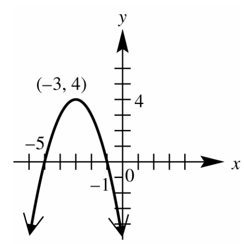
$$f(x) = -3x^2 + 24x - 46$$
$$= -3(x^2 - 8x + 16 - 16) - 46$$
$$\text{Note: } \left[\tfrac{1}{2}(-8)\right]^2 = (-4)^2 = 16$$
$$= -3(x^2 - 8x + 16) + 48 - 46$$
$$= -3(x-4)^2 + 2$$



This equation is of the form $y = a(x-h)^2 + k$, with $h = 4$, $k = 2$, and $a = -3$. The graph opens downward and is narrower than $y = x^2$. The vertex is $(4,2)$. It is a translation of the graph $y = -3x^2$, 4 units to the right and 2 units up. The axis is the vertical line $x = 4$. Additional points on the graph are $(3,-1)$ and $(5,-1)$. The domain is $(-\infty, \infty)$. Since the largest value of y is 2 and the graph opens downward, the range is $(-\infty, 2]$.

25. $f(x) = -x^2 - 6x - 5$; Rewrite by completing the square on x.

$$f(x) = -x^2 - 6x - 5 = -(x^2 + 6x + 9 - 9) - 5$$
$$\text{Note: } \left[\tfrac{1}{2}(6)\right]^2 = 3^2 = 9$$
$$= -(x^2 + 6x + 9) + 9 - 5 = -(x+3)^2 + 4$$
$$= -1 \cdot [x - (-3)]^2 + 4$$



This equation is of the form $y = a(x-h)^2 + k$, with $h = -3$, $k = 4$, and $a = -1$. The graph opens downward and has the same shape as $y = x^2$. The vertex is $(-3,4)$. It is a translation of the graph $y = -x^2$, 3 units to the left and 4 units up. The axis is the vertical line $x = -3$. Additional points on the graph are $(-2,3)$ and $(-1,0)$. The domain is $(-\infty, \infty)$. Since the largest value of y is 4 and the graph opens downward, the range is $(-\infty, 4]$.

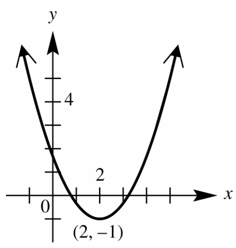
26. $f(x)=\frac{2}{3}x^2-\frac{8}{3}x+\frac{5}{3}$; Rewrite by completing the square on x.

$f(x)=\frac{2}{3}x^2-\frac{8}{3}x+\frac{5}{3}=\frac{2}{3}(x^2-4x+4-4)+\frac{5}{3}$

Note: $\left[\frac{1}{2}(-4)\right]^2=(-2)^2=4$

$=\frac{2}{3}(x^2-4x+4)-\frac{8}{3}+\frac{5}{3}=\frac{2}{3}(x-2)^2-1$

$=\frac{2}{3}(x-2)^2+(-1)$



$f(x)=\frac{2}{3}x^2-\frac{8}{3}x+\frac{5}{3}$

This equation is of the form $y=a(x-h)^2+k$, with $h=2,\ k=-1,$ and $a=\frac{2}{3}$. The graph opens upward and is wider than $y=x^2$. The vertex is $(2,-1)$. It is a translation of the graph $y=\frac{2}{3}x^2$, 2 units to the right and 1 unit down. The axis is the vertical line $x=2$. Additional points on the graph are $\left(1,-\frac{1}{3}\right)$ and $\left(3,-\frac{1}{3}\right)$. The domain is $(-\infty,\infty)$. Since the smallest value of y is -1 and the graph opens upward, the range is $[-1,\infty)$.

27. The minimum value of $f(x)$ is $f(-3)=3$.

28. $f(x)$ is as small as possible ($y=3$) at the vertex, where $x=-3$.

29. There are no real solutions to the equation $f(x)=1$ since the value of $f(x)$ is never less than 3.

30. To find the number of solutions to the equation $f(x)=4$, consider the intersection of the horizontal line $y=4$ with the graph of $y=f(x)$. There are two intersection points, so there are two solutions to the equation $f(x)=4$.

31. $a<0,\ b^2-4ac=0$

The correct choice is E. $a<0$ indicates that the parabola will open downward, while $b^2-4ac=0$ indicates that the graph will have exactly one x-intercept.

32. $a>0,\ b^2-4ac<0$

The correct choice is A. $a>0$ indicates that the parabola will open upward, while $b^2-4ac<0$ indicates that the graph will have no x-intercepts.

33. $a<0,\ b^2-4ac<0$

The correct choice is D. $a<0$ indicates that the parabola will open downward, while $b^2-4ac<0$ indicates that the graph will have no x-intercepts.

34. $a<0,\ b^2-4ac>0$

The correct choice is F. $a<0$ indicates that the parabola will open downward, while $b^2-4ac>0$ indicates that the graph will have two x-intercepts.

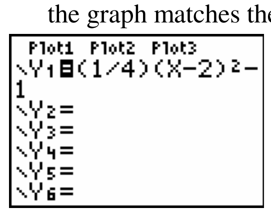
35. $a>0,\ b^2-4ac>0$

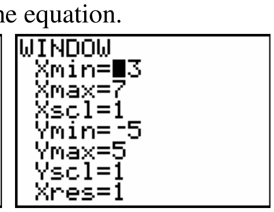
The correct choice is C. $a>0$ indicates that the parabola will open upward, while $b^2-4ac>0$ indicates that the graph will have two x-intercepts.

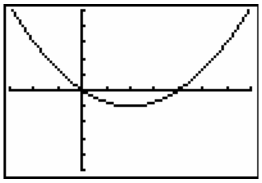
36. $a>0,\ b^2-4ac=0$

The correct choice is B. $a>0$ indicates that the parabola will open upward, while $b^2-4ac=0$ indicates that the graph will have exactly one x-intercept.

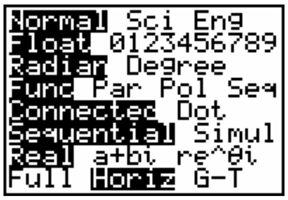
37. The vertex of the parabola in the figure is (2, −1) and the y-intercept is 0. The equation takes the form $f(x)=a(x-2)^2-1$. When $x=0, f(x)=0,$ so $0=a(0-2)^2-1\Rightarrow$ $0=4a-1\Rightarrow a=\frac{1}{4}$. The equation is $f(x)=\frac{1}{4}(x-2)^2-1$. This function may also be written as $f(x)=\frac{1}{4}(x-2)^2-1$ $=\frac{1}{4}\left(x^2-4x+4\right)-1=\frac{1}{4}x^2-x$. Graphing this function on a graphing calculator shows that the graph matches the equation.



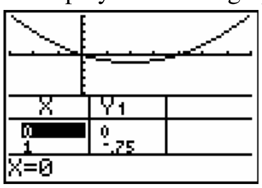




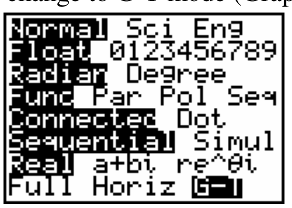
This graph can be displayed in other modes. If you press the MODE button, the following screen will appear. Toggle down to the last row and change FULL to Horiz.



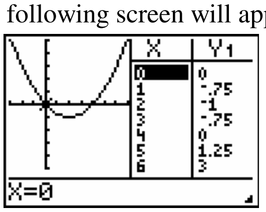
Hit 2nd then GRAPH (TABLE) to get the table to display below the graph.



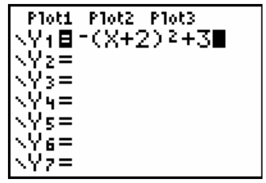
If you press the MODE button again, you can change to G-T mode (Graph-Table).

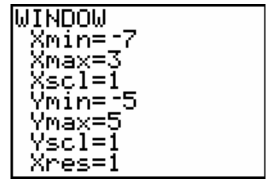


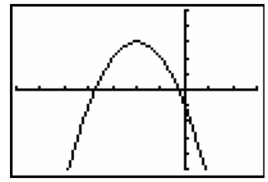
When you press the Graph button, the following screen will appear.



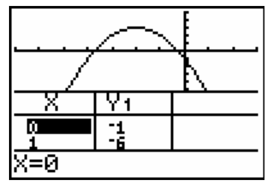
38. The vertex of the parabola in the figure is $(-2, 3)$ and the y-intercept is -1. The equation takes the form $f(x) = a(x+2)^2 + 3$. When $x = 0, f(x) = -1$, so $-1 = a(0+2)^2 + 3 \Rightarrow$ $-1 = 4a + 3 \Rightarrow a = -1$. The equation is $f(x) = -(x+2)^2 + 3$. This function may also be written as $f(x) = -(x+2)^2 + 3$ $= -(x^2 + 4x + 4) + 3 = -x^2 - 4x - 1$. Graphing this function on a graphing calculator shows that the graph matches the equation.



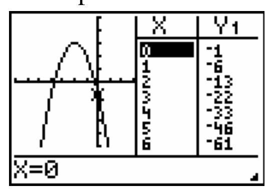




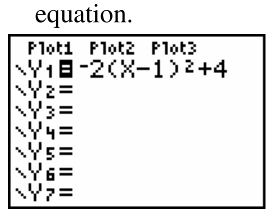
In Horizontal mode

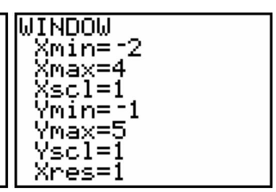


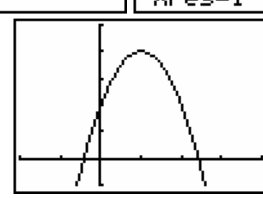
In Graph-Table mode



39. The vertex of the parabola in the figure is $(1, 4)$ and the y-intercept is 2. The equation takes the form $f(x) = a(x-1)^2 + 4$. When $x = 0, f(x) = 2$, so $2 = a(0-1)^2 + 4 \Rightarrow$ $2 = a + 4 \Rightarrow a = -2$. The equation is $f(x) = -2(x-1)^2 + 4$. This function may also be written as $f(x) = -2(x^2 - 2x + 1) + 4$ $= -2x^2 + 4x - 2 + 4 = -2x^2 + 4x + 2$. Graphing this function on a graphing calculator shows that the graph matches the equation.







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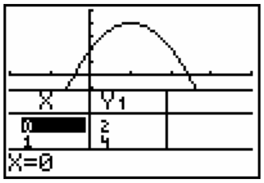
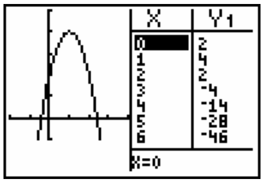
(*continued from page 257*)

In Horizontal mode



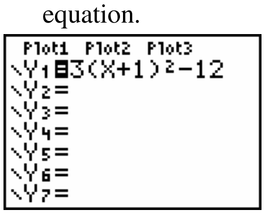
In Graph-Table mode

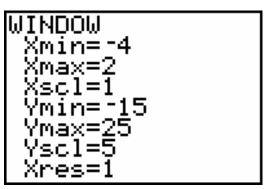


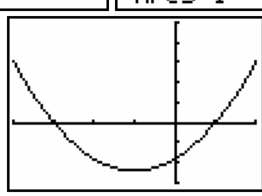
40. The vertex of the parabola is (–1, –12). The equation takes the form $f(x) = a(x+1)^2 - 12$.
When $x = 1$, $f(x) = 0$, so
$0 = a(1+1)^2 - 12 = 4a - 12 \Rightarrow$
$4a = 12 \Rightarrow a = 3.$ The equation is
$f(x) = 3(x+1)^2 - 12.$
We may also write this equation as:

$$f(x) = 3(x^2 + 2x + 1) - 12$$
$$= 3x^2 + 6x + 3 - 12 = 3x^2 + 6x - 9$$

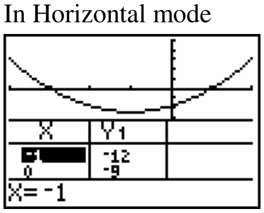
Graphing this function on a graphing calculator shows that the graph matches the equation.



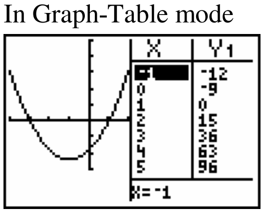




In Horizontal mode



In Graph-Table mode



41. Quadratic; the points lie in a pattern that suggests a parabola opening downward, so $a < 0$.

42. Linear; the points lie in pattern that suggesting a positive slope.

43. Quadratic; the points lie in a pattern suggesting a parabola opening upward, so $a > 0$.

44. Quadratic; the points lie in a pattern suggesting a parabola opening downward, so $a < 0$.

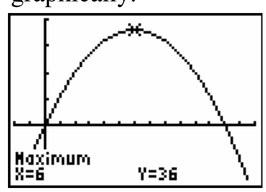
45. Linear; the points lie in pattern that suggesting a positive slope.

46. Quadratic; the points lie in a pattern suggesting a parabola opening downwards, so $a < 0$.

47. Let x = one number. Then $12 - x$ = the other number. Now find the maximum of $x(12 - x)$ by finding the vertex of the function:

$$f(x) = x(12 - x) = 12x - x^2 = -(x^2 - 12x)$$
$$= -(x^2 - 12x + 36) + 36 \quad \text{complete the square}$$
$$= -(x - 6)^2 + 36$$

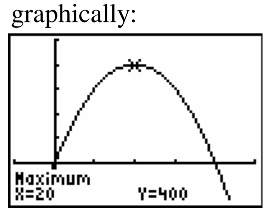
The vertex occurs at the vertex. So the two numbers are 6 and 12 – 6 = 6. Confirm graphically:



48. Let x = one number. Then $40 - x$ = the other number. Now find the maximum of $x(40 - x)$ by finding the vertex of the function:

$$f(x) = x(40 - x) = 40x - x^2 = -(x^2 - 40x)$$
$$= -(x^2 - 40x + 400) + 400 \quad \text{complete the square}$$
$$= -(x - 20)^2 + 400$$

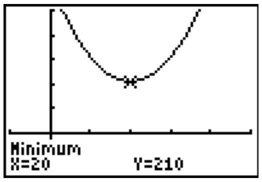
The vertex occurs at the vertex. So the two numbers are 20 and 40 – 20 = 20. Confirm graphically:



49. To find the number of units to be sold to minimize her costs, find the vertex of the function:

$$C(x) = x^2 - 40x + 610$$
$$= (x^2 - 40x + 400) + 610 - 400 \quad \text{complete the square}$$
$$= (x - 20)^2 + 210$$

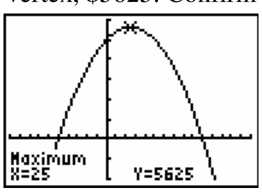
The vertex occurs at $x = 20$, so she should sell 20 units to minimize her costs. The minimum cost is the value of the function at the vertex, \$210. Confirm graphically:



50. To find the number of unsold seats which produce maximum revenue, find the vertex of the function:

$$R(x) = -x^2 + 50x + 5000$$
$$\text{complete the square}$$
$$= -(x^2 - 50x + 625) + 5000 + 625$$
$$= -(x - 25)^2 + 5625$$

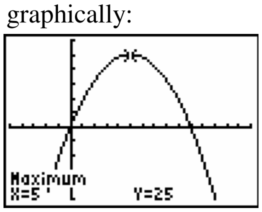
The vertex occurs at $x = 25$, which means that there are 25 unsold seats. The maximum revenue is the value of the function at the vertex, \$5625. Confirm graphically:



51. To find the amount of rainfall that will maximize the number of mosquitos, find the vertex of the function:

$$M(x) = 10x - x^2 = -(x^2 - 10x)$$
$$= -(x^2 - 10x + 25) + 25 \quad \text{complete the square}$$
$$= -(x - 5)^2 + 25$$

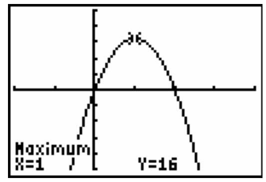
The vertex occurs at $x = 5$, which means that the maximum number of mosquitoes occurs when there are 5 inches of rain. The maximum number of mosquitoes is the value of the function at the vertex, 25 million. Confirm graphically:



52. To find the number of seconds the object will take to reach its maximum height, find the vertex of the function:

$$s(t) = -16t^2 + 32t = -16(t^2 - 2t)$$
$$= -16(t^2 - 2t + 1) + 16 \quad \text{complete the square}$$
$$= -16(t - 1)^2 + 16$$

The object will reach its maximum height at $x = 1$ second. The maximum height is the value of the function at the vertex, 16 feet. Confirm graphically:



53. **(a)** Since $v_0 = 200$, and $s_0 = 50$, and $s(t) = -16t^2 + v_0t + s_0$ we have $s(t)$ or $f(t) = -16t^2 + 200t + 50$.

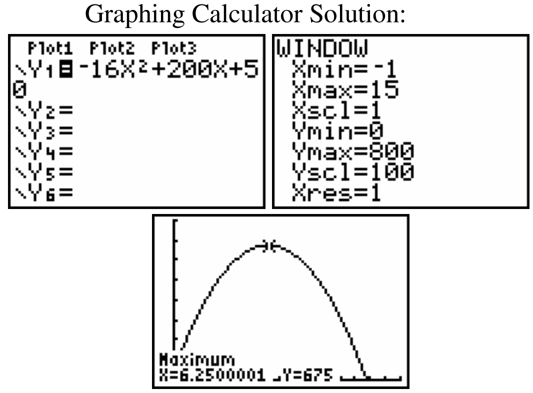
(b) Algebraic Solution:
Find the coordinates of the vertex of the parabola. Using the vertex formula with $a = -16$ and $b = 200$,

$$x = -\frac{b}{2a} = -\frac{200}{2(-16)} = 6.25 \text{ and}$$

$$y = -16(6.25)^2 + 200(6.25) + 50 = 675.$$

The vertex is (6.25, 675). Since $a < 0$, this is the maximum point.

Graphing Calculator Solution:



Thus, the number of seconds to reach maximum height is 6.25 seconds. The maximum height is 675 ft.

(c) Algebraic Solution:
To find the time interval in which the rocket will be more than 300 ft above ground level, solve the inequality $-16t^2 + 200t + 50 > 300$:

(*continued on next page*)

(*continued from page 259*)

$-16t^2+200t+50>300 \Rightarrow$
$-16t^2+200t-250>0 \Rightarrow$
$-8t^2+100t-125>0$

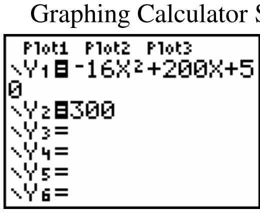
Solve the corresponding equation $-8t^2+100t-125=0$.
Use the quadratic formula with $a=-8$, $b=100$, and $c=-125$.

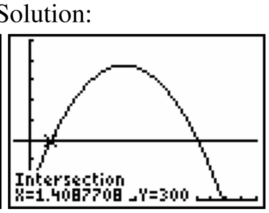
$$t=\frac{-100\pm\sqrt{100^2-4(-8)(-125)}}{2(-8)}$$
$$=\frac{-100\pm\sqrt{10{,}000-4000}}{-16}$$
$$=\frac{-100\pm\sqrt{6000}}{-16}$$
$$t=\frac{-100+\sqrt{6000}}{-16}\approx 1.4 \text{ or}$$
$$t=\frac{-100-\sqrt{6000}}{-16}\approx 11.1$$

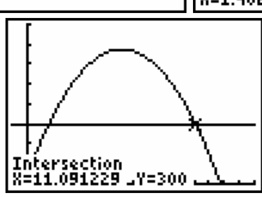
The values 1.4 and 11.1 divide the number line into three intervals: $(-\infty, 1.4)$, $(1.4, 11.1)$, and $(11.1, \infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $-16t^2+200t+50>300$ True or False?
$(-\infty, 1.4)$	0	$-16\cdot 0^2+200\cdot 0+50\overset{?}{>}300$ $50>300$ False
$(1.4, 11.1)$	2	$-16\cdot 2^2+200\cdot 2+50\overset{?}{>}300$ $386>300$ True
$(11.1, \infty)$	12	$-16\cdot 12^2+200\cdot 12+50\overset{?}{>}300$ $146>300$ False

Graphing Calculator Solution:







The rocket will be more than 300 ft above the ground between 1.4 sec and 11.1 sec.

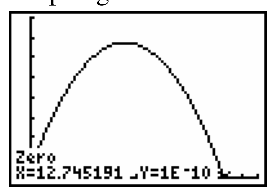
(d) Algebraic Solution:
To find the number of seconds for the toy rocket to hit the ground, let $f(t)=0$ and solve for t.

$-16t^2+200t+50=0$

Use the quadratic formula with $a=-16$, $b=200$, and $c=50$.

$$t=\frac{-200\pm\sqrt{200^2-4(-16)(50)}}{2(-16)}$$
$$=\frac{-200\pm\sqrt{40{,}000+3200}}{-32}$$
$$=\frac{-200\pm\sqrt{43{,}200}}{-32}$$
$$t=\frac{-200+\sqrt{43{,}200}}{-32}\approx -.25 \text{ or}$$
$$t=\frac{-200-\sqrt{43{,}200}}{-32}\approx 12.75$$

We reject the negative solution.
Graphing Calculator Solution:



It will take approximately 12.75 seconds for the toy rocket to hit the ground.

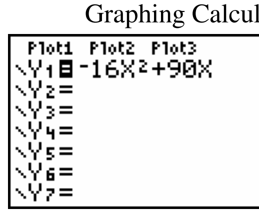
54. **(a)** Since $v_0=90$, and $s_0=0$, and $s(t)=-16t^2+v_0t+s_0$ we have $s(t)$ or $f(t)=-16t^2+90t$.

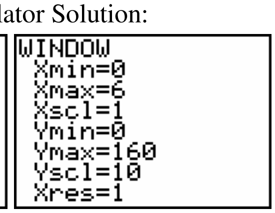
(b) Algebraic Solution:
Find the coordinates of the vertex of the parabola. Using the vertex formula with $a=-16$ and $b=90$,

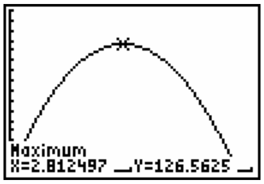
$x=-\frac{b}{2a}=-\frac{90}{2(-16)}=2.8125$ and

$y=-16(2.8125)^2+90(2.8125)=126.5625.$

The vertex is $(2.8125, 126.5625)$. Since $a<0$, this is the maximum point.
Graphing Calculator Solution:







Thus, the number of seconds to reach maximum height is 2.8125 seconds. The maximum height is 126.5625 ft.

(c) Algebraic Solution:
To find the time interval in which the rock will be more than 120 ft above ground level, solve the inequality

$$-16t^2 + 90t > 120 \Rightarrow$$
$$-16t^2 + 90t - 120 > 0 \Rightarrow$$
$$-8t^2 + 45t - 60 > 0$$

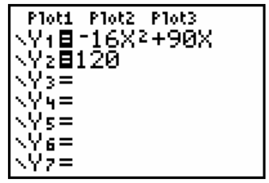
Solve the corresponding equation
$-8t^2 + 45t - 60 = 0.$
Use the quadratic formula with $a = -8$, $b = 45$, and $c = -60$.

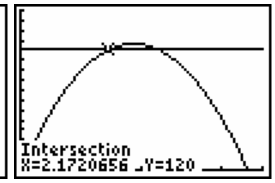
$$t = \frac{-45 \pm \sqrt{45^2 - 4(-8)(-60)}}{2(-8)}$$
$$= \frac{-45 \pm \sqrt{2025 - 1920}}{-16} = \frac{-45 \pm \sqrt{105}}{-16}$$
$$t = \frac{-45 + \sqrt{105}}{-16} \approx 2.2 \text{ or}$$
$$t = \frac{-45 - \sqrt{105}}{-16} \approx 3.5$$

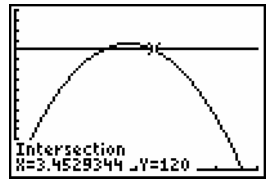
The values 2.2 and 3.5 divide the number line into three intervals: $(-\infty, 2.2)$, $(2.2, 3.5)$, and $(3.5, \infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $-16t^2 + 90t > 120$ True or False?
$(-\infty, 2.2)$	0	$-16 \cdot 0^2 + 90 \cdot 0 \overset{?}{>} 120$ $0 > 120$ False
$(2.2, 3.5)$	3	$-16 \cdot 3^2 + 90 \cdot 3 \overset{?}{>} 120$ $126 > 120$ True
$(3.5, \infty)$	4	$-16 \cdot 4^2 + 90 \cdot 4 \overset{?}{>} 120$ $104 > 120$ False

Graphing Calculator Solution:







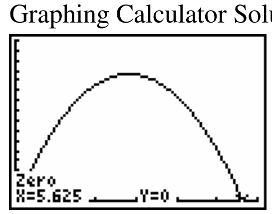
The rock will be more than 120 ft above the ground between 2.2 sec and 3.5 sec.

(d) Algebraic Solution:
To find the number of seconds it will take the rock to hit the ground, let $f(t) = 0$ and solve for t. $-16t^2 + 90t = 0$
Factor and set each factor equal to zero.

$$-16t^2 + 90t = 0 \Rightarrow -2t(8t - 45) = 0$$
$$-2t = 0 \Rightarrow t = 0 \quad \text{or}$$
$$8t - 45 = 0 \Rightarrow t = \tfrac{45}{8} = 5.625$$

Since we are seeking the time in which the rock returns to the ground, we reject the zero solution.

Graphing Calculator Solution:



It will take 5.625 seconds for the rock to return to the ground.

55. (a) The length of the other side would be $640 - 2x$.

(b) In order for the two lengths to be positive, $0 < x < 320$.

(c) $A(x) = x(640 - 2x) = 640x - 2x^2$
$= -2x^2 + 640x$

(d) Algebraic Solution:
Solve the inequality
$30,000 < -2x^2 + 640x < 40,000.$ Treat this as two inequalities,
$-2x^2 + 640x > 30,000$ and
$-2x^2 + 640x < 40,000.$

(*continued on next page*)

(*continued from page 261*)

For $-2x^2+640x>30,000,$ we solve the corresponding equation
$-2x^2+640x=30,000.$
This equation is equivalent to
$-2x^2+640x-30,000=0$ or
$x^2-320x+15,000=0.$
Use the quadratic formula with $a=1$, $b=-320$, and $c=15,000$.

$$x=\frac{-(-320)\pm\sqrt{(-320)^2-4(1)(15,000)}}{2(1)}$$
$$=\frac{320\pm\sqrt{102,400-60,000}}{2}$$
$$=\frac{320\pm\sqrt{42,400}}{2}$$
$$x=\frac{320-\sqrt{42,400}}{2}\approx 57.04 \text{ or}$$
$$x=\frac{320+\sqrt{42,400}}{2}\approx 262.96$$

The values 57.04 and 262.96 divide the number line into three intervals: $(-\infty,57.04)$, $(57.04,262.96)$, and $(262.96,\infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $-2x^2+640x>30,000$ True or False?
$(-\infty,57.04)$	0	$-2\cdot 0^2+640\cdot 0\overset{?}{>}30,000$ $0>30,000$ False
$(57.04,262.96)$	60	$-2\cdot 60^2+640\cdot 60\overset{?}{>}30,000$ $31,200>30,000$ True
$(262.96,\infty)$	300	$-2\cdot 300^2+640\cdot 300\overset{?}{>}30,000$ $12,000>30,000$ False

Thus, the first inequality is satisfied when the measure of x is in the interval $(57.04,262.96)$.

For $-2x^2+640x<40,000,$ we solve the corresponding equation
$-2x^2+640x=40,000.$
This equation is equivalent to
$-2x^2+640x-40,000=0$ or
$x^2-320x+20,000=0$

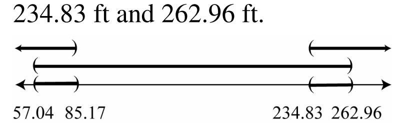
Use the quadratic formula with $a=1$, $b=-320$, and $c=20,000$.

$$x=\frac{-(-320)\pm\sqrt{(-320)^2-4(1)(20,000)}}{2(1)}$$
$$=\frac{320\pm\sqrt{102,400-80,000}}{2}$$
$$=\frac{320\pm\sqrt{22,400}}{2}$$
$$x=\frac{320-\sqrt{22,400}}{2}\approx 85.17 \text{ or}$$
$$x=\frac{320+\sqrt{22,400}}{2}\approx 234.83$$

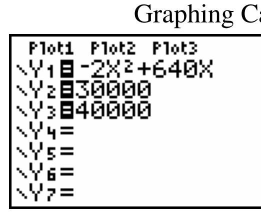
The values 85.17 and 234.83 divide the number line into three intervals: $(-\infty,85.17)$, $(85.17,234.83)$, and $(234.83,\infty)$. Use a test point in each interval to determine where the inequality is satisfied.

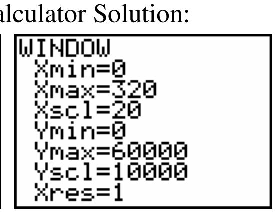
Interval	Test Value	Is $-2x^2+640x<40,000$ True or False?
$(-\infty,85.17)$	0	$-2\cdot 0^2+640\cdot 0\overset{?}{<}40,000$ $0<40,000$ True
$(85.17,234.83)$	90	$-2\cdot 90^2+640\cdot 90\overset{?}{<}40,000$ $41,400<40,000$ False
$(234.83,\infty)$	300	$-2\cdot 300^2+640\cdot 300\overset{?}{<}40,000$ $12,000<40,000$ True

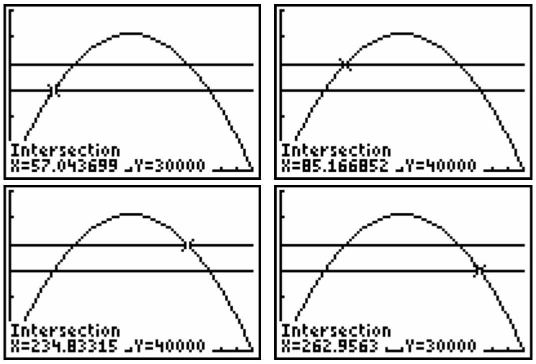
Thus, the second inequality is satisfied when the measure of x is in the interval $(-\infty,85.17)$ or $(234.83,\infty)$. We must now seek the intersection of the intervals $(57.04,262.96)$, and $(-\infty,85.17)$ or $(234.83,\infty)$. If we use the real number line as an aid, we can see that the solution would be between 57.04 ft and 85.17 ft or 234.83 ft and 262.96 ft.



Graphing Calculator Solution:







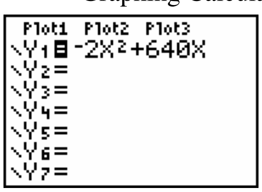
We can see from the graphs that the quadratic function lies between the lines when x is between 57.04 ft and 85.17 ft or 234.83 ft and 262.96 ft.

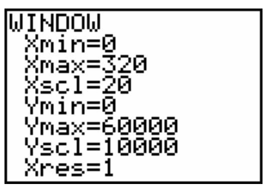
(e) Algebraic Solution:
Find the coordinates of the vertex of the parabola, $A(x) = -2x^2 + 640x$. Using the vertex formula with $a = -2$ and $b = 640$. We have

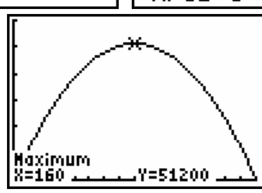
$x = -\frac{b}{2a} = -\frac{640}{2(-2)} = 160$ and

$y = -2(160)^2 + 640(160) = 51,200.$

Since $a < 0$, this is the maximum point.
Graphing Calculator Solution:







Thus, the length of the two parallel sides would be 160 ft and the third side would be $640 - 2(160) = 620 - 320 = 320$ ft.

The maximum area would be 51,200 ft^2.

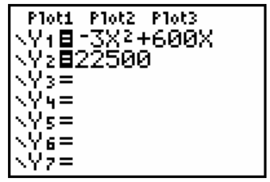
56. (a) The length of the remaining piece would be $600 - 3x$.

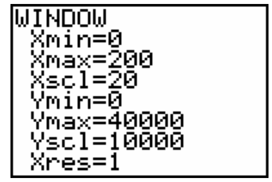
(b) $A(x) = x(600 - 3x) = 600x - 3x^2$
$= -3x^2 + 600x$
and in order for the two lengths to be positive, $0 < x < 200$.

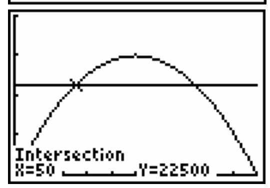
(c) Algebraic Solution:
Solve the equation
$-3x^2 + 600x = 22,500.$

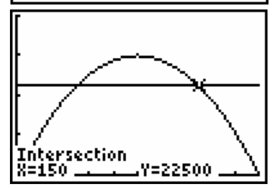
$0 = 3x^2 - 600x + 22,500$
$0 = x^2 - 200x + 7,500$
$0 = (x - 50)(x - 150)$
$x - 50 = 0 \Rightarrow x = 50$ or
$x - 150 = 0 \Rightarrow x = 150$

Graphing Calculator Solution:





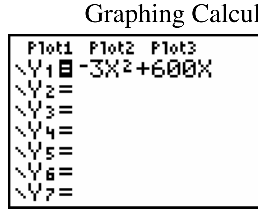


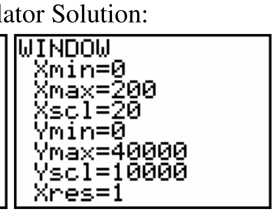


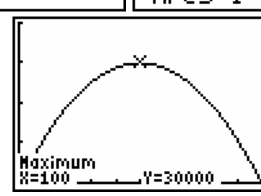
If $x = 50$ ft then the length of the non-parallel side is
$600 - 3(50) = 600 - 150 = 450$ ft. If $x = 150$ ft then the length of the non-parallel side is
$600 - 3(150) = 600 - 450 = 150$ ft.

(d) Algebraic Solution:
Find the coordinates of the vertex of the parabola, $A(x) = -3x^2 + 600x$. Using the vertex formula with $a = -3$ and $b = 600$, $x = -\frac{b}{2a} = -\frac{600}{2(-3)} = 100$ and

$y = -3(100)^2 + 600(100) = 30,000.$ Since $a < 0$, this is the maximum point.
Graphing Calculator Solution:







We are seeking the y-coordinate of the vertex as the maximum area. The maximum area is therefore 30,000 ft^2.

57. **(a)** The length of the original piece of cardboard would be $2x$.

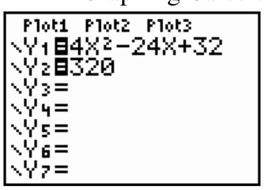
(b) The length of the rectangular box would be $2x-4$ and the width would be $x-4$, where $x>4$.

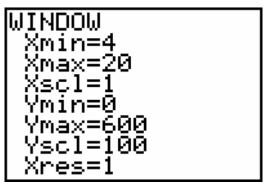
(c) $V(x)=(2x-4)(x-4)(2)$
$=(2x^2-12x+16)(2)$
$=4x^2-24x+32$

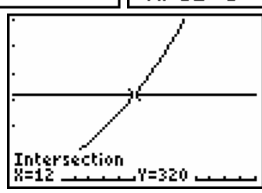
(d) Algebraic Solution:
Solve the equation $4x^2-24x+32=320$.

$$4x^2-24x+32=320$$
$$4x^2-24x-288=0$$
$$x^2-6x-72=0$$
$$(x+6)(x-12)=0$$

$x+6=0 \Rightarrow x=-6$ or
$x-12=0 \Rightarrow x=12$
We discard the negative solution.
Graphing Calculator Solution:







If $x=12$, then the dimensions of the bottom of the box will be $12-4=8$ in by $2(12)-4=24-4=20$ in.

(e) Algebraic Solution:
Solve the inequality
$400<4x^2-24x+32<500$. Treat this as two inequalities, $4x^2-24x+32>400$ and $4x^2-24x+32<500$.
For $4x^2-24x+32>400$, we solve the corresponding equation
$4x^2-24x+32=400$.
This equation is equivalent to
$4x^2-24x-368=0$ or $x^2-6x-92=0$
Use the quadratic formula with $a = 1$, $b = -6$, and $c = -92$.

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-92)}}{2(1)}$$
$$=\frac{6\pm\sqrt{36+368}}{2}=\frac{6\pm\sqrt{404}}{2}$$
$$=\frac{6\pm2\sqrt{101}}{2}=3\pm\sqrt{101}$$

$x=3-\sqrt{101}\approx-7.0$ or
$x=3+\sqrt{101}\approx13.0$
Since $x>4$, we need only check the intervals: $(4,13.0)$ and $(13.0,\infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $4x^2-24x+32>400$ True or False?
$(4,13.0)$	5	$4\cdot5^2-24\cdot5+32\overset{?}{>}400$ $12>400$ False
$(13.0,\infty)$	14	$4\cdot14^2-24\cdot14+32\overset{?}{>}400$ $480>400$ True

Thus, the first inequality is satisfied when the length, x, is in the interval $(13.0,\infty)$.

For $4x^2-24x+32<500$, we solve the corresponding equation
$4x^2-24x+32=500$. This equation is equivalent to $4x^2-24x-468=0$ or $x^2-6x-117=0$. Use the quadratic formula with $a = 1$, $b = -6$, and $c = -117$.

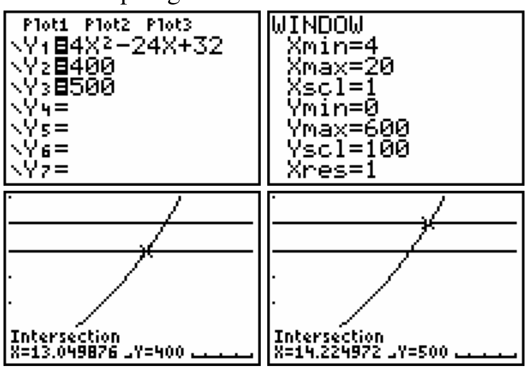
$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-117)}}{2(1)}$$
$$=\frac{6\pm\sqrt{36+468}}{2}=\frac{6\pm\sqrt{504}}{2}$$
$$=\frac{6\pm2\sqrt{126}}{2}=3\pm\sqrt{126}$$

$x=3-\sqrt{126}\approx-8.2$ or
$x=3+\sqrt{126}\approx14.2$
Since $x>4$, we need only check the intervals: $(4,14.2)$ and $(14.2,\infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $4x^2-24x+32<500$ True or False?
$(4,14.2)$	5	$4\cdot5^2-24\cdot5+32\overset{?}{<}500$ $12<500$ True
$(14.2,\infty)$	15	$4\cdot15^2-24\cdot15+32\overset{?}{<}500$ $572<500$ False

Thus, the second inequality is satisfied when the length, x, is in the interval $(4,14.2)$. We must now seek the intersection of the intervals $(13.0,\infty)$ and $(4,14.2)$. This intersection is $(13,14.2)$.

Graphing Calculator Solution:



We can see from the graphs that the quadratic function lies between the lines when x is between 13.0 in and 14.2 in. Thus, the volume will be between 400 in^3 and 500 in^3. when the original width is between 13.0 in. and 14.2 in.

58. **(a)** The length of the original piece of sheet metal would be $2.5x$.

(b) Since the length of the box would be $2.5x-6$ and the width would be $x-6$, we must have $x>6$ in order to ensure positive measures.

(c)
$$\begin{aligned}V(x)&=(2.5x-6)(x-6)(3)\\&=(2.5x^2-21x+36)(3)\\&=7.5x^2-63x+108\end{aligned}$$

(d) Algebraic Solution:
Solve the inequality $600<7.5x^2-63x+108<800$. Treat this as two inequalities, $7.5x^2-63x+108>600$ and $7.5x^2-63x+108<800$.

For $7.5x^2-63x+108>600$, we solve the corresponding equation $7.5x^2-63x+108=600$. This equation is equivalent to $7.5x^2-63x-492=0$ or $2.5x^2-21x-164=0$ Use the quadratic formula with $a=2.5$, $b=-21$, and $c=-164$.

$$x=\frac{-(-21)\pm\sqrt{(-21)^2-4(2.5)(-164)}}{2(2.5)}=\frac{21\pm\sqrt{441+1640}}{5}=\frac{21\pm\sqrt{2081}}{5}$$
$$x=\frac{21-\sqrt{2081}}{5}\approx-4.9 \text{ or}$$
$$x=\frac{21+\sqrt{2081}}{5}\approx13.3$$

Since $x>6$, we need only check the intervals: $(6,13.3)$ and $(13.3,\infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $7.5x^2-63x+108>600$ True or False?
$(6,13.3)$	7	$7.5\cdot7^2-63\cdot7+108\overset{?}{>}600$ $34.5>600$ False
$(13.3,\infty)$	14	$7.5\cdot14^2-63\cdot14+108\overset{?}{>}600$ $696>600$ True

Thus, the first inequality is satisfied when the length, x, is in the interval $(13.3,\infty)$.

For $7.5x^2-63x+108<800$, we solve the corresponding equation $7.5x^2-63x+108=800$. This equation is equivalent to $7.5x^2-63x-692=0$. Use the quadratic formula with $a=7.5$, $b=-63$, and $c=-692$.

$$x=\frac{-(-63)\pm\sqrt{(-63)^2-4(7.5)(-692)}}{2(7.5)}=\frac{63\pm\sqrt{3969+20,760}}{15}=\frac{63\pm\sqrt{24,729}}{15}$$
$$x=\frac{63-\sqrt{24,729}}{15}\approx-6.3 \text{ or}$$
$$x=\frac{63+\sqrt{24,729}}{15}\approx14.7$$

(*continued on next page*)

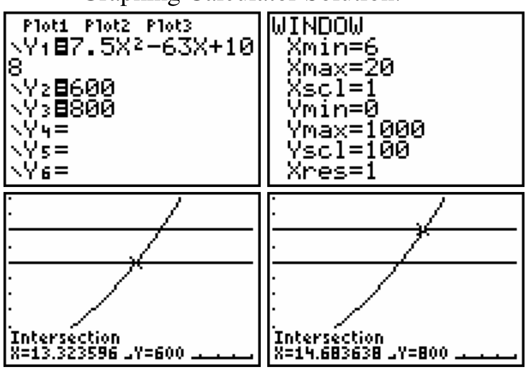
(*continued from page 265*)

Since $x > 6,$ we need only check the intervals $(6, 14.7)$ and $(14.7, \infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $7.5x^2 - 63x + 108 < 800$ True or False?
$(6, 14.7)$	7	$7.5 \cdot 7^2 - 63 \cdot 7 + 108 \overset{?}{<} 800$ $34.5 < 800$ True
$(14.7, \infty)$	15	$7.5 \cdot 15^2 - 63 \cdot 15 + 108 \overset{?}{<} 800$ $850.5 < 800$ False

Thus, the second inequality is satisfied when the length, x, is in the interval $(6, 14.7)$. We must now seek the intersection of the intervals $(13.3, \infty)$ and $(6, 14.7)$. This intersection is $(13.3, 14.7)$.

Graphing Calculator Solution:



We can see from the graphs that the quadratic function lies between the lines when x is between 13.3 in and 14.7 in. Thus, when the original width is between 13.3 in and 14.7 in, the volume will be between 600 in^3 and 800 in^3.

59. $h(x) = -.5x^2 + 1.25x + 3$

(a) Find $h(x)$ when $x = 2$.

$$h(2) = -.5(2)^2 + 1.25(2) + 3$$
$$= -.5(4) + 1.25(2) + 3$$
$$= -2 + 2.5 + 3 = 3.5$$

When the distance from the base of the stump was 2 ft, the frog was 3.5 ft high.

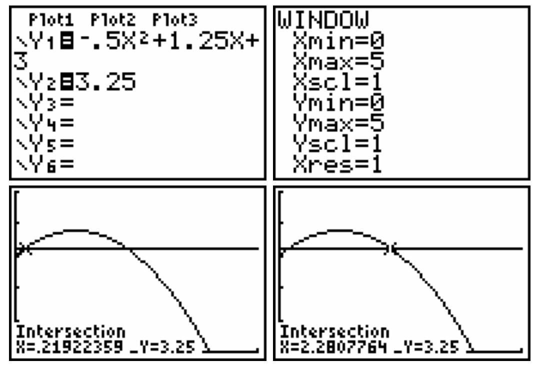
(b) Algebraic Solution:
Find x when $h(x) = 3.25$.

$$3.25 = -0.5x^2 + 1.25x + 3$$
$$0.5x^2 - 1.25x + 0.25 = 0$$
$$2x^2 - 5x + 1 = 0$$

Use the quadratic formula with $a = 2$, $b = -5$, and $c = 1$.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{25-8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$
$$x = \frac{5 - \sqrt{17}}{4} \approx .2 \text{ or } x = \frac{5 + \sqrt{17}}{4} \approx 2.3$$

Graphing Calculator Solution:

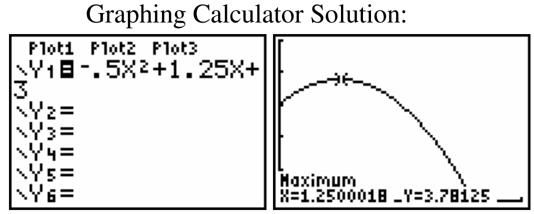


The frog was 3.25 ft above the ground when he was approximately .2 ft from the stump (on the way up) and 2.3 ft from the stump (on the way down).

(c) Algebraic Solution:
Since the parabola opens downward, the vertex is the maximum point. Use the vertex formula to find the x-coordinate of the vertex of $h(x) = -0.5x^2 + 1.25x + 3$.

$$x = -\frac{b}{2a} = -\frac{1.25}{2(-0.5)} = -\frac{1.25}{-1} = 1.25$$

Graphing Calculator Solution:



The frog reached its highest point at 1.25 ft from the stump.

(d) The maximum height is the y-coordinate of the vertex.

$$y = h(1.25)$$
$$= -.5(1.25)^2 + 1.25(1.25) + 3 = 3.78125$$

The maximum height reached by the frog was approximately 3.78 ft. This agrees with the graphing calculator solution by interpreting the y-coordinate of the vertex as the maximum height.

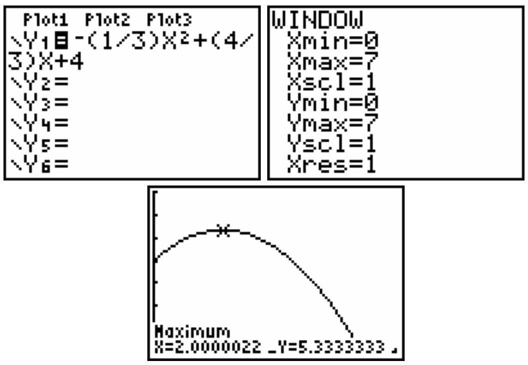
60. $h(x) = -\frac{1}{3}x^2 + \frac{4}{3}x + 4$

(a) Algebraic Solution:

Since the parabola opens downward, the vertex is the maximum point. Use the vertex formula to find the x-coordinate of the vertex of $h(x) = -\frac{1}{3}x^2 + \frac{4}{3}x + 4$.

$$x = -\frac{b}{2a} = -\frac{\frac{4}{3}}{2\left(-\frac{1}{3}\right)} = -\frac{\frac{4}{3}}{-\frac{2}{3}} = -\frac{4}{3}\left(-\frac{3}{2}\right) = 2$$

Graphing Calculator Solution:



The frog reached its highest point at 2 ft from the stump.

(b) The maximum height is the y-coordinate of the vertex.

$$y = h(2) = -\frac{1}{3}(2)^2 + \frac{4}{3}(2) + 4 = \frac{16}{3} = 5\frac{1}{3}$$

The maximum height reached by the frog was approximately $5\frac{1}{3}$ ft. This agrees with the graphing calculator solution by interpreting the y-coordinate of the vertex as the maximum height.

61. $y = \dfrac{-16x^2}{.434v^2} + 1.15x + 8$

(a) Let $y = 10$ and $x = 15$.

$$10 = \frac{-16(15)^2}{.434v^2} + 1.15(15) + 8$$
$$10 = \frac{-16(225)}{.434v^2} + 1.15(15) + 8$$
$$10 = \frac{-3600}{.434v^2} + 17.25 + 8$$
$$10 = \frac{-3600}{.434v^2} + 25.25$$
$$\frac{3600}{.434v^2} = 15.25$$
$$3600 = 6.6185v^2 \Rightarrow \frac{3600}{6.6185} = v^2$$
$$\pm\sqrt{\frac{3600}{6.6185}} = v \Rightarrow v \approx \pm 23.32$$

Since v represents a velocity, only the positive square root is meaningful. The basketball should have an initial velocity of 23.32 ft per sec.

(b) $y = \dfrac{-16x^2}{.434(23.32)^2} + 1.15x + 8$

Algebraic Solution:

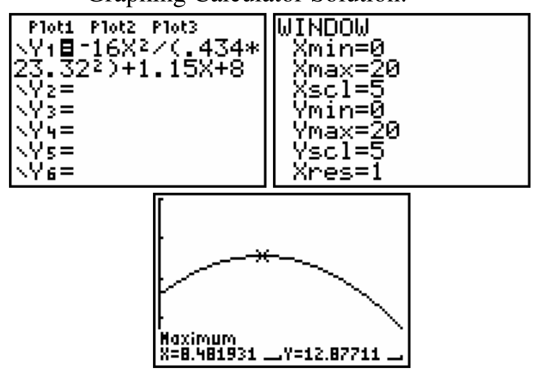
Since the parabola opens downward, the vertex is the maximum point. Use the vertex formula to find the x-coordinate of the vertex.

$$x = -\frac{b}{2a} = -\frac{1.15}{2\left(\frac{-16}{.434(23.32)^2}\right)} \approx 8.482$$

To find the y-coordinate of the vertex, evaluate the quadratic function when $x = 8.482$.

$$y = \frac{-16(8.482)^2}{.434(23.32)^2} + 1.15(8.482) + 8 \approx 12.88$$

Graphing Calculator Solution:



The y-coordinate of the vertex represents the maximum height of the basketball, which is approximately 12.88 ft.

62. $y = \dfrac{-16x^2}{.117v^2} + 2.75x + 3$

(a) Let $y = 10$ and $x = 15$.

$$10 = \frac{-16(15)^2}{.117v^2} + 2.75(15) + 3$$
$$10 = \frac{-16(225)}{.117v^2} + 2.75(15) + 3$$
$$10 = \frac{-3600}{.117v^2} + 41.25 + 3$$
$$10 = \frac{-3600}{.117v^2} + 44.25$$

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(*continued from page 267*)

$$\frac{3600}{.117v^2} = 34.25 \Rightarrow 3600 = 4.00725v^2$$
$$\frac{3600}{4.00725} = v^2$$
$$\pm\sqrt{\frac{3600}{4.00725}} = v \Rightarrow v \approx \pm 29.97$$

Since v represents a velocity, only the positive square root is meaningful. The basketball should have an initial velocity of 29.97 ft per sec.

(b) $y = \dfrac{-16x^2}{.117(29.97)^2} + 2.75x + 3$

Algebraic Solution:

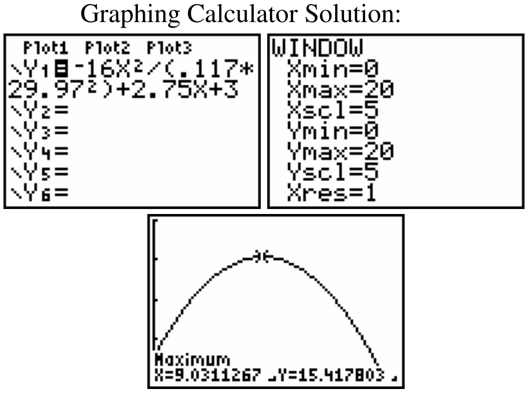
Since the parabola opens downward, the vertex is the maximum point. Use the vertex formula to find the x-coordinate of the vertex.

$$x = -\frac{b}{2a} = -\frac{2.75}{2\left(\frac{-16}{.117(29.97)^2}\right)} \approx 9.031$$

To find the y-coordinate of the vertex, evaluate the quadratic function when $x = 9.0311$.

$$y = \frac{-16(9.031)^2}{.117(29.97)^2} + 2.75(9.031) + 3 \approx 15.42$$

Graphing Calculator Solution:



The y-coordinate of the vertex represents the maximum height of the basketball, which is approximately 15.42 ft. The underhand shot produces a higher arc.

63. $f(x) = .0222x^2 + .0716x + 31.8$

(a) The year 2009 implies $x = 16$

$$f(16) = .0222(16)^2 + .0716(16) + 31.8 \approx 38.6\%$$

Approximately 38.6% is predicted for 2005.

(b) No, because according to the model, the number of births to unmarried mothers should rise after 2004 since the coefficient of the x^2 term is positive. It is not realistic to assume they will decrease, based on the trend seen in the period 1990-2000.

64. $f(x) = -348.2x^2 + 1402x + 13,679$

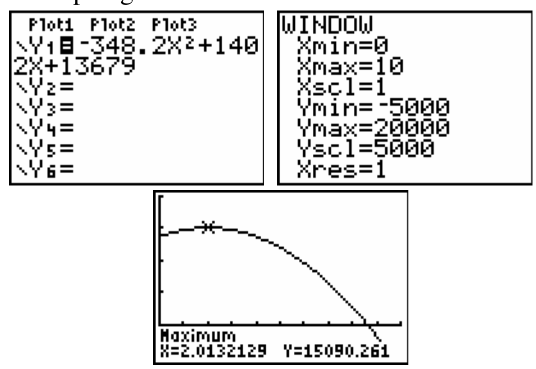
We seek the x-coordinate of the vertex.

Algebraic Solution:

Since the parabola opens upward, the vertex is the minimum point. Use the vertex formula to find the x-coordinate of the vertex.

$$x = -\frac{b}{2a} = -\frac{-1402}{2(-348.2)} = -\frac{-1402}{-696.4} \approx 2.01$$

Graphing Calculator Solution:



The percent of freshman planning to get a computer science degree reached its maximum approximately 2.01 years after 1998, which rounds to the year 2000.

65. $f(x) = -132.1x^2 + 1439x + 41,648$

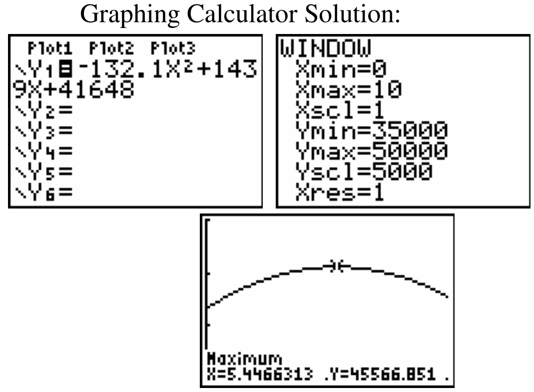
We seek the x-coordinate of the vertex.

Algebraic Solution:

Since the parabola opens upward, the vertex is the minimum point. Use the vertex formula to find the x-coordinate of the vertex.

$$x = -\frac{b}{2a} = -\frac{1439}{2(-132.1)} = -\frac{1439}{-264.2} \approx 5.4$$

Graphing Calculator Solution:



Based on the model, the median family income reach its maximum about 5.4 years after 1995, which rounds to the year 2000.

66. $f(x) = .0232x^2 - 2.28x + 60.0$

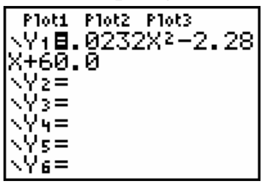
Algebraic Solution:
Since the parabola opens upward, the vertex is the minimum point. Use the vertex formula to find the x-coordinate of the vertex.

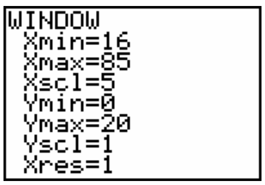
$$x = -\frac{b}{2a} = -\frac{-2.28}{2(.0232)} = -\frac{-2.28}{.0464} \approx 49.14$$

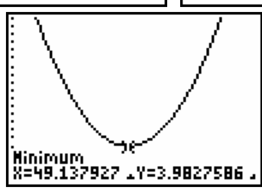
The minimum accident rate is the y-coordinate of the vertex.

$$f(49.14) = .0232(49.14)^2 - 2.28(49.14) + 60.0 \approx 3.98$$

Graphing Calculator Solution:

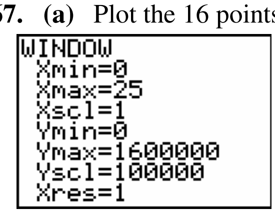


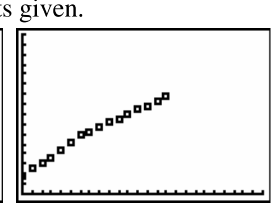




The minimum accident rate of 3.98 occurs at age 49.

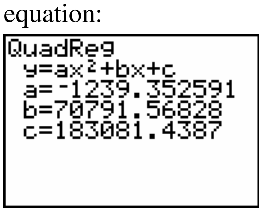
67. **(a)** Plot the 16 points given.





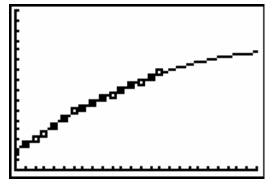
(b) A quadratic function would model the data better because the data increases at a different rate each year.

(c) Use the quadratic regression function on the graphing calculator to find the equation:



The equation is
$f(x) = -1239x^2 + 70,792x + 183,081$

(d) Plotting the points together with $f(x)$, we see that f models the data almost exactly.



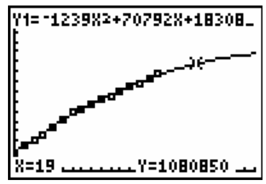
(e) $x = 19$ corresponds to the year 2009, and $x = 20$ corresponds to the year 2010.

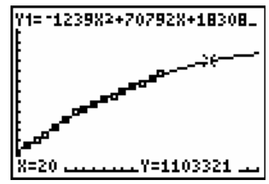
Algebraic Solution:

$$f(19) \approx -1239(19)^2 + 70,792(19) + 183,081 \approx 1,080,850$$

$$f(20) \approx -1239(20)^2 + 70,792(20) + 183,081 \approx 1,103,321$$

Graphing Calculator Solution:



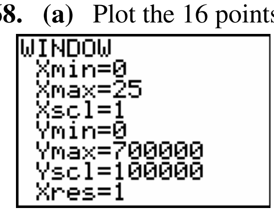


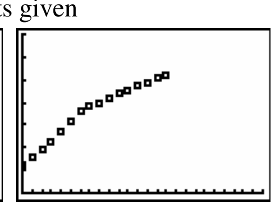
In the year 2009, approximately 1,080,850 people will have been diagnosed with AIDS since 1990. In the year 2010, approximately 1,103,321 people will have been diagnosed with AIDS since 1990.

(f) The number of new cases in the year 2010 will be approximately

$$f(20) - f(19) \approx 1,103,321 - 1,080,850 \approx 22,471$$

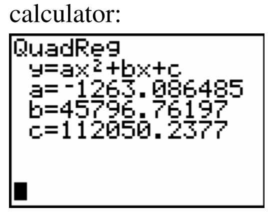
68. **(a)** Plot the 16 points given





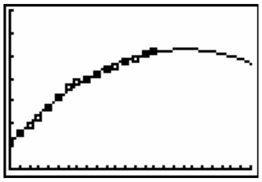
(b) A quadratic function would model the data better because the data increases at a different rate each year.

(c) Find the equation by using the quadratic regression function on the graphing calculator:



The equation is
$g(x) = -1263x^2 + 45,797x + 112,050$.

(d) Plotting the points together with $g(x)$, we see that g models the data almost exactly.



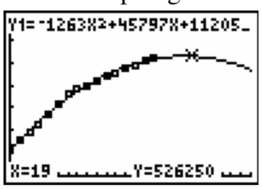
(e) $x = 19$ corresponds to the year 2009, and $x = 20$ corresponds to the year 2010.

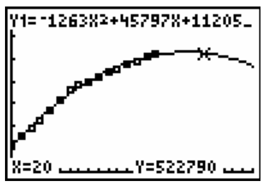
Algebraic Solution:

$$g(19) \approx -1263(19)^2 + 45,797(19) + 112,050 \approx 526,250$$

$$g(20) \approx -1263(20)^2 + 45,797(20) + 112,050 \approx 522,790$$

Graphing Calculator Solution:

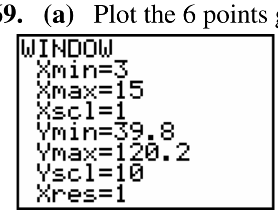


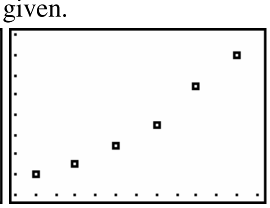


The number of deaths in the year 2009 will be approximately 526,250. the number of deaths in the year 2010 will be approximately 522,790.

(f) The data suggests that a quadratic model is a good representation of the data; however, for x values to the right of the vertex, it is no longer appropriate because the number of deaths continues to grow, not fall as in the model.

69. **(a)** Plot the 6 points given.





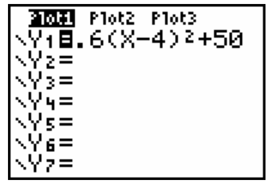
(b) Let the point (4, 50), be the vertex. Then, $f(x) = a(x-4)^2 + 50.$

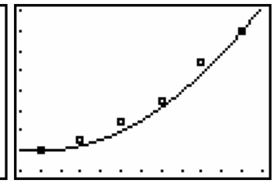
Next let the point (14, 110) lie on the graph of the function and solve for a.

$$f(14) = a(14-4)^2 + 50 = 110$$
$$a(10)^2 + 50 = 110$$
$$100a + 50 = 110$$
$$100a = 60 \Rightarrow a = \tfrac{60}{100} = .6$$

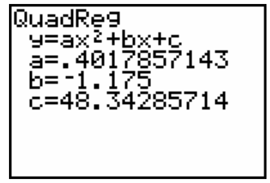
Thus, $f(x) = .6(x-4)^2 + 50.$

(c) Plotting the points together with $f(x)$, we see that there is a relatively good fit.





(d) The quadratic regression curve is $g(x) = .402x^2 - 1.175x + 48.343$



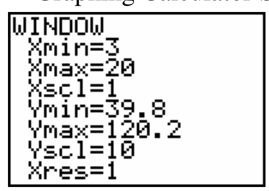
(e) $x = 16$ corresponds to the year 2006.

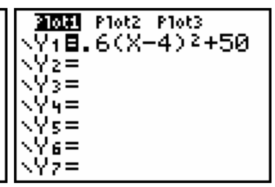
Algebraic Solution:

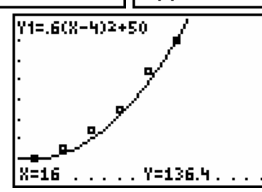
$$f(16) = .6(16-4)^2 + 50 = 136.4$$

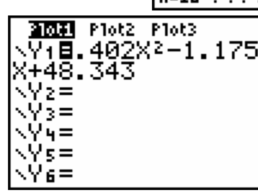
$$g(16) = .402(16)^2 - 1.175(16) + 48.343 \approx 132.5$$

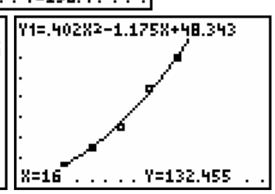
Graphing Calculator Solution:











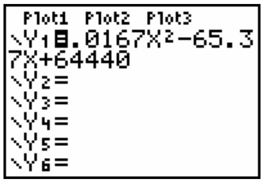
The graphing calculator agrees with the above calculations. In the year 2006, approximately 136.4 (thousand) are predicted by f to be over 100 in 2006. In the year 2006, approximately 132.5 (thousand) are predicted by g to be over 100 in 2006.

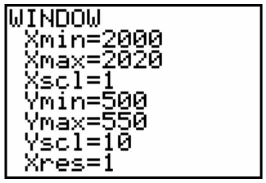
70. $f(x) = .0167x^2 - 65.37x + 64,440$

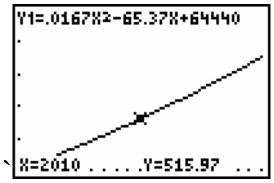
Algebraic Solution:

$$f(2010) = .0167(2010)^2 - 65.37(2010) + 64,440 = 515.97$$

Graphing Calculator Solution:

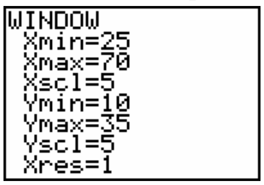


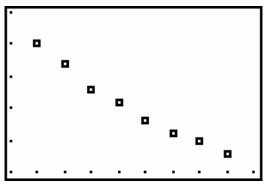




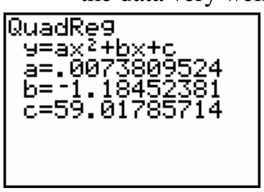
Approximately 516 ppm will be the concentration in 2010.

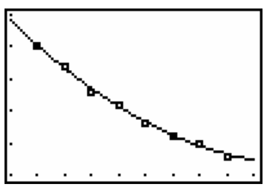
71. **(a)** Plot the 8 points given.





(b) $g(x) = .0074x^2 - 1.185x + 59.02$ models the data very well.

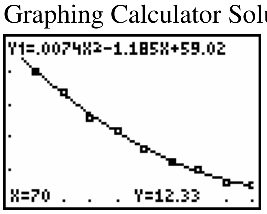




(c) Algebraic Solution:

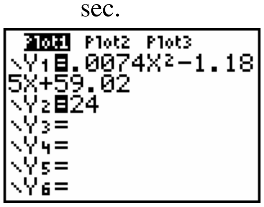
$$g(70) = .0074(70)^2 - 1.185(70) + 59.02 = 12.33$$

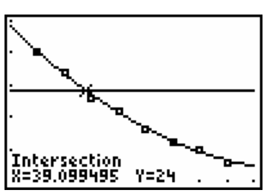
Graphing Calculator Solution:



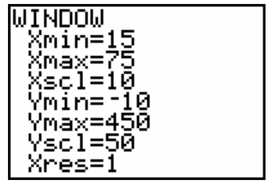
With an initial speed of 70 mph, the coast-down time would be 12.33 sec.

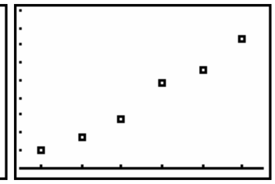
(d) A speed of approximately 39.1 mph corresponds to a coast-down time of 24 sec.



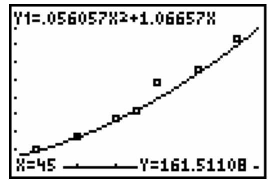


72. **(a)** Plot the 6 points given.





(b) $f(45) \approx 161.5$ ft is the stopping distance when the speed is 45 mph.



(c) Answers will vary. The model is quite good, although the stopping distances are a little low for the higher speeds.

73. $y = x^2 - 10x + c$

An x-intercept occurs where $y = 0$, or $0 = x^2 - 10x + c$. There will be exactly one x-intercept if this equation has exactly one solution, or if the discriminant is zero.

$b^2 - 4ac = 0 \Rightarrow (-10)^2 - 4(1)c = 0 \Rightarrow$
$100 - 4c = 0 \Rightarrow 100 = 4c \Rightarrow c = 25$

74. $y = ax^2 - 8x + 4$

There will be no x-intercepts if the equation $0 = ax^2 - 8x + 4$ has no solution. This happens when the discriminant is less than zero.

$b^2 - 4ac < 0 \Rightarrow (-8)^2 - 4(a)(4) < 0 \Rightarrow$
$64 - 16a < 0 \Rightarrow -16a < -64 \Rightarrow a > 4$

Thus $ax^2 - 8x + 4$ has no x-intercepts if $a > 4$.

75. x-intercepts 2 and 5, and y-intercept 5

Since we have x-intercepts 2 and 5, f has linear factors of $x - 2$ and $x - 5$.

$f(x) = a(x-2)(x-5) = a(x^2 - 7x + 10)$

Since the y-intercept is 5, $f(0) = 5$.

$f(x) = a(0^2 - 7(0) + 10) = 5 \Rightarrow$

$10a = 5 \Rightarrow a = \frac{1}{2}$

The required quadratic function is

$f(x) = \frac{1}{2}(x^2 - 7x + 10) = \frac{1}{2}x^2 - \frac{7}{2}x + 5.$

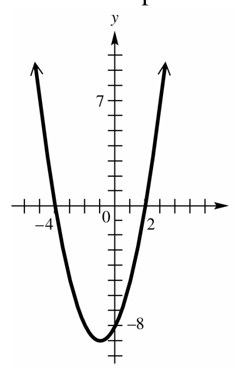
76. x-intercepts 1 and –2, and y-intercept 4
Since we have x-intercepts 1 and –2, f has linear factors of $x-1$ and $x-(-2)=x+2$.
$f(x)=a(x-1)(x+2)=a(x^2+x-2)$
Since the y-intercept is 4, $f(0) = 4$.
$f(x)=a(0^2+0-2)=4\Rightarrow -2a=4\Rightarrow a=-2$
The required quadratic function is
$f(x)=-2(x^2+x-2)=-2x^2-2x+4.$

77. Use the distance formula,
$d(P,\ R)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, to find the distance between the points
$P(x,2x)$ and $R(1,\ 7)$, where P is any point on the line $y = 2x$.
$d(P,\ R)=\sqrt{(x-1)^2+(2x-7)^2}$
$=\sqrt{(x^2-2x+1)+(4x^2-28x+49)}$
$=\sqrt{5x^2-30x+50}$
Consider the equation $y=5x^2-30x+50.$ This is the equation of a parabola opening upward, so the expression $5x^2-30x+50$ has a minimum value, which is the y-value of the vertex. Complete the square to find the vertex.
$y=5x^2-30x+50=5(x^2-6x)+50$
$=5(x^2-6x+9-9)+50$
$=5(x^2-6x+9)-45+50=5(x-3)^2+5$
The vertex is (3, 5), so the minimum value of $5x^2-30x+50$ is 5 when $x = 3$. Thus, the minimum value of $\sqrt{5x^2-30x+50}$ is $\sqrt{5}$ when $x = 3$. The point on the line $y = 2x$ for which $x = 3$ is (3, 6). Thus, the closest point on the line $y = 2x$ to the point (1, 7) is the point (3, 6).

78. Consider the quadratic function written in the form $y=a(x-h)^2+k.$ The quadratic function has vertex (5, 3), so $h = 5$ and $k = 3$. Thus $y=a(x-5)^2+3.$ If the quadratic function has a zero at $x = 2$ then $0=a(2-5)^2+3.$ Solve for a.
$0=a(2-5)^2+3\Rightarrow -3=a(-3)^2\Rightarrow$
$-3=9a\Rightarrow -\frac{1}{3}=a$
Substitute $-\frac{1}{3}$ for a and solve for the remaining zero of the function.
$0=-\frac{1}{3}(x-5)^2+3\Rightarrow 0=-(x-5)^2+9$
$(x-5)^2=9\Rightarrow x-5=\pm3\Rightarrow$
$x-5=-3\Rightarrow x=2$ or $x-5=3\Rightarrow x=8$
The other zero of the function is 8.

79. Graph the function $f(x)=x^2+2x-8$.
Complete the square to find the vertex, (–1, –9).
$y=x^2-2x-8=(x^2-2x)-8$
$=(x^2-2x+1-1)-8$
$=(x^2-2x+1)-1-8=(x-1)^2-9$
From the graph (and table), we see that the x-intercepts are – 4 and 2.
Use a table of values to find points on the graph.

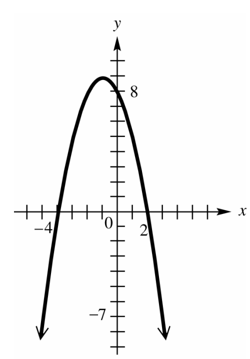
x	y
–5	7
–4	0
–3	–5
–2	–8
–1	–9
0	–8
1	–5
2	0
3	7



$f(x)=x^2+2x-8$

80. The solution set of the inequality $x^2+2x-8<0$ consists of all x-values for which the graph of f lies below the x-axis. By examining the graph of the function $f(x)=x^2+2x-8$ from Exercise 79, we see that the graph lies below the x-axis when x is between the – 4 and 2. Thus, the solution set is the open interval (– 4, 2).

81. Graph $g(x)=-f(x)=-x^2-2x+8.$ The graph of g is obtained by reflecting the graph of f across the x-axis.



$g(x)=-f(x)=-x^2-2x+8$

82. The solution set of the inequality $-x^2 - 2x - 8 > 0$ consists of all x values for which the graph of g lies above the x-axis. By examining the graph of $g(x) = -x^2 - 2x + 8$ from Exercise 81, we see that the graph lies above the x-axis when x is between –4 and 2. Thus, the solution set is the open interval (–4, 2).

83. The two solution sets are the same, the open interval (– 4, 2).

84. Answers will vary.

Section 3.2: Synthetic Division

Connections (page 326)

1. To find $f(-2+i)$ use synthetic division with $f(x) = x^3 - 4x^2 + 2x - 29i$.

$$\begin{array}{r|rrrr} -2+i & 1 & -4 & 2 & -29i \\ & & -2+i & 11-8i & -18+29i \\ \hline & 1 & -6+i & 13-8i & -18 \end{array}$$

The remainder is –18 so by the remainder theorem $f(-2+i) = -18$.

2. Use synthetic division to check if i is a zero of $f(x) = x^3 + 2ix^2 + 2x + i$.

$$\begin{array}{r|rrrr} i & 1 & 2i & 2 & i \\ & & i & -3 & -i \\ \hline & 1 & 3i & -1 & 0 \end{array}$$

Since the remainder is 0, $f(i) = 0$ and i is a zero of $f(x)$. To check $-i$, perform synthetic division.

$$\begin{array}{r|rrrr} -i & 1 & 2i & 2 & i \\ & & -i & 1 & -3i \\ \hline & 1 & i & 3 & -2i \end{array}$$

Since the remainder is $-2i$, $f(-i) = -2i$ and $-i$ is not a zero of $f(x)$.

3. Answers will vary.
One example is the function $f(x) = x^2 + 1$.

Exercises

1. Since $x + 1$ is in the form $x + k$, $k = -1$.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 11 & 9 \\ & & -1 & -2 & -9 \\ \hline & 1 & 2 & 9 & 0 \end{array}$$

$$\frac{x^3 + 3x^2 + 11x + 9}{x+1} = x^2 + 2x + 9$$

2. Since $x + 2$ is in the form $x + k$, $k = -2$

$$\begin{array}{r|rrrr} -2 & 1 & 7 & 13 & 6 \\ & & -2 & -10 & -6 \\ \hline & 1 & 5 & 3 & 0 \end{array}$$

$$\frac{x^3 + 7x^2 + 13x + 6}{x+2} = x^2 + 5x + 3$$

3. Express $x+1$ in the form $x - k$ by writing it as $x - (-1)$. Thus $k = -1$.

$$\begin{array}{r|rrrrr} -1 & 5 & 5 & 2 & -1 & -3 \\ & & -5 & 0 & -2 & 3 \\ \hline & 5 & 0 & 2 & -3 & 0 \end{array}$$

$$\frac{5x^4 + 5x^3 + 2x^2 - x - 3}{x+1} = 5x^3 + 2x - 3$$

4. Since $x - 2$ is in the form $x - k$, $k = 2$.

$$\begin{array}{r|rrrrr} 2 & 2 & -1 & -7 & 7 & -10 \\ & & 4 & 6 & -2 & 10 \\ \hline & 2 & 3 & -1 & 5 & 0 \end{array}$$

$$\frac{2x^4 - x^3 - 7x^2 + 7x - 10}{x-2} = 2x^3 + 3x^2 - x + 5$$

5. Express $x+4$ in the form $x - k$ by writing it as $x - (-4)$. Thus $k = -4$.

$$\begin{array}{r|rrrrr} -4 & 1 & 4 & 2 & 9 & 4 \\ & & -4 & 0 & -8 & -4 \\ \hline & 1 & 0 & 2 & 1 & 0 \end{array}$$

$$\frac{x^4 + 4x^3 + 2x^2 + 9x + 4}{x+4} = x^3 + 2x + 1$$

6. Express $x+3$ in the form $x - k$ by writing it as $x - (-3)$. Thus $k = -3$.

$$\begin{array}{r|rrrrr} -3 & 1 & 5 & 4 & -3 & 9 \\ & & -3 & -6 & 6 & -9 \\ \hline & 1 & 2 & -2 & 3 & 0 \end{array}$$

$$\frac{x^4 + 5x^3 + 4x^2 - 3x + 9}{x+3} = x^3 + 2x^2 - 2x + 3$$

7. Express $x+2$ in the form $x-k$ by writing it as $x-(-2)$. Thus $k=-2$.

$$\begin{array}{r|rrrrrr} -2 & 1 & 3 & 2 & 2 & 3 & 1\\ & & -2 & -2 & 0 & -4 & 2\\ \hline & 1 & 1 & 0 & 2 & -1 & 3\end{array}$$

$$\frac{x^5+3x^4+2x^3+2x^2+3x+1}{x+2} = x^4+x^3+2x-1+\frac{3}{x+2}$$

8. Express $x+2$ in the form $x-k$ by writing it as $x-(-2)$. Thus $k=-2$. Since the x^5 – term is missing, include a 0 as its coefficient.

$$\begin{array}{r|rrrrrrr} -2 & 1 & 0 & -3 & 2 & -6 & -5 & 3\\ & & -2 & 4 & -2 & 0 & 12 & -14\\ \hline & 1 & -2 & 1 & 0 & -6 & 7 & -11\end{array}$$

$$\frac{x^6-3x^4+2x^3-6x^2-5x+3}{x+2} = x^5-2x^4+x^3-6x+7+\frac{-11}{x+2}$$

9. Since $x-2$ is in the form $x-k$, $k=2$.

$$\begin{array}{r|rrrr} 2 & -9 & 8 & -7 & 2\\ & & -18 & -20 & -54\\ \hline & -9 & -10 & -27 & -52\end{array}$$

$$\frac{-9x^3+8x^2-7x+2}{x-2} = -9x^2-10x-27+\frac{-52}{x-2}$$

10. Express $x+1$ in the form $x-k$ by writing it as $x-(-1)$. Thus $k=-1$. The *x*-term is missing, so include a 0 as its coefficient.

$$\begin{array}{r|rrrrr} -1 & -11 & 2 & -8 & 0 & -4\\ & & 11 & -13 & 21 & -21\\ \hline & -11 & 13 & -21 & 21 & -25\end{array}$$

$$\frac{-11x^4+2x^3-8x^2-4}{x+1} = -11x^3+13x^2-21x+21+\frac{-25}{x+1}$$

11. Since $x-\frac{1}{3}$ is in the form $x-k$, $k=\frac{1}{3}$.

$$\begin{array}{r|rrrr} \frac{1}{3} & \frac{1}{3} & -\frac{2}{9} & \frac{1}{27} & 1\\ & & \frac{1}{9} & -\frac{1}{27} & 0\\ \hline & \frac{1}{3} & -\frac{1}{9} & 0 & 1\end{array}$$

$$\frac{\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{1}{27}x+1}{x-\frac{1}{3}} = \frac{1}{3}x^2-\frac{1}{9}x+\frac{1}{x-\frac{1}{3}}$$

12. Express $x+\frac{1}{2}$ in the form $x-k$ by writing it as $x-\left(-\frac{1}{2}\right)$. Thus $k=-\frac{1}{2}$.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{8}\\ & & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8}\\ \hline & 1 & \frac{1}{2} & \frac{1}{4} & 0\end{array}$$

$$\frac{x^3+x^2+\frac{1}{2}x+\frac{1}{8}}{x+\frac{1}{2}} = x^2+\frac{1}{2}x+\frac{1}{4}$$

13. Since $x-2$ is in the form $x-k$, $k=2$. The constant term is missing, so include a 0.

$$\begin{array}{r|rrrrr} 2 & 1 & -3 & -4 & 12 & 0\\ & & 2 & -2 & -12 & 0\\ \hline & 1 & -1 & -6 & 0 & 0\end{array}$$

$$\frac{x^4-3x^3-4x^2+12x}{x-2} = x^3-x^2-6x$$

14. Express $x+1$ in the form $x-k$ by writing it as $x-(-1)$. Thus $k=-1$. Since the constant term is missing, include a 0.

$$\begin{array}{r|rrrrr} -1 & 1 & 5 & -6 & 2 & 0\\ & & -1 & -4 & 10 & -12\\ \hline & 1 & 4 & -10 & 12 & -12\end{array}$$

$$\frac{x^4+5x^3-6x^2+2x}{x+1} = x^3+4x^2-10x+12+\frac{-12}{x+1}$$

15. Since $x-1$ is in the form $x-k$, $k=1$. The x^2 - and *x*-terms are missing, so include 0's as their coefficients.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1\\ & & 1 & 1 & 1\\ \hline & 1 & 1 & 1 & 0\end{array}$$

$$\frac{x^3-1}{x-1} = x^2+x+1$$

16. $x-1$ is in the form $x-k$, so $k=1$. Since the x^3, x^2 and $x-$ terms are missing, include 0's as their coefficients.

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$\frac{x^4-1}{x-1} = x^3+x^2+x+1$$

17. Express $x+1$ in the form $x-k$ by writing it as $x-(-1)$. Thus $k=-1$. Since the x^4, x^3, x^2 and $x-$terms are missing, include 0's as their coefficients.

$$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\frac{x^5+1}{x+1} = x^4-x^3+x^2-x+1$$

18. Express $x+1$ in the form $x-k$ by writing it as $x-(-1)$. Thus $k=-1$. Since the x^6, x^5, x^4, x^3, x^2 and $x-$terms are missing, include 0's as their coefficients.

$$\begin{array}{r|rrrrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\frac{x^7+1}{x+1} = x^6-x^5+x^4-x^3+x^2-x+1$$

19. $f(x)=2x^3+x^2+x-8;\ k=-1$
Use synthetic division to write the polynomial in the form $f(x)=(x-k)q(x)+r$.

$$\begin{array}{r|rrrr} -1 & 2 & 1 & 1 & -8 \\ & & -2 & 1 & -2 \\ \hline & 2 & -1 & 2 & -10 \end{array}$$

$$\begin{aligned} f(x) &= [x-(-1)](2x^2-x+2)-10 \\ &= (x+1)(2x^2-x+2)-10 \end{aligned}$$

20. $f(x)=2x^3+3x^2-16x+10;\ k=-4$
Use synthetic division to write the polynomial in the form $f(x)=(x-k)q(x)+r$.

$$\begin{array}{r|rrrr} -4 & 2 & 3 & -16 & 10 \\ & & -8 & 20 & -16 \\ \hline & 2 & -5 & 4 & -6 \end{array}$$

$$\begin{aligned} f(x) &= [x-(-4)](2x^2-5x+4)+(-6) \\ &= (x+4)(2x^2-5x+4)-6 \end{aligned}$$

21. $f(x)=x^3+4x^2+5x+2;\ k=-2$

$$\begin{array}{r|rrrr} -2 & 1 & 4 & 5 & 2 \\ & & -2 & -4 & -2 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$\begin{aligned} f(x) &= [x-(-2)](x^2+2x+1)+0 \\ &= (x+2)(x^2+2x+1)+0 \end{aligned}$$

22. $f(x)=-x^3+x^2+3x-2;\ k=2$

$$\begin{array}{r|rrrr} 2 & -1 & 1 & 3 & -2 \\ & & -2 & -2 & 2 \\ \hline & -1 & -1 & 1 & 0 \end{array}$$

$$f(x)=(x-2)(-x^2-x+1)+0$$

23. $f(x)=4x^4-3x^3-20x^2-x;\ k=3$

$$\begin{array}{r|rrrrr} 3 & 4 & -3 & -20 & -1 & 0 \\ & & 12 & 27 & 21 & 60 \\ \hline & 4 & 9 & 7 & 20 & 60 \end{array}$$

$$f(x)=(x-3)(4x^3+9x^2+7x+20)+60$$

24. $f(x)=2x^4+x^3-15x^2+3x;\ k=-3$

$$\begin{array}{r|rrrrr} -3 & 2 & 1 & -15 & 3 & 0 \\ & & -6 & 15 & 0 & -9 \\ \hline & 2 & -5 & 0 & 3 & -9 \end{array}$$

$$\begin{aligned} f(x) &= [x-(-3)](2x^3-5x^2+3)+(-9) \\ &= (x+3)(2x^3-5x^2+3)-9 \end{aligned}$$

25. $f(x)=3x^4+4x^3-10x^2+15;\ k=-1$

$$\begin{array}{r|rrrrr} -1 & 3 & 4 & -10 & 0 & 15 \\ & & -3 & -1 & 11 & -11 \\ \hline & 3 & 1 & -11 & 11 & 4 \end{array}$$

$$\begin{aligned} f(x) &= [x-(-1)](3x^3+x^2-11x+11)+4 \\ &= (x+1)(3x^3+x^2-11x+11)+4 \end{aligned}$$

26. $f(x) = -5x^4 + x^3 + 2x^2 + 3x + 1;\ k = 1$

$$\begin{array}{r|rrrrr} 1 & -5 & 1 & 2 & 3 & 1 \\ & & -5 & -4 & -2 & 1 \\ \hline & -5 & -4 & -2 & 1 & 2 \end{array}$$

$f(x) = (x-1)(-5x^3 - 4x^2 - 2x + 1) + 2$

27. $f(x) = x^2 + 5x + 6;\ k = -2$

$$\begin{array}{r|rrr} -2 & 1 & 5 & 6 \\ & & -2 & -6 \\ \hline & 1 & 3 & 0 \end{array}$$

$f(-2) = 0$

28. $f(x) = x^2 - 4x - 5;\ k = 5$

$$\begin{array}{r|rrr} 5 & 1 & -4 & -5 \\ & & 5 & 5 \\ \hline & 1 & 1 & 0 \end{array}$$

$f(5) = 0$

29. $f(x) = 2x^2 - 3x - 3;\ k = 2$

$$\begin{array}{r|rrr} 2 & 2 & -3 & -3 \\ & & 4 & 2 \\ \hline & 2 & 1 & -1 \end{array}$$

$f(2) = -1$

30. $f(x) = -x^3 + 8x^2 + 63;\ k = 4$

$$\begin{array}{r|rrrr} 4 & -1 & 8 & 0 & 63 \\ & & -4 & 16 & 64 \\ \hline & -1 & 4 & 16 & 127 \end{array}$$

$f(4) = 127$

31. $f(x) = x^3 - 4x^2 + 2x + 1;\ k = -1$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 2 & 1 \\ & & -1 & 5 & -7 \\ \hline & 1 & -5 & 7 & -6 \end{array}$$

$f(-1) = -6$

32. $f(x) = 2x^3 - 3x^2 - 5x + 4;\ k = 2$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -5 & 4 \\ & & 4 & 2 & -6 \\ \hline & 2 & 1 & -3 & -2 \end{array}$$

$f(2) = -2$

33. $f(x) = 2x^5 - 10x^3 - 19x^2 - 50;\ k = 3$

$$\begin{array}{r|rrrrrr} 3 & 2 & 0 & -10 & -19 & 0 & -50 \\ & & 6 & 18 & 24 & 15 & 45 \\ \hline & 2 & 6 & 8 & 5 & 15 & -5 \end{array}$$

$f(3) = -5$

34. $f(x) = x^4 + 6x^3 + 9x^2 + 3x - 3;\ k = 4$

$$\begin{array}{r|rrrrr} 4 & 1 & 6 & 9 & 3 & -3 \\ & & 4 & 40 & 196 & 796 \\ \hline & 1 & 10 & 49 & 199 & 793 \end{array}$$

$f(4) = 793$

35. $f(x) = 6x^4 + x^3 - 8x^2 + 5x + 6;\ k = \frac{1}{2}$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 6 & 1 & -8 & 5 & 6 \\ & & 3 & 2 & -3 & 1 \\ \hline & 6 & 4 & -6 & 2 & 7 \end{array}$$

$f\left(\frac{1}{2}\right) = 7$

36. $f(x) = 6x^3 - 31x^2 - 15x;\ k = -\frac{1}{2}$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 6 & -31 & -15 & 0 \\ & & -3 & 17 & -1 \\ \hline & 6 & -34 & 2 & -1 \end{array}$$

$f\left(-\frac{1}{2}\right) = -1$

37. $f(x) = x^2 - 5x + 1;\ k = 2 + i$

$$\begin{array}{r|rrr} 2+i & 1 & -5 & 1 \\ & & 2+i & -7-i \\ \hline & 1 & -3+i & -6-i \end{array}$$

$f(2+i) = -6 - i$

38. $f(x) = x^2 - x + 3;\ k = 3 - 2i$

$$\begin{array}{r|rrr} 3-2i & 1 & -1 & 3 \\ & & 3-2i & 2-10i \\ \hline & 1 & 2-2i & 5-10i \end{array}$$

$f(3-2i) = 5 - 10i$

39. $f(x) = x^2 + 4; k = 2i$

$$\begin{array}{r|rrr} -2i & 1 & 0 & 4 \\ & & -2i & -4 \\ \hline & 1 & 0-2i & 0 \end{array}$$

$f(2i) = 0$

40. $f(x)=2x^2+10; k=\sqrt{5}i$

$$\begin{array}{r|rrr} \sqrt{5}i & 2 & 0 & 10 \\ & & 2\sqrt{5}i & -10 \\ \hline & 2 & 2\sqrt{5}i & 0 \end{array}$$

$f\left(\sqrt{5}i\right)=0$

41. To determine if $k=2$ is a zero of $f(x)=x^2+2x-8$, divide synthetically.

$$\begin{array}{r|rrr} 2 & 1 & 2 & -8 \\ & & 2 & 8 \\ \hline & 1 & 4 & 0 \end{array}$$

Yes, 2 is a zero of $f(x)$ because $f(2)=0$.

42. To determine if $k=-5$ is a zero of $f(x)=x^2+4x-5$, divide synthetically.

$$\begin{array}{r|rrr} -5 & 1 & 4 & -5 \\ & & -5 & 5 \\ \hline & 1 & -1 & 0 \end{array}$$

Yes, -5 is a zero of $f(x)$ because $f(-5)=0$.

43. To determine if $k=2$ is a zero of $f(x)=x^3-3x^2+4x-4$, divide synthetically.

$$\begin{array}{r|rrrr} 2 & 1 & -3 & 4 & -4 \\ & & 2 & -2 & 4 \\ \hline & 1 & -1 & 2 & 0 \end{array}$$

Yes, 2 is a zero of $f(x)$ because $f(2)=0$.

44. To determine if $k=-3$ is a zero of $f(x)=x^3+2x^2-x+6$, divide synthetically.

$$\begin{array}{r|rrrr} -3 & 1 & 2 & -1 & 6 \\ & & -3 & 3 & -6 \\ \hline & 1 & -1 & 2 & 0 \end{array}$$

Yes, -3 is a zero of $f(x)$ because $f(-3)=0$.

45. To determine if $k=1$ is a zero of $f(x)=2x^3-6x^2-9x+4$, divide synthetically.

$$\begin{array}{r|rrrr} 1 & 2 & -6 & -9 & 4 \\ & & 2 & -4 & -13 \\ \hline & 2 & -4 & -13 & -9 \end{array}$$

No, 1 is not a zero of $f(x)$ because $f(1)=-9$.

46. To determine if $k=1$ is a zero of $f(x)=2x^3+9x^2-16x+12$, divide synthetically.

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -16 & 12 \\ & & 2 & 11 & -5 \\ \hline & 2 & 11 & -5 & 7 \end{array}$$

No, 1 is not a zero of $f(x)$ because $f(1)=7$.

47. To determine if $k=0$ is a zero of $f(x)=x^3+7x^2+10x$, divide synthetically.

$$\begin{array}{r|rrrr} 0 & 1 & 7 & 10 & 0 \\ & & 0 & 0 & 0 \\ \hline & 1 & 7 & 2 & 0 \end{array}$$

Yes, 0 is a zero of $f(x)$ because $f(0)=0$.

48. To determine if $k=0$ is a zero of $f(x)=2x^3-3x^2-5x$, divide synthetically.

$$\begin{array}{r|rrrr} 0 & 2 & -3 & -5 & 0 \\ & & 0 & 0 & 0 \\ \hline & 2 & -3 & -5 & 0 \end{array}$$

Yes, 0 is a zero of $f(x)$ because $f(0)=0$.

49. To determine if $k=-\frac{3}{2}$ is a zero of $f(x)=2x^4+3x^3-8x^2-2x+15$, divide synthetically.

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 2 & 3 & -8 & -2 & 15 \\ & & -3 & 0 & 12 & -15 \\ \hline & 2 & 0 & -8 & 10 & 0 \end{array}$$

Yes, $-\frac{3}{2}$ is a zero of $f(x)$ because $f\left(-\frac{3}{2}\right)=0$.

50. To determine if $k=-\frac{4}{3}$ is a zero of $f(x)=3x^4+2x^3-5x+10$, divide synthetically.

$$\begin{array}{r|rrrrr} -\frac{4}{3} & 3 & 2 & 0 & -5 & 10 \\ & & -4 & \frac{8}{3} & -\frac{32}{9} & \frac{308}{27} \\ \hline & 3 & -2 & \frac{8}{3} & -\frac{77}{9} & \frac{578}{27} \end{array}$$

No, $-\frac{4}{3}$ is not a zero of $f(x)$ because $f\left(-\frac{4}{3}\right)=\frac{578}{27}$.

51. To determine if $k=\frac{2}{5}$ is a zero of $f(x)=5x^4+2x^3-x+3,$ divide synthetically.

$$\begin{array}{r|rrrrr} \frac{2}{5} & 5 & 2 & 0 & -1 & 3 \\ & & 2 & \frac{8}{5} & \frac{16}{25} & -\frac{18}{125} \\ \hline & 5 & 4 & \frac{8}{5} & -\frac{9}{25} & \frac{357}{125} \end{array}$$

No, $\frac{2}{5}$ is not a zero of $f(x)$ because $f\left(\frac{2}{5}\right)=\frac{357}{125}.$

52. To determine if $k=\frac{1}{2}$ is a zero of $f(x)=16x^4+4x^2-2,$ divide synthetically.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 16 & 0 & 4 & 0 & -2 \\ & & 8 & 4 & 4 & 2 \\ \hline & 16 & 8 & 8 & 4 & 0 \end{array}$$

Yes, $\frac{1}{2}$ is a zero of $f(x)$ because $f\left(\frac{1}{2}\right)=0.$

53. To determine if $k=1-i$ is a zero of $f(x)=x^2-2x+2,$ divide synthetically.

$$\begin{array}{r|rrr} 1-i & 1 & -2 & 2 \\ & & 1-i & -2 \\ \hline & 1 & -1-i & 0 \end{array}$$

Yes, $1-i$ is a zero of $f(x)$ because $f(1-i)=0.$

54. To determine if $k=2-i$ is a zero of $f(x)=x^2-4x+5,$ divide synthetically.

$$\begin{array}{r|rrr} 2-i & 1 & -4 & 5 \\ & & 2-i & -5 \\ \hline & 1 & -2-i & 0 \end{array}$$

Yes, $2-i$ is a zero of $f(x)$ because $f(2-i)=0.$

55. To determine if $k=2+i$ is a zero of $f(x)=x^2+3x+4,$ divide synthetically.

$$\begin{array}{r|rrr} 2+i & 1 & 3 & 4 \\ & & 2+i & 9+7i \\ \hline & 1 & 5+i & 13+7i \end{array}$$

No, $2+i$ is not a zero of $f(x)$ because $f\left(2+i\right)=13+7i.$

56. To determine if $k=1-2i$ is a zero of $f(x)=x^2-3x+5,$ divide synthetically.

$$\begin{array}{r|rrr} 1-2i & 1 & -3 & 5 \\ & & 1-2i & -6+2i \\ \hline & 1 & -2-2i & -1+2i \end{array}$$

No, $1-2i$ is not a zero of $f(x)$ because $f\left(1-2i\right)=-1+2i.$

57. To determine if $k=1+i$ is a zero of $f(x)=x^3+3x^2-x+1,$ divide synthetically.

$$\begin{array}{r|rrrr} 1+i & 1 & 3 & -1 & 1 \\ & & 1+i & 3+5i & -3+7i \\ \hline & 1 & 4+i & 2+5i & -2+7i \end{array}$$

No, $1+i$ is not a zero of $f(x)$ because $f\left(1+i\right)=-2+7i.$

58. To determine if $k=2-i$ is a zero of $f(x)=2x^3-x^2+3x-5,$ divide synthetically.

$$\begin{array}{r|rrrr} 2-i & 2 & -1 & 3 & -5 \\ & & 4-2i & 4-7i & 7-21i \\ \hline & 2 & 3-2i & 7-7i & 2-21i \end{array}$$

No, $2-i$ is not a zero of $f(x)$ because $f\left(2-i\right)=2-21i.$

59.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -1 & 2 \\ & & -2 & 8 & -14 \\ \hline & 1 & -4 & 7 & -12 \end{array}$$

$f(-2)=-12$

The coordinates of the corresponding point are $(-2,-12).$

60.

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -1 & 2 \\ & & -1 & 3 & -2 \\ \hline & 1 & -3 & 2 & 0 \end{array}$$

$f(-1)=0$

The coordinates of the corresponding point are $(-1, 0).$

61.

$$\begin{array}{r|rrrr} 0 & 1 & -2 & -1 & 2 \\ & & 0 & 0 & 0 \\ \hline & 1 & -2 & -1 & 2 \end{array}$$

$f(0)=2$

The coordinates of the corresponding point are $(0, 2).$

62. $\begin{array}{r|rrrr} 1 & 1 & -2 & -1 & 2 \\ & & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$

$f(1) = 0$

The coordinates of the corresponding point are (1, 0).

63. $\begin{array}{r|rrrr} \frac{3}{2} & 1 & -2 & -1 & 2 \\ & & \frac{3}{2} & -\frac{3}{4} & -\frac{21}{8} \\ \hline & 1 & -\frac{1}{2} & -\frac{7}{4} & -\frac{5}{8} \end{array}$

$f\left(\frac{3}{2}\right) = -\frac{5}{8}$

The coordinates of the corresponding point are $\left(\frac{3}{2}, -\frac{5}{8}\right)$.

64. $\begin{array}{r|rrrr} 2 & 1 & -2 & -1 & 2 \\ & & 2 & 0 & -2 \\ \hline & 1 & 0 & -1 & 0 \end{array}$

$f(2) = 0$

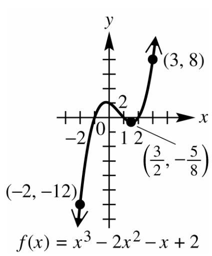
The coordinates of the corresponding point are (2, 0).

65. $\begin{array}{r|rrrr} 3 & 1 & -2 & -1 & 2 \\ & & 3 & 3 & 6 \\ \hline & 1 & 1 & 2 & 8 \end{array}$

$f(3) = 8$

The coordinates of the corresponding point are (3, 8).

66.



Section 3.3: Zeros of Polynomial Functions

Connections (page 336)

1. $x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{\frac{-n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$

   can be used to solve a cubic equation $x^3 + mx = n$.

   For $x^3 + 9x = 26$, $m = 9$ and $n = 26$.

   Substitute

   $$\begin{aligned} x &= \sqrt[3]{\frac{26}{2} + \sqrt{\left(\frac{26}{2}\right)^2 + \left(\frac{9}{3}\right)^3}} - \sqrt[3]{-\frac{26}{2} + \sqrt{\left(\frac{26}{2}\right)^2 + \left(\frac{9}{3}\right)^3}} \\ &= \sqrt[3]{13 + \sqrt{13^2 + 3^3}} - \sqrt[3]{-13 + \sqrt{13^2 + 3^3}} \\ &= \sqrt[3]{13 + \sqrt{169 + 27}} - \sqrt[3]{-13 + \sqrt{169 + 27}} \\ &= \sqrt[3]{13 + 14} - \sqrt[3]{-13 + 14} \\ &= \sqrt[3]{27} - \sqrt[3]{1} = 3 - 1 = 2 \end{aligned}$$

Exercises

1. Since $x - 1$ is a factor of $f(x) = x^6 - x^4 + 2x^2 - 2$, we are assured that $f(1) = 0$. This statement is justified by the factor theorem; therefore, it is true.

2. Since $f(1) = 0$ for $f(x) = x^6 - x^4 + 2x^2 - 2$, we are assured that $x - 1$ is a factor of $f(x)$. This statement is justified by the factor theorem; therefore, it is true.

3. For the function $f(x) = (x+2)^4(x-3)$, 2 is a zero of multiplicity 4. To find the zero, set the factor equal to 0: $x + 2 = 0 \Rightarrow x = -2$
   2 is not a zero of the function; therefore, the statement is false. (It would be true to say that –2 is a zero of multiplicity 4.)

4. Since $2 + 3i$ is a zero of $f(x) = x^2 - 4x + 13$, we are assured that $2 - 3i$ is also a zero. By the conjugate zeros theorem, if a complex number is a zero of $f(x)$, then its conjugate is also a zero of $f(x)$. Therefore, this statement is true.

5. $x^3 - 5x^2 + 3x + 1$; $x - 1$

   Let $f(x) = x^3 - 5x^2 + 3x + 1$. By the factor theorem, $x - 1$ will be a factor of $f(x)$ if and only if $f(1) = 0$. Use synthetic division and the remainder theorem.

   $\begin{array}{r|rrrr} 1 & 1 & -5 & 3 & 1 \\ & & 1 & -4 & -1 \\ \hline & 1 & -4 & -1 & 0 \end{array}$

   Since $f(1) = 0$, $x - 1$ is a factor of $f(x)$.

6. $x^3 + 6x^2 - 2x - 7$; $x + 1$

   Let $f(x) = x^3 + 6x^2 - 2x - 7$. By the factor theorem, $x + 1$ will be a factor of $f(x)$ if and only if $f(-1) = 0$. Use synthetic division and the remainder theorem.

(*continued on next page*)

(*continued from page 279*)

$$\begin{array}{r|rrrr} -1 & 1 & 6 & -2 & -7 \\ & & -1 & -5 & 7 \\ \hline & 1 & 5 & -7 & 0 \end{array}$$

Since $f(-1) = 0$, $x+1$ is a factor of $f(x)$.

7. $2x^4 + 5x^3 - 8x^2 + 3x + 13;\ x+1$

Let $f(x) = 2x^4 + 5x^3 - 8x^2 + 3x + 13$. By the factor theorem, $x + 1$ will be a factor of $f(x)$ if and only if $f(-1) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrrr} -1 & 2 & 5 & -8 & 3 & 13 \\ & & -2 & -3 & 11 & -14 \\ \hline & 2 & 3 & -11 & 14 & -1 \end{array}$$

Since the remainder is -1, $f(-1) = -1$, so $x + 1$ is not a factor of $f(x)$.

8. $-3x^4 + x^3 - 5x^2 + 2x + 4;\ x-1$

Let $f(x) = -3x^4 + x^3 - 5x^2 + 2x + 4$. By the factor theorem, $x - 1$ will be a factor of $f(x)$ if and only if $f(1) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrrr} 1 & -3 & 1 & -5 & 2 & 4 \\ & & -3 & -2 & -7 & -5 \\ \hline & -3 & -2 & -7 & -5 & -1 \end{array}$$

Since the remainder is -1, $f(1) = -1$, so $x - 1$ is not a factor of $f(x)$.

9. $-x^3 + 3x - 2;\ x+2$

Let $f(x) = -x^3 + 3x - 2$. By the factor theorem, $x + 2$ will be a factor of $f(x)$ if and only if $f(-2) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrr} -2 & -1 & 0 & 3 & -2 \\ & & 2 & -4 & 2 \\ \hline & -1 & 2 & -1 & 0 \end{array}$$

Since $f(-2) = 0$, $x+2$ is a factor of $f(x)$.

10. $-2x^3 + x^2 - 63;\ x+3$

Let $f(x) = -2x^3 + x^2 - 63$. By the factor theorem, $x + 3$ will be a factor of $f(x)$ if and only if $f(-3) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrr} -3 & -2 & 1 & 0 & -63 \\ & & 6 & -21 & 63 \\ \hline & -2 & 7 & -21 & 0 \end{array}$$

Since $f(-3) = 0$, $x+3$ is a factor of $f(x)$.

11. $4x^2 + 2x + 54;\ x-4$

Let $f(x) = 4x^2 + 2x + 54$. By the factor theorem, $x - 4$ will be a factor of $f(x)$ if and only if $f(4) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrr} 4 & 4 & 2 & 54 \\ & & 16 & 72 \\ \hline & 4 & 18 & 126 \end{array}$$

Since the remainder is 126, $f(4) = 126$, so $x - 4$ is not a factor of $f(x)$.

12. $5x^2 - 14x + 10;\ x+2$

Let $f(x) = 5x^2 - 14x + 10$. By the factor theorem, $x + 2$ will be a factor of $f(x)$ if and only if $f(-2) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrr} -2 & 5 & -14 & 10 \\ & & -10 & 48 \\ \hline & 5 & -24 & 58 \end{array}$$

Since the remainder is 58, $f(-2) = 58$, so $x+2$ is not a factor of $f(x)$.

13. $x^3 + 2x^2 - 3;\ x-1$

Let $f(x) = x^3 + 2x^2 - 1$. By the factor theorem, $x - 1$ will be a factor of $f(x)$ if and only if $f(1) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrr} 1 & 1 & 2 & 0 & -3 \\ & & 1 & 3 & 3 \\ \hline & 1 & 3 & 3 & 0 \end{array}$$

Since $f(1) = 0$, $x-1$ is a factor of $f(x)$.

14. $2x^3 + x + 2;\ x+1$

Let $f(x) = 2x^3 + x + 2$. By the factor theorem, $x + 1$ will be a factor of $f(x)$ if and only if $f(-1) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrr} -1 & 2 & 0 & 1 & 2 \\ & & -2 & 2 & -3 \\ \hline & 2 & -2 & 3 & -1 \end{array}$$

Since $f(-1) = -1$, $x + 1$ is not a factor of $f(x)$.

15. $2x^4 + 5x^3 - 2x^2 + 5x + 6;\ x + 3$

Let $f(x) = 2x^4 + 5x^3 - 2x^2 + 5x + 6$. By the factor theorem, $x + 3$ will be a factor of $f(x)$ if and only if $f(-3) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrrr} -3 & 2 & 5 & -2 & 5 & 6 \\ & & -6 & 3 & -3 & -6 \\ \hline & 2 & -1 & 1 & 2 & 0 \end{array}$$

Since $f(-3) = 0$, $x + 3$ is a factor of $f(x)$.

16. $5x^4 + 16x^3 - 15x^2 + 8x + 16;\ x + 4$

Let $f(x) = 5x^4 + 16x^3 - 15x^2 + 8x + 16$. By the factor theorem, $x + 4$ will be a factor of $f(x)$ if and only if $f(-4) = 0$. Use synthetic division and the remainder theorem.

$$\begin{array}{r|rrrrr} -4 & 5 & 16 & -15 & 8 & 16 \\ & & -20 & 16 & -4 & -16 \\ \hline & 5 & -4 & 1 & 4 & 0 \end{array}$$

Since $f(-4) = 0$, $x + 4$ is a factor of $f(x)$.

17. $f(x) = 2x^3 - 3x^2 - 17x + 30;\ k = 2$

Since 2 is a zero of $f(x)$, $x - 2$ is a factor.

Divide $f(x)$ by $x - 2$.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & 0 \end{array}$$

Thus, $f(x) = (x - 2)(2x^2 + x - 15)$
$= (x - 2)(2x - 5)(x + 3)$.

18. $f(x) = 2x^3 - 3x^2 - 5x + 6;\ k = 1$

Since 1 is a zero of $f(x)$, $x - 1$ is a factor.
Divide $f(x)$ by $x - 1$.

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -5 & 6 \\ & & 2 & -1 & -6 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

Thus, $f(x) = (x - 1)(2x^2 - x - 6)$
$= (x - 1)(2x + 3)(x - 2)$

19. $f(x) = 6x^3 + 13x^2 - 14x + 3;\ k = -3$

Since -3 is a zero of $f(x)$, $x + 3$ is a factor.

Divide $f(x)$ by $x + 3$.

$$\begin{array}{r|rrrr} -3 & 6 & 13 & -14 & 3 \\ & & -18 & 15 & -3 \\ \hline & 6 & -5 & 1 & 0 \end{array}$$

Thus, $f(x) = (x + 3)(6x^2 - 5x + 1)$
$= (x + 3)(3x - 1)(2x - 1)$

20. $f(x) = 6x^3 + 17x^2 - 63x + 10;\ k = -5$

Since -5 is a zero of $f(x)$, $x + 5$ is a factor.

Divide $f(x)$ by $x + 5$.

$$\begin{array}{r|rrrr} -5 & 6 & 17 & -63 & 10 \\ & & -30 & 65 & -10 \\ \hline & 6 & -13 & 2 & 0 \end{array}$$

Thus, $f(x) = (x + 5)(6x^2 - 13x + 2)$
$= (x + 5)(6x - 1)(x - 2)$

21. $f(x) = 6x^3 + 25x^2 + 3x - 4;\ k = -4$

Since -4 is a zero of $f(x)$, $x + 4$ is a factor.

Divide $f(x)$ by $x + 4$.

$$\begin{array}{r|rrrr} -4 & 6 & 25 & 3 & -4 \\ & & -24 & -4 & 4 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

Thus, $f(x) = (x + 4)(6x^2 + x - 1)$
$= (x + 4)(3x - 1)(2x + 1)$

22. $f(x) = 8x^3 + 50x^2 + 47x - 15;\ k = -5$

Since -5 is a zero of $f(x)$, $x + 5$ is a factor.

Divide $f(x)$ by $x + 5$.

$$\begin{array}{r|rrrr} -5 & 8 & 50 & 47 & -15 \\ & & -40 & -50 & 15 \\ \hline & 8 & 10 & -3 & 0 \end{array}$$

Thus, $f(x) = (x + 5)(8x^2 + 10x - 3)$
$= (x + 5)(4x - 1)(2x + 3)$.

23. $f(x) = x^3 + (7 - 3i)x^2 + (12 - 21i)x - 36i$
$k = 3i$

Since $3i$ is a zero of $f(x)$, $x - 3i$ is a factor.

Divide $f(x)$ by $x - 3i$.

$$\begin{array}{r|rrrr} 3i & 1 & 7-3i & 12-21i & -36i \\ & & 3i & 21i & 36i \\ \hline & 1 & 7 & 12 & 0 \end{array}$$

Thus, $f(x) = (x - 3i)(x^2 + 7x + 12)$
$= (x - 3i)(x + 4)(x + 3)$

24. $f(x) = 2x^3 + (3+2i)x^2 + (1+3i)x + i;\ k = -i$
Since $-i$ is a zero of $f(x)$, $x+i$ is a factor.
Divide $f(x)$ by $x+i$.

$$\begin{array}{r|rrrr} -i & 2 & 3+2i & 1+3i & i \\ & & -2i & -3i & -i \\ \hline & 2 & 3 & 1 & 0 \end{array}$$

Thus, $f(x) = (x+i)(2x^2+3x+1)$
$= (x+i)(2x+1)(x+1)$

25. $f(x) = 2x^3 + (3-2i)x^2 + (-8-5i)x + (3+3i)$
$k = 1+i$
Since $1+i$ is a zero of $f(x)$, $x-(1+i)$ is a factor. Divide $f(x)$ by $x-(1+i)$.

$$\begin{array}{r|rrrr} 1+i & 2 & 3-2i & -8-5i & 3+3i \\ & & 2+2i & 5+5i & -3-3i \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Thus, $f(x) = \left[x-(1+i)\right](2x^2+5x-3)$
$= \left[x-(1+i)\right](2x-1)(x+3)$

26. $f(x) = 6x^3 + (19-6i)x^2 + (16-7i)x + (4-2i)$
$k = -2+i$
Since $-2+i$ is a zero of $f(x)$, $x-(-2+i)$ is a factor. Divide $f(x)$ by $x-(-2+i)$.

$$\begin{array}{r|rrrr} -2+i & 6 & 19-6i & 16-7i & 4-2i \\ & & -12+6i & -14+7i & -4+2i \\ \hline & 6 & 7 & 2 & 0 \end{array}$$

Thus, $f(x) = \left[x-(-2+i)\right](6x^2+7x+2)$
$= \left[x-(-2+i)\right](3x+2)(2x+1)$

27. $f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12;\ k = -2$
(multiplicity 2)
Since -2 is a zero of $f(x)$, $x+2$ is a factor.
Divide $f(x)$ by $x+2$.

$$\begin{array}{r|rrrrr} -2 & 1 & 2 & -7 & -20 & -12 \\ & & -2 & 0 & 14 & 12 \\ \hline & 1 & 0 & -7 & -6 & 0 \end{array}$$

Thus, $f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12$
$= (x+2)(x^3 - 7x - 6)$

Since -2 has multiplicity 2, divide the quotient polynomial by $x+2$.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -7 & -6 \\ & & -2 & 4 & 6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

Thus, $f(x) = (x+2)(x^3 - 7x - 6)$
$= (x+2)^2(x^2 - 2x - 3)$
$= (x+2)^2(x+1)(x-3)$

28. $f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5;\ k = -1$
(multiplicity 3)
Since -1 is a zero of $f(x)$, $x+1$ is a factor.
Divide $f(x)$ by $x+1$.

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -9 & -13 & -5 \\ & & -2 & 1 & 8 & 5 \\ \hline & 2 & -1 & -8 & -5 & 0 \end{array}$$

$f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5$
$= (x+1)(2x^3 - x^2 - 8x - 5)$

Since -1 has multiplicity 3, divide the quotient polynomial by $x+1$.

$$\begin{array}{r|rrrr} -1 & 2 & -1 & -8 & -5 \\ & & -2 & 3 & 5 \\ \hline & 2 & -3 & -5 & 0 \end{array}$$

$f(x) = (x+1)(2x^3 - x^2 - 8x - 5)$
$= (x+1)^2(2x^2 - 3x - 5)$
$= (x+1)^2(x+1)(2x-5)$
$= (x+1)^3(2x-5)$

29. $f(x) = x^3 - x^2 - 4x - 6;\ 3$
Since 3 is a zero, first divide $f(x)$ by $x - 3$.

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -4 & -6 \\ & & 3 & 6 & 6 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

This gives $f(x) = (x-3)(x^2+2x+2)$. Since x^2+2x+2 cannot be factored, use the quadratic formula with $a = 1$, $b = 2$, and $c = 2$ to find the remaining two zeros.

$$x = \frac{-2 \pm \sqrt{4-4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{4-8}}{2}$$
$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

The remaining zeros are $-1 \pm i$.

30. $f(x)=x^3+4x^2-5;\ 1$

Since 1 is a zero, first divide $f(x)$ by $x-1$.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 0 & -5 \\ & & 1 & 5 & 5 \\ \hline & 1 & 5 & 5 & 0 \end{array}$$

Thus we have $f(x)=(x-1)(x^2+5x+5)$.

Since x^2+5x+5 cannot be factored, use the quadratic formula with $a = 1$, $b = 5$, and $c = 5$ to find the remaining two zeros.

$$x=\frac{-5\pm\sqrt{25-4(1)(5)}}{2}=\frac{-5\pm\sqrt{25-20}}{2}=\frac{-5\pm\sqrt{5}}{2}$$

The remaining zeros are $\frac{-5\pm\sqrt{5}}{2}$.

31. $f(x)=x^3-7x^2+17x-15;\ 2-i$

Since $2-i$ is also a zero, first divide $f(x)$ by $x-(2-i)$.

$$\begin{array}{r|rrrr} 2-i & 1 & -7 & 17 & -15 \\ & & 2-i & -11+3i & 15 \\ \hline & 1 & -5-i & 6+3i & 0 \end{array}$$

By the conjugate zeros theorem, $2+i$ is also a zero, so divide the quotient polynomial from the first synthetic division by $x-(2+i)$.

$$\begin{array}{r|rrr} 2+i & 1 & -5-i & 6+3i \\ & & 2+i & -6-3i \\ \hline & 1 & -3 & 0 \end{array}$$

This gives $f(x)=[x-(2-i)][x-(2+i)](x-3)$. The remaining zeros are $2+i$ and 3.

32. $f(x)=4x^3+6x^2-2x-1;\ \frac{1}{2}$

Since $\frac{1}{2}$ is a zero, first divide $f(x)$ by $x-\frac{1}{2}$.

$$\begin{array}{r|rrrr} \frac{1}{2} & 4 & 6 & -2 & -1 \\ & & 2 & 4 & 1 \\ \hline & 4 & 8 & 2 & 0 \end{array}$$

This gives
$$\begin{aligned} f(x)&=\left(x-\tfrac{1}{2}\right)\left(4x^2+8x+2\right)\\ &=\left(x-\tfrac{1}{2}\right)\cdot 2\cdot(2x^2+4x+1)\\ &=(2x-1)(2x^2+4x+1) \end{aligned}$$

Since $2x^2+4x+1$ cannot be factored, use the quadratic formula with $a = 2$, $b = 4$, and $c = 1$ to find the other two zeros.

$$x=\frac{-4\pm\sqrt{16-4(2)(1)}}{2(2)}=\frac{-4\pm\sqrt{16-8}}{4}=\frac{-4\pm\sqrt{8}}{4}=\frac{-4\pm2\sqrt{2}}{4}=\frac{-2\pm\sqrt{2}}{2}$$

The remaining zeros are $\frac{-2\pm\sqrt{2}}{2}$.

33. $f(x)=x^4+5x^2+4;\ -i$

Since $-i$ is a zero, first divide $f(x)$ by $x+i$.

$$\begin{array}{r|rrrrr} -i & 1 & 0 & 5 & 0 & 4 \\ & & -i & -1 & -4i & -4 \\ \hline & 1 & -i & 4 & -4i & 0 \end{array}$$

By the conjugate zeros theorem, i is also a zero, so divide the quotient polynomial from the first synthetic division by $x-i$.

$$\begin{array}{r|rrrr} i & 1 & -i & 4 & -4i \\ & & i & 0 & 4i \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

The remaining zeros will be zeros of the new quotient polynomial, x^2+4. Find the remaining zeros by using the square root property.

$$x^2+4=0\Rightarrow x^2=-4\Rightarrow x=\pm\sqrt{-4}\Rightarrow x=\pm 2i$$

The other zeros are i and $\pm 2i$.

34. $f(x)=x^4+10x^3+27x^2+10x+26;\ i$

Since i is a zero, first divide $f(x)$ by $x-i$.

$$\begin{array}{r|rrrrr} i & 1 & 10 & 27 & 10 & 26 \\ & & i & -1+10i & -10+26i & -26 \\ \hline & 1 & 10+i & 26+10i & 26i & 0 \end{array}$$

By the conjugate zeros theorem, $-i$ is also a zero, so divide the quotient polynomial from the first synthetic division by $x+i$.

$$\begin{array}{r|rrrr} -i & 1 & 10+i & 26+10i & 26i \\ & & -i & -10i & -26i \\ \hline & 1 & 10 & 26 & 0 \end{array}$$

The remaining zeros will be the zeros of the new quotient polynomial, $x^2+10x+26$. Since $x^2+10x+26$ cannot be factored, use the quadratic formula to find these zeros.

$$x=\frac{-10\pm\sqrt{100-4(1)(26)}}{2(1)}=\frac{-10\pm\sqrt{100-104}}{2}=\frac{-10\pm\sqrt{-4}}{2}=\frac{-10\pm2i}{2}=-5\pm i$$

The other zeros are $-i$ and $-5\pm i$.

35. **(a)** $f(x) = x^3 - 2x^2 - 13x - 10$

p must be a factor of $a_0 = -10$ and q must be a factor of $a_3 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 5, \pm 10$ and q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are ± 1, $\pm 2, \pm 5, \pm 10$.

(b) The remainder theorem shows that –1 is a zero.

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -13 & -10 \\ & & -1 & 3 & 10 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

The new quotient polynomial will be $x^2 - 3x - 10$.

$x^2 - 3x - 10 = 0$
$(x+2)(x-5) = 0$
$x + 2 = 0 \Rightarrow x = -2$ or $x - 5 = 0 \Rightarrow x = 5$

The rational zeros are –1, –2, and 5.

(c) Since the three zeros are –1, –2, and 5, the factors are $x + 1$, $x + 2$, and $x - 5$.
$f(x) = (x+1)(x+2)(x-5)$

36. **(a)** $f(x) = x^3 + 5x^2 + 2x - 8$

p must be a factor of $a_0 = -8$ and q must be a factor of $a_3 = 1$. Thus, p can be ± 1, $\pm 2, \pm 4, \pm 8$ and q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2$, $\pm 4, \pm 8$.

(b) The remainder theorem shows that – 4 is a zero.

$$\begin{array}{r|rrrr} -4 & 1 & 5 & 2 & -8 \\ & & -4 & -4 & 8 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

The new quotient polynomial will be $x^2 + x - 2$.

$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$x + 2 = 0 \Rightarrow x = -2$ or $x - 1 = 0 \Rightarrow x = 1$

The rational zeros are – 4, –2, and 1.

(c) Since the three zeros are – 4, –2, and 1, the factors are $x + 4$, $x + 2$, and $x - 1$.
$f(x) = (x+4)(x+2)(x-1)$

37. **(a)** $f(x) = x^3 + 6x^2 - x - 30$

p must be a factor of $a_0 = -30$ and q must be a factor of $a_3 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15$, ± 30 and q can be ± 1. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10$, $\pm 15, \pm 30$.

(b) The remainder theorem shows that –5 is a zero.

$$\begin{array}{r|rrrr} -5 & 1 & 6 & -1 & -30 \\ & & -5 & -5 & 30 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

The new quotient polynomial will be $x^2 + x - 6$.

$x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$x + 3 = 0 \Rightarrow x = -3$ or $x - 2 = 0 \Rightarrow x = 2$

The rational zeros are –5, –3, and 2.

(c) Since the three zeros are –5, –3, and 2, the factors are $x + 5$, $x + 3$, and $x - 2$.
$f(x) = (x+5)(x+3)(x-2)$

38. **(a)** $f(x) = x^3 - x^2 - 10x - 8$

p must be a factor of $a_0 = -8$ and q must be a factor of $a_3 = 1$. Thus, p can be ± 1, $\pm 2, \pm 4, \pm 8$ and q can be ± 1. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 4, \pm 8$.

(b) The remainder theorem shows that –2 is a zero.

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -10 & -8 \\ & & -2 & 6 & 8 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

The new quotient polynomial will be $x^2 - 3x - 4$.

$x^2 - 3x - 4 = 0$
$(x+1)(x-4) = 0$
$x + 1 = 0 \Rightarrow x = -1$ or $x - 4 = 0 \Rightarrow x = 4$

The rational zeros are –2, –1, and 4.

(c) Since the three zeros are –2, –1, and 4, the factors are $x + 2$, $x + 1$, and $x - 4$.
$f(x) = (x+2)(x+1)(x-4)$

39. **(a)** $f(x) = 6x^3 + 17x^2 - 31x - 12$

p must be a factor of $a_0 = -12$ and q must be a factor of $a_3 = 6$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ and q can be $\pm 1, \pm 2, \pm 3, \pm 6$. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{1}{6}$.

(b) The remainder theorem shows that -4 is a zero.

$$\begin{array}{r|rrrr} -4 & 6 & 17 & -31 & -12 \\ & & -24 & 28 & 12 \\ \hline & 6 & -7 & -3 & 0 \end{array}$$

The new quotient polynomial is $6x^2 - 7x - 3$.

$6x^2 - 7x - 3 = 0 \Rightarrow (3x+1)(2x-3) = 0$

$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$ or

$2x - 3 = 0 \Rightarrow x = \frac{3}{2}$

The rational zeros are $-4, -\frac{1}{3}$, and $\frac{3}{2}$.

(c) Since the three zeros are $-4, -\frac{1}{3}$, and $\frac{3}{2}$, the factors are $x + 4$, $3x + 1$, and $2x - 3$. $f(x) = (x+4)(3x+1)(2x-3)$

40. **(a)** $f(x) = 15x^3 + 61x^2 + 2x - 8$

p must be a factor of $a_0 = -8$ and q must be a factor of $a_3 = 15$. Thus, p can be $\pm 1, \pm 2, \pm 4, \pm 8$ and q can be $\pm 1, \pm 3, \pm 5, \pm 15$. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm\frac{8}{5}, \pm\frac{1}{15}, \pm\frac{2}{15}, \pm\frac{4}{15}, \pm\frac{8}{15}$.

(b) The remainder theorem shows that -4 is zero.

$$\begin{array}{r|rrrr} -4 & 15 & 61 & 2 & -8 \\ & & -60 & -4 & 8 \\ \hline & 15 & 1 & -2 & 0 \end{array}$$

The new quotient polynomial is $15x^2 + x - 2$.

$15x^2 + x - 2 = 0 \Rightarrow (5x+2)(3x-1) = 0$

$5x + 2 = 0$ or $3x - 1 = 0$

$x = -\frac{2}{5}$ $\quad$ $x = \frac{1}{3}$

The rational zeros are $-4, -\frac{2}{5}$, and $\frac{1}{3}$.

(c) Since the three zeros are $-4, -\frac{2}{5}$, and $\frac{1}{3}$, the factors are $x + 4$, $5x + 2$, and $3x - 1$. $f(x) = (x+4)(5x+2)(3x-1)$

41. **(a)** $f(x) = 24x^3 + 40x^2 - 2x - 12$

p must be a factor of $a_0 = -12$ and q must be a factor of $a_3 = 24$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ and q can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{6}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{12}, \pm\frac{1}{24}$.

(b) The remainder theorem shows that $-\frac{3}{2}$ is a zero.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 24 & 40 & -2 & -12 \\ & & -36 & -6 & 12 \\ \hline & 24 & 4 & -8 & 0 \end{array}$$

The new quotient polynomial is $24x^2 + 4x - 8$.

$24x^2 + 4x - 8 = 0 \Rightarrow 4(6x^2 + x - 2) = 0$
$\Rightarrow 4(3x+2)(2x-1) = 0$

$3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$ or

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

The rational zeros are $-\frac{3}{2}, -\frac{2}{3}$, and $\frac{1}{2}$.

(c) Since the three rational zeros are $-\frac{3}{2}, -\frac{2}{3}$, and $\frac{1}{2}$, the factors are $3x + 2$, $2x + 3$, and $2x - 1$.

$f(x) = 2(2x+3)(3x+2)(2x-1)$

Note: Since $-\frac{3}{2}$ is a zero,

$$\begin{aligned} f(x) &= 24x^3 + 40x^2 - 2x - 12 \\ &= \left[x - \left(-\tfrac{3}{2}\right)\right]\left(24x^2 + 4x - 8\right) \\ &= \left(x + \tfrac{3}{2}\right)\left[4(6x^2 + x + 2)\right] \\ &= 4\left(x + \tfrac{3}{2}\right)(3x+2)(2x-1) \\ &= 2\left[2\left(x + \tfrac{3}{2}\right)\right](3x+2)(2x-1) \\ &= 2(2x+3)(3x+2)(2x-1) \end{aligned}$$

42. (a) $f(x) = 24x^3 + 80x^2 + 82x + 24$
p must be a factor of $a_0 = 24$ and q must be a factor of $a_3 = 24$. Thus, p can be $\pm 1, \pm 2, \pm 3, 4, \pm 6, \pm 8, \pm 12, \pm 24,$ and q can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{6}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{12}, \pm\frac{1}{24}$.

(b) The remainder theorem shows that $-\frac{3}{2}$ is a zero.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 24 & 80 & 82 & 24 \\ & & -36 & -66 & -24 \\ \hline & 24 & 44 & 16 & 0 \end{array}$$

The new quotient polynomial is $24x^2 + 44x + 16$.

$$24x^2 + 44x + 16 = 0$$
$$4(6x^2 + 11x + 4) = 0$$
$$4(3x + 4)(2x + 1) = 0$$
$$3x + 4 = 0 \Rightarrow x = -\frac{4}{3} \quad \text{or}$$
$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

The rational zeros are $-\frac{3}{2}, -\frac{4}{3}$, and $-\frac{1}{2}$.

(c) Since the three zeros are $-\frac{3}{2}, -\frac{4}{3}$, and $-\frac{1}{2}$, the factors are $2x + 3$, $3x + 4$, and $2x + 1$.
$f(x) = 2(2x + 3)(3x + 4)(2x + 1)$

Note: Since $-\frac{3}{2}$ is a zero,

$$\begin{aligned} f(x) &= 24x^3 + 80x^2 + 82x + 24 \\ &= \left[x - \left(-\tfrac{3}{2}\right)\right]\left(24x^2 + 44x + 16\right) \\ &= \left(x + \tfrac{3}{2}\right)\left[4(6x^2 + 11x + 4)\right] \\ &= 4\left(x + \tfrac{3}{2}\right)(3x + 4)(2x + 1) \\ &= 2\left[2\left(x + \tfrac{3}{2}\right)\right](3x + 4)(2x + 1) \\ &= 2(2x + 3)(3x + 4)(2x + 1) \end{aligned}$$

43. $f(x) = 7x^3 + x$
To find the zeros, let $f(x) = 0$ and factor the binomial. $7x^3 + x = 0 \Rightarrow x(7x^2 + 1) = 0$
Set each factor equal to zero and solve for x.

$$x = 0 \quad \text{or} \quad 7x^2 + 1 = 0$$
$$7x^2 = -1$$
$$x^2 = -\frac{1}{7}$$
$$x = \pm\sqrt{-\frac{1}{7}} = \pm\frac{\sqrt{7}}{7}$$

The zeros are 0 and $\pm\frac{\sqrt{7}}{7}i$.

44. $f(x) = (x + 1)^2(x - 1)^3(x^2 - 10)$
To find the zeros, let $f(x) = 0$.
Set each factor equal to zero and solve for x.
$(x + 1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$, multiplicity 2
$(x - 1)^3 = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$, multiplicity 3
$x^2 - 10 = 0 \Rightarrow x^2 = 10 \Rightarrow x = \pm\sqrt{10}$
The zeros are –1 (multiplicity 2), 1 (multiplicity 3), and $\pm\sqrt{10}$.

45. $f(x) = 3(x - 2)(x + 3)(x^2 - 1)$
To find the zeros, let $f(x) = 0$.
Set each factor equal to zero and solve for x.
$x - 2 = 0 \Rightarrow x = 2$
$x + 3 = 0 \Rightarrow x = -3$
$x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$
The zeros are 2, –3, 1, and –1.

46. $f(x) = 5x^2\left(x + 1 - \sqrt{2}\right)(2x + 5) = 0$
To find the zeros, let $f(x) = 0$. Set each factor equal to zero and solve for x.
$5x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$, multiplicity 2
$x + 1 - \sqrt{2} = 0 \Rightarrow x = -1 + \sqrt{2}$
$2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$
The zeros are 0 (multiplicity 2), $-1 + \sqrt{2}$, and $-\frac{5}{2}$.

47. $f(x) = (x^2 + x - 2)^5(x - 1 + \sqrt{3})^2$
To find the zeros, let $f(x) = 0$.
Set each factor equal to zero and solve for x.
$(x^2 + x - 2)^5 = 0 \Rightarrow x^2 + x - 2 = 0 \Rightarrow$
$(x + 2)(x - 1) = 0 \Rightarrow$
$x = -2$, multiplicity 5 or $x = 1$, multiplicity 5
$(x - 1 + \sqrt{3})^2 = 0 \Rightarrow x - 1 + \sqrt{3} = 0 \Rightarrow$
$x = 1 - \sqrt{3}$, multiplicity 2
The zeros are –2 (multiplicity 5), 1 (multiplicity 5) and $1 - \sqrt{3}$ (multiplicity 2).

48. $f(x) = (7x-2)^3(x^2+9)^2 = 0$
To find the zeros, let $f(x) = 0$.
Set each factor equal to zero and solve for x.
$(7x-2)^3 = 0 \Rightarrow 7x-2 = 0 \Rightarrow$
$x = \frac{2}{7}$, multiplicity 3
$(x^2+9)^2 = 0 \Rightarrow x^2+9 = 0 \Rightarrow$
$x^2 = -9 \Rightarrow x = \pm\sqrt{-9} = \pm 3i$, multiplicity 2
The zeros are $\frac{2}{7}$ (multiplicity 3), $3i$ (multiplicity 2), and $-3i$ (multiplicity 2).

49. Zeros of –3, 1, and 4, $f(2) = 30$
These three zeros give $x-(-3) = x+3$, $x-1$, and $x-4$ as factors of $f(x)$. Since $f(x)$ is to be degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form $f(x) = a(x+3)(x-1)(x-4)$ for some real number a. To find a, use the fact that $f(2) = 30$.
$f(2) = a(2+3)(2-1)(2-4) = 30 \Rightarrow$
$a(5)(1)(-2) = 30 \Rightarrow -10a = 30 \Rightarrow$
$a = -3$
Thus,
$$\begin{aligned} f(x) &= -3(x+3)(x-1)(x-4) \\ &= -3(x^2+2x-3)(x-4) \\ &= -3(x^3-2x^2-11x+12) \\ &= -3x^3+6x^2+33x-36 \end{aligned}$$

50. Zeros of 1, –1, and 0; $f(2) = 3$
These three zeros give $x-1$, $x-(-1) = x+1$, and $x-0 = x$ as factors of $f(x)$. Since $f(x)$ is to be degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form $f(x) = a(x-1)(x+1)x$ for some real number a. To find a, use the fact that $f(2) = 3$.
$f(2) == a(2-1)(2+1)(2) = 3 \Rightarrow$
$a(1)(3)(2) = 3 \Rightarrow 6a = 3 \Rightarrow a = \frac{1}{2}$
Thus,
$$\begin{aligned} f(x) &= \tfrac{1}{2}(x-1)(x+1)x \\ &= \tfrac{1}{2}(x^2-1)x = \tfrac{1}{2}x^3 - \tfrac{1}{2}x \end{aligned}$$

51. Zeros of – 2, 1, and 0; $f(-1) = -1$
These three zeros give $x-(-2) = x+2$, $x-1$, and $x-0 = x$ as factors of $f(x)$. Since $f(x)$ is to be degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form $f(x) = a(x+2)(x-1)x$ for some real number a. To find a, use the fact that $f(-1) = -1$.
$f(-1) = a(-1+2)(-1-1)(-1) = -1 \Rightarrow$
$a(1)(-2)(-1) = -1 \Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}$
Thus,
$$\begin{aligned} f(x) &= -\tfrac{1}{2}(x+2)(x-1)x \\ &= -\tfrac{1}{2}(x^2+x-2)x = -\tfrac{1}{2}x^3 - \tfrac{1}{2}x^2 + x \end{aligned}$$

52. Zeros of 2, –3, and 5; $f(3) = 6$
These three zeros give $x-2$, $x-(-3) = x+3$, and $x-5$ as factors of $f(x)$. Since $f(x)$ is to be degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form $f(x) = a(x-2)(x+3)(x-5)$. for some real number a. To find a, use the fact that $f(3) = 6$.
$f(3) = a(3-2)(3+3)(3-5) = 6 \Rightarrow$
$a(1)(6)(-2) = 6 \Rightarrow -12a = 6 \Rightarrow a = -\frac{1}{2}$
Thus,
$$\begin{aligned} f(x) &= -\tfrac{1}{2}(x-2)(x+3)(x-5) \\ &= -\tfrac{1}{2}(x^2+x-6)(x-5) \\ &= -\tfrac{1}{2}(x^3-4x^2-11x+30) \\ &= -\tfrac{1}{2}x^3+2x^2+\tfrac{11}{2}x-15 \end{aligned}$$

53. Zero of –3 having multiplicity 3; $f(3) = 36$
These three zeros give $x-(-3) = x+3$, $x-(-3) = x+3$, and $x-(-3) = x+3$ as factors of $f(x)$. Since $f(x)$ is to be degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form $f(x) = a(x+3)(x+3)(x+3) = a(x+3)^3$ for some real number a. To find a, use the fact that $f(3) = 36$.
$f(3) = a(3+3)^3 = 36 \Rightarrow a(6)^3 = 36 \Rightarrow$
$216a = 36 \Rightarrow a = \frac{1}{6}$
Thus,
$$\begin{aligned} f(x) &= \tfrac{1}{6}(x+3)^3 = \tfrac{1}{6}(x+3)^2(x+3) \\ &= \tfrac{1}{6}(x^2+6x+9)(x+3) \\ &= \tfrac{1}{6}(x^3+9x^2+27x+27) \\ &= \tfrac{1}{6}x^3+\tfrac{3}{2}x^2+\tfrac{9}{2}x+\tfrac{9}{2} \end{aligned}$$

54. Zero of 4 having multiplicity 2 and zero of 2 having multiplicity 1; $f(1)=-18$

These three zeros give $x-4$, $x-4$, and $x-2$ as factors of $f(x)$. Since $f(x)$ is to be degree 3, these are the only possible factors by the number of zeros theorem. Therefore, $f(x)$ has the form

$$f(x)=a(x-4)(x-4)(x-2)=a(x-4)^2(x-2)$$

for some real number a. To find a, use the fact that $f(1)=-18$.

$$f(1)=a(1-4)^2(1-2)=-18\Rightarrow$$
$$a(-3)^2(-1)=-18\Rightarrow -9a=-18\Rightarrow a=2$$

Thus,
$$\begin{aligned}f(x)&=2(x-4)^2(x-2)\\&=2(x^2-8x+16)(x-2)\\&=2(x^3-10x^2+32x-32)\\&=2x^3-20x^2+64x-64\end{aligned}$$

In Exercises 55–72, we must find a polynomial of least degree with real coefficients having the given zeros. For each of these exercises, other answers are possible.

55. $5+i$ and $5-i$

$$\begin{aligned}f(x)&=\left[x-(5+i)\right]\left[x-(5-i)\right]\\&=(x-5-i)(x-5+i)\\&=\left[(x-5)-i\right]\left[(x-5)+i\right]\\&=(x-5)^2-i^2\\&=\left(x^2-10x+25\right)-i^2\\&=x^2-10x+25+1=x^2-10x+26\end{aligned}$$

56. $7-2i$ and $7+2i$

$$\begin{aligned}f(x)&=\left[x-(7-2i)\right]\left[x-(7+2i)\right]\\&=(x-7+2i)(x-7-2i)\\&=\left[(x-7)+2i\right]\left[(x-7)-2i\right]\\&=(x-7)^2-(2i)^2\\&=\left(x^2-14x+49\right)-4i^2\\&=x^2-14x+49+4=x^2-14x+53\end{aligned}$$

57. 2 and $1+i$

By the conjugate zeros theorem, $1-i$ must also be a zero.

$$\begin{aligned}f(x)&=(x-2)\left[x-(1+i)\right]\left[x-(1-i)\right]=(x-2)(x-1-i)(x-1+i)=(x-2)\left[(x-1)-i\right]\left[(x-1)+i\right]\\&=(x-2)\left[(x-1)^2-i^2\right]=(x-2)\left[\left(x^2-2x+1\right)-i^2\right]=(x-2)\left(x^2-2x+1+1\right)\\&=(x-2)\left(x^2-2x+2\right)=x^3-4x^2+6x-4\end{aligned}$$

58. -3, 2, $-i$, and $2+i$

By the conjugate zeros theorem, i and $2-i$ must also be zeros.

$$\begin{aligned}f(x)&=(x+3)(x-2)(x+i)(x-i)\cdot\left[x-(2+i)\right]\left[x-(2-i)\right]\\&=\left[(x+3)(x-2)\right]\left[(x+i)(x-i)\right]\cdot\left[(x-2-i)(x-2+i)\right]\\&=\left(x^2+x-6\right)\left(x^2-i^2\right)\cdot\left[(x-2)-i\right]\left[(x-2)+i\right]\end{aligned}$$

$$\begin{aligned}f(x)&=\left(x^2+x-6\right)\left(x^2+1\right)\left[(x-2)^2-i^2\right]\\&=\left(x^4+x^3-5x^2+x-6\right)\cdot\left[\left(x^2-4x+4\right)-i^2\right]\\&=\left(x^4+x^3-5x^2+x-6\right)\left(x^2-4x+4+1\right)=\left(x^4+x^3-5x^2+x-6\right)\left(x^2-4x+5\right)\\&=x^6-3x^5-4x^4+26x^3-35x^2+29x-30\end{aligned}$$

59. $1+\sqrt{2}$, $1-\sqrt{2}$, and 1

$$\begin{aligned}f(x)&=\left[x-\left(1+\sqrt{2}\right)\right]\left[x-\left(1-\sqrt{2}\right)\right](x-1)=\left(x-1-\sqrt{2}\right)\left(x-1+\sqrt{2}\right)(x-1)\\&=\left[(x-1)-\sqrt{2}\right]\left[(x-1)+\sqrt{2}\right](x-1)=\left[(x-1)^2-\left(\sqrt{2}\right)^2\right](x-1)=\left(x^2-2x+1-2\right)(x-1)\\&=\left(x^2-2x-1\right)(x-1)=x^3-3x^2+x+1\end{aligned}$$

60. $1-\sqrt{3},\ 1+\sqrt{3},$ and 1

$$\begin{aligned} f(x) &= \left[x-\left(1-\sqrt{3}\right)\right]\left[x-\left(1+\sqrt{3}\right)\right](x-1) \\ &= \left(x-1+\sqrt{3}\right)\left(x-1-\sqrt{3}\right)(x-1) \\ &= \left[(x-1)+\sqrt{3}\right]\left[(x-1)-\sqrt{3}\right](x-1) \\ &= \left[(x-1)^2-\left(\sqrt{3}\right)^2\right](x-1) \\ &= \left(x^2-2x+1-3\right)(x-1) \\ &= \left(x^2-2x-2\right)(x-1) = x^3-3x^2+2 \end{aligned}$$

61. $2+i,\ 2-i,\ 3,$ and -1

$$\begin{aligned} f(x) &= \left[x-(2+i)\right]\left[x-(2-i)\right](x-3)(x+1) \\ &= \left[(x-2-i)(x-2+i)\right]\left[(x-3)(x+1)\right] \\ &= \left[(x-2)-i\right]\left[(x-2)+i\right]\left(x^2-2x-3\right) \\ &= \left[(x-2)^2-i^2\right]\left(x^2-2x-3\right) \\ &= \left[\left(x^2-4x+4\right)-i^2\right]\left(x^2-2x-3\right) \\ &= \left(x^2-4x+4+1\right)\left(x^2-2x-3\right) \\ &= \left(x^2-4x+5\right)\left(x^2-2x-3\right) \\ &= x^4-6x^3+10x^2+2x-15 \end{aligned}$$

62. $3 + 2i$, -1, and 2

By the conjugate zeros theorem, $3 - 2i$ must also be a zero.

$$\begin{aligned} f(x) &= \left[x-(3+2i)\right]\left[x-(3-2i)\right]\cdot(x+1)(x-2) = \left[(x-3)-2i\right]\left[(x-3)+2i\right]\cdot\left[(x+1)(x-2)\right] \\ &= \left[(x-3)^2-(2i)^2\right]\left(x^2-x-2\right) = \left(x^2-6x+9+4\right)\left(x^2-x-2\right) = \left(x^2-6x+13\right)\left(x^2-x-2\right) \\ &= x^4-7x^3+17x^2-x-26 \end{aligned}$$

63. 2 and $3 + i$

By the conjugate zeros theorem, $3 - i$ must also be a zero.

$$\begin{aligned} f(x) &= (x-2)\left[x-(3+i)\right]\left[x-(3-i)\right] = (x-2)\left[(x-3)-i\right]\left[(x-3)+i\right] = (x-2)\left[(x-3)^2-i^2\right] \\ &= (x-2)\left(x^2-6x+9+1\right) = (x-2)\left(x^2-6x+10\right) = x^3-8x^2+22x-20 \end{aligned}$$

64. -1 and $4-2i$

By the conjugate zeros theorem, $4+2i$ must also be a zero.

$$\begin{aligned} f(x) &= (x+1)\left[x-(4-2i)\right]\left[x-(4+2i)\right] = (x+1)(x-4+2i)(x-4-2i) = (x+1)\left[(x-4)+2i\right]\left[(x-4)-2i\right] \\ &= (x+1)\left[(x-4)^2-(2i)^2\right] = (x+1)\left(x^2-8x+16+4\right) = (x+1)\left(x^2-8x+20\right) = x^3-7x^2+12x+20 \end{aligned}$$

65. $1-\sqrt{2},\ 1+\sqrt{2},$ and $1-i$

By the conjugate zeros theorem, $1+i$ must also be a zero.

$$\begin{aligned} f(x) &= \left[x-\left(1-\sqrt{2}\right)\right]\left[x-\left(1+\sqrt{2}\right)\right]\cdot\left[x-(1-i)\right]\left[x-(1+i)\right] \\ &= \left(x-1+\sqrt{2}\right)\left(x-1-\sqrt{2}\right)\cdot(x-1+i)(x-1-i) = \left[(x-1)+\sqrt{2}\right]\left[(x-1)-\sqrt{2}\right]\cdot\left[(x-1)+i\right]\left[(x-1)-i\right] \\ &= \left[(x-1)^2-\left(\sqrt{2}\right)^2\right]\left[(x-1)^2-i^2\right] = \left(x^2-2x+1-2\right)\left[\left(x^2-2x+1\right)-i^2\right] = \left(x^2-2x-1\right)\left(x^2-2x+1+1\right) \\ &= \left(x^2-2x-1\right)\left(x^2-2x+2\right) = x^4-4x^3+5x^2-2x-2 \end{aligned}$$

66. $2+\sqrt{3},\ 2-\sqrt{3},$ and $2+3i$

By the conjugate zeros theorem, $2-3i$ must also be a zero.

$$\begin{aligned} f(x) &= \left[x-\left(2+\sqrt{3}\right)\right]\left[x-\left(2-\sqrt{3}\right)\right]\cdot\left[x-(2+3i)\right]\left[x-(2-3i)\right] \\ &= \left(x-2-\sqrt{3}\right)\left(x-2+\sqrt{3}\right)\cdot(x-2-3i)(x-2+3i) \\ &= \left[(x-2)-\sqrt{3}\right]\left[(x-2)+\sqrt{3}\right]\cdot\left[(x-2)-3i\right]\left[(x-2)+3i\right] = \left[(x-2)^2-\left(\sqrt{3}\right)^2\right]\left[(x-2)^2-(3i)^2\right] \\ &= \left(x^2-4x+4-3\right)\left[\left(x^2-4x+4\right)-9i^2\right] = \left(x^2-4x+1\right)\left(x^2-4x+4+9\right) = \left(x^2-4x+1\right)\left(x^2-4x+13\right) \\ &= x^4-8x^3+30x^2-56x+13 \end{aligned}$$

67. $2-i$ and $6-3i$

By the conjugate zeros theorem, $2+i$ and $6+3i$ must also be zeros.

$$\begin{aligned}f(x)&=\left[x-(2-i)\right]\left[x-(2+i)\right]\cdot\left[x-(6-3i)\right]\left[x-(6+3i)\right]=\left[(x-2+i)(x-2-i)\right]\cdot\left[(x-6+3i)(x-6-3i)\right]\\&=\left[(x-2)+i\right]\left[(x-2)-i\right]\cdot\left[(x-6)+3i\right]\left[(x-6)-3i\right]=\left[(x-2)^2-i^2\right]\left[(x-6)^2-(3i)^2\right]\\&=\left[\left(x^2-4x+4\right)-i^2\right]\cdot\left[\left(x^2-12x+36\right)-9i^2\right]=\left(x^2-4x+4+1\right)\left(x^2-12x+36+9\right)\\&=\left(x^2-4x+5\right)\left(x^2-12x+45\right)=x^4-16x^3+98x^2-240x+225\end{aligned}$$

68. $5+i$ and $4-i$

By the conjugate zeros theorem, $5-i$ and $4+i$ must also be zeros.

$$\begin{aligned}f(x)&=\left[x-(5+i)\right]\left[x-(5-i)\right]\cdot\left[x-(4-i)\right]\left[x-(4+i)\right]=\left[(x-5-i)(x-5+i)\right]\cdot\left[(x-4+i)(x-4-i)\right]\\&=\left[(x-5)-i\right]\left[(x-5)+i\right]\cdot\left[(x-4)+i\right]\left[(x-4)-i\right]=\left[(x-5)^2-i^2\right]\left[(x-4)^2-i^2\right]\\&=\left[\left(x^2-10x+25\right)-i^2\right]\cdot\left[\left(x^2-8x+16\right)-i^2\right]=\left(x^2-10x+25+1\right)\left(x^2-8x+16+1\right)\\&=\left(x^2-10x+26\right)\left(x^2-8x+17\right)=x^4-18x^3+123x^2-378x+442\end{aligned}$$

69. 4, $1-2i$, and $3+4i$

By the conjugate zeros theorem, $1+2i$ and $3-4i$ must also be zeros.

$$\begin{aligned}f(x)&=(x-4)\left[x-(1-2i)\right]\left[x-(1+2i)\right]\cdot\left[x-(3+4i)\right]\left[x-(3-4i)\right]\\&=(x-4)(x-1+2i)(x-1-2i)\cdot(x-3-4i)(x-3+4i)\\&=(x-4)\left[(x-1)+2i\right]\left[(x-1)-2i\right]\cdot\left[(x-3)-4i\right]\left[(x-3)+4i\right]\\&=(x-4)\left[(x-1)^2-(2i)^2\right]\left[(x-3)^2-(4i)^2\right]=(x-4)\left(x^2-2x+1-4i^2\right)\cdot\left(x^2-6x+9-16i^2\right)\\&=(x-4)\left(x^2-2x+1+4\right)\left(x^2-6x+9+16\right)=(x-4)\left(x^2-2x+5\right)\left(x^2-6x+25\right)\\&=(x-4)\left(x^4-8x^3+42x^2-80x+125\right)=x^5-12x^4+74x^3-248x^2+445x-500\end{aligned}$$

70. $-1, 1+\sqrt{2}, 1-\sqrt{2}$, and $1+4i$

By the conjugate zeros theorem, $1-4i$ must also be a zero.

$$\begin{aligned}f(x)&=(x+1)\left[x-\left(1+\sqrt{2}\right)\right]\left[x-\left(1-\sqrt{2}\right)\right]\cdot\left[x-(1+4i)\right]\left[x-(1-4i)\right]\\&=(x+1)\left(x-1-\sqrt{2}\right)\left(x-1+\sqrt{2}\right)\cdot(x-1-4i)(x-1+4i)\\&=(x+1)\left[(x-1)-\sqrt{2}\right]\left[(x-1)+\sqrt{2}\right]\cdot\left[(x-1)-4i\right]\left[(x-1)+4i\right]\\&=(x+1)\left[(x-1)^2-\left(\sqrt{2}\right)^2\right]\cdot\left[(x-1)^2-(4i)^2\right]=(x+1)\left(x^2-2x+1-2\right)\cdot\left[\left(x^2-2x+1\right)-16i^2\right]\\&=(x+1)\left(x^2-2x-1\right)\left(x^2-2x+1+16\right)=(x+1)\left(x^2-2x-1\right)\left(x^2-2x+17\right)\\&=(x+1)\left(x^4-4x^3+20x^2-32x-17\right)=x^5-3x^4+16x^3-12x^2-49x-17\end{aligned}$$

71. $1 + 2i$, 2 (multiplicity 2).
By the conjugate zeros theorem, $1 - 2i$ must also be a zero.

$$\begin{aligned} f(x) &= (x-2)^2\left[x-(1+2i)\right]\left[x-(1-2i)\right] \\ &= \left(x^2-4x+4\right)\left[(x-1)-2i\right]\left[(x-1)+2i\right] \\ &= \left(x^2-4x+4\right)\left[(x-1)^2-(2i)^2\right] \\ &= \left(x^2-4x+4\right)\left(x^2-2x+1-4i^2\right) \\ &= \left(x^2-4x+4\right)\left(x^2-2x+1+4\right) \\ &= \left(x^2-4x+4\right)\left(x^2-2x+5\right) \\ &= x^4-6x^3+17x^2-28x+20 \end{aligned}$$

72. $2 + i$, -3 (multiplicity 2)
By the conjugate zeros theorem, $2 - i$ is also a zero.

$$\begin{aligned} f(x) &= (x+3)^2\left[x-(2+i)\right]\left[x-(2-i)\right] \\ &= \left(x^2+6x+9\right)\left[(x-2)-i\right]\left[(x-2)+i\right] \\ &= \left(x^2+6x+9\right)\left[(x-2)^2-i^2\right] \\ &= \left(x^2+6x+9\right)\left(x^2-4x+4+1\right) \\ &= \left(x^2+6x+9\right)\left(x^2-4x+5\right) \\ &= x^4+2x^3-10x^2-6x+45 \end{aligned}$$

73. $f(x) = 2x^3 - 4x^2 + 2x + 7$

$f(x) = \underbrace{2x^3 -}_{1}\underbrace{4x^2 +}_{2} 2x + 7$ has 2 variations in sign. f has either 2 or 2 – 2 = 0 positive real zeros.

$f(-x) = -2x^3 - 4x^2 \underbrace{-2x +}_{1} 7$ has 1 variation in sign. f has 1 negative real zero.

74. $f(x) = x^3 + 2x^2 + x - 10$

$f(x) = x^3 + 2x^2 \underbrace{+x -}_{1} 10$ has 1 variation in sign. f has 1 positive real zero.

$f(-x) = \underbrace{-x^3 +}_{1}\underbrace{2x^2 -}_{2} x - 10$ has 2 variations in sign. f has 2 or 2 – 2 = 0 negative real zeros.

75. $f(x) = 5x^4 + 3x^2 + 2x - 9$

$f(x) = 5x^4 + 3x^2 \underbrace{+2x -}_{1} 9$ has 1 variation in sign. f has 1 positive real zero.

$f(-x) = 5x^4 \underbrace{+3x^2 -}_{1} 2x - 9$ has 1 variation in sign. f has 1 negative real zero.

76. $f(x) = 3x^4 + 2x^3 - 8x^2 - 10x - 1$

$f(x) = 3x^4 \underbrace{+2x^3 -}_{1} 8x^2 - 10x - 1$ has 1 variation in sign. f has 1 positive real zero.

$f(-x) = \underbrace{3x^4 -}_{1} 2x^3 \underbrace{-8x^2 +}_{2}\underbrace{10x -}_{3} 1$ has 3 variations in sign. f has 3 or 3 – 2 = 1 negative real zeros.

77. $f(x) = x^5 + 3x^4 - x^3 + 2x + 3$

$f(x) = x^5 \underbrace{+3x^4 -}_{1}\underbrace{x^3 +}_{2} 2x + 3$ has 2 variations in sign. f has 2 or 2 – 2 = 0 positive real zeros.

$f(-x) = \underbrace{-x^5 +}_{1} 3x^4 \underbrace{+x^3 -}_{2}\underbrace{2x +}_{3} 3$ has 3 variations in sign. f has 3 or 3 – 2 = 1 negative real zeros.

78. $f(x) = 2x^5 - x^4 + x^3 - x^2 + x + 5$

$f(x) = \underbrace{2x^5 -}_{1}\underbrace{x^4 +}_{2}\underbrace{x^3 -}_{3}\underbrace{x^2 +}_{4} x + 5$ has 4 variations in sign. f has 4 or 4 – 2 = 2 or 2 – 2 = 0 positive real zeros.

$f(-x) = -2x^5 - x^4 - x^3 - x^2 \underbrace{-x +}_{1} 5$ has 1 variation in sign. f has 1 negative real zero.

79. $f(x) = x^4 + 2x^3 - 3x^2 + 24x - 180$

p must be a factor of $a_0 = -180$ and q must be a factor of $a_4 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9, \pm 10, \pm 12, \pm 15, \pm 18, \pm 20, \pm 30, \pm 36, \pm 45, \pm 60, \pm 90$, and ± 180. q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9, \pm 10, \pm 12, \pm 15, \pm 18, \pm 20, \pm 30, \pm 36, \pm 45, \pm 60, \pm 90$, and ± 180.
Using the remainder theorem and synthetic division, we find that one zero is $x = -5$.

$$\begin{array}{r|rrrrr} -5 & 1 & 2 & -3 & 24 & -180 \\ & & -5 & 15 & -60 & 180 \\ \hline & 1 & -3 & 12 & -36 & 0 \end{array}$$

Setting the quotient $x^3 - 3x^2 + 12x - 36$ equal to zero and factoring by grouping, we have:

$$\begin{aligned} 0 &= x^3 - 3x^2 + 12x - 36 \\ &= x^2(x-3) + 12(x-3) \\ &= (x^2+12)(x-3) \Rightarrow \end{aligned}$$

$x^2 + 12 = 0 \Rightarrow x^2 = -12 \Rightarrow x = \sqrt{-12} \Rightarrow$
$x = \pm 2i\sqrt{3}$ or $x - 3 = 0 \Rightarrow x = 3$

Thus, the zeros of $f(x)$ are $\left\{-5, 3, \pm 2i\sqrt{3}\right\}$.

80. $f(x) = x^3 - x^2 - 8x + 12$

p must be a factor of $a_0 = 12$ and q must be a factor of $a_3 = 1$. Thus, p can be ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12. q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12. Using the remainder theorem and synthetic division, we find that one zero is $x = -3$.

$$\begin{array}{r|rrrr} -3 & 1 & -1 & -8 & 12 \\ & & -3 & 12 & -12 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

Setting the quotient $x^2 - 4x + 4$ equal to zero and then factoring we have:

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0 \Rightarrow x = 2$ (with multiplicity 2)

Thus, the zeros of $f(x)$ are $\{-3, 2, 2\}$.

81. $f(x) = x^4 + x^3 - 9x^2 + 11x - 4$

p must be a factor of $a_0 = -4$ and q must be a factor of $a_4 = 1$. Thus, p can be ± 1, ± 2,and ± 4. q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are ± 1, ± 2, and ± 4. Using the remainder theorem and synthetic division, we find that one zero is $x = -4$.

$$\begin{array}{r|rrrrr} -4 & 1 & 1 & -9 & 11 & -4 \\ & & -4 & 12 & -12 & 4 \\ \hline & 1 & -3 & 3 & -1 & 0 \end{array}$$

Setting the quotient $x^3 - 3x^2 + 3x - 1$ equal to zero and factoring by grouping, we have:

$x^3 - 3x^2 + 3x - 1 = 0$

$(x-1)^3 = 0 \Rightarrow x = 1$ (with multiplicity 3)

Thus, the zeros of $f(x)$ are $\{-4, 1, 1, 1\}$.

82. $f(x) = x^3 - 14x + 8$

p must be a factor of $a_0 = 8$ and q must be a factor of $a_3 = 1$. Thus, p can be ± 1, ± 2, ± 4, or ± 8. q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are ± 1, ± 2, ± 4, and ± 8. Using the remainder theorem and synthetic division, we find that one zero is $x = -4$.

$$\begin{array}{r|rrrr} -4 & 1 & 0 & -14 & 8 \\ & & -4 & 16 & -8 \\ \hline & 1 & -4 & 2 & 0 \end{array}$$

Setting the quotient $x^2 - 4x + 2$ equal to zero and using the quadratic equation with $a = 1$, $b = -4$, and $c = 2$ to solve this, we have:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

Thus, the zeros of $f(x)$ are $\left\{-4, 2 \pm \sqrt{2}\right\}$.

83. $f(x) = 2x^5 + 11x^4 + 16x^3 + 15x^2 + 36x$

$a_0 = 0$, so 0 is a zero of the function. Factoring, we have

$2x^5 + 11x^4 + 16x^3 + 15x^2 + 36x$
$= x(2x^4 + 11x^3 + 16x^2 + 15x + 36)$

To find a zero of the quotient, $2x^4 + 11x^3 + 16x^2 + 15x + 36$, p must be a factor of $a_0 = 36$ and q must be a factor of $a_4 = 2$. Thus, p can be ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18,or ± 36. q can be ± 1 or ± 2. The possible rational zeros, $\frac{p}{q}$, are

$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12$ ± 18, or ± 36. Using the remainder theorem and synthetic division, we find that one zero is $x = -3$.

$$\begin{array}{r|rrrrrr} -3 & 2 & 11 & 16 & 15 & 36 & 0 \\ & & -6 & -15 & -3 & -36 & 0 \\ \hline & 2 & 5 & 1 & 12 & 0 & 0 \end{array}$$

Now find a zero of the quotient $2x^3 + 5x^2 + x + 12$. p must be a factor of $a_0 = 12$ and q must be a factor of $a_3 = 2$. Thus, p can be ± 1, ± 2, ± 3, ± 4, ± 6, or ± 12. q can be ± 1 or ± 2. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm 1,$ $\pm 2, \pm 3, \pm 4, \pm 6,$ or ± 12. Using the remainder theorem and synthetic division, we find that one zero is $x = -3$

$$\begin{array}{r|rrrr} -3 & 2 & 5 & 1 & 12 \\ & & -6 & 3 & -12 \\ \hline & 2 & -1 & 4 & 0 \end{array}$$

Setting the quotient $2x^2 - x + 4$ equal to zero and using the quadratic equation with $a = 2$, $b = -1$, and $c = 4$ to solve this, we have:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(4)}}{2(2)}$$
$$= \frac{1 \pm \sqrt{1-32}}{4} = \frac{1 \pm i\sqrt{31}}{4}$$

Thus, the zeros of $f(x)$ are

$$\left\{-3, -3, 0, \frac{1 \pm i\sqrt{31}}{4}\right\}.$$

84. $f(x) = 3x^3 - 9x^2 - 31x + 5$

p must be a factor of $a_0 = 5$ and q must be a factor of $a_3 = 3$. Thus, p can be $\pm$ 1 or $\pm$ 5. q can be $\pm$ 1 or ± 3. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 5, \pm \frac{1}{3}$, or $\pm \frac{5}{3}$. Using the remainder theorem and synthetic division, we find that one zero is $x = 5$

$$\begin{array}{r|rrrr} 5 & 3 & -9 & -31 & 5 \\ & & 15 & 30 & -5 \\ \hline & 3 & 6 & -1 & 0 \end{array}$$

Setting the quotient $3x^2 + 6x - 1$ equal to zero and using the quadratic equation with $a = 3$, $b = 6$, and $c = -1$ to solve this, we have:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-1)}}{2(3)} = \frac{-6 \pm \sqrt{36+12}}{6}$$
$$= \frac{-6 \pm \sqrt{48}}{6} = \frac{-6 \pm 4\sqrt{3}}{6} = \frac{-3 \pm 2\sqrt{3}}{3}$$

Thus, the zeros of $f(x)$ are $\left\{5, \frac{-3 \pm 2\sqrt{3}}{3}\right\}$.

85. $f(x) = x^5 - 6x^4 + 14x^3 - 20x^2 + 24x - 16$

p must be a factor of $a_0 = -16$ and q must be a factor of $a_3 = 2$. Thus, p can be $\pm$ 1, $\pm$ 2, $\pm$ 4, $\pm$ 8, or $\pm$ 16. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are $\pm$ 1, $\pm$ 2, $\pm$ 4, $\pm$ 8, or $\pm$ 16. Using the remainder theorem and synthetic division, we find that one zero is $x = 2$

$$\begin{array}{r|rrrrrr} 2 & 1 & -6 & 14 & -20 & 24 & -16 \\ & & 2 & -8 & 12 & -16 & 16 \\ \hline & 1 & -4 & 6 & -8 & 8 & 0 \end{array}$$

To find a zero of the quotient, $x^4 - 4x^3 + 6x^2 - 8x + 8$, p must be a factor of $a_0 = 8$ and q must be a factor of $a_4 = 1$. Thus, p can be $\pm$ 1, $\pm$ 2, $\pm$ 4, or $\pm$ 8. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are $\pm$ 1, $\pm$ 2, $\pm$ 4, or $\pm$ 8.

Using the remainder theorem and synthetic division, we find that one zero is $x = 2$.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 6 & -8 & 8 \\ & & 2 & -4 & 4 & -8 \\ \hline & 1 & -2 & 2 & -4 & 0 \end{array}$$

Now find a zero of the quotient $x^3 - 2x^2 + 2x - 4$. p must be a factor of $a_0 = -4$ and q must be a factor of $a_3 = 1$. Thus, p can be $\pm$ 1, $\pm$ 2, $\pm$ 3, or $\pm$ 4. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are $\pm$ 1, $\pm$ 2, $\pm$ 3, or $\pm$ 4. Using the remainder theorem and synthetic division, we find that one zero is $x = 2$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 2 & -4 \\ & & 2 & 0 & 4 \\ \hline & 1 & 0 & 2 & 0 \end{array}$$

Setting the quotient $x^2 + 2$ equal to zero, we have: $x^2 + 2 = 0 \Rightarrow x^2 = -2 \Rightarrow x = \pm i\sqrt{2}$

Thus, the zeros of $f(x)$ are $\left\{2, 2, 2, \pm i\sqrt{2}\right\}$.

86. $f(x) = 9x^4 + 30x^3 + 241x^2 + 720x + 600$

p must be a factor of $a_0 = 600$ and q must be a factor of $a_4 = 9$. Thus, p can be $\pm$ 1, $\pm$ 2, $\pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 25, \pm 30, \pm 40, \pm 50, \pm 60, \pm 75, \pm 100, \pm 120, \pm 150, \pm 200, \pm 300$, or $\pm$ 600. q can be $\pm$ 1, ± 3, or ± 9.

The possible rational zeros, $\frac{p}{q}$, are $\pm$ 1, $\pm$ 2, $\pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 25, \pm 30, \pm 40, \pm 50, \pm 60, \pm 75, \pm 100, \pm 120, \pm 150, \pm 200, \pm 300, \pm 600, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{5}{3}, \pm \frac{8}{3}, \pm \frac{10}{3}, \pm \frac{20}{3}, \pm \frac{25}{3}, \pm \frac{40}{3}, \pm \frac{50}{3}, \pm \frac{100}{3},. \pm \frac{200}{3}, \pm \frac{1}{9}, \pm \frac{2}{9}, \pm \frac{4}{9}, \pm \frac{5}{9}, \pm \frac{8}{9}, \pm \frac{10}{9}, \pm \frac{20}{9}, \pm \frac{25}{9}, \pm \frac{40}{9}, \pm \frac{50}{9}, \pm \frac{100}{9}$, or $\pm \frac{200}{9}$. Using the remainder theorem and synthetic division, we find that one zero is $x = -\frac{5}{3}$.

$$\begin{array}{r|rrrrr} -\frac{5}{3} & 9 & 30 & 241 & 720 & 600 \\ & & -15 & -25 & -360 & -600 \\ \hline & 9 & 15 & 216 & 360 & 0 \end{array}$$

(*continued on next page*)

(*continued from page 293*)

Setting the quotient, $9x^3+15x^2+216x+360$, equal to zero and factoring by grouping, we have $9x^3+15x^2+216x+360=0$

$$3x^2(3x+5)+72(3x+5)=0$$
$$(3x^2+72)(3x+5)=0 \Rightarrow$$
$$3x^2+72=0 \Rightarrow x^2=-24 \Rightarrow x=\pm 2i\sqrt{6} \text{ or}$$
$$3x+5=0 \Rightarrow x=-\tfrac{5}{3}$$

Thus, the zeros of $f(x)$ are $\left\{-\frac{5}{3},-\frac{5}{3},\pm 2i\sqrt{6}\right\}$.

87. $f(x)=2x^4-x^3+7x^2-4x-4$

p must be a factor of $a_0=-4$ and q must be a factor of $a_4=2$. Thus, p can be ± 1, ± 2,and ± 4. q can be ± 1 or ± 2. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{2}$, ± 1, ± 2, and ± 4. Using the remainder theorem and synthetic division, we find that one zero is $x = 1$.

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & 7 & -4 & -4 \\ & & 2 & 1 & 8 & 4 \\ \hline & 2 & 1 & 8 & 4 & 0 \end{array}$$

Setting the quotient $2x^3+x^2+8x+4$ equal to zero and factoring by grouping, we have:

$$2x^3+x^2+8x+4=0$$
$$x^2(2x+1)+4(2x+1)=0$$
$$(x^2+4)(2x+1)=0 \Rightarrow$$
$$x^2+4=0 \Rightarrow x^2=-4 \Rightarrow x=\pm 2i \text{ or}$$
$$2x+1=0 \Rightarrow x=-\tfrac{1}{2}$$

Thus, the zeros of $f(x)$ are $\left\{-\frac{1}{2},1,\pm 2i\right\}$.

88. $f(x)=32x^4-188x^3+261x^2+54x-27$

p must be a factor of $a_0=-27$ and q must be a factor of $a_4=32$. Thus, p can be $\pm1,\pm3,\pm9$, or ± 27. q can be $\pm1,\pm2,\pm4,\pm8,\pm16,$ or ± 32. The possible rational zeros, $\frac{p}{q}$, are $\pm1,\pm\frac{1}{2}$, $\pm\frac{1}{4},\pm\frac{1}{8},\pm\frac{1}{16},\pm\frac{1}{32},\pm3,\pm\frac{3}{2},\pm\frac{3}{4},\pm\frac{3}{8},\pm\frac{3}{16}$, $\pm\frac{3}{32},\pm9,\pm\frac{9}{2},\pm\frac{9}{4},\pm\frac{9}{8},\pm\frac{9}{16},\pm\frac{9}{32},\pm27,\pm\frac{27}{2}$, $\pm\frac{27}{4},\pm\frac{27}{8},\pm\frac{27}{16}$, or $\pm\frac{27}{32}$. Using the remainder theorem and synthetic division, we find that one zero is $x = 3$.

$$\begin{array}{r|rrrrr} 3 & 32 & -188 & 261 & 54 & -27 \\ & & 96 & -276 & -45 & 27 \\ \hline & 32 & -92 & -15 & 9 & 0 \end{array}$$

Now find a zero of the quotient $32x^3-92x^2-15x+9$. p must be a factor of $a_0=9$ and q must be a factor of $a_3=32$. Thus, p can be $\pm1,\pm3,$ or ± 9. q can be $\pm1,\pm2,\pm4,\pm8,\pm16,$ or ± 32. The possible rational zeros, $\frac{p}{q}$, are $\pm1,\pm\frac{1}{2}$, $\pm\frac{1}{4},\pm\frac{1}{8},\pm\frac{1}{16},\pm\frac{1}{32},\pm3,\pm\frac{3}{2},\pm\frac{3}{4},\pm\frac{3}{8},\pm\frac{3}{16}$, $\pm\frac{3}{32},\pm9,\pm\frac{9}{2},\pm\frac{9}{4},\pm\frac{9}{8},\pm\frac{9}{16}$, or $\pm\frac{9}{32}$. Using the remainder theorem and synthetic division, we find that one zero is $x = 3$.

$$\begin{array}{r|rrrr} 3 & 32 & -92 & -15 & 9 \\ & & 96 & 12 & -9 \\ \hline & 32 & 4 & -3 & 0 \end{array}$$

Setting the quotient $32x^2+4x-3$ equal to zero and solving by factoring, we have

$$32x^2+4x-3=0$$
$$(8x+3)(4x-1)=0 \Rightarrow$$
$$8x+3=0 \Rightarrow x=-\tfrac{3}{8} \text{ or}$$
$$4x-1=0 \Rightarrow x=\tfrac{1}{4}$$

Thus, the zeros of $f(x)$ are $\left\{-\frac{3}{8},\frac{1}{4},3,3\right\}$.

89. $f(x)=5x^3-9x^2+28x+6$

p must be a factor of $a_0=6$ and q must be a factor of $a_3=5$. Thus, p can be $\pm1,\pm2,\pm3,$ or ± 6. q can be ± 1 or ± 5. The possible rational zeros, $\frac{p}{q}$, are $\pm1,\pm2,\pm3,\pm6,\pm\frac{1}{5},\pm\frac{2}{5},\pm\frac{3}{5}$, or $\pm\frac{6}{5}$. Using the remainder theorem and synthetic division, we find that one zero is $x=-\frac{1}{5}$

$$\begin{array}{r|rrrr} -\frac{1}{5} & 5 & -9 & 28 & 6 \\ & & -1 & 2 & -6 \\ \hline & 5 & -10 & 30 & 0 \end{array}$$

Setting the quotient $5x^2-10x+30=5(x^2-2x+6)$ equal to zero and using the quadratic equation with $a = 1$, $b = -2$, and $c = 6$ to solve this, we have:

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(6)}}{2(1)}=\frac{2\pm\sqrt{4-24}}{2}$$
$$=\frac{2\pm\sqrt{-20}}{2}=\frac{2\pm 2i\sqrt{5}}{2}=1\pm i\sqrt{5}$$

Thus, the zeros of $f(x)$ are $\left\{-\frac{1}{5},1\pm i\sqrt{5}\right\}$.

90. $f(x) = 4x^3 + 3x^2 + 8x + 6$

p must be a factor of $a_0 = 6$ and q must be a factor of $a_3 = 4$. Thus, p can be $\pm 1, \pm 2, \pm 3,$ or ± 6. q can be $\pm 1, \pm 2,$ or ± 4. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{1}{4},$ $\pm \frac{3}{2}$, or $\pm \frac{3}{4}$. Using the remainder theorem and synthetic division, we find that one zero is $x = -\frac{3}{4}$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 4 & 3 & 8 & 6 \\ & & -3 & 0 & -6 \\ \hline & 4 & 0 & 8 & 0 \end{array}$$

Setting the quotient $4x^2 + 8$ equal to zero and solving, we have:

$4x^2 + 8 = 0 \Rightarrow x^2 = -2 \Rightarrow x = \pm i\sqrt{2}$

Thus, the zeros of $f(x)$ are $\left\{-\frac{3}{4}, \pm i\sqrt{2}\right\}$.

91. $f(x) = x^4 + 29x^2 + 100$

Letting $u = x^2$, we have $x^4 + 29x^2 + 100 = u^2 + 29u + 100$. Set the function equal to zero, and solve for u using the quadratic formula with $a = 1$, $b = 29$ and $c = 100$.

$$u = \frac{-29 \pm \sqrt{29^2 - 4(1)(100)}}{2(1)}$$
$$= \frac{-29 \pm \sqrt{441}}{2} = \frac{-29 \pm 21}{2} = -25 \text{ or } u = -4$$

$u = -25 \Rightarrow x^2 = -25 \Rightarrow x = \pm 5i$

$u = -4 \Rightarrow x^2 = -4 \Rightarrow x = \pm 2i$

Thus, the zeros of $f(x)$ are $\{\pm 2i, \pm 5i\}$.

92. $f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$

p must be a factor of $a_0 = 1$ and q must be a factor of $a_4 = 1$. Thus, p can be ± 1, and. q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are ± 1. Using the remainder theorem and synthetic division, we find that one zero is $x = -1$.

$$\begin{array}{r|rrrrr} -1 & 1 & 4 & 6 & 4 & 1 \\ & & -1 & -3 & -3 & -1 \\ \hline & 1 & 3 & 3 & 1 & 0 \end{array}$$

Setting the quotient $x^3 + 3x^2 + 3x + 1$ equal to zero and factoring, we have:

$x^3 + 3x^2 + 3x + 1 = 0$

$(x+1)^3 = 0$

$x = -1$ (with multiplicity 3)

Thus, the zeros of $f(x)$ are $\{-1, -1, -1, -1\}$.

93. $f(x) = x^4 + 2x^2 + 1$

Setting $f(x)$ equal to 0, then factoring to solve for x, we have

$x^4 + 2x^2 + 1 = 0$

$(x^2 + 1)^2 = 0$

$x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow$

$x = \pm i$ (with multiplicity 2)

Thus, the zeros of $f(x)$ are $\{\pm i, \pm i\}$.

94. $f(x) = x^4 - 8x^3 + 24x^2 - 32x + 16$

p must be a factor of $a_0 = 16$ and q must be a factor of $a_4 = 1$. Thus, p can be ± 1, ± 2, ± 4, ± 8, or ± 16. q can be ± 1. The possible rational zeros, $\frac{p}{q}$, are ± 1, ± 2, ± 4, ± 8, or ± 16. Using the remainder theorem and synthetic division, we find that one zero is $x = 2$.

$$\begin{array}{r|rrrrr} 2 & 1 & -8 & 24 & -32 & 16 \\ & & 2 & -12 & 24 & -16 \\ \hline & 1 & -6 & 12 & -8 & 0 \end{array}$$

Now find a zero of the quotient $x^3 - 6x^2 + 12x - 8$. p must be a factor of $a_0 = -8$ and q must be a factor of $a_3 = 1$. Thus, p can be ± 1, ± 2, ± 4, or ± 8, while q can be ± 1. Using the remainder theorem and synthetic division, we find that one zero is $x = 2$.

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

Setting the quotient $x^2 - 4x + 4$ equal to zero and solving for x, we have

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x - 2 = 0 \Rightarrow x = 2$ (with multiplicity 2)

Thus, the zeros of $f(x)$ are $\{2, 2, 2, 2\}$.

95. $f(x) = x^4 - 6x^3 + 7x^2$

Setting $f(x)$ equal to zero and solving for x, we have $x^4 - 6x^3 + 7x^2 = 0 \Rightarrow x^2(x^2 - 6x + 7) = 0 \Rightarrow$

$x^2 = 0 \Rightarrow$

$x = 0$ (with multiplicity 2) or

$x^2 - 6x + 7 = 0 \Rightarrow$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} = \frac{6 \pm \sqrt{36 - 28}}{2} = \frac{6 \pm \sqrt{8}}{2} = 3 \pm \sqrt{2}$$

Thus, the zeros of $f(x)$ are $\left\{0, 0, 3 \pm \sqrt{2}\right\}$.

96. $f(x) = 4x^4 - 65x^2 + 16$

Letting $u = x^2$, we have

$4x^4 - 65x^2 + 16 = 4u^2 - 65u + 16$

Set the function equal to zero, and solve for u using the quadratic formula with $a = 4$, $b = -65$ and $c = 16$.

$$u = \frac{-(-65) \pm \sqrt{(-65)^2 - 4(4)(16)}}{2(4)} = \frac{65 \pm \sqrt{3969}}{8} = \frac{65 \pm 63}{8} = 16 \text{ or } u = \tfrac{1}{4}$$

$u = 16 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

$u = \tfrac{1}{4} \Rightarrow x^2 = \tfrac{1}{4} \Rightarrow x = \pm \tfrac{1}{2}$

Thus, the zeros of $f(x)$ are $\left\{\pm \tfrac{1}{2}, \pm 4\right\}$.

97. $f(x) = x^4 - 8x^3 + 29x^2 - 66x + 72$

p must be a factor of $a_0 = 72$ and q must be a factor of $a_4 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 18, \pm 24, \pm 36,$ or ± 72. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 18, \pm 24, \pm 36,$ or ± 72.

Using the remainder theorem and synthetic division, we find that one zero is $x = 3$.

3)	1	−8	29	−66	72
		3	−15	42	−72
	1	−5	14	−24	0

Now find a zero of the quotient $x^3 - 5x^2 + 14x - 24$. p must be a factor of $a_0 = -24$ and q must be a factor of $a_3 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12,$ or ± 24, while q can be $\pm$ 1.

Using the remainder theorem and synthetic division, we find that one zero is $x = 3$.

3)	1	−5	14	−24
		3	−6	24
	1	−2	8	0

Setting the quotient $x^2 - 2x + 8$ equal to zero and solving for x using the quadratic formula with $a = 1$, $b = -2$, and $c = 8$, we have

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(8)}}{2(1)} = \frac{2 \pm \sqrt{4 - 32}}{2} = \frac{2 \pm \sqrt{-28}}{2} = 1 \pm i\sqrt{7}$$

Thus, the zeros of $f(x)$ are $\left\{3, 3, 1 \pm i\sqrt{7}\right\}$.

98. $f(x) = 12x^4 - 43x^3 + 50x^2 + 38x - 12$

p must be a factor of $a_0 = -12$ and q must be a factor of $a_4 = 12$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6,$ or ± 12. q can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6,$ or ± 12. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \tfrac{1}{2}, \pm \tfrac{1}{3}, \pm \tfrac{1}{4}, \pm \tfrac{1}{6}, \pm \tfrac{1}{12}, \pm \tfrac{2}{3}, \pm \tfrac{3}{2}, \pm \tfrac{3}{4},$ or $\pm \tfrac{4}{3}$. Using the remainder theorem and synthetic division, we find that one zero is $x = \tfrac{1}{4}$.

$\tfrac{1}{4}$)	12	−43	50	38	−12
		3	−10	10	12
	12	−40	40	48	0

Now find a zero of the quotient, $12x^3 - 40x^2 + 40x + 48$. p must be a factor of $a_0 = 48$ and q must be a factor of $a_3 = 12$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24$ or ± 48 while q can be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6,$ or ± 12.

The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm 16, \pm 24, \pm 48, \pm \tfrac{1}{2}, \pm \tfrac{1}{3}, \pm \tfrac{1}{4}, \pm \tfrac{1}{6}, \pm \tfrac{1}{12}, \pm \tfrac{2}{3}, \pm \tfrac{3}{2}, \pm \tfrac{3}{4}, \pm \tfrac{4}{3}, \pm \tfrac{8}{3},$ or $\pm \tfrac{16}{3}$.

Using the remainder theorem and synthetic division, we find that one zero is $x = -\tfrac{2}{3}$.

$-\tfrac{2}{3}$)	12	−40	40	48
		−8	32	−48
	12	−48	72	0

Set the quotient $12x^2 - 48x + 72 = 12(x^2 - 4x + 6)$ equal to zero and solve for x using the quadratic formula with $a = 1$, $b = -4$, and $c = 6$:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(6)}}{2(1)} = \frac{4 \pm \sqrt{16-24}}{2} = \frac{4 \pm \sqrt{-8}}{2} = 2 \pm i\sqrt{2}$$

Thus, the zeros of $f(x)$ are $\left\{-\frac{2}{3}, \frac{1}{4}, 2 \pm i\sqrt{2}\right\}$.

99. $f(x) = x^6 - 9x^4 - 16x^2 + 144$

Let $u = x^2$. Then $x^6 - 9x^4 - 16x^2 + 144 \Rightarrow u^3 - 9u^2 - 16u + 144$. Set $f(u)$ equal to 0, then factor by grouping to solve for u:

$$u^3 - 9u^2 - 16u + 144 = 0$$
$$u^2(u-9) - 16(u-9) = 0$$
$$(u^2 - 16)(u-9) = 0 \Rightarrow$$
$$u^2 - 16 = 0 \Rightarrow u^2 = 16 \Rightarrow u = \pm 4 \text{ or}$$
$$u - 9 = 0 \Rightarrow u = 9$$
$$u = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$
$$u = -4 \Rightarrow x^2 = -4 \Rightarrow x = \pm 2i$$
$$u = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

Thus, the zeros of $f(x)$ are $\{\pm 2, \pm 3, \pm 2i\}$.

100. $f(x) = x^6 - x^5 - 26x^4 + 44x^3 + 91x^2 - 139x + 30$

p must be a factor of $a_0 = 30$ and q must be a factor of $a_6 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15,$ or ± 30. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15,$ or ± 30. Using the remainder theorem and synthetic division, we find that one zero is $x = 1$.

1)	1	−1	−26	44	91	−139	30
		1	0	−26	18	109	−30
	1	0	−26	18	109	−30	0

Now find a zero of the quotient, $x^5 + 26x^4 + 18x^2 + 109x - 30$. p must be a factor of $a_0 = 30$ and q must be a factor of $a_5 = 1$. Thus, p can be $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15,$ or ± 30. q can be $\pm$ 1.

The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15,$ or ± 30. Using the remainder theorem and synthetic division, we find that one zero is $x = -2$.

−2)	1	0	−26	18	109	−30
		−2	4	44	−124	30
	1	−2	−22	62	−15	0

Now find a zero of the quotient, $x^4 - 2x^3 - 22x^2 + 62x - 15$. p must be a factor of $a_0 = -15$ and q must be a factor of $a_4 = 1$. Thus, p can be $\pm 1, \pm 3, \pm 5,$ or ± 15. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are $\pm 1, \pm 3, \pm 5,$ or ± 15. Using the remainder theorem and synthetic division, we find that one zero is $x = 3$.

3)	1	−2	−22	62	−15
		3	3	−57	15
	1	1	−19	5	0

Now find a zero of the quotient, $x^3 + x^2 - 19x + 5$. p must be a factor of $a_0 = 5$ and q must be a factor of $a_3 = 1$. Thus, p can be ± 1 or ± 5. q can be $\pm$ 1. The possible rational zeros, $\frac{p}{q}$, are ± 1 or ± 5. Using the remainder theorem and synthetic division, we find that one zero is $x = -5$.

−5)	1	1	−19	5
		−5	20	−5
	1	−4	1	0

Set the quotient $x^2 - 4x + 1$ equal to zero and solve for x using the quadratic formula with $a = 1$, $b = -4$, and $c = 1$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Thus, the zeros of $f(x)$ are $\{1, -2, 3, -5, 2 \pm \sqrt{3}\}$

For exercises 101−104, let $c = a + bi$ and $d = m + ni$.

101.
$$\overline{c+d} = \overline{(a+bi)+(m+ni)} = \overline{(a+m)+(b+n)i} = (a+m)-(b+n)i$$
$$= a + m - bi - ni = (a - bi) + (m - ni)$$
$$= \overline{c} + \overline{d}$$

102. $\overline{cd} = \overline{(a+bi)(m+ni)}$
$= \overline{am + ani + bmi + bni^2}$
$= \overline{am + ani + bmi - bn}$
$= \overline{(am-bn)+(an+bm)i}$
$= (am-bn)-(an+bm)i$
$\overline{c}\cdot\overline{d} = \overline{a+bi}\cdot\overline{m+ni} = (a-bi)(m-ni)$
$= am - ani - bmi + bni^2$
$= am - ani - bmi - bn$
$= (am-bn)-(an+bm)i$
Therefore, $\overline{cd} = \overline{c}\cdot\overline{d}$

103. If a is a real number then a is of the form $a+0\cdot i.$ $\overline{a} = \overline{a+0\cdot i} = a - 0i = a.$ Therefore, the statement is true.

104. $\overline{c^2} = \overline{(a+bi)^2} = \overline{a^2+2abi+b^2i^2}$
$= \overline{a^2+2abi-b^2} = \overline{(a^2-b^2)+2abi}$
$= (a^2-b^2)-2abi$
$(\overline{c})^2 = (\overline{a+bi})^2 = (a-bi)^2 = a^2-2abi+b^2i^2$
$= a^2-2abi-b^2 = (a^2-b^2)-2abi$

Therefore, $\overline{c^2} = (\overline{c})^2.$

Section 3.4: Polynomial Functions: Graphs, Applications, and Models

1. $y = x^3 - 3x^2 - 6x + 8$
The range of an odd-degree polynomial is $(-\infty, \infty)$. The y-intercept of the graph is 8. The graph fitting these criteria is A.

2. $y = x^4 + 7x^3 - 5x^2 - 75x$
A polynomial of degree 4 has at most 3 turning points. A polynomial of even degree will have a range of the form $(-\infty, k]$ or else $[k, \infty)$ for some real number k. The graph fitting these criteria is B.

3. Since graph C crosses the x-axis at one point, the graph has one real zero.

4. $y = -x^3 + 9x^2 - 27x + 17$
The range of an odd-degree polynomial is $(-\infty, \infty)$. The y-intercept of the graph is 17. The graph fitting these criteria is C.

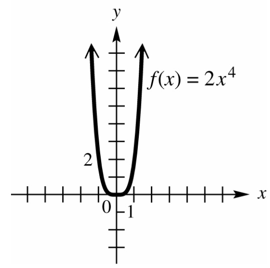
5. A polynomial of degree 3 can have at most 2 turning points. Graphs B and D have more than 2 turning points, so they cannot be graphs of cubic polynomial functions.

6. Graph B is a polynomial of even degree, which has a minimum point. It appears from the graph that the range is $[-100, \infty)$. This is the only graph whose range is *not* $(-\infty, \infty)$.

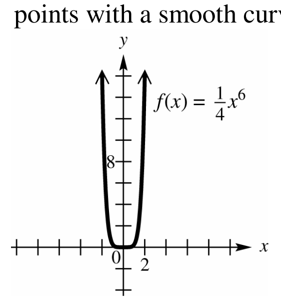
7. Since graph B touches the x-axis at –5, the function has 2 real zeros of –5. The two other real zeros are where the graph crosses the x-axis, at 0 and 3.
$f(x) = x^4 + 7x^3 - 5x^2 - 75x = x(x+5)^2(x-3)$

8. Since graph D touches the x-axis at –1, the function has 2 real zeros of –1. The three other real zeros are where the graph crosses the x-axis, at –6, 3, and 5.
$f(x) = -x^5 + 36x^3 - 22x^2 - 147x - 90$
$= -(x+6)(x+1)^2(x-3)(x-5)$

9. $f(x) = 2x^4$ is in the form $f(x) = ax^n$.
$|a| = 2 > 1,$ so the graph is narrower than $f(x) = x^4.$ It includes the points $(-2, 32),$ $(-1, 2),$ $(0, 0),$ $(1, 2),$ and $(2, 32)$. Connect these points with a smooth curve.

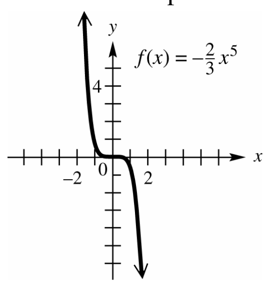


10. $f(x) = \frac{1}{4}x^6$ is in the form $f(x) = ax^n$.
$|a| = \frac{1}{4} < 1,$ so the graph is broader than that of $f(x) = x^6$ but has the same general shape. It includes the points $(-2, 16),$ $\left(-1, \frac{1}{4}\right),$ $(0, 0),$ $\left(1, \frac{1}{4}\right),$ and $(2, 16)$. Connect these points with a smooth curve.



11. $f(x) = -\frac{2}{3}x^5$ is in the form $f(x) = ax^n$.

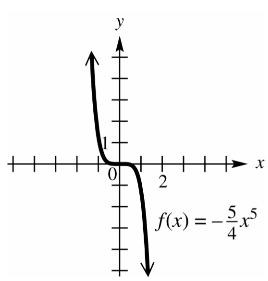
$|a| = \frac{2}{3} < 1,$ so the graph is broader than that of $f(x) = x^5$. Since $a = -\frac{2}{3}$ is a negative, the graph is the reflection of $f(x) = \frac{2}{3}x^5$ about the *x*-axis. It includes the points $\left(-2, \frac{64}{3}\right),$ $\left(-1, \frac{2}{3}\right),$ $(0, 0),$ $\left(1, -\frac{2}{3}\right),$ and $\left(2, -\frac{64}{3}\right).$ Connect these points with a smooth curve.



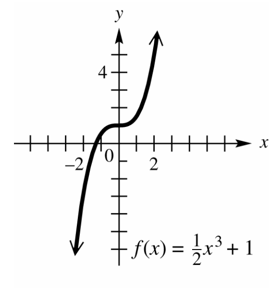
12. $f(x) = -\frac{5}{4}x^5$ is in the form $f(x) = ax^n$.

$|a| = \frac{5}{4} > 1,$ so the graph is narrower than $f(x) = x^5$. The negative sign causes the graph to be the reflection of the graph of $f(x) = \frac{5}{4}x^5$ about the *x*-axis.

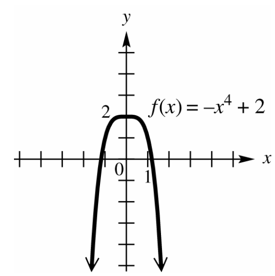
The graph includes the points $(-2, 40),$ $\left(-1, \frac{5}{4}\right),$ $(0, 0),$ $\left(1, -\frac{5}{4}\right)$ and $(2, -40).$



13. $f(x) = \frac{1}{2}x^3 + 1$ is in the form $f(x) = ax^n + k,$ with $|a| = \frac{1}{2} < 0$ and $k = 1.$ The graph of $f(x) = \frac{1}{2}x^3 + 1$ looks like $y = x^3$ but is broader and is translated 1 unit up. The graph includes the points $(-2, -3),$ $\left(-1, \frac{1}{2}\right),$ $(0, 1),$ $\left(1, \frac{3}{2}\right),$ and $(2, 5).$

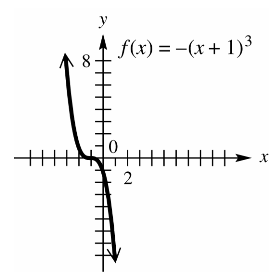


14. $f(x) = -x^4 + 2$ is in the form $f(x) = ax^n + k,$ with $|a| = 1$ and $k = 2.$ The graph is the same as $f(x) = x^4$ when reflected about the *x*-axis and translated 2 units up. The graph includes the points $(-2, -14),$ $(-1, 1),$ $(0, 2),$ $(1, 1),$ and $(2, -14).$



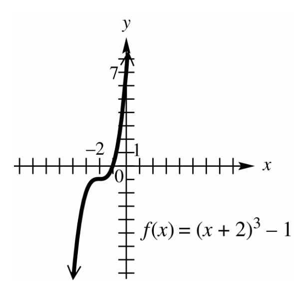
15. $f(x) = -(x+1)^3 = -[x - (-1)]^3$

The graph can be obtained by reflecting the graph of $f(x) = x^3$ about the *x*-axis and then translating it 1 unit to the left.

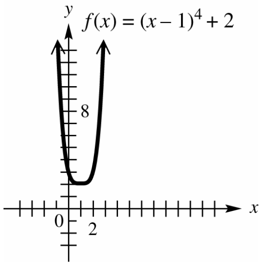


16. $f(x) = (x+2)^3 - 1 = [x - (-2)]^3 + (-1)$

The graph is the graph of $f(x) = x^3$ translated 1 unit down and 2 units to the left.

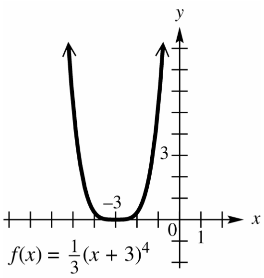


17. $f(x)=(x-1)^4+2$ This graph has the same shape as $y=x^4$, but is translated 1 unit to the right and 2 units up.



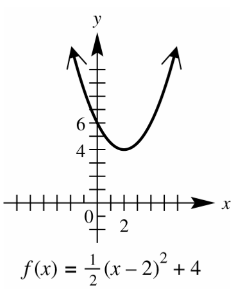
18. $f(x)=\frac{1}{3}(x+3)^4$ is in the form $f(x)=a(x-h)^n$, with $|a|=\frac{1}{3}<1$.

The graph is broader than that of $f(x)=x^4$. Since $h=-3$, the graph has been translated 3 units to the left.

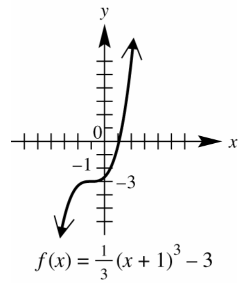


19. $f(x)=\frac{1}{2}(x-2)^2+4$ is in the form $f(x)=a(x-h)^n+k$, with $|a|=\frac{1}{2}<1$.

The graph is broader than that of $f(x)=x^2$. Since $h=2$, the graph has been translated 2 units to the right. Also, since $k=4$, the graph has been translated 4 units up.



20. $f(x)=\frac{1}{3}(x+1)^3-3=\frac{1}{3}\left[x-(-1)\right]^3+(-3)$ is in the form $f(x)=a(x-h)^n+k$, with $|a|=\frac{1}{3}<1$. The graph is broader than that of $f(x)=x^3$. Since $h=-1$, the graph has been translated 1 units to the left. Also, since $k=-3$, the graph has been translated 3 units down.



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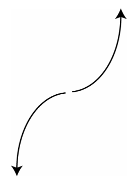

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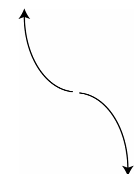

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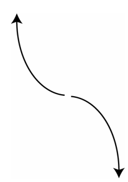

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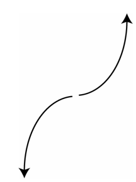

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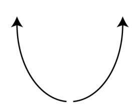

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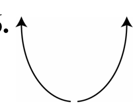
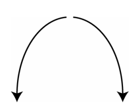
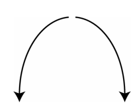


27.



28.



29. $f(x)=x^3+5x^2+2x-8$

Step 1: p must be a factor of $a_0=-8$ and q must be a factor of $a_3=1$. Thus, p can be ±1, ±2, ±4, ±8 and q can be ±1. The possible rational zeros, $\frac{p}{q}$, are ±1, ±2, ±4, and ±8. The remainder theorem shows that – 4 is a zero.

−4)	1	5	2	−8
		−4	−4	8
	1	1	−2	0

The new quotient polynomial is $x^2 + x - 2$.

$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x + 2 = 0 \Rightarrow x = -2 \quad \text{or} \quad x - 1 = 0 \Rightarrow x = 1$$

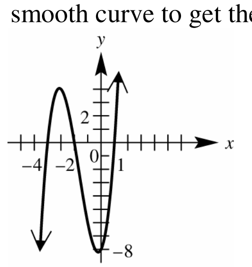
The rational zeros are – 4, –2, and 1. Since the three zeros are – 4, –2, and 1, the factors are $x + 4$, $x + 2$, and $x - 1$ and thus $f(x) = (x+4)(x+2)(x-1)$.

Step 2: $f(0) = -8$, so plot $(0, -8)$.

Step 3: The x-intercepts divide the x-axis into four intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -4)$	–5	–18	Negative	Below
$(-4, -2)$	–3	4	Positive	Above
$(-2, 1)$	0	–8	Negative	Below
$(1, \infty)$	2	24	Positive	Above

Plot the x-intercepts, y-intercept and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = x^3 + 5x^2 + 2x - 8$

30. $f(x) = x^3 + 3x^2 - 13x - 15$

Step 1: p must be a factor of $a_0 = -15$ and q must be a factor of $a_3 = 1$. Thus, p can be ±1, ±3, ±5, ±15 and q can be ±1. The possible rational zeros, $\frac{p}{q}$, are ±1, ±3, ±5, and ±15.

The remainder theorem shows that – 5 is a zero.

$$\begin{array}{r|rrrr} -5 & 1 & 3 & -13 & -15 \\ & & -5 & 10 & 15 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

The new quotient polynomial is $x^2 - 2x - 3$.

$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x + 1 = 0 \Rightarrow x = -1 \quad \text{or} \quad x - 3 = 0 \Rightarrow x = 3$$

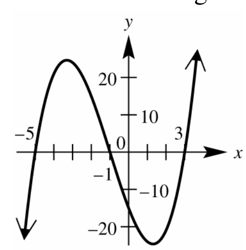
The rational zeros are – 5, –1, and 3. Since the three zeros are – 5, –1, and 3, the factors are $x + 5$, $x + 1$, and $x - 3$ and thus $f(x) = (x+5)(x+1)(x-3)$.

Step 2: $f(0) = -15$, so plot $(0, -15)$.

Step 3: The x-intercepts divide the x-axis into four intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -5)$	–6	–45	Negative	Below
$(-5, -1)$	–2	15	Positive	Above
$(-1, 3)$	0	–15	Negative	Below
$(3, \infty)$	4	45	Positive	Above

Plot the x-intercepts, y-intercept and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = x^3 + 3x^2 - 13x - 15$

31. $f(x) = 2x(x-3)(x+2)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$2x = 0 \Rightarrow x = 0$ or $x - 3 = 0 \Rightarrow x = 3$ or $x + 2 = 0 \Rightarrow x = -2$

The three zeros, –2, 0, and 3, divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0, 0)$.

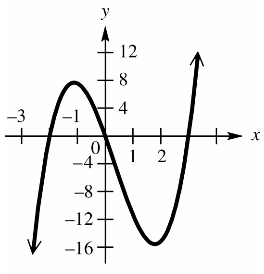
Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	–3	–36	Negative	Below
$(-2, 0)$	–1	8	Positive	Above
$(0, 3)$	1	–12	Negative	Below
$(3, \infty)$	4	48	Positive	Above

(*continued on next page*)

(*continued from page 301*)

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = 2x(x-3)(x+2)$

32. $f(x) = x^2(x+1)(x-1)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x^2 = 0$ or $x+1=0$ or $x-1=0$
$x=0$ $\quad x=-1$ $\quad x=1$

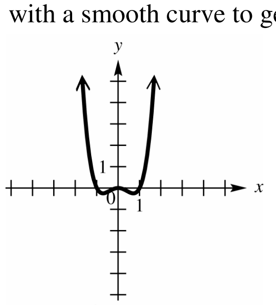
The zeros are –1, 0, and 1; divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0)=0$, so plot $(0,0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	–2	12	Positive	Above
$(-1, 0)$	$-\frac{1}{2}$	$-\frac{3}{16}$	Negative	Below
$(0, 1)$	$\frac{1}{2}$	$-\frac{3}{16}$	Negative	Below
$(1, \infty)$	2	12	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x^2(x+1)(x-1)$

33. $f(x) = x^2(x-2)(x+3)^2$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x^2 = 0 \Rightarrow x = 0$ or $x-2=0 \Rightarrow x=2$ or
$(x+3)^2 = 0 \Rightarrow x+3=0 \Rightarrow x=-3$

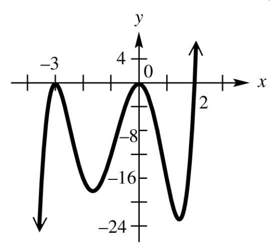
The zeros are –3, 0, and 2; divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0)=0$, so plot $(0,0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–4	–96	Negative	Below
$(-3, 0)$	–1	–12	Negative	Below
$(0, 2)$	1	–16	Negative	Below
$(2, \infty)$	3	324	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x^2(x-2)(x+3)^2$

34. $f(x) = x^2(x-5)(x+3)(x-1)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x^2 = 0$ or $x-5=0$ or $x+3=0$ or $x-1=0$
$x=0$ $\quad x=5$ $\quad x=-3$ $\quad x=1$

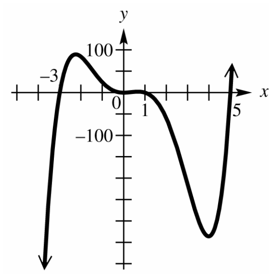
The zeros are –3, 0, 1, and 5, which divide the x-axis into five regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0)=0$, so plot $(0,0)$.

Step 3: The x-intercepts divide the x-axis into five intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	-4	-720	Negative	Below
$(-3, 0)$	-1	24	Positive	Above
$(0, 1)$	$\frac{1}{2}$	$\frac{63}{32} \approx 1.97$	Positive	Above
$(1, 5)$	2	-60	Negative	Below
$(5, \infty)$	6	1620	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x^2(x-5)(x+3)(x-1)$

35. $f(x) = (3x-1)(x+2)^2$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$3x - 1 = 0 \Rightarrow x = \frac{1}{3}$ or $x + 2 = 0 \Rightarrow x = -2$

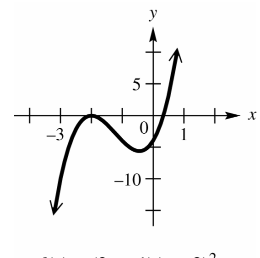
The zeros are –2 and $\frac{1}{3}$, which divide the x-axis into three regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = -4,$ so plot $(0, -4)$.

Step 3: The x-intercepts divide the x-axis into three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-3	-10	Negative	Below
$\left(-2, \frac{1}{3}\right)$	0	-4	Negative	Below
$\left(\frac{1}{3}, \infty\right)$	1	18	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = (3x-1)(x+2)^2$

36. $f(x) = (4x+3)(x+2)^2$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$4x + 3 = 0 \Rightarrow x = -\frac{3}{4}$ or $x + 2 = 0 \Rightarrow x = -2$

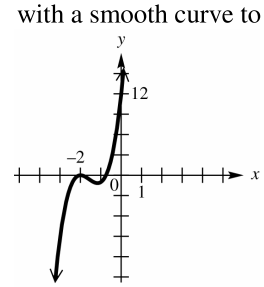
The zeros are –2 and $-\frac{3}{4}$, which divide the x-axis into three regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 12,$ so plot $(0, 12)$.

Step 3: The x-intercepts divide the x-axis into three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-3	-9	Negative	Below
$\left(-2, -\frac{3}{4}\right)$	-1	-1	Negative	Below
$\left(-\frac{3}{4}, \infty\right)$	0	12	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = (4x+3)(x+2)^2$

37. $f(x) = x^3 + 5x^2 - x - 5$

Step 1: Find the zeros of the function by factoring by grouping.

$$f(x) = x^3 + 5x^2 - x - 5 = x^2(x+5) - 1(x+5)$$
$$= (x+5)(x^2-1) = (x+5)(x+1)(x-1)$$

$x + 5 = 0 \Rightarrow x = -5$ or $x + 1 = 0 \Rightarrow x = -1$ or $x - 1 = 0 \Rightarrow x = 1$

(*continued on next page*)

(*continued from page 303*)

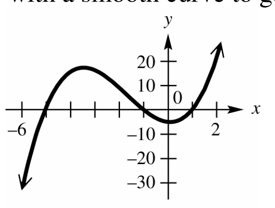
The zeros are –5, –1, and 1, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = -5$, so plot $(0, -5)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -5)$	–6	–35	Negative	Below
$(-5, -1)$	–2	9	Positive	Above
$(-1, 1)$	0	–5	Negative	Below
$(1, \infty)$	2	21	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = x^3 + 5x^2 - x - 5$

38. $f(x) = x^3 + x^2 - 36x - 36$

Step 1: Find the zeros of the function by factoring by grouping.

$$f(x) = x^3 + x^2 - 36x - 36$$
$$= x^2(x+1) - 36(x+1)$$
$$= (x+1)(x^2 - 36) = (x+1)(x-6)(x+6)$$

$x + 1 = 0$ or $x - 6 = 0$ or $x + 6 = 0$

$x = -1$ $\quad x = 6$ $\quad x = -6$

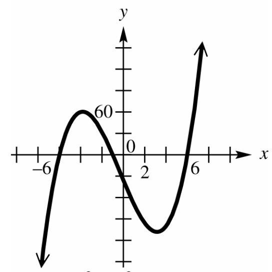
The zeros are –6, –1, and 6, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = -36$, so plot $(0, -36)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -6)$	–7	–78	Negative	Below
$(-6, -1)$	–4	60	Positive	Above
$(-1, 6)$	0	–36	Negative	Below
$(6, \infty)$	7	104	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x^3 + x^2 - 36x - 36$

39. $f(x) = x^3 - x^2 - 2x$

Step 1: Find the zeros of the function by factoring out the common factor, x, and then factoring the resulting quadratic factor. Set each factor equal to 0 and solve the resulting equations.

$$f(x) = x^3 - x^2 - 2x$$
$$= x(x^2 - x - 2) = x(x+1)(x-2)$$

$x = 0$ or $x + 1 = 0$ or $x - 2 = 0$

$x = -1$ $\quad x = 2$

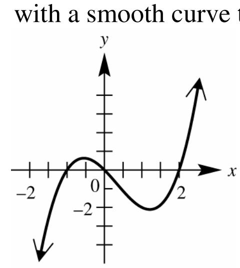
The zeros are –1, 0, and 2, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0, 0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	–2	–8	Negative	Below
$(-1, 0)$	$-\frac{1}{2}$	$\frac{5}{8}$	Positive	Above
$(0, 2)$	1	–2	Negative	Below
$(2, \infty)$	3	12	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x^3 - x^2 - 2x$

40. $f(x) = 3x^4 + 5x^3 - 2x^2$

Step 1: Find the zeros of the function by factoring out the common factor, *x*, and then factoring the resulting quadratic factor. Set each factor equal to 0 and solve the resulting equations.

$$f(x) = 3x^4 + 5x^3 - 2x^2 = x^2(3x^2 + 5x - 2)$$
$$= x^2(x+2)(3x-1)$$

$x^2 = 0 \Rightarrow x = 0$ or $x + 2 = 0 \Rightarrow x = -2$ or $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

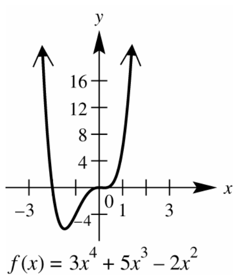
The zeros are –2, 0, and $\frac{1}{3}$, which divide the *x*-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0,0)$.

Step 3: The *x*-intercepts divide the *x*-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty, -2)$	-3	90	Positive	Above
$(-2, 0)$	-1	-4	Negative	Below
$\left(0, \frac{1}{3}\right)$	$\frac{1}{6}$	$-\frac{13}{432} \approx -.03$	Negative	Below
$\left(\frac{1}{3}, \infty\right)$	1	6	Positive	Above

Plot the *x*-intercepts, *y*-intercept (which is also an *x*-intercept in this exercise), and test points with a smooth curve to get the graph.



41. $f(x) = 2x^3(x^2 - 4)(x - 1)$

Step 1: Find the zeros of the function by factoring the difference of two squares in the polynomial. Set each factor equal to 0 and solve the resulting equations.

$$f(x) = 2x^3(x^2 - 4)(x - 1)$$
$$= 2x^3(x+2)(x-2)(x-1)$$

$2x^3 = 0 \Rightarrow x = 0$ or $x + 2 = 0 \Rightarrow x = -2$ or $x - 2 = 0 \Rightarrow x = 2$ or $x - 1 = 0 \Rightarrow x = 1$

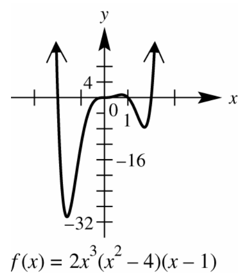
The zeros are –2, 0, 1, and 2, which divide the *x*-axis into five regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0,0)$.

Step 3: The *x*-intercepts divide the *x*-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty, -2)$	-3	1080	Positive	Above
$(-2, 0)$	-1	-12	Negative	Below
(0, 1)	$\frac{1}{2}$	$\frac{15}{32} \approx .5$	Positive	Above
(1, 2)	$\frac{3}{2}$	$-\frac{189}{32} \approx -5.9$	Negative	Below
$(2, \infty)$	3	540	Positive	Above

Plot the *x*-intercepts, *y*-intercept (which is also an *x*-intercept in this exercise), and test points with a smooth curve to get the graph.



42. $f(x) = x^2(x-3)^3(x+1)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x^2 = 0 \Rightarrow x = 0$ or $(x-3)^3 = 0 \Rightarrow x = 3$ or $x + 1 = 0 \Rightarrow x = -1$

The zeros are –1, 0, and 3, which divide the *x*-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0,0)$.

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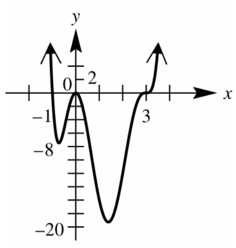
(*continued from page 305*)

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	-2	500	Positive	Above
$(-1, 0)$	$-\frac{1}{2}$	$-\frac{343}{64} \approx -5.4$	Negative	Below
$(0, 3)$	1	-16	Negative	Below
$(3, \infty)$	4	80	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x^2(x-3)^3(x+1)$

43. $f(x) = 2x^3 - 5x^2 - x + 6$

Step 1: Find the zeros of the function. p must be a factor of $a_0 = 6$ and q must be a factor of $a_3 = 2$. Thus, p can be $\pm1, \pm2, \pm3$, or ±6 and q can be ±1 or ±2. The possible rational zeros, $\frac{p}{q}$, are $\pm1, \pm\frac{3}{2}$ $\pm2, \pm3$, or ±6. The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrr} 2 & 2 & -5 & -1 & 6 \\ & & 4 & -2 & -6 \\ \hline & 2 & -1 & -3 & 0 \end{array}$$

The new quotient polynomial is $2x^2 - x - 3$.

$$2x^2 - x - 3 = 0$$
$$(2x-3)(x+1) = 0$$
$$2x - 3 = 0 \Rightarrow x = \tfrac{3}{2} \quad \text{or} \quad x + 1 = 0 \Rightarrow x = -1$$

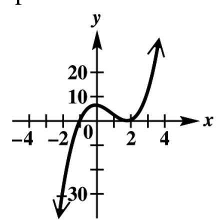
The rational zeros are -1, $\frac{3}{2}$, and 2. Since the three zeros are -1, $\frac{3}{2}$, and 2, the factors are $x + 1$, $x - \frac{3}{2}$, and $x - 2$ and thus $f(x) = (x+1)\left(x - \frac{3}{2}\right)(x-2)$.

Step 2: $f(0) = 6$, so plot $(0, 6)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	-2	-28	Negative	Below
$\left(-1, \frac{3}{2}\right)$	1	2	Positive	Above
$\left(\frac{3}{2}, 2\right)$	$\frac{7}{4}$	$-\frac{11}{32}$	Negative	Below
$(2, \infty)$	3	12	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.



$f(x) = 2x^3 - 5x^2 - x + 6$

44. $f(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$

Step 1: Find the zeros of the function. p must be a factor of $a_0 = -2$ and q must be a factor of $a_4 = 2$. Thus, p can be ±1 or ±2 and q can be ±1 or ±2. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{2}$, ±1, or ±2. The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrrr} 2 & 2 & 1 & -6 & -7 & -2 \\ & & 4 & 10 & 8 & 2 \\ \hline & 2 & 5 & 4 & 1 & 0 \end{array}$$

The new quotient polynomial is $2x^3 + 5x^2 + 4x + 1$. Find the zeros of this function. p must be a factor of $a_0 = 1$ and q must be a factor of $a_3 = 2$. Thus, p can be ±1 and q can be ±1 or ±2. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{2}$ or ±1. The remainder theorem shows that -1 is a zero.

$$\begin{array}{r|rrrr} -1 & 2 & 5 & 4 & 1 \\ & & -2 & -3 & -1 \\ \hline & 2 & 3 & 1 & 0 \end{array}$$

The new quotient polynomial is $2x^2 + 3x + 1$.

$$2x^2 + 3x + 1 = 0$$
$$(2x+1)(x+1) = 0$$
$$2x + 1 = 0 \Rightarrow x = -\tfrac{1}{2} \quad \text{or} \quad x + 1 = 0 \Rightarrow x = -1$$

The rational zeros are $-1, -1, -\frac{1}{2}$, and 2.

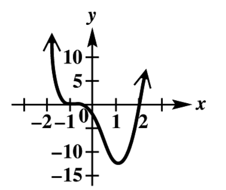
Since the four zeros are $-1, -1, -\frac{1}{2}$, and 2, the factors are $x+1$, $x+1$, $x+\frac{1}{2}$, and $x-2$, and thus $f(x)=(x+1)^2\left(x+\frac{1}{2}\right)(x-2)$.

Step 2: $f(0)=-2$, so plot $(0,-2)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-1)$	-2	12	Positive	Above
$\left(-1,-\frac{1}{2}\right)$	$-\frac{3}{4}$	$\frac{11}{128}$	Positive	Above
$\left(-\frac{1}{2},2\right)$	1	-12	Negative	Below
$(2,\infty)$	3	112	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.



$f(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$

45. $f(x)=3x^4-7x^3-6x^2+12x+8$

Step 1: Find the zeros of the function.
p must be a factor of $a_0=8$ and q must be a factor of $a_4=3$. Thus, p can be ±1, ±2, ±4, or ±8, and q can be ±1 or ±3. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{3},\pm\frac{2}{3},\pm1,\pm\frac{4}{3},\pm2,\pm\frac{8}{3},\pm4,$ or ±8. The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrrr} 2 & 3 & -7 & -6 & 12 & 8 \\ & & 6 & -2 & -16 & -8 \\ \hline & 3 & -1 & -8 & -4 & 0 \end{array}$$

The new quotient polynomial is $3x^3-x^2-8x-4$. Find the zeros of this function. p must be a factor of $a_0=-4$ and q must be a factor of $a_3=3$. Thus, p can be ±1, ±2, or ±4, and q can be ±1 or ±3.

The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{3},\pm\frac{2}{3},\pm1,\pm\frac{4}{3},\pm2,$ or ±4. The remainder theorem shows that −1 is a zero.

$$\begin{array}{r|rrrr} -1 & 3 & -1 & -8 & -4 \\ & & -3 & 4 & 4 \\ \hline & 3 & -4 & -4 & 0 \end{array}$$

The new quotient polynomial is $3x^2-4x-4$.

$3x^2-4x-4=0$
$(3x+2)(x-2)=0$
$3x+2=0 \Rightarrow x=-\frac{2}{3}$ or $x-2=0 \Rightarrow x=2$

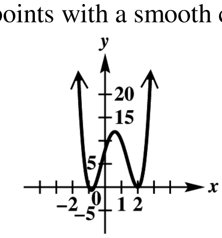
The rational zeros are -1, $-\frac{2}{3}$, 2, and 2. Since the four zeros are -1, $-\frac{2}{3}$, 2, and 2, the factors are $x+1$, $x+\frac{2}{3}$, $x-2$, and $x-2$, and thus $f(x)=(x+1)\left(x+\frac{2}{3}\right)(x-2)^2$.

Step 2: $f(0)=8$, so plot $(0,8)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-1)$	-2	64	Positive	Above
$\left(-1,-\frac{2}{3}\right)$	$-\frac{5}{6}$	$-\frac{289}{432}$	Negative	Below
$\left(-\frac{2}{3},2\right)$	1	10	Positive	Above
$(2,\infty)$	3	44	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.



$f(x) = 3x^4 - 7x^3 - 6x^2 + 12x + 8$

46. $f(x)=x^4+3x^3-3x^2-11x-6$

Step 1: Find the zeros of the function.
p must be a factor of $a_0=-6$ and q must be a factor of $a_4=1$. Thus, p can be ±1, ±2, ±3, or ±6, and q can be ±1. The possible rational zeros, $\frac{p}{q}$, are $\pm1,\pm2,\pm3,$ or ±6.

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The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrrr} 2 & 1 & 3 & -3 & -11 & -6 \\ & & 2 & 10 & 14 & 6 \\ \hline & 1 & 5 & 7 & 3 & 0 \end{array}$$

The new quotient polynomial is x^3+5x^2+7x+3. Find the zeros of this function. p must be a factor of $a_0=3$ and q must be a factor of $a_3=1$. Thus, p can be ±1 or ±3, and q can be ±1. The possible rational zeros, $\frac{p}{q}$, are ±1 or ±3. The remainder theorem shows that −1 is a zero.

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 7 & 3 \\ & & -1 & -4 & -3 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

The new quotient polynomial is x^2+4x+3.

$$x^2+4x+3=0$$
$$(x+3)(x+1)=0$$
$$x=3=0 \Rightarrow x=-3 \quad \text{or} \quad x+1=0 \Rightarrow x=-1$$

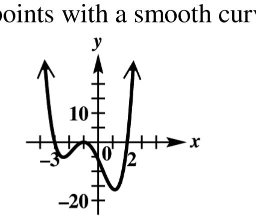
The rational zeros are – 3, −1, −1, and 2. Since the four zeros are – 3, −1, −1, and 2, the factors are $x+3$, $x+1$, $x+1$, and $x-2$, and thus $f(x)=(x+3)(x+1)^2(x-2)$.

Step 2: $f(0)=-6$, so plot $(0,-6)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	– 4	54	Positive	Above
$(-3, -1)$	−2	−4	Negative	Below
$(-1, 2)$	1	−16	Negative	Below
$(2, \infty)$	3	96	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.



$f(x)=x^4+3x^3-3x^2-11x-6$

47. $f(x)=2x^2-7x+4$; 2 and 3

Use synthetic division to find $f(2)$ and $f(3)$.

$$\begin{array}{r|rrr} 2 & 2 & -7 & 4 \\ & & 4 & -6 \\ \hline & 2 & -3 & -2 \end{array} \qquad \begin{array}{r|rrr} 3 & 2 & -7 & 4 \\ & & 6 & -3 \\ \hline & 2 & -1 & 1 \end{array}$$

Since $f(2)=-2$ is negative and $f(3)=1$ is positive, there is a zero between 2 and 3.

48. $f(x)=3x^2-x-4$; 1 and 2

Use synthetic division to find $f(1)$ and $f(2)$.

$$\begin{array}{r|rrr} 1 & 3 & -1 & -4 \\ & & 3 & 2 \\ \hline & 3 & 2 & -2 \end{array} \qquad \begin{array}{r|rrr} 2 & 3 & -1 & -4 \\ & & 6 & 10 \\ \hline & 3 & 5 & 6 \end{array}$$

Since $f(1)=-2$ is negative and $f(2)=6$ is positive, there is a zero between 1 and 2.

49. $f(x)=2x^3-5x^2-5x+7$; 0 and 1

Since $f(0)=7$ can easily be determined, use synthetic division only to find $f(1)$.

$$\begin{array}{r|rrrr} 1 & 2 & -5 & -5 & 7 \\ & & 2 & -3 & -8 \\ \hline & 2 & -3 & -8 & -1 \end{array}$$

Since $f(0)=7$ is positive and $f(1)=-1$ is negative, there is a zero between 0 and 1.

50. $f(x)=2x^3-9x^2+x+20$; 2 and 2.5

Use synthetic division to find $f(2)$ and $f(2.5)$.

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 1 & 20 \\ & & 4 & -10 & -18 \\ \hline & 2 & -5 & -9 & 2 \end{array} \qquad \begin{array}{r|rrrr} 2.5 & 2 & -9 & 1 & 20 \\ & & 5 & -10 & -22.5 \\ \hline & 2 & -4 & -9 & -2.5 \end{array}$$

Since $f(2)=2$ is positive and $f(2.5)=-2.5$ is negative, there is a zero between 2 and 2.5.

51. $f(x)=2x^4-4x^2+4x-8$; 1 and 2

Use synthetic division to find $f(1)$ and $f(2)$.

$$\begin{array}{r|rrrrr} 1 & 2 & 0 & -4 & 4 & -8 \\ & & 2 & 2 & -2 & 2 \\ \hline & 2 & 2 & -2 & 2 & -6 \end{array} \qquad \begin{array}{r|rrrrr} 2 & 2 & 0 & -4 & 4 & -8 \\ & & 4 & 8 & 8 & 24 \\ \hline & 2 & 4 & 4 & 12 & 16 \end{array}$$

Since $f(1)=-6$ is negative and $f(2)=16$ is positive, there is a zero between 1 and 2.

52. $f(x)=x^4-4x^3-x+3$; .5 and 1

Use synthetic division to find $f(1)$ and $f(.5)$

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 0 & -1 & 3 \\ & & 1 & -3 & -3 & -4 \\ \hline & 1 & -3 & -3 & -4 & -1 \end{array}$$

.5)	1	−4	0	−1	3
		.5	−1.75	−.875	−.9375
	1	−3.5	−1.75	−1.875	2.0625

Since $f(1) = -1$ is negative and $f(.5) = 2.0625$ is positive, there is a zero between .5 and 1.

53. $f(x) = x^4 + x^3 - 6x^2 - 20x - 16$; 3.2 and 3.3

Use synthetic division to find $f(3.2)$ and $f(3.3)$.

3.2)	1	1	−6	−20	−16
		3.2	13.44	23.808	12.1856
	1	4.2	7.44	3.808	−3.8144

3.3)	1	1	−6	−20	−16
		3.3	14.19	27.027	23.1891
	1	4.3	8.19	7.027	7.1891

Since $f(3.2) = -3.8144$ is negative and $f(3.3) = 7.1892$ is positive, there is a zero between 3.2 and 3.3.

54. $f(x) = x^4 - 2x^3 - 2x^2 - 18x + 5$; 3.7 and 3.8

Use synthetic division to find $f(3.7)$ and $f(3.8)$.

3.7)	1	−2	−2	−18	5
		3.7	6.29	15.873	−7.8699
	1	1.7	4.29	−2.127	−2.8699

3.8)	1	−2	−2	−18	5
		3.8	6.84	18.392	1.4896
	1	1.8	4.84	.392	6.4896

Since $f(3.7) = -2.8699$ is negative and $f(3.8) = 6.4896$ is positive, there is a zero between 3.7 and 3.8.

55. $f(x) = x^4 - 4x^3 - 20x^2 + 32x + 12$; -1 and 0

Since $f(0) = 12$ can easily be determined, use synthetic division only to find $f(-1)$.

−1)	1	−4	−20	32	12
		−1	5	15	−47
	1	−5	−15	47	−35

Since $f(-1) = -35$ is negative and $f(0) = 12$ is positive, there is a zero between -1 and 0.

56. $f(x) = x^5 + 2x^4 + x^3 + 3$; -1.8 and -1.7

Use synthetic division to find $f(-1.8)$ and $f(-1.7)$.

−1.8)	1	2	1	0	0	3
		−1.8	−.36	−1.152	2.0736	−3.73248
	1	.2	.64	−1.152	2.0736	−.73248

−1.7)	1	2	1	0	0	3
		−1.7	−.51	−.833	1.4161	−2.40737
	1	.3	.49	−.833	1.4161	.59263

Since $f(-1.8) = -.73248$ is negative and $f(-1.7) = .59263$ is positive, there is a zero between -1.8 and -1.7.

57. $f(x) = x^4 - x^3 + 3x^2 - 8x + 8$; no real zero greater than 0.

Since $f(x)$ has real coefficients and the leading coefficient, 1, is positive, use the boundedness theorem. Divide $f(x)$ synthetically by $x - 2$. Since $2 > 0$ and all numbers in the last row are nonnegative, $f(x)$ has no zero greater than 2.

2)	1	−1	3	−8	8
		2	2	10	4
	1	1	5	2	12

58. $f(x) = 2x^5 - x^4 + 2x^3 - 2x^2 + 4x - 4$; no real zero greater than 1.

Since $f(x)$ has real coefficients and the leading coefficient, 2, is positive, use the boundedness theorem. Divide $f(x)$ synthetically by $x - 1$. Since $1 > 0$ and all numbers in the last row are nonnegative, $f(x)$ has no zero greater than 1.

1)	2	−1	2	−2	4	−4
		2	1	3	1	5
	2	1	3	1	5	1

59. $f(x) = x^4 + x^3 - x^2 + 3$; no real zero less than −2.

Since $f(x)$ has real coefficients, use the boundedness theorem. Divide $f(x)$ synthetically by $x + 2 = x - (-2)$. Since $-2 < 0$ and the numbers in the last row alternate in sign, $f(x)$ has no zero less than -2.

−2)	1	1	−1	0	3
		−2	2	−2	4
	1	−1	1	−2	7

60. $f(x) = x^5 + 2x^3 - 2x^2 + 5x + 5$; no real zero less than -1.
Since $f(x)$ has real coefficients, use the boundedness theorem. Divide $f(x)$ synthetically by $x + 1 = x - (-1)$. Since $-1 < 0$ and the numbers in the last row alternate in sign, $f(x)$ has no zero less than -1.

-1	1	0	2	-2	5	5
		-1	1	-3	5	-10
	1	-1	3	-5	10	-5

61. $f(x) = 3x^4 + 2x^3 - 4x^2 + x - 1$; no real zero greater than 1
Since $f(x)$ has real coefficients and the leading coefficient, 3, is positive, use the boundedness theorem. Divide $f(x)$ synthetically by $x - 1$. Since $1 > 0$ and all numbers in the last row are nonnegative, $f(x)$ has no zero greater than 1.

1	3	2	-4	1	-1
		3	5	1	2
	3	5	1	2	1

62. $f(x) = 3x^4 + 2x^3 - 4x^2 + x - 1$; no real zero less than -2
Since $f(x)$ has real coefficients, use the boundedness theorem. Divide $f(x)$ synthetically by $x + 2 = x - (-2)$. Since $-2 < 0$ and the numbers in the last row alternate in sign, $f(x)$ has no zero less than -2.

-2	3	2	-4	1	-1
		-6	8	-8	14
	3	-4	4	-7	13

63. $f(x) = x^5 - 3x^3 + x + 2$; no real zero greater than 2.
Since $f(x)$ has real coefficients and the leading coefficient, 1, is positive, use the boundedness theorem. Divide $f(x)$ synthetically by $x - 2$. Since $2 > 0$ and all numbers in the last row are nonnegative, $f(x)$ has no zero greater than 2.

2	1	0	-3	0	1	2
		2	4	2	4	10
	1	2	1	2	5	12

64. $f(x) = x^5 - 3x^3 + x + 2$; no real zero less than -3.
Since $f(x)$ has real coefficients, use the boundedness theorem. Divide $f(x)$ synthetically by $x + 3 = x - (-3)$. Since $-3 < 0$ and the numbers in the last row alternate in sign, $f(x)$ has no zero less than -3.

-3	1	0	-3	0	1	2
		-3	9	-18	54	-165
	1	-3	6	-18	55	-163

65. The graph shows that the zeros are –6, 2, and 5. The polynomial function has the form $f(x) = a(x+6)(x-2)(x-5)$. Since (0, 30) is on the graph, $f(0) = 30$.
$f(0) = a(0+6)(0-2)(0-5) \Rightarrow 30 = 60a \Rightarrow \frac{1}{2} = a$
A cubic polynomial that has the graph shown is $f(x) = \frac{1}{2}(x+6)(x-2)(x-5)$ or $f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 - 16x + 30$.

66. The graph shows that the zeros are –5 and 3. There is a turning point at 3, so 3 is a zero of multiplicity 2. The polynomial function has the form $f(x) = a(x-3)^2(x+5)$. Since (0, 9) is on the graph, $f(0) = 9$.
$f(0) = a(0-3)^2(0+5) \Rightarrow 9 = 45a \Rightarrow \frac{1}{5} = a$
A cubic polynomial that has the graph shown is $f(x) = \frac{1}{5}(x-3)^2(x+5)$ or $f(x) = \frac{1}{5}x^3 - \frac{1}{5}x^2 - \frac{21}{5}x + 9$.

67. The graph shows that the zeros are -1 and 1. There is a turning point at both points. Since the graph crosses and is tangent to the x-axis at both $x = -1$ and $x = 1$, these are zeros with odd multiplicity. This has to be 3 in order to find a polynomial of least degree. The polynomial function has the form $f(x) = a(x-1)^3(x+1)^3$. Since $(0, -1)$ is on the graph, $f(0) = -1$.
$f(0) = -1 = a(0-1)^3(0+1)^3 \Rightarrow a = 1$

So the function is

$$f(x) = (x-1)^3(x+1)^3$$
$$= (x^3 - 3x^2 + 3x - 1)(x^3 + 3x^2 + 3x + 1)$$
$$= x^6 - 3x^4 + 3x^2 - 1$$

68. The graph shows that the zeros are -1 and 1. The graph crosses and is tangent to the x-axis at $x = 1$, so this is a zero with odd multiplicity. This has to be 3 in order to find a polynomial of least degree. The polynomial function has the form $f(x) = a(x-1)^3(x+1)$. Since (0, 2) is on the graph, $f(0) = 2$.

$f(0) = 2 = a(0-1)^3(0+1) \Rightarrow 2 = -a \Rightarrow a = -2$

So the function is

$$f(x) = -2(x-1)^3(x+1)$$
$$= -2(x^3 - 3x^2 + 3x - 1)(x+1)$$
$$= -2x^4 + 4x^3 - 4x + 2$$

69. The graph shows that the zeros are -3 and 3. The graph is tangent to the x-axis at both $x = -3$ and $x = 3$, so these are zeros with even multiplicity. This has to be 2 in order to find a polynomial of least degree. The polynomial function has the form

$f(x) = a(x-3)^2(x+3)^2$. Since (0, 81) is on the graph, $f(0) = 81$.

$f(0) = 81 = a(0-3)^2(0+3)^2 \Rightarrow 81 = 81a \Rightarrow$
$a = 1$.

So the function is

$$f(x) = (x-3)^2(x+3)^2$$
$$= (x^2 - 6x + 9)(x^2 + 6x + 9)$$
$$= x^4 - 18x^2 + 81$$

70. The graph shows that the zeros are -1 and 2. The graph is tangent to the x-axis at $x = -1$, so this is a zero with even multiplicity. This has to be 2 in order to find a polynomial of least degree. The polynomial function has the form $f(x) = a(x+1)^2(x-2)$. Since (0, 4) is on the graph, $f(0) = 4$.

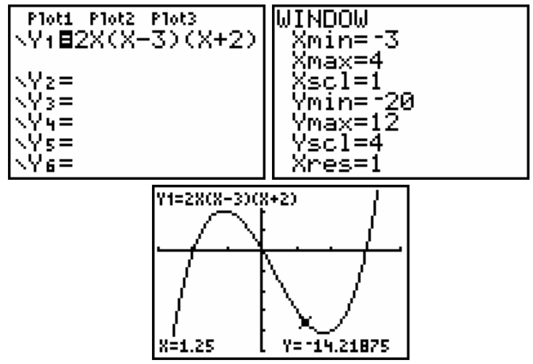
$f(0) = 4 = a(0+1)^2(0-2) \Rightarrow 4 = -2a \Rightarrow$
$a = -2$.

So the function is

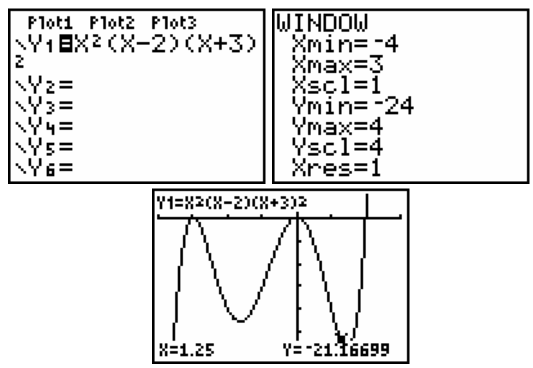
$$f(x) = -2(x-1)^2(x-2)$$
$$= -2(x^2 - 2x + 1)(x-2)$$
$$= -2x^3 + 6x + 4$$

71. $f(1.25) \approx -14.21875$

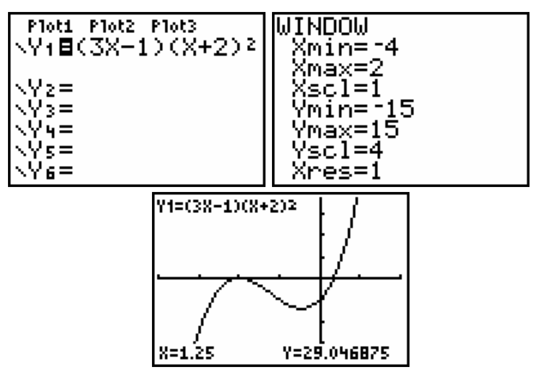
In order to have the calculator give the third screen, enter your function and the window, then press the GRAPH button. Press the TRACE button, then type in 1.25 then hit the ENTER key.



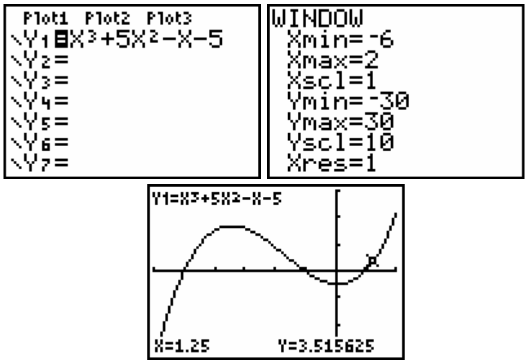
72. $f(1.25) \approx -21.16699$



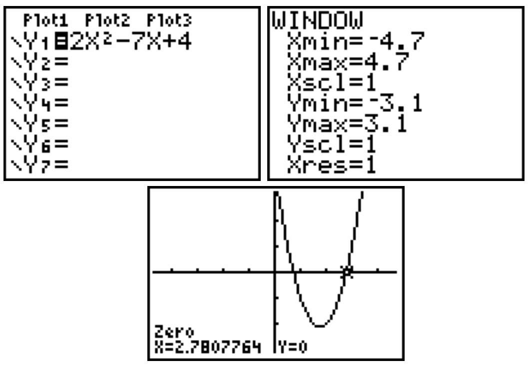
73. $f(1.25) \approx 29.046875$



74. $f(1.25) \approx 3.515625$

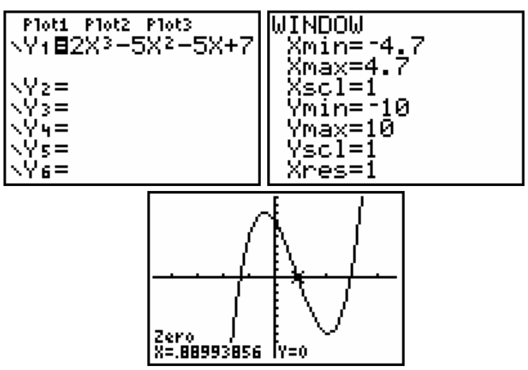


75. $f(x) = 2x^2 - 7x + 4$; 2 and 3



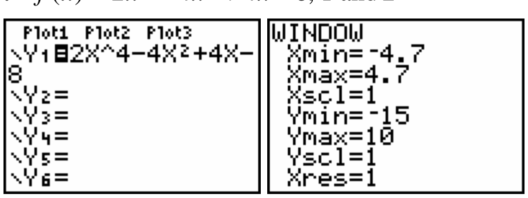
The real zero between 2 and 3 is approximately 2.7807764.

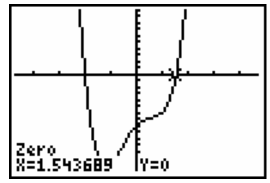
76. $f(x) = 2x^3 - 5x^2 - 5x + 7$; 0 and 1



The real zero between 0 and 1 is approximately .88993856.

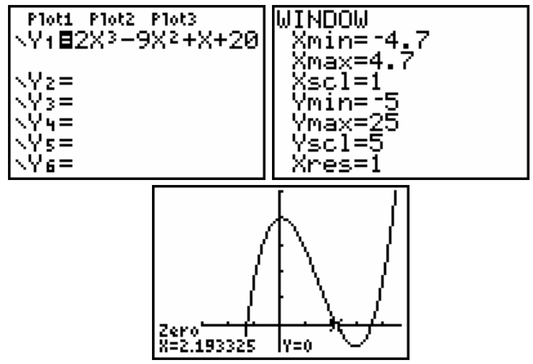
77. $f(x) = 2x^4 - 4x^2 + 4x - 8$; 1 and 2





The real zero between 1 and 2 is approximately 1.543689.

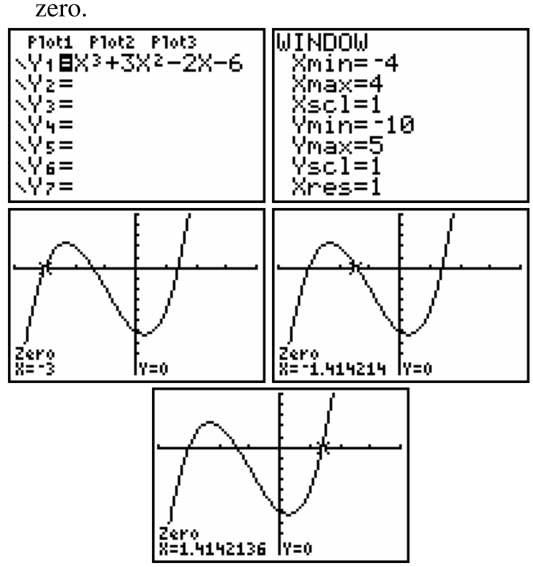
78. $f(x) = 2x^3 - 9x^2 + x + 20$; 2 and 2.5



The real zero between 2 and 2.5 is approximately 2.193325.

79. $f(x) = x^3 + 3x^2 - 2x - 6$

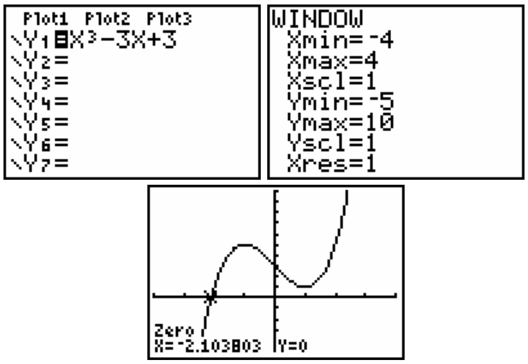
The highest degree term is x^3, so the graph will have end behavior similar to the graph of $f(x) = x^3$, which is downward at the left and upward at the right. There is at least one real zero because the polynomial is of odd degree. There are at most three real zeros because the polynomial is third-degree. A graphing calculator can be used to approximate each zero.



The graphs show that the zeros are approximately –3.0, –1.4 and 1.4.

80. $f(x) = x^3 - 3x + 3$

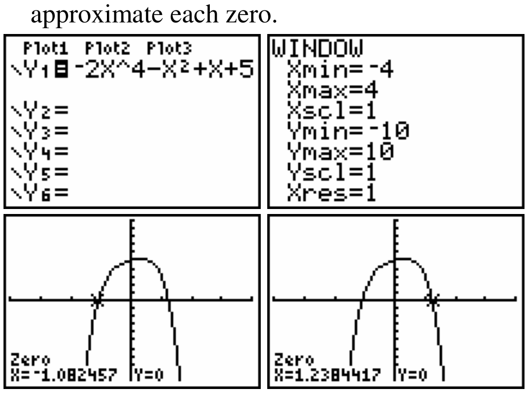
The highest degree term is x^3, so the graph will have end behavior similar to the graph of $f(x) = x^3$. The end behavior is downward at the left and upward at the right. There are at most three real zeros, since the polynomial is third-degree. A graphing calculator can be used to approximate each zero.



The graph shows that the negative zero is approximately –2.1. Since this is a cubic polynomial with two turning points, both of which lie above the *x*-axis, there can be no other zeros.

81. $f(x) = -2x^4 - x^2 + x + 5$

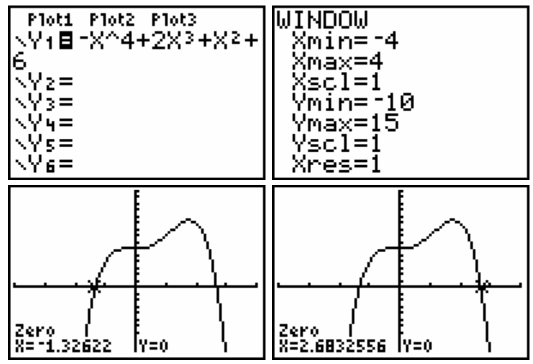
The highest degree term is $-2x^4$ so the graph will have the same end behavior as the graph of $f(x) = -x^4$, which is downward at both the left and the right. Since $f(0) = 5 > 0$, the end behavior and the intermediate value theorem tell us that there must be at least one zero on each side of the *y*-axis, that is, at least one negative and one positive zero. A graphing calculator can be used to approximate each zero.



The graphs show that the only zeros are approximately –1.1 and 1.2

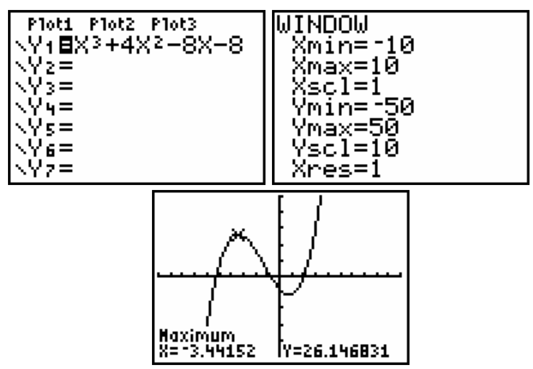
82. $f(x) = -x^4 + 2x^3 + 3x^2 + 6$

The highest degree term is $-x^4$, so the graph will have end behavior similar to the graph of $f(x) = -x^4$, which is negative for all values of *x* with large absolute values. The end behavior is downward at the left and the right. There are at most four real zeros, since the polynomial is fourth-degree. Since $f(0) = 6$, a positive number, there will be at least one negative and one positive zero by the intermediate value theorem. A graphing calculator can be used to approximate each zero.



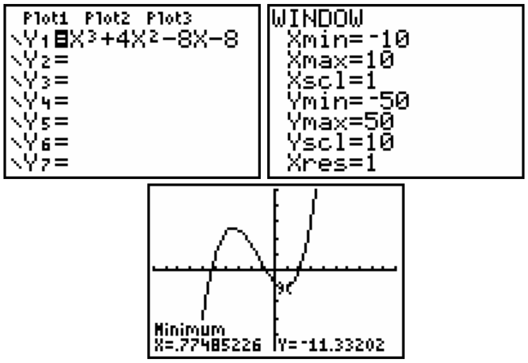
The graphs show that the only zeros are approximately –1.5 and 3.1.

83. $f(x) = x^3 + 4x^2 - 8x - 8; [-3.8, -3]$



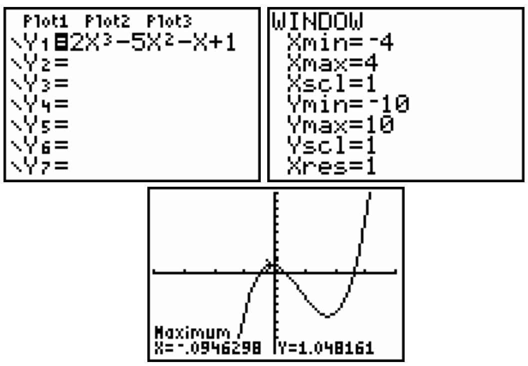
The turning point is (–3.44, 26.15).

84. $f(x) = x^3 + 4x^2 - 8x - 8; [-.3, 1]$



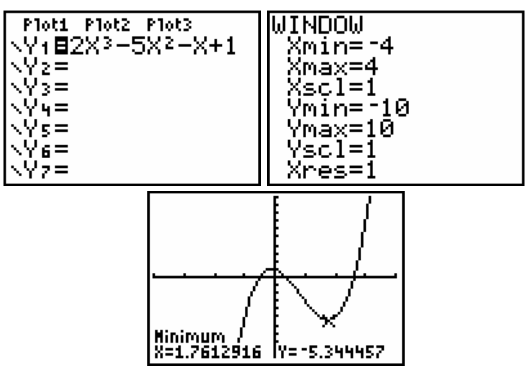
The turning point is (.77, –11.33).

85. $f(x) = 2x^3 - 5x^2 - x + 1; [-1, 0]$



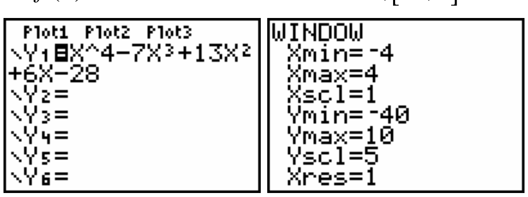
The turning point is (–.09, 1.05).

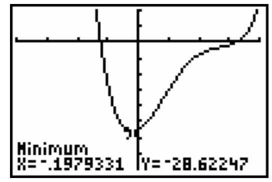
86. $f(x) = 2x^3 - 5x^2 - x + 1; [1.4, 2]$



The turning point is (1.76, –5.34).

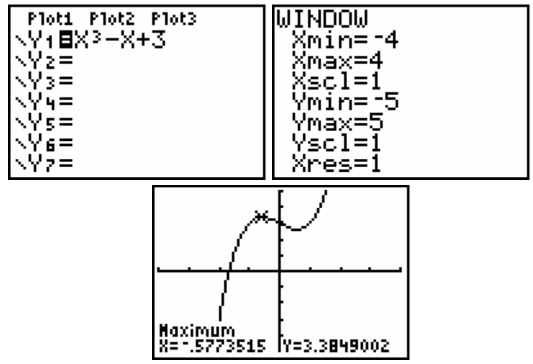
87. $f(x) = x^4 - 7x^3 + 13x^2 + 6x - 28; [-1, 0]$





The turning point is (–.20, –28.62).

88. $f(x) = x^3 - x + 3; [-1, 0]$



The turning point is (–.58, 3.38).

89. Answers will vary.

90. Since the two real zeros, 2 and 3, yield linear factors of $x - 2$ and $x - 3$, they can be synthetically divided into

$f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12.$

Start with the factor $x - 2$.

$$\begin{array}{r|rrrrr} 2 & 1 & -7 & 18 & -22 & 12 \\ & & 2 & -10 & 16 & -12 \\ \hline & 1 & -5 & 8 & -6 & 0 \end{array}$$

Now synthetically divide $x - 3$ into the quotient polynomial.

$$\begin{array}{r|rrrr} 3 & 1 & -5 & 8 & -6 \\ & & 3 & -6 & 6 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

We now have

$$f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12 = (x-2)(x-3)(x^2 - 2x + 2)$$

Apply the quadratic formula to the equation $x^2 - 2x + 2 = 0$ to find the non-real complex zeros where $a = 1$, $b = -2$, and $c = 2$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Thus the two non-real complex zeros are $1 - i$ and $1 + i$.

91. $f(x) = x^3 - 3x^2 - 6x + 8 = (x-4)(x-1)(x+2)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x - 4 = 0 \Rightarrow x = 4$ or $x - 1 = 0 \Rightarrow x = 1$ or $x + 2 = 0 \Rightarrow x = -2$

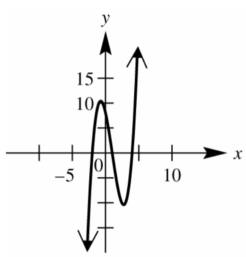
The zeros are –2, 1, and 4, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 8,$ so plot $(0, 8).$

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	–3	–28	Negative	Below
$(-2, 1)$	0	8	Positive	Above
$(1, 4)$	2	–8	Negative	Below
$(4, \infty)$	5	28	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = x^3 - 3x^2 - 6x + 8$
$= (x-4)(x-1)(x+2)$

(a) $f(x) = 0$ has the solution set $\{-2, 1, 4\}$.

(b) $f(x) < 0$ has the solution set $(-\infty, -2) \cup (1, 4)$.

(c) $f(x) > 0$ has the solution set $(-2, 1) \cup (4, \infty)$.

92. $f(x) = x^3 + 4x^2 - 11x - 30$
$= (x-3)(x+2)(x+5)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x - 3 = 0 \Rightarrow x = 3$ or $x + 2 = 0 \Rightarrow x = -2$ or $x + 5 = 0 \Rightarrow x = -5$

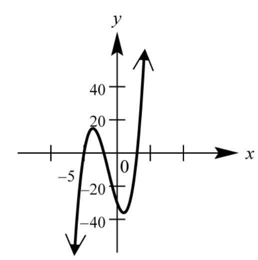
The zeros are –5, –2, and 3, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = -30,$ so plot $(0, -30).$

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -5)$	–6	–36	Negative	Below
$(-5, -2)$	–3	12	Positive	Above
$(-2, 3)$	0	–30	Negative	Below
$(3, \infty)$	4	54	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = x^3 + 4x^2 - 11x - 30$
$= (x-3)(x+2)(x+5)$

(a) $f(x) = 0$ has the solution set $\{-5, -2, 3\}$.

(b) $f(x) < 0$ has the solution set $(-\infty, -5) \cup (-2, 3)$.

(c) $f(x) > 0$ has the solution set $(-5, -2) \cup (3, \infty)$.

93. $f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$
$= (x-3)^2(2x+5)(x-1)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$(x-3)^2 = 0 \Rightarrow x = 3$ or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$ or $x - 1 = 0 \Rightarrow x = 1$

The zeros are $-\frac{5}{2} = -2.5,$ 1, and 3, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

(*continued on next page*)

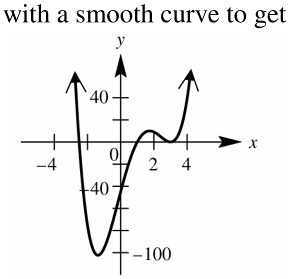
(*continued from page 315*)

Step 2: $f(0) = -45$, so plot $(0, -45)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2.5)$	–3	144	Positive	Above
$(-2.5, 1)$	0	–45	Negative	Below
$(1, 3)$	2	9	Positive	Above
$(3, \infty)$	4	39	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$
$= (x-3)^2(2x+5)(x-1)$

(a) $f(x) = 0$ has the solution set $\{-2.5, 1, 3 \text{ (multiplicity 2)}\}$.

(b) $f(x) < 0$ has the solution set $(-2.5, 1)$.

(c) $f(x) > 0$ has the solution set $(-\infty, -2.5) \cup (1, 3) \cup (3, \infty)$.

94. $f(x) = 4x^4 + 27x^3 - 42x^2 - 445x - 300$
$= (x+5)^2(4x+3)(x-4)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$(x+5)^2 = 0$ or $4x + 3 = 0$ or $x - 4 = 0$
$x = -5$ $\quad x = -\frac{3}{4}$ $\quad x = 4$

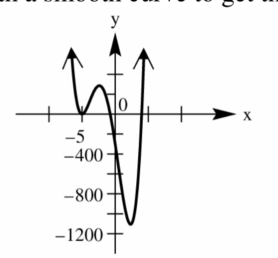
The zeros are –5, $-\frac{3}{4} = -.75$, and 4, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = -300$, so plot $(0, -300)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -5)$	–6	210	Positive	Above
$(-5, -.75)$	–1	80	Positive	Above
$(-.75, 4)$	0	–300	Negative	Below
$(4, \infty)$	5	2300	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = 4x^4 + 27x^3 - 42x^2 - 445x - 300$
$= (x+5)^2(4x+3)(x-4)$

(a) $f(x) = 0$ has the solution set $\{-5 \text{ (muliplicity 2)}, -.75, 4\}$.

(b) $f(x) < 0$ has the solution set $(-.75, 4)$.

(c) $f(x) > 0$ has the solution set $(-\infty, -5) \cup (-5, -.75) \cup (4, \infty)$.

95. $f(x) = -x^4 - 4x^3 + 3x^2 + 18x$
$= x(2-x)(x+3)^2$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x = 0$ or $2 - x = 0 \Rightarrow 2 = x$ or
$(x+3)^2 = 0 \Rightarrow x = -3$

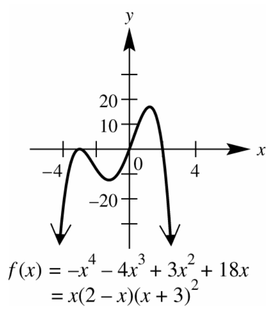
The zeros are 0, 2, and -3, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0, 0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–4	–24	Negative	Below
$(-3, 0)$	–1	–12	Negative	Below
$(0, 2)$	1	16	Positive	Above
$(2, \infty)$	3	–108	Negative	Below

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



(a) $f(x) = 0$ has the solution set $\{-3 \text{ (multiplicity 2)}, 0, 2\}$.

(b) $f(x) \geq 0$ has the solution set $\{-3\} \cup [0, 2]$.

(c) $f(x) \leq 0$ has the solution set $(-\infty, 0] \cup [2, \infty)$.

96. $f(x) = -x^4 + 2x^3 + 8x^2 = x^2(4 - x)(x + 2)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x^2 = 0 \Rightarrow x = 0$ or $4 - x = 0 \Rightarrow 4 = x$ or
$x + 2 = 0 \Rightarrow x = -2$

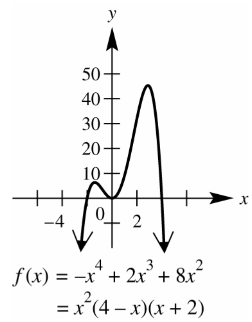
The zeros are –2, 0, and 4, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0, 0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	–3	–63	Negative	Below
$(-2, 0)$	–1	5	Positive	Above
$(0, 4)$	1	9	Positive	Above
$(4, \infty)$	5	–175	Negative	Below

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



(a) $f(x) = 0$ has the solution set $\{-2, 0 \text{ (multiplicity 2)}, 4\}$.

(b) $f(x) \geq 0$ has the solution set $[-2, 4]$.

(c) $f(x) \leq 0$ has the solution set $(-\infty, -2] \cup \{0\} \cup [4, \infty)$.

97. **(a)** In order for the length, width and height of the box to be positive quantities, $0 < x < 6$.

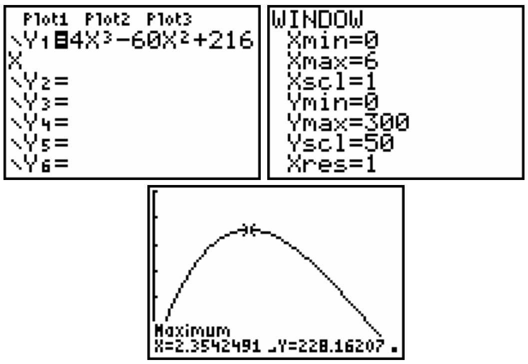
(b) The width of the rectangular base of the box would be $12 - 2x$. The length would be $18 - 2x$ and the height would be x. The volume would be the product of these three measures; therefore,

$$V(x) = x(18 - 2x)(12 - 2x) = 4x^3 - 60x^2 + 216x$$

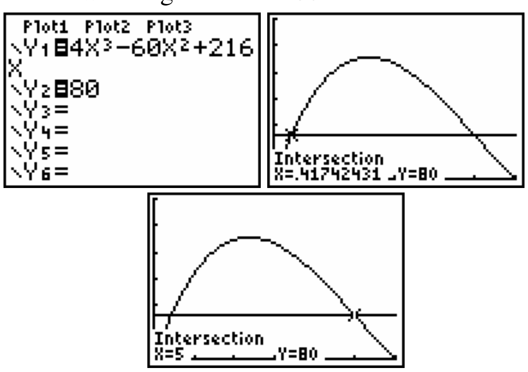
(*continued on next page*)

(*continued from page 318*)

(c) When x is approximately 2.35 in. the maximum volume will be approximately 228.16 in.^{3}.



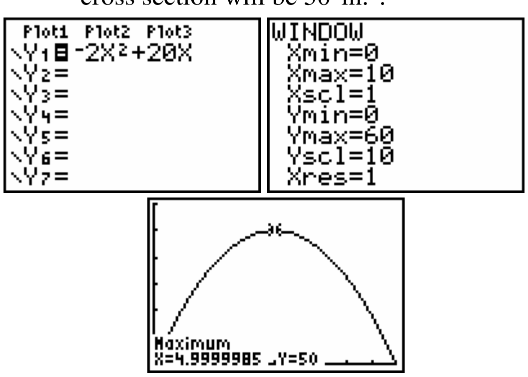
(d) When x is between .42 in. (approximately) and 5 in., the volume will be greater than 80 in.^{3}.



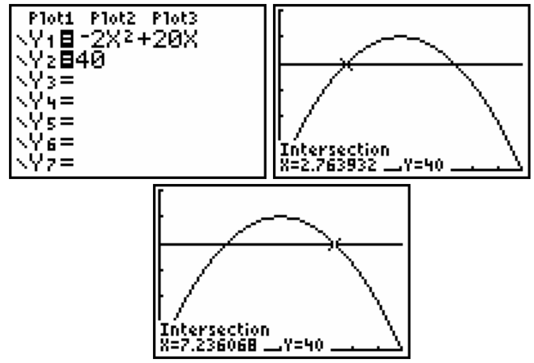
98. (a) In order for the width and height of the gutter to be positive quantities, $0 < x < 10$.

(b) $A(x) = x(20-2x) = -2x^2 + 20x$

(c) When x is 5 in. the maximum area of the cross section will be 50 in.^{2}.



(d) The area of a cross section will be less than 40 in.^{2} when x is between 0 in. and 2.76 in. or between 7.24 in. and 10 in.



99. (a) length of the leg = $x - 1$. The domain is $x > 1$ or $(1, \infty)$.

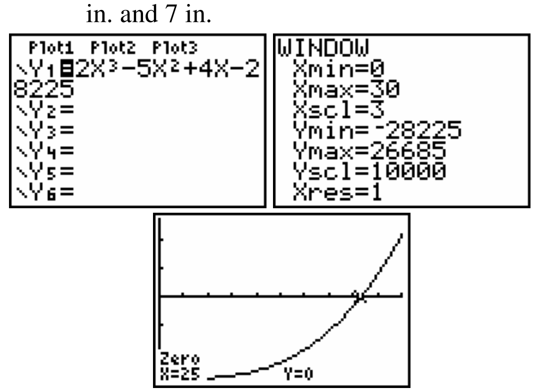
(b) By the Pythagorean theorem,
$$a^2 + b^2 = c^2 \Rightarrow a^2 + (x-1)^2 = x^2 \Rightarrow$$
$$a^2 = x^2 - (x-1)^2 \Rightarrow a = \sqrt{x^2 - (x-1)^2}$$
Thus, the length of the other leg is $\sqrt{x^2 - (x-1)^2}$.

(c) $A = \frac{1}{2}bh \Rightarrow 84 = \frac{1}{2}(x-1)\left(\sqrt{x^2-(x-1)^2}\right)$

Multiply by 2: $168 = (x-1)\left(\sqrt{x^2-(x-1)^2}\right)$

Square both sides.
$$28,224 = (x-1)^2\left[x^2 - (x-1)^2\right]$$
$$28,224 = (x^2 - 2x + 1)\cdot\left[x^2 - (x^2 - 2x + 1)\right]$$
$$28,224 = (x^2 - 2x + 1)(2x - 1)$$
$$28,224 = 2x^3 - 5x^2 + 4x - 1$$
$$2x^3 - 5x^2 + 4x - 28,225 = 0$$

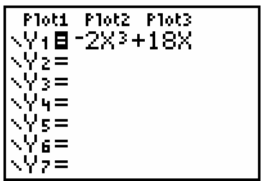
(d) Solving this cubic equation graphically, we obtain $x = 25$. If $x = 25$, $x - 1 = 24$, and
$$\sqrt{x^2-(x-1)^2} = \sqrt{625-576} = \sqrt{49} = 7.$$
The hypotenuse is 25 in.; the legs are 24 in. and 7 in.

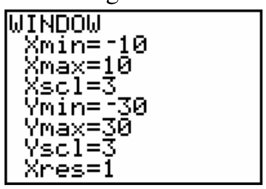


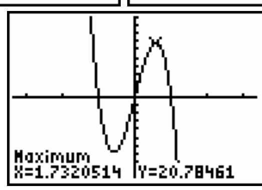
100. Since the point $C(x, y)$ lies on the parabola $y = 9 - x^2$, we may write the coordinates of this point as $C(x, 9 - x^2)$. Examine the dimensions of the rectangle. The width is $2x$ and the length is $9 - x^2$. The area of this rectangle is

$$A(x) = 2x(9 - x^2) = 18x - 2x^3 = -2x^3 + 18x.$$

Also, since $C(x, y)$ lies in the first quadrant, we know we are looking for a positive value of x in which there is a turning point. Notice that for a positive value of x, the maximum point will represent the desired value of x and the maximum area of the rectangle.







The x-coordinate of the maximum point is approximately 1.732. Therefore, the value of x that maximizes the area of the rectangle is approximately 1.732.

101. Use the following volume formulas:

$$V_{\text{cylinder}} = \pi r^2 h \text{ and}$$

$$V_{\text{hemisphere}} = \tfrac{1}{2}V_{\text{sphere}} = \tfrac{1}{2}\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{2}{3}\pi r^3$$

$$\pi r^2 h + 2\left(\tfrac{2}{3}\pi r^3\right) = \text{Total volume of tank } (V)$$

Let $V = 144\pi$, $h = 12$, and $r = x$.

$$\pi x^2(12) + \tfrac{4}{3}\pi x^3 = 144\pi$$
$$\tfrac{4}{3}x^3 + 12x^2 = 144$$
$$\tfrac{4}{3}x^3 + 12x^2 - 144 = 0$$
$$4x^3 + 36x^2 - 432 = 0$$
$$x^3 + 9x^2 - 108 = 0$$

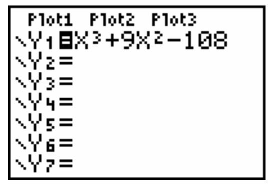
The end behavior of the graph of $f(x) = x^3 + 9x^2 - 108$, together with the negative y-intercept, tells us that this cubic polynomial must have one positive zero. (We are not interested in negative zeros because x represents the radius.)

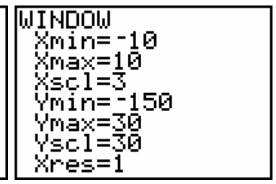
Algebraic Solution:
Use synthetic division or a graphing calculator to locate this zero.

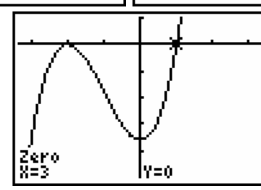
$$\begin{array}{r|rrrr} 3 & 1 & 9 & 0 & -108 \\ & & 3 & 36 & 108 \\ \hline & 1 & 12 & 36 & 0 \end{array}$$

Note that the bottom line of the synthetic division above implies the polynomial function $g(x) = x^2 + 12x + 36 = (x + 6)^2$, which has the zero -6 (multiplicity 2).

Graphing Calculator Solution:

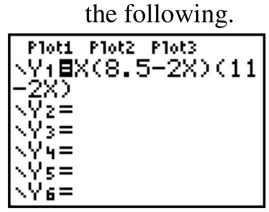


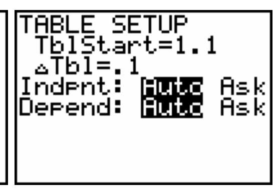


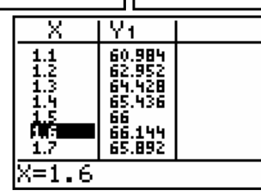


A radius of 3 feet would cause the volume of the tank to be 144π ft^3.

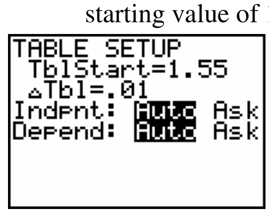
102. **(a)** Using the table feature with a starting value of 1.1 and a change of .1, we have the following.

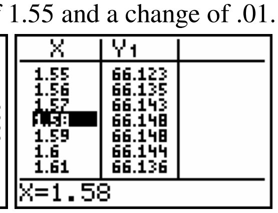






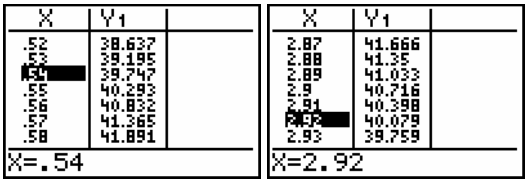
This would imply a maximum volume of about 66.14 in.^{3}. To get a higher accuracy, change the table setup to a starting value of 1.55 and a change of .01.





This would imply a maximum volume of about 66.15 in.^{3}.

(b) With a change of .01, toggle up/down to obtain the following screens.



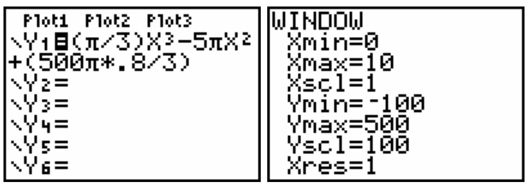
This would imply that the volume would be greater than 40 in.^{3} when x is between .54 in. and 2.92 in.

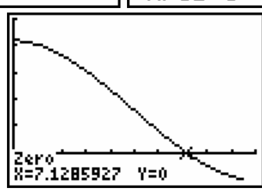
103. $f(x)=\frac{\pi}{3}x^3-5\pi x^2+\frac{500\pi d}{3}$

Use a graphing calculator for the three parts of this exercise.

(a) When $d=.8$ we have

$f(x)=\frac{\pi}{3}x^3-5\pi x^2+\frac{500\pi(.8)}{3}.$

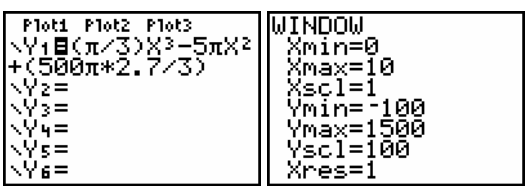


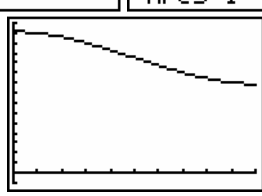


The approximate depth is 7.13 cm. The ball floats partly above the surface.

(b) When $d=2.7$ we have

$f(x)=\frac{\pi}{3}x^3-5\pi x^2+\frac{500\pi(2.7)}{3}.$

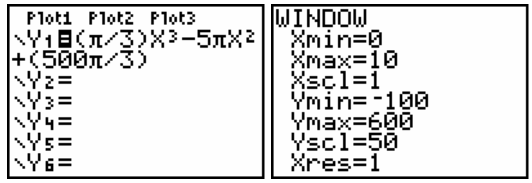


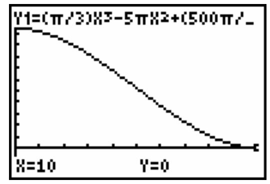


There is no x-intercept in this window. The sphere is more dense than water and sinks below the surface.

(c) When $d=1$ we have

$f(x)=\frac{\pi}{3}x^3-5\pi x^2+\frac{500\pi}{3}.$



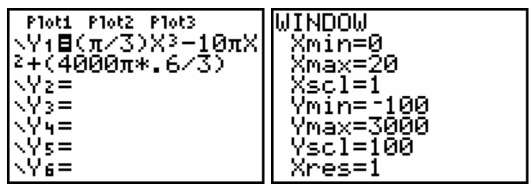


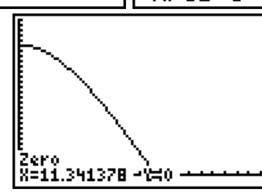
By tracing on the curve we see that the approximate depth is 10 cm. The balloon is submerged with its top even with the surface.

104. $f(x)=\frac{\pi}{3}x^3-10\pi x^2+\frac{4000\pi d}{3}$

Use a graphing calculator for this exercise. When $d=.6$ we have

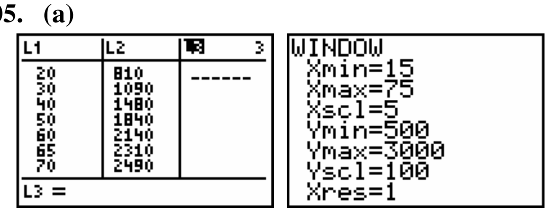
$f(x)=\frac{\pi}{3}x^3-10\pi x^2+\frac{4000\pi(.6)}{3}.$



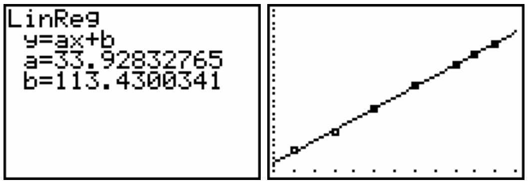


The approximate depth is 11.34 cm.

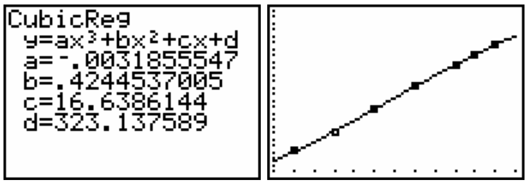
105. **(a)**



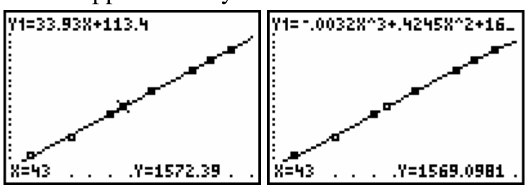
(b) The best-fitting linear function would be $y = 33.93x + 113.4$.



(c) The best-fitting cubic function would be $y = -.0032x^3 + .4245x^2 + 16.64x + 323.1$.

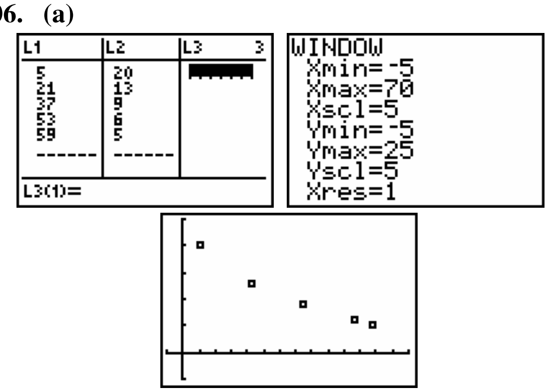


(d) linear: approximately 1572 ft; cubic: approximately 1569 ft

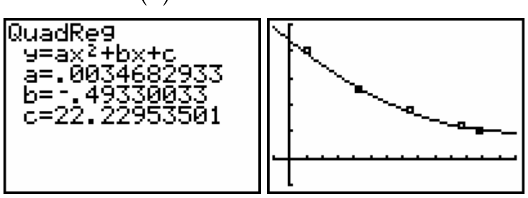


(e) The cubic function appears slightly better because only one data point is not on the curve.

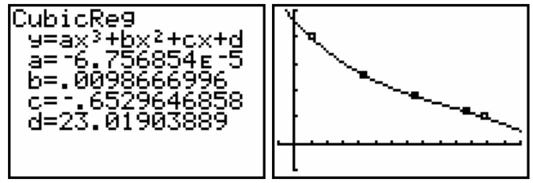
106. **(a)**



(b) The best-fitting quadratic function would be $C(x) = .0035x^2 - .49x + 22$

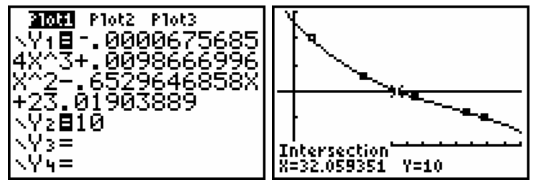


(c) The best-fitting cubic function would be $C(x) = -.000068x^3 + .00987x^2 - .653x + 23$

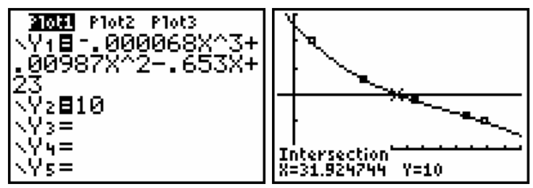


(d) The cubic function is a slightly better fit.

(e) Using the a, b, c and d values found by the calculator in part c, then the solution is $0 \le x < 32.06$.

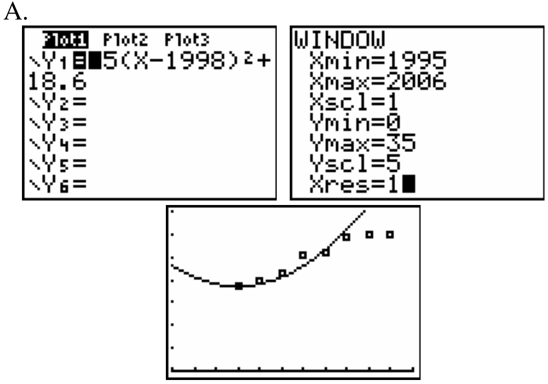


Using the approximated values of a, b, c and d values stated in part c, then the solution is $0 \le x < 31.92$.

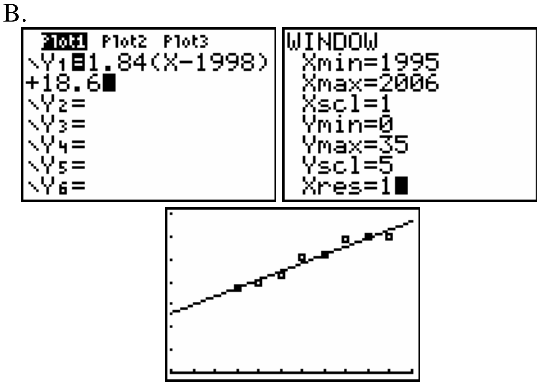


107. The function in B, $g(x) = 1.84(x - 1998) + 18.6$ provides the best model.

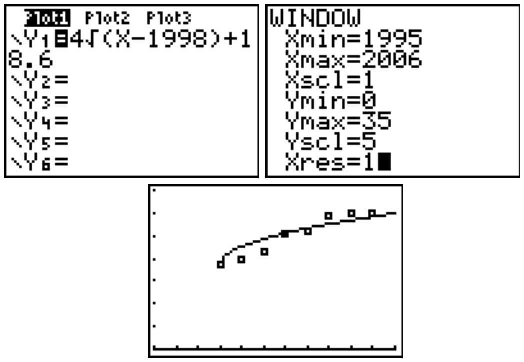
A.



B.



C.



108. **(a)** If the length of a pendulum increases, so does the period of oscillation, T.

(b) Answers will vary.

(c) For $k = \dfrac{L}{T^n}$, we have

$k = \dfrac{1.0}{1.11^n}$ and $k = \dfrac{2.0}{1.57^n}$. Try different values of n.

For $n = 2, k = \dfrac{1.0}{1.11^2} \approx .81$ and

$k = \dfrac{2.0}{1.57^2} \approx .81.$

When $n = 2$, k is constant. $k \approx .81$

(d) $k \approx .81,\ n = 2,\ L = 5$

$5 = .81T^2 \Rightarrow 6.1728 = T^2 \Rightarrow T \approx 2.48$

For a pendulum with length 5 ft, the value of T is 2.48 sec.

(e) If $L = 2L$, then $2L = kT^2 \Rightarrow \sqrt{\dfrac{2L}{k}} = T.$

Thus, T increases by a factor of $\sqrt{2} \approx 1.414$.

Summary Exercises on Polynomial Functions, Zeros, and Graphs

1. $f(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$

(a) $f(x) = x^4 \underbrace{+3x^3 -}_{1} 3x^2 - 11x - 6$ has 1 variation in sign. f has 1 positive real zero.

$f(-x) = \underbrace{x^4 -}_{1} 3x^3 \underbrace{-3x^2 +}_{2} \underbrace{11x -}_{3} 6$ has 3 variations in sign. f has 3 or 3 – 2 = 1 negative real zeros.

(b) p must be a factor of $a_0 = -6$ and q must be a factor of $a_4 = 1$. Thus, p can be ±1, ±2, ±3, ±6 and q can be ±1. The possible zeros, $\frac{p}{q}$, are $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$.

(c) The remainder theorem shows that 2 is zero.

$$\begin{array}{r|rrrrr} 2 & 1 & 3 & -3 & -11 & -6 \\ & & 2 & 10 & 14 & 6 \\ \hline & 1 & 5 & 7 & 3 & 0 \end{array}$$

The new quotient polynomial is $x^3 + 5x^2 + 7x + 3$. Since f has 1 positive real zero, try negative values. The remainder theorem shows that -3 is zero.

$$\begin{array}{r|rrrr} -3 & 1 & 5 & 7 & 3 \\ & & -3 & -6 & -3 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

The new quotient polynomial is $x^2 + 2x + 1$. Factor this polynomial and set equal to zero to find the remaining zeros.

$x^2 + 2x + 1 = 0 \Rightarrow (x+1)^2 = 0 \Rightarrow$
$x + 1 = 0 \Rightarrow x = -1$

The rational zeros are $-3,\ -1$ (multiplicity 2), and 2.

(d) All zeros have been found, and they are all rational.

(e) All zeros have been found, and they are all real.

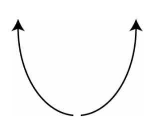
(f) The x-intercepts occur when $x = -3, -1,$ and 2.

(g) $f(0) = -6$

(h)
$$\begin{array}{r|rrrrr} 4 & 1 & 3 & -3 & -11 & -6 \\ & & 4 & 28 & 100 & 356 \\ \hline & 1 & 7 & 25 & 89 & 350 \end{array}$$

The corresponding point on the graph is $(4, 350)$.

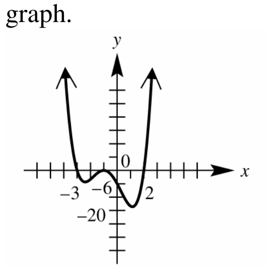
(i)



(j) The x-intercepts divide the x-axis into four intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–4	54	Positive	Above
$(-3, -1)$	–2	–4	Negative	Below
$(-1, 2)$	0	–6	Negative	Below
$(2, \infty)$	3	96	Positive	Above

Plot the x-intercepts, y-intercept and test points with a smooth curve to get the graph.



$f(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$

2. $f(x) = -2x^5 + 5x^4 + 34x^3 - 30x^2 - 84x + 45$

(a) $f(x) = \underbrace{-2x^5 + 5x^4}_{1} \underbrace{+34x^3 - 30x^2}_{2} \underbrace{-84x + 45}_{3}$

has 3 variations in sign. f has 3 or 3 – 2 = 1 positive real zeros.

$f(-x) = 2x^5 \underbrace{+5x^4 - 34x^3}_{1} \underbrace{-30x^2 + 84x}_{2} + 45$

has 2 variations in sign. f has 2 or 2 – 2 = 0 negative real zeros.

(b) p must be a factor of $a_0 = 45$ and q must be a factor of $a_5 = -2$. Thus, p can be ±1, ±3, ±5, ±9, ±15, ±45 and q can be $\pm 1, \pm 2$.

The possible zeros, $\frac{p}{q}$, are

$\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45,$

$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2}.$

(c) The remainder theorem shows that 5 is zero.

$$\begin{array}{r|rrrrrr} 5 & -2 & 5 & 34 & -30 & -84 & 45 \\ & & -10 & -25 & 45 & 75 & -45 \\ \hline & -2 & -5 & 9 & 15 & -9 & 0 \end{array}$$

The new quotient polynomial will be $-2x^4 - 5x^3 + 9x^2 + 15x - 9$.

Some of the possible rational zeros listed above (such as $\pm 15, \pm 45, \pm\frac{15}{2}$, and $\pm\frac{45}{2}$ are not possible rational roots of the new quotient polynomial. The remainder theorem shows that -3 is a zero.

$$\begin{array}{r|rrrrr} -3 & -2 & -5 & 9 & 15 & -9 \\ & & 6 & -3 & -18 & 9 \\ \hline & -2 & 1 & 6 & -3 & 0 \end{array}$$

The new quotient polynomial is $-2x^3 + x^2 + 6x - 3$. Of the original list of possible rational zeros, the only ones to now consider are $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. The remainder theorem shows that $\frac{1}{2}$ is zero.

$$\begin{array}{r|rrrr} \frac{1}{2} & -2 & 1 & 6 & -3 \\ & & -1 & 0 & 3 \\ \hline & -2 & 0 & 6 & 0 \end{array}$$

After examining the other possible zeros, it can be determined that the only rational zeros are $-3, \frac{1}{2}$, and 5.

(d) Based on the part c, the new quotient polynomial is $-2x^2 + 6$. Set this polynomial equal to zero to find the remaining zeros.

$-2x^2 + 6 = 0 \Rightarrow 6 = 2x^2 \Rightarrow$

$3 = x^2 \Rightarrow x = \pm\sqrt{3}$

The other real zeros are $-\sqrt{3}$ and $\sqrt{3}$.

(e) All zeros have been found, and they are all real.

(f) The x-intercepts occur when $x = -3, -\sqrt{3}, \frac{1}{2}, \sqrt{3},$ and 5.

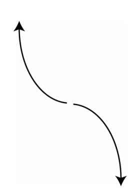
(g) $f(0) = 45$

(h)

$$\begin{array}{r|rrrrrr} 4 & -2 & 5 & 34 & -30 & -84 & 45 \\ & & -8 & -12 & 88 & 232 & 592 \\ \hline & -2 & -3 & 22 & 58 & 148 & 637 \end{array}$$

The corresponding point on the graph is $(4, 637)$.

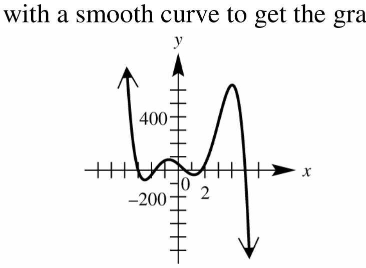
(i)



(j) The x-intercepts divide the x-axis into six intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–4	1053	Positive	Above
$\left(-3, -\sqrt{3}\right)$	–2	–35	Negative	Below
$\left(-\sqrt{3}, \frac{1}{2}\right)$	0	45	Positive	Above
$\left(\frac{1}{2}, \sqrt{3}\right)$	1	–32	Negative	Below
$\left(\sqrt{3}, 5\right)$	2	45	Positive	Above
$(5, \infty)$	6	–3267	Negative	Below

Plot the x-intercepts, y-intercept and test points (the y-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = -2x^5 + 5x^4 + 34x^3 - 30x^2 - 84x + 45$

3. $f(x) = 2x^5 - 10x^4 + x^3 - 5x^2 - x + 5$

(a) $f(x) = \underbrace{2x^5 - 10x^4}_{1} \underbrace{+ x^3}_{2} \underbrace{- 5x^2}_{3} \underbrace{- x + 5}_{4}$ has 4 variations in sign. f has 4 or 4 – 2 = 2 or 2 – 2 = 0 positive real zeros.

$f(-x) = -2x^5 - 10x^4 - x^3 \underbrace{-5x^2 + x}_{1} + 5$

has 1 variation in sign. f has 1 negative real zero.

(b) p must be a factor of $a_0 = 5$ and q must be a factor of $a_5 = 2$. Thus, p can be ±1, ±5 and q can be ±1, ±2. The possible zeros, $\frac{p}{q}$, are ±1, ±5, $\pm\frac{1}{2}$, $\pm\frac{5}{2}$.

(c) The remainder theorem shows that 5 is zero.

$$\begin{array}{r|rrrrrr} 5 & 2 & -10 & 1 & -5 & -1 & 5 \\ & & 10 & 0 & 5 & 0 & -5 \\ \hline & 2 & 0 & 1 & 0 & -1 & 0 \end{array}$$

The new quotient polynomial will be $2x^4 + x^2 - 1$. At this point, if we try to determine which of the possible rational zero will result in a remainder of zero, we will find that none of them work. We have found the only rational zero of our polynomial function. The rational zero is 5.

(d) Since the new quotient polynomial is $2x^4 + x^2 - 1$, we can factor this polynomial and set equal to zero to find the remaining zeros.

$2x^4 + x^2 - 1 = 0 \Rightarrow \left(x^2 + 1\right)\left(2x^2 - 1\right) = 0$

Since $x^2 + 1 = 0$ does not yield any real solutions, we will examine $2x^2 - 1 = 0$ first.

$2x^2 - 1 = 0 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

The other real zeros are $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$.

(e) Examining $x^2 + 1 = 0$, we have $x^2 = -1 \Rightarrow x = \pm\sqrt{-1} = \pm i$. Thus, the complex zeros are $-i$ and i.

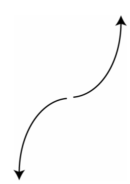
(f) The x-intercepts occur when $x = -\frac{\sqrt{2}}{2} \approx -.71$, $\frac{\sqrt{2}}{2} \approx .71$, and 5.

(g) $f(0) = 5$

(h)
$$\begin{array}{r|rrrrrr} 4 & 2 & -10 & 1 & -5 & -1 & 5 \\ & & 8 & -8 & -28 & -132 & -532 \\ \hline & 2 & -2 & -7 & -33 & -133 & -527 \end{array}$$

The corresponding point on the graph is $(4, -527)$.

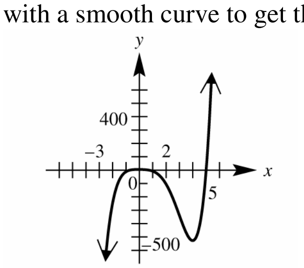
(i)



(j) The x-intercepts divide the x-axis into four intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty, -\frac{\sqrt{2}}{2}\right)$	–1	–12	Negative	Below
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	0	5	Positive	Above
$\left(\frac{\sqrt{2}}{2}, 5\right)$	1	–8	Negative	Below
$(5, \infty)$	6	2627	Positive	Above

Plot the x-intercepts, y-intercept and test points (the y-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = 2x^5 - 10x^4 + x^3 - 5x^2 - x + 5$

4. $f(x) = 3x^4 - 4x^3 - 22x^2 + 15x + 18$

(a) $f(x) = \underbrace{3x^4 - 4}_{1}x^3 \underbrace{-22x^2 +}_{2}15x + 18$ has 2 variations in sign. f has 2 or 2 – 2 = 0 positive real zeros.

$f(-x) = 3x^4 \underbrace{+4x^3 - }_{1}22x^2 \underbrace{-15x +}_{2}18$ has 2 variations in sign. f has 2 or 2 – 2 = 0 negative real zeros.

(b) p must be a factor of $a_0 = 18$ and q must be a factor of $a_4 = 3$. Thus, p can be ±1, ±2, ±3, ±6, ±9, ±18 and q can be ±1, ±3. The possible zeros, $\frac{p}{q}$, are

$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm\frac{1}{3}, \pm\frac{2}{3}$.

(c) The remainder theorem shows that 3 is zero.

$$\begin{array}{r|rrrrr} 3 & 3 & -4 & -22 & 15 & 18 \\ & & 9 & 15 & -21 & -18 \\ \hline & 3 & 5 & -7 & -6 & 0 \end{array}$$

The new quotient polynomial will be $3x^3 + 5x^2 - 7x - 6$. The remainder theorem shows that $-\frac{2}{3}$ is zero.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 5 & -7 & -6 \\ & & -2 & -2 & 6 \\ \hline & 3 & 3 & -9 & 0 \end{array}$$

The new quotient polynomial is $3x^2 + 3x - 9 = 3\left(x^2 + x - 3\right)$. At this point, if we try to determine which of the possible rational zero will result in a remainder of zero, we will find that none of them work. We have found the only rational zeros of our polynomial function. The rational zeros are $-\frac{2}{3}$ and 3.

(d) To find the remaining real zeros we must solve $x^2 + x - 3 = 0$. To solve this equations, we use the quadratic formula with $a = 1, b = 1$, and $c = -3$.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2(1)} = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

Thus, the other real zeros are $\frac{-1-\sqrt{13}}{2}$ and $\frac{-1+\sqrt{13}}{2}$.

(e) All zeros have been found, and they are all real.

(f) The x-intercepts occur when $x = \frac{-1-\sqrt{13}}{2} \approx -2.30, -\frac{2}{3} \approx -.67, \frac{-1+\sqrt{13}}{2} \approx 1.30$, and 3.

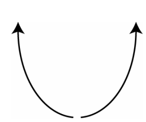
(g) $f(0) = 18$

(h)

$$\begin{array}{r|rrrrr} 4 & 3 & -4 & -22 & 15 & 18 \\ & & 12 & 32 & 40 & 220 \\ \hline & 3 & 8 & 10 & 55 & 238 \end{array}$$

The corresponding point on the graph is $(4, 238)$.

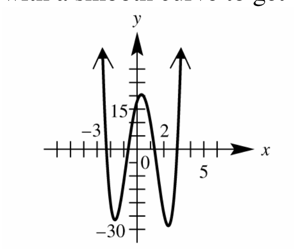
(i)



(j) The x-intercepts divide the x-axis into five intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty, \frac{-1-\sqrt{13}}{2}\right)$	–3	126	Positive	Above
$\left(\frac{-1-\sqrt{13}}{2}, -\frac{2}{3}\right)$	–1	–12	Negative	Below
$\left(-\frac{2}{3}, \frac{-1+\sqrt{13}}{2}\right)$	0	18	Positive	Above
$\left(\frac{-1+\sqrt{13}}{2}, 3\right)$	2	–24	Negative	Below
$(3, \infty)$	4	238	Positive	Above

Plot the x-intercepts, y-intercept and test points (the y-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = 3x^4 - 4x^3 - 22x^2 + 15x + 18$

5. $f(x) = -2x^4 - x^3 + x + 2$

(a) $f(x) = -2x^4 \underbrace{-x^3 +}_{1} x + 2$ has 1 variation in sign. f has 1 positive real zero.
$f(x) = \underbrace{-2x^4 +}_{1}\underbrace{x^3 -}_{2}\underbrace{x +}_{3}2$ has 3 variations in sign. f has 3 or 3 – 2 = 1 negative real zeros.

(b) p must be a factor of $a_0 = 2$ and q must be a factor of $a_4 = -2$. Thus, p can be ± 1, ± 2 and q can be $\pm 1, \pm 2$. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2, \pm\frac{1}{2}$.

(c) The remainder theorem shows that -1 is zero.

$$\begin{array}{r|rrrrr} -1 & -2 & -1 & 0 & 1 & 2 \\ & & 2 & -1 & 1 & -2 \\ \hline & -2 & 1 & -1 & 2 & 0 \end{array}$$

The new quotient polynomial will be $-2x^3 + x^2 - x + 2$. Since the signs are alternating in the bottom row of the synthetic division, we know that -1 is a lower bound. The remainder theorem shows that 1 is zero.

$$\begin{array}{r|rrrr} 1 & -2 & 1 & -1 & 2 \\ & & -2 & -1 & -2 \\ \hline & -2 & -1 & -2 & 0 \end{array}$$

The new quotient polynomial is $-2x^2 - x - 2 = -\left(2x^2 + x + 2\right)$. At this point, if we try to determine which of the possible rational zeros will result in a remainder of zero, we will find that none of them work. We have found the only rational zeros of our polynomial function. The rational zeros are -1 and 1.

(d) If we examine the discriminant of $2x^2 + x + 2$, namely $1^2 - 4(2)(2) = 1 - 16 = -15$, we see that it is negative, which implies there are no other real zeros.

(e) To find the remaining complex zeros, we must solve $2x^2 + x + 2 = 0$. To solve this equation, we use the quadratic formula with $a = 2, b = 1$, and $c = 2$.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(2)}}{2(2)} = \frac{-1 \pm \sqrt{1-16}}{4} = \frac{-1 \pm \sqrt{-15}}{4} = \frac{-1 \pm i\sqrt{15}}{4}$$

Thus, the other complex zeros are $-\frac{1}{4} + \frac{\sqrt{15}}{4}i$ and $-\frac{1}{4} - \frac{\sqrt{15}}{4}i$

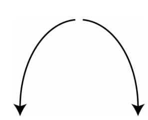
(f) The x-intercepts occur when $x = -1$ and 1.

(g) $f(0) = 2$

(h)
$$\begin{array}{r|rrrrr} 4 & -2 & -1 & 0 & 1 & 2 \\ & & -8 & -36 & -144 & -572 \\ \hline & -2 & -9 & -36 & -143 & -570 \end{array}$$

The corresponding point on the graph is $(4, -570)$.

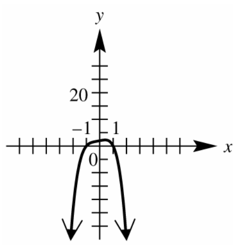
(i)



(j) The x-intercepts divide the x-axis into three intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	-2	-24	Negative	Below
$(-1, 1)$	0	2	Positive	Above
$(1, \infty)$	2	-36	Negative	Below

Plot the x-intercepts, y-intercept and test points (the y-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = -2x^4 - x^3 + x + 2$

6. $f(x) = 4x^5 + 8x^4 + 9x^3 + 27x^2 + 27x$
$= x(4x^4 + 8x^3 + 9x^2 + 27x + 27)$

(a) $g(x) = 4x^4 + 8x^3 + 9x^2 + 27x + 27$ has 0 variations in sign. f has 0 positive real zeros.

$g(-x) = \underbrace{4x^4 - 8x^3}_{1} \underbrace{+ 9x^2}_{2} \underbrace{- 27x}_{3} \underbrace{+ 27}_{4}$ has 4 variations in sign. f has 4 or 4 – 2 = 2 or 2 – 2 = 0 negative real zeros.

(b) $f(x) = 4x^5 + 8x^4 + 9x^3 + 27x^2 + 27x$
$= x(4x^4 + 8x^3 + 9x^2 + 27x + 27)$

p must be a factor of $a_0 = 27$ and q must be a factor of $a_4 = 4.$ Thus, p can be ±1, ±3, ±9, ±27 and q can be ±1, ±2, ±4. The possible zeros, $\frac{p}{q}$, are

$\pm 1, \pm 3, \pm 9, \pm 27, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm\frac{27}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{9}{4}, \pm\frac{27}{4}.$

(c) Because of the factorizations of f, we know that 0 is a rational zero. The remainder theorem shows that $-\frac{3}{2}$ is a zero.

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 4 & 8 & 9 & 27 & 27 \\ & & -6 & -3 & -9 & -27 \\ \hline & 4 & 2 & 6 & 18 & 0 \end{array}$$

The new quotient polynomial will be $4x^3 + 2x^2 + 6x + 18 = 2(2x^3 + x^2 + 3x + 9)$

The remainder theorem shows that $-\frac{3}{2}$ is a zero again.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 2 & 1 & 3 & 9 \\ & & -3 & 3 & -9 \\ \hline & 2 & -2 & 6 & 0 \end{array}$$

The new quotient polynomial is $2x^2 - 2x + 6 = 2(x^2 - x + 3).$ At this point, if we try to determine which of the possible rational zeros will result in a remainder of zero, we will find that none of them work. We have found the only rational zeros of our polynomial function. The rational zeros are 0 and $-\frac{3}{2}$ (multiplicity 2).

(d) If we examine the discriminant of $x^2 - x + 3,$ namely $(-1)^2 - 4(1)(3) = 1 - 12 = -11,$ we see that it is negative, which implies there are no other real zeros.

(e) To find the remaining complex zeros, we must solve $x^2 - x + 3 = 0.$ To solve this equation, we use the quadratic formula with $a = 1, b = -1,$ and $c = 3.$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{1-12}}{2} = \frac{1 \pm \sqrt{-11}}{2} = \frac{1 \pm i\sqrt{11}}{2}$$

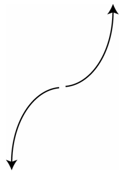
Thus, the other complex zeros are $\frac{1}{2} + \frac{\sqrt{11}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{11}}{2}i$

(f) The x-intercepts occur when $x = -\frac{3}{2} = -1.5$ and 0.

(g) $f(0) = 0$

(h)
$$\begin{array}{r|rrrrrr} 4 & 4 & 8 & 9 & 27 & 27 & 0 \\ & & 16 & 96 & 420 & 1788 & 7260 \\ \hline & 4 & 24 & 105 & 447 & 1815 & 7260 \end{array}$$

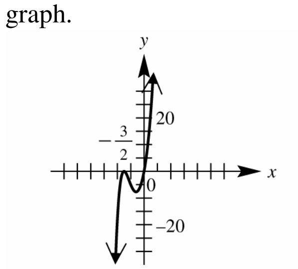
The corresponding point on the graph is $(4, 7260).$

(i) 

(j) The x-intercepts divide the x-axis into three intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty,-\frac{3}{2}\right)$	–2	–18	Negative	Below
$\left(-\frac{3}{2},0\right)$	–1	–5	Negative	Below
$(0, \infty)$	1	75	Positive	Above

Plot the x-intercepts, y-intercept and test points with a smooth curve to get the graph.



$f(x) = 4x^5 + 8x^4 + 9x^3 + 27x^2 + 27x$

7. $f(x) = 3x^4 - 14x^2 - 5$

(a) $f(x) = \underbrace{3x^4 - 14x^2}_{1} - 5$ has 1 variation in sign. f has 1 positive real zero.

$f(-x) = \underbrace{3x^4 - 14x^2}_{1} - 5$ has 1 variation in sign. f has 1 negative real zeros.

(b) p must be a factor of $a_0 = -5$ and q must be a factor of $a_4 = 3$. Thus, p can be ±1, ±5 and q can be $\pm1, \pm3$. The possible zeros, $\frac{p}{q}$, are ±1, ±5, $\pm\frac{1}{3}$, $\pm\frac{5}{3}$.

(c) If we try to determine which of the possible rational zeros will result in a remainder of zero, we will find that none of them work. There are no rational zeros.

(d) We can factor the polynomial, $3x^4 - 14x^2 - 5$, and set equal to zero to the remaining zeros.

$$3x^4 - 14x^2 - 5 = 0$$
$$\left(3x^2+1\right)\left(x^2-5\right) = 0$$

Since $3x^2 + 1 = 0$ does not yield any real solutions, we will examine $x^2 - 5 = 0$ first.

$x^2 - 5 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

The real zeros are $-\sqrt{5}$ and $\sqrt{5}$.

(e) Examining $3x^2 + 1 = 0$, we have

$3x^2 = -1 \Rightarrow x^2 = -\frac{1}{3} \Rightarrow x = \pm\sqrt{-\frac{1}{3}} = \pm i\frac{\sqrt{3}}{3}$.

Thus, the complex zeros are $-\frac{\sqrt{3}}{3}i$ and $\frac{\sqrt{3}}{3}i$.

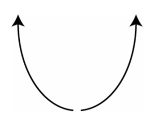
(f) The x-intercepts occur when $x = -\sqrt{5} \approx -2.24$ and $\sqrt{5} \approx 2.24$.

(g) $f(0) = -5$

(h)

$$\begin{array}{r|rrrrr} 4 & 3 & 0 & -14 & 0 & -5 \\ & & 12 & 48 & 136 & 544 \\ \hline & 3 & 12 & 34 & 136 & 539 \end{array}$$

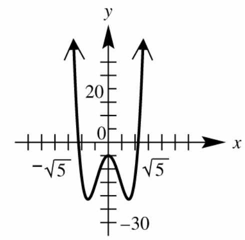
The corresponding point on the graph is $(4, 539)$.

(i) 

(j) The x-intercepts divide the x-axis into three intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty,-\sqrt{5}\right)$	–3	112	Positive	Above
$\left(-\sqrt{5},\sqrt{5}\right)$	0	–5	Negative	Below
$\left(\sqrt{5},\infty\right)$	3	112	Positive	Above

Plot the x-intercepts, y-intercept and test points (the y-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = 3x^4 - 14x^2 - 5$

8. $f(x) = -x^5 - x^4 + 10x^3 + 10x^2 - 9x - 9$

(a) $f(x) = -x^5 \underbrace{-x^4 +}_{1} 10x^3 \underbrace{+10x^2 -}_{2} 9x - 9$

has 2 variations in sign. *f* has 2 or $2 - 2 = 0$ positive real zeros.

$f(-x) = \underbrace{x^5 -}_{1} x^4 \underbrace{-10x^3 +}_{2} 10x^2 \underbrace{+9x -}_{3} 9$

has 3 variations in sign. *f* has 3 or $3 - 2 = 1$ negative real zeros.

(b) p must be a factor of $a_0 = -9$ and q must be a factor of $a_5 = -1$. Thus, p can be ±1, ±3, ±9 and q can be ±1. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 3, \pm 9$.

(c) The remainder theorem shows that -3 is zero.

$$\begin{array}{r|rrrrrr} -3 & -1 & -1 & 10 & 10 & -9 & -9 \\ & & 3 & -6 & -12 & 6 & 9 \\ \hline & -1 & 2 & 4 & -2 & -3 & 0 \end{array}$$

The new quotient polynomial will be $-x^4 + 2x^3 + 4x^2 - 2x - 3$. The remainder theorem shows that 1 is a zero.

$$\begin{array}{r|rrrrr} 1 & -1 & 2 & 4 & -2 & -3 \\ & & -1 & 1 & 5 & 3 \\ \hline & -1 & 1 & 5 & 3 & 0 \end{array}$$

The remainder theorem shows that 1 is a zero.

$$\begin{array}{r|rrrr} 3 & -1 & 1 & 5 & 3 \\ & & -3 & -6 & -3 \\ \hline & -1 & -2 & -1 & 0 \end{array}$$

The new quotient polynomial is $-x^2 - 2x - 1 = -\left(x^2 + 2x + 1\right)$. Factor $x^2 + 2x + 1$ and set equal to zero to find the remaining zeros.

$$x^2 + 2x + 1 = 0 \Rightarrow (x+1)^2 = 0 \Rightarrow$$
$$x + 1 = 0 \Rightarrow x = -1$$

The rational zeros are $-3, -1$ (multiplicity 2), 1, and 3.

(d) All zeros have been found, and they are all rational.

(e) All zeros have been found, and they are all real.

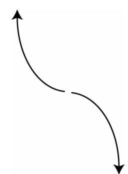
(f) The *x*-intercepts occur when $x = -3, -1, 1,$ and 3.

(g) $f(0) = -9$

(h)
$$\begin{array}{r|rrrrrr} 4 & -1 & -1 & 10 & 10 & -9 & -9 \\ & & -4 & -20 & -40 & -120 & -516 \\ \hline & -1 & -5 & -10 & -30 & -129 & -525 \end{array}$$

The corresponding point on the graph is $(4, -525)$.

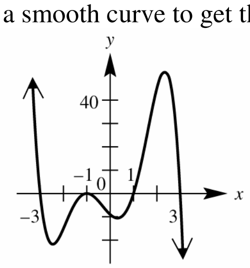
(i)



(j) The *x*-intercepts divide the *x*-axis into five intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty, -3)$	–4	315	Positive	Above
$(-3, -1)$	–2	–15	Negative	Below
$(-1, 1)$	0	–9	Negative	Below
$(-1, 3)$	2	45	Positive	Above
$(3, \infty)$	4	–525	Negative	Below

Plot the *x*-intercepts, *y*-intercept and test points (the *y*-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = -x^5 - x^4 + 10x^3 + 10x^2 - 9x - 9$

9. $f(x) = -3x^4 + 22x^3 - 55x^2 + 52x - 12$

(a) $f(x) = \underbrace{-3x^4 + 22x^3}_{1} \underbrace{- 55x^2}_{2,3} \underbrace{+ 52x - 12}_{4}$

has 4 variations in sign. *f* has 4 or $4 - 2 = 2$ or $2 - 2 = 0$ positive real zeros.

$f(-x) = -3x^4 - 22x^3 - 55x^2 - 52x - 12$

has 0 variations in sign. *f* has 0 negative real zeros.

(b) *p* must be a factor of $a_0 = -12$ and *q* must be a factor of $a_4 = -3$. Thus, *p* can be ±1, ±2, ±3, ±4, ±6, ±12 and *q* can be ±1, ±3. The possible zeros, $\frac{p}{q}$, are

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$.

(c) The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrrr} 2 & -3 & 22 & -55 & 52 & -12 \\ & & -6 & 32 & -46 & 12 \\ \hline & -3 & 16 & -23 & 6 & 0 \end{array}$$

The new quotient polynomial will be $-3x^3 + 16x^2 - 23x + 6$. The remainder theorem shows that 3 is zero.

$$\begin{array}{r|rrrr} 3 & -3 & 16 & -23 & 6 \\ & & -9 & 21 & -6 \\ \hline & -3 & 7 & -2 & 0 \end{array}$$

The new quotient polynomial is $-3x^2 + 7x - 2 = -(3x^2 - 7x + 2)$. Factor $3x^2 - 7x + 2$ and set equal to zero to find the remaining zeros.

$3x^2 - 7x + 2 = 0 \Rightarrow (3x - 1)(x - 2) = 0$

$3x - 1 = 0 \Rightarrow x = \frac{1}{3}$ or $x - 2 = 0 \Rightarrow x = 2$

The rational zeros are $\frac{1}{3}$, 2 (multiplicity 2), and 3.

(d) All zeros have been found, and they are all rational.

(e) All zeros have been found, and they are all real.

(f) The *x*-intercepts occur when $x = \frac{1}{3}$, 2, and 3.

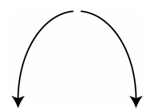
(g) $f(0) = -12$

(h)

$$\begin{array}{r|rrrrr} 4 & -3 & 22 & -55 & 52 & -12 \\ & & -12 & 40 & -60 & -32 \\ \hline & -3 & 10 & -15 & -8 & -44 \end{array}$$

The corresponding point on the graph is $(4, -44)$.

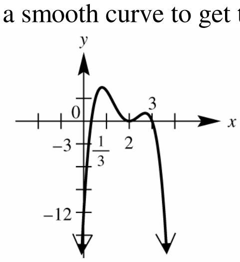
(i)



(j) The *x*-intercepts divide the *x*-axis into four intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$\left(-\infty, \frac{1}{3}\right)$	0	−12	Negative	Below
$\left(\frac{1}{3}, 2\right)$	1	4	Positive	Above
(2, 3)	$\frac{5}{2}$	$\frac{13}{16}$	Positive	Above
(3, ∞)	4	−44	Negative	Below

Plot the *x*-intercepts, *y*-intercept and test points (the *y*-intercept is also a test point) with a smooth curve to get the graph.



$f(x) = -3x^4 + 22x^3 - 55x^2 + 52x - 12$

10. Exercise 2: $-\sqrt{3} \approx -1.732$ and $\sqrt{3} \approx 1.732$

Exercise 3: $x = -\frac{\sqrt{2}}{2} \approx -.707$ and $\frac{\sqrt{2}}{2} \approx .707$

Exercise 4: $\frac{-1-\sqrt{13}}{2} \approx -2.303$ and

$\frac{-1+\sqrt{13}}{2} \approx 1.303$

Exercise 7: $-\sqrt{5} \approx -2.236$ and $\sqrt{5} \approx 2.236$

Section 3.5: Rational Functions: Graphs, Applications, and Models

1. $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

2. $f(x) = \frac{1}{x^2}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(0, \infty)$

3. $f(x) = \frac{1}{x}$

Increasing: nowhere
Decreasing: $(-\infty, 0) \cup (0, \infty)$
Constant: nowhere

4. $f(x) = \frac{1}{x^2}$

Increasing: $(-\infty, 0)$

Decreasing: $(0, \infty)$

Constant: nowhere

5. $y = \frac{1}{x-3} + 2$

Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 2$

6. $y = \frac{1}{(x+2)^2} - 4$

Vertical asymptote: $x = -2$
Horizontal asymptote: $y = -4$

7. $f(x) = \frac{1}{x^2}$ is an even function. It exhibits symmetry with respect to y-axis.

8. $f(x) = \frac{1}{x}$ is an odd function. It exhibits symmetry with respect to the origin.

9. Graphs A, B, and C have a domain of $(-\infty, 3) \cup (3, \infty)$.

10. Graph B has the range $(-\infty, 3) \cup (3, \infty)$.

11. Graph A has a range of $(-\infty, 0) \cup (0, \infty)$.

12. Graphs C and D have the range $(0, \infty)$.

13. Graph A has a single solution to the equation $f(x) = 3$.

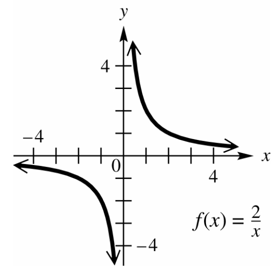
14. The range of graph B is $(-\infty, 3) \cup (3, \infty)$.

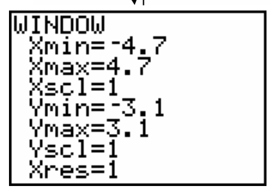
15. Graphs A, C, and D have the x-axis as a horizontal asymptote.

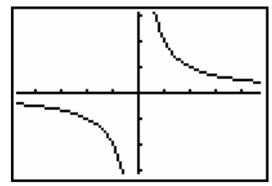
16. Graph C is symmetric with respect to the vertical line $x = 3$. (Note that graph D is not symmetric with respect to the vertical line $x = 0$ (the y-axis) because of the "hole" in this graph.)

17. $f(x) = \frac{2}{x}$

To obtain the graph of $f(x) = \frac{2}{x}$, stretch the graph of $f(x) = \frac{1}{x}$, vertically by a factor of 2. Just as with the graph of $f(x) = \frac{1}{x}$, $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote.





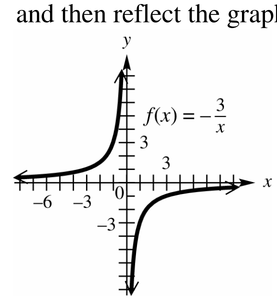


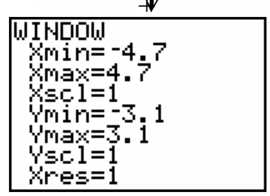
Domain: $(-\infty, 0) \cup (0, \infty)$

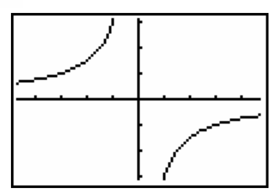
Range: $(-\infty, 0) \cup (0, \infty)$

18. $f(x) = -\frac{3}{x}$

To obtain the graph of $f(x) = -\frac{3}{x}$, stretch the graph of $f(x) = \frac{1}{x}$, vertically by a factor of 3 and then reflect the graph across the x-axis.





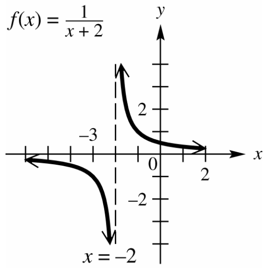


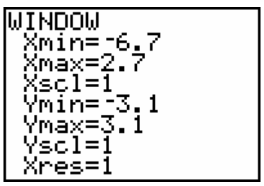
Domain: $(-\infty, 0) \cup (0, \infty)$

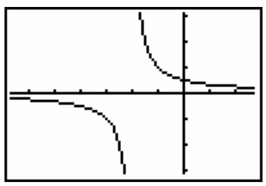
Range: $(-\infty, 0) \cup (0, \infty)$

19. $f(x)=\frac{1}{x+2}$

To obtain the graph of $f(x)=\frac{1}{x+2}$ shift the graph of $f(x)=\frac{1}{x}$, to the left 2 units. Just as with $f(x)=\frac{1}{x}$, $y=0$ is the horizontal asymptote, but this graph has $x=-2$ as its vertical asymptote (this affects the domain).





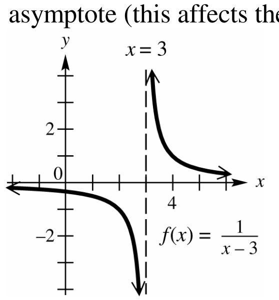


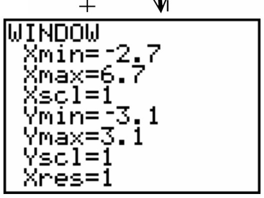
Domain: $(-\infty,-2)\cup(-2,\infty)$

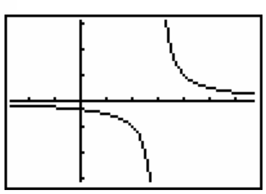
Range: $(-\infty,0)\cup(0,\infty)$

20. $f(x)=\frac{1}{x-3}$

To obtain the graph of $f(x)=\frac{1}{x-3}$ shift the graph of $f(x)=\frac{1}{x}$, right 3 units. Just as with $f(x)=\frac{1}{x}$, $y=0$ is the horizontal asymptote, but this graph has $x=3$ as its vertical asymptote (this affects the domain).





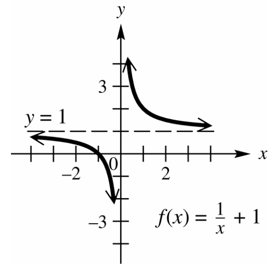


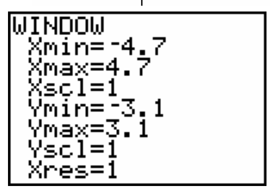
Domain: $(-\infty,3)\cup(3,\infty)$

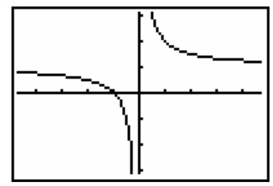
Range: $(-\infty,0)\cup(0,\infty)$

21. $f(x)=\frac{1}{x}+1$

To obtain the graph of $f(x)=\frac{1}{x}+1$, translate the graph of $f(x)=\frac{1}{x}$, 1 unit up. Just as with $f(x)=\frac{1}{x}$, $x=0$ is the vertical asymptote, but this graph has $y=1$ as its horizontal asymptote (this affects the range).





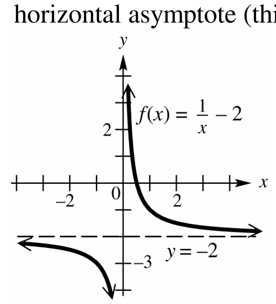


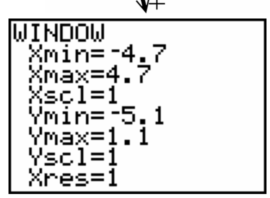
Domain: $(-\infty,0)\cup(0,\infty)$

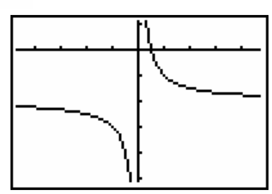
Range: $(-\infty,1)\cup(1,\infty)$

22. $f(x)=\frac{1}{x}-2$

To obtain the graph of $f(x)=\frac{1}{x}-2$, translate the graph of $f(x)=\frac{1}{x}$, 2 units down. Just as with $f(x)=\frac{1}{x}$, $x=0$ is the vertical asymptote, but this graph has $y=-2$ as its horizontal asymptote (this affects the range).





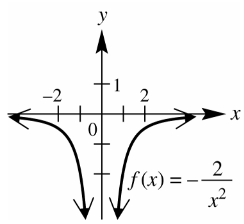


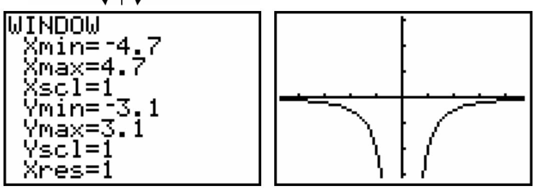
Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(-\infty,-2)\cup(-2,\infty)$

23. $f(x) = -\frac{2}{x^2}$

To obtain the graph of $f(x) = -\frac{2}{x^2}$, stretch the graph of $f(x) = \frac{1}{x^2}$, by a factor of 2, and then reflect the graph will be reflected across the *x*-axis.



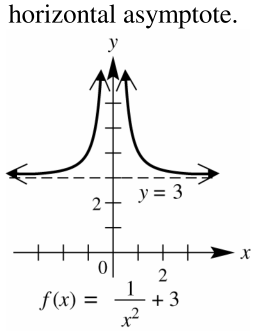


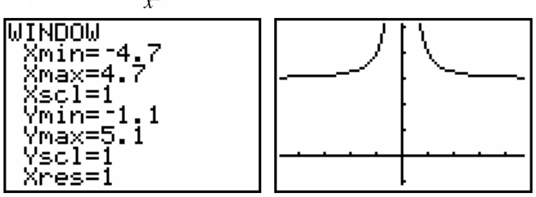
Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(-\infty,0)$

24. $f(x) = \frac{1}{x^2} + 3$

To obtain the graph of $f(x) = \frac{1}{x^2} + 3$, translate the graph of $f(x) = \frac{1}{x^2}$, up 3 units. Just as with $f(x) = \frac{1}{x^2}$, $x = 0$ is the vertical asymptote, but this graph has $y = 3$ as its horizontal asymptote.



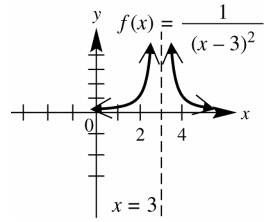


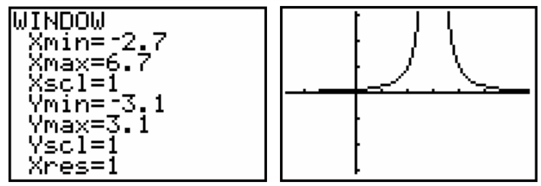
Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(3,\infty)$

25. $f(x) = \frac{1}{(x-3)^2}$

To obtain the graph of $f(x) = \frac{1}{(x-3)^2}$,shift the graph of $f(x) = \frac{1}{x^2}$, 3 units to the right. Just as with $f(x) = \frac{1}{x^2}$, $y = 0$ is the horizontal asymptote, but this graph has $x = 3$ as its vertical asymptote.



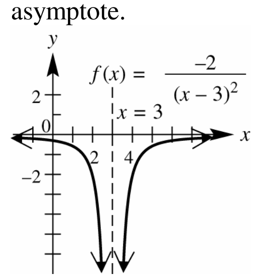


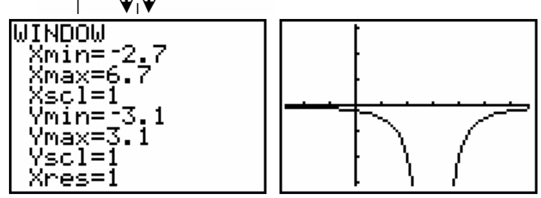
Domain: $(-\infty,3)\cup(3,\infty)$

Range: $(0,\infty)$

26. $f(x) = \frac{-2}{(x-3)^2}$

To obtain the graph of $f(x) = \frac{-2}{(x-3)^2}$ shift the graph of $f(x) = \frac{1}{x^2}$, 3 units to the right, stretch the graph by a factor of 2, then reflect the graph across the *x*-axis. Just as with $f(x) = \frac{1}{x^2}$, $y = 0$ is the horizontal asymptote, but this graph has $x = 3$ as its vertical asymptote.



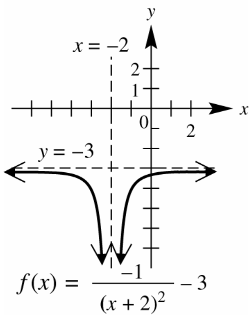


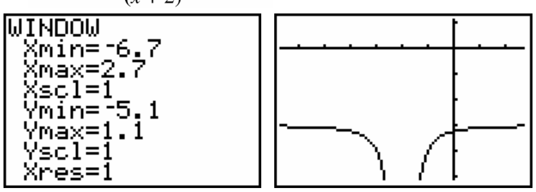
Domain: $(-\infty,3)\cup(3,\infty)$

Range: $(-\infty,0)$

27. $f(x)=\frac{-1}{(x+2)^2}-3$

To obtain the graph of $f(x)=\frac{-1}{(x+2)^2}-3$, shift the graph of $f(x)=\frac{1}{x^2}$, 2 units to the left, reflect the graph across the x-axis, and then shfit the graph 3 units down. Unlike the graph of $f(x)=\frac{1}{x^2}$, the vertical asymptote of $f(x)=\frac{-1}{(x+2)^2}-3$ will be $x=-2$ and the horizontal asymptote is $y=-3$.



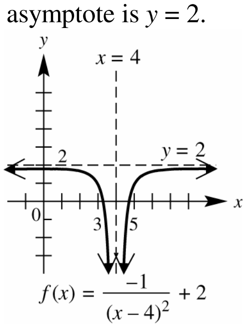


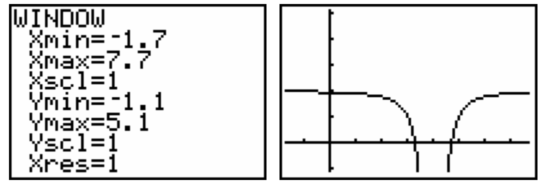
Domain: $(-\infty,-2)\cup(-2,\infty)$

Range: $(-\infty,-3)$

28. $f(x)=\frac{-1}{(x-4)^2}+2$

To obtain the graph of $f(x)=\frac{-1}{(x-4)^2}+2$, shift the graph of $f(x)=\frac{1}{x^2}$ 4 units to the right, then reflect the graph across the x-axis, and shift it up 2 units. Unlike the graph of $f(x)=\frac{1}{x^2}$, the vertical asymptote of $f(x)=\frac{-1}{(x-4)^2}+2$ is $x=4$ and the horizontal asymptote is $y=2$.





Domain: $(-\infty,4)\cup(4,\infty)$; range: $(-\infty,2)$

29. D. The graph of $f(x)=\frac{x+7}{x+1}$ has the vertical asymptote at $x=-1$.

30. B. The graph of $f(x)=\frac{x+10}{x+2}$ has a y-intercept of 5 since $f(0)=\frac{0+10}{0+2}=\frac{10}{2}=5$.

31. G. The graph of $f(x)=\frac{1}{x+4}$ has the x-axis as its horizontal asymptote, and the y-axis is not its vertical asymptote. The line $x=-4$ is its vertical asymptote.

32. H. The graph of $f(x)=\frac{-3}{x^2}$ has the x-axis as its horizontal asymptote, and the y-axis is its vertical asymptote.

33. E. The graph of $f(x)=\frac{x^2-16}{x+4}$ has a "hole" in its graph located at $x=-4$ since $f(x)=\frac{x^2-16}{x+4}=\frac{(x+4)(x-4)}{x+4}=x-4,\ x\neq-4$.

34. C. The graph of $f(x)=\frac{4x+3}{x-7}$ has a horizontal asymptote at $y=4$.

35. F. The graph of $f(x)=\frac{x^2+3x+4}{x-5}$ has an oblique asymptote since $f(x)=x+8+\frac{44}{x-5}$.

36. A. The graph of $f(x)=\frac{x+3}{x-6}$ has an x-intercept of -3 since $f(-3)=\frac{-3+3}{-3+6}=\frac{0}{3}=0$.

37. $f(x)=\frac{3}{x-5}$. To find the vertical asymptote, set the denominator equal to zero.
$x-5=0\Rightarrow x=5$
The equation of the vertical asymptote is $x=5$. To find the horizontal asymptote, divide each term by the largest power of x in the expression.

$$f(x)=\frac{\frac{3}{x}}{\frac{x}{x}-\frac{5}{x}}=\frac{\frac{3}{x}}{1-\frac{5}{x}}$$

As $|x|\to\infty$, $\frac{1}{x}$ approaches 0, thus $f(x)$ approaches $\frac{0}{1-0}=\frac{0}{1}=0$. The line $y=0$ (that is, the x-axis) is the horizontal asymptote.

38. $f(x)=\frac{-6}{x+9}$

To find the vertical asymptote, set the denominator equal to zero.

$x+9=0\Rightarrow x=-9$

The equation of the vertical asymptote is $x=-9$. To find the horizontal asymptote divide each term by the largest power of x in the expression.

$$f(x)=\frac{\frac{-6}{x}}{\frac{x}{x}+\frac{9}{x}}=\frac{\frac{-6}{x}}{1+\frac{9}{x}}$$

As $|x|\to\infty$, $\frac{1}{x}$ approaches 0, and the value of $f(x)$ approaches $\frac{0}{1+0}=\frac{0}{1}=0$. The line $y=0$ (that is, the x-axis) is the horizontal asymptote.

39. $f(x)=\frac{4-3x}{2x+1}$

To find the vertical asymptote, set the denominator equal to zero.

$2x+1=0\Rightarrow x=-\frac{1}{2}$

The equation of the vertical asymptote is $x=-\frac{1}{2}$. To find the horizontal asymptote, divide each term by the largest power of x in the expression.

$$f(x)=\frac{\frac{4}{x}-\frac{3x}{x}}{\frac{2x}{x}+\frac{1}{x}}=\frac{\frac{4}{x}-3}{2+\frac{1}{x}}$$

As $|x|\to\infty$, $\frac{1}{x}$ approaches 0, so $f(x)$ approaches $\frac{0-3}{2+0}=-\frac{3}{2}$. The equation of the horizontal asymptote is $y=-\frac{3}{2}$.

40. $f(x)=\frac{2x+6}{x-4}$

To find the vertical asymptote, set the denominator equal to zero.

$x-4=0\Rightarrow x=4$

The equation of the vertical asymptote is $x=4$. To find the horizontal asymptote, divide each term by the largest power of x in the expression.

$$f(x)=\frac{\frac{2x}{x}+\frac{6}{x}}{\frac{x}{x}-\frac{4}{x}}=\frac{2+\frac{6}{x}}{1-\frac{4}{x}}$$

As $|x|\to\infty$, $\frac{1}{x}$ approaches 0, so $f(x)$ approaches $\frac{2+0}{1-0}=\frac{2}{1}=2$. The equation of the horizontal asymptote is $y=2$.

41. $f(x)=\frac{x^2-1}{x+3}$

The vertical asymptote is $x=-3$, found by solving $x+3=0$. Since the numerator is of degree exactly one more than the denominator, there is no horizontal asymptote, but there may be an oblique asymptote. To find it, divide the numerator by the denominator.

$$\begin{array}{r|rrr} -3 & 1 & 0 & -1 \\ & & -3 & 9 \\ \hline & 1 & -3 & 8 \end{array}$$

Thus, $f(x)=\frac{x^2-1}{x+3}=x-3+\frac{8}{x+3}$.

For very large values of $|x|$, $\frac{8}{x+3}$ is close to 0, and the graph approaches the line $y=x-3$.

42. $f(x)=\frac{x^2+4}{x-1}$

The vertical asymptote is $x=1$, found by solving $x-1=0$. Since the numerator is of degree exactly one more than the denominator, there is no horizontal asymptote, but there may be an oblique asymptote. To find it, divide the numerator by the denominator.

$$\begin{array}{r|rrr} 1 & 1 & 0 & 4 \\ & & 1 & 1 \\ \hline & 1 & 1 & 5 \end{array}$$

Thus, $f(x)=\frac{x^2+4}{x-1}=x+1+\frac{5}{x-1}$.

For very large values of $|x|$, $\frac{5}{x-1}$ is close to 0, and the graph approaches the line $y=x+1$.

43. $f(x)=\frac{(x-3)(x+1)}{(x+2)(2x-5)}$

To find the vertical asymptotes, set the denominator equal to zero and solve.

$x+2=0\Rightarrow x=-2$ and $2x-5=0\Rightarrow x=\frac{5}{2}$

Thus, the vertical asymptotes are $x=-2$ and $x=\frac{5}{2}$.

To determine the horizontal asymptote, first multiply the factors in the numerator and denominator to get $f(x)=\frac{x^2-2x-3}{2x^2-x-10}$. Divide the numerator and denominator by x^2.

$$f(x)=\frac{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{x}{x^2}-\frac{10}{x^2}}=\frac{1-\frac{2}{x}-\frac{3}{x^2}}{2-\frac{1}{x}-\frac{10}{x^2}}$$

As $|x|\to\infty$, $\frac{1}{x}$ and $\frac{1}{x^2}$ approach 0, so $f(x)$ approaches $\frac{1-0-0}{2-0-0}=\frac{1}{2}$. Thus, the equation of the horizontal asymptote is $y=\frac{1}{2}$.

44. $f(x)=\frac{3(x+2)(x-4)}{(5x-1)(x-5)}$

To find the vertical asymptotes, set the denominator equal to zero and solve.

$5x-1=0 \Rightarrow x=\frac{1}{5}$ and $x-5=0 \Rightarrow x=5$

Thus, the vertical asymptotes are $x=\frac{1}{5}$ and $x=5$.

To determine the horizontal asymptote, first multiply the factors in the numerator and denominator to get $f(x)=\frac{3x^2-6x-24}{5x^2-26x+5}$. Divide the numerator and denominator by x^2.

$$f(x)=\frac{\frac{3x^2}{x^2}-\frac{6x}{x^2}-\frac{24}{x^2}}{\frac{5x^2}{x^2}-\frac{26x}{x^2}+\frac{5}{x^2}}=\frac{3-\frac{6}{x}-\frac{24}{x^2}}{5-\frac{26}{x}+\frac{5}{x^2}}$$

As $|x|\to\infty$, $\frac{1}{x}$ and $\frac{1}{x^2}$ approach 0, so $f(x)$ approaches $\frac{3-0-0}{5-0+0}=\frac{3}{5}$. Thus, the equation of the horizontal asymptote is $y=\frac{3}{5}$.

45. $f(x)=\frac{x^2+1}{x^2+9}$

To find the vertical asymptotes, set the denominator equal to zero and solve.

$x^2+9=0 \Rightarrow x^2=-9 \Rightarrow x=\pm\sqrt{-9}=\pm 3i$

Thus, there are no vertical asymptotes.

To determine the horizontal asymptote, divide the numerator and denominator by x^2.

$$f(x)=\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}}=\frac{1+\frac{1}{x^2}}{1+\frac{9}{x^2}}$$

As $|x|\to\infty$, $\frac{1}{x^2}$ approaches 0, so $f(x)$ approaches $\frac{1+0}{1+0}=\frac{1}{1}=1$. Thus, the equation of the horizontal asymptote is $y=1$.

46. $f(x)=\frac{4x^2+25}{x^2+9}$

To find the vertical asymptotes, set the denominator equal to zero and solve.

$x^2+9=0 \Rightarrow x^2=-9 \Rightarrow x=\pm\sqrt{-9}=\pm 3i$

Thus, there are no vertical asymptotes. To determine the horizontal asymptote, divide the numerator and denominator by x^2.

$$f(x)=\frac{\frac{4x^2}{x^2}+\frac{25}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}}=\frac{4+\frac{25}{x^2}}{1+\frac{9}{x^2}}$$

As $|x|\to\infty$, $\frac{1}{x^2}$ approaches 0, so $f(x)$ approaches $\frac{4+0}{1+0}=\frac{4}{1}=4$. Thus, the equation of the horizontal asymptote is $y=4$.

47. **(a)** Translating $y=\frac{1}{x}$ three units to the right gives $y=\frac{1}{x-3}$. Translating $y=\frac{1}{x-3}$ two units up yields

$y=\frac{1}{x-3}+2=\frac{1}{x-3}+\frac{2x-6}{x-3}=\frac{2x-5}{x-3}$. So $f(x)=\frac{2x-5}{x-3}$.

(b) $f(x)=\frac{2x-5}{x-3}$ has a zero when $2x-5=0$ or $x=\frac{5}{2}$.

(c) $f(x)=\frac{2x-5}{x-3}$ has a horizontal asymptote at $y=2$ and a vertical asymptote at $x=3$.

48. **(a)** Translating $y=-\frac{1}{x^2}$ three units to the left and one unit up yields

$$y=-\frac{1}{(x+3)^2}+1=\frac{-1}{(x+3)^2}+\frac{x^2+6x+9}{(x+3)^2}$$
$$=\frac{x^2+6x+8}{(x+3)^2}=\frac{x^2+6x+8}{x^2+6x+9}$$

So $f(x)=\frac{x^2+6x+8}{x^2+6x+9}$.

(b) $f(x)=\frac{x^2+6x+8}{x^2+6x+9}=\frac{(x+4)(x+2)}{(x+3)^2}$ has zeros where $(x+4)(x+2)=0$.

$x+4=0 \Rightarrow x=-4$ and

$x+2=0 \Rightarrow x=-2$

Thus *f* has zeros of -4 and -2.

(c) $f(x)=\frac{x^2+6x+8}{x^2+6x+9}=\frac{x^2+6x+8}{(x+3)^2}$ has a horizontal asymptote at $y=1$ and a vertical asymptote at $x=-3$.

49. **(a)** $f(x)=x+1+\frac{x^2-x}{x^4+1}$ has an oblique asymptote of $y=x+1$.

(b) In order to determine when the function intersects the oblique asymptote, we must determine the values of that that make $\frac{x^2-x}{x^4+1}=0$. Since

$\frac{x^2-x}{x^4+1}=\frac{x(x-1)}{x^4+1}$, and $x(x-1)=0$ when $x=0$ or $x=1$, the function crosses its asymptote at $x=0$ and $x=1$.

(c) For large values of x, $\frac{x^2-x}{x^4+1} > 0$. Thus as $x \to \infty$, the function approaches its asymptote from above.

50. (a) $f(x) = \frac{1}{(x-2)^2}$
Notice that no matter what value x takes on $(x \neq 2)$, $(x-2)^2$ will be greater than zero. That means $f(x) > 0$, which is graph C.

(b) $f(x) = \frac{1}{x-2}$
Notice if $x > 2$, $f(x) > 0$. If $x < 2$, $x - 2 < 0$, and $f(x) < 0$. This is graph A.

(c) $f(x) = \frac{-1}{x-2}$
Notice if $x > 2$, $f(x) < 0$. If $x < 2$, $f(x) > 0$. This is graph B.

(d) $f(x) = \frac{-1}{(x-2)^2}$
Notice that no matter what value x takes on $(x \neq 2)$, $(x-2)^2$ will be greater than zero. Thus, $f(x) < 0$. This is graph D.

51. Function A, because the denominator can never be equal to 0.

52. Function C, because the degree of the numerator is greater than the degree of the denominator.

53. From the graph, the vertical asymptote is $x = 2$, the horizontal asymptote is $y = 4$, and there is no oblique asymptote. The function is defined for all real numbers x such that $x \neq 2$, therefore the domain is $(-\infty, 2) \cup (2, \infty)$.

54. From the graph, the vertical asymptotes are $x = -4$ and $x = 4$, the horizontal asymptote is $y = 2$, and there is no oblique asymptote. The function is defined for all real numbers x such that $x \neq \pm 4$, therefore the domain is $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$.

55. From the graph, the vertical asymptotes are $x = -2$ and $x = 2$, the horizontal asymptote is $y = -4$, and there is no oblique asymptote. The function is defined for all real numbers x such that $x \neq \pm 2$, therefore the domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

56. From the graph, the vertical asymptote is $x = -4$, the horizontal asymptote is $y = -2$, and there is no oblique asymptote. The function is defined for all real numbers x such that $x \neq -4$, therefore the domain is $(-\infty, -4) \cup (-4, \infty)$.

57. From the graph, there is no vertical asymptote, the horizontal asymptote is $y = 0$, and there is no oblique asymptote. The function is defined for all x; therefore, the domain is $(-\infty, \infty)$.

58. From the graph, the vertical asymptote is $x = 0$, the horizontal asymptote is $y = 0$, and there is no oblique asymptote.The function is defined for all $x \neq 0$, therefore the domain is $(-\infty, 0) \cup (0, \infty)$.

59. From the graph, the vertical asymptote is $x = -1$, there is no horizontal asymptote, and the oblique asymptote passes through the points (0,−1) and (1, 0). Thus, the equation of the oblique asymptote is $y = x - 1$. The function is defined for all $x \neq -1$, therefore the domain is $(-\infty, -1) \cup (-1, \infty)$.

60. From the graph, the vertical asymptote is $x = \frac{1}{2}$, there is no horizontal asymptote, and the oblique asymptote passes through the points $\left(-\frac{1}{2}, 0\right)$ and $(0, 1)$. Thus, the equation of the oblique asymptote is $y = 2x + 1$.The function is defined for all $x \neq \frac{1}{2}$, therefore the domain is $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

For exercises 61−100, follow the 7 steps outlined on page 365 to sketch the graph. The solutions for exercises 61 and 62 show all of the steps. The solutions for exercises 63−100 are abbreviated.

61. *Step 1*: Find the vertical asymptote by setting the denominator equal to 0: $x - 4 = 0 \Rightarrow x = 4$
Step 2: Find the horizontal asymptote. Since the numerator and denominator have the same degree, the horizontal asymptote has equation $y = \frac{a_n}{b_n} = \frac{1}{1} = 1$. There is no oblique asymptote.
Step 3: Find the y-intercept: $f(0) = \frac{0+1}{0-4} = -\frac{1}{4}$
Step 4: Find the x-intercept:
$\frac{x+1}{x-4} = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$

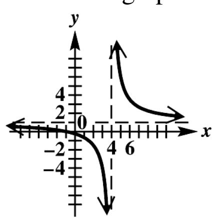
(*continued on next page*)

(*continued from page 337*)

Step 5: Determine whether the graph will intersect the horizontal asymptote:
$f(x)=\frac{x+1}{x-4}=1 \Rightarrow x+1=x-4 \Rightarrow 0=-5 \Rightarrow$
the graph does not intersect its horizontal asymptote.
Step 6: Plot a point in each of the intervals $(-\infty,-1),(-1,4),$ and $(4,\infty)$.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	–5	$\frac{4}{9}$	Positive	Above
$(-1, 4)$	1	−1	Negative	Below
$(4,\infty)$	5	6	Positive	Above

Step 7: Plot the intercepts and test points to sketch the graph.

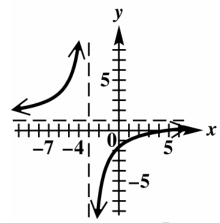


$f(x)=\frac{x+1}{x-4}$

62. *Step 1*: Find the vertical asymptote by setting the denominator equal to 0:
$x+3=0 \Rightarrow x=-3$
Step 2: Find the horizontal asymptote. Since the numerator and denominator have the same degree, the horizontal asymptote has equation
$y=\frac{a_n}{b_n}=\frac{1}{1}=1$. There is no oblique asymptote.
Step 3: Find the y-intercept: $f(0)=\frac{0-5}{0+3}=-\frac{5}{3}$
Step 4: Find the x-intercept:
$\frac{x-5}{x+3}=0 \Rightarrow x-5=0 \Rightarrow x=5$
Step 5: Determine whether the graph will intersect the horizontal asymptote:
$f(x)=\frac{x-5}{x+3}=1 \Rightarrow x-5=x+3 \Rightarrow 0=8 \Rightarrow$ the graph does not intersect its horizontal asymptote.
Step 6: Plot a point in each of the intervals $(-\infty,-3),(-3,5),$ and $(5,\infty)$.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–5	5	Positive	Above
$(-3, 5)$	1	−1	Negative	Below
$(5,\infty)$	6	$\frac{1}{9}$	Positive	Above

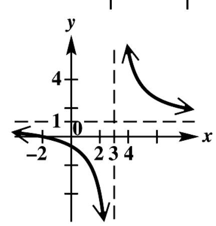
Step 7: Sketch the graph



$f(x)=\frac{x-5}{x+3}$

63. $f(x)=\frac{x+2}{x-3}$
vertical asymptote: $x = 3$
horizontal asymptote: $y = 1$
y-intercept: $-\frac{2}{3}$; x-intercept: −2
$f(x)$ does not intersect the horizontal asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	–5	$\frac{3}{8}$	Positive	Above
$(-2, 3)$	1	$-\frac{3}{2}$	Negative	Below
$(3,\infty)$	5	$\frac{7}{2}$	Positive	Above

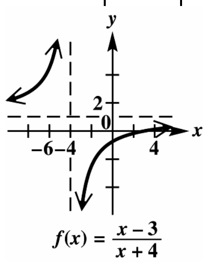


$f(x)=\frac{x+2}{x-3}$

64. $f(x) = \frac{x-3}{x+4}$

vertical asymptote: $x = -4$
horizontal asymptote: $y = 1$
y-intercept: $-\frac{3}{4}$; x-intercept: 3
$f(x)$ does not intersect the horizontal asymptote

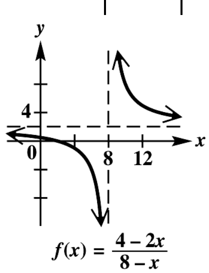
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -4)$	–5	8	Positive	Above
$(-4, 3)$	1	$-\frac{2}{5}$	Negative	Below
$(3, \infty)$	5	$\frac{2}{9}$	Positive	Above



65. $f(x) = \frac{4-2x}{8-x}$

vertical asymptote: $x = 8$
horizontal asymptote: $y = 2$
y-intercept: $\frac{1}{2}$; x-intercept: 2
$f(x)$ does not intersect the horizontal asymptote

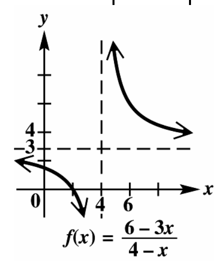
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, 2)$	–5	$\frac{14}{13}$	Positive	Above
$(2, 8)$	4	–1	Negative	Below
$(8, \infty)$	10	8	Positive	Above



66. $f(x) = \frac{6-3x}{4-x}$

vertical asymptote: $x = 4$
horizontal asymptote: $y = 3$
y-intercept: $\frac{3}{2}$; x-intercept: 2
$f(x)$ does not intersect the horizontal asymptote

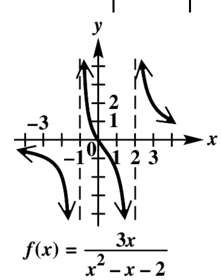
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, 2)$	–5	$\frac{7}{3}$	Positive	Above
$(2, 4)$	3	–3	Negative	Below
$(4, \infty)$	5	9	Positive	Above



67. $f(x) = \frac{3x}{(x+1)(x-2)}$

vertical asymptotes: $x = -1$, $x = 2$
horizontal asymptote: $y = 0$
y-intercept: 0; x-intercept: 0
$f(x)$ intersects the horizontal asymptote

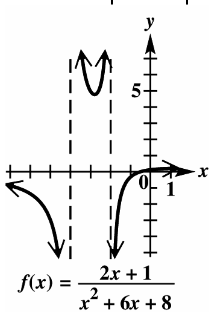
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	–5	$-\frac{15}{28}$	Negative	Below
$(-1, 0)$	$-\frac{1}{2}$	$\frac{6}{5}$	Positive	Above
$(0, 2)$	1	$-\frac{3}{2}$	Negative	Below
$(2, \infty)$	5	$\frac{5}{6}$	Positive	Above



68. $f(x) = \frac{2x+1}{(x+2)(x+4)}$

vertical asymptotes: $x = -2$, $x = -4$
horizontal asymptote: $y = 0$
y-intercept: $\frac{1}{8}$; x-intercept: $-\frac{1}{2}$
$f(x)$ intersects the horizontal asymptote

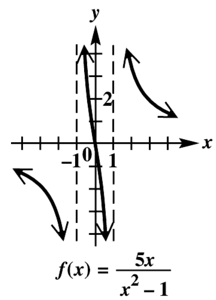
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -4)$	-5	-3	Negative	Below
$(-4, -2)$	-3	5	Positive	Above
$\left(-2, -\frac{1}{2}\right)$	-1	$-\frac{1}{3}$	Negative	Below
$\left(-\frac{1}{2}, \infty\right)$	1	$\frac{1}{5}$	Positive	Above



69. $f(x) = \frac{5x}{x^2-1}$

vertical asymptotes: $x = -1$, $x = 1$
horizontal asymptote: $y = 0$
y-intercept: 0; x-intercept: 0
$f(x)$ intersects the horizontal asymptote

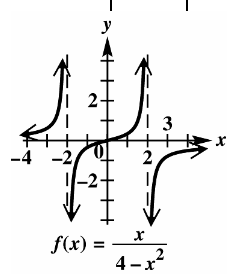
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	-2	$-\frac{10}{3}$	Negative	Below
$(-1, 0)$	$-\frac{1}{2}$	$\frac{10}{3}$	Positive	Above
$(0, 1)$	$\frac{1}{2}$	$-\frac{10}{3}$	Negative	Below
$(1, \infty)$	2	$\frac{10}{3}$	Positive	Above



70. $f(x) = \frac{x}{4-x^2}$

vertical asymptotes: $x = -2$, $x = 2$
horizontal asymptote: $y = 0$
y-intercept: 0; x-intercept: 0
$f(x)$ intersects the horizontal asymptote

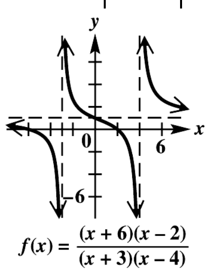
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-3	$\frac{3}{5}$	Positive	Above
$(-2, 0)$	-1	$-\frac{1}{3}$	Negative	Below
$(0, 2)$	1	$\frac{1}{3}$	Positive	Above
$(2, \infty)$	3	$-\frac{3}{5}$	Negative	Below



71. $f(x)=\frac{(x+6)(x-2)}{(x+3)(x-4)}$

vertical asymptotes: $x=-3$, $x=4$
horizontal asymptote: $y=1$
y-intercept: 1; x-intercepts: -6, 2
$f(x)$ intersects the horizontal asymptote

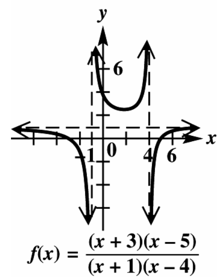
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-6)$	-8	$\frac{1}{3}$	Positive	Above
$(-6,-3)$	-4	$-\frac{3}{2}$	Negative	Below
$(-3,2)$	1	$\frac{7}{12}$	Positive	Above
$(2,4)$	3	$-\frac{3}{2}$	Negative	Below
$(4,\infty)$	6	$\frac{8}{3}$	Positive	Above



72. $f(x)=\frac{(x+3)(x-5)}{(x+1)(x-4)}$

vertical asymptotes: $x=-1$, $x=4$
horizontal asymptote: $y=1$
y-intercept: $\frac{15}{4}$; x-intercepts: -3, 5
$f(x)$ intersects the horizontal asymptote

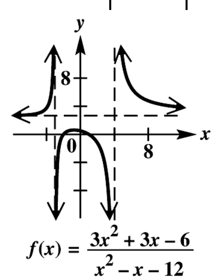
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-3)$	-4	$\frac{3}{8}$	Positive	Above
$(-3,-1)$	-2	$-\frac{7}{6}$	Negative	Below
$(-1,4)$	1	$\frac{8}{3}$	Positive	Above
$(4,5)$	$\frac{9}{2}$	$-\frac{15}{11}$	Negative	Below
$(5,\infty)$	6	$\frac{9}{14}$	Positive	Above



73. $f(x)=\frac{3x^2+3x-6}{x^2-x-12}$

vertical asymptotes: $x=-3$, $x=4$
horizontal asymptote: $y=3$
y-intercept: $\frac{1}{2}$; x-intercepts: -2, 1
$f(x)$ intersects the horizontal asymptote

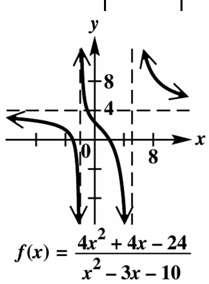
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-3)$	-4	$\frac{15}{4}$	Positive	Above
$(-3,-2)$	$-\frac{5}{2}$	$-\frac{21}{13}$	Negative	Below
$(-2,1)$	-1	$\frac{3}{5}$	Positive	Above
$(1,4)$	2	$-\frac{6}{5}$	Negative	Below
$(4,\infty)$	6	$\frac{20}{3}$	Positive	Above



74. $f(x)=\frac{4x^2+4x-24}{x^2-3x-10}$

vertical asymptotes: $x=-2$, $x=5$
horizontal asymptote: $y=4$
y-intercept: $\frac{12}{5}$; x-intercepts: -3, 2
$f(x)$ intersects the horizontal asymptote

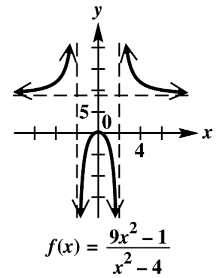
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-3)$	-4	$\frac{4}{3}$	Positive	Above
$(-3,-2)$	$-\frac{5}{2}$	$-\frac{12}{5}$	Negative	Below
$(-2,2)$	-1	4	Positive	Above
$(2,5)$	3	$-\frac{12}{5}$	Negative	Below
$(5,\infty)$	6	18	Positive	Above



75. $f(x)=\frac{9x^2-1}{x^2-4}$

vertical asymptotes: $x=-2$, $x=2$
horizontal asymptote: $y=9$
y-intercept: $\frac{1}{4}$; x-intercepts: $-\frac{1}{3},\frac{1}{3}$
$f(x)$ does not intersect the horizontal asymptote

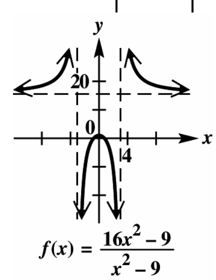
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-2)$	-3	16	Positive	Above
$\left(-2,-\frac{1}{3}\right)$	-1	$-\frac{8}{3}$	Negative	Below
$\left(-\frac{1}{3},\frac{1}{3}\right)$	$\frac{1}{4}$	$\frac{1}{9}$	Positive	Above
$\left(\frac{1}{3},2\right)$	1	$-\frac{8}{3}$	Negative	Below
$(2,\infty)$	3	16	Positive	Above



76. $f(x)=\frac{16x^2-9}{x^2-9}$

vertical asymptotes: $x=-3$, $x=3$
horizontal asymptote: $y=16$
y-intercept: 1; x-intercepts: $-\frac{3}{4},\frac{3}{4}$
$f(x)$ does not intersect the horizontal asymptote

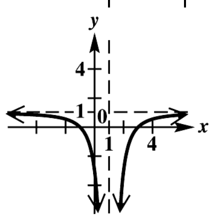
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-3)$	-6	21	Positive	Above
$\left(-3,-\frac{3}{4}\right)$	-2	-11	Negative	Below
$\left(-\frac{3}{4},\frac{3}{4}\right)$	$\frac{1}{2}$	$\frac{4}{7}$	Positive	Above
$\left(\frac{3}{4},3\right)$	1	$-\frac{7}{8}$	Negative	Below
$(3,\infty)$	6	21	Positive	Above



77. $f(x)=\frac{(x-3)(x+1)}{(x-1)^2}$

vertical asymptote: $x = 1$
horizontal asymptote: $y = 1$
y-intercept: -3; x-intercepts: -1, 3
$f(x)$ intersects the horizontal asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	-2	$\frac{5}{9}$	Positive	Above
$(-1, 1)$	0	-3	Negative	Below
$(1, 3)$	2	-3	Negative	Below
$(3, \infty)$	4	$\frac{5}{9}$	Positive	Above

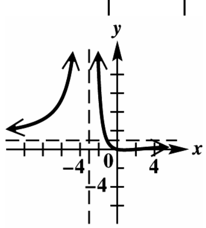


$f(x)=\frac{(x-3)(x+1)}{(x-1)^2}$

78. $f(x)=\frac{x(x-2)}{(x+3)^2}$

vertical asymptote: $x = -3$
horizontal asymptote: $y = 1$
y-intercept: 0; x-intercepts: 0, 2
$f(x)$ intersects the horizontal asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	-4	24	Positive	Above
$(-3, 0)$	-2	8	Positive	Above
$(0, 2)$	1	$-\frac{1}{16}$	Negative	Below
$(2, \infty)$	4	$\frac{8}{49}$	Positive	Above

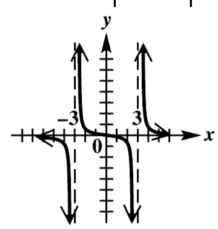


$f(x)=\frac{x(x-2)}{(x+3)^2}$

79. $f(x)=\frac{x}{x^2-9}$

vertical asymptotes: $x = -3$, $x = 3$
horizontal asymptote: $y = 0$
y-intercept: 0; x-intercept: 0
$f(x)$ intersects the horizontal asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	-4	$-\frac{4}{7}$	Negative	Below
$(-3, 0)$	-1	$\frac{1}{8}$	Positive	Above
$(0, 3)$	1	$-\frac{1}{8}$	Negative	Below
$(3, \infty)$	4	$\frac{4}{7}$	Positive	Above

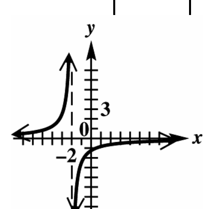


$f(x)=\frac{x}{x^2-9}$

80. $f(x)=\frac{-5}{2x+4}$

vertical asymptote: $x = -2$
horizontal asymptote: $y = 0$
y-intercept: $-\frac{5}{4}$; x-intercept: none
$f(x)$ does not intersect the horizontal asymptote

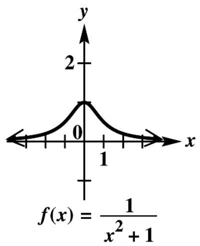
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-4	$\frac{5}{4}$	Positive	Above
$(-2, \infty)$	4	$-\frac{5}{12}$	Negative	Below



$f(x)=\frac{-5}{2x+4}$

81. $f(x) = \frac{1}{x^2+1}$

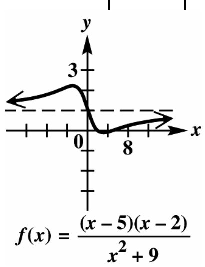
vertical asymptote: none
horizontal asymptote: $y = 0$
y-intercept: 1; x-intercept: none
$f(x)$ does not intersect the horizontal asymptote



82. $f(x) = \frac{(x-5)(x-2)}{x^2+9}$

vertical asymptote: none
horizontal asymptote: $y = 1$
y-intercept: $\frac{10}{9}$; x-intercepts: 2, 5
$f(x)$ intersects the horizontal asymptote

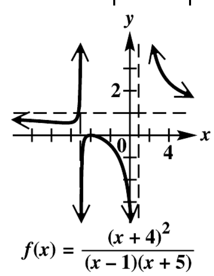
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, 2)$	1	$\frac{2}{5}$	Positive	Above
$(2, 5)$	3	$-\frac{1}{9}$	Negative	Below
$(5, \infty)$	6	$\frac{4}{45}$	Positive	Above



83. $f(x) = \frac{(x+4)^2}{(x-1)(x+5)}$

vertical asymptotes: $x = -5$, $x = 1$
horizontal asymptote: $y = 1$
y-intercept: $-\frac{16}{5}$; x-intercept: -4
$f(x)$ intersects the horizontal asymptote

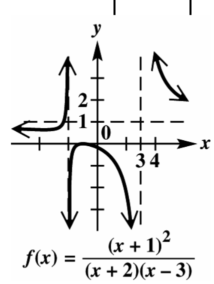
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -5)$	-6	$\frac{4}{7}$	Positive	Above
$(-5, -4)$	$-\frac{9}{2}$	$-\frac{1}{11}$	Negative	Below
$(-4, 1)$	-2	$-\frac{4}{9}$	Negative	Below
$(1, \infty)$	2	$\frac{36}{7}$	Positive	Above



84. $f(x) = \frac{(x+1)^2}{(x+2)(x-3)}$

vertical asymptotes: $x = -2$, $x = 3$
horizontal asymptote: $y = 1$
y-intercept: $-\frac{1}{6}$; x-intercept: -1
$f(x)$ intersects the horizontal asymptote

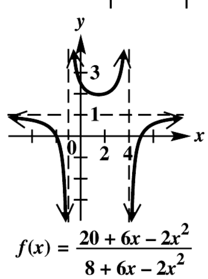
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-4	$\frac{9}{14}$	Positive	Above
$(-2, -1)$	$-\frac{3}{2}$	$-\frac{1}{9}$	Negative	Below
$(-1, 3)$	2	$-\frac{9}{4}$	Negative	Below
$(3, \infty)$	4	$\frac{25}{6}$	Positive	Above



85. $f(x) = \frac{20+6x-2x^2}{8+6x-2x^2}$

vertical asymptotes: $x = -1$, $x = 4$
horizontal asymptote: $y = 1$
y-intercept: $\frac{5}{2}$; x-intercepts: -2, 5
$f(x)$ does not intersect the horizontal asymptote

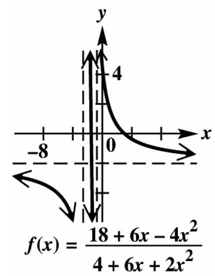
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-4	$\frac{3}{4}$	Positive	Above
$(-2, -1)$	$-\frac{3}{2}$	$-\frac{13}{11}$	Negative	Below
$(-1, 4)$	2	2	Positive	Above
$(4, 5)$	$\frac{9}{2}$	$-\frac{13}{11}$	Negative	Below
$(5, \infty)$	6	$\frac{4}{7}$	Positive	Above



86. $f(x) = \frac{18+6x-4x^2}{4+6x+2x^2}$

vertical asymptotes: $x = -2$, $x = -1$
horizontal asymptote: $y = -2$
y-intercept: $\frac{9}{2}$; x-intercepts: $-\frac{3}{2}$, 3
$f(x)$ intersects the horizontal asymptote

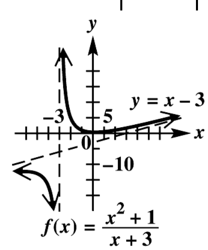
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-4	$-\frac{35}{6}$	Negative	Below
$\left(-2, -\frac{3}{2}\right)$	$-\frac{7}{4}$	$\frac{38}{3}$	Positive	Above
$\left(-\frac{3}{2}, -1\right)$	$-\frac{5}{4}$	$-\frac{34}{3}$	Negative	Below
$(-1, 3)$	2	$\frac{7}{12}$	Positive	Above
$(3, \infty)$	4	$-\frac{11}{30}$	Negative	Below



87. $f(x) = \frac{x^2+1}{x+3}$

vertical asymptote: $x = -3$
oblique asymptote: $y = x - 3$
y-intercept: $\frac{1}{3}$; x-intercepts: none
$f(x)$ does not intersect the oblique asymptote

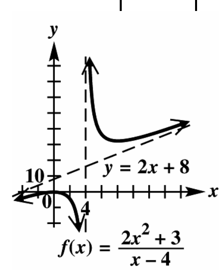
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	-4	-17	Negative	Below
$(-3, \infty)$	4	$\frac{17}{7}$	Positive	Above



88. $f(x) = \frac{2x^2+3}{x-4}$

vertical asymptote: $x = 4$
oblique asymptote: $y = 2x + 8$
y-intercept: $-\frac{3}{4}$; x-intercepts: none
$f(x)$ does not intersect the oblique asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, 4)$	-2	$-\frac{11}{6}$	Negative	Below
$(4, \infty)$	6	$\frac{75}{2}$	Positive	Above



89. $f(x) = \frac{x^2+2x}{2x-1}$

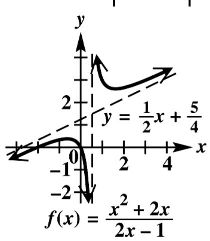
vertical asymptote: $x = \frac{1}{2}$

oblique asymptote: $y = \frac{x}{2} + \frac{5}{4}$

y-intercept: 0; x-intercepts: −2, 0

$f(x)$ does not intersect the oblique asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	−3	$-\frac{3}{7}$	Negative	Below
$(-2, 0)$	−1	$\frac{1}{3}$	Positive	Above
$\left(0, \frac{1}{2}\right)$	$\frac{1}{4}$	$-\frac{9}{8}$	Negative	Below
$\left(\frac{1}{2}, \infty\right)$	2	$\frac{8}{3}$	Positive	Above



90. $f(x) = \frac{x^2-x}{x+2}$

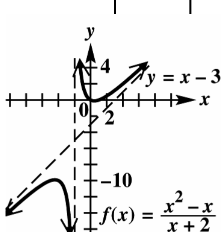
vertical asymptote: $x = -2$

oblique asymptote: $y = x - 3$

y-intercept: 0; x-intercepts: 0, 1

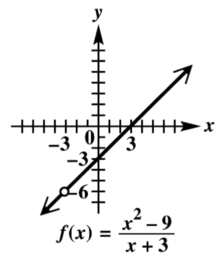
$f(x)$ does not intersect the oblique asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	−3	−12	Negative	Below
$(-2, 0)$	−1	2	Positive	Above
$(0, 1)$	$\frac{1}{2}$	$-\frac{1}{10}$	Negative	Below
$(1, \infty)$	2	$\frac{1}{2}$	Positive	Above



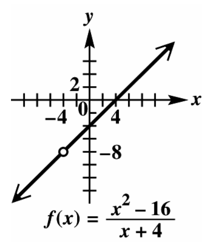
91. $f(x) = \frac{x^2-9}{x+3}$

The function degenerates into the line $f(x) = x - 3, x \neq -3$



92. $f(x) = \frac{x^2-16}{x+4}$

The function degenerates into the line $f(x) = x - 4, x \neq -4$



93. $f(x) = \frac{2x^2-5x-2}{x-2}$

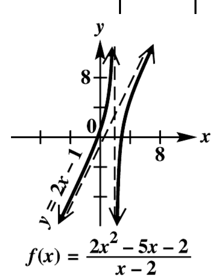
vertical asymptote: $x = 2$

oblique asymptote: $y = 2x - 1$

y-intercept: 1; x-intercepts: $\approx -.4, \approx 2.9$

$f(x)$ does not intersect the oblique asymptote

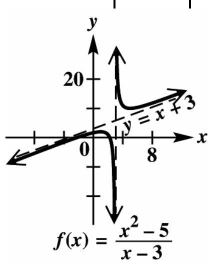
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -.4)$	−2	−4	Negative	Below
$(-.4, 2)$	1	5	Positive	Above
$(2, 2.9)$	2.5	−4	Negative	Below
$(2.9, \infty)$	4	5	Positive	Above



94. $f(x)=\frac{x^2-5}{x-3}$

vertical asymptote: $x = 3$
oblique asymptote: $y = x + 3$
y-intercept: $\frac{5}{3}$; x-intercepts: $\approx -2.2, \approx 2.2$
$f(x)$ does not intersect the oblique asymptote

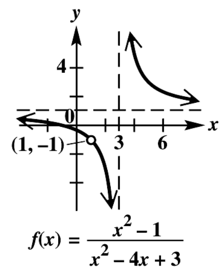
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2.2)$	-4	$-\frac{11}{7}$	Negative	Below
$(-2.2, 2.2)$	1	2	Positive	Above
$(2.2, 3)$	2.5	-2.5	Negative	Below
$(3, \infty)$	4	11	Positive	Above



95. $f(x)=\frac{x^2-1}{x^2-4x+3}$

vertical asymptote: $x = 3$
horizontal asymptote: $y = 1$
y-intercept: $-\frac{1}{3}$; x-intercept: -1
$f(x)$ is not defined for $x = 1$, so there is a hole in the graph.
$f(x)$ does not intersect the horizontal asymptote

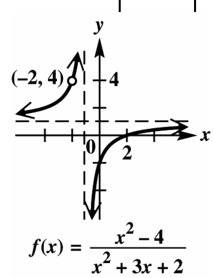
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	-4	$\frac{3}{7}$	Positive	Above
$(-1, 1)$	$\frac{1}{2}$	$-\frac{3}{5}$	Negative	Below
$(1, 3)$	2	-3	Negative	Below
$(3, \infty)$	4	5	Positive	Above



96. $f(x)=\frac{x^2-4}{x^2+3x+2}$

vertical asymptote: $x = -1$
horizontal asymptote: $y = 1$
y-intercept: -2; x-intercept: 2
$f(x)$ is not defined for $x = -2$, so there is a hole in the graph.
$f(x)$ does not intersect the horizontal asymptote

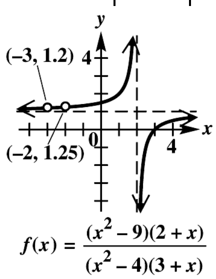
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-4	2	Positive	Above
$(-2, -1)$	$-\frac{3}{2}$	7	Positive	Above
$(-1, 2)$	$\frac{1}{2}$	-1	Negative	Below
$(2, \infty)$	4	$\frac{2}{5}$	Positive	Above



97. $f(x)=\frac{(x^2-9)(2+x)}{(x^2-4)(3+x)}$

vertical asymptote: $x = 2$
horizontal asymptote: $y = 1$
y-intercept: $\frac{3}{2}$; x-intercept: 3
$f(x)$ is not defined for $x = -3$ and $x = -2$, so there are two holes in the graph.
$f(x)$ does not intersect the horizontal asymptote

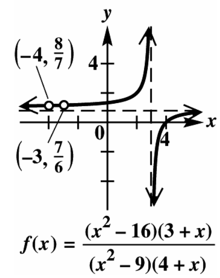
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	-4	$\frac{7}{6}$	Positive	Above
$(-3, -2)$	$-\frac{5}{2}$	$\frac{11}{9}$	Positive	Above
$(-2, 2)$	1	2	Positive	Above
$(2, 3)$	$\frac{5}{2}$	-1	Negative	Below
$(3, \infty)$	4	$\frac{1}{2}$	Positive	Above



98. $f(x)=\frac{(x^2-16)(3+x)}{(x^2-9)(4+x)}$

vertical asymptote: $x = 3$
horizontal asymptote: $y = 1$
y-intercept: $\frac{4}{3}$; x-intercept: 4
$f(x)$ is not defined for $x = -4$ and $x = -3$, so there are two holes in the graph.
$f(x)$ does not intersect the horizontal asymptote

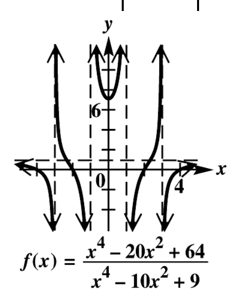
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -4)$	-5	$\frac{9}{8}$	Positive	Above
$(-4, -3)$	$-\frac{7}{2}$	$\frac{15}{13}$	Positive	Above
$(-3, 3)$	1	$\frac{3}{2}$	Positive	Above
$(3, 4)$	$\frac{7}{2}$	-1	Negative	Below
$(4, \infty)$	5	$\frac{1}{2}$	Positive	Above



99. $f(x)=\frac{x^4-20x^2+64}{x^4-10x^2+9}$

vertical asymptotes: $x = -3$, $x = -1$, $x = 1$, $x = 3$
horizontal asymptote: $y = 1$
y-intercept: $\frac{64}{9}$; x-intercepts: $-4, -2, 2, 4$
$f(x)$ intersects the horizontal asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -4)$	-5	.49	Positive	Above
$(-4, -3)$	-3.5	$-.85$	Negative	Below
$(-3, -2)$	-2.5	1.52	Positive	Above
$(-2, -1)$	-1.5	-2.85	Negative	Below
$(-1, 1)$	.5	9	Positive	Above
$(1, 2)$	1.5	-2.85	Negative	Below
$(2, 3)$	2.5	1.52	Positive	Above
$(3, 4)$	3.5	$-.85$	Negative	Below
$(4, \infty)$	5	.49	Positive	Above



100. $f(x) = \frac{x^4 - 5x^2 + 4}{x^4 - 24x^2 + 108}$

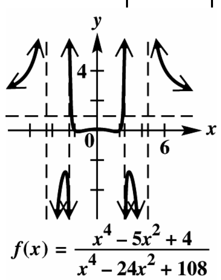
vertical asymptotes: $x = -3\sqrt{2}, x = -\sqrt{6},$
$x = \sqrt{6}, x = 3\sqrt{2}$

horizontal asymptote: $y = 1$

y-intercept: $\frac{1}{27}$; x-intercepts: −2, −1, 1, 2

$f(x)$ intersects the horizontal asymptote

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3\sqrt{2})$	−6	2.07	Positive	Above
$(-3\sqrt{2}, -\sqrt{6})$	−4	−9	Negative	Below
$(-\sqrt{6}, -2)$	−2.25	.36	Positive	Above
(−2, −1)	−1.5	−.04	Negative	Below
(−1, 1)	.5	.03	Positive	Above
(1, 2)	1.5	−.04	Negative	Below
$(2, \sqrt{6})$	2.25	.36	Positive	Above
$(\sqrt{6}, 3\sqrt{2})$	4	−9	Negative	Below
$(3\sqrt{2}, \infty)$	5	2.07	Positive	Above



101. The graph has a vertical asymptote, $x = 2$, so $x - 2$ is the denominator of the function. There is a "hole" in the graph at $x = -2$, so $x + 2$ is in the denominator and numerator also. The x-intercept is 3, so that when $f(x) = 0$, $x = 3$. This condition exists if $x - 3$ is a factor of the numerator. Putting these conditions together, we have a possible function $f(x) = \frac{(x-3)(x+2)}{(x-2)(x+2)}$ or $f(x) = \frac{x^2 - x - 6}{x^2 - 4}$.

102. The graph has a vertical asymptote $x = -2$, so $x + 2$ is the denominator of the function. The x-intercepts are 0 and −4, so that when $f(x) = 0$, $x = 0$ or $x = -4$. Such conditions would exist if x and $x + 4$ were factors of the numerator. The horizontal asymptote is $y = 2$, so the numerator and denominator have the same degree. Since the numerator will have degree 2, we must make the denominator also have degree 2. Also, from the horizontal asymptote, we have $y = 2 = \frac{a_n}{b_n} = \frac{2}{1}$. Putting these conditions together, we have a possible function $f(x) = \frac{2x(x+4)}{(x+2)^2}$ or $f(x) = \frac{2x^2 + 8x}{x^2 + 4x + 4}$.

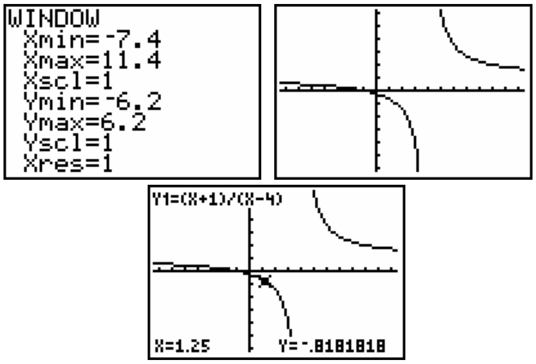
103. The graph has vertical asymptotes at $x = 4$ and $x = 0$, so $x - 4$ and x are factors in the denominator of the function. The only x-intercept is 2, so that when $f(x) = 0$, $x = 2$. This condition exists if $x - 2$ is a factor of the numerator. The graph has a horizontal asymptote $y = 0$, so the degree of the denominator is larger than the degree of the numerator. Putting these conditions together, we have a possible function
$f(x) = \frac{x-2}{x(x-4)}$ or $f(x) = \frac{x-2}{x^2 - 4x}$.

104. This is the graph of a rational function. The horizontal asymptote is $y = 0$. Therefore, the numerator has a lower degree than the denominator. Since there is no vertical asymptote, the denominator must be quadratic with no real solutions. The y-intercept is −4, so that $f(0) = -4$. Putting these conditions together, we have a possible function
$f(x) = \frac{-4}{x^2 + 1}$.

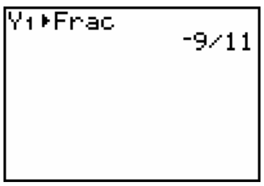
105. Several answers are possible. One answer is
$f(x) = \frac{(x-3)(x+1)}{(x-1)^2}$.

106. Several answers are possible. One answer is
$f(x) = \frac{(x-1)(x-3)}{x(x-2)}$.

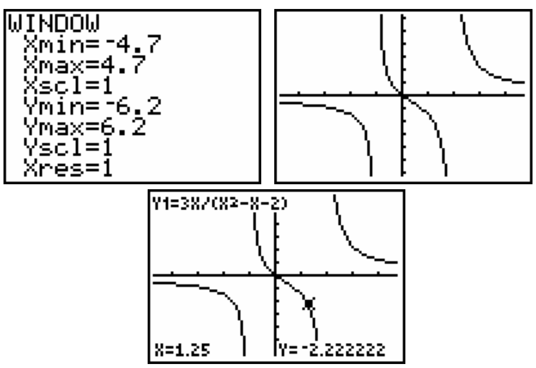
107. $f(x)=\frac{x+1}{x-4}$



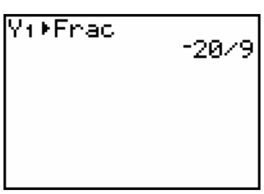
From the last screen we can see that $f(1.25)=-.\overline{81}$. If you wanted the fractional equivalent, you can go back to your homescreen and obtain Y_1 from the VARS menu. Thus $f(1.25)=-\frac{9}{11}$.



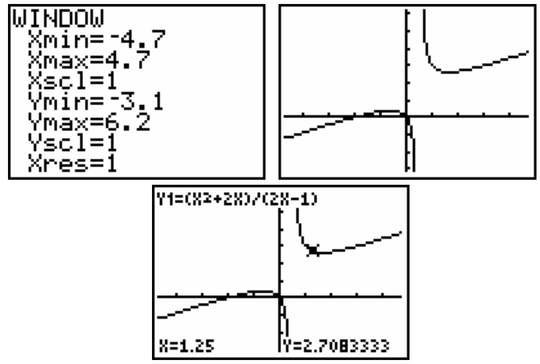
108. $f(x)=\frac{3x}{x^2-x-2}$



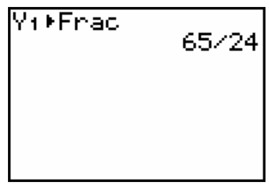
From the last screen we can see that $f(1.25)=-2.\overline{2}$. If you wanted the fractional equivalent, you can go back to your homescreen and obtain Y_1 from the VARS menu. Thus $f(1.25)=-\frac{20}{9}$.



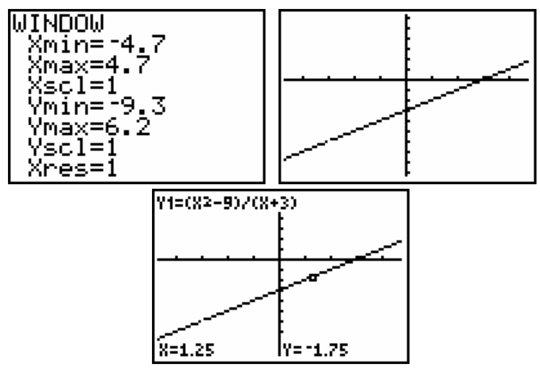
109. $f(x)=\frac{x^2+2x}{2x-1}$



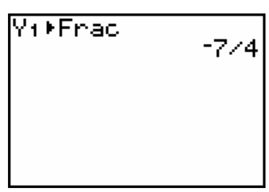
From the last screen we can see that $f(1.25)=-2.708\overline{3}$. If you wanted the fractional equivalent, you can go back to your homescreen and obtain Y_1 from the VARS menu. Thus $f(1.25)=\frac{65}{24}$.



110. $f(x)=\frac{x^2-9}{x+3}$



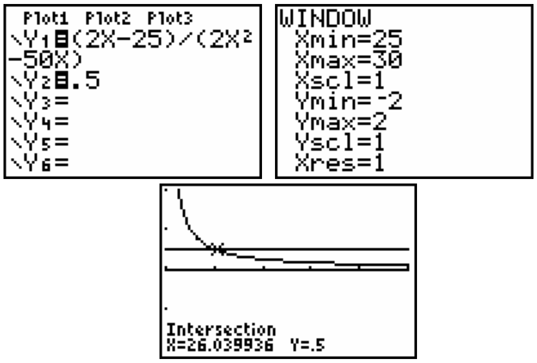
From the last screen we can see that $f(1.25)=-1.75$. If you wanted the fractional equivalent, you can go back to your homescreen and obtain Y_1 from the VARS menu. Thus $f(1.25)=-\frac{7}{4}$.



111. **(a)** $T(r) = \frac{2r-k}{2r^2-2kr}; k = 25 \Rightarrow$

$T(r) = \frac{2r-25}{2r^2-2(25)r} = \frac{2r-25}{2r^2-50r}$

Graph $Y_1 = \frac{2x-25}{2x^2-50x}$ and $Y_2 = .5$ (since 30 sec = .5 min) on the same screen.



Using the "intersect" option in the CALC menu, we find that the graphs intersect at $x \approx 26,$ which represents $r \approx 26$ in the given function. Therefore, there must be an average admittance rate of 26 vehicles per minute.

(b) $\frac{26}{5.3} \approx 4.9$ or 5 parking attendants must be on duty to keep the wait less than 30 seconds.

112. **(a)** Answers will vary. If the average number of people served is less than 9, the graph of $f(x)$ will be below the x-axis. This would correspond to a negative average waiting time, which is meaningless.

(b) If the average time to serve a customer is 5 minutes, then $\frac{60}{5} = 12$ customers can be served in an hour.

(c) If $x = 12,$

$$f(12) = \frac{9}{12(12-9)} = \frac{9}{12(3)} = \frac{9}{36} = \frac{1}{4} \text{ hr} = 15 \text{ min.}$$

A customer will have to wait 15 minutes on average.

(d) Let $f(x) = \frac{1}{8}.$

$$\frac{1}{8} = \frac{9}{x(x-9)}$$

$$x(x-9)(8)\left(\frac{1}{8}\right) = \left[\frac{9}{x(x-9)}\right](x)(x-9)(8)$$

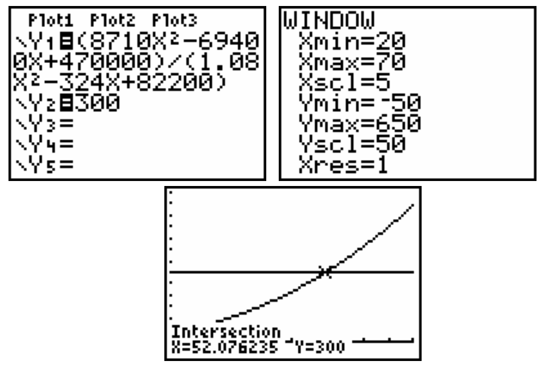
$$x^2 - 9x = 72 \Rightarrow x^2 - 9x - 72 = 0$$

Solve for x using the quadratic formula where $a = 1,$ $b = -9,$ and $c = -72.$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-72)}}{2(1)} = \frac{9 \pm \sqrt{81+288}}{2} = \frac{9 \pm \sqrt{369}}{2} \approx 4.25 \text{ min}$$

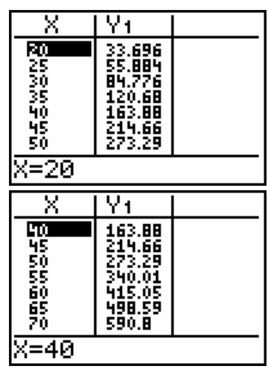
The employee must serve each customer in 4.25 minutes to cut the waiting time in half. An assistant could be hired to handle menial tasks, or a more efficient register system could be installed.

113. **(a)** Graph $y = d(x) = \frac{8710x^2 - 69,400x + 470,000}{1.08x^2 - 324x + 82,200}$ and $y = 300$ on the same calculator screen.



The graphs intersect when $x \approx 52.1$ miles per hour.

(b)



x	$d(x)$	x	$d(x)$
20	**34**	50	**273**
25	**56**	55	**340**
30	**85**	60	**415**
35	**121**	65	**499**
40	**164**	70	**591**
45	**215**		

(c) By comparing values in the table generated in part b, it appears that when the speed is doubled, the stopping distance is more than doubled.

(d) If the stopping distance doubled whenever the speed doubled, then there would be a linear relationship between the speed and the stopping distance.

114. (a) The ratios are listed in the table below.

Year	Deaths/Cases	Year	Deaths/Cases
1990	**.620**	1998	**.595**
1991	**.623**	1999	**.583**
1992	**.607**	2000	**.575**
1993	**.610**	2001	**.567**
1994	**.612**	2002	**.559**
1995	**.622**	2003	**.551**
1996	**.616**	2004	**.544**
1997	**.604**	2005	**.536**

(b) The values of the ratio stay about the same from 1990–1995, and then they steadily decrease.

(c) In exercise 3.1.67,

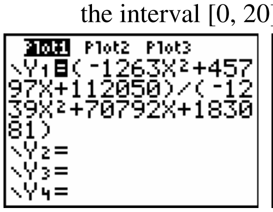
$f(x) = -1239x^2 + 70,792x + 183,081$

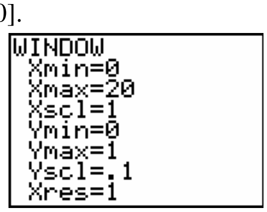
In exercises 3.1.68,

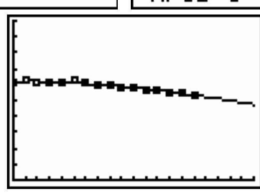
$g(x) = -1263x^2 + 45,797x + 112,050$

Let $h(x) = \frac{g(x)}{f(x)} = \frac{-1263x^2 + 45,797x + 112,050}{-1239x^2 + 70,792x + 183,081}$

Let $x = 0$ represent 1990. Graph $h(x)$ on the interval [0, 20].

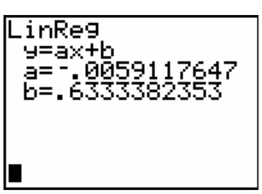






The model predicts the ratios in the table quite well.

(d)



$k(x) = -.0059x + .633$

(e) Using this model, we predict that the ratio of deaths to cases in 2006 was $k(16) = -.0059(16) + .633 = .5386$. Then solve

$\frac{25,000,000}{n} = .5386 \Rightarrow n = 46,416,636$

The model predicts that the cumulative number of AIDS cases in 2006 was 46,416,636.

115. $R(x) = \frac{80x - 8000}{x - 110}$

(a) $R(55) = \frac{80(55) - 8000}{55 - 110} \approx \$65.5$ tens of millions

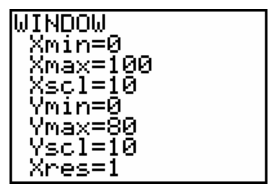
(b) $R(60) = \frac{80(60) - 8000}{60 - 110} = \$64$ tens of millions

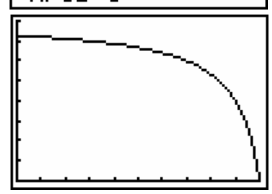
(c) $R(70) = \frac{80(70) - 8000}{70 - 110} = \$60$ tens of millions

(d) $R(90) = \frac{80(90) - 8000}{90 - 110} = \$40$ tens of millions

(e) $R(100) = \frac{80(100) - 8000}{100 - 110} = \$0$

(f)





116. $R(x) = \frac{60x - 6000}{x - 120}$

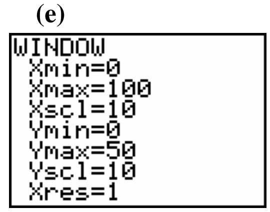
(a) $R(50) = \frac{60(50) - 6000}{50 - 120} = \$42.9$ tens of millions

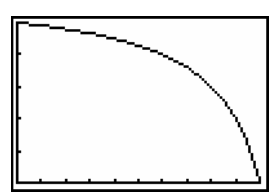
(b) $R(60) = \frac{60(60) - 6000}{60 - 120} = \$40$ tens of millions

(c) $R(80) = \frac{60(80) - 6000}{80 - 120} = \$30$ tens of millions

(d) $R(100) = \frac{60(100) - 6000}{100 - 120} = \$0$

(e)





117. Since the degree of the numerator equals the degree of the denominator in $f(x)=\frac{x^4-3x^3-21x^2+43x+60}{x^4-6x^3+x^2+24x-20}$, the graph has a horizontal asymptote at $y=\frac{1}{1}=1$.

118. Use synthetic division where $g(x)=x^4-3x^3-21x^2+43x+60$ and $k=-4$.

$$\begin{array}{r|rrrrr} -4 & 1 & -3 & -21 & 43 & 60 \\ & & -4 & 28 & -28 & -60 \\ \hline & 1 & -7 & 7 & 15 & 0 \end{array}$$

Now use synthetic division where $h(x)=x^3-7x^2+7x+15$ and $k=-1$.

$$\begin{array}{r|rrrr} -1 & 1 & -7 & 7 & 15 \\ & & -1 & 8 & -15 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The resulting polynomial quotient is $x^2-8x+15$, which factors to $(x-3)(x-5)$. Thus, the complete factorization of the numerator would be as follows.

$$\begin{aligned}&[x-(-4)][x-(-1)](x-3)(x-5)\\&=(x+4)(x+1)(x-3)(x-5)\end{aligned}$$

119. **(a)** Use synthetic division where $g(x)=x^4-6x^3+x^2+24x-20$ and $k=1$.

$$\begin{array}{r|rrrrr} 1 & 1 & -6 & 1 & 24 & -20 \\ & & 1 & -5 & -4 & 20 \\ \hline & 1 & -5 & -4 & 20 & 0 \end{array}$$

Now use synthetic division where $h(x)=x^3-5x^2-4x+20$ and $k=2$.

$$\begin{array}{r|rrrr} 2 & 1 & -5 & -4 & 20 \\ & & 2 & -6 & -20 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

The resulting polynomial quotient is $x^2-3x-10$, which factors to $(x+2)(x-5)$. Thus, the complete factorization of the denominator would be $(x-1)(x-2)(x+2)(x-5)$.

(b) $f(x)=\frac{(x+4)(x+1)(x-3)(x-5)}{(x-1)(x-2)(x+2)(x-5)}$

120. **(a)** $x-5$ is common factor to both the numerator and denominator.

(b) The value of x that makes $x-5=0$, namely $x=5$, will yield a "hole" in the graph of *f*. Note: When the common factor of $x-5$ is canceled, $g(x)=\frac{(x+4)(x+1)(x-3)}{(x-1)(x-2)(x+2)}$ agrees with *f* except at one point, which is the "hole" in *f*. To find the point, evaluate *g* at 5.

$$g(5)=\frac{(5+4)(5+1)(5-3)}{(5-1)(5-2)(5+2)}=\frac{(9)(6)(2)}{(4)(3)(7)}=\frac{108}{84}=\frac{9}{7}$$

121. Although $x-5$ is a factor of the numerator, it will not yield an *x*-intercept because it is also a factor of the denominator. The other three factors of the numerator, namely $(x+4)$, $(x+1)$, and $(x-3)$ will yield *x*-intercepts of -4, -1, and 3 because any of these will yield $f(x)=0$.

122. The *y*-intercept occurs at

$$f(0)=\frac{0^4-3(0^3)-21(0^2)+43(0)+60}{0^4-6(0^3)+0^2+24(0)-20}=\frac{60}{-20}=-3.$$

123. Although $x-5$ is a factor of the denominator, it will not yield a vertical asymptote because it is also a factor of the numerator. The other three factors of the denominator (namely $(x-1)$, $(x-2)$, and $(x+2)$) will yield vertical asymptotes of $x=1$, $x=2$, and $x=-2$ because any of these will yield a denominator of zero.

124. To determine where the function intersects the horizontal asymptote, we must solve the equation $\frac{x^4-3x^3-21x^2+43x+60}{x^4-6x^3+x^2+24x-20}=1$. However, since we know that the numerator and denominator have a common factor of $x-5$, it would be easier to solve the equation $\frac{(x+4)(x+1)(x-3)}{(x-1)(x-2)(x+2)}=1$.

$$\begin{aligned}\frac{(x+4)(x+1)(x-3)}{(x-1)(x-2)(x+2)}&=1\\ \frac{x^3+2x^2-11x-12}{x^3-x^2-4x+4}&=1\\ x^3+2x^2-11x-12&=x^3-x^2-4x+4\\ 3x^2-7x-16&=0\end{aligned}$$

(*continued on next page*)

(*continued from page 353*)

Solve for x using the quadratic formula where $a = 3,\ b = -7,$ and $c = -16.$

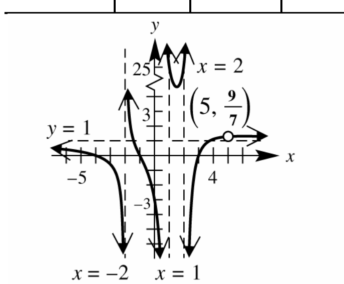
$$x = \frac{-(-7)\pm\sqrt{(-7)^2-4(3)(-16)}}{2(3)} = \frac{7\pm\sqrt{49+192}}{6} = \frac{7\pm\sqrt{241}}{6} \Rightarrow$$

$$x = \frac{7-\sqrt{241}}{6} \approx -1.42 \text{ or } x = \frac{7+\sqrt{241}}{6} \approx 3.75$$

Thus the points of intersection are $\left(\frac{7-\sqrt{241}}{6}, 1\right)$ and $\left(\frac{7+\sqrt{241}}{6}, 1\right).$

125. The vertical asymptotes are $x = 1,\ x = 2,$ and $x = -2$ and the x-intercepts occur at $-4,\ -1,$ and $3.$ We should also consider the "hole" in the graph which occurs at $x = 5.$ This divides the real number line into eight intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -4)$	-5	$\frac{16}{63}$	Positive	Above
$(-4, -2)$	-3	$-\frac{3}{5}$	Negative	Below
$(-2, -1)$	$-\frac{3}{2}$	$\frac{9}{7}$	Positive	Above
$(-1, 1)$	0	-3	Negative	Below
$(1, 2)$	$\frac{3}{2}$	$\frac{165}{7}$	Positive	Above
$(2, 3)$	$\frac{5}{2}$	$-\frac{91}{27}$	Negative	Below
$(3, 5)$	4	$\frac{10}{9}$	Positive	Above
$(5, \infty)$	6	$\frac{21}{16}$	Positive	Above



$$f(x) = \frac{x^4-3x^3-21x^2+43x+60}{x^4-6x^3+x^2+24x-20}$$

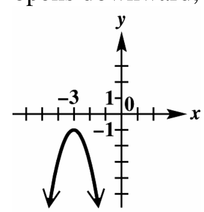
126. **(a)** $(-4,-2)\cup(-1,1)\cup(2,3)$

(b) $(-\infty,-4)\cup(-2,-1)\cup(1,2)\cup(3,5)\cup(5,\infty)$

Chapter 3 Quiz
(Sections 3.1–3.5)

1. **(a)** $f(x) = -2(x+3)^2 - 1$

This equation is of the form $y = a(x-h)^2 + k,$ with $h = -3,$ $k = -1,$ and $a = -2.$ The graph opens downward and is narrower than $y = x^2.$ It is a horizontal translation of the graph of $y = -2x^2,$ 3 units to the left and 1 unit down. The vertex is $(-3, -1)$. The axis is $x = -3$. The domain is $(-\infty, \infty)$. Since the largest value of y is -1 and the graph opens downward, the range is $(-\infty, -1]$.

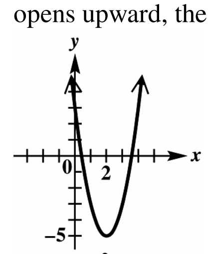


$f(x) = -2(x+3)^2 - 1$

(b) $f(x) = 2x^2 - 8x + 3$; Rewrite by completing the square on x

$$f(x) = 2x^2 - 8x + 3$$
$$= 2(x^2 - 4x + 4) + (3 - 8)$$
$$\text{Note: } 2\left[\tfrac{1}{2}(-4)\right]^2 = 2(-2)^2 = 8$$
$$= 2(x-2)^2 - 5$$

This equation is of the form $y = a(x-h)^2 + k,$ with $h = 2,$ $k = -5,$ and $a = 2.$ The graph opens upward and is narrower than $y = x^2.$ It is a horizontal translation of the graph of $y = 2x^2,$ 2 units to the right and 5 units down. The vertex is $(2, -5)$. The axis is $x = 2$. The domain is $(-\infty, \infty)$. Since the smallest value of y is -5 and the graph opens upward, the range is $[-5, \infty)$.



$f(x) = 2x^2 - 8x + 3$

2. (a) Since $v_0 = 64$, and $s_0 = 200$, and $s(t) = -16t^2 + v_0 t + s_0$ we have $s(t) = -16t^2 + 64t + 200$.

(b) Algebraic Solution:

To find the time interval in which the ball will be more than 240 ft above ground level, solve the inequality

$-16t^2 + 64t + 200 > 240$:

$-16t^2 + 64t - 40 > 0$

$-2t^2 + 8t - 5 > 0$

Solve the corresponding equation

$-2t^2 + 8t - 5 = 0$

Use the quadratic formula with $a = -2$, $b = 8$, and $c = -5$.

$$t = \frac{-8 \pm \sqrt{8^2 - 4(-2)(-5)}}{2(-2)} = \frac{-8 \pm \sqrt{64 - 40}}{-4} = \frac{-8 \pm \sqrt{24}}{-4} = \frac{-8 \pm 2\sqrt{6}}{-4}$$

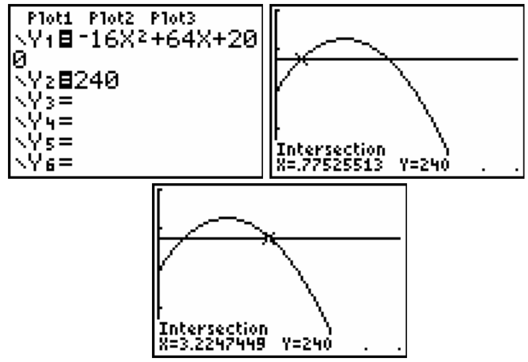
$$t = \frac{-8 + 2\sqrt{6}}{-4} \approx .78 \text{ or}$$

$$t = \frac{-8 - 2\sqrt{6}}{-4} \approx 3.22$$

The values .78 and 3.22 divide the number line into three intervals: $(-\infty, .78)$, $(.78, 3.22)$, and $(3.22, \infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $-16t^2 + 64t + 200 > 240$ True or False?
$(-\infty, .78)$	0	$-16 \cdot 0^2 + 64 \cdot 0 + 200 \overset{?}{>} 240$ $200 > 300$ False
$(.78, 3.22)$	1	$-16 \cdot 1^2 + 64 \cdot 1 + 200 \overset{?}{>} 240$ $248 > 240$ True
$(3.22, \infty)$	5	$-16 \cdot 5^2 + 64 \cdot 5 + 200 \overset{?}{>} 240$ $120 > 300$ False

Graphing Calculator Solution:



The ball will be more than 240 ft above the ground between approximately.78 sec and 3.22 sec.

3. $f(x) = 2x^4 + x^3 - 3x + 4$; $k = 2$

Use synthetic division to write the polynomial in the form $f(x) = (x - k)q(x) + r$. Since the x^2 is missing, include 0 as its coefficient.

2)	2	1	0	−3	4
		4	10	20	34
	2	5	10	17	38

$f(x) = (x - 2)(2x^3 + 5x^2 + 10x + 17) + 38$

The remainder is 38, so $k = 2$ is not a zero of the function. $k(2) = 38$.

4. $f(x) = x^2 - 4x + 5$; $k = 2 + i$

Use synthetic division to write the polynomial in the form $f(x) = (x - k)q(x) + r$.

$2+i$)	1	−4	5
		$2+i$	−5
	1	$-2+i$	0

$f(x) = [x - (2+i)][x - 2 + i] + 0$
$= [x - 2 - i][x - 2 + i] + 0$

The remainder is 0, so $k = 2 + i$ is a zero of the function.

5. Since $3 - i$ is a factor, $3 + i$ is also a factor. So

$f(x)$
$= [(x - (3 - i)][(x - (3 + i)][(x - (-2)](x - 3)$
$= (x - 3 + i)(x - 3 - i)(x + 2)(x - 3)$
$= (x^2 - 6x + 1)(x^2 - x - 6)$
$= x^4 - 7x^3 + 10x^2 + 26x - 60$

6. $f(x) = x(x-2)^3(x+2)^2$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x = 0$ or $(x-2)^3 = 0 \Rightarrow x = 2$ or
$(x+2)^2 = 0 \Rightarrow x = -2$

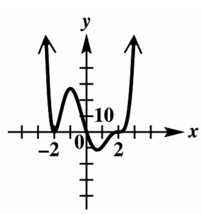
Note that $x = -2$ is a zero of multiplicity 2, so the graph is tangent to the x-axis at $x = -2$. Also note that $x = 2$ is a zero of multiplicity 3, so the graph crosse the x-axis at $x = 2$. The three zeros, –2, 0, and 2, divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0$, so plot $(0, 0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	–3	375	Positive	Above
$(-2, 0)$	–1	27	Positive	Above
$(0, 2)$	1	–9	Negative	Below
$(2, \infty)$	3	75	Positive	Above

Plot the x-intercepts, y-intercept (which is also an x-intercept in this exercise), and test points with a smooth curve to get the graph.



$f(x) = x(x-2)^3(x+2)^2$

7. $f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$

Step 1: p must be a factor of $a_0 = -45$ and q must be a factor of $a_4 = 2$. Thus, p can be ± 1, ± 3, ± 5, ± 9, ± 15, or ± 45, and q can be ± 1 or ± 2. The possible rational zeros, $\frac{p}{q}$, are

$\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm 3, \pm\frac{9}{2}, \pm 5, \pm\frac{15}{2}, \pm 15, \pm\frac{45}{2}$, or ± 45. The remainder theorem shows that 1 is a zero.

1)	2	−9	−5	57	−45
		2	−7	−12	45
	2	−7	−12	45	0

The new quotient polynomial is
$2x^3 - 7x^2 - 12x + 45$.

p must be a factor of $a_0 = -45$ and q must be a factor of $a_3 = 2$. Thus, p can be ± 1, ± 3, ± 5, ± 9, ± 15, or ± 45, and q can be ± 1 or ± 2. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{2}, \pm 1$,

$\pm\frac{3}{2}, \pm\frac{5}{2}, \pm 3, \pm\frac{9}{2}, \pm 5, \pm\frac{15}{2}, \pm 15, \pm\frac{45}{2}$, or ± 45.

The remainder theorem shows that 3 is a zero.

3)	2	−7	−12	45
		6	−3	−45
	2	−1	−15	0

The new quotient polynomial is

$2x^2 - x - 15 = 0$
$(2x+5)(x-3) = 0$
$2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$ or $x - 3 = 0 \Rightarrow x = 3$

The rational zeros are $-\frac{5}{2}$, 1, and 3. Note that 3 is a zero of multiplicity 2, so the graph is tangent to the x-axis at $x = 3$. The factors are $2x + 5$, $x - 1$, and $(x-3)^2$ and thus

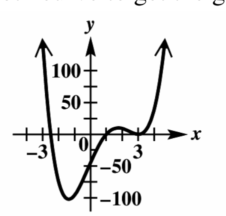
$f(x) = (2x+5)(x-1)(x-3)^2$.

Step 2: $f(0) = -45$, so plot $(0, -45)$.

Step 3: The x-intercepts divide the x-axis into four intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty, -\frac{5}{2}\right)$	–3	144	Positive	Above
$\left(-\frac{5}{2}, 1\right)$	–1	–96	Negative	Below
$(1, 3)$	2	9	Positive	Above
$(3, \infty)$	4	39	Positive	Above

Plot the x-intercepts, y-intercept and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$
$= (x-3)^2(2x+5)(x-1)$

8. $f(x) = -4x^5 + 16x^4 + 13x^3 - 76x^2 - 3x + 18$

Step 1: p must be a factor of $a_0 = 18$ and q must be a factor of $a_5 = -4$. Thus, p can be ±1, ±2, ±3, ±6, ±9, or ±18, and q can be ±1 ±2, or ±4. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 6, \pm 9, \pm\frac{9}{2}$, $\pm\frac{9}{4}$, or ± 18. The remainder theorem shows that 3 is a zero.

$$\begin{array}{r|rrrrrr} 3 & -4 & 16 & 13 & -76 & -3 & 18 \\ & & -12 & 12 & 75 & -3 & -18 \\ \hline & -4 & 4 & 25 & -1 & -6 & 0 \end{array}$$

The new quotient polynomial is $-4x^4 + 4x^3 + 25x^2 - x - 6$. Find the zeros of this function. p must be a factor of $a_0 = -6$ and q must be a factor of $a_4 = -4$. Thus, p can be ±1, ±2, ±3, or ±6, and q can be ±1 ±2, or ±4. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3,$ or ± 6. The remainder theorem shows that −2 is a zero.

$$\begin{array}{r|rrrrr} -2 & -4 & 4 & 25 & -1 & -6 \\ & & 8 & -24 & -2 & 6 \\ \hline & -4 & 12 & 1 & -3 & 0 \end{array}$$

The new quotient polynomial is $-4x^3 + 12x^2 + x - 3$. Find the zeros of this function. p must be a factor of $a_0 = -3$ and q must be a factor of $a_4 = -4$. Thus, p can be ±1 or ±3, and q can be ±1 ±2, or ±4. The possible rational zeros, $\frac{p}{q}$, are $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1,$ or ± 3. The remainder theorem shows that 3 is a zero.

$$\begin{array}{r|rrrr} 3 & -4 & 12 & 1 & -3 \\ & & -12 & 0 & 3 \\ \hline & -4 & 0 & 1 & 0 \end{array}$$

The new quotient polynomial is $-4x^2 + 1$. Set this equal to 0 and solve by factoring:

$$-4x^2 + 1 = 0 \Rightarrow -\left(4x^2 - 1\right) = 0 \Rightarrow$$
$$-\left(2x+1\right)\left(2x-1\right) = 0 \Rightarrow x = \pm\frac{1}{2}$$

The zeros are $\pm\frac{1}{2}$, −2, and 3. Note the 3 is a zero of multiplicity 2, so the graph is tangent to the x-axis at $x = 3$. Thus, the factored form of the function is

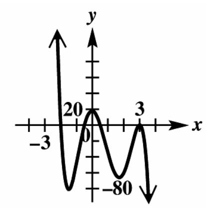
$$-\left(x+2\right)\left(2x+1\right)\left(2x-1\right)\left(x-3\right)^2.$$

Step 2: $f\left(0\right) = 18$, so plot $\left(0, 18\right)$.

Step 3: The x-intercepts divide the x-axis into five intervals:

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	−3	1260	Positive	Above
$\left(-2, -\frac{1}{2}\right)$	−1	−48	Negative	Below
$\left(-\frac{1}{2}, \frac{1}{2}\right)$	0	18	Positive	Above
$\left(\frac{1}{2}, 3\right)$	2	−60	Negative	Below
$(3, \infty)$	4	−378	Negative	Below

Plot the x-intercepts, y-intercept and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = -4x^5 + 16x^4 + 13x^3 - 76x^2 - 3x + 18$
$= -(x+2)(2x+1)(2x-1)(x-3)^2$

9. $f(x) = \dfrac{3x+1}{x^2 + 7x + 10} = \dfrac{3x+1}{(x+5)(x+2)}$

Step 1: The graph has vertical asymptotes where the denominator equals zero, that is, at $x = -5$ and $x = -2$.

Step 2: Since the degree of the numerator is less than the degree of the denominator, the graph has a horizontal asymptote at $y = 0$ (the x-axis).

Step 3: The y-intercept is $f(0) = \frac{3(0)+1}{0^2+7(0)+10} = \frac{1}{10}$

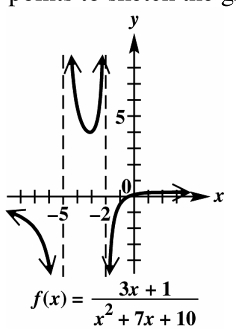
Step 4: Find any x-intercepts by solving $f(x) = 0$: $\frac{3x+1}{x^2+7x+10} = \frac{3x+1}{(x+2)(x+5)} = 0 \Rightarrow$ $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$

Step 5: The graph will intersect the horizontal asymptote, $y = 0$, when $f(x) = 0$. From step 4, that is at $x = -\frac{1}{3}$.

Step 6: Plot a point in each of the intervals determined by the x-intercept and the vertical asymptotes. There are four intervals, $(-\infty, -5), (-5, -2), \left(-2, -\frac{1}{3}\right),$ and $\left(-\frac{1}{3}, \infty\right)$.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-5)$	–6	−4.24	Negative	Below
$(-5,-2)$	–4	5.5	Positive	Above
$\left(-2,-\frac{1}{3}\right)$	–1	–.5	Negative	Below
$\left(-\frac{1}{3},\infty\right)$	3	.25	Positive	Above

Use the asymptotes, intercepts, and these points to sketch the graph.



10. $f(x) = \dfrac{x^2+2x+1}{x-1} = \dfrac{(x+1)^2}{x-1}$

Step 1: The graph has a vertical asymptote where the denominator equals zero, that is, at $x = 1$.

Step 2: Since the degree of the numerator is one more than the degree of the denominator, the graph has an oblique asymptote. Divide the numerator by the denominator:

$$\begin{array}{r|rrr} 1 & 1 & 2 & 1 \\ & & 1 & 3 \\ \hline & 1 & 3 & 4 \end{array}$$

$f(x) = \frac{x^2+2x+1}{x-1} = x+3+\frac{4}{x-1}$

The oblique asymptote is $y = x + 3$.
There is no horizontal asymptote.

Step 3: The y-intercept is

$f(0) = \frac{0^2+2(0)+1}{0-1} = -1$

Step 4: Find any x-intercepts by solving

$f(x) = 0: \frac{x^2+2x+1}{x-1} = 0 \Rightarrow x^2+2x+1 = 0 \Rightarrow$

$(x+1)^2 = 0 \Rightarrow x = -1$

Step 5: The graph will intersect the oblique asymptote when $f(x) = x+3$.

$\frac{x^2+2x+1}{x-1} = x+3$

$x^2+2x+1 = (x+3)(x-1)$

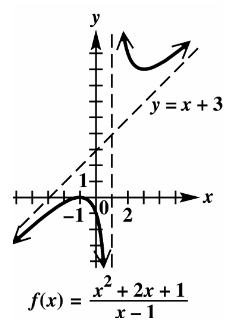
$x^2+2x+1 = x^2+2x-3 \Rightarrow 1 = -3$

This is a false statement, so the graph does not intersect the oblique asymptote.

Step 6: Plot a point in each of the intervals determined by the x-intercept and the vertical asymptote. There are three intervals, $(-\infty,-1), (-1,1),$ and $(1,\infty)$.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-1)$	–3	−1	Negative	Below
$(-1,1)$	$\frac{1}{2}$	$-\frac{9}{2}$	Negative	Below
$(1,\infty)$	2	9	Positive	Above

Use the asymptotes, intercepts, these points, and the general behavior of the graph near its asymptotes to sketch the graph.



Section 3.6: Variation

1. $C = 2\pi r$, where C is the circumference of a circle of radius r
The circumference of a circle varies directly as (or is proportional to) its radius.

2. $d = \frac{1}{5}s$, where d is the approximate distance (in miles) from a storm and s is the number of seconds between seeing lightning and hearing thunder
The distance from a storm varies directly as (or is proportional to) the number of seconds between seeing lightning and hearing thunder.

3. $r = \frac{d}{t}$, where r is the speed when traveling d miles in t hours
The average speed varies directly as (or is proportional to) the distance traveled and inversely as the time.

4. $d = \frac{1}{4\pi nr^2}$, where d is the distance a gas atom of radius r travels between collisions and n is the number of atoms per unit volume
The average distance a gas atom travels between collisions varies inversely as the square of its radius and the number of atoms per unit volume.

5. $s = kx^3$, where s is the strength of a muscle of length x
The strength of a muscle varies directly as (or is proportional to) the cube of its length.

6. $f = \frac{mv^2}{r}$, where f is the centripetal force of an object of mass m moving along a circle of radius r at velocity v.
The centripetal force of an object varies directly as (or is proportional to) its mass and the square of its velocity and inversely as the radius of the circle it moves along.

7. y varies directly as x, $y = kx$, is a straight-line model. It matches graph C.

8. y varies inversely as x, $y = \frac{k}{x}$, matches graph B.

9. y varies directly as the second power of x, $y = kx^2$, matches graph A.

10. x varies directly as the second power of y, $x = ky^2$ or $y = \sqrt{\frac{x}{k}}$, matches graph D.

11. *Step 1*: $y = kx$
Step 2: Substitute $x = 4$ and $y = 20$ to find k.
$20 = k(4) \Rightarrow k = 5$
Step 3: $y = 5x$
Step 4: Now find y when $x = -6$.
$y = 5(-6) = -30$

12. *Step 1*: $y = kx$
Step 2: Substitute $x = 30$ and $y = 9$ to find k.
$9 = k(30) \Rightarrow k = \frac{9}{30} = \frac{3}{10}$
Step 3: $y = \frac{3}{10}x$
Step 4: Now find y when $x = 40$.
$y = \frac{3}{10}(40) = 12$

13. *Step 1*: $m = kxy$
Step 2: Substitute $m = 10$, $x = 2$, and $y = 14$ to find k.
$10 = k(4)(7) \Rightarrow 10 = 28k \Rightarrow k = \frac{5}{14}$
Step 3: $m = \frac{5}{14}xy$
Step 4: Now find m when $x = 11$ and $y = 8$.
$m = \frac{5}{14}(11)(8) = \frac{440}{14} = \frac{220}{7}$

14. *Step 1*: $m = kzp$
Step 2: Substitute $m = 10$, $z = 2$, and $p = 7.5$ to find k.
$10 = k(2)(7.5) \Rightarrow 10 = 15k \Rightarrow k = \frac{10}{15} = \frac{2}{3}$
Step 3: $m = \frac{2}{3}zp$
Step 4: Now find m when $z = 5$ and $p = 7$.
$m = \frac{2}{3}(5)(7) = \frac{70}{3}$

15. *Step 1*: $y = \frac{k}{x}$
Step 2: Substitute $x = 3$ and $y = 10$ to find k.
$10 = \frac{k}{3} \Rightarrow k = 30$
Step 3: $y = \frac{30}{x}$
Step 4: Now find y when $x = 20$.
$y = \frac{30}{20} = \frac{3}{2}$

16. *Step 1*: $y = \frac{k}{x}$
Step 2: Substitute $x = \frac{1}{4}$ & $y = 20$ to find k.
$20 = \frac{k}{\frac{1}{4}} \Rightarrow 20 = 4k \Rightarrow k = 5$
Step 3: $y = \frac{5}{x}$
Step 4: Now find y when $x = 15$.
$y = \frac{5}{15} = \frac{1}{3}$

17. *Step 1*: $r=\frac{km^2}{s}$

Step 2: Substitute $r=12$, $m=6$, and $s=4$ to find *k*.

$12=\frac{k\cdot 6^2}{4}\Rightarrow 12=\frac{36k}{4}\Rightarrow 12=9k\Rightarrow k=\frac{4}{3}$

Step 3: $r=\frac{4}{3}\cdot\frac{m^2}{s}=\frac{4m^2}{3s}$

Step 4: Now find *r* when $m=6$ and $s=20$.

$r=\frac{4(6^2)}{3(20)}=\frac{4(36)}{60}=\frac{144}{60}=\frac{12}{5}$

18. *Step 1*: $p=\frac{kz^2}{r}$

Step 2: Substitute $p=\frac{32}{5}$, $z=4$, and $r=10$ to find *k*.

$\frac{32}{5}=\frac{k(4^2)}{10}\Rightarrow\frac{32}{5}=\frac{16k}{10}$

$320=80k\Rightarrow\frac{320}{80}=k\Rightarrow k=4$

Step 3: $p=\frac{4z^2}{r}$

Step 4: Now find *p* when $z=3$ and $r=36$.

$p=\frac{4(3^2)}{36}=\frac{4(9)}{36}=\frac{36}{36}=1$

19. *Step 1*: $a=\frac{kmn^2}{y^3}$

Step 2: Substitute $a=9$, $m=4$, $n=9$, and $y=3$ to find *k*.

$9=\frac{k(4)(9^2)}{3^3}\Rightarrow 9=\frac{k(4)(81)}{27}$

$9=\frac{324k}{27}\Rightarrow 9=12k$

$k=\frac{3}{4}$

Step 3: $a=\frac{3}{4}\cdot\frac{mn^2}{y^3}=\frac{3mn^2}{4y^3}$

Step 4: Now find *a* when $m=6$, $n=2$, and $y=5$.

$a=\frac{3(6)(2^2)}{4(5^3)}=\frac{3(6)(4)}{4(125)}=\frac{18}{125}$

20. *Step 1*: $y=\frac{kx}{m^2r^2}$

Step 2: Substitute $y=\frac{5}{3}$, $x=1$, $m=2$, and $r=3$ to find *k*.

$\frac{5}{3}=\frac{k(1)}{(2^2)(3^2)}\Rightarrow\frac{5}{3}=\frac{k}{(4)(9)}$

$\frac{5}{3}=\frac{k}{36}\Rightarrow 180=3k$

$k=60$

Step 3: $y=\frac{60x}{m^2r^2}$

Step 4: Now find *y* when $x=3$, $m=1$, and $r=8$.

$y=\frac{60(3)}{(1^2)(8^2)}=\frac{180}{(1)(64)}=\frac{180}{64}=\frac{45}{16}$

21. *Step 1*: Let C be the circumference of the circle (in inches), *r* is the radius of the circle (in inches). $C=kr$

Step 2: Substitute $C=43.96$ and $r=7$ to find *k*.

$43.96=k(7)\Rightarrow k=6.28$

Step 3: $C=6.28r$

Step 4: Now find *C* when $r=11$.

$C=6.28(11)=69.08$

A radius of 11 in. yields a circumference of 69.08 in.

22. *Step 1*: Let p be the pressure (in psi); *d* is the depth (in inches). $p=kd$

Step 2: Substitute $p=50$ and $d=10$ to find *k*.

$50=k(10)\Rightarrow k=5$

Step 3: $p=5d$

Step 4: Now find *p* when $d=15$.

$p=5(15)=75$

A pressure of 75 psi occurs at a depth of 15 in.

23. *Step 1*: Let R be the resistance (in ohms); *t* is the temperature (in degrees Kelvin, K). $R=kt$

Step 2: Substitute $R=646$ and $t=190$ to find *k*: $646=k(190)\Rightarrow k=\frac{646}{190}=\frac{17}{5}$

Step 3: $R=\frac{17}{5}t$

Step 4: Now find *R* when $t=250$.

$R=\frac{17}{5}(250)=850$

A resistance of 850 ohms occurs at 250K.

24. *Step 1*: Let d be the distance a person can see (in km), *h* is the height above Earth (in meters). $d=k\sqrt{h}$

Step 2: Substitute $d=18$ and $h=144$ to find *k*.

$18=k\sqrt{144}\Rightarrow 18=12k\Rightarrow k=\frac{18}{12}=\frac{3}{2}$

Step 3: $d=\frac{3}{2}\sqrt{h}$

Step 4: Now find *d* when $h=64$.

$d=\frac{3}{2}\sqrt{64}=\frac{3}{2}(8)=12$

A person can see 12 km from a point 64 m above earth.

25. *Step 1*: Let e be weight on earth (in lbs). m is weight on moon (in lbs). $e = km$
Step 2: Substitute $e = 200$ and $m = 32$ to find k: $200 = k(32) \Rightarrow k = \frac{200}{32} = 6.25$
Step 3: $e = 6.25m$
Step 4: Now find m when $e = 50$.
$60 = 6.25m \Rightarrow m = \frac{50}{6.25} = 8$
The dog would weigh 8 lbs on the moon.

26. *Step 1*: Let w be the amount of water (in gal). D is the diameter of pipe (in inches). $w = kD^2$
Step 2: Substitute $w = 200$ and $D = 6$ to find k.
$200 = k(6^2) \Rightarrow 200 = 36k \Rightarrow k = \frac{200}{36} = \frac{50}{9}$
Step 3: $w = \frac{50}{9}D^2$
Step 4: Now find w when $D = 12$.
$w = \frac{50}{9}(12^2) = \frac{50}{9}(144) = 800$
A 12-in. diameter pipe would allow 800 gallons of water to flow.

27. *Step 1*: Let d be the distance the spring stretches (in inches), f is the force applied (in lbs). $d = kf$
Step 2: Substitute $d = 8$ & $f = 15$ to find k.
$8 = k(15) \Rightarrow k = \frac{8}{15}$
Step 3: $d = \frac{8}{15}f$
Step 4: Now find d when $f = 30$.
$d = \frac{8}{15}(30) = 16$
The spring will stretch 16 in.

28. *Step 1*: Let c be the current (in amps). R is the resistance (in ohms). $c = \frac{k}{R}$
Step 2: Substitute $c = 50$ and $R = 10$ to find k.
$50 = \frac{k}{10} \Rightarrow k = 500$
Step 3: $c = \frac{500}{R}$
Step 4: Now find c when $R = 5$.
$c = \frac{500}{5} = 100$: The current will be 100 amps.

29. *Step 1*: Let v be the speed of the pulley (in rpm). D is the diameter of the pulley (in inches). $v = \frac{k}{D}$
Step 2: Substitute $v = 150$ and $D = 3$ to find k.
$150 = \frac{k}{3} \Rightarrow k = 450$
Step 3: $v = \frac{450}{d}$
Step 4: Now find v when $D = 5$: $v = \frac{450}{5} = 90$
The 5-in. diameter pulley will have a speed of 90 revolutions per minute.

30. *Step 1*: Let w be the weight of the object (in lbs). d is the distance from the center of Earth (in mi). $w = \frac{k}{d^2}$
Step 2: Substitute $w = 90$ and $d = 8000$ to find k.
$90 = \frac{k}{8000^2} \Rightarrow 90 = \frac{k}{64{,}000{,}000}$
$k = 5{,}760{,}000{,}000$
Step 3: $w = \frac{5{,}760{,}000{,}000}{d^2}$
Step 4: Now find w when $d = 12{,}000$.
$w = \frac{5{,}760{,}000{,}000}{12{,}000^2} = \frac{5{,}760{,}000{,}000}{144{,}000{,}000} = 40$
An object will weigh 40 lbs when it is 12,000 mi from the center of Earth.

31. *Step 1*: Let R be the resistance (in ohms), D is the diameter (in inches). $R = \frac{k}{D^2}$
Step 2: Substitute $D = .01$ and $R = .4$ to find k.
$.4 = \frac{k}{.01^2} \Rightarrow .4 = \frac{k}{.0001} \Rightarrow k = .00004$
Step 3: $R = \frac{.00004}{D^2}$
Step 4: Now find R when $D = .03$.
$R = \frac{.00004}{.03^2} = \frac{.00004}{.0009} \approx .0444$
The resistance is approximately .0444 ohm.

32. *Step 1*: Let I be the illumination produced (in candela). d is the distance from the source (in meters). $I = \frac{k}{d^2}$
Step 2: Substitute $I = 70$ & $d = 5$ to find k.
$70 = \frac{k}{5^2} \Rightarrow 70 = \frac{k}{25} \Rightarrow k = 1750$
Step 3: $I = \frac{1750}{d^2}$
Step 4: Now find I when $d = 12$.
$I = \frac{1750}{12^2} = \frac{1750}{144} = \frac{875}{72}$
The illumination is $\frac{875}{72}$ candela.

33. *Step 1*: Let i be the interest (in dollars). p is the principal (in dollars). t is the time (in years). $i = kpt$
Step 2: Substitute $i = 110$, $p = 1000$, and $t = 2$ to find k.
$110 = k(1000)(2) \Rightarrow 110 = 2000k$
$k = \frac{110}{2000} = .055$ (5.5%)
Step 3: $i = .055pt$
Step 4: Now find i when $p = 5000$ and $t = 5$.
$i = .055(5000)(5) = 1375$
The amount of interest is $1375.

34. *Step 1*: Let V be the volume (in L), t is the temperature (in degrees Kelvin, K), p is the pressure (in newtons per square centimeter).
$V = \frac{kt}{p}$
Step 2: Substitute $v = 1.3$, $t = 300$, and $p = 18$ to find k.
$1.3 = \frac{k(300)}{18} \Rightarrow 23.4 = 300k \Rightarrow k = \frac{23.4}{300} = .078$
Step 3: $V = \frac{.078t}{p}$
Step 4: Now find V when $t = 340$ and $p = 24$.
$V = \frac{.078(340)}{24} = \frac{26.52}{24} = 1.105$
The volume is 1.105 L.

35. *Step 1*: Let F be the force of the wind (in lbs), A is the area (in ft^2), v is the velocity of the wind (in mph). $F = kAv^2$
Step 2: Substitute $F = 50$, $A = \frac{1}{2}$, and $v = 40$ to find k.
$50 = k\left(\frac{1}{2}\right)\left(40^2\right) = k\left(\frac{1}{2}\right)(1600)$
$50 = 800k \Rightarrow k = \frac{50}{800} = \frac{1}{16}$
Step 3: $F = \frac{1}{16}Av^2$
Step 4: Now find F when $v = 80$, and $A = 2$.
$F = \frac{1}{16}(2)\left(80^2\right) = \frac{1}{16}(2)(6400) = 800$
The force would be 800 pounds.

36. *Step 1*: Let V be the volume (in cm^3). r is the radius (in cm), h is the height (in cm).
$V = kr^2h$
Step 2: Substitute $V = 300$, $r = 3$, and $h = 10.62$ to find k.
$300 = k\left(3^2\right)(10.62) \Rightarrow 300 = k(9)(10.62)$
$300 = 95.58k \Rightarrow k = \frac{300}{95.58} \approx 3.1387$
Step 3: $V = 3.1387r^2h$
Step 4: Now find V when $h = 15.92$ and $r = 4$.
$V = 3.1387\left(4^2\right)(15.92)$
$= 3.1387(16)(15.92) \approx 799.5$
The volume is 799.5 cm^3.

37. *Step 1*: Let L be the load (in metric tons), D is the diameter (in meters), h is the height (in meters). $L = \frac{kD^4}{h^2}$
Step 2: Substitute $h = 9$, $D = 1$, and $L = 8$ to find k: $8 = \frac{k\left(1^4\right)}{9^2} \Rightarrow 8 = \frac{k}{81} \Rightarrow k = 648$
Step 3: $L = \frac{648D^4}{h^2}$
Step 4: Now find L when $D = \frac{2}{3}$ and $h = 12$.
$L = \frac{648\left(\frac{2}{3}\right)^4}{12^2} = \frac{648\left(\frac{16}{81}\right)}{144} = \frac{128}{144} = \frac{8}{9}$
A column 12 m high and $\frac{2}{3}$ m in diameter will support $\frac{8}{9}$ metric ton.

38. *Step 1*: Let L be the load (in kg), w is the width (in cm), h is the height (in cm), l is the length between supports (in meters). $L = \frac{kwh^2}{l}$
Step 2: Substitute $L = 400$, $w = 12$, $h = 15$, and $l = 8$ to find k.
$400 = \frac{k(12)\left(15^2\right)}{8} \Rightarrow 400 = \frac{k(12)(225)}{8}$
$400 = \frac{2700k}{8} \Rightarrow 3200 = 2700k \Rightarrow k = \frac{3200}{2700} = \frac{32}{27}$
Step 3: $L = \frac{32}{27} \cdot \frac{wh^2}{l} = \frac{32wh^2}{27l}$
Step 4: Now find L when $w = 24$, $h = 8$, and $l = 16$: $L = \frac{32(24)\left(8^2\right)}{27(16)} = \frac{32(24)(64)}{432} = \frac{49,152}{432} = \frac{1024}{9}$
The maximum load is $\frac{1024}{9}$ kg.

39. *Step 1*: Let p be the period of pendulum (in sec), l is the length of pendulum (in cm), a is the acceleration due to gravity $\left(\text{in } \frac{\text{cm}}{\text{sec}^2}\right)$.
$p = \frac{k\sqrt{l}}{\sqrt{a}}$
Step 2: Substitute $p = 6\pi$, $l = 289$, and $a = 980$ to find k.
$6\pi = \frac{k\sqrt{289}}{\sqrt{980}} \Rightarrow 6\pi = \frac{17k}{14\sqrt{5}}$
$84\sqrt{5}\pi = 17k \Rightarrow k = \frac{84\sqrt{5}\pi}{17}$
Step 3: $p = \frac{84\sqrt{5}\pi}{17} \cdot \frac{\sqrt{l}}{\sqrt{a}} = \frac{84\sqrt{5}\pi\sqrt{l}}{17\sqrt{a}}$
Step 4: Now find p when $l = 121$ and $a = 980$.
$p = \frac{84\pi\sqrt{5}\sqrt{121}}{17\sqrt{980}} = \frac{84\pi\sqrt{5}(11)}{17\left(14\sqrt{5}\right)} = \frac{924\pi\sqrt{5}}{238\sqrt{5}} = \frac{66\pi}{17}$
The period is $\frac{66\pi}{17}$ sec.

40. *Step 1*: Let N be the number of calls, d is the distance (in mi). $N = \frac{kp_1p_2}{d}$
Step 2: Substitute $N = 10,000$, $d = 500$, $p_1 = 50,000$, and $p_2 = 125,000$ to find k.
$10,000 = \frac{k(50,000)(125,000)}{500}$
$10,000 = \frac{6,250,000,000k}{500}$
$5,000,000 = 6,250,000,000k$
$k = \frac{5,000,000}{6,250,000,000} = .0008$

Step 3: $N = \frac{.0008 p_1 p_2}{d}$

Step 4: Now find *N* when $D = 800$, $p_1 = 20,000$ and $p_2 = 80,000$.

$N = \cdot \frac{.0008(20,000)(80,000)}{800} = 1600$

There are 1600 calls between the cities.

41. *Step 1*: Let *B* be the BMI, *w* is the weight (in lbs), *h* is the height (in inches). $B = \frac{kw}{h^2}$

Step 2: Substitute $w = 177$, $B = 24$, and $h = 72$ (6 feet) to find *k*.

$24 = \frac{k(177)}{72^2} \Rightarrow 24 = \frac{177k}{5184} \Rightarrow 124,416 = 177k$

$k = \frac{124,416}{177} = \frac{41,472}{59}$

Step 3: $B = \frac{41,472}{59} \cdot \frac{w}{h^2} = \frac{41,472w}{59h^2}$

Step 4: Now find *B* when $w = 130$ and $h = 66$.

$B = \frac{41,472(130)}{59(66^2)} = \frac{5,391,360}{59(4356)} = \frac{5,391,360}{257,004} \approx 20.98$

The BMI would be approximately 21.

42. *Step 1*: Let *R* be the resistance, *l* is the length, *r* is the radius. $R = \frac{kl}{r^4}$

Step 2: Substitute $R = 25$, $l = 12$, and $r = .2$ to find *k*.

$25 = \frac{k(12)}{.0016} \Rightarrow .04 = 12k \Rightarrow k = \frac{.04}{12} = \frac{4}{1200} = \frac{1}{300}$

Step 3: $R = \frac{1}{300} \cdot \frac{l}{r^4} = \frac{l}{300r^4}$

Step 4: Now find *R* when $r = .3$ and $l = 12$.

$R = \frac{12}{300(.3^4)} = \frac{12}{300(.0081)} = \frac{12}{2.43} \approx 4.94$

The resistance would be 4.94.

43. *Step 1*: Let *R* be the radiation, *t* is the temperature (in degrees Kelvin, K). $R = kt^4$:

Step 2: Substitute $R = 213.73$ and $t = 293$ to find *k*.

$213.73 = k(293^4) \Rightarrow k = \frac{213.73}{293^4} \approx 2.9 \times 10^{-8}$

Step 3: $R = (2.9 \times 10^{-8})t^4$

Step 4: Now find *R* when $t = 335$.

$R = (2.9 \times 10^{-8}) \cdot 335^4 \approx 365.24$

The radiation of heat would be 365.24.

44. *Step 1*: Let *Y* be the yield of the bomb (in kilotons), *d* is the distance (in km). $d = k\sqrt[3]{Y}$

Step 2: Substitute $Y = 100$ and $d = 3$ to find *k*.

$3 = k\sqrt[3]{100} \Rightarrow k = \frac{3}{\sqrt[3]{100}}$

Step 3: $d = \frac{3}{\sqrt[3]{100}} \cdot \sqrt[3]{Y}$

Step 4: Now find *d* when $Y = 1500$.

$d = \frac{3}{\sqrt[3]{100}} \cdot \sqrt[3]{1500} = 3\sqrt[3]{15} \approx 7.4$

The distance is approximately 7.4 km.

45. *Step 1*: Let *p* be the person's pelidisi, *w* is the person's weight (in g), *h* is the person's sitting height (in cm). $p = \frac{k\sqrt[3]{w}}{h}$

Step 2: Substitute $w = 48,820$, $h = 78.7$, and $p = 100$ to find *k*.

$100 = \frac{k\sqrt[3]{48,820}}{78.7} \Rightarrow 7870 = \sqrt[3]{48,820}k$

$k = \frac{7870}{\sqrt[3]{48,820}} \approx 215.33$

Step 3: $p = \frac{215.33\sqrt[3]{w}}{h}$

Step 4: Now find *p* when $w = 54,430$ and $h = 88.9$.

$p = \frac{215.33\sqrt[3]{54,430}}{88.9} \approx 92$

This person's pelidisi is 92. The individual is undernourished since his pelidisi is below 100.

46. $L = \frac{25F^2}{st}$

(a) Here, $L = 500$, $s = 200$, and $t = \frac{1}{250}$.

Substitute these values into the formula and solve for *F*.

$500 = \frac{25F^2}{200 \cdot \frac{1}{250}} \Rightarrow 500 = \frac{25F^2}{.8}$

$400 = 25F^2 \Rightarrow 16 = F^2 \Rightarrow \pm 4 = F$

A negative value of *F* is not meaningful in this problem. The appropriate *F*-stop is 4.

(b) Here, $L = 125$, $F = 2$, and $s = 200$.

Substitute these values into the formula and solve for *t*.

$125 = \frac{25(2^2)}{200t} \Rightarrow 125 = \frac{25(4)}{200t} = \frac{100}{200t}$

$125 = \frac{1}{2t} \Rightarrow 250t = 1 \Rightarrow t = \frac{1}{250}$

A shutter speed of $\frac{1}{250}$ sec should be used.

47. For $k > 0$, if y varies directly as x, when x increases, y <u>increases</u>, and when x decreases, y <u>decreases</u>.

48. For $k > 0$, if y varies inversely as x, when x increases, y <u>decreases</u>, and when x decreases, y <u>increases</u>.

49. $y = \frac{k}{x}$

If x is doubled, then $y_1 = \frac{k}{2x} = \frac{1}{2} \cdot \frac{k}{x} = \frac{1}{2} y.$

Thus, y is half as large as it was before.

50. $y = kx$

If x is halved, then we have the following.

$y_1 = k\left(\frac{1}{2}x\right) = \frac{1}{2} \cdot kx = \frac{1}{2} y$

Thus, y is half as large as it was before.

51. $y = kx$

If x is replaced by $\frac{1}{3}x$, then we have

$y_1 = k\left(\frac{1}{3}x\right) = \frac{1}{3} \cdot kx = \frac{1}{3} y.$ Thus, y is one-third as large as it was before.

52. $y = \frac{k}{x}$

If x is replaced by $3x$, then we have the following.

$y_1 = \frac{k}{3x} = \frac{1}{3} \cdot \frac{k}{x} = \frac{1}{3} y$

Thus, y is one-third as large as it was before.

53. $p = \frac{kr^3}{t^2}$

If r is replaced by $\frac{1}{2}r$ and t is replaced by $2t$, then we have the following.

$$p_1 = \frac{k\left(\frac{1}{2}r\right)^3}{(2t)^2} = \frac{k\left(\frac{1}{8}r^3\right)}{4t^2} = \frac{1}{8} \cdot \frac{kr^3}{4t^2} = \frac{kr^3}{32t^2} = \frac{1}{32} \cdot \frac{kr^3}{t^2}$$

Thus, so p is $\frac{1}{32}$ as large as it was before.

54. $m = kp^2q^4$

If p is replaced by $2p$ and q is replaced by $3q$ then

$m_1 = k(2p)^2(3q)^4 = k\left(4p^2\right)\left(81q^4\right)$

$m_1 = 324kp^2q^4 = 324m$

Thus, if p doubles and q triples, m is 324 times as large as it was before.

Chapter 3: Review Exercises

1. $f(x) = 3(x+4)^2 - 5$

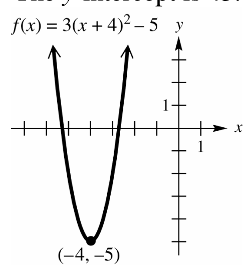
Since $f(x) = 3(x+4)^2 - 5$ $= 3\left[x - (-4)\right]^2 + (-5)$, the function has the form $f(x) = a(x-h)^2 + k$ with $a = 3$, $h = -4$, and $k = -5$. The graph is a parabola that opens upward with vertex $(h, k) = (-4, -5)$. The axis is $x = h \Rightarrow x = -4$. To find the x-intercepts, let $f(x) = 0$.

$$3(x+4)^2 - 5 = 0 \Rightarrow 3(x+4)^2 = 5 \Rightarrow$$
$$(x+4)^2 = \frac{5}{3} \Rightarrow x + 4 = \pm\sqrt{\frac{5}{3}} \Rightarrow$$
$$x = -4 \pm \sqrt{\frac{5}{3}} = -4 \pm \frac{\sqrt{15}}{3} = \frac{-12 \pm \sqrt{15}}{3}$$

Thus, the x-intercepts are $\frac{-12 \pm \sqrt{15}}{3}$ or approximately –5.3 and –2.7. To find the y-intercept, let $x = 0$.

$f(0) = 3(0+4)^2 - 5 = 3(16) - 5 = 48 - 5 = 43$

The y-intercept is 43.



The domain is $(-\infty, \infty)$. Since the lowest point on the graph is $(-4, -5)$, the range is $[-5, \infty)$.

2. $f(x) = -\frac{2}{3}(x-6)^2 + 7$

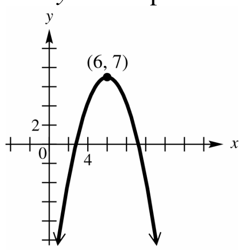
This function has the form $f(x) = a(x-h)^2 + k$, with $a = -\frac{2}{3}$, $h = 6$, and $k = 7$. The graph is a parabola that opens downward with vertex $(h, k) = (6, 7)$. The axis is $x = h \Rightarrow x = 6$. To find the x-intercepts, let $f(x) = 0$.

$$-\frac{2}{3}(x-6)^2 + 7 = 0 \Rightarrow -\frac{2}{3}(x-6)^2 = -7$$
$$\frac{2}{3}(x-6)^2 = 7 \Rightarrow (x-6)^2 = 7 \cdot \frac{3}{2} = \frac{21}{2}$$
$$x - 6 = \pm\sqrt{\frac{21}{2}}$$
$$x = 6 \pm \sqrt{\frac{21}{2}} = 6 \pm \frac{\sqrt{42}}{2} = \frac{12 \pm \sqrt{42}}{2}$$

Thus, the x-intercepts are $\frac{12\pm\sqrt{42}}{2}$ or approximately 2.8 and 9.2. To find the y-intercept, let $x=0$.

$$f(0)=-\tfrac{2}{3}(0-6)^2+7$$
$$=-\tfrac{2}{3}(36)+7=-24+7=-17$$

The y-intercept is –17.



$f(x)=-\frac{2}{3}(x-6)^2+7$

The domain is $(-\infty,\infty)$. Since the highest on the graph is $(6,7)$, the range is $(-\infty,7]$.

3. $f(x)=-3x^2-12x-1$

Complete the square so the function has the form $f(x)=a(x-h)^2+k$.

$$f(x)=-3x^2-12x-1=-3\left(x^2+4x\right)-1$$
$$=-3\left(x^2+4x+4-4\right)-1$$
$$f(x)=-3(x+2)^2+12-1$$
$$=-3\left[x-(-2)\right]^2+11$$

This parabola is in now in standard form with $a=-3$, $h=-2$, and $k=11$. It opens downward with vertex $(h,\,k)=(-2,11)$. The axis is $x=h\Rightarrow x=-2$. To find the x-intercepts, let $f(x)=0$.

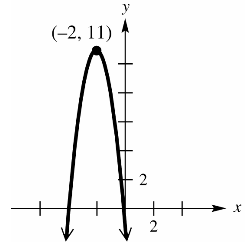
$$-3(x+2)^2+11=0\Rightarrow -3(x+2)^2=-11$$
$$(x+2)^2=\tfrac{11}{3}\Rightarrow x+2=\pm\sqrt{\tfrac{11}{3}}$$
$$x=-2\pm\sqrt{\tfrac{11}{3}}=-2\pm\tfrac{\sqrt{33}}{3}$$
$$=\tfrac{-6\pm\sqrt{33}}{3}$$

Thus, the x-intercepts are $\frac{-6\pm\sqrt{33}}{3}$ or approximately -3.9 and $-.1$.
To find the y-intercept, let $x=0$.

$$f(0)=-3\left(0^2\right)-12(0)-1=-1$$

The y-intercept is –1.

$f(x)=-3x^2-12x-1$



The domain is $(-\infty,\infty)$. Since the highest on the graph is $(-2,11)$, the range is $(-\infty,11]$.

4. $f(x)=4x^2-4x+3$

Complete the square so the function has the form $f(x)=a(x-h)^2+k$.

$$f(x)=4x^2-4x+3=4\left(x^2-x\right)+3$$
$$=4\left(x^2-x+\tfrac{1}{4}-\tfrac{1}{4}\right)+3$$
$$=4\left(x-\tfrac{1}{2}\right)^2-1+3=4\left(x-\tfrac{1}{2}\right)^2+2$$

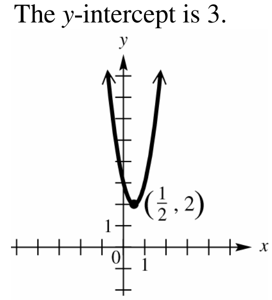
This parabola is in now in standard form with $a=4$, $h=\frac{1}{2}$, and $k=2$. It opens upward with vertex $(h,\,k)=\left(\frac{1}{2},2\right)$. The axis is $x=h\Rightarrow x=\frac{1}{2}$. To find the x-intercepts, let $f(x)=0$.

$$4\left(x-\tfrac{1}{2}\right)^2+2=0\Rightarrow 4\left(x-\tfrac{1}{2}\right)^2=-2$$
$$\left(x-\tfrac{1}{2}\right)^2=-\tfrac{1}{2}\Rightarrow x-\tfrac{1}{2}=\pm\sqrt{-\tfrac{1}{2}}$$
$$x-\tfrac{1}{2}=\pm i\tfrac{\sqrt{2}}{2}\Rightarrow x=\tfrac{1\pm i\sqrt{2}}{2}$$

Therefore, the graph has no x-intercepts. To find the y-intercept, let $x=0$.

$$f(0)=4\left(0^2\right)-4(0)+3=3$$

The y-intercept is 3.



$f(x)=4x^2-4x+3$

The domain is $(-\infty,\infty)$. Since the highest on the graph is $\left(\frac{1}{2},2\right)$, the range is $[2,\infty)$.

5. $f(x) = a(x-h)^2 + k;\ a > 0$
The graph is a parabola that opens upward. The coordinates of the lowest point of the graph are represented by the vertex, (h, k).

6. $f(x) = a(x-h)^2 + k,\ a > 0$
To find the y-intercept by let
$f(0) = a(0-h)^2 + k = ah^2 + k$
The y-intercept is $ah^2 + k$.

7. For the graph to have one or more x-intercepts, $f(x) = 0$ must have real number solutions.

$$a(x-h)^2 + k = 0 \Rightarrow a(x-h)^2 = -k$$
$$(x-h)^2 = \frac{-k}{a} \Rightarrow x - h = \pm\sqrt{\frac{-k}{a}}$$
$$x = h \pm \sqrt{\frac{-k}{a}}$$

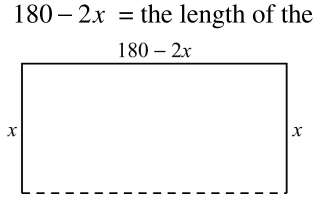
These solutions are real only if $\frac{-k}{a} \geq 0$. Since a > 0, this condition is equivalent to $-k \geq 0$ or $k \leq 0$. The x-intercepts for these conditions are given by $h \pm \sqrt{\frac{-k}{a}}$.

8. If a is positive, the graph of $y = ax^2 + bx + c$ is a parabola that opens upward, so the y-coordinate of the lowest point of the graph is the y-coordinate of the vertex, k, in the equation $y = a(x-h)^2 + k$. Complete the square to find the vertex.

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$
$$\text{Note } \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$$
$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$
$$= a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \left(c - \frac{b^2}{4a}\right)$$

The smallest value is $c - \frac{b^2}{4a}$.

9. Let x = the width of the rectangular region.
$180 - 2x$ = the length of the region.



Since $A = LW$, we have $A(x) = (180 - 2x)x = -2x^2 + 180x.$ Since the graph of $A(x)$ is a parabola that opens downward, the maximum area is the y-coordinate of the vertex.

$$A(x) = -2x^2 + 180x = -2\left(x^2 - 90x\right)$$
$$= -2\left(x^2 - 90x + 2025\right) + 4050$$
$$= -2(x-45)^2 + 4050$$

The x-coordinate of the vertex, $x = 45$, is the width that gives the maximum area. The length will be $L = 180 - 2(45) = 90.$ Thus, the dimensions of the region are $90 \text{ m} \times 45 \text{ m}$.

10. $s(t) = -16t^2 + 800t + 600$

(a) To find what height the projectile was fired, we can find what the height was at $t = 0$ seconds.
$s(0) = -16\left(0^2\right) + 800(0) + 600 = 600$
The initial height was 600 ft.
Note: Since this function is of the form $s(t) = -16t^2 + v_0 t + s_0$, the constant term represents the initial height.

(b) Since the graph of $s(t)$ is a parabola that opens downward, the time at which the projectile will reach its maximum height will be the x-coordinate of the vertex. Using the formula that the x-coordinate of the vertex is $x = -\frac{b}{2a}$, where $a = -16 \text{ and } b = 800,$ we have
$x = -\frac{b}{2a} = -\frac{800}{2(-16)} = 25.$
Thus, the projectile will reach its maximum height at 25 sec.

(c) The maximum height is the y-coordinate of the vertex. This will occur at $s(25)$.
$$s(25) = -16\left(25^2\right) + 800(25) + 600$$
$$= 10{,}600$$
The maximum height of the projectile is 10,600 ft.

(d) Algebraic Solution:
To find the time interval in which the projectile will be more than 5000 ft above ground, solve the inequality
$$-16t^2 + 800t + 600 > 5000$$
$$-16t^2 + 800t - 4400 > 0$$
$$t^2 - 50t + 275 < 0$$
Solve the corresponding equation
$t^2 - 50t + 275 = 0.$
Use the quadratic formula with a = 1, b = –50, and c = 275.

$$t=\frac{-(-50)\pm\sqrt{(-50)^2-4(1)(275)}}{2(1)}$$
$$=\frac{50\pm\sqrt{2500-1100}}{2}=\frac{50\pm\sqrt{1400}}{2}$$
$$=\frac{50\pm10\sqrt{14}}{2}=25\pm5\sqrt{14}$$

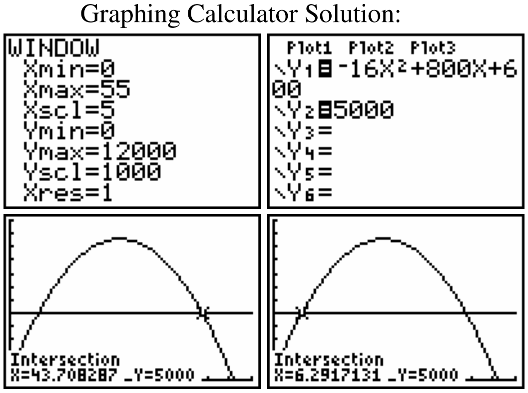
$t=25-5\sqrt{14}\approx6.3$ or

$t=25+5\sqrt{14}\approx43.7$

The values 6.3 and 43.7 divide the number line into three intervals: $(-\infty, 6.3)$, $(6.3, 43.7)$, and $(43.7, \infty)$. Use a test point in each interval to determine where the inequality is satisfied.

Interval	Test Value	Is $-16t^2+800t+600>5000$ True or False?
$(-\infty, 6.3)$	0	$-16(0^2)+800(0)+600\overset{?}{>}5000$ $600>5000$ False
$(6.3, 43.7)$	10	$-16(10^2)+800(10)+600\overset{?}{>}5000$ $7000>5000$ True
$(43.7, \infty)$	50	$-16(50^2)+800(50)+600\overset{?}{>}5000$ $600>5000$ False

Graphing Calculator Solution:



The projectile will be more than 5000 ft above the ground between 6.3 sec and 43.7 sec.

(e) Algebraic Solution:

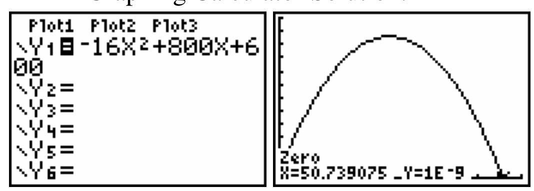
To find the time in which the projectile will be in the air, we must solve determine at what time the projectile returns to the earth. To do this we the solve $-16t^2+800t+600=0.$

Since $-16t^2+800t+600=0\Rightarrow$ $2t^2-100t-75=0,$ use the quadratic formula with $a = 2$, $b = -100$, and $c = -75$.

$$t=\frac{-(-100)\pm\sqrt{(-100)^2-4(2)(-75)}}{2(2)}$$
$$=\frac{100\pm\sqrt{10{,}000+600}}{4}$$
$$=\frac{100\pm\sqrt{10{,}600}}{4}$$
$$=\frac{100\pm10\sqrt{106}}{4}=\frac{50\pm5\sqrt{106}}{2}$$
$$t=\frac{50-5\sqrt{106}}{2}\approx-.7 \text{ or}$$
$$t=\frac{50+5\sqrt{106}}{2}\approx50.7$$

We reject the negative solution.

Graphing Calculator Solution:



The projectile will be in the air for 50.7 sec.

11. Because $V(x)$ has different equations on two different intervals, it is a piecewise-defined function.

$$V(x)=\begin{cases}2x^2-32x+150 & \text{if } 1\le x<8\\ 31x-226 & \text{if } 8\le x\le12\end{cases}$$

Note: Since $2\cdot8^2-32\cdot8+150=22$ and $31\cdot8-226=22,$ the point (8, 22) can be found using either rule, and the graph will have no breaks.

(a) In January, $x=1.$

$V(x)=2x^2-32x+150$

$V(1)=2(1)^2-32(1)+150=120$

(b) In May, $x=5.$

$V(x)=2x^2-32x+150$

$V(5)=2(5)^2-32(5)+150=40$

(c) In August, $x=8.$

$V(x)=31x-226$

$V(8)=31(8)-226=22$

(d) In October, $x=10.$

$V(x)=31x-226$

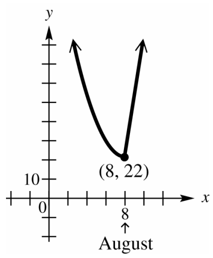
$V(10)=31(10)-226=84$

(e) In December, $x = 12.$

$V(x) = 31x - 226$

$V(12) = 31(12) - 226 = 146$

(f)



The graph shows a minimum point at (8, 22). Thus, in August $(x = 8)$ the fewest volunteers are available.

12. **(a)** Since $(0, 353)$ is the vertex, we have

$f(x) = a(x - 0)^2 + 353 = ax^2 + 353.$

Next we solve for a using the point $(285, 2000)$.

$2000 = a(285^2) + 353$

$2000 = a(81,225) + 353$

$1647 = 81,225a \Rightarrow a = \frac{1647}{81,225} \approx .0203$

Thus, we have $f(x) = .0203x^2 + 353.$

(b) To predict the amount of carbon in 2300, we need to evaluate the function found in part b, where $x = 310.$

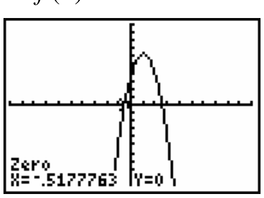
$f(310) = .0203(310^2) + 353 = 2303.83$

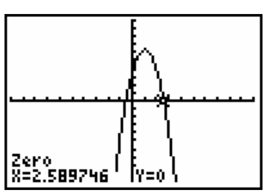
Thus, about 2304 ppm is predicted in 2300.

13. $f(x) = -2.64x^2 + 5.47x + 3.54$

The discriminant is $b^2 - 4ac$ in the standard quadratic equation $y = ax^2 + bx + c.$ Here we have $a = -2.64$, $b = 5.47$, and $c = 3.54$. Thus, $b^2 - 4ac = (5.47)^2 - 4(-2.64)(3.54) = 67.3033.$ Because the discriminant is 67.3033, a positive number, there are two x-intercepts.

14. $f(x) = -2.64x^2 + 5.47x + 3.54$



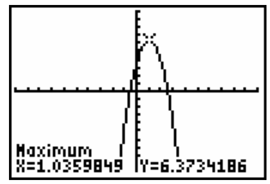


From the preceding calculator screens, we see that (approximating to the nearest hundredth) the solution set is $\{-.52, 2.59\}$.

15. **(a)** $f(x) > 0$ on the open interval $(-.52, 2.59)$.

(b) $f(x) < 0$ on $(-\infty, -.52) \cup (2.59, \infty)$.

16. Use "maximum" from the CALC menu to find the coordinates of the vertex.



From this screen, we see that the coordinates of the vertex, to the nearest hundredth, are (1.04, 6.37).

17. $\dfrac{x^3 + x^2 - 11x - 10}{x - 3}$

Since $x - 3$ is in the form $x - k$, $k = 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 1 & -11 & -10 \\ & & 3 & 12 & 3 \\ \hline & 1 & 4 & 1 & -7 \end{array}$$

$$\frac{x^3 + x^2 - 11x - 10}{x - 3} = x^2 + 4x + 1 + \frac{-7}{x - 3}$$

18. $\dfrac{3x^3 + 8x^2 + 5x + 10}{x + 2}$

Express $x + 2$ in the form $x - k$ by writing it as $x - (-2)$. Thus $k = -2$.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & 10 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & 8 \end{array}$$

$$\frac{3x^3 + 8x^2 + 5x + 10}{x + 2} = 3x^2 + 2x + 1 + \frac{8}{x + 2}$$

19. $\dfrac{2x^3 - x + 6}{x + 4}$

Express $x + 4$ in the form $x - k$ by writing it as $x - (-4)$. Thus we have $k = -4$. Since the x^2 – term is missing, include a 0 as its coefficient.

$$\begin{array}{r|rrrr} -4 & 2 & 0 & -1 & 6 \\ & & -8 & 32 & -124 \\ \hline & 2 & -8 & 31 & -118 \end{array}$$

$$\frac{2x^3 - x + 6}{x + 4} = 2x^2 - 8x + 31 + \frac{-118}{x + 4}$$

20. $\dfrac{\frac{1}{3}x^3 - 2x^2 - 9x + 4}{x+3}$

Express $x+3$ in the form $x-k$ by writing it as $x-(-3)$. Thus $k=-3$.

$$\begin{array}{r|rrrr} -3 & \frac{1}{3} & -2 & -9 & 4 \\ & & -1 & 9 & 0 \\ \hline & \frac{1}{3} & -3 & 0 & 4 \end{array}$$

$$\frac{\frac{1}{3}x^3 - 2x^2 - 9x + 4}{x+3} = \frac{1}{3}x^2 - 3x + \frac{4}{x+3}$$

21. $f(x) = 5x^3 - 3x^2 + 2x - 6;\ k = 2$

Use synthetic division to write the polynomial in the form $f(x) = (x-k)q(x) + r$.

$$\begin{array}{r|rrrr} 2 & 5 & -3 & 2 & -6 \\ & & 10 & 14 & 32 \\ \hline & 5 & 7 & 16 & 26 \end{array}$$

$$f(x) = (x-2)(5x^2 + 7x + 16) + 26$$

22. $f(x) = -3x^3 + 5x - 6;\ k = -1$

Use synthetic division to write the polynomial in the form $f(x) = (x-k)q(x) + r$.

$$\begin{array}{r|rrrr} -1 & -3 & 0 & 5 & -6 \\ & & 3 & -3 & -2 \\ \hline & -3 & 3 & 2 & -8 \end{array}$$

$$\begin{aligned} f(x) &= [x-(-1)](-3x^2 + 3x + 2) + (-8) \\ &= (x+1)(-3x^2 + 3x + 2) - 8 \end{aligned}$$

23. $f(x) = -x^3 + 5x^2 - 7x + 1;$ find $f(2)$.

$$\begin{array}{r|rrrr} 2 & -1 & 5 & -7 & 1 \\ & & -2 & 6 & -2 \\ \hline & -1 & 3 & -1 & -1 \end{array}$$

The synthetic division shows that $f(2) = -1$.

24. $f(x) = 2x^3 - 3x^2 + 7x - 12;$ find $f(2)$.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 7 & -12 \\ & & 4 & 2 & 18 \\ \hline & 2 & 1 & 9 & 6 \end{array}$$

The synthetic division shows that $f(2) = 6$.

25. $f(x) = 5x^4 - 12x^2 + 2x - 8;$ find $f(2)$.

$$\begin{array}{r|rrrrr} 2 & 5 & 0 & -12 & 2 & -8 \\ & & 10 & 20 & 16 & 36 \\ \hline & 5 & 10 & 8 & 18 & 28 \end{array}$$

The synthetic division shows that $f(2) = 28$.

26. $f(x) = x^5 + 4x^2 - 2x - 4;$ find $f(2)$.

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 4 & -2 & -4 \\ & & 2 & 4 & 8 & 24 & 44 \\ \hline & 1 & 2 & 4 & 12 & 22 & 40 \end{array}$$

The synthetic division shows that $f(2) = 40$.

27. $f(x) = x^3 + 2x^2 + 3x + 2;\ k = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & 3 & 2 \\ & & -1 & -1 & -2 \\ \hline & 1 & 1 & 2 & 0 \end{array}$$

Since $f(-3) = 0$, we have that -3 is a zero of the function.

28. $f(x) = \frac{1}{4}x^3 + \frac{3}{4}x^2 - \frac{1}{2}x + 6;\ k = -4$

$$\begin{array}{r|rrrr} -4 & \frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 6 \\ & & -1 & 1 & -2 \\ \hline & \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 4 \end{array}$$

Since $f(-4) = 4$, we have that -4 is not a zero of the function.

29. By the conjugate zeros theorem, $7 - 2i$ is also a zero.

30. If $f(x)$ has a zero at $x = -3$, then $f(-3) = 0$. D is a true statement.

In Exercises 31 through 34, other answers are possible.

31. Zeros: –1, 4, 7

$$\begin{aligned} f(x) &= [x-(-1)](x-4)(x-7) \\ &= (x+1)(x-4)(x-7) \\ &= (x+1)[(x-4)(x-7)] \\ &= (x+1)(x^2 - 11x + 28) \\ &= x^3 - 10x^2 + 17x + 28 \end{aligned}$$

32. Zeros: 8, 2, 3

$$\begin{aligned} f(x) &= (x-8)(x-2)(x-3) \\ &= [(x-8)(x-2)](x-3) \\ &= (x^2 - 10x + 16)(x-3) \\ &= x^3 - 13x^2 + 46x - 48 \end{aligned}$$

33. Zeros: $\sqrt{3}, -\sqrt{3}$, 2, 3.

$$\begin{aligned} f(x) &= (x-\sqrt{3})(x+\sqrt{3})(x-2)(x-3) \\ &= [(x-\sqrt{3})(x+\sqrt{3})][(x-2)(x-3)] \\ &= (x^2 - 3)(x^2 - 5x + 6) \\ &= x^4 - 5x^3 + 3x^2 + 15x - 18 \end{aligned}$$

34. Zeros: $-2+\sqrt{5},\ -2-\sqrt{5},\ -2,\ 1$

$$\begin{aligned} f(x) &= \left[x-\left(-2+\sqrt{5}\right)\right]\left[x-\left(-2-\sqrt{5}\right)\right] \cdot \left[x-(-2)\right](x-1) \\ &= \left[(x+2)-\sqrt{5}\right]\left[(x+2)+\sqrt{5}\right] \cdot (x+2)(x-1) \\ &= \left[(x+2)^2-\left(\sqrt{5}\right)^2\right]\left[(x+2)(x-1)\right] \\ &= \left[(x+2)^2-5\right]\left(x^2+x-2\right) \\ &= \left(x^2+4x+4-5\right)\left(x^2+x-2\right) \\ &= \left(x^2+4x-1\right)\left(x^2+x-2\right) \\ &= x^4+5x^3+x^2-9x+2 \end{aligned}$$

35. $f(x)=2x^3-9x^2-6x+5$

p must be a factor of $a_0=5$ and q must be a factor of $a_3=2$. Thus, p can be $\pm1, \pm5,$ and q can be $\pm1, \pm2$. The possible zeros, $\frac{p}{q}$, are $\pm1, \pm5, \pm\frac{1}{2}, \pm\frac{5}{2}$. The remainder theorem shows that $\frac{1}{2}$ is a zero.

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -9 & -6 & 5 \\ & & 1 & -4 & -5 \\ \hline & 2 & -8 & -10 & 0 \end{array}$$

The new quotient polynomial is therefore $2x^2-8x-10$.

$$\begin{aligned} 2x^2-8x-10&=0 \\ x^2-4x-5&=0 \\ (x+1)(x-5)&=0 \end{aligned}$$

$x+1=0 \Rightarrow x=-1$ or $x-5=0 \Rightarrow x=5$

The rational zeros are $\frac{1}{2}$, -1, and 5.

36. $f(x)=8x^4-14x^3-29x^2-4x+3$

p must be a factor of $a_0=3$ and q must be a factor of $a_4=8$. Thus, p can be $\pm1, \pm3,$ and q can be $\pm1, \pm2, \pm4, \pm8$. The possible zeros, $\frac{p}{q}$, are $\pm1, \pm3, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{3}{8}$. The remainder theorem shows that 3 is a zero.

$$\begin{array}{r|rrrrr} 3 & 8 & -14 & -29 & -4 & 3 \\ & & 24 & 30 & 3 & -3 \\ \hline & 8 & 10 & 1 & -1 & 0 \end{array}$$

The new quotient polynomial is therefore $8x^3+10x^2+x-1$. The remainder theorem shows that -1 is a zero.

$$\begin{array}{r|rrrr} -1 & 8 & 10 & 1 & -1 \\ & & -8 & -2 & 1 \\ \hline & 8 & 2 & -1 & 0 \end{array}$$

The new quotient polynomial is therefore $8x^2+2x-1$.

$$\begin{aligned} 8x^2+2x-1&=0 \\ (4x-1)(2x+1)&=0 \end{aligned}$$

$4x-1=0 \Rightarrow x=\frac{1}{4}$ or $2x+1=0 \Rightarrow x=-\frac{1}{2}$

The rational zeros are 3, -1, $\frac{1}{4}$, and $-\frac{1}{2}$.

37. $f(x)=3x^3-8x^2+x+2$

(a) To show there is a zero between -1 and 0, we need to find $f(-1)$ and $f(0)$. Find $f(-1)$ by synthetic division and $f(0)$ by evaluating the function.

$$\begin{array}{r|rrrr} -1 & 3 & -8 & 1 & 2 \\ & & -3 & 11 & -12 \\ \hline & 3 & -11 & 12 & -10 \end{array}$$

$f(0)=3\left(0^3\right)-8\left(0^2\right)+(0)+2=2$

Since $f(-1)=-10<0$ and $f(0)=2>0$, there must be a zero between -1 and 0.

(b) To show there is a zero between 2 and 3, we need to find $f(2)$ and $f(3)$. Find these by synthetic division.

$$\begin{array}{r|rrrr} 2 & 3 & -8 & 1 & 2 \\ & & 6 & -4 & -6 \\ \hline & 3 & -2 & -3 & -4 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 3 & -8 & 1 & 2 \\ & & 9 & 3 & 12 \\ \hline & 3 & 1 & 4 & 14 \end{array}$$

Since $f(2)=-4<0$ and $f(3)=14>0$, there must be a zero between 2 and 3.

38. $f(x)=4x^3-37x^2+50x+60$

(a) To show there is a zero between 2 and 3, we need to find $f(2)$ and $f(3)$. Find these by synthetic division.

$$\begin{array}{r|rrrr} 2 & 4 & -37 & 50 & 60 \\ & & 8 & -58 & -16 \\ \hline & 4 & -29 & -8 & 44 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 4 & -37 & 50 & 60 \\ & & 12 & -75 & -75 \\ \hline & 4 & -25 & -25 & -15 \end{array}$$

Since $f(2) = 44 > 0$ and $f(3) = -15 < 0,$ there must be a zero between 2 and 3.

(b) To show there is a zero between 7 and 8, we need to find $f(7)$ and $f(8)$. Find these by synthetic division.

$$\begin{array}{r|rrrr} 7 & 4 & -37 & 50 & 60 \\ & & 28 & -63 & -91 \\ \hline & 4 & -9 & -13 & -31 \end{array}$$

$$\begin{array}{r|rrrr} 8 & 4 & -37 & 50 & 60 \\ & & 32 & -40 & 80 \\ \hline & 4 & -5 & 10 & 140 \end{array}$$

Since $f(7) = -31 < 0$ and $f(8) = 140 > 0,$ there must be a zero between 7 and 8.

39. $f(x) = 6x^4 + 13x^3 - 11x^2 - 3x + 5$

(a) To show that there is no zero greater than 1, we must show that 1 is an upper bound.

$$\begin{array}{r|rrrrr} 1 & 6 & 13 & -11 & -3 & 5 \\ & & 6 & 19 & 8 & 5 \\ \hline & 6 & 19 & 8 & 5 & 10 \end{array}$$

Since the 1 is positive and the bottom row has all positive numbers, there is no zero greater than 1.

(b) To show that there is no zero less than -3, we must show that -3 is an lower bound.

$$\begin{array}{r|rrrrr} -3 & 6 & 13 & -11 & -3 & 5 \\ & & -18 & 15 & -12 & 45 \\ \hline & 6 & -5 & 4 & -15 & 50 \end{array}$$

Since the -3 is negative and the bottom row alternates in sign, there is no zero less than -3.

40. $f(x) = x^3 + 3x^2 - 4x - 2$

$f(x) = x^3 \underbrace{+3x^2 -}_{1} 4x - 2$ has 1 variation in sign. *f* has 1 positive real zero.

$f(-x) = \underbrace{-x^3 +}_{1} 3x^2 \underbrace{+4x -}_{2} 2$ has 2 variations in sign. *f* has 2 or 2 – 2 = 0 negative real zeros.

41. To determine if $x+1$ is a factor of $f(x) = x^3 + 2x^2 + 3x + 2,$ we can find $f(-1)$ by synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 2 & 3 & 2 \\ & & -1 & -1 & -2 \\ \hline & 1 & 1 & 2 & 0 \end{array}$$

Since $f(-1) = 0,$ $x+1$ is a factor of $f(x).$

42. Since $-1 - 3i$ is a zero, by the conjugate zeros theorem, $-1 + 3i$ is also a zero.

$$\begin{aligned} f(x) &= a(x-3)(x-1) \\ &\quad \cdot\left[x-(-1-3i)\right]\left[x-(-1+3i)\right] \\ &= a\left[(x-3)(x-1)\right] \\ &\quad \cdot\left[(x+1)+3i\right]\left[(x+1)-3i\right] \\ &= a\left(x^2-4x+3\right)\left[(x+1)^2-(3i)^2\right] \\ &= a\left(x^2-4x+3\right)\left(x^2+2x+1-9i^2\right) \\ &= a\left(x^2-4x+3\right)\left(x^2+2x+1+9\right) \\ &= a\left(x^2-4x+3\right)\left(x^2+2x+10\right) \\ &= a\left(x^4-2x^3+5x^2-34x+30\right) \end{aligned}$$

Since $f(2) = -36,$ we can now find a.

$-36 = a\left(2^4 - 2\cdot 2^3 + 5\cdot 2^2 - 34\cdot 2 + 30\right)$

$-36 = a(-18) \Rightarrow 2 = a$

Thus, the polynomial function is

$$\begin{aligned} f(x) &= 2\left(x^4-2x^3+5x^2-34x+30\right) \\ &= 2x^4-4x^3+10x^2-68x+60. \end{aligned}$$

43. Since $f(x) = a\left[x-(-2)\right](x-1)(x-4)$ $= a(x+2)(x-1)(x-4),$ we can solve for a.

$16 = a(2+2)(2-1)(2-4)$

$16 = -8a \Rightarrow a = -2$

The polynomial function is $f(x) = -2(x+2)(x-1)(x-4).$ It can be expanded as follows.

$$\begin{aligned} f(x) &= -2(x+2)(x-1)(x-4) \\ &= -2\left[(x+2)(x-1)\right](x-4) \\ &= -2\left(x^2+x-2\right)(x-4) \\ &= -2\left(x^3-3x^2-6x+8\right) \\ &= -2x^3+6x^2+12x-16 \end{aligned}$$

44. $f(x) = x^4 - 3x^3 - 8x^2 + 22x - 24$; $1+i$ is a zero

Use synthetic division where $k = 1+i$.

$$\begin{array}{r|rrrrr} 1+i & 1 & -3 & -8 & 22 & -24 \\ & & 1+i & -3-i & -10-12i & 24 \\ \hline & 1 & -2+i & -11-i & 12-12i & 0 \end{array}$$

Since 1 + *i* is a zero, 1 − *i* is also a zero. Synthetically divide $1-i$ on the quotient polynomial.

$$\begin{array}{r|rrrr} 1-i & 1 & -2+i & -11-i & 12-12i \\ & & 1-i & -1+i & -12+12i \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

The quotient polynomial is $x^2 - x - 12 = (x-4)(x+3)$, which has zeros of 4 and -3. Thus, all the zeros are $1+i, 1-i, 4,$ and -3.

45. $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$; 1 and $-2i$ are zeros.

Use synthetic division where $k = 1$.

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & 7 & -4 & -4 \\ & & 2 & 1 & 8 & 4 \\ \hline & 2 & 1 & 8 & 4 & 0 \end{array}$$

Synthetically divide $-2i$ into the quotient polynomial.

$$\begin{array}{r|rrrr} -2i & 2 & 1 & 8 & 4 \\ & & -4i & -8-2i & -4 \\ \hline & 2 & 1-4i & -2i & 0 \end{array}$$

Since $-2i$ is a zero, $2i$ is also a zero. Synthetically divide $2i$ into the quotient polynomial.

$$\begin{array}{r|rrr} 2i & 2 & 1-4i & -2i \\ & & 4i & 2i \\ \hline & 2 & 1 & 0 \end{array}$$

Since the quotient polynomial represented is $2x+1$, we can set it equal to zero to find the remaining zero.

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

Thus, all the zeros are $1, -\frac{1}{2}$ and $\pm 2i$.

46. $x-4$ is a factor of $f(x) = x^3 - 2x^2 + sx + 4$ if 4 is a zero.

$$\begin{array}{r|rrrr} 4 & 1 & -2 & s & 4 \\ & & 4 & 8 & 4s+32 \\ \hline & 1 & 2 & s+8 & 4s+36 \end{array}$$

To ensure that 4 is a zero, the remainder, $4s+36$, must equal zero.

$4s + 36 = 0 \Rightarrow 4s = -36 \Rightarrow s = -9$

Thus, the value of *s* is -9.

47. To determine the value of s such that when the polynomial $x^3 - 3x^2 + sx - 4$ is divided by $x-2$, the remainder is 5, we perform synthetic division.

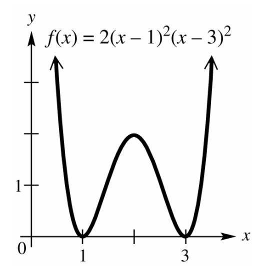
$$\begin{array}{r|rrrr} 2 & 1 & -3 & s & -4 \\ & & 2 & -2 & 2s-4 \\ \hline & 1 & -1 & s-2 & 2s-8 \end{array}$$

To ensure that 5 is the remainder set $2s-8$ equal to 5 and solve.

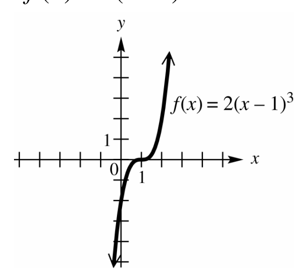
$2s - 8 = 5 \Rightarrow 2s = 13 \Rightarrow s = \frac{13}{2}$

Thus, the value of s is $\frac{13}{2}$.

48. A fourth-degree polynomial function having exactly two distinct real zeros is any function such that the polynomial can be factored as $a(x-b)^2(x-c)^2$, where *a*, *b*, and *c* are real numbers. One example is $f(x) = 2(x-1)^2(x-3)^2$.



49. Any polynomial that can be factored as $a(x-b)^3$, where *a* and *b* are real numbers, will be a cubic polynomial function having exactly one real zero. One example is $f(x) = 2(x-1)^3$.



50. **(a)** Since $f(x) = x^5 - 9x^2$ is a fifth degree polynomial function, it has a maximum of four turning points.

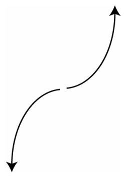
(b) Since $f(x) = 4x^3 - 6x^2 + 2$ is a third degree polynomial function, it has a maximum of two turning points.

51. The polynomial has a leading term of $10x^7$.

(a) The domain is $(-\infty, \infty)$.

(b) The range is $(-\infty, \infty)$.

(c) As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.



(d) There are at most 7 zeros.

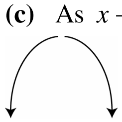
(e) There are at most 6 turning points.

52. The polynomial has a leading term of $-9x^6$.

(a) The domain is $(-\infty, \infty)$.

(b) The range is $(-\infty, M]$, where M is the greatest value assumed by the function.

(c) As $x \to \pm\infty$, $f(x) \to -\infty$.



(d) There are at most 6 zeros.

(e) There are at most 5 turning points.

53. $f(x) = (x-2)^2(x-5)$ is a cubic polynomial with positive y-values for $x > 5$, so it matches graph C.

54. $f(x) = -(x-2)^2(x-5)$ is the vertical reflection of graph C, so it matches graph D.

55. $f(x) = (x-2)^2(x-5)^2$ is a quartic polynomial that is always greater than or equal to 0, so it matches graph E.

56. $f(x) = (x-2)(x-5)$ is a quadratic polynomial that opens up, so it matches graph A.

57. $f(x) = -(x-2)(x-5)$ is a quadratic polynomial that opens down, so it matches graph B.

58. $f(x) = -(x-2)^2(x-5)^2$ is the vertical reflection of graph E, so it matches graph F.

59. $f(x) = (x-2)^2(x+3)$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$(x-2)^2 = 0 \Rightarrow x = 2$ or $x + 3 = 0 \Rightarrow x = -3$

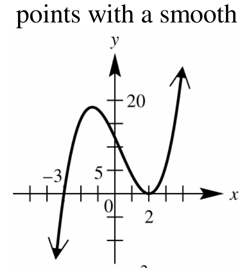
The zeros are –3 and 2, which divide the x-axis into three regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = (0-2)^2(0+3) = 4 \cdot 3 = 12$, so plot $(0, 12)$.

Step 3: The x-intercepts divide the x-axis into three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–4	–36	Negative	Below
$(-3, 2)$	1	4	Positive	Above
$(2, \infty)$	3	6	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.



$f(x) = (x-2)^2(x+3)$

60. $f(x) = -2x^3 + 7x^2 - 2x - 3$

Step 1: The first step is to find the zeros of the polynomial function. p must be a factor of $a_0 = -3$ and q must be a factor of $a_3 = -2$. Thus, p can be $\pm1, \pm3$, and q can be $\pm1, \pm2$. The possible zeros, $\frac{p}{q}$, are $\pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

The remainder theorem shows that $-\frac{1}{2}$ is a zero.

$$\begin{array}{r|rrrr} -\frac{1}{2} & -2 & 7 & -2 & -3 \\ & & 1 & -4 & 3 \\ \hline & -2 & 8 & -6 & 0 \end{array}$$

The new quotient polynomial therefore is $-2x^2 + 8x - 6$.

(*continued on next page*)

(*continued from page 373*)

$-2x^2+8x-6=0 \Rightarrow x^2-4x+3=0$

$(x-1)(x-3)=0$

$x-1=0 \Rightarrow x=1$ or $x-3=0 \Rightarrow x=3$

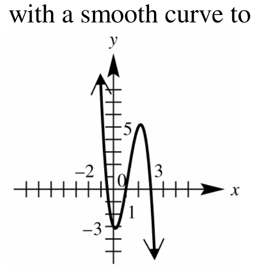
The rational zeros are $-\frac{1}{2}$, 1, and 3, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0)=-3$, so plot $(0,-3)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty,-\frac{1}{2}\right)$	–1	8	Positive	Above
$\left(-\frac{1}{2},1\right)$	0	–3	Negative	Below
$(1,3)$	2	5	Positive	Above
$(3,\infty)$	4	–27	Negative	Below

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x)=-2x^3+7x^2-2x-3$

61. $f(x)=2x^3+x^2-x$

Step 1: Factor the polynomial function and set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$2x^3+x^2-x=x\left(2x^2+x-1\right)$

$=x(x+1)(2x-1)$

$x=0$ or $x+1=0 \Rightarrow x=-1$ or

$2x-1=0 \Rightarrow x=\frac{1}{2}$

The zeros are –1, 0, and $\frac{1}{2}$, which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

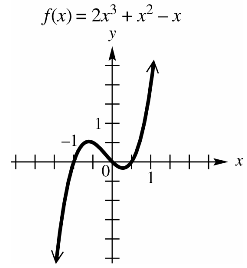
Step 2: $f(0)=0$, so plot $(0,0)$.

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-1)$	–2	–10	Negative	Below
$(-1,0)$	$-\frac{1}{2}$	$\frac{1}{2}$	Positive	Above
$\left(0,\frac{1}{2}\right)$	$\frac{1}{4}$	$-\frac{5}{32}$	Negative	Below
$\left(\frac{1}{2},\infty\right)$	1	2	Positive	Above

Plot the x-intercepts, y-intercept (the y-intercept is also an x-intercept), and test points with a smooth curve to get the graph.

$f(x)=2x^3+x^2-x$



62. $f(x)=x^4-3x^2+2$

Step 1: Factor the polynomial function and set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$x^4-3x^2+2=\left(x^2-2\right)\left(x^2-1\right)$

$=\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)(x+1)(x-1)$

$x+\sqrt{2}=0 \Rightarrow x=-\sqrt{2} \approx -1.4$ or

$x-\sqrt{2}=0 \Rightarrow x=\sqrt{2} \approx 1.4$ or

$x+1=0 \Rightarrow x=-1$ or $x-1=0 \Rightarrow x=1$

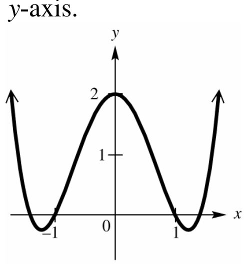
The zeros are $-\sqrt{2}$, –1, 1, and $\sqrt{2}$, which divide the x-axis into five regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0)=2$, so plot $(0,2)$.

Step 3: The x-intercepts divide the x-axis into five intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty,-\sqrt{2}\right)$	–2	6	Positive	Above
$\left(-\sqrt{2},-1\right)$	–1.2	–.25	Negative	Below
$(-1,1)$	0	2	Positive	Above
$\left(1,\sqrt{2}\right)$	1.2	–.25	Negative	Below
$\left(\sqrt{2},\infty\right)$	2	6	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph. Note that the graph is symmetric with respect to the y-axis.



$f(x) = x^4 - 3x^2 + 2$

63. $f(x) = x^4 + x^3 - 3x^2 - 4x - 4$

Step 1: The first step is to find the zeros of the polynomial function. p must be a factor of $a_0 = -4$ and q must be a factor of $a_4 = 1$. Thus, p can be $\pm 1, \pm 2,$ or ± 4, and q can be ± 1. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2,$ or ± 4. The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrrr} 2 & 1 & 1 & -3 & -4 & -4 \\ & & 2 & 6 & 6 & 4 \\ \hline & 1 & 3 & 3 & 2 & 0 \end{array}$$

The new quotient polynomial therefore is $x^3 + 3x^2 + 3x + 2.$

p must be a factor of $a_0 = 2$ and q must be a factor of $a_3 = 1$. Thus, p can be $\pm 1,$ or ± 2, and q can be ± 1. The possible zeros, $\frac{p}{q}$, are $\pm 1,$ or ± 2. The remainder theorem shows that -2 is a zero.

$$\begin{array}{r|rrrr} -2 & 1 & 3 & 3 & 2 \\ & & -2 & -2 & -2 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

The new quotient polynomial therefore is $x^2 + x + 1$. Setting this equal to 0 and using the quadratic formula with $a = 1$, $b = 1$, and $c = 1$, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-4}}{2}$$
$$= -\tfrac{1}{2} \pm 2i$$

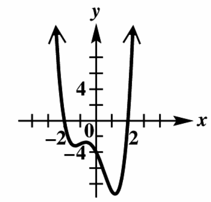
The rational zeros are -2 and 2, which divide the x-axis into three regions.

Step 2: $f(0) = -4,$ so plot $(0, -4)$.

Step 3: The x-intercepts divide the x-axis into three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	–3	35	Positive	Above
$(-2, 2)$	–1	–3	Negative	Below
$(2, \infty)$	3	65	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.



$f(x) = x^4 + x^3 - 3x^2 - 4x - 4$

64. $f(x) = -2x^4 + 7x^3 - 4x^2 - 4x$

Step 1: The first step is to find the zeros of the polynomial function.

$f(x) = -2x^4 + 7x^3 - 4x^2 - 4x$
$= x(-2x^3 + 7x^2 - 4x - 4) \Rightarrow x = 0$ is a zero.

Now find the zeros of the other factor. p must be a factor of $a_3 = -4$ and q must be a factor of $a_3 = -2.$ Thus, p can be $\pm 1, \pm 2,$ or ± 4, and q can be ± 1 or $\pm 2.$ The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 2,$ or ± 4. The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrr} 2 & -2 & 7 & -4 & -4 \\ & & -4 & 6 & 4 \\ \hline & -2 & 3 & 2 & 0 \end{array}$$

The new quotient polynomial therefore is $-2x^2 + 3x + 2$. Factor to find the zeros:

$$-2x^2 + 3x + 2 = -(2x^2 - 3x - 2)$$
$$= -(2x + 1)(x - 2)$$

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$
$x - 2 = 0 \Rightarrow x = 2$

The rational zeros are $-\frac{1}{2}$, 0 and 2 (with multiplicity 2), which divide the x-axis into four regions.

Step 2: $f(0) = 0,$ so plot $(0, 0)$.

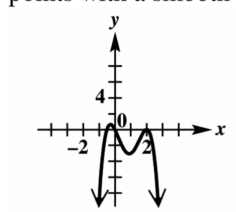
(*continued on next page*)

(*continued from page 375*)

Step 3: The x-intercepts divide the x-axis into four intervals.

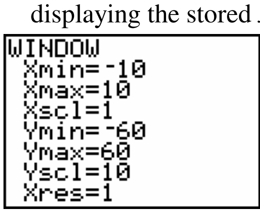
Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty, -\frac{1}{2}\right)$	–1	–9	Negative	Below
$\left(-\frac{1}{2}, 0\right)$	$-\frac{1}{2}$	$\frac{81}{128}$	Positive	Above
(0, 2)	1	–3	Negative	Below
$(2, \infty)$	3	–21	Negative	Below

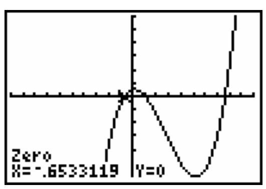
Plot the x-intercepts, y-intercept, and test points with a smooth curve to get the graph.

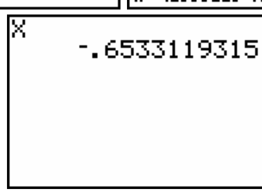


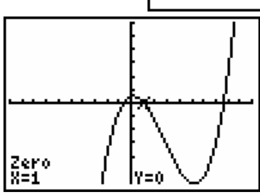
$f(x) = -2x^4 + 7x^3 - 4x^2 - 4x$

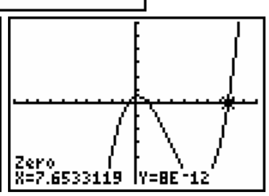
65. Using the calculator, we find that the real zeros are 7.6533119, 1, and –.6533119. For the two approximations of the zeros, a higher accuracy can be found by going to the home screen and displaying the stored x-value.

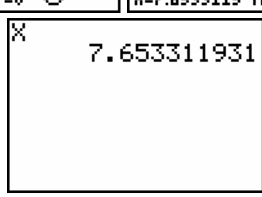




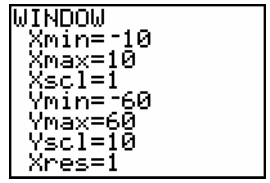


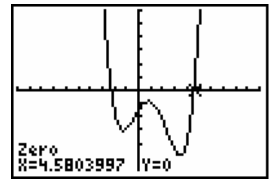


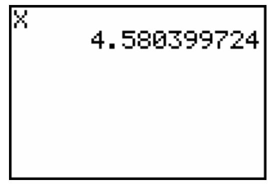


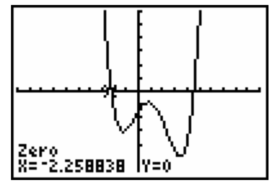


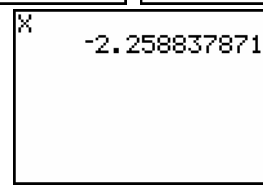
66. Using the calculator, we find that the real zeros are 4.5803997 and –2.258838. For these two approximations of the zeros, a higher accuracy can be found by going to the home screen and displaying the stored x-value.



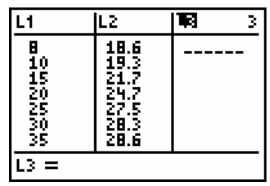


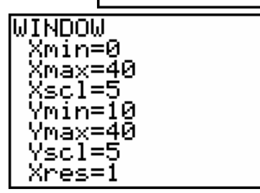


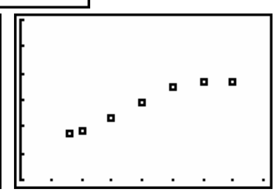




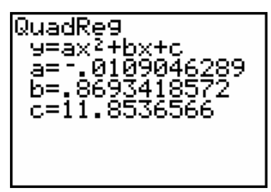
67. **(a)**



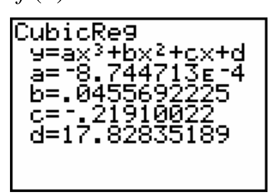




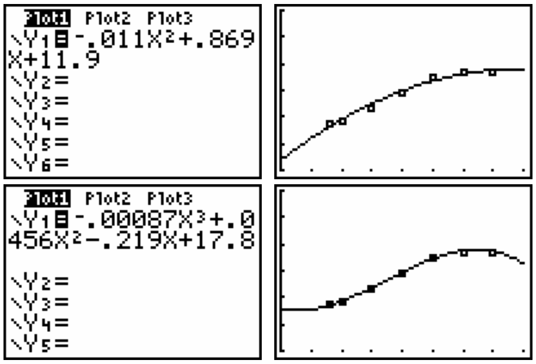
(b) $f(x) = -.011x^2 + .869x + 11.9$



(c) $f(x) = -.00087x^3 + .0456x^2 - .219x + 17.8$



(d)



(e) Both functions approximate the data well. The quadratic function is probably better for prediction, because it is unlikely that the percent of out-of-pocket spending would decrease after 2025 (as the cubic function shows) unless changes were made in Medicare law.

68. $V_{\text{cube}} = x^3$

The volume of the solid with the top sliced off is $V(x) = x \cdot x(x-2)$. Since the volume is 32 in.^{3}, we need to solve the equation

$32 = x^3 - 2x^2 \Rightarrow 0 = x^3 - 2x^2 - 32.$

Algebraic Solution:

p must be a factor of $a_0 = -32$ and q must be a factor of $a_3 = 1$. Thus, p can be $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$, and q can be ±1. The possible zeros, $\frac{p}{q}$, are $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$. The remainder theorem shows that 4 is a zero.

$$\begin{array}{r|rrrr} 4 & 1 & -2 & 0 & -32 \\ & & 4 & 8 & 32 \\ \hline & 1 & 2 & 8 & 0 \end{array}$$

The new quotient polynomial is $x^2 + 2x + 8$. If we examine the discriminant, $b^2 - 4ac$, we will see that the quotient polynomial cannot be factored further over the reals.

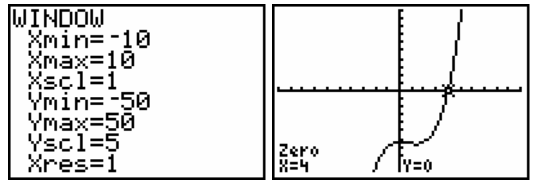
$b^2 - 4ac = 2^2 - 4(1)(8) = 4 - 32 = -28 < 0$

Since the discriminant is negative, 4 is our only real solution to the equation

$x^3 - 2x^2 - 32 = 0.$

Graphing Calculator Solution:

Graph $Y_1 = x^3 - 2x^2 - 32$ on a graphing calculator.



The graph will show that 4 is the only real zero.

Since the only real solution is 4, the dimensions of the original cube are 4 in. × 4 in. × 4 in.

69. $V_{\text{box}} = LWH$

If we let $L = x + 11$, $W = 3x$, and $H = x$, we have the volume of the rectangular box is $V(x) = (x+11)(3x)x$. Since the volume is 720 in.^{3}, we need to solve the equation

$720 = (x+11)(3x)x \Rightarrow 720 = 3x^3 + 33x^2$

$0 = 3x^3 + 33x^2 - 720 \Rightarrow x^3 + 11x^2 - 240 = 0$

Algebraic Solution:

p must be a factor of $a_0 = -240$ and q must be a factor of $a_3 = 1$. Including the $\pm$, there are 40 factors of p and 2 factors of q. There are 80 possible zeros, $\frac{p}{q}$. Because this list is so long, we will simply show that 4 is a zero by the remainder theorem.

$$\begin{array}{r|rrrr} 4 & 1 & 11 & 0 & -240 \\ & & 4 & 60 & 240 \\ \hline & 1 & 15 & 60 & 0 \end{array}$$

The new quotient polynomial is $x^2 + 15x + 60$. If we examine the discriminant, $b^2 - 4ac$, we will see that the quotient polynomial cannot be factored further over the reals

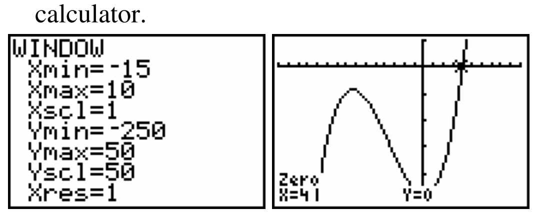
$b^2 - 4ac = 15^2 - 4(1)(60)$
$= 225 - 240 = -15 < 0$

Since the discriminant is negative, 4 is our only real solution to the equation

$x^3 + 11x^2 - 240 = 0.$

Graphing Calculator Solution:

Graph $Y_1 = x^3 + 11x^2 - 240$ on a graphing calculator.



(*continued on next page*)

(*continued from page 377*)

The graph will show that 4 is the only real zero. Since the only real solution is 4, the dimensions of the rectangular box are 4 in. by 3(4) = 12 in. by $4+11=15$ in. or $12 \text{ in.} \times 4 \text{ in.} \times 15 \text{ in.}$

70. The Intermediate Value Theorem on page 347 of your text pertains to polynomials. The function $f(x)=\frac{1}{x}$ is a rational function, not a polynomial function. So the example does not contradict the theorem.

71. $f(x)=\frac{4}{x-1}$

Step 1: The graph has a vertical asymptote where $x-1=0 \Rightarrow x=1$

Step 2: Since the degree of the numerator (which is considered degree zero) is less than the degree of the denominator, the graph has a horizontal asymptote at $y=0$ (the x-axis).

Step 3: The y-intercept is $f(0)=\frac{4}{0-1}=\frac{4}{-1}=-4.$

Step 4: Any x-intercepts are found by solving $f(x)=0: \ \frac{4}{x-1}=0 \Rightarrow 4=0$

This is a false statement, so there is no x-intercept.

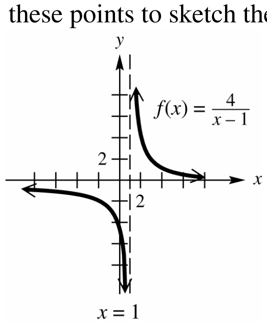
Step 5: The graph will intersect the horizontal asymptote when $\frac{4}{x-1}=0 \Rightarrow 4=0$

This is a false statement, so the graph does not intersect the horizontal asymptote.

Step 6: Since the vertical asymptote is $x=1$ and there is no x-intercept, we must determine values in two intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, 1)$	–2	$-\frac{4}{3}$	Negative	Below
$(1, \infty)$	2	4	Positive	Above

Step 7: Use the asymptotes, intercepts, and these points to sketch the graph.



72. $f(x)=\frac{4x-2}{3x+1}$

Step 1: The graph has a vertical asymptote where $3x+1=0 \Rightarrow x=-\frac{1}{3}.$

Step 2: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote at $y=\frac{4}{3}.$

Step 3: The y-intercept is $f(0)=\frac{4(0)-2}{3(0)+1}=\frac{0-2}{0+1}=\frac{-2}{1}=-2.$

Step 4: Any x-intercepts are found by solving $f(x)=0.$

$\frac{4x-2}{3x+1}=0 \Rightarrow 4x-2=0 \Rightarrow 4x=2 \Rightarrow x=\frac{1}{2}$

The only x-intercept is $\frac{1}{2}.$

Step 5: The graph will intersect the horizontal asymptote when $\frac{4x-2}{3x+1}=\frac{4}{3}.$

$\frac{4x-2}{3x+1}=\frac{4}{3} \Rightarrow 3(4x-2)=4(3x+1)$

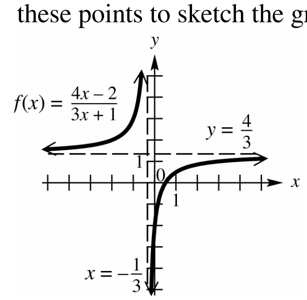
$12x-6=12x+4 \Rightarrow -6=4$

This is a false statement, so the graph does not intersect the horizontal asymptote.

Step 6: Since the vertical asymptote is $x=-\frac{1}{3}$ and the x-intercept occurs at $\frac{1}{2},$ we must determine values in three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty,-\frac{1}{3}\right)$	–1	3	Positive	Above
$\left(-\frac{1}{3},\frac{1}{2}\right)$	0	–2	Negative	Below
$\left(\frac{1}{2},\infty\right)$	1	$\frac{1}{2}$	Positive	Above

Step 7: Use the asymptotes, intercepts, and these points to sketch the graph.



73. $f(x)=\frac{6x}{x^2+x-2}=\frac{6x}{(x+2)(x-1)}$

Step 1: The graph has a vertical asymptote where $(x+2)(x-1)=0,$ that is, when $x=-2$ and $x=1.$

Step 2: Since the degree of the numerator is less than the degree of the denominator, the graph has a horizontal asymptote at $y=0$ (the x-axis).

Step 3: The y-intercept is $f(0)=\frac{6(0)}{0^2+0-2}=\frac{0}{0+0-2}=\frac{0}{-2}=0.$

Step 4: Any x-intercepts are found by solving $f(x)=0.$

$\frac{6x}{x^2+x-2}=0\Rightarrow 6x=0\Rightarrow x=0$

The only x-intercept is 0.

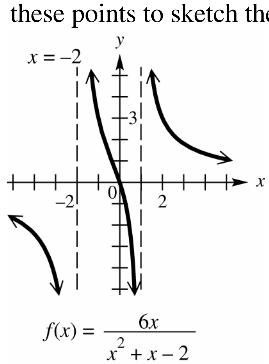
Step 5: The graph will intersect the horizontal asymptote when $\frac{6x}{x^2+x-2}=0.$

$\frac{6x}{x^2+x-2}=0\Rightarrow 6x=0\Rightarrow x=0$

Step 6: Since the vertical asymptotes are $x=-2$ and $x=1$ and the x-intercept occurs at $0,$ we must determine values in four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-2)$	-3	$-\frac{9}{2}$	Negative	Below
$(-2,0)$	-1	3	Positive	Above
$(0,1)$	$\frac{1}{2}$	$-\frac{12}{5}$	Negative	Below
$(1,\infty)$	2	3	Positive	Above

Step 7: Use the asymptotes, intercepts, and these points to sketch the graph.



74. $f(x)=\frac{2x}{x^2-1}=\frac{2x}{(x+1)(x-1)}$

Step 1: The graph has a vertical asymptote where $(x+1)(x-1)=0,$ that is, when $x=-1$ and $x=1.$

Step 2: Since the degree of the numerator is less than the degree of the denominator, the graph has a horizontal asymptote at $y=0$ (the x-axis).

Step 3: The y-intercept is $f(0)=\frac{2(0)}{0^2-1}=\frac{0}{0-1}=\frac{0}{-1}=0.$

Step 4: Any x-intercepts are found by solving $f(x)=0:\ \frac{2x}{x^2-1}=0\Rightarrow 2x=0\Rightarrow x=0$

The only x-intercept is 0.

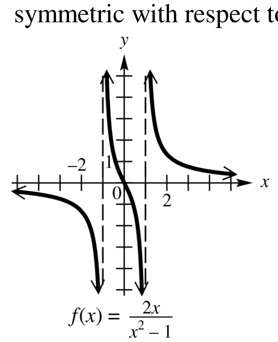
Step 5: The graph will intersect the horizontal asymptote when $\frac{2x}{x^2-1}=0.$

$\frac{2x}{x^2-1}=0\Rightarrow 2x=0\Rightarrow x=0$

Step 6: Since the vertical asymptotes are $x=-1$ and $x=1$ and the x-intercept occurs at $0,$ we must determine values in four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty,-1)$	-2	$-\frac{4}{3}$	Negative	Below
$(-1,0)$	$-\frac{1}{2}$	$\frac{4}{3}$	Positive	Above
$(0,1)$	$\frac{1}{2}$	$-\frac{4}{3}$	Negative	Below
$(1,\infty)$	2	$\frac{4}{3}$	Positive	Above

Step 7: Use the asymptotes, intercepts, and these points to sketch the graph. Note: This function is an odd function, thus the graph is symmetric with respect to the origin.



75. $f(x) = \frac{x^2+4}{x+2}$

Step 1: The graph has a vertical asymptote where $x + 2 = 0 \Rightarrow x = -2$.

Step 2: Since the degree of the numerator is one more than the degree of the denominator, the graph has an oblique asymptote. Divide $x^2 + 4$ by $x + 2$.

$$\begin{array}{r|rrr} -2 & 1 & 0 & 4 \\ & & -2 & 4 \\ \hline & 1 & -2 & 8 \end{array}$$

$f(x) = \frac{x^2+4}{x+2} = x - 2 + \frac{8}{x+3}$

The oblique asymptote is the line $y = x - 2$.

Step 3: The *y*-intercept is

$f(0) = \frac{0^2+4}{0+2} = \frac{0+4}{2} = \frac{4}{2} = 2$.

Step 4: Any *x*-intercepts are found by solving $f(x) = 0$.

$\frac{x^2+4}{x+2} = 0 \Rightarrow x^2 + 4 = 0 \Rightarrow x = \pm 2i$

There are no *x*-intercepts.

Step 5: The graph will intersect the oblique asymptote when $\frac{x^2+4}{x+2} = x - 2$.

$\frac{x^2+4}{x+2} = x - 2 \Rightarrow x^2 + 4 = (x+2)(x-2)$

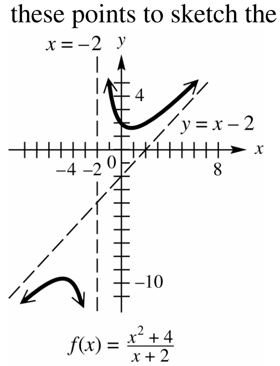
$x^2 + 4 = x^2 - 4 \Rightarrow 4 = -4$

This is a false statement, so the graph does not intersect the oblique asymptote.

Step 6: Since the vertical asymptote is $x = -2$ and there are no *x*-intercepts, we must determine values in two intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty, -2)$	–3	−13	Negative	Below
$(-2, \infty)$	3	$\frac{13}{5}$	Positive	Above

Step 7: Use the asymptotes, *y*-intercept, and these points to sketch the graph.



76. $f(x) = \frac{x^2-1}{x} = \frac{(x+1)(x-1)}{x}$

Step 1: The graph has a vertical asymptote when $x = 0$.

Step 2: Since the degree of the numerator is one more than the degree of the denominator, $f(x)$ has an oblique asymptote. Since $f(x) = \frac{x^2-1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$, the oblique asymptote is $y = x$.

Step 3: Since $f(0) = \frac{0^2-1}{0} = \frac{-1}{0}$ is undefined, there is no *y*-intercept

Step 4: Any *x*-intercepts are found by solving $f(x) = 0$.

$\frac{(x+1)(x-1)}{x} = 0 \Rightarrow (x+1)(x-1) = 0$

$x + 1 = 0 \Rightarrow x = -1$ or $x - 1 = 0 \Rightarrow x = 1$

Step 5: The graph will intersect the oblique asymptote when $\frac{x^2-1}{x} = x$.

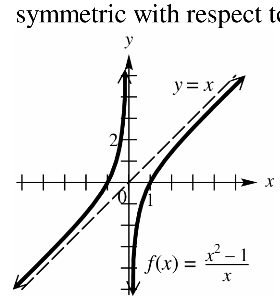
$\frac{x^2-1}{x} = x \Rightarrow x^2 - 1 = x^2 \Rightarrow -1 = 0$

This is a false statement, so the graph does not intersect the oblique asymptote.

Step 6: Since the vertical asymptote is $x = 0$ and *x*-intercepts of -1 and 1, we must determine values in four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$(-\infty, -1)$	–2	$-\frac{3}{2}$	Negative	Below
$(-1, 0)$	$-\frac{1}{2}$	$\frac{3}{2}$	Positive	Above
$(0, 1)$	$\frac{1}{2}$	$-\frac{3}{2}$	Negative	Below
$(1, \infty)$	2	$\frac{3}{2}$	Positive	Above

Step 7: Use the asymptotes, intercepts, and these points to sketch the graph. Note: This function is an odd function, thus the graph is symmetric with respect to the origin.



77. $f(x)=\frac{-2}{x^2+1}$

Step 1: The graph has a vertical asymptote where $x^2+1=0.$ Since we have $x^2+1=0 \Rightarrow x=\pm i,$ there are no vertical asymptotes.

Step 2: Since the degree of the numerator (which is considered degree zero) is less than the degree of the denominator, the graph has a horizontal asymptote at $y=0$ (the x-axis).

Step 3: The y-intercept is $f(0)=\frac{-2}{0^2+1}=\frac{-2}{0+1}=\frac{-2}{1}=-2.$

Step 4: Any x-intercepts are found by solving $f(x)=0.$

$\frac{-2}{x^2+1}=0 \Rightarrow -2=0$

This is a false statement, so there is no x-intercept.

Step 5: The graph will intersect the horizontal asymptote when $\frac{-2}{x^2+1}=0.$

$\frac{-2}{x^2+1}=0 \Rightarrow -2=0$

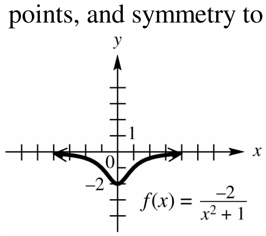
This is a false statement, so the graph does not intersect the horizontal asymptote.

Step 6: Since there are no vertical asymptotes and no x-intercept, we can see from the y-intercept that the function is always negative and, therefore, below the x-axis. Because of the limited information we have, we should explore symmetry and calculate a few more points on the graph.

x	y
-3	$-\frac{1}{5}$
$\frac{1}{2}$	$-\frac{8}{5}$
1	-1

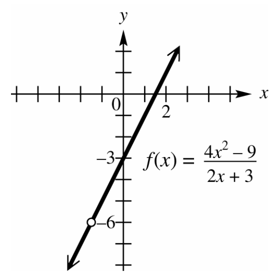
Since $f(-x)=\frac{-2}{(-x)^2+1}=\frac{-2}{x^2+1}=f(x),$ the graph is symmetric about the y-axis.

Step 7: Use the asymptote, y-intercept, these points, and symmetry to sketch the graph.

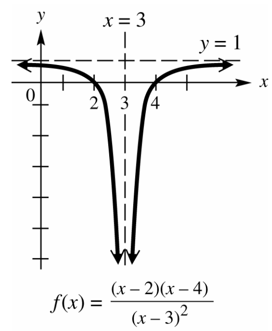


78. $f(x)=\frac{4x^2-9}{2x+3}$

Since reduction can occur after factoring, $f(x)=\frac{4x^2-9}{2x+3}=\frac{(2x+3)(2x-3)}{2x+3}=2x-3\left(x\neq-\frac{3}{2}\right),$ the graph of this function will be the same as the graph of $y=2x-3$ (a straight line), with the exception of the point with x-value $-\frac{3}{2}$. A "hole" appears in the graph at $\left(-\frac{3}{2},-6\right)$.

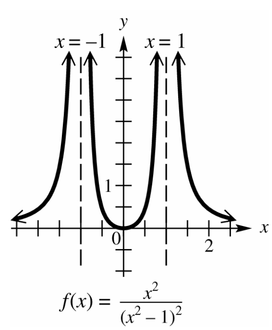


79. (a)



(b) The graph has a vertical asymptote $x=3$, so $x-3$ is in the denominator of the function. The x-intercepts are 2 and 4, so that when $f(x)=0,$ $x=2$ or $x=4$. This would exist if $x-2$ and $x-4$ were factors of the numerator. The horizontal asymptote is $y=1$, so the numerator and denominator have the same degree. Since the numerator will have degree 2, we must make the denominator also have degree 2. Putting these conditions together, we get a possible function $f(x)=\frac{(x-2)(x-4)}{(x-3)^2}$.

80. (a)

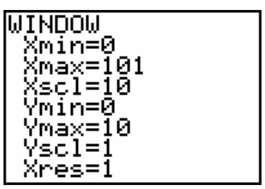


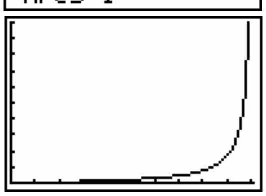
(b) If $x = -1$ and $x = 1$ are vertical asymptotes, $(x-1)(x+1) = x^2 - 1$ is a factor of the denominator. If the *x*-axis is a horizontal asymptote, the numerator has a lower degree than the denominator. If 0 is an *x*-intercept, $f(0) = 0$, so the numerator must be 0 if $x = 0$. The function is never negative, so only even-numbered powers of x should appear. Putting these conditions together, we get a possible function $f(x) = \frac{x^2}{(x^2-1)^2}$.

81. The graph has a vertical asymptote $x = 1$, so $x - 1$ is in the denominator of the function. The *x*-intercept is 2, so that when $f(x) = 0$, $x = 2$. This would exist if $x - 2$ was a factor of the numerator. The horizontal asymptote is $y = -3$, so the numerator and denominator have the same degree. They both have degree 1. Also, from the horizontal asymptote, we have $y = -3 = \frac{a_n}{b_n} = \frac{-3}{1}$. Putting these conditions together, we have a possible function $f(x) = \frac{-3(x-2)}{x-1}$ or $f(x) = \frac{-3x+6}{x-1}$.

82. $C(x) = \frac{10x}{49(101-x)}$

(a)

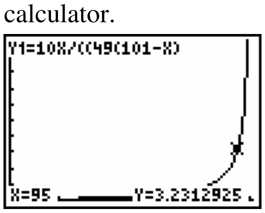




(b) To find the cost of earning 95 points, find $C(95)$.

$C(95) = \frac{10(95)}{49(101-95)} \approx 3.23$

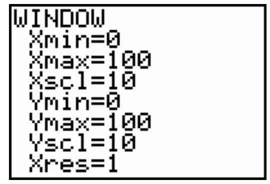
This can also be found using the graphing calculator.

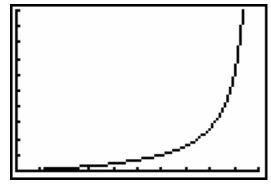


To earn 95 points, an owner would expect to pay approximately \$3.23 thousand.

83. $C(x) = \frac{6.7x}{100-x}$

(a)

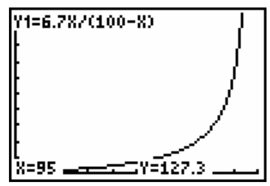




(b) To find the cost to remove 95% of the pollutant, find $C(95)$.

$C(95) = \frac{6.7(95)}{100-95} = 127.3$

This can also be found using the graphing calculator.



It would cost \$127.3 thousand to remove 95% of the pollutant.

84. *Step 1*: $x = ky$

Step 2: Substitute $x = 12$ and $y = 4$ to find *k*.

$12 = k(4) \Rightarrow k = 3$

Step 3: $x = 3y$

Step 4: Now find *x* when $y = 12$.

$x = 3(12) = 36$

85. *Step 1*: $x = ky$

Step 2: Substitute $x = 20$ and $y = 14$ to find *k*. $20 = k(14) \Rightarrow k = \frac{20}{14} = \frac{10}{7}$

Step 3: $x = \frac{10}{7}y$

Step 4: Now find *y* when $x = 50$.

$50 = \frac{10}{7}y \Rightarrow 350 = 10y \Rightarrow y = 35$

86. *Step 1*: $z = \frac{k}{w}$

Step 2: Substitute $z = 10$ and $w = \frac{1}{2}$ to find *k*.

$10 = \frac{k}{\frac{1}{2}} \Rightarrow 10 = 2k \Rightarrow k = 5$

Step 3: $z = \frac{5}{w}$

Step 4: Now find *z* when $w = 10$.

$z = \frac{5}{10} = .5$

87. *Step 1*: $t = \frac{k}{s}$

Step 2: Substitute $t = 3$ & $s = 5$ to find *k*.

$3 = \frac{k}{5} \Rightarrow k = 15$

Step 3: $t = \frac{15}{s}$

Step 4: Now find *s* when $t = 20$.

$20 = \frac{15}{s} \Rightarrow 20s = 15 \Rightarrow s = \frac{15}{20} = \frac{3}{4}$

88. *Step 1*: $p = kqr^2$

Step 2: Substitute $p = 100$, $q = 2$, and $r = 3$ to find *k*.

$100 = k(2)(3^2) \Rightarrow 100 = 18k \Rightarrow k = \frac{100}{18} = \frac{50}{9}$

Step 3: $p = \frac{50}{9}qr^2$

Step 4: Now find *p* when $q = 5$ and $r = 2$.

$p = \frac{50}{9}(5)(2^2) = \frac{1000}{9} = 111\frac{1}{9}$

89. *Step 1*: $f = kg^2h$

Step 2: Substitute $f = 50$, $g = 5$, and $h = 4$ to find *k*.

$50 = k(5^2)(4) \Rightarrow 50 = 100k \Rightarrow k = \frac{50}{100} = \frac{1}{2}$

Step 3: $f = \frac{1}{2}g^2h$

Step 4: Now find *f* when $g = 3$ and $h = 6$.

$f = \frac{1}{2}(3^2)(6) = 27$

90. *Step 1*: Let p be the pressure (in kg per m^2). *d* is the distance (in meters). $p = kd$

Step 2: Substitute $p = 60$ & $d = 4$ to find *k*.

$60 = k(4) \Rightarrow k = 15$

Step 3: $p = 15d$

Step 4: Now find *p* when $d = 10$.

$p = 15(10) = 150$

A pressure of 150 kg per m^2 occurs at a depth of 10 m.

91. *Step 1*: Let *f* be the force to keep car from skidding (in lbs), *r* is the radius of curve (in ft), *w* is the weight of car (in lbs), *s* is the speed of the car (in mph). $f = \frac{kws^2}{r}$

Step 2: *f* = 3000, *w* = 2000, *r* = 500, and *s* = 30 to find *k*.

$3000 = \frac{k(2000)(30^2)}{500} = \frac{1{,}800{,}000k}{500}$

$3000 = 3600k \Rightarrow k = \frac{3000}{3600} = \frac{5}{6}$

Step 3: $f = \frac{5}{6} \cdot \frac{ws^2}{r} = \frac{5ws^2}{6r}$

Step 4: Now find *f* when *w* = 2000, *r* = 800, and *s* = 60

$f = \frac{5(2000)(60^2)}{6(800)} = 7500$

A force of 7500 lb is needed.

92. *Step 1*: Let *p* be the power, *v* is the wind velocity (in km per hr). $p = kv^3$

Step 2: Substitute *p* = 10,000 and *v* = 10 to find *k*.

$10{,}000 = k \cdot 10^3 \Rightarrow 10{,}000 = 1000k \Rightarrow k = 10$

Step 3: $p = 10v^3$

Step 4: Now find *p* when $v = 15$.

$p = 10 \cdot 15^3 = 33{,}750$

33,750 units of power are produced.

Chapter 3: Test

1. $f(x) = -2x^2 + 6x - 3$

Complete the square so the function has the form $f(x) = a(x-h)^2 + k$.

$f(x) = -2x^2 + 6x - 3 = -2(x^2 - 3x) - 3$

$= -2\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) - 3$

Note: $\left[\frac{1}{2}(-3)\right]^2 = \frac{9}{4}$

$= -2\left(x - \frac{3}{2}\right)^2 + \frac{9}{2} - 3 = -2\left(x - \frac{3}{2}\right)^2 + \frac{3}{2}$

This parabola is in now in standard form with $a = -2$, $h = \frac{3}{2}$, and $k = \frac{3}{2}$. It opens downward with vertex $(h, k) = \left(\frac{3}{2}, \frac{3}{2}\right)$. The axis is $x = h \Rightarrow x = \frac{3}{2}$. To find the *x*-intercepts, let $f(x) = 0$.

$-2\left(x - \frac{3}{2}\right)^2 + \frac{3}{2} = 0 \Rightarrow -2\left(x - \frac{3}{2}\right)^2 = -\frac{3}{2}$

$\left(x - \frac{3}{2}\right)^2 = \frac{3}{4} \Rightarrow x - \frac{3}{2} = \pm\sqrt{\frac{3}{4}}$

$x = \frac{3}{2} \pm \sqrt{\frac{3}{4}} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} = \frac{3 \pm \sqrt{3}}{2}$

Thus the *x*-intercepts are $\frac{3 \pm \sqrt{3}}{2}$ or approximately .63 and 2.37.

Note: If you use the quadratic formula to solve $-2x^2 + 6x - 3 = 0$ where $a = -2, b = 6,$ and $c = -3,$ you will arrive at the equivalent answer of *x*-intercepts: $\frac{-3 \pm \sqrt{3}}{-2}$.

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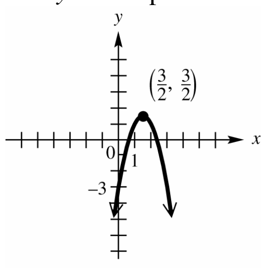
To find the y-intercept, let $x = 0$.

$f(0) = -2(0^2) + 6(0) - 3 = -3$

The y-intercept is –3.



$f(x) = -2x^2 + 6x - 3$

The domain is $(-\infty, \infty)$. Since the highest on the graph is $\left(\frac{3}{2}, \frac{3}{2}\right)$, the range is $\left(-\infty, \frac{3}{2}\right]$.

2. $f(x) = -.3857x^2 + 1.2829x + 11.329$

The year 2003 corresponds to $x = 5$ in this model.

$f(5) = -.3857(5^2) + 1.2829(5) + 11.329$
$= 8.101$

Approximately 8.1 million tickets would have been sold in 2003 according to the model.

3. $\dfrac{3x^3 + 4x^2 - 9x + 6}{x + 2}$

Express $x + 2$ in the form $x - k$ by writing it as $x - (-2)$. Thus $k = -2$.

$$\begin{array}{r|rrrr} -2 & 3 & 4 & -9 & 6 \\ & & -6 & 4 & 10 \\ \hline & 3 & -2 & -5 & 16 \end{array}$$

Thus,

$$\frac{3x^3 + 4x^2 - 9x + 6}{x + 2} = 3x^2 - 2x - 5 + \frac{16}{x + 2}$$

4. $\dfrac{2x^3 - 11x^2 + 28}{x - 5}$

Since $x - 5$ is in the form $x - k$, $k = 5$.

$$\begin{array}{r|rrrr} 5 & 2 & -11 & 0 & 28 \\ & & 10 & -5 & -25 \\ \hline & 2 & -1 & -5 & 3 \end{array}$$

$$\frac{2x^3 - 11x^2 + 28}{x - 5} = 2x^2 - x - 5 + \frac{3}{x - 5}$$

5. $f(x) = 2x^3 - 9x^2 + 4x + 8;\ k = 5$

$$\begin{array}{r|rrrr} 5 & 2 & -9 & 4 & 8 \\ & & 10 & 5 & 45 \\ \hline & 2 & 1 & 9 & 53 \end{array}$$

Thus, $f(5) = 53$.

6. $6x^4 - 11x^3 - 35x^2 + 34x + 24;\ x - 3$

Let $f(x) = 6x^4 - 11x^3 - 35x^2 + 34x + 24$. By the factor theorem, $x - 3$ will be a factor of $f(x)$ only if $f(3) = 0$.

$$\begin{array}{r|rrrrr} 3 & 6 & -11 & -35 & 34 & 24 \\ & & 18 & 21 & -42 & -24 \\ \hline & 6 & 7 & -14 & -8 & 0 \end{array}$$

Since $f(3) = 0$, $x - 3$ is a factor of $f(x)$. The other factor is $6x^3 + 7x^2 - 14x - 8$.

7. $f(x) = x^3 + 8x^2 + 25x + 26;\ -2$

Since -2 is a zero, first divide $f(x)$ by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 1 & 8 & 25 & 26 \\ & & -2 & -12 & -26 \\ \hline & 1 & 6 & 13 & 0 \end{array}$$

This gives $f(x) = (x + 2)(x^2 + 6x + 13)$. Since $x^2 + 6x + 13$ cannot be factored, use the quadratic formula with $a = 1$, $b = 6$, and $c = 13$ to find the remaining two zeros.

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 52}}{2}$$
$$= \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

The zeros are $-2, -3 - 2i$, and $-3 + 2i$.

8. Zeros of –1, 2 and i; $f(3) = 80$

By the conjugate zeros theorem, $-i$ is also a zero. The polynomial has the following form.

$f(x) = a(x + 1)(x - 2)(x - i)(x + i)$

Use the condition $f(3) = 80$ to find a.

$80 = a(3 + 1)(3 - 2)(3 - i)(3 + i)$
$80 = a(4)(1)\left[(3 - i)(3 + i)\right]$
$80 = a\left[(4)(1)\right]\left[3^2 - i^2\right]$
$80 = a(4)\left[9 - (-1)\right]$
$80 = a(4)(10) \Rightarrow 80 = 40a \Rightarrow a = 2$

Thus we have the following.

$$f(x) = 2(x+1)(x-2)(x-i)(x+i)$$
$$= 2[(x+1)(x-2)][(x-i)(x+i)]$$
$$= 2(x^2 - x - 2)(x^2 + 1)$$
$$= 2(x^4 - x^3 - x^2 - x - 2)$$
$$= 2x^4 - 2x^3 - 2x^2 - 2x - 4$$

9. Since $f(x) = x^4 + 8x^2 + 12$ $= (x^2+6)(x^2+2),$ the zeros are $\pm i\sqrt{6}$ and $\pm i\sqrt{2}$. Moreover, $f(x) > 0$ for all *x*; the graph never crosses or touches the *x*-axis, so $f(x)$ has no real zeros.

10. **(a)** $f(x) = x^3 - 5x^2 + 2x + 7$; find $f(1)$ and $f(2)$ synthetically.

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 2 & 7 \\ & & 1 & -4 & -2 \\ \hline & 1 & -4 & -2 & 5 \end{array}$$

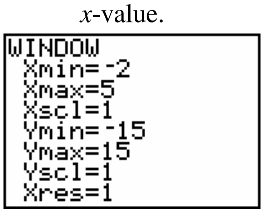
$$\begin{array}{r|rrrr} 2 & 1 & -5 & 2 & 7 \\ & & 2 & -6 & -8 \\ \hline & 1 & -3 & -4 & -1 \end{array}$$

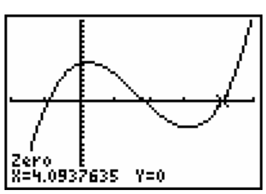
By the intermediate value theorem, since $f(1) = 5 > 0$ and $f(2) = -1 < 0$, there must be at least one real zero between 1 and 2.

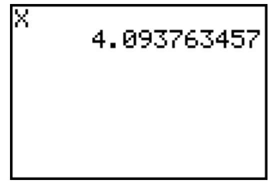
(b) $f(x) = \underbrace{x^3 - 5x^2}_{1} \underbrace{+ 2x}_{2} + 7$ has 2 variations in sign. *f* has 2 or $2 - 2 = 0$ positive real zeros.

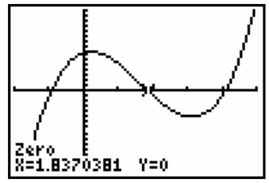
$f(-x) = -x^3 - 5x^2 \underbrace{-2x + 7}_{1}$ has 1 variation in sign. *f* has 1 negative real zero.

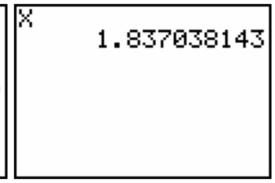
(c) Using the calculator, we find that the real zeros are 4.0937635, 1.8370381, and –.9308016. For these approximations, a higher accuracy can be found by going to the home screen and displaying the stored *x*-value.

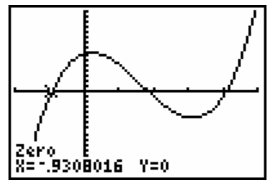


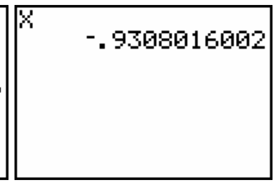




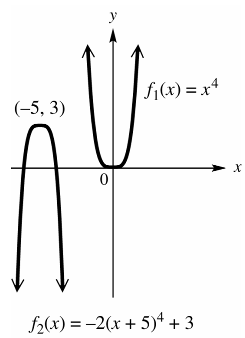








11. To obtain the graph of f_2, shift the graph of f_1 5 units to the left, stretch by a factor of 2, reflect across the *x*-axis, and shift 3 units up.



12. $f(x) = -x^7 + x - 4$

Since $f(x)$ is of odd degree and the sign of a_n is negative, the end behavior is . The correct graph is C.

13. $f(x) = x^3 - 5x^2 + 3x + 9$

Step 1: The first step is to find the zeros of the polynomial function. *p* must be a factor of $a_0 = 9$ and *q* must be a factor of $a_3 = 1$. Thus, *p* can be $\pm 1, \pm 3, \pm 9$ and *q* can be ± 1. The possible zeros, $\frac{p}{q}$, are $\pm 1, \pm 3, \pm 9$. The remainder theorem shows that 3 is a zero.

$$\begin{array}{r|rrrr} 3 & 1 & -5 & 3 & 9 \\ & & 3 & -6 & -9 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

The new quotient polynomial is $x^2 - 2x - 3$.

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$x^2 - 2x - 3 = 0 \Rightarrow (x+1)(x-3) = 0$

$x + 1 = 0 \Rightarrow x = -1$ or $x - 3 = 0 \Rightarrow x = 3$

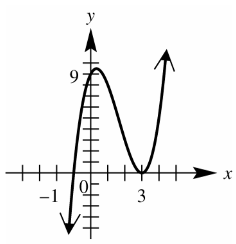
The rational zeros are − 1 and 3 (multiplicity 2) which divide the x-axis into three regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 9,$ so plot $(0, 9).$

Step 3: The x-intercepts divide the x-axis into three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -1)$	–2	–25	Negative	Below
$(-1, 3)$	0	9	Positive	Above
$(3, \infty)$	4	5	Positive	Above

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



14. $f(x) = 2x^2(x-2)^2$

Step 1: Set each factor equal to 0 and solve the resulting equations to find the zeros of the function.

$2x^2 = 0 \Rightarrow x = 0$ or $(x-2)^2 = 0 \Rightarrow x = 2$

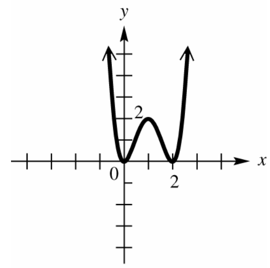
The zeros are 0 (multiplicity 2) and 2 (multiplicity 2), which divide the x-axis into three regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 0,$ so plot $(0, 0).$

Step 3: The x-intercepts divide the x-axis into three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, 0)$	–1	18	Positive	Above
$(0, 2)$	1	2	Positive	Above
$(2, \infty)$	3	18	Positive	Above

Plot the x-intercepts, y-intercept (the y-intercept is an x-intercept), and test points with a smooth curve to get the graph.



15. $f(x) = -x^3 - 4x^2 + 11x + 30$

Step 1: The first step is to find the zeros of the polynomial function. p must be a factor of $a_0 = 30$ and q must be a factor of $a_3 = -1.$ Thus, p can be $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$ and q can be $\pm1.$ The possible zeros, $\frac{p}{q}$, are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30.$ The remainder theorem shows that 3 is a zero.

$$\begin{array}{r|rrrr} 3 & -1 & -4 & 11 & 30 \\ & & -3 & -21 & -30 \\ \hline & -1 & -7 & -10 & 0 \end{array}$$

The new quotient polynomial is $-x^2 - 7x - 10.$

$-x^2 - 7x - 10 = 0 \Rightarrow x^2 + 7x + 10 = 0$

$(x+5)(x+2) = 0$

$x + 5 = 0 \Rightarrow x = -5$ or $x + 2 = 0 \Rightarrow x = -2$

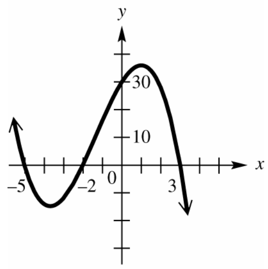
The rational zeros are $-5,\ -2,$ and $3,$ which divide the x-axis into four regions. Test a point in each region to find the sign of $f(x)$ in that region.

Step 2: $f(0) = 30,$ so plot $(0, 30).$

Step 3: The x-intercepts divide the x-axis into four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -5)$	–6	36	Positive	Above
$(-5, -2)$	–3	–12	Negative	Below
$(-2, 3)$	0	30	Positive	Above
$(3, \infty)$	4	–54	Negative	Below

Plot the x-intercepts, y-intercept, and test points (the y-intercept is one of the test points) with a smooth curve to get the graph.



$f(x) = -x^3 - 4x^2 + 11x + 30$

16. The zeros are –3 and 2. Since the graph of *f* touches the *x*-axis at 2, the zero 2 has multiplicity 2. Thus, $f(x) = a(x-2)^2(x+3)$. Also, since the point $(0, 24)$ is on the graph, we have $f(0) = 24$.

$$24 = a(0-2)^2(0+3) \Rightarrow 24 = a(-2)^2(3)$$
$$24 = a(4)(3) \Rightarrow 24 = 12a \Rightarrow a = 2$$

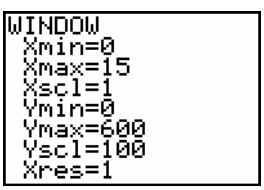
The polynomial function is the following.

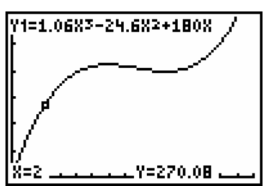
$$\begin{aligned} f(x) &= 2(x-2)^2(x+3) \\ &= 2(x^2 - 4x + 4)(x+3) \\ &= 2(x^3 - x^2 - 8x + 12) \\ &= 2x^3 - 2x^2 - 16x + 24 \end{aligned}$$

17. $f(t) = 1.06t^3 - 24.6t^2 + 180t$

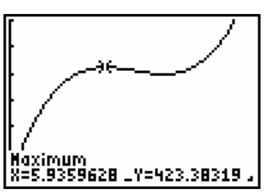
(a) $f(2) = 1.06(2^3) - 24.6(2)^2 + 180(2)$
$= 270.08$

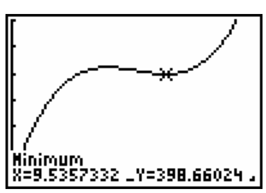
This can also be found using the graphing calculator.





(b) From the graph we see that the amount of change is increasing from $t = 0$ to $t = 5.9$ and from $t = 9.5$ to $t = 15$ and decreasing from $t = 5.9$ to $t = 9.5$.





18. $f(x) = \frac{3x-1}{x-2}$

Step 1: The graph has a vertical asymptote where $x - 2 = 0$, that is, when $x = 2$.

Step 2: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote at $y = \frac{3}{1} = 3$.

Step 3: The *y*-intercept is

$f(0) = \frac{3(0)-1}{0-2} = \frac{0-1}{-2} = \frac{-1}{-2} = \frac{1}{2}$.

Step 4: Any *x*-intercepts are found by solving $f(x) = 0$.

$\frac{3x-1}{x-2} = 0 \Rightarrow 3x - 1 = 0 \Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3}$

The only *x*-intercept is $\frac{1}{3}$.

Step 5: The graph will intersect the horizontal asymptote when $\frac{3x-1}{x-2} = 3$.

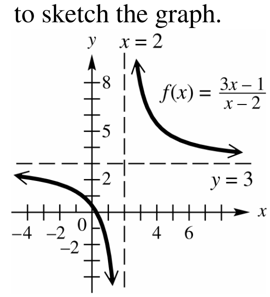
$\frac{3x-1}{x-2} = 3 \Rightarrow 3x - 1 = 3(x-2)$
$3x - 1 = 3x - 6 \Rightarrow -1 = -6$

This is a false statement, so the graph does not intersect the horizontal asymptote.

Step 6: Since the vertical asymptote is $x = 2$ and the *x*-intercept occurs at $\frac{1}{3}$, we must determine values in three intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below *x*-Axis
$\left(-\infty, \frac{1}{3}\right)$	0	$\frac{1}{2}$	Positive	Above
$\left(\frac{1}{3}, 2\right)$	1	–2	Negative	Below
$(2, \infty)$	3	8	Positive	Above

Step 7: Use the asymptotes, intercepts, and test points (the *y*-intercept is one of the test points) to sketch the graph.



19. $f(x) = \frac{x^2-1}{x^2-9} = \frac{(x+1)(x-1)}{(x+3)(x-3)}$

Step 1: The graph has a vertical asymptote where $(x+3)(x-3) = 0$, that is, when $x = -3$ and $x = 3$.

Step 2: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote at $y = \frac{1}{1} = 1$.

Step 3: The y-intercept is $f(0) = \frac{0^2-1}{0^2-9} = \frac{-1}{-9} = \frac{1}{9}$.

Step 4: Any x-intercepts are found by solving $f(x) = 0$.

$\frac{(x+1)(x-1)}{(x+3)(x-3)} = 0 \Rightarrow (x+1)(x-1) = 0$

$x = -1$ or $x = 1$

The x-intercepts are -1 and 1.

Step 5: The graph will intersect the horizontal asymptote when $\frac{x^2-1}{x^2-9} = 1$.

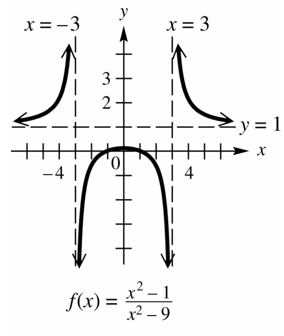
$\frac{x^2-1}{x^2-9} = 1 \Rightarrow x^2 - 1 = x^2 - 9 \Rightarrow -1 = -9$

This is a false statement, so the graph does not intersect the horizontal asymptote.

Step 6: Since the vertical asymptotes are $x = -3$ and $x = 3$, and the x-intercepts occur at -1 and 1, we must determine values in five intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	-4	$\frac{15}{7}$	Positive	Above
$(-3, -1)$	-2	$-\frac{3}{5}$	Negative	Below
$(-1, 1)$	0	$\frac{1}{9}$	Positive	Above
$(1, 3)$	2	$-\frac{3}{5}$	Negative	Below
$(3, \infty)$	4	$\frac{15}{7}$	Positive	Above

Step 7: Use the asymptotes, intercepts, and test points (the y-intercept is one of the test points) to sketch the graph. Note: This function is an even function, thus the graph is symmetric with respect to the y-axis.



20. $f(x) = \frac{2x^2 + x - 6}{x - 1}$

(a) Since the degree of the numerator is one more than the degree of the denominator, the graph has an oblique asymptote.

Divide $2x^2 + x - 6$ by $x - 1$.

$$\begin{array}{r|rrr} 1 & 2 & 1 & -6 \\ & & 2 & 3 \\ \hline & 2 & 3 & -3 \end{array}$$

$f(x) = \frac{2x^2+x-6}{x-1} = 2x + 3 - \frac{3}{x-1}$

The oblique asymptote is the line $y = 2x + 3$.

(b) To find the x-intercepts, let $f(x) = 0$.

$\frac{2x^2+x-6}{x-1} = 0 \Rightarrow 2x^2 + x - 6 = 0$

$(x+2)(2x-3) = 0$

$x = -2$ or $x = \frac{3}{2}$

The x-intercepts are -2 and $\frac{3}{2}$.

(c) The y-intercept is

$f(0) = \frac{2(0^2)+0-6}{0-1} = \frac{2(0)+0-6}{-1}$
$= \frac{0+0-6}{-1} = \frac{-6}{-1} = 6.$

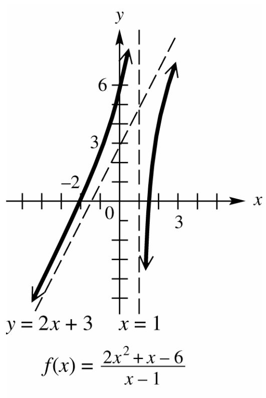
(d) To find the vertical asymptote, set the denominator equal to zero and solve for x.

$x - 1 = 0 \Rightarrow x = 1$

The equation of the vertical asymptote is $x = 1$.

(e) Use the information from (a)–(d) and a few additional points to graph the function. Since the vertical asymptote is $x = 1$, and the x-intercepts occur at -2 and $\frac{3}{2}$, we must determine values in four intervals.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-3	$-\frac{9}{4}$	Negative	Below
$(-2,1)$	0	6	Positive	Above
$\left(1, \frac{3}{2}\right)$	$\frac{5}{4}$	$-\frac{13}{2}$	Negative	Below
$\left(\frac{3}{2}, \infty\right)$	2	4	Positive	Above



21. *Step 1*: $y = k\sqrt{x}$

Step 2: Substitute $y = 12$ and $x = 4$ to find k.

$12 = k\sqrt{4} \Rightarrow 12 = 2k \Rightarrow k = 6$

Step 3: $y = 6\sqrt{x}$

Step 4: Now find y when $x = 100$.

$y = 6\sqrt{100} = 6 \cdot 10 = 60$

22. *Step 1*: Let w be the weight of the object (in kg), d is the distance from the center of Earth (in km). $w = \frac{k}{d^2}$

Step 2: Substitute $w = 90$ and $d = 6400$ to find k.

$90 = \frac{k}{6400^2} \Rightarrow k = 3,686,400,000$

Step 3: $w = \frac{3,686,400,000}{d^2}$

Step 4: Now find w when $d = 800 + 6400 = 7200$.

$w = \frac{3,686,400,000}{7200^2} = \frac{3,686,400,000}{51,840,000} = \frac{640}{9} \approx 71.1$

The man weighs $\frac{640}{9}$ kg or approximately 71.1 kg.

Chapter 3: Quantitative Reasoning

1 part per million corresponds to $\text{DMF} = 200 + \frac{100}{1} = 200 + 100 = 300$ incidences per 100 examinees. To find the level of fluoride that would correspond to a DMF count of 250, we must solve $250 = 200 + \frac{100}{x}$.

$250 = 200 + \frac{100}{x} \Rightarrow 50 = \frac{100}{x}$

$50x = 100 \Rightarrow x = 2$ ppm

Chapter 4

INVERSE, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS

Section 4.1: Inverse Functions

1. Yes, it is one-to-one, because every number in the list of registered passenger cars is used only once.

2. It is not one-to-one because both Illinois and Wisconsin are paired with the same range element, 40.

3. This is a one-to-one function since every horizontal line intersects the graph in no more than one point.

4. This function is not one-to-one because there are infinitely many horizontal lines that intersect the graph in two points.

5. This is a one-to-one function since every horizontal line intersects the graph in no more than one point.

6. This function is one-to-one because every horizontal line will intersect the graph in exactly one point.

7. This is not a one-to-one function since there is a horizontal line that intersects the graph in more than one point. (Here a horizontal line intersects the curve at an infinite number of points.)

8. This function is not one-to-one because one horizontal line (the same line as the given graph) intersects the graph in infinitely many points.

9. $y = 2x - 8$

   Using the definition of a one-to-one function, we have $f(a) = f(b) \Rightarrow 2a - 8 = 2b - 8 \Rightarrow$ $2a = 2b \Rightarrow a = b$. So the function is one-to-one.

10. $y = 4x + 20$

    Using the definition of a one-to-one function, we have $f(a) = f(b) \Rightarrow 4a + 20 = 4b + 20 \Rightarrow$ $4a = 4b \Rightarrow a = b$. So the function is one-to-one.

11. $y = \sqrt{36 - x^2}$

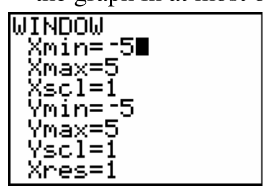
    If $x = 6$, $y = \sqrt{36 - 6^2} = \sqrt{36 - 36} = \sqrt{0} = 0$.
    If $x = -6$,
    $y = \sqrt{36 - (-6)^2} = \sqrt{36 - 36} = \sqrt{0} = 0$.
    Since two different values of x lead to the same value of y, the function is not one-to-one.

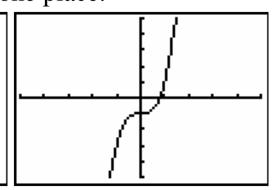
12. $y = -\sqrt{100 - x^2}$

    If $x = 10$, $y = -\sqrt{100 - 10^2} = -\sqrt{100 - 100}$
    $= -\sqrt{0} = -0 = 0$. If $x = -10$,
    $y = -\sqrt{100 - (-10)^2} = -\sqrt{100 - 100}$
    $= -\sqrt{0} = -0 = 0$. Since two different values of x lead to the same value of y, the function is not one-to-one.

13. $y = 2x^3 - 1$

    Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in at most one place.



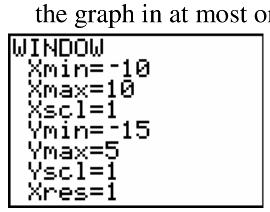


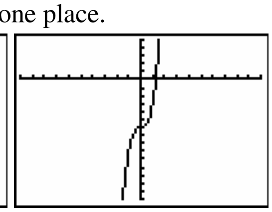
Another way of showing that a function is one-to-one is to assume that you have two equal y-values $(f(a) = f(b))$ and show that they must have come from the same x-value $(a = b)$.

$f(a) = f(b) \Rightarrow 2a^3 - 1 = 2b^3 - 1 \Rightarrow$
$2a^3 = 2b^3 \Rightarrow a^3 = b^3 \Rightarrow \sqrt[3]{a^3} = \sqrt[3]{b^3} \Rightarrow a = b$
So, the function is one-to-one.

14. $y = 3x^3 - 6$

    Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in at most one place.





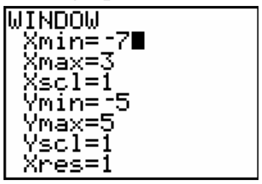
Another way of showing that a function is one-to-one is to assume that you have two equal y-values $\left(f\left(a\right)=f\left(b\right)\right)$ and show that they must have come from the same x-value $\left(a=b\right)$.

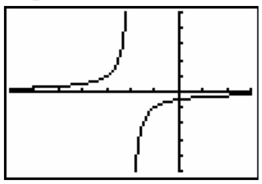
$f\left(a\right)=f\left(b\right)\Rightarrow 3a^3-6=3b^3-6\Rightarrow$
$3a^3=3b^3\Rightarrow a^3=b^3\Rightarrow \sqrt[3]{a^3}=\sqrt[3]{b^3}\Rightarrow a=b$

So, the function is one-to-one.

15. $y=-\frac{1}{x+2}$

Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in at most one place.





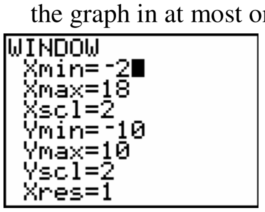
We could also show that $f\left(a\right)=f\left(a\right)$ implies $a=b$.

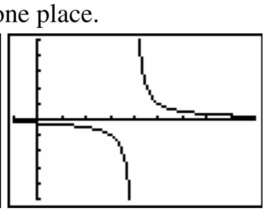
$f\left(a\right)=f\left(b\right)\Rightarrow -\frac{1}{a+2}=-\frac{1}{b+2}\Rightarrow$
$b+2=a+2\Rightarrow b=a$

So, the function is one-to-one.

16. $y=\frac{4}{x-8}$

Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in at most one place.





We could also show that $f\left(a\right)=f\left(a\right)$ implies $a=b$.

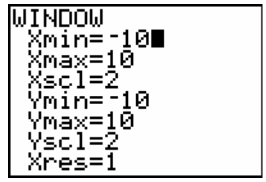
$f\left(a\right)=f\left(b\right)\Rightarrow \frac{4}{a-8}=\frac{4}{b-8}\Rightarrow$
$4(b-8)=4(a-8)\Rightarrow b-8=a-8\Rightarrow a=b$

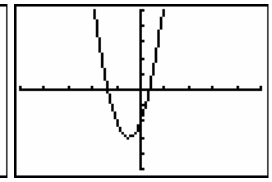
So, the function is one-to-one.

17. $y=2(x+1)^2-6$

Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in two places, except a horizontal line through the vertex.

For example, $f(0)=2(0+1)^2-6=-4$ and $f(-2)=2(-2+1)^2-6=-4$





So, the function is not one-to-one.

18. For a constant function defined over the set of real numbers, $f(x)=c$ for all values of c. Therefore, the function is not one-to-one.

19. In order for a function to have an inverse, it must be one-to-one.

20. If f and g are inverses, then $(f\circ g)(x)=\underline{x}$, and $\underline{(g\circ f)(x)}=x$.

21. The domain of f is equal to the range of f^{-1}, and the range of f is equal to the domain of f^{-1}.

22. If the point (a, b) lies on the graph of f, and f has an inverse, then the point $\underline{(b, a)}$ lies on the graph of f^{-1}.

23. It is false that if $f\left(x\right)=x^2$, then $f^{-1}\left(x\right)=\sqrt{x}$ because f is not a one-to-one function and, hence, does not have an inverse.

24. If the function f has an inverse, then the graph of f^{-1} may be obtained by reflecting the graph of f across the line with equation $\underline{y=x}$.

25. If a function f has an inverse and $f\left(-3\right)=6$, then $f^{-1}\left(6\right)=\underline{-3}$.

26. If $f\left(-4\right)=16$ and $f\left(4\right)=16$, then f does not have an inverse because it is not one-to-one.

27. Answers will vary. A polynomial of even degree has end behavior pointing in the same direction. This would indicate that the function would not pass the horizontal line test and cannot have an inverse.

28. Answers will vary. A polynomial of odd degree has end behavior pointing in opposite directions. Depending on whether or not the function has turning points, the function may or may not be one-to-one. If the function has turning points, then it is not one-to one. If it does not have turning points, then it is not one-to-one.

29. The inverse operation of tying your shoelaces would be untying your shoelaces, since untying "undoes" tying.

30. The inverse operation of starting a car would be stopping a car, since stopping "undoes" starting.

31. The inverse operation of entering a room would be leaving a room, since leaving "undoes" entering.

32. The inverse operation of climbing the stairs would be descending the stairs, since descending "undoes" climbing.

33. The inverse operation of screwing in a light bulb would be unscrewing the light bulb.

34. The inverse operation of filling a cup would be emptying a cup, since emptying "undoes" filling.

35. For each point (x, y) for the first function, there is a point (y, x) for the second function, so $f(x)$ and $g(x)$ are inverses of each other.

36. The point $(-2,-8)$ is on $f(x)$, but the point $(-8,-2)$ is not on $g(x)$ (there are other examples), so the functions are not inverses of each other.

37. The point $(3,5)$ is on $f(x)$, but the point $(5,3)$ is not on $g(x)$ (there is another example), so the functions are not inverses of each other.

38. These functions are inverses since their graphs are symmetric with respect to the line $y = x$.

39. These functions are inverses since their graphs are symmetric with respect to the line $y = x$.

40. These functions are not inverses since their graphs are not symmetric with respect to the line $y = x$.

41. $f(x) = 2x+4, g(x) = \frac{1}{2}x-2$

$(f \circ g)(x) = 2\left(\frac{1}{2}x-2\right)+4 = x-4+4 = x$

$(g \circ f)(x) = \frac{1}{2}(2x+4)-2 = x+2-2 = x$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

42. $f(x) = 3x+9, g(x) = \frac{1}{3}x-3$

$(f \circ g)(x) = 3\left(\frac{1}{3}x-3\right)+9 = x-9+9 = x$

$(g \circ f)(x) = \frac{1}{3}(3x+9)-3 = x+3-3 = x$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

43. $f(x) = -3x+12, g(x) = -\frac{1}{3}x-12$

$$(f \circ g)(x) = -3\left(-\frac{1}{3}x-12\right)+12 = x+36+12 = x+48$$

Since $(f \circ g)(x) \neq x$, the functions are not inverses. It is not necessary to check $(g \circ f)(x)$.

44. $f(x) = -4x+2, g(x) = -\frac{1}{4}x-2$

$$(f \circ g)(x) = -4\left(-\frac{1}{4}x-2\right)+2 = x+8+2 = x+10$$

Since $(f \circ g)(x) \neq x$, the functions are not inverses. It is not necessary to check $(g \circ f)(x)$.

45. $f(x) = \frac{x+1}{x-2},\ g(x) = \frac{2x+1}{x-1}$

$$(f \circ g)(x) = \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} = \frac{\frac{2x+1+x-1}{x-1}}{\frac{2x+1-2(x-1)}{x-1}} = \frac{3x}{3} = x$$

$$(g \circ f)(x) = \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1} = \frac{\frac{2x+2}{x-2}+1}{\frac{x+1-(x-2)}{x-2}} = \frac{\frac{2x+2+x-2}{x-2}}{\frac{x+1-(x-2)}{x-2}} = \frac{3x}{3} = x$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

46. $f(x) = \frac{x-3}{x+4},\ g(x) = \frac{4x+3}{1-x}$

$$(f \circ g)(x) = \frac{\frac{4x+3}{1-x}-3}{\frac{4x+3}{1-x}+4} = \frac{\frac{4x+3-3(1-x)}{1-x}}{\frac{4x+3+4(1-x)}{x-1}} = \frac{4x+3-3+3x}{4x+3+4-4x} = \frac{7x}{7} = x$$

$$(g \circ f)(x) = \frac{4\left(\frac{x-3}{x+4}\right)+3}{1-\frac{x-3}{x+4}} = \frac{\frac{4x-12}{x+4}+3}{\frac{x+4-(x-3)}{x+4}}$$
$$= \frac{\frac{4x-12+3(x+4)}{x+4}}{\frac{7}{x+4}} = \frac{\frac{4x-12+3x+12}{x+4}}{\frac{7}{x+4}}$$
$$= \frac{7x}{7} = x$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

47. $f(x) = \frac{2}{x+6}$, $g(x) = \frac{6x+2}{x}$

$$(f \circ g)(x) = f[g(x)] = \frac{2}{\frac{6x+2}{x}+6} = \frac{2}{\frac{6x+2+6x}{x}}$$
$$= \frac{2}{1} \cdot \frac{x}{12x+2} = \frac{2x}{12x+2} = \frac{x}{6x+1} \neq x$$

Since $(f \circ g)(x) \neq x$, the functions are not inverses. It is not necessary to check $(g \circ f)(x)$.

48. $f(x) = \frac{-1}{x+1}$, $g(x) = \frac{1-x}{x}$

$$(f \circ g)(x) = \frac{-1}{\frac{1-x}{x}+1} = \frac{-1}{\frac{1-x+x}{x}}$$
$$= \frac{-1}{\frac{1}{x}} = -1 \cdot \frac{x}{1} = -x \neq x$$

Since $(f \circ g)(x) \neq x$, the functions are not inverses. It is not necessary to check $(g \circ f)(x)$.

49. $f(x) = x^2 + 3$, domain $[0, \infty)$;
$g(x) = \sqrt{x-3}$, domain $[3, \infty)$

$$(f \circ g)(x) = f\left(\sqrt{x-3}\right) = \left(\sqrt{x-3}\right)^2 + 3 = x$$
$$(g \circ f)(x) = g\left(x^2+3\right) = \sqrt{x^2+3-3}$$
$$= \sqrt{x^2} = |x| = x \text{ for } [0, \infty)$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

50. $f(x) = \sqrt{x+8}$, domain $[-8, \infty)$;
$g(x) = x^2 - 8$, domain $[0, \infty)$

$$(f \circ g)(x) = \sqrt{x^2-8+8} = \sqrt{x^2} = x \text{ for } [0, \infty)$$
$$(g \circ f)(x) = \left(\sqrt{x+8}\right)^2 - 8$$
$$= x + 8 - 8 = x \text{ for } [-8, \infty)$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

51. Since each y-value corresponds to only one x-value, this function is one-to one and has an inverse. The inverse is: $\{(6,-3),(1,2),(8,5)\}$.

52. Since each y-value corresponds to only one x-value, this function is one-to one and has an inverse. The inverse is:
$\left\{(-1,3),(0,5),(5,0),\left(\frac{2}{3},4\right)\right\}$.

53. Since the y-value -3 corresponds to two different x-values, this function is not one-to-one.

54. Since the y-value -4 (as well as -8) corresponds to two different x-values, this function is not one-to-one.

55. $y = 3x - 4$

The function, $f(x) = 3x - 4$, is one-to-one.

(a) *Step 1*: Interchange x and y: $x = 3y - 4$

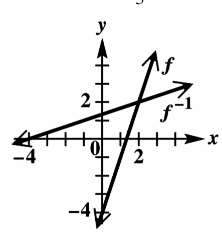
Step 2: Solve for y.

$x = 3y - 4 \Rightarrow x + 4 = 3y \Rightarrow \frac{x+4}{3} = y \Rightarrow$
$y = \frac{x+4}{3} = \frac{1}{3}x + \frac{4}{3}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$

(b) The graph of the original function, $f(x) = 3x - 4$, is a line with slope 3 and y-intercept –4. Since $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$, the graph of the inverse function is a line with slope $\frac{1}{3}$ and y-intercept $\frac{4}{3}$.



(c) For both $f(x)$ and $f^{-1}(x)$, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.

56. $y = 4x - 5$

The function, $f(x) = 4x - 5$, is one-to-one.

(a) *Step 1*: Interchange x and y: $x = 4y - 5$

Step 2: Solve for y.

$x = 4y - 5 \Rightarrow x + 5 = 4y \Rightarrow \frac{x+5}{4} = y \Rightarrow$
$y = \frac{x+5}{4} = \frac{1}{4}x + \frac{5}{4}$

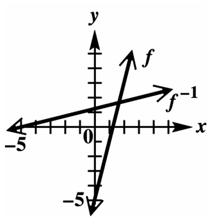
(*continued on next page*)

(*continued from page 393*)

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x)=\frac{1}{4}x+\frac{5}{4}$

(b) The graph of the original function, $f(x)=4x-5,$ is a line with slope 4 and y-intercept –5. Since $f^{-1}(x)=\frac{1}{4}x+\frac{5}{4},$ the graph of the inverse function is a line with slope $\frac{1}{4}$ and y-intercept $\frac{5}{4}$.



(c) For both $f(x)$ and $f^{-1}(x)$, the domain is (–∞, ∞) and the range is (–∞, ∞).

57. $f(x)=-4x+3$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y=-4x+3$

$x=-4y+3$

Step 2: Solve for y.

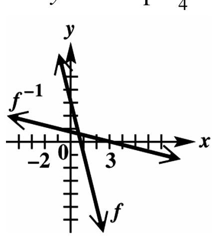
$x=-4y+3\Rightarrow x-3=-4y\Rightarrow\frac{x-3}{-4}=y\Rightarrow$

$y=\frac{x-3}{-4}=-\frac{1}{4}x+\frac{3}{4}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x)=-\frac{1}{4}x+\frac{3}{4}$

(b) The graph of the original function, $f(x)=-4x+3,$ is a line with slope –4 and y-intercept 3. Since $f^{-1}(x)=-\frac{1}{4}x+\frac{3}{4},$ the graph of the inverse function is a line with slope $-\frac{1}{4}$ and y-intercept $\frac{3}{4}$.



(c) For both $f(x)$ and $f^{-1}(x)$, the domain is (–∞, ∞) and the range is (–∞, ∞).

58. $f(x)=-6x-8$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y=-6x-8\Rightarrow x=-6y-8$

Step 2: Solve for y.

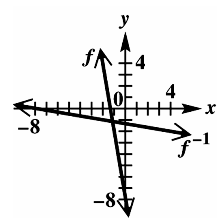
$x=-6y-8\Rightarrow x+8=-6y\Rightarrow\frac{x+8}{-6}=y\Rightarrow$

$y=\frac{x+8}{-6}=-\frac{1}{6}x-\frac{4}{3}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x)=-\frac{1}{6}x-\frac{4}{3}$

(b) The graph of the original function, $f(x)=-6x-8,$ is a line with slope –6 and y-intercept –8. Since $f^{-1}(x)=-\frac{1}{6}x-\frac{4}{3},$ the graph of the inverse function is a line with slope $-\frac{1}{6}$ and y-intercept $-\frac{4}{3}$.



(c) For both $f(x)$ and $f^{-1}(x)$, the domain is (–∞, ∞) and the range is (–∞, ∞).

59. $f(x)=x^3+1$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y=x^3+1\Rightarrow x=y^3+1$

Step 2: Solve for y.

$x=y^3+1\Rightarrow x-1=y^3\Rightarrow$

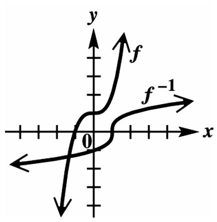
$\sqrt[3]{x-1}=y\Rightarrow y=\sqrt[3]{x-1}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x)=\sqrt[3]{x-1}$

(b) Tables of ordered pairs will be helpful in drawing the graphs of these functions.

x	$f(x)$	x	$f^{-1}(x)$
–2	–7	–7	–2
–1	0	0	–1
0	1	1	0
1	2	2	1
2	9	9	2



(c) For both $f(x)$ and $f^{-1}(x)$, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.

60. $f(x) = -x^3 - 2$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with *y* and interchange *x* and *y*.

$y = -x^3 - 2 \Rightarrow x = -y^3 - 2$

Step 2: Solve for *y*.

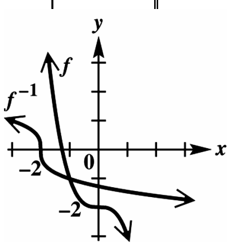
$x = -y^3 - 2 \Rightarrow x + 2 = -y^3 \Rightarrow$
$-x - 2 = y^3 \Rightarrow \sqrt[3]{-x-2} = y \Rightarrow$
$y = \sqrt[3]{-x-2}$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{-x-2}$

(b) Tables of ordered pairs will be helpful in drawing the graphs of these functions.

x	$f(x)$	x	$f^{-1}(x)$
–2	6	6	–2
–1	–1	–1	–1
0	–2	–2	0
1	–3	–3	1
2	–10	–10	2



(c) For both $f(x)$ and $f^{-1}(x)$, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.

61. $y = x^2$

This is not a one-to-one function since two different *x*-values can correspond to the same *y*-value ($2^2 = 4$ and $(-2)^2 = 4$, for example), so this function is not one-to-one. Thus, the function has no inverse function.

62. $y = -x^2 + 2$

This is not a one-to-one function since two different *x*-values can correspond to the same *y*-value ($-2^2 + 2 = -4 + 2 = -2$ and $-(-2)^2 + 2 = -4 + 2 = -2,$ for example), so this function is not one-to-one. Thus, the function has no inverse function.

63. $y = \frac{1}{x}$

The function, $f(x) = \frac{1}{x}$, is one-to-one.

(a) *Step 1*: Interchange *x* and *y*.

$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Step 2: Solve for *y*.

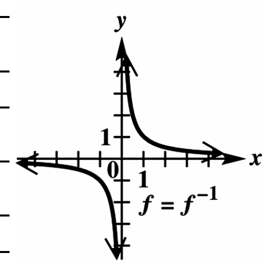
$x = \frac{1}{y} \Rightarrow xy = 1 \Rightarrow y = \frac{1}{x}$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{1}{x}$

(b) Tables of ordered pairs will be helpful in drawing the graph of this function $\left(\text{in this case, } f(x) = f^{-1}(x)\right)$.

x	$f(x) = f^{-1}(x)$
–2	$-\frac{1}{2}$
–1	–1
$-\frac{1}{2}$	–2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



(c) For both $f(x)$ and $f^{-1}(x)$, the domain and range are both $(-\infty, 0) \cup (0, \infty)$.

64. $y = \frac{4}{x}$

The function, $f(x) = \frac{4}{x}$, is one-to-one.

(a) *Step 1*: Interchange *x* and *y*.

$y = \frac{4}{x} \Rightarrow x = \frac{4}{y}$

Step 2: Solve for *y*.

$x = \frac{4}{y} \Rightarrow xy = 4 \Rightarrow y = \frac{4}{x}$

Step 3: Replace *y* with $f^{-1}(x)$.

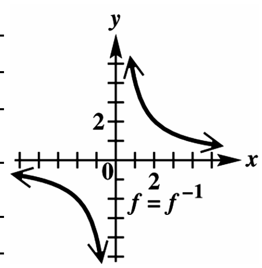
$f^{-1}(x) = \frac{4}{x}$

(*continued on next page*)

(*continued from page 395*)

(b) Tables of ordered pairs will be helpful in drawing the graph of this function $\left(\text{in this case } f(x) = f^{-1}(x)\right)$.

x	$f(x) = f^{-1}(x)$
-2	-2
-1	-4
$-\frac{1}{2}$	-8
$\frac{1}{2}$	8
1	8
2	2



(c) For both $f(x)$ and $f^{-1}(x)$, the domain and range are both $(-\infty, 0) \cup (0, \infty)$.

65. $f(x) = \frac{1}{x-3}$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y = \frac{1}{x-3} \Rightarrow x = \frac{1}{y-3}$

Step 2: Solve for y.

$x = \frac{1}{y-3} \Rightarrow x(y-3) = 1 \Rightarrow xy - 3x = 1 \Rightarrow$
$xy = 1 + 3x \Rightarrow y = \frac{1+3x}{x}$

Step 3: Replace y with $f^{-1}(x)$.

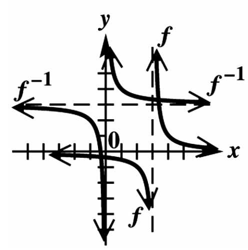
$f^{-1}(x) = \frac{1+3x}{x}$

(b) To graph $f(x) = \frac{1}{x-3}$, we can determine that there are no x-intercepts. The y-intercept is $f(0) = \frac{1}{0-3} = -\frac{1}{3}$. There is a vertical aymptote when the denominator is zero, $x - 3 = 0$, which implies $x = 3$ is the vertical asymptote. Also, the horizontal asymptote is $y = 0$ since the degree of the numerator is less than the denominator. Examining the following intervals, we have test points which will be helpful in drawing the graph of *f* as well as f^{-1}.

Interval	Test Point	Value of *f*(*x*)	Sign of *f*(*x*)	Graph Above or Below *x*-Axis
$(-\infty, 3)$	2	-1	Negative	Below
$(3, \infty)$	4	1	Positive	Above

Plot the vertical asymptote, *y*-intercept, and test points with a smooth curve to get the graph of *f*.

To graph $f^{-1}(x) = \frac{1+3x}{x}$, we can determine that the x-intercept occurs when $1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$. There is no y-intercept since 0 is not in the domain f^{-1}. There is a vertical aymptote when the denominator is zero, namely $x = 0$. Also, since the degree of the numerator is the same as the denominator, the horizontal asymptote is $y = \frac{3}{1} = 3$. Using this information along with the points $(-1, 2)$ and $(1, 4)$, we can sketch f^{-1}.



(c) Domain of $f =$ range of $f^{-1} = (-\infty, 3) \cup (3, \infty)$; Domain of $f^{-1} =$ range of $f = (-\infty, 0) \cup (0, \infty)$

66. $f(x) = \frac{1}{x+2}$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y = \frac{1}{x+2} \Rightarrow x = \frac{1}{y+2}$

Step 2: Solve for y.

$x = \frac{1}{y+2} \Rightarrow x(y+2) = 1 \Rightarrow xy + 2x = 1 \Rightarrow$
$xy = 1 - 2x \Rightarrow y = \frac{1-2x}{x}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \frac{1-2x}{x}$

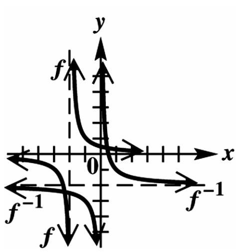
(b) To graph $f(x) = \frac{1}{x+2}$, we can determine that there are no x-intercepts. The y-intercept is $f(0) = \frac{1}{0+2} = \frac{1}{2}$. There is a vertical aymptote when the denominator is zero, $x + 2 = 0$, which implies $x = -2$ is the vertical asymptote.

Also, the horizontal asymptote is $y = 0$ since the degree of the numerator is less than the denominator. Examining the following intervals, we have test points which will be helpful in drawing the graph of *f* as well as f^{-1}.

Interval	Test Point	Value of *f*(*x*)	Sign of *f*(*x*)	Graph Above or Below *x*-Axis
$(-\infty,-2)$	–3	–1	Negative	Below
$(-2,\infty)$	–1	1	Positive	Above

Plot the vertical asymptote, *y*-intercept, and test points with a smooth curve to get the graph of *f*.

To graph $f^{-1}(x)=\frac{1-2x}{x}$, we can determine that the *x*-intercept occurs when $1-2x=0 \Rightarrow x=\frac{1}{2}$. There is no *y*-intercept since 0 is not in the domain f^{-1}. There is a vertical aymptote when the denominator is zero, namely $x = 0$. Also, since the degree of the numerator is the same as the denominator, the horizontal asymptote is $y=\frac{-2}{1}=-2$. Using this information along with the points $(-1,-3)$ and $(1,-1)$, we can sketch f^{-1}.



(c) Domain of f = range of $f^{-1}=(-\infty,-2)\cup(-2,\infty)$;

Domain of f^{-1} = range of $f=(-\infty,0)\cup(0,\infty)$

67. $f(x)=\frac{x+1}{x-3}$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with *y* and interchange *x* and *y*.

$y=\frac{x+1}{x-3} \Rightarrow x=\frac{y+1}{y-3}$

Step 2: Solve for *y*.

$x=\frac{y+1}{y-3} \Rightarrow x(y-3)=y+1 \Rightarrow$
$xy-3x=y+1 \Rightarrow xy-y=3x+1 \Rightarrow$
$y(x-1)=3x+1 \Rightarrow y=\frac{3x+1}{x-1}$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x)=\frac{3x+1}{x-1}$

(b) To graph $f(x)=\frac{x+1}{x-3}$, find the *x*-intercept: $\frac{x+1}{x-3}=0 \Rightarrow x+1=0 \Rightarrow x=-1$. The *y*-intercept is $f(0)=\frac{0+1}{0-3}=-\frac{1}{3}$. There is a vertical aymptote when the denominator is zero, $x-3=0 \Rightarrow x=3$ is the vertical asymptote. Since the degree of the numerator equals the degree of the denominator, the horizontal asymptote is $y=\frac{1}{1}=1$. Examining the following intervals, we have test points which will be helpful in drawing the graph of *f* as well as f^{-1}.

Interval	Test Point	Value of *f*(*x*)	Sign of *f*(*x*)	Graph Above or Below *x*-Axis
$(-\infty,-\frac{1}{3})$	–3	$\frac{1}{3}$	Positive	Above
$(-\frac{1}{3},1)$	$\frac{1}{2}$	$-\frac{3}{5}$	Negative	Below
$(1,\infty)$	2	–3	Negative	Below

Plot the vertical asymptote, *y*-intercept, and test points with a smooth curve to get the graph of *f*.

To graph $f^{-1}(x)=\frac{3x+1}{x-1}$ we can determine that the *x*-intercept occurs when $3x+1=0 \Rightarrow x=-\frac{1}{3}$. The *y*-intercept is $f(0)=\frac{3(0)+1}{0-1}=-1$. There is a vertical aymptote when the denominator is zero, $x-1=0 \Rightarrow x=1$. Also, since the degree of the numerator is the same as the denominator, the horizontal asymptote is $y=\frac{3}{1}=3$. Examining the following intervals, we have test points which will be helpful in drawing the graph of *f* as well as f^{-1}.

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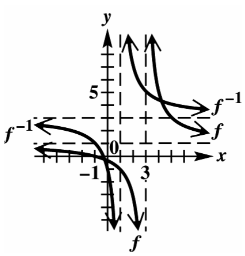
(*continued from page 397*)

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$\left(-\infty, -\frac{1}{3}\right)$	-2	$\frac{5}{3}$	Positive	Above
$\left(-\frac{1}{3}, 1\right)$	$\frac{1}{2}$	-5	Negative	Below
$(1, \infty)$	2	7	Positive	Above

Plot the vertical asymptote, y-intercept, and test points with a smooth curve to get the graph of f^{-1}.



(c) Domain of $f =$ range of $f^{-1} = (-\infty, 3) \cup (3, \infty)$;

Domain of $f^{-1} =$ range of $f = (-\infty, 1) \cup (1, \infty)$

68. $f(x) = \frac{x+2}{x-1}$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y = \frac{x+2}{x-1} \Rightarrow x = \frac{y+2}{y-1}$

Step 2: Solve for y.

$x = \frac{y+2}{y-1} \Rightarrow x(y-1) = y + 2 \Rightarrow$
$xy - x = y + 2 \Rightarrow xy - y = x + 2 \Rightarrow$
$y(x-1) = x + 2 \Rightarrow y = \frac{x+2}{x-1}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \frac{x+2}{x-1}$. So $f(x) = \frac{x+2}{x-1} = f^{-1}(x)$

(b) To graph $f(x) = \frac{x+2}{x-1}$, find the x-intercept: $\frac{x+2}{x-1} = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$.

The y-intercept is $f(0) = \frac{0+2}{0-1} = -2$.

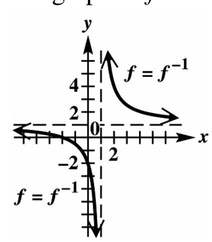
There is a vertical aymptote when the denominator is zero, $x - 1 = 0 \Rightarrow x = 1$ is the vertical asymptote.

Since the degree of the numerator equals the degree of the denominator, the horizontal asymptote is $y = \frac{1}{1} = 1$.

Examining the following intervals, we have test points which will be helpful in drawing the graph of f as well as f^{-1}.

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -2)$	-4	$\frac{2}{5}$	Positive	Above
$(-2, 1)$	-1	$-\frac{1}{2}$	Negative	Below
$(1, \infty)$	2	4	Positive	Above

Plot the vertical asymptote, y-intercept, and test points with a smooth curve to get the graph of f.



(c) Domain of $f =$ range of $f^{-1} = (-\infty, 1) \cup (1, \infty)$;

Domain of $f^{-1} =$ range of $f = (-\infty, 1) \cup (1, \infty)$

69. $f(x) = \sqrt{6 + x}$

This function is one-to-one.

(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.

$y = \sqrt{6 + x} \Rightarrow x = \sqrt{6 + y}$

Step 2: Solve for y. In this problem we must consider that the range of f will be the domain of f^{-1}.

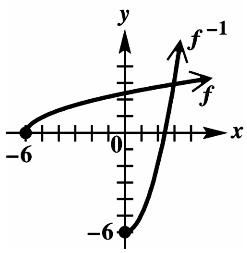
$x = \sqrt{6 + y}$
$x^2 = \left(\sqrt{6 + y}\right)^2$, for $x \geq 0$
$x^2 = 6 + y$, for $x \geq 0$
$x^2 - 6 = y$, for $x \geq 0$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = x^2 - 6$, for $x \geq 0$

(b) Tables of ordered pairs will be helpful in drawing the graphs of these functions.

x	$f(x)$	x	$f^{-1}(x)$
–6	0	0	–6
–5	1	1	–5
–2	2	2	–2
3	3	3	3



(c) Domain of f = range of f^{-1} = [–6, ∞);
Range of f = domain of f^{-1} = [0, ∞)

70. $f(x) = -\sqrt{x^2 - 16},\ x \geq 4$
This function is one-to-one.

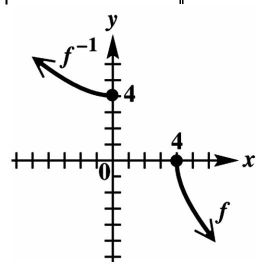
(a) *Step 1*: Replace $f(x)$ with y and interchange x and y.
$y = -\sqrt{x^2 - 16} \Rightarrow x = -\sqrt{y^2 - 16}$
Step 2: Solve for y. In this problem we must consider that the range of f will be the domain of f^{-1}. We must also consider that the domain of f will be the range of f^{-1}.

$$x = -\sqrt{y^2 - 16}$$
$$x^2 = \left(-\sqrt{y^2 - 16}\right)^2, \text{ for } x \leq 0$$
(restriction due to range of f)
$$x^2 = y^2 - 16, \text{ for } x \leq 0$$
$$x^2 + 16 = y^2, \text{ for } x \leq 0$$
$$\sqrt{x^2 + 16} = y, \text{ for } x \leq 0$$
(positive square root due to the domain of f)

Step 3: Replace y with $f^{-1}(x)$.
$f^{-1}(x) = \sqrt{x^2 + 16}$, for $x \leq 0$

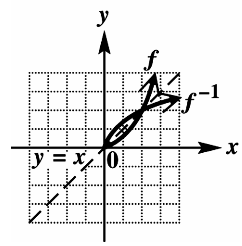
(b) Tables of ordered pairs will be helpful in drawing the graphs of these functions.

x	$f(x)$	x	$f^{-1}(x)$
4	0	0	4
5	−3	−3	5
6	$-2\sqrt{5} \approx -4.5$	$-2\sqrt{5} \approx -4.5$	6
7	$-\sqrt{33} \approx -5.7$	$-\sqrt{33} \approx -5.7$	7

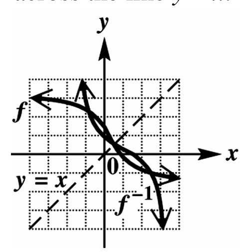


(c) Domain of f = range of f^{-1} = [4, ∞);
Range of f = domain of f^{-1} = (–∞, 0]

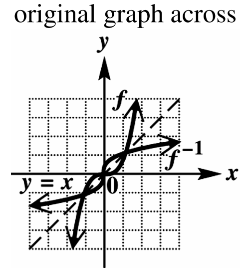
71. Draw the mirror image of the original graph across the line $y = x$.



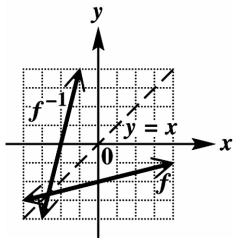
72. Draw the mirror image of the original graph across the line $y = x$.



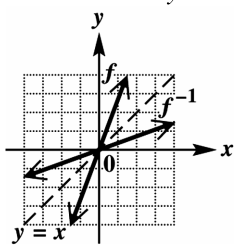
73. Carefully draw the mirror image of the original graph across the line $y = x$.



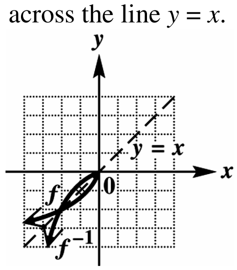
74. Draw the mirror image of the original graph across the line $y = x$.



75. Draw the mirror image of the original graph across the line $y = x$.



76. Draw the mirror image of the original graph across the line $y = x$.



77. To find $f^{-1}(4)$, find the point with y-coordinate equal to 4. That point is (4, 4). The graph of f^{-1} contains (4, 4). Hence $f^{-1}(4) = 4$.

78. To find $f^{-1}(2)$, find the point with y-coordinate equal to 2. That point is (3, 2). The graph of f^{-1} contains (2, 3). Hence $f^{-1}(2) = 3$.

79. To find $f^{-1}(0)$, find the point with y-coordinate equal to 0. That point is $(2,0)$. The graph of f^{-1} contains $(0,2)$. Hence $f^{-1}(0) = 2$.

80. To find $f^{-1}(-2)$, find the point with y-coordinate equal to –2. That point is $(0,-2)$. The graph of f^{-1} contains $(-2,0)$. Hence $f^{-1}(-2) = 0$.

81. To find $f^{-1}(-3)$, find the point with y-coordinate equal to -3. That point is $(-2,-3)$. The graph of f^{-1} contains $(-3,-2)$. Hence $f^{-1}(-3) = -2$.

82. To find $f^{-1}(-4)$, find the point with y-coordinate equal to -4. That point is $(-4,-4)$. The graph of f^{-1} contains $(-4,-4)$. Hence $f^{-1}(-4) = -4$.

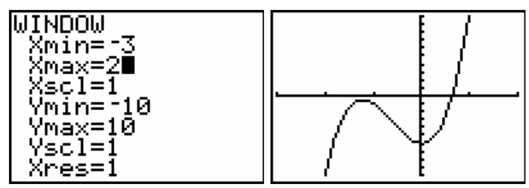
83. $f^{-1}(1000)$ represents the number of dollars required to build 1000 cars.

84. $f^{-1}(5)$ represents the radius of a sphere with a volume of 5 cu inches.

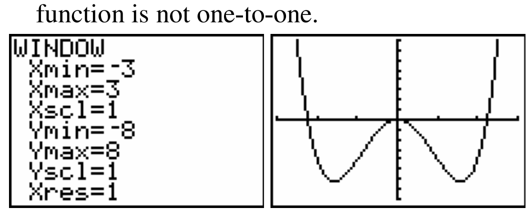
85. If a line has slope a, the slope of its reflection in the line $y = x$ will be reciprocal of a, which is $\frac{1}{a}$.

86. $f^{-1}(f(2)) = f^{-1}(3) = 2.$

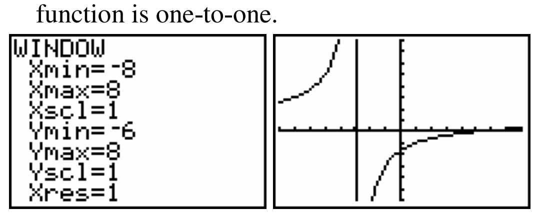
87. The horizontal line test will show that this function is not one-to-one.



88. The horizontal line test will show that this function is not one-to-one.



89. The horizontal line test will show that this function is one-to-one.



Find the equation of f^{-1}.

Step 1: Replace $f(x) = \frac{x-5}{x+3}$ with y and interchange x and y.

$y = \frac{x-5}{x+3} \Rightarrow x = \frac{y-5}{y+3}$

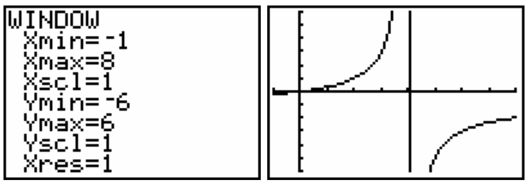
Step 2: Solve for *y*.

$x = \frac{y-5}{y+3} \Rightarrow x(y+3) = y-5 \Rightarrow$
$xy + 3x = y - 5 \Rightarrow xy - y = -5 - 3x \Rightarrow$
$y(x-1) = -5 - 3x \Rightarrow y = \frac{-5-3x}{x-1}$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{-5-3x}{x-1}$

90. The horizontal line test will show that this function is one-to-one.



Find the equation of f^{-1}.

Step 1: Replace $f(x) = \frac{-x}{x-4}$ with *y* and interchange *x* and *y*.

$y = \frac{-x}{x-4} \Rightarrow x = \frac{-y}{y-4}$

Step 2: Solve for *y*.

$x = \frac{-y}{y-4} \Rightarrow x(y-4) = -y \Rightarrow xy - 4x = -y$
$xy + y = 4x \Rightarrow y(x+1) = 4x \Rightarrow y = \frac{4x}{x+1}$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{4x}{x+1}$

91. Given $f(x) = 3x - 2$, find $f^{-1}(x)$.

Step 1: Replace $f(x) = 3x - 2$ with *y* and interchange *x* and *y*.

$y = 3x - 2 \Rightarrow x = 3y - 2$

Step 2: Solve for *y*.

$x = 3y - 2 \Rightarrow x + 2 = 3y \Rightarrow \frac{x+2}{3} = y$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{x+2}{3}$

37; $f^{-1}(37) = \frac{37+2}{3} = \frac{39}{3} = 13$; M	25; $f^{-1}(25) = \frac{25+2}{3} = \frac{27}{3} = 9$; I
19; $f^{-1}(19) = \frac{19+2}{3} = \frac{21}{3} = 7$; G	61; $f^{-1}(61) = \frac{61+2}{3} = \frac{63}{3} = 21$; U
13; $f^{-1}(13) = \frac{13+2}{3} = \frac{15}{3} = 5$; E	34; $f^{-1}(34) = \frac{34+2}{3} = \frac{36}{3} = 12$; L
22; $f^{-1}(22) = \frac{22+2}{3} = \frac{24}{3} = 8$; H	1; $f^{-1}(1) = \frac{1+2}{3} = \frac{3}{3} = 1$; A
55; $f^{-1}(55) = \frac{55+2}{3} = \frac{57}{3} = 19$; S	1; $f^{-1}(1) = \frac{1+2}{3} = \frac{3}{3} = 1$; A
52; $f^{-1}(52) = \frac{52+2}{3} = \frac{54}{3} = 18$; R	52; $f^{-1}(52) = \frac{52+2}{3} = \frac{54}{3} = 18$; R
25; $f^{-1}(25) = \frac{25+2}{3} = \frac{27}{3} = 9$; I	64; $f^{-1}(64) = \frac{64+2}{3} = \frac{66}{3} = 22$; V
13; $f^{-1}(13) = \frac{13+2}{3} = \frac{15}{3} = 5$; E	10; $f^{-1}(10) = \frac{10+2}{3} = \frac{12}{3} = 4$; D

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92. Given $f(x) = 2x - 9$, find $f^{-1}(x)$.

Step 1: Replace $f(x) = 2x - 9$ with *y* and interchange *x* and *y*.

$y = 2x - 9 \Rightarrow x = 2y - 9$

Step 2: Solve for *y*.

$x = 2y - 9 \Rightarrow x + 9 = 2y \Rightarrow \frac{x+9}{2} = y$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{x+9}{2}$

−5; $f^{-1}(-5) = \frac{-5+9}{2} = \frac{4}{2} = 2$; B	9; $f^{-1}(9) = \frac{9+9}{2} = \frac{18}{2} = 9$; I
5; $f^{-1}(5) = \frac{5+9}{2} = \frac{14}{2} = 7$; G	5; $f^{-1}(5) = \frac{5+9}{2} = \frac{14}{2} = 7$; G
9; $f^{-1}(9) = \frac{9+9}{2} = \frac{18}{2} = 9$; I	27; $f^{-1}(27) = \frac{27+9}{2} = \frac{36}{2} = 18$; R
15; $f^{-1}(15) = \frac{15+9}{2} = \frac{24}{2} = 12$; L	29; $f^{-1}(29) = \frac{29+9}{2} = \frac{38}{2} = 19$; S
−1; $f^{-1}(-1) = \frac{-1+9}{2} = \frac{8}{2} = 4$; D	21; $f^{-1}(21) = \frac{21+9}{2} = \frac{30}{2} = 15$; O
19; $f^{-1}(19) = \frac{19+9}{2} = \frac{28}{2} = 14$; N	31; $f^{-1}(31) = \frac{31+9}{2} = \frac{40}{2} = 20$; T
−3; $f^{-1}(-3) = \frac{-3+9}{2} = \frac{6}{2} = 3$; C	27; $f^{-1}(27) = \frac{27+9}{2} = \frac{36}{2} = 18$; R
41; $f^{-1}(41) = \frac{41+9}{2} = \frac{50}{2} = 25$; Y	

The message is BIG GIRLS DON'T CRY.

93. Given $f(x) = x^3 - 1$, we have the following.

S = 19; $f(19) = 19^3 - 1 = 6859 - 1 = 6858$	E = 5; $f(5) = 5^3 = 125 - 1 = 124$
N = 14; $f(14) = 14^3 - 1 = 2744 - 1 = 2743$	D = 4; $f(4) = 4^3 - 1 = 64 - 1 = 63$
H = 8; $f(8) = 8^3 - 1 = 512 - 1 = 511$	E = 5; $f(5) = 5^3 - 1 = 125 - 1 = 124$
L = 12; $f(12) = 12^3 - 1 = 1728 - 1 = 1727$	P = 16; $f(16) = 16^3 - 1 = 4096 - 1 = 4095$

Given $f(x) = x^3 - 1$, find $f^{-1}(x)$.

Step 1: Replace $f(x) = x^3 - 1$ with y and interchange x and y.

$y = x^3 - 1 \Rightarrow x = y^3 - 1$

Step 2: Solve for y.

$x = y^3 - 1 \Rightarrow x + 1 = y^3 \Rightarrow \sqrt[3]{x+1} = y$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{x+1}$

94. Given $f(x) = (x+1)^3$, we have the following.

S = 19; $f(19) = (19+1)^3 = 20^3 = 8000$	A = 1; $f(1) = (1+1)^3 = 2^3 = 8$
I = 9; $f(9) = (9+1)^3 = 10^3 = 1000$	L = 12; $f(12) = (12+1)^3 = 13^3 = 2197$
O = 15; $f(15) = (15+1)^3 = 16^3 = 4096$	R = 18; $f(18) = (18+1)^3 = 19^3 = 6859$
B = 2; $f(2) = (2+1)^3 = 3^3 = 27$	E = 5; $f(5) = (5+1)^3 = 6^3 = 216$
W = 23; $f(23) = (23+1)^3 = 24^3 = 13{,}824$	A = 1; $f(1) = (1+1)^3 = 2^3 = 8$
R = 18; $f(18) = (18+1)^3 = 19^3 = 6859$	E = 5; $f(5) = (5+1)^3 = 6^3 = 216$

Given $f(x) = (x+1)^3$, find $f^{-1}(x)$.

Step 1: Replace $f(x) = (x+1)^3$ with y and interchange x and y.

$y = (x+1)^3 \Rightarrow x = (y+1)^3$

Step 2: Solve for y.

$x = (y+1)^3 \Rightarrow \sqrt[3]{x} = y + 1 \Rightarrow \sqrt[3]{x} - 1 = y$

Step 3: Replace y with $f^{-1}(x)$.

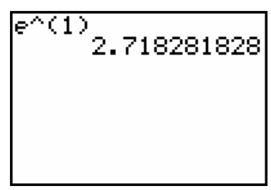
$f^{-1}(x) = \sqrt[3]{x} - 1$

95. Answers will vary. In the encoding/decoding process, you want to ensure that one value to be encoded will yield one encoded value (so the encoding relation itself must be a function). In order to ensure that one encoded value will yield only one decoded value, your decoding relation must also be a function. To ensure this, the original encoding function must be one-to one.

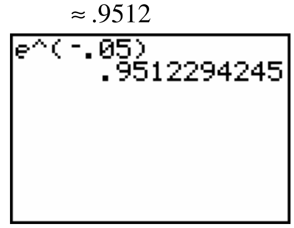
Section 4.2: Exponential Functions

Connections (page 427)

1. $e^1 \approx 1 + 1 + \frac{1^2}{2\cdot1} + \frac{1^3}{3\cdot2\cdot1} + \frac{1^4}{4\cdot3\cdot2\cdot1} + \frac{1^5}{5\cdot4\cdot3\cdot2\cdot1}$
$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \approx 2.717$



2. $e^{-.05} \approx 1 + (-.05) + \frac{(-.05)^2}{2\cdot1} + \frac{(-.05)^3}{3\cdot2\cdot1} + \frac{(-.05)^4}{4\cdot3\cdot2\cdot1} + \frac{(-.05)^5}{5\cdot4\cdot3\cdot2\cdot1}$
$= 1 - .05 + \frac{.0025}{2} - \frac{.000125}{6} + \frac{.00000625}{24} - \frac{.0000003125}{120}$
$\approx .9512$



3. $\frac{x^6}{6\cdot5\cdot4\cdot3\cdot2\cdot1}$

Exercises

1. $f(x)=3^x$
$f(2)=3^2=9$

2. $f(x)=3^x$
$f(3)=3^3=27$

3. $f(x)=3^x$
$f(-2)=3^{-2}=\frac{1}{3^2}=\frac{1}{9}$

4. $f(x)=3^x$
$f(-3)=3^{-3}=\frac{1}{3^3}=\frac{1}{27}$

5. $g(x)=\left(\frac{1}{4}\right)^x$
$g(2)=\left(\frac{1}{4}\right)^2=\frac{1}{16}$

6. $g(x)=\left(\frac{1}{4}\right)^x$
$g(3)=\left(\frac{1}{4}\right)^3=\frac{1}{64}$

7. $g(x)=\left(\frac{1}{4}\right)^x$
$g(-2)=\left(\frac{1}{4}\right)^{-2}=4^2=16$

8. $g(x)=\left(\frac{1}{4}\right)^x$
$g(-3)=\left(\frac{1}{4}\right)^{-3}=4^3=64$

9. $f(x)=3^x$
$f\left(\frac{3}{2}\right)=3^{3/2}\approx 5.196$

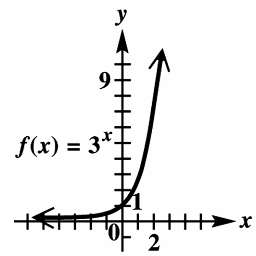
10. $g(x)=\left(\frac{1}{4}\right)^x$
$g\left(\frac{3}{2}\right)=\left(\frac{1}{4}\right)^{3/2}=\frac{1}{\left(\sqrt{4}\right)^3}=\frac{1}{2^3}=\frac{1}{8}$

11. $g(x)=\left(\frac{1}{4}\right)^x$
$g(2.34)=\left(\frac{1}{4}\right)^{2.34}\approx .039$

12. $f(x)=3^x$
$f(1.68)=3^{1.68}\approx 6.332$

13. The y-intercept of $f(x)=3^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

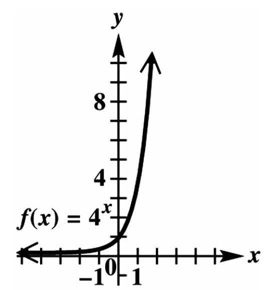
x	$f(x)$
-2	$\frac{1}{9}\approx .1$
-1	$\frac{1}{3}\approx .3$
$-\frac{1}{2}$	$\approx .6$
0	1
$\frac{1}{2}$	≈ 1.7
1	3
2	9



Plot these points and draw a smooth curve through them. This is an increasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one.

14. The y-intercept of $f(x)=4^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

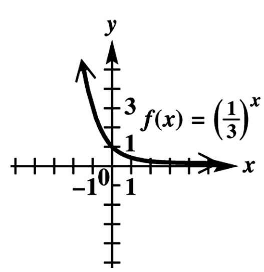
x	$f(x)$
-2	.0625
-1	.25
$-\frac{1}{2}$	.5
0	1
$\frac{1}{2}$	2
1	4
2	16



Plot these points and draw a smooth curve through them. This is an increasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one.

15. The y-intercept of $f(x)=\left(\frac{1}{3}\right)^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

x	$f(x)$
-2	9
-1	3
$-\frac{1}{2}$	≈ 1.7
0	1
$\frac{1}{2}$	$\approx .6$
1	$\frac{1}{3}\approx .3$
2	$\frac{1}{9}\approx .1$



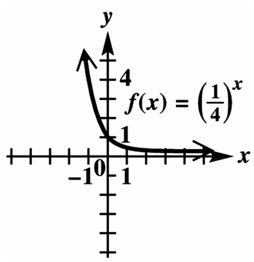
(*continued on next page*)

(*continued from page 403*)

Plot these points and draw a smooth curve through them. This is a decreasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one. Note: Since $f(x)=\left(\frac{1}{3}\right)^x=\left(3^{-1}\right)^x=3^{-x}$, the graph of $f(x)=\left(\frac{1}{3}\right)^x$ is the reflection of the graph of $f(x)=3^x$ (Exercise 13) about the y-axis.

16. The y-intercept of $f(x)=\left(\frac{1}{4}\right)^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

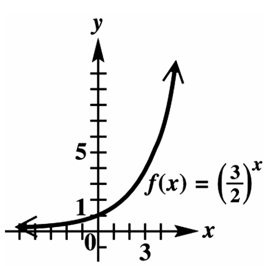
x	$f(x)$
-2	16
-1	4
$-\frac{1}{2}$	2
0	1
$\frac{1}{2}$	.5
1	.25
2	.0625



Plot these points and draw a smooth curve through them. This is a decreasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one. Note: Since $f(x)=\left(\frac{1}{4}\right)^x=\left(4^{-1}\right)^x=4^{-x}$, the graph of $f(x)=\left(\frac{1}{4}\right)^x$ is the reflection of the graph of $f(x)=4^x$ (Exercise 14) about the y-axis.

17. The y-intercept of $f(x)=\left(\frac{3}{2}\right)^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

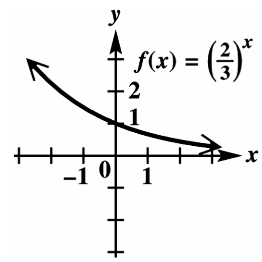
x	$f(x)$
-2	$\approx .4$
-1	$\approx .7$
$-\frac{1}{2}$	$\approx .8$
0	1
$\frac{1}{2}$	≈ 1.2
1	1.5
2	2.25



Plot these points and draw a smooth curve through them. This is an increasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one.

18. The y-intercept of $f(x)=\left(\frac{2}{3}\right)^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

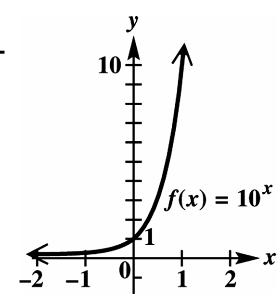
x	$f(x)$
-2	2.25
-1	1.5
$-\frac{1}{2}$	≈ 1.2
0	1
$\frac{1}{2}$	$\approx .8$
1	$\approx .7$
2	$\approx .4$



Plot these points and draw a smooth curve through them. This is a decreasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one. Note: Since $f(x)=\left(\frac{2}{3}\right)^x=\left[\left(\frac{3}{2}\right)^{-1}\right]^x=\left(\frac{3}{2}\right)^{-x}$, the graph of $f(x)=\left(\frac{2}{3}\right)^x$ is the reflection of the graph of $f(x)=\left(\frac{3}{2}\right)^x$ (Exercise 17) about the y-axis.

19. The y-intercept of $f(x)=10^x$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

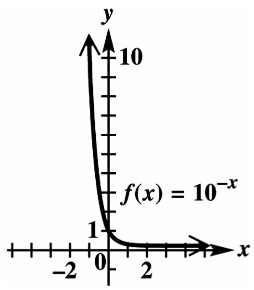
x	$f(x)$
-2	.01
-1	.1
$-\frac{1}{2}$	$\approx .3$
0	1
$\frac{1}{2}$	≈ 3.2
1	10
2	100



Plot these points and draw a smooth curve through them. This is an increasing function. The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$ and is one-to-one.

20. The y-intercept of $f(x) = 10^{-x}$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

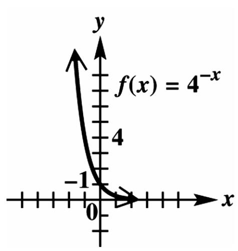
x	$f(x)$
-2	100
-1	10
$-\frac{1}{2}$	≈ 3.2
0	1
$\frac{1}{2}$	$\approx .3$
1	.1
2	.01



Plot these points and draw a smooth curve through them. This is a decreasing function. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$ and is one-to-one. Note: The graph of $f(x) = 10^{-x}$ is the reflection of the graph of $f(x) = 10^{x}$ (Exercise 19) about the y-axis.

21. The y-intercept of $f(x) = 4^{-x}$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

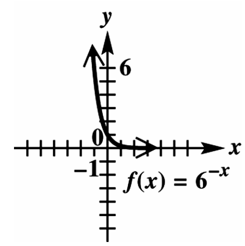
x	$f(x)$
-2	16
-1	4
$-\frac{1}{2}$	2
0	1
$\frac{1}{2}$	.5
1	.25
2	.0625



Plot these points and draw a smooth curve through them. This is a decreasing function. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$ and is one-to-one. Note: The graph of $f(x) = 4^{-x}$ is the reflection of the graph of $f(x) = 4^{x}$ (Exercise 14) about the y-axis.

22. The y-intercept of $f(x) = 6^{-x}$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

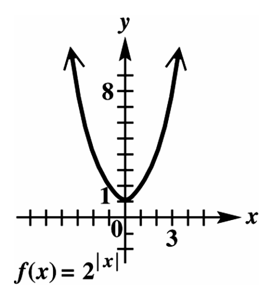
x	$f(x)$
-2	36
-1	6
$-\frac{1}{2}$	≈ 2.4
0	1
$\frac{1}{2}$	$\approx .4$
1	$\frac{1}{6} \approx .2$
2	$\frac{1}{36} \approx .03$



Plot these points and draw a smooth curve through them. This is an increasing function. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$ and is one-to-one.

23. The y-intercept of $f(x) = 2^{|x|}$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

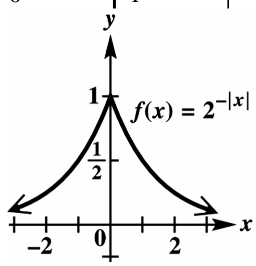
x	$f(x)$
-2	4
-1	2
$-\frac{1}{2}$	≈ 1.4
0	1
$\frac{1}{2}$	≈ 1.4
1	2
2	4



Plot these points and draw a smooth curve through them. The domain is $(-\infty, \infty)$ and the range is $[1, \infty)$ and is not one-to-one. Note: For $x < 0$, $|x| = -x$, so the graph is the same as that of $f(x) = 2^{-x}$. For $x \geq 0$, we have $|x| = x$, so the graph is the same as that of $f(x) = 2^{x}$. Since $|-x| = |x|$, the graph is symmetric with respect to the y-axis.

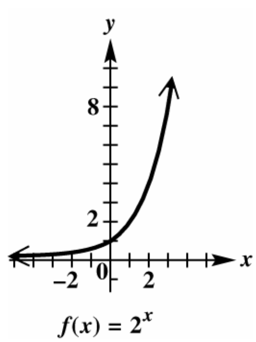
24. The y-intercept of $f(x) = 2^{-|x|}$ is 1, and the x-axis is a horizontal asymptote. Make a table of values.

x	$f(x)$	x	$f(x)$
-2	$\frac{1}{4} = .25$	$\frac{1}{2}$	$\approx .7$
-1	$\frac{1}{2} = .5$	1	$\frac{1}{2} = .5$
$-\frac{1}{2}$	$\approx .7$	2	$\frac{1}{4} = .25$
0	1		

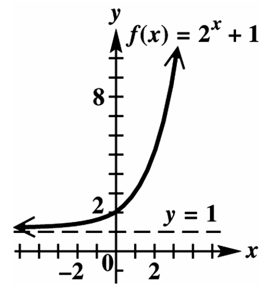


Plot these points and draw a smooth curve through them. The domain is $(-\infty, \infty)$ and the range is $(0, 1]$ and is not one-to-one. Note: For $x < 0$, $|x| = -x$, so the graph is the same as that of $f(x) = 2^{-(-x)} = 2^x$. For $x \geq 0$, we have $|x| = x$, so the graph is the same as that of $f(x) = 2^{-x}$. Since $|-x| = |x|$, the graph is symmetric with respect to the y-axis.

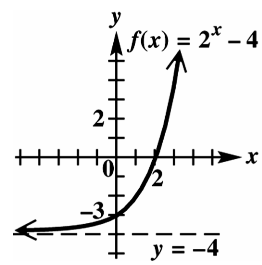
For Exercises 25–28, refer to the following graph of $f(x) = 2^x$.



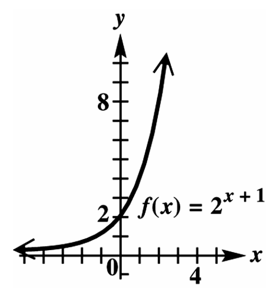
25. The graph of $f(x) = 2^x + 1$ is obtained by translating the graph of $f(x) = 2^x$ up 1 unit.



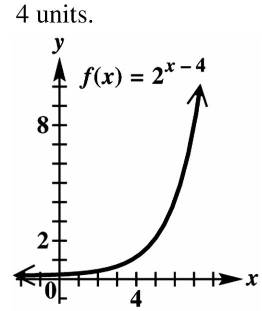
26. The graph of $f(x) = 2^x - 4$ is obtained by translating the graph of $f(x) = 2^x$ down 4 units.



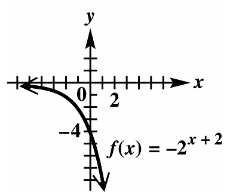
27. Since $f(x) = 2^{x+1} = 2^{x-(-1)}$, the graph is obtained by translating the graph of $f(x) = 2^x$ to the left 1 unit.



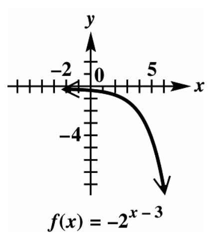
28. The graph of $f(x) = 2^{x-4}$ is obtained by translating the graph of $f(x) = 2^x$ to the right 4 units.



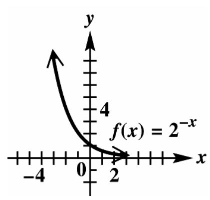
29. The graph of $f(x) = -2^{x+2}$ is obtained by translating the graph of $f(x) = 2^x$ to the left 2 units and then reflecting the graph across the x-axis.



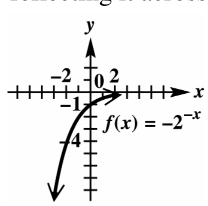
30. The graph of $f(x) = -2^{x-3}$ is obtained by translating the graph of $f(x) = 2^x$ to the right 3 units and then reflecting the graph across the x-axis.



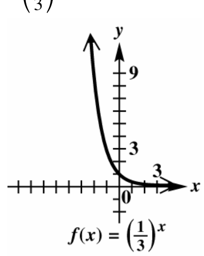
31. The graph of $f(x) = 2^{-x}$ is obtained by reflecting the graph across the y-axis.



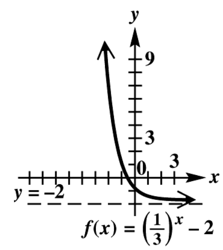
32. The graph of $f(x) = -2^{-x}$ is obtained by reflecting the graph across the y-axis and then reflecting it across the x-axis.



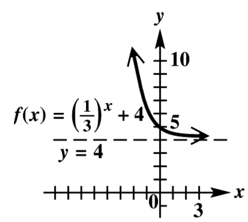
For Exercises 33–40, refer to the following graph of $f(x) = \left(\frac{1}{3}\right)^x$.



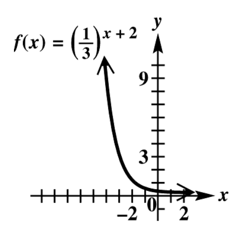
33. The graph of $f(x) = \left(\frac{1}{3}\right)^x - 2$ obtained by translating the graph of $f(x) = \left(\frac{1}{3}\right)^x$ down 2 units.



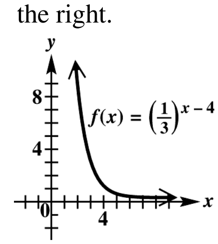
34. The graph of $f(x) = \left(\frac{1}{3}\right)^x + 4$ is obtained by translating the graph of $f(x) = \left(\frac{1}{3}\right)^x$ up 4 units.



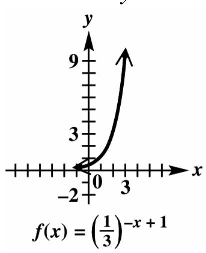
35. Since $f(x) = \left(\frac{1}{3}\right)^{x+2} = \left(\frac{1}{3}\right)^{x-(-2)}$, the graph is obtained by translating the graph of $f(x) = \left(\frac{1}{3}\right)^x$ 2 units to the left.



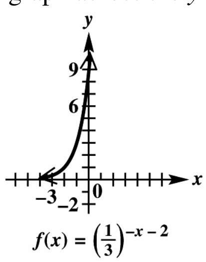
36. The graph of $f(x) = \left(\frac{1}{3}\right)^{x-4}$ is obtained by translating the graph of $f(x) = \left(\frac{1}{3}\right)^x$ 4 units to the right.



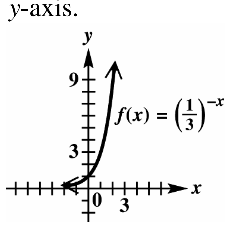
37. The graph of $f(x)=\left(\frac{1}{3}\right)^{-x+1}$ is obtained by translating the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$ left one unit and then reflecting the resulting graph across the y-axis.



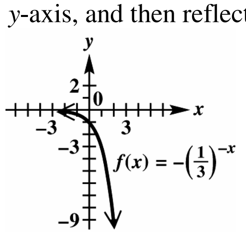
38. The graph of $f(x)=\left(\frac{1}{3}\right)^{-x-2}$ is obtained by translating the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$ two units to the right and then reflecting the resulting graph across the y-axis.



39. The graph of $f(x)=\left(\frac{1}{3}\right)^{-x}$ is obtained by reflecting the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$ across the y-axis.



40. The graph of $f(x)=-\left(\frac{1}{3}\right)^{-x}$ is obtained by reflecting the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$ across the y-axis, and then reflecting it across the x-axis.



41. The graph of $f(x)=a^{-x}$ is the same as $g(x)=\underline{\left(\frac{1}{a}\right)^{x}}$.

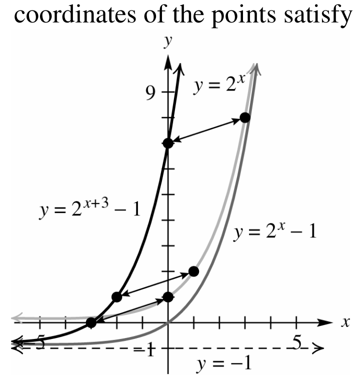
42. If $a > 1$, the graph of $f(x)=a^{x}$ <u>rises</u> from left to right. If $0 < a < 1$, then the graph of $g(x)=a^{x}$ <u>falls</u> from left to right.

43. Since the horizontal asymptote is $y = -1$, the graph has been shifted down one unit. So the general form of the equation is $f(x)=a^{x+b}-1$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for b:

$1=2^{-2+b}-1\Rightarrow 2=2^{-2+b}\Rightarrow 2^{1}=2^{-2+b}\Rightarrow$
$1=-2+b\Rightarrow 3=b$

So, the equation is $f(x)=2^{x+3}-1$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph up one unit and right three units to transform the given graph into the graph of $y=2^{x}$, it goes through the points (3, 8), (1, 2), and (0, 1), which is the y-intercept. $8=2^{3}$, so $a = 2$, and the equation is $f(x)=2^{x+3}-1$. Verify by checking that the coordinates of the points satisfy the equation.

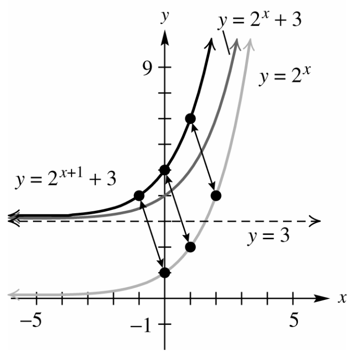


44. Since the horizontal asymptote is $y = 3$, the graph has been shifted up three units. So the general form of the equation is $f(x)=a^{x+b}+3$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for b:

$4=2^{-1+b}+3\Rightarrow 1=2^{-1+b}\Rightarrow 2^{0}=2^{-1+b}\Rightarrow$
$0=-1+b\Rightarrow 1=b$

So, the equation is $f(x)=2^{x+1}+3$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph down three units and right one unit to transform the given graph into the graph of $y = 2^x$, it goes through the points (−1, 1), (0, 2), and (1, 4). The y-intercept is (0, 2). $2 = 2^1$, so a = 2, and the equation is $f(x) = 2^{x+1} + 3$. Verify by checking that the coordinates of the points satisfy the equation.

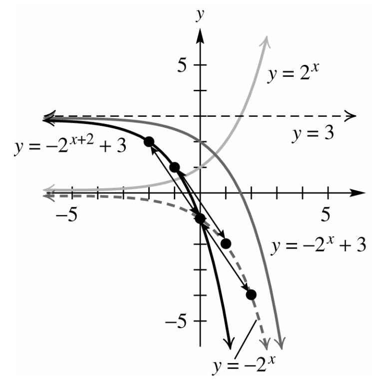


45. Since the horizontal asymptote is y = 3, the graph has been shifted up three units. The graph has also been reflected across the x-axis. So the general form of the equation is $f(x) = -a^{x+b} + 3$. The base is either 2 or 3, so try a = 2. Then substitute the coordinates of a point in the equation and solve for b:

$-1 = -2^{0+b} + 3 \Rightarrow -4 = -2^b \Rightarrow 4 = 2^b \Rightarrow$
$2^2 = 2^b \Rightarrow 2 = b$

So, the equation is $f(x) = -2^{x+2} + 3$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph down three units and right two units to transform the given graph into the graph of $y = -\left(2^x\right)$, it goes through the points (0, −1), (1, −2), and (2, −4). The y-intercept is (0, −1). $-2 = -\left(2^1\right)$, so a = 2, and the equation is $f(x) = 2^{x+1} + 3$. Verify by checking that the coordinates of the points satisfy the equation.

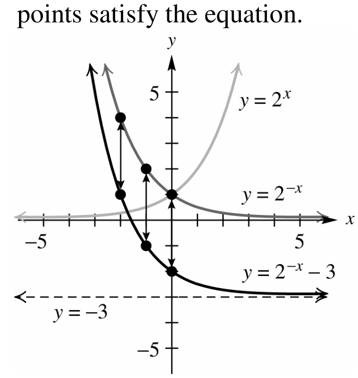


46. Since the horizontal asymptote is y = −3, the graph has been shifted down three units. The graph has also been reflected across the y-axis. So the general form of the equation is $f(x) = a^{-x+b} - 3$. The base is either 2 or 3, so try a = 2. Then substitute the coordinates of a point in the equation and solve for b:

$1 = 2^{-(-2)+b} - 3 \Rightarrow 4 = 2^{2+b} \Rightarrow$
$2^2 = 2^{2+b} \Rightarrow 2 = 2 + b \Rightarrow 0 = b$

So, the equation is $f(x) = 2^{-x} - 3$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph up three units to transform the given graph into the graph of $y = 2^{-x}$, it goes through the points (−2, 4), (−1, 2), and (0, 1). The y-intercept is (0, 1). $4 = (-2)^2$, so a = 2, and the equation is $f(x) = 2^{-x} - 3$. Verify by checking that the coordinates of the points satisfy the equation.

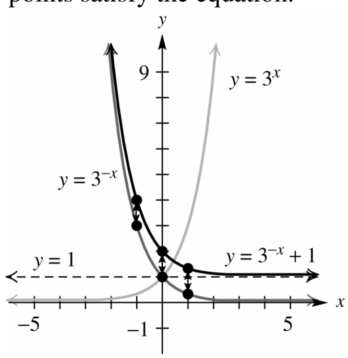


47. Since the horizontal asymptote is $y = 1$, the graph has been shifted up one unit. The graph has also been reflected across the y-axis. So the general form of the equation is $f(x) = a^{-x+b} + 1$. The base is either 2 or 3, so try $a = 3$. Then substitute the coordinates of a point in the equation and solve for b:

$4 = 3^{-(-1)+b} + 1 \Rightarrow 3 = 3^{1+b} \Rightarrow 3^1 = 3^{1+b} \Rightarrow$
$1 = 1 + b \Rightarrow 0 = b$

So, the equation is $f(x) = 3^{-x} + 1$. Verify by checking that the coordinates of the other two points satisfy the equation.

Alternate solution: Working backward and shifting the graph down one unit to transform the given graph into the graph of $y = 3^{-x}$, it goes through the points (−1, 3), (0, 1), and $\left(1, \frac{1}{3}\right)$. The y-intercept is (0, 1). $3 = 3^{-(-1)}$, so $a = 3$, and the equation is $f(x) = 3^{-x} + 1$. Verify by checking that the coordinates of the points satisfy the equation.

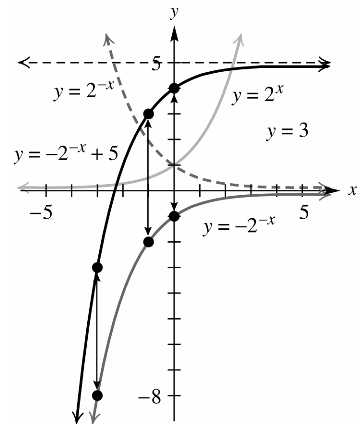


48. Since the horizontal asymptote is $y = 5$, the graph has been shifted up five units. The graph has also been reflected across the x-axis and the y-axis. So the general form of the equation is $f(x) = -a^{-x+b} + 5$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for b:

$3 = -2^{-(-1)+b} + 5 \Rightarrow -2 = -2^{1+b} \Rightarrow$
$2 = 2^{1+b} \Rightarrow 2^1 = 2^{1+b} \Rightarrow 1 = 1 + b \Rightarrow 0 = b$

So, the equation is $f(x) = -2^{-x} + 5$. Verify by checking that the coordinates of the points satisfy the equation.

The figure below compares the graphs of $y = 2^x, y = 2^{-x}, y = -2^{-x}$, and $y = -2^{-x} + 5$.



49. $4^x = 2 \Rightarrow \left(2^2\right)^x = 2^1 \Rightarrow 2^{2x} = 2^1 \Rightarrow$
$2x = 1 \Rightarrow x = \frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

50. $125^r = 5 \Rightarrow \left(5^3\right)^r = 5^1 \Rightarrow 5^{3r} = 5^1 \Rightarrow$
$3r = 1 \Rightarrow r = \frac{1}{3}$
Solution set: $\left\{\frac{1}{3}\right\}$

51. $\left(\frac{1}{2}\right)^k = 4 \Rightarrow \left(2^{-1}\right)^k = 2^2 \Rightarrow 2^{-k} = 2^2$
$-k = 2 \Rightarrow k = -2$
Solution set: $\{-2\}$

52. $\left(\frac{2}{3}\right)^x = \frac{9}{4} \Rightarrow \left(\frac{2}{3}\right)^x = \left(\frac{3}{2}\right)^2 \Rightarrow$
$\left(\frac{2}{3}\right)^x = \left[\left(\frac{2}{3}\right)^{-1}\right]^2 \Rightarrow \left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2} \Rightarrow x = -2$
Solution set: $\{-2\}$

53. $2^{3-y} = 8 \Rightarrow 2^{3-y} = 2^3 \Rightarrow 3 - y = 3 \Rightarrow$
$-y = 0 \Rightarrow y = 0$
Solution set: {0}

54. $5^{2p+1} = 25 \Rightarrow 5^{2p+1} = 5^2 \Rightarrow 2p + 1 = 2 \Rightarrow$
$2p = 1 \Rightarrow p = \frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

55. $e^{4x-1} = \left(e^2\right)^x \Rightarrow e^{4x-1} = e^{2x} \Rightarrow 4x - 1 = 2x \Rightarrow$
$-1 = -2x \Rightarrow \frac{1}{2} = x$
Solution set: $\left\{\frac{1}{2}\right\}$

56. $e^{3-x}=\left(e^3\right)^{-x}\Rightarrow e^{3-x}=e^{-3x}\Rightarrow$
$3-x=-3x\Rightarrow 3=-2x\Rightarrow -\frac{3}{2}=x$
Solution set: $\left\{-\frac{3}{2}\right\}$

57. $27^{4z}=9^{z+1}\Rightarrow\left(3^3\right)^{4z}=\left(3^2\right)^{z+1}\Rightarrow$
$3^{3(4z)}=3^{2(z+1)}\Rightarrow 3^{12z}=3^{2z+2}\Rightarrow$
$12z=2z+2\Rightarrow 10z=2\Rightarrow z=\frac{1}{5}$
Solution set: $\left\{\frac{1}{5}\right\}$

58. $32^{t+1}=16^{1-t}\Rightarrow\left(2^5\right)^{t+1}=\left(2^4\right)^{1-t}\Rightarrow$
$2^{5t+5}=2^{4(1-t)}\Rightarrow 2^{5t+5}=2^{4-4t}\Rightarrow$
$5t+5=4-4t\Rightarrow 9t=-1\Rightarrow t=-\frac{1}{9}$
Solution set: $\left\{-\frac{1}{9}\right\}$

59. $4^{x-2}=2^{3x+3}\Rightarrow\left(2^2\right)^{x-2}=2^{3x+3}\Rightarrow$
$2^{2(x-2)}=2^{3x+3}\Rightarrow 2^{2x-4}=2^{3x+3}\Rightarrow$
$2x-4=3x+3\Rightarrow -4=x+3\Rightarrow -7=x$
Solution set: $\{-7\}$

60. $2^{6-3x}=8^{x+1}\Rightarrow 2^{6-3x}=\left(2^3\right)^{x+1}\Rightarrow$
$2^{6-3x}=2^{3(x+1)}\Rightarrow 2^{6-3x}=2^{3x+3}\Rightarrow$
$6-3x=3x+3\Rightarrow 6=6x+3\Rightarrow$
$3=6x\Rightarrow \frac{1}{2}=x$
Solution set: $\left\{\frac{1}{2}\right\}$

61. $\left(\frac{1}{e}\right)^{-x}=\left(\frac{1}{e^2}\right)^{x+1}\Rightarrow\left(e^{-1}\right)^{-x}=\left(e^{-2}\right)^{x+1}\Rightarrow$
$e^x=e^{-2(x+1)}\Rightarrow e^x=e^{-2x-2}\Rightarrow$
$x=-2x-2\Rightarrow 3x=-2\Rightarrow x=-\frac{2}{3}$
Solution set: $\left\{-\frac{2}{3}\right\}$

62. $e^{k-1}=\left(\frac{1}{e^4}\right)^{k+1}\Rightarrow e^{k-1}=\left(e^{-4}\right)^{k+1}\Rightarrow$
$e^{k-1}=e^{-4(k+1)}\Rightarrow e^{k-1}=e^{-4k-4}\Rightarrow$
$k-1=-4k-4\Rightarrow 5k-1=-4\Rightarrow$
$5k=-3\Rightarrow k=-\frac{3}{5}$
Solution set: $\left\{-\frac{3}{5}\right\}$

63. $\left(\sqrt{2}\right)^{x+4}=4^x\Rightarrow\left(2^{1/2}\right)^{x+4}=\left(2^2\right)^x\Rightarrow$
$2^{(1/2)(x+4)}=2^{2x}\Rightarrow 2^{(1/2)x+2}=2^{2x}\Rightarrow$
$\frac{1}{2}x+2=2x\Rightarrow 2=\frac{3}{2}x\Rightarrow\frac{2}{3}\cdot 2=x\Rightarrow x=\frac{4}{3}$
Solution set: $\left\{\frac{4}{3}\right\}$

64. $\left(\sqrt[3]{5}\right)^{-x}=\left(\frac{1}{5}\right)^{x+2}\Rightarrow\left(5^{1/3}\right)^{-x}=\left(5^{-1}\right)^{x+2}\Rightarrow$
$5^{-(1/3)x}=5^{-(x+2)}\Rightarrow 5^{-(1/3)x}=5^{-x-2}\Rightarrow$
$-\frac{1}{3}x=-x-2\Rightarrow\frac{2}{3}x=-2\Rightarrow x=-2\cdot\frac{3}{2}=-3$
Solution set: $\{-3\}$

65. $\frac{1}{27}=b^{-3}\Rightarrow 3^{-3}=b^{-3}\Rightarrow b=3$
Alternate solution:
$\frac{1}{27}=b^{-3}\Rightarrow\frac{1}{27}=\frac{1}{b^3}\Rightarrow 27=b^3\Rightarrow b=\sqrt[3]{27}=3$
Solution set: $\{3\}$

66. $\frac{1}{81}=k^{-4}\Rightarrow\frac{1}{81}=\frac{1}{k^4}\Rightarrow 81=k^4\Rightarrow$
$k=\pm\sqrt[4]{81}=\pm 3$
Solution set: $\{-3,3\}$

67. $r^{2/3}=4\Rightarrow\left(r^{2/3}\right)^{3/2}=4^{3/2}\Rightarrow r=\left(\pm\sqrt{4}\right)^3\Rightarrow$
$r=\left(\pm 2\right)^3\Rightarrow r=\pm 8$
Recall from Chapter 1 that it is necessary to check all proposed solutions in the original equation when you raise both sides to a power.
Check $r=-8$.
$4=r^{2/3}$
$4\stackrel{?}{=}(-8)^{2/3}$
$4=\left(\sqrt[3]{-8}\right)^2\Rightarrow 4=(-2)^2\Rightarrow 4=4$
This is a true statement. -8 is a solution.
Check $r=8$.
$4=r^{2/3}$
$4\stackrel{?}{=}8^{2/3}$
$4=\left(\sqrt[3]{8}\right)^2\Rightarrow 4=2^2\Rightarrow 4=4$
This is a true statement. 8 is a solution.
Solution set: $\{-8,8\}$

68. $z^{5/2}=32\Rightarrow\left(z^{5/2}\right)^{2/5}=32^{2/5}\Rightarrow z=32^{2/5}\Rightarrow$
$z=\left(\sqrt[5]{32}\right)^2\Rightarrow z=2^2\Rightarrow z=4$
Recall from Chapter 1 that it is necessary to check all proposed solutions in the original equation when you raise both sides to a power.
Check $z=4$.
$z^{5/2}=32$
$4^{5/2}\stackrel{?}{=}32$
$\left(\sqrt{4}\right)^5=32\Rightarrow 2^5=32\Rightarrow 32=32$
This is a true statement. 4 is a solution.
Solution set: $\{4\}$

69. $x^{5/3} = -243 \Rightarrow \left(x^{5/3}\right)^{3/5} = (-243)^{3/5} \Rightarrow$

$x = (-3)^3 = -27$

Recall from Chapter 1 that it is necessary to check all proposed solutions in the original equation when you raise both sides to a power.

Check $z = -27$.

$$x^{5/3} = -243$$
$$(-27)^{5/3} \stackrel{?}{=} -243$$
$$(-3)^5 = -243$$

This is a true statement. −27 is a solution.
Solution set: {−27}

70. There will be no real solutions. If n is even, then $n = 2m$. $a^{1/n} = a^{1/(2m)} = a^{(1/2)(1/m)}$ $= \left(\sqrt{a}\right)^{(1/m)}$. Since $a < 0$, $\sqrt{a}$ is complex. Therefore $\left(\sqrt{a}\right)^{(1/m)}$ is also complex. Thus, there are no real solutions.

71. (a) Use the compound interest formula to find the future value, $A = P\left(1+\frac{r}{m}\right)^{tm}$, given $m = 2$, $P = 8906.54$, $r = .05$, and $t = 9$.

$$A = P\left(1+\frac{r}{m}\right)^{tm} = (8906.54)\left(1+\frac{.05}{2}\right)^{9(2)}$$
$$= (8906.54)(1+.025)^{18} \approx 13{,}891.16276$$

Rounding to the nearest cent, the future value is \$13,891.16. The amount of interest would be
$13,891.16 − $8906.54 = $4984.62.

(b) Use the continuous compounding interest formula to find the future value, $A = Pe^{rt}$, given $P = 8906.54$, $r = .05$, and $t = 9$.

$$A = Pe^{rt} = 8906.54e^{.05(9)} = 8906.54e^{.45}$$
$$\approx 8906.54(1.568312) \approx 13{,}968.23521$$

Rounding to the nearest cent, the future value is \$13,968.24. The amount of interest would be
$13,968.24 − $8906.54 = $5061.70.

72. (a) Use the compound interest formula to find the future value, $A = P\left(1+\frac{r}{m}\right)^{tm}$, given $m = 4$, $P = 56{,}780$, $r = .053$, and $t = \frac{23}{4}$.

$$A = P\left(1+\frac{r}{m}\right)^{tm} = (56{,}780)\left(1+\frac{.053}{4}\right)^{23}$$
$$= (56{,}780)(1+.01325)^{23} \approx 76{,}855.9462$$

Rounding to the nearest cent, the future value is \$76,855.95. The amount of interest would be
$76,855.95 − $56,780 = $20,075.95.

(b) Use the continuous compounding interest formula to find the future value, $A = Pe^{rt}$, given $P = 56{,}780$, $r = .053$, and $t = 15$.

$$A = Pe^{rt} = 56{,}780e^{.053(15)} = 56{,}780e^{.795}$$
$$\approx 56{,}780(2.214441) = 125{,}735.9598$$

Rounding to the nearest cent, the future value is \$125,735.96. The amount of interest would be
$125,735.96 − $56,780 = $68,955.96.

73. Use the compound interest formula to find the present amount, $A = P\left(1+\frac{r}{n}\right)^{tn}$, given $n = 4$, $A = 25{,}000$, $r = .06$, and $t = \frac{11}{4}$.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$
$$25{,}000 = P\left(1+\frac{.06}{4}\right)^{(11/4)(4)}$$
$$25{,}000 = P(1.015)^{11}$$
$$P = \frac{25{,}000}{(1.015)^{11}} \approx \$21{,}223.33083$$

Rounding to the nearest cent, the present value is \$21,223.33.

74. Use the compound interest formula to find the present amount, $A = P\left(1+\frac{r}{n}\right)^{tn}$, given $n = 12$, $A = 45{,}000$, $r = .036$, and $t = 1$.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$
$$45{,}000 = P\left(1+\frac{.036}{12}\right)^{1(12)}$$
$$45{,}000 = P(1.003)^{12}$$
$$P = \frac{45{,}000}{(1.003)^{12}} \approx \$43{,}411.15267$$

Rounding to the nearest cent, the present value is \$43,411.15.

75. Use the compound interest formula to find the present value, $A = P\left(1+\frac{r}{n}\right)^{tn}$, given $n = 4$, $A = 5,000$, $r = .035$, and $t = 10$.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$
$$5,000 = P\left(1+\frac{.035}{4}\right)^{10(4)}$$
$$5,000 = P(1.00875)^{40}$$
$$P = \frac{5,000}{(1.00875)^{40}} \approx \$3528.808535$$

Rounding to the nearest cent, the present value is \$3528.81.

76. Use the compound interest formula to find the interest rate, $A = P\left(1+\frac{r}{n}\right)^{tn}$, given $n = 12$, $A = 65,325$, $P = 65,000$, and $t = \frac{6}{12} = \frac{1}{2}$.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$
$$65,325 = 65,000\left(1+\frac{r}{12}\right)^{(1/2)(12)}$$
$$65,325 = 65,000\left(1+\frac{r}{12}\right)^{6}$$
$$1.005 = \left(1+\frac{r}{12}\right)^{6} \Rightarrow (1.005)^{\frac{1}{6}} = 1+\frac{r}{12} \Rightarrow$$
$$(1.005)^{\frac{1}{6}} - 1 = \frac{r}{12} \Rightarrow 12\left[(1.005)^{\frac{1}{6}} - 1\right] = r$$
$$r \approx .0099792301$$

The interest rate, to the nearest tenth, is 1.0%.

77. Use the compound interest formula to find the interest rate, $A = P\left(1+\frac{r}{n}\right)^{tn}$, given $n = 4$, $A = 1500$, $P = 1200$, and $t = 5$.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$
$$1500 = 1200\left(1+\frac{r}{4}\right)^{5(4)}$$
$$1500 = 1200\left(1+\frac{r}{4}\right)^{20}$$
$$1.25 = \left(1+\frac{r}{4}\right)^{20}$$
$$(1.25)^{\frac{1}{20}} = 1+\frac{r}{4} \Rightarrow (1.25)^{\frac{1}{20}} - 1 = \frac{r}{4} \Rightarrow$$
$$4\left[(1.25)^{\frac{1}{20}} - 1\right] = r \Rightarrow r \approx .044878604$$

The interest rate, to the nearest tenth, is 4.5%.

78. Use the compound interest formula to find the interest rate, $A = P\left(1+\frac{r}{n}\right)^{tn}$, given $n = 4$, $A = 8400$, $P = 5000$, and $t = 8$.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$
$$8400 = 5000\left(1+\frac{r}{4}\right)^{8(4)} \Rightarrow$$
$$8400 = 5000\left(1+\frac{r}{4}\right)^{32} \Rightarrow 1.68 = \left(1+\frac{r}{4}\right)^{32}$$
$$(1.68)^{\frac{1}{32}} = 1+\frac{r}{4} \Rightarrow (1.68)^{\frac{1}{32}} - 1 = \frac{r}{4} \Rightarrow$$
$$4\left[(1.68)^{\frac{1}{32}} - 1\right] = r \Rightarrow r \approx .0653777543$$

The interest rate, to the nearest tenth, is 6.5%.

79. For each bank we need to calculate $\left(1+\frac{r}{n}\right)^{n}$. Since the base, $1+\frac{r}{n}$, is greater than 1, we need only compare the three values calculated to determine which bank will yield the least amount of interest. It is understood that the amount of time, t, and the principal, P, are the same for all three banks.

Bank A: Calculate $\left(1+\frac{r}{n}\right)^{n}$ where $n = 1$ and $r = .064$.

$$\left(1+\frac{.064}{1}\right)^{1} = (1+.064)^{1} = (1.064)^{1} = 1.064$$

Bank B: Calculate $\left(1+\frac{r}{n}\right)^{n}$ where $n = 12$ and $r = .063$.

$$\left(1+\frac{.063}{12}\right)^{12} = (1+.00525)^{12} = (1.00525)^{12} \approx 1.064851339$$

Bank C: Calculate $\left(1+\frac{r}{n}\right)^{n}$ where $n = 4$ and $r = .0635$.

$$\left(1+\frac{.0635}{4}\right)^{4} = (1+.015875)^{4} = (1.015875)^{4} \approx 1.06502816$$

Bank A will charge you the least amount of interest, even though it has the highest stated rate.

80. Given $P = 10,000$, $r = .05$, and $t = 10$, the compound interest formula, $A = P\left(1+\frac{r}{n}\right)^{tn}$, becomes $A = 10,000\left(1+\frac{.05}{n}\right)^{10n}$.

(a) $m = 1 \Rightarrow A = 10,000\left(1+\frac{.05}{1}\right)^{10(1)}$
$$= 10,000(1+.05)^{10} \approx 16,288.94627$$

Rounding to the nearest cent, the future value is \$16,288.95.

(b) $m = 4 \Rightarrow A = 10,000\left(1+\frac{.05}{4}\right)^{10(4)}$

$= 10,000(1+.0125)^{40}$

$\approx 16,436.19463$

Rounding to the nearest cent, the future value is \$16,436.19.

(c) $m = 12 \Rightarrow A = 10,000\left(1+\frac{.05}{12}\right)^{10(12)}$

$= 10,000\left(1+\frac{.05}{12}\right)^{120}$

$\approx 16,470.09498$

Rounding to the nearest cent, the future value is \$16,470.09.

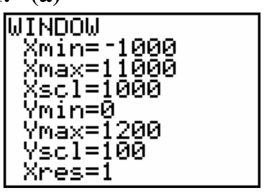
(d) $m = 365 \Rightarrow A = 10,000\left(1+\frac{.05}{365}\right)^{10(365)}$

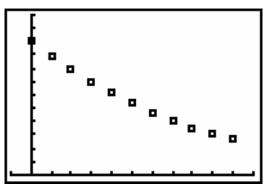
$= 10,000\left(1+\frac{.05}{365}\right)^{3650}$

$\approx 16,486.64814$

Rounding to the nearest cent, the future value is \$16,486.65.

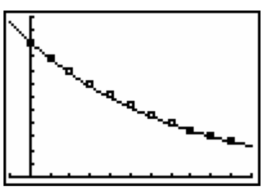
81. (a)





(b) From the graph above, we can see that the data are not linear but exponentially decreasing.

(c)



(d) $P(x) = 1013e^{-.0001341x}$

$P(1500) = 1013e^{-.0001341(1500)}$

$\approx 1013(.817790) \approx 828$

$P(11,000) = 1013e^{-.0001341(11,000)}$

$\approx 1013(.228756) \approx 232$

When the altitude is 1500 m, the function P gives a pressure of 828 mb, which is less than the actual value of 846 mb. When the altitude is 11,000 m, the function P gives a pressure of 232 mb, which is more than the actual value of 227 mb.

82. $y = 6079e^{.0126x}$

(a) 2006 is 6 years after 2000, so use $x = 6$.

$y = 6079e^{.0126(6)} = 6079e^{.0756}$

$\approx 6079(1.078531075) \approx 6556$

The function gives a population of about 6556 million, which differs from the actual value by 6556 − 6555 = 1 million.

(b) 2010 is 10 years after 2000, so use $x = 10$

$y = 6079e^{.0126(10)} = 6079e^{.126}$

$\approx 6079(1.134282168) \approx 6895$

The function gives a population of about 6895 million in 2010.

(c) 2025 is 25 years after 2000, so use $x = 25$.

$y = 6079e^{.0126(25)} = 6079e^{.315}$

$\approx 6079(1.370259311) \approx 8330$

The function gives a population of about 8330 million in 2025.

(d) Answers will vary.

83. (a) Evaluate $T = 50,000(1+.06)^n$ where $n = 4$.

$T = 50,000(1+.06)^4$

$= 50,000(1.06)^4 \approx 63,123.848$

Total population after 4 years is about 63,000.

(b) Evaluate $T = 30,000(1+.12)^n$ where $n = 3$.

$T = 30,000(1+.12)^3$

$= 30,000(1.12)^3 = 42,147.84$

There would be about 42,000 deer after 3 years.

(c) Evaluate $T = 45,000(1+.08)^n$ where $n = 5$.

$T = 45,000(1+.08)^5$

$= 45,000(1.08)^5 \approx 66,119.76346$

There would be about 66,000 deer after 5 years. Thus, we can expect about $66,000 - 45,000 = 21,000$ additional deer after 5 years.

84. $p(t) = 250 - 120(2.8)^{-.5t}$

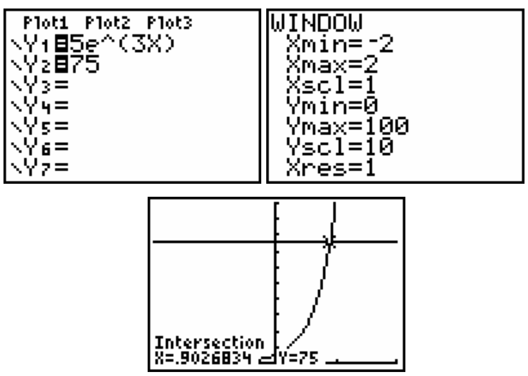
(a) $p(2) = 250 - 120(2.8)^{-.5(2)}$
$= 250 - 120(2.8)^{-1} \approx 207.1428571$
After 2 months, a person will type about 207 symbols per minute.

(b) $p(4) = 250 - 120(2.8)^{-.5(4)}$
$= 250 - 120(2.8)^{-2} \approx 234.6938776$
After 4 months, a person will type about 235 symbols per minute.

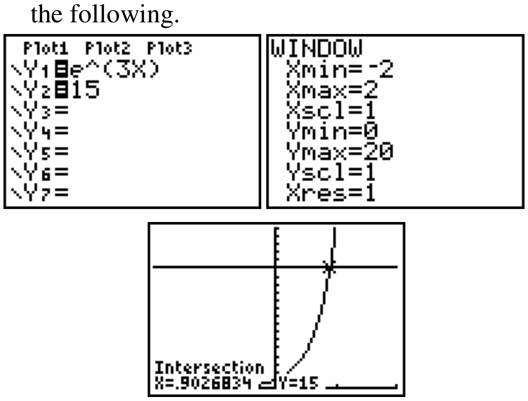
(c) $p(10) = 250 - 120(2.8)^{-.5(10)}$
$= 250 - 120(2.8)^{-5} \approx 249.3027459$
After 10 months, a person will type about 249 symbols per minute.

(d) The number of symbols approaches 250.

85. $5e^{3x} = 75$

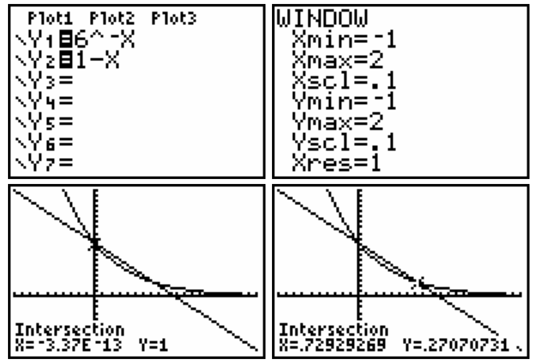


Since $5e^{3x} = 75 \Rightarrow e^{3x} = 15$ we could also do the following.



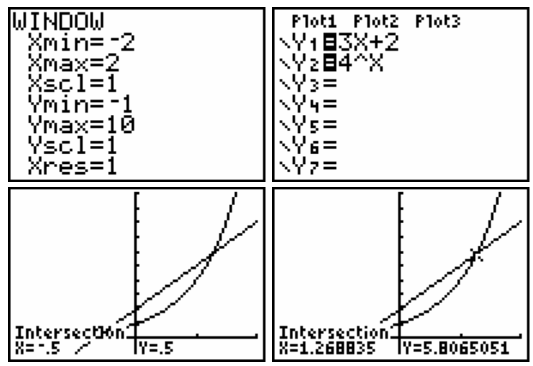
Solution set: {.9}

86. $6^{-x} = 1 - x$
We can see from the following screens that there are two solutions.



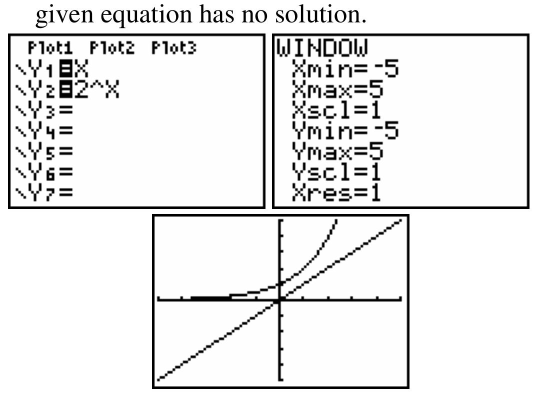
Solution set: {0, .7}

87. $3x + 2 = 4^x$
We can see from the following screens that there are two solutions.



Solution set: {–.5, 1.3}

88. $x = 2^x$
Use a graphing calculator to graph the line and the exponential function on the same screen. These two graphs do not intersect, so the given equation has no solution.



Solution set: $\varnothing$

89. Answers will vary.

90. $f(x) = a^x$ and $f(3) = 27$ together imply that $a^3 = 27$, so $a = 3$ and $f(x) = 3^x$.

(a) $f(1) = 3^1 = 3$

(b) $f(-1) = 3^{-1} = \frac{1}{3}$

(c) $f(2) = 3^2 = 9$

(d) $f(0) = 3^0 = 1$

91. If the graph of the exponential function $f(x) = a^x$ contains the point (3, 8), we have $a^3 = 8$. This implies $a = \sqrt[3]{8} = 2$. Thus, the equation which satisfies the given condition is $f(x) = 2^x$.

92. If the graph of the exponential function $f(x) = a^x$ contains the point (3, 125), we have $a^3 = 125$. This implies $a = \sqrt[3]{125} = 5$. Thus, the equation which satisfies the given condition is $f(x) = 5^x$.

93. If the graph of the exponential function $f(x) = a^x$ contains the point $(-3, 64)$, we have $a^{-3} = 64$. This implies $a^3 = \frac{1}{64} \Rightarrow a = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}$. Thus, the equation which satisfies the given condition is $f(x) = \left(\frac{1}{4}\right)^x$.

94. If the graph of the exponential function $f(x) = a^x$ contains the point (–2, 36) we have $a^{-2} = 36$. This implies $a^2 = \frac{1}{36} \Rightarrow a = \sqrt{\frac{1}{36}} = \frac{1}{6}$. Thus, the equation which satisfies the given condition is $f(x) = \left(\frac{1}{6}\right)^x$.

95. $f(t) = 3^{2t+3} = 3^{2t} \cdot 3^3 = 27 \cdot \left(3^2\right)^t = 27 \cdot 9^t$

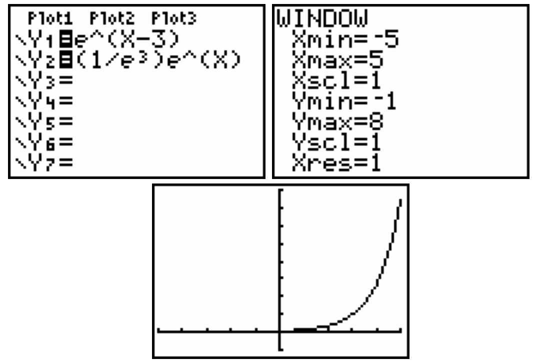
96. $f(t) = 2^{3t+2} = 2^{3t} \cdot 2^2 = 4 \cdot \left(2^3\right)^t = 4 \cdot 8^t$

97. $f(t) = \left(\frac{1}{3}\right)^{1-2t} = \left(\frac{1}{3}\right)^1 \left(\frac{1}{3}\right)^{-2t}$
$= \left(\frac{1}{3}\right)(3)^{2t} = \left(\frac{1}{3}\right)\left(3^2\right)^t = \left(\frac{1}{3}\right)9^t$

98. $f(t) = \left(\frac{1}{2}\right)^{1-2t} = \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{-2t}$
$= \left(\frac{1}{2}\right)(2)^{2t} = \left(\frac{1}{2}\right)\left(2^2\right)^t = \left(\frac{1}{2}\right)4^t$

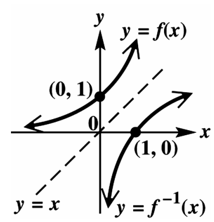
99. Answers will vary.

100. Since $y = e^{x-3} = e^x \cdot e^{-3} = \frac{1}{e^3}e^x, C = \frac{1}{e^3}$ or $C = e^{-3}$.



101. Yes; $f(x) = a^x$ is a one-to-one function. Therefore, an inverse function exists for *f*.

102. Since $f(x) = a^x$ has an inverse, the graph of $f^{-1}(x)$ will be the reflection of *f* across the line $y = x$.



103. Since $f(x) = a^x$ has an inverse, we find it as follows: $y = a^x \Rightarrow x = a^y$

104. If $a = 10$, the equation for $f^{-1}(x)$ will be given by $x = 10^y$.

105. If $a = e$, the equation for $f^{-1}(x)$ will be given by $x = e^y$.

106. If the point (p, q) is on the graph of *f*, then the point (q, p) is in the graph of f^{-1}.

Section 4.3: Logarithmic Functions

Connections (page 441)

1.
$$\begin{array}{r} \log_{10} 458.3 \approx 2.661149857 \\ +\log_{10} 294.6 \approx 2.469232743 \\ \hline \approx 5.130382600 \\ 10^{5.130382600} \approx 135{,}015.18 \end{array}$$
A calculator gives $(458.3)(294.6) = 135{,}015.18$.

2. Answers will vary.

Exercises

1. **(a)** C; $\log_2 16 = 4$ because $2^4 = 16$.

(b) A; $\log_3 1 = 0$ because $3^0 = 1$.

(c) E; $\log_{10} .1 = -1$ because $10^{-1} = .1$.

(d) B; $\log_2 \sqrt{2} = \frac{1}{2}$ because $2^{1/2} = \sqrt{2}$.

(e) F; $\log_e \left(\frac{1}{e^2}\right) = -2$ because $e^{-2} = \frac{1}{e^2}$.

(f) D; $\log_{1/2} 8 = -3$ because $\left(\frac{1}{2}\right)^{-3} = 8$.

2. **(a)** F; $\log_3 81 = 4$ because $3^4 = 16$.

(b) B; $\log_3 \frac{1}{3} = -1$ because $3^{-1} = \frac{1}{3}$.

(c) A; $\log_{10} .01 = -2$ because $10^{-2} = .01$.

(d) D; $\log_6 \sqrt{6} = \frac{1}{2}$ because $6^{1/2} = \sqrt{6}$.

(e) C; $\log_e 1 = 0$ because $e^0 = 1$.

(f) E; $\log_3 27^{3/2} = \frac{9}{2}$ because $3^{9/2} = \left(3^3\right)^{3/2} = 27^{3/2}$.

3. $3^4 = 81$ is equivalent to $\log_3 81 = 4$.

4. $2^5 = 32$ is equivalent to $\log_2 32 = 5$.

5. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$ is equivalent to $\log_{2/3} \frac{27}{8} = -3$.

6. $10^{-4} = .0001$ is equivalent to $\log_{10} .0001 = -4$.

7. $\log_6 36 = 2$ is equivalent to $6^2 = 36$.

8. $\log_5 5 = 1$ is equivalent to $5^1 = 5$.

9. $\log_{\sqrt{3}} 81 = 8$ is equivalent to $\left(\sqrt{3}\right)^8 = 81$.

10. $\log_4 \frac{1}{64} = -3$ is equivalent to $4^{-3} = \frac{1}{64}$.

11. Answers will vary.

12. $\log_a 1 = 0$ for all real numbers a, because $a^0 = 1$, $(a \neq 0)$ for all real numbers a.

13. $x = \log_5 \frac{1}{625} \Rightarrow 5^x = \frac{1}{625} \Rightarrow 5^x = \frac{1}{5^4} \Rightarrow$
$5^x = 5^{-4} \Rightarrow x = -4$
Solution set: $\{-4\}$

14. $x = \log_3 \frac{1}{81} \Rightarrow 3^x = \frac{1}{81} \Rightarrow 3^x = \frac{1}{3^4} \Rightarrow$
$3^x = 3^{-4} \Rightarrow x = -4$
Solution set: $\{-4\}$

15. $\log_x \frac{1}{32} = 5 \Rightarrow x^5 = \frac{1}{32} \Rightarrow x^5 = \frac{1}{2^5} = \left(\frac{1}{2}\right)^5 \Rightarrow$
$x = \frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

16. $\log_x \frac{27}{64} = 3 \Rightarrow x^3 = \frac{27}{64} \Rightarrow x^3 = \frac{3^3}{4^3} = \left(\frac{3}{4}\right)^3 \Rightarrow$
$x = \frac{3}{4}$
Solution set: $\left\{\frac{3}{4}\right\}$

17. $x = \log_8 \sqrt[4]{8} \Rightarrow 8^x = \sqrt[4]{8} \Rightarrow 8^x = 8^{1/4} \Rightarrow x = \frac{1}{4}$
Solution set: $\left\{\frac{1}{4}\right\}$

18. $x = \log_{100} 10 \Rightarrow x = \log_{100} 10 \Rightarrow$
$100^x = 10 \Rightarrow 10^{2x} = 10 \Rightarrow$
$2x = 1 \Rightarrow x = \frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

19. $x = 3^{\log_3 8}$
Writing as a logarithmic equation, we have
$\log_3 8 = \log_3 x \Rightarrow x = 8$
Using the Theorem of Inverses on page 440, we can directly state that $x = 8$.
Solution set: $\{8\}$

20. $x = 12^{\log_{12} 5}$
Writing as a logarithmic equation, we have
$\log_{12} 5 = \log_{12} x \Rightarrow x = 5$
Using the Theorem of Inverses on page 440, we can directly state that $x = 5$.
Solution set: $\{5\}$

21. $x = 2^{\log_2 9}$
Writing as a logarithmic equation, we have
$\log_2 9 = \log_2 x \Rightarrow x = 9$
Using the Theorem of Inverses on page 440, we can directly state that $x = 9$.
Solution set: $\{9\}$

22. $x = 8^{\log_8 11}$
Writing as a logarithmic equation, we have $\log_8 11 = \log_8 x \Rightarrow x = 11$
Using the Theorem of Inverses on page 440, we can directly state that $x = 11$.
Solution set: $\{11\}$

23. $\log_x 25 = -2 \Rightarrow x^{-2} = 25 \Rightarrow x^{-2} = 5^2 \Rightarrow$
$\left(x^{-2}\right)^{-1/2} = \left(5^2\right)^{-1/2} \Rightarrow x = 5^{-1}$
Do not include a $\pm$ since the base, x, cannot be negative.
$x = \frac{1}{5}$
Solution set: $\left\{\frac{1}{5}\right\}$

24. $\log_x \frac{1}{16} = -2 \Rightarrow x^{-2} = \frac{1}{16} \Rightarrow x^{-2} = \frac{1}{2^4} \Rightarrow$
$x^{-2} = 2^{-4} \Rightarrow \left(x^{-2}\right)^{-1/2} = \left(2^{-4}\right)^{-1/2} \Rightarrow x = 2^2$
Do not include a $\pm$ since the base, x, cannot be negative.
$x = 4$
Solution set: $\{4\}$

25. $\log_4 x = 3 \Rightarrow 4^3 = x \Rightarrow 64 = x$
Solution set: $\{64\}$

26. $\log_2 x = -1 \Rightarrow 2^{-1} = x \Rightarrow \frac{1}{2} = x$
Solution set: $\left\{\frac{1}{2}\right\}$

27. $x = \log_4 \sqrt[3]{16} \Rightarrow 4^x = \sqrt[3]{16} \Rightarrow 4^x = (16)^{1/3} \Rightarrow$
$4^x = \left(4^2\right)^{1/3} \Rightarrow 4^x = 4^{2/3} \Rightarrow x = \frac{2}{3}$
Solution set: $\left\{\frac{2}{3}\right\}$

28. $x = \log_5 \sqrt[4]{25} \Rightarrow 5^x = \sqrt[4]{25} \Rightarrow 5^x = (25)^{1/4} \Rightarrow$
$5^x = \left(5^2\right)^{1/4} \Rightarrow 5^x = 5^{2/4} \Rightarrow 5^x = 5^{1/2} \Rightarrow x = \frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

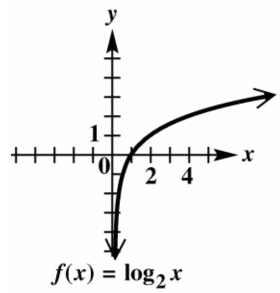
29. $\log_9 x = \frac{5}{2} \Rightarrow 9^{5/2} = x \Rightarrow (3)^5 = 243 = x$
Note that we do not include $\sqrt{9} = -3$ because logarithms are not defined for negative numbers.
Solution set: $\{243\}$

30. $\log_4 x = \frac{7}{2} \Rightarrow x = 4^{7/2} = 2^7 = 128$
Note that we do not include $\sqrt{4} = -2$ because logarithms are not defined for negative numbers.
Solution set: $\{128\}$

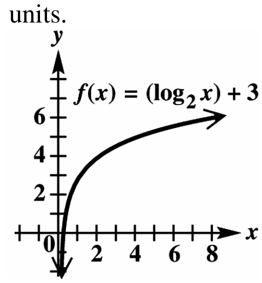
31. Answers will vary.

32. y (2.2319281) represents the exponent to which 2 must be raised in order to obtain x (5).

For Exercises 33–35, refer to the following graph of $f(x) = \log_2 x$.

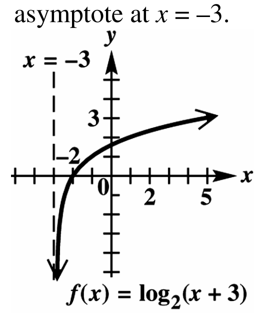


33. The graph of $f(x) = (\log_2 x) + 3$ is obtained by translating the graph of $f(x) = \log_2 x$ up 3 units.



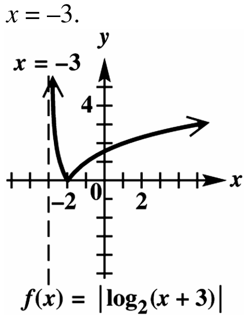
Domain: $(0, \infty)$; range: $(-\infty, \infty)$

34. The graph of $f(x) = \log_2(x + 3)$ is obtained by translating the graph of $f(x) = \log_2 x$ to the left 3 units. The graph has a vertical asymptote at $x = -3$.



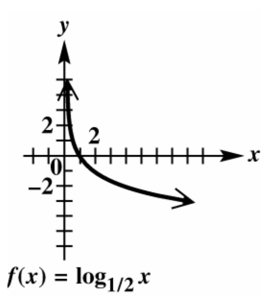
Domain: $(-3, \infty)$; range: $(-\infty, \infty)$

35. To find the graph of $f(x) = \left|\log_2(x+3)\right|$, translate the graph of $f(x) = \log_2 x$ to the left 3 units to obtain the graph of $\log_2(x+3)$. (See Exercise 34.) For the portion of the graph where $f(x) \geq 0$, that is, where $x \geq -2$, use the same graph as in 34. For the portion of the graph in 34 where $f(x) < 0$, $-3 < x < -2$, reflect the graph about the x-axis. In this way, each negative value of $f(x)$ on the graph in 34 is replaced by its opposite, which is positive. The graph has a vertical asymptote at $x = -3$.

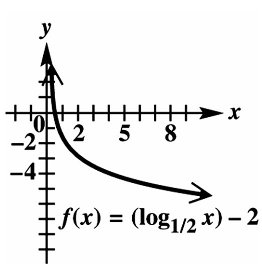


Domain: $(-3, \infty)$; range: $[0, \infty)$

For Exercises 36–38, refer to the following graph of $f(x) = \log_{1/2} x$.

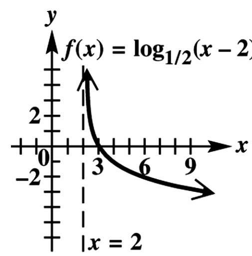


36. The graph of $f(x) = \left(\log_{1/2} x\right) - 2$ is obtained by translating the graph of $f(x) = \log_{1/2} x$ down 2 units.



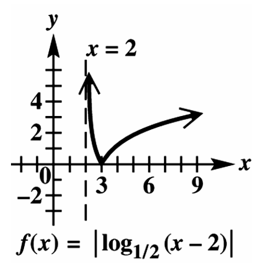
Domain: $(0, \infty)$; range: $(-\infty, \infty)$

37. The graph of $f(x) = \log_{1/2}(x-2)$ is obtained by translating the graph of $f(x) = \log_{1/2} x$ to the right 2 units. The graph has a vertical asymptote at $x = 2$.



Domain: $(2, \infty)$; range: $(-\infty, \infty)$

38. To find the graph of $f(x) = \left|\log_{1/2}(x-2)\right|$ translate the graph of $f(x) = \log_{1/2} x$ to the right 2 units to obtain the graph of $\log_{1/2}(x-2)$. (See Exercise 37.) For the portion of the graph where $f(x) \geq 0$, that is, where $2 < x \leq 3$, use the same graph as in 37. For the portion of the graph in 37 where $f(x) < 0$, $x > 3$, reflect the graph about the x-axis. In this way, each negative value of $f(x)$ on the graph in 37 is replaced by its opposite, which is positive. The graph has a vertical asymptote at $x = 2$.



Domain: $(2, \infty)$; range: $[0, \infty)$

39. Because $f(x) = \log_2 x$ has a vertical asymptote, which is the y-axis (the line $x = 0$), x-intercept of 1, and is increasing, the correct choice is the graph in E.

40. Because $f(x) = \log_2 2x$ has a vertical asymptote, which is the y-axis (the line $x = 0$), has an x-intercept when $2x = 1 \Rightarrow x = \frac{1}{2}$, and is increasing, the correct choice is the graph in D.

41. Because $f(x) = \log_2 \frac{1}{x} = \log_2 x^{-1} = -\log_2 x,$ it has a vertical asymptote, which is the y-axis (the line $x = 0$), has an x-intercept 1, and is the reflection of $f(x) = \log_2 x$ across the x-axis, it is decreasing and the correct choice is the graph in B.

42. Because $f(x) = \log_2 \frac{x}{2}$ has a vertical asymptote, which is the y-axis (the line $x = 0$), has an x-intercept when $\frac{x}{2} = 1 \Rightarrow x = 2,$ and is increasing, the correct choice is the graph in C.

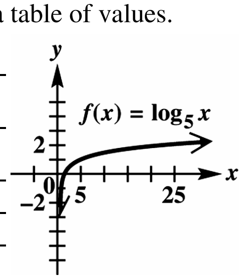
43. Because $f(x) = \log_2 (x-1)$ represents the horizontal shift of $f(x) = \log_2 x$ to the right 1 unit, the function has a vertical asymptote which is the line $x = 1,$ has an x-intercept when $x - 1 = 1 \Rightarrow x = 2,$ and is increasing, the correct choice is the graph in F.

44. Because $f(x) = \log_2 (-x)$ represents a reflection of $f(x) = \log_2 x$ over the y-axis, it has a vertical asymptote, which is the y-axis (the line $x = 0$), and passes through $(-1, 0),$ the correct choice is the graph in A.

45. $f(x) = \log_5 x$

Since $f(x) = y = \log_5 x,$ we can write the exponential form as $x = 5^y$ to find ordered pairs that satisfy the equation. It is easier to choose values for y and find the corresponding values of x. Make a table of values.

x	$y = \log_5 x$
$\frac{1}{25} = .04$	-2
$\frac{1}{5} = .2$	-1
1	0
5	1
25	2

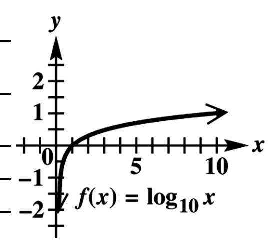


The graph can also be found by reflecting the graph of $f(x) = 5^x$ about the line $y = x$. The graph has the y-axis as a vertical asymptote.

46. $f(x) = \log_{10} x$

Since $f(x) = y = \log_{10} x,$ we can write the exponential form as $x = 10^y$ to find ordered pairs that satisfy the equation. It is easier to choose values for y and find the corresponding values of x. Make a table of values.

x	$y = \log_{10} x$
$\frac{1}{100} = .01$	-2
$\frac{1}{10} = .1$	-1
1	0
10	1
100	2

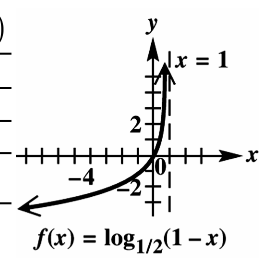


The graph can also be found by reflecting the graph of $f(x) = 10^x$ about the line $y = x$. The graph has the y-axis as a vertical asymptote.

47. $f(x) = \log_{1/2} (1-x)$

Since $f(x) = y = \log_{1/2} (1-x),$ we can write the exponential form as $1 - x = \left(\frac{1}{2}\right)^y \Rightarrow x = 1 - \left(\frac{1}{2}\right)^y$ to find ordered pairs that satisfy the equation. It is easier to choose values for y and find the corresponding values of x. Make a table of values.

x	$y = \log_{1/2} (1-x)$
-3	-2
-1	-1
0	0
$\frac{1}{2} = .5$	1
$\frac{3}{4} = .75$	2

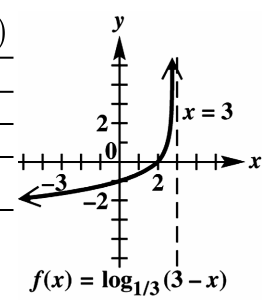


The graph has the line $x = 1$ as a vertical asymptote.

48. $f(x) = \log_{1/3} (3-x)$

Since $f(x) = y = \log_{1/3} (3-x),$ we can write the exponential form as $3 - x = \left(\frac{1}{3}\right)^y \Rightarrow x = 3 - \left(\frac{1}{3}\right)^y$ to find ordered pairs that satisfy the equation. It is easier to choose values for y and find the corresponding values of x. Make a table of values.

x	$y = \log_{1/3} (3-x)$
-6	-2
0	-1
2	0
$\frac{8}{3} \approx 2.7$	1
$\frac{26}{9} \approx 2.9$	2

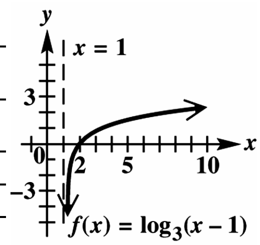


The graph has the line $x = 3$ as a vertical asymptote.

49. $f(x) = \log_3(x-1)$

Since $f(x) = y = \log_3(x-1)$, we can write the exponential form as $x - 1 = 3^y \Rightarrow x = 3^y + 1$ to find ordered pairs that satisfy the equation. It is easier to choose values for *y* and find the corresponding values of *x*. Make a table of values.

x	$y = \log_3(x-1)$
$\frac{10}{9} \approx 1.1$	-2
$\frac{4}{3} \approx 1.3$	-1
2	0
4	1
10	2

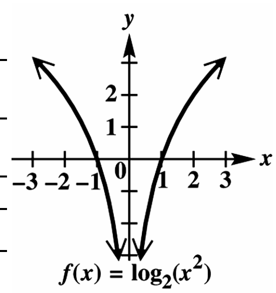


The vertical asymptote will be $x = 1$.

50. $f(x) = \log_2(x^2)$

Since $f(x) = y = \log_2(x^2)$, we can write the exponential form as $x^2 = 2^y$ to find ordered pairs that satisfy the equation. It is easier to choose values for *y* and find the corresponding values of *x*. Make a table of values.

x	$y = \log_2(x^2)$
$\pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} = \pm.5$	-2
$\pm\sqrt{\frac{1}{2}} \approx \pm.7$	-1
± 1	0
$\pm\sqrt{2} \approx \pm 1.4$	1
± 2	2



The graph has the *y*-axis as a vertical asymptote and is symmetric with respect to the *y*-axis.

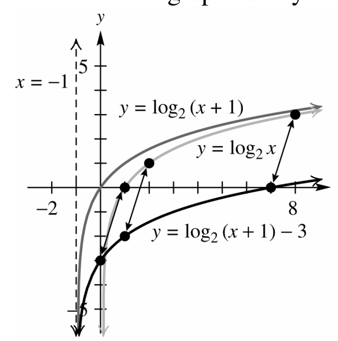
51. Since the vertical asymptote is $x = -1$, the graph has been shifted left one unit. So the general form of the equation is $f(x) = \log_a(x+1) + k$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for *k*:

$-2 = \log_2(1+1) + k \Rightarrow -2 - k = \log_2 2 \Rightarrow$
$2^{-2-k} = 2 \Rightarrow 2^{-2-k} = 2^1 \Rightarrow$
$-2 - k = 1 \Rightarrow -3 = k$

So, the equation is $f(x) = \log_2(x+1) - 3$.

Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph up three units and right one unit to transform the given graph into the graph of $y = \log_2 x$, it goes through the points (1, 0), which is the *x*-intercept, (2, 1), and (8, 3). $3 = \log_2 8$, so $a = 2$, and the equation is $f(x) = \log_2(x+1) - 3$. Verify by checking that the coordinates of the points shown on the graph satisfy the equation.



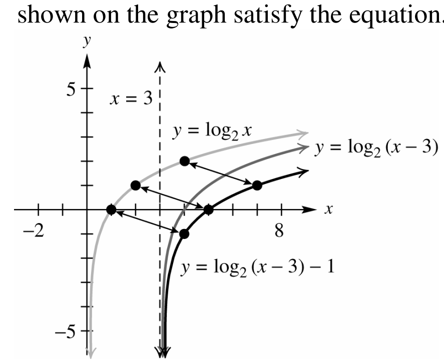
52. Since the vertical asymptote is $x = 3$, the graph has been shifted right three units, and $b = 3$. So the general form of the equation is $f(x) = \log_a(x-3) + k$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for *k*:

$-1 = \log_2(4-3) + k \Rightarrow -1 - k = \log_2 1 \Rightarrow$
$2^{-1-k} = 1 \Rightarrow 2^{-1-k} = 2^0 \Rightarrow$
$-1 - k = 1 \Rightarrow -1 = k$

So, the equation is $f(x) = \log_2(x-3) - 1$.

Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph up one unit and left three units to transform the given graph into the graph of $y = \log_2 x$, it goes through the points (1, 0), which is the *x*-intercept, (2, 1), and (4, 2). $2 = \log_2 4$, so $a = 2$, and the equation is $f(x) = \log_2(x-3) - 1$. Verify by checking that the coordinates of the points shown on the graph satisfy the equation.

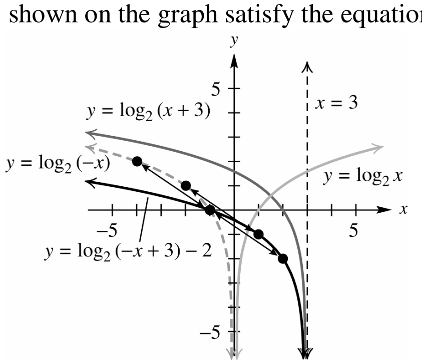


53. The graph has been reflected across the y-axis, so the general form of the equation is $f(x) = \log_a(-x-b)+k$. Since the vertical asymptote of the graph is $x = 3$, the graph has been shifted right three units, and $b = -3$. So the general form of the equation is $f(x) = \log_a(-x+3)+k$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for k:

$-1 = \log_2(-1+3)+k \Rightarrow -1-k = \log_2 2 \Rightarrow$
$2^{-1-k} = 2 \Rightarrow 2^{-1-k} = 2^1 \Rightarrow$
$-1-k = 1 \Rightarrow -2 = k$

So, the equation is $f(x) = \log_2(-x+3)-2$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph up two units and left three units to transform the given graph into the graph of $y = \log_2 x$, it goes through the points (−1, 0), which is the x-intercept, (−2, 1), and (−4, 2). $2 = \log_2[-(-4)]$, so $a = 2$, and the equation is $f(x) = \log_2(-x+3)-2$. Verify by checking that the coordinates of the points shown on the graph satisfy the equation.

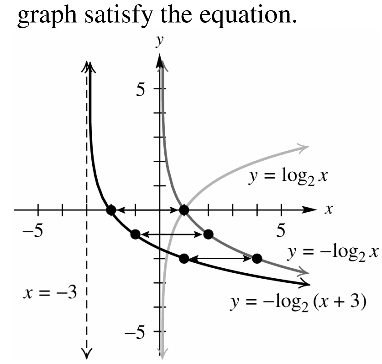


54. The graph has been reflected across the x-axis, so the general form of the equation is $f(x) = -\log_a(x-b)+k$. Since the vertical asymptote of the original graph is $x = -3$, the graph has been shifted left three units, and $b = -3$. So the general form of the equation is $f(x) = \log_a(-x+3)+k$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for k:

$-2 = -\log_2(1+3)+k \Rightarrow$
$-(-2-k) = \log_2 4 \Rightarrow 2^{2+k} = 4 \Rightarrow$
$2^{2+k} = 2^2 \Rightarrow 2+k = 2 \Rightarrow k = 0$

So, the equation is $f(x) = -\log_2(x+3)$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph right three units, it goes through the points (1, 0), which is the x-intercept, (2, −1), and (4, −2). $-2 = -\log_2 4$, so $a = 2$, and the equation is $f(x) = -\log_2(x+3)$. Verify by checking that the coordinates of the points shown on the graph satisfy the equation.

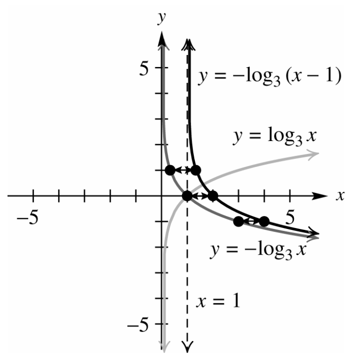


55. The graph has been reflected across the x-axis, so the general form of the equation is $f(x) = -\log_a(x-b)+k$. Since the vertical asymptote of the original graph is $x = 1$, the graph has been shifted right one unit and $b = 1$. So the general form of the equation is $f(x) = -\log_a(x-1)+k$. The base is either 2 or 3, so try $a = 3$. Then substitute the coordinates of a point in the equation and solve for k:

$-1 = -\log_3(4-1)+k \Rightarrow$
$-(-1-k) = \log_3 3 \Rightarrow 3^{1+k} = 3 \Rightarrow$
$3^{1+k} = 3^1 \Rightarrow 1+k = 1 \Rightarrow k = 0$

So, the equation is $f(x) = -\log_3(x-1)$. Verify that the coordinates of other two points given satisfy the equation.

Alternate solution: Working backward and shifting the graph left one unit, it goes through the points (1, 0), which is the x-intercept, $\left(\frac{1}{3}, 1\right)$, and (3, −1). $-1 = -\log_3 3$, so $a = 3$, and the equation is $f(x) = -\log_3(x-1)$. Verify by checking that the coordinates of the points shown on the graph satisfy the equation.



56. The graph has been reflected across the x-axis, and then reflected across the y-axis, so the general form of the equation is $f(x) = -\log_a(-x-b)+k$. Since the vertical asymptote of the original graph is $x = 5$, the graph has been shifted right five units. So the general form of the equation is $f(x) = -\log_a(-x+5)$. The base is either 2 or 3, so try $a = 2$. Then substitute the coordinates of a point in the equation and solve for k:

$-1 = -\log_2(-3+5)+k \Rightarrow$

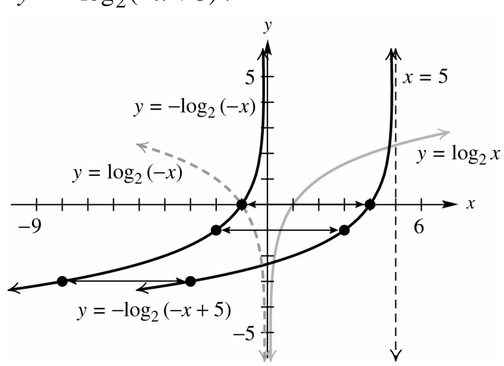
$-(-1-k) = \log_2 2 \Rightarrow 2^{1+k} = 2 \Rightarrow$

$2^{1+k} = 2^1 \Rightarrow 1+k = 1 \Rightarrow k = 0$

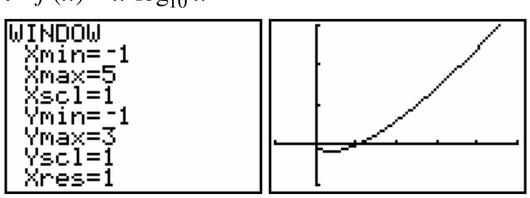
So, the equation is $f(x) = -\log_2(-x+5)$.

Verify that the coordinates of other two points given satisfy the equation.

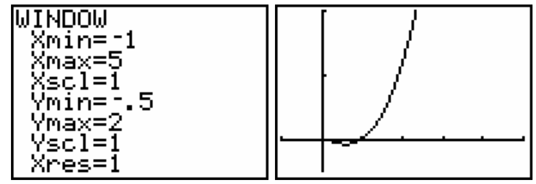
The figure below compares the graphs of $y = \log_2 x$, $y = \log_2(-x)$, $y = -\log_2(-x)$, and $y = -\log_2(-x+5)$.



57. $f(x) = x\log_{10} x$



58. $f(x) = x^2\log_{10} x$



59. $\log_2 \frac{6x}{y} = \log_2 6x - \log_2 y$
$= \log_2 6 + \log_2 x - \log_2 y$

60. $\log_3 \frac{4p}{q} = \log_3 4p - \log_3 q$
$= \log_3 4 + \log_3 p - \log_3 q$

61. $\log_5 \frac{5\sqrt{7}}{3} = \log_5 5\sqrt{7} - \log_5 3$
$= \log_5 5 + \log_5 \sqrt{7} - \log_5 3$
$= 1 + \log_5 7^{1/2} - \log_5 3$
$= 1 + \frac{1}{2}\log_5 7 - \log_5 3$

62. $\log_2 \frac{2\sqrt{3}}{5} = \log_2 2\sqrt{3} - \log_2 5$
$= \log_2 2 + \log_2 \sqrt{3} - \log_2 5$
$= 1 + \log_2 3^{1/2} - \log_2 5 = 1 + \frac{1}{2}\log_2 3 - \log_2 5$

63. $\log_4(2x+5y)$
Since this is a sum, none of the logarithm properties apply, so this expression cannot be simplified.

64. $\log_6(7m+3q)$
Since this is a sum, none of the logarithm properties apply, so this expression cannot be simplified.

65. $\log_m \sqrt{\frac{5r^3}{z^5}} = \log_m \left(\frac{5r^3}{z^5}\right)^{1/2} = \frac{1}{2}\log_m \frac{5r^3}{z^5}$
$= \frac{1}{2}\left(\log_m 5r^3 - \log_m z^5\right)$
$= \frac{1}{2}\left(\log_m 5 + \log_m r^3 - \log_m z^5\right)$
$= \frac{1}{2}\left(\log_m 5 + 3\log_m r - 5\log_m z\right)$

66. $\log_p \sqrt[3]{\frac{m^5n^4}{t^2}} = \log_p \left(\frac{m^5n^4}{t^2}\right)^{1/3} = \frac{1}{3}\log_p \left(\frac{m^5n^4}{t^2}\right)$
$= \frac{1}{3}\left(\log_p m^5 + \log_p n^4 - \log_p t^2\right)$
$= \frac{1}{3}\left(5\log_p m + 4\log_p n - 2\log_p t\right)$

67. $\log_2 \frac{ab}{cd} = \log_2(ab) - \log_2(cd)$
$= \log_2 a + \log_2 b - \left(\log_2 c + \log_2 d\right)$
$= \log_2 a + \log_2 b - \log_2 c - \log_2 d$

68. $\log_2 \frac{xy}{tqr} = \log_2(xy) - \log_2(tqr)$
$= \log_2 x + \log_2 y - (\log_2 t + \log_2 q + \log_2 r)$
$= \log_2 x + \log_2 y - \log_2 t - \log_2 q - \log_2 r$

69. $\log_3 \frac{\sqrt{x}\sqrt[3]{y}}{w^2\sqrt{z}} = \log_3\left(x^{1/2}y^{1/3}\right) - \log_3\left(w^2 z^{1/2}\right)$
$= \log_3 x^{1/2} + \log_3 y^{1/3} - \left(\log_3 w^2 + \log_3 z^{1/2}\right)$
$= \frac{1}{2}\log_3 x + \frac{1}{3}\log_3 y - \left(2\log_3 w + \frac{1}{2}\log_3 z\right)$
$= \frac{1}{2}\log_3 x + \frac{1}{3}\log_3 y - 2\log_3 w - \frac{1}{2}\log_3 z$

70. $\log_4 \frac{\sqrt[3]{a}\sqrt[4]{b}}{\sqrt{c}\sqrt[3]{d^2}} = \log_4\left(a^{1/3}b^{1/4}\right) - \log_4\left(c^{1/2}d^{2/3}\right)$
$= \log_4 a^{1/3} + \log_4 b^{1/4} - \left(\log_4 c^{1/2} + \log_4 d^{2/3}\right)$
$= \frac{1}{3}\log_4 a + \frac{1}{4}\log_4 b - \left(\frac{1}{2}\log_4 c + \frac{2}{3}\log_4 d\right)$
$= \frac{1}{3}\log_4 a + \frac{1}{4}\log_4 b - \frac{1}{2}\log_4 c - \frac{2}{3}\log_4 d$

71. $\log_a x + \log_a y - \log_a m = \log_a xy - \log_a m$
$= \log_a \frac{xy}{m}$

72. $\log_b k + \log_b m - \log_b a = \log_b(km) - \log_b a$
$= \log_b \frac{km}{a}$

73. $\log_a m - \log_a n - \log_a t$
$= \log_a m - (\log_a n + \log_a t)$
$= \log_a m - \log_a(nt) = \log_a \frac{m}{nt}$

74. $\log_b p - \log_b q - \log_b r$
$= \log_b p - (\log_b q + \log_b r)$
$= \log_b p - \log_b(qr) = \log_b \frac{p}{qr}$

75. $2\log_m a - 3\log_m b^2 = \log_m a^2 - \log_m\left(b^2\right)^3$
$= \log_m a^2 - \log_m b^6$
$= \log_m \frac{a^2}{b^6}$

76. $\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3$
$= \log_y\left(p^3q^4\right)^{1/2} - \log_y\left(p^4q^3\right)^{2/3}$
$= \log_y \frac{\left(p^3q^4\right)^{1/2}}{\left(p^4q^3\right)^{2/3}} = \log_y \frac{p^{3/2}q^2}{p^{8/3}q^2} = \log_y \frac{p^{9/6}}{p^{16/6}}$
$= \log_y\left(p^{(9/6)-(16/6)}\right) = \log_y\left(p^{-7/6}\right)$

77. $2\log_a(z+1) + \log_a(3z+2)$
$= \log_a(z+1)^2 + \log_a(3z+2)$
$= \log_a\left[(z+1)^2(3z+2)\right]$

78. $\log_b(2y+5) - \frac{1}{2}\log_b(y+3)$
$= \log_b(2y+5) - \log_b(y+3)^{1/2}$
$= \log_b \frac{2y+5}{(y+3)^{1/2}} = \log_b \frac{2y+5}{\sqrt{y+3}}$

79. $-\frac{2}{3}\log_5 5m^2 + \frac{1}{2}\log_5 25m^2$
$= \log_5\left(5m^2\right)^{-2/3} + \log_5\left(25m^2\right)^{1/2}$
$= \log_5\left[\left(5m^2\right)^{-2/3} \cdot \left(25m^2\right)^{1/2}\right]$
$= \log_5\left(5^{-2/3}m^{-4/3} \cdot 5m\right)$
$= \log_5\left(5^{-2/3} \cdot 5^1 \cdot m^{-4/3} \cdot m^1\right)$
$= \log_5\left(5^{1/3} \cdot m^{-1/3}\right) = \log_5 \frac{5^{1/3}}{m^{1/3}} = \log_5 \sqrt[3]{\frac{5}{m}}$

80. $-\frac{3}{4}\log_3 16p^4 - \frac{2}{3}\log_3 8p^3$
$= \log_3\left(16p^4\right)^{-3/4} - \log_3\left(8p^3\right)^{2/3}$
$= \log_3\left[\frac{\left(16p^4\right)^{-3/4}}{\left(8p^3\right)^{2/3}}\right] = \log_3 \frac{16^{-3/4}p^{-3}}{4p^2} = \log_3 \frac{2^{-3}}{4p^2 \cdot p^3}$
$= \log_3 \frac{1}{2^3 \cdot 4p^{2+3}} = \log_3 \frac{1}{8 \cdot 4p^5} = \log_3 \frac{1}{32p^5}$

81. $\log_{10} 6 = \log_{10}(2 \cdot 3) = \log_{10} 2 + \log_{10} 3$
$= .3010 + .4771 = .7781$

82. $\log_{10} 12 = \log_{10}\left(3 \cdot 2^2\right) = \log_{10} 3 + 2\log_{10} 2$
$= .4771 + 2(.3010) = .4771 + .6020$
$= 1.0791$

83. $\log_{10} \frac{3}{2} = \log_{10} 3 - \log_{10} 2$
$= .4771 - .3010 = .1761$

84. $\log_{10} \frac{2}{9} = \log_{10} 2 - \log_{10} 9 = \log_{10} 2 - \log_{10}\left(3^2\right)$
$= \log_{10} 2 - 2\log_{10} 3$
$= .3010 - 2(.4771) = -.6532$

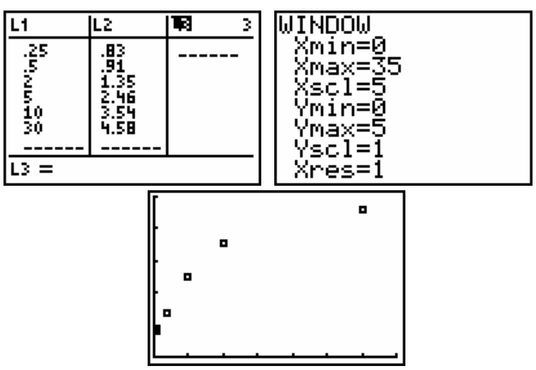
85. $\log_{10}\frac{9}{4}=\log_{10}9-\log_{10}4=\log_{10}3^2-\log_{10}2^2$
$=2\log_{10}3-2\log_{10}2$
$=2(.4771)-2(.3010)=.9542-.6020$
$=.3522$

86. $\log_{10}\frac{20}{27}=\log_{10}20-\log_{10}27$
$=\log_{10}(2\cdot 10)-\log_{10}3^3$
$=\log_{10}2+\log_{10}10-3\log_{10}3$
$=.3010+1-3(.4771)$
$=.3010+1-1.4313=-.1303$

87. $\log_{10}\sqrt{30}=\log_{10}30^{1/2}=\frac{1}{2}\log_{10}30$
$=\frac{1}{2}\log_{10}(10\cdot 3)$
$=\frac{1}{2}(\log_{10}10+\log_{10}3)=\frac{1}{2}(1+.4771)$
$=\frac{1}{2}(1.4771)\approx .7386$

88. $\log_{10}36^{1/3}=\frac{1}{3}\log_{10}36=\frac{1}{3}\log_{10}6^2$
$=\frac{2}{3}\log_{10}(2\cdot 3)=\frac{2}{3}(\log_{10}2+\log_{10}3)$
$=\frac{2}{3}(.3010+.4771)=\frac{2}{3}(.7781)$
$\approx .5187$

89. **(a)** If the x-values are representing years, 3 months is $\frac{3}{12}=\frac{1}{4}=.25$ yr and 6 months is $\frac{6}{12}=\frac{1}{2}=.5$ yr.



(b) A logarithmic function will model the data best.

90. **(a)** From the graph, $\log_3 .3=-1.1$

(b) From the graph, $\log_3 .8=-.2$

91. $f(x)=\log_a x$ and $f(3)=2$

$2=\log_a 3\Rightarrow a^2=3\Rightarrow (a^2)^{1/2}=3^{1/2}\Rightarrow a=\sqrt{3}$

(There is no $\pm$ because a must be positive and not equal to 1.) We now have $f(x)=\log_{\sqrt{3}}x$.

(a) $f\left(\frac{1}{9}\right)=\log_{\sqrt{3}}\frac{1}{9}\Rightarrow y=\log_{\sqrt{3}}\frac{1}{9}\Rightarrow$
$\left(\sqrt{3}\right)^y=\frac{1}{9}\Rightarrow \left(3^{1/2}\right)^y=\frac{1}{3^2}\Rightarrow$
$3^{y/2}=3^{-2}\Rightarrow \frac{y}{2}=-2\Rightarrow y=-4$

(b) $f(27)=\log_{\sqrt{3}}27\Rightarrow y=\log_{\sqrt{3}}27\Rightarrow$
$\left(\sqrt{3}\right)^y=27\Rightarrow \left(3^{1/2}\right)^y=3^3\Rightarrow$
$3^{y/2}=3^3\Rightarrow \frac{y}{2}=3\Rightarrow y=6$

(c) $f(9)=\log_{\sqrt{3}}9\Rightarrow y=\log_{\sqrt{3}}9\Rightarrow$
$\left(\sqrt{3}\right)^y=9\Rightarrow \left(3^{1/2}\right)^y=3^2\Rightarrow$
$3^{y/2}=3^2\Rightarrow \frac{y}{2}=2\Rightarrow y=4$

(d) $f\left(\frac{\sqrt{3}}{3}\right)=\log_{\sqrt{3}}\left(\frac{\sqrt{3}}{3}\right)\Rightarrow y=\log_{\sqrt{3}}\left(\frac{\sqrt{3}}{3}\right)\Rightarrow$
$\left(\sqrt{3}\right)^y=\frac{\sqrt{3}}{3}\Rightarrow \left(3^{1/2}\right)^y=\frac{3^{1/2}}{3^1}\Rightarrow$
$\left(3^{1/2}\right)^y=3^{1/2-1}=3^{-1/2}\Rightarrow$
$3^{y/2}=3^{-1/2}\Rightarrow \frac{y}{2}=-\frac{1}{2}\Rightarrow y=-1$

92. **(a)** $100^{\log 3}=10^{2\log 3}=10^{\log 3^2}=10^{\log 9}=9$

(b) $\log_{10}.01^3=\log_{10}\left(10^{-2}\right)^3$
$=\log_{10}10^{-6}=-6$

(c) $\log_{10}.0001^5=\log_{10}\left(10^{-4}\right)^5$
$=\log_{10}10^{-20}=-20$

(d) $1000^{\log_{10}5}=10^{3\log_{10}5}=10^{\log_{10}5^3}$
$=10^{\log_{10}125}=125$

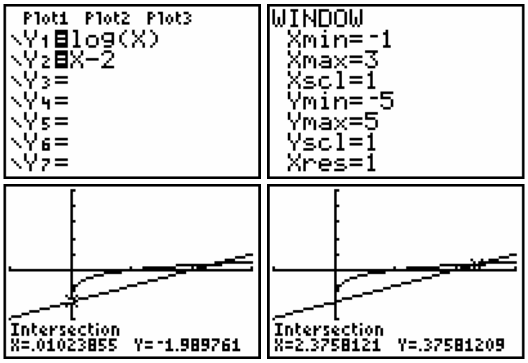
93. In the formula $A=P\left(1+\frac{r}{n}\right)^{tn}$ we substitute $A=2P$ since we want the present value to be doubled in the future. Thus, we need to solve for t in the equation $2P=P\left(1+\frac{r}{n}\right)^{tn}$.

$2P=P\left(1+\frac{r}{n}\right)^{tn}\Rightarrow 2=\left(1+\frac{r}{n}\right)^{tn}$
$2^{\frac{1}{tn}}=\left[\left(1+\frac{r}{n}\right)^{tn}\right]^{\frac{1}{tn}}\Rightarrow 2^{\frac{1}{tn}}=\left(1+\frac{r}{n}\right)$
$\log_2\left(1+\frac{r}{n}\right)=\frac{1}{tn}\Rightarrow tn=\frac{1}{\log_2\left(1+\frac{r}{n}\right)}$
$t=\frac{1}{n\log_2\left(1+\frac{r}{n}\right)}\Rightarrow t=\frac{1}{\log_2\left(1+\frac{r}{n}\right)^n}$

94. (5, 4) is equivalent to stating that when $x=5$ we have $y=4$. So for $f(x)=\log_a x$ or $y=\log_a x$, we have $4=\log_a 5$.

95. $\log_{10} x = x - 2$

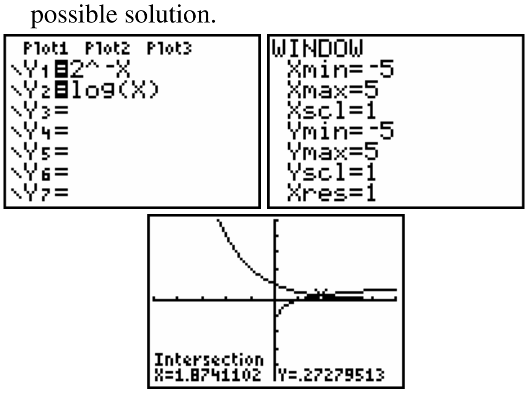
The x-coordinates of the intersection points will be the solutions to the given equation. There are two points of intersection, hence there are two possible solutions.



Solution set: {.01, 2.38}

96. $2^{-x} = \log_{10} x$

The x-coordinate of the intersection point will be the solution to the given equation. There is one point of intersection, hence there is one possible solution.



Solution set: {1.87}

97. Prove that $\log_a \frac{x}{y} = \log_a x - \log_a y$.

Let $m = \log_a x$ and $n = \log_a y$. Changing to exponential form we have $a^m = x$ and $a^n = y$. Since $\frac{x}{y} = \frac{a^m}{a^n}$ we have $\frac{x}{y} = a^{m-n}$. Changing to logarithmic form, we have $\log_a \frac{x}{y} = m - n$. Substituting for m and n we have $\log_a \frac{x}{y} = \log_a x - \log_a y$.

98. Prove: $\log_a x^r = r\log_a x$.

Let $m = \log_a x$. Changing to exponential form we have $a^m = x$. Raising both sides to the power r, we have $\left(a^m\right)^r = x^r$. This implies $x^r = a^{mr}$. Changing to logarithmic form, we have $\log_a x^r = mr$. Substituting for m we have $\log_a x^r = r\log_a x$.

Summary Exercises on Inverse, Exponential, and Logarithmic Functions

1. $f(x) = 3x - 4,\ g(x) = \frac{1}{3}x + \frac{4}{3}$

$(f \circ g)(x) = f\left[g(x)\right] = 3\left(\frac{1}{3}x + \frac{4}{3}\right) - 4$ Since
$= x + 4 - 4 = x$

$(g \circ f)(x) = g\left[f(x)\right] = \frac{1}{3}(3x - 4) + \frac{4}{3}$
$= \frac{3x}{3} = x$

$(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

2. $f(x) = 8 - 5x,\ g(x) = 8 + \frac{1}{5}x$

$(f \circ g)(x) = f[g(x)] = 8 - 5\left(8 + \frac{1}{5}x\right)$
$= 8 - 40 - x = -32 - x \neq x$

Since $(f \circ g)(x) \neq x$, the functions are not inverses. It is not necessary to check $(g \circ f)(x)$.

3. $f(x) = 1 + \log_2 x,\ g(x) = 2^{x-1}$

$(f \circ g)(x) = f\left[g(x)\right] = f\left(2^{x-1}\right) = 1 + \log_2 2^{x-1}$
$= 1 + (x-1)\log_2 2 = 1 + x - 1 = x$

$(g \circ f)(x) = g\left[f(x)\right] = g\left(1 + \log_2 x\right)$
$= 2^{(1+\log_2 x - 1)} = 2^{\log_2 x} = x$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

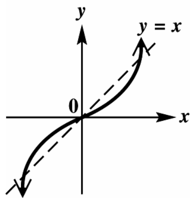
4. $f(x) = 3^{x/5} - 2,\ g(x) = 5\log_3(x+2)$

$(f \circ g)(x) = f\left[g(x)\right] = f\left(5\log_3(x+2)\right)$
$= 3^{\frac{1}{5}\left[5\log_3(x+2)\right]} - 2$
$= 3^{\log_3(x+2)} - 2 = (x+2) - 2 = x$

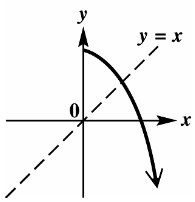
$(g \circ f)(x) = g\left[f(x)\right] = g\left(3^{x/5} - 2\right)$
$= 5\log_3\left[\left(3^{x/5} - 2\right) + 2\right]$
$= 5\log_3\left(3^{x/5}\right) = 5 \cdot \frac{x}{5} = x$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, these functions are inverses.

5. Since any horiztontal line when passed through the graph of this function will touch the graph in at most one place, the function is one-to-one. A sketch of the graph of the inverse function is as follows.

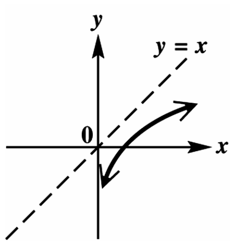


6. Since any horiztontal line when passed through the graph of this function will touch the graph in at most one place, the function is one-to-one. A sketch of the graph of the inverse function is as follows.



7. Since a horizontal line when passed through the graph of this function will touch the graph in more than one place, the function is not one-to-one.

8. Since any horiztontal line when passed through the graph of this function will touch the graph in at most one place, the function is one-to-one. A sketch of the graph of the inverse function is as follows.



9. Because $f(x) = \log_3(x+2)$ has a vertical asymptote when $x+2=0 \Rightarrow x=-2$, x-intercept when $x+2=1 \Rightarrow x=-1$, and is increasing, the correct choice is the graph in B.

10. The function $f(x) = 5 - 2^x$ is a reflection about the x-axis of $f(x) = 2^x$, shifted up 5 units. It has a y-intercept of $f(0) = 5 - 2^0 = 5 - 1 = 4$ and a horizontal asymptote of $y = 5$. The correct choice is the graph in D.

11. Because $f(x) = \log_2(5-x)$ has a vertical asymptote when $5 - x = 0 \Rightarrow x = 5$, x-intercept when $5 - x = 1 \Rightarrow x = 4$. The correct choice is the graph in C.

12. The function $f(x) = 3^x - 2$ is the graph of $f(x) = 3^x$, shifted down 2 units. It has a y-intercept of $f(0) = 3^0 - 2 = 1 - 2 = -1$ and a horizontal asymptote of $y = -2$. The correct choice is the graph in A.

13. The functions in Exercises 9 and 12 are inverses. The functions in Exercises 10 and 11 are inverses.

14. $f(x) = \log_5 x$

This function is one-to-one.

Step 1: Replace $f(x)$ with y and interchange x and y. $y = \log_5 x \Rightarrow x = \log_5 y$

Step 2: Solve for y. $x = \log_5 y \Rightarrow 5^x = y$

Step 3: Replace y with $f^{-1}(x)$. $f^{-1}(x) = 5^x$

15. $f(x) = 3x - 6$

This function is one-to-one.

Step 1: Replace $f(x)$ with y and interchange x and y. $y = 3x - 6 \Rightarrow x = 3y - 6$

Step 2: Solve for y.

$x = 3y - 6 \Rightarrow x + 6 = 3y \Rightarrow \frac{x+6}{3} = y$

Step 3: Replace y with $f^{-1}(x)$:

$f^{-1}(x) = \frac{x+6}{3} = \frac{1}{3}x + 2$

For both $f(x)$ and $f^{-1}(x)$, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.

16. $f(x) = 2(x+1)^3$

This function is one-to-one.

Step 1: Replace $f(x)$ with y and interchange x and y. $y = 2(x+1)^3 \Rightarrow x = 2(y+1)^3$

Step 2: Solve for y.

$x = 2(y+1)^3 \Rightarrow \frac{x}{2} = (y+1)^3 \Rightarrow$

$\sqrt[3]{\frac{x}{2}} = y + 1 \Rightarrow \sqrt[3]{\frac{x}{2}} - 1 = y$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{\frac{x}{2}} - 1$

For both $f(x)$ and $f^{-1}(x)$, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.

17. $f(x) = 3x^2$

If $x = 1$, $f(1) = 3(1)^2 = 3(1) = 3$. Also if $x = -1$, $f(-1) = 3(-1)^2 = 3(1) = 3$.

Since two different values of *x* lead to the same value of *y*, the function is not one-to-one.

18. $f(x) = \frac{2x-1}{5-3x}$

This function is one-to-one.

Step 1: Replace $f(x)$ with *y* and interchange *x* and *y*. $y = \frac{2x-1}{5-3x} \Rightarrow x = \frac{2y-1}{5-3y}$

Step 2: Solve for *y*.

$$x = \frac{2y-1}{5-3y} \Rightarrow x(5-3y) = 2y-1 \Rightarrow$$
$$5x - 3xy = 2y - 1 \Rightarrow 5x + 1 = 2y + 3xy \Rightarrow$$
$$5x + 1 = (2+3x)y \Rightarrow y = \frac{5x+1}{2+3x}$$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{5x+1}{2+3x}$

Domain of f = range of $f^{-1} = \left(-\infty, \frac{5}{3}\right) \cup \left(\frac{5}{3}, \infty\right)$;

Domain of f^{-1} = range of $f = \left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$

19. $f(x) = \sqrt[3]{5 - x^4}$

If $x = 1$, $f(1) = \sqrt[3]{5 - 1^4} = \sqrt[3]{5-1} = \sqrt[3]{4}$. Also if $x = -1$, $f(-1) = \sqrt[3]{5 - (-1)^4} = \sqrt[3]{5-1} = \sqrt[3]{4}$.

Since two different values of *x* lead to the same value of *y*, the function is not one-to-one.

20. $f(x) = \sqrt{x^2 - 9},\ x \ge 3$

This function is one-to-one.

Step 1: Replace $f(x)$ with *y* and interchange *x* and *y*. $y = \sqrt{x^2 - 9} \Rightarrow x = \sqrt{y^2 - 9}$

Step 2: Solve for *y*. In this problem we must consider that the range of *f* will be the domain of f^{-1}. We must also consider that the domain of *f* will be the range of f^{-1}.

$$x = \sqrt{y^2 - 9}$$
$$x^2 = \left(\sqrt{y^2-9}\right)^2, \text{ for } x \ge 0$$

(restriction because of range of *f*)

$$x^2 = y^2 - 9, \text{ for } x \ge 0$$
$$x^2 + 9 = y^2, \text{ for } x \ge 0$$
$$\sqrt{x^2 + 9} = y, \text{ for } x \ge 0$$

(positive square root because of the domain of *f*)

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \sqrt{x^2 + 9}$, for $x \ge 0$

Domain of *f* = range of $f^{-1} = [3, \infty)$

Range of *f* = domain of $f^{-1} = [0, \infty)$

21. $\left(\frac{1}{10}\right)^{-3} = 1000$ is equivalent to $\log_{1/10} 1000 = -3$.

22. $a^b = c$ is equivalent to $\log_a c = b$.

23. $\left(\sqrt{3}\right)^4 = 9$ is equivalent to $\log_{\sqrt{3}} 9 = 4$.

24. $4^{-3/2} = \frac{1}{8}$ is equivalent to $\log_4 \frac{1}{8} = -\frac{3}{2}$.

25. $2^x = 32$ is equivalent to $\log_2 32 = x$.

26. $27^{4/3} = 81$ is equivalent to $\log_{27} 81 = \frac{4}{3}$.

27. $3x = 7^{\log_7 6} \Rightarrow 3x = 6 \Rightarrow x = 2$

Solution set: $\{2\}$

28. $x = \log_{10} .001 \Rightarrow 10^x = .001 \Rightarrow 10^x = 10^{-3} \Rightarrow x = -3$

Solution set: $\{-3\}$

29. $x = \log_6 \frac{1}{216} \Rightarrow 6^x = \frac{1}{216} \Rightarrow 6^x = 6^{-3} \Rightarrow x = -3$

Solution set: $\{-3\}$

30. $\log_x 5 = \frac{1}{2} \Rightarrow x^{1/2} = 5 \Rightarrow \left(x^{1/2}\right)^2 = 5^2 \Rightarrow x = 25$

Recall from Chapter 1 that it is necessary to check all proposed solutions in the original equation when you raise both sides to a power.

Check $x = 25$.

$$\log_x 5 = \frac{1}{2}$$
$$\log_{25} 5 \stackrel{?}{=} \frac{1}{2}$$
$$\log_{25} \sqrt{25} = \frac{1}{2} \Rightarrow \log_{25} 25^{1/2} = \frac{1}{2} \Rightarrow$$
$$\frac{1}{2}\log_{25} 25 = \frac{1}{2} \Rightarrow \frac{1}{2} \cdot 1 = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$$

This is a true statement.

Solution set: $\{25\}$

31. $\log_{10} .01 = x$
$10^x = .01 \Rightarrow 10^x = 10^{-2} \Rightarrow x = -2$
Solution set: $\{-2\}$

32. $\log_x 3 = -1 \Rightarrow x^{-1} = 3 \Rightarrow x^{-1} = \left(\frac{1}{3}\right)^{-1} \Rightarrow$
$x = \frac{1}{3}$
Solution set: $\left\{\frac{1}{3}\right\}$

33. $\log_x 1 = 0 \Rightarrow x^0 = 1$
This is a true statement for all real numbers greater than 0, excluding 1.
Solution set: $(0,1) \cup (1,\infty)$

34. $x = \log_2 \sqrt{8} \Rightarrow 2^x = \sqrt{8} \Rightarrow 2^x = 8^{1/2} \Rightarrow$
$2^x = \left(2^3\right)^{1/2} \Rightarrow 2^x = 2^{3(1/2)} \Rightarrow$
$2^x = 2^{3/2} \Rightarrow x = \frac{3}{2}$
Solution set: $\left\{\frac{3}{2}\right\}$

35. $\log_x \sqrt[3]{5} = \frac{1}{3} \Rightarrow x^{1/3} = \sqrt[3]{5} \Rightarrow x^{1/3} = 5^{1/3} \Rightarrow$
$\left(x^{1/3}\right)^3 = \left(5^{1/3}\right)^3 \Rightarrow x = 5$
Recall from Chapter 1 that it is necessary to check all proposed solutions in the original equation when you raise both sides to a power.
Check $x = 5$.

$\log_x \sqrt[3]{5} = \frac{1}{3}$
$\log_5 \sqrt[3]{5} \stackrel{?}{=} \frac{1}{3}$
$\log_5 5^{1/3} = \frac{1}{3} \Rightarrow \frac{1}{3}\log_5 5 = \frac{1}{3} \Rightarrow \frac{1}{3} \cdot 1 = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{1}{3}$
This is a true statement.
Solution set: $\{5\}$

36. $\log_{1/3} x = -5 \Rightarrow \frac{1}{3}^{-5} = x \Rightarrow 3^5 = 243 = x$
Solution set: $\{243\}$

37. $\log_{10}\left(\log_2 2^{10}\right) = x \Rightarrow$
$\log_{10}\left(10\log_2 2\right) = x \Rightarrow \log_{10}\left(10 \cdot 1\right) = x \Rightarrow$
$\log_{10} 10 = x \Rightarrow x = 1$
Solution set: $\{1\}$

38. $x = \log_{4/5} \frac{25}{16} \Rightarrow \left(\frac{4}{5}\right)^x = \frac{25}{16} \Rightarrow \left(\frac{4}{5}\right)^x = \left(\frac{16}{25}\right)^{-1} \Rightarrow$
$\left(\frac{4}{5}\right)^x = \left[\left(\frac{4}{5}\right)^2\right]^{-1} \Rightarrow \left(\frac{4}{5}\right)^x = \left(\frac{4}{5}\right)^{-2} \Rightarrow x = -2$
Solution set: $\{-2\}$

39. $2x - 1 = \log_6 6^x \Rightarrow 2x - 1 = x \Rightarrow$
$-1 = -x \Rightarrow 1 = x$
Solution set: $\{1\}$

40. $x = \sqrt{\log_{1/2} \frac{1}{16}} \Rightarrow x^2 = \left(\sqrt{\log_{1/2} \frac{1}{16}}\right)^2 \Rightarrow$
$x^2 = \log_{1/2} \frac{1}{16} \Rightarrow \left(\frac{1}{2}\right)^{x^2} = \frac{1}{16} \Rightarrow$
$\left(\frac{1}{2}\right)^{x^2} = \left(\frac{1}{2}\right)^4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
Reject the negative solution.
Solution set: $\{2\}$

41. $2^x = \log_2 16 \Rightarrow 2^{\left(2^x\right)} = 16 \Rightarrow 2^{\left(2^x\right)} = 2^4 \Rightarrow$
$2^x = 4 \Rightarrow 2^x = 2^2 \Rightarrow x = 2$
Solution set: $\{2\}$

42. $\log_3 x = -2 \Rightarrow 3^{-2} = x \Rightarrow \frac{1}{3^2} = x \Rightarrow x = \frac{1}{9}$
Solution set: $\left\{\frac{1}{9}\right\}$

43. $\left(\frac{1}{3}\right)^{x+1} = 9^x \Rightarrow \left(3^{-1}\right)^{x+1} = \left(3^2\right)^x \Rightarrow$
$3^{(-1)(x+1)} = 3^{2x} \Rightarrow 3^{-x-1} = 3^{2x} \Rightarrow$
$-x - 1 = 2x \Rightarrow x = -\frac{1}{3}$
Solution set: $\left\{-\frac{1}{3}\right\}$

44. $5^{2x-6} = 25^{x-3} \Rightarrow 5^{2x-6} = \left(5^2\right)^{x-3} \Rightarrow$
$5^{2x-6} = 5^{2x-6} \Rightarrow 2x - 6 = 2x - 6$
This statement is true for every value of x.
Solution set: $\{(-\infty,\infty)\}$

Section 4.4: Evaluating Logarithms and the Change-of-Base Theorem

1. For $f(x) = a^x$, where $a > 0$, the function is *increasing* over its entire domain.
2. For $g(x) = \log_a x$, where $a > 1$, the function is *increasing* over its entire domain.
3. $f(x) = 5^x$
   This function is one-to-one.
   Step 1: Replace $f(x)$ with y and interchange x and y. $y = 5^x \Rightarrow x = 5^y$
   Step 2: Solve for y.
   $x = 5^y \Rightarrow y = \log_5 x$

(*continued on next page*)

(*continued from page 429*)

Step 3: Replace y with $f^{-1}(x)$.
$f^{-1}(x) = \log_5 x$

4. Since $4^{\log_4 11} = 11$, the exponent to which 4 must be raised to obtain 11 is $\log_4 11$.

5. A base e logarithm is called a natural logarithm, while a base 10 logarithm is called a common logarithm.

6. $\log_3 12 = \frac{\ln 12}{\ln 3}$

7. $\log_2 0$ is undefined because there is no power of 2 that yields a result of 0. In other words, the equation $2^x = 0$ has no solution.

8. Let $\log_2 12 = x$. This implies $2^x = 12$. Since $2^3 = 8$ and $2^4 = 16$, $\log_2 12$ must lie between 3 and 4.

9. $\log 8 = .90308999$

10. $\ln 2.75 = 1.0116009$

11. $\log 53 \approx 1.7243$

12. $\log 87 \approx 1.9395$

13. $\log .0013 \approx -2.8861$

14. $\log .078 \approx -1.1079$

15. $\ln 53 \approx 3.9703$

16. $\ln 87 \approx 4.4659$

17. $\ln .0013 \approx -6.6454$

18. $\ln .078 \approx -2.5510$

19. $\log\left(3.1\times 10^4\right) \approx 4.4914$

20. $\log\left(8.6\times 10^3\right) \approx 3.9345$

21. $\log\left(5.0\times 10^{-6}\right) \approx -5.3010$

22. $\log\left(6.6\times 10^{-7}\right) \approx -6.1805$

23. $\ln\left(6\times e^4\right) \approx 5.7918$

24. $\ln\left(9\times e^4\right) \approx 5.1972$

25. $\log(387\times 23) \approx 3.9494$

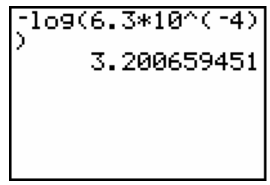
26. $\log \frac{584}{296} = .2951$

27. $\log(387\times 23) = \log 387 + \log 23$ by the product property of logarithms.

28. $\log \frac{584}{296} = \log 584 - \log 296$ by the quotient property of logarithms.

29. Grapefruit, $f(x) = \sqrt[3]{2x-7}$

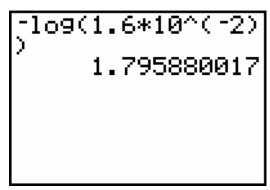
$$\text{pH} = -\log\left[H_3 0^+\right] = -\log\left(6.3\times 10^{-4}\right)$$
$$= -\left(\log 6.3 + \log 10^{-4}\right) = -(.7793 - 4)$$
$$= -.7993 + 4 \approx 3.2$$



The answer is rounded to the nearest tenth because it is customary to round pH values to the nearest tenth. The pH of grapefruit is 3.2.

30. Limes, $y = \log_{1/3} x$

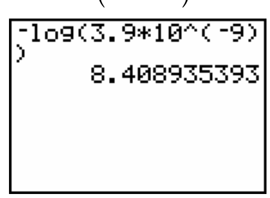
$$\text{pH} = -\log\left[H_3 0^+\right] = -\log\left(1.6\times 10^{-2}\right)$$
$$= -\left(\log 1.6 + \log 10^{-2}\right) = -(.2041 - 2)$$
$$= -(-1.7959) \approx 1.8$$



The pH of limes is 1.8.

31. Crackers, $f(x) = \sqrt[3]{2x-7}$

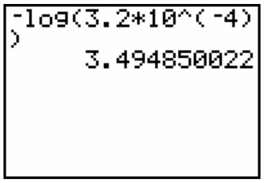
$$\text{pH} = -\log\left[H_3 0^+\right] = -\log\left(3.9\times 10^{-9}\right)$$
$$= -\left(\log 3.9 + \log 10^{-9}\right) = -(.59106 - 9)$$
$$= -(-8.409) \approx 8.4$$



The answer is rounded to the nearest tenth because it is customary to round pH values to the nearest tenth. The pH of crackers is 8.4.

32. Sodium hydroxide (lye), 3.2×10^{-4}

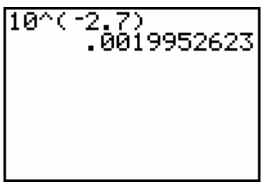
$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right] = -\log\left(3.2\times10^{-14}\right)\\ &= -\left(\log 3.2+\log 10^{-14}\right) = -\left(.5052-14\right)\\ &= -\left(-13.49\right)\approx 13.5\end{aligned}$$



The pH of sodium hydroxide is 13.5.

33. Soda pop, 2.7

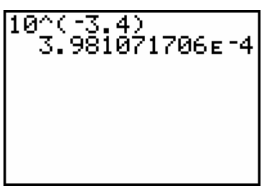
$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right]\\ 2.7 &= -\log\left[\text{H}_3 0^+\right]\\ -2.7 &= \log\left[\text{H}_3 0^+\right] \Rightarrow \left[\text{H}_3 0^+\right] = 10^{-2.7}\end{aligned}$$



$\left[\text{H}_3 0^+\right]\approx 2.0\times10^{-3}$

34. Wine, 3.4

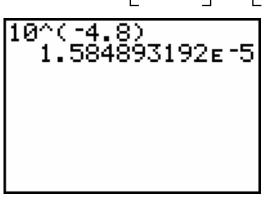
$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right]\\ 3.4 &= -\log\left[\text{H}_3 0^+\right]\\ -3.4 &= \log\left[\text{H}_3 0^+\right] \Rightarrow \left[\text{H}_3 0^+\right] = 10^{-3.4}\end{aligned}$$



$\left[\text{H}_3 0^+\right]\approx 4.0\times10^{-4}$

35. Beer, 4.8

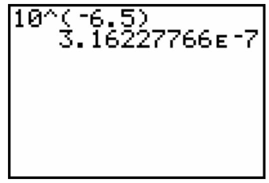
$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right]\\ 4.8 &= -\log\left[\text{H}_3 0^+\right]\\ -4.8 &= \log\left[\text{H}_3 0^+\right] \Rightarrow \left[\text{H}_3 0^+\right] = 10^{-4.8}\end{aligned}$$



$\left[\text{H}_3 0^+\right]\approx 1.6\times10^{-5}$

36. Drinking water, 6.5

$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right]\\ 6.5 &= -\log\left[\text{H}_3 0^+\right]\\ -6.5 &= \log\left[\text{H}_3 0^+\right] \Rightarrow \left[\text{H}_3 0^+\right] = 10^{-6.5}\end{aligned}$$



$\left[\text{H}_3 0^+\right]\approx 3.2\times10^{-7}$

37. Wetland, 2.49×10^{-5}

$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right] = -\log\left(2.49\times10^{-5}\right)\\ &= -\left(\log 2.49+\log 10^{-5}\right)\\ &= -\log 2.49-\left(-5\right) = -\log 2.49+5\\ \text{pH} &\approx 4.6\end{aligned}$$

Since the pH is between 4.0 and 6.0, it is a poor fen.

38. Wetland, 6.22×10^{-5}

$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right] = -\log\left(6.22\times10^{-5}\right)\\ &= -\left(\log 6.22+\log 10^{-5}\right)\\ &= -\log 6.22-\left(-5\right) = -\log 6.22+5\\ \text{pH} &\approx 4.2\end{aligned}$$

Since the pH is between 4.0 and 6.0, it is a poor fen.

39. Wetland, 2.49×10^{-2}

$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right] = -\log\left(2.49\times10^{-2}\right)\\ &= -\left(\log 2.49+\log 10^{-2}\right)\\ &= -\log 2.49-\left(-2\right) = -\log 2.49+2\\ \text{pH} &\approx 1.6\end{aligned}$$

Since the pH is 3.0 or less, it is a bog.

40. Wetland, 3.14×10^{-2}

$$\begin{aligned}\text{pH} &= -\log\left[\text{H}_3 0^+\right] = -\log\left(3.14\times10^{-2}\right)\\ &= -\left(\log 3.14+\log 10^{-2}\right)\\ &= -\log 3.14-\left(-2\right) = -\log 3.14+2\\ \text{pH} &\approx 1.5\end{aligned}$$

Since the pH is 3.0 or less, it is a bog.

41. Wetland, 2.49×10^{-7}

$$\begin{aligned}\text{pH} &= -\log\left[H_3 0^+\right] = -\log\left(2.49\times10^{-7}\right)\\ &= -\left(\log 2.49 + \log 10^{-7}\right)\\ &= -\log 2.49 - (-7) = -\log 2.49 + 7\end{aligned}$$

$\text{pH} \approx 6.6$

Since the pH is greater than 6.0, it is a rich fen.

42. Wetland, 5.86×10^{-7}

$$\begin{aligned}\text{pH} &= -\log\left[H_3 0^+\right] = -\log\left(5.86\times10^{-7}\right)\\ &= -\left(\log 5.86 + \log 10^{-7}\right)\\ &= -\log 5.86 - (-7) = -\log 5.86 + 7\end{aligned}$$

$\text{pH} \approx 6.2$

Since the pH is greater than 6.0, it is a rich fen.

43. **(a)** log 398.4 ≈ 2.60031933

(b) log 39.84 ≈ 1.60031933

(c) log 3.984 ≈ .6003193298

(d) The whole number parts will vary, but the decimal parts will be the same.

44. $\log 25,000 = 4.3979$. The pattern continues because each number in the pattern is of the form 2.5×10^n, n = 1, 2, 3, … The whole number portion of the logarithm (called the *characteristic*) is the power of 10.

45. $d = 10\log\frac{I}{I_0}$, where d is the decibel rating.

(a) $d = 10\log\frac{100I_0}{I_0}$
$= 10\log_{10}100 = 10(2) = 20$

(b) $d = 10\log\frac{1000I_0}{I_0}$
$= 10\log_{10}1000 = 10(3) = 30$

(c) $d = 10\log\frac{100,000I_0}{I_0}$
$= 10\log_{10}100,000 = 10(5) = 50$

(d) $d = 10\log\frac{1,000,000I_0}{I_0}$
$= 10\log_{10}1,000,000 = 10(6) = 60$

(e) $I = 2I_0$
$d = 10\log\frac{2I_0}{I_0} = 10\log 2 \approx 3.0103$

The described rating is increased by about 3 decimals.

46. $d = 10\log\frac{I}{I_0}$, where d is the decibel rating.

(a) $d = 10\log\frac{115I_0}{I_0}$
$= 10\log 115 \approx 21$

(b) $d = 10\log\frac{9,500,000I_0}{I_0}$
$= 10\log 9,500,000 \approx 70$

(c) $d = 10\log\frac{1,200,000,000I_0}{I_0}$
$= 10\log 1,200,000,000 \approx 91$

(d) $d = 10\log\frac{895,000,000,000I_0}{I_0}$
$= 10\log 895,000,000,000 \approx 120$

(e) $d = 10\log\frac{109,000,000,000,000I_0}{I_0}$
$= 10\log 109,000,000,000,000 \approx 140$

47. $r = \log_{10}\frac{I}{I_0}$, where r is the Richter scale rating of an earthquake.

(a) $r = \log_{10}\frac{1000I_0}{I_0} = \log_{10}1000 = 3$

(b) $r = \log_{10}\frac{1,000,000I_0}{I_0} = \log_{10}1,000,000 = 6$

(c) $r = \log_{10}\frac{100,000,000I_0}{I_0}$
$= \log_{10}100,000,000 = 8$

48. From exercise 47, the magnitude of an earthquake, measured on the Richter scale, is $r = \log_{10}\frac{I}{I_0}$, where I is the amplitude registered on a seismograph 100 km fro the epicenter of the earthquake, and I_0 is the amplitude of an earthquake of a certain small size. So, $9.1 = \log_{10}\frac{I}{I_0} \Rightarrow \frac{I}{I_0} = 10^{9.1} \Rightarrow$
$I = 10^{9.1}I_0 = 1,258,925,412I_0$

49. From exercise 47, the magnitude of an earthquake, measured on the Richter scale, is $r = \log_{10}\frac{I}{I_0}$, where I is the amplitude registered on a seismograph 100 km fro the epicenter of the earthquake, and I_0 is the amplitude of an earthquake of a certain small size. So, $8.6 = \log_{10}\frac{I}{I_0} \Rightarrow \frac{I}{I_0} = 10^{8.6} \Rightarrow$
$I = 10^{8.6}I_0 \approx 398,107,171I_0$

50. $\dfrac{10^{9.1}}{10^{8.6}} = 10^{.5} \approx 3.16$

The force of the 2004 earthquake was about 3.16 times greater than the force of the 2005 earthquake.

51. The year 2009 is represented by 109.

$f(109) = -85.4 + 32.4\ln 109 \approx 66.6$ million

We must assume that the model continues to be logarithmic.

52. **(a)** $f(t) = 74.61 + 3.84\ln t,\ t \geq 1$

In the year 2002, $t = 8$

$f(8) = 74.61 + 3.84\ln 8 \approx 82.5951$

Thus, the percent of freshmen entering college in 2002 who performed volunteer work during their last year of high school will be about 82.595%. This is exceptionally close to the percent shown in the graph and to the actual percent of 82.6%.

(b) Answers will vary.

53. If $a = .36$, then

$S(n) = a\ln\left(1+\frac{n}{a}\right) = .36\ln\left(1+\frac{n}{.36}\right).$

(a) $S(100) = .36\ln\left(1+\frac{100}{.36}\right) \approx 2.0269 \approx 2$

(b) $S(200) = .36\ln\left(1+\frac{200}{.36}\right) \approx 2.2758 \approx 2$

(c) $S(150) = .36\ln\left(1+\frac{150}{.36}\right) \approx 2.1725 \approx 2$

(d) $S(10) = .36\ln\left(1+\frac{10}{.36}\right) \approx 1.2095 \approx 1$

54. If $a = .88$, then

$S(n) = a\ln\left(1+\frac{n}{a}\right) = .88\ln\left(1+\frac{n}{.88}\right).$

(a) $S(50) = .88\ln\left(1+\frac{50}{.88}\right) \approx 3.5704 \approx 4$

(b) $S(100) = .88\ln\left(1+\frac{100}{.88}\right) \approx 4.1728 \approx 4$

(c) $S(250) = .88\ln\left(1+\frac{250}{.88}\right) \approx 4.9745 \approx 5$

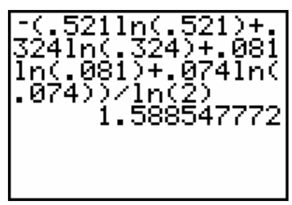
55. The index of diversity H for 2 species is given by $H = -\left[P_1\log_2 P_1 + P_2\log_2 P_2\right]$. When $P_1 = \frac{50}{100} = .5$ and $P_2 = \frac{50}{100} = .5$ we have $H = -\left[.5\log_2 .5 + .5\log_2 .5\right]$.

Since $\log_2 .5 = \log_2 \frac{1}{2} = \log_2 2^{-1} = -1$, we have $H = -\left[.5(-1) + .5(-1)\right] = -(-1) = 1$. Thus, the index of diversity is 1.

56.
$$H = \begin{bmatrix} P_1\log_2 P_1 + P_2\log_2 P_2 \\ + P_3\log_2 P_3 + P_4\log_2 P_4 \end{bmatrix} = -\begin{bmatrix} .521\log_2 .521 + .324\log_2 .324 \\ +.081\log_2 .081 \\ +.074\log_2 .074 \end{bmatrix}$$

We need the change-of-base theorem to calculate each term. Using the natural log we have the following.

$$H = -\begin{bmatrix} .521\log_2 .521 + .324\log_2 .324 \\ +.081\log_2 .081 \\ +.074\log_2 .074 \end{bmatrix}$$
$$= -\begin{bmatrix} .521\frac{\ln .521}{\ln 2} + .324\frac{\ln .324}{\ln 2} + .081\frac{\ln .081}{\ln 2} \\ +.074\frac{\ln .074}{\ln 2} \end{bmatrix}$$
$$= -\begin{bmatrix} .521\ln .521 + .324\ln .324 \\ +.081\ln .081 \\ +.074\ln .074 \end{bmatrix} \Big/ \ln 2$$



The index of diversity approximately1.59.

57. $T(k) = 1.03k\ln\frac{C}{C_0}$

Since $10 \leq k \leq 16$ and $\frac{C}{C_0} = 2$, the range for $T = 1.03k\ln\frac{C}{C_0}$ will be between $T(10)$ and $T(16)$. Since $T(10) = 1.03(10)\ln 2 \approx 7.1$ and $T(16) = 1.03(16)\ln 2 \approx 11.4$, the predicted increased global temperature due to the greenhouse effect from a doubling of the carbon dioxide in the atmosphere is between 7°F and 11°F.

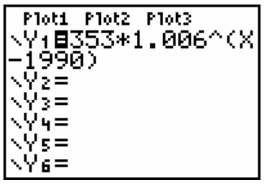
58. (a) $T(C) = 6.489 \ln \frac{C}{280}$ and

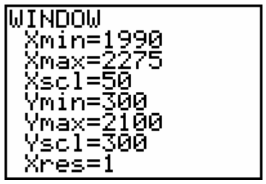
$C(x) = 353(1.006)^{x-1990}$

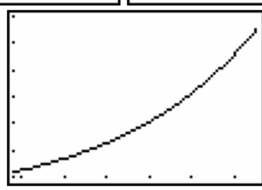
To find T as a function of x, we need to find $T[C(x)]$.

$$T[C(x)] = T(x)$$
$$= 6.489 \ln \left[\frac{353(1.006)^{x-1990}}{280} \right]$$

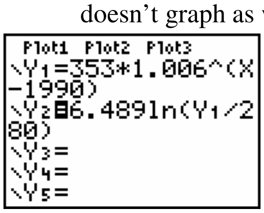
(b)

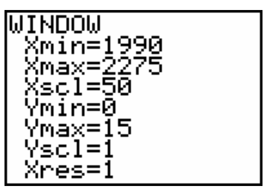


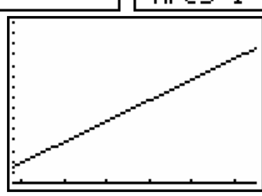




To graph T, you can either type the function in directly, or use Y_1 as defined before. From the VARS menu, choose Y-VARS, choose Function, then choose Y_1 to be inserted into Y_2. Be sure to press the equal sign on Y_1 so that it doesn't graph as well.



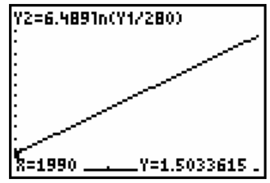


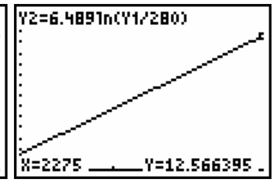


C is an exponential function, and T is a linear function over the same time period. While the carbon dioxide levels in the atmosphere increase at an exponential rate, the average global temperature will rise at a linear rate.

(c) The slope of the graph of T can be approximated using two points. Since $T(1990) \approx 1.5033615$ and $T(2275) \approx 12.566395$, the slope is

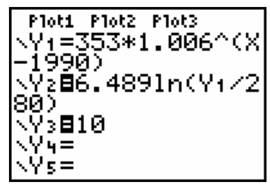
$$m = \frac{12.566395 - 1.5033615}{2275 - 1990} = .0388.$$

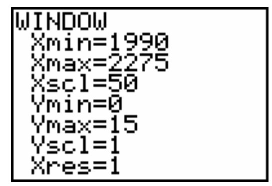


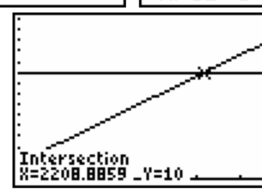


This means that temperature is expected to rise at an average rate of .04°F per year from 1990 to 2275.

(d) Graph $y = T(x)$ and $y = 10$ on the same screen.







Their graphs intersect when $x \approx 2208.9$

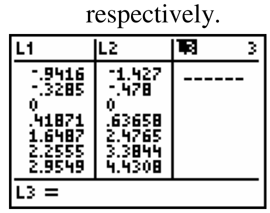
$C(2208.9) = 353(1.006)^{2208.9-1990}$

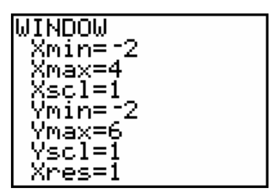
≈ 1308 ppm. Since $C_0 = 280$, this would be about $\frac{1308}{280} \approx 4.67$ times above preindustrial carbon dioxide levels.

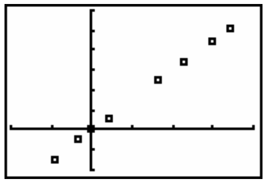
59. $t = (1.26 \times 10^9) \frac{\ln[1+8.33(.103)]}{\ln 2} \approx 1.126 \times 10^9$

The rock sample is approximately 1.126 billion years old.

60. (a) In L_1 and L_2 enter the natural D and P, respectively.







If we let $x = \ln D$ and $y = \ln P$, then the data points appear to be linear. Note: There is one more point graphed that is not shown in the list, namely (3.4045, 5.1047).

(b) Choose two points from the list and find the equation through them. If we use (0, 0), representing Earth and $(2.9549, 4.4308)$ representing Uranus, we obtain $m = \dfrac{4.4308 - 0}{2.9549 - 0} \approx 1.5.$ Since the y-intercept is 0, the equation is $y = 1.5x$ or $\ln P = 1.5 \ln D$. Note: Since the points lie approximately but not exactly on a line, a slightly different equation will be found if a different pair of points is used.

(c) For Pluto, $D = 39.5$, so $\ln P = 1.5 \ln D = 1.5 \ln 39.5.$ Solve this equation for P.
$P = e^{1.5 \ln 39.5} = e^{\ln 39.5^{1.5}} = (39.5)^{1.5} \approx 248.3$
The linear equation predicts that the period of the planet Pluto is 248.3 years, which is very close to the true value of 248.5 years.

For Exercises 61−72, as noted on page 452 of the text, the solutions will be evaluated at the intermediate steps to four decimal places. However, the final answers are obtained without rounding the intermediate steps.

61. $\log_2 5 = \frac{\ln 5}{\ln 2} \approx \frac{1.6094}{.6931} \approx 2.3219$
We could also have used the common logarithm. $\log_2 5 = \frac{\log 5}{\log 2} \approx \frac{.6990}{.3010} \approx 2.3219$

62. $\log_2 9 = \frac{\ln 9}{\ln 2} \approx \frac{2.1972}{.6931} \approx 3.1699$
We could also have used the common logarithm. $\log_2 9 = \frac{\log 9}{\log 2} \approx \frac{.9542}{.3010} \approx 3.1699$

63. $\log_8 .59 = \frac{\log .59}{\log 8} \approx \frac{-.2291}{.9031} \approx -.2537$
We could also have used the natural logarithm. $\log_8 .59 = \frac{\ln .59}{\ln 8} \approx \frac{-.5276}{2.0794} \approx -.2537$

64. $\log_8 .71 = \frac{\log .71}{\log 8} \approx \frac{-.1487}{.9031} \approx -.1647$
We could also have used the natural logarithm. $\log_8 .71 = \frac{\ln .71}{\ln 8} \approx \frac{-.3425}{2.0794} \approx -.1647$

65. $\log_{1/2} 3 = \frac{\log 3}{\log[1/2]} \approx \frac{.4771}{-.3010} \approx -1.5850$
We could also have used the natural logarithm. $\log_{1/2} 3 = \frac{\ln 3}{\ln[1/2]} \approx \frac{1.0986}{-.6931} \approx -1.5850$

66. $\log_{1/3} 2 = \frac{\log 2}{\log[1/3]} \approx \frac{.3010}{-.4771} \approx -.6309$
We could also have used the natural logarithm. $\log_{1/3} 2 = \frac{\ln 2}{\ln[1/3]} \approx \frac{.6931}{-1.0986} \approx -.6309$

67. $\log_\pi e = \frac{\ln e}{\ln \pi} \approx \frac{1}{1.1447} \approx .8736$
We could also have used the common logarithm. $\log_\pi e = \frac{\log e}{\log \pi} \approx \frac{.4343}{.4971} \approx .8736$

68. Since $\sqrt{2} = 2^{1/2}$, we have
$\log_\pi \sqrt{2} = \frac{\ln 2^{1/2}}{\ln \pi} = \frac{\frac{1}{2}\ln 2}{\ln \pi} \approx \frac{.3466}{1.1447} \approx .3028$
We could also have used the common logarithm.
$\log_\pi \sqrt{2} = \frac{\log \sqrt{2}}{\log \pi} = \frac{\frac{1}{2}\ln 2}{\ln \pi} \approx \frac{.1505}{.4971} \approx .3028$

69. Since $\sqrt{13} = 13^{1/2}$, we have
$\log_{\sqrt{13}} 12 = \frac{\ln 12}{\ln \sqrt{13}} = \frac{\ln 12}{\frac{1}{2}\ln 13} \approx \frac{2.4849}{1.2825} \approx 1.9376.$
The required logarithm can also be found by entering $\ln \sqrt{13}$ directly into the calculator. We could also have used the common logarithm.
$\log_{\sqrt{13}} 12 = \frac{\log 12}{\log \sqrt{13}} = \frac{\log 12}{\frac{1}{2}\log 13} \approx \frac{1.0792}{.5570} \approx 1.9376.$

70. Since $\sqrt{19} = 19^{1/2}$, we have
$\log_{\sqrt{19}} 5 = \frac{\ln 5}{\ln \sqrt{19}} = \frac{\ln 5}{\frac{1}{2}\ln 19} \approx \frac{1.6094}{1.4722} \approx 1.0932$
The required logarithm can also be found by entering $\ln \sqrt{19}$ directly into the calculator. We could also have used the common logarithm.
$\log_{\sqrt{19}} 5 = \frac{\log 5}{\log \sqrt{19}} = \frac{\log 5}{\frac{1}{2}\log 19} \approx \frac{.6990}{.6394} \approx 1.0932$

71. $\log_{.32} 5 = \frac{\log 5}{\log .32} \approx \frac{.6990}{-.4949} \approx -1.4125$
We could also have used the natural logarithm. $\log_{.32} 5 = \frac{\ln 5}{\ln .32} \approx \frac{1.6094}{-1.1394} \approx -1.4125$

72. $\log_{.91} 8 = \frac{\log 8}{\log .91} \approx \frac{.9031}{-.0410} \approx -22.0488$
We could also have used the natural logarithm. $\log_{.91} 8 = \frac{\ln 8}{\ln .91} \approx \frac{2.0794}{-.0943} \approx -22.0488$

73. $\ln\left(b^4\sqrt{a}\right) = \ln\left(b^4 a^{1/2}\right) = \ln b^4 + \ln a^{1/2}$
$= 4\ln b + \frac{1}{2}\ln a = 4v + \frac{1}{2}u$

74. $\ln \frac{a^3}{b^2} = \ln a^3 - \ln b^2 = 3\ln a - 2\ln b = 3u - 2v$

75. $\ln\sqrt{\frac{a^3}{b^5}} = \ln\left(\frac{a^3}{b^5}\right)^{1/2} = \ln\left(\frac{a^{3/2}}{b^{5/2}}\right) = \ln a^{3/2} - \ln b^{5/2}$
$= \frac{3}{2}\ln a - \frac{5}{2}\ln b = \frac{3}{2}u - \frac{5}{2}v$

76. $\ln\left(\sqrt[3]{a} \cdot b^4\right) = \ln\left(a^{1/3} \cdot b^4\right) = \ln a^{1/3} + \ln b^4$
$= \frac{1}{3}\ln a + 4\ln b = \frac{1}{3}u + 4v$

77. $g(x) = e^x$

(a) $g(\ln 4) = e^{\ln 4} = 4$

(b) $g\left[\ln\left(5^2\right)\right] = e^{\ln 5^2} = 5^2$ or 25

(c) $g\left[\ln\left(\frac{1}{e}\right)\right] = e^{\ln(1/e)} = \frac{1}{e}$

78. $f(x) = 3^x$

(a) $f(\log_3 2) = 3^{\log_3 2} = 2$

(b) $f\left[\log_3(\ln 3)\right] = 3^{\log_3(\ln 3)} = \ln 3$

(c) $f\left[\log_3(2\ln 3)\right] = 3^{\log_3(2\ln 3)}$
$= 2\ln 3$ or $\ln 9$

79. $f(x) = \ln x$

(a) $f\left(e^6\right) = \ln e^6 = 6$

(b) $f\left(e^{\ln 3}\right) = \ln e^{\ln 3} = \ln 3$

(c) $f\left(e^{2\ln 3}\right) = \ln e^{2\ln 3} = 2\ln 3$ or $\ln 9$.

80. $f(x) = \log_2 x$

(a) $f\left(2^7\right) = \log_2 2^7 = 7$

(b) $f\left(2^{\log_2 2}\right) = \log_2 2^{\log_2 2} = \log_2 2 = 1$

(c) $f\left(2^{2\log_2 2}\right) = \log_2 2^{2\log_2 2}$
$= 2\log_2 2 = 2 \cdot 1 = 2$

81. $2\ln 3x = \ln(3x)^2 = \ln\left(3^2 x^2\right) = \ln 9x^2$
It is equivalent to D.

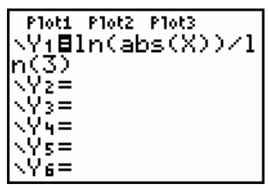
82. $\ln(4x) - \ln(2x) = \ln\frac{4x}{2x} = \ln\frac{4}{2} = \ln 2$
It is equivalent to D.

83. $f(x) = \ln|x|$
The domain of *f* is all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$ and the range is $(-\infty, \infty)$.
Since $f(-x) = \ln|-x| = \ln|x| = f(x)$, this is an even function and symmetric with respect to the *y*-axis.

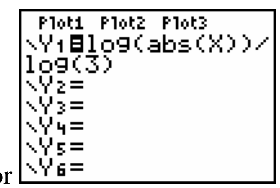
84. $f(x) = \log_3|x|$

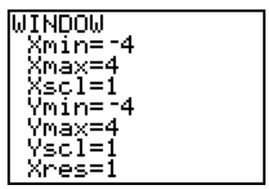
(a) The domain of *f* is all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$.

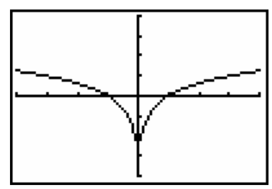
(b) To graph $f(x) = \log_3|x|$, we must use the change-of-base theorem to graph either $Y_1 = \frac{\ln|x|}{\ln 3}$ or $Y_1 = \frac{\log|x|}{\log 3}$.



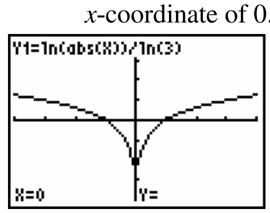
or

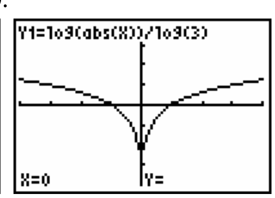






(c) The graph appears to show a point with *x*-value zero, indicating a domain of $(-\infty, \infty)$, which is incorrect. If we trace on either function (ln or log) we will see that there is not a *y*-coordinate, given the *x*-coordinate of 0.





85. $f(x) = \ln e^2 x = \ln e^2 + \ln x = 2 + \ln x$
$f(x) = \ln e^2 x$ is a vertical shift of the graph of $g(x) = \ln x$, 2 units up.

86. $f(x) = \ln \frac{x}{e} = \ln x - \ln e = \ln x - 1$

$f(x) = \ln \frac{x}{e}$ is a vertical shift of the graph of $g(x) = \ln x$, 1 unit down.

87. $f(x) = \ln \frac{x}{e^2} = \ln x - 2\ln e = \ln x - 2$

$f(x) = \ln \frac{x}{e^2}$ is a vertical shift of the graph of $g(x) = \ln x$, 2 units down.

88. In the last line, $2 < 1$, the sign should have flipped because the value of $\log \frac{1}{3} \approx -.4771$ is negative and whenever you divide (or multiply) both sides of an inequality by a negative value, the sign flips.

Chapter 4 Quiz
(Sections 4.1–4.4)

1. *Step 1*: Replace $f(x)$ with *y* and interchange *x* and *y*: $y = \sqrt[3]{3x-6} \Rightarrow x = \sqrt[3]{3y-6}$

Step 2: Solve for *y*.

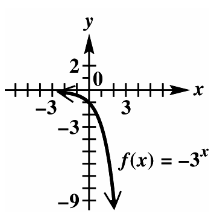
$x = \sqrt[3]{3y-6} \Rightarrow x^3 = 3y - 6 \Rightarrow \frac{x^3+6}{3} = y$

Step 3: Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \frac{x^3+6}{3}$

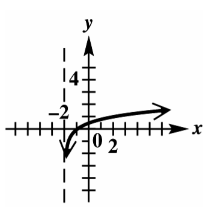
2. $4^{2x+1} = 8^{3x-6} \Rightarrow \left(2^2\right)^{2x+1} = \left(2^3\right)^{3x-6} \Rightarrow$
$2^{2(2x+1)} = 2^{3(3x-6)} \Rightarrow 2^{4x+2} = 2^{9x-18} \Rightarrow$
$4x + 2 = 9x - 18 \Rightarrow -5x = -20 \Rightarrow x = 4$
Solution set: $\{4\}$

3.



Domain: $(-\infty, \infty)$; range: $(-\infty, 0)$

4.



$f(x) = \log_4 (x + 2)$

Domain: $(-2, \infty)$; range: $(-\infty, \infty)$

5. Use the compound interest formula to find the amount in the account, $A = P\left(1 + \frac{r}{n}\right)^{tn}$, given $n = 4$, $P = 2000$, $r = .036$, and $t = 2$.

$A = P\left(1 + \frac{r}{n}\right)^{tn} = (2000)\left(1 + \frac{.036}{4}\right)^{2(4)}$
$= (2000)(1 + .009)^8 \approx 2148.62$

Rounding to the nearest cent, there will be \$2148.62 after 2 years.

6. **(a)** $\log 34.56 \approx 1.5386$

(b) $\ln 34.56 \approx 3.5427$

7. The expression $\log_6 25$ represents the exponent to which 6 must be raised in order to obtain 25.

8. **(a)** Writing as a logarithmic equation, we have

$x = 3^{\log_3 4} \Rightarrow \log_3 x = \log_3 4 \Rightarrow x = 4$
Solution set: $\{4\}$

(b) $\log_x 25 = 2 \Rightarrow x^2 = 25 \Rightarrow x^2 = 5^2 \Rightarrow x = 5$
Solution set: $\{5\}$

(c) $\log_4 x = -2 \Rightarrow 4^{-2} = x \Rightarrow \frac{1}{16} = x$

Solution set: $\left\{\frac{1}{16}\right\}$

9. $\log_3 \frac{\sqrt{x} \cdot y}{pq^4} = \log_3\left(\sqrt{x} \cdot y\right) - \log_3\left(pq^4\right)$
$= \log_3\left(x^{1/2}\right) + \log_3 y - \left(\log_3 p + \log_3 q^4\right)$
$= \frac{1}{2}\log_3 x + \log_3 y - \log_3 p - 4\log_3 q$

10. Given $\log_b 9 = 3.1699$ and $\log_b 5 = 2.3219$

$\log_b 225 = \log_b (9 \cdot 25) = \log_b \left(9 \cdot 5^2\right)$
$= \log_b 9 + 2\log_b 5$
$= 3.1699 + 2(2.3219) = 7.8137$

11. $\log_3 40 = \frac{\log 40}{\log 3} \approx \frac{1.6021}{.4771} \approx 3.3578$

We could also have used the natural logarithm. $\log_3 40 = \frac{\ln 40}{\ln 3} \approx \frac{3.689}{1.0986} \approx 3.3578$

12. $f(x) = 4^x$

$f(\log_4 12) = 4^{\log_4 12} = 12$

Section 4.5: Exponential and Logarithmic Equations

1. Since x is the exponent to which 7 must be raised in order to obtain 19, the solution is $\log_7 19$ or $\frac{\log 19}{\log 7}$ or $\frac{\ln 19}{\ln 7}$.

2. Since x is the exponent to which 3 must be raised in order to obtain 10, the solution is $\log_3 10$ or $\frac{\log 10}{\log 3} = \frac{1}{\log 3}$ or $\frac{\ln 10}{\ln 3}$.

3. Since x is the exponent to which $\frac{1}{2}$ must be raised in order to obtain 12, the solution is $\log_{1/2} 12$ or $\frac{\log 12}{\log\left(\frac{1}{2}\right)}$ or $\frac{\ln 12}{\ln\left(\frac{1}{2}\right)}$.

4. Since x is the exponent to which $\frac{1}{3}$ must be raised in order to obtain 4, the solution is $\log_{1/3} 4$ or $\frac{\log 4}{\log\left(\frac{1}{3}\right)}$ or $\frac{\ln 4}{\ln\left(\frac{1}{3}\right)}$.

5. $3^x = 7$
$\ln 3^x = \ln 7$
$x \ln 3 = \ln 7$
$x = \frac{\ln 7}{\ln 3} \approx 1.771$
Solution set: $\{1.771\}$

6. $5^x = 13$
$\ln 5^x = \ln 13$
$x \ln 5 = \ln 13$
$x = \frac{\ln 13}{\ln 5} \approx 1.594$
Solution set: $\{1.594\}$

7. $\left(\frac{1}{2}\right)^x = 5 \Rightarrow \ln\left(\frac{1}{2}\right)^x = \ln 5 \Rightarrow x \ln\frac{1}{2} = \ln 5$
$x = \frac{\ln 5}{\ln \frac{1}{2}} \approx -2.322$
Solution set: $\{-2.322\}$

8. $\left(\frac{1}{3}\right)^x = 6 \Rightarrow \ln\left(\frac{1}{3}\right)^x = \ln 6 \Rightarrow x \ln\frac{1}{3} = \ln 6$
$x = \frac{\ln 6}{\ln \frac{1}{3}} \approx -1.631$
Solution set: $\{-1.631\}$

9. $.8^x = 4 \Rightarrow \ln\left(.8^x\right) = \ln 4 \Rightarrow x \ln .8 = \ln 4 \Rightarrow$
$x = \frac{\ln 4}{\ln .8} \approx -6.213$
Solution set: $\{-6.213\}$

10. $.6^x = 3 \Rightarrow \ln\left(.6^x\right) = \ln 3 \Rightarrow x \ln .6 = \ln 3 \Rightarrow$
$x = \frac{\ln 3}{\ln .6} \approx -2.151$
Solution set: $\{-2.151\}$

11. $4^{x-1} = 3^{2x} \Rightarrow \ln\left(4^{x-1}\right) = \ln\left(3^{2x}\right) \Rightarrow$
$(x-1)\ln 4 = 2x \ln 3$
$x \ln 4 - \ln 4 = 2x \ln 3$
$x \ln 4 - 2x \ln 3 = \ln 4 \Rightarrow x(\ln 4 - 2\ln 3) = \ln 4 \Rightarrow$
$x = \frac{\ln 4}{\ln 4 - 2\ln 3} \approx -1.710$
Solution set: $\{-1.710\}$

12. $2^{x+3} = 5^x \Rightarrow \ln\left(2^{x+3}\right) = \ln\left(5^x\right) \Rightarrow$
$(x+3)\ln 2 = x \ln 5 \Rightarrow x \ln 2 + 3\ln 2 = x \ln 5$
$x \ln 2 - x \ln 5 = -3\ln 2$
$x(\ln 2 - \ln 5) = -3\ln 2$
$x = \frac{-3\ln 2}{\ln 2 - \ln 5} \approx 2.269$
Solution set: $\{2.269\}$

13. $6^{x+1} = 4^{2x-1} \Rightarrow \ln\left(6^{x+1}\right) = \ln\left(4^{2x-1}\right)$
$(x+1)\ln 6 = (2x-1)\ln 4$
$x \ln 6 + \ln 6 = 2x \ln 4 - \ln 4$
$\ln 6 + \ln 4 = 2x \ln 4 - x \ln 6$
$\ln 6 + \ln 4 = x(2\ln 4 - \ln 6)$
$x = \frac{\ln 6 + \ln 4}{2\ln 4 - \ln 6} \approx 3.240$
Solution set: $\{3.240\}$

14. $3^{x-4} = 7^{2x+5}$
$\ln\left(3^{x-4}\right) = \ln\left(7^{2x+5}\right)$
$(x-4)\ln 3 = (2x+5)\ln 7$
$x \ln 3 - 4\ln 3 = 2x \ln 7 + 5\ln 7$
$x \ln 3 - 2x \ln 7 = 4\ln 3 + 5\ln 7$
$x(\ln 3 - 2\ln 7) = 4\ln 3 + 5\ln 7$
$x = \frac{4\ln 3 + 5\ln 7}{\ln 3 - 2\ln 7} \approx -5.057$
Solution set: $\{-5.057\}$

15. $e^{x^2} = 100 \Rightarrow \ln\left(e^{x^2}\right) = \ln 100$
$x^2 = \ln 100 \Rightarrow x = \pm\ln 100 = \pm 2.146$
Solution set: $\{\pm 2.146\}$

16. $e^{x^4} = 1000 \Rightarrow \ln\left(e^{x^4}\right) = \ln 1000$
$x^4 = \ln 1000 \Rightarrow x = \pm\sqrt[4]{\ln 1000} = \pm 1.621$
Solution set: $\{\pm 1.621\}$

17. $e^{3x-7} \cdot e^{-2x} = 4e \Rightarrow e^{x-7} = 4e$
$\ln\left(e^{x-7}\right) = \ln(4e) \Rightarrow (x-7)\ln e = \ln 4 + \ln e$
$x - 7 = \ln 4 + 1 \Rightarrow x = \ln 4 + 8 \approx 9.386$
Solution set: $\{9.386\}$

18. $e^{1-3x} \cdot e^{5x} = 2e \Rightarrow e^{1+2x} = 2e$
$\ln\left(e^{1+2x}\right) = \ln(2e) \Rightarrow (1+2x)\ln e = \ln 2 + \ln e$
$1+2x = \ln 2 + 1 \Rightarrow x = \frac{\ln 2}{2} \approx .347$
Solution set: $\{.347\}$

19. $\left(\frac{1}{3}\right)^x = -3$ has no solution since $\frac{1}{3}$ raised to any power is positive.
Solution set: $\varnothing$

20. $\left(\frac{1}{9}\right)^x = -9$ has no solution since $\frac{1}{9}$ raised to any power is positive.
Solution set: $\varnothing$

21. $.05(1.15)^x = 5 \Rightarrow 1.15^x = \frac{5}{.05} = 100 \Rightarrow$
$\log\left(1.15^x\right) = \log 100 \Rightarrow x \log 1.15 = \log 100 \Rightarrow$
$x = \frac{\log 100}{\log 1.15} = \frac{2}{\log 1.15} \approx 32.950$
Solution set: $\{32.950\}$

22. $1.2(.9)^x = .6 \Rightarrow .9^x = \frac{.6}{1.2} = .5 \Rightarrow$
$\ln\left(.9^x\right) = \ln .5 \Rightarrow x \ln .9 = \ln .5 \Rightarrow$
$x = \frac{\ln .5}{\ln .9} \approx 6.579$
Solution set: $\{6.579\}$

23. $3(2)^{x-2} + 1 = 100 \Rightarrow 3(2)^{x-2} = 99 \Rightarrow$
$2^{x-2} = 33 \Rightarrow \ln\left(2^{x-2}\right) = \ln 33 \Rightarrow$
$(x-2)\ln 2 = \ln 33 \Rightarrow x = \frac{\ln 33}{\ln 2} + 2 \approx 7.044$
Solution set: $\{7.044\}$

24. $5(1.2)^{3x-2} + 1 = 7 \Rightarrow 5(1.2)^{3x-2} = 6 \Rightarrow$
$1.2^{3x-2} = 1.2 \Rightarrow 3x - 2 = 1 \Rightarrow x = 1$
Solution set: $\{1\}$

25. $2(1.05)^x + 3 = 10 \Rightarrow 2(1.05)^x = 7 \Rightarrow$
$1.05^x = 3.5 \Rightarrow \ln\left(1.05^x\right) = \ln 3.5 \Rightarrow$
$x \ln 1.05 = \ln 3.5 \Rightarrow x = \frac{\ln 3.5}{\ln 1.05} \approx 25.677$
Solution set: $\{25.677\}$

26. $3(1.4)^x - 4 = 60 \Rightarrow 3(1.4)^x = 64 \Rightarrow$
$1.4^x = \frac{64}{3} \Rightarrow \ln\left(1.4^x\right) = \ln\left(\frac{64}{3}\right) \Rightarrow$
$x \ln 1.4 = \ln 64 - \ln 3$
$x = \frac{\ln 64 - \ln 3}{\ln 1.4} \approx 9.095$
Solution set: $\{9.095\}$

27. $5(1.015)^{x-1980} = 8 \Rightarrow 1.015^{(x-1980)} = 1.6$
$\ln\left(1.015^{(x-1980)}\right) = \ln 1.6$
$(x-1980)\ln 1.015 = \ln 1.6$
$x = \frac{\ln 1.6}{\ln 1.015} + 1980 \approx 2011.568$
Solution set: $\{2011.568\}$

28. $30 - 3(.75)^{x-1} = 29 \Rightarrow 3(.75)^{x-1} = 1$
$.75^{(x-1)} = \frac{1}{3} \Rightarrow \ln\left(.75^{(x-1)}\right) = \ln\left(\frac{1}{3}\right)$
$(x-1)\ln .75 = \ln 1 - \ln 3$
$x = \frac{\ln 1 - \ln 3}{\ln .75} + 1 \approx 4.819$
Solution set: $\{4.819\}$

29. $5 \ln x = 10 \Rightarrow \ln x = 2 \Rightarrow e^2 = x$
Solution set: $\left\{e^2\right\}$

30. $3 \log x = 2 \Rightarrow \log x = \frac{2}{3} \Rightarrow 10^{2/3} = x \Rightarrow \sqrt[3]{100}$
Solution set: $\left\{\sqrt[3]{100}\right\}$

31. $\ln(4x) = 1.5 \Rightarrow 4x = e^{1.5} \Rightarrow x = \frac{e^{1.5}}{4}$
Solution set: $\left\{\frac{e^{1.5}}{4}\right\}$

32. $\ln(2x) = 5 \Rightarrow e^5 = 2x \Rightarrow \frac{e^5}{2} = x$
Solution set: $\left\{\frac{e^5}{2}\right\}$

33. $\log(2-x) = .5 \Rightarrow 2 - x = 10^{.5} \Rightarrow 2 - x = \sqrt{10}$
$x = 2 - \sqrt{10}$
Solution set: $\left\{2 - \sqrt{10}\right\}$

34. $\ln(1-x) = \frac{1}{2} \Rightarrow e^{1/2} = 1 - x \Rightarrow x = 1 - \sqrt{e}$
Solution set: $\left\{1 - \sqrt{e}\right\}$

35. $\log_6(2x+4) = 2 \Rightarrow 2x + 4 = 6^2$
$2x + 4 = 36 \Rightarrow 2x = 32 \Rightarrow x = 16$
Solution set: $\{16\}$

36. $\log_5(8-3x) = 3 \Rightarrow 8 - 3x = 5^3 = 8 - 3x = 125$
$-3x = 117 \Rightarrow x = -39$
Solution set: $\{-39\}$

37. $\log_4\left(x^3 + 37\right) = 3 \Rightarrow x^3 + 37 = 4^3$
$x^3 = 27 \Rightarrow x = 3$
Solution set: $\{3\}$

38. $\log_7\left(x^3 + 65\right) = 0 \Rightarrow x^3 + 65 = 7^0 \Rightarrow x^3 = -64$
$x = -4$
Solution set: $\{-4\}$

39. $\ln x + \ln x^2 = 3 \Rightarrow \ln\left(x \cdot x^2\right) = 3 \Rightarrow \ln x^3 = 3$
$x^3 = e^3 \Rightarrow x = e$
Solution set: $\{e\}$

40. $\log x + \log x^2 = 3 \Rightarrow \log\left(x \cdot x^2\right) = 3$

$$\log x^3 = 3 \Rightarrow x^3 = 10^3 \Rightarrow x = 10$$

Solution set: {10}

41. $\log x + \log(x-21) = 2 \Rightarrow \log\left[x(x-21)\right] = 2$

$$\log\left(x^2 - 21x\right) = 2 \Rightarrow x^2 - 21x = 10^2$$
$$x^2 - 21x - 100 = 0 \Rightarrow (x-25)(x+4) = 0 \Rightarrow$$
$$x = 25 \text{ or } x = -4$$

Since the negative solution $(x = -4)$ is not in the domain of log x, it must be discarded.
Solution set: {25}

42. $\log x + \log(3x-13) = 1$

$$\log_{10}\left[x(3x-13)\right] = 1$$
$$x(3x-13) = 10^1 \Rightarrow 3x^2 - 13x = 10$$
$$3x^2 - 13x - 10 = 0 \Rightarrow (3x+2)(x-5) = 0$$
$$3x + 2 = 0 \Rightarrow x = -\tfrac{2}{3} \quad \text{or} \quad x - 5 = 0 \Rightarrow x = 5$$

Since the negative solution $\left(x = -\frac{2}{3}\right)$ is not in the domain of log x, it must be discarded.
Solution set: {5}

43.

$$\log(x+25) = 1 + \log(2x-7)$$
$$\log(x+25) - \log(2x-7) = 1$$
$$\log_{10}\tfrac{x+25}{2x-7} = 1 \Rightarrow \tfrac{x+25}{2x-7} = 10^1$$
$$x + 25 = 10(2x-7)$$
$$x + 25 = 20x - 70$$
$$25 = 19x - 70$$
$$95 = 19x \Rightarrow 5 = x$$

Solution set: $\{5\}$

44.

$$\log(11x+9) = 3 + \log(x+3)$$
$$\log(11x+9) - \log(x+3) = 3$$
$$\log_{10}\tfrac{11x+9}{x+3} = 3 \Rightarrow \tfrac{11x+9}{x+3} = 10^3$$
$$11x + 9 = 1000(x+3)$$
$$11x + 9 = 1000x + 3000$$
$$9 = 989x + 3000$$
$$-2991 = 989x$$
$$x = -\tfrac{2991}{989} \approx -3.0243$$

Since $x \approx -3.0243$, $x + 3$ is negative. Therefore $\log(x+3)$ is not defined. This proposed solution must be discarded.
Solution set: $\varnothing$

45. $\ln(4x-2) - \ln 4 = -\ln(x-2)$

$$\ln\tfrac{4x-2}{4} = -\ln(x-2) \Rightarrow \tfrac{4x-2}{4} = \tfrac{1}{x-2}$$
$$(4x-2)(x-2) = 4 \Rightarrow 4x^2 - 10x + 4 = 4$$
$$4x^2 - 10x = 0 \Rightarrow 2x(2x-5) = 0 \Rightarrow$$
$$2x = 0 \Rightarrow x = 0 \text{ or } 2x - 5 = 0 \Rightarrow x = \tfrac{5}{2} = 2.5$$

Since $x = 0$ is not in the domain of ln $(x - 2)$, it must be discarded.
Solution set: {2.5}

46. $\ln(5+4x) - \ln(3+x) = \ln 3$

$$\ln\tfrac{5+4x}{3+x} = \ln 3 \Rightarrow \tfrac{5+4x}{3+x} = 3$$
$$5 + 4x = 3(3+x)$$
$$5 + 4x = 9 + 3x \Rightarrow x + 5 = 9$$
$$x = 4$$

Solution set: {4}

47. $\log_5(x+2) + \log_5(x-2) = 1$

$$\log_5\left[(x+2)(x-2)\right] = 1 \Rightarrow x^2 - 4 = 5^1$$
$$x^2 - 9 = 0$$
$$(x-3)(x+3) = 0 \Rightarrow x = \pm 3$$

−3 is not in the domain, so reject it.
Solution set: {3}

48. $\log_2(x-7) + \log_2 x = 3 \Rightarrow$

$$\log_2\left[x(x-7)\right] = 3 \Rightarrow x^2 - 7x = 2^3 \Rightarrow$$
$$x^2 - 7x - 8 = 0 \Rightarrow (x-8)(x+1) = 0 \Rightarrow$$
$$x - 8 = 0 \Rightarrow x = 8 \text{ or } x + 1 = 0 \Rightarrow x = -1$$

Since the negative solution $(x = -1)$ is not in the domain of log x, it must be discarded.
Solution set: {8}

49. $\log_2(2x-3) + \log_2(x+1) = 1$

$$\log_2\left[(2x-3)(x+1)\right] = 1$$
$$2x^2 - x - 3 = 2^1$$
$$2x^2 - x - 5 = 0$$
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-5)}}{2(2)} = \frac{1 \pm \sqrt{41}}{4}$$

Since the negative solution $\left(x = \frac{1-\sqrt{41}}{4}\right)$ is not in the domain of log $(x + 1)$, it must be discarded.

Solution set: $\left\{\frac{1+\sqrt{41}}{4}\right\}$

50. $\log_5(3x+2)+\log_5(x-1)=1$

$$\log_5[(3x+2)(x-1)]=1$$
$$3x^2-x-2=5^1$$
$$3x^2-x-7=0$$
$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(3)(-7)}}{2(3)}=\frac{1\pm\sqrt{85}}{6}$$

Since the negative solution $\left(x=\frac{1-\sqrt{85}}{6}\right)$ is not in the domain of log $(x-1)$, it must be discarded.

Solution set: $\left\{\frac{1+\sqrt{85}}{6}\right\}$

51. $\ln e^x-2\ln e=\ln e^4\Rightarrow x-2=4\Rightarrow x=6$
Solution set: {6}

52. $\ln e^x-\ln e^3=\ln e^3\Rightarrow x-3=3\Rightarrow x=6$
Solution set: {6}

53. $\log_2(\log_2 x)=1\Rightarrow\log_2 x=2^1\Rightarrow$
$\log_2 x=2\Rightarrow x=2^2\Rightarrow x=4$
Solution set: {4}

54.
$$\log x=\sqrt{\log x}$$
$$(\log x)^2=\left(\sqrt{\log x}\right)^2\Rightarrow(\log x)^2=\log x$$
$$(\log x)^2-\log x=0\Rightarrow\log x(\log x-1)=0$$
$$\log_{10}x=0 \quad\text{or}\quad \log_{10}x-1=0$$
$$x=10^0 \qquad \log_{10}x=1$$
$$x=1 \qquad x=10^1=10$$

Since the work involves squaring both sides, both proposed solutions must be checked in the original equation.

Check $x=1$.

$$\log x=\sqrt{\log x}$$
$$\log 1\stackrel{?}{=}\sqrt{\log 1}$$
$$0=\sqrt{0}$$
$$0=0$$

This is a true statement; therefore, 1 is a solution.

Check $x=10$.

$$\log x=\sqrt{\log x}$$
$$\log 10\stackrel{?}{=}\sqrt{\log 10}$$
$$1=\sqrt{1}$$
$$1=1$$

This is a true statement; therefore, 10 is a solution.
Solution set: {1, 10}

55. $\log x^2=(\log x)^2\Rightarrow 2\log x=(\log x)^2\Rightarrow$
$(\log x)^2-2\log x=0\Rightarrow\log x(\log x-2)=0$

$$\log_{10}x=0 \quad\text{or}\quad \log_{10}x-2=0$$
$$x=10^0 \qquad \log_{10}x=2$$
$$x=1 \qquad x=10^2=100$$

Solution set: {1, 100}

56. $\log_2\sqrt{2x^2}=\frac{3}{2}\Rightarrow\sqrt{2x^2}=2^{3/2}\Rightarrow$
$\left(\sqrt{2x^2}\right)^2=\left(2^{3/2}\right)^2\Rightarrow 2x^2=2^3\Rightarrow$
$2x^2=8\Rightarrow x^2=4\Rightarrow x=\pm 2$

Since the solution involves squaring both sides, both proposed solutions must be checked in the original equation.

Check $x=-2$.

$$\log_2\sqrt{2x^2}=\tfrac{3}{2}$$
$$\log_2\sqrt{2(-2)^2}\stackrel{?}{=}\tfrac{3}{2}$$
$$\log_2\sqrt{2(4)}=\tfrac{3}{2}$$
$$\log_2\sqrt{8}=\tfrac{3}{2}$$
$$\log_2\sqrt{2^3}=\tfrac{3}{2}$$
$$\log_2 2^{3/2}=\tfrac{3}{2}$$
$$\tfrac{3}{2}=\tfrac{3}{2}$$

This is a true statement; therefore, -2 is a solution.

Check $x=2$.

$$\log_2\sqrt{2x^2}=\tfrac{3}{2}$$
$$\log_2\sqrt{2(2)^2}\stackrel{?}{=}\tfrac{3}{2}$$
$$\log_2\sqrt{2(4)}=\tfrac{3}{2}$$
$$\log_2\sqrt{8}=\tfrac{3}{2}$$
$$\log_2\sqrt{2^3}=\tfrac{3}{2}$$
$$\log_2 2^{3/2}=\tfrac{3}{2}$$
$$\tfrac{3}{2}=\tfrac{3}{2}$$

This is a true statement; therefore, 2 is a solution.
Solution set: {–2, 2}

57. Answers will vary.One should not immediately reject a negative answer when solving equations involving logarithms. One must examine what happens to an answer when substituted back into the original equation. An answer (whether negative, positive, or zero) must not allow a nonpositive value in the argument of the logarithm. If it does, regardless of its sign, it must be rejected.

58. $\log_a(4x-7)+\log_a(x^2+4)=0$

$\log_a t$ is defined when t is in the interval $(0, \infty)$, but it is undefined when t is in the interval $(-\infty, 0]$. Since x^2+4 is positive for all values of x, $\log_a(x^2+4)$ will be defined for all values of x. However, $\log_a(4x-7)$ will be undefined when $4x-7$ is in the interval $(-\infty, 0]$. $4x-7 \le 0 \Rightarrow 4x \le 7 \Rightarrow x \le \frac{7}{4}$

Thus, the numbers in the interval $\left(-\infty, \frac{7}{4}\right]$ could not be solutions of the given equation.

59. $p = a + \dfrac{k}{\ln x}$, for x

$$p - a = \frac{k}{\ln x}$$
$$(\ln x)(p-a) = k$$
$$\ln x = \frac{k}{p-a}$$
$$x = e^{k/(p-a)}$$

60. $r = p - k\ln t$, for t

$$r - p = -k\ln t$$
$$p - r = k\ln t$$
$$\frac{p-r}{k} = \ln t$$
$$e^{(p-r)/k} = t$$
$$t = e^{(p-r)/k}$$

61. $T = T_0 + (T_1 - T_0)10^{-kt}$, for t

$$T - T_0 = (T_1 - T_0)10^{-kt}$$
$$\frac{T-T_0}{T_1-T_0} = 10^{-kt}$$
$$\log_{10}\left(\frac{T-T_0}{T_1-T_0}\right) = -kt$$
$$\frac{\log\left(\frac{T-T_0}{T_1-T_0}\right)}{-k} = \frac{-kt}{-k}$$
$$t = -\frac{1}{k}\log\left(\frac{T-T_0}{T_1-T_0}\right)$$

62. $A = \dfrac{pi}{1-(1+i)^{-n}}$, for n

$$A\left(1-(1+i)^{-n}\right) = pi$$
$$1-(1+i)^{-n} = \frac{pi}{A}$$
$$1 - \frac{pi}{A} = (1+i)^{-n}$$
$$\log\left(1-\frac{pi}{A}\right) = -n\log(1+i)$$
$$-\frac{\log\left(1-\frac{pi}{A}\right)}{\log(1+i)} = -\frac{\log\left(\frac{A-pi}{A}\right)}{\log(1+i)} = n$$

63. $I = \frac{E}{R}\left(1-e^{-Rt/2}\right)$, for t

$$RI = R\left[\frac{E}{R}\left(1-e^{-Rt/2}\right)\right]$$
$$RI = E\left(1-e^{-Rt/2}\right)$$
$$RI = E - Ee^{-Rt/2}$$
$$RI + Ee^{-Rt/2} = E$$
$$Ee^{-Rt/2} = E - RI$$
$$\frac{Ee^{-Rt/2}}{E} = \frac{E-RI}{E}$$
$$e^{-Rt/2} = 1 - \frac{RI}{E}$$
$$\ln e^{-Rt/2} = \ln\left(1-\frac{RI}{E}\right)$$
$$-\frac{Rt}{2} = \ln\left(1-\frac{RI}{E}\right)$$
$$-\frac{2}{R}\left(-\frac{Rt}{2}\right) = -\frac{2}{R}\ln\left(1-\frac{RI}{E}\right)$$
$$t = -\frac{2}{R}\ln\left(1-\frac{RI}{E}\right)$$

64. $y = \frac{K}{1+ae^{-bx}}$, for b

$$y\left(1+ae^{-bx}\right) = K$$
$$1 + ae^{-bx} = \frac{K}{y}$$
$$ae^{-bx} = \frac{K}{y} - 1 = \frac{K-y}{y}$$
$$e^{-bx} = \frac{K-y}{ay}$$
$$-bx = \ln\left(\frac{K-y}{ay}\right)$$
$$b = \frac{\ln\left(\frac{K-y}{ay}\right)}{-x}$$

65. $y = A + B\left(1-e^{-Cx}\right)$, for x

$$y - A = B\left(1-e^{-Cx}\right)$$
$$\frac{y-A}{B} = 1 - e^{-Cx}$$
$$\frac{y-A}{B} - 1 = \frac{y-A-B}{B} = -e^{-Cx}$$
$$\frac{A+B-y}{B} = e^{-Cx}$$
$$\ln\left(\frac{A+B-y}{B}\right) = -Cx$$
$$\frac{\ln\left(\frac{A+B-y}{B}\right)}{-C} = x$$

66. $m = 6 - 2.5\log\left(\frac{M}{M_0}\right)$, for M

$$m = 6 - 2.5\log\left(\frac{M}{M_0}\right)$$
$$m - 6 = -2.5\log\left(\frac{M}{M_0}\right)$$
$$\frac{6-m}{2.5} = \log\left(\frac{M}{M_0}\right)$$
$$10^{(6-m)/2.5} = \frac{M}{M_0}$$
$$10^{(6-m)/2.5}\cdot M_0 = M$$

67. $\log A = \log B - C\log x$, for A

$$\log A = \log B - C\log x$$
$$\log A = \log \tfrac{B}{x^C}$$
$$A = \tfrac{B}{x^C}$$

68. $d = 10\log\left(\frac{I}{I_0}\right)$, for I

$$d = 10\log\left(\tfrac{I}{I_0}\right)$$
$$\tfrac{d}{10} = \log\left(\tfrac{I}{I_0}\right)$$
$$10^{d/10} = \tfrac{I}{I_0}$$
$$I_0 \cdot 10^{d/10} = I$$

69. $A = P\left(1+\frac{r}{n}\right)^{nt}$, for t

$$\tfrac{A}{P} = \left(1+\tfrac{r}{n}\right)^{nt}$$
$$\log\left(\tfrac{A}{P}\right) = nt\log\left(1+\tfrac{r}{n}\right)$$
$$\frac{\log\left(\frac{A}{P}\right)}{n\log\left(1+\frac{r}{n}\right)} = t$$

70. $D = 160 + 10\log x$, for x

$$D - 160 = 10\log x$$
$$\tfrac{D-160}{10} = \log x$$
$$10^{(D-160)/10} = 10^{(D/10)-16} = x$$

71. $A = P\left(1+\frac{r}{n}\right)^{tn}$

To solve for A, substitute P = 10,000, r = .03, n = 4, and t = 5.

$$A = 10{,}000\left(1+\tfrac{.03}{4}\right)^{(5)(4)}$$
$$A = 10{,}000(1.0075)^{20} \approx 11{,}611.84$$

There will be \$11,611.84 in the account.

72. $A = Pe^{rt}$

To solve for A, substitute P = 5000, r = .04, and t = 8.

$$A = 5000e^{(.04)(8)}$$
$$A = 5000e^{.32} \approx 6885.64$$

There will be \$6885.64 in the account.

73. $A = P\left(1+\frac{r}{n}\right)^{tn}$

To solve for t, substitute A = 30,000, P = 27,000, r = .04, and n = 4.

$$30{,}000 = 27{,}000\left(1+\tfrac{.04}{4}\right)^{t(4)}$$
$$\tfrac{30{,}000}{27{,}000} = (1+.01)^{4t} \Rightarrow \tfrac{10}{9} = 1.01^{4t}$$
$$\ln\tfrac{10}{9} = \ln\left(1.01^{4t}\right) \Rightarrow \ln\tfrac{10}{9} = 4t\ln 1.01$$
$$t = \frac{\ln\frac{10}{9}}{4\ln 1.01} \approx 2.6$$

To the nearest tenth of a year, Tom will be ready to buy a car in 2.6 yr.

74. $A = P\left(1+\frac{r}{n}\right)^{tn}$

To solve for t, substitute A = 2063, P = 1786, r = .026, and n = 12.

$$2063 = 1786\left(1+\tfrac{.026}{12}\right)^{t(12)}$$
$$\tfrac{2063}{1786} = \left(1+\tfrac{.026}{12}\right)^{12t} \Rightarrow \tfrac{2063}{1786} \approx 1.002167^{12t}$$
$$\ln\tfrac{2063}{1786} \approx \ln\left(1.002167^{12t}\right)$$
$$\ln\tfrac{2063}{1786} = 12t\ln 1.002167$$
$$t = \frac{\ln\frac{2063}{1786}}{12\ln 1.002167} \approx 5.55$$

To the nearest hundredth, t = 5.55 yr.

75. $A = P\left(1+\frac{r}{n}\right)^{tn}$

To solve for r, substitute A = 2500, P = 2000, t = 3.5, and n = 2.

$$2500 = 2000\left(1+\tfrac{r}{2}\right)^{(3.5)(2)}$$
$$1.25 = \left(1+\tfrac{r}{2}\right)^{7} \Rightarrow \sqrt[7]{1.25} = 1+\tfrac{r}{2}$$
$$\sqrt[7]{1.25} - 1 = \tfrac{r}{2} \Rightarrow r = 2\left(\sqrt[7]{1.25}-1\right)$$
$$r \approx .0648$$

The interest rate is about 6.48%.

76. $A = P\left(1+\frac{r}{n}\right)^{tn}$

To solve for r, substitute A = 20,000, P = 16,000, t = 5.25, and n = 4.

$$20{,}000 = 16{,}000\left(1+\tfrac{r}{4}\right)^{(5.25)(4)}$$
$$1.25 = \left(1+\tfrac{r}{4}\right)^{21} \Rightarrow \sqrt[21]{1.25} = 1+\tfrac{r}{4}$$
$$\sqrt[21]{1.25} - 1 = \tfrac{r}{4} \Rightarrow r = 4\left(\sqrt[21]{1.25}-1\right)$$
$$r \approx .0427$$

The interest rate is about 4.27%.

77. **(a)** $f(3000) = 86.3\ln 3000 - 680 \approx 10.9$

At 3000 ft, about 10.9% of the moisture falls as snow.

(b) $f(4000) = 86.3\ln 4000 - 680 \approx 35.8$

At 4000 ft, about 35.8% of the moisture falls as snow.

(c) $f(7000) = 86.3\ln 7000 - 680 \approx 84.1$

At 7000 ft, about 84.1% of the moisture falls as snow.

78. Note that x refers to the number of thousands of catalogs sent.

(a) $T(5) = 5000\log(5+1) \approx 3891$
The total sales are about \$3891.

(b) $T(24) = 5000\log(24+1) \approx 6990$
The total sales are about \$6990.

(c) $T(49) = 5000\log(49+1) \approx 8495$
The total sales are about \$8495.

79. Double the 2000 value is $2(7,990) = 15,980.$

$$f(x) = 8160(1.06)^x$$
$$15,980 = 8160(1.06)^x$$
$$\tfrac{15,980}{8160} = 1.06^x \Rightarrow \tfrac{47}{24} = 1.06^x$$
$$\ln\tfrac{47}{24} = \ln 1.06^x \Rightarrow \ln\tfrac{47}{24} = x\ln 1.06$$
$$\frac{\ln\frac{47}{24}}{\ln 1.06} = x$$
$$x \approx 11.53$$

During 2011, the cost of a year's tuition, room and board, and fees at a public university will be double the cost in 2000.

80. $f(t) = 11.65\left(1 - e^{-t/1.27}\right)$

(a) At the finish line $t = 9.86$.
$$f(9.86) = 11.65\left(1 - e^{-9.86/1.27}\right) \approx 11.6451$$
He was running approximately 11.6451 m per sec as he crossed the finish line.

(b)
$$10 = 11.65\left(1 - e^{-t/1.27}\right)$$
$$\tfrac{10}{11.65} = 1 - e^{-t/1.27}$$
$$e^{-t/1.27} = 1 - \tfrac{10}{11.65}$$
$$-\tfrac{t}{1.27} = \ln\left(1 - \tfrac{10}{11.65}\right)$$
$$t = -1.27\ln\left(1 - \tfrac{10}{11.65}\right) \approx 2.4823$$
After 2.4823 sec, he was running at a rate of 10 m per sec.

81. $f(x) = \dfrac{25}{1 + 1364.3e^{-x/9.316}}$

(a) In 1997, $x = 97$.
$$f(97) = \frac{25}{1 + 1364.3e^{-97/9.316}} \approx 24$$
In 1997, about 24% of U.S. children lived in a home without a father.

(b)
$$10 = \frac{25}{1 + 1364.3e^{-x/9.316}}$$
$$10\left(1 + 1364.3e^{-x/9.316}\right) = 25$$
$$10 + 13,643e^{-x/9.316} = 25$$
$$13,643e^{-x/9.316} = 15$$
$$e^{-x/9.316} = \tfrac{15}{13,643}$$
$$-\tfrac{x}{9.316} = \ln\tfrac{15}{13,643}$$
$$x = -9.316\ln\tfrac{15}{13,643}$$
$$\approx 63.47$$
During 1963, 10% of U.S. children lived in a home without a father.

82. $f(x) = -301\ln\frac{x}{207}$

(a) The left side is a reflection of the right side across the vertical axis of the tower; the graph of $f(-x)$ is exactly the reflection of the graph of $f(x)$ across the y-axis.

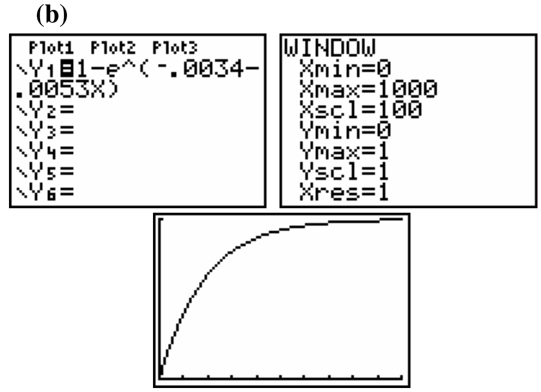
(b) x is half the length. We have $x = \frac{15.7488}{2} = 7.8744$. The height is $f(7.8744) = -301\ln\frac{7.8744}{207} \approx 984$ ft.

(c) Let $y = 500$ and solve for x.
$$500 = -301\ln\tfrac{x}{207} \Rightarrow -\tfrac{500}{301} = \ln\tfrac{x}{207}$$
$$e^{-500/301} = \tfrac{x}{207} \Rightarrow x = 207e^{-500/301} \approx 39$$
The height is 500 feet, about 39 feet from the center.

83. $\ln(1-P) = -.0034 - .0053T$

(a) Change this equation to exponential form, then isolate P.
$$1 - P = e^{-.0034-.0053T}$$
$$P(T) = 1 - e^{-.0034-.0053T}$$

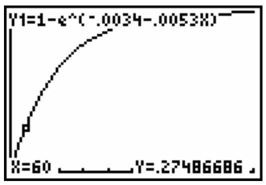
(b)



From the graph one can see that initially there is a rapid reduction of carbon dioxide emissions. However, after a while there is little benefit in raising taxes further.

(c) $P(60) = 1 - e^{-.00340-.0053(60)}$

$\approx .275$ or 27.5%



The reduction in carbon emissions from a tax of $60 per ton of carbon is 27.5%.

(d) We must determine T when $P = .5$.

$$.5 = 1 - e^{-.0034-.0053T}$$
$$.5 - 1 = -e^{-.0034-.0053T}$$
$$-.5 = -e^{-.0034-.0053T}$$
$$.5 = e^{-.0034-.0053T}$$
$$\ln .5 = -.0034 - .0053T$$
$$\ln .5 + .0034 = -.0053T$$
$$T = \frac{\ln .5 + .0034}{-.0053} \approx 130.14$$

The value T = $130.14 will give a 50% reduction in carbon emissions.

84. **(a)** Substituting $\frac{C}{C_0} = 2$ in $R = 6.3\ln\frac{C}{C_0}$, we have $R = 6.3\ln 2 \approx 4.4$ w/m^2.

(b) $T(R) = 1.03R$

$T(4.4) = 1.03(4.4) \approx 4.5°\text{F}$

While this is less than that predicted by Arrhenius in 1896, his values are still consistent with some current computer models.

85. *Step 1*: Replace $f(x) = e^{x+1} - 4$ with y and interchange x and y.

$y = e^{x+1} - 4 \Rightarrow x = e^{y+1} - 4$

Step 2: Solve for y.

$$x = e^{y+1} - 4 \Rightarrow x + 4 = e^{y+1}$$
$$\ln(x+4) = y + 1 \Rightarrow \ln(x+4) - 1 = y$$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \ln(x+4) - 1$

Domain: $(-4, \infty)$; range: $(-\infty, \infty)$

86. *Step 1*: Replace $f(x) = 2\ln 3x$ with y and interchange x and y.

$y = 2\ln 3x \Rightarrow x = 2\ln 3y$

Step 2: Solve for y.

$x = 2\ln 3y \Rightarrow \frac{x}{2} = \ln 3y \Rightarrow 3y = e^{x/2}$

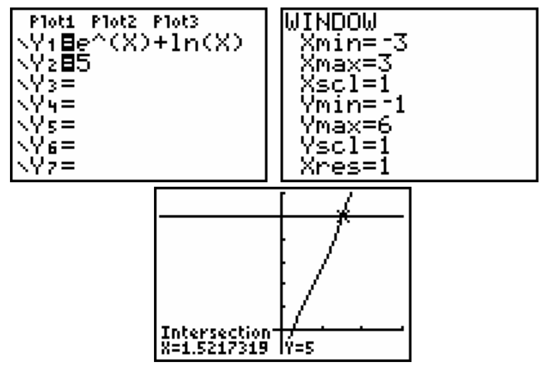
$y = \frac{1}{3}e^{x/2}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \frac{1}{3}e^{x/2}$

Domain: $(-\infty, \infty)$; range: $(0, \infty)$

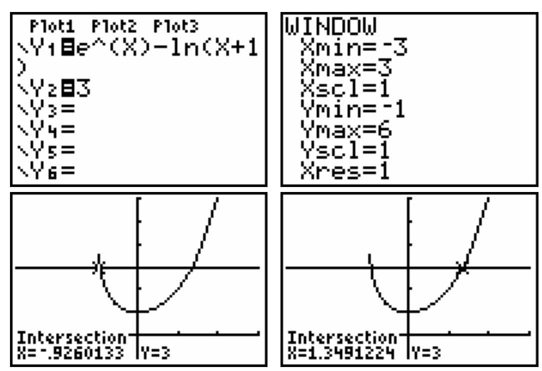
87. $e^x + \ln x = 5$



The two graphs intersect at approximately (1.52, 5). The x-coordinate of this point is the solution of the equation.

Solution set: {1.52}

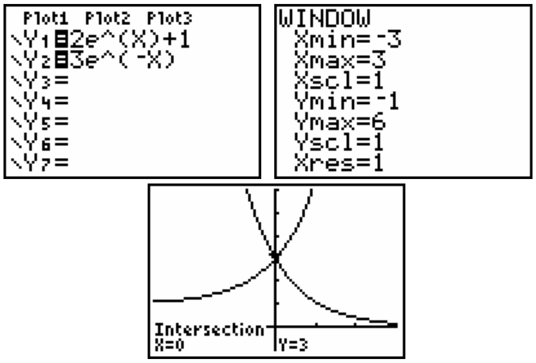
88. $e^x - \ln(x+1) = 3$



The two graphs intersect at approximately $(-.93, 3)$ and $(1.35, 3)$. The x-coordinate of these points represent the solution of the equation.

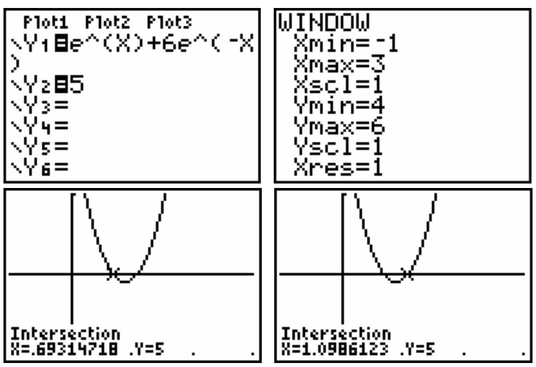
Solution set: {−.93, 1.35}

89. $2e^x + 1 = 3e^{-x}$



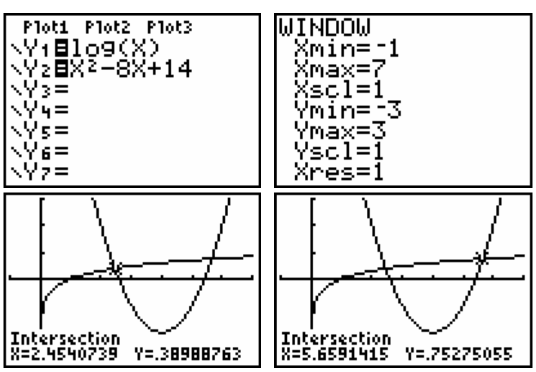
The two curves intersect at the point (0, 3). The x-coordinate of this point is the solution of the equation.
Solution set: {0}

90. $e^x + 6e^{-x} = 5$



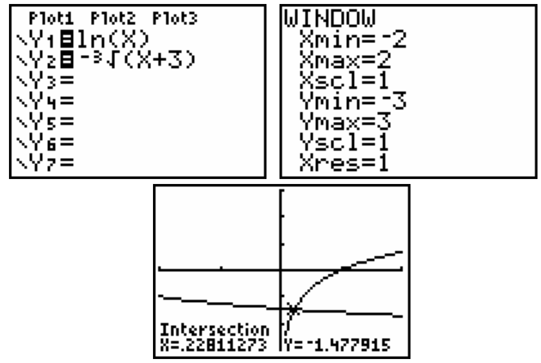
The two graphs intersect at approximately $(.69, 5)$ and $(1.10, 5)$. The x-coordinate of these points represent the solution of the equation.
Solution set: {.69, 1.10}

91. $\log x = x^2 - 8x + 14$



The two graphs intersect at approximately $(2.45, .39)$ and $(5.66, .75)$. The x-coordinate of these points represent the solution of the equation.
Solution set: {2.45, 5.66}

92. $\ln x = -\sqrt[3]{x+3}$



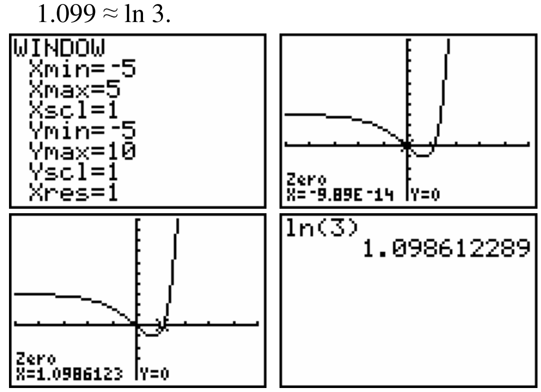
The two graphs intersect at approximately $(.23, -1.48)$. The x-coordinate of this point is the solution of the equation.
Solution set: {.23}

93. The expression e^{2x} is equivalent to $\left(e^x\right)^2$ by the power rule for exponents, $\left(a^m\right)^n = a^{mn}$.

94. $(e^x)^2 - 4e^x + 3 = 0$
$(e^x - 1)(e^x - 3) = 0$

95. $(e^x - 1)(e^x - 3) = 0$
$e^x - 1 = 0 \Rightarrow e^x = 1 \Rightarrow \ln e^x = \ln 1 \Rightarrow x = 0$ or
$e^x - 3 = 0 \Rightarrow e^x = 3 \Rightarrow \ln e^x = \ln 3 \Rightarrow x = \ln 3$
Solution set: $\{0, \ln 3\}$

96. The graph intersects the x-axis at 0 and $1.099 \approx \ln 3$.



Section 4.6: Applications and Models of Exponential Growth and Decay

1. The equation $2e^{.02x} = 6$ represents an initial amount of 2 and a final amount 6 (triple that of 2). The correct choice is B.

2. The equation $2e^{.02x} = 3$ represents a final amount of 3 (million). The correct choice is D.

3. The equation $y = 2e^{.02(3)}$ represents the amount after 3 yr. The correct choice is C.

4. The equation $y = 2e^{.02(1/3)}$ represents that amount after $\frac{1}{3}$ yr = 4 months. The correct choice is A.

5. $A(t) = 500e^{-.032t}$

(a) $t = 4 \Rightarrow A(4) = 500e^{-.032(4)} \approx 440$
After 4 years, about 440 g remain.

(b) $t = 8 \Rightarrow A(8) = 500e^{-.032(8)} \approx 387$
After 8 years, about 387 g remain.

(c) $t = 20 \Rightarrow A(20) = 500e^{-.032(20)} \approx 264$
After 20 years, about 264 g remain.

(d) Find t when $A(t) = 250$.

$$250 = 500e^{-.032t}$$
$$.5 = e^{-.032t}$$
$$\ln .5 = \ln e^{-.032t} \Rightarrow \ln .5 = -.032t$$
$$t = \frac{\ln .5}{-.032} \approx 21.66$$

The half-life is about 21.66 yr.

6. $A(t) = 500e^{-.053t}$

(a) $t = 4 \Rightarrow A(4) = 500e^{-.053(4)} \approx 404$
After 4 yr, about 404 g remain.

(b) $t = 8 \Rightarrow A(8) = 500e^{-.053(8)} \approx 327$
After 8 yr, about 327 g remain.

(c) $t = 20 \Rightarrow A(20) = 500e^{-.053(20)} \approx 173$
After 20 yr, about 173 g remain.

(d) Find t when $A(t) = 250$.

$$250 = 500e^{-.053t}$$
$$.5 = e^{-.053t}$$
$$\ln .5 = \ln e^{-.053t} \Rightarrow \ln .5 = -.053t$$
$$t = \frac{\ln .5}{-.053} \approx 13.08$$

The half-life is about 13.08 yr.

7. $A(t) = A_0e^{-.00043t}$

Find t when $A(t) = \frac{1}{2}A_0$.

$$\frac{1}{2}A_0 = A_0e^{-.00043t}$$
$$\frac{1}{2} = e^{-.00043t} \Rightarrow \ln\frac{1}{2} = \ln e^{-.00043t}$$
$$\ln\frac{1}{2} = -.00043t \Rightarrow t = \frac{\ln\frac{1}{2}}{-.00043} \approx 1611.97$$

The half-life is about 1611.97 yr.

8. $A(t) = A_0e^{-.087t}$

Find t when $A(t) = .25A_0$.

$$.25A_0 = A_0e^{-.087t}$$
$$.25 = e^{-.087t}$$
$$\ln .25 = \ln e^{-.087t} \Rightarrow \ln .25 = -.087t$$
$$t = \frac{\ln .25}{-.087} \approx 15.93$$

It will take about 15.93 days to decay to 25% of the initial amount.

9. First find the given values to find y_0 and then k: $12 = y_0e^{k(0)} \Rightarrow 12 = y_0$

$$y = 12e^{kt} \Rightarrow 6 = 12e^{4k} \Rightarrow .5 = e^{4k} \Rightarrow$$
$$\ln .5 = 4k \Rightarrow \frac{\ln .5}{4} = k \Rightarrow k \approx -.173$$

The exponential decay equation is
$y = 12e^{-.173t}$

To find the amount present after 7 years, let $t = 7$. $y = 12e^{-.173(7)} \approx 3.57$

After 7 years, about 3.57 g of the substance will be present.

10. $M = 6 - \frac{5}{2}\log\frac{I}{I_0}$

Find I when $M = 1$

$$1 = 6 - \frac{5}{2}\log\frac{I}{I_0} \Rightarrow -5 = -\frac{5}{2}\log\frac{I}{I_0}$$
$$2 = \log\frac{I}{I_0} \Rightarrow \frac{I}{I_0} = 10^2 \Rightarrow \frac{I}{I_0} = 100$$
$$I = 100\ I_0$$

Find I when $M = 3$.

$$3 = 6 - \frac{5}{2}\log\frac{I}{I_0} \Rightarrow -3 = -\frac{5}{2}\log\frac{I}{I_0} \Rightarrow 1.2 = \log\frac{I}{I_0}$$
$$\frac{I}{I_0} = 10^{1.2} \Rightarrow I = 10^{1.2}\ I_0 \approx 15.85I_0$$

Now find the ratio of light intensities.

$$\frac{100\,I_0}{15.85\,I_0} \approx 6.3$$

The light intensity of a magnitude 1 star is about 6.3 times as great as that of a magnitude 3 star.

11. From Example 5, we have the amount of radiocarbon present after t years is given by $y = y_0e^{-(\ln 2)(1/5700)t}$, where y_0 is the initial amount present. Letting $y = \frac{1}{3}y_0$, solve for t.

$$\frac{1}{3}y_0 = y_0e^{-(\ln 2)(1/5700)t}$$
$$\frac{1}{3} = e^{-(\ln 2)(1/5700)t}$$
$$\ln\frac{1}{3} = \ln e^{-(\ln 2)(1/5700)t}$$
$$\ln\frac{1}{3} = -\frac{\ln 2}{5700}t \Rightarrow -\frac{5700}{\ln 2}\ln\frac{1}{3} = t$$
$$-\frac{5700}{\ln 2}(\ln 1 - \ln 3) = t \Rightarrow t = \frac{5700\ln 3}{\ln 2} \approx 9034.29$$

The Egyptian died about 9000 yr ago.

12. From Example 5, we have the amount of radiocarbon present after t years is given by $y = y_0 e^{-(\ln 2)(1/5700)t}$, where y_0 is the initial amount present. If we let $y = .60y_0$, we can solve for t.

$$.60y_0 = y_0 e^{-(\ln 2)(1/5700)t}$$
$$.60 = e^{-(\ln 2)(1/5700)t}$$
$$\ln .60 = \ln e^{-(\ln 2)(1/5700)t}$$
$$\ln .60 = -\tfrac{\ln 2}{5700}t \Rightarrow -\tfrac{5700}{\ln 2}\ln .60 = t$$
$$t = -\tfrac{5700 \ln .60}{\ln 2} \approx 4200.70$$

The sample was about 4200 yr old.

13. Since $y = y_0 e^{-(\ln 2)(1/5700)t}$, where y_0 is the initial amount present. If we let $y = .15y_0$, we can solve for t.

$$.15y_0 = y_0 e^{-(\ln 2)(1/5700)t}$$
$$.15 = e^{-(\ln 2)(1/5700)t}$$
$$\ln .15 = \ln e^{-(\ln 2)(1/5700)t}$$
$$\ln .15 = -\tfrac{\ln 2}{5700}t \Rightarrow -\tfrac{5700}{\ln 2}\ln .15 = t$$
$$t = -\tfrac{5700 \ln .15}{\ln 2} \approx 15{,}600.70$$

The paintings are about 15,600 yr old.

14. $A(t) = 10e^{.0095t}$

Solve $A(t) = 15$ for t.

$$15 = 10e^{.0095t} \Rightarrow 1.5 = e^{.0095t}$$
$$\ln 1.5 = \ln e^{.0095t} \Rightarrow \ln 1.5 = .0095t$$
$$t = \tfrac{\ln 1.5}{.0095} \approx 42.68$$

15 g of the chemical will dissolve at about 43°C.

15. **(a)** A point on the graph of $f(x) = A_0 a^{x-1950}$ is $(1950, .05)$.

Since $f(1950) = A_0 a^{1950-1950} = .05$, we have $A_0 a^0 = .05 \Rightarrow A_0 = .05$. Thus, we have the function $f(x) = .05a^{x-1950}$.

Since the point $(2000, .35)$ is also on the graph of the function, we have

$$f(2000) = .05a^{2000-1950} = .35 \Rightarrow$$
$$a^{50} = 7 \Rightarrow a = \sqrt[50]{7} \approx 1.04$$

Finally, we have the function

$$f(x) = .05(1.04)^{x-1950}.$$

(b) If you think of $f(x) = .05(1.04)^{x-1950}$ as a function in the form of $A = P\left(1+\frac{r}{n}\right)^{tn}$ where $n = 1$ (compounded annually), we have

$$f(x) = .05(1+.04)^{x-1950} = P(1+r)^t.$$

Thus, the average annual percent increase would be about 4%.

16. Given $t = T\dfrac{\ln\left[1+8.33\left(\frac{A}{K}\right)\right]}{\ln 2}$, find t when $\frac{A}{K} = .175$ and $T = 1.26\times 10^9$.

$$t = 1.26\times 10^9 \cdot \frac{\ln\left[1+8.33(.175)\right]}{\ln 2} \approx 1.635\times 10^9$$

The rock sample is about 1.635 billion yr old.

17. From Example 6, we have the temperature, $f(t)$, of a body at time t after being introduced into an environment having constant temperature T_0 is $f(t) = T_0 + Ce^{-kt}$, where C and k are constants. From the given information, when $t = 0$, $T_0 = 0$, and the temperature of the water is $f(0) = 100$.

$$100 = 0 + Ce^{-0k} \Rightarrow 100 = C$$

Thus, we have $f(t) = 0 + 100e^{-kt} = 100e^{-kt}$.

Also, when $t = \frac{24}{60} = \frac{2}{5}$ hr, $f\left(\frac{2}{5}\right) = 50$. Using this information, we have

$$50 = 100e^{-(2/5)k} \Rightarrow \tfrac{1}{2} = e^{-(2/5)k}$$
$$\ln\tfrac{1}{2} = \ln e^{-(2/5)k} \Rightarrow \ln\tfrac{1}{2} = -\tfrac{2}{5}k$$
$$\ln 1 - \ln 2 = -\tfrac{2}{5}k \Rightarrow \ln 2 = \tfrac{2}{5}k$$
$$k = \tfrac{5}{2}\ln 2 \approx 1.733$$

Thus, the model is $f(t) = 100e^{-1.733t}$.

To find the temperature after $\frac{96}{60} = \frac{8}{5}$ hrs, we find $f\left(\frac{8}{5}\right)$. Since $f\left(\frac{8}{5}\right) = 100e^{-1.733(8/5)} \approx 6.25$, the temperature after 96 minutes is about 6.25° C. Note: We could have used the exact value of k to perform the calculations.

$$f\left(\tfrac{8}{5}\right) = 100e^{-\left(\frac{5}{2}\ln 2\right)\left(\frac{8}{5}\right)} = 100e^{-4\ln 2}$$
$$= 100e^{\ln\frac{1}{16}} = 100\cdot\tfrac{1}{16} = 6.25$$

18. From Example 6, we have the temperature, $f(t)$, of a body at time t after being introduced into an environment having constant temperature T_0 is $f(t) = T_0 + Ce^{-kt}$, where C and k are constants. From the given information, when $t = 0$, $T_0 = 50$, and the temperature of the metal is $f(0) = 300.$

$300 = 50 + Ce^{-0k} \Rightarrow 250 = C$ Thus, we have $f(t) = 50 + 250e^{-kt}$. Also, when $t = \frac{4}{60} = \frac{1}{15}$ hr, $f\left(\frac{1}{15}\right) = 175$. Using this information we have the following.

$$175 = 50 + 250e^{-(1/15)k} \Rightarrow 125 = 250e^{-(1/15)k}$$
$$\tfrac{1}{2} = e^{-(1/15)k} \Rightarrow \ln\tfrac{1}{2} = \ln e^{-(1/15)k}$$
$$\ln\tfrac{1}{2} = -\tfrac{1}{15}k \Rightarrow \ln 1 - \ln 2 = -\tfrac{1}{15}k$$
$$\ln 2 = \tfrac{1}{15}k \Rightarrow k = 15\ln 2 \approx 10.397$$

Thus, the model is $f(t) = 50 + 250e^{-10.397t}$.

To find the temperature after $\frac{12}{60} = \frac{1}{5}$ hrs, we find $f\left(\frac{1}{5}\right)$.

Since $f\left(\frac{1}{5}\right) = 50 + 250e^{-10.397(1/5)} \approx 81.25$, the temperature after 12 minutes is about $81.25°$C. Note: We could have used the exact value of k to perform the calculations.

$$f\left(\tfrac{1}{5}\right) = 50 + 250e^{-(15\ln 2)\left(\frac{1}{5}\right)} = 50 + 250e^{-3\ln 2}$$
$$= 50 + 250e^{\ln\frac{1}{8}} = 50 + 250 \cdot \tfrac{1}{8}$$
$$= 50 + 31.25 = 81.25$$

19. Given P = 60,000 and t = 5, substitute r = .07 and n = 4 into the compound interest formula, $A = P\left(1+\frac{r}{n}\right)^{tn}$. We have

$$A = 60{,}000\left(1+\tfrac{.07}{4}\right)^{5(4)}$$
$$= 60{,}000(1.0175)^{20} \approx 84{,}886.692.$$

The interest from this investment would be \$84,886.69 – \$60,000 = \$24,886.69.
Given P = 60,000 and t = 5, substitute r = .0675 into the continuous compounding formula, $A = Pe^{rt}$.

$$A = 60{,}000e^{.0675(5)} = 60{,}000e^{.3375} \approx 84{,}086.377$$

The interest from this investment would be \$84,086.38 – \$60,000 = \$24,086.38.

(a) The investment that offers 7% compounded quarterly will earn more interest than the investment that offers 6.75% compounded continuously.

(b) Note that \$24,886.69 – \$24,086.38 = \$800.31. The investment that offers 7% compounded quarterly will earn \$800.32 more in interest. Note: Keeping all digits in the calculator rather than rounding the two amounts of interest to the nearest cent before subtracting them will yield a final answer of 800.315242 ≈ \$800.32. This discrepancy is insignificant.

20.
$$A = Pe^{rt}$$
$$A = 80{,}000,\ P = 60{,}000,\ r = .0675$$
$$80{,}000 = 60{,}000e^{.0675t}$$
$$\tfrac{4}{3} = e^{.0675t}$$
$$\ln\tfrac{4}{3} = \ln e^{.0675t}$$
$$\ln\tfrac{4}{3} = .0675t$$
$$t = \frac{\ln\frac{4}{3}}{.0675} \approx 4.3$$

With the continuous compounding plan, it will take about 4.3 yr for Russ's \$60,000 to grow to \$80,000.

21.
$$A = Pe^{rt}$$
$$2P = Pe^{.025t}$$
$$2 = e^{.025t}$$
$$\ln 2 = .025t$$
$$27.73 \approx t$$

The doubling time is about 27.73 yr if interest is compounded continuously.

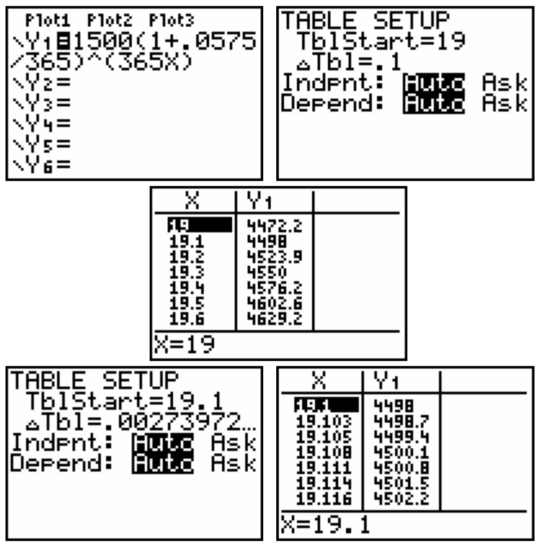
22. From Example 2 we see that the time t required for an investment to double is given by $t = \frac{\ln 2}{r}$, where r represents the interest rate. Thus, t is inversely proportional to r, so when the interest rate is tripled, the time required for an investment to double will be divided by 3.

23.
$$A = Pe^{rt}$$
$$3P = Pe^{.05t} \Rightarrow 3 = e^{.05t} \Rightarrow \ln 3 = \ln e^{.05t}$$
$$\ln 3 = .05t \Rightarrow t = \tfrac{\ln 3}{.05} \approx 21.97$$

It will take about 21.97 years for the investment to triple.

24. Enter $Y_1 = 1500\left(1+\frac{.0575}{365}\right)^{365t}$

Using the TABLE feature, we are seeking, $Y_1 = 3(1500) = 4500$.



When $t \approx 19.1078$ we are fairly close to 4500. Since .1078(365) ≈ 39, investment will triple in about 19 yr, 39 days.

Solve the equation for t with $y = 4500$.

$$4500 = 1500\left(1+\frac{.0575}{365}\right)^{365t}$$
$$3 = \left(1+\frac{.0575}{365}\right)^{365t}$$
$$\ln 3 = \ln\left(1+\frac{.0575}{365}\right)^{365t}$$
$$\ln 3 = 365t\ln\left(1+\frac{.0575}{365}\right)$$
$$t = \frac{\ln 3}{365\ln\left(1+\frac{.0575}{365}\right)} \approx 19.1078$$

This confirms the value found using the calculator.

25. (a) 1969 is represented by $t = 4$

$$M(4) = 434e^{-.08(4)} \approx 315$$

There were 315 continuously serving members in 1969.

(b) 1973 is represented by $t = 8$

$$M(8) = 434e^{-.08(8)} \approx 229$$

There were 229 continuously serving members in 1973.

(c) 1979 is represented by $t = 14$

$$M(14) = 434e^{-.08(14)} \approx 142$$

There were 142 continuously serving members in 1973.

26. $M(t) = 338 = 434e^{-.08t}$

$$\frac{338}{434} = e^{-.08t} \Rightarrow \ln\left(\frac{338}{434}\right) = -.08t$$
$$t = \frac{\ln 338 - \ln 434}{-.08} \approx 3$$

$t = 3$ represents 1968, so there were 338 continuously serving members in 1968.

27. (a) A point associated with the graph of $f(x) = P_0a^{x-2000}$ is $(2000,1)$. Since $f(2000) = P_0a^{2000-2000} = 1$, we have $P_0a^0 = 1 \Rightarrow P_0 = 1$. Thus, we have $f(x) = a^{x-2000}$. Since the point $(2025,1.4)$ is projected to be on the graph, we have the following.

$$f(2025) = a^{2025-2000} = 1.4$$
$$a^{25} = 1.4 \Rightarrow a = \sqrt[25]{1.4} \approx 1.01355$$

(b) From part (a) we have $f(x) = (1.01355)^{x-2000}$. To find the population projected for 2010, we must find $f(2010)$. Thus

$$f(2010) = (1.01355)^{2010-2000}$$
$$= (1.01355)^{10} \approx 1.14 \text{ billion is}$$

the estimated population in 2010.

(c) We must solve $(1.01355)^{x-2000} = 1.5$ for x.

$$(1.01355)^{x-2000} = 1.5$$
$$\ln(1.01355)^{x-2000} = \ln 1.5$$
$$(x-2000)\ln(1.01355) = \ln 1.5$$
$$x - 2000 = \frac{\ln 1.5}{\ln(1.01355)}$$
$$x = 2000 + \frac{\ln 1.5}{\ln(1.01355)}$$
$$\approx 2030.13$$

In 2030, it is projected that the population will reach 1.5 billion.

28. $P(t) = P_0e^{-.04t}$

(a) If $t = 1$, then

$$P(1) = 1{,}000{,}000e^{-.04(1)} \approx 960{,}789.44.$$

The population after 1 year is about 961,000.

(b) Find t when $P(1) = 750,000$.

$750,000 = 1,000,000e^{-.04t} \Rightarrow .75 = e^{-.04t}$
$\ln .75 = \ln e^{-.04t} \Rightarrow \ln .75 = -.04t \Rightarrow$
$t = \frac{\ln .75}{-.04} \approx 7.2$

It takes about 7.2 yr for the population to be reduced to 750,000.

(c) Find t when $P(t) = \frac{1}{2}P_0$.

$\frac{1}{2}P_0 = P_0e^{-.04t} \Rightarrow \frac{1}{2} = e^{-.04t} \Rightarrow$
$\ln\frac{1}{2} = -.04t \Rightarrow t = \frac{\ln\frac{1}{2}}{-.04} \approx 17.3$

It will take about 17.3 yr for the population to decline to half the initial number.

29. $f(x) = 14.621e^{.141x}$

(a) $t = 4$ represents 2004.

$f(4) = 14.621e^{.141(4)} \approx 25.7$

In 2004, gaming revenues were about \$25.7 billion.

(b) Find t when $f(t) = 22.3$.

$22.3 = 14.621e^{.141x} \Rightarrow \frac{22.3}{14.621} = e^{.141x}$
$\ln\left(\frac{22.3}{14.621}\right) = .141x \Rightarrow \frac{\ln\left(\frac{22.3}{14.621}\right)}{.141} = x$
$x \approx 3$

Revenues reached \$22.3 billion in 2003.

30. $A(t) = 283.84e^{.0647t}$

In 2004, $t = 14$, so

$A(14) = 283.84e^{.0647(14)} \approx 702$

In 2004, personal consumption expenditures will be about \$702 billion.

31. $L = 9 + 2e^{.15t}$

(a) In 1988, $t = 6$, so $L = 9 + 2e^{.15(6)} \approx 13.92$

(b) In 1998, $t = 16$, so
$L = 9 + 2e^{.15(16)} \approx 31.05$

(c) In 2008, $t = 26$, so
$L = 9 + 2e^{.15(26)} \approx 107.8$

32. $n \approx -7600 \log r$

(a) $n \approx -7600 \log .9 \approx 347.76$
n is about 350 years.

(b) $n \approx -7600 \log .3 \approx 3973.88$
n is about 4000 years.

(c) $n \approx -7600 \log .5 \approx 2287.83$
About 2300 years have elapsed.

33. $f(t) = 15,000e^{-.05t}$

(a) At the beginning of the epidemic, $t = 0$.
$f(0) = 15,000e^{-.05(0)} = 15,000$
At the beginning of the epidemic, 15,000 people were susceptible.

(b) After 10 days, $t = 10$.
$f(0) = 15,000e^{-.05(10)} \approx 9098$
After 10 days, approximately 9098 people were susceptible.

(c) After 3 weeks, $t = 21$.
$f(21) = 15,000e^{-.05(21)} \approx 5249$
After three weeks, approximately 5249 people were susceptible.

34. If the original number of susceptible people is y_0, then the half-life is the amount of time needed to decrease to $\frac{y_0}{2}$ people.

$f(t) = \frac{y_0}{2} = y_0e^{-.05t}$
$\frac{y_0/2}{y_0} = \frac{1}{2} = e^{-.05t} \Rightarrow \ln\left(\frac{1}{2}\right) = -.05t$
$\frac{\ln\left(\frac{1}{2}\right)}{-.05} = t \Rightarrow t \approx 14$

The initial number of people susceptible will decrease to half its amount in about 14 days.

35. $f(t) = 500e^{.1t}$

(a) $f(2) = 500e^{.1(2)} \approx 611$
At two days, the bacteria count is approximately 611 million.

(b) $f(4) = 500e^{.1(4)} \approx 746$
At four days, the bacteria count is approximately 746 million.

(c) $f(7) = 500e^{.1(7)} \approx 1007$
At one week (seven days), the bacteria count is approximately 1007 million.

36. If the original bacteria count is y_0, then the doubling time is the amount of time needed to increase to $2y_0$ bacteria.

$f(t) = 2y_0 = y_0e^{.1t}$
$\frac{2y_0}{y_0} = 2 = e^{.1t} \Rightarrow \ln 2 = .1t$
$\frac{\ln 2}{.1} = t \Rightarrow t \approx 6.9$

The bacteria count will double in about 6.9 days.

37. $f(t) = 200(.90)^{t-1}$

Find t when $f(t) = 50$.

$$50 = 200(.90)^{t-1}$$
$$.25 = (.90)^{t-1}$$
$$\ln .25 = \ln\left[(.90)^{t-1}\right]$$
$$\ln .25 = (t-1)\ln .90$$
$$t - 1 = \frac{\ln .25}{\ln .90} \Rightarrow t = 1 + \frac{\ln .25}{\ln .90} \approx 14.2$$

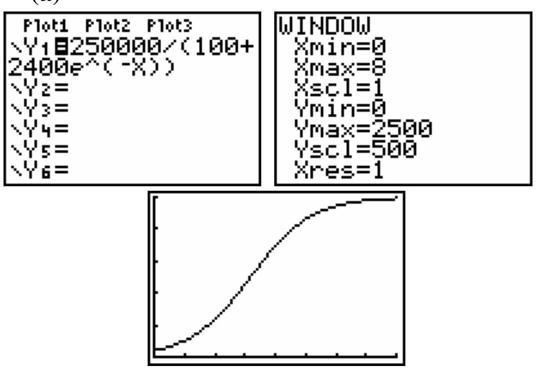
The dose will reach a level of 50 mg in about 14.2 hr.

38. $G(t) = \frac{MG_0}{G_0 + (M - G_0)e^{-kMt}}$; When

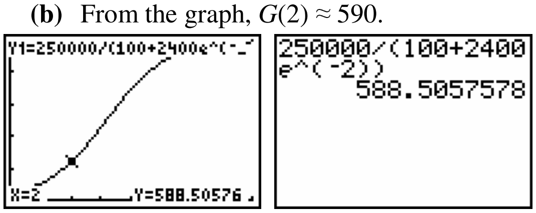
$G_0 = 100$, $M = 2500$, and $k = .0004$ we have

$$G(t) = \frac{(2500)(100)}{100 + (2500-100)e^{-(.0004)(2500)t}} = \frac{250{,}000}{100 + 2400e^{-t}}$$

(a)

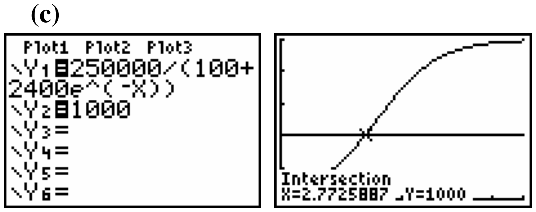


(b) From the graph, $G(2) \approx 590$.



$$G(2) = \frac{250{,}000}{100 + 2400e^{-2}} \approx 589$$

(c)



From the graph, $t \approx 2.8$.

Algebraically we have

$$1000 = \frac{250{,}000}{100 + 2400e^{-t}}$$
$$1000 = \frac{2500}{1 + 24e^{-t}}$$
$$1000\left(1 + 24e^{-t}\right) = 2500$$
$$1000 + 24{,}000e^{-t} = 2500$$
$$24{,}000e^{-t} = 1500$$
$$e^{-t} = .0625$$
$$\ln e^{-t} = \ln .0625$$
$$-t = \ln .0625$$
$$t = -\ln .0625 \approx 2.7726$$

39. $A(t) = 100e^{.024t}$

We want to find the year in which the CPI will be 175.

$$175 = 100e^{.024t} \Rightarrow 1.75 = e^{.024t}$$
$$\ln 1.75 = \ln e^{.024t} \Rightarrow \ln 1.75 = .024t$$
$$t = \frac{\ln 1.75}{.024} \approx 23.3$$

Twenty-three years after 1990, or in 2013, costs were 75% than in 1990.

40. $S(t) = S_0 e^{-at}$

(a) $S(t) = 50{,}000e^{-.10t}$

$$S(1) = 50{,}000e^{-.10(1)} = 50{,}000e^{-.1} \approx 45{,}200$$
$$S(3) = 50{,}000e^{-.10(3)} = 50{,}000e^{-.3} \approx 37{,}000$$

(b) $S(t) = 80{,}000e^{-.05t}$

$$S(2) = 80{,}000e^{-.05(2)} = 80{,}000e^{-.1} \approx 72{,}400$$
$$S(10) = 80{,}000e^{-.05(10)} = 80{,}000e^{-.5} \approx 48{,}500$$

41. $S(t) = 50{,}000e^{-.1t}$

Find t when $S(t) = 25{,}000$.

$$25{,}000 = 50{,}000e^{-.1t} \Rightarrow .5 = e^{-.1t}$$
$$\ln .5 = \ln e^{-.1t} \Rightarrow \ln .5 = -.1t \Rightarrow t = \frac{\ln .5}{-.1} \approx 6.9$$

It will take about 6.9 yr for sales to fall to half the initial sales.

42. Use the formula for continuous compounding with $r = .06$, $P = 4$ and $A = 3(2) = 6$.

$$12 = 4e^{.06t} \Rightarrow 3 = e^{.06t} \Rightarrow \ln 3 = \ln e^{.06t}$$
$$\ln 3 = .06t \Rightarrow t = \frac{\ln 3}{.06} \approx 18.3$$

The cost will triple in about 18.3 yr.

43. Use the formula for continuous compounding with $r = .06$.

$$A = Pe^{rt} \Rightarrow 2P = Pe^{.06t} \Rightarrow 2 = e^{.06t}$$
$$\ln 2 = \ln e^{.06t} \Rightarrow \ln 2 = .06t \Rightarrow t = \frac{\ln 2}{.06} \approx 11.6$$

It will take about 11.6 yr before twice as much electricity is needed.

44. Use the formula for continuous compounding with $r = .02$.

$$A = Pe^{rt} \Rightarrow 2P = Pe^{.02t} \Rightarrow 2 = e^{.02t}$$
$$\ln 2 = \ln e^{.02t} \Rightarrow \ln 2 = .02t \Rightarrow t = \frac{\ln 2}{.02} \approx 34.7 \approx 35$$

It will take about 35 yr before twice as much electricity is needed.

45. $f(x) = \frac{.9}{1+271e^{-.122x}}$

(a) $f(25) = \frac{.9}{1+271e^{-.122(25)}} = \frac{.9}{1+271e^{-3.05}} \approx .065$

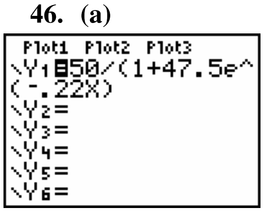
$f(65) = \frac{.9}{1+271e^{-.122(65)}} = \frac{.9}{1+271e^{-7.93}} \approx .820$

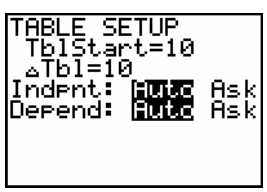
Among people age 25, 6.5% have some CHD, while among people age 65, 82% have some CHD.

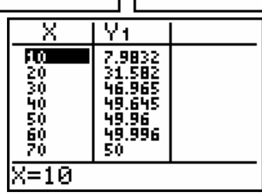
(b)
$$.50 = \frac{.9}{1+271e^{-.122x}}$$
$$.50\left(1+271e^{-.122x}\right) = .9$$
$$.5 + 135.5e^{-.122x} = .9$$
$$135.5e^{-.122x} = .4$$
$$e^{-.122x} = \frac{.4}{135.5}$$
$$\ln e^{-.122x} = \ln \frac{.4}{135.5}$$
$$-.122x = \ln \frac{.4}{135.5}$$
$$x = \frac{\ln \frac{.4}{135.5}}{-.122} \approx 47.75$$

At about 48, the likelihood of coronary heart disease is 50%.

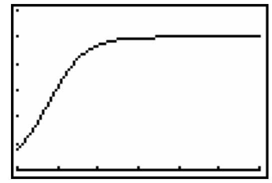
46. (a)



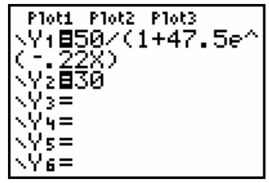


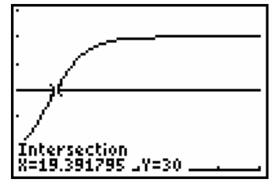


(b) The horizontal asymptote is $y = 50$. It tells us that this tree cannot grow taller than 50 ft.



(c) We can see from the graphing calculator that after about 19.4 yr, the tree is 30 ft tall.





This agrees with the algebraic solution.

$$30 = \frac{50}{1+47.5e^{-.22x}}$$
$$30\left(1+47.5e^{-.22x}\right) = 50$$
$$30 + 1425e^{-.22x} = 50$$
$$1425e^{-.22x} = 20$$
$$e^{-.22x} = \frac{20}{1425}$$
$$\ln e^{-.22x} = \ln \frac{20}{1425}$$
$$-.22x = \ln \frac{20}{1425}$$
$$x = \frac{\ln \frac{20}{1425}}{-.22} \approx 19.4$$

Summary Exercises on Functions: Domains and Defining Equations

1. $f(x) = 3x - 6$

Domain: $(-\infty, \infty)$

2. $f(x) = \sqrt{2x-7}$

The domain is the set of all real numbers such that $2x - 7 \geq 0 \Rightarrow x \geq \frac{7}{2}$. Domain: $\left[\frac{7}{2}, \infty\right)$

3. $f(x) = |x+4|$

Domain: $(-\infty, \infty)$

4. $f(x) = \frac{x+2}{x-6}$

The domain is the set of all real numbers such that $x - 6 \neq 0 \Rightarrow x \neq 6$

Domain: $(-\infty, 6) \cup (6, \infty)$

5. $f(x) = \frac{-2}{x^2+7}$

The domain is the set of all real numbers such that $x^2 + 7 \neq 0 \Rightarrow$ there is no real solution.

Domain: $(-\infty, \infty)$

6. $f(x) = \sqrt{x^2 - 9}$
The domain is the set of all real numbers such that $x^2 - 9 \ge 0 \Rightarrow x^2 \ge 9 \Rightarrow x \le -3$ or $x \ge 3$
Domain: $(-\infty, -3] \cup [3, \infty)$

7. $f(x) = \frac{x^2+7}{x^2-9}$
The domain is the set of all real numbers such that $x^2 - 9 \ne 0 \Rightarrow x \ne -3$ or $x \ne 3$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

8. $f(x) = \sqrt[3]{x^3 + 7x - 4}$
The domain is the set of all real numbers such that $x^3 + 7x - 4$ is real. Domain: $(-\infty, \infty)$

9. $f(x) = \log_5 \left(16 - x^2\right)$
The domain is the set of all real numbers such that $16 - x^2 > 0 \Rightarrow 16 > x^2 \Rightarrow 4 > x$ and $-4 < x$.
Domain: $(-4, 4)$

10. $f(x) = \log\left(\frac{x+7}{x-3}\right)$
The domain is the set of all real numbers such that $\frac{x+7}{x-3} > 0$. $\frac{x+7}{x-3} = 0$ when $x = -7$, and $\frac{x+7}{x-3}$ is undefined when $x = 3$. Use these values to determine the intervals where $\frac{x+7}{x-3} > 0$:

Interval	Test Point	Value of $\frac{x+7}{x-3}$	Sign of $\frac{x+7}{x-3}$
$(-\infty, -7)$	-8	$\frac{1}{11}$	Positive
$(-7, 3)$	0	$-\frac{7}{3}$	Negative
$(3, \infty)$	4	11	Positive

Domain: $(-\infty, -7) \cup (3, \infty)$

11. $f(x) = \sqrt{x^2 - 7x - 8}$
The domain is the set of all real numbers such that $x^2 - 7x - 8 \ge 0$. Solve the equation to find the test intervals: $x^2 - 7x - 8 = 0 \Rightarrow$ $(x - 8)(x + 1) = 0 \Rightarrow x = 8$ or $x = -1$

Interval	Test Point	Value of $x^2 - 7x - 8$	Sign of $x^2 - 7x - 8$
$(-\infty, -1)$	-2	10	Positive
$(-1, 8)$	0	-8	Negative
$(8, \infty)$	10	22	Positive

Domain: $(-\infty, -1] \cup [8, \infty)$

12. $f(x) = 2^{1/x}$
The domain is the set of all real numbers such that $\frac{1}{x}$ is defined, or $x \ne 0$
Domain: $(-\infty, 0) \cup (0, \infty)$

13. $f(x) = \frac{1}{2x^2 - x + 7}$
The domain is the set of all real numbers such that $2x^2 - x + 7 \ne 0$
$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(7)}}{2(2)} = \frac{1 \pm \sqrt{-55}}{4} \Rightarrow$ there are no real solutions. Domain: $(-\infty, \infty)$

14. $f(x) = \frac{x^2-25}{x+5}$
The domain is the set of all real numbers such that $x + 5 \ne 0 \Rightarrow x \ne -5$
Domain: $(-\infty, -5) \cup (-5, \infty)$

15. $f(x) = \sqrt{x^3 - 1}$
The domain is the set of all real numbers such that $x^3 - 1 \ge 0 \Rightarrow x^3 \ge 1 \Rightarrow x \ge 1$
Domain: $[1, \infty)$

16. $f(x) = \ln\left|x^2 - 5\right|$
The domain is the set of all real numbers such that $\left|x^2 - 5\right| \ne 0 \Rightarrow x^2 - 5 \ne 0 \Rightarrow x^2 \ne 5 \Rightarrow$
$x \ne \sqrt{5}$ or $x \ne -\sqrt{5}$
Domain: $\left(-\infty, -\sqrt{5}\right) \cup \left(-\sqrt{5}, \sqrt{5}\right) \cup \left(\sqrt{5}, \infty\right)$

17. $f(x) = e^{x^2 + x + 4}$
The domain is the set of values such that $x^2 + x + 4$ is real. Domain: $(-\infty, \infty)$

18. $f(x) = \frac{x^3-1}{x^2-1}$
The domain is the set of all real numbers such that $\frac{x^3-1}{x^2-1}$ is defined, or $x^2 - 1 \ne 0 \Rightarrow$
$x^2 \ne 1 \Rightarrow x \ne 1$ or $x \ne -1$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

19. $f(x) = \sqrt{\frac{-1}{x^3-1}}$
The domain is the set of all real numbers such that $\sqrt{\frac{-1}{x^3-1}}$ is defined or $\frac{-1}{x^3-1} \ge 0$. $\frac{-1}{x^3-1}$ is defined for $x^3 - 1 \ne 0 \Rightarrow x^3 \ne 1 \Rightarrow x \ne 1$

Interval	Test Point	Value of $\frac{-1}{x^3-1}$	Sign of $\frac{-1}{x^3-1}$
$(-\infty, 1)$	0	1	Positive
$(1, \infty)$	2	$-\frac{1}{7}$	Negative

Domain: $(-\infty, 1)$

20. $f(x) = \sqrt[3]{\frac{1}{x^3-8}}$

The domain is the set of all real numbers such that $\frac{1}{x^3-8}$ is defined, or

$x^3 - 8 \neq 0 \Rightarrow x^3 \neq 8 \Rightarrow x \neq 2$

Domain: $(-\infty, 2) \cup (2, \infty)$

21. $f(x) = \ln\left(x^2 + 1\right)$

The domain is the set of all real numbers such that $x^2 + 1 > 0$. Domain: $(-\infty, \infty)$

22. $f(x) = \sqrt{(x-3)(x+2)(x-4)}$

The domain is the set of all real numbers such that $(x-3)(x+2)(x-4) \geq 0$. Solve to find the test intervals: $(x-3)(x+2)(x-4) = 0 \Rightarrow$ $x = 3$ or $x = -2$ or $x = 4$.

Interval	Test Point	Value of $(x-3)(x+2)(x-4)$	Sign
$(-\infty, -2)$	-3	-42	Negative
$(-2, 3)$	0	24	Positive
$(3, 4)$	3.5	-1.375	Negative
$(4, \infty)$	5	14	Positive

Domain: $[-2, 3] \cup [4, \infty)$

23. $f(x) = \log\left(\frac{x+2}{x-3}\right)^2$

Since $\left(\frac{x+2}{x-3}\right)^2 \geq 0$ for all real numbers, the domain of *f*(*x*) is the set of all real numbers such that $\frac{x+2}{x-3} \neq 0$. $\frac{x+2}{x-3} = 0$ when $x = -2$, and $\frac{x+2}{x-3}$ is undefined when $x = 3$.

Domain: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

24. $f(x) = \sqrt[12]{(4-x)^2(x+3)}$

The domain is the set of all real numbers such that $(4-x)^2(x+3) \geq 0$. Solve the equation to find the test intervals: $(4-x)^2(x+3) = 0 \Rightarrow$ $x = 4$ or $x = -3$.

Interval	Test Point	Value of $(4-x)^2(x+3)$	Sign
$(-\infty, -3)$	-4	-64	Negative
$(-3, 4)$	0	48	Positive
$(4, \infty)$	5	8	Positive

Domain: $[-3, 4] \cup [4, \infty)$ or $[-3, \infty)$

25. $f(x) = e^{|1/x|}$

The domain is the set of all real numbers such that $\frac{1}{x}$ is defined, or $x \neq 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

26. $f(x) = \frac{1}{|x^2-7|}$

The domain is the set of all real numbers such that $\frac{1}{|x^2-7|}$ is defined, or $x^2 - 7 \neq 0 \Rightarrow$

$x^2 \neq 7 \Rightarrow x \neq \pm\sqrt{7}$

Domain: $\left(-\infty, -\sqrt{7}\right) \cup \left(-\sqrt{7}, \sqrt{7}\right) \cup \left(\sqrt{7}, \infty\right)$

27. $f(x) = x^{100} - x^{50} + x^2 + 5$; Domain: $(-\infty, \infty)$

28. $f(x) = \sqrt{-x^2 - 9}$

The domain is the set of all real numbers such that $-x^2 - 9 \geq 0 \Rightarrow -x^2 \geq 9$, which is impossible. The function is not defined for any values of *x*, thus the domain is $\varnothing$.

29. $f(x) = \sqrt[4]{16 - x^4}$

The domain is the set of all real numbers such that $16 - x^4 \geq 0$. Solve the equation to find the test intervals: $16 - x^4 = 0 \Rightarrow$

$(2-x)(2+x)(4+x^2) = 0 \Rightarrow x = 2$ or $x = -2$

Interval	Test Point	Value of $16 - x^4$	Sign of $16 - x^4$
$(-\infty, -2)$	-3	-65	Negative
$(-2, 2)$	0	16	Positive
$(2, \infty)$	3	-65	Negative

Domain: $[-2, 2]$

30. $f(x) = \sqrt[3]{16 - x^4}$

Because the root index is odd, the domain is the set of all real numbers for which $16 - x^4$ is real. Domain: $(-\infty, \infty)$

31. $f(x)=\sqrt{\frac{x^2-2x-63}{x^2+x-12}}$
The domain is the set of real numbers such that $\frac{x^2-2x-63}{x^2+x-12}\ge 0$. $\frac{x^2-2x-63}{x^2+x-12}$ is not defined for $x^2+x-12=0\Rightarrow(x+4)(x-3)=0\Rightarrow$ $x\ne -4$ or $x\ne 3$. Solve $\frac{x^2-2x-63}{x^2+x-12}=0$ to find the test intervals: $\frac{x^2-2x-63}{x^2+x-12}=0\Rightarrow$ $x^2-2x-63=0\Rightarrow(x-9)(x+7)=0\Rightarrow x=9$ or $x=-7$.

Interval	Test Point	Value of $\frac{x^2-2x-63}{x^2+x-12}$	Sign
(−∞, −7)	−10	$\frac{19}{26}$	Positive
(−7, −4)	−5	$-\frac{7}{2}$	Negative
(−4, 3)	0	$\frac{21}{4}$	Positive
(3, 9)	5	$-\frac{8}{3}$	Negative
(9, ∞)	10	$\frac{17}{98}$	Positive

Domain: $(-\infty,-7]\cup(-4,3)\cup[9,\infty)$

32. $f(x)=\sqrt[5]{5-x}$
Because the root index is odd, the domain is the set of all real numbers for which $5-x$ is real. Domain: $(-\infty,\infty)$

33. $f(x)=\left|\sqrt{5-x}\right|$
The domain is the set of real numbers such that $5-x\ge 0\Rightarrow 5\ge x$
Domain: $(-\infty,5]$

34. $f(x)=\sqrt{\frac{-1}{x-3}}$
The domain is the set of real numbers such that $\frac{-1}{x-3}\ge 0$ and $x-3\ne 0\Rightarrow x\ne 3$

Interval	Test Point	Value of $\frac{-1}{x-3}$	Sign
(−∞, 3)	0	$\frac{1}{3}$	Positive
(3, ∞)	5	$-\frac{1}{2}$	Negative

Domain: $(-\infty,3)$

35. $f(x)=\log\left|\frac{1}{4-x}\right|$
The domain is the set of real numbers such that $\left|\frac{1}{4-x}\right|>0\Rightarrow\frac{1}{4-x}>0\Rightarrow 4-x>0\Rightarrow 4>x$ or $-\frac{1}{4-x}<0\Rightarrow -4+x<0\Rightarrow x<4$
Domain: $(-\infty,4)\cup(4,\infty)$

36. $f(x)=6^{x^2-9}$
The domain is the set of real numbers such that x^2-9 is a real number.
Domain: $(-\infty,\infty)$

37. $f(x)=6^{\sqrt{x^2-25}}$
The domain is the set of real numbers such that $\sqrt{x^2-25}$ is a real number.
$x^2-25\ge 0\Rightarrow x^2\ge 25\Rightarrow x\ge 5$ or $x\le -5$
Domain: $(-\infty,-5]\cup[5,\infty)$

38. $f(x)=6^{\sqrt[3]{x^2-25}}$
The domain is the set of real numbers such that $\sqrt[3]{x^2-25}$ is a real number. Because the the root index is odd, the domain is the set of all real numbers for which x^2-25 is real.
Domain: $(-\infty,\infty)$

39. $f(x)=\ln\left(\frac{-3}{(x+2)(x-6)}\right)$
The domain is the set of real numbers such that $\frac{-3}{(x+2)(x-6)}>0$ and $(x+2)(x-6)\ne 0\Rightarrow$ $x\ne -2$ or $x\ne 6$.

Interval	Test Point	Value of $\frac{-3}{(x+2)(x-6)}$	Sign of $\frac{-3}{(x+2)(x-6)}$
(−∞, −2)	−3	$-\frac{1}{3}$	Negative
(−2, 6)	0	$\frac{1}{4}$	Positive
(6, ∞)	7	$-\frac{1}{3}$	Negative

Domain: $(-2,6)$

40. $f(x)=\frac{-2}{\log x}$
The domain is the set of real numbers such that $\frac{-2}{\log x}$ is defined. $\frac{-2}{\log x}$ is not defined for $x\le 0$ and for $\log x=0\Rightarrow x=1$.
Domain: $(0,1)\cup(1,\infty)$

41. Choice A can be written as a function of x.
$3x+2y=6 \Rightarrow y=f(x)=-\frac{3}{2}x+3$

42. Choice B can be written as a function of x.
$x^2+y-2=0 \Rightarrow y=f(x)=-x^2+2$

43. Choice C can be written as a function of x.
$x^3+y^3=5 \Rightarrow y=f(x)=\sqrt[3]{5-x^3}$

44. Choice D can be written as a function of x.
$x=10^y \Rightarrow \log x=y=f(x)$

45. Choice A can be written as a function of x.
$x=\frac{2-y}{y+3} \Rightarrow y=f(x)=\frac{2-3x}{x+1}$

46. Choice B can be written as a function of x.
$e^{y+2}=x \Rightarrow y=f(x)=\ln x-2$

47. Choice D can be written as a function of x.
$2x=\frac{1}{y^3} \Rightarrow y=f(x)=\sqrt[3]{\frac{1}{2x}}$

48. Choice C can be written as a function of x.
$x=\frac{1}{y} \Rightarrow y=f(x)=\frac{1}{x}$

49. Choice C can be written as a function of x.
$\frac{x}{4}-\frac{y}{9}=0 \Rightarrow y=f(x)=\frac{9x}{4}$

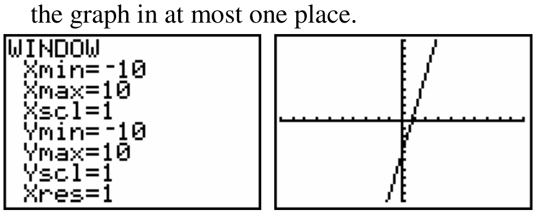
50. Choice B can be written as a function of x.
$y-\sqrt{(x+2)^2}=0 \Rightarrow y=f(x)=\sqrt{(x+2)^2}$

Chapter 4: Review Exercises

1. This is not a one-to-one function since a horizontal line can intersect the graph in more than one point.

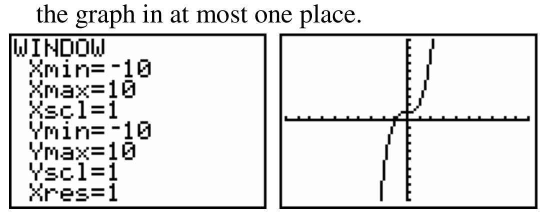
2. This is a one-to-one function since every horizontal line intersects the graph in no more than one point.

3. $y=5x-4$
Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in at most one place.



If we attempt to find the inverse function, we see that this function can be found.
Step 1: Interchange x and y. $x=5y-4$
Step 2: Solve for y.
$x=5y-4 \Rightarrow x+4=5y \Rightarrow y=\frac{x+4}{5}$
Step 3: Replace y with $f^{-1}(x)$. $f^{-1}(x)=\frac{x+4}{5}$
Also, an acceptable way of showing that a function is one-to-one is to assume that you have two equal y-values $\left(f(x_1)=f(x_2)\right)$ and show that they must have come from the same x-value $(x_1=x_2)$.
$f(x_1)=f(x_2) \Rightarrow 5x_1-4=5x_2-4 \Rightarrow$
$5x_1=5x_2 \Rightarrow x_1=x_2$
So, the function is one-to-one.

4. $y=x^3+1$
Looking at this function graphed on a TI-83, we can see that it appears that any horizontal line passed through the function will intersect the graph in at most one place.



If we attempt to find the inverse function, we see that this function can be found.
Step 1: Interchange x and y. $x=y^3+1$
Step 2: Solve for y.
$x=y^3+1 \Rightarrow x-1=y^3 \Rightarrow y=\sqrt[3]{x-1}$
Step 3: Replace y with $f^{-1}(x)$.
$f^{-1}(x)=\sqrt[3]{x-1}$
Also, an acceptable way of showing that a function is one-to-one is to assume that you have two equal y-values $\left(f(x_1)=f(x_2)\right)$ and show that they must have come from the same x-value $(x_1=x_2)$.
$f(x_1)=f(x_2) \Rightarrow \sqrt[3]{x_1-1}=\sqrt[3]{x_2-1} \Rightarrow$
$\sqrt[3]{x_1-1}=\sqrt[3]{x_2-1} \Rightarrow \left(\sqrt[3]{x_1-1}\right)^3=\left(\sqrt[3]{x_2-1}\right)^3 \Rightarrow$
$x_1-1=x_2-1 \Rightarrow x_1=x_2$
So, the function is one-to-one.

5. $y = (x+3)^2$

If $x = -2$, $y = (-2+3)^2 = 1^2 = 1$.

If $x = -4$, $y = (-4+3)^2 = (-1)^2 = 1$.

Since two different values of x lead to the same value of y, the function is not one-to-one.

6. $y = \sqrt{3x^2 + 2}$

If $x = 1$, $y = \sqrt{3(1)^2 + 2} = \sqrt{3(1)+2}$ $= \sqrt{3+2} = \sqrt{5}$. If $x = -1$, $y = \sqrt{3(-1)^2 + 2}$ $= \sqrt{3(1)+2} = \sqrt{3+2} = \sqrt{5}$. Since two different values of x lead to the same value of y, the function is not one-to-one.

7. $f(x) = x^3 - 3$

This function is one-to-one.

Step 1: Replace $f(x)$ with y and interchange x and y. $y = x^3 - 3 \Rightarrow x = y^3 - 3$

Step 2: Solve for y.

$x = y^3 - 3 \Rightarrow x + 3 = y^3 \Rightarrow y = \sqrt[3]{x+3}$

Step 3: Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{x+3}$

8. $f(x) = \sqrt{25 - x^2}$

If $x = 4$, $y = \sqrt{25 - 4^2} = \sqrt{25-16} = \sqrt{9} = 3$.

If $x = -4$, $y = \sqrt{25 - (-4)^2} = \sqrt{25-16}$ $= \sqrt{9} = 3$. Since two different values of x lead to the same value of y, the function is not one-to-one. Thus, f has no inverse function.

9. $f^{-1}(\$50{,}000)$ represents the number of years after 2004 required for the investment to reach \$50,000.

10. The two graphs are reflections of each other across the line $y = x$; thus, they are inverses of each other.

11. To have an inverse, a function must be a <u>one-to-one</u> function.

12. Yes. In addition, if the function is one-to-one and has an inverse, the function will have at most one x-intercept and the inverse function will have at most one y-intercept.

13. $y = \log_{.3} x$

The point (1, 0) is on the graph of every function of the form $y = \log_a x$, so the correct choice must be either B or C. Since the base is $a = .3$ and $0 < .3 < 1$, $y = \log_{.3} x$ is a decreasing function, and so the correct choice must be B.

14. $y = e^x$

The point (0, 1) is on the graph since $e^0 = 1$, so the correct choice must be either A or D. Since the base is e and $e > 1$, $y = e^x$ is an increasing function, and so the correct choice must be A.

15. $y = \ln x = \log_e x$

The point (1, 0) is on the graph of every function of the form $y = \log_a x$, so the correct choice must be either B or C. Since the base is $a = e$ and $e > 1$, $y = \ln x$ is an increasing function, and so the correct choice must be C.

16. $y = (.3)^x$

The point (0, 1) is on the graph since $(.3)^0 = 1$, so the correct choice must be either A or D. Since the base is .3 and $0 < .3 < 1$, $y = (.3)^x$ is a decreasing function, and so the correct choice must be D.

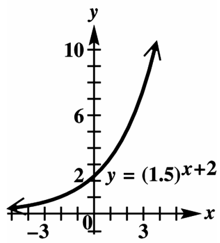
17. $2^5 = 32$ is written in logarithmic form as $\log_2 32 = 5$.

18. $100^{1/2} = 10$ is written in logarithmic form as $\log_{100} 10 = \frac{1}{2}$.

19. $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$ is written in logarithmic form as $\log_{3/4} \frac{4}{3} = -1$.

20. The y-intercept of $f(x) = (1.5)^{x+2}$ is $f(0) = (1.5)^{0+2} = 1.5^2 = 2.25$ and the x-axis is a horizontal asymptote. Make a table of values, plot the points, and draw a smooth curve through them.

x	$f(x)$
–4	≈ .44
–3	≈ .67
–2.5	≈ .82
–2	1
–1.5	≈ 1.22
–1	1.5
0	2.25



21. $\log_3 4$ is the logarithm with the base 3 of 4. ($\log_4 3$ would be the logarithm with the base 4 of 3.)

22. $\log_9 27 = \frac{3}{2}$ is written in exponential form as $9^{3/2} = 27.$

23. $\log 1000 = 3$ is written in exponential form as $10^3 = 1000.$

24. $\ln\sqrt{e} = \frac{1}{2}$ is written in exponential form as $e^{1/2} = \sqrt{e}.$

25. Let $f(x) = \log_a x$ be the required function. Then $f(81) = 4 \Rightarrow \log_a 81 = 4 \Rightarrow a^4 = 81 \Rightarrow a^4 = 3^4 \Rightarrow a = 3.$ The base is 3.

26. Let $f(x) = a^x$ be the required function. Then $f(-4) = \frac{1}{16} \Rightarrow a^{-4} = \frac{1}{16} \Rightarrow a^{-4} = 2^{-4} \Rightarrow a = 2.$ The base is 2.

27.
$$\begin{aligned}\log_3 \tfrac{mn}{5r} &= \log_3 mn - \log_3 5r\\ &= \log_3 m + \log_3 n - (\log_3 5 + \log_3 r)\\ &= \log_3 m + \log_3 n - \log_3 5 - \log_3 r\end{aligned}$$

28.
$$\begin{aligned}&\log_5\left(x^2 y^4 \sqrt[5]{m^3 p}\right)\\ &= \log_5 x^2 y^4 \left(m^3 p\right)^{1/5}\\ &= \log_5 x^2 + \log_5 y^4 + \log_5 \left(m^3 p\right)^{1/5}\\ &= 2\log_5 x + 4\log_5 y + \tfrac{1}{5}\left(\log_5 m^3 p\right)\\ &= 2\log_5 x + 4\log_5 y + \tfrac{1}{5}\left(\log_5 m^3 + \log_5 p\right)\\ &= 2\log_5 x + 4\log_5 y + \tfrac{1}{5}\left(3\log_5 m + \log_5 p\right)\end{aligned}$$

29. $\log_7(7k + 5r^2)$
Since this is the logarithm of a sum, this expression cannot be simplified.

30. $\log 45.6 \approx 1.6590$

31. $\log .0411 \approx -1.3862$

32. $\ln 470 \approx 6.1527$

33. $\ln 144{,}000 \approx 11.8776$

34. To find $\log_3 769$, use the change-of-base theorem. We have
$\log_3 769 = \frac{\log 769}{\log 3} = \frac{\ln 769}{\ln 3} \approx 6.0486.$

35. To find $\log_{2/3} \frac{5}{8}$, use the change-of-base theorem. We have
$\log_{2/3} \frac{5}{8} = \frac{\log\frac{5}{8}}{\log\frac{2}{3}} = \frac{\ln\frac{5}{8}}{\ln\frac{2}{3}} \approx 1.1592.$

36. $8^x = 32 \Rightarrow \left(2^3\right)^x = 2^5 \Rightarrow 2^{3x} = 2^5 \Rightarrow 3x = 5 \Rightarrow x = \frac{5}{3}$
Solution set: $\left\{\frac{5}{3}\right\}$

37. $16^{x+4} = 8^{3x-2} \Rightarrow \left(2^4\right)^{x+4} = \left(2^3\right)^{3x-2} \Rightarrow$
$2^{4x+16} = 2^{9x-6} \Rightarrow 4x + 16 = 9x - 6 \Rightarrow$
$22 = 5x \Rightarrow \frac{22}{5} = x$
Solution set: $\left\{\frac{22}{5}\right\}$

38. $4^x = 12 \Rightarrow \log\left(4^x\right) = \log 12 \Rightarrow$
$x\log 4 = \log 12 \Rightarrow x = \frac{\log 12}{\log 4} \approx 1.792$

39.
$$\begin{aligned}3^{2x-5} = 13 &\Rightarrow \ln 3^{2x-5} = \ln 13 \Rightarrow\\ (2x-5)\ln 3 = \ln 13 &\Rightarrow 2x - 5 = \tfrac{\ln 13}{\ln 3} \Rightarrow\\ 2x &= 5 + \tfrac{\ln 13}{\ln 3}\\ x &= \tfrac{1}{2}\left(5 + \tfrac{\ln 13}{\ln 3}\right) \approx 3.667\end{aligned}$$
or
$$\begin{aligned}3^{2x-5} = 13 &\Rightarrow \ln 3^{2x-5} = \ln 13\\ (2x-5)\ln 3 &= \ln 13\\ 2x\ln 3 - 5\ln 3 &= \ln 13\\ x\ln 3^2 - \ln 3^5 &= \ln 13\\ x\ln 9 - \ln 243 &= \ln 13\\ x\ln 9 &= \ln 13 + \ln 243\\ x\ln 9 &= \ln 3159 \Rightarrow x = \tfrac{\ln 3159}{\ln 9} \approx 3.667\end{aligned}$$
Solution set: {3.667}

40.
$$\begin{aligned}2^{x+3} = 5^x &\Rightarrow \ln 2^{x+3} = \ln 5^x\\ (x+3)\ln 2 &= x\ln 5\\ x\ln 2 + 3\ln 2 &= x\ln 5\\ x\ln 2 - x\ln 5 &= -3\ln 2\\ x(\ln 2 - \ln 5) &= -\ln 2^3\\ x\left(\ln\tfrac{2}{5}\right) = -\ln 8 &\Rightarrow x = \frac{-\ln 8}{\ln\frac{2}{5}} \approx 2.269\end{aligned}$$
Solution set: {2.269}

41. $6^{x+3} = 4^x \Rightarrow \ln 6^{x+3} = \ln 4^x \Rightarrow$
$(x+3)\ln 6 = x\ln 4$
$x\ln 6 + 3\ln 6 = x\ln 4$
$x\ln 6 - x\ln 4 = -3\ln 6$
$x(\ln 6 - \ln 4) = -3\ln 6 \Rightarrow x\left(\ln \frac{6}{4}\right) = -\ln 6^3 \Rightarrow$
$x\left(\ln \frac{3}{2}\right) = -\ln 216$
$x = \dfrac{-\ln 216}{\ln \frac{3}{2}} \approx -13.257$
Solution set: {−13.257}

42. $e^{x-1} = 4 \Rightarrow \ln e^{x-1} = \ln 4 \Rightarrow x - 1 = \ln 4 \Rightarrow$
$x = \ln 4 + 1 \approx 2.386$
Solution set: {2.386}

43. $e^{2-x} = 12 \Rightarrow \ln e^{2-x} = \ln 12 \Rightarrow$
$2 - x = \ln 12 \Rightarrow -x = -2 + \ln 12 \Rightarrow$
$x = 2 - \ln 12 \approx -.485$
Solution set: {−.485}

44. $2e^{5x+2} = 8 \Rightarrow e^{5x+2} = 4 \Rightarrow$
$\ln e^{5x+2} = \ln 4 \Rightarrow 5x + 2 = \ln 4 \Rightarrow$
$5x = \ln 4 - 2 \Rightarrow x = \frac{1}{5}(\ln 4 - 2) \approx -.123$
Solution set: {−.123}

45. $10e^{3x-7} = 5 \Rightarrow e^{3x-7} = \frac{1}{2} \Rightarrow$
$\ln e^{3x-7} = \ln \frac{1}{2} \Rightarrow 3x - 7 = \ln \frac{1}{2} \Rightarrow$
$3x = \ln \frac{1}{2} + 7 \Rightarrow x = \frac{1}{3}\left(\ln \frac{1}{2} + 7\right) \approx 2.102$
Solution set: {2.102}

46. $5^{x+2} = 2^{2x-1}$
$\ln 5^{x+2} = \ln 2^{2x-1}$
$(x+2)\ln 5 = (2x-1)\ln 2$
$x\ln 5 + 2\ln 5 = 2x\ln 2 - \ln 2$
$x\ln 5 + \ln 5^2 = x\ln 2^2 - \ln 2$
$x\ln 5 + \ln 25 = x\ln 4 - \ln 2$
$\ln 25 + \ln 2 = x\ln 4 - x\ln 5$
$\ln 25 + \ln 2 = x(\ln 4 - \ln 5)$
$\frac{\ln 25 + \ln 2}{\ln 4 - \ln 5} = x$
$x = \frac{\ln 50}{\ln \frac{4}{5}} \approx -17.531$
Solution set: {−17.531}

47. $6^{x-3} = 3^{4x+1}$
$\ln 6^{x-3} = \ln 3^{4x+1}$
$(x-3)\ln 6 = (4x+1)\ln 3$
$x\ln 6 - 3\ln 6 = 4x\ln 3 + \ln 3$
$x\ln 6 - \ln 6^3 = x\ln 3^4 + \ln 3$
$x\ln 6 - \ln 216 = x\ln 81 + \ln 3$
$x\ln 6 - x\ln 81 = \ln 3 + \ln 216$
$x(\ln 6 - \ln 81) = \ln 3 + \ln 216$
$x = \frac{\ln 3 + \ln 216}{\ln 6 - \ln 81} = \frac{\ln(3\cdot 216)}{\ln \frac{6}{81}} = \frac{\ln 648}{\ln \frac{2}{27}}$
$x \approx -2.487$
Solution set: {−2.487}

48. $e^{8x} \cdot e^{2x} = e^{20} \Rightarrow e^{10x} = e^{20} \Rightarrow$
$10x = 20 \Rightarrow x = 2$
Solution set: {2}

49. $e^{6x} \cdot e^{x} = e^{21} \Rightarrow e^{7x} = e^{21} \Rightarrow 7x = 21 \Rightarrow x = 3$
Solution set: {3}

50. $100(1.02)^{x/4} = 200 \Rightarrow 1.02^{x/4} = 2 \Rightarrow$
$\ln\left(1.02^{x/4}\right) = \ln 2 \Rightarrow \frac{x}{4}\ln(1.02) = \ln 2 \Rightarrow$
$x\ln(1.02) = 4\ln 2 \Rightarrow x\ln(1.02) = \ln 2^4 \Rightarrow$
$x\ln(1.02) = \ln 16 \Rightarrow$
$x = \dfrac{\ln 16}{\ln(1.02)} \approx 140.011$
Solution set: {140.011}

51. $3\ln x = 13 \Rightarrow \ln x = \frac{13}{3} \Rightarrow x = e^{13/3}$
Solution set: $\left\{e^{13/3}\right\}$

52. $\ln(5x) = 16 \Rightarrow 5x = e^{16} \Rightarrow x = \frac{e^{16}}{5}$
Solution set: $\left\{\frac{e^{16}}{5}\right\}$

53. $\log(2x+7) = .25 \Rightarrow 2x + 7 = 10^{.25} = \sqrt[4]{10} \Rightarrow$
$x = \frac{\sqrt[4]{10} - 7}{2}$
Solution set: $\left\{\frac{\sqrt[4]{10} - 7}{2}\right\}$

54. $\ln x + \ln x^3 = 12 \Rightarrow \ln x + 3\ln x = 12 \Rightarrow$
$4\ln x = 12 \Rightarrow \ln x = 3 \Rightarrow x = e^3$
Solution set: $\left\{e^3\right\}$

55. $\log x + \log(13 - 3x) = 1 \Rightarrow$
$\log[x(13 - 3x)] = 1 \Rightarrow 13x - 3x^2 = 10^1 \Rightarrow$
$13x - 3x^2 = 10 \Rightarrow 3x^2 - 13x + 10 = 0 \Rightarrow$
$(3x - 10)(x - 1) = 0 \Rightarrow x = \frac{10}{3}$ or $x = 1$
Solution set: $\left\{1, \frac{10}{3}\right\}$

56. $\log_7(3x+2) - \log_7(x-2) = 1 \Rightarrow$
$\log_7 \frac{3x+2}{x-2} = 1 \Rightarrow \frac{3x+2}{x-2} = 7^1 \Rightarrow$
$3x + 2 = 7(x - 2) \Rightarrow 3x + 2 = 7x - 14 \Rightarrow$
$-4x = -16 \Rightarrow x = 4$
Solution set: {4}

57. $\ln(6x) - \ln(x+1) = \ln 4$

$$\ln\frac{6x}{x+1} = \ln 4 \Rightarrow \frac{6x}{x+1} = 4$$
$$6x = 4(x+1)$$
$$6x = 4x+4$$
$$2x = 4 \Rightarrow x = 2$$

Solution set: {2}

58. $\log_{16}\sqrt{x+1} = \frac{1}{4} \Rightarrow \sqrt{x+1} = 16^{1/4}$

$$\left(\sqrt{x+1}\right)^2 = \left(16^{1/4}\right)^2 \Rightarrow x+1 = 16^{1/2}$$
$$x+1 = \sqrt{16} = 4 \Rightarrow x = 3$$

Since the solution involves squaring both sides, the proposed solution must be checked in the original equation.

Check $x = 3$.

$$\log_{16}\sqrt{x+1} = \frac{1}{4}$$
$$\log_{16}\sqrt{3+1} \stackrel{?}{=} \frac{1}{4} \Rightarrow \log_{16}\sqrt{4} = \frac{1}{4}$$
$$\log_{16} 4^{1/2} = \frac{1}{4} \Rightarrow \frac{1}{2}\log_{16} 4 = \frac{1}{4}$$
$$\frac{1}{2}\log_{16}\sqrt{16} = \frac{1}{4} \Rightarrow \frac{1}{2}\log_{16} 16^{1/2} = \frac{1}{4}$$
$$\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \Rightarrow \frac{1}{4} = \frac{1}{4}$$

This is a true statement. Therefore, 3 is a solution.

Solution set: {3}

59. $\ln\left[\ln\left(e^{-x}\right)\right] = \ln 3$

$$\ln(-x) = \ln 3 \Rightarrow -x = 3 \Rightarrow x = -3$$

Solution set: {–3}

60. $S = a\ln\left(1+\frac{n}{a}\right)$ for n

$$\frac{S}{a} = \ln\left(1+\frac{n}{a}\right)$$
$$e^{S/a} = 1+\frac{n}{a}$$
$$e^{S/a} - 1 = \frac{n}{a}$$
$$n = a\left(e^{S/a} - 1\right)$$

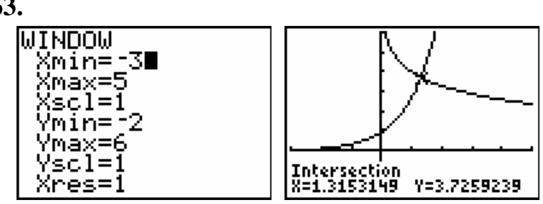
61. $\frac{d}{10} = \log\left(\frac{I}{I_0}\right) \Rightarrow 10^{d/10} = \frac{I}{I_0} \Rightarrow$

$$I_0\left(10^{d/10}\right) = I \Rightarrow I_0 = \frac{I}{10^{d/10}}$$

62. $D = 200 + 100\log x$, for x

$$\frac{D-200}{100} = \frac{D}{100} - 2 = \log x \Rightarrow 10^{\frac{D}{100}-2} = x$$

63.



Solution set: {1.315}

64. (a) $6.6 = \log_{10}\frac{I}{I_0}$

$$\frac{I}{I_0} = 10^{6.6}$$
$$I = I_0 10^{6.6} \approx 3{,}981{,}071.71 I_0$$

The magnitude was about $4{,}000{,}000 I_0$.

(b) $6.5 = \log_{10}\frac{I}{I_0}$

$$10^{6.5} = \frac{I}{I_0} \Rightarrow I_0 10^{6.5} = I$$
$$I \approx 3{,}162{,}277.66 I_0$$

The magnitude was about $3{,}200{,}000 I_0$.

(c) Consider the ratio of the magnitudes.

$$\frac{4{,}000{,}000\ I_0}{3{,}200{,}000\ I_0} = \frac{40}{32} = \frac{5}{4} = 1.25$$

The earthquake with a measure of 6.6 was about 1.25 times as great.

65. (a) $8.3 = \log_{10}\frac{I}{I_0}$

$$\frac{I}{I_0} = 10^{8.3}$$
$$I = 10^{8.3} I_0 \approx 199{,}526{,}231.5 I_0$$

The magnitude was about $200{,}000{,}000 I_0$.

(b) $7.1 = \log_{10}\frac{I}{I_0}$

$$\frac{I}{I_0} = 10^{7.1}$$
$$I = 10^{7.1} I_0 \approx 12{,}589{,}254.12 I_0$$

The magnitude was about $13{,}000{,}000 I_0$.

(c) $\frac{200{,}000{,}000 I_0}{13{,}000{,}000 I_0} = \frac{200}{13} \approx 15.38$

The 1906 earthquake had a magnitude more than 15 times greater than the 1989 earthquake. Note: If the more precise values found in parts (a) and (b) were used, the 1906 earthquake had a magnitude of almost 16 times greater than the 1989 earthquake.

66. For 89 decibels, we have

$$89 = 10\ \log\frac{I}{I_0} \Rightarrow 8.9 = \log\frac{I}{I_0}$$
$$\frac{I}{I_0} = 10^{8.9} \Rightarrow I = 10^{8.9} I_0.$$

For 86 decibels, we have

$$86 = 10\log\frac{I}{I_0}$$
$$\frac{I}{I_0} = 10^{8.6} \Rightarrow I = 10^{8.6} I_0.$$

To compare these intensities, find their ratio.

$$\frac{10^{8.9} I_0}{10^{8.6} I_0} = 10^{8.9-8.6} = 10^{.3} \approx 2$$

From this calculation, we see that 89 decibels is about twice as loud as 86 decibels. This is a 100% increase.

67. Substitute $A = 5760$, $P = 3500$, $t = 10$, $n = 1$ into the formula $A = P\left(1+\frac{r}{n}\right)^{tn}$.

$$5760 = 3500\left(1+\frac{r}{1}\right)^{10(1)}$$
$$\frac{288}{175} = (1+r)^{10}$$
$$\left(\frac{288}{175}\right)^{1/10} = 1+r$$
$$\left(\frac{288}{175}\right)^{1/10} - 1 = r$$
$$r \approx .051$$

The annual interest rate, to the nearest tenth, is 5.1%.

68. Substitute $P = 48{,}000$, $A = 58{,}344$, $r = .05$, and $n = 2$ into the formula $A = P\left(1+\frac{r}{n}\right)^{tn}$.

$$58{,}344 = 48{,}000\left(1+\frac{.05}{2}\right)^{t(2)}$$
$$58{,}344 = 48{,}000(1.025)^{2t}$$
$$1.2155 = (1.025)^{2t}$$
$$\ln 1.2155 = \ln(1.025)^{2t}$$
$$\ln 1.2155 = 2t\ln 1.025$$
$$t = \frac{\ln 1.2155}{2\ln 1.025} \approx 4.0$$

\$48,000 will increase to \$58,344 in about 4.0 yr.

69. First, substitute $P = 10{,}000$, $r = .08$, $t = 12$, and $n = 1$ into the formula $A = P\left(1+\frac{r}{n}\right)^{tn}$.

$$A = 10{,}000\left(1+\frac{.08}{1}\right)^{12(1)}$$
$$= 10{,}000(1.08)^{12} \approx 25{,}181.70$$

After the first 12 yr, there would be about \$25,181.70 in the account. To finish off the 21-year period, substitute $P = 25{,}181.70$, $r = .10$, $t = 9$, and $n = 2$ into the formula $A = P\left(1+\frac{r}{n}\right)^{tn}$.

$$A = 25{,}181.70\left(1+\frac{.10}{2}\right)^{9(2)} = 25{,}181.70(1+.05)^{18}$$
$$= 25{,}181.70(1.05)^{18} \approx 60{,}602.76$$

At the end of the 21-year period, about \$60,606.76 would be in the account. Note: If it was possible to transfer the money accrued after the 12 years to the new account without rounding, the amount after the 21-year period would be \$60,606.77. The difference is not significant.

70. First, substitute $P = 12{,}000$, $r = .05$, $t = 8$, and $n = 1$ into the formula $A = P\left(1+\frac{r}{n}\right)^{tn}$.

$$A = 12{,}000\left(1+\frac{.05}{1}\right)^{8(1)} = 12{,}000(1.05)^{8}$$
$$\approx 17{,}729.47$$

After the first 8 yr, there would be \$17,729.47 in the account. To finish off the 14-year period, substitute $P = 17{,}729.47$, $r = .06$, $t = 6$, and $n = 1$ into the formula $A = P\left(1+\frac{r}{n}\right)^{tn}$.

$$A = 17{,}729.47\left(1+\frac{.06}{1}\right)^{6(1)}$$
$$= 17{,}729.47(1.06)^{6} \approx 25{,}149.59$$

At the end of the 14-year period, \$25,149.59 would be in the account.

71. To find t, substitute $a = 2$, $P = 1$, and $r = .04$ into $A = Pe^{rt}$ and solve.

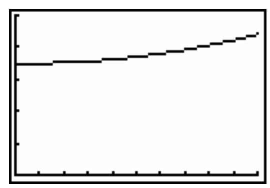
$$2 = 1\cdot e^{.04t} \Rightarrow 2 = e^{.04t} \Rightarrow \ln 2 = \ln e^{.04t} \Rightarrow$$
$$\ln 2 = .04t \Rightarrow t = \frac{\ln 2}{.04} \approx 17.3$$

It would take about 17.3 yr.

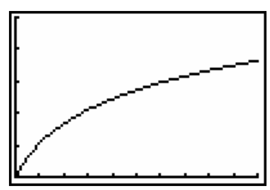
72. For each of the following parts the window is as follows.

```
WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=0
 Ymax=500
 Yscl=100
 Xres=1
```

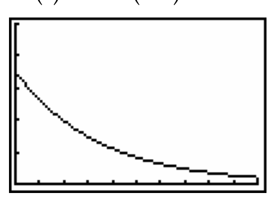
(a) $A(t) = t^2 - t + 350$



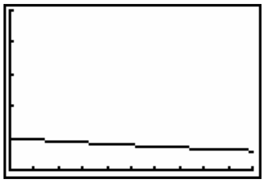
(b) $A(t) = 350\log(t+1)$



(c) $A(t) = 350(.75)^t$



(d) $A(t) = 100(.95)^t$



Function (c) best describes $A(t)$.

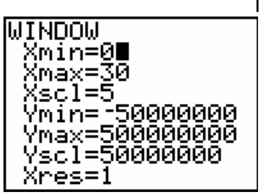
73. Double the 2003 total payoff value is $2(152.7) = 305.4.$ Using the function $f(x) = 93.54e^{.16x}$, we solve for x when $f(x) = 305.4.$

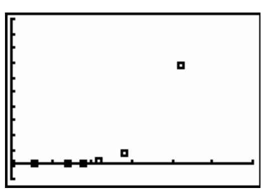
$$93.54e^{.16x} = 305.4 \Rightarrow e^{.16x} = \frac{305.4}{93.54} \Rightarrow$$
$$\ln e^{.16x} = \ln \frac{305.4}{93.54} \Rightarrow .16x = \ln \frac{305.4}{93.54} \Rightarrow$$
$$x = \frac{\ln \frac{305.4}{93.54}}{.16} \approx 7.40$$

Since x represents the number of years since 2000, in 2007 the total payoff value will be double of 2003.

74. (a) Plot the year on the x-axis and the number of processors on the y-axis. Let $x = 0$ correspond to the year 1986.

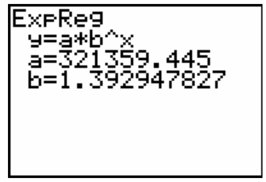
Year	Transistors
$1986 - 1986 = 0$	275,000
$1989 - 1986 = 3$	1,200,000
$1993 - 1986 = 7$	3,300,000
$1995 - 1986 = 9$	5,500,000
$1997 - 1986 = 11$	9,500,000
$2000 - 1986 = 14$	42,000,000
$2007 - 1986 = 21$	336,000,000

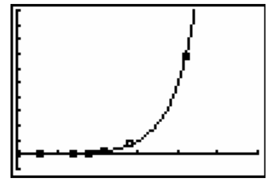




(b) The data are clearly not linear and do not level off like a logarithmic function. The data are increasing at a faster rate as x increases. Of the three choices, an exponential function will describe this data best.

(c) Using the exponential regression feature on the TI graphing calculator, we have if $f(x) = a(b)^x$, then $f(x) \approx 321,359(1.393)^x$. Other answers are possible using the techniques described in this chapter.





(d) Since $2010 - 1986 = 24,$ we can predict the number of transistors on a chip in the year 2010 by evaluating $f(24)$.

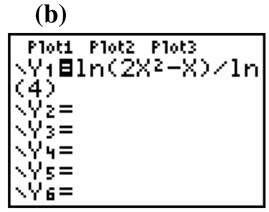
$$f(24) \approx 321,359(1.393)^{24} \approx 915,835,052$$

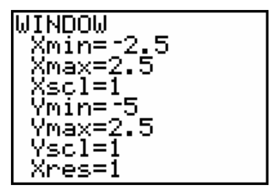
There will be approximately 916,000,000 transistors on a chip in the year 2005.

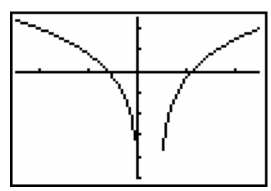
75. $f(x) = \log_4(2x^2 - x)$

(a) Use the change-of-base theorem with base e to write the function as $f(x) = \frac{\ln(2x^2 - x)}{\ln 4}$.

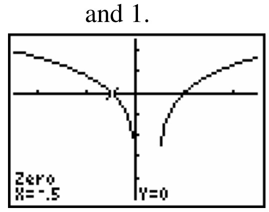
(b)

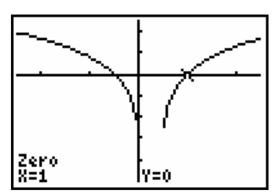






(c) From the graph, the x-intercepts are $-\frac{1}{2}$ and 1.





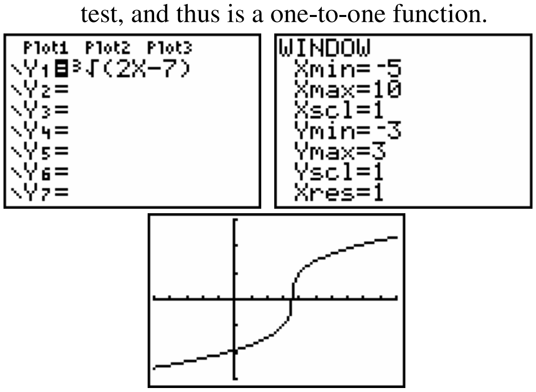
(d) From the graph, the vertical asymptotes are $x = 0$ and $x = \frac{1}{2}$. This be can verified algebraically. The vertical asymptotes will occur when $2x^2 - x = 0$.
$2x^2 - x = 0 \Rightarrow x(2x-1) = 0 \Rightarrow$
$x = 0$ or $x = \frac{1}{2}$

(e) To make a y-intercept, $x = 0$ must be in the domain, which is not the case here.

Chapter 4: Test

1. (a) $f(x) = \sqrt[3]{2x-7}$
Since it is a cube root, $2x - 7$ may be any real number.
Domain: $(-\infty, \infty)$
Since the cube root of any real number is also any real number.
Range: $(-\infty, \infty)$

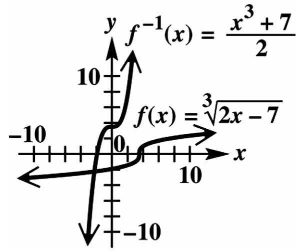
(b) $f(x) = \sqrt[3]{2x-7}$
The graph of f passes the horizontal line test, and thus is a one-to-one function.



(c) *Step 1*: Replace $f(x)$ with y and interchange x and y.
$y = \sqrt[3]{2x-7} \Rightarrow x = \sqrt[3]{2y-7}$
Step 2: Solve for y.
$x = \sqrt[3]{2y-7} \Rightarrow x^3 = \left(\sqrt[3]{2y-7}\right)^3 \Rightarrow$
$x^3 = 2y - 7 \Rightarrow x^3 + 7 = 2y \Rightarrow \frac{x^3+7}{2} = y$
Step 3: Replace y with $f^{-1}(x)$.
$f^{-1}(x) = \frac{x^3+7}{2}$

(d) Since the domain and range of f are $(-\infty, \infty)$, the domain and range of f^{-1} are also $(-\infty, \infty)$.

(e)



These graphs are reflections of each other across the line $y = x$.

2. (a) $y = \log_{1/3} x$
The point (1, 0) is on the graph of every function of the form $y = \log_a x$, so the correct choice must be either B or C. Since the base is $a = \frac{1}{3}$ and $0 < \frac{1}{3} < 1$, $y = \log_{1/3} x$ is a decreasing function, and so the correct choice must be B.

(b) $y = e^x$
The point (0, 1) is on the graph since $e^0 = 1$, so the correct choice must be either A or D. Since the base is e and $e > 1$, $y = e^x$ is an increasing function, and so the correct choice must be A.

(c) $y = \ln x$ or $y = \log_e x$
The point (1, 0) is on the graph of every function of the form $y = \log_a x$, so the correct choice must be B or C. Since the base is $a = e$ and $e > 1$, $y = \ln x$ is an increasing function, and the correct choice must be C.

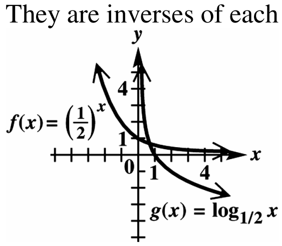
(d) $y = \left(\frac{1}{3}\right)^x$
The point (0, 1) is on the graph since $\left(\frac{1}{3}\right)^0 = 1$, so the correct choice must be either A or D. Since the base is $\frac{1}{3}$ and $0 < \frac{1}{3} < 1$, $y = \left(\frac{1}{3}\right)^x$ is a decreasing function, and so the correct choice must be D.

3. $\left(\frac{1}{8}\right)^{2x-3} = 16^{x+1} \Rightarrow \left(2^{-3}\right)^{2x-3} = \left(2^4\right)^{x+1} \Rightarrow$
$2^{-3(2x-3)} = 2^{4(x+1)} \Rightarrow 2^{-6x+9} = 2^{4x+4} \Rightarrow$
$-6x + 9 = 4x + 4 \Rightarrow -10x + 9 = 4 \Rightarrow$
$-10x = -5 \Rightarrow x = \frac{1}{2}$
Solution set: $\left\{\frac{1}{2}\right\}$

4. (a) $4^{3/2} = 8$ is written in logarithmic form as $\log_4 8 = \frac{3}{2}$.

(b) $\log_8 4 = \frac{2}{3}$ is written in exponential form as $8^{2/3} = 4$.

5. They are inverses of each other.



6. $\log_7 \frac{x^2 \sqrt[4]{y}}{z^3} = \log_7 x^2 + \log_7 \sqrt[4]{y} - \log_7 z^3$
$= \log_7 x^2 + \log_7 y^{1/4} - \log_7 z^3$
$= 2\log_7 x + \frac{1}{4}\log_7 y - 3\log_7 z$

7. $\log 2388 \approx 3.3780$

8. $\ln 2388 \approx 7.7782$

9. $\log_9 13 = \frac{\ln 13}{\ln 9} = \frac{\log 13}{\log 9} \approx 1.1674$

10. $\log_x \frac{9}{16} = 2 \Rightarrow x^2 = \frac{9}{16} \Rightarrow x = \pm\frac{3}{4}$
Since the negative solution is not in the domain, it must be discarded.
Solution set: $\left\{\frac{3}{4}\right\}$

11. $9^x = 4 \Rightarrow x\log 9 = \log 4 \Rightarrow x = \frac{\log 4}{\log 9} \approx .631$
Solution set: $\{.631\}$

12.
$$2^{x+1} = 3^{x-4}$$
$$(x+1)\log 2 = (x-4)\log 3$$
$$x\log 2 + \log 2 = x\log 3 - 4\log 3$$
$$x\log 2 - x\log 3 = -\log 2 - 4\log 3$$
$$x(\log 2 - \log 3) = -\log 2 - 4\log 3$$
$$x = \frac{-\log 2 - 4\log 3}{\log 2 - \log 3} \approx 12.548$$
Solution set: $\{12.548\}$

13.
$$e^{.4x} = 4^{x-2} \Rightarrow .4x = (x-2)\ln 4 \Rightarrow$$
$$.4x = x\ln 4 - 2\ln 4$$
$$.4x - x\ln 4 = -2\ln 4 \Rightarrow x(.4 - \ln 4) = -2\ln 4$$
$$x = \frac{-2\ln 4}{.4 - \ln 4} \approx 2.811$$
Solution set: $\{2.811\}$

14. $\log_2 x + \log_2(x+2) = 3 \Rightarrow \log_2[x(x+2)] = 3$
$$x^2 + 2x = 2^3 \Rightarrow x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0 \Rightarrow x = -4 \text{ or } x = 2$$
Since the negative solution is not in the domain it must be discarded.
Solution set: $\{2\}$

15. $\ln x - 4\ln 3 = \ln\left(\frac{1}{5}x\right)$
$\ln x - \ln 3^4 = \ln \frac{x}{5}$
$\ln x - \ln 81 = \ln \frac{x}{5} \Rightarrow \ln \frac{x}{81} = \ln \frac{x}{5}$
$\frac{x}{81} = \frac{x}{5} \Rightarrow \frac{5}{81} = 1 \Rightarrow$ there is no solution.
Solution set: $\varnothing$

16. $\log_3(x+1) - \log_3(x-3) = 2$
$$\log_3 \frac{x+1}{x-3} = 2$$
$$\frac{x+1}{x-3} = 3^2 \Rightarrow \frac{x+1}{x-3} = 9$$
$$x+1 = 9(x-3)$$
$$x+1 = 9x - 27$$
$$-8x = -28 \Rightarrow x = \frac{28}{8} = \frac{7}{2}$$
Solution set: $\left\{\frac{7}{2}\right\}$

17. Answers will vary.
$\log_5 27$ is the exponent to which 5 must be raised in order to obtain 27. To approximate $\log_5 27$ on your calculator, use the change-of-base formula; $\log_5 27 = \frac{\log 27}{\log 5} = \frac{\ln 27}{\ln 5} \approx 2.048$.

18. $v(t) = 176(1 - e^{-.18t})$
Find the time t at which $v(t) = 147$.
$$147 = 176(1 - e^{-.18t}) \Rightarrow \frac{147}{176} = 1 - e^{-.18t} \Rightarrow$$
$$-e^{-.18t} = \frac{147}{176} - 1 \Rightarrow -e^{-.18t} = -\frac{29}{176}$$
$$e^{-.18t} = \frac{29}{176} \Rightarrow \ln e^{-.18t} = \ln \frac{29}{176} \Rightarrow$$
$$-.18t = \ln \frac{29}{176} \Rightarrow t = \frac{\ln \frac{29}{176}}{-.18} \approx 10.02$$
It will take the skydiver about 10 sec to attain the speed of 147 ft per sec (100 mph).

19. (a) Substitute $P = 5000$, $A = 18{,}000$, $r = .068$, and $n = 12$ into the formula $A = P\left(1 + \frac{r}{n}\right)^{tn}$.
$$18{,}000 = 5000\left(1 + \frac{.068}{12}\right)^{t(12)}$$
$$3.6 = \left(1 + \frac{.068}{12}\right)^{12t}$$
$$\ln 3.6 = \ln\left(1 + \frac{.068}{12}\right)^{12t}$$
$$\ln 3.6 = 12t\ln\left(1 + \frac{.068}{12}\right)$$
$$t = \frac{\ln 3.6}{12\ln\left(1 + \frac{.068}{12}\right)} \approx 18.9$$
It will take about 18.9 years.

(b) Substitute $P = 5000$, $A = 18{,}000$, and $r = .068$, and into the formula $A = Pe^{rt}$.

$18{,}000 = 5000e^{.068t} \Rightarrow 3.6 = e^{.068t} \Rightarrow$

$\ln 3.6 = \ln e^{.068t} \Rightarrow \ln 3.6 = .068t \Rightarrow$

$t = \frac{\ln 3.6}{.068} \approx 18.8$

It will take about 18.8 years.

20. Substitute $A = 3P$ and $r = .068$ into the continuous compounding formula $A = Pe^{rt}$, then solve for t:

$A = Pe^{rt}$

$3P = Pe^{.068t} \Rightarrow 3 = e^{.068t} \Rightarrow \ln 3 = \ln e^{.068t} \Rightarrow$

$\ln 3 = .068t \Rightarrow \dfrac{\ln 3}{.068} = t \Rightarrow t \approx 16.16$

It will take about 16.2 years for any amount of money to triple at 6.8% annual interest.

21. $A(t) = 600e^{-.05t}$

(a) $A(12) = 600e^{-.05(12)} = 600e^{-.6} \approx 329.3$

The amount of radioactive material present after 12 days is about 329.3 g.

(b) Since $A(0) = 600e^{-.05(0)} = 600e^{0} = 600$ g is the amount initially present, we seek to find t when $A(t) = \frac{1}{2}(600) = 300$ g is present.

$300 = 600e^{-.05t} \Rightarrow .5 = e^{-.05t}$

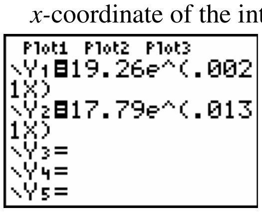
$\ln .5 = \ln e^{-.05t} \Rightarrow \ln .5 = -.05t$

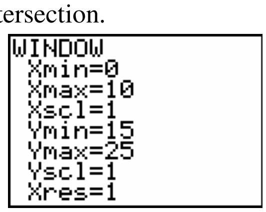
$t = \frac{\ln .5}{-.05} \approx 13.9$

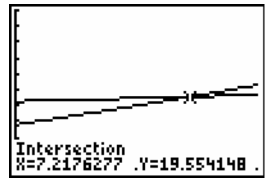
The half-life of the material is about 13.9 days.

22. Let $x = 0$ correspond to the year 2005. The population of New York (in millions) can be approximated by $y = 19.26e^{.0021x}$. The population of Florida (in millions) can be approximated by $y = 17.79e^{.0131x}$.

Graphing Calculator Solution:
Graph the two functions on the same screen and use the "intersect" option to find the x-coordinate of the intersection.







Algebraic Solution:
Set the two y-values equal to each other and solve for x.

$19.26e^{.0021x} = 17.79e^{.0131x}$

$\frac{19.26}{17.79} = \frac{e^{.0131x}}{e^{.0021x}}$

$\frac{19.26}{17.79} = e^{.0131x-.0021x}$

$\frac{19.26}{17.79} = e^{.011x}$

$\ln \frac{19.26}{17.79} = \ln e^{.011x}$

$\ln \frac{19.26}{17.79} = .011x$

$x = \frac{\ln \frac{19.26}{17.79}}{.011} \approx 7.2$

$x \approx 7.2$ corresponds to the middle of the year 2012. During 2012 the populations will be equal.

Chapter 4: Quantitative Reasoning

1. Since you are taxed on the entire amount when it is withdrawn at 60, calculate 60% (40% goes to taxes) of $A = P\left(1+\frac{r}{m}\right)^{tm}$ when $P = 3000$, $r = .08$, $m = 1$ and $t = 60 - 25 = 35$.

$.60\left[(3000)\left(1+\frac{.08}{1}\right)^{35(1)}\right]$

$= \left[.60(3000)\right]\left(1+\frac{.08}{1}\right)^{35(1)}$

$= 1800(1.08)^{35} \approx 26{,}613.62$

$\$26{,}613.62$ will remain after taxes are paid.

2. Since you are taxed on the money and the annual interest, calculate $A = P\left(1+\frac{r}{m}\right)^{tm}$ when $P = 1800$, $r = .048$, $m = 1$ and $t = 35$.

$1800(1+.048)^{35(1)} = 1800(1.048)^{35} \approx 9287.90$

$\$9287.90$ will be available at age 60.

3. Since $26{,}613.62 - 9287.90 = 17{,}325.72$, to the nearest dollar, $17,326 will be additionally earned with the IRA.

4. Since $.60\left[(3000)\left(1+\frac{.08}{1}\right)^{35(1)}\right]$

$= \left[.60(3000)\right]\left(1+\frac{.08}{1}\right)^{35(1)}$, this is the same calculation.

Chapter 5

TRIGONOMETRIC FUNCTIONS

Section 5.1: Angles

1. $30°$

(a) $90° - 30° = 60°$

(b) $180° - 30° = 150°$

2. $60°$

(a) $90° - 60° = 30°$

(b) $180° - 60° = 120°$

3. $45°$

(a) $90° - 45° = 45°$

(b) $180° - 45° = 135°$

4. $18°$

(a) $90° - 18° = 72°$

(b) $180° - 18° = 162°$

5. $54°$

(a) $90° - 54° = 36°$

(b) $180° - 54° = 126°$

6. $89°$

(a) $90° - 89° = 1°$

(b) $180° - 89° = 91°$

7. $1°$

(a) $90° - 1° = 89°$

(b) $180° - 1° = 179°$

8. $10°$

(a) $90° - 10° = 80°$

(b) $180° - 10° = 170°$

9. $14°20'$

(a) $90° - 14°20' = 89°60' - 14°20' = 75°40'$

(b) $180° - 14°20' = 179°60' - 14°20' = 165°40'$

10. $39°50'$

(a) $90° - 39°50' = 89°60' - 39°50' = 50°10'$

(b) $180° - 39°50' = 179°60' - 39°50' = 140°10'$

11. $20°10'30''$

(a) $90° - 20°10'30'' = 89°59'60'' - 20°10'30'' = 69°49'30''$

(b) $180° - 20°10'30'' = 179°59'60'' - 20°10'30'' = 159°49'30''$

12. $50°40'50''$

(a) $90° - 50°40'50'' = 89°59'60'' - 50°40'50'' = 39°19'10''$

(b) $180° - 50°40'50'' = 179°59'60'' - 50°40'50'' = 129°19'10''$

13. The two angles form a straight angle.
$7x + 11x = 180 \Rightarrow 18x = 180 \Rightarrow x = 10$
The measures of the two angles are
$(7x)° = [7(10)]° = 70°$ and
$(11x)° = [11(10)]° = 110°$.

14. The two angles form a straight angle.
$$(20x + 10) + (3x + 9) = 180$$
$$23x + 19 = 180$$
$$23x = 161 \Rightarrow x = 7$$
The measures of the two angles are
$(20x + 10)° = [20(7) + 10]° = 150°$ and
$(3x + 9)° = [3(7) + 9]° = 30°$.

15. The two angles form a right angle.
$4y + 2y = 90 \Rightarrow 6y = 90 \Rightarrow y = 15$
The two angles have measures of
$(4y)° = [4(15)]° = 60°$ and
$(2y)° = [2(15)]° = 30°$.

16. The two angles form a right angle.
$$(5k + 5) + (3k + 5) = 90 \Rightarrow 8k + 10 = 90 \Rightarrow$$
$$8k = 80 \Rightarrow k = 10$$
The measures of the two angles are
$(5k + 5)° = [5(10) + 5]° = (50 + 5)° = 55°$ and
$(3k + 5)° = [3(10) + 5]° = (30 + 5)° = 35°$.

17. The two angles form a straight angle.
$(-4x)+(-14x)=180 \Rightarrow -18x=180 \Rightarrow$
$x=-10$
The measures of the two angles are
$(-4x)° = [-4(-10)]° = 40°$ and
$(-14x)° = [-14(-10)]° = 140°.$

18. The two angles form a straight angle.
$9x+9x=180$
$18x=180$
$x=10$
The measure of each of the angles is
$9 \cdot 10° = 90°$.

19. The sum of the measures of two supplementary angles is 180°.
$(10x+7)+(7x+3)=180$
$17x+10=180$
$17x=170 \Rightarrow x=10$
The measures of the two angles are
$(10x+7)° = [10(10)+7]° = (100+7)° = 107°$
and $(7x+3)° = [7(10)+3]°$
$= (70+3)° = 73°$.

20. The sum of the measures of two supplementary angles is 180°.
$(6x-4)+(8x-12)=180 \Rightarrow 14x-16=180 \Rightarrow$
$14x=196 \Rightarrow x=14$
The measures of the two angles are
$(6x-4)° = [6(14)-4]° = (84-4)° = 80°$
and $(8x-12)° = [8(14)-12]°$
$= (112-12)° = 100°$.

21. The sum of the measures of two complementary angles is 90°.
$(9x+6)+3x=90 \Rightarrow 12x+6=90 \Rightarrow$
$12x=84 \Rightarrow x=7$
The measures of the two angles are
$(9x+6)° = [9(7)+6]° = (63+6)° = 69°$ and
$(3x)° = [3(7)]° = 21°.$

22. The sum of the measures of two complementary angles is 90°.
$(3x-5)+(6x-40)=90 \Rightarrow 9x-45=90 \Rightarrow$
$9x=135 \Rightarrow x=15$
The measures of the two angles are
$(3x-5)° = [3(15)-5]° = (45-5)° = 40°$ and
$(6x-40)° = [6(15)-40]° = (90-40)° = 50°.$

23. Let x = the measure of the angle.
If the angle is its own complement, then we have $x+x=90 \Rightarrow 2x=90 \Rightarrow x=45$
Thus, a 45° angle is its own complement.

24. Let x = the measure of the angle.
If the angle is its own supplement, then we have $x+x=180 \Rightarrow 2x=180 \Rightarrow x=90$
Thus, a 90° angle is its own supplement.

25. $\frac{25 \text{ minutes}}{60 \text{ minutes}} = \frac{x}{360°}$
$$x = \frac{25}{60}(360) = 25(6) = 150°$$

26. Since the minute hand is $\frac{3}{4}$ the way around, the hour hand is $\frac{3}{4}$ of the way between the 1 and 2. Thus, the hour hand is located 8.75 minutes past 12. The minute hand is 15 minutes before the 12. The smaller angle formed by the hands of the clock can be found by solving the proportion
$\frac{(15+8.75)\text{ minutes}}{60 \text{ minutes}} = \frac{x}{360°}.$
$\frac{(15+8.75)\text{ minutes}}{60 \text{ minutes}} = \frac{x}{360°} \Rightarrow \frac{23.75}{60} = \frac{x}{360} \Rightarrow$
$x = \frac{23.75}{60}(360) = 23.75(6) = 142.5° = 142°30'$

27. At 15 minutes after the hour, the minute hand is $\frac{1}{4}$ the way around, so the hour hand is $\frac{1}{4}$ of the way between the 3 and 4. Thus, the hour hand is located 16.25 minutes past 12. The minute hand is 15 minutes after the 12. The smaller angle formed by the hands of the clock can be found by solving the proportion
$\frac{(16.25-15)\text{ minutes}}{60 \text{ minutes}} = \frac{x}{360°}.$
$\frac{(16.25-15)\text{ minutes}}{60 \text{ minutes}} = \frac{x}{360°} \Rightarrow \frac{1.25}{60} = \frac{x}{360} \Rightarrow$
$x = \frac{1.25}{60}(360) = 1.25(6) = 7.5° = 7°30'$

28. At 9:00, the minute hand is on the twelve, or 60 minutes after 12, and the hour hand is on the 9, or 45 minutes after 12. The smaller angle formed by the hands of the clock can be found by solving the proportion
$\frac{(60-45)\text{ minutes}}{60 \text{ minutes}} = \frac{x}{360°}.$

$$\frac{(60-45)\text{ minutes}}{60\text{ minutes}}=\frac{x}{360°}\Rightarrow\frac{15}{60}=\frac{x}{360}\Rightarrow$$
$$x=\frac{15}{60}(360)=15(6)=90°$$

29. If an angle measures x degrees and two angles are complementary if their sum is 90°, then the complement of an angle of $x°$ is $(90-x)°$.

30. If an angle measures x degrees and two angles are supplementary if their sum is 180°, then the supplement of an angle of $x°$ is $(180-x)°$.

31. The first negative angle coterminal with x between 0° and 60° is $(x-360)°$.

32. The first positive angle coterminal with x between 0° and –60° is $(x+360)°$.

33.
$$\begin{array}{r} 62°\ 18' \\ +21°\ 41' \\ \hline 83°\ 59' \end{array}$$

34.
$$\begin{array}{r} 75°\ 15' \\ +\ 83°\ 32' \\ \hline 158°\ 47' \end{array}$$

35. $71°18'-47°\ 29'=70°\ 78'-47°\ 29'$
$$\begin{array}{r} 70°\ 78' \\ -47°\ 29' \\ \hline 23°\ 49' \end{array}$$

36. $47°\ 23'-73°\ 48'=-(73°\ 48'-47°\ 23')$

Since $\begin{array}{r} 73°\ 48' \\ -47°\ 23' \\ \hline 26°\ 25' \end{array}$, we have

$$\begin{aligned}47°\ 23'-73°\ 48'&=-(73°\ 48'-47°\ 23')\\&=-26°\ 25'\end{aligned}$$

37. $90°-51°\ 28'=89°\ 60'-51°\ 28'$
$$\begin{array}{r} 89°\ 60' \\ -51°\ 28' \\ \hline 38°\ 32' \end{array}$$

38. $90°-17°\ 13'=89°\ 60'-17°\ 13'$
$$\begin{array}{r} 89°\ 60' \\ -17°\ 13' \\ \hline 72°\ 47' \end{array}$$

39. $180°-119°\ 26'=179°\ 60'-119°\ 26'$
$$\begin{array}{r} 179°\ 60' \\ -119°\ 26' \\ \hline 60°\ 34' \end{array}$$

40. $180°-124°\ 51'=179°\ 60'-124°\ 51'$
$$\begin{array}{r} 179°\ 60' \\ -124°\ 51' \\ \hline 55°\ \ 9' \end{array}$$

41. $$\begin{aligned}26°20'+18°17'-14°10'&=44°37'-14°10'\\&=30°27'\end{aligned}$$

42. $$\begin{aligned}55°30'+12°44'-8°15'&=67°74'-8°15'\\&=59°59'\end{aligned}$$

43. $90°-72°\ 58'\ 11''=89°\ 59'\ 60''-72°\ 58'\ 11''$
$$\begin{array}{r} 89°\ 59'\ 60'' \\ -72°\ 58'\ 11'' \\ \hline 17°\ \ 1'\ 49'' \end{array}$$

44. $90°-36°18'47''=89°59'60''-36°18'47''$
$$\begin{array}{r} 89°59'60'' \\ -36°18'47'' \\ \hline 53°41'13'' \end{array}$$

45. $35°30'=35°+\frac{30}{60}°=35°+.5°=35.5°$

46. $82°30'=82°+\frac{30}{60}°=82°+.5°=82.5°$

47. $112°15'=112°+\frac{15}{60}°=112°+.25°=112.25°$

48. $133°45'=133°+\frac{45}{60}°=133°+.75°=133.75°$

49. $-60°12'=-\left(60°+\frac{12}{60}°\right)=-(60°+.2°)=-60.2°$

50. $-70°48'=-\left(70°+\frac{48}{60}°\right)=-(70°+.8°)=-70.8°$

51. $20°54'00''=20°+\frac{54}{60}°=20°+.900°=20.9°$

52. $38°\ 42'00''=38°+\frac{42}{60}°=38.7°$

53. $$\begin{aligned}91°35'54''&=91°+\tfrac{35}{60}°+\tfrac{54}{3600}°\\&\approx 91°+.5833°+.0150°\approx 91.598°\end{aligned}$$

54. $$\begin{aligned}34°\ 51'\ 35''&=34°+\tfrac{51}{60}°+\tfrac{35}{3600}°\\&\approx 34°+.8500°+.0097°\\&\approx 34.860°\end{aligned}$$

55. $$\begin{aligned}274°18'59''&=274°+\tfrac{18}{60}°+\tfrac{59}{3600}°\\&\approx 274°+.3000°+.0164°\\&\approx 274.316°\end{aligned}$$

56. $165°\ 51'\ 9'' = 165° + \frac{51}{60}° + \frac{9}{3600}°$
$= 165° + .8500° + .0025°$
$= 165.853°$

57. $39.25° = 39° + .25° = 39° + .25(60')$
$= 39° + 15' + 0'' = 39°15'00''$

58. $46.75° = 46° + .75° = 46° + .75(60')$
$= 46° + 45' + 0'' = 46°45'00''$

59. $126.76° = 126° + .76° = 126° + .76(60')$
$= 126° + 45.6' = 126° + 45' + .6'$
$= 126° + 45' + .6(60'')$
$= 126° + 45' + 36'' = 126°45'36''$

60. $174.255° = 174° + .255° = 174° + .255(60')$
$= 174° + 15.3' = 174° + 15' + .3'$
$= 174° + 15' + .3(60'')$
$= 174° + 15' + 18'' = 174°15'18''$

61. $-18.515° = -(18° + .515°)$
$= -(18° + .515(60'))$
$= -(18° + 30.9') = -(18° + 30' + .9')$
$= -(18° + 30' + .9(60''))$
$= -(18° + 30' + 54'') = -18°30'54''$

62. $-25.485° = -(25° + .485°)$
$= -(25° + .485(60'))$
$= -(25° + 29.1') = -(25° + 29' + .1')$
$= -(25° + 29' + .1(60''))$
$= -(25° + 29' + 6'') = -25°29'6''$

63. $31.4296° = 31° + .4296° = 31° + .4296(60')$
$= 31° + 25.776' = 31° + 25' + .776'$
$= 31° + 25' + .776(60'')$
$= 31°25'46.56'' \approx 31°\ 25'\ 47''$

64. $59.0854° = 59° + .0854° = 59° + .0854(60')$
$= 59° + 5.124' = 59° + 5' + .124'$
$= 59° + 5' + .124(60'')$
$= 59°\ 5'\ 7.44'' \approx 59°5'\ 7''$

65. $89.9004° = 89° + .9004° = 89° + .9004(60')$
$= 89° + 54.024' = 89° + 54' + .024'$
$= 89° + 54' + .024(60'')$
$= 89°54'1.44'' \approx 89°54'1''$

66. $102.3771° = 102° + .3771° = 102° + .3771(60')$
$= 102° + 22.626'$
$= 102° + 22' + .626'$
$= 120° + 22' + .626(60'')$
$= 102°\ 22'\ 37.56'' \approx 102°\ 22'\ 38''$

67. $178.5994° = 178° + .5994°$
$= 178° + .5994(60')$
$= 178° + 35.964'$
$= 178° + 35' + .964'$
$= 178° + 35' + .964(60'')$
$= 178°35'57.84'' \approx 178°35'58''$

68. $122.6853° = 122° + .6853° = 122° + .6853(60')$
$= 122° + 41.118'$
$= 122° + 41' + .118'$
$= 122° + 41' + .118(60'')$
$= 122°\ 41'\ 7.08'' \approx 122°\ 41'\ 7''$

69. $32°$ is coterminal with $360° + 32° = 392°$.

70. $86°$ is coterminal with $360° + 86° = 446°$.

71. $26°30'$ is coterminal with
$360° + 26°30' = 386°30'$.

72. $58°40'$ is coterminal with
$360° + 58°40' = 418°40'$.

73. $-40°$ is coterminal with $360° + (-40°) = 320°$.

74. $-98°$ is coterminal with $360° + (-98°) = 262°$.

75. $-125°$ is coterminal with
$360° + (-125°) = 235°$.

76. $-203°$ is coterminal with
$360° + (-203°) = 157°$.

77. $361°$ is coterminal with $361° - 360° = 1°$.

78. $541°$ is coterminal with $541° - 360° = 181°$.

79. $-361°$ is coterminal with
$-361° + 2(360°) = 359°$.

80. $-541°$ is coterminal with
$-541° + 2(360°) = 179°$.

81. $539°$ is coterminal with $539° - 360° = 179°$.

82. $699°$ is coterminal with $699° - 360° = 339°$.

83. $850°$ is coterminal with
$850° - 2(360°) = 850° - 720° = 130°$.

84. $1000°$ is coterminal with
$1000° - 2 \cdot 360° = 1000° - 720° = 280°$.

85. $5280°$ is coterminal with
$5280° - 14 \cdot 360° = 5280° - 5040° = 240°$.

86. $8440°$ is coterminal with
$8440° - 23 \cdot 360° = 8440° - 8280° = 160°$.

87. −5280° is coterminal with
$-5280° + 15 \cdot 360° = -5280° + 5400° = 120°$.

88. −8440° is coterminal with
$-8440° + 24 \cdot 360° = -8440° + 8640° = 200°$.

In exercises 89−92, answers may vary.

89. 90° is coterminal with
$90° + 360° = 450°$
$90° + 2(360°) = 810°$
$90° - 360° = -270°$
$90° - 2(360°) = -630°$

90. 180° is coterminal with
$180° + 360° = 540°$
$180° + 2(360°) = 900°$
$180° - 360° = -180°$
$180° - 2(360°) = -540°$

91. 0° is coterminal with
$0° + 360° = 360°$
$0° + 2(360°) = 720°$
$0° - 360° = -360°$
$0° - 2(360°) = -720°$

92. 270° is coterminal with
$270° + 360° = 630°$
$270° + 2(360°) = 990°$
$270° - 360° = -90°$
$270° - 2(360°) = -450°$

93. 30°
A coterminal angle can be obtained by adding an integer multiple of 360°.
$30° + n \cdot 360°$

94. 45°
A coterminal angle can be obtained by adding an integer multiple of 360°.
$45° + n \cdot 360°$

95. 135°
A coterminal angle can be obtained by adding an integer multiple of 360°.
$135° + n \cdot 360°$

96. 225°
A coterminal angle can be obtained by adding an integer multiple of 360°.
$225° + n \cdot 360°$

97. −90°
A coterminal angle can be obtained by adding an integer multiple of 360°.
$-90° + n \cdot 360°$

98. −180°
A coterminal angle can be obtained by adding integer multiple of 360°.
$-180° + n \cdot 360°$

99. 0°
A coterminal angle can be obtained by adding integer multiple of 360°.
$0° + n \cdot 360° = n \cdot 360°$

100. 360°
A coterminal angle can be obtained by adding integer multiple of 360°.
$360° + n \cdot 360°$, or $n \cdot 360°$

101. The answers to Exercises 99 and 100 give the same set of angles since 0° is coterminal with $0° + 360° = 360°$.

102. **A.** $360° + r°$ is coterminal with $r°$ because you are adding an integer multiple of 360° to $r°$, $r° + 1 \cdot 360°$.

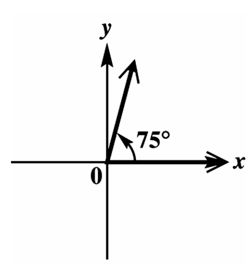
B. $r° - 360°$ is coterminal with $r°$ because you are adding an integer multiple of 360° to $r°$, $r° + (-1) \cdot 360°$.

C. $360° - r°$ is not coterminal with $r°$ because you are not adding an integer multiple of 360° to $r°$.
$360° - r° \neq r° + n \cdot 360°$ for an integer value n.

D. $r° + 180°$ is not coterminal with $r°$ because you are not adding an integer multiple of 360° to $r°$.
$r° + 180° \neq r° + n \cdot 360°$ for an integer value n. You are adding $\frac{1}{2}(360°)$.
Thus, choices C and D are not coterminal with $r°$.

For Exercises 103−114, angles other than those given are possible.

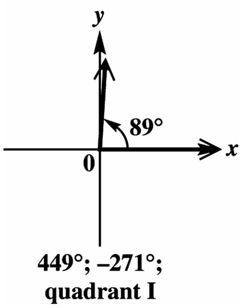
103.



435°; –285°; quadrant I

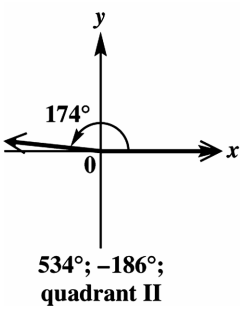
75° is coterminal with 75° + 360° = 435° and 75° – 360° = –285°. These angles are in quadrant I.

104.



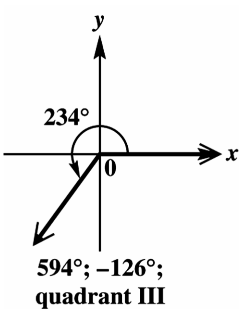
89° is coterminal with $89° + 360° = 449°$ and $89° - 360° = -271°$. These angles are in quadrant I.

105.



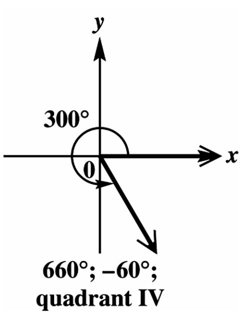
174° is coterminal with $174° + 360° = 534°$ and $174° - 360° = -186°$. These angles are in quadrant II.

106.



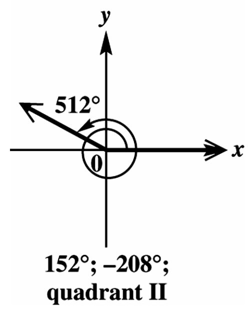
234° is coterminal with $234° + 360° = 594°$ and $234° - 360° = -126°$. These angles are in quadrant III.

107.



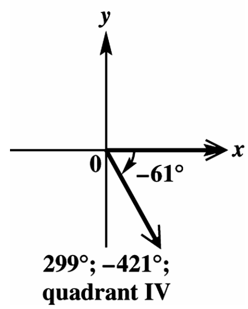
300° is coterminal with $300° + 360° = 660°$ and $300° - 360° = -60°$. These angles are in quadrant IV.

108.



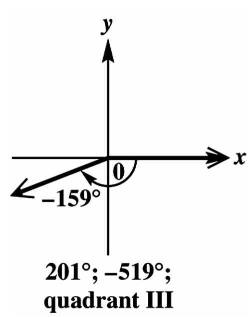
512° is coterminal with $512° - 360° = 152°$ and $512° - 2 \cdot 360° = -208°$. These angles are in quadrant II.

109.



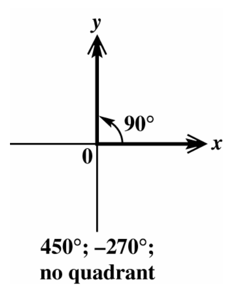
–61° is coterminal with $-61° + 360° = 299°$ and $-61° - 360° = -421°$. These angles are in quadrant IV.

110.



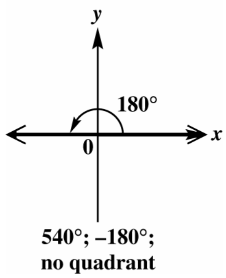
–159° is coterminal with $-159° + 360° = 201°$ and $-159° - 360° = -519°$. These angles are in quadrant III.

111.



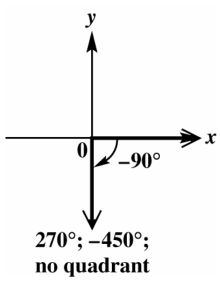
90° is coterminal with $90° + 360° = 450°$ and $90° - 360° = -270°$. These angles are not in a quadrant.

112.



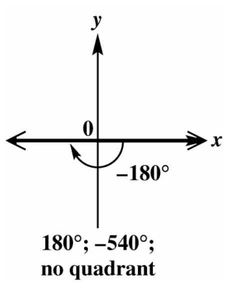
$180°$ is coterminal with $180° + 360° = 540°$ and $180° - 360° = -180°$. These angles are not in a quadrant.

113.



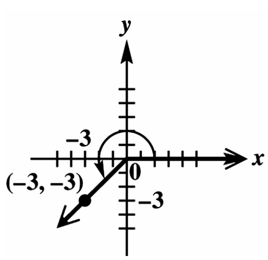
$-90°$ is coterminal with $-90° + 360° = 270°$ and $-90° - 360° = -450°$. These angles are not in a quadrant.

114.



$-180°$ is coterminal with $-180° + 360° = 180°$ and $-180° - 360° = -540°$. These angles are not in a quadrant.

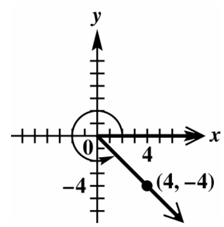
115.



Points: $(0,0)$ and $(-3,-3)$

$$r = \sqrt{(-3-0)^2 + (-3-0)^2}$$
$$= \sqrt{(-3)^2 + (-3)^2}$$
$$= \sqrt{9+9} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$$

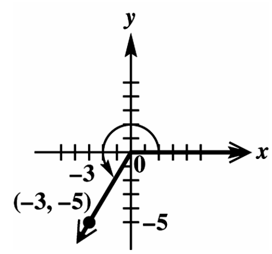
116.



Points: $(0,0)$ and $(4,-4)$

$$r = \sqrt{(4-0)^2 + (-4-0)^2}$$
$$= \sqrt{(4)^2 + (-4)^2}$$
$$= \sqrt{16+16} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

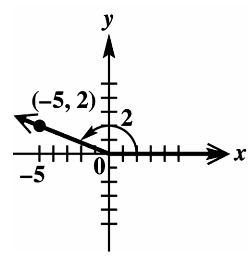
117.



Points: $(0,0)$ and $(-3,-5)$

$$r = \sqrt{(-3-0)^2 + (-5-0)^2}$$
$$= \sqrt{(-3)^2 + (-5)^2} = \sqrt{9+25} = \sqrt{34}$$

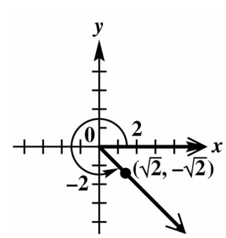
118.



Points: $(0,0)$ and $(-5,2)$

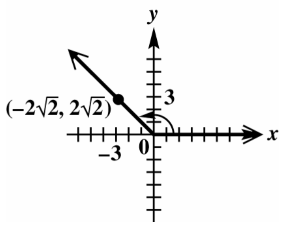
$$r = \sqrt{(-5-0)^2 + (2-0)^2}$$
$$= \sqrt{(-5)^2 + 2^2} = \sqrt{25+4} = \sqrt{29}$$

119.



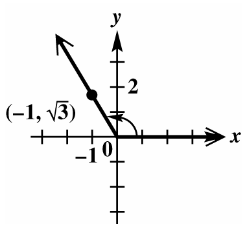
Points: $(0,0)$ and $\left(\sqrt{2}, -\sqrt{2}\right)$

$$r = \sqrt{\left(\sqrt{2}-0\right)^2 + \left(-\sqrt{2}-0\right)^2}$$
$$= \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2$$

120. 

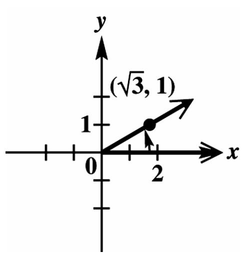
Points: $(0,0)$ and $\left(-2\sqrt{2}, 2\sqrt{2}\right)$

$$r = \sqrt{\left(-2\sqrt{2}-0\right)^2 + \left(2\sqrt{2}-0\right)^2}$$
$$= \sqrt{\left(-2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} = \sqrt{8+8} = \sqrt{16} = 4$$

121. 

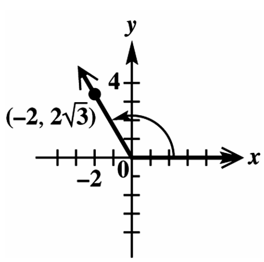
Points: $(0,0)$ and $\left(-1, \sqrt{3}\right)$

$$r = \sqrt{(-1-0)^2 + \left(\sqrt{3}-0\right)^2}$$
$$= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

122. 

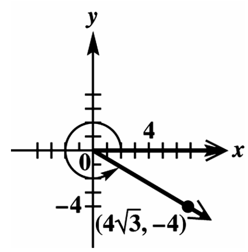
Points: $(0,0)$ and $\left(\sqrt{3}, 1\right)$

$$r = \sqrt{\left(\sqrt{3}-0\right)^2 + (1-0)^2}$$
$$= \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

123. 

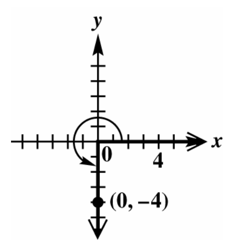
Points: $(0,0)$ and $\left(-2, 2\sqrt{3}\right)$

$$r = \sqrt{(-2-0)^2 + \left(2\sqrt{3}-0\right)^2}$$
$$= \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$$

124. 

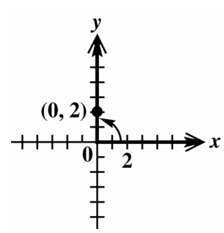
Points: $(0,0)$ and $\left(4\sqrt{3}, -4\right)$

$$r = \sqrt{\left(4\sqrt{3}-0\right)^2 + (-4-0)^2}$$
$$= \sqrt{\left(4\sqrt{3}\right)^2 + (-4)^2} = \sqrt{48+16} = \sqrt{64} = 8$$

125. 

Points: $(0,0)$ and $(0,-4)$

$$r = \sqrt{(0-0)^2 + (-4-0)^2}$$
$$= \sqrt{(0)^2 + (-4)^2} = \sqrt{0+16} = \sqrt{16} = 4$$

126. 

Points: $(0,0)$ and $(0,2)$

$$r = \sqrt{(0-0)^2 + (2-0)^2}$$
$$= \sqrt{(0)^2 + (2)^2} = \sqrt{0+4} = \sqrt{4} = 2$$

127. 45 revolutions per min $= \frac{45}{60}$ revolution per sec
$= \frac{3}{4}$ revolution per sec
A turntable will make $\frac{3}{4}$ revolution in 1 sec.

128. 90 revolutions per min $= \frac{90}{60}$ revolutions per min $= 1.5$ revolutions per sec
A windmill will make 1.5 revolutions in 1 sec.

129. 600 rotations per min

$$\begin{aligned} &= \tfrac{600}{60} \text{ rotations per sec} \\ &= 10 \text{ rotations per sec} \\ &= 5 \text{ rotations per } \tfrac{1}{2} \text{ sec} \\ &= 5(360°) \text{ per } \tfrac{1}{2} \text{ sec} \\ &= 1800° \text{ per } \tfrac{1}{2} \text{ sec} \end{aligned}$$

A point on the edge of the tire will move 1800° in $\frac{1}{2}$ sec.

130. If the propeller rotates 1000 times per minute, then it rotates $\frac{1000}{60} = 16\frac{2}{3}$ times per sec. Each rotation is 360°, so the total number of degrees a point rotates in 1 sec is
$(360°)\left(16\frac{2}{3}\right) = 6000°$.

131. 75° per min

$$\begin{aligned} 75° \text{ per min} &= 75°(60) \text{ per hr} = 4500° \text{ per hr} \\ &= \tfrac{4500°}{360°} \text{ rotations per hr} \\ &= 12.5 \text{ rotations per hr} \end{aligned}$$

The pulley makes 12.5 rotations in 1 hr.

132. First, convert $74.25°$ to degrees and minutes. Find the difference between this measurement and $74°20'$.

$74.25° = 74° + .25(60') = 74°15'$, so

$74°20' - 74°15' = 5'$

Next, convert $74°20'$ to decimal degrees. Find the difference between this measurement and $74.25°$, rounded to the nearest hundredth of a degree.

$74°20' = 74° + \frac{20}{60}° \approx 74.333°$, so

$74.333° - 74.250° = .083° \approx .08°$

The difference in measurements is $5'$ to the nearest minute or .08° to the nearest hundredth of a degree.

133. The earth rotates 360° in 24 hr. 360° is equal to $360(60') = 21{,}600'$.

$$\frac{24 \text{ hr}}{21{,}600'} = \frac{x}{1'} \Rightarrow$$

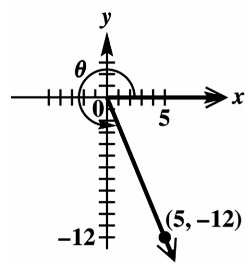
$$\begin{aligned} x &= \frac{24}{21{,}600} \text{hr} = \frac{24}{21{,}600}(60 \text{ min}) \\ &= \frac{1}{15} \text{ min} = \frac{1}{15}(60 \text{ sec}) = 4 \text{ sec} \end{aligned}$$

It should take the motor 4 sec to rotate the telescope through an angle of 1 min.

134. Since we have five central angles that comprise a full circle, we have
$5(2\theta) = 360° \Rightarrow 10\theta = 360° \Rightarrow \theta = 36°$
The angle of each point of the five-pointed star measures 36°.

Section 5.2: Trigonometric Functions

1.



$(5,-12)$
$x = 5$, $y = -12$, and

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{5^2 + (-12)^2} \\ &= \sqrt{25 + 144} = \sqrt{169} = 13 \end{aligned}$$

$$\sin\theta = \frac{y}{r} = \frac{-12}{13} = -\frac{12}{13}; \ \cos\theta = \frac{x}{r} = \frac{5}{13}$$

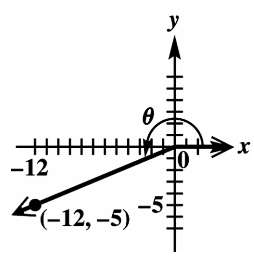
$$\tan\theta = \frac{y}{x} = \frac{-12}{5} = -\frac{12}{5}$$

$$\cot\theta = \frac{x}{y} = \frac{5}{-12} = -\frac{5}{12}$$

$$\sec\theta = \frac{r}{x} = \frac{13}{5}$$

$$\csc\theta = \frac{r}{y} = \frac{13}{-12} = -\frac{13}{12}$$

2.



$(-12,-5)$
$x = -12$, $y = -5$, and

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + (-12)^2} \\ &= \sqrt{25 + 144} = \sqrt{169} = 13 \end{aligned}$$

$$\sin\theta = \frac{y}{r} = \frac{-5}{13} = -\frac{5}{13}$$

$$\cos\theta = \frac{x}{r} = \frac{-12}{13} = -\frac{12}{13}$$

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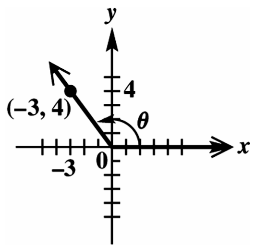
$$\tan\theta = \frac{y}{x} = \frac{-5}{-12} = \frac{5}{12}$$

$$\cot\theta = \frac{x}{y} = \frac{-12}{-5} = \frac{12}{5}$$

$$\sec\theta = \frac{r}{x} = \frac{13}{-12} = -\frac{13}{12}$$

$$\csc\theta = \frac{r}{y} = \frac{13}{-5} = -\frac{13}{5}$$

3.



$(-3, 4)$

$x = -3$, $y = 4$ and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

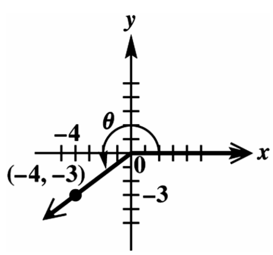
$$\sin\theta = \frac{y}{r} = \frac{4}{5};\ \cos\theta = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

$$\cot\theta = \frac{x}{y} = \frac{-3}{4} = -\frac{3}{4}$$

$$\sec\theta = \frac{r}{x} = \frac{5}{-3} = -\frac{5}{3};\ \csc\theta = \frac{r}{y} = \frac{5}{4}$$

4.



$(-4, -3)$

$x = -4$, $y = -3$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$$

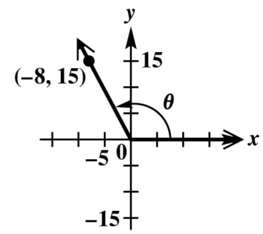
$$\tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$$

$$\cot\theta = \frac{x}{y} = \frac{-4}{-3} = \frac{4}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{5}{-4} = -\frac{5}{4}$$

$$\csc\theta = \frac{r}{y} = \frac{5}{-3} = -\frac{5}{3}$$

5.



$(-8, 15)$

$x = -8$, $y = 15$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-8)^2 + 15^2} = \sqrt{64+225} = \sqrt{289} = 17$$

$$\sin\theta = \frac{y}{r} = \frac{15}{17}$$

$$\cos\theta = \frac{x}{r} = \frac{-8}{17} = -\frac{8}{17}$$

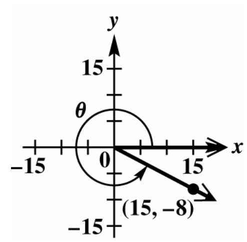
$$\tan\theta = \frac{y}{x} = \frac{15}{-8} = -\frac{15}{8}$$

$$\cot\theta = \frac{x}{y} = \frac{-8}{15} = -\frac{8}{15}$$

$$\sec\theta = \frac{r}{x} = \frac{17}{-8} = -\frac{17}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15}$$

6.



$(15, -8)$

$x = 15$, $y = -8$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{15^2 + (-8)^2} = \sqrt{225+64} = \sqrt{289} = 17$$

$$\sin\theta = \frac{y}{r} = \frac{-8}{17} = -\frac{8}{17}$$

$$\cos\theta = \frac{x}{r} = \frac{15}{17} = \frac{15}{17}$$

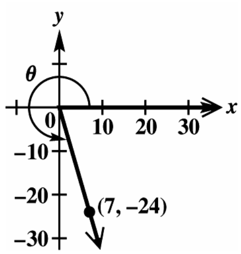
$$\tan\theta = \frac{y}{x} = \frac{-8}{15} = -\frac{8}{15}$$

$\cot\theta = \frac{x}{y} = \frac{15}{-8} = -\frac{15}{8}$

$\sec\theta = \frac{r}{x} = \frac{17}{15}$

$\csc\theta = \frac{r}{y} = \frac{17}{-8} = -\frac{17}{8}$

7.



$(7, -24)$

$x = 7$, $y = -24$, and

$r = \sqrt{x^2 + y^2} = \sqrt{7^2 + (-24)^2}$
$= \sqrt{49 + 576} = \sqrt{625} = 25$

$\sin\theta = \frac{y}{r} = \frac{-24}{25} = -\frac{24}{25}$

$\cos\theta = \frac{x}{r} = \frac{7}{25}$

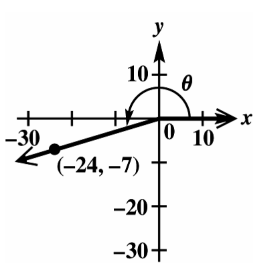
$\tan\theta = \frac{y}{x} = \frac{-24}{7} = -\frac{24}{7}$

$\cot\theta = \frac{x}{y} = \frac{7}{-24} = -\frac{7}{24}$

$\sec\theta = \frac{r}{x} = \frac{25}{7}$

$\csc\theta = \frac{r}{y} = \frac{25}{-24} = -\frac{25}{24}$

8.



$(-24, -7)$

$x = -24$, $y = -7$, and

$r = \sqrt{x^2 + y^2} = \sqrt{(-24)^2 + (-7)^2}$
$= \sqrt{576 + 49} = \sqrt{625} = 25$

$\sin\theta = \frac{y}{r} = \frac{-7}{25} = -\frac{7}{25}$

$\cos\theta = \frac{x}{r} = \frac{-24}{25} = -\frac{24}{25}$

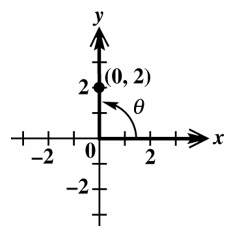
$\tan\theta = \frac{y}{x} = \frac{-7}{-24} = \frac{7}{24}$

$\cot\theta = \frac{x}{y} = \frac{-24}{-7} = \frac{24}{7}$

$\sec\theta = \frac{r}{x} = \frac{25}{-24} = -\frac{25}{24}$

$\csc\theta = \frac{r}{y} = \frac{25}{-7} = -\frac{25}{7}$

9.



$(0, 2)$

$x = 0$, $y = 2$, and

$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 2^2} = \sqrt{0 + 4} = \sqrt{4} = 2$

$\sin\theta = \frac{y}{r} = \frac{2}{2} = 1$

$\cos\theta = \frac{x}{r} = \frac{0}{2} = 0$

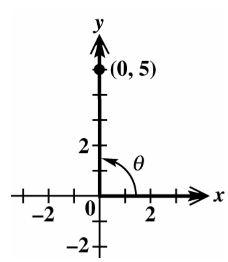
$\tan\theta = \frac{y}{x} = \frac{2}{0}$ undefined

$\cot\theta = \frac{x}{y} = \frac{0}{2} = 0$

$\sec\theta = \frac{r}{x} = \frac{2}{0}$ undefined

$\csc\theta = \frac{r}{y} = \frac{2}{2} = 1$

10.



$(0, 5)$

$x = 0$, $y = 5$, and

$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 5^2} = \sqrt{0 + 5} = \sqrt{25} = 5$

$\sin\theta = \frac{y}{r} = \frac{5}{5} = 1$

$\cos\theta = \frac{x}{r} = \frac{0}{5} = 0$

$\tan\theta = \frac{y}{x} = \frac{5}{0}$ undefined

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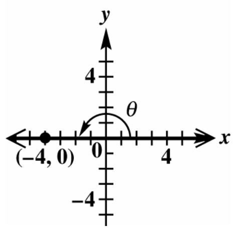
$\cot\theta = \frac{x}{y} = \frac{0}{5} = 0$

$\sec\theta = \frac{r}{x} = \frac{5}{0}$ undefined

$\csc\theta = \frac{r}{y} = \frac{5}{5} = 1$

11.



$(-4, 0)$

$x = -4$, $y = 2$, and

$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 0^2}$
$= \sqrt{16 + 0} = \sqrt{16} = 4$

$\sin\theta = \frac{y}{r} = \frac{0}{4} = 0$

$\cos\theta = \frac{x}{r} = \frac{-4}{4} = -1$

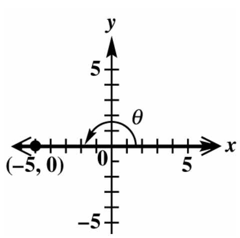
$\tan\theta = \frac{y}{x} = \frac{0}{-4} = 0$

$\cot\theta = \frac{x}{y} = \frac{-4}{0}$ undefined

$\sec\theta = \frac{r}{x} = \frac{4}{-4} = -1$

$\csc\theta = \frac{r}{y} = \frac{4}{0}$ undefined

12.



$(-5, 0)$

$x = -5$, $y = 0$, and

$r = \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + 0^2}$
$= \sqrt{25 + 0} = \sqrt{25} = 5$

$\sin\theta = \frac{y}{r} = \frac{0}{5} = 0$

$\cos\theta = \frac{x}{r} = \frac{-5}{5} = -1$

$\tan\theta = \frac{y}{x} = \frac{0}{-5} = 0$

$\cot\theta = \frac{x}{y} = \frac{-5}{0}$ undefined

$\sec\theta = \frac{r}{x} = \frac{5}{-5} = -1$

$\csc\theta = \frac{r}{y} = \frac{5}{0}$ undefined

13. Answers will vary.For any nonquadrantal angle θ, a point on the terminal side of θ will be of the form (x, y) where $x, y \neq 0$.

Now $\sin\theta = \frac{y}{r}$ and $\csc\theta = \frac{r}{y}$ both exist and are simply reciprocals of each other, and hence will have the same sign.

14. Answers will vary. For any point on the terminal side of a nonquadrantal θ, a right triangle can be formed by connecting the point to the origin and connecting the point to the *x*-axis. The value of *r* can be interpreted as the length of the hypotenuse of this right triangle.

In Exercises 15−22, $r = \sqrt{x^2 + y^2}$, which is positive.

15. In quadrant II, *x* is negative, so $\frac{x}{r}$ is negative.

16. In quadrant III, *y* is negative, so $\frac{y}{r}$ is negative.

17. In quadrant IV, *x* is positive and *y* is negative, so $\frac{y}{x}$ is negative.

18. In quadrant IV, *x* is positive and *y* is negative, so $\frac{x}{y}$ is negative.

19. In quadrant II, *y* is positive, so $\frac{y}{r}$ is positive.

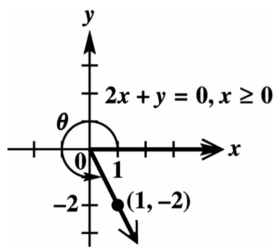
20. In quadrant III, *x* is negative, so $\frac{x}{r}$ is negative.

21. In quadrant IV, *x* is positive, so $\frac{x}{r}$ is positive.

22. In quadrant IV, *y* is negative, so $\frac{y}{r}$ is negative.

23. Since $x \ge 0$, the graph of the line $2x + y = 0$ is shown to the right of the y-axis. A point on this line is $(1, -2)$ since $2(1) + (-2) = 0$. The corresponding value of r is

$r = \sqrt{1^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$.



$$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{5}} = -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{1} = -2$$

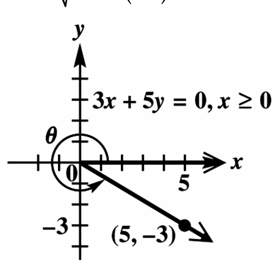
$$\cot\theta = \frac{x}{y} = \frac{1}{-2} = -\frac{1}{2}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$$

24. Since $x \ge 0$, the graph of the line $3x + 5y = 0$ is shown to the right of the y-axis. A point on this graph is $(5, -3)$ since $3(5) + 5(-3) = 0$. The corresponding value of r is

$r = \sqrt{5^2 + (-3)^2} = \sqrt{25+9} = \sqrt{34}$.



$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{34}} = -\frac{3}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34}$$

$$\cos\theta = \frac{x}{r} = \frac{5}{\sqrt{34}} = \frac{5}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{5} = -\frac{3}{5}$$

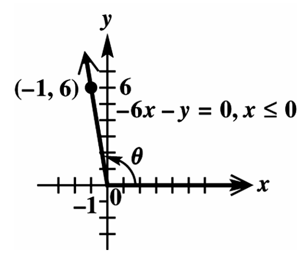
$$\cot\theta = \frac{x}{y} = \frac{5}{-3} = -\frac{5}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{5}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{34}}{-3} = -\frac{\sqrt{34}}{3}$$

25. Since $x \le 0$, the graph of the line $-6x - y = 0$ is shown to the left of the y-axis. A point on this graph is $(-1, 6)$ since $-6(-1) - 6 = 0$. The corresponding value of r is

$r = \sqrt{(-1)^2 + 6^2} = \sqrt{1+36} = \sqrt{37}$.



$$\sin\theta = \frac{y}{r} = \frac{6}{\sqrt{37}} = \frac{6}{\sqrt{37}} \cdot \frac{\sqrt{37}}{\sqrt{37}} = \frac{6\sqrt{37}}{37}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{37}} = -\frac{1}{\sqrt{37}} \cdot \frac{\sqrt{37}}{\sqrt{37}} = -\frac{\sqrt{37}}{37}$$

$$\tan\theta = \frac{y}{x} = \frac{6}{-1} = -6$$

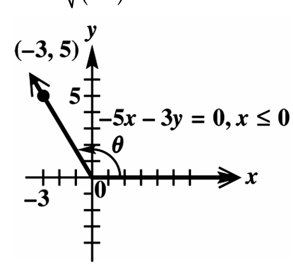
$$\cot\theta = \frac{x}{y} = \frac{-1}{6} = -\frac{1}{6}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{37}}{-1} = -\sqrt{37}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{37}}{6}$$

26. Since $x \le 0$, the graph of the line $-5x - 3y = 0$ is shown to the left of the y-axis. A point on this line is $(-3, 5)$ since $-5(-3) - 3(5) = 0$. The corresponding value of r is

$r = \sqrt{(-3)^2 + 5^2} = \sqrt{9+25} = \sqrt{34}$.



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$$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{34}} = \frac{5}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{34}} = -\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34}$$

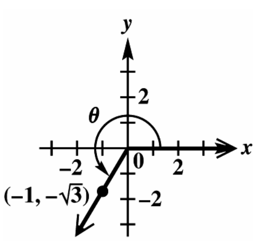
$$\tan\theta = \frac{y}{x} = \frac{5}{-3} = -\frac{5}{3}$$

$$\cot\theta = \frac{x}{y} = \frac{-3}{5} = -\frac{3}{5}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{-3} = -\frac{\sqrt{34}}{3}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{34}}{5}$$

27. Since $x \le 0$, the graph of the line $-\sqrt{3}x + y = 0$ is shown to the left of the y-axis. A point on this line is $\left(-1, -\sqrt{3}\right)$ since $-\sqrt{3}\left(-1\right) - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$ The corresponding value of r is $r = \sqrt{\left(-1\right)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$.



$-\sqrt{3}x + y = 0, x \le 0$

$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{2} = -\frac{1}{2}$$

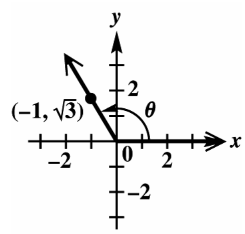
$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\cot\theta = \frac{x}{y} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{2}{-1} = -2$$

$$\csc\theta = \frac{r}{y} = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

28. Since $x \le 0$, the graph of the line $\sqrt{3}x + y = 0$ is shown to the left of the y-axis. A point on this line is $\left(-1, \sqrt{3}\right)$ since $\sqrt{3}\left(-1\right) + \sqrt{3} = -\sqrt{3} + \sqrt{3} = 0$ The corresponding value of r is $r = \sqrt{\left(-1\right)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$.



$\sqrt{3}x + y = 0, x \le 0$

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{2} = -\frac{1}{2}$$

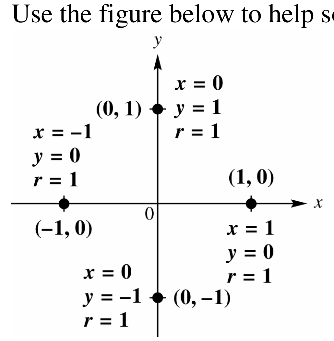
$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\cot\theta = \frac{x}{y} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{2}{-1} = -2$$

$$\csc\theta = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Use the figure below to help solve exercises 29–46.



29. $\cos 90°$

$$\cos 90° = \frac{x}{r} = \frac{0}{1} = 0$$

30. $\sin 90°$

$$\sin 90° = \frac{y}{r} = \frac{1}{1} = 1$$

31. $\tan 180°$

$\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0$

32. $\cot 90°$

$\cot 90° = \frac{x}{y} = \frac{0}{1} = 0$

33. $\sec 180°$

$\sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$

34. $\csc 270°$

$\csc 270° = \frac{r}{y} = \frac{1}{-1} = -1$

35. $\sin(-270°)$

The quadrantal angle $\theta = -270°$ is coterminal with $-270° + 360° = 90°$.

$\sin(-270°) = \sin 90 = \frac{y}{r} = \frac{1}{1} = 1$

36. $\cos(-90°)$

The quadrantal angle $\theta = -90°$ is coterminal with $-90° + 360° = 270°$.

$\cos(-90°) = \cos 270° = \frac{x}{r} = \frac{0}{1} = 0$

37. $\cot 540°$

The quadrantal angle $\theta = 540°$ is coterminal with $540° - 360° = 180°$.

$\cot 540° = \cot 180° = \frac{x}{y} = \frac{-1}{0}$ undefined

38. $\tan 450°$

The quadrantal angle $\theta = 450°$ is coterminal with $450° - 360° = 90°$.

$\tan 450° = \tan 90° = \frac{y}{x} = \frac{1}{0}$ undefined

39. $\csc(-450°)$

The quadrantal angle $\theta = -450°$ is coterminal with $720° - 450° = 270°$.

$\csc(-450°) = \csc 270° = \frac{r}{y} = \frac{1}{-1} = -1$

40. $\sec(-540°)$

The quadrantal angle $\theta = -540°$ is coterminal with $720° - 540° = 180°$.

$\sec(-540°) = \sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$

41. $\cos 90° + 3\sin 270°$

$\cos 90° = \frac{x}{r} = \frac{0}{1} = 0$

$\sin 270° = \frac{y}{r} = \frac{-1}{1} = -1$

$\cos 90° + 3\sin 270° = 0 + 3(-1) = -3$

42. $\tan 0° - 6\sin 90°$

$\tan 0° = \frac{y}{x} = \frac{0}{1} = 0$

$\sin 90° = \frac{y}{r} = \frac{1}{1} = 1$

$\tan 0° - 6\sin 90° = 0 - 6(1) = -6$

43. $3\sec\ 180° - 5\tan\ 360°$

$\sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$ and

$\tan 360° = \tan 0° = \frac{y}{x} = \frac{0}{1} = 0$

$$\begin{aligned} 3\sec\ 180° - 5\tan\ 360° &= 3(-1) - 5(0) \\ &= -3 - 0 = -3 \end{aligned}$$

44. $4\csc 270° + 3\cos 180°$

$\csc 270° = \frac{r}{y} = \frac{1}{-1} = -1$ and

$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1$

$$\begin{aligned} 4\csc 270° + 3\cos 180° &= 4(-1) + 3(-1) \\ &= -4 + (-3) = -7 \end{aligned}$$

45. $\tan 360° + 4\sin 180° + 5\cos^2 180°$

$\tan 360° = \tan 0° = \frac{y}{x} = \frac{0}{1} = 0,$

$\sin 180° = \frac{y}{r} = \frac{0}{1} = 0,$ and

$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1$

$$\begin{aligned} &\tan 360° + 4\sin 180° + 5\cos^2 180° \\ &\quad = 0 + 4(0) + 5(-1)^2 = 0 + 0 + 5(1) = 5 \end{aligned}$$

46. $2\sec 0° + 4\cot^2 90° + \cos 360°$

$\sec 0° = \frac{r}{x} = \frac{1}{1} = 1,$ $\cot 90° = \frac{x}{y} = \frac{0}{1} = 0,$ and

$\cos 360° = \cos 0° = \frac{x}{r} = \frac{1}{1} = 1$

$$\begin{aligned} &2\sec 0° + 4\cot^2 90° + \cos 360° \\ &\quad = 2(1) + 4(0)^2 + 1 = 2 + 4(0) + 1 \\ &\quad = 2 + 0 + 1 = 3 \end{aligned}$$

47. $\cos\left[(2n+1)\cdot 90°\right]$

This angle is a quadrantal angle whose terminal side lies on either the positive part of the y-axis or the negative part of the y-axis. Any point on these terminal sides would have the form $(0,k)$, where k is any real number, $k \neq 0$.

$$\cos\left[(2n+1)\cdot 90°\right] = \frac{x}{r} = \frac{0}{\sqrt{0^2+k^2}} = \frac{0}{\sqrt{k^2}} = \frac{0}{|k|} = 0$$

48. $\sin\left[n\cdot 180°\right]$

This angle is a quadrantal angle whose terminal side lies on either the positive part of the x-axis or the negative part of the x-axis. Any point on these terminal sides would have the form $(k,0)$, where k is any real number, $k \neq 0$.

$$\sin\left[n\cdot 180°\right] = \frac{y}{r} = \frac{0}{\sqrt{0^2+k^2}} = \frac{0}{\sqrt{k^2}} = \frac{0}{|k|} = 0$$

49. $\tan\left[n\cdot 180°\right]$

The angle is a quadrantal angle whose terminal side lies on either the positive part of the x-axis or the negative part of the x-axis. Any point on these terminal sides would have the form $(k, 0)$, where k is any real number, $k \neq 0$.

$$\tan\left[n\cdot 180°\right] = \frac{y}{x} = \frac{0}{k} = 0$$

50. $\tan\left[(2n+1)\cdot 90°\right]$

This angle is a quadrantal angle whose terminal side lies on either the positive part of the y-axis or the negative part of the y-axis. Any point on these terminal sides would have the form $(0,k)$, where k is any real number, $k \neq 0$.

$$\tan\left[(2n+1)\cdot 90°\right] = \frac{y}{x} = \frac{k}{0} \text{ undefined}$$

51. $\sin\left[270° + n\cdot 360°\right]$

This angle is a quadrantal angle that is coterminal with $\theta = 270°$. $\sin 270° = -1$, so $\sin\left[270° + n\cdot 360°\right] = -1$.

52. $\cot\left[n\cdot 180°\right]$

The angle is a quadrantal angle whose terminal side lies on either the positive part of the x-axis or the negative part of the x-axis. Any point on these terminal sides would have the form $(k, 0)$, where k is any real number, $k \neq 0$.

$$\cot\left[n\cdot 180°\right] = \frac{x}{y} = \frac{k}{0} \text{ undefined}$$

53. $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

54. $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{5}{8}} = \frac{8}{5}$

55. $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{3}{7}} = -\frac{7}{3}$

56. $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{8}{43}} = -\frac{43}{8}$

57. $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{5}$

58. $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{18}$

59. $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{\sqrt{8}}{2}} = \frac{2}{\sqrt{8}} = \frac{2}{2\sqrt{2}} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$

60. $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{\sqrt{24}}{3}} = \frac{3}{\sqrt{24}} = \frac{3}{2\sqrt{6}} = \frac{3\sqrt{6}}{12} = \frac{\sqrt{6}}{4}$

61. $\sin\theta = \frac{1}{\csc\theta} = \frac{1}{1.42716327} \approx .70069071$

62. $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{9.80425133} \approx .10199657$

63. No; Since $\sin\theta = \frac{1}{\csc\theta}$, if $\sin\theta > 0$, then $\csc\theta > 0$.

64. The value of $\cos\theta$ cannot exceed 1.

65. A 74° angle in standard position lies in quadrant I, so all its trigonometric functions are positive.

66. An 84° angle in standard position lies in quadrant I, so all its trigonometric functions are positive.

67. A 218° angle in standard position lies in quadrant III, so its sine, cosine, secant, and cosecant are negative, while its tangent and cotangent are positive.

68. A 195° angle in standard position lies in quadrant III, so its sine, cosine, secant, and cosecant are negative, while its tangent and cotangent are positive.

69. A 178° angle in standard position lies in quadrant II, so its sine and cosecant are positive, while its cosine, secant, tangent and cotangent are negative.

70. A 125° angle in standard position lies in quadrant II, so its sine and cosecant are positive, while its cosine, secant, tangent and cotangent are negative.

71. A −80° angle in standard position lies in quadrant IV, so its cosine and secant are positive, while its sine, cosecant, tangent and cotangent are negative.

72. A −15° angle in standard position lies in quadrant IV, so its cosine and secant are positive, while its sine, cosecant, tangent and cotangent are negative.

73. Since $\sin\theta > 0$, $\csc\theta$ is also greater than 0. The functions are greater than 0 (positive) in quadrants I and II.

74. Since $\cos\theta > 0$, $\sec\theta$ is also greater than 0. The functions are greater than 0 (positive) in quadrants I and IV.

75. $\tan\theta < 0$ in quadrants II and IV, while $\cos\theta < 0$ in quadrants II and III. Both conditions are met only in quadrant II.

76. $\cos\theta < 0$ in quadrants II and III, while $\sin\theta < 0$ in quadrants III and IV. Both conditions are met only in quadrant III.

77. $\sec\theta > 0$ in quadrants I and IV, while $\csc\theta > 0$ in quadrants I and II. Both conditions are met only in quadrant I.

78. $\csc\theta > 0$ in quadrants I and II, while $\cot\theta > 0$ in quadrants I and III. Both conditions are met only in quadrant I.

79. Since $\sin\theta < 0$, $\csc\theta$ is also less than 0. The functions are less than 0 (negative) in quadrants III and IV.

80. Since $\tan\theta < 0$, $\cot\theta$ is also less than 0. The functions are less that 0 (negative) in quadrants II and IV.

81. Impossible because the range of $\sin\theta$ is [−1, 1].

82. Impossible because the range of $\sin\theta$ is [−1, 1].

83. Possible because the range of $\cos\theta$ is [−1, 1].

84. Possible because the range of $\cos\theta$ is [−1, 1].

85. Possible because the range of $\tan\theta$ is $(-\infty, \infty)$.

86. Possible because the range of $\cot\theta$ is $(-\infty, \infty)$.

87. Impossible because the range of $\sec\theta$ is $(-\infty, -1] \cup [1, \infty)$.

88. Impossible because the range of $\sec\theta$ is $(-\infty, -1] \cup [1, \infty)$.

89. Possible because the range of $\csc\theta$ is $(-\infty, -1] \cup [1, \infty)$.

90. $\tan\theta = 2$ is possible because the range of $\tan\theta$ is $(-\infty, \infty)$. However, when $\tan\theta = 2$, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{2}$. Thus, $\tan\theta = 2$ and $\cot\theta = -2$ is impossible.

91. $\cos\theta = -2$ is impossible because the range of $\cos\theta$ is [−1, 1].

92. If $\cos\theta = \frac{1}{2}$, then $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{2}} = 2$.

Thus, there is no angle θ such that $\cos\theta = \frac{1}{2}$ and $\sec\theta = -2$.

93. If $\sin\theta = \frac{3}{5}$, then $y = 3$ and $r = 5$. So
$r^2 = x^2 + y^2 \Rightarrow 5^2 = x^2 + 3^2 \Rightarrow 25 = x^2 + 9 \Rightarrow$
$16 = x^2 \Rightarrow \pm 4 = x$. θ is in quadrant II, so $x = -4$. Therefore, $\cos\theta = -\frac{4}{5}$. Alternatively, use the identity $\sin^2\theta + \cos^2\theta = 1$:

$$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1 \Rightarrow$$

$$\frac{9}{25} + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{16}{25} \Rightarrow \cos\theta = \pm\frac{4}{5}$$

θ is in quadrant II, so $\cos\theta$ is negative.
Thus, $\cos\theta = -\frac{4}{5}$.

94. If $\cos\theta = \frac{4}{5}$, then $x = 4$ and $r = 5$. So
$r^2 = x^2 + y^2 \Rightarrow 5^2 = 4^2 + y^2 \Rightarrow$
$25 = 16 + y^2 \Rightarrow 9 = y^2 \Rightarrow \pm 3 = y$
θ is in quadrant IV, so $y = -3$. Therefore, $\sin\theta = -\frac{3}{5}$. Alternatively, use the identity
$\sin^2\theta + \cos^2\theta = 1$: $\sin^2\theta + \left(\frac{4}{5}\right)^2 = 1 \Rightarrow$
$\sin^2\theta + \frac{16}{25} = 1 \Rightarrow \sin^2\theta = \frac{9}{25} \Rightarrow \sin\theta = \pm\frac{3}{5}$
θ is in quadrant IV, so $\sin\theta$ is negative. Thus, $\sin\theta = -\frac{3}{5}$.

95. If $\cot\theta = -\frac{1}{2}$ and θ is in quadrant IV, then $x = 1$ and $y = -2$. So
$r^2 = x^2 + y^2 \Rightarrow r^2 = 1^2 + (-2)^2 \Rightarrow$
$r^2 = 1 + 4 \Rightarrow r^2 = 5 \Rightarrow r = \sqrt{5}$
Therefore, $\csc\theta = \frac{r}{y} = -\frac{\sqrt{5}}{2}$. Alternatively, use the identity $1 + \cot^2\theta = \csc^2\theta$:
$1 + \left(-\frac{1}{2}\right)^2 = \csc^2\theta \Rightarrow 1 + \frac{1}{4} = \csc^2\theta \Rightarrow$
$\frac{5}{4} = \csc^2\theta \Rightarrow \pm\frac{\sqrt{5}}{2} = \csc\theta$. θ is in quadrant IV, so $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{5}}{2}$

96. If $\tan\theta = \frac{\sqrt{7}}{3}$ and θ is in quadrant III, then $x = -3$ and $y = -\sqrt{7}$. So $r^2 = x^2 + y^2 \Rightarrow$
$r^2 = (-3)^2 + \left(-\sqrt{7}\right)^2 \Rightarrow r^2 = 9 + 7 \Rightarrow$
$r^2 = 16 \Rightarrow r = 4$. Therefore, $\sec\theta = \frac{r}{x} = -\frac{4}{3}$.
Alternatively, use the identity
$\tan^2\theta + 1 = \sec^2\theta$:
$\left(\frac{\sqrt{7}}{3}\right)^2 + 1 = \sec^2\theta \Rightarrow \frac{7}{9} + 1 = \sec^2\theta \Rightarrow$
$\frac{16}{9} = \sec^2\theta \Rightarrow \pm\frac{4}{3} = \sec\theta$
θ is in quadrant III, so $\sec\theta$ is negative.
Thus $\sec\theta = -\frac{4}{3}$

97. If $\sin\theta = \frac{1}{2}$, then $y = 1$ and $r = 2$. So
$r^2 = x^2 + y^2 \Rightarrow 2^2 = x^2 + 1^2 \Rightarrow 4 = x^2 + 1 \Rightarrow$
$3 = x^2 \Rightarrow \pm\sqrt{3} = x$
θ is in quadrant II, so $x = -\sqrt{3}$. Therefore,
$\tan\theta = -\frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.
Alternatively, using the identity $\sin^2\theta + \cos^2\theta = 1$ gives
$\left(\frac{1}{2}\right)^2 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow$
$\cos\theta = \pm\frac{\sqrt{3}}{2}$. Since θ is in quadrant II, $\cos\theta = -\frac{\sqrt{3}}{2}$. Then, using the identity
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$
$= -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

98. If $\csc\theta = -2$, then $r = 2$ and $y = -1$. So
$r^2 = x^2 + y^2 \Rightarrow 2^2 = x^2 + (-1)^2 \Rightarrow$
$4 = x^2 + 1 \Rightarrow 3 = x^2 \Rightarrow \pm\sqrt{3} = x$
θ is in quadrant III, so $x = -\sqrt{3}$. Therefore,
$\cot\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$. Alternatively, using the identity $1 + \cot^2\theta = \csc^2\theta$ gives
$1 + \cot^2\theta = (-2)^2 \Rightarrow \cot^2\theta = 3 \Rightarrow$
$\cot\theta = \pm\sqrt{3}$. Since θ is in quadrant III, $\cot\theta = \sqrt{3}$.

99. Using the identity $1 + \cot^2\theta = \csc^2\theta$ gives
$1 + \cot^2\theta = (-3.5891420)^2 \Rightarrow$
$\cot^2\theta = 11.8819403 \Rightarrow \cot\theta \approx \pm 3.44701905$
Since θ is in quadrant III,
$\cot\theta \approx 3.44701905$.

100. Using the identity $\sin^2 \theta + \cos^2 \theta = 1$. gives

$$(.49268329)^2 + \cos^2 \theta = 1$$
$$.24273682 + \cos^2 \theta \approx 1$$
$$\cos^2 \theta \approx .75726318$$
$$\cos \theta \approx \pm .87020870$$

Since θ is in quadrant II, $\cos \theta < 0$; therefore, $\cos \theta \approx -.87020870$. Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, we have

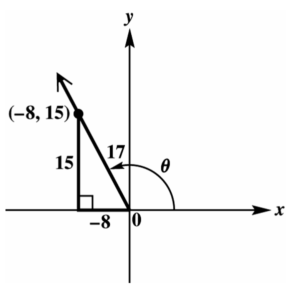
$$\tan \theta \approx \frac{.49268329}{-.87020870} \approx -.56616682.$$

For Exercises 101−106, remember that *r* is always positive.

101. $\tan \theta = -\frac{15}{8} = \frac{15}{-8}$, with θ in quadrant II

$\tan \theta = \frac{y}{x}$ and θ is in quadrant II, so let $y = 15$ and $x = -8$.

$$x^2 + y^2 = r^2 \Rightarrow (-8)^2 + 15^2 = r^2 \Rightarrow$$
$$64 + 225 = r^2 \Rightarrow 289 = r^2 \Rightarrow r = 17$$



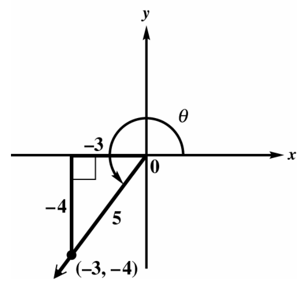
$$\sin \theta = \frac{y}{r} = \frac{15}{17}; \ \cos \theta = \frac{x}{r} = \frac{-8}{17} = -\frac{8}{17}$$
$$\tan \theta = \frac{y}{x} = \frac{15}{-8} = -\frac{15}{8}$$
$$\cot \theta = \frac{x}{y} = \frac{-8}{15} = -\frac{8}{15}$$
$$\sec \theta = \frac{r}{x} = \frac{17}{-8} = -\frac{17}{8}; \ \csc \theta = \frac{r}{y} = \frac{17}{15}$$

102. $\cos \theta = -\frac{3}{5} = \frac{-3}{5}$, with θ in quadrant III

$\cos \theta = \frac{x}{r}$ and θ in quadrant III, so let $x = -3$, $r = 5$.

$$x^2 + y^2 = r^2 \Rightarrow (-3)^2 + y^2 = 5^2 \Rightarrow$$
$$9 + y^2 = 25 \Rightarrow y^2 = 16 \Rightarrow y = \pm\sqrt{16} \Rightarrow$$
$$y = \pm 4$$

θ is in quadrant III, so $y = -4$.



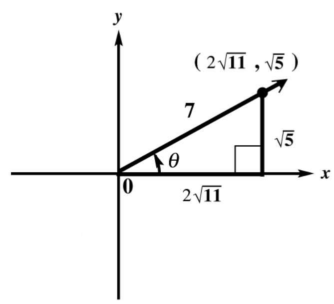
$$\sin \theta = \frac{y}{r} = \frac{-4}{5} = -\frac{4}{5}; \ \cos \theta = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$$
$$\tan \theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}; \ \cot \theta = \frac{x}{y} = \frac{-3}{-4} = \frac{3}{4}$$
$$\sec \theta = \frac{r}{x} = \frac{5}{-3} = -\frac{5}{3}; \ \csc \theta = \frac{r}{y} = \frac{5}{-4} = -\frac{5}{4}$$

103. $\sin \theta = \frac{\sqrt{5}}{7}$, with θ in quadrant I

$\sin \theta = \frac{y}{r}$ and θ in quadrant I, so let $y = \sqrt{5}, r = 7$.

$$x^2 + y^2 = r^2 \Rightarrow x^2 + \left(\sqrt{5}\right)^2 = 7^2 \Rightarrow$$
$$x^2 + 5 = 49 \Rightarrow x^2 = 44 \Rightarrow$$
$$x = \pm\sqrt{44} \Rightarrow x = \pm 2\sqrt{11}$$

θ is in quadrant I, so $x = 2\sqrt{11}$.



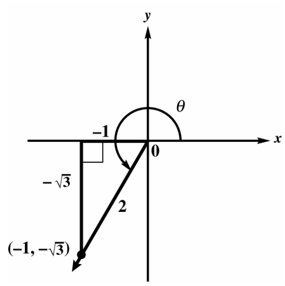
Drawing not to scale

$$\sin \theta = \frac{y}{r} = \frac{\sqrt{5}}{7}$$
$$\cos \theta = \frac{x}{r} = \frac{2\sqrt{11}}{7}$$
$$\tan \theta = \frac{y}{x} = \frac{\sqrt{5}}{2\sqrt{11}} = \frac{\sqrt{5}}{2\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{\sqrt{55}}{22}$$
$$\cot \theta = \frac{x}{y} = \frac{2\sqrt{11}}{\sqrt{5}} = \frac{2\sqrt{11}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{55}}{5}$$
$$\sec \theta = \frac{r}{x} = \frac{7}{2\sqrt{11}} = \frac{7}{2\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{7\sqrt{11}}{22}$$
$$\csc \theta = \frac{r}{y} = \frac{7}{\sqrt{5}} = \frac{7}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{7\sqrt{5}}{5}$$

104. $\tan\theta = \sqrt{3} = \dfrac{-\sqrt{3}}{-1}$

$\tan\theta = \dfrac{y}{x}$ and θ is in quadrant III, so let $y = -\sqrt{3}$ and $x = -1$.

$$x^2 + y^2 = r^2 \Rightarrow (-1)^2 + \left(-\sqrt{3}\right)^2 = r^2 \Rightarrow$$
$$1 + 3 = r^2 \Rightarrow 4 = r^2 \Rightarrow r = 2$$

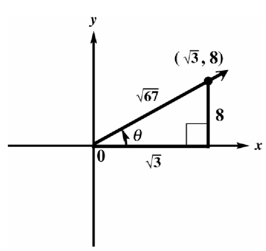


$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\cos\theta = \frac{x}{r} = \frac{-1}{2} = -\frac{1}{2}$$
$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$
$$\cot\theta = \frac{x}{y} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\sec\theta = \frac{r}{x} = \frac{2}{-1} = -2$$
$$\csc\theta = \frac{r}{y} = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

105. $\cot\theta = \dfrac{\sqrt{3}}{8}$, with θ in quadrant I

$\cot\theta = \dfrac{x}{y}$ and θ is in quadrant I, so let $x = \sqrt{3}$ and $y = 8$.

$$x^2 + y^2 = r^2 \Rightarrow \left(\sqrt{3}\right)^2 + (8)^2 = r^2 \Rightarrow$$
$$3 + 64 = r^2 \Rightarrow 67 = r^2 \Rightarrow \sqrt{67} = r$$



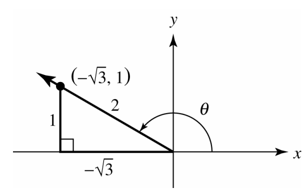
$$\sin\theta = \frac{y}{r} = \frac{8}{\sqrt{67}} = \frac{8}{\sqrt{67}} \cdot \frac{\sqrt{67}}{\sqrt{67}} = \frac{8\sqrt{67}}{67}$$
$$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{\sqrt{67}} = \frac{\sqrt{3}}{\sqrt{67}} \cdot \frac{\sqrt{67}}{\sqrt{67}} = \frac{\sqrt{3}\sqrt{67}}{67} = \frac{\sqrt{201}}{67}$$
$$\tan\theta = \frac{y}{x} = \frac{8}{\sqrt{3}} = \frac{8}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$$
$$\cot\theta = \frac{x}{y} = \frac{\sqrt{3}}{8}$$
$$\sec\theta = \frac{r}{x} = \frac{\sqrt{67}}{\sqrt{3}} = \frac{\sqrt{67}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{67}\sqrt{3}}{3} = \frac{\sqrt{201}}{3}$$
$$\csc\theta = \frac{r}{y} = \frac{\sqrt{67}}{8}$$

106. $\csc\theta = 2$, with θ in quadrant II

$\csc\theta = \dfrac{r}{y}$ and θ is in quadrant II, so let $r = 2$ and $y = 1$.

$$x^2 + y^2 = r^2 \Rightarrow x^2 + 1^2 = 2^2 \Rightarrow$$
$$x^2 + 1 = 4 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

θ is in quadrant II, so $x = -\sqrt{3}$



$$\sin\theta = \frac{y}{r} = \frac{1}{2};\ \cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\cot\theta = \frac{x}{y} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$
$$\sec\theta = \frac{r}{x} = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$
$$\csc\theta = \frac{r}{y} = \frac{2}{1} = 2$$

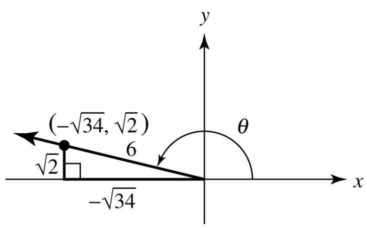
107. $\sin\theta = \frac{\sqrt{2}}{6}$, given that $\cos\theta < 0$

$\sin\theta$ is positive and $\cos\theta$ is negative when θ is in quadrant II.

$\sin\theta = \frac{y}{r}$ and θ in quadrant II, so let $y = \sqrt{2}, r = 6.$

$x^2 + y^2 = r^2 \Rightarrow x^2 + \left(\sqrt{2}\right)^2 = 6^2 \Rightarrow$
$x^2 + 2 = 36 \Rightarrow x^2 = 34 \Rightarrow$
$x = \pm\sqrt{34}$

θ is in quadrant II, so $x = -\sqrt{34}.$



$\sin\theta = \frac{y}{r} = \frac{\sqrt{2}}{6}$; $\cos\theta = \frac{x}{r} = -\frac{\sqrt{34}}{6}$

$\tan\theta = \frac{y}{x} = -\frac{\sqrt{2}}{\sqrt{34}} = -\frac{\sqrt{2}}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = -\frac{\sqrt{68}}{34} = -\frac{2\sqrt{17}}{34} = -\frac{\sqrt{17}}{17}$

$\cot\theta = \frac{x}{y} = -\frac{\sqrt{34}}{\sqrt{2}} = -\frac{\sqrt{34}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{68}}{2} = -\frac{2\sqrt{17}}{2} = -\sqrt{17}$

$\sec\theta = \frac{r}{x} = -\frac{6}{\sqrt{34}} = -\frac{6}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = -\frac{6\sqrt{34}}{34} = -\frac{3\sqrt{34}}{17}$

$\csc\theta = \frac{r}{y} = \frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

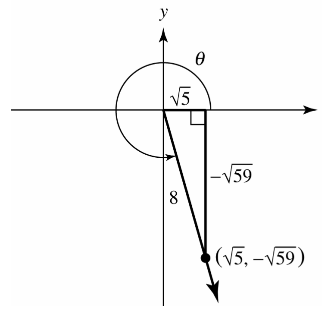
108. $\cos\theta = \frac{\sqrt{5}}{8}$, given that $\tan\theta < 0$

$\cos\theta$ is positive and $\tan\theta$ is negative when θ is in quadrant IV.

$\cos\theta = \frac{x}{r}$ and θ in quadrant IV, so let $x = \sqrt{5}, r = 8.$

$x^2 + y^2 = r^2 \Rightarrow \left(\sqrt{5}\right)^2 + y^2 = 8^2 \Rightarrow$
$5 + y^2 = 64 \Rightarrow y^2 = 59 \Rightarrow y = \pm\sqrt{59}$

θ is in quadrant IV, so $y = -\sqrt{59}.$



$\sin\theta = \frac{y}{r} = -\frac{\sqrt{59}}{8}$; $\cos\theta = \frac{x}{r} = \frac{\sqrt{5}}{8}$

$\tan\theta = \frac{y}{x} = -\frac{\sqrt{59}}{\sqrt{5}} = -\frac{\sqrt{59}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{295}}{5}$

$\cot\theta = \frac{x}{y} = -\frac{\sqrt{5}}{\sqrt{59}} = -\frac{\sqrt{5}}{\sqrt{59}} \cdot \frac{\sqrt{59}}{\sqrt{59}} = -\frac{\sqrt{295}}{59}$

$\sec\theta = \frac{r}{x} = \frac{8}{\sqrt{5}} = \frac{8}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{8\sqrt{5}}{5}$

$\csc\theta = \frac{r}{y} = -\frac{8}{\sqrt{59}} = -\frac{8}{\sqrt{59}} \cdot \frac{\sqrt{59}}{\sqrt{59}} = -\frac{8\sqrt{59}}{59}$

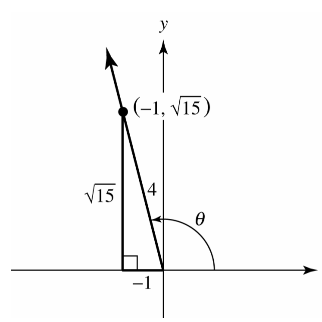
109. $\sec\theta = -4$, given that $\sin\theta > 0$

$\sec\theta$ is negative and $\sin\theta$ is positive when θ is in quadrant II.

$\sec\theta = \frac{r}{x}$ and θ in quadrant II, so let $x = -1, r = 4.$

$x^2 + y^2 = r^2 \Rightarrow (-1)^2 + y^2 = 4^2 \Rightarrow$
$1 + y^2 = 16 \Rightarrow y^2 = 15 \Rightarrow y = \pm\sqrt{15}$

θ is in quadrant II, so $y = \sqrt{15}.$



$\sin\theta = \frac{y}{r} = \frac{\sqrt{15}}{4}$; $\cos\theta = \frac{x}{r} = -\frac{1}{4}$

$\tan\theta = \frac{y}{x} = -\frac{\sqrt{15}}{1} = -\sqrt{15}$

$\cot\theta = \frac{x}{y} = -\frac{1}{\sqrt{15}} = -\frac{1}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = -\frac{\sqrt{15}}{15}$

(*continued on next page*)

(*continued from page 487*)

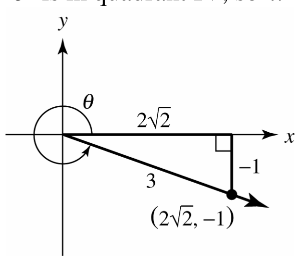
$\sec\theta = \dfrac{r}{x} = -4$

$\csc\theta = \dfrac{r}{y} = \dfrac{4}{\sqrt{15}} = \dfrac{4}{\sqrt{15}} \cdot \dfrac{\sqrt{15}}{\sqrt{15}} = \dfrac{4\sqrt{15}}{15}$

110. $\csc\theta = -3$, given that $\cos\theta > 0$
$\csc\theta$ is negative and $\cos\theta$ is positive when θ is in quadrant IV.
$\csc\theta = \dfrac{r}{y}$ and θ in quadrant IV, so let $y = -1, r = 3$.

$x^2 + y^2 = r^2 \Rightarrow x^2 + (-1)^2 = 3^2 \Rightarrow$
$x^2 + 1 = 9 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$

θ is in quadrant IV, so $x = 2\sqrt{2}$.



$\sin\theta = \dfrac{y}{r} = -\dfrac{1}{3}$

$\cos\theta = \dfrac{x}{r} = \dfrac{2\sqrt{2}}{3}$

$\tan\theta = \dfrac{y}{x} = -\dfrac{1}{2\sqrt{2}} = -\dfrac{1}{2\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = -\dfrac{\sqrt{2}}{4}$

$\cot\theta = \dfrac{x}{y} = -\dfrac{2\sqrt{2}}{1} = -2\sqrt{2}$

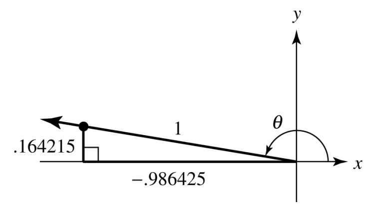
$\sec\theta = \dfrac{r}{x} = \dfrac{3}{2\sqrt{2}} = \dfrac{3}{2\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{3\sqrt{2}}{4}$

$\csc\theta = \dfrac{r}{y} = -3$

111. $\sin\theta = .164215 = \dfrac{.164215}{1}$, with θ in quadrant II
$\sin\theta = \dfrac{y}{r}$ and θ is in quadrant II, let $y = .164215$ and $r = 1$.

$x^2 + y^2 = r^2 \Rightarrow x^2 + (.164215)^2 = 1^2 \Rightarrow$
$x^2 + .026966 = 1 \Rightarrow x^2 = .973034 \Rightarrow$
$x \approx \pm\sqrt{.973034} \Rightarrow x \approx \pm .986425$

θ is in quadrant II, so, $x = -.986425$.



$\sin\theta = \dfrac{y}{r} = \dfrac{.164215}{1} = .164215$

$\cos\theta = \dfrac{x}{r} \approx \dfrac{-.986425}{1} = -.986425$

$\tan\theta = \dfrac{y}{x} \approx \dfrac{.164215}{-.986425} \approx -.166475$

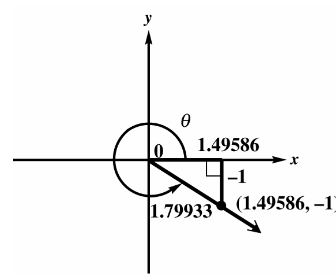
$\cot\theta = \dfrac{x}{y} \approx \dfrac{-.986425}{.164215} \approx -6.00691$

$\sec\theta = \dfrac{r}{x} \approx \dfrac{1}{-.986425} \approx -1.01376$

$\csc\theta = \dfrac{r}{y} = \dfrac{1}{.164215} \approx 6.08958$

112. $\cot\theta = -1.49586 = \dfrac{1.49586}{-1}$, with θ in quadrant IV
$\cot\theta = \dfrac{x}{y}$ and θ is in quadrant IV, let $x = 1.49586$ and $y = -1$.

$x^2 + y^2 = r^2$
$(1.49586)^2 + (-1)^2 = r^2 \Rightarrow 2.23760 + 1 \approx r^2$
$3.23760 \approx r^2 \Rightarrow r \approx 1.79933$



$\sin\theta = \dfrac{y}{r} \approx \dfrac{-1}{1.79933} \approx -.555762$

$\cos\theta = \dfrac{x}{r} \approx \dfrac{1.49586}{1.79933} \approx .831342$

$\tan\theta = \dfrac{y}{x} \approx \dfrac{-1}{1.49586} \approx -.668512$

$\cot\theta = \dfrac{x}{y} \approx \dfrac{1.49586}{-1} \approx -1.49586$

$\sec\theta = \dfrac{r}{x} \approx \dfrac{1.79933}{1.49586} \approx 1.20287$

$\csc\theta = \dfrac{r}{y} \approx \dfrac{1.79933}{-1} \approx -1.79933$

113. $x^2 + y^2 = r^2 \Rightarrow \frac{x^2 + y^2}{y^2} = \frac{r^2}{y^2} \Rightarrow$

$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2} \Rightarrow \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2 \Rightarrow$

$1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2$

Since $\cot\theta = \frac{x}{y}$ and $\csc\theta = \frac{r}{y}$, we have

$1 + (\cot\theta)^2 = (\csc\theta)^2$ or $1 + \cot^2\theta = \csc^2\theta$.

114. $\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \div \frac{y}{r} = \frac{x}{r} \cdot \frac{r}{y} = \frac{x}{y} = \cot\theta$

115. The statement is false. For example, $\sin 180° + \cos 180° = 0 + (-1) = -1 \neq 1$.

116. The statement is false since $-1 \le \sin\theta \le 1$ for all θ.

117. $90° < \theta < 180° \Rightarrow 180° < 2\theta < 360°$, so 2θ lies in either quadrant III or IV. Thus, $\sin 2\theta$ is negative.

118. $90° < \theta < 180° \Rightarrow 45° < \frac{\theta}{2} < 90°$, so $\frac{\theta}{2}$ lies in quadrant I. Thus, $\tan\frac{\theta}{2}$ is positive.

119. $90° < \theta < 180° \Rightarrow 270° < \theta + 180° < 360°$, so $\theta + 180°$ lies in quadrant IV. Thus, $\cot(\theta + 180°)$ is negative.

120. $90° < \theta < 180° \Rightarrow -90° > -\theta > -180° \Rightarrow$ $-180° < \theta < -90°$, so $-\theta$ lies in quadrant III ($-180°$ is coterminal with 180°, and $-90°$ is coterminal with 270°.) Thus, $\cos(-\theta)$ is negative.

121. $-90° < \theta < 90° \Rightarrow -45° < \frac{\theta}{2} < 45°$, so $\frac{\theta}{2}$ lies in either quadrant I or quadrant IV. Thus $\cos\frac{\theta}{2}$ is positive.

122. $-90° < \theta < 90° \Rightarrow 90° < \theta + 180° < 270°$, so $\theta + 180°$ lies in either quadrant II or quadrant III. Thus $\cos(\theta + 180°)$ is negative.

123. $-90° < \theta < 90° \Rightarrow 90° > -\theta > -90° \Rightarrow$ $-90° < -\theta < 90°$, so $-\theta$ lies in either quadrant I or quadrant IV. Thus $\sec(-\theta)$ is positive.

124. $-90° < \theta < 90° \Rightarrow -270° < \theta - 180° < -90°$, so $\theta - 180°$ lies in either quadrant II or quadrant III. Thus $\sec(\theta - 180°)$ is negative.

Section 5.3: EvaluatingTrigonometric Functions

1. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{21}{29}$

$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{20}{29}$

$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{21}{20}$

2. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{45}{53}$

$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{28}{53}$

$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{45}{28}$

3. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{n}{p}$

$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{m}{p}$

$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{n}{m}$

4. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{k}{z}$

$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{y}{z}$

$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{k}{y}$

For Exercises 5−10, refer to the Function Values of Special Angles chart on page 518 of the text.

5. C; $\sin 30° = \frac{1}{2}$

6. H; $\cos 45° = \frac{\sqrt{2}}{2}$

7. B; $\tan 45° = 1$

8. G; $\sec 60° = \dfrac{1}{\cos 60°} = \dfrac{1}{\frac{1}{2}} = 2$

9. E; $\csc 60° = \dfrac{1}{\sin 60°} = \dfrac{1}{\frac{\sqrt{3}}{2}} = \dfrac{2}{\sqrt{3}}$
$= \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

10. A; $\cot 30° = \dfrac{\cos 30°}{\sin 30°} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \dfrac{\sqrt{3}}{2} \cdot \dfrac{2}{1} = \sqrt{3}$

11. $a = 5, b = 12$
$c^2 = a^2 + b^2 \Rightarrow c^2 = 5^2 + 12^2 \Rightarrow c^2 = 169 \Rightarrow$
$c = 13$
$\sin B = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{12}{13}$
$\cos B = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{5}{13}$
$\tan B = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{b}{a} = \dfrac{12}{5}$
$\cot B = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{a}{b} = \dfrac{5}{12}$
$\sec B = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{c}{a} = \dfrac{13}{5}$
$\csc B = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{c}{b} = \dfrac{13}{12}$

12. $a = 3, b = 5$
$c^2 = a^2 + b^2 \Rightarrow c^2 = 3^2 + 5^2 \Rightarrow c^2 = 34 \Rightarrow$
$c = \sqrt{34}$
$\sin B = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{5}{\sqrt{34}}$
$= \dfrac{5}{\sqrt{34}} \cdot \dfrac{\sqrt{34}}{\sqrt{34}} = \dfrac{5\sqrt{34}}{34}$
$\cos B = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{3}{\sqrt{34}}$
$= \dfrac{3}{\sqrt{34}} \cdot \dfrac{\sqrt{34}}{\sqrt{34}} = \dfrac{3\sqrt{34}}{34}$
$\tan B = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{b}{a} = \dfrac{5}{3}$
$\cot B = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{a}{b} = \dfrac{3}{5}$
$\sec B = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{c}{a} = \dfrac{\sqrt{34}}{3}$
$\csc B = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{c}{b} = \dfrac{\sqrt{34}}{5}$

13. $a = 6, c = 7$
$c^2 = a^2 + b^2 \Rightarrow 7^2 = 6^2 + b^2 \Rightarrow$
$49 = 36 + b^2 \Rightarrow 13 = b^2 \Rightarrow \sqrt{13} = b$
$\sin B = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{\sqrt{13}}{7}$
$\cos B = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{6}{7}$
$\tan B = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{b}{a} = \dfrac{\sqrt{13}}{6}$
$\cot B = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{a}{b} = \dfrac{6}{\sqrt{13}}$
$= \dfrac{6}{\sqrt{13}} \cdot \dfrac{\sqrt{13}}{\sqrt{13}} = \dfrac{6\sqrt{13}}{13}$
$\sec B = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{c}{a} = \dfrac{7}{6}$
$\csc B = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{c}{b} = \dfrac{7}{\sqrt{13}}$
$= \dfrac{7}{\sqrt{13}} \cdot \dfrac{\sqrt{13}}{\sqrt{13}} = \dfrac{7\sqrt{13}}{13}$

14. $b = 7, c = 12$
$c^2 = a^2 + b^2 \Rightarrow 12^2 = a^2 + 7^2 \Rightarrow$
$144 = a^2 + 49 \Rightarrow 95 = a^2 \Rightarrow \sqrt{95} = a$
$\sin B = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{7}{12}$
$\cos B = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{\sqrt{95}}{12}$
$\tan B = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{b}{a} = \dfrac{7}{\sqrt{95}}$
$= \dfrac{7}{\sqrt{95}} \cdot \dfrac{\sqrt{95}}{\sqrt{95}} = \dfrac{7\sqrt{95}}{95}$
$\cot B = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{a}{b} = \dfrac{\sqrt{95}}{7}$
$\sec B = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{c}{a} = \dfrac{12}{\sqrt{95}}$
$= \dfrac{12}{\sqrt{95}} \cdot \dfrac{\sqrt{95}}{\sqrt{95}} = \dfrac{12\sqrt{95}}{95}$
$\csc B = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{c}{b} = \dfrac{12}{7}$

15. $\sin\theta = \cos(90° - \theta); \cos\theta = \sin(90° - \theta);$
$\tan\theta = \cot(90° - \theta); \cot\theta = \tan(90° - \theta);$
$\sec\theta = \csc(90° - \theta); \csc\theta = \sec(90° - \theta)$

16. $\cot 73° = \tan(90° - 73°) = \tan 17°$

17. $\sec 39° = \csc(90° - 39°) = \csc 51°$

18. $\sin 27° = \cos(90° - 27°) = \cos 63°$

19. $\sec(\theta + 15°) = \csc[90° - (\theta + 15°)]$
$= \csc(75° - \theta)$

20. $\cos(\alpha + 20°) = \sin[90° - (\alpha + 20°)]$
$= \sin(70° - \alpha)$

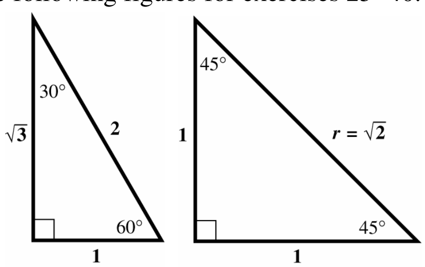
21. $\cot(\theta - 10°) = \tan[90° - (\theta - 10°)]$
$= \tan(100° - \theta)$

22. $\tan 25.4° = \cot(90° - 25.4°) = \cot 64.6°$

23. $\sin 38.7° = \cos(90° - 38.7°) = \cos 51.3°$

24. $\csc 49.9° = \sec(90° - 49.9°) = \sec 40.1°$

Use the following figures for exercises 25–40.



25. $\tan 30° = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{1}{\sqrt{3}}$
$= \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

26. $\cot 30° = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

27. $\sin 30° = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{1}{2}$

28. $\cos 30° = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{\sqrt{3}}{2}$

29. $\sec 30° = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{2}{\sqrt{3}}$
$= \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

30. $\csc 30° = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{2}{1} = 2$

31. $\csc 45° = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{\sqrt{2}}{1} = \sqrt{2}$

32. $\sec 45° = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{\sqrt{2}}{1} = \sqrt{2}$

33. $\cos 45° = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

34. $\cot 45° = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{1}{1} = 1$

35. $\tan 45° = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{1}{1} = 1$

36. $\sin 45° = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

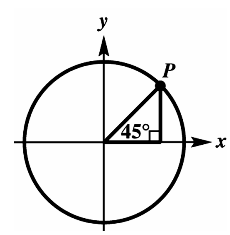
37. $\sin 60° = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{\sqrt{3}}{2}$

38. $\cos 60° = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{1}{2}$

39. $\tan 60° = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

40. $\csc 60° = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{2}{\sqrt{3}}$
$= \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

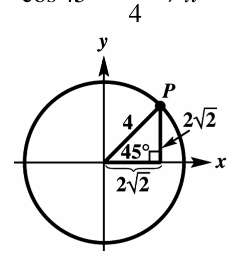
41.



42. $\sin 45° = \dfrac{y}{4} \Rightarrow y = 4\sin 45° = 4 \cdot \dfrac{\sqrt{2}}{2} = 2\sqrt{2}$

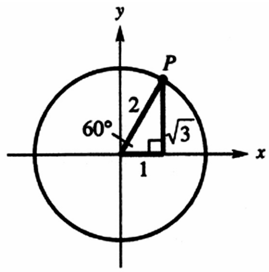
and

$\cos 45° = \dfrac{x}{4} \Rightarrow x = 4\cos 45° = 4 \cdot \dfrac{\sqrt{2}}{2} = 2\sqrt{2}$



43. The legs of the right triangle provide the coordinates of P, $\left(2\sqrt{2}, 2\sqrt{2}\right)$.

44.



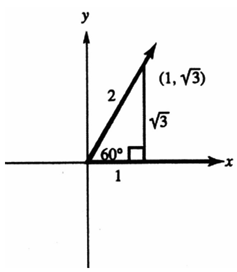
$\sin 60° = \frac{y}{2} \Rightarrow y = 2\sin 60° = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

and $\cos 60° = \frac{x}{2} \Rightarrow x = 2\cos 60° = 2 \cdot \frac{1}{2} = 1$

The legs of the right triangle provide the coordinates of P. P is $\left(1, \sqrt{3}\right)$.

45. .7071067812 is a rational approximation for the exact value $\frac{\sqrt{2}}{2}$ (an irrational value).

46.

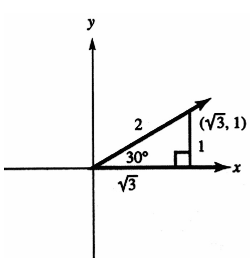


The line passes through $(0,0)$ and $\left(1, \sqrt{3}\right)$.

The slope is change in y over the change in x.

Thus, $m = \frac{\sqrt{3}}{1} = \sqrt{3}$ and the equation of the line is $y = \sqrt{3}x$.

47.



The line passes through (0, 0) and $\left(\sqrt{3}, 1\right)$.

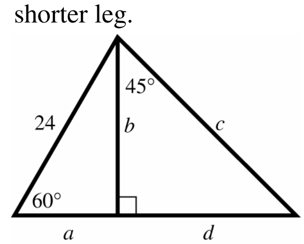
The slope is change in y over the change in x.

Thus, $m = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ and the equation of the line is $y = \frac{\sqrt{3}}{3}x$.

48. One point on the line $y = \frac{\sqrt{3}}{3}x$, is the origin $(0,0)$. Let (x, y) be any other point on this line. Then, by the definition of slope, $m = \frac{y-0}{x-0} = \frac{y}{x} = \frac{\sqrt{3}}{3}$, but also, by the definition of tangent, $\tan\theta = \frac{\sqrt{3}}{3}$. Because $\tan 30° = \frac{\sqrt{3}}{3}$, the line $y = \frac{\sqrt{3}}{3}x$ makes a 30° angle with the positive x-axis. (See Exercise 47).

49. One point on the line $y = \sqrt{3}x$ is the origin $(0,0)$. Let (x, y) be any other point on this line. Then, by the definition of slope, $m = \frac{y-0}{x-0} = \frac{y}{x} = \sqrt{3}$, but also, by the definition of tangent, $\tan\theta = \sqrt{3}$. Because $\tan 60° = \sqrt{3}$, the line $y = \sqrt{3}x$ makes a 60° angle with the positive x-axis (See exercise 46).

50. Apply the relationships between the lengths of the sides of a 30º–60º right triangle first to the triangle on the left to find the values of a and b. In the 30º–60º right triangle, the side opposite the 30º angle is $\frac{1}{2}$ the length of the hypotenuse. The longer leg is $\sqrt{3}$ times the shorter leg.



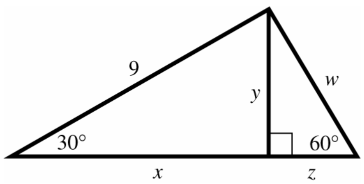
$a = \frac{1}{2}(24) = 12$ and $b = a\sqrt{3} = 12\sqrt{3}$

Apply the relationships between the lengths of the sides of a 45º–45º right triangle next to the triangle on the right to find the values of d and c. In the 45º–45º right triangle, the sides opposite the 45º angles measure the same. The hypotenuse is $\sqrt{2}$ times the measure of a leg.

$d = b = 12\sqrt{3}$ and

$c = d\sqrt{2} = \left(12\sqrt{3}\right)\left(\sqrt{2}\right) = 12\sqrt{6}$

51. Apply the relationships between the lengths of the sides of a $30° - 60°$ right triangle first to the triangle on the left to find the values of y and x, and then to the triangle on the right to find the values of z and w. In the $30° - 60°$ right triangle, the side opposite the $30°$ angle is $\frac{1}{2}$ the length of the hypotenuse. The longer leg is $\sqrt{3}$ times the shorter leg.



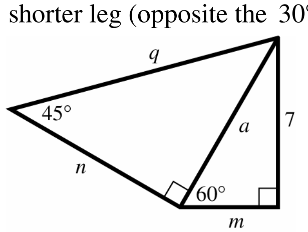
Thus, we have

$$y = \frac{1}{2}(9) = \frac{9}{2} \text{ and } x = y\sqrt{3} = \frac{9\sqrt{3}}{2}$$

$$y = z\sqrt{3}, \text{ so } z = \frac{y}{\sqrt{3}} = \frac{\frac{9}{2}}{\sqrt{3}} = \frac{9\sqrt{3}}{6} = \frac{3\sqrt{3}}{2},$$

$$\text{and } w = 2z, \text{ so } w = 2\left(\frac{3\sqrt{3}}{2}\right) = 3\sqrt{3}$$

52. Apply the relationships between the lengths of the sides of a 30º–60º right triangle first to the triangle on the right to find the values of m and a. In the 30º–60º right triangle, the side opposite the 60º angle is $\sqrt{3}$ times as long as the side opposite to the $30°$ angle. The length of the hypotenuse is 2 times as long as the shorter leg (opposite the $30°$ angle).



Thus, we have

$$7 = m\sqrt{3} \Rightarrow m = \frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{3} \text{ and}$$

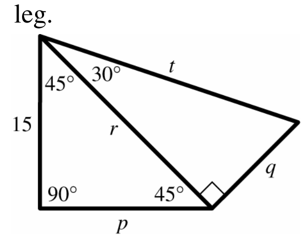
$$a = 2m \Rightarrow a = 2\left(\frac{7\sqrt{3}}{3}\right) = \frac{14\sqrt{3}}{3}$$

Apply the relationships between the lengths of the sides of a 45º–45º right triangle next to the triangle on the left to find the values of n and q. In the 45º–45º right triangle, the sides opposite the 45º angles measure the same. The hypotenuse is $\sqrt{2}$ times the measure of a leg.

Thus, we have $n = a = \frac{14\sqrt{3}}{3}$ and

$$q = n\sqrt{2} = \left(\frac{14\sqrt{3}}{3}\right)\sqrt{2} = \frac{14\sqrt{6}}{3}.$$

53. Apply the relationships between the lengths of the sides of a $45° - 45°$ right triangle to the triangle on the left to find the values of p and r. In the $45° - 45°$ right triangle, the sides opposite the $45°$ angles measure the same. The hypotenuse is $\sqrt{2}$ times the measure of a leg.



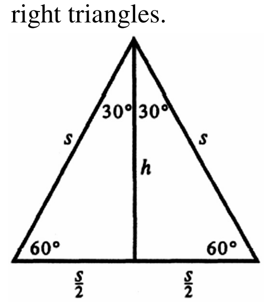
Thus, we have $p = 15$ and $r = p\sqrt{2} = 15\sqrt{2}$

Apply the relationships between the lengths of the sides of a $30° - 60°$ right triangle next to the triangle on the right to find the values of q and t. In the $30° - 60°$ right triangle, the side opposite the $60°$ angle is $\sqrt{3}$ times as long as the side opposite to the $30°$ angle. The length of the hypotenuse is 2 times as long as the shorter leg (opposite the $30°$ angle). Thus, we have $r = q\sqrt{3} \Rightarrow$

$$q = \frac{r}{\sqrt{3}} = \frac{15\sqrt{2}}{\sqrt{3}} = \frac{15\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = 5\sqrt{6} \text{ and}$$

$$t = 2q = 2\left(5\sqrt{6}\right) = 10\sqrt{6}$$

54. Let h be the height of the equilateral triangle. h bisects the base, s, and forms two 30°–60° right triangles.



The formula for the area of a triangle is $A = \frac{1}{2}bh.$ In this triangle, $b = s$. The height h of the triangle is the side opposite the 60° angle in either 30°–60° right triangle.

(*continued on next page*)

(*continued from page 493*)

The side opposite the 30° angle is $\frac{s}{2}$. The height is $\sqrt{3}\cdot\frac{s}{2}. = \frac{s\sqrt{3}}{2}$ So the area of the entire triangle is $A = \frac{1}{2}s\left(\frac{s\sqrt{3}}{2}\right) = \frac{s^2\sqrt{3}}{4}$.

55. Since $A = \frac{1}{2}bh,$ we have
$A = \frac{1}{2}\cdot s\cdot s = \frac{1}{2}s^2$ or $A = \frac{s^2}{2}$.

56. Yes, the third angle can be found by subtracting the given acute angle from 90°, and the remaining two sides can be found using a trigonometric function involving the known angle and side.

57. C; $180° - 98° = 82°$
(98° is in quadrant II)

58. F; $212° - 180° = 32°$
(212° is in quadrant III)

59. A; $-135° + 360° = 225°$ and $225° - 180° = 45°$
(225° is in quadrant III)

60. B; $-60° + 360° = 300°$ and $360° - 300° = 60°$
(300° is in quadrant IV)

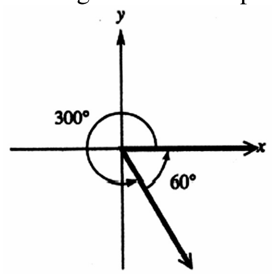
61. D; $750° - 2\cdot 360° = 30°$
(30° is in quadrant I)

62. B; $480° - 360° = 120°$ and $180° - 120° = 60°$
(120° is in quadrant II)

63. 2 is a good choice for *r* because in a $30° - 60°$ right triangle, the hypotenuse is twice the length of the shorter side (the side opposite to the $30°$ angle). By choosing 2, one avoids introducing a fraction (or decimal) when determining the length of the shorter side. Choosing any even positive integer for *r* would have this result; however, 2 is the most convenient value.

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
64.	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
65.	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
66.	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
67.	120°	$\frac{\sqrt{3}}{2}$	$\cos 120° = -\cos 60° = -\frac{1}{2}$	$-\sqrt{3}$	$\cot 120° = -\cot 60° = -\frac{\sqrt{3}}{3}$	$\sec 120° = -\sec 60° = -2$	$\frac{2\sqrt{3}}{3}$
68.	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\tan 135° = -\tan 45° = -1$	$\cot 135° = -\cot 45° = -1$	$-\sqrt{2}$	$\sqrt{2}$
69.	150°	$\sin 150° = \sin 30° = \frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$\cot 150° = -\cot 30° = -\sqrt{3}$	$\sec 150° = -\sec 30° = -\frac{2\sqrt{3}}{3}$	2
70.	210°	$-\frac{1}{2}$	$\cos 210° = -\cos 30° = -\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\sec 210° = -\sec 30° = -\frac{2\sqrt{3}}{3}$	−2
71.	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\tan 240° = \tan 60° = \sqrt{3}$	$\cot 240° = \cot 60° = \frac{\sqrt{3}}{3}$	−2	$-\frac{2\sqrt{3}}{3}$

72. To find the reference angle for $300°$, sketch this angle in standard position.



The reference angle is $360° - 300° = 60°$. Since 300° lies in quadrant IV, the sine, tangent, cotangent, and cosecant are negative.

$\sin 300° = -\sin 60° = -\frac{\sqrt{3}}{2}$

$\cos 300° = \cos 60° = \frac{1}{2}$

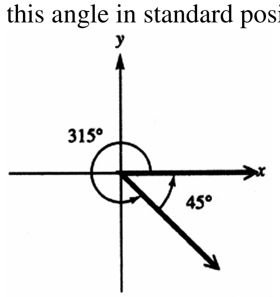
$\tan 300° = -\tan 60° = -\sqrt{3}$

$\cot 300° = -\cot 60° = -\frac{\sqrt{3}}{3}$

$\sec 300° = \sec 60° = 2$

$\csc 300° = -\csc 60° = -\frac{2\sqrt{3}}{3}$

73. To find the reference angle for $315°$, sketch this angle in standard position.



The reference angle is $360° - 315° = 45°$. Since 315° lies in quadrant IV, the sine, tangent, cotangent, and cosecant are negative.

$\sin 315° = -\sin 45° = -\frac{\sqrt{2}}{2}$

$\cos 315° = \cos 45° = \frac{\sqrt{2}}{2}$

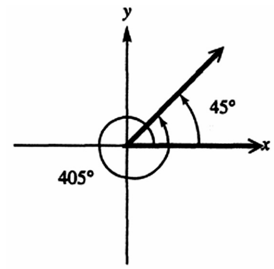
$\tan 315° = -\tan 45° = -1$

$\cot 315° = -\cot 45° = -1$

$\sec 315° = \sec 45° = \sqrt{2}$

$\csc 315° = -\csc 45° = -\sqrt{2}$

74. To find the reference angle for $405°$, sketch this angle in standard position.



The reference angle for $405°$ is $405° - 360° = 45°$. Because $405°$ lies in quadrant I, the values of all of its trigonometric functions will be positive, so these values will be identical to the trigonometric function values for $45°$. See the Function Values of Special Angles table on page 518.)

$\sin 405° = \sin 45° = \frac{\sqrt{2}}{2}$

$\cos 405° = \cos 45° = \frac{\sqrt{2}}{2}$

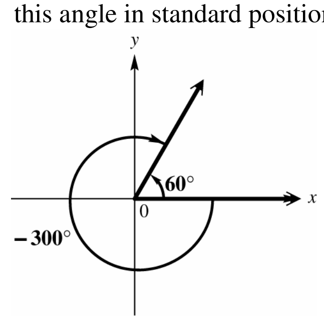
$\tan 405° = \tan 45° = 1$

$\cot 405° = \cot 45° = 1$

$\sec 405° = \sec 45° = \sqrt{2}$

$\csc 405° = \csc 45° = \sqrt{2}$

75. To find the reference angle for $-300°$, sketch this angle in standard position.



The reference angle for $-300°$ is $-300° + 360° = 60°$. Because $-300°$ lies in quadrant I, the values of all of its trigonometric functions will be positive, so these values will be identical to the trigonometric function values for $60°$. See the Function Values of Special Angles table on page 518.)

$\sin(-300°) = \sin 60° = \frac{\sqrt{3}}{2}$

$\cos(-300°) = \cos 60° = \frac{1}{2}$

$\tan(-300°) = \tan 60° = \sqrt{3}$

$\cot(-300°) = \cot 60° = \frac{\sqrt{3}}{3}$

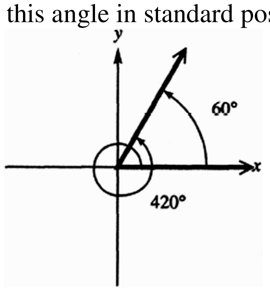
(*continued on next page*)

(*continued from page 495*)

$\sec(-300°) = \sec 60° = 2$

$\csc(-300°) = \csc 60° = \dfrac{2\sqrt{3}}{3}$

76. To find the reference angle for 420º, sketch this angle in standard position.



The reference angle for 420º is 420º – 360º = 60º. Because 420º lies in quadrant I, the values of all of its trigonometric functions will be positive, so these values will be identical to the trigonometric function values for 60°. See the Function Values of Special Angles table on page 54.)

$\sin(420°) = \sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos(420°) = \cos 60° = \dfrac{1}{2}$

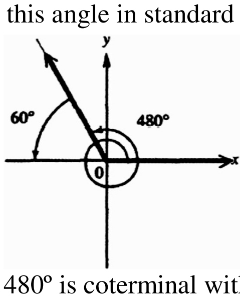
$\tan(420°) = \tan 60° = \sqrt{3}$

$\cot(420°) = \cot 60° = \dfrac{\sqrt{3}}{3}$

$\sec(420°) = \sec 60° = 2$

$\csc(420°) = \csc 60° = \dfrac{2\sqrt{3}}{3}$

77. To find the reference angle for 480º, sketch this angle in standard position.



480º is coterminal with 480º – 360º = 120º. The reference angle is 180º – 120º = 60º. Because 480º lies in quadrant II, the cosine, tangent, cotangent, and secant are negative.

$\sin(480°) = \sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos(480°) = -\cos 60° = -\dfrac{1}{2}$

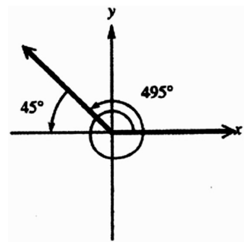
$\tan(480°) = -\tan 60° = -\sqrt{3}$

$\cot(480°) = -\cot 60° = -\dfrac{\sqrt{3}}{3}$

$\sec(80°) = -\sec 60° = -2$

$\csc(480°) = \csc 60° = \dfrac{2\sqrt{3}}{3}$

78. To find the reference angle for 495º, sketch this angle in standard position.



495º is coterminal with 495º – 360º = 135º. The reference angle is 180º – 135º = 45º. Since 495° lies in quadrant II, the cosine, tangent, cotangent, and secant are negative.

$\sin 495° = \sin 45° = \dfrac{\sqrt{2}}{2}$

$\cos 495° = -\cos 45° = -\dfrac{\sqrt{2}}{2}$

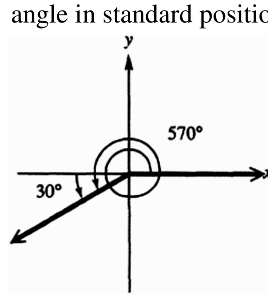
$\tan 495° = -\tan 45° = -1$

$\cot 495° = -\cot 45° = -1$

$\sec 495° = -\sec 45° = -\sqrt{2}$

$\csc 495° = \csc 45° = \sqrt{2}$

79. To find the reference angle for 570º sketch this angle in standard position.



570° is coterminal with 570º – 360º = 210º. The reference angle is 210º – 180º = 30º. Since 570º lies in quadrant III, the sine, cosine, secant, and cosecant are negative.

$\sin 570° = -\sin 30° = -\dfrac{1}{2}$

$\cos 570° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$

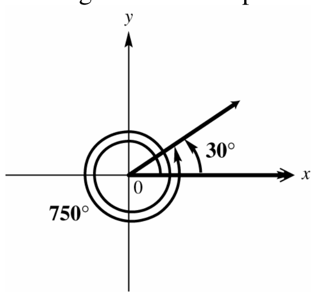
$\tan 570° = \tan 30° = \dfrac{\sqrt{3}}{3}$

$\cot 570° = \cot 30° = \sqrt{3}$

$\sec 570° = -\sec 30° = -\dfrac{2\sqrt{3}}{3}$

$\csc 570° = -\csc 30° = -2$

80. To find the reference angle for $750°$, sketch this angle in standard position.



$750°$ is coterminal with $30°$ because $750° - 2 \cdot 360° = 750° - 720° = 30°$. Since $750°$ lies in quadrant I, the values of all of its trigonometric functions will be positive, so these values will be identical to the trigonometric function values for $30°$.

$\sin 750° = \sin 30° = \frac{1}{2}$

$\cos 750° = \cos 30° = \frac{\sqrt{3}}{2}$

$\tan 750° = \tan 30° = \frac{\sqrt{3}}{3}$

$\cot 750° = \cot 30° = \sqrt{3}$

$\sec 750° = \sec 30° = \frac{2\sqrt{3}}{3}$

$\csc 750° = \csc 30° = 2$

81. 1305° is coterminal with $1305° - 3 \cdot 360° = 1305° = 1080° = 225°$. The reference angle is 225° – 180° = 45°. Since 1305° lies in quadrant III, the sine, cosine, and secant and cosecant are negative.

$\sin 1305° = -\sin 45° = -\frac{\sqrt{2}}{2}$

$\cos 1305° = -\cos 45° = -\frac{\sqrt{2}}{2}$

$\tan 1305° = \tan 45° = 1$

$\cot 1305° = \cot 45° = 1$

$\sec 1305° = -\sec 45° = -\sqrt{2}$

$\csc 1305° = -\csc 45° = -\sqrt{2}$

82. 1500° is coterminal with $1500° - 4 \cdot 360° = 1500° - 1440° = 60°$. Because 1500° lies in quadrant I, the values of all of its trigonometric functions will be positive, so these values will be identical to the trigonometric function values for $60°$.

$\sin(420°) = \sin 60° = \frac{\sqrt{3}}{2}$

$\cos(420°) = \cos 60° = \frac{1}{2}$

$\tan(420°) = \tan 60° = \sqrt{3}$

$\cot(420°) = \cot 60° = \frac{\sqrt{3}}{3}$

$\sec(420°) = \sec 60° = 2$

$\csc(420°) = \csc 60° = \frac{2\sqrt{3}}{3}$

83. $2670°$ is coterminal with $2670° - 7 \cdot 360° = 2670° + 2520° = 150°$. The reference angle is $180° - 150° = 30°$. Since $2670°$ lies in quadrant II, the cosine, tangent, cotangent, and secant are negative.

$\sin 2670° = \sin 30° = \frac{1}{2}$

$\cos 2670° = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\tan 2670° = -\tan 30° = -\frac{\sqrt{3}}{3}$

$\cot 2670° = -\cot 30° = -\sqrt{3}$

$\sec 2670° = -\sec 30° = -\frac{2\sqrt{3}}{3}$

$\csc 2670° = \csc 30° = 2$

84. $-390°$ is coterminal with $-390° + 2 \cdot 360° = -390° + 720° = 330°$. The reference angle is $360° - 330° = 30°$. Since $-390°$ lies in quadrant IV, the sine, tangent, cotangent, and cosecant are negative.

$\sin(-390°) = -\sin 30° = -\frac{1}{2}$

$\cos(-390°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\tan(-390°) = -\tan 30° = -\frac{\sqrt{3}}{3}$

$\cot(-390°) = -\cot 30° = -\sqrt{3}$

$\sec(-390°) = \sec 30° = \frac{2\sqrt{3}}{3}$

$\csc(-390°) = -\csc 30° = -2$

85. $-510°$ is coterminal with $-510°+2\cdot 360° = -510°+720° = 210°$. The reference angle is $210°-180° = 30°$. Since $-510°$ lies in quadrant III, the sine, cosine, and secant and cosecant are negative.

$\sin(-510°) = -\sin 30° = -\frac{1}{2}$

$\cos(-510°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\tan(-510°) = \tan 30° = \frac{\sqrt{3}}{3}$

$\cot(-510°) = \cot 30° = \sqrt{3}$

$\sec(-510°) = -\sec 30° = -\frac{2\sqrt{3}}{3}$

$\csc(-510°) = -\csc 30° = -2$

86. $-1020°$ is coterminal with $-1020°+3\cdot 360° = -1020°+1080° = 60°$. Because $-1020°$ lies in quadrant I, the values of all of its trigonometric functions will be positive, so these values will be identical to the trigonometric function values for $60°$.

$\sin(420°) = \sin 60° = \frac{\sqrt{3}}{2}$

$\cos(420°) = \cos 60° = \frac{1}{2}$

$\tan(420°) = \tan 60° = \sqrt{3}$

$\cot(420°) = \cot 60° = \frac{\sqrt{3}}{3}$

$\sec(420°) = \sec 60° = 2$

$\csc(420°) = \csc 60° = \frac{2\sqrt{3}}{3}$

87. $-1290°$ is coterminal with $-1290°+4\cdot 360° = -1290°+1440° = 150°$. The reference angle is $180°-150° = 30°$. Since $-1290°$ lies in quadrant II, the cosine, tangent, cotangent, and secant are negative.

$\sin 2670° = \sin 30° = \frac{1}{2}$

$\cos 2670° = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\tan 2670° = -\tan 30° = -\frac{\sqrt{3}}{3}$

$\cot 2670° = -\cot 30° = -\sqrt{3}$

$\sec 2670° = -\sec 30° = -\frac{2\sqrt{3}}{3}$

$\csc 2670° = \csc 30° = 2$

88. $-855°$ is coterminal with $-855°+3\cdot 360° = -855°+1080° = 225°$. The reference angle is $225° - 180° = 45°$. Since $-855°$ lies in quadrant III, the sine, cosine, and secant and cosecant are negative.

$\sin(-855°) = -\sin 45° = -\frac{\sqrt{2}}{2}$

$\cos(-855°) = -\cos 45° = -\frac{\sqrt{2}}{2}$

$\tan(-855°) = \tan 45° = 1$

$\cot(-855°) = \cot 45° = 1$

$\sec(-855°) = -\sec 45° = -\sqrt{2}$

$\csc(-855°) = -\csc 45° = -\sqrt{2}$

89. $-1860°$ is coterminal with $-1860°+6\cdot 360° = -1860°+2160° = 300°$. The reference angle is $360° - 300° = 60°$. Since $-1860°$ lies in quadrant IV, the sine, tangent, cotangent, and cosecant are negative.

$\sin(-1860°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

$\cos(-1860°) = \cos 60° = \frac{1}{2}$

$\tan(-1860°) = -\tan 60° = -\sqrt{3}$

$\cot(-1860°) = -\cot 60° = -\frac{\sqrt{3}}{3}$

$\sec(-1860°) = \sec 60° = 2$

$\csc(-1860°) = -\csc 60° = -\frac{2\sqrt{3}}{3}$

90. Since $1305°$ is coterminal with an angle of $1305°-3\cdot 360° = 1305°-1080° = 225°$, it lies in quadrant III. Its reference angle is $225°-180° = 45°$. Since the sine is negative in quadrant III, we have

$\sin 1305° = -\sin 45° = -\frac{\sqrt{2}}{2}.$

91. Since $-510°$ is coterminal with an angle of $-510°+2\cdot 360° = -510°+720° = 210°$, it lies in quadrant III. Its reference angle is $210°-180° = 30°$. Since the cosine is negative in quadrant III, we have

$\cos(-510°) = -\cos 30° = -\frac{\sqrt{3}}{2}.$

92. Since $-1020°$ is coterminal with an angle of $-1020°+3\cdot 360° = -1020°+1080° = 60°$, it lies in quadrant I. Because $-1020°$ lies in quadrant I, the values of all of its trigonometric functions will be positive, so $\tan(-1020°) = \tan 60° = \sqrt{3}.$

93. Since $1500°$ is coterminal with an angle of $1500° - 4 \cdot 360° = 1500° - 1440° = 60°$, it lies in quadrant I. Because $1500°$ lies in quadrant I, the values of all of its trigonometric functions will be positive, so

$$\sin 1500° = \sin 60° = \frac{\sqrt{3}}{2}.$$

94. Since $-495°$ is coterminal with an angle of $-495° + 2 \cdot 360° = -495° + 720° = 225°$, it lies in quadrant III. Its reference angle is $225° - 180° = 45°$. Since the secant is negative in quadrant III, we have $\sec(-495°) = -\sec 45° = -\sqrt{2}$.

95. Since $-855°$ is coterminal with $-855° + 3 \cdot 360° = -855° + 1080° = 225°$, it lies in quadrant III. Its reference angle is $225° - 180° = 45°$. Since the cosecant is negative in quadrant III, we have. $\csc(-855°) = -\csc 45° = -\sqrt{2}$

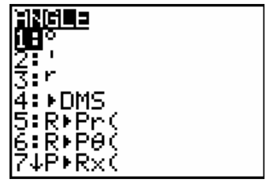
96. Since $2280°$ is coterminal with $2280° - 6 \cdot 360° = 2280° - 2160° = 120°$, it lies in quadrant II. Its reference angle is $180° - 120° = 60°$. Since the cotangent is negative in quadrant II, we have

$$\cot 2280° = -\cot 60° = -\frac{\sqrt{3}}{3}.$$

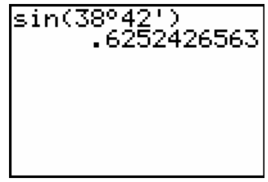
97. Since $3015°$ is coterminal with $3015° - 8 \cdot 360° = 3015° - 2880° = 135°$, it lies in quadrant II. Its reference angle is $180° - 135° = 45°$. Since the tangent is negative in quadrant II, we have $\tan 3015° = -\tan 45° = -1$.

98. The CAUTION on page 521 suggests verifying that a calculator is in degree mode by finding sin 90°. If the calculator is in degree mode, the display should be 1.

For Exercises 99–112, be sure your calculator is in degree mode. If your calculator accepts angles in degrees, minutes, and seconds, it is not necessary to change angles to decimal degrees. Keystroke sequences may vary on the type and/or model of calculator being used. Screens shown will be from a TI-83 Plus calculator. To obtain the degree (°) and (′) symbols, go to the ANGLE menu (2nd APPS). The calculation for decimal degrees is indicated for calculators that do not accept degree, minutes, and seconds.

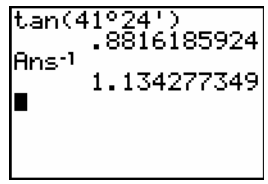


99. $\sin 38°42' \approx .6252427$



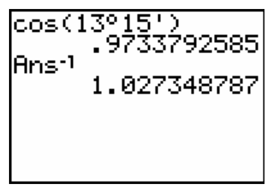
$38°42' = \left(38 + \frac{42}{60}\right)° = 38.7°$

100. $\cot 41°24' \approx 1.1342773$



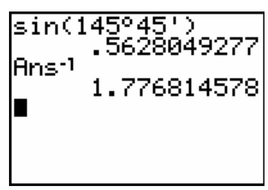
$42°24' = \left(42 + \frac{24}{60}\right)° = 41.4°$

101. $\sec 13°15' \approx 1.0273488$



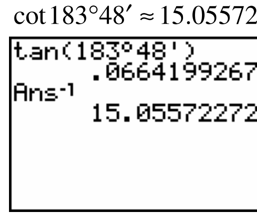
$13°15' = \left(13 + \frac{15}{60}\right)° = 13.25°$

102. $\csc 145°45' \approx 1.7768146$



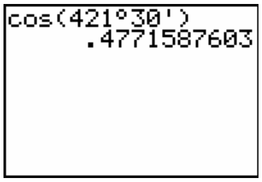
$145°45' = \left(145 + \frac{45}{60}\right)° = 145.75°$

103. $\cot 183°48' \approx 15.055723$



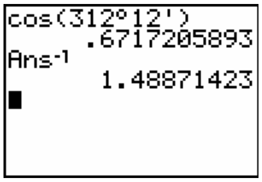
$183°48' = \left(183 + \frac{48}{60}\right)° = 183.8°$

104. $\cos 421°30' \approx .4771588$



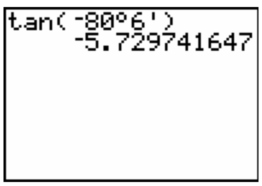
$421°30' = \left(421 + \frac{30}{60}\right)° = 421.5°$

105. $\sec 312°12' \approx 1.4887142$



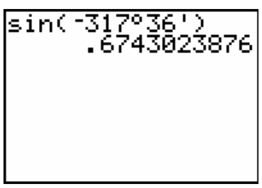
$312°12' = \left(312 + \frac{12}{60}\right)° = 312.2°$

106. $\tan(-80°6') \approx -5.7297416$



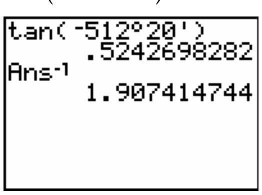
$(-80°6') = -\left(80 + \frac{6}{60}\right)° = -80.1°$

107. $\sin(-317°36') \approx .6743024$



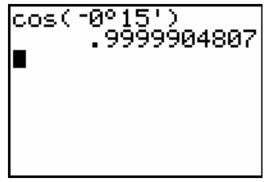
$-317°36' = -\left(317 + \frac{36}{60}\right)° = -317.6°$

108. $\cot(-512°20') \approx 1.9074147$



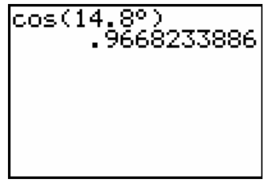
$-512°20' = -\left(512 + \frac{20}{60}\right)° = -512.333333°$

109. $\cos(-15') \approx .9999905$

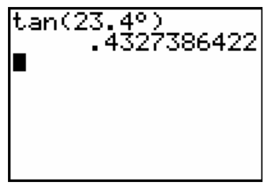


$-15' = -\left(\frac{15}{60}\right)° = -.25°$

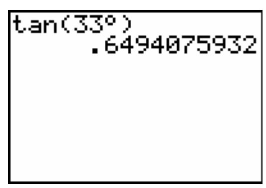
110. $\frac{1}{\sec 14.8°} = \cos 14.8° \approx .9668234$



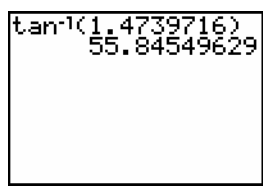
111. $\frac{1}{\cot 23.4°} = \tan 23.4° \approx .9668234$



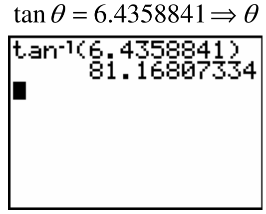
112. $\frac{\sin 33°}{\cos 33°} = \tan 33° \approx .6494076$



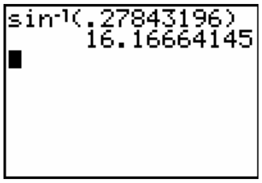
113. $\tan\theta = 1.4739716 \Rightarrow \theta \approx 55.845496°$



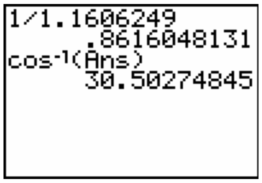
114. $\tan\theta = 6.4358841 \Rightarrow \theta \approx 81.168073°$



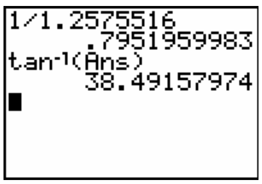
115. $\sin\theta = .27843196 \Rightarrow \theta \approx 16.166641°$



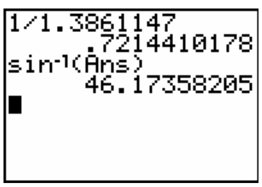
116. $\sec\theta = 1.1606249 \Rightarrow \theta \approx 30.502748°$



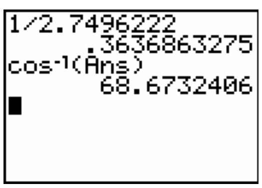
117. $\cot\theta = 1.2575516 \Rightarrow \theta \approx 38.491580°$



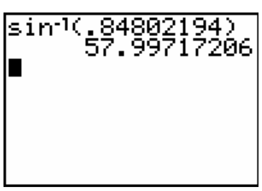
118. $\csc\theta = 1.3861147 \Rightarrow \theta \approx 46.173582°$



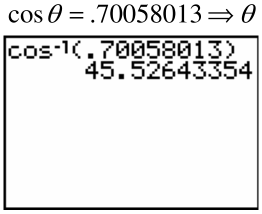
119. $\sec\theta = 2.7496222 \Rightarrow \theta \approx 68.673241°$



120. $\sin\theta = .84802194 \Rightarrow \theta \approx 57.997172°$



121. $\cos\theta = .70058013 \Rightarrow \theta \approx 45.526434°$



122. The range for $\sin\theta$ is $[-1, 1]$, so there is no value of θ for which $\sin\theta = 2$.

123. $\sin\theta = \frac{1}{2}$

Since $\sin\theta$ is positive, θ must lie in quadrants I or II. Since one angle, namely $30°$, lies in quadrant I, that angle is also the reference angle, θ'. The angle in quadrant II will be $180° - \theta' = 180° - 30° = 150°$.

124. $\cos\theta = \frac{\sqrt{3}}{2}$

Since $\cos\theta$ is positive, θ must lie in quadrants I or IV. Since one angle, namely $30°$, lies in quadrant I, that angle is also the reference angle, θ'. The angle in quadrant IV will be $360° - \theta' = 360° - 30° = 330°$.

125. $\tan\theta = -\sqrt{3}$

Since $\tan\theta$ is negative, θ must lie in quadrants II or IV. Since the absolute value of $\tan\theta$ is $\sqrt{3}$, the reference angle, θ' must be $60°$. The quadrant II angle θ equals $180° - \theta' = 180° - 60° = 120°$, and the quadrant IV angle θ equals $360° - \theta' = 360° - 60° = 300°$.

126. $\sec\theta = -\sqrt{2}$

Since $\sec\theta$ is negative, θ must lie in quadrants II or III. Since the absolute value of $\sec\theta$ is $\sqrt{2}$, the reference angle, θ' must be $45°$. The quadrant II angle θ equals $180° - \theta' = 180° - 45° = 135°$, and the quadrant III angle θ equals $180° + \theta' = 180° + 45° = 225°$.

127. $\cos\theta = \frac{\sqrt{2}}{2}$

Since $\cos\theta$ is positive, θ must lie in quadrants I or IV. Since one angle, namely $45°$, lies in quadrant I, that angle is also the reference angle, θ'. The angle in quadrant IV will be $360° - \theta' = 360° - 45° = 315°$.

128. $\cot\theta = -\frac{\sqrt{3}}{3}$
Since $\cot\theta$ is negative, θ must lie in quadrants II or IV. Since the absolute value of $\cot\theta$ is $\frac{\sqrt{3}}{3}$, the reference angle, θ' must be $60°$. The quadrant II angle θ equals $180° - \theta' = 180° - 60° = 120°$, and the quadrant IV angle θ equals $360° - \theta' = 360° - 60° = 300°$.

129. $\csc\theta = -2$
Since $\csc\theta$ is negative, θ must lie in quadrants III or IV. The absolute value of $\csc\theta$ is 2, so the reference angle, θ', is $30°$. The angle in quadrant III will be $180° + \theta' = 180° + 30° = 210°$, and the quadrant IV angle is $360° - \theta' = 360° - 30° = 330°$.

130. $\sin\theta = -\frac{\sqrt{3}}{2}$
Since $\sin\theta$ is negative, θ must lie in quadrants III or IV. The absolute value of $\sin\theta$ is $\frac{\sqrt{3}}{2}$, so the reference angle, θ', is $60°$. The angle in quadrant III will be $180° + \theta' = 180° + 60° = 240°$, and the quadrant IV angle is $360° - \theta' = 360° - 60° = 300°$.

131. $\tan\theta = \frac{\sqrt{3}}{3}$
Since $\tan\theta$ is positive, θ must lie in quadrants I or III. Since one angle, namely $30°$, lies in quadrant I, that angle is also the reference angle, θ'. The angle in quadrant III will be $180° + \theta' = 180° + 30° = 210°$.

132. $\cos\theta = -\frac{1}{2}$
Since $\cos\theta$ is negative, θ must lie in quadrants II or III. Since the absolute value of $\cos\theta$ is $\frac{1}{2}$, the reference angle, θ' must be $60°$. The quadrant II angle θ equals $180° - \theta' = 180° - 60° = 120°$, and the quadrant III angle θ equals $180° + \theta' = 180° + 60° = 240°$.

133. $\csc\theta = -\sqrt{2}$
Since $\csc\theta$ is negative, θ must lie in quadrants III or IV. Since the absolute value of $\csc\theta$ is $\sqrt{2}$, the reference angle, θ' must be $45°$. The quadrant III angle θ equals $180° + \theta' = 180° + 45° = 225°$. and the quadrant IV angle θ equals $360° - \theta' = 360° - 45° = 315°$.

134. $\cot\theta = -1$
Since $\cot\theta$ is negative, θ must lie in quadrants II or IV. Since the absolute value of $\cot\theta$ is 1 the reference angle, θ' must be $45°$. The quadrant II angle θ equals $180° - \theta' = 180° - 45° = 135°$. and the quadrant IV angle θ equals $360° - \theta' = 360° - 45° = 315°$.

135. For Auto A, calculate $70 \cdot \cos 10° \approx 68.94$.
Auto A's reading is approximately 68.94 mph.
For Auto B, calculate $70 \cdot \cos 20° \approx 65.78$.
Auto B's reading is approximately 65.78 mph.

136. $F = W\sin\theta$
$F = 2400 \sin\left(-2.4°\right) \approx -100.5$ lb
F is negative because the car is traveling downhill.

137. $F = W\sin\theta$
$F = 2100\sin 1.8° \approx 65.96$ lb

138. $F = W\sin\theta$
$-145 = W\sin(-3°) \Rightarrow \frac{-145}{\sin(-3°)} = W \Rightarrow$
$W \approx 2771$ lb

139. $F = W\sin\theta$
$-130 = 2600\sin\theta \Rightarrow \frac{-130}{2600} = \sin\theta \Rightarrow$
$-.05 = \sin\theta \Rightarrow \theta = \sin^{-1}\left(-.05\right) \approx -2.87°$

140. $F = W\sin\theta$
$F = 2200\sin 2° \approx 76.77889275$ lb
$F = 2000\sin 2.2° \approx 76.77561818$ lb
The 2200-lb car on a 2° uphill grade has the greater grade resistance.

141. $R = \dfrac{V^2}{g(f + \tan\theta)}$

(a) Since 45 mph = 66 ft/sec,
$V = 66$, $\theta = 3°$, $g = 32.2$, and $f = .14$, we have

$$R = \frac{V^2}{g(f+\tan\theta)} = \frac{66^2}{32.2(.14 + \tan 3°)} \approx 703 \text{ ft}$$

(b) Since there are 5280 ft in one mile and 3600 sec in one min, we have

$$70 \text{ mph} = 70 \text{ mph} \cdot \frac{1 \text{ hr}}{3600 \text{ sec}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} = 102\frac{2}{3} \text{ ft per sec} \approx 102.67 \text{ ft per sec}$$

Since $V = 102.67$, $\theta = 3°$, $g = 32.2$, and $f = .14$, we have

$$R = \frac{V^2}{g(f+\tan\theta)} \approx \frac{102.67^2}{32.2(.14 + \tan 3°)} \approx 1701 \text{ ft}$$

(c) Intuitively, increasing θ would make it easier to negotiate the curve at a higher speed much like is done at a race track. Mathematically, a larger value of θ (acute) will lead to a larger value for $\tan\theta$. If $\tan\theta$ increases, then the ratio determining R will *decrease*. Thus, the radius can be smaller and the curve sharper if θ is increased.

$$R = \frac{V^2}{g(f+\tan\theta)} = \frac{66^2}{32.2(.14 + \tan 4°)} \approx 644 \text{ ft}$$

$$R = \frac{V^2}{g(f+\tan\theta)} \approx \frac{102.67^2}{32.2(.14 + \tan 4°)} \approx 1559 \text{ ft}$$

As predicted, both values are less.

142. From Exercise 113, $R = \dfrac{V^2}{g(f + \tan\theta)}$.

Solving for V we have

$$R = \frac{V^2}{g(f+\tan\theta)} \Rightarrow V^2 = Rg(f + \tan\theta) \Rightarrow V = \sqrt{Rg(f+\tan\theta)}$$

Since $R = 1150$, $\theta = 2.1°$, $g = 32.2$, and $f = .14$, we have

$$V = \sqrt{Rg(f+\tan\theta)} = \sqrt{1150(32.2)(.14 + \tan 2.1°)} \approx 80.9 \text{ ft/sec}$$

80.9 ft/sec · 3600 sec/hr · 1 mi/5280 ft ≈ 55 mph, so it should have a 55 mph speed limit

Chapter 5 Quiz
(Sections 5.1–5.3)

1. 19°

(a) $90° - 19° = 71°$

(b) $180° - 19° = 161°$

2. The two angles form a straight angle.

$$(3x+5)+(5x+15) = 180 \Rightarrow 8x + 20 = 180 \Rightarrow 8x = 160 \Rightarrow x = 20$$

The measures of the two angles are $(3x+5)° = [3(20)+5]° = 65°$ and $(5x+15)° = [5(20)+15]° = 115°$.

3. The two angles form a right angle.

$$(5x-1) + 2x = 90 \Rightarrow 7x - 1 = 90 \Rightarrow 7x = 91 \Rightarrow x = 13$$

The measures of the two angles are $(5x-1)° = [5(13)-1] = 64°$ and $2x = 2(13) = 26°$.

4. (a) $77°12'09'' = 77° + \frac{12}{60}° + \frac{9}{3600}° = 77° + .2° + .0025° = 77.2025°$

(b) $22.0250° = 22° + .0250(60') = 22° + 1.5' = 22° + 01' + .5(60'') = 22° + 01' + 30'' = 22°1'30''$

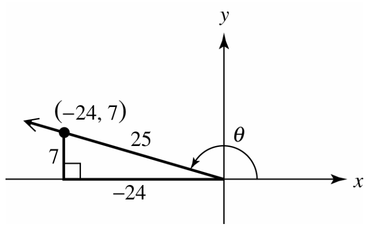
5. (a) 410° is coterminal with 410° − 360° = 50°.

(b) −60° is coterminal with −60° + 360° = 300°.

(c) 890° is coterminal with 890° − 2(360°) = 890° − 720° = 170°.

(d) 57º is coterminal with 57º + 360º = 417º.

6. 300 rotations per min = $\frac{300}{60} = 5$ rotations per sec = 5(360°) per sec = 1800° per sec
The edge of the flywheel will move 1800° in 1 second.

7. 

$x = -24$, $y = 7$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-24)^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25$$

$$\sin\theta = \frac{y}{r} = \frac{7}{25} = \frac{7}{25}$$
$$\cos\theta = \frac{x}{r} = \frac{-24}{25} = -\frac{24}{25}$$
$$\tan\theta = \frac{y}{x} = \frac{7}{-24} = -\frac{7}{24}$$
$$\cot\theta = \frac{x}{y} = \frac{-24}{7} = -\frac{24}{7}$$
$$\sec\theta = \frac{r}{x} = \frac{25}{-24} = -\frac{25}{24}$$
$$\csc\theta = \frac{r}{y} = \frac{25}{7} = \frac{25}{7}$$

8. $\sin A = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{24}{40} = \dfrac{3}{5}$

$\cos A = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{32}{40} = \dfrac{4}{5}$

$\tan A = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{24}{32} = \dfrac{3}{4}$

9.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

10. $\sin 30° = \dfrac{w}{36} \Rightarrow w = 36\sin 30° = 36 \cdot \dfrac{1}{2} = 18$

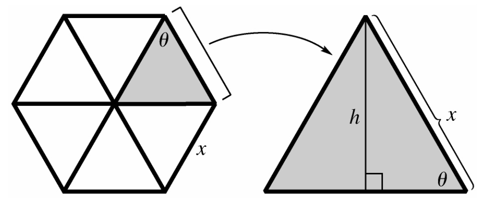
$\cos 30° = \dfrac{x}{36} \Rightarrow x = 36\cos 30° = 36 \cdot \dfrac{\sqrt{3}}{2} = 18\sqrt{3}$

$\tan 45° = \dfrac{w}{y} \Rightarrow 1 = \dfrac{18}{y} \Rightarrow y = 18$

$\sin 45° = \dfrac{w}{z} \Rightarrow \dfrac{\sqrt{2}}{2} = \dfrac{18}{z} \Rightarrow z = \dfrac{36}{\sqrt{2}} = 18\sqrt{2}$

11. The height of one of the six equilateral triangles from the solar cell is

$\sin\theta = \dfrac{h}{x} \Rightarrow h = x\sin\theta$.



Thus, the area of each of the triangles is $A = \frac{1}{2}bh = \frac{1}{2}x^2\sin\theta$. So, the area of the solar cell is $A = 6 \cdot \frac{1}{2}x^2\sin\theta = 3x^2\sin\theta$.

12. 180º − 135º = 45º, so the reference angle is 45º. The original angle (135º) lies in quadrant II, so the sine and cosecant are positive, while the remaining trigonometric functions are negative.

$\sin 135° = \dfrac{\sqrt{2}}{2}$; $\cos 135° = -\dfrac{\sqrt{2}}{2}$

$\tan 135° = -1$; $\cot 135° = -1$

$\sec 135° = -\sqrt{2}$; $\csc 135° = \sqrt{2}$

13. −150º is coterminal with 360º − 150º = 210º. Since this lies in quadrant III, the reference angle is 210º − 180º = 30º. In quadrant III, the tangent and cotangent functions are positive, while the remaining trigonometric functions are negative.

$\sin(-150°) = -\sin 30° = -\dfrac{1}{2}$

$\cos(-150°) = -\cos 30° = -\dfrac{\sqrt{3}}{2}$

$\tan(-150°) = \tan 30° = \dfrac{\sqrt{3}}{3}$

$\cot(-150°) = \cot 30° = \sqrt{3}$

$\sec(-150°) = -\sec 30° = -\dfrac{2\sqrt{3}}{3}$

$\csc(-150°) = -\csc 30° = -2$

14. 1020º is coterminal with 1020º − 720º = 300º. Since this lies in quadrant IV, the reference angle is 360º − 300º = 60º. In quadrant IV, the cosine and secant are positive, while the remaining trigonometric functions are negative.

$\sin 1020° = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

$\cos 1020° = \cos 60° = \dfrac{1}{2}$

$\tan 1020° = -\tan 60° = -\sqrt{3}$

$\cot 1020° = -\cot 60° = -\frac{\sqrt{3}}{3}$

$\sec 1020° = \sec 60° = 2$

$\csc 1020° = -\csc 60° = -\frac{2\sqrt{3}}{3}$

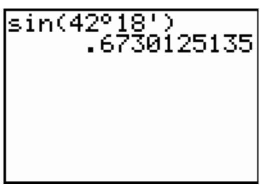
15. $\sin\theta = \frac{\sqrt{3}}{2}$

Since $\sin\theta$ is positive, θ must lie in quadrants I or II, and the reference angle, θ', is $60°$. The angle in quadrant I is 60º, while the angle in quadrant II is $180° - \theta' = 180° - 60° = 120°$.

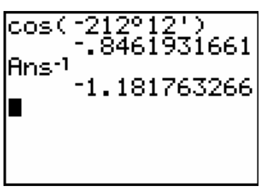
16. $\sec\theta = -\sqrt{2}$

Since $\sec\theta$ is negative, θ must lie in quadrants II or III. Since the absolute value of $\sec\theta$ is $\sqrt{2}$, the reference angle, θ' must be $45°$. The quadrant II angle θ equals $180° - \theta' = 180° - 45° = 135°$, and the quadrant III angle θ equals $180° + \theta' = 180° + 45° = 225°$.

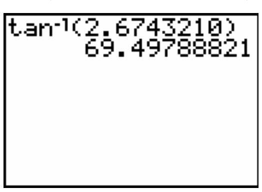
17. $\sin 42°18' \approx .67301251$



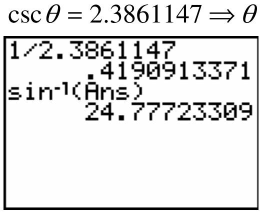
18. $\sec(-212°12') \approx -1.18176327$



19. $\tan\theta = 2.6743210 \Rightarrow \theta \approx 69.497888°$



20. $\csc\theta = 2.3861147 \Rightarrow \theta \approx 24.777233°$



Section 5.4: Solving Right Triangles

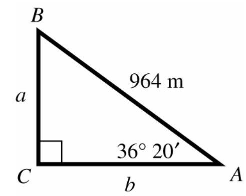
1. 22,894.5 to 22,895.5

2. 28,999.5 to 29,000.5

3. 8958.5 to 8959.5

4. Answers will vary.
No; the number of points scored will be a whole number.

5. $A = 36°20'$, $c = 964$ m

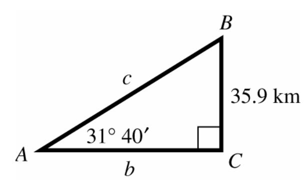


$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$
$B = 90° - 36°20'$
$= 89°60' - 36°20' = 53°40'$

$\sin A = \frac{a}{c} \Rightarrow \sin 36°20' = \frac{a}{964} \Rightarrow$
$a = 964 \sin 36°20' \approx 571$ m (rounded to three significant digits)

$\cos A = \frac{b}{c} \Rightarrow \cos 36°20' = \frac{b}{964} \Rightarrow$
$b = 964 \cos 36°20' \approx 777$ m (rounded to three significant digits)

6. $A = 31°40'$, $a = 35.9$ km

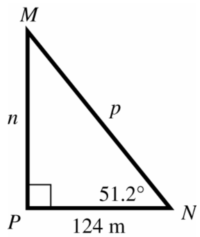


$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$
$B = 90° - 31°40'$
$= 89°60' - 31°40' = 58°20'$

$\sin A = \frac{a}{c} \Rightarrow \sin 31°40' = \frac{35.9}{c} \Rightarrow$
$c = \frac{35.9}{\sin 31°40'} \approx 68.4$ km (rounded to three significant digits)

$\tan A = \frac{a}{b} \Rightarrow \tan 31°40' = \frac{35.9}{b} \Rightarrow$
$b = \frac{35.9}{\tan 31°40'} \approx 58.2$ km (rounded to three significant digits)

7. $N = 51.2°$, $m = 124$ m

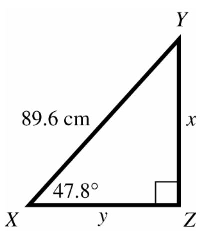


$M + N = 90° \Rightarrow M = 90° - N \Rightarrow$
$M = 90° - 51.2° = 38.8°$

$\tan N = \frac{n}{m} \Rightarrow \tan 51.2° = \frac{n}{124} \Rightarrow$
$n = 124 \tan 51.2° \approx 154$ m (rounded to three significant digits)

$\cos N = \frac{m}{p} \Rightarrow \cos 51.2° = \frac{124}{p} \Rightarrow$
$p = \frac{124}{\cos 51.2°} \approx 198$ m (rounded to three significant digits)

8. $X = 47.8°$, $z = 89.6$ cm

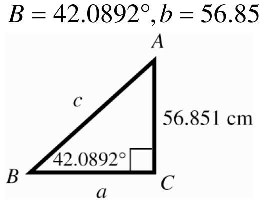


$Y + X = 90° \Rightarrow Y = 90° - X \Rightarrow$
$Y = 90° - 47.8° = 42.2°$

$\sin X = \frac{x}{z} \Rightarrow \sin 47.8° = \frac{x}{89.6} \Rightarrow$
$x = 89.6 \sin 47.8° \approx 66.4$ cm (rounded to three significant digits)

$\cos X = \frac{y}{z} \Rightarrow \cos 47.8° = \frac{y}{89.6} \Rightarrow$
$y = 89.6 \cos 47.8° \approx 60.2$ cm (rounded to three significant digits)

9. $B = 42.0892°, b = 56.851$

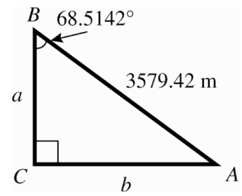


$A + B = 90° \Rightarrow A = 90° - B \Rightarrow$
$A = 90° - 42.0892° = 47.9108°$

$\sin B = \frac{b}{c} \Rightarrow \sin 42.0892° = \frac{56.851}{c} \Rightarrow$
$c = \frac{56.851}{\sin 42.0892°} \approx 84.816$ cm
(rounded to five significant digits)

$\tan B = \frac{b}{a} \Rightarrow \tan 42.0892° = \frac{56.851}{a} \Rightarrow$
$a = \frac{56.851}{\tan 42.0892°} \approx 62.942$ cm
(rounded to five significant digits)

10. $B = 68.5142°, c = 3579.42$

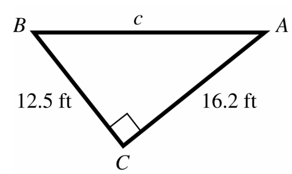


$A + B = 90° \Rightarrow A = 90° - B \Rightarrow$
$A = 90° - 68.5142° = 21.4858°$

$\sin B = \frac{b}{c} \Rightarrow \sin 68.5142° = \frac{b}{3579.42}$
$b = 3579.42 \sin 68.5142° \approx 3330.68$ m
(rounded to six significant digits)

$\cos B = \frac{a}{c} \Rightarrow \cos 68.5142° = \frac{a}{3579.42} \Rightarrow$
$a = 3579.42 \cos 68.5142° \approx 1311.04$ m
(rounded to six significant digits)

11. $a = 12.5, b = 16.2$

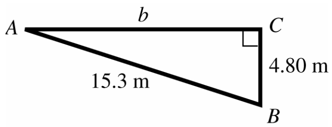


Using the Pythagorean theorem, we have
$a^2 + b^2 = c^2 \Rightarrow 12.5^2 + 16.2^2 = c^2 \Rightarrow$
$418.69 = c^2 \Rightarrow c \approx 20.5$ ft (rounded to three significant digits)

$\tan A = \frac{a}{b} \Rightarrow \tan A = \frac{12.5}{16.2} \Rightarrow$
$A = \tan^{-1} \frac{12.5}{16.2} \approx 37.6540°$
$\approx 37° + (.6540 \cdot 60)' \approx 37°39' \approx 37°40'$
(rounded to three significant digits)

$\tan B = \frac{b}{a} \Rightarrow \tan B = \frac{16.2}{12.5} \Rightarrow$
$B = \tan^{-1} \frac{16.2}{12.5} \approx 52.3460°$
$\approx 52° + (.3460 \cdot 60)' \approx 52°21'$
$\approx 52°20'$ (rounded to three significant digits)

12. $a = 4.80, c = 15.3$



Using the Pythagorean theorem, we have

$$a^2 + b^2 = c^2 \Rightarrow 4.80^2 + b^2 = 15.3^2 \Rightarrow$$

$$4.80^2 + b^2 = 15.3^2$$

$$b^2 = 15.3^2 - 4.80^2 = 211.05$$

$b \approx 14.5$ m (rounded to three significant digits)

$$\sin A = \frac{a}{b} \Rightarrow \sin A = \frac{4.80}{15.3} \Rightarrow$$

$$A = \sin^{-1}\frac{4.80}{15.3} \approx 18.2839°$$

$$\approx 18° + (.2839 \cdot 60)' \approx 18°17' \approx 18°20'$$

(rounded to three significant digits)

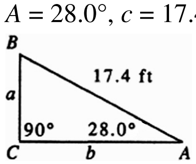
$$\cos B = \frac{a}{c} \Rightarrow \cos B = \frac{4.80}{15.3} \Rightarrow$$

$$B = \cos^{-1}\frac{4.80}{15.3} \approx 71.7161°$$

$$\approx 71° + (.7161 \cdot 60)' \approx 71°43' \approx 71°40'$$

(rounded to three significant digits)

13. $A = 28.0°$, $c = 17.4$ ft



$$A + B = 90°$$

$$B = 90° - A$$

$$B = 90° - 28.0° = 62.0°$$

$$\sin A = \frac{a}{c} \Rightarrow \sin 28.0° = \frac{a}{17.4} \Rightarrow$$

$a = 17.4 \sin 28.0° \approx 8.17$ ft (rounded to three significant digits)

$$\cos A = \frac{b}{c} \Rightarrow \cos 28.00° = \frac{b}{17.4} \Rightarrow$$

$b = 17.4 \cos 28.00° \approx 15.4$ ft (rounded to three significant digits)

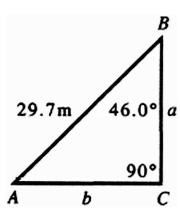
14. $B = 46.0°$, $c = 29.7$ m

$$A + B = 90°$$

$$A = 90° - B$$

$$A = 90° - 46.0°$$

$$= 44.0°$$



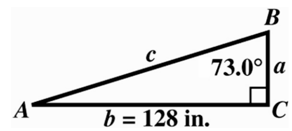
$$\cos B = \frac{a}{c} \Rightarrow \cos 46.0° = \frac{a}{29.7} \Rightarrow$$

$a = 29.7 \cos 46.0° \approx 20.6$ m (rounded to three significant digits)

$$\sin B = \frac{b}{c} \Rightarrow \sin 46.0° = \frac{b}{29.7} \Rightarrow$$

$b = 29.7 \sin 46.0° \approx 21.4$ m (rounded to three significant digits)

15. Solve the right triangle with $B = 73.0°$, $b = 128$ in. and $C = 90°$



$$A + B = 90° \Rightarrow A = 90° - B \Rightarrow$$

$$A = 90° - 73.0° = 17.0°$$

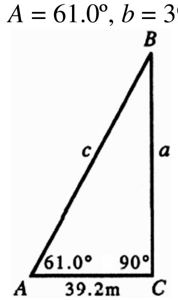
$$\tan B° = \frac{b}{a} \Rightarrow \tan 73.0° = \frac{128}{a} \Rightarrow$$

$a = \frac{128}{\tan 73.0°} \Rightarrow a = 39.1$ in (rounded to three significant digits)

$$\sin B° = \frac{b}{c} \Rightarrow \sin 73.0° = \frac{128}{c} \Rightarrow$$

$c = \frac{128}{\sin 73.0°} \Rightarrow c = 134$ in (rounded to three significant digits)

16. $A = 61.0°$, $b = 39.2$ cm



$$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$$

$$B = 90° - 61.0° = 29.0°$$

$$\tan A = \frac{a}{b} \Rightarrow \tan 61.0° = \frac{a}{39.2} \Rightarrow$$

$a = 39.2 \tan 61.0 \approx 70.7$ cm

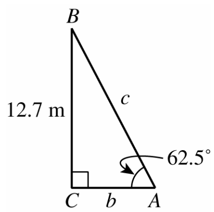
(rounded to three significant digits)

$$\cos A = \frac{b}{c} \Rightarrow \cos 61.0° = \frac{39.2}{c} \Rightarrow$$

$c = \frac{39.2}{\cos 61.0°} \approx 80.9$ cm

(rounded to three significant digits)

17. $A = 62.5°$, $a = 12.7$ m



$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$
$B = 90° - 62.5° = 27.5°$

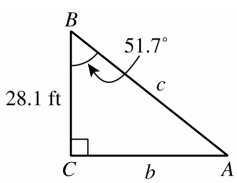
$\tan A = \frac{a}{b} \Rightarrow \tan 62.5° = \frac{12.7}{b} \Rightarrow$

$b = \frac{12.7}{\tan 62.5°} \approx 6.61$ m (rounded to three significant digits)

$\sin A = \frac{a}{c} \Rightarrow \sin 62.5° = \frac{12.7}{c} \Rightarrow$

$c = \frac{12.7}{\sin 62.5°} \approx 14.3$ m (rounded to three significant digits)

18. $B = 51.7°$, $a = 28.1$ ft



$A + B = 90° \Rightarrow B = 90° - B \Rightarrow$
$A = 90° - 51.7° = 38.3°$

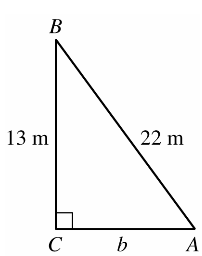
$\tan B = \frac{b}{a} \Rightarrow \tan 51.7° = \frac{b}{28.1} \Rightarrow$

$b = 28.1 \tan 51.7° \approx 35.6$ ft (rounded to three significant digits)

$\cos B = \frac{a}{c} \Rightarrow \cos 51.7° = \frac{28.1}{c} \Rightarrow$

$c = \frac{28.1}{\cos 51.7°} \approx 45.3$ ft (rounded to three significant digits)

19. $a = 13$ m, $c = 22$ m



$c^2 = a^2 + b^2 \Rightarrow 22^2 = 13^2 + b^2 \Rightarrow$
$484 = 169 + b^2 \Rightarrow 315 = b^2 \Rightarrow b \approx 18$ m
(rounded to two significant digits)

We will determine the measurements of both A and B by using the sides of the right triangle. In practice, once you find one of the measurements, subtract it from $90°$ to find the other.

$\sin A = \frac{a}{c} \Rightarrow \sin A = \frac{13}{22} \Rightarrow$

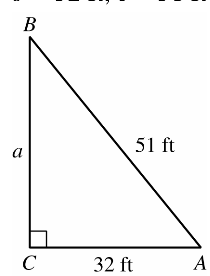
$A \approx \sin^{-1}\left(\frac{13}{22}\right) \approx 36.2215° \approx 36°$ (rounded to two significant digits)

$\cos B = \frac{b}{c} \Rightarrow \cos B = \frac{13}{22} \Rightarrow$

$B \approx \cos^{-1}\left(\frac{13}{22}\right) \approx 53.7784° \approx 54°$

(rounded to two significant digits)

20. $b = 32$ ft, $c = 51$ ft



$c^2 = a^2 + b^2 \Rightarrow 51^2 = a^2 + 32^2 \Rightarrow$
$2601 = a^2 + 1024 \Rightarrow 1577 = a^2 \Rightarrow a \approx 40$ ft
(rounded to two significant digits)

We will determine the measurements of both A and B by using the sides of the right triangle. In practice, once you find one of the measurements, subtract it from $90°$ to find the other.

$\cos A = \frac{b}{c} \Rightarrow \cos A = \frac{32}{51} \Rightarrow$

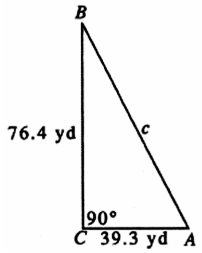
$A \approx \cos^{-1}\left(\frac{32}{51}\right) \approx 51.1377° \approx 51°$

(rounded to two significant digits)

$\sin B = \frac{b}{c} \Rightarrow \sin B = \frac{32}{51} \Rightarrow$

$B \approx \sin^{-1}\left(\frac{32}{51}\right) \approx 38.8623° \approx 39°$ (rounded to two significant digits)

21. $a = 76.4$ yd, $b = 39.3$ yd



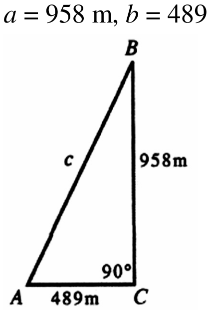
$$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2 + b^2}$$
$$= \sqrt{(76.4)^2 + (39.3)^2} = \sqrt{5836.96 + 1544.49}$$
$$= \sqrt{7381.45} \approx 85.9 \text{ yd}$$ (rounded to three significant digits)

We will determine the measurements of both A and B by using the sides of the right triangle. In practice, once you find one of the measurements, subtract it from $90°$ to find the other.

$$\tan A = \frac{a}{b} \Rightarrow \tan A = \frac{76.4}{39.3} \Rightarrow$$
$$A \approx \tan^{-1}\left(\frac{76.4}{39.3}\right) \approx 62.7788°$$
$$\approx 62° + (.7788 \cdot 60)' \approx 62°47' \approx 62°50'$$
(rounded to three significant digits)

$$\tan B = \frac{b}{a} \Rightarrow \tan B = \frac{39.3}{76.4} \Rightarrow$$
$$B \approx \tan^{-1}\left(\frac{39.3}{76.4}\right) \approx 27.2212°$$
$$\approx 27° + (.2212 \cdot 60)' \approx 27°13' \approx 27°10'$$
(rounded to three significant digits)

22. $a = 958$ m, $b = 489$ m



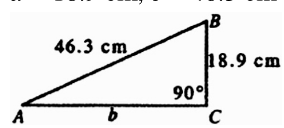
$$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{958^2 + 489^2}$$
$$= \sqrt{917{,}764 + 239{,}121} = \sqrt{1{,}156{,}885}$$
$$\approx 1075.565887 \approx 1080$$ (rounded to three significant digits)

We will determine the measurements of both A and B by using the sides of the right triangle. In practice, once you find one of the measurements, subtract it from $90°$ to find the other.

$$\tan A = \frac{a}{b} \Rightarrow \tan A = \frac{958}{489} \Rightarrow$$
$$A \approx \tan^{-1}\left(\frac{958}{489}\right) \approx 62.9585°$$
$$\approx 63° + (.9585 \cdot 60)' \approx 62°58' \approx 63°00'$$
(rounded to three significant digits)

$$\tan B = \frac{b}{a} \Rightarrow \tan B = \frac{489}{958} \Rightarrow$$
$$B \approx \tan^{-1}\left(\frac{489}{958}\right) \approx 27.0415°$$
$$\approx 27° + (.0415 \cdot 60)' \approx 27°02' \approx 27°00'$$
(rounded to three significant digits)

23. $a = 18.9$ cm, $c = 46.3$ cm

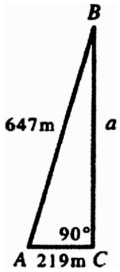


$$c^2 = a^2 + b^2 \Rightarrow 46.3^2 = 18.9^2 + b^2 \Rightarrow$$
$$2143.69 = 357.21 + b^2 \Rightarrow 1786.48 = b^2 \Rightarrow$$
$b \approx 42.3$ cm (rounded to three significant digits)

$$\sin A = \frac{a}{c} \Rightarrow \sin A = \frac{18.9}{46.3} \Rightarrow$$
$$A \approx \sin^{-1}\left(\frac{18.9}{46.3}\right) \approx 24.09227°$$
$$\approx 24° + (.09227 \cdot 60)' \approx 24°06' \approx 24°10'$$
(rounded to three significant digits)

$$\cos B = \frac{a}{c} \Rightarrow \cos B = \frac{18.9}{46.3} \Rightarrow$$
$$B \approx \cos^{-1}\left(\frac{18.9}{46.3}\right) \approx 65.9077°$$
$$\approx 65° + (.9077 \cdot 60)' \approx 65°54' \approx 65°50'$$
(rounded to three significant digits)

24. $b = 219$ cm, $c = 647$ m



$$\begin{aligned} c^2 &= a^2 + b^2 \\ 647^2 &= a^2 + 219^2 \\ 418{,}609 &= a^2 + 47{,}961 \\ 370{,}648 &= a^2 \\ a &\approx 609 \text{ m} \end{aligned}$$

(rounded to three significant digits)

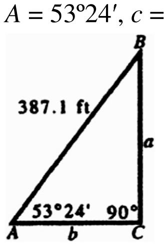
$$\cos A = \frac{b}{c} \Rightarrow \cos A = \frac{219}{647} \Rightarrow$$
$$A \approx \cos^{-1}\left(\frac{219}{647}\right) \approx 70.2154°$$
$$\approx 70° + (.2154 \cdot 60)' \approx 70°13' \approx 70°10'$$

(rounded to three significant digits)

$$\sin B = \frac{b}{c} \Rightarrow \sin B = \frac{219}{647} \Rightarrow$$
$$B \approx \sin^{-1}\left(\frac{219}{647}\right) \approx 19.7846°$$
$$\approx 19° + (.7846 \cdot 60)' \approx 19°47' \approx 19°50'$$

(rounded to three significant digits)

25. $A = 53°24'$, $c = 387.1$ ft

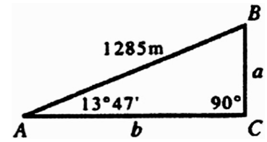


$$\begin{aligned} A + B &= 90° \\ B &= 90° - A \\ B &= 90° - 53°24' \\ &= 89°60' - 53°24' \\ &= 36°36' \end{aligned}$$

$$\sin A = \frac{a}{c} \Rightarrow \sin 53°24' = \frac{a}{387.1} \Rightarrow$$
$a = 387.1 \sin 53°24' \approx 310.8$ ft (rounded to four significant digits)

$$\cos A = \frac{b}{c} \Rightarrow \cos 53°24' = \frac{b}{387.1} \Rightarrow$$
$b = 387.1 \cos 53°24' \approx 230.8$ ft (rounded to four significant digits)

26. $A = 13°47'$, $c = 1285$ m

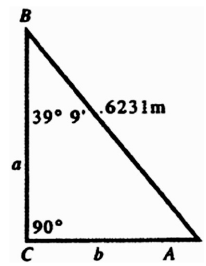


$$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$$
$$B = 90° - 13°47' = 89°60' - 13°47' = 76°13'$$

$$\sin A = \frac{a}{c} \Rightarrow \sin 13°47' = \frac{a}{1285} \Rightarrow$$
$a = 1285 \sin 13°47' \approx 306.2$ m (rounded to four significant digits)

$$\cos A = \frac{b}{c} \Rightarrow \cos 13°47' = \frac{b}{1285} \Rightarrow$$
$b = 1285 \cos 13°47' \approx 1248$ m (rounded to four significant digits)

27. $B = 39°09'$, $c = .6231$ m

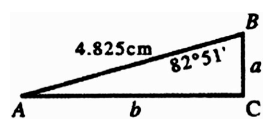


$$\begin{aligned} A + B &= 90° \\ B &= 90° - A \\ B &= 90° - 39°09' \\ &= 89°60' - 39°09' \\ &= 50°51' \end{aligned}$$

$$\sin B = \frac{b}{c} \Rightarrow \sin 39°09' = \frac{b}{.6231} \Rightarrow$$
$b = .6231 \sin 39°09' \approx .3934$ m (rounded to four significant digits)

$$\cos B = \frac{a}{c} \Rightarrow \cos 39°09' = \frac{a}{.6231} \Rightarrow$$
$a = .6231 \cos 39°09' \approx .4832$ m (rounded to four significant digits)

28. $B = 82°51'$, $c = 4.825$ cm



$$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$$
$$B = 90° - 82°51' = 89°60' - 82°51' = 7°9'$$

$$\sin B = \frac{b}{c} \Rightarrow \sin 82°51' = \frac{b}{4.825} \Rightarrow$$
$b = 4.825 \sin 82°51' \approx 4.787$ m (rounded to four significant digits)

$$\cos B = \frac{a}{c} \Rightarrow \cos 82°51' = \frac{a}{4.825} \Rightarrow$$
$a = 4.825 \cos 82°51' \approx .6006$ m (rounded to four significant digits)

29. Answers will vary. The angle of elevation and the angle of depression are measured between the line of sight and a horizontal line. So, in the diagram, lines *AD* and *CB* are both horizontal. Hence, they are parallel. The line formed by *AB* is a transversal and angles *DAB* and *ABC* are alternate interior angle and thus have the same measure.

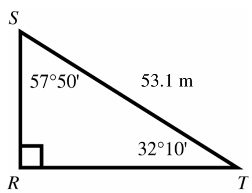
30. The angle of depression is measured between the line of sight and a horizontal line. This angle is measured between the line of sight and a vertical line.

31. $\sin 43°50' = \dfrac{d}{13.5}$

$d = 13.5 \sin 43°50' \approx 9.3496000$

The ladder goes up the wall 9.35 m. (rounded to three significant digits)

32. $T = 32°\ 10'$ and $S = 57°50'$



Since

$S + T = 32°\ 10' + 57°50' = 89°60' = 90°$,

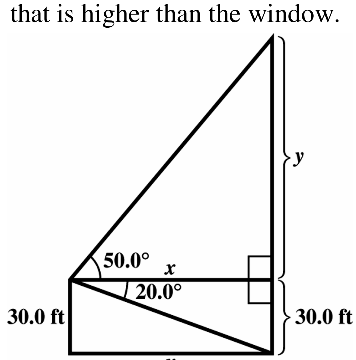
triangle *RST* is a right triangle. Thus, we have

$\tan 32°10' = \dfrac{RS}{53.1}$

$RS = 53.1 \tan 32°10' \approx 33.395727$

The distance across the lake is 33.4 m. (rounded to three significant digits)

33. Let *x* represent the horizontal distance between the two buildings and *y* represent the height of the portion of the building across the street that is higher than the window.



$$\tan 20.0° = \frac{30.0}{x} \Rightarrow x = \frac{30.3}{\tan 20.0°} \approx 82.4$$

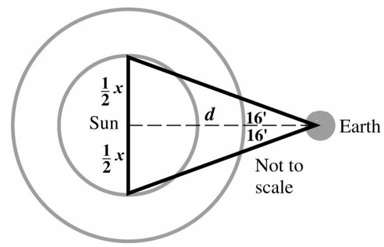
$$\tan 50.0° = \frac{y}{x} \Rightarrow$$

$$y = x \tan 50.0° = \left(\frac{30.0}{\tan 20.0°}\right) \tan 50.0°$$

$$\begin{aligned} \text{height} &= y + 30.0 \\ &= \left(\frac{30.0}{\tan 20.0°}\right) \tan 50.0° + 30.0 \\ &\approx 128.2295 \end{aligned}$$

The height of the building across the street is about 128 ft. (rounded to three significant digits)

34. Let *x* = the diameter of the sun.



Since the included angle is $32'$, $\frac{1}{2}(32') = 16'$.

We will use this angle, *d*, and half of the diameter to set up the following equation.

$$\frac{\frac{1}{2}x}{92{,}919{,}800} = \tan 16' \Rightarrow$$

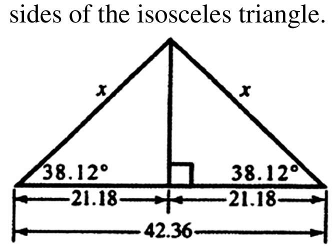
$$x = 2(92{,}919{,}800)(\tan 16')$$

$$\approx 864{,}943.0189$$

The diameter of the sun is about 864,900 mi. (rounded to four significant digits)

35. The altitude of an isosceles triangle bisects the base as well as the angle opposite the base. The two right triangles formed have interior angles which have the same measure. The lengths of the corresponding sides also have the same measure. Since the altitude bisects the base, each leg (base) of the right triangles is $\frac{42.36}{2} = 21.18$ in.

Let *x* = the length of each of the two equal sides of the isosceles triangle.



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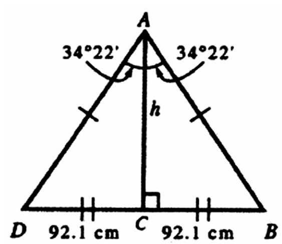
$$\cos 38.12° = \frac{21.18}{x} \Rightarrow x\cos 38.12° = 21.18 \Rightarrow$$
$$x = \frac{21.18}{\cos 38.12°} \approx 26.921918$$

The length of each of the two equal sides of the triangle is 26.92 in. (rounded to four significant digits)

36. The altitude of an isosceles triangle bisects the base as well as the angle opposite the base. The two right triangles formed have interior angles which have the same measure. The lengths of the corresponding sides also have the same measure. Since the altitude bisects the base, each leg (base) of the right triangles are $\frac{184.2}{2} = 92.10$ cm. Each angle opposite to the base of the right triangles measures $\frac{1}{2}(68°44') = 34°22'$.
Let h = the altitude.

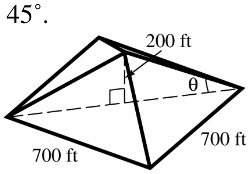


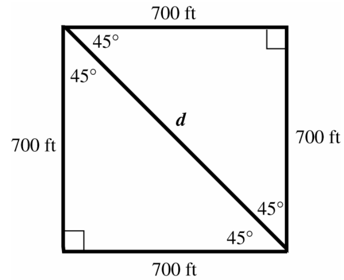
In triangle *ABC*,

$$\tan 34°22' = \frac{92.10}{h} \Rightarrow h\tan 34°22' = 92.10 \Rightarrow$$
$$h = \frac{92.10}{\tan 34°22'} \approx 134.67667$$

The altitude of the triangle is 134.7 cm. (rounded to four significant digits)

37. In order to find the angle of elevation, θ, we need to first find the length of the diagonal of the square base. The diagonal forms two isosceles right triangles. Each angle formed by a side of the square and the diagonal measures 45°.

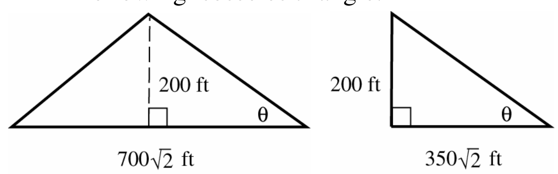




By the Pythagorean theorem,

$$700^2 + 700^2 = d^2 \Rightarrow 2\cdot 700^2 = d^2 \Rightarrow$$
$$d = \sqrt{2\cdot 700^2} \Rightarrow d = 700\sqrt{2}$$

Thus, length of the diagonal is $700\sqrt{2}$ ft. To to find the angle, θ, we consider the following isosceles triangle.



The height of the pyramid bisects the base of this triangle and forms two right triangles. We can use one of these triangles to find the angle of elevation, θ.

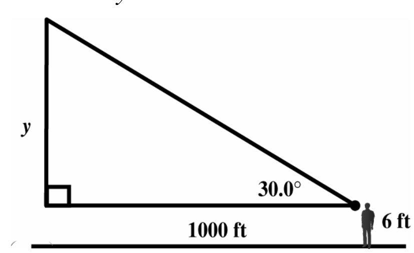
$$\tan\theta = \frac{200}{350\sqrt{2}} \approx .4040610178$$
$$\theta \approx \tan^{-1}(.4040610178) \approx 22.0017$$

Rounding this figure to two significant digits, we have $\theta \approx 22°$.

38. Let y = the height of the spotlight (this measurement starts 6 feet above ground)

$$\tan 30.0° = \frac{y}{1000}$$
$$y = 1000\cdot\tan 30.0° \approx 577.3502$$

Rounding this figure to three significant digits, we have $y \approx 577$.

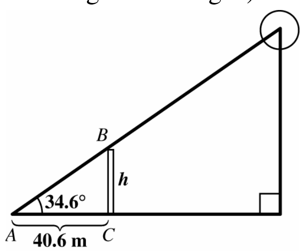


However, the observer's eye-height is 6 feet from the ground, so the cloud ceiling is $577 + 6 = 583$ ft.

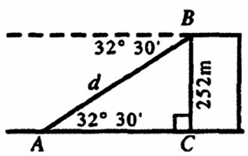
39. Let h represent the height of the tower. In triangle ABC we have

$$\tan 34.6° = \frac{h}{40.6}$$
$$h = 40.6 \tan 34.6° \approx 28.0081$$

The height of the tower is 28.0 m. (rounded to three significant digits)



40. Let d = the distance from the top B of the building to the point on the ground A.



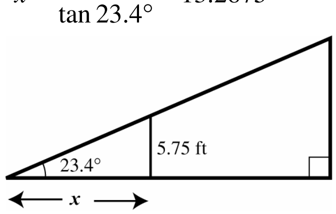
In triangle ABC,

$$\sin 32°30' = \frac{252}{d}$$
$$d = \frac{252}{\sin 32°30'} \approx 469.0121$$

The distance from the top of the building to the point on the ground is 469 m. (rounded to three significant digits)

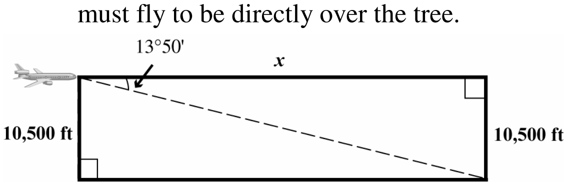
41. Let x = the length of the shadow cast by Diane Carr.

$$\tan 23.4° = \frac{5.75}{x}$$
$$x \tan 23.4° = 5.75$$
$$x = \frac{5.75}{\tan 23.4°} \approx 13.2875$$



The length of the shadow cast by Diane Carr is 13.3 ft. (rounded to three significant digits)

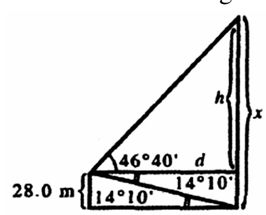
42. Let x = the horizontal distance that the plan must fly to be directly over the tree.



$$\tan 13°50' = \frac{10,500}{x}$$
$$x \tan 13°50' = 10,500$$
$$x = \frac{10,500}{\tan 13°50'} \approx 42,641.2351$$

The horizontal distance that the plan must fly to be directly over the tree is 42,600 ft. (rounded to three significant digits)

43. Let x = the height of the taller building; h = the difference in height between the shorter and taller buildings; d = the distance between the buildings along the ground.



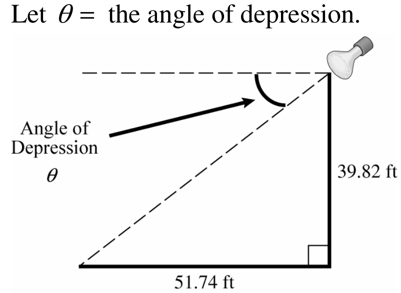
$$\frac{28.0}{d} = \tan 14°10' \Rightarrow 28.0 = d \tan 14°10' \Rightarrow$$
$$d = \frac{28.0}{\tan 14°10'} \approx 110.9262493 \text{ m}$$

(We hold on to these digits for the intermediate steps.) To find h, solve

$$\frac{h}{d} = \tan 46°40'$$
$$h = d \tan 46°40' \approx (110.9262493)\tan 46°40'$$
$$\approx 117.5749$$

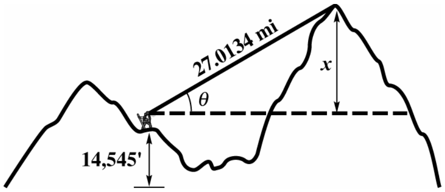
Thus, the value of h rounded to three significant digits is 118 m. Since $x = h + 28.0 = 118 + 28.0 \approx 146\text{ m}$, the height of the taller building is 146 m.

44. Let $\theta =$ the angle of depression.



$$\tan\theta = \frac{39.82}{51.74} \approx .7696173174$$
$$\theta = \tan^{-1}(.7696173174)$$
$$\theta \approx 37.58° \approx 37°35'$$

45. **(a)** Let x = the height of the peak above 14,545 ft.



Since the diagonal of the right triangle formed is in miles, we must first convert this measurement to feet. Since there are 5280 ft in one mile, we have the length of the diagonal is

$$27.0134(5280) = 142,630.752.$$

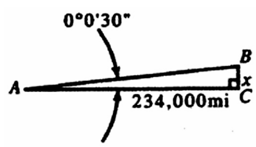
Find the value of x by solving

$$\sin 5.82° = \frac{x}{142,630.752}$$
$$x = 142,630.752 \sin 5.82° \approx 14,463.2674$$

Thus, the value of x rounded to five significant digits is 14,463 ft. Thus, the total height is about 14,545 + 14,463 = 29,008 ≈ 29,000 ft.

(b) The curvature of the earth would make the peak appear shorter than it actually is. Initially the surveyors did not think Mt. Everest was the tallest peak in the Himalayas. It did not look like the tallest peak because it was farther away than the other large peaks.

46. Let x = the distance from the assigned target.



In triangle ABC, we have

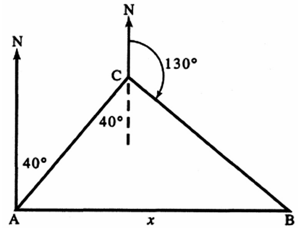
$$\tan 0°0'30'' = \frac{x}{234,000}$$
$$x = 234,000 \tan 0°0'30'' \approx 34.0339$$

The distance from the assigned target is 34.0 mi. (rounded to three significant digits)

47. Let x = the distance the plane is from its starting point. In the figure, the measure of angle ACB is

$$40° + (180° - 130°) = 40° + 50° = 90°.$$

Therefore, triangle ACB is a right triangle.



Since $d = rt$, the distance traveled in 1.5 hr is (1.5 hr)(110 mph) = 165 mi. The distance traveled in 1.3 hr is (1.3 hr)(110 mph) = 143 mi.

Using the Pythagorean theorem, we have

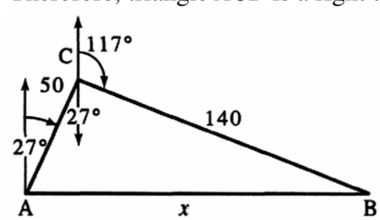
$$x^2 = 165^2 + 143^2 \Rightarrow x^2 = 27,225 + 20,449 \Rightarrow$$
$$x^2 = 47,674 \Rightarrow x \approx 218.3438$$

The plane is 220 mi from its starting point. (rounded to two significant digits)

48. Let x = the distance from the starting point. In the figure, the measure of angle ACB is

$$27° + (180° - 117°) = 27° + 63° = 90°.$$

Therefore, triangle ACB is a right triangle.

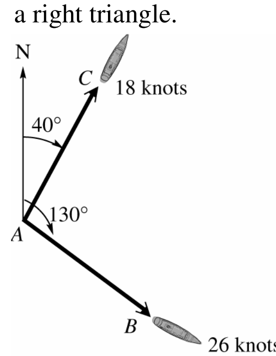


Applying the Pythagorean theorem, we have

$$x^2 = 50^2 + 140^2 \Rightarrow x^2 = 2500 + 19,600 \Rightarrow$$
$$x^2 = 22,100 \Rightarrow x = \sqrt{22,100} \approx 148.6607$$

The distance of the end of the trip from the starting point is 150 km. (rounded to two significant digits)

49. Let x = distance the ships are apart. In the figure, the measure of angle CAB is $130° - 40° = 90°$. Therefore, triangle CAB is a right triangle.

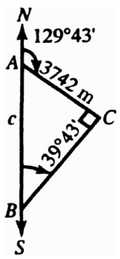


Since $d = rt$, the distance traveled by the first ship in 1.5 hr is
(1.5 hr)(18 knots) = 27 nautical mi and the second ship is
(1.5hr)(26 knots) = 39 nautical mi.
Applying the Pythagorean theorem, we have

$x^2 = 27^2 + 39^2 \Rightarrow x^2 = 729 + 1521 \Rightarrow$
$x^2 = 2250 \Rightarrow x = \sqrt{2250} \approx 47.4342$

The ships are 47 nautical mi apart. (rounded to 2 significant digits)

50. Let C = the location of the ship, and let c = the distance between the lighthouses.



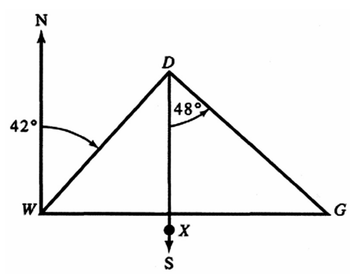
$m\angle BAC = 180° - 129°43'$
$= 179°60' - 129°43' = 50°17'$

Since $50°17' + 39°43' = 90°$, we have a right triangle. Thus,

$$\sin 39°43' = \frac{3742}{c} \Rightarrow c \sin 39°43' = 3742 \Rightarrow$$
$$c = \frac{3742}{\sin 39°43'} \approx 5856.1020$$

The distance between the lighthouses is 5856 m (rounded to four significant digits).

51. Draw triangle *WDG* with *W* representing Winston-Salem, *D* representing Danville, and *G* representing Goldsboro. Name any point *X* on the line due south from *D*.



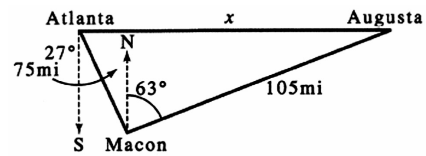
Since the bearing from *W* to *D* is 42° (equivalent to N 42° E), angle *WDX* measures 42°. Since angle *XDG* measures 48°, the measure of angle *D* is 42° + 48° = 90°. Thus, triangle *WDG* is a right triangle.
Using $d = rt$ and the Pythagorean theorem, we have

$$WG = \sqrt{(WD)^2 + (DG)^2}$$
$$= \sqrt{[65(1.1)]^2 + [65(1.8)]^2}$$

$$WG = \sqrt{71.5^2 + 117^2} = \sqrt{5112.25 + 13,689}$$
$$= \sqrt{18,8001.25} \approx 137$$

The distance from Winston-Salem to Goldsboro is approximately 140 mi. (rounded to two significant digits)

52. Let x = the distance from Atlanta to Augusta.



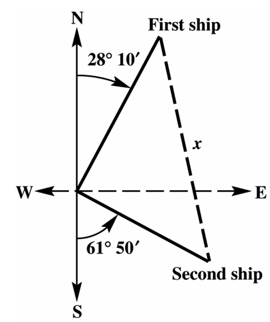
The line from Atlanta to Macon makes an angle of $27° + 63° = 90°$, with the line from Macon to Augusta. Since $d = rt$, the distance from Atlanta to Macon is $60\left(1\frac{1}{4}\right) = 75$ mi.
The distance from Macon to Augusta is $60\left(1\frac{3}{4}\right) = 105$ mi.

Use the Pythagorean theorem to find x:

$x^2 = 75^2 + 105^2 \Rightarrow x^2 = 5635 + 11,025 \Rightarrow$
$x^2 = 16,650 \approx 129.0349$

The distance from Atlanta to Augusta is 130 mi. (rounded to two significant digits)

53. Let x = distance between the two ships.

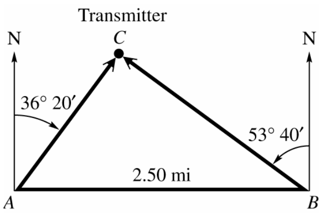


The angle between the bearings of the ships is $180° - (28°10' + 61°50') = 90°$. The triangle formed is a right triangle. The distance traveled at 24.0 mph is
(4 hr) (24.0 mph) = 96 mi. The distance traveled at 28.0 mph is
(4 hr)(28.0 mph) = 112 mi.
Applying the Pythagorean theorem we have

$x^2 = 96^2 + 112^2 \Rightarrow x^2 = 9216 + 12,544 \Rightarrow$
$x^2 = 21,760 \Rightarrow x = \sqrt{21,760} \approx 147.5127$

The ships are 148 mi apart. (rounded to three significant digits)

54. Let C = the location of the transmitter; a = the distance of the transmitter from B.

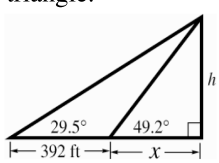


The measure of angle CBA is
$90° - 53°40' = 89°60' - 53°40' = 36°20'$.
The measure of angle CAB is
$90° - 36°20' = 89°60' - 36°20' = 53°40'$.
Since $A + B = 90°$, so $C = 90°$. Thus, we have

$$\sin A = \frac{a}{2.50} \Rightarrow \sin 53°40' = \frac{a}{2.50} \Rightarrow$$
$$a = 2.50 \sin 53°40' \approx 2.0140$$

The distance of the transmitter from B is 2.01 mi. (rounded to 3 significant digits)

55. Algebraic solution:
Let x = the side adjacent to 49.2º in the smaller triangle.



In the larger right triangle, we have

$$\tan 29.5° = \frac{h}{392 + x} \Rightarrow h = (392 + x)\tan 29.5° .$$

In the smaller right triangle, we have

$$\tan 49.2° = \frac{h}{x} \Rightarrow h = x \tan 49.2° .$$

Substituting, we have

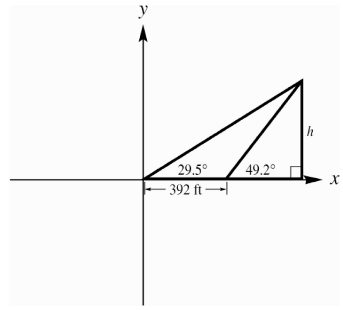
$$x \tan 49.2° = (392 + x)\tan 29.5°$$
$$x \tan 49.2° = 392 \tan 29.5° + x \tan 29.5°$$
$$x \tan 49.2° - x \tan 29.5° = 392 \tan 29.5°$$
$$x(\tan 49.2° - \tan 29.5°) = 392 \tan 29.5°$$
$$x = \frac{392 \tan 29.5°}{\tan 49.2° - \tan 29.5°}$$

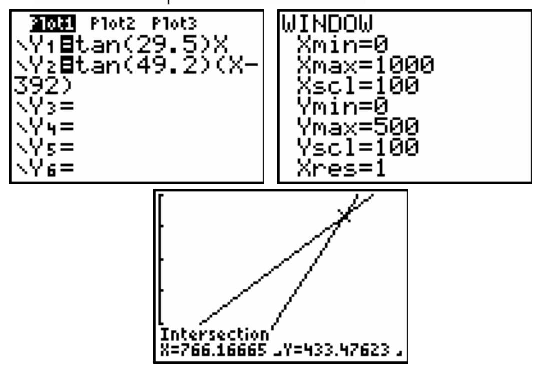
Now substitute this expression for x in the equation for the smaller triangle to obtain

$$h = x \tan 49.2°$$
$$h = \frac{392 \tan 29.5°}{\tan 49.2° - \tan 29.5°} \cdot \tan 49.2°$$

$\approx 433.4762 \approx 433$ ft (rounded to three significant digits.

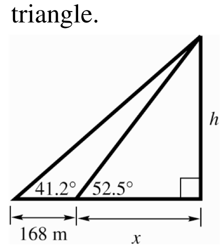
Graphing calculator solution:
The first line considered is $y = (\tan 29.5°)x$ and the second is $y = (\tan 29.5°)(x - 392)$.





The height of the triangle is 433 ft (rounded to three significant digits.

56. Algebraic solution:
Let x = the side adjacent to 52.5º in the smaller triangle.



In the larger right triangle, we have

$$\tan 41.2° = \frac{h}{168 + x} \Rightarrow h = (168 + x)\tan 41.2° .$$

In the smaller right triangle, we have

$$\tan 52.5° = \frac{h}{x} \Rightarrow h = x \tan 52.5° .$$

Substituting, we have

$$x \tan 52.5° = (168 + x)\tan 41.2°$$
$$x \tan 52.5° = 168 \tan 41.2° + x \tan 41.2°$$
$$x \tan 52.5° - x \tan 41.2° = 168 \tan 41.2°$$
$$x(\tan 52.5° - \tan 41.2°) = 168 \tan 41.2°$$
$$x = \frac{168 \tan 41.2°}{\tan 52.5° - \tan 41.2°}$$

Now substitute this expression for x in the equation for the smaller triangle to obtain

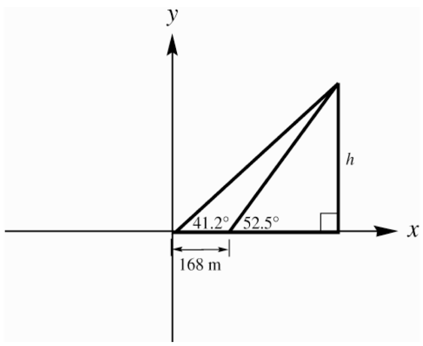
$$h = x\tan 52.5°$$

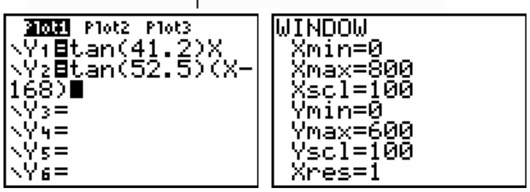
$$h = \frac{168\tan 41.2°}{\tan 52.5° - \tan 41.2°}\cdot \tan 52.5°$$

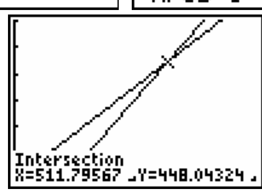
$\approx 448.0432 \approx 448$ m (rounded to three significant digits.

Graphing calculator solution:

The first line considered is $y = (\tan 41.2°)x$ and the second is $y = (\tan 52.5°)(x - 168)$.

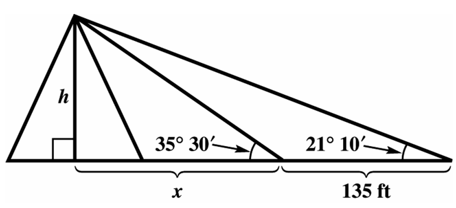






The height of the triangle is 448 m (rounded to three significant digits.

57. Let x = the distance from the closer point on the ground to the base of height h of the pyramid.



In the larger right triangle, we have

$$\tan 21°10' = \frac{h}{135 + x} \Rightarrow h = (135 + x)\tan 21°10'$$

In the smaller right triangle, we have

$$\tan 35°30' = \frac{h}{x} \Rightarrow h = x\tan 35°30'.$$

Substitute for h in this equation, and solve for x to obtain the following.

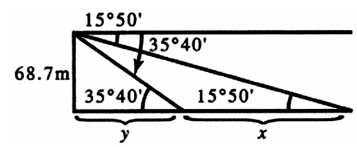
$$(135 + x)\tan 21°10' = x\tan 35°30'$$
$$135\tan 21°10' + x\tan 21°10' = x\tan 35°30'$$
$$135\tan 21°10' = x\tan 35°30' - x\tan 21°10'$$
$$135\tan 21°10' = x(\tan 35°30' - \tan 21°10')$$
$$\frac{135\tan 21°10'}{\tan 35°30' - \tan 21°10'} = x$$

Substitute for x in the equation for the smaller triangle.

$$h = \frac{135\tan 21°10'}{\tan 35°30' - \tan 21°10'}\tan 35°30'$$
$$\approx 114.3427$$

The height of the pyramid is 114 ft. (rounded to three significant digits)

58. Let x = the distance traveled by the whale as it approaches the tower; y = the distance from the tower to the whale as it turns.



$$\frac{68.7}{y} = \tan 35°40' \Rightarrow 68.7 = y\tan 35°40' \Rightarrow$$

$$y = \frac{68.7}{\tan 35°40'} \text{ and } \frac{68.7}{x + y} = \tan 15°50'$$

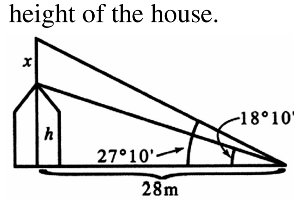
$$68.7 = (x + y)\tan 15°50'$$

$$x + y = \frac{68.7}{\tan 15°50'} \Rightarrow x = \frac{68.7}{\tan 15°50'} - y$$

$$x = \frac{68.7}{\tan 15°50'} - \frac{68.7}{\tan 35°40'} \approx 146.5190$$

The whale traveled 147 m as it approached the lighthouse. (rounded to three significant digits)

59. Let x = the height of the antenna; h = the height of the house.



In the smaller right triangle, we have

$$\tan 18°10' = \frac{h}{28} \Rightarrow h = 28\tan 18°10'.$$

(*continued on next page*)

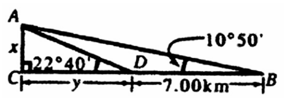
(*continued from page 517*)

In the larger right triangle, we have

$$\tan 27°10' = \frac{x+h}{28} \Rightarrow x+h = 28\tan 27°10' \Rightarrow$$
$$x = 28\tan 27°10' - h$$
$$x = 28\tan 27°10' - 28\tan 18°10'$$
$$\approx 5.1816$$

The height of the antenna is 5.18 m. (rounded to three significant digits)

60. Let x = the height of Mt. Whitney above the level of the road; y = the distance shown in the figure below.



In triangle ADC,

$$\tan 22°40' = \frac{x}{y} \Rightarrow y\tan 22°40' = x \Rightarrow$$
$$y = \frac{x}{\tan 22°40'}. \quad (1)$$

In triangle ABC

$$\tan 10°50' = \frac{x}{y+7.00}$$
$$(y+7.00)\tan 10°50' = x$$
$$y\tan 10°50' + 7.00\tan 10°50' = x$$
$$\frac{x - 7.00\tan 10°50'}{\tan 10°50'} = y \qquad (2)$$

Setting equations 1 and 2 equal, we have

$$\frac{x}{\tan 22°40'} = \frac{x - 7.00\tan 10°50'}{\tan 10°50'}$$
$$x\tan 10°50' = x\tan 22°40' - 7.00(\tan 10°50')(\tan 22°40')$$
$$7.00(\tan 10°50')(\tan 22°40') = x\tan 22°40' - x\tan 10°50'$$
$$7.00(\tan 10°50')(\tan 22°40') = x(\tan 22°40' - \tan 10°50')$$
$$x = \frac{7.00(\tan 10°50')(\tan 22°40')}{\tan 22°40' - \tan 10°50'}$$
$$x \approx 2.4725$$

The height of the top of Mt. Whitney above road level is 2.47 km. (rounded to three significant digits)

61. (a) From the figure in the text,

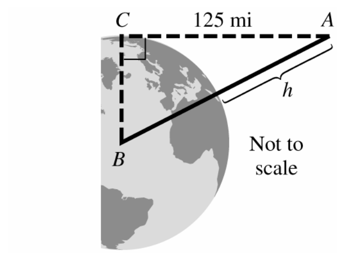
$$d = \frac{b}{2}\cot\frac{\alpha}{2} + \frac{b}{2}\cot\frac{\beta}{2} = \frac{b}{2}\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right)$$

(b) Using the result of part (a), let $\alpha = 37'48''$, $\beta = 42'3''$, and $b = 2.000$

$$d = \frac{b}{2}\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right) \Rightarrow$$
$$d = \frac{2.000}{2}\left(\cot\frac{37'48''}{2} + \cot\frac{42'3''}{2}\right)$$
$$= \cot .315° + \cot .3504166667$$
$$\approx 345.3951$$

The distance between the two point P and Q is about 345.4 cm.

62. Let h = the minimum height above the surface of the earth so a pilot at A can see an object on the horizon at C.

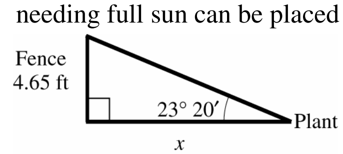


Using the Pythagorean theorem, we have

$$\left(4.00\times 10^3 + h\right)^2 = \left(4.00\times 10^3\right)^2 + 125^2$$
$$(4000+h)^2 = 4000^2 + 125^2$$
$$(4000+h)^2 = 16{,}000{,}000 + 15{,}625$$
$$(4000+h)^2 = 16{,}015{,}625$$
$$4000 + h = \sqrt{16{,}015{,}625}$$
$$h = \sqrt{16{,}015{,}625} - 4000$$
$$\approx 4001.9526 - 4000 = 1.9526$$

The minimum height above the surface of the earth would be 1.95 mi. (rounded to 3 significant digits)

63. Let x = the minimum distance that a plant needing full sun can be placed from the fence.



$$\tan 23°20' = \frac{4.65}{x} \Rightarrow x\tan 23°20' = 4.65 \Rightarrow$$
$$x = \frac{4.65}{\tan 23°20'} \approx 10.7799$$

The minimum distance is 10.8 ft. (rounded to three significant digits)

64. $\tan A = \dfrac{1.0837}{1.4923} \approx .7261944649$

$A \approx \tan^{-1}(.7261944649) \approx 35.987°$

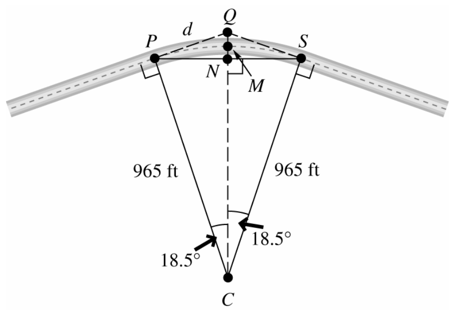
$\approx 35°59.2' \approx 35°59'10''$

$\tan B = \dfrac{1.4923}{1.0837} \approx 1.377041617$

$B \approx \tan^{-1}(1.377041617) \approx 54.013°$

$\approx 54°00.8' \approx 54°00'50''$

65. **(a)** If $\theta = 37°$, then $\dfrac{\theta}{2} = \dfrac{37°}{2} = 18.5°$.

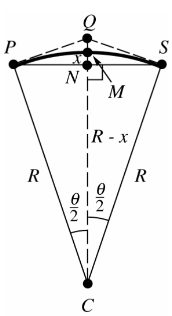


To find the distance between P and Q, d, we first note that angle QPC is a right angle. Hence, triangle QPC is a right triangle and we can solve

$$\tan 18.5° = \frac{d}{965}$$
$$d = 965 \tan 18.5° \approx 322.8845$$

The distance between P and Q, is 320 ft. (rounded to two significant digits)

(b) Since we are dealing with a circle, the distance between M and C is R. If we let x be the distance from N to M, then the distance from C to N will be $R - x$.



Since triangle CNP is a right triangle, we can set up the following equation.

$$\cos\frac{\theta}{2} = \frac{R-x}{R} \Rightarrow R\cos\frac{\theta}{2} = R - x \Rightarrow$$
$$x = R - R\cos\frac{\theta}{2} \Rightarrow x = R\left(1 - \cos\frac{\theta}{2}\right)$$

66. **(a)** $\theta \approx \dfrac{57.3S}{R} = \dfrac{57.3(336)}{600} = 32.088°$

$d = R\left(1 - \cos\dfrac{\theta}{2}\right)$

$= 600(1 - \cos 16.044°) \approx 23.3702$ ft

The distance is 23 ft. (rounded to two significant digits)

(b) $\theta \approx \dfrac{57.3S}{R} = \dfrac{57.3(485)}{600} = 46.3175°$

$d = R\left(1 - \cos\dfrac{\theta}{2}\right)$

$= 600(1 - \cos 23.15875°)$

≈ 48.3488

The distance is 48 ft. (rounded to two significant digits)

(c) The faster the speed, the more land needs to be cleared on the inside of the curve.

Chapter 5 Review Exercises

1. The complement of 35° is $90° - 35° = 55°$. The supplement of 35° is $180° - 35° = 145°$.

2. $-51°$ is coterminal with $360° + (-51°) = 309°$

3. $-174°$. is coterminal with $-174° + 360° = 186°$

4. 792° is coterminal with 792° − 2(360°) = 72°.

5. The sum of the measures of the interior angles of a triangle is 180°.

$60° + y + 90° = 180°$

$150° + y = 180° \Rightarrow y = 30°$

Substitute 30° for y, and solve for x:

$30° + (x + y) + 90° = 180°$

$30° + (x + 30°) + 90° = 180°$

$150° + x = 180° \Rightarrow x = 30°$

Thus, $x = y = 30°$.

6. 320 rotations per min = $\frac{320}{60}$ rotations per sec

$= \frac{16}{3}$ rotations per sec

$= \frac{16}{3} \cdot \frac{2}{3} = \frac{32}{9}$ rotations per $\frac{2}{3}$ sec

$= \frac{32}{9}(360°)$ per $\frac{2}{3}$ sec = 1280° per $\frac{2}{3}$ sec

A point of the edge of the pulley will move 1280° in $\frac{2}{3}$ sec.

7. 650 rotations per min = $\frac{650}{60}$ rotations per sec

= $\frac{65}{6}$ rotations per sec

= 26 rotations per 2.4 sec

= $26(360°)$ per 2.4 sec = 9360° per 2.4 sec

A point of the edge of the propeller will move 9360° in 2.4 sec.

8. $$47°25'11'' = 47° + \tfrac{25}{60}° + \tfrac{11}{3600}° \approx 47° + .4167° + .0031° \approx 47.420°$$

9. $$119°8'3'' = 119° + \tfrac{8}{60}° + \tfrac{3}{3600}° \approx 119° + .1333° + .0008° \approx 119.134°$$

10. $$\begin{aligned} -61.5034° &= -[61° + .5034(60')] \\ &= -[61° + 30.204'] \\ &= -[61° + 30' + .204(60'')] \\ &= -[61° + 30' + 12.24''] \\ &= -61°30'12.24'' \approx -61°30'12'' \end{aligned}$$

11. $$\begin{aligned} 275.1005 &= 275° + .1005(60') = 275° + 6.03' \\ &= 275°\ 6' + .03' = 275°6' + .03(60'') \\ &= 275°6'1.8'' \approx 275°6'2'' \end{aligned}$$

12. The two indicated angles are vertical angles. Hence, their measures are equal.
$9x + 4 = 12x - 14 \Rightarrow 18 = 3x \Rightarrow x = 6$
9(6) + 4 = 58, and 12(6) − 14 = 58, so the measure of each of the two angles is 58°.

13. The three angles are the interior angles of a triangle. Hence, the sum of their measures is 180°.
$$\begin{aligned} 4x + (5x+5) + (4x-20) &= 180 \\ 13x - 15 &= 180 \\ 13x = 195 &\Rightarrow x = 15 \end{aligned}$$
4(15) = 60, 5(15) + 5 = 80, and 4(15) − 20 = 40, so the measures of the angles are 40°, 60°, and 80°.

14. $x = -3$, $y = -3$ and
$$r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$
$$\sin\theta = \frac{y}{r} = -\frac{3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$
$$\cos\theta = \frac{x}{r} = \frac{-3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$
$$\tan\theta = \frac{y}{x} = \frac{-3}{-3} = 1$$
$$\cot\theta = \frac{x}{y} = \frac{-3}{-3} = 1$$
$$\sec\theta = \frac{r}{x} = \frac{3\sqrt{2}}{-3} = -\sqrt{2}$$
$$\csc\theta = \frac{r}{y} = \frac{3\sqrt{2}}{-3} = -\sqrt{2}$$

15. $x = 1,\ y = -\sqrt{3}$ and
$$r = \sqrt{x^2+y^2} = \sqrt{1^2+\left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$$
$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$
$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$
$$\cot\theta = \frac{x}{y} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec\theta = \frac{r}{x} = \frac{2}{1} = 2$$
$$\csc\theta = \frac{r}{y} = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

16. For the quadrantal angle, 180°, we will choose the point $(-1, 0)$, so $x = -1$, $y = 0$, and $r = 1$.
$$\sin 180° = \frac{y}{r} = \frac{0}{1} = 0$$
$$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1$$
$$\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0$$
$$\cot 180° = \frac{x}{y} = \frac{-1}{0} \text{ undefined}$$
$$\sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$$
$$\csc 180° = \frac{r}{y} = \frac{1}{0} \text{ undefined}$$

17. $(3, -4)$
$x = 3$, $y = -4$ and
$$r = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$
$$\sin\theta = \frac{y}{r} = \frac{-4}{5} = -\frac{4}{5}$$
$$\cos\theta = \frac{x}{r} = \frac{3}{5}$$

$\tan\theta = \frac{y}{x} = \frac{-4}{3} = -\frac{4}{3}$

$\cot\theta = \frac{x}{y} = \frac{3}{-4} = -\frac{3}{4}$

$\sec\theta = \frac{r}{x} = \frac{5}{3}$

$\csc\theta = \frac{r}{y} = \frac{5}{-4} = -\frac{5}{4}$

18. $(9,-2)$

$x = 9,\ y = -2,$ and $r = \sqrt{9^2+(-2)^2} = \sqrt{81+4} = \sqrt{85}$

$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{85}} = -\frac{2}{\sqrt{85}}\cdot\frac{\sqrt{85}}{\sqrt{85}} = -\frac{2\sqrt{85}}{85}$

$\cos\theta = \frac{x}{r} = \frac{9}{\sqrt{85}} = \frac{9}{\sqrt{85}}\cdot\frac{\sqrt{85}}{\sqrt{85}} = \frac{9\sqrt{85}}{85}$

$\tan\theta = \frac{y}{x} = \frac{-2}{9} = -\frac{2}{9}$

$\cot\theta = \frac{x}{y} = \frac{9}{-2} = -\frac{9}{2}$

$\sec\theta = \frac{r}{x} = \frac{\sqrt{85}}{9}$

$\csc\theta = \frac{r}{y} = \frac{\sqrt{85}}{-2} = -\frac{\sqrt{85}}{2}$

19. $\left(6\sqrt{3},-6\right)$

$x = 6\sqrt{3}$, $y = -6$, and

$r = \sqrt{\left(6\sqrt{3}\right)^2+\left(-6\right)^2} = \sqrt{108+36} = \sqrt{144} = 12$

$\sin\theta = \frac{y}{r} = \frac{-6}{12} = -\frac{1}{2}$

$\cos\theta = \frac{x}{r} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x} = \frac{-6}{6\sqrt{3}} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$\cot\theta = \frac{x}{y} = \frac{6\sqrt{3}}{-6} = -\sqrt{3}$

$\sec\theta = \frac{r}{x} = \frac{12}{6\sqrt{3}} = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\csc\theta = \frac{r}{y} = \frac{12}{-6} = -2$

20. Since the terminal side of the angle is defined by $5x-3y=0, x\ge 0$, a point on this terminal side would be (3, 5), since 5(3) − 3(5) = 0. Then $r = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$.

$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{34}} = \frac{5}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$

$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{34}} = \frac{3}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$

$\tan\theta = \frac{y}{x} = \frac{5}{3}$; $\cot\theta = \frac{x}{y} = \frac{3}{5}$

$\sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{3}$; $\csc\theta = \frac{r}{y} = \frac{\sqrt{34}}{5}$

θ	**21.** 180º	**22.** −90º
$\sin\theta$	0	−1
$\cos\theta$	−1	0
$\tan\theta$	0	undefined
$\cot\theta$	undefined	0
$\sec\theta$	−1	undefined
$\csc\theta$	undefined	−1

23. $\cos\theta = -\frac{5}{8}$ given that θ is in quadrant III

$\cos\theta = \frac{x}{r} = \frac{-5}{8}$ and θ in quadrant III, so let $x = -5$, $r = 8$.

$x^2+y^2=r^2 \Rightarrow (-5)^2+y^2=8^2 \Rightarrow$
$25+y^2=64 \Rightarrow y^2=39 \Rightarrow y=\pm\sqrt{39}$

θ is in quadrant III, so $y=-\sqrt{39}$.

$\sin\theta = \frac{y}{r} = \frac{-\sqrt{39}}{8} = -\frac{\sqrt{39}}{8}$

$\cos\theta = \frac{x}{r} = \frac{-5}{8} = -\frac{5}{8}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{39}}{-5} = \frac{\sqrt{39}}{5}$

$\cot\theta = \frac{x}{y} = \frac{-5}{-\sqrt{39}} = \frac{5}{\sqrt{39}}\cdot\frac{\sqrt{39}}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$

$\sec\theta = \frac{r}{x} = \frac{8}{-5} = -\frac{8}{5}$

$\csc\theta = \frac{r}{y} = \frac{8}{-\sqrt{39}} = -\frac{8}{\sqrt{39}}$
$= -\frac{8}{\sqrt{39}}\cdot\frac{\sqrt{39}}{\sqrt{39}} - \frac{8\sqrt{39}}{39}$

24. $\tan\theta = 2$ given that θ is in quadrant III
θ in quadrant III $\Rightarrow x < 0, y < 0$, so

$$\tan\theta = \frac{y}{x} = 2 = \frac{-2}{-1} \Rightarrow x = -1, y = -2$$

$x^2 + y^2 = r^2 \Rightarrow (-1)^2 + (-2)^2 = r^2 \Rightarrow$
$1 + 4 = r^2 \Rightarrow r^2 = 5 \Rightarrow r = \sqrt{5}$, since r is positive.

$$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{5}} = -\frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{5}} = -\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{-1} = 2$$

$$\cot\theta = \frac{x}{y} = \frac{-1}{-2} = \frac{1}{2}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$$

25. $\sec\theta = -\sqrt{5}$ given that θ is in quadrant II
θ in quadrant II $\Rightarrow x < 0, y > 0$, so

$$\sec\theta = -\sqrt{5} = \frac{r}{x} = \frac{\sqrt{5}}{-1} \Rightarrow x = -1, r = \sqrt{5}$$

$x^2 + y^2 = r^2 \Rightarrow (-1)^2 + y^2 = \left(\sqrt{5}\right)^2 \Rightarrow$
$1 + y^2 = 5 \Rightarrow y^2 = 4 \Rightarrow y = 2$

$$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{5}} = -\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{2}{-1} = -2$$

$$\cot\theta = \frac{x}{y} = \frac{-1}{2} = -\frac{1}{2}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{2}$$

26. $\sin\theta = -\frac{2}{5}$ given that θ is in quadrant III
θ in quadrant III $\Rightarrow x < 0, y < 0$, so

$$\sin\theta = \frac{y}{r} = -\frac{2}{5} = \frac{-2}{5} \Rightarrow y = -2, r = 5$$

$x^2 + y^2 = r^2 \Rightarrow x^2 + (-2)^2 = 5^2 \Rightarrow$
$x^2 + 4 = 25 \Rightarrow x^2 = 21 \Rightarrow x = -\sqrt{21}$

$$\sin\theta = \frac{y}{r} = -\frac{2}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-\sqrt{21}}{5} = -\frac{\sqrt{21}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{-\sqrt{21}} = \frac{2}{\sqrt{21}}\cdot\frac{\sqrt{21}}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

$$\cot\theta = \frac{x}{y} = \frac{-\sqrt{21}}{-2} = \frac{\sqrt{21}}{2}$$

$$\sec\theta = \frac{r}{x} = \frac{5}{-\sqrt{21}} = -\frac{5}{\sqrt{21}}\cdot\frac{\sqrt{21}}{\sqrt{21}} = -\frac{5\sqrt{21}}{21}$$

$$\csc\theta = \frac{r}{y} = \frac{5}{-2} = -\frac{5}{2}$$

27. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{60}{61}$

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{11}{61}$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{60}{11}$$

$$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{11}{60}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{61}{11}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{61}{60}$$

28. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{40}{58} = \frac{20}{29}$

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{42}{58} = \frac{21}{29}$$

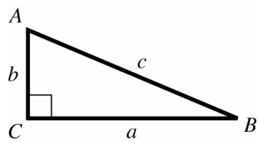
$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{40}{42} = \frac{20}{21}$$

$$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{42}{40} = \frac{21}{20}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{58}{42} = \frac{29}{21}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{58}{40} = \frac{29}{20}$$

29. The sum of the measures of angles A and B is $90°$, and, thus, they are complementary angles. Since sine and cosine are cofunctions, we have $\sin B = \cos(90° - B) = \cos A$.



30. A 120º angle lies in quadrant II, so the reference angle is $180° - 120° = 60°$. Since $120°$ is in quadrant II, the cosine, tangent, cotangent, and secant are negative.

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$$
$$\cos 120° = -\cos 60° = -\frac{1}{2}$$
$$\tan 120° = -\tan 60° = -\sqrt{3}$$
$$\cot 120° = -\cot 60° = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec 120° = -\sec 60° = -2$$
$$\csc 120° = \csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

31. 1020º is coterminal with $1020° - 2 \cdot 360° = 300°$. The reference angle is $360° - 300° = 60°$. Because $1020°$ lies in quadrant IV, the sine, tangent, cotangent, and cosecant are negative.

$$\sin 1020° = -\sin 60° = -\frac{\sqrt{3}}{2}$$
$$\cos 1020° = \cos 60° = \frac{1}{2}$$
$$\tan 1020° = -\tan 60° = -\sqrt{3}$$
$$\cot 1020° = -\cot 60° = -\frac{\sqrt{3}}{3}$$
$$\sec 1020° = \sec 60° = 2$$
$$\csc 1020° = -\csc 60° = -\frac{2\sqrt{3}}{3}$$

32. –225° is coterminal with –225° + 360° = 135°. This angle lies in quadrant II. The reference angle is $180° - 135° = 45°$. Since –225° is in quadrant II, the cosine, tangent, cotangent, and secant are negative.

$$\sin(-225°) = \sin 45° = \frac{\sqrt{2}}{2}$$
$$\cos(-225°) = -\cos 45° = -\frac{\sqrt{2}}{2}$$
$$\tan(-225°) = -\tan 45° = -1$$
$$\cot(-225°) = -\cot 45° = -1$$
$$\sec(-225°) = -\sec 45° = -\sqrt{2}$$
$$\csc(-225°) = \csc 45° = \sqrt{2}$$

33. –1470° is coterminal with $-1470° + 5 \cdot 360° = 330°$. This angle lies in quadrant IV. The reference angle is $360° - 330° = 30°$. Since –1470° is in quadrant IV, the sine, tangent, cotangent, and cosecant are negative.

$$\sin(-1470°) = -\sin 30° = -\frac{1}{2}$$
$$\cos(-1470°) = \cos 30° = \frac{\sqrt{3}}{2}$$
$$\tan(-1470°) = -\tan 30° = -\frac{\sqrt{3}}{3}$$
$$\cot(-1470°) = -\cot 30° = -\sqrt{3}$$
$$\sec(-1470°) = \sec 30° = \frac{2\sqrt{3}}{3}$$
$$\csc(-1470°) = -\csc 30° = -2$$

34. $\sin\theta = -\frac{1}{2}$

Since $\sin\theta$ is negative, θ must lie in quadrants III or IV. The absolute value of $\sin\theta$ is $\frac{1}{2}$, so the reference angle, θ', is 30º. The angle in quadrant III will be $180° + \theta' = 180° + 30° = 210°$, and the quadrant IV angle is $360° - \theta' = 360° - 30° = 330°$.

35. $\cos\theta = -\frac{1}{2}$

Since $\cos\theta$ is negative, θ must lie in quadrants II or III. Since the absolute value of $\cos\theta$ is $\frac{1}{2}$, the reference angle, θ' must be $60°$. The quadrant II angle θ equals $180° - \theta' = 180° - 60° = 120°$, and the quadrant III angle θ equals $180° + \theta' = 180° + 60° = 240°$.

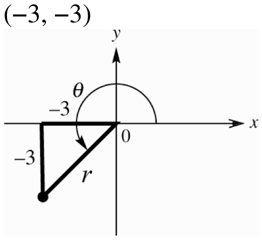
36. $\cot\theta = -1$

Since $\cot\theta$ is negative, θ must lie in quadrants II or IV. Since the absolute value of $\cot\theta$ is 1 the reference angle, θ' must be $45°$. The quadrant II angle θ equals $180° - \theta' = 180° - 45° = 135°$. and the quadrant IV angle θ equals $360° - \theta' = 360° - 45° = 315°$.

37. $\sec\theta = -\frac{2\sqrt{3}}{3}$

Since $\sec\theta$ is negative, θ must lie in quadrants II or III. Since the absolute value of $\sec\theta$ is $\frac{2\sqrt{3}}{3}$, the reference angle, θ' must be 30º. The quadrant II angle θ equals $180° - \theta' = 180° - 30° = 150°$, and the quadrant III angle θ equals $180° + \theta' = 180° + 30° = 210°$.

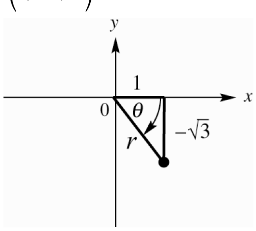
38. (a) $(-3, -3)$



The distance from the origin is r:

$$r = \sqrt{x^2 + y^2} \Rightarrow r = \sqrt{(-3)^2 + (-3)^2} \Rightarrow$$
$$r = \sqrt{9+9} \Rightarrow r = \sqrt{18} \Rightarrow r = 3\sqrt{2}$$
$$\sin\theta = \frac{y}{r} = -\frac{3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$
$$\cos\theta = \frac{x}{r} = -\frac{3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$
$$\tan\theta = \frac{y}{x} = \frac{-3}{-3} = 1$$

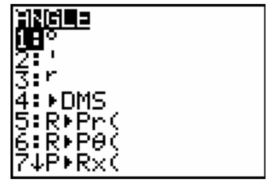
(b) $\left(1, -\sqrt{3}\right)$



The distance from the origin is r:

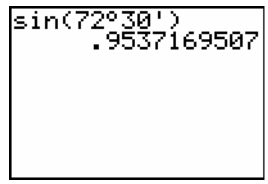
$$r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{1+3} = \sqrt{4} = 2$$
$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\cos\theta = \frac{x}{r} = \frac{1}{2}; \ \tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

For the exercises in this section, be sure your calculator is in degree mode. If your calculator accepts angles in degrees, minutes, and seconds, it is not necessary to change angles to decimal degrees. Keystroke sequences may vary on the type and/or model of calculator being used. Screens shown will be from a TI-83 Plus calculator. To obtain the degree (°) and (′) symbols, go to the ANGLE menu (2nd APPS).



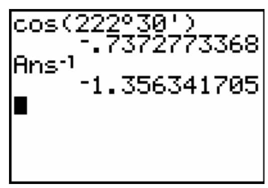
For Exercises 39–40, the calculation for decimal degrees is indicated for calculators that do not accept degree, minutes, and seconds.

39. $\sin 72°30' \approx .95371695$



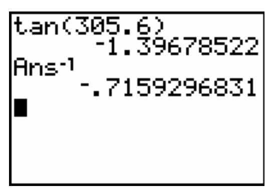
$$72°30' = \left(72 + \tfrac{30}{60}\right)° = 72.5°$$

40. $\sec 222°30' \approx -1.3563417$

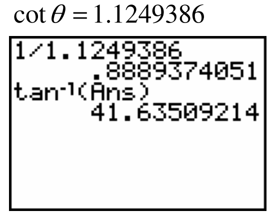


$$222°30' = \left(222 + \tfrac{30}{60}\right)° = 222.5°$$

41. $\cot 305.6° \approx -.71592968$

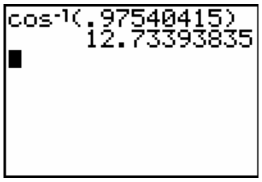


42. $\cot\theta = 1.1249386$



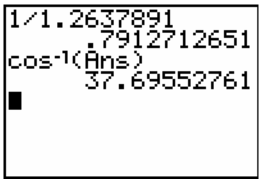
$\theta \approx 41.635092°$

43. $\cos\theta = .97540415$



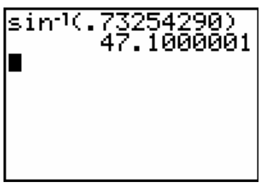
$\theta \approx 12.733938°$

44. $\sec\theta = 1.2637891$

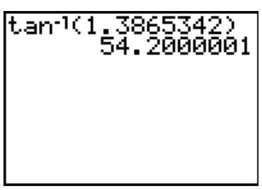


$\theta \approx 37.695528°$

45. Since the value of $\sin\theta$ is positive in quadrants I and II, the two angles in $[0°, 360°)$ are approximately $47.1°$ and $180° - 47.1° = 132.9°$.



46. Since the value of $\tan\theta$ is positive in quadrants I and III, the two angles in $[0°, 360°)$ are approximately 54.2º and 180º − 54.2º = 234.2º



47. $A = 58°\ 30',\ c = 748$

$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$

$B = 90° - 58°30' = 89°60' - 58°30'$
$= 31°30'$

$\sin A = \dfrac{a}{c} \Rightarrow \sin 58°30' = \dfrac{a}{748} \Rightarrow$

$a = 748 \sin 58°30' \approx 638$ (rounded to three significant digits)

$\cos A = \dfrac{b}{c} \Rightarrow \cos 58°30' = \dfrac{b}{748} \Rightarrow$

$b = 748 \cos 58°30' \approx 391$ (rounded to three significant digits)

48. $a = 129.70,\ b = 368.10$

$c = \sqrt{a^2 + b^2} \Rightarrow c = \sqrt{129.70^2 + 368.10^2}$
≈ 390.28 (rounded to five significant digits)

$\tan A = \dfrac{a}{b} = \dfrac{129.70}{368.10} \Rightarrow$

$A = \tan^{-1}\left(\dfrac{129.70}{368.10}\right) \approx 19.41°$

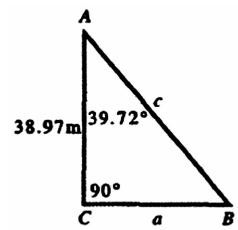
$\approx 19° + (.41 \cdot 60)' \approx 19°25'$

$\tan B = \dfrac{b}{a} = \dfrac{368.10}{129.70} \Rightarrow$

$B = \tan^{-1}\left(\dfrac{368.10}{129.70}\right) \approx 70.59°$

$\approx 70° + (.59 \cdot 60)' \approx 70°35'$

49. $A = 39.72°,\ b = 38.97$ m



$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$

$B = 90° - 39.72° = 50.28°$

$\tan A = \dfrac{a}{b} \Rightarrow \tan 39.72° = \dfrac{a}{38.97} \Rightarrow$

$a = 38.97 \tan 39.72° \approx 32.38$ m (rounded to four significant digits)

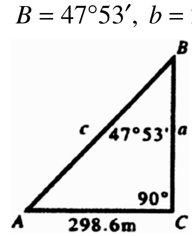
$\cos A = \dfrac{b}{c} \Rightarrow \cos 39.72° = \dfrac{38.97}{c} \Rightarrow$

$c \cos 39.72° = 38.97 \Rightarrow$

$c = \dfrac{38.97}{\cos 39.72°} \approx 50.66$ m

(rounded to four significant digits)

50. $B = 47°53',\ b = 298.6$ m



$A + B = 90° \Rightarrow A = 90° - B \Rightarrow$

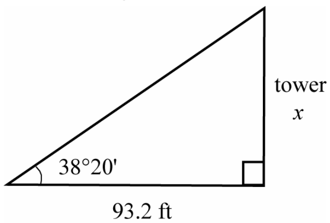
$A = 90° - 47°53' = 89°60' - 47°53'$
$= 42°7'$

(*continued on next page*)

(*continued from page 525*)

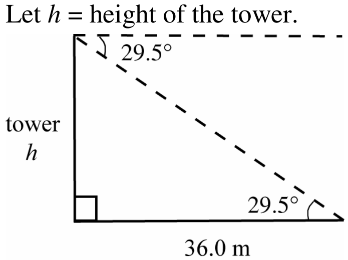
$$\tan B = \frac{b}{a} \Rightarrow \tan 47°53' = \frac{298.6}{a} \Rightarrow$$
$$a \tan 47°53' = 298.6 \Rightarrow$$
$$a = \frac{298.6}{\tan 47°53'} \approx 270.0 \text{ m}$$
(rounded to four significant digits)
$$\sin B = \frac{b}{c} \Rightarrow \sin 47°53' = \frac{298.6}{c} \Rightarrow$$
$$c \sin 47°53' = 298.6 \Rightarrow$$
$$c = \frac{298.6}{\sin 47°53'} \approx 402.5 \text{ m}$$
(rounded to four significant digits)

51. Let x = height of the tower.



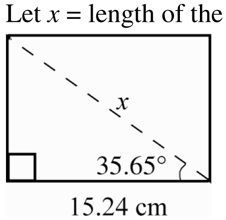
$$\tan 38°20' = \frac{x}{93.2}$$
$$x = 93.2 \tan 38°20'$$
$$x \approx 73.6930$$
The height of the tower is 73.7 ft. (rounded to three significant digits)

52. Let h = height of the tower.



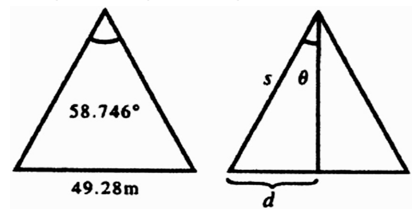
$$\tan 29.5° = \frac{h}{36.0}$$
$$h = 36.0 \tan 29.5°$$
$$h \approx 20.3678$$
The height of the tower is 20.4 m. (rounded to three significant digits)

53. Let x = length of the diagonal



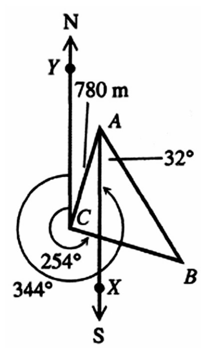
$$\cos 35.65° = \frac{15.24}{x}$$
$$x = \frac{15.24}{\cos 35.65°} \approx 18.7548$$
The length of the diagonal is 18.75 cm (rounded to three significant digits).

54. Let x = the length of the equal sides of an isosceles triangle.
Divide the isosceles triangle into two congruent right triangles



$$d = \frac{1}{2}(49.28) = 24.64 \text{ and}$$
$$\theta = \frac{1}{2}(58.746°) = 29.373°$$
$$\sin\theta = \frac{d}{s} \Rightarrow \sin 29.373° = \frac{24.64}{s} \Rightarrow$$
$$s = \frac{24.64}{\sin 29.373°} \approx 50.2352$$
Each side is 50.24 m long (rounded to 4 significant digits).

55. Draw triangle ABC and extend the north-south lines to a point X south of A and S to a point Y, north of C.



Angle ACB = 344° – 254° = 90°, so ABC is a right triangle.
Angle BAX = 32° since it is an alternate interior angle to 32°.
Angle YCA = 360° – 344° = 16°
Angle XAC = 16° since it is an alternate interior angle to angle YCA.
Angle BAC = 32° + 16° = 48°.

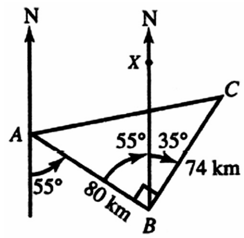
In triangle ABC,

$$\cos A = \frac{AC}{AB} \Rightarrow \cos 48° = \frac{780}{AB} \Rightarrow$$

$$AB \cos 48° = 780 \Rightarrow AB = \frac{780}{\cos 48°} \approx 1165.6917$$

The distance from A to B is 1200 m. (rounded to two significant digits)

56. Draw triangle ABC and extend north-south lines from points A and B. Angle ABX is 55° (alternate interior angles of parallel lines cut by a transversal have the same measure) so Angle ABC is 55° + 35° = 90°.



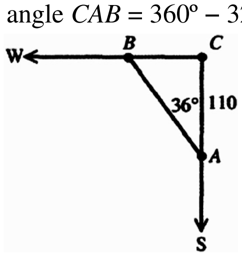
Since angle ABC is a right angle, use the Pythagorean theorem to find the distance from A to C.

$$(AC)^2 = 80^2 + 74^2 \Rightarrow (AC)^2 = 6400 + 5476 \Rightarrow$$

$$(AC)^2 = 11{,}876 \Rightarrow AC = \sqrt{11{,}876} \approx 108.9771$$

It is 110 km from A to C. (rounded to two significant digits)

57. Suppose A is the car heading south at 55 mph, B is the car heading west, and point C is the intersection from which they start. After two hours, using $d = rt$, $AC = 55(2) = 110$. Angle ACB is a right angle, so triangle ACB is a right triangle. The bearing of A from B is 324º, so angle CAB = 360º − 324º = 36º.



$$\cos \angle CAB = \frac{AC}{AB} \Rightarrow \cos 36° = \frac{110}{AB} \Rightarrow$$

$$AB = \frac{110}{\cos 36°} \approx 135.9675$$

The distance from A to B is about 140 mi (rounded to two significant digits).

58. $h = R\left(\frac{1}{\cos\left(\frac{180T}{P}\right)} - 1\right)$

(a) Let R = 3955 mi, T = 25 min, P = 140 min.

$$h = R\left(\frac{1}{\cos\left(\frac{180T}{P}\right)} - 1\right)$$

$$h = 3955\left(\frac{1}{\cos\left(\frac{180 \cdot 25}{140}\right)} - 1\right) \approx 715.9424$$

The height of the satellite is approximately 716 mi.

(b) Let R = 3955 mi, T = 30 min, P = 140 min.

$$h = R\left(\frac{1}{\cos\left(\frac{180T}{P}\right)} - 1\right)$$

$$h = 3955\left(\frac{1}{\cos\left(\frac{180 \cdot 30}{140}\right)} - 1\right) \approx 1103.6349$$

The height of the satellite is approximately 1104 mi.

Chapter 5 Test

1. 67°

(a) complement: 90° − 67° = 23°

(b) supplement: 180° − 67° = 113°

2. **(a)**
$$74°18'36'' = 74° + \tfrac{18}{60}° + \tfrac{36}{3600}° \approx 74° + .3° + .01° = 74.31°$$

(b)
$$\begin{aligned} 45.2025° &= 45° + .2025° \\ &= 45° + .2025(60') \\ &= 45° + 12.15' \\ &= 45° + 12' + .15' \\ &= 45° + 12' + .15(60'') \\ &= 45° + 12' + 09'' \\ &= 45°12'9'' \end{aligned}$$

3. **(a)** 390° is coterminal with 390° − 360° = 30°.

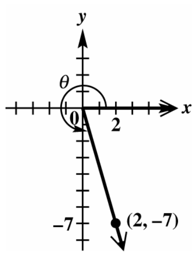
(b) −80° is coterminal with −80° + 360° = 280°.

(c) 810° is coterminal with 810° − 2(360°) = 810° − 720° = 90°.

4. $\dfrac{450(360°)}{1\text{ min}} = \dfrac{450(360°)}{60\text{ sec}} = \dfrac{450(6°)}{\text{sec}}$
$= 2700°/\text{sec}$

A point on the tire rotates 2700° in one second.

5.



(2, −7)
$x = 2,\ y = -7$

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{53}$$

$$\sin\theta = \frac{y}{r} = \frac{-7}{\sqrt{53}} = -\frac{7}{\sqrt{53}}\cdot\frac{\sqrt{53}}{\sqrt{53}} = -\frac{7\sqrt{53}}{53}$$

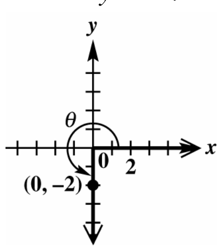
$$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{53}} = \frac{2}{\sqrt{53}}\cdot\frac{\sqrt{53}}{\sqrt{53}} = \frac{2\sqrt{53}}{53}$$

$$\tan\theta = \frac{y}{x} = \frac{-7}{2} = -\frac{7}{2};\ \cot\theta = \frac{x}{y} = \frac{2}{-7} = -\frac{2}{7}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{53}}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{53}}{-7} = -\frac{\sqrt{53}}{7}$$

6.



(0, −2)
$x = 0,\ y = -2$

$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-2)^2} = \sqrt{0 + 4} = \sqrt{4} = 2$$

$$\sin\theta = \frac{y}{r} = \frac{-2}{2} = -1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{2} = 0$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{0}\ \text{undefined}$$

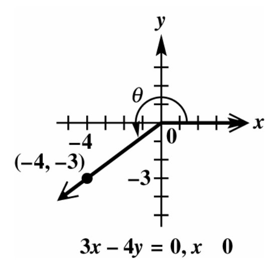
$$\cot\theta = \frac{x}{y} = \frac{0}{-2} = 0$$

$$\sec\theta = \frac{r}{x} = \frac{2}{0}\ \text{undefined}$$

$$\csc\theta = \frac{r}{y} = \frac{2}{-2} = -1$$

7. Since $x \le 0$, the graph of the line $3x - 4y = 0$ is shown to the left of the y-axis. A point on this graph is $(-4, -3)$ since $3(-4) - 4(-3) = 0$. The corresponding value of r is

$$r = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$



(−4, −3)
$x = -4,\ y = -3$

$$\sin\theta = \frac{y}{r} = -\frac{3}{5};\ \cos\theta = \frac{x}{r} = -\frac{4}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{4};\ \cot\theta = \frac{x}{y} = \frac{4}{3}$$

$$\sec\theta = \frac{r}{x} = -\frac{5}{4};\ \csc\theta = \frac{r}{y} = -\frac{5}{3}$$

8.

θ	90º	–360º	630º
$\sin\theta$	1	0	1
$\cos\theta$	0	1	0
$\tan\theta$	undefined	0	Undefined
$\cot\theta$	0	undefined	0
$\sec\theta$	undefined	1	Undefined
$\csc\theta$	1	undefined	1

9. **(a)** $\cos\theta > 0$ in quadrants I and IV, while $\tan\theta > 0$ in quadrants I and III. So, both conditions are met only in quadrant I.

(b) $\sin\theta < 0$ in quadrants III and IV. Since $\csc\theta$ is the reciprocal of $\sin\theta < 0$, $\csc\theta < 0$ also in quadrants III and IV. Thus, both conditions are met in quadrants III and IV.

(c) $\cot\theta > 0$ in quadrants I and III, while $\cos < 0$ in quadrants II and III. Both conditions are met only in quadrant III.

10. $\sin\theta = \frac{3}{7}$ with θ in quadrant II

θ in quadrant II $\Rightarrow x < 0, y > 0$

$\sin\theta = \frac{y}{r} = \frac{3}{7} \Rightarrow y = 3, r = 7$

$r^2 = x^2 + y^2 \Rightarrow 7^2 = x^2 + 3^2 \Rightarrow 49 = x^2 + 9 \Rightarrow$
$x^2 = 40 \Rightarrow x = -\sqrt{40} = -2\sqrt{10}$

$\cos\theta = \frac{x}{r} = \frac{-2\sqrt{10}}{7} = -\frac{2\sqrt{10}}{7}$

$\tan\theta = \frac{y}{x} = \frac{3}{-2\sqrt{10}} = -\frac{3}{2\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{20}$

$\cot\theta = \frac{x}{y} = \frac{-2\sqrt{10}}{7}$

$\sec\theta = \frac{r}{x} = \frac{7}{-2\sqrt{10}} = -\frac{7}{2\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{7\sqrt{10}}{20}$

$\csc\theta = \frac{r}{y} = \frac{7}{3}$

11. $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{12}{13}$

$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{5}{13}$

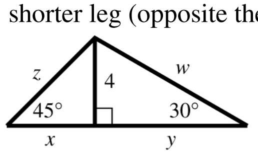
$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{12}{5}$

$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{5}{12}$

$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{13}{5}$

$\csc A = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{13}{12}$

12. Apply the relationships between the lengths of the sides of a $30° - 60°$ right triangle first to the triangle on the right to find the values of *y* and *w*. In the $30° - 60°$ right triangle, the side opposite the $60°$ angle is $\sqrt{3}$ times as long as the side opposite to the $30°$ angle. The length of the hypotenuse is 2 times as long as the shorter leg (opposite the $30°$ angle).



Thus, we have $y = 4\sqrt{3}$ and $w = 2(4) = 8$.

Apply the relationships between the lengths of the sides of a $45° - 45°$ right triangle next to the triangle on the left to find the values of *x* and *z*.

In the $45° - 45°$ right triangle, the sides opposite the $45°$ angles measure the same. The hypotenuse is $\sqrt{2}$ times the measure of a leg. Thus, we have $x = 4$ and $z = 4\sqrt{2}$

13. A 240º angle lies in quadrant III, so the reference angle is $240° - 180° = 60°$. Since 240° is in quadrant III, the sine, cosine, secant, and cosecant are negative.

$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$

$\cos 240° = -\cos 60° = -\frac{1}{2}$

$\tan 240° = \tan 60° = \sqrt{3}$

$\cot 240° = \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sec 240° = -\sec 60° = -2$

$\csc 240° = -\csc 60° = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

14. –135° is coterminal with –135° + 360° = 225°. This angle lies in quadrant III. The reference angle is $225° - 180° = 45°$. Since –135° is in quadrant III, the sine, cosine, secant, and cosecant are negative.

$\sin(-135°) = -\sin 45° = -\frac{\sqrt{2}}{2}$

$\cos(-35°) = -\cos 45° = -\frac{\sqrt{2}}{2}$

$\tan(-135°) = \tan 45° = 1$

$\cot(-135°) = \cot 45° = 1$

$\sec(-135°) = -\sec 45° = -\sqrt{2}$

$\csc(-135°) = -\csc 45° = -\sqrt{2}$

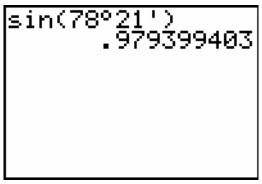
15. $\cos\theta = -\frac{\sqrt{2}}{2}$

Since $\cos\theta$ is negative, θ must lie in quadrant II or quadrant III. The absolute value of $\cos\theta$ is $\frac{\sqrt{2}}{2}$, so $\theta' = 45°$. The quadrant II angle θ equals $180° - \theta' = 180° - 45° = 135°$, and the quadrant III angle θ equals $180° + \theta' = 180° + 45° = 225°$.

16. $\csc\theta = -\dfrac{2\sqrt{3}}{3}$

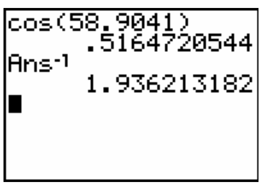
Since $\csc\theta$ is negative, θ must lie in quadrant III or quadrant IV. The absolute value of $\csc\theta$ is $\dfrac{2\sqrt{3}}{3}$, so $\theta' = 60°$. The quadrant III angle θ equals $180° + \theta' = 180° + 60° = 240°$, and the quadrant IV angle θ equals $360° - \theta' = 360° - 60° = 300°$.

17. **(a)** $\sin 78°21' \approx .97939940$

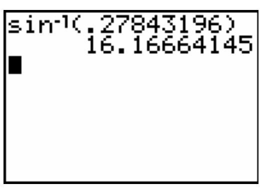


$78°21' = \left(78 + \frac{21}{60}\right)° = 78.35°$

(b) $\sec 58.9041° \approx 1.9362432$

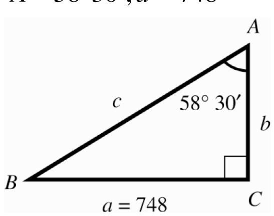


18. $\sin\theta = .27843196$



$\theta \approx 16.16664145°$

19. $A = 58°30', a = 748$



$A + B = 90° \Rightarrow B = 90° - A \Rightarrow$
$B = 90° - 58°30' = 31°30'$

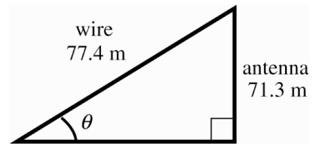
$\tan A = \dfrac{a}{b} \Rightarrow \tan 58°30' = \dfrac{748}{b} \Rightarrow$

$b = \dfrac{748}{\tan 58°30'} \approx 458$ (rounded to three significant digits

$\sin A = \dfrac{a}{c} \Rightarrow \sin 58°30' = \dfrac{748}{c} \Rightarrow$

$c = \dfrac{748}{\sin 58°30'} \approx 877$ (rounded to three significant digits

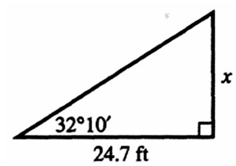
20. Let θ = the measure of the angle that the guy wire makes with the ground.



$\sin\theta = \dfrac{71.3}{77.4}$

$\theta = \sin^{-1}\left(\dfrac{71.3}{77.4}\right) \approx 67.1° \approx 67°10'$

21. Let x = the height of the flagpole.

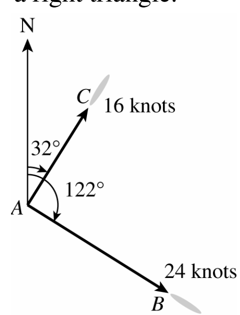


$\tan 32°10' = \dfrac{x}{24.7}$

$x = 24.7 \tan 32°10' \approx 15.5344$

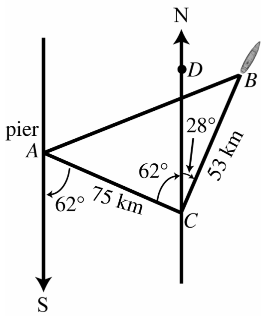
The flagpole is approximately 15.5 ft high. (rounded to three significant digits)

22. Let x = distance the ships are apart. In the figure, the measure of angle CAB is $122° - 32° = 90°$. Therefore, triangle CAB is a right triangle.



Since $d = rt$, the distance traveled by the first ship in 2.5 hr is
(2.5 hr)(16 knots) = 40 nautical mi and the second ship is
(2.5hr)(24 knots) = 60 nautical mi.
Applying the Pythagorean theorem, we have
$x^2 = 40^2 + 60^2 \Rightarrow x^2 = 1600 + 3600 \Rightarrow$
$x^2 = 5200 \Rightarrow x = \sqrt{5200} \approx 72.111$
The ships are 72 nautical mi apart. (rounded to 2 significant digits)

23. Draw triangle ACB and extend north-south lines from points A and C. Angle ACD is 62° (alternate interior angles of parallel lines cut by a transversal have the same measure), so Angle ACB is 62° + 28° = 90°.

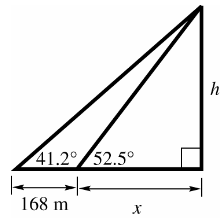


Since angle ACB is a right angle, use the Pythagorean theorem to find the distance from A to B.

$$(AB)^2 = 75^2 + 53^2 \Rightarrow (AB)^2 = 5625 + 2809 \Rightarrow$$
$$(AB)^2 = 8434 \Rightarrow AB = \sqrt{8434} \approx 91.8368$$

It is 92 km from the pier to the boat, rounded to two significant digits.

24. Let x = the side adjacent to 52.5° in the smaller triangle.



In the larger triangle, we have

$$\tan 41.2° = \frac{h}{168 + x} \Rightarrow h = (168 + x)\tan 41.2°.$$

In the smaller triangle, we have

$$\tan 52.5° = \frac{h}{x} \Rightarrow h = x\ \tan 52.5°.$$

Substitute for h in this equation to solve for x.

$$(168 + x)\tan 41.2° = x\tan 52.5°$$
$$168\tan 41.2° + x\tan 41.2° = x\tan 52.5°$$
$$168\tan 41.2° = x\tan 52.5° - x\tan 41.2°$$
$$168\tan 41.2° = x(\tan 52.5° - \tan 41.2°)$$
$$\frac{168\tan 41.2°}{\tan 52.5° - \tan 41.2°} = x$$

Substituting for x in the equation for the smaller triangle gives

$$h = x\tan 52.5°$$
$$h = \frac{168\tan 41.2°\tan 52.5°}{\tan 52.5° - \tan 41.2°} \approx 448.0432$$

The height of the triangle is approximately 448 m. (rounded to three significant digits)

Chapter 5: Quantitative Reasoning

$$D = \frac{v^2\sin\theta\cos\theta + v\cos\theta\sqrt{(v\sin\theta)^2 + 64h}}{32}$$

All answers are rounded to four significant digits.

1. Since $v = 44$ ft per sec and $h = 7$ ft, we have $D = \dfrac{44^2\sin\theta\cos\theta + 44\cos\theta\sqrt{(44\sin\theta)^2 + 64\cdot 7}}{32}$

If $\theta = 40°$, $D = \dfrac{1936\sin 40\cos 40 + 44\cos 40\sqrt{(44\sin 40)^2 + 448}}{32} \approx 67.00$ ft.

If $\theta = 42°$, $D = \dfrac{1936\sin 42\cos 42 + 44\cos 42\sqrt{(44\sin 42)^2 + 448}}{32} \approx 67.14$ ft

If $\theta = 45°$, $D = \dfrac{1936\sin 45\cos 45 + 44\cos 45\sqrt{(44\sin 45)^2 + 448}}{32} \approx 66.84$ ft

As θ increases, D increases and then decreases.

2. Since $h = 7$ ft and $\theta = 42°$, we have $D = \dfrac{v^2 \sin 42 \cos 42 + v \cos 42 \sqrt{(v \sin 42)^2 + 64h}}{32}$

If $v = 43$, $D = \dfrac{43^2 \sin 42 \cos 42 + 43 \cos 42 \sqrt{(43 \sin 42)^2 + 448}}{32} \approx 64.40$ ft

If $v = 44$, $D = \dfrac{44^2 \sin 42 \cos 42 + 44 \cos 42 \sqrt{(44 \sin 42)^2 + 448}}{32} \approx 67.14$ ft

If $v = 45$, $D = \dfrac{45^2 \sin 42 \cos 42 + 45 \cos 42 \sqrt{(45 \sin 42)^2 + 448}}{32} \approx 69.93$ ft

As v increases, D increases.

3. The velocity affects the distance more. The shot-putter should concentrate on achieving as large a value of v as possible.

Chapter 6

THE CIRCULAR FUNCTIONS AND THEIR GRAPHS

Section 6.1: Radian Measure

1. Since θ is in quadrant I, $0 < \theta < \frac{\pi}{2}$. Since $\frac{\pi}{2} \approx 1.57$, 1 is the only integer value in the interval. Thus, the radian measure of θ is 1 radian.

2. Since θ is in quadrant II, $\frac{\pi}{2} < \theta < \pi$. Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, 2 and 3 are the only integers in the interval. Since θ is closer to $\frac{\pi}{2}$, the radian measure of θ is 2 radians.

3. Since θ is in quadrant II, $\frac{\pi}{2} < \theta < \pi$. Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, 2 and 3 are the only integers in the interval. Since θ is closer to π, the radian measure of θ is 3 radians.

4. Since θ is an angle in quadrant IV drawn in a clockwise direction, $-\frac{\pi}{2} < \theta < 0$. Also $-\frac{\pi}{2} \approx -1.57$, and –1 is the only integer in the interval. Thus, the radian measure of θ is –1 radian.

5. Since θ is an angle in quadrant III drawn in a clockwise direction, $-\pi < \theta < -\frac{\pi}{2}$. Also $-\pi \approx -3.14$ and $-\frac{\pi}{2} \approx -1.57$, so –2 and –3 are the only integers in the interval. θ is closer to $-\pi$, so the radian measure of θ is –3 radians.

6. Since θ is an angle in quadrant III drawn in a clockwise direction, $-\pi < \theta < -\frac{\pi}{2}$. Also $-\pi \approx -3.14$ and $-\frac{\pi}{2} \approx -1.57$, so –2 and –3 are the only integers in the interval. θ is closer to $-\frac{\pi}{2}$, so the radian measure of θ is –2 radians.

7. $60° = 60\left(\frac{\pi}{180} \text{ radian}\right) = \frac{\pi}{3}$ radians

8. $30° = 30\left(\frac{\pi}{180} \text{ radian}\right) = \frac{\pi}{6}$ radians

9. $90° = 90\left(\frac{\pi}{180} \text{ radian}\right) = \frac{\pi}{2}$ radians

10. $120° = 120\left(\frac{\pi}{180} \text{ radian}\right) = \frac{2\pi}{3}$ radians

11. $150° = 150\left(\frac{\pi}{180} \text{ radian}\right) = \frac{5\pi}{6}$ radians

12. $270° = 270\left(\frac{\pi}{180} \text{ radian}\right) = \frac{3\pi}{2}$ radians

13. $-300° = -300\left(\frac{\pi}{180} \text{ radian}\right) = -\frac{5\pi}{3}$ radians

14. $-315° = -315\left(\frac{\pi}{180} \text{ radian}\right) = -\frac{7\pi}{4}$ radians

15. $450° = 450\left(\frac{\pi}{180} \text{ radian}\right) = \frac{5\pi}{2}$ radians

16. $480° = 480\left(\frac{\pi}{180} \text{ radian}\right) = \frac{8\pi}{3}$ radians

17. $1800° = 1800°\left(\frac{\pi}{180°} \text{ radian}\right) = 10\pi$ radians

18. $-3600° = -3600°\left(\frac{\pi}{180°} \text{ radian}\right)$
$= -20\pi$ radians

19. Multiply the degree measure by $\frac{\pi}{180}$ radian and reduce. Your answer will be in radians. Leave the answer as a multiple of π, unless otherwise directed.

20. Multiply the radian measure by $\frac{180°}{\pi}$ and reduce. Your answer will be in degrees.

21.–24. Answers will vary.

25. $\frac{\pi}{3} = \frac{\pi}{3}\left(\frac{180°}{\pi}\right) = 60°$

26. $\frac{8\pi}{3} = \frac{8\pi}{3}\left(\frac{180°}{\pi}\right) = 480°$

27. $\frac{7\pi}{4} = \frac{7\pi}{4}\left(\frac{180°}{\pi}\right) = 315°$

28. $\frac{2\pi}{3} = \frac{2\pi}{3}\left(\frac{180°}{\pi}\right) = 120°$

29. $\frac{11\pi}{6} = \frac{11\pi}{6}\left(\frac{180°}{\pi}\right) = 330°$

30. $\frac{15\pi}{4} = \frac{15\pi}{4}\left(\frac{180°}{\pi}\right) = 675°$

31. $-\frac{\pi}{6} = -\frac{\pi}{6}\left(\frac{180°}{\pi}\right) = -30°$

32. $-\frac{8\pi}{5} = -\frac{8\pi}{5}\left(\frac{180°}{\pi}\right) = -288°$

33. $\frac{7\pi}{10} = \frac{7\pi}{10}\left(\frac{180°}{\pi}\right) = 126°$

34. $\frac{11\pi}{15} = \frac{11\pi}{15}\left(\frac{180°}{\pi}\right) = 132°$

35. $-\frac{4\pi}{15} = -\frac{4\pi}{15}\left(\frac{180°}{\pi}\right) = -48°$

36. $-\frac{7\pi}{20} = -\frac{7\pi}{20}\left(\frac{180°}{\pi}\right) = -63°$

37. $\frac{17\pi}{20} = \frac{17\pi}{20}\left(\frac{180°}{\pi}\right) = 153°$

38. $\frac{11\pi}{30} = \frac{11\pi}{30}\left(\frac{180°}{\pi}\right) = 66°$

39. $-5\pi = -5\pi\left(\frac{180°}{\pi}\right) = -900°$

40. $15\pi = 15\pi\left(\frac{180°}{\pi}\right) = 2700°$

41. $39° = 39\left(\frac{\pi}{180}\text{ radian}\right) \approx .68\text{ radian}$

42. $74° = 74\left(\frac{\pi}{180}\text{ radian}\right) \approx 1.29\text{ radians}$

43. $42.5° = 42.5\left(\frac{\pi}{180}\text{ radian}\right) \approx .742\text{ radians}$

44. $264.9° = 264.9\left(\frac{\pi}{180}\text{ radian}\right) \approx 4.623\text{ radians}$

45. $139°10' = \left(139 + \frac{10}{60}\right)°$
$\approx 139.1666667\left(\frac{\pi}{180}\text{ radian}\right)$
$\approx 2.43\text{ radians}$

46. $174°50' = \left(174 + \frac{50}{60}\right)°$
$\approx 174.8333333\left(\frac{\pi}{180}\text{ radian}\right)$
$\approx 3.05\text{ radians}$

47. $64.29° = 64.29\left(\frac{\pi}{180}\text{ radian}\right) \approx 1.122\text{ radians}$

48. $85.04° = 85.04\left(\frac{\pi}{180}\text{ radian}\right) \approx 1.484\text{ radians}$

49. $56°25' = \left(56 + \frac{25}{60}\right)°$
$\approx 56.41666667\left(\frac{\pi}{180}\text{ radian}\right)$
$\approx .9847\text{ radians}$

50. $122°37' = \left(122 + \frac{37}{60}\right)°$
$\approx 122.61666667\left(\frac{\pi}{180}\text{ radian}\right)$
$\approx 2.140\text{ radians}$

51. $47.6925° = 47.6925\left(\frac{\pi}{180}\text{ radian}\right)$
$\approx .832391\text{ radian}$

52. $23.0143° = 23.0143\left(\frac{\pi}{180}\text{ radian}\right)$
$\approx .401675\text{ radian}$

53. $2\text{ radians} = 2\left(\frac{180°}{\pi}\right) \approx 114.591559°$
$= 114° + (.591559 \cdot 60)'$
$\approx 114°35'$

54. $5 \text{ radians} = 5\left(\frac{180°}{\pi}\right) \approx 286.4788976°$
$= 286° + (.4788976 \cdot 60)'$
$\approx 286°29'$

55. $1.74 \text{ radians} = 1.74\left(\frac{180°}{\pi}\right) \approx 99.69465635°$
$= 99° + (.69465635 \cdot 60)'$
$\approx 99°42'$

56. $3.06 \text{ radians} = 3.06\left(\frac{180°}{\pi}\right) \approx 175.3250853°$
$= 175° + (.3250853 \cdot 60)'$
$\approx 175°20'$

57. $.3417 \text{ radian} = .3417\left(\frac{180°}{\pi}\right) \approx 19.57796786°$
$= 19° + (.57796786 \cdot 60)'$
$= 19°35'$

58. $9.84763 \text{ radian} = 9.84763\left(\frac{180°}{\pi}\right)$
$\approx 564.2276372°$
$= 564° + (.2276372 \cdot 60)'$
$\approx 564°14'$

59. $-5.01095 \text{ radian} = -5.01095\left(\frac{180°}{\pi}\right)$
$\approx -287.1062864°$
$= -\left[287° + (.1062864 \cdot 60)'\right]$
$\approx -287°6'$

60. $-3.47189 \text{ radians} = -3.47189\left(\frac{180°}{\pi}\right)$
$\approx -198.9246439°$
$= -\left[198° + (.9246439 \cdot 60)'\right]$
$\approx -198°55'$

61. Begin the calculation with the blank next to 30º, and then proceed counterclockwise from there.

$30° = 30\left(\frac{\pi}{180} \text{ radian}\right) = \frac{\pi}{6} \text{ radians}$

$\frac{\pi}{4} \text{ radians} = \frac{\pi}{4}\left(\frac{180°}{\pi}\right) = 45°$

$60° = 60\left(\frac{\pi}{180} \text{ radian}\right) = \frac{\pi}{3} \text{ radians}$

$\frac{2\pi}{3} \text{ radians} = \frac{2\pi}{3}\left(\frac{180°}{\pi}\right) = 120°$

$\frac{3\pi}{4} \text{ radians} = \frac{3\pi}{4}\left(\frac{180°}{\pi}\right) = 135°$

$150° = 150\left(\frac{\pi}{180} \text{ radian}\right) = \frac{5\pi}{6} \text{ radians}$

$180° = 180\left(\frac{\pi}{180} \text{ radian}\right) = \pi \text{ radians}$

$210° = 210\left(\frac{\pi}{180} \text{ radian}\right) = \frac{7\pi}{6} \text{ radians}$

$225° = 225\left(\frac{\pi}{180} \text{ radian}\right) = \frac{5\pi}{4} \text{ radians}$

$\frac{4\pi}{3} \text{ radians} = \frac{4\pi}{3}\left(\frac{180°}{\pi}\right) = 240°$

$\frac{5\pi}{3} \text{ radians} = \frac{5\pi}{3}\left(\frac{180°}{\pi}\right) = 300°$

$315° = 315\left(\frac{\pi}{180} \text{ radian}\right) = \frac{7\pi}{4} \text{ radians}$

$330° = 330\left(\frac{\pi}{180} \text{ radian}\right) = \frac{11\pi}{6} \text{ radians}$

62. Answers will vary.

63. $r = 4,\ \theta = \frac{\pi}{2}$
$s = r\theta = 4\left(\frac{\pi}{2}\right) = 2\pi$

64. $r = 12,\ \theta = \frac{\pi}{3}$
$s = r\theta = 12\left(\frac{\pi}{3}\right) = 4\pi$

65. $r = 16,\ \theta = \frac{5\pi}{4}$
$s = r\theta = 16\left(\frac{5\pi}{4}\right) = 20\pi$

66. $s = 6\pi,\ \theta = \frac{3\pi}{4}$
$s = r\theta \Rightarrow r = \frac{s}{\theta} = \frac{6\pi}{\frac{3\pi}{4}} = 6\pi \cdot \frac{4}{3\pi} = 8$

67. $s = 3\pi,\ \theta = \frac{\pi}{2}$
$s = r\theta \Rightarrow r = \frac{s}{\theta} = \frac{3\pi}{\frac{\pi}{2}} = 3\pi \cdot \frac{2}{\pi} = 6$

68. $s = 14\pi,\ \theta = \frac{7\pi}{4}$

$s = r\theta \Rightarrow r = \frac{s}{\theta} = \frac{14\pi}{\frac{7\pi}{4}} = 14\pi \cdot \frac{4}{7\pi} = 8$

69. $r = 3,\ s = 3$

$s = r\theta \Rightarrow \theta = \frac{s}{r} = \frac{3}{3} = 1$

70. $s = 6,\ r = 4$

$s = r\theta \Rightarrow \theta = \frac{s}{r} = \frac{6}{4} = \frac{3}{2}$ or 1.5

71. $s = 20,\ r = 10$

$s = r\theta \Rightarrow \theta = \frac{s}{r} = \frac{20}{10} = 2$

72. Answers will vary.

73. $r = 12.3$ cm, $\theta = \frac{2\pi}{3}$ radians

$s = r\theta = 12.3\left(\frac{2\pi}{3}\right) = 8.2\pi \approx 25.8$ cm

74. $r = .892$ cm, $\theta = \frac{11\pi}{10}$ radians

$s = r\theta = .892\left(\frac{11\pi}{10}\right)$
$= .9812\pi$ cm ≈ 3.08 cm

75. $r = 1.38$ ft, $\theta = \frac{5\pi}{6}$ radians

$s = r\theta = 1.38\left(\frac{5\pi}{6}\right)$
$= 1.15\pi$ ft ≈ 3.61 ft (rounded to three significant digits)

76. $r = 3.24$ mi, $\theta = \frac{7\pi}{6}$ radians

$s = r\theta = 3.24\left(\frac{7\pi}{6}\right) = 3.78\pi$ mi ≈ 11.9 mi

(rounded to three significant digits)

77. $r = 4.82$ m, $\theta = 60°$

Converting θ to radians, we have

$\theta = 60° = 60\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{3}$ radians.

Thus, the arc is

$s = r\theta = 4.82\left(\frac{\pi}{3}\right) = \frac{4.82\pi}{3} \approx 5.05$ m.

(rounded to three significant digits)

78. $r = 71.9$ cm, $\theta = 135°$

Converting θ to radians, we have

$\theta = 135° = 135\left(\frac{\pi}{180}\text{ radian}\right) = \frac{3\pi}{4}$ radians.

Thus, the arc is

$s = r\theta = 71.9\left(\frac{3\pi}{4}\right)\text{cm} = \frac{215.7\pi}{4}\text{ cm} \approx 169$ cm.

(rounded to three significant digits)

79. $r = 15.1$ in., $\theta = 210°$

Converting θ to radians, we have

$\theta = 210° = 210\left(\frac{\pi}{180}\text{ radian}\right) = \frac{7\pi}{6}$ radians.

Thus, the arc is

$s = r\theta = 15.1\left(\frac{7\pi}{6}\right) = \frac{105.7\pi}{6} \approx 55.3$ in.

(rounded to three significant digits)

80. $r = 12.4$ ft, $\theta = 330°$

Converting θ to radians, we have

$\theta = 330° = 330\left(\frac{\pi}{180}\text{ radian}\right) = \frac{11\pi}{6}$ radians.

Thus, the arc is

$s = r\theta = 12.4\left(\frac{11\pi}{6}\right) = \frac{136.4\pi}{6} \approx 71.4$ ft

(rounded to three significant digits)

81. The formula for arc length is $s = r\theta$. Substituting $2r$ for r we obtain $s = (2r)\theta = 2(r\theta)$. The length of the arc is doubled.

82. Recall that π radians = 180°. If θ is measured in degrees, then the formula becomes

$s = r \cdot \frac{\theta\pi}{180} \Rightarrow s = \frac{\pi r\theta}{180}$

For Exercises 83–88, note that since 6400 has two significant digits and the angles are given to the nearest degree, we can have only two significant digits in the answers.

83. 9° N, 40° N

$\theta = 40° - 9° = 31° = 31\left(\frac{\pi}{180}\text{ radian}\right)$
$= \frac{31\pi}{180}$ radian

$s = r\theta = 6400\left(\frac{31\pi}{180}\right) \approx 3500$ km

84. 36° N, 49°N

$$\theta = 49° - 36° = 13° = 13\left(\frac{\pi}{180}\text{ radian}\right) = \frac{13\pi}{180}\text{ radian}$$

$$s = r\theta = 6400\left(\frac{13\pi}{180}\right) \approx 1500\text{ km}$$

85. 41° N, 12° S
12° S = –12° N

$$\theta = 41° - (-12°) = 53° = 53\left(\frac{\pi}{180}\text{ radian}\right) = \frac{53\pi}{180}\text{ radian}$$

$$s = r\theta = 6400\left(\frac{53\pi}{180}\right) \approx 5900\text{ km}$$

86. 45° N, 34° S
34° S = –34° N

$$\theta = 45° - (-34°) = 79° = 79\left(\frac{\pi}{180}\text{ radian}\right) = \frac{79\pi}{180}\text{ radians}$$

$$s = r\theta = 6400\left(\frac{79\pi}{180}\right) \approx 8800\text{ km}$$

87. r = 6400 km, s = 1200 km

$$s = r\theta \Rightarrow 1200 = 6400\theta \Rightarrow \theta = \frac{1200}{6400} = \frac{3}{16}$$

Converting $\frac{3}{16}$ radian to degrees, we have $\theta = \frac{3}{16}\left(\frac{180°}{\pi}\right) \approx 11°$. The north-south distance between the two cities is 11°.
Let x = the latitude of Madison.
$x - 33° = 11° \Rightarrow x = 44°$ N
The latitude of Madison is 44° N .

88. r = 6400 km, s = 1100 km

$$s = r\theta \Rightarrow 1100 = 6400\theta \Rightarrow \theta = \frac{1100}{6400} = \frac{11}{64}$$

Converting $\frac{11}{64}$ radian to degrees we have $\theta = \frac{11}{64}\cdot\frac{180°}{\pi} \approx 10°$. The north-south distance between the two cities is 10°.
Let x = the latitude of Toronto.
$x - 33° = 10° \Rightarrow x = 43°$ N
The latitude of Toronto is 43° N.

89. The arc length on the smaller gear is

$$s = r\theta = 3.7\left(300\cdot\frac{\pi}{180}\right) = 3.7\left(\frac{5\pi}{3}\right) = \frac{18.5\pi}{3}\text{ cm}$$

An arc with this length on the larger gear corresponds to an angle measure θ where

$$s = r\theta \Rightarrow \frac{18.5\pi}{3} = 7.1\theta \Rightarrow \frac{18.5\pi}{21.3} = \theta \Rightarrow$$

$$\theta = \frac{18.5\pi}{21.3}\cdot\frac{180}{\pi} \approx 156°$$

The larger gear will rotate through approximately 156º.

90. The arc length on the smaller gear is

$$s = r\theta = 4.8\left(315\cdot\frac{\pi}{180}\right) = 4.8\left(\frac{7\pi}{4}\right) = 8.4\pi\text{ in.}$$

An arc with this length on the larger gear corresponds to an angle measure θ where

$$s = r\theta \Rightarrow 8.4\pi = 7.1\theta \Rightarrow \frac{8.4\pi}{7.1} = \theta \Rightarrow$$

$$\theta = \frac{8.4\pi}{7.1}\cdot\frac{180}{\pi} \approx 213°$$

The larger gear will rotate through approximately 213º.

91. A rotation of

$\theta = 60.0\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{3}$ radians on the smaller wheel moves through an arc length of $s = r\theta = 5.23\left(\frac{\pi}{3}\right) = \frac{5.23\pi}{3}$ cm. (holding on to more digits for the intermediate steps)
Since both wheels move together, the larger wheel moves $\frac{5.23\pi}{3}$ 5.48 cm, which rotates it through an angle θ, where

$$\frac{5.23\pi}{3} = 8.16\theta$$

$$\theta = \frac{5.23\pi}{24.48}\text{ radian} = \frac{5.23\pi}{24.48}\left(\frac{180°}{\pi}\right) \approx 38.5°$$

The larger wheel rotates through 38.5°.

92. A rotation of

$\theta = 150\left(\frac{\pi}{180}\text{ radian}\right) = \frac{5\pi}{6}$ radians on the smaller wheel moves through an arc length of

$$s = r\theta = 6.84\left(\frac{5\pi}{6}\right) = 5.7\pi\text{ cm.}$$

(*continued on next page*)

(*continued from page 537*)

Since both wheels move together, the larger wheel moves 5.7π cm, which rotates it through an angle θ, where

$$5.7\pi = 12.46\theta$$
$$\theta = \frac{5.7\pi}{12.46} \text{ radian} = \frac{5.7\pi}{12.46}\left(\frac{180°}{\pi}\right) \approx 82.3°$$

The larger wheel rotates through 82.3°.

93. The arc length s represents the distance traveled by a point on the rim of a wheel. Since the two wheels rotate together, s will be the same for both wheels.
For the smaller wheel,

$$\theta = 80° = 80\left(\frac{\pi}{180}\right) = \frac{4\pi}{9} \text{ radians and}$$
$$s = r\theta = 11.7\left(\frac{4\pi}{9}\right) = 5.2\pi \text{ cm.}$$

For the larger wheel,

$$\theta = 50° = 50\left(\frac{\pi}{180}\text{ radian}\right) = \frac{5\pi}{18} \text{ radian.}$$

Thus, we can solve

$$s = r\theta \Rightarrow 5.2\pi = r\left(\frac{5\pi}{18}\right) \Rightarrow$$
$$r = 5.2\pi \cdot \frac{18}{5\pi} = 18.72$$

The radius of the larger wheel is 18.7 cm. (rounded to 3 significant digits)

94. The arc length s represents the distance traveled by a point on the rim of a wheel. Since the two wheels rotate together, s will be the same for both wheels.
For the smaller wheel,

$$\theta = 120° = 120\left(\frac{\pi}{180}\right) = \frac{2\pi}{3} \text{ radians and}$$
$$s = r\theta = 14.6\left(\frac{2\pi}{3}\right) = \frac{29.2\pi}{3} \text{ in.}$$

For the larger wheel,

$$\theta = 60° = 60\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{3} \text{ radian. Thus,}$$

we can solve $s = r\theta \Rightarrow \frac{29.2\pi}{3} = r\left(\frac{\pi}{3}\right) \Rightarrow$

$$r = \frac{29.2\pi}{3} \cdot \frac{3}{\pi} = 29.2$$

The radius of the larger wheel is 29.2 in. (rounded to 3 significant digits)

95. **(a)** The number of inches lifted is the arc length in a circle with $r = 9.27$ in. and $\theta = 71°50'$.

$$71°50' = \left(71 + \tfrac{50}{60}\right)\left(\frac{\pi}{180°}\right)$$
$$s = r\theta \Rightarrow$$
$$s = 9.27\left(71 + \tfrac{50}{60}\right)\left(\frac{\pi}{180°}\right) \approx 11.6221$$

The weight will rise 11.6 in. (rounded to three significant digits)

(b) When the weight is raised 6 in., we have $s = r\theta \Rightarrow 6 = 9.27\theta \Rightarrow$

$$\theta = \frac{6}{9.27}\text{ radian} = \frac{6}{9.27}\left(\frac{180°}{\pi}\right)$$
$$\approx 37.0846° = 37° + .0846(60') \approx 37°5'$$

The pulley must be rotated through $37°5'$.

96. To find the radius of the pulley, first convert 51.6° to radians.

$$\theta = 51.6° = 51.6\left(\frac{\pi}{180}\right) = \frac{51.6\pi}{180} \text{ radians}$$

Now substitute this value of θ and $s = 11.4$ cm into the equation $s = r\theta$ and solve for r.

$$s = r\theta \Rightarrow 11.4 = r\left(\frac{51.6\pi}{180}\right) \Rightarrow$$
$$r = 11.4 \cdot \frac{180}{51.6\pi} \approx 12.6584$$

The radius of the pulley is 12.7 cm. (rounded to three significant digits)

97. A rotation of

$$\theta = 180\left(\frac{\pi}{180}\text{ radian}\right) = \pi \text{ radians. The chain}$$

moves a distance equal to half the arc length of the larger gear. So, for the large gear and pedal, $s = r\theta \Rightarrow 4.72\pi$. Thus, the chain moves 4.72π in. The small gear rotates through an angle as follows.

$$\theta = \frac{s}{r} \Rightarrow \theta = \frac{4.72\pi}{1.38} \approx 3.42\pi$$

θ for the wheel and θ for the small gear are the same, or 3.42π. So, for the wheel, we have

$$s = r\theta \Rightarrow r = 13.6(3.42\pi) \approx 146.12$$

The bicycle will move 146 in. (rounded to three significant digits)

98. **(a)** In one hour, the car travels 55 mi. The radius is given in inches, so convert 55 mi to inches:

$$s = 55 \text{ mi} = 55(5280) \text{ ft} = 290,400 \text{ ft}$$
$$= 290,400(12) \text{ in.} = 3,484,800 \text{ in.}$$

Solving for the radius, we have
$s = r\theta \Rightarrow 3,484,800 \text{ in.} = (14 \text{ in.})\theta \Rightarrow$
$\theta = \frac{3,484,800}{14} \approx 248,914.29$ radians
Each rotation is 2π radians. Thus, we have $\frac{\theta}{2\pi} = \frac{248,914.29}{2\pi} \approx 39,615.94$
Thus, the number of rotations is 39,616 (rounded to the nearest whole rotation).

(b) Find s for the 16-in. wheel.
$s = r\theta \Rightarrow$
$s \approx (16 \text{ in.})(248,914.29) = 3,982,628.64$ in.
$(3,982,628.64 \text{ in.})\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)\left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right) \approx 62.9$ mi
The car with the 16-in. tires has gone 62.9 mi in one hour, so its speed is 62.9 mph. Yes, the driver deserves a ticket.

99. Let t = the length of the train.
t is approximately the arc length subtended by $3°\ 20'$. First convert $\theta = 3°20'$ to radians.
$\theta = 3°20' = \left(3 + \frac{20}{60}\right)° = 3\frac{1}{3}°$
$= \left(3\frac{1}{3}\right)\left(\frac{\pi}{180} \text{ radian}\right) = \left(\frac{10}{3}\right)\left(\frac{\pi}{180} \text{ radian}\right)$
$= \frac{\pi}{54}$ radian
The length of the train is
$t = r\theta \Rightarrow t = 3.5\left(\frac{\pi}{54}\right) \approx .20$ km long.
(rounded to two significant digits)

100. Let r = the distance of the boat.
The height of the mast, 32 ft, is approximately the arc length subtended by $2°\ 10'$. First convert $\theta = 2°10'$ to radians.
$\theta = 2°10' = \left(2 + \frac{10}{60}\right)° = 2\frac{1}{6}°$
$= \left(2\frac{1}{6}\right)\left(\frac{\pi}{180} \text{ radian}\right) = \left(\frac{13}{6}\right)\left(\frac{\pi}{180} \text{ radian}\right)$
$= \frac{13\pi}{1080}$ radian
We must now find the radius, r.
$s = r\theta \Rightarrow r = \frac{s}{\theta} \Rightarrow$
$r = \frac{32}{\frac{13\pi}{1080}} = 32 \cdot \frac{1080}{13\pi} \approx 846.2146$
The boat is about 850 ft away. (rounded to two significant digits)

101. $r = 6, s = 2\pi$
$s = r\theta \Rightarrow 2\pi = 6\theta \Rightarrow \theta = \frac{2\pi}{6} = \frac{\pi}{3}$
$A = \frac{1}{2}r^2\theta \Rightarrow$
$A = \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right) = \frac{1}{2}(36)\left(\frac{\pi}{3}\right) = 6\pi$

102. $r = 8, s = 4\pi$
$s = r\theta \Rightarrow 4\pi = 8\theta \Rightarrow \theta = \frac{4\pi}{8} = \frac{\pi}{2}$
$A = \frac{1}{2}r^2\theta \Rightarrow$
$A = \frac{1}{2}(8)^2\left(\frac{\pi}{2}\right) = \frac{1}{2}(64)\left(\frac{\pi}{2}\right) = 16\pi$

103. $r = 12, s = 12\pi$
$s = r\theta \Rightarrow 12\pi = 12\theta \Rightarrow \theta = \frac{12\pi}{12} = \pi$
$A = \frac{1}{2}r^2\theta \Rightarrow$
$A = \frac{1}{2}(12)^2\pi = 72\pi$

104. $r = 10, s = 15\pi$
$s = r\theta \Rightarrow 15\pi = 10\theta \Rightarrow \theta = \frac{15\pi}{10} = \frac{3\pi}{2}$
$A = \frac{1}{2}r^2\theta \Rightarrow$
$A = \frac{1}{2}(10)^2\left(\frac{3\pi}{2}\right) = \frac{1}{2}(100)\left(\frac{3\pi}{2}\right) = 75\pi$

105. $A = 6\pi$ sq units, $r = 6$
$A = \frac{1}{2}r^2\theta \Rightarrow 6\pi = \frac{1}{2}(6)^2\theta \Rightarrow$
$6\pi = \frac{1}{2}(36)\theta \Rightarrow 6\pi = 18\theta \Rightarrow$
$\theta = \frac{6\pi}{18} = \frac{\pi}{3}$ radian
$\frac{\pi}{3}$ radian $= \frac{\pi}{3} \cdot \frac{180}{\pi} = 60°$
The measure of the central angle is 60º.

106. $A = 96\pi$ sq units, $r = 12$
$A = \frac{1}{2}r^2\theta \Rightarrow 96\pi = \frac{1}{2}(12)^2\theta \Rightarrow$
$96\pi = \frac{1}{2}(144)\theta \Rightarrow 96\pi = 72\theta \Rightarrow$
$\theta = \frac{96\pi}{72} = \frac{4\pi}{3}$ radian
$\frac{4\pi}{3}$ radian $= \frac{4\pi}{3} \cdot \frac{180}{\pi} = 240°$
The measure of the central angle is 240º.

107. $A = 3$ sq units, $r = 2$

$$A = \frac{1}{2}r^2\theta \Rightarrow 3 = \frac{1}{2}(2)^2\theta \Rightarrow 3 = \frac{1}{2}(4)\theta \Rightarrow$$
$$3 = 2\theta \Rightarrow \theta = \frac{3}{2} = 1.5 \text{ radians}$$

108. $A = 8$ sq units, $r = 4$

$$A = \frac{1}{2}r^2\theta \Rightarrow 8 = \frac{1}{2}(4)^2\theta \Rightarrow 8 = \frac{1}{2}(16)\theta \Rightarrow$$
$$8 = 8\theta \Rightarrow \theta = 1 \text{ radian}$$

In Exercises 109–116, we will be rounding to the nearest tenth.

109. $r = 29.2$ m, $\theta = \frac{5\pi}{6}$ radians

$$A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(29.2)^2\left(\frac{5\pi}{6}\right) \Rightarrow$$
$$A = \frac{1}{2}(852.64)\left(\frac{5\pi}{6}\right) \approx 1116.1032$$

The area of the sector is 1116.1 m^2. (1120 m^2 rounded to three significant digits)

110. $r = 59.8$ km, $\theta = \frac{2\pi}{3}$ radians

$$A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(59.8)^2\left(\frac{2\pi}{3}\right) \Rightarrow$$
$$A = \frac{1}{2}(3576.04)\left(\frac{2\pi}{3}\right) \approx 3744.8203$$

The area of the sector is 3744.8 km^2.

(3740 km^2 rounded to three significant digits)

111. $r = 30.0$ ft, $\theta = \frac{\pi}{2}$ radians

$$A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(30.0)^2\left(\frac{\pi}{2}\right) \Rightarrow$$
$$A = \frac{1}{2}(900)\left(\frac{\pi}{2}\right) = 225\pi \approx 706.8583$$

The area of the sector is 706.9 ft^2. (707 ft^2 rounded to three significant digits)

112. $r = 90.0$ yd, $\theta = \frac{5\pi}{6}$ radians

$$A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(90.0)^2\left(\frac{5\pi}{6}\right) \Rightarrow$$
$$A = \frac{1}{2}(8100)\left(\frac{2\pi}{3}\right) = 3375\pi \approx 10,602.8752$$

The area of the sector is $10,602.9 \text{ yd}^2$.

($106,00 \text{ yd}^2$ rounded to three significant digits)

113. $r = 12.7$ cm, $\theta = 81°$

The formula $A = \frac{1}{2}r^2\theta$ requires that θ be measured in radians. Converting 81° to radians, we have

$$\theta = 81\left(\frac{\pi}{180} \text{ radian}\right) = \frac{9\pi}{20} \text{ radians. Since}$$
$$A = \frac{1}{2}(12.7)^2\left(\frac{9\pi}{20}\right) = \frac{1}{2}(161.29)\left(\frac{9\pi}{20}\right)$$

≈ 114.0092, the area of the sector is 114.0 cm^2. (114 cm^2 rounded to three significant digits)

114. $r = 18.3$ m, $\theta = 125°$

The formula $A = \frac{1}{2}r^2\theta$ requires that θ be measured in radians. Converting 125° to radians, we have

$$\theta = 125\left(\frac{\pi}{180} \text{ radian}\right) = \frac{25\pi}{36} \text{ radians. Since}$$
$$A = \frac{1}{2}(18.3)^2\left(\frac{25\pi}{36}\right) = \frac{1}{2}(334.89)\left(\frac{25\pi}{36}\right)$$

≈ 365.3083, the area of the sector is 365.3 m^2. (365 m^2 rounded to three significant digits)

115. $r = 40.0$ mi, $\theta = 135°$

The formula $A = \frac{1}{2}r^2\theta$ requires that θ be measured in radians. Converting 135° to radians, we have

$$\theta = 135\left(\frac{\pi}{180} \text{ radian}\right) = \frac{3\pi}{4} \text{ radians.}$$
$$A = \frac{1}{2}(40.0)^2\left(\frac{3\pi}{4}\right) = \frac{1}{2}(1600)\left(\frac{3\pi}{4}\right)$$
$$= 600\pi \approx 1884.9556$$

The area of the sector is 1885.0 mi^2.

(1880 mi^2 rounded to three significant digits)

116. $r = 90.0$ km, $\theta = 270°$

The formula $A = \frac{1}{2}r^2\theta$ requires that θ be measured in radians. Converting 270° to radians, we have

$$\theta = 270\left(\frac{\pi}{180} \text{ radian}\right) = \frac{3\pi}{2} \text{ radians.}$$

$$A=\frac{1}{2}(90.0)^2\left(\frac{3\pi}{2}\right)=\frac{1}{2}(8100)\left(\frac{3\pi}{2}\right)$$
$$=6075\pi\approx 19{,}085.1754$$

The area of the sector is 19,085.2 km^2. (19,100 km^2 rounded to three significant digits)

117. $A=16\text{ in.}^2,\ r=3.0\text{ in.}$

$$A=\frac{1}{2}r^2\theta\Rightarrow 16=\frac{1}{2}(3)^2\theta\Rightarrow 16=\frac{9}{2}\theta\Rightarrow$$
$$\theta=16\cdot\frac{2}{9}=\frac{32}{9}\approx 3.6\text{ radians}$$

(rounded to two significant digits)

118. $A=64\text{ m}^2,\ \theta=\frac{\pi}{6}$ radian

$$A=\frac{1}{2}r^2\theta\Rightarrow 64=\frac{1}{2}r^2\left(\frac{\pi}{6}\right)\Rightarrow 64=\frac{\pi}{12}r^2\Rightarrow$$
$$r^2=64\cdot\frac{12}{\pi}\Rightarrow r^2=\frac{768}{\pi}\Rightarrow r=\sqrt{\frac{768}{\pi}}\approx 16\text{ m}$$

(rounded to two significant digits)

119. **(a)** The central angle in degrees measures $\frac{360°}{27}=13\frac{1}{3}°$. Converting to radians, we have the following.

$$13\tfrac{1}{3}°=\left(13\tfrac{1}{3}\right)\left(\frac{\pi}{180}\text{ radian}\right)$$
$$=\left(\frac{40}{3}\right)\left(\frac{\pi}{180}\text{ radian}\right)=\frac{2\pi}{27}\text{ radian}$$

(b) Since $C=2\pi r$, and r = 76 ft, we have $C=2\pi(76)=152\pi\approx 477.5221$. The circumference is about 478 ft.

(c) Since $r=76$ ft and $\theta=\frac{2\pi}{27}$, we have

$$s=r\theta=76\left(\frac{2\pi}{27}\right)=\frac{152\pi}{27}\approx 17.6860.$$

Thus, the length of the arc is 17.7 ft. Note: If this measurement is approximated to be $\frac{160}{9}$, then the approximated value would be 17.8 ft, rounded to three significant digits.

(d) Area of sector with $r=76$ ft and $\theta=\frac{2\pi}{27}$ is as follows.

$$A=\frac{1}{2}r^2\theta\Rightarrow A=\frac{1}{2}\left(76^2\right)\frac{2\pi}{27}\Rightarrow$$
$$A=\frac{1}{2}(5776)\frac{2\pi}{27}=\frac{5776\pi}{27}\approx 672\text{ ft}^2$$

120. The area cleaned is the area of the sector "wiped" by the total area and blade minus the area "wiped" by the arm only. We must first convert 95° to radians.

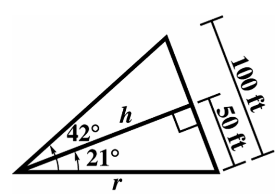
$$95°=(95)\left(\frac{\pi}{180}\text{ radian}\right)=\frac{19\pi}{36}\text{ radians}$$

Since 10 – 7 = 3, the arm was 3 in. long. Thus, we have

$$A_{\text{arm only}}=\frac{1}{2}(3)^2\left(\frac{19\pi}{36}\right)=\frac{1}{2}(9)\left(\frac{19\pi}{36}\right)$$
$$=\frac{19\pi}{8}\approx 7.4613\text{ in.}^2$$
$$A_{\text{arm and blade}}=\frac{1}{2}(10)^2\left(\frac{19\pi}{36}\right)=\frac{1}{2}(100)\left(\frac{19\pi}{36}\right)$$
$$=\frac{475\pi}{18}\approx 82.9031\text{ in.}^2$$

Since $82.9031-7.4613=75.4418$, the area of the region cleaned was about 75.4 in.^2.

121. **(a)**



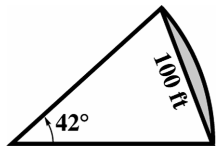
The triangle formed by the central angle and the chord is isosceles. Therefore, the bisector of the central angle is also the perpendicular bisector of the chord.

$$\sin 21°=\frac{50}{r}\Rightarrow r=\frac{50}{\sin 21°}\approx 140\text{ ft}$$

(b) $r=\frac{50}{\sin 21°};\ \theta=42°$

Converting θ to radians, we have $42\left(\frac{\pi}{180}\text{ radians}\right)=\frac{7\pi}{30}$ radians. Solving for the arc length, we have

$$s=r\theta\Rightarrow$$
$$s=\frac{50}{\sin 21°}\cdot\frac{7\pi}{30}=\frac{35\pi}{3\sin 21°}\approx 102\text{ ft}$$

(c) 

The area of the portion of the circle can be found by subtracting the area of the triangle from the area of the sector. From the figure in part (a), we have

$\tan 21° = \frac{50}{h}$ so $h = \frac{50}{\tan 21°}$.

$$A_{\text{sector}} = \frac{1}{2}r^2\theta \Rightarrow$$
$$A_{\text{sector}} = \frac{1}{2}\left(\frac{50}{\sin 21°}\right)^2\left(\frac{7\pi}{30}\right) \approx 7135 \text{ ft}^2$$

and

$$A_{\text{triangle}} = \frac{1}{2}bh \Rightarrow$$
$$A_{\text{triangle}} = \frac{1}{2}(100)\left(\frac{50}{\tan 21°}\right) \approx 6513 \text{ ft}^2$$

The area bounded by the arc and the chord is $7135 - 6513 = 622$ ft^2.

122. If the land area is circular, the area of a circle is $A = \pi r^2$, and we have $950{,}000 = \pi r^2 \Rightarrow$

$$r^2 = \frac{950{,}000}{\pi}$$
$$r = \sqrt{\frac{950{,}000}{\pi}} \approx 549.9040$$

Thus, the radius is 550 m. (rounded to two significant digits)

If the land area is a 35 ° sector of a circle, find the radius by first converting $\theta = 35°$ to radians, giving $35\left(\frac{\pi}{180}\text{ radian}\right) = \frac{7\pi}{36}$ radian.

The area of a sector is $A = \frac{1}{2}r^2\theta$, so

$$A = \frac{1}{2}r^2\theta \Rightarrow$$
$$950{,}000 = \frac{1}{2}r^2\left(\frac{7\pi}{36}\right)$$
$$950{,}000 = r^2\left(\frac{7\pi}{72}\right)$$
$$r^2 = 950{,}000 \cdot \frac{72}{7\pi} = \frac{68{,}400{,}000}{7\pi}$$
$$r = \sqrt{\frac{68{,}400{,}000}{7\pi}} \approx 1763.6163$$

Thus, the radius is 1800 m. (rounded to two significant digits)

123. Use the Pythagorean theorem to find the hypotenuse of the triangle, which is also the radius of the sector of the circle.

$$r^2 = 30^2 + 40^2 \Rightarrow r^2 = 900 + 1600 \Rightarrow$$
$$r^2 = 2500 \Rightarrow r = 50$$

The total area of the lot is the sum of the areas of the triangle and the sector.

Converting $\theta = 60°$ to radians, we have

$$60\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{3}\text{ radians.}$$

$$A_{\text{triangle}} = \frac{1}{2}bh = \frac{1}{2}(30)(40) = 600 \text{ yd}^2$$
$$A_{\text{sector}} = \frac{1}{2}r^2\theta = \frac{1}{2}(50)^2\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2}(2500)\left(\frac{\pi}{3}\right) = \frac{1250\pi}{3}\text{ yd}^2$$

Total area

$$A_{\text{triangle}} + A_{\text{sector}} = 600 + \frac{1250\pi}{3} \approx 1908.9969$$

or 1900 yd^2, rounded to two significant digits.

124. Converting $\theta = 1' = \left(\frac{1}{60}\right)°$ to radians, we have

$\frac{1}{60}\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{10{,}800}$ radian. Solving for the arc length, we have

$$s = r\theta \Rightarrow s = 3963 \cdot \frac{\pi}{10{,}800} = \frac{11\pi}{30} \approx 1.1519.$$

Thus, there are approximately 1.15 statute miles in 1 nautical mile (rounded to two decimal places.)

125. Converting $\theta = 7°12' = \left(7 + \frac{12}{60}\right)° = 7.2°$ to radians, we have

$$7.2\left(\frac{\pi}{180}\text{ radian}\right) = \frac{7.2\pi}{180} = \frac{\pi}{25}\text{ radian.}$$

Solving for the radius with the arc length formula, we have

$$s = r\theta \Rightarrow 496 = r \cdot \frac{\pi}{25} \Rightarrow$$
$$r = 496 \cdot \frac{25}{\pi} = \frac{12{,}400}{\pi} \approx 3947.0426$$

Thus, the radius is approximately 3950 mi. (rounded to three significant digits) Using this approximate radius, we can find the circumference of the Earth. Since $C = 2\pi r$, we have $C \approx 2\pi(3950) \approx 24{,}800$. Thus, the approximate circumference is 24,800 mi. (rounded to three significant digits)

126. The central angle in degrees measures $.517°$. Converting to radians, we have

$$.517° = (.517)\left(\frac{\pi}{180}\text{ radian}\right) = \frac{.517\pi}{180}\text{ radian}$$

$$s = r\theta \Rightarrow s = 238,900\left(\frac{.517\pi}{180}\right) \approx 2155.6788$$

Recall, from page 115 of your text (above Exercises 37−38), for very small central angles, there is little difference between the arc and the inscribed chord. Thus, the diameter of the moon is approximately 2156 mi. (rounded to four significant digits)

127. If we let $r' = 2r$, then

$$A_{\text{sector}} = \frac{1}{2}(r')^2\theta = \frac{1}{2}(2r)^2\theta \quad \text{Thus, the area,}$$
$$= \frac{1}{2}(4r^2)\theta = 4\left(\frac{1}{2}r^2\theta\right)$$

$\frac{1}{2}r^2\theta$, is quadrupled.

128. $A_{\text{sector}} = \frac{1}{2}r^2\theta$, where θ is in radians.

Thus, $A_{\text{sector}} = \frac{1}{2}r^2\frac{\theta}{180} = \frac{\pi r^2\theta}{360}$,

θ is in degrees.

Section 6.2: The Unit Circle and Circular Functions

Connections (page 576)

1. Answers will vary.
2. Answers will vary.

Exercises

1. An angle of $\theta = \frac{\pi}{2}$ radians intersects the unit circle at the point $(0,1)$.

(a) $\sin\theta = y = 1$

(b) $\cos\theta = x = 0$

(c) $\tan\theta = \frac{y}{x} = \frac{1}{0}$; undefined

2. An angle of $\theta = \pi$ radians intersects the unit circle at the point $(-1,0)$.

(a) $\sin\theta = y = 0$

(b) $\cos\theta = x = -1$

(c) $\tan\theta = \frac{y}{x} = \frac{0}{-1} = 0$

3. An angle of $\theta = 2\pi$ radians intersects the unit circle at the point $(1,0)$.

(a) $\sin\theta = y = 0$

(b) $\cos\theta = x = 1$

(c) $\tan\theta = \frac{y}{x} = \frac{0}{1} = 0$

4. An angle of $\theta = 3\pi$ radians intersects the unit circle at the point $(-1,0)$.

(a) $\sin\theta = y = 0$

(b) $\cos\theta = x = -1$

(c) $\tan\theta = \frac{y}{x} = \frac{0}{-1} = 0$

5. An angle of $\theta = -\pi$ radians intersects the unit circle at the point $(-1,0)$.

(a) $\sin\theta = y = 0$

(b) $\cos\theta = x = -1$

(c) $\tan\theta = \frac{y}{x} = \frac{0}{-1} = 0$

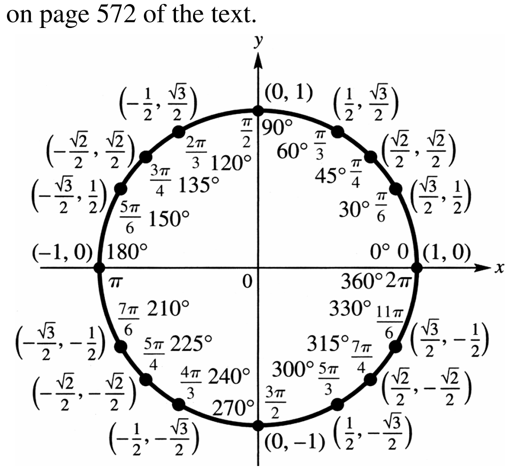
6. An angle of $\theta = -\frac{3\pi}{2}$ radians intersects the unit circle at the point $(0,1)$.

(a) $\sin\theta = y = 1$

(b) $\cos\theta = x = 0$

(c) $\tan\theta = \frac{y}{x} = \frac{1}{0}$; undefined

For Exercises 7−22, use the following copy of Figure 12 on page 572 of the text.



7. Since $\frac{7\pi}{6}$ is in quadrant III, the reference angle is $\frac{7\pi}{6}-\pi=\frac{7\pi}{6}-\frac{6\pi}{6}=\frac{\pi}{6}$. In quadrant III, the sine is negative. Thus, $\sin\frac{7\pi}{6}=-\sin\frac{\pi}{6}=-\frac{1}{2}$. Converting $\frac{7\pi}{6}$ to degrees, we have $\frac{7\pi}{6}=\frac{7}{6}(180°)=210°$. The reference angle is $210°-180°=30°$. Thus, $\sin\frac{7\pi}{6}=\sin 210°=-\sin 30°=-\frac{1}{2}$.

8. Since $\frac{5\pi}{3}$ is in quadrant IV, the reference angle is $2\pi-\frac{5\pi}{3}=\frac{6\pi}{3}-\frac{5\pi}{3}=\frac{\pi}{3}$. In quadrant IV, the cosine is positive. Thus, $\cos\frac{5\pi}{3}=\cos\frac{\pi}{3}=\frac{1}{2}$.
Converting $\frac{5\pi}{3}$ to degrees, we have $\frac{5\pi}{3}=\frac{5}{3}(180°)=300°$. The reference angle is $360°-300°=60°$. Thus, $\cos\frac{5\pi}{3}=\cos 300°=\cos 60°=\frac{1}{2}$.

9. Since $\frac{3\pi}{4}$ is in quadrant II, the reference angle is $\pi-\frac{3\pi}{4}=\frac{4\pi}{4}-\frac{3\pi}{4}=\frac{\pi}{4}$. In quadrant II, the tangent is negative. Thus, $\tan\frac{3\pi}{4}=-\tan\frac{\pi}{4}=-1$.
Converting $\frac{3\pi}{4}$ to degrees, we have $\frac{3\pi}{4}=\frac{3}{4}(180°)=135°$. The reference angle is $180°-135°=45°$. Thus, $\tan\frac{3\pi}{4}=\tan 135°=-\tan 45°=-1$.

10. Since $\frac{2\pi}{3}$ is in quadrant II, the reference angle is $\pi-\frac{2\pi}{3}=\frac{3\pi}{3}-\frac{2\pi}{3}=\frac{\pi}{3}$. In quadrant II, the secant is negative. Thus, $\sec\frac{2\pi}{3}=-\sec\frac{\pi}{3}=-2$.
Converting $\frac{2\pi}{3}$ to degrees, we have $\frac{2\pi}{3}=\frac{2}{3}(180°)=120°$. The reference angle is $180°-120°=60°$. Thus, $\sec\frac{2\pi}{3}=\sec 120°=-\sec 60°=-2$.

11. Since $\frac{11\pi}{6}$ is in quadrant IV, the reference angle is $2\pi-\frac{11\pi}{6}=\frac{12\pi}{6}-\frac{11\pi}{6}=\frac{\pi}{6}$. In quadrant IV, the cosecant is negative. Thus, $\csc\frac{11\pi}{6}=-\csc\frac{\pi}{6}=-2$.
Converting $\frac{11\pi}{6}$ to degrees, we have $\frac{11\pi}{6}=\frac{11}{6}(180°)=330°$. The reference angle is $360°-330°=30°$. Thus, $\csc\frac{11\pi}{6}=\csc 330°=-\csc 30°=-2$.

12. Since $\frac{5\pi}{6}$ is in quadrant II, the reference angle is $\pi-\frac{5\pi}{6}=\frac{6\pi}{6}-\frac{5\pi}{6}=\frac{\pi}{6}$. In quadrant II, the cotangent is negative. Thus, $\cot\frac{5\pi}{6}=-\cot\frac{\pi}{6}=-\sqrt{3}$.
Converting $\frac{5\pi}{6}$ to degrees, we have $\frac{5\pi}{6}=\frac{5}{6}(180°)=150°$. The reference angle is $180°-150°=30°$. Thus, $\cot\frac{5\pi}{6}=\cot 150°=-\cot 30°=-\sqrt{3}$.

13. $-\frac{4\pi}{3}$ is coterminal with $-\frac{4\pi}{3}+2\pi=-\frac{4\pi}{3}+\frac{6\pi}{3}=\frac{2\pi}{3}$. Since $\frac{2\pi}{3}$ is in quadrant II, the reference angle is $\pi-\frac{2\pi}{3}=\frac{3\pi}{3}-\frac{2\pi}{3}=\frac{\pi}{3}$. In quadrant II, the cosine is negative. Thus, $\cos\left(-\frac{4\pi}{3}\right)=\cos\frac{2\pi}{3}=-\cos\frac{\pi}{3}=-\frac{1}{2}$.

Converting $\frac{2\pi}{3}$ to degrees, we have

$\frac{2\pi}{3} = \frac{2}{3}(180°) = 120°$. The reference angle is $180° - 120° = 60°$. Thus,

$$\cos\left(-\frac{4\pi}{3}\right) = \cos\frac{2\pi}{3} = \cos 120°$$
$$= -\cos 60° = -\frac{1}{2}$$

14. $\frac{17\pi}{3}$ is coterminal with

$\frac{17\pi}{3} - 2(2\pi) = \frac{17\pi}{3} - 4\pi = \frac{17\pi}{3} - \frac{12\pi}{3} = \frac{5\pi}{3}$.

Since $\frac{5\pi}{3}$ is in quadrant IV, the reference angle is $2\pi - \frac{5\pi}{3} = \frac{6\pi}{3} - \frac{5\pi}{3} = \frac{\pi}{3}$. In quadrant IV, the tangent is negative. Thus,

$\tan\frac{17\pi}{3} = \tan\frac{5\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$.

Converting $\frac{5\pi}{3}$ to degrees, we have

$\frac{5\pi}{3} = \frac{5}{3}(180°) = 300°$. The reference angle is $360° - 300° = 60°$. Thus,

$\tan\frac{5\pi}{3} = \tan 300° = -\tan 60° = -\sqrt{3}$.

15. Since $\frac{7\pi}{4}$ is in quadrant IV, the reference angle is $2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}$. In quadrant IV, the cosine is positive. Thus,

$\cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Converting $\frac{7\pi}{4}$ to degrees, we have $\frac{7\pi}{4} = \frac{7}{4}(180°) = 315°$. The reference angle is $360° - 315° = 45°$. Thus,

$\cos\frac{7\pi}{4} = \cos 315° = \cos 45° = \frac{\sqrt{2}}{2}$.

16. Since $\frac{5\pi}{4}$ is in quadrant III, the reference angle is $\frac{5\pi}{4} - \pi = \frac{5\pi}{4} - \frac{4\pi}{4} = \frac{\pi}{4}$. In quadrant III, the secant is negative. Thus,

$\sec\frac{5\pi}{4} = -\sec\frac{\pi}{4} = -\sqrt{2}$.

Converting $\frac{5\pi}{4}$ to degrees, we have

$\frac{5\pi}{4} = \frac{5}{4}(180°) = 225°$. The reference angle is $225° - 180° = 45°$. Thus,

$\sec\frac{5\pi}{4} = \sec 225° = -\sec 45° = -\sqrt{2}$.

17. $-\frac{4\pi}{3}$ is coterminal with

$-\frac{4\pi}{3} + 2\pi = -\frac{4\pi}{3} + \frac{6\pi}{3} = \frac{2\pi}{3}$. Since $\frac{2\pi}{3}$ is in quadrant II, the reference angle is

$\pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}$. In quadrant II, the sine is positive. Thus,

$\sin\left(-\frac{4\pi}{3}\right) = \sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Converting $\frac{2\pi}{3}$ to degrees, we have

$\frac{2\pi}{3} = \frac{2}{3}(180°) = 120°$. The reference angle is $180° - 120° = 60°$. Thus,

$\sin\left(-\frac{4\pi}{3}\right) = \sin\frac{2\pi}{3} = \sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$

18. $-\frac{5\pi}{6}$ is coterminal with

$-\frac{5\pi}{6} + 2\pi = -\frac{5\pi}{6} + \frac{12\pi}{6} = \frac{7\pi}{6}$. Since $\frac{7\pi}{6}$ is in quadrant III, the reference angle is

$\frac{7\pi}{6} - \pi = \frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$. In quadrant III, the sine is negative. Thus,

$\sin\left(-\frac{5\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$.

Converting $\frac{7\pi}{6}$ to degrees, we have

$\frac{7\pi}{6} = \frac{7}{6}(180°) = 210°$. The reference angle is $210° - 180° = 30°$. Thus,

$$\sin\left(-\frac{5\pi}{6}\right) = \sin\frac{7\pi}{6} = \sin 210°$$
$$= -\sin 30° = -\frac{1}{2}$$

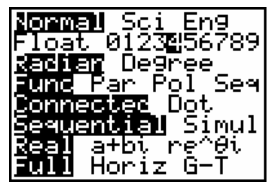
19. $\frac{23\pi}{6}$ is coterminal with $\frac{23\pi}{6} - 2\pi = \frac{23\pi}{6} - \frac{12\pi}{6} = \frac{11\pi}{6}$. Since $\frac{11\pi}{6}$ is in quadrant IV, the reference angle is $2\pi - \frac{11\pi}{6} = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6}$. In quadrant IV, the secant is positive. Thus,
$\sec\frac{23\pi}{6} = \sec\frac{11\pi}{6} = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$. Converting $\frac{11\pi}{6}$ to degrees, we have
$\frac{11\pi}{6} = \frac{11}{6}(180°) = 330°$. The reference angle is $360° - 330° = 30°$. Thus,
$\sec\frac{23\pi}{6} = \sec\frac{11\pi}{6} = \sec 330° = \sec 30° = \frac{2\sqrt{3}}{3}$.

20. $\frac{13\pi}{3}$ is coterminal with
$\frac{13\pi}{3} - 2(2\pi) = \frac{13\pi}{3} - 4\pi = \frac{13\pi}{3} - \frac{12\pi}{3} = \frac{\pi}{3}$.
Since $\frac{\pi}{3}$ is in quadrant I, we have
$\csc\frac{13\pi}{3} = \csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$. Converting $\frac{\pi}{3}$ to degrees, we have $\frac{\pi}{3} = \frac{1}{3}(180°) = 60°$. Thus,
$\csc\frac{\pi}{3} = \csc 60° = \frac{2\sqrt{3}}{3}$.

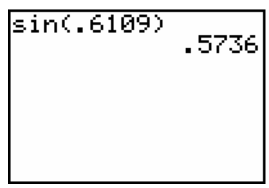
21. Since $\frac{5\pi}{6}$ is in quadrant II, the reference angle is $\pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}$. In quadrant II, the tangent is negative. Thus,
$\tan\frac{5\pi}{6} = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$.
Converting $\frac{5\pi}{6}$ to degrees, we have
$\frac{5\pi}{6} = \frac{5}{6}(180°) = 150°$. The reference angle is $180° - 150° = 30°$. Thus,
$\tan\frac{5\pi}{6} = \tan 150° = -\tan 30° = -\frac{\sqrt{3}}{3}$.

22. Since $\frac{3\pi}{4}$ is in quadrant II, the reference angle is $\pi - \frac{3\pi}{4} = \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4}$. In quadrant II, the cosine is negative. Thus,
$\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$. Converting $\frac{3\pi}{4}$ to degrees, we have $\frac{3\pi}{4} = \frac{3}{4}(180°) = 135°$.
The reference angle is $180° - 135° = 45°$.
Thus, $\tan\frac{3\pi}{4} = \cos 135° = -\cos 45° = -\frac{\sqrt{2}}{2}$.

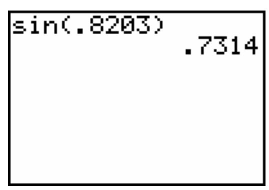
For Exercises 23−34, 45−54, and 55−66, your calculator must be set in radian mode. Keystroke sequences may vary based on the type and/or model of calculator being used. As in Example 3, we will set the calculator to show four decimal digits.



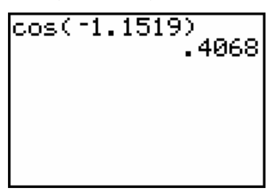
23. $\sin .6109 \approx .5736$



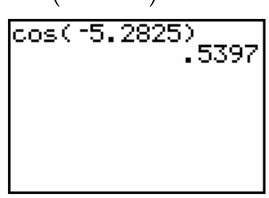
24. $\sin .8203 \approx .7314$



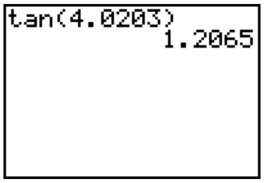
25. $\cos(-1.1519) \approx .4068$



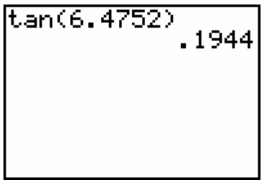
26. $\cos(-5.2825) \approx .5397$



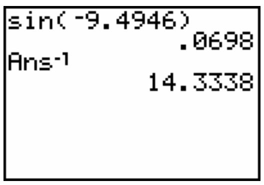
27. $\tan 4.0203 \approx 1.2065$



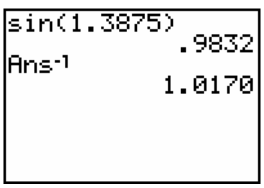
28. $\tan 6.4752 \approx .1944$



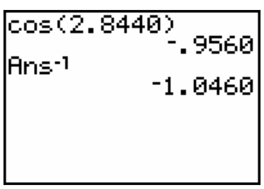
29. $\csc(-9.4946) \approx 14.3338$



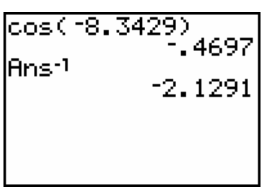
30. $\csc 1.3875 \approx 1.0170$



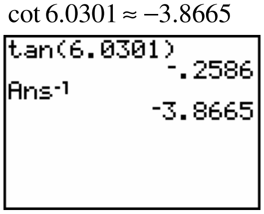
31. $\sec 2.8440 \approx -1.0460$



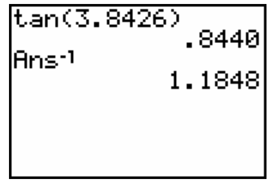
32. $\sec(-8.3429) \approx -2.1291$



33. $\cot 6.0301 \approx -3.8665$



34. $\cot 3.8426 \approx 1.1848$



35. $\cos .8 \approx .7$

36. $\cos .6 \approx .8$

37. $\sin 2 \approx .9$

38. $\sin 4 \approx -.75$

39. $\sin 3.8 \approx -.6$

40. $\cos 3.2 \approx -1.0$

41. $\cos\theta = -.65 \Rightarrow x = -.65 \Rightarrow \theta \approx 2.3$ radians or $\theta \approx 4.0$ radians

42. $\sin\theta = -.95 \Rightarrow y = -.95 \Rightarrow \theta \approx 4.4$ radians or $\theta \approx 5.0$ radians

43. $\sin\theta = -.7 \Rightarrow y = .7 \Rightarrow \theta \approx .8$ radians or $\theta \approx 2.4$ radians

44. $\cos\theta = .3 \Rightarrow x = .3 \Rightarrow \theta \approx 1.3$ radians or $\theta \approx 5.0$ radians

45. $\cos 2$

$\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, so $\frac{\pi}{2} < 2 < \pi$. Thus, an angle of 2 radians is in quadrant II. (The figure for Exercises 35–38 also shows that 2 radians is in quadrant II.) Because values of the cosine function are negative in quadrant II, cos 2 is negative.

46. $\sin(-1)$

$-\frac{\pi}{2} \approx -1.57$, so $-\frac{\pi}{2} < -1 < 0$. Thus, an angle of -1 radian is in quadrant IV. Because values of the sine function are negative in quadrant IV, $\sin(-1)$ is negative.

47. $\sin 5$

$\frac{3\pi}{2} \approx 4.71$ and $2\pi \approx 6.28$, so $\frac{3\pi}{2} < 5 < 2\pi$. Thus, an angle of 5 radians is in quadrant IV. (The figure for Exercises 35 – 38 also shows that 5 radians is in quadrant IV.) Because values of the sine function are negative in quadrant IV, sin 5 is negative.

48. $\cos 6$

$\frac{3\pi}{2} \approx 4.71$ and so $\frac{3\pi}{2} < 6 < 2\pi$. Thus, an angle of 6 radians is in quadrant IV. (The figure for Exercises 35−38 also shows that 6 radians is in quadrant IV.) Because values of the cosine function are positive in quadrant IV, cos 6 is positive.

49. $\tan 6.29$

$2\pi \approx 6.28$ and

$2\pi + \frac{\pi}{2} = \frac{4\pi}{2} + \frac{\pi}{2} = \frac{5\pi}{2} \approx 7.85$, so

$2\pi < 6.29 < \frac{5\pi}{2}$. Notice that 2π is coterminal with 0 and $\frac{5\pi}{2}$ is coterminal with $\frac{\pi}{2}$. Thus, an angle of 6.29 radians is in quadrant I. Because values of the tangent function are positive in quadrant I, $\tan 6.29$ is positive.

50. $\tan(-6.29)$

$-2\pi - \frac{\pi}{2} = -\frac{4\pi}{2} - \frac{\pi}{2} = -\frac{5\pi}{2} \approx -7.85$ and

$-2\pi \approx -6.28$, so $-\frac{5\pi}{2} < -6.29 < -2\pi$. Notice that $\frac{5\pi}{2}$ is coterminal with $\frac{3\pi}{2}$ and -2π is coterminal with 0. Thus, an angle of -6.29 is in quadrant IV. Because values of the tangent function are negative in quadrant IV, $\tan(-6.29)$ is negative.

51. $\sin\theta = y = \frac{\sqrt{2}}{2}$; $\cos\theta = x = \frac{\sqrt{2}}{2}$

$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$; $\cot\theta = \frac{x}{y} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

$\sec\theta = \frac{1}{x} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

$\csc\theta = \frac{1}{y} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

52. $\sin\theta = y = \frac{8}{17}$; $\cos\theta = x = -\frac{15}{17}$

$\tan\theta = \frac{y}{x} = \frac{\frac{8}{17}}{-\frac{15}{17}} = \frac{8}{17}\left(-\frac{17}{15}\right) = -\frac{8}{15}$

$\cot\theta = \frac{x}{y} = \frac{-\frac{15}{17}}{\frac{8}{17}} = -\frac{15}{17}\left(\frac{17}{8}\right) = -\frac{15}{8}$

$\sec\theta = \frac{1}{x} = \frac{1}{-\frac{15}{17}} = -\frac{17}{15}$; $\csc\theta = \frac{1}{y} = \frac{1}{\frac{8}{17}} = \frac{17}{8}$

53. $\sin\theta = y = -\frac{12}{13}$; $\cos\theta = x = \frac{5}{13}$

$\tan\theta = \frac{y}{x} = \frac{-\frac{12}{13}}{\frac{5}{13}} = -\frac{12}{13}\left(\frac{13}{5}\right) = -\frac{12}{5}$

$\cot\theta = \frac{x}{y} = \frac{\frac{5}{13}}{-\frac{12}{13}} = \frac{5}{13}\left(-\frac{13}{12}\right) = -\frac{5}{12}$

$\sec\theta = \frac{1}{x} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$; $\csc\theta = \frac{1}{y} = \frac{1}{-\frac{12}{13}} = -\frac{13}{12}$

54. $\sin\theta = y = -\frac{1}{2}$; $\cos\theta = x = -\frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

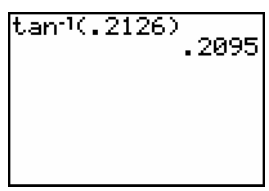
$\cot\theta = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(-\frac{2}{1}\right) = \sqrt{3}$

$\sec\theta = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$

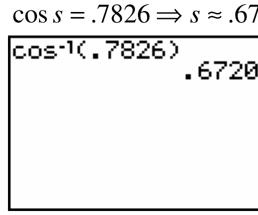
$= -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

$\csc\theta = \frac{1}{y} = \frac{1}{-\frac{1}{2}} = 1 \cdot \left(-\frac{2}{1}\right) = -2$

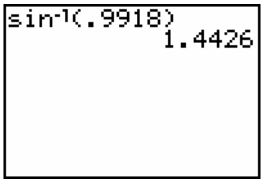
55. $\tan s = .2126 \Rightarrow s \approx .2095$



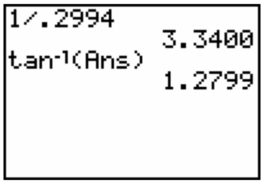
56. $\cos s = .7826 \Rightarrow s \approx .6720$



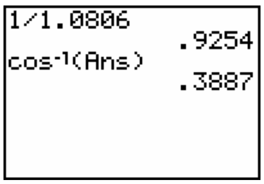
57. $\sin s = .9918 \Rightarrow s \approx 1.4426$



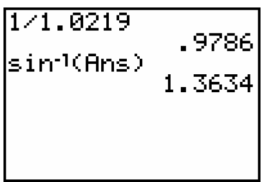
58. $\cot s = .2994 \Rightarrow s \approx 1.2799$



59. $\sec s = 1.0806 \Rightarrow s \approx .3887$



60. $\csc s = 1.0219 \Rightarrow s \approx 1.3634$



61. $\left[\frac{\pi}{2}, \pi\right]$; $\sin s = \frac{1}{2}$

Recall that $\sin\frac{\pi}{6} = \frac{1}{2}$ and in quadrant II, sin s is positive. Therefore,

$$\sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{5\pi}{6} = \frac{1}{2}, \text{ so } s = \frac{5\pi}{6}.$$

62. $\left[\frac{\pi}{2}, \pi\right]$; $\cos s = -\frac{1}{2}$

Recall that $\cos\frac{\pi}{3} = \frac{1}{2}$ and in quadrant II, cos s is negative. Therefore,

$$\cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}, \text{ so } s = \frac{2\pi}{3}.$$

63. $\left[\pi, \frac{3\pi}{2}\right]$; $\tan s = \sqrt{3}$

Recall that $\tan\frac{\pi}{3} = \sqrt{3}$ and in quadrant III, tan s is positive. Therefore,

$$\tan\left(\pi + \frac{\pi}{3}\right) = \tan\frac{4\pi}{3} = \sqrt{3}, \text{ so } s = \frac{4\pi}{3}.$$

64. $\left[\pi, \frac{3\pi}{2}\right]$; $\sin s = -\frac{1}{2}$

Recall that $\sin\frac{\pi}{6} = \frac{1}{2}$ and in quadrant III, sin s is negative. Therefore,

$$\sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6} = -\frac{1}{2}, \text{ so } s = \frac{7\pi}{6}.$$

65. $\left[\frac{3\pi}{2}, 2\pi\right]$; $\tan s = -1$

Recall that $\tan\frac{\pi}{4} = 1$ and in quadrant IV, tan s is negative. Therefore,

$$\tan\left(2\pi - \frac{\pi}{4}\right) = \tan\frac{7\pi}{4} = -1, \text{ so } s = \frac{7\pi}{4}.$$

66. $\left[\frac{3\pi}{2}, 2\pi\right]$; $\cos s = \frac{\sqrt{3}}{2}$

Recall that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and in quadrant IV, cos s is positive. Therefore,

$$\cos\left(2\pi - \frac{\pi}{6}\right) = \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}, \text{ so } s = \frac{11\pi}{6}.$$

67. The circumference of the unit circle is 2π.
$\omega = 1$ radian per sec, $\theta = 2\pi$ radians

$$\omega = \frac{\theta}{t} \Rightarrow 1 = \frac{2\pi}{t} \Rightarrow t = 2\pi \text{ sec}$$

68. The circumference of the unit circle is 2π.
$v = 1$ unit per sec, $s = 2\pi$ units

$$v = \frac{s}{t} \Rightarrow 1 = \frac{2\pi}{t} \Rightarrow t = 2\pi \text{ sec}$$

69. $r = 20$ cm, $\omega = \frac{\pi}{12}$ radian per sec, $t = 6$ sec

(a) $\omega = \frac{\theta}{t} \Rightarrow \frac{\pi}{12} = \frac{\theta}{6} \Rightarrow \theta = \frac{\pi}{2}$ radians

(b) $s = r\theta \Rightarrow s = 20 \cdot \frac{\pi}{2} = 10\pi$ cm

(c) $v = \frac{r\theta}{t} \Rightarrow v = \frac{20 \cdot \frac{\pi}{2}}{6} = \frac{10\pi}{6} = \frac{5\pi}{3}$ cm per sec

70. $r = 30$ cm, $\omega = \frac{\pi}{10}$ radian per sec, $t = 4$ sec

(a) $\omega = \frac{\theta}{t} \Rightarrow \frac{\pi}{10} = \frac{\theta}{4} \Rightarrow \theta = \frac{4\pi}{10} = \frac{2\pi}{5}$ radians

(b) $s = r\theta \Rightarrow s = 30 \cdot \frac{2\pi}{5} = 12\pi$ cm

(c) $v = \frac{r\theta}{t} \Rightarrow v = \frac{30 \cdot \frac{2\pi}{5}}{4} = \frac{12\pi}{4} = 3\pi$ cm per sec

71. $\omega = \frac{2\pi}{3}$ radians per sec, $t = 3$ sec

$\omega = \frac{\theta}{t} \Rightarrow \frac{2\pi}{3} = \frac{\theta}{3} \Rightarrow \theta = 2\pi$ radians

72. $\omega = \frac{\pi}{4}$ radians per min, $t = 5$ min

$\omega = \frac{\theta}{t} \Rightarrow \frac{\pi}{4} = \frac{\theta}{5} \Rightarrow \theta = \frac{5\pi}{4}$ radians

73. $\theta = \frac{3\pi}{4}$ radians, $t = 8$ sec

$\omega = \frac{\theta}{t} \Rightarrow \theta = \frac{\frac{3\pi}{4}}{8} = \frac{3\pi}{4} \cdot \frac{1}{8} = \frac{3\pi}{32}$ radian per sec

74. $\theta = \frac{2\pi}{5}$ radians, $t = 10$ sec

$\omega = \frac{\theta}{t} \Rightarrow \omega = \frac{\frac{2\pi}{5}}{10} = \frac{2\pi}{5} \cdot \frac{1}{10} = \frac{\pi}{25}$ radian per sec

75. $\theta = \frac{2\pi}{9}$ radian, $\omega = \frac{5\pi}{27}$ radian per min

$\omega = \frac{\theta}{t} \Rightarrow \frac{5\pi}{27} = \frac{\frac{2\pi}{9}}{t} \Rightarrow \frac{5\pi}{27} = \frac{2\pi}{9t} \Rightarrow$
$45\pi t = 54\pi \Rightarrow t = \frac{54\pi}{45\pi} = \frac{6}{5}$ min

76. $\theta = \frac{3\pi}{8}$ radians, $\omega = \frac{\pi}{24}$ radian per min

$\omega = \frac{\theta}{t} \Rightarrow \frac{\pi}{24} = \frac{\frac{3\pi}{8}}{t} \Rightarrow \frac{\pi}{24} = \frac{3\pi}{8t} \Rightarrow$
$8\pi t = 72\pi \Rightarrow t = \frac{72\pi}{8\pi} = 9$ min

77. $\theta = 3.871142$ radians, $t = 21.4693$ sec
$\omega = \frac{\theta}{t}$
$\omega = \frac{3.871142}{21.4693} \approx .180311$ radian per sec

78. $\omega = .90674$ radian per min, $t = 11.876$ min
$\omega = \frac{\theta}{t} \Rightarrow .90674 = \frac{\theta}{11.876} \Rightarrow$
$\theta = (.90674)(11.876) \approx 10.768$ radians

79. $r = 12$ m, $\omega = \frac{2\pi}{3}$ radians per sec

$v = r\omega \Rightarrow v = 12\left(\frac{2\pi}{3}\right) = 8\pi$ m per sec

80. $r = 8$ m, $\omega = \frac{9\pi}{5}$ radians per sec

$v = r\omega \Rightarrow v = 8\left(\frac{9\pi}{5}\right) = \frac{72\pi}{5}$ cm per sec

81. $v = 9$ m per sec, $r = 5$ m

$v = r\omega \Rightarrow 9 = 5\omega \Rightarrow \omega = \frac{9}{5}$ radians per sec

82. $v = 18$ ft per sec, $r = 3$ ft
$v = r\omega \Rightarrow 18 = 3\omega \Rightarrow \omega = 6$ radians per sec

83. $v = 107.692$ m per sec, $r = 58.7413$ m
$v = r\omega \Rightarrow 107.692 = 58.7413\omega \Rightarrow$
$\omega = \frac{107.692}{58.7413} \approx 1.83333$ radians per sec

84. $r = 24.93215$ cm, $\omega = .372914$ radian per sec
$v = r\omega$
$v = (24.93215)(.372914) \approx 9.29755$ cm per sec

85. $r = 6$ cm, $\omega = \frac{\pi}{3}$ radians per sec, $t = 9$ sec

$s = r\omega t \Rightarrow s = 6\left(\frac{\pi}{3}\right)(9) = 18\pi$ cm

86. $r = 9$ yd, $\omega = \frac{2\pi}{5}$ radians per sec, $t = 12$ sec

$s = r\omega t \Rightarrow s = 9\left(\frac{2\pi}{5}\right)(12) = \frac{216\pi}{5}$ yd

87. $s = 6\pi$ cm, $r = 2$ cm, $\omega = \frac{\pi}{4}$ radians per sec

$s = r\omega t \Rightarrow 6\pi = 2\left(\frac{\pi}{4}\right)t \Rightarrow 6\pi = \left(\frac{\pi}{2}\right)t \Rightarrow$

$t = 6\pi\left(\frac{2}{\pi}\right) = 12$ sec

88. $s = \dfrac{12\pi}{5}$ m, $r = \dfrac{3}{2}$ m, $\omega = \dfrac{2\pi}{5}$ radians per sec

$$s = r\omega t \Rightarrow \frac{12\pi}{5} = \frac{3}{2}\left(\frac{2\pi}{5}\right)t \Rightarrow$$
$$\frac{12\pi}{5} = \frac{3\pi}{5}t \Rightarrow t = \frac{12\pi}{5} \cdot \frac{5}{3\pi} = 4 \text{ sec}$$

89. $s = \dfrac{3\pi}{4}$ km, $r = 2$ km, $t = 4$ sec

$$s = r\omega t \Rightarrow \frac{3\pi}{4} = 2\omega \cdot 4 \Rightarrow$$
$$\frac{3\pi}{4} = 8\omega \Rightarrow \omega = \frac{3\pi}{4} \cdot \frac{1}{8} = \frac{3\pi}{32} \text{ radian per sec}$$

90. $s = \dfrac{8\pi}{9}$ m, $r = \dfrac{4}{3}$ m, $t = 12$ sec

$$s = r\omega t \Rightarrow \frac{8\pi}{9} = \frac{4}{3}\omega \cdot 12 \Rightarrow$$
$$\frac{8\pi}{9} = 16\omega \Rightarrow \omega = \frac{8\pi}{9} \cdot \frac{1}{16} = \frac{\pi}{18} \text{ radian per sec}$$

91. The hour hand of a clock moves through an angle of 2π radians (one complete revolution) in 12 hours, so
$$\omega = \frac{\theta}{t} = \frac{2\pi}{12} = \frac{\pi}{6} \text{ radian per hr.}$$

92. The line makes 300 revolutions per minute. Each revolution is 2π radians, so we have
$$\omega = 2\pi(300) = 600\pi \text{ radians per min}$$

93. The minute hand makes one revolution per hour. Each revolution is 2π radians, so we have $\omega = 2\pi(1) = 2\pi$ radians per hr . There are 60 minutes in 1 hour, so $\omega = \dfrac{2\pi}{60} = \dfrac{\pi}{30}$ radians per min.

94. The second hand makes one revolution per minute. Each revolution is 2π radians, so we have $\omega = 2\pi(1) = 2\pi$ radians per min . There are 60 seconds in 1 min, so $\omega = \dfrac{2\pi}{60} = \dfrac{\pi}{30}$ radians per sec.

95. The minute hand of a clock moves through an angle of 2π radians in 60 min, and at the tip of the minute hand, $r = 7$ cm, so we have
$$v = \frac{r\theta}{t} \Rightarrow v = \frac{7(2\pi)}{60} = \frac{7\pi}{30} \text{ cm per min}$$

96. The second hand makes one revolution per minute. Each revolution is 2π radians, and at the tip of the second hand, $r = 28$ mm, so we have $v = r\omega \Rightarrow v = 28 \cdot 2\pi = 56\pi$ mm per min. There are 60 seconds in 1 min, so
$$v = \frac{56\pi}{60} = \frac{14\pi}{15} \text{ mm per sec.}$$

97. The flywheel making 42 rotations per min turns through an angle $42(2\pi) = 84\pi$ radians in 1 minute with $r = 2$ m. So,
$$v = \frac{r\theta}{t} \Rightarrow v = \frac{2(84\pi)}{1} = 168\pi \text{ m per min}$$

98. The point on the tread of the tire is rotating 35 times per min. Each rotation is 2π radians. Thus, we have
$\omega = 35(2\pi) = 70\pi$ radians per min. Since
$v = r\omega$, we have
$v = 18(70\pi) = 1260\pi$ cm per min.

99. At 500 rotations per min, the propeller turns through an angle of $\theta = 500(2\pi) = 1000\pi$ radians in 1 min with $r = \dfrac{3}{2} = 1.5$ m, we have
$$v = \frac{r\theta}{t} \Rightarrow v = \frac{1.5(1000\pi)}{1} = 1500\pi \text{ m per min.}$$

100. The point on the edge of the gyroscope is rotating 680 times per min. Each rotation is 2π radians.
$\omega = 680(2\pi) = 1360\pi$ radians per min, so
$v = r\omega \Rightarrow$
$v = 83(1360\pi) = 112{,}880\pi$ cm per min

101. At 215 revolutions per minute, the bicycle tire is moving $215\ (2\pi) = 430\pi$ radians per min. This is the angular velocity ω. The linear velocity of the bicycle is
$v = r\omega = 13(430\pi) = 5590\pi$ in. per min.
Convert this to miles per hour:
$$v = \frac{5590\pi \text{ in.}}{\text{min}} \cdot \frac{60 \text{ min}}{\text{hr}} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 16.6 \text{ mph}$$

102. Mars will make one full rotation (of 2π radians) during the course of one day. Thus,
$$2\pi \text{ radians}\left(\frac{1 \text{ hr}}{0.2552 \text{ radian}}\right) \approx 24.62 \text{ hr}$$

103. **(a)** $\theta = \dfrac{1}{365}(2\pi) = \dfrac{2\pi}{365}$ radian

(b) $\omega = \frac{2\pi}{365}$ radian per day

$= \frac{2\pi}{365} \cdot \frac{1}{24}$ radian per hr

$= \frac{\pi}{4380}$ radian per hr

(c) $v = r\omega$

$v = (93{,}000{,}000)\left(\frac{\pi}{4380}\right) \approx 67{,}000$ mph

104. **(a)** The earth completes one revolution per day, so it turns through $\theta = 2\pi$ radians in time
$t = 1$ day $= 24$ hr. So, we have:

$\omega = \frac{\theta}{t}$

$\omega = \frac{2\pi}{1} = 2\pi$ radians per day

$= \frac{2\pi}{24}$ radian per hr $= \frac{\pi}{12}$ radian per hr

(b) At the poles, $r = 0$, so $v = r\omega = 0$.

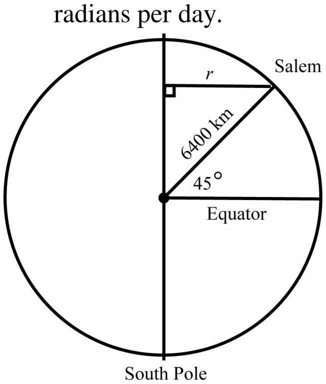
(c) At the equator, $r = 6400$ km. So,

$v = r\omega \Rightarrow$

$v = 6400(2\pi) = 12{,}800\pi$ km per day

$= \frac{12{,}800\pi}{24}$ km per hr $\approx 533\pi$ km per hr

(d) Salem rotates about the axis in a circle of radius r at an angular velocity $\omega = 2\pi$ radians per day.



$\sin 45° = \frac{r}{6400} \Rightarrow$

$r = 6400 \sin 45° = 6400\left(\frac{\sqrt{2}}{2}\right)$

$= 3200\sqrt{2}$ km

$v = r\omega \Rightarrow$

$v = 3200\sqrt{2}\,(2\pi)$

$\approx 9050\pi$ km per day

$\approx 28{,}000$ km per day

$v = \frac{9050\pi}{24}$ km per hr

$\approx 377\pi$ km per hr

≈ 1200 km per hr

105. **(a)** Since $s = 56$ cm of belt go around in $t = 18$ sec, the linear velocity is

$v = \frac{s}{t} \Rightarrow v = \frac{56}{18} = \frac{28}{9} \approx 3.1$ cm per sec

(b) Since the 56 cm belt goes around in 18 sec, we have

$v = r\omega \Rightarrow \frac{56}{18} = (12.96)\omega \Rightarrow$

$\frac{28}{9} = (12.96)\omega \Rightarrow$

$\omega = \frac{\frac{28}{9}}{12.96} \approx .24$ radian per sec

106. The larger pulley rotates 25 times in 36 sec or $\frac{25}{36}$ times per sec. Thus, its angular velocity is

$\omega = \frac{25}{36}(2\pi) = \frac{25\pi}{18}$ radians per sec. The linear velocity of the belt is

$v = r\omega \Rightarrow v = 15\left(\frac{25\pi}{18}\right) = \frac{125\pi}{6}$ cm per sec

To find the angular velocity of the smaller pulley, use $v = \frac{125\pi}{6}$ cm per sec and $r = 8$ cm.

$v = r\omega \Rightarrow \frac{125\pi}{6} = 8\omega \Rightarrow$

$\omega = \frac{125\pi}{6}\left(\frac{1}{8}\right) = \frac{125\pi}{48}$ radians per sec

107. $\omega = (152)(2\pi) = 304\pi$ radians per min

$= \frac{304\pi}{60}$ radians per sec

$= \frac{76\pi}{15}$ radians per sec

$v = r\omega \Rightarrow 59.4 = r\left(\frac{76\pi}{15}\right) \Rightarrow$

$r = 59.4\left(\frac{15}{76\pi}\right) \approx 3.73$ cm

108. Let s = the length of the track on the arc. First, converting $40°$ to radians, we have

$$40° = 40\left(\frac{\pi}{180}\text{radian}\right) = \frac{2\pi}{9}\text{radian.}$$

The length of track, s, is the arc length when $\theta = \frac{2\pi}{9}$ and $r = 1800$ ft.

$$s = r\theta \Rightarrow s = 1800\left(\frac{2\pi}{9}\right) = 400\pi \text{ ft}$$

Expressing the velocity $v = 30$ mph in ft per sec, we have

$$v = 30 \text{ mph} = \frac{30}{3600} \text{ mi per sec}$$
$$= \frac{(30)5280}{3600} \text{ ft per sec} = 44 \text{ ft per sec}$$
$$v = \frac{s}{t} \Rightarrow 44 = \frac{400\pi}{t} \Rightarrow 44t = 400\pi \Rightarrow$$
$$t = \frac{400\pi}{44} = \frac{100\pi}{11} \approx 29 \text{ sec}$$

109. In one minute, the propeller makes 5000 revolutions. Each revolution is 2π radians, so we have $5000(2\pi) = 10{,}000\pi$ radians per min. There are 60 sec in a minute, so

$$\omega = \frac{10{,}000\pi}{60} = \frac{500\pi}{3} \approx 523.6 \text{ radians per sec}$$

110. $r = 5\,\text{ft};\ \omega = 25$ radians per sec

$$v = r\omega \Rightarrow v = 5(25) = 125 \text{ ft per sec}$$

Section 6.3: Graphs of the Sine and Cosine Functions

1. $y = \sin x$

The graph is a sinusoidal curve with amplitude 1 and period 2π. Since sin 0 = 0, the point (0, 0) is on the graph. This matches with graph G.

2. $y = \cos x$

The graph is a sinusoidal curve with amplitude 1 and period 2π. Since cos 0 = 1, the point (0, 1) is on the graph. This matches with graph A.

3. $y = -\sin x$

The graph is a sinusoidal curve with amplitude 1 and period 2π. Bcause $a = -1$, the graph is a reflection of $y = \sin x$ in the x-axis. This matches with graph E.

4. $y = -\cos x$

The graph is a sinusoidal curve with amplitude 1 and period 2π. Because $a = -1$, the graph is a reflection of $y = \cos x$ in the x-axis. This matches with graph D.

5. $y = \sin 2x$

The graph is a sinusoidal curve with amplitude 1 and period π. Since $\sin(2\cdot 0) = \sin 0 = 0$, the point (0, 0) is on the graph. This matches with graph B.

6. $y = \cos 2x$

The graph is a sinusoidal curve with amplitude 1 and period π. Since $\cos(2\cdot 0) = \cos 0 = 1$, the point (0, 1) is on the graph. This matches with graph H.

7. $y = 2\sin x$

The graph is a sinusoidal curve with amplitude 2 and period 2π. Since $2\sin 0 = 2\cdot 0 = 0$ and $2\sin\pi = 2\cdot 1 = 2$, the points (0, 0) and $(\pi, 2)$, are on the graph. This matches with graph F.

8. $y = 2\cos x$

The graph is a sinusoidal curve with amplitude 2 and period 2π. Since $2\cos 0 = 2\cdot 1 = 2$, the point (0, 2) is on the graph. This matches with graph C.

9. $y = \sin 3x$

The graph is a sinusoidal curve with amplitude 1 and period $\frac{2\pi}{3}$. Since $\sin(3\cdot 0) = \sin 0 = 0$, the point (0, 0) is on the graph. This matches with graph D.

10. $y = \cos 3x$

The graph is a sinusoidal curve with amplitude 1 and period $\frac{2\pi}{3}$. Since $\cos(3\cdot 0) = \cos 0 = 1$, the point (0, 1) is on the graph. This matches with graph B.

11. $y = 3\cos x$

The graph is a sinusoidal curve with amplitude 3 and period 2π. Since $3\cos 0 = 3\cdot 1 = 3$, the point (0, 3) is on the graph. This matches with graph C.

12. $y = 3\sin x$

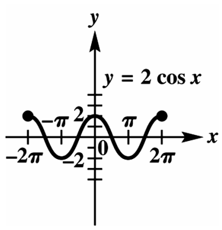
The graph is a sinusoidal curve with amplitude 3 and period 2π. Since $3\sin 0 = 3 \cdot 0 = 0$ and $3\sin\frac{\pi}{2} = 3 \cdot 1 = 3$, the points (0, 0) and $\left(\frac{\pi}{2}, 3\right)$ are on the graph. This matches with graph A.

13. $y = 2\cos x$

Amplitude: $|2| = 2$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	–1	0	1
$2\cos x$	2	0	–2	0	2

This table gives five values for graphing one period of the function. Repeat this cycle for the interval $[-2\pi, 0]$.

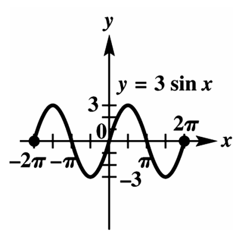


14. $y = 3\sin x$

Amplitude: $|3| = 3$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	–1	0
$3\sin x$	0	3	0	–3	0

This table gives five values for graphing one period of $y = 3\sin x$. Repeat this cycle for the interval $[-2\pi, 0]$.

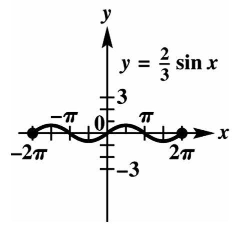


15. $y = \frac{2}{3}\sin x$

Amplitude: $\left|\frac{2}{3}\right| = \frac{2}{3}$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	–1	0
$\frac{2}{3}\sin x$	0	$\frac{2}{3} \approx .7$	0	$-\frac{2}{3} \approx -.7$	0

This table gives five values for graphing one period of $y = \frac{2}{3}\sin x$. Repeat this cycle for the interval $[-2\pi, 0]$.

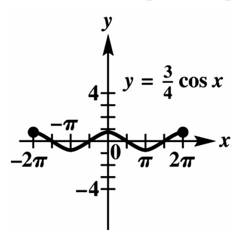


16. $y = \frac{3}{4}\cos x$

Amplitude: $\left|\frac{3}{4}\right| = \frac{3}{4}$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	–1	0	1
$\frac{3}{4}\cos x$	$\frac{3}{4}$	0	$-\frac{3}{4}$	0	$\frac{3}{4}$

This table gives five values for graphing one period of $y = \frac{3}{4}\cos x$. Repeat this cycle for the interval $[-2\pi, 0]$.

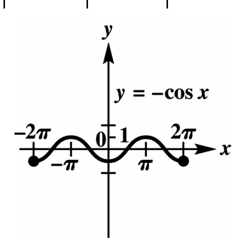


17. $y = -\cos x$

Amplitude: $|-1| = 1$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	–1	0	1
$-\cos x$	–1	0	1	0	–1

This table gives five values for graphing one period of $y = -\cos x$. Repeat this cycle for the interval $[-2\pi, 0]$.

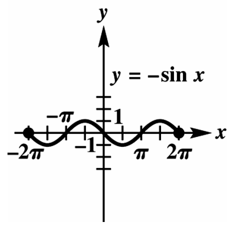


18. $y = -\sin x$

Amplitude: $|-1| = 1$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	–1	0
$-\sin x$	0	–1	0	1	0

This table gives five values for graphing one period of $y = -\sin x$. Repeat this cycle for the interval $[-2\pi, 0]$.

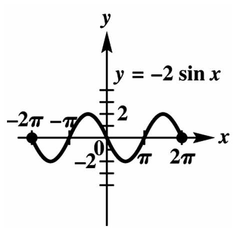


19. $y = -2\sin x$

Amplitude: $|-2| = 2$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	–1	0
$-2\sin x$	0	–2	0	2	0

This table gives five values for graphing one period of $y = -2\sin x$. Repeat this cycle for the interval $[-2\pi, 0]$.

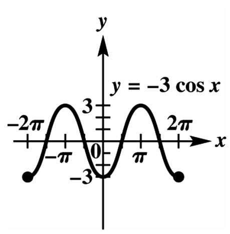


20. $y = -3\cos x$

Amplitude: $|-3| = 3$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	–1	0	1
$-3\cos x$	–3	0	3	0	–3

This table gives five values for graphing one period of $y = -3\cos x$. Repeat this cycle for the interval $[-2\pi, 0]$.

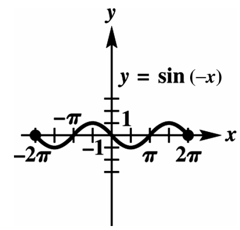


21. $y = \sin(-x)$

Amplitude: 1

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$-x$	0	$-\frac{\pi}{2}$	$-\pi$	$-\frac{3\pi}{2}$	-2π
$\sin(-x)$	0	–1	0	1	0

This table gives five values for graphing one period of $y = \sin(-x)$. Repeat this cycle for the interval $[-2\pi, 0]$.



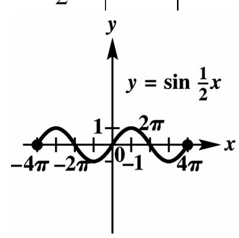
22. The graph of $y = \sin(-x)$ is the same as the graph of $y = -\sin x$ because the sine function is an odd function.

23. $y = \sin\frac{1}{2}x$

Period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$ and amplitude: $|1| = 1$

Divide the interval $[0, 4\pi]$ into four equal parts to get x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-4\pi, 0]$.

x	0	π	2π	3π	4π
$\frac{1}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{1}{2}x$	0	1	0	–1	0



24. $y = \sin\frac{2}{3}x$

Period: $\frac{2\pi}{\frac{2}{3}} = 2\pi \cdot \frac{3}{2} = 3\pi$ and amplitude: $|1| = 1$

Divide the interval $[0, 3\pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-3\pi, 0]$.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$\frac{2}{3}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{2}{3}x$	0	1	0	–1	0

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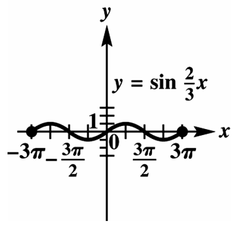
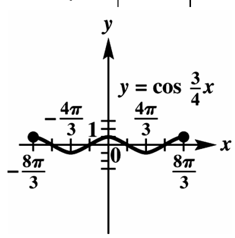


25. $y = \cos \frac{3}{4} x$

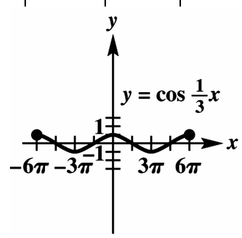
Period: $\frac{2\pi}{\frac{3}{4}} = 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3}$ and amplitude: $|1| = 1$

Divide the interval $\left[0, \frac{8\pi}{3}\right]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $\left[-\frac{8\pi}{3}, 0\right]$.

x	0	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π	$\frac{8\pi}{3}$
$\frac{3}{4}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \frac{3}{4}x$	1	0	-1	0	1



Exercise 25



Exercise 26

26. $y = \cos \frac{1}{3} x$

Period: $\frac{2\pi}{\frac{1}{3}} = 2\pi \cdot 3 = 6\pi$ and amplitude: $|1| = 1$

Divide the interval $[0, 6\pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-6\pi, 0]$.

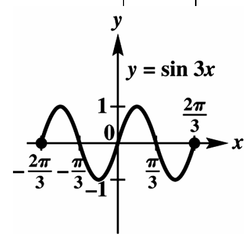
x	0	$\frac{3\pi}{2}$	3π	$\frac{9\pi}{2}$	6π
$\frac{1}{3}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \frac{1}{3}x$	1	0	-1	0	1

27. $y = \sin 3x$

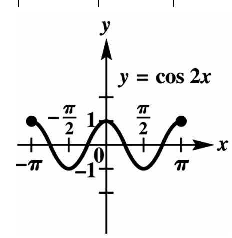
Period: $\frac{2\pi}{3}$ and amplitude: $|1| = 1$

Divide the interval $\left[0, \frac{2\pi}{3}\right]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $\left[-\frac{2\pi}{3}, 0\right]$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 3x$	0	1	0	-1	0



Exercise 27



Exercise 28

28. $y = \cos 2x$

Period: $\frac{2\pi}{2} = \pi$ and amplitude: $|1| = 1$

Divide the interval $[0, \pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-\pi, 0]$.

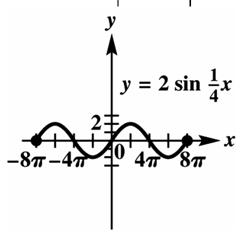
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 2x$	1	0	-1	0	1

29. $y = 2\sin\frac{1}{4}x$

Period: $\frac{2\pi}{\frac{1}{4}} = 2\pi \cdot \frac{4}{1} = 8\pi$ and amplitude: $|2| = 2$

Divide the interval $[0, 8\pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-8\pi, 0]$.

x	0	2π	4π	6π	8π
$\frac{1}{4}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{1}{4}x$	0	1	0	−1	0
$2\sin\frac{1}{4}x$	0	2	0	−2	0

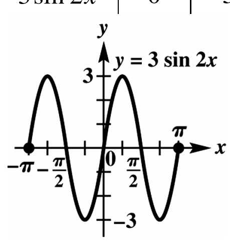


30. $y = 3 \sin 2x$

Period: $\frac{2\pi}{2} = \pi$ and amplitude: $|3| = 3$

Divide the interval $[0, \pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-\pi, 0]$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2x$	0	1	0	−1	0
$3\sin 2x$	0	3	0	−3	0

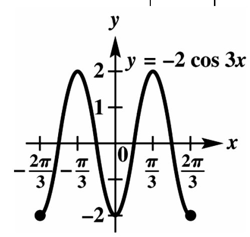


31. $y = -2 \cos 3x$

Period: $\frac{2\pi}{3}$ and amplitude: $|-2| = 2$

Divide the interval $\left[0, \frac{2\pi}{3}\right]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $\left[-\frac{2\pi}{3}, 0\right]$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 3x$	1	0	−1	0	1
$-2\cos 3x$	−2	0	2	0	−2

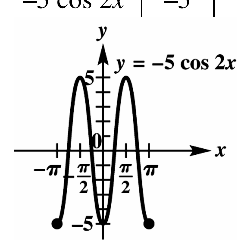


32. $y = -5 \cos 2x$

Period: $\frac{2\pi}{2} = \pi$ and amplitude: $|-5| = 5$

Divide the interval $[0, \pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-\pi, 0]$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 2x$	1	0	−1	0	1
$-5\cos 2x$	−5	0	5	0	−5

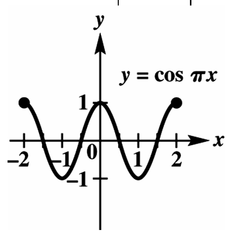


33. $y = \cos \pi x$

Period: $\frac{2\pi}{\pi} = 2$ and amplitude: $|1| = 1$

Divide the interval [0, 2] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−2, 0].

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \pi x$	1	0	-1	0	1

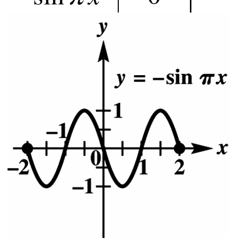


34. $y = -\sin \pi x$

Period: $\frac{2\pi}{\pi} = 2$ and amplitude: $|-1| = 1$

Divide the interval [0, 2] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−2, 0].

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \pi x$	0	1	0	-1	0
$-\sin \pi x$	0	-1	0	1	0

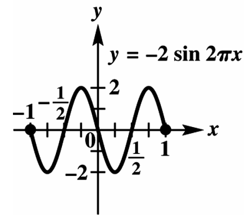


35. $y = -2\sin 2\pi x$

Period: $\frac{2\pi}{2\pi} = 1$ and amplitude: $|-2| = 2$

Divide the interval [0, 1] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−1, 0].

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$2\pi x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2\pi x$	0	1	0	-1	0
$-2\sin \pi x$	0	-2	0	2	0

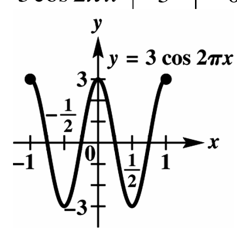


36. $y = 3\cos 2\pi x$

Period: $\frac{2\pi}{2\pi} = 1$ and amplitude: $|3| = 3$

Divide the interval [0, 1] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−1, 0].

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$2\pi x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 2\pi x$	1	0	-1	0	1
$3\cos 2\pi x$	3	0	-3	0	3



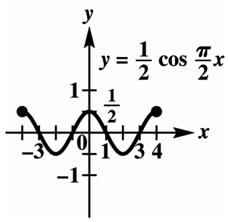
37. $y = \frac{1}{2}\cos \frac{\pi}{2} x$

Period: $\frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$ and amplitude:

$\left|\frac{1}{2}\right| = \frac{1}{2}$

Divide the interval [0, 4] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−4, 0].

x	0	1	2	3	4
$\frac{\pi}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{\pi}{2}x$	1	0	−1	0	1
$\frac{1}{2}\cos\frac{\pi}{2}x$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$

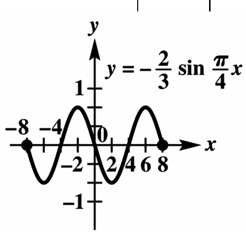


38. $y = -\frac{2}{3}\cos\frac{\pi}{4}x$

Period: $\frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$ and amplitude: $\left|-\frac{2}{3}\right| = \frac{2}{3}$

Divide the interval [0, 8] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−8, 0].

x	0	2	4	6	8
$\frac{\pi}{4}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{\pi}{4}x$	0	1	0	−1	0
$-\frac{2}{3}\sin\frac{\pi}{4}x$	0	$-\frac{2}{3}$	0	$\frac{2}{3}$	0

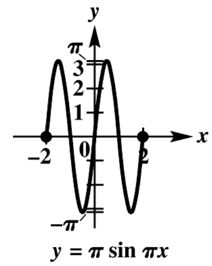


39. $y = \pi\sin\pi x$

Period: $\frac{2\pi}{\pi} = 2$ and amplitude: $|\pi| = \pi$

Divide the interval [0, 2] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−2, 0].

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\pi x$	0	1	0	−1	0
$\pi\sin\pi x$	0	π	0	π	0

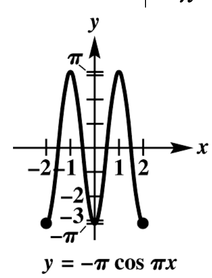


40. $y = -\pi\cos\pi x$

Period: $\frac{2\pi}{\pi} = 2$ and amplitude: $|-\pi| = \pi$

Divide the interval [0, 2] into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval [−2, 0].

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\pi x$	1	0	−1	0	1
$-\pi\cos\pi x$	$-\pi$	0	π	0	$-\pi$



41. The amplitude is $\frac{1}{2}\left[2-(-2)\right] = \frac{1}{2}(4) = 2$, so $a = 2$. One complete cycle of the graph is achieved in π units, so the period $\pi = \frac{2\pi}{b} \Rightarrow b = \frac{2\pi}{\pi} = 2$. Comparing the given graph with the general sine and cosine curves, we see that this graph is a cosine curve. Substituting $a = 2$ and $b = 2$, the function is $y = 2\cos 2x$. Verify by confirming minimum and maximum points and x-intercepts from the graph:

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$(0,2) \Rightarrow 2 = 2\cos(2\cdot 0) = 2\cos 0 = 2\cdot 1 = 2$

$\left(\frac{\pi}{4},0\right) \Rightarrow 0 = 2\cos\left(2\cdot\frac{\pi}{4}\right) = 2\cos\frac{\pi}{2} = 2\cdot 0 = 0$

$\left(\frac{\pi}{2},-2\right) \Rightarrow 0 = 2\cos\left(2\cdot\frac{\pi}{2}\right) = 2\cos\pi$
$= 2(-1) = -2$

$\left(\frac{3\pi}{4},0\right) \Rightarrow 0 = 2\cos\left(2\cdot\frac{3\pi}{4}\right)$
$= 2\cos\frac{3\pi}{2} = 2\cdot 0 = 0$

$(\pi,2) \Rightarrow 2 = 2\cos(2\pi) = 2(1) = 2$

42. The amplitude is $\frac{1}{2}\left[2-(-2)\right] = \frac{1}{2}(4) = 2$, so $a = 2$. One complete cycle of the graph is achieved in π units, so the period $\pi = \frac{2\pi}{b} \Rightarrow b = \frac{2\pi}{\pi} = 2$. Comparing the given graph with the general sine and cosine curves, we see that this graph is the reflection of the sine curve in the x-axis. Thus, $a = -2$. Substituting $a = -2$ and $b = 2$, the function is $y = -2\sin 2x$. Verify by confirming minimum and maximum points and x-intercepts from the graph:

$(0,0) \Rightarrow 0 = -2\sin(2\cdot 0) = -2\sin 0 = -2\cdot 0 = 0$

$\left(\frac{\pi}{4},-2\right) \Rightarrow -2 = -2\sin\left(2\cdot\frac{\pi}{4}\right) = -2\sin\frac{\pi}{2}$
$= -2(1) = -2$

$\left(\frac{\pi}{2},0\right) \Rightarrow 0 = -2\sin\left(2\cdot\frac{\pi}{2}\right) = -2\sin\pi$
$= -2(0) = 0$

$\left(\frac{3\pi}{4},2\right) \Rightarrow 2 = -2\sin\left(2\cdot\frac{3\pi}{4}\right)$
$= -2\sin\frac{3\pi}{2} = -2(-1) = 2$

$(\pi,0) \Rightarrow 2 = -2\sin(2\pi) = -2(0) = 0$

43. The amplitude is $\frac{1}{2}\left[3-(-3)\right] = \frac{1}{2}(6) = 3$, so $a = 3$. One-half of a cycle of the graph is achieved in 2π units, so the period is $2\cdot 2\pi = 4\pi$ and $4\pi = \frac{2\pi}{b} \Rightarrow b = \frac{2\pi}{4\pi} = \frac{1}{2}$. Comparing the given graph with the general sine and cosine curves, we see that this graph is the reflection of the cosine curve in the x-axis. Thus, $a = -3$.

Substituting $a = -3$ and $b = \frac{1}{2}$, the function is $y = -3\cos\frac{1}{2}x$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$(0,-3) \Rightarrow -3 = -3\cos\left(\frac{1}{2}\cdot 0\right) = -3\cos 0$
$= -3\cdot 1 = -3$

$(\pi,0) \Rightarrow 0 = -3\cos\left(\frac{1}{2}\cdot\pi\right) = -3\cos\frac{\pi}{2}$
$= -3(0) = 0$

$(2\pi,3) \Rightarrow 3 = -3\cos\left(\frac{1}{2}\cdot 2\pi\right) = -3\cos\pi$
$= -3(-1) = 3$

44. The amplitude is $\frac{1}{2}\left[3-(-3)\right] = \frac{1}{2}(6) = 3$, so $a = 3$. One-half of a cycle of the graph is achieved in 2π units, so the period is $2\cdot 2\pi = 4\pi$ and $4\pi = \frac{2\pi}{b} \Rightarrow b = \frac{2\pi}{4\pi} = \frac{1}{2}$. Comparing the given graph with the general sine and cosine curves, we see that this graph is a cosine curve Substituting $a = 3$ and $b = \frac{1}{2}$, the function is $y = 3\cos\frac{1}{2}x$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$(0,3) \Rightarrow 3 = 3\cos\left(\frac{1}{2}\cdot 0\right) = 3\cos 0 = 3\cdot 1 = 3$

$(\pi,0) \Rightarrow 0 = 3\cos\left(\frac{1}{2}\cdot\pi\right) = 3\cos\frac{\pi}{2}$
$= 3(0) = 0$

$(2\pi,-3) \Rightarrow -3 = 3\cos\left(\frac{1}{2}\cdot 2\pi\right) = 3\cos\pi$
$= 3(-1) = -3$

45. The amplitude is $\frac{1}{2}\left[3-(-3)\right] = \frac{1}{2}(6) = 3$, so $a = 3$. One complete cycle of the graph is achieved in $\frac{\pi}{2}$ units, so the period $\frac{\pi}{2} = \frac{2\pi}{b} \Rightarrow b = 2\pi\cdot\frac{2}{\pi} = 4$. Comparing the given graph with the general sine and cosine curves, we see that this graph is a sine curve. Substituting $a = 3$ and $b = 4$, the function is $y = 3\sin 4x$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$(0,0) \Rightarrow 0 = 3\sin(4\cdot 0) = 3\sin 0 = 3\cdot 0 = 0$

$\left(\frac{\pi}{8},3\right) \Rightarrow 3 = 3\sin\left(4\cdot\frac{\pi}{8}\right) = 3\sin\frac{\pi}{2}$
$= 3(1) = 3$

$$\left(\frac{\pi}{4},0\right) \Rightarrow 0 = 3\sin\left(2\cdot\frac{\pi}{4}\right) = 3\sin\frac{\pi}{2} = 3(0) = 0$$

$$\left(\frac{3\pi}{8},-3\right) \Rightarrow -3 = 3\sin\left(4\cdot\frac{3\pi}{8}\right) = 3\sin\frac{3\pi}{2} = 3(-1) = -3$$

$$\left(\frac{\pi}{2},0\right) \Rightarrow 0 = 3\sin\left(2\cdot\frac{\pi}{2}\right) = 3\sin\pi = 3(0) = 0$$

46. The amplitude is $\frac{1}{2}\left[3-(-3)\right] = \frac{1}{2}(6) = 3$, so $a = 3$. One complete cycle of the graph is achieved in $\frac{\pi}{2}$ units, so the period $\frac{\pi}{2} = \frac{2\pi}{b} \Rightarrow b = 2\pi\cdot\frac{2}{\pi} = 4$. Comparing the given graph with the general sine and cosine curves, we see that this graph is the reflection of the cosine curve in the x-axis. Thus, $a = -3$. Substituting $a = -3$ and $b = 4$, the function is $y = -3\cos 4x$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$$(0,-3) \Rightarrow -3 = -3\cos(4\cdot 0) = -3\cos 0 = -3\cdot 1 = -3$$

$$\left(\frac{\pi}{8},0\right) \Rightarrow 0 = -3\cos\left(4\cdot\frac{\pi}{8}\right) = -3\cos\frac{\pi}{2} = -3(0) = 0$$

$$\left(\frac{\pi}{4},3\right) \Rightarrow 3 = -3\cos\left(4\cdot\frac{\pi}{4}\right) = -3\cos\pi = -3(-1) = 3$$

$$\left(\frac{3\pi}{8},0\right) \Rightarrow 0 = -3\cos\left(4\cdot\frac{3\pi}{8}\right) = -3\cos\frac{3\pi}{2} = -3(0) = 0$$

$$\left(\frac{\pi}{2},-3\right) \Rightarrow -3 = -3\cos\left(4\cdot\frac{\pi}{2}\right) = -3\cos 2\pi = -3(1) = -3$$

47. (a) The highest temperature is 80°; the lowest is 50°.

(b) The amplitude is $\frac{1}{2}(80-50) = \frac{1}{2}(30) = 15.$

(c) The period is about 35,000 yr.

(d) The trend of the temperature now is downward.

48. (a) The amplitude is $\frac{1}{2}(120-80) = \frac{1}{2}(40) = 20.$

(b) Since the period is .8 sec, there are $\frac{1}{.8} = 1.25$ beats per sec and the pulse rate is $60(1.25) = 75$ beats per min.

49. (a) The latest time that the animals begin their evening activity is 8:00 P.M., the earliest time is 4:00 P.M. So, 4:00 $\le y \le$ 8:00. Since there is a difference of 4 hr in these times, the amplitude is $\frac{1}{2}(4) = 2$ hr.

(b) The length of this period is 1 yr.

50. (a) The amplitude of the graph is $\frac{1}{3}$; the period is $\frac{3}{2}$. Since $\frac{2\pi}{k} = \frac{3}{2}$, $k = \frac{4\pi}{3}$. The equation is $y = \frac{1}{3}\sin\frac{4\pi t}{3}$.

(b) It takes $\frac{3}{2}$ sec for a complete movement of the arm.

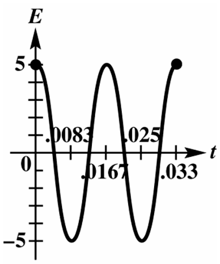
51. $E = 5\cos 120\pi r$

(a) Amplitude: $|5| = 5$ and period: $\frac{2\pi}{120\pi} = \frac{1}{60}$ sec

(b) Since the period is $\frac{1}{60}$, one cycle is completed in $\frac{1}{60}$ sec. Therefore, in 1 sec, 60 cycles are completed.

(c) $t = 0$,
$E = 5\cos 120\pi(0) = 5\cos 0 = 5(1) = 5$
$t = .03$,
$E = 5\cos 120\pi(.03) = 5\cos 3.6\pi \approx 1.545$
$t = .06$,
$E = 5\cos 120\pi(.06) = 5\cos 7.2\pi \approx -4.045$
$t = .09$,
$E = 5\cos 120\pi(.09) = 5\cos 10.8\pi 0 \approx -4.045$
$t = .12$,
$E = 5\cos 120\pi(.12) = 5\cos 14.4\pi \approx 1.545$

(d)



$E = 5 \cos 120\pi t$

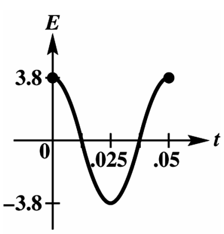
52. $E = 3.8 \cos 40\pi t$

(a) Amplitude: 3.8 and Period: $\dfrac{2\pi}{40\pi} = \dfrac{1}{20}$

(b) Frequency $= \dfrac{1}{\text{period}}$ = number of cycles per second = 20

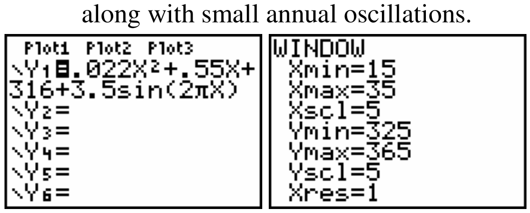
(c) $t = .02,$
$$E = 3.8\cos 40\pi(.02) = 3.8\cos .8\pi \approx -3.074$$
$t = .04,$
$$E = 3.8\cos 40\pi(.04) = 3.8\cos 1.6\pi \approx 1.174$$
$t = .08,$
$$E = 3.8\cos 40\pi(.08) = 3.8\cos 3.2\pi \approx -3.074$$
$t = .12,$
$$E = 3.8\cos 40\pi(.12) = 3.8\cos 4.8\pi \approx -3.074$$
$t = .14,$ $E = 3.8\cos 40\pi(.14) = 3.8\cos 5.6\pi \approx 1.174$

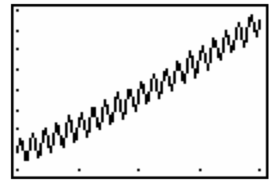
(d)



$E = 3.8 \cos 40\pi t$

53. (a) The graph has a general upward trend along with small annual oscillations.





(b) The seasonal variations are caused by the term $3.5\sin 2\pi x.$ The maximums will occur when $2\pi x = \dfrac{\pi}{2} + 2n\pi,$ where n is an integer. Since x cannot be negative, n cannot be negative. This is equivalent to

$$2\pi x = \frac{\pi}{2} + 2n\pi,\ n = 0, 1, 2, \ldots$$
$$2x = \frac{1}{2} + 2n,\ n = 0, 1, 2, \ldots$$
$$x = \frac{1}{4} + n,\ n = 0, 1, 2, \ldots$$
$$x = \frac{4n+1}{4},\ n = 0, 1, 2, \ldots$$
$$x = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \ldots$$

Since x is in years, $x = \dfrac{1}{4}$ corresponds to April when the seasonal carbon dioxide levels are maximum.

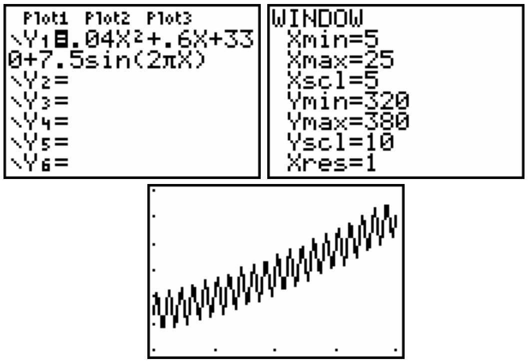
The minimums will occur when $2\pi x = \dfrac{3\pi}{2} + 2n\pi,$ where n is an integer. Since x cannot be negative, n cannot be negative. This is equivalent to

$$2\pi x = \frac{3\pi}{2} + 2n\pi,\ n = 0, 1, 2, \ldots$$
$$2x = \frac{3}{2} + 2n,\ n = 0, 1, 2, \ldots$$
$$x = \frac{3}{4} + n,\ n = 0, 1, 2, \ldots$$
$$x = \frac{4n+3}{4},\ n = 0, 1, 2, \ldots$$
$$x = \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \ldots$$

This is $\dfrac{1}{2}$ yr later, which corresponds to October.

(c) Answers will vary.

54. (a) The graph of C has a general upward trend similar to L (in Exercise 53) except that both the carbon dioxide levels and the seasonal oscillations are larger for C than L.



(b) Answers will vary.

(c) To solve this problem, horizontally translate the graph of C a distance of 1970 units to the right. The new C function can be written as

$$C(x) = .04(x-1970)^2 + .6(x-1970)+330 + 7.5\sin\left[2\pi(x-1970)\right]$$

where x is the actual year
This function would now be valid for $1970 \le x \le 1995$.

55. $T(x) = 37\sin\left[\frac{2\pi}{365}(x-101)\right]+25$

(a) March 15 (day 74)

$$T(74) = 37\sin\left[\frac{2\pi}{365}(74-101)\right]+25 \approx 8.4° \approx 8°$$

(b) April 5 (day 95)

$$T(95) = 37\sin\left[\frac{2\pi}{365}(95-101)\right]+25 \approx 21.1° \approx 21°$$

(c) Day 200

$$T(200) = 37\sin\left[\frac{2\pi}{365}(200-101)\right]+25 \approx 61.67° \approx 62°$$

(d) June 25 is day 176.
$(31+28+31+30+31+25=176)$

$$T(176) = 37\sin\left[\frac{2\pi}{365}(176-101)\right]+25 \approx 60.56° \approx 61°$$

(e) October 1 is day 274.
$31+28+31+30+31 +30+31+31+30+1=274$

$$T(274) = 37\sin\left[\frac{2\pi}{365}(274-101)\right]+25 \approx 31.02° \approx 31°$$

(f) December 31 is day 365.

$$T(365) = 37\sin\left[\frac{2\pi}{365}(365-101)\right]+25 \approx -11.48° \approx -11°$$

56. $\Delta S = .034(1367)\sin\left[\frac{2\pi(82.5-N)}{365.25}\right]$

(a) $N = 80$,

$$\Delta S = .034(1367)\sin\left[\frac{2\pi(82.5-80)}{365.25}\right] \approx 1.998 \text{ watts per m}^2$$

(b) $N = 1268$,

$$\Delta S = .034(1367)\sin\left[\frac{2\pi(82.5-1268)}{365.25}\right] \approx -46.461 \text{ watts per m}^2$$

(c) Since the greatest value the sine function can be is 1, the maximum of $\Delta S = .034(1367)\sin\left[\frac{2\pi(82.5-N)}{365.25}\right]$ is $.034(1367) = 46.478$ watts per m^2.

(d) $\Delta S = .034(1367)\sin\left[\frac{2\pi(82.5-N)}{365.25}\right]=0$ when $\sin\left[\frac{2\pi(82.5-N)}{365.25}\right]=0.$ This can occur when $\frac{2\pi(82.5-N)}{365.25}=0$ or $N=82.5.$ Other answers are possible. Since N represents a day number, which should be a natural number, we might interpret day 82.5 as noon on the 82^{nd} day.

57. The graph repeats each day, so the period is 24 hours.

58. The amplitude is approximately $\frac{1}{2}(2.6-.2)=\frac{1}{2}(2.4)=1.2$.

59. On January 20, low tide was at approximately 6 P.M., with height approximately .2 ft.

60. On January 20 at Kahului, low tide was at approximately 6 P.M. + 1:19, or 7:19 P.M., with height approximately .2 − .2 = 0 ft.

61. On January 22, high tide was at approximately 2 A.M., with height approximately 2.6 feet.

62. On January 22 at Lahaina, high tide was at approximately 2 A.M. + 1:18, or 3:18 A.M., with height approximately 2.6 − .2 = 2.4 feet.

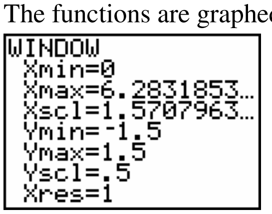
63. $-1 \le y \le 1$
Amplitude: 1
Period: 8 squares $= 8(30°) = 240°$ or $\dfrac{4\pi}{3}$

64. $-1 \le y \le 1$
Amplitude: 1
Period: 4 squares $= 4(30°) = 120°$ or $\dfrac{2\pi}{3}$

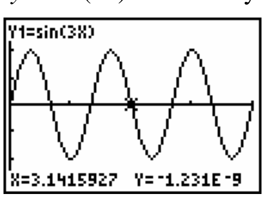
65. **(a)** No, we can't say that $\sin bx = b\sin x$. If b is not zero, then the period of $y = \sin bx$ is $\dfrac{2\pi}{|b|}$, and the amplitude is 1. The period of $y = b\sin x$ is 2π, and the amplitude is $|b|$.

(b) No, we can't say that $\cos bx = b\cos x$. If b is not zero, then the period of $y = \cos bx$ is $\dfrac{2\pi}{|b|}$, and the amplitude is 1. The period of $y = b\cos x$ is 2π, and the amplitude is $|b|$.

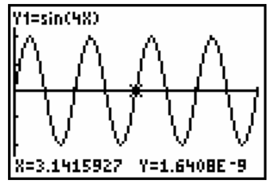
66. The functions are graphed in the window



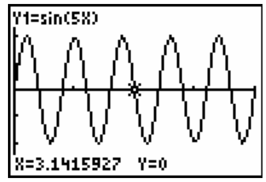
$y = \sin(3x)$ shows 3 cycles:



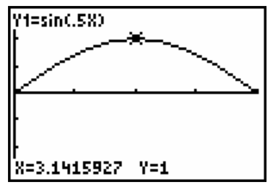
$y = \sin(4x)$ shows 4 cycles:



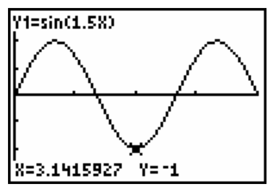
$y = \sin(5x)$ shows 5 cycles:



$y = \sin\left(\dfrac{1}{2}x\right)$ shows $\dfrac{1}{2}$ cycle:



$y = \sin\left(\dfrac{3}{2}x\right)$ shows $\dfrac{3}{2}$ cycles:



Section 6.4: Translations of the Graphs of the Sine and Cosine Functions

1. $y = \sin\left(x - \dfrac{\pi}{4}\right)$ is the graph of $y = \sin x$, shifted to the right $\dfrac{\pi}{4}$ unit. This matches choice D.

2. $y = \sin\left(x + \dfrac{\pi}{4}\right) = \sin\left[x - \left(-\dfrac{\pi}{4}\right)\right]$ is the graph of $y = \sin x$, shifted to the left $\dfrac{\pi}{4}$ unit. This matches choice G.

3. $y = \cos\left(x - \dfrac{\pi}{4}\right)$ is the graph of $y = \cos x$, shifted to the right $\dfrac{\pi}{4}$ unit. This matches choice H.

4. $y = \cos\left(x + \frac{\pi}{4}\right) = \cos\left[x - \left(-\frac{\pi}{4}\right)\right]$ is the graph of $y = \cos x$, shifted to the left $\frac{\pi}{4}$ unit. This matches choice A.

5. $y = 1 + \sin x$ is the graph of $y = \sin x$, translated vertically 1 unit up. This matches choice B.

6. $y = -1 + \sin x$ is the graph of $y = \sin x$, translated vertically 1 unit down. This matches choice E.

7. $y = 1 + \cos x$ is the graph of $y = \cos x$, translated vertically 1 unit up. This matches choice F.

8. $y = -1 + \cos x$ is the graph of $y = \cos x$, translated vertically 1 unit down. This matches choice C.

9. $y = \cos\left(x - \frac{\pi}{4}\right)$ is the graph of $y = \cos x$, shifted to the right $\frac{\pi}{4}$ unit. This matches choice C.

10. $y = \sin\left(x - \frac{\pi}{4}\right)$ is the graph of $y = \sin x$, shifted to the right $\frac{\pi}{4}$ unit. This matches choice B.

11. $y = 1 + \sin x$ is the graph of $y = \sin x$, translated vertically 1 unit up. This matches choice A.

12. $y = -1 + \cos x$ is the graph of $y = \cos x$, translated vertically 1 unit down. This matches choice D.

13. The graph of $y = \sin x + 1$ is the graph of $y = \sin x$ translated vertically 1 unit up, while the graph of $y = \sin(x+1)$ is the graph of $y = \sin x$ shifted horizontally 1 unit left.

14. $y = \sin x + 1$

15. $y = 3\sin(2x - 4) = 3\sin\left[2(x - 2)\right]$

The amplitude = $|3| = 3$, period = $\frac{2\pi}{2} = \pi$, and phase shift = 2. This matches choice B.

16. $y = 2\sin(3x - 4) = 2\sin\left[3\left(x - \frac{4}{3}\right)\right]$

The amplitude = $|2| = 2$, period = $\frac{2\pi}{3}$, and phase shift = $\frac{4}{3}$. This matches choice D.

17. $y = 4\sin(3x - 2) = 4\sin\left[3\left(x - \frac{2}{3}\right)\right]$

The amplitude = $|4| = 4$, period = $\frac{2\pi}{3}$, and phase shift = $\frac{2}{3}$. This matches choice C.

18. $y = 2\sin(4x - 3) = 2\sin\left[4\left(x - \frac{3}{4}\right)\right]$

The amplitude = $|2| = 2$, period = $\frac{2\pi}{4} = \frac{\pi}{2}$, and phase shift = $\frac{3}{4}$. This matches choice A.

19. This is a sine curve that has been shifted one unit down, so the equation is $y = -1 + \sin x$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$(0,-1) \Rightarrow -1 = -1 + \sin 0 = -1 + 0 = -1$

$\left(\frac{\pi}{2}, 0\right) \Rightarrow 0 = -1 + \sin\frac{\pi}{2} = -1 + 1 = 0$

$(\pi,-1) \Rightarrow -1 = -1 + \sin\pi = -1 + 0 = -1$

$\left(\frac{3\pi}{2}, -2\right) \Rightarrow -2 = -1 + \sin\frac{3\pi}{2} = -1 + (-1) = -2$

$(2\pi,-1) \Rightarrow -1 = 1 + \sin(2\pi) = -1 + 0 = -1$

20. This is a cosine curve that has been shifted two units up, so the equation is $y = 2 + \cos x$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$(0,3) \Rightarrow 3 = 2 + \cos 0 = 2 + 1 = 3$

$\left(\frac{\pi}{2}, 2\right) \Rightarrow 2 = 2 + \cos\frac{\pi}{2} = 2 + 0 = 2$

$(\pi,1) \Rightarrow 1 = 2 + \cos\pi = 2 + (-1) = 1$

$\left(\frac{3\pi}{2}, 2\right) \Rightarrow 2 = 2 + \cos\frac{3\pi}{2} = 2 + 0 = 2$

$(2\pi,3) \Rightarrow 3 = 2 + \cos(2\pi) = 2 + 1 = 3$

21. The maximum is at $\left(\frac{\pi}{3}, 1\right)$, so this is a cosine curve that has been shifted $\frac{\pi}{3}$ units to the right. Thus, the equation is $y = \cos\left(x - \frac{\pi}{3}\right)$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$\left(\frac{\pi}{3}, 1\right) \Rightarrow 1 = \cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right) = \cos 0 = 1$

$\left(\frac{4\pi}{3}, -1\right) \Rightarrow -1 = \cos\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) = \cos \pi = -1$

$\left(\frac{7\pi}{3}, 1\right) \Rightarrow 1 = \cos\left(\frac{7\pi}{3} - \frac{\pi}{3}\right) = \cos 2\pi = 1$

$\left(\frac{10\pi}{3}, -1\right) \Rightarrow -1 = \cos\left(\frac{10\pi}{3} - \frac{\pi}{3}\right) = \cos 3\pi = -1$

$\left(\frac{13\pi}{3}, 1\right) \Rightarrow 1 = \cos\left(\frac{13\pi}{3} - \frac{\pi}{3}\right) = \cos 4\pi = 1$

22. The x-intercepts are at $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$. So this is a cosine curve that has been shifted $\frac{\pi}{6}$ units to the right. Thus, the equation is $y = \cos\left(x - \frac{\pi}{6}\right)$. Verify by confirming minimum and maximum points and x–intercepts from the graph:

$\left(\frac{\pi}{6}, 1\right) \Rightarrow 1 = \cos\left(\frac{\pi}{6} - \frac{\pi}{6}\right) = \cos 0 = 1$

$\left(\frac{2\pi}{3}, 0\right) \Rightarrow 0 = \cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) = \cos \frac{\pi}{2} = 0$

$\left(\frac{7\pi}{6}, -1\right) \Rightarrow -1 = \cos\left(\frac{7\pi}{6} - \frac{\pi}{6}\right) = \cos \pi = -1$

$\left(\frac{5\pi}{3}, 0\right) \Rightarrow 0 = \cos\left(\frac{5\pi}{3} - \frac{\pi}{6}\right) = \cos \frac{3\pi}{2} = 0$

$\left(\frac{13\pi}{6}, 1\right) \Rightarrow 1 = \cos\left(\frac{13\pi}{6} - \frac{\pi}{6}\right) = \cos 2\pi = 1$

23. $y = 2\sin(x - \pi)$

amplitude: $|2| = 2$; period: $\frac{2\pi}{1} = 2\pi$; There is no vertical translation. The phase shift is π units to the right.

24. $y = \frac{2}{3}\sin\left(x + \frac{\pi}{2}\right) = \frac{2}{3}\sin\left[x - \left(-\frac{\pi}{2}\right)\right]$

amplitude: $\left|\frac{2}{3}\right| = \frac{2}{3}$; period: $\frac{2\pi}{1} = 2\pi$; There is no vertical translation. The phase shift is $\frac{\pi}{2}$ units to the left.

25. $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right) = 4\cos\frac{1}{2}\left[x - (-\pi)\right]$

amplitude: $|4| = 4$; period: $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$; There is no vertical translation. The phase shift is π units to the left.

26. $y = \frac{1}{2}\sin\left(\frac{1}{2}x + \pi\right) = \frac{1}{2}\sin\frac{1}{2}\left[x - (-2\pi)\right]$

amplitude: $\left|\frac{1}{2}\right| = \frac{1}{2}$; period: $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$; There is no vertical translation. The phase shift is 2π units to the left.

27. $y = 3\cos\frac{\pi}{2}\left(x - \frac{1}{2}\right)$

amplitude: $|3| = 3$; period: $\frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$; There is no vertical translation. The phase shift is $\frac{1}{2}$ unit to the right.

28. $y = -\cos\pi\left(x - \frac{1}{3}\right)$

amplitude: $|-1| = 1$; period: $\frac{2\pi}{\pi} = 2$; There is no vertical translation. The phase shift is $\frac{1}{3}$ unit to the right.

29. $y = 2 - \sin\left(3x - \frac{\pi}{5}\right) = -\sin 3\left(x - \frac{\pi}{15}\right) + 2$

amplitude: $|-1| = 1$; period: $\frac{2\pi}{3}$; The vertical translation is 2 units up. The phase shift is $\frac{\pi}{15}$ unit to the right

30. $y = -1 + \frac{1}{2}\cos(2x - 3\pi) = \frac{1}{2}\cos 2\left(x - \frac{3\pi}{2}\right) - 1$

amplitude: $\left|\frac{1}{2}\right| = \frac{1}{2}$; period: $\frac{2\pi}{2} = \pi$; The vertical translation is 1 unit down. The phase shift is $\frac{3\pi}{2}$ units to the right.

31. $y = \cos\left(x - \frac{\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

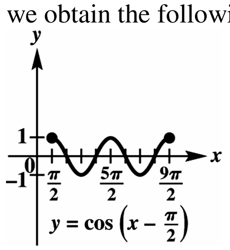
$0 \le x - \frac{\pi}{2} \le 2\pi \Rightarrow 0 + \frac{\pi}{2} \le x \le 2\pi + \frac{\pi}{2} \Rightarrow$
$\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

Step 2: Divide the period into four equal parts to get the following *x*-values: $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, $\frac{5\pi}{2}$

Step 3: Evaluate the function for each of the five *x*-values

x	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
$x - \frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\left(x - \frac{\pi}{2}\right)$	1	0	−1	0	1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 1. The period is 2π. There is no vertical translation. The phase shift is $\frac{\pi}{2}$ unit to the right.

32. $y = \sin\left(x - \frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

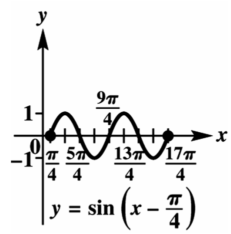
$0 \le x - \frac{\pi}{4} \le 2\pi \Rightarrow 0 + \frac{\pi}{4} \le x \le 2\pi + \frac{\pi}{4} \Rightarrow$
$\frac{\pi}{4} \le x \le \frac{9\pi}{4}$

Step 2: Divide the period into four equal parts to get the following *x*-values: $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, $\frac{9\pi}{4}$

Step 3: Evaluate the function for each of the five *x*-values

x	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$
$x - \frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\left(x - \frac{\pi}{4}\right)$	0	1	0	−1	0

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 1. The period is 2π. There is no vertical translation. The phase shift is $\frac{\pi}{4}$ unit to the right.

33. $y = \sin\left(x + \frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

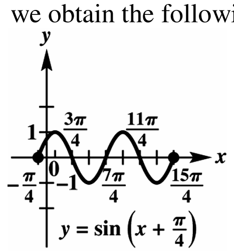
$0 \le x + \frac{\pi}{4} \le 2\pi \Rightarrow 0 - \frac{\pi}{4} \le x \le 2\pi - \frac{\pi}{4} \Rightarrow$
$-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$

Step 2: Divide the period into four equal parts to get the following *x*-values: $-\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$x+\frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\left(x+\frac{\pi}{4}\right)$	0	1	0	-1	0

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 1. The period is 2π. There is no vertical translation. The phase shift is $\frac{\pi}{4}$ unit to the left.

34. $y=\cos\left(x-\frac{\pi}{3}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

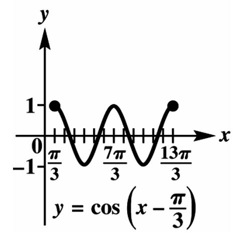
$0 \le x-\frac{\pi}{3} \le 2\pi \Rightarrow 0+\frac{\pi}{3} \le x \le 2\pi+\frac{\pi}{3} \Rightarrow$
$\frac{\pi}{3} \le x \le \frac{7\pi}{3}$

Step 2: Divide the period into four equal parts to get the following x-values: $\frac{\pi}{3}$, $\frac{5\pi}{6}$, $\frac{4\pi}{3}$, $\frac{11\pi}{6}$, $\frac{7\pi}{3}$

Step 3: Evaluate the function for each of the five x-values.

x	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$
$x-\frac{\pi}{3}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\left(x-\frac{\pi}{3}\right)$	1	0	-1	0	1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 1. The period is 2π. There is no vertical translation. The phase shift is $\frac{\pi}{3}$ units to the right

35. $y=2\cos\left(x-\frac{\pi}{3}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

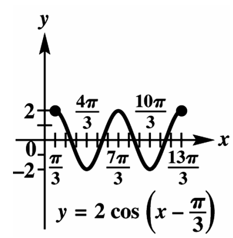
$0 \le x-\frac{\pi}{3} \le 2\pi \Rightarrow 0+\frac{\pi}{3} \le x \le 2\pi+\frac{\pi}{3} \Rightarrow$
$\frac{\pi}{3} \le x \le \frac{7\pi}{3}$

Step 2: Divide the period into four equal parts to get the following x-values $\frac{\pi}{3}$, $\frac{5\pi}{6}$, $\frac{4\pi}{3}$, $\frac{11\pi}{6}$, $\frac{7\pi}{3}$

Step 3: Evaluate the function for each of the five x-values.

x	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$
$x-\frac{\pi}{3}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\left(x-\frac{\pi}{3}\right)$	1	0	-1	0	1
$2\cos\left(x-\frac{\pi}{3}\right)$	2	0	-2	0	2

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 2. The period is 2π. There is no vertical translation. The phase shift is $\frac{\pi}{3}$ units to the right.

36. $y = 3\sin\left(x - \frac{3\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

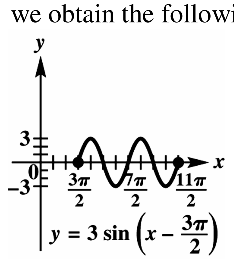
$$0 \le x - \frac{3\pi}{2} \le 2\pi \Rightarrow$$
$$0 + \frac{3\pi}{2} \le x \le 2\pi + \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} \le x \le \frac{7\pi}{2}$$

Step 2: Divide the period into four equal parts to get the following x-values: $\frac{3\pi}{2}$, 2π, $\frac{5\pi}{2}$, 3π, $\frac{7\pi}{2}$

Step 3: Evaluate the function for each of the five x-values.

x	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$
$x - \frac{3\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\left(x - \frac{3\pi}{2}\right)$	0	1	0	−1	0
$3\sin\left(x - \frac{3\pi}{2}\right)$	0	3	0	−3	0

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 3. The period is 2π. There is no vertical translation. The phase shift is $\frac{3\pi}{2}$ units to the right.

37. $y = \frac{3}{2}\sin 2\left(x + \frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

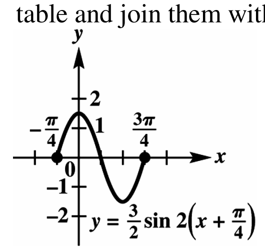
$$0 \le 2\left(x + \frac{\pi}{4}\right) \le 2\pi \Rightarrow 0 \le x + \frac{\pi}{4} \le \pi \Rightarrow$$
$$-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{\pi}{4}$, 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$2\left(x + \frac{\pi}{4}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2\left(x + \frac{\pi}{4}\right)$	0	1	0	−1	0
$\frac{3}{2}\sin 2\left(x + \frac{\pi}{4}\right)$	0	$\frac{3}{2}$	0	$-\frac{3}{2}$	0

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is $\frac{3}{2}$. The period is $\frac{2\pi}{2} = \pi$. There is no vertical translation. The phase shift is $\frac{\pi}{4}$ unit to the left.

38. $y = -\frac{1}{2}\cos 4\left(x + \frac{\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

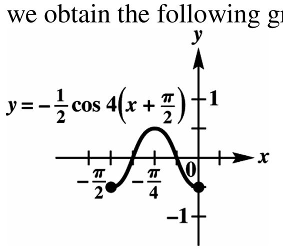
$$0 \le 4\left(x + \frac{\pi}{2}\right) \le 2\pi \Rightarrow 0 \le x + \frac{\pi}{2} \le \frac{\pi}{2} \Rightarrow$$
$$-\frac{\pi}{2} \le x \le 0$$

Step 2: Divide the period into four equal parts to get the following *x*-values: $-\frac{\pi}{2}$, $-\frac{3\pi}{8}$, $-\frac{\pi}{4}$, $-\frac{\pi}{8}$, 0

Step 3: Evaluate the function for each of the five *x*-values

x	$-\frac{\pi}{2}$	$-\frac{3\pi}{8}$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0
$4\left(x+\frac{\pi}{2}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 4\left(x+\frac{\pi}{2}\right)$	1	0	-1	0	1
$-\frac{1}{2}\cos 4\left(x+\frac{\pi}{2}\right)$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$. The period is $\frac{2\pi}{4} = \frac{\pi}{2}$. There is no vertical translation. The phase shift is $\frac{\pi}{2}$ unit to the left.

39. $y = -4\sin(2x-\pi) = -4\sin 2\left(x-\frac{\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

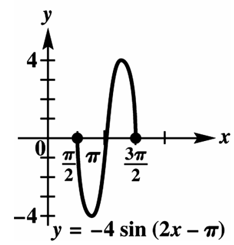
$0 \le 2\left(x-\frac{\pi}{2}\right) \le 2\pi \Rightarrow 0 \le x-\frac{\pi}{2} \le \frac{2\pi}{2} \Rightarrow$
$0 \le x-\frac{\pi}{2} \le \pi \Rightarrow \frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Step 2:Divide the period into four equal parts to get the following *x*-values: $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$

Step 3: Evaluate the function for each of the five *x*-values

x	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$2\left(x-\frac{\pi}{2}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2\left(x-\frac{\pi}{2}\right)$	0	1	0	-1	0
$-4\sin 2\left(x-\frac{\pi}{2}\right)$	0	-4	0	4	0

Steps 4 and 5:Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is $|-4|$, which is 4. The period is $\frac{2\pi}{2}$, which is π. There is no vertical translation. The phase shift is $\frac{\pi}{2}$ units to the right

40. $y = 3\cos(4x+\pi) = 3\cos 4\left(x+\frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

$0 \le 4\left(x+\frac{\pi}{4}\right) \le 2\pi \Rightarrow 0 \le x+\frac{\pi}{4} \le \frac{\pi}{2} \Rightarrow$
$0-\frac{\pi}{4} \le x \le \frac{\pi}{2}-\frac{\pi}{4} \Rightarrow -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

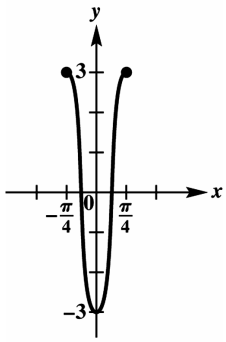
Step 2: Divide the period into four equal parts to get the following *x*-values: $-\frac{\pi}{4}$, $-\frac{\pi}{8}$, 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$

Step 3: Evaluate the function for each of the five *x*-values.

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$4\left(x+\frac{\pi}{4}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$\cos 4\left(x+\frac{\pi}{4}\right)$	1	0	−1	0	1
$3\cos 4\left(x+\frac{\pi}{4}\right)$	3	0	−3	0	3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



$y = 3\cos(4x + \pi)$

The amplitude is 3. The period is $\frac{2\pi}{4}$, which is $\frac{\pi}{2}$. There is no vertical translation. The phase shift is $\frac{\pi}{4}$ unit to the left.

41. $y = \frac{1}{2}\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right) = \frac{1}{2}\cos\frac{1}{2}\left(x - \frac{\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

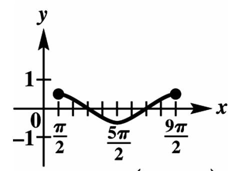
$$0 \le \frac{1}{2}\left(x - \frac{\pi}{2}\right) \le 2\pi \Rightarrow 0 \le x - \frac{\pi}{2} \le 4\pi \Rightarrow$$
$$\frac{\pi}{2} \le x \le \frac{8\pi}{2} + \frac{\pi}{2} \Rightarrow \frac{\pi}{2} \le x \le \frac{9\pi}{2}$$

Step 2: Divide the period into four equal parts to get the following x-values: $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

Step 3: Evaluate the function for each of the five x-values.

x	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$\frac{7\pi}{2}$	$\frac{9\pi}{2}$
$\frac{1}{2}\left(x-\frac{\pi}{2}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{1}{2}\left(x-\frac{\pi}{2}\right)$	1	0	−1	0	1
$\frac{1}{2}\cos\frac{1}{2}\left(x-\frac{\pi}{2}\right)$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



$y = \frac{1}{2}\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

The amplitude is $\frac{1}{2}$. The period is $\frac{2\pi}{\frac{1}{2}}$, which is 4π. There is no vertical translation. The phase shift is $\frac{\pi}{2}$ units to the right.

42. $y = -\frac{1}{4}\sin\left(\frac{3}{4}x + \frac{\pi}{8}\right) = -\frac{1}{4}\sin\frac{3}{4}\left(x + \frac{\pi}{6}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

$$0 \le \frac{3}{4}\left(x + \frac{\pi}{6}\right) \le 2\pi \Rightarrow 0 \le x + \frac{\pi}{6} \le \frac{8\pi}{3} \Rightarrow$$
$$-\frac{\pi}{6} \le x \le \frac{16\pi}{6} - \frac{\pi}{6} \Rightarrow -\frac{\pi}{6} \le x \le \frac{15\pi}{6}$$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{15\pi}{6}$

Step 3: Evaluate the function for each of the five x-values.

x	$-\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{15\pi}{6}$
$\frac{3}{4}\left(x+\frac{\pi}{6}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{3}{4}\left(x+\frac{\pi}{6}\right)$	0	1	0	−1	0
$-\frac{1}{4}\sin\frac{3}{4}\left(x+\frac{\pi}{6}\right)$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	0

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.

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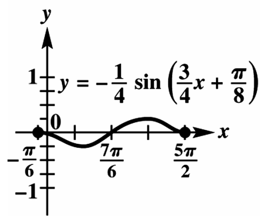
(*continued from page 571*)



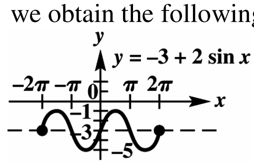
The amplitude is $\left|-\frac{1}{4}\right|$, which is $\frac{1}{4}$. The period is $\frac{2\pi}{\frac{3}{4}}$, which is $\frac{8\pi}{3}$. There is no vertical translation. The phase shift if $\frac{\pi}{6}$ units to the left.

43. $y = -3 + 2\sin x$

Step 1: The period is 2π.
Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π
Step 3: Evaluate the function for each of the five x-values:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	−1	0
$2\sin x$	0	2	0	−2	0
$-3 + 2\sin x$	−3	−1	−3	−5	−3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is 2. The vertical translation is 3 units down. There is no phase shift.

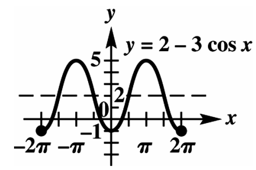
44. $y = 2 - 3\cos x$

Step 1: The period is 2π.
Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π

Evaluate the function for each of the five x-values:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	−1	0	1
$-3\cos x$	−3	0	3	0	−3
$2 - 3\cos x$	−1	2	5	2	−1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $|-3| = 3$. The vertical translation is 2 units up. There is no phase shift.

45. $y = -1 - 2\cos 5x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

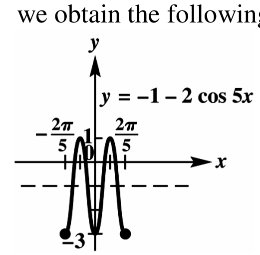
$0 \le 5x \le 2\pi \Rightarrow 0 \le x \le \frac{2\pi}{5}$

Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{\pi}{10}$, $\frac{\pi}{5}$, $\frac{3\pi}{10}$, $\frac{2\pi}{5}$

Step 3: Evaluate the function for each of the five x-values.

x	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$
$5x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 5x$	1	0	−1	0	1
$-2\cos 5x$	−2	0	2	0	−2
$-1 - 2\cos 5x$	−3	−1	1	−1	−3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The period is $\frac{2\pi}{5}$. The amplitude is $|-2|$, which is 2. The vertical translation is 1 unit down. There is no phase shift.

46. $y = 1 - \frac{2}{3}\sin\frac{3}{4}x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

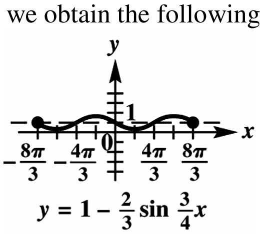
$0 \le \frac{3}{4}x \le 2\pi \Rightarrow 0 \le x < \frac{4}{3}\cdot 2\pi \Rightarrow 0 \le x \le \frac{8\pi}{3}$

Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, 2π, $\frac{8\pi}{3}$

Step 3: Evaluate the function for each of the five x-values.

x	0	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π	$\frac{8\pi}{3}$
$\frac{3}{4}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{3}{4}x$	0	1	0	−1	0
$-\frac{2}{3}\sin\frac{3}{4}x$	0	$-\frac{2}{3}$	0	$\frac{2}{3}$	0
$1-\frac{2}{3}\sin\frac{3}{4}x$	1	$\frac{1}{3}$	1	$\frac{5}{3}$	1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $\left|-\frac{2}{3}\right|$, which is $\frac{2}{3}$. The period is $\frac{2\pi}{\frac{3}{4}}$, which is $\frac{8\pi}{3}$. The vertical translation is 1 unit up. There is no phase shift.

47. $y = 1 - 2\cos\frac{1}{2}x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

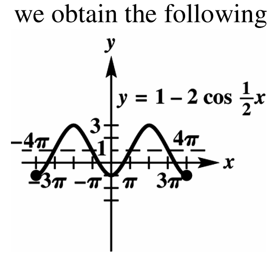
$0 \le \frac{1}{2}x \le 2\pi \Rightarrow 0 \le x \le 4\pi$

Step 2: Divide the period into four equal parts to get the following x-values: 0, π, 2π, 3π, 4π

Step 3: Evaluate the function for each of the five x-values.

x	0	π	2π	3π	4π
$\frac{1}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{1}{2}x$	1	0	−1	0	1
$-2\cos\frac{1}{2}x$	−2	0	2	0	−2
$1-2\cos\frac{1}{2}x$	−1	1	3	1	−1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $|-2|$, which is 2. The period is $\frac{2\pi}{\frac{1}{2}}$, which is 4π. The vertical translation is 1 unit up. There is no phase shift.

48. $y = -3 + 3\sin\frac{1}{2}x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

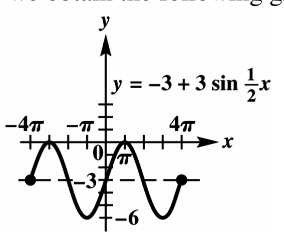
$0 \le \frac{1}{2}x \le 2\pi \Rightarrow 0 \le x \le 4\pi$

Step 2: Divide the period into four equal parts to get the following x-values: 0, π, 2π, 3π, 4π

Step 3: Evaluate the function for each of the five x-values.

x	0	π	2π	3π	4π
$\frac{1}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{1}{2}x$	0	1	0	-1	0
$3\sin\frac{1}{2}x$	0	3	0	-3	0
$-3+3\sin\frac{1}{2}x$	-3	0	-3	-6	-3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $|3|$, which is 3. The period is $\frac{2\pi}{\frac{1}{2}}$, which is 4π. The vertical translation is 3 units down. There is no phase shift.

49. $y=-2+\frac{1}{2}\sin 3x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

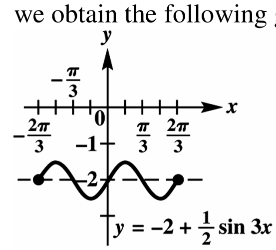
$0\le 3x\le 2\pi \Rightarrow 0\le x\le \frac{2\pi}{3}$

Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$

Step 3: Evaluate the function for each of the five x-values.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 3x$	0	1	0	-1	0
$\frac{1}{2}\sin 3x$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
$-2+\frac{1}{2}\sin 3x$	-2	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-2

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $\left|\frac{1}{2}\right|=\frac{1}{2}$. The period is $\frac{2\pi}{3}$.

The vertical translation is 2 units down. There is no phase shift.

50. $y=1+\frac{2}{3}\cos\frac{1}{2}x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

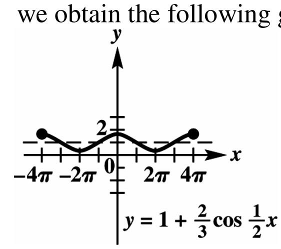
$0\le \frac{1}{2}x\le 2\pi \Rightarrow 0\le x\le 4\pi$

Step 2: Divide the period into four equal parts to get the following x-values: 0, π, 2π, 3π, 4π

Step 3: Evaluate the function for each of the five x-values.

x	0	π	2π	3π	4π
$\frac{1}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{1}{2}x$	1	0	-1	0	1
$\frac{2}{3}\cos\frac{1}{2}x$	$\frac{2}{3}$	0	$-\frac{2}{3}$	0	$\frac{2}{3}$
$1+\frac{2}{3}\cos\frac{1}{2}x$	$\frac{5}{3}$	1	$\frac{1}{3}$	1	$\frac{5}{3}$

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $\left|\frac{2}{3}\right|=\frac{2}{3}$. The period is 4π.

The vertical translation is 1 unit up. There is no phase shift.

51. $y = -3 + 2\sin\left(x + \frac{\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

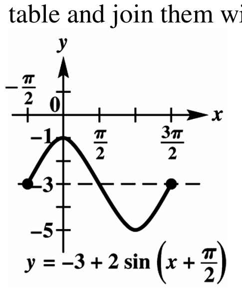
$0 \le x + \frac{\pi}{2} \le 2\pi \Rightarrow 0 - \frac{\pi}{2} \le x \le 2\pi - \frac{\pi}{2} \Rightarrow$
$-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$x+\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\left(x+\frac{\pi}{2}\right)$	0	1	0	−1	0
$2\sin\left(x+\frac{\pi}{2}\right)$	0	2	0	−2	0
$-3+2\sin\left(x+\frac{\pi}{2}\right)$	−3	−1	−3	−5	−3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is $|2|$, which is 2.The period is 2π. The vertical translation is 3 units down. The phase shift is $\frac{\pi}{2}$ units to the left.

52. $y = 4 - 3\cos(x - \pi)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

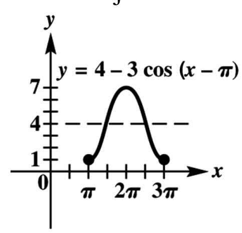
$0 \le x - \pi \le 2\pi \Rightarrow \pi \le x \le 3\pi$

Step 2: Divide the period into four equal parts to get the following x-values: π, $\frac{3\pi}{2}$, 2π, $\frac{5\pi}{2}$, 3π

Step 3: Evaluate the function for each of the five x-values.

x	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
$x-\pi$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos(x-\pi)$	1	0	−1	0	1
$-3\cos(x-\pi)$	−3	0	3	0	−3
$4-3\cos(x-\pi)$	1	4	7	4	1

Steps 4 and 5: Plot the points found in the table and join then with a sinusoidal curve.



The amplitude is $|-3|$, which is 3. The period is 2π. The vertical translation is 4 units up. The phase shift is π units to the right.

53. $y = \frac{1}{2} + \sin 2\left(x + \frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

$0 \le 2\left(x + \frac{\pi}{4}\right) \le 2\pi \Rightarrow 0 \le x + \frac{\pi}{4} \le \frac{2\pi}{2} \Rightarrow$
$0 \le x + \frac{\pi}{4} \le \pi \Rightarrow -\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{\pi}{4}$, 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$

Step 3: Evaluate the function for each of the five x-values.

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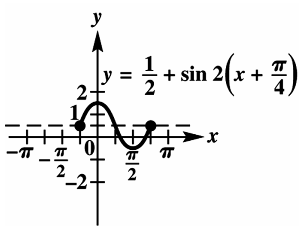
x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$2\left(x+\frac{\pi}{4}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2\left(x+\frac{\pi}{4}\right)$	0	1	0	-1	0
$\frac{1}{2}+\sin 2\left(x+\frac{\pi}{4}\right)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is $|1|$, which is 1. The period is $\frac{2\pi}{2}$, which is π. The vertical translation is $\frac{1}{2}$ unit up. The phase shift is $\frac{\pi}{4}$ units to the left.

54. $y=-\frac{5}{2}+\cos 3\left(x-\frac{\pi}{6}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

$$0 \le 3\left(x-\frac{\pi}{6}\right) \le 2\pi \Rightarrow 0 \le x-\frac{\pi}{6} \le \frac{2\pi}{3} \Rightarrow$$
$$\frac{\pi}{6} \le x \le \frac{4\pi}{6}+\frac{\pi}{6} \Rightarrow \frac{\pi}{6} \le x \le \frac{5\pi}{6}$$

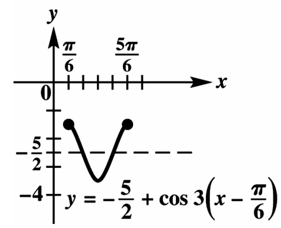
Step 2: Divide the period into four equal parts to get the following x-values: $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{5\pi}{6}$

Step 3: Evaluate the function for each of the five x-values.

x	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
$3\left(x-\frac{\pi}{6}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 3\left(x-\frac{\pi}{6}\right)$	1	0	-1	0	1

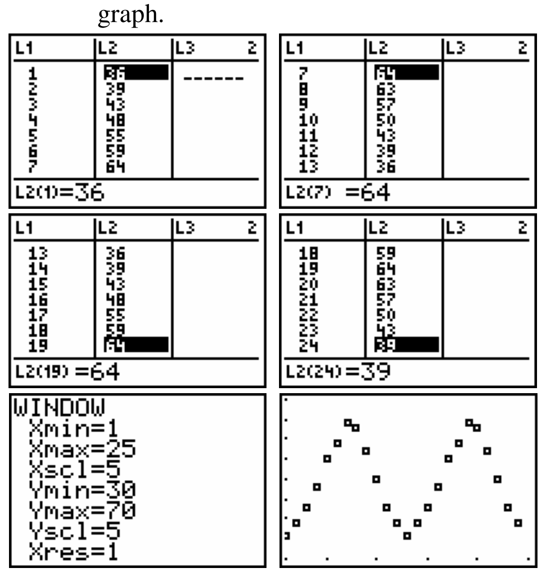
x	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
$-\frac{5}{2}+\cos 3\left(x-\frac{\pi}{6}\right)$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.

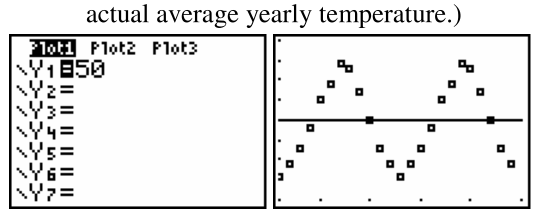


The amplitude is $|1|$, which is 1. The period is $\frac{2\pi}{3}$. The vertical translation is $\frac{5}{2}$ units down. The phase shift is $\frac{\pi}{6}$ units to the right.

55. (a) Let January correspond to $x = 1$, February to $x = 2$, … , and December of the second year to $x = 24$. Yes, the data appear to outline the graph of a translated sine graph.



(b) The sine graph is vertically centered around the line $y = 50$. This line represents the average yearly temperature in Vancouver of 50°F. (This is also the actual average yearly temperature.)



(c) The amplitude of the sine graph is approximately 14 since the average monthly high is 64, the average monthly low is 36, and $\frac{1}{2}(64-36)=\frac{1}{2}(28)=14.$ The period is 12 since the temperature cycles every twelve months. Let $b=\frac{2\pi}{12}=\frac{\pi}{6}.$ One way to determine the phase shift is to use the following technique. The minimum temperature occurs in January.
Thus, when $x = 1$, $b(x – d)$ must equal $\left(-\frac{\pi}{2}\right)+2\pi n,$ where n is an integer, since the sine function is minimum at these values. Solving for d, we have

$$\frac{\pi}{6}(1-d)=-\frac{\pi}{2}\Rightarrow 1-d=\frac{6}{\pi}\left(-\frac{\pi}{2}\right)\Rightarrow$$
$$1-d=-3\Rightarrow -d=-4\Rightarrow d=4$$

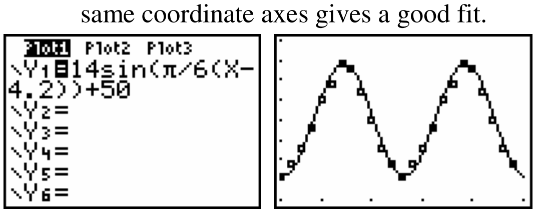
This can be used as a first approximation. Trial and error with a calculator leads to $d = 4.2$

(d) Let $f(x) = a \sin b(x – d) + c$. Since the amplitude is 14, let $a = 14$. The period is equal to 1 yr or 12 mo, so $b=\frac{\pi}{6}.$ The average of the maximum and minimum temperatures is

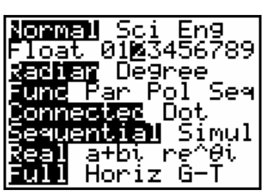
$\frac{1}{2}(64+36)=\frac{1}{2}(100)=50$. Thus,

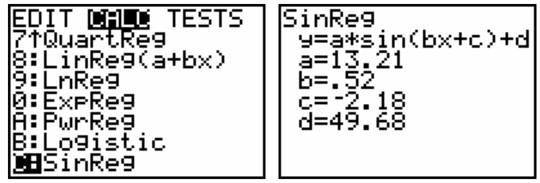
$$f(x)=14\sin\left[\frac{\pi}{6}(x-4.2)\right]+50$$

(e) Plotting the data with $f(x)=14\sin\left[\frac{\pi}{6}(x-4.2)\right]+50$ on the same coordinate axes gives a good fit.



(f)





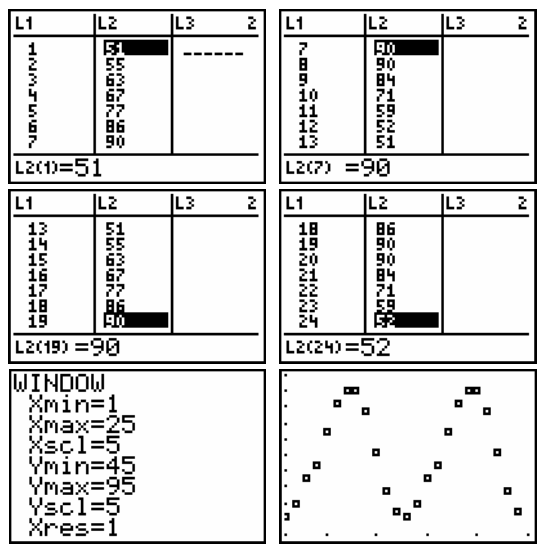
From the sine regression we have

$$y=13.21\sin(.52x-2.18)+49.68$$
$$=13.21\sin\left[.52(x-4.19)\right]+49.68$$

56. (a) We can predict the average yearly temperature by finding the mean of the average monthly temperatures:

$$\frac{51+55+63+67+77+86+50+90+84+71+59+52}{12}$$
$$=\frac{845}{12}\approx 70.4°\text{F},$$ which is very close to the actual value of 70°F.

(b) Let January correspond to $x = 1$, February to $x = 2$, … , and December of the second year to $x = 24$.



(c) Let the amplitude a be $\frac{1}{2}(90-51)=\frac{1}{2}(39)=19.5.$ Since the period is 12, let $b=\frac{\pi}{6}.$ Let $c=\frac{1}{2}(90+51)=\frac{1}{2}(141)=70.5.$
The minimum temperature occurs in January. Thus, when $x=1,$ $b(x – d)$ must equal an odd multiple of π since the cosine function is minimum at these values.

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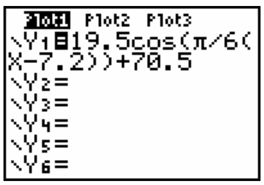
Solving for d, we have

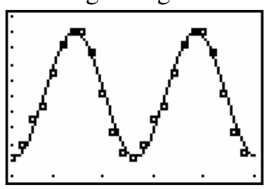
$$\frac{\pi}{6}(1-d) = -\pi \Rightarrow 1-d = \frac{6}{\pi}(-\pi) \Rightarrow$$
$$1-d=-6 \Rightarrow -d=-7 \Rightarrow d=7$$

d can be adjusted slightly to give a better visual fit. Try $d = 7.2$. Thus, we have

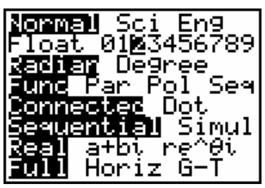
$$f(x) = a\cos b(x-d)+c$$
$$= 19.5\cos\left[\frac{\pi}{6}(x-7.2)\right]+70.5$$

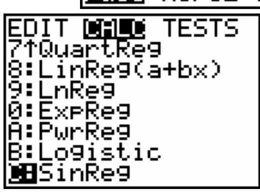
(d) Plotting the data with $f(x) = 19.5\cos\left[\frac{\pi}{6}(x-7.2)\right]+70.5$ on the same coordinate axes give a good fit.

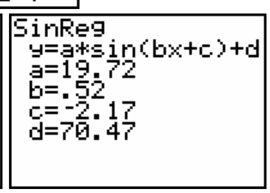




(e)







From the sine regression we have

$$y = 19.72\sin(.52x-2.17)+70.47$$
$$= 19.72\sin\left[.52(x-4.17)\right]+70.47$$

Chapter 6 Quiz

(Sections 6.1–6.4)

1. $225° = 225\left(\frac{\pi}{180}\text{ radian}\right) = \frac{5\pi}{4}$ radians

2. $-\frac{7\pi}{6} = -\frac{7\pi}{6}\left(\frac{180°}{\pi}\right) = -210°$

3. $r = 300,\ s = 450$

$$s = r\theta \Rightarrow 450 = 300\theta \Rightarrow \theta = \frac{450}{300} = 1.5$$

4. From exercise 5, $\theta = 1.5$

$$A = \frac{1}{2}r^2\theta \Rightarrow$$
$$A = \frac{1}{2}(300)^2(1.5) = 67{,}500\text{ in}^2$$

5. $-\frac{5\pi}{6}$ is coterminal with $-\frac{5\pi}{6}+2\pi = -\frac{5\pi}{6}+\frac{12\pi}{6} = \frac{7\pi}{6}$. Since $\frac{7\pi}{6}$ is in quadrant III, the reference angle is $\frac{7\pi}{6}-\pi = \frac{7\pi}{6}-\frac{6\pi}{6} = \frac{\pi}{6}$.

In quadrant III, the sine is negative. Thus, $\sin\left(-\frac{5\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$.

Converting $\frac{7\pi}{6}$ to degrees, we have $\frac{7\pi}{6} = \frac{7}{6}(180°) = 210°$. The reference angle is $210°-180° = 30°$. Thus,

$$\sin\left(-\frac{5\pi}{6}\right) = \sin\frac{7\pi}{6} = \sin 210°$$
$$= -\sin 30° = -\frac{1}{2}$$

6. 3π is coterminal with $3\pi - 2\pi = \pi$. So $\tan 3\pi = \tan\pi = 0$

Converting 3π to degrees, we have $3\pi\left(\frac{180}{\pi}\right) = 540°$. The reference angle is $540° - 360° = 180°$. Thus $\tan 3\pi = \tan 540° = \tan 180° = 0$.

7. $\left[\frac{\pi}{2},\pi\right]$; $\sin s = \frac{\sqrt{3}}{2}$

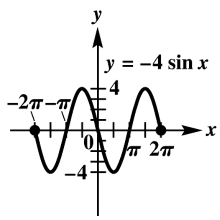
Recall that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and in quadrant II, $\sin s$ is positive. Therefore, $\sin\left(\pi-\frac{\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, and thus, $s = \frac{2\pi}{3}$.

8. $y = -4 \sin x$

Amplitude: $|-4| = 4$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	−1	0
$-4\sin x$	0	−4	0	4	0

This table gives five values for graphing one period of $y = -4 \sin x$. Repeat this cycle for the interval $[-2\pi, 0]$.

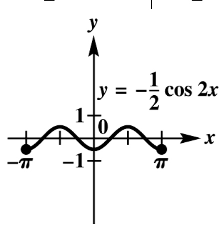


9. $y = -\frac{1}{2}\cos 2x$

Period: $\frac{2\pi}{2} = \pi$ and amplitude: $\left|-\frac{1}{2}\right| = \frac{1}{2}$

Divide the interval $[0, \pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Make a table. Repeat this cycle for the interval $[-\pi, 0]$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 2x$	1	0	−1	0	1
$-\frac{1}{2}\cos 2x$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$



10. $y = -2\cos\left(x + \frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

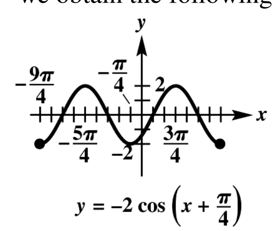
$$0 \le x + \frac{\pi}{4} \le 2\pi \Rightarrow 0 - \frac{\pi}{4} \le x \le 2\pi - \frac{\pi}{4} \Rightarrow$$
$$-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$x+\frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\left(x+\frac{\pi}{4}\right)$	1	0	−1	0	1
$-2\cos\left(x+\frac{\pi}{4}\right)$	−2	0	2	0	−2

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



The amplitude is 2. The period is 2π.

11. $y = 2 + \sin(2x - \pi) = 2 + \sin 2\left(x - \frac{\pi}{2}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

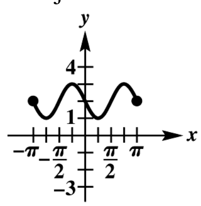
$$0 \le 2\left(x - \frac{\pi}{2}\right) \le 2\pi \Rightarrow 0 \le x - \frac{\pi}{2} \le \frac{2\pi}{2} \Rightarrow$$
$$0 \le x - \frac{\pi}{2} \le \pi \Rightarrow \frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

Step 2:Divide the period into four equal parts to get the following x-values: $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$

Step 3: Evaluate the function for each of the five x-values

x	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$2\left(x-\frac{\pi}{2}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2\left(x-\frac{\pi}{2}\right)$	0	1	0	−1	0
$2+\sin 2\left(x-\frac{\pi}{2}\right)$	2	3	2	1	2

Steps 4 and 5:Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is $|1|$, which is 1. The period is $\frac{2\pi}{2}$, which is π.

12. The amplitude is $\frac{1}{2}[1-(-1)]=\frac{1}{2}(2)=1$, so a = 1. One complete cycle of the graph is achieved in π units, so the period $\pi=\frac{2\pi}{b}\Rightarrow b=\frac{2\pi}{\pi}=2$. Comparing the given graph with the general sine and cosine curves, we see that this graph is a cosine curve. Substituting a = 1 and b = 2, the function is $y=\cos 2x$. Verify by confirming minimum and maximum points and x-intercepts from the graph:

$(0,1)\Rightarrow 1=\cos(2\cdot 0)=\cos 0=1$

$\left(\frac{\pi}{4},0\right)\Rightarrow 0=\cos\left(2\cdot\frac{\pi}{4}\right)=\cos\frac{\pi}{2}=0$

$\left(\frac{\pi}{2},-1\right)\Rightarrow -1=\cos\left(2\cdot\frac{\pi}{2}\right)=\cos\pi=-1$

$\left(\frac{3\pi}{4},0\right)\Rightarrow 0=\cos\left(2\cdot\frac{3\pi}{4}\right)=\cos\frac{3\pi}{2}=0$

$(\pi,1)\Rightarrow 1=\cos(2\pi)=1$

13. The amplitude is $\frac{1}{2}[1-(-1)]=\frac{1}{2}(2)=1$, so a = 1. One complete cycle of the graph is achieved in 2π units, so the period $2\pi=\frac{2\pi}{b}\Rightarrow b=\frac{2\pi}{2\pi}=1$.

Comparing the given graph with the general sine and cosine curves, we see that this graph is a sine curve reflected in the x-axis. So a = −1. Substituting a = −1 and b = 1, the function is $y=-\sin x$. Verify by confirming minimum and maximum points and x-intercepts from the graph: $(0,0)\Rightarrow 0=-\sin 0=0$

$\left(\frac{\pi}{2},-1\right)\Rightarrow -1=-\sin\frac{\pi}{2}=-1$

$(\pi,0)\Rightarrow 0=-\sin\pi=0$

$\left(\frac{3\pi}{2},1\right)\Rightarrow 1=-\sin\frac{3\pi}{2}=-1$

$(2\pi,0)\Rightarrow 0=-\sin 2\pi=0$

Exercises 14 and 15 refer to the function

$f(x)=12\sin\left[\frac{\pi}{6}(x-3.9)\right]+72$

14. April is represented by x = 4.

$f(4)=12\sin\left[\frac{\pi}{6}(4-3.9)\right]+72\approx 73°\text{F}$

15. From the equation, we have a = 12 and c = 72. Thus, the average monthly temperature is 72ºF. The lowest monthly average temperature is 72 − 12 = 60ºF. The highest monthly average temperature is 72 + 12 = 84ºF.

Section 6.5: Graphs of the Tangent, Cotangent, Secant, and Cosecant Functions

1. $y=-\tan x$
The graph is the reflection of the graph of $y=\tan x$ about the x-axis. This matches with graph C.

2. $y=-\cot x$
The graph is the reflection of the graph of $y=\cot x$ about the x-axis. This matches with graph A.

3. $y=\tan\left(x-\frac{\pi}{4}\right)$

The graph is the graph of $y=\tan x$ shifted $\frac{\pi}{4}$ units to the right. This matches with graph B.

4. $y=\cot\left(x-\frac{\pi}{4}\right)$

The graph is the graph of $y=\cot x$ shifted $\frac{\pi}{4}$ units to the right. This matches with graph D.

5. $y = \cot\left(x + \frac{\pi}{4}\right)$

The graph is the graph of $y = \cot x$ shifted $\frac{\pi}{4}$ units to the left. This matches with graph F.

6. $y = \tan\left(x + \frac{\pi}{4}\right)$

The graph is the graph of $y = \tan x$ shifted $\frac{\pi}{4}$ units to the left. This matches with graph E.

7. $y = \tan 4x$

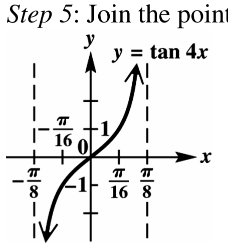
Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{4}$. Tangent has asymptotes of the form $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$. Therefore, the asymptotes for $y = \tan 4x$ are $4x = -\frac{\pi}{2} \Rightarrow x = -\frac{\pi}{8}$ and $4x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{8}$.

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $-\frac{\pi}{8}, -\frac{\pi}{16}, 0, \frac{\pi}{16}, \frac{\pi}{8}$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(-\frac{\pi}{16}, -1\right)$, $(0,0)$, $\left(\frac{\pi}{16}, 1\right)$

Step 5: Join the points with a smooth curve.



8. $y = \tan \frac{1}{2}x$

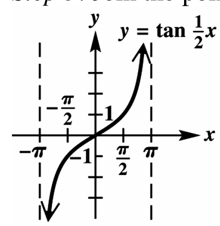
Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\frac{\pi}{b}$, so the period for this function is 2π. Tangent has asymptotes of the form $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$. Therefore, the asymptotes for $y = \tan \frac{1}{2}x$ are $\frac{1}{2}x = -\frac{\pi}{2} \Rightarrow x = -\pi$ and $\frac{1}{2}x = \frac{\pi}{2} \Rightarrow x = \pi$

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(-\frac{\pi}{2}, -1\right)$, $(0,0)$, $\left(\frac{\pi}{2}, 1\right)$

Step 5: Join the points with a smooth curve.



9. $y = 2 \tan x$

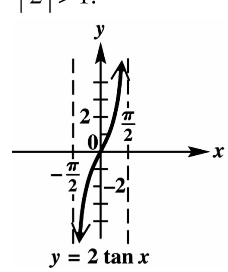
Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\frac{\pi}{b}$, so the period for this function is π. Tangent has asymptotes of the form $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$. Therefore, the asymptotes for $y = 2\tan x$ are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(-\frac{\pi}{4}, -2\right)$, $(0,0)$, $\left(\frac{\pi}{4}, 2\right)$

Step 5: Join the points with a smooth curve. The graph is "stretched" because $a = 2$ and $|2| > 1$.



10. $y = 2 \cot x$

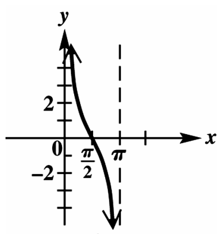
Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\frac{\pi}{b}$, so the period for this function is π. Cotangent has asymptotes of the form $bx = 0$ and $bx = \pi$. The asymptotes for $y = 2 \cot x$ are $x = 0$ and $x = \pi$.

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(\frac{\pi}{4}, 2\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{4}, -2\right)$

Step 5: Join the points with a smooth curve. The graph is "stretched" because $a = 2$ and $|2| > 1$.



y = 2 cot x

11. $y = 2 \tan \frac{1}{4} x$

Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\frac{\pi}{b}$, so the period for this function is 4π. Tangent has asymptotes of the form $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$. Therefore, the asymptotes for $y = 2 \tan \frac{1}{4} x$ are

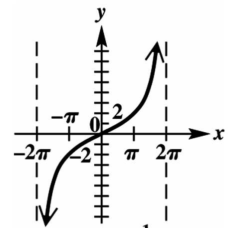
$\frac{1}{4} x = -\frac{\pi}{2} \Rightarrow x = -2\pi$ and $\frac{1}{4} x = \frac{\pi}{2} \Rightarrow x = 2\pi$

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $-2\pi, -\pi, 0, \pi, 2\pi$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have: $(-\pi, -2), (0,0), (\pi, 2)$

Step 5: Join the points with a smooth curve.



y = 2 tan ¼x

12. $y = \frac{1}{2} \cot x$

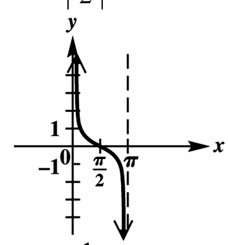
Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\frac{\pi}{b}$, so the period for this function is π. Cotangent has asymptotes of the form $bx = 0$ and $bx = \pi$. The asymptotes for $y = \frac{1}{2} \cot x$ are $x = 0$ and $x = \pi$.

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{4}, -\frac{1}{2}\right)$

Step 5: Join the points with a smooth curve. The graph is "compressed" because $a = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$.



y = ½ cot x

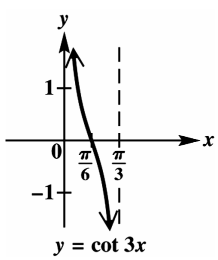
13. $y = \cot 3x$

Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{3}$. Cotangent has asymptotes of the form $bx = 0$ and $bx = \pi$.

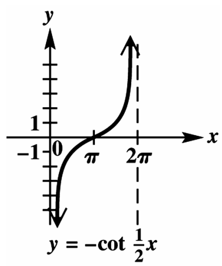
The asymptotes for $y = \cot 3x$ are

$3x = 0 \Rightarrow x = 0$ and $3x = \pi \Rightarrow x = \dfrac{\pi}{3}$

Step 2: Sketch the two vertical asymptotes found in Step 1.
Step 3: Divide the interval into four equal parts: $0, \dfrac{\pi}{12}, \dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}$
Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(\dfrac{\pi}{12}, 1\right), \left(\dfrac{\pi}{6}, 0\right), \left(\dfrac{\pi}{4}, -1\right)$
Step 5: Join the points with a smooth curve.



Exercise 13



Exercise 14

14. $y = -\cot \dfrac{1}{2}x$

Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\dfrac{\pi}{b}$, so the period for this function is 2π. Cotangent has asymptotes of the form $bx = 0$ and $bx = \pi$. The asymptotes for $y = -\cot \dfrac{1}{2}x$ are

$\dfrac{1}{2}x = 0 \Rightarrow x = 0$ and $\dfrac{1}{2}x = \pi \Rightarrow x = 2\pi$

Step 2: Sketch the two vertical asymptotes found in Step 1.
Step 3: Divide the interval into four equal parts: $0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$
Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(\dfrac{\pi}{2}, -1\right), (\pi, 0), \left(\dfrac{3\pi}{2}, 1\right)$
Step 5: Join the points with a smooth curve. The graph is the reflection of the graph of $y = \cot \dfrac{1}{2}x$ about the x-axis.

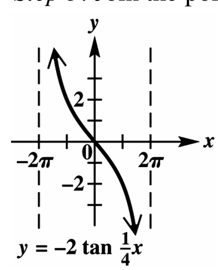
15. $y = -2\tan \dfrac{1}{4}x$

Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\dfrac{\pi}{b}$, so the period for this function is 4π. Tangent has asymptotes of the form $bx = -\dfrac{\pi}{2}$ and $bx = \dfrac{\pi}{2}$.

Therefore, the asymptotes for $y = -2\tan \dfrac{1}{4}x$ are

$\dfrac{1}{4}x = -\dfrac{\pi}{2} \Rightarrow x = -2\pi$ and $\dfrac{1}{4}x = \dfrac{\pi}{2} \Rightarrow x = 2\pi$

Step 2: Sketch the two vertical asymptotes found in Step 1.
Step 3: Divide the interval into four equal parts: $-2\pi, -\pi, 0, \pi, 2\pi$
Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $(-\pi, 2), (0, 0), (\pi, -2)$
Step 5: Join the points with a smooth curve.



16. $y = 3\tan \dfrac{1}{2}x$

Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\dfrac{\pi}{b}$, so the period for this function is 2π. Tangent has asymptotes of the form $bx = -\dfrac{\pi}{2}$ and $bx = \dfrac{\pi}{2}$.

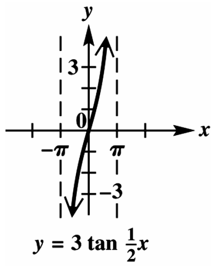
Therefore, the asymptotes for $y = 3\tan \dfrac{1}{2}x$ are

$\dfrac{1}{2}x = -\dfrac{\pi}{2} \Rightarrow x = -\pi$ and $\dfrac{1}{2}x = \dfrac{\pi}{2} \Rightarrow x = \pi$.

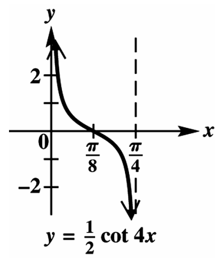
Step 2: Sketch the two vertical asymptotes found in Step 1.
Step 3: Divide the interval into four equal parts: $-\pi, -\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}, \pi$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(-\frac{\pi}{2},-3\right),\ (0,0),\ \left(\frac{\pi}{2},3\right)$

Step 5: Join the points with a smooth curve.



Exercise 16



Exercise 17

17. $y=\frac{1}{2}\cot 4x$

Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{4}$. Cotangent has asymptotes of the form $bx=0$ and $bx=\pi$. The asymptotes for $y=\frac{1}{2}\cot 4x$ are

$$4x=0\Rightarrow x=0 \text{ and } 4x=\pi\Rightarrow x=\frac{\pi}{4}$$

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $0,\frac{\pi}{16},\frac{\pi}{8},\frac{3\pi}{16},\frac{\pi}{4}$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(\frac{\pi}{16},\frac{1}{2}\right),\ \left(\frac{\pi}{8},0\right),\ \left(\frac{3\pi}{16},-\frac{1}{2}\right)$

Step 5: Join the points with a smooth curve.

18. $y=-\frac{1}{2}\cot 2x$

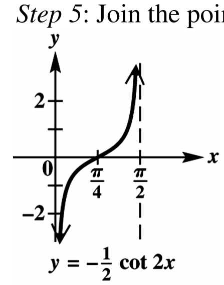
Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{2}$. Cotangent has asymptotes of the form $bx=0$ and $bx=\pi$. The asymptotes for $y=-\frac{1}{2}\cot 2x$ are $2x=0\Rightarrow x=0$ and $2x=\pi\Rightarrow x=\frac{\pi}{2}$

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $0,\frac{\pi}{8},\frac{\pi}{4},\frac{3\pi}{8},\frac{\pi}{2}$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(\frac{\pi}{8},-\frac{1}{2}\right),\ \left(\frac{\pi}{4},0\right),\ \left(\frac{3\pi}{8},\frac{1}{2}\right)$

Step 5: Join the points with a smooth curve.



19. $y=\tan(2x-\pi)=\tan 2\left(x-\frac{\pi}{2}\right)$

Period: $\frac{\pi}{b}=\frac{\pi}{2}$

Vertical translation: none

Phase shift (horizontal translation): $\frac{\pi}{2}$ units to the right

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y=\tan x$ occur at $-\frac{\pi}{2},\ \frac{\pi}{2},$ and $\frac{3\pi}{2}$, so use the following equations to locate asymptotes:

$2\left(x-\frac{\pi}{2}\right)=-\frac{\pi}{2},\ 2\left(x-\frac{\pi}{2}\right)=\frac{\pi}{2}$, and $2\left(x-\frac{\pi}{2}\right)=\frac{3\pi}{2}$

Solve each of these equations:

$$2\left(x-\frac{\pi}{2}\right)=-\frac{\pi}{2}\Rightarrow x-\frac{\pi}{2}=-\frac{\pi}{4}\Rightarrow x=\frac{\pi}{4}$$

$$2\left(x-\frac{\pi}{2}\right)=\frac{\pi}{2}\Rightarrow x-\frac{\pi}{2}=\frac{\pi}{4}\Rightarrow x=\frac{3\pi}{4}$$

$$2\left(x-\frac{\pi}{2}\right)=\frac{3\pi}{2}\Rightarrow x-\frac{\pi}{2}=\frac{3\pi}{4}\Rightarrow x=\frac{5\pi}{4}$$

Divide the interval $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ into four equal parts to obtain the following key x-values:

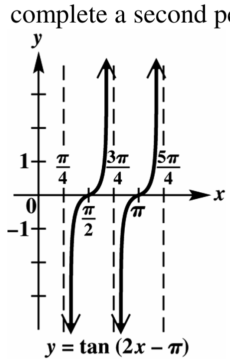
first-quarter value: $\frac{3\pi}{8}$; middle value: $\frac{\pi}{2}$;

third-quarter value: $\frac{5\pi}{8}$

Evaluating the given function at these three key x-values gives the points

$\left(\frac{3\pi}{8},-1\right)$, $\left(\frac{\pi}{2},0\right)$, $\left(\frac{5\pi}{8},1\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x=\frac{3\pi}{4}$ and $x=\frac{5\pi}{4}$ to complete a second period of the graph.



20. $y=\tan\left(\frac{x}{2}+\pi\right)=\tan\frac{1}{2}(x+2\pi)$

Period: $\frac{\pi}{b}=\frac{\pi}{\frac{1}{2}}=2\pi$

Vertical translation: none

Phase shift (horizontal translation): 2π units to the left

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y=\tan x$ occur at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{5\pi}{2}$, so use the following equations to locate asymptotes.

$\frac{1}{2}(x+2\pi)=\frac{\pi}{2}$, $\frac{1}{2}(x+2\pi)=\frac{3\pi}{2}$, and

$\frac{1}{2}(x+2\pi)=\frac{5\pi}{2}$

Solve each of these equations:

$\frac{1}{2}(x+2\pi)=\frac{\pi}{2}\Rightarrow x+2\pi=\pi\Rightarrow x=-\pi$

$\frac{1}{2}(x+2\pi)=\frac{3\pi}{2}\Rightarrow x+2\pi=3\pi\Rightarrow x=\pi$

$\frac{1}{2}(x+2\pi)=\frac{5\pi}{2}\Rightarrow x+2\pi=5\pi\Rightarrow x=3\pi$

Divide the interval $(\pi,3\pi)$ into four equal parts to obtain the following key x-values:

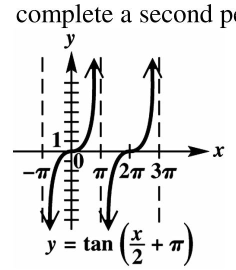
first-quarter value: $\frac{3\pi}{2}$; middle value: 2π;

third-quarter value: $\frac{5\pi}{2}$

Evaluating the given function at these three key x-values gives the points.

$\left(\frac{3\pi}{2},-1\right)$, $(2\pi,0)$, $\left(\frac{5\pi}{2},1\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x=\pi$ and $x=3\pi$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x=-\pi$ and $x=\pi$ to complete a second period of the graph.



21. $y=\cot\left(3x+\frac{\pi}{4}\right)=\cot 3\left(x+\frac{\pi}{12}\right)$

Period: $\frac{\pi}{b}=\frac{\pi}{3}$

Vertical translation: none

Phase shift (horizontal translation): $\frac{\pi}{12}$ unit to the left

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y=\cot x$ occur at multiples of π, so use the following equations to locate asymptotes:

$3\left(x+\frac{\pi}{12}\right)=0$, $3\left(x+\frac{\pi}{12}\right)=\pi$, and

$3\left(x+\frac{\pi}{12}\right)=2\pi$

Solve each of these equations:

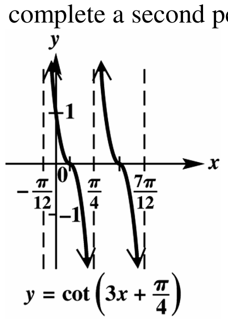
$3\left(x+\frac{\pi}{12}\right)=0 \Rightarrow x+\frac{\pi}{12}=0 \Rightarrow x=-\frac{\pi}{12}$

$3\left(x+\frac{\pi}{12}\right)=\pi \Rightarrow x+\frac{\pi}{12}=\frac{\pi}{3} \Rightarrow x=\frac{\pi}{3}-\frac{\pi}{12}=\frac{\pi}{4}$

$3\left(x+\frac{\pi}{12}\right)=2\pi \Rightarrow x+\frac{\pi}{12}=\frac{2\pi}{3} \Rightarrow$
$x=\frac{2\pi}{3}-\frac{\pi}{12} \Rightarrow x=\frac{7\pi}{12}$

Divide the interval $\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$ into four equal parts to obtain the following key x-values: first-quarter value: $\frac{\pi}{3}$; middle value: $\frac{5\pi}{12}$; third-quarter value: $\frac{\pi}{2}$. Evaluating the given function at these three key x-values gives the points. $\left(\frac{\pi}{3},1\right)$, $\left(\frac{5\pi}{12},0\right)$, $\left(\frac{\pi}{2},-1\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x=\frac{\pi}{4}$ and $x=\frac{7\pi}{12}$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x=-\frac{\pi}{12}$ and $x=\frac{\pi}{4}$ to complete a second period of the graph.



22. $y=\cot\left(2x-\frac{3\pi}{2}\right)=\cot 2\left(x-\frac{3\pi}{4}\right)$

Period: $\frac{\pi}{b}=\frac{\pi}{2}$

Vertical translation: none

Phase shift (horizontal translation): $\frac{3\pi}{4}$ units to the right

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y=\cot x$ occur at multiples of π, so use the following equations to locate asymptotes.

$2\left(x-\frac{3\pi}{4}\right)=-\pi,\ 2\left(x-\frac{3\pi}{4}\right)=0,$ and $2\left(x-\frac{3\pi}{4}\right)=\pi$

Solve each of these equations:

$2\left(x-\frac{3\pi}{4}\right)=-\pi \Rightarrow x-\frac{3\pi}{4}=-\frac{\pi}{2} \Rightarrow$
$x=-\frac{\pi}{2}+\frac{3\pi}{4} \Rightarrow x=\frac{\pi}{4}$

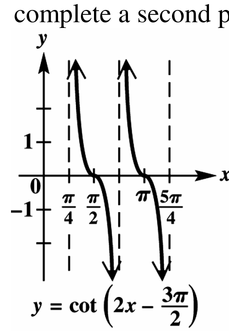
$2\left(x-\frac{3\pi}{4}\right)=0 \Rightarrow x-\frac{3\pi}{4}=0 \Rightarrow x=\frac{3\pi}{4}$

$2\left(x-\frac{3\pi}{4}\right)=\pi \Rightarrow x-\frac{3\pi}{4}=\frac{\pi}{2} \Rightarrow$
$x=\frac{\pi}{2}+\frac{3\pi}{4}=\frac{5\pi}{4}$

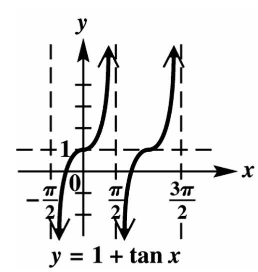
Divide the interval $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ into four equal parts to obtain the following key x-values: first-quarter value: $\frac{3\pi}{8}$; middle value: $\frac{\pi}{2}$; third-quarter value: $\frac{5\pi}{8}$

Evaluating the given function at these three key x-values gives the points $\left(\frac{3\pi}{8},1\right)$, $\left(\frac{\pi}{2},0\right)$, $\left(\frac{5\pi}{8},-1\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x=\frac{3\pi}{4}$ and $x=\frac{5\pi}{4}$ to complete a second period of the graph.



Exercise 22



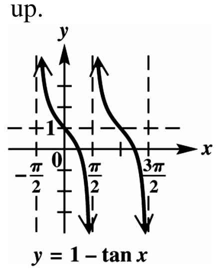
Exercise 23

23. $y = 1 + \tan x$

This is the graph of $y = \tan x$ translated vertically 1 unit up.

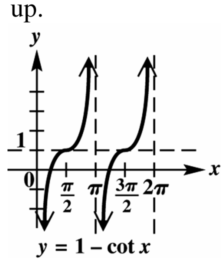
24. $y = 1 - \tan x$

This is the graph of $y = \tan x$, reflected over the x-axis and then translated vertically 1 unit up.

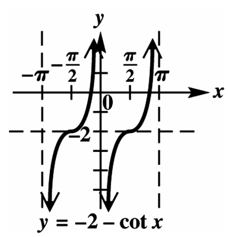


25. $y = 1 - \cot x$

This is the graph of $y = \cot x$ reflected about the x-axis and then translated vertically 1 unit up.



Exercise 25



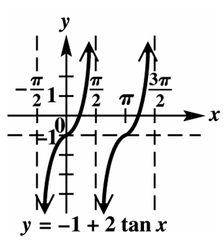
Exercise 26

26. $y = -2 - \cot x$

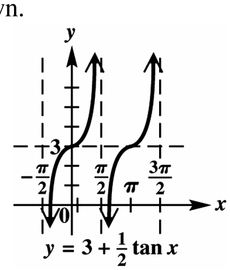
This is the graph of $y = \cot x$ reflected about the x-axis and then translated vertically 2 units down.

27. $y = -1 + 2\tan x$

This is the graph of $y = 2\tan x$ translated vertically 1 unit down.



Exercise 27



Exercise 28

28. $y = 3 + \frac{1}{2}\tan x$

This is the graph of $y = \frac{1}{2}\tan x$ translated vertically 3 units up.

29. $y = -1 + \frac{1}{2}\cot(2x - 3\pi) = -1 + \frac{1}{2}\cot 2\left(x - \frac{3\pi}{2}\right)$

Period: $\frac{\pi}{b} = \frac{\pi}{2}$.

Vertical translation: 1 unit down

Phase shift (horizontal translation): $\frac{3\pi}{2}$ units to the right

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y = \cot x$ occur at multiples of π, use the following equations to locate asymptotes:

$2\left(x - \frac{3\pi}{2}\right) = -2\pi$, $2\left(x - \frac{3\pi}{2}\right) = -\pi$, and $2\left(x - \frac{3\pi}{2}\right) = 0$

Solve each of these equations:

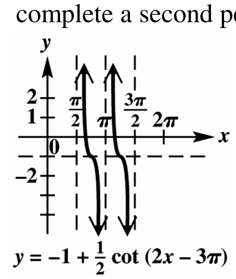
$$2\left(x - \frac{3\pi}{2}\right) = -2\pi \Rightarrow x - \frac{3\pi}{2} = -\pi \Rightarrow$$
$$x = -\pi + \frac{3\pi}{2} = \frac{\pi}{2}$$

$$2\left(x - \frac{3\pi}{2}\right) = -\pi \Rightarrow x - \frac{3\pi}{2} = -\frac{\pi}{2} \Rightarrow$$
$$x = -\frac{\pi}{2} + \frac{3\pi}{2} \Rightarrow x = \frac{2\pi}{2} = \pi$$

$$2\left(x - \frac{3\pi}{2}\right) = 0 \Rightarrow x - \frac{3\pi}{2} = 0 \Rightarrow x = \frac{3\pi}{2}$$

Divide the interval $\left(\frac{\pi}{2}, \pi\right)$ into four equal parts to obtain the following key x-values: first-quarter value: $\frac{5\pi}{8}$; middle value: $\frac{3\pi}{4}$; third-quarter value: $\frac{7\pi}{8}$. Evaluating the given function at these three key x-values gives the points. $\left(\frac{5\pi}{8}, -\frac{1}{2}\right)$, $\left(\frac{3\pi}{4}, -1\right)$, $\left(\frac{7\pi}{8}, -\frac{3}{2}\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x = \frac{\pi}{2}$ and $x = \pi$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x = \pi$ and $x = \frac{3\pi}{2}$ to complete a second period of the graph.



30. $y = -2 + 3\tan(4x + \pi) = -2 + 3\tan 4\left(x + \frac{\pi}{4}\right)$

Period: $\frac{\pi}{b} = \frac{\pi}{4}$

Vertical translation: 2 units down

Phase shift (horizontal translation): $\frac{\pi}{4}$ unit to the left

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y = \tan x$ occur at $-\frac{\pi}{2}, \frac{\pi}{2}$, and $\frac{3\pi}{2}$, use the following equations to locate asymptotes:

$4\left(x + \frac{\pi}{4}\right) = -\frac{\pi}{2}$, $4\left(x + \frac{\pi}{4}\right) = \frac{\pi}{2}$, and $4\left(x + \frac{\pi}{4}\right) = \frac{3\pi}{2}$

Solve each of these equations.:

$4\left(x + \frac{\pi}{4}\right) = -\frac{\pi}{2} \Rightarrow x + \frac{\pi}{4} = -\frac{\pi}{8} \Rightarrow x = -\frac{3\pi}{8}$

$4\left(x + \frac{\pi}{4}\right) = \frac{\pi}{2} \Rightarrow x + \frac{\pi}{4} = \frac{\pi}{8} \Rightarrow x = -\frac{\pi}{8}$

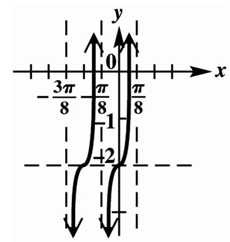
$4\left(x + \frac{\pi}{4}\right) = \frac{3\pi}{2} \Rightarrow x + \frac{\pi}{4} = \frac{3\pi}{8} \Rightarrow x = \frac{\pi}{8}$

Divide the interval $\left(-\frac{3\pi}{8}, -\frac{\pi}{8}\right)$ into four equal parts to obtain the following key x-values: first-quarter value: $-\frac{5\pi}{16}$; middle value: $-\frac{\pi}{4}$; third-quarter value: $-\frac{3\pi}{16}$

Evaluating the given function at these three key x-values gives the points.

$\left(-\frac{5\pi}{16}, -5\right)$, $\left(-\frac{\pi}{4}, -2\right)$, $\left(-\frac{3\pi}{16}, 1\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x = -\frac{3\pi}{8}$ and $x = -\frac{\pi}{8}$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x = -\frac{\pi}{8}$ and $x = \frac{\pi}{8}$ to complete a second period of the graph.



$y = -2 + 3\tan(4x + \pi)$

31. $y = 1 - 2\cot 2\left(x + \frac{\pi}{2}\right)$

Period: $\frac{\pi}{b} = \frac{\pi}{2}$

Vertical translation: 1 unit up

Phase shift (horizontal translation): $\frac{\pi}{2}$ unit to the left

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y = \cot x$ occur at multiples of π, so use the following equations to locate asymptotes.

$2\left(x + \frac{\pi}{2}\right) = 0$, $2\left(x + \frac{\pi}{2}\right) = \pi$, and $2\left(x + \frac{\pi}{2}\right) = 2\pi$

Solve each of these equations:

$2\left(x + \frac{\pi}{2}\right) = 0 \Rightarrow x + \frac{\pi}{2} = 0 \Rightarrow x = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$

$2\left(x + \frac{\pi}{2}\right) = \pi \Rightarrow x + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow$
$x = \frac{\pi}{2} - \frac{\pi}{2} \Rightarrow x = 0$

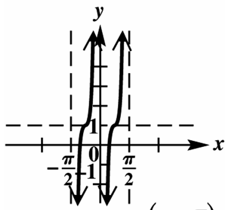
$2\left(x + \frac{\pi}{2}\right) = 2\pi \Rightarrow x + \frac{\pi}{2} = \pi \Rightarrow$
$x = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Divide the interval $\left(0, \frac{\pi}{2}\right)$ into four equal parts to obtain the following key x-values: first-quarter value: $\frac{\pi}{8}$; middle value: $\frac{\pi}{4}$; third-quarter value: $\frac{3\pi}{8}$

Evaluating the given function at these three key x-values gives the points

$\left(\frac{\pi}{8}, -1\right)$, $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{3\pi}{8}, 3\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x = 0$ and $x = \frac{\pi}{2}$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x = -\frac{\pi}{2}$ and $x = 0$ to complete a second period of the graph.



$y = 1 - 2\cot 2\left(x + \frac{\pi}{2}\right)$

32. $y = \frac{2}{3}\tan\left(\frac{3}{4}x - \pi\right) - 2 = -2 + \frac{2}{3}\tan\frac{3}{4}\left(x - \frac{4\pi}{3}\right)$

Period is $\frac{\pi}{b} = \frac{\pi}{\frac{3}{4}} = \frac{4}{3}\pi = \frac{4\pi}{3}$

Vertical translation: 2 units down

Phase shift (horizontal translation): $\frac{4\pi}{3}$ units to the right

Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Asymptotes of the graph $y = \tan x$ occur at $-\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$, so use the following equations to locate asymptotes:

$\frac{3}{4}\left(x - \frac{4\pi}{3}\right) = -\frac{\pi}{2}$, $\frac{3}{4}\left(x - \frac{4\pi}{3}\right) = \frac{\pi}{2}$, and $\frac{3}{4}\left(x - \frac{4\pi}{3}\right) = \frac{3\pi}{2}$

Solve each of these equations:

$$\frac{3}{4}\left(x - \frac{4\pi}{3}\right) = -\frac{\pi}{2} \Rightarrow x - \frac{4\pi}{3} = -\frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}$$

$$\frac{3}{4}\left(x - \frac{4\pi}{3}\right) = \frac{\pi}{2} \Rightarrow x - \frac{4\pi}{3} = \frac{2\pi}{3} \Rightarrow x = 2\pi$$

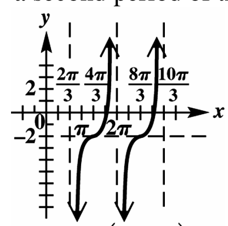
$$\frac{3}{4}\left(x - \frac{4\pi}{3}\right) = \frac{3\pi}{2} \Rightarrow x - \frac{4\pi}{3} = 2\pi \Rightarrow x = \frac{10\pi}{3}$$

Divide the interval $\left(\frac{2\pi}{3}, 2\pi\right)$ into four equal parts to obtain the following key x-values: first-quarter value: π middle value: $\frac{4\pi}{3}$; third-quarter value: $\frac{5\pi}{3}$.

Evaluating the given function at these three key x-values gives the points $\left(\pi, -\frac{8}{3}\right)$, $\left(\frac{4\pi}{3}, -2\right)$, $\left(\frac{5\pi}{3}, -\frac{4}{3}\right)$

Connect these points with a smooth curve and continue to graph to approach the asymptote $x = \frac{2\pi}{3}$ and $x = 2\pi$ to complete one period of the graph.

Sketch the identical curve between the asymptotes $x = 2\pi$ and $x = \frac{10\pi}{3}$ to complete a second period of the graph.



$y = \frac{2}{3}\tan\left(\frac{3}{4}x - \pi\right) - 2$

33. Since the asymptotes are at $-\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$, this is a tangent function of the form $y = a\tan x$. The graph passes through the point $\left(\frac{\pi}{4}, -2\right)$. Substituting these values into the generic equation gives

$$y = a\tan x \Rightarrow -2 = a\tan\frac{\pi}{4} \Rightarrow -2 = a\cdot 1 \Rightarrow -2 = a$$

Thus, the equation of the graph is $y = -2\tan x$.

34. Since the asymptotes occur at multiples of π, this is a cotangent function of the form $y = a\cot x$. The graph passes through the point $\left(\frac{\pi}{4}, -2\right)$. Substituting these values into the generic equation gives

$$y = a\cot x \Rightarrow -2 = a\cot\frac{\pi}{4} \Rightarrow -2 = a\cdot 1 \Rightarrow -2 = a$$

Thus, the equation of the graph is $y = -2\cot x$.

35. Since the asymptotes occur at 0, $\frac{\pi}{3}$, and $\frac{2\pi}{3}$ this is a cotangent function of the form $y = a\cot bx$. The period of the function is $\frac{\pi}{3}$, so we have $\frac{\pi}{b} = \frac{\pi}{3} \Rightarrow b = 3$.

(*continued on next page*)

(*continued from page 589*)

The graph passes through the point $\left(\frac{\pi}{12},1\right)$. Substituting these values into the generic equation gives

$$y = a\cot bx \Rightarrow 1 = a\cot\left(3\cdot\frac{\pi}{12}\right) \Rightarrow$$

$$1 = a\cot\frac{\pi}{4} \Rightarrow 1 = a\cdot 1 \Rightarrow 1 = a$$

Thus, the equation of the graph is $y = \cot 3x$.

36. Since the asymptotes occur at $-\frac{\pi}{6}, \frac{\pi}{6}$, and $\frac{\pi}{2}$, this is a tangent function of the form $y = a\tan bx$. The period of the function is $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$, so we have $\frac{\pi}{b} = \frac{\pi}{3} \Rightarrow b = 3$.

The graph passes through the point $\left(\frac{\pi}{12},1\right)$. Substituting these values into the generic equation gives

$$y = a\tan bx \Rightarrow 1 = a\tan\left(3\cdot\frac{\pi}{12}\right) \Rightarrow$$

$$1 = a\tan\frac{\pi}{4} \Rightarrow 1 = a\cdot 1 \Rightarrow 1 = a$$

Thus, the equation of the graph is $y = \tan 3x$.

37. True; $\frac{\pi}{2}$ is the smallest positive value where $\cos\frac{\pi}{2} = 0$. Since $\tan\frac{\pi}{2} = \frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}$, $\frac{\pi}{2}$ is the smallest positive value where the tangent function is undefined. Thus, $k = \frac{\pi}{2}$ is the smallest positive value for which $x = k$ is an asymptote for the tangent function.

38. False; $\cot\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0$. The smallest such number is π.

39. The function $\tan x$ has a period of π, so it repeats four times over the interval $(-2\pi, 2\pi]$. Since its range is $(-\infty,\infty)$, $\tan x = c$ has four solutions for every value of c.

40. The domain of the tangent function, $y = \tan x$, is $\left\{x \,\middle|\, x \neq \frac{\pi}{2} + n\pi, \text{ where } n \text{ is any integer}\right\}$, and the range is $(-\infty,\infty)$. For the function

$$f(x) = -4\tan(2x+\pi) = -4\tan 2\left(x+\frac{\pi}{2}\right),$$

the period is $\frac{\pi}{2}$. Therefore, the domain is $\left\{x \,\middle|\, x \neq \frac{\pi}{4} + \frac{\pi}{2}n, \text{ where } n \text{ is any integer}\right\}$. This can also be written as $\left\{x \,\middle|\, x \neq (2n+1)\frac{\pi}{4}, \text{ where } n \text{ is any integer}\right\}$.

The range remains $(-\infty,\infty)$.

41. $\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$,

$\left\{x \,\middle|\, x \neq (2n+1)\frac{\pi}{4}, \text{ where } n \text{ is any integer}\right\}$.

42. $\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\cot x$,

$\{x \mid x \neq n\pi, \text{ where } n \text{ is any integer}\}$.

43. $d = 4\tan 2\pi t$

(a) $d = 4\tan 2\pi(0) = 4\tan 0 \approx 4(0) = 0$ m

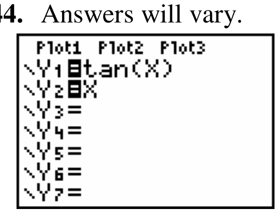
(b) $d = 4\tan 2\pi(.4) = 4\tan .8\pi$
$\approx 4(-.7265) \approx -2.9$ m

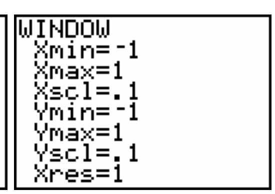
(c) $d = 4\tan 2\pi(.8) = 4\tan 1.6\pi$
$\approx 4(-3.0777) \approx -12.3$ m

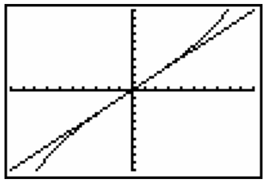
(d) $d = 4\tan 2\pi(1.2) = 4\tan 2.4\pi$
$\approx 4(3.0777) \approx 12.3$ m

(e) $t = .25$ leads to $\tan\frac{\pi}{2}$, which is undefined.

44. Answers will vary.







45. $y = -\csc x$
The graph is the reflection of the graph of $y = \csc x$ about the x-axis. This matches with graph B.

46. $y = -\sec x$
The graph is the reflection of the graph of $y = \sec x$ about the x-axis. This matches with graph C.

47. $y = \sec\left(x - \frac{\pi}{2}\right)$

The graph is the graph of $y = \sec x$ shifted $\frac{\pi}{2}$ units to the right. This matches with graph D.

48. $y = \csc\left(x + \frac{\pi}{2}\right)$

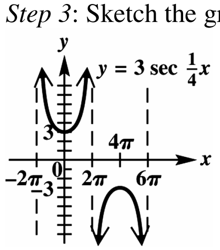
The graph is the graph of $y = \csc x$ shifted $\frac{\pi}{2}$ units to the left. This matches with graph A.

49. $y = 3\sec\frac{1}{4}x$

Step 1: Graph the corresponding reciprocal function $y = 3\cos\frac{1}{4}x$. The period is $\frac{2\pi}{\frac{1}{4}} = 2\pi \cdot \frac{4}{1} = 8\pi$ and its amplitude is $|3| = 3$. One period is in the interval $0 \le x \le 8\pi$. Dividing the interval into four equal parts gives us the following key points: $(0,1)$, $(2\pi,0)$, $(4\pi,-1)$, $(6\pi,0)$, $(8\pi,1)$

Step 2: The vertical asymptotes of $y = \sec\frac{1}{4}x$ are at the x-intercepts of $y = \cos\frac{1}{4}x$, which are $x = 2\pi$ and $x = 6\pi$. Continuing this pattern to the left, we also have a vertical asymptote of $x = -2\pi$.

Step 3: Sketch the graph.

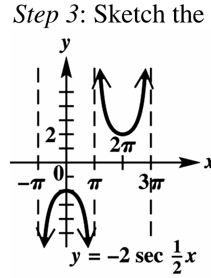


50. $y = -2\sec\frac{1}{2}x$

Step 1: Graph the corresponding reciprocal function $y = -2\cos\frac{1}{2}x$. The period is $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$ and its amplitude is $|-2| = 2$. One period is in the interval $0 \le x \le 4\pi$. Dividing the interval into four equal parts gives us the following key points: $(0,-2)$, $(\pi,0)$, $(2\pi,2)$, $(3\pi,0)$, $(4\pi,-2)$

Step 2: The vertical asymptotes of $y = -2\sec\frac{1}{2}x$ are at the x-intercepts of $y = -2\cos\frac{1}{2}x$, which are $x = \pi$ and $x = 3\pi$. Continuing this pattern to the left, we also have a vertical asymptote of $x = -\pi$.

Step 3: Sketch the graph.

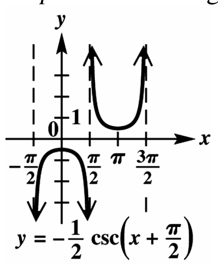


51. $y = -\frac{1}{2}\csc\left(x + \frac{\pi}{2}\right)$

Step 1: Graph the corresponding reciprocal function $y = -\frac{1}{2}\sin\left(x + \frac{\pi}{2}\right)$. The period is 2π and its amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$. One period is in the interval $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.
Dividing the interval into four equal parts gives us the following key points: $\left(-\frac{\pi}{2},0\right)$, $\left(0,-\frac{1}{2}\right)$, $\left(\frac{\pi}{2},0\right)$, $\left(\pi,\frac{1}{2}\right)$, $\left(\frac{3\pi}{2},0\right)$

Step 2: The vertical asymptotes of $y=-\frac{1}{2}\csc\left(x+\frac{\pi}{2}\right)$ are at the *x*-intercepts of $y=-\frac{1}{2}\sin\left(x+\frac{\pi}{2}\right)$, which are $x=-\frac{\pi}{2}$, $x=\frac{\pi}{2}$, and $x=\frac{3\pi}{2}$.

Step 3: Sketch the graph.

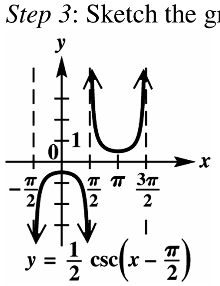


52. $y=\frac{1}{2}\csc\left(x-\frac{\pi}{2}\right)$

Step 1: Graph the corresponding reciprocal function $y=\frac{1}{2}\sin\left(x-\frac{\pi}{2}\right)$. The period is 2π and its amplitude is $\left|\frac{1}{2}\right|=\frac{1}{2}$. One period is in the interval $\frac{\pi}{2}\le x\le\frac{5\pi}{2}$. Dividing the interval into four equal parts us the following key points: $\left(\frac{\pi}{2},0\right)$, $\left(\pi,\frac{1}{2}\right)$, $\left(\frac{3\pi}{2},0\right)$, $\left(2\pi,-\frac{1}{2}\right)$, $\left(\frac{5\pi}{2},0\right)$

Step 2: The vertical asymptotes of $y=\frac{1}{2}\csc\left(x-\frac{\pi}{2}\right)$ are at the *x*-intercepts of $y=\frac{1}{2}\sin\left(x-\frac{\pi}{2}\right)$, which are $x=\frac{\pi}{2}$, $x=\frac{3\pi}{2}$, and $x=\frac{5\pi}{2}$.

Step 3: Sketch the graph.

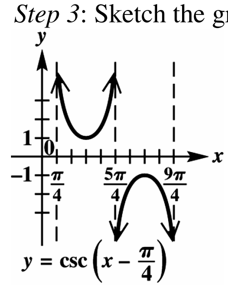


53. $y=\csc\left(x-\frac{\pi}{4}\right)$

Step 1: Graph the corresponding reciprocal function $y=\sin\left(x-\frac{\pi}{4}\right)$ The period is 2π and its amplitude is $|1|=1$. One period is in the interval $\frac{\pi}{4}\le x\le\frac{9\pi}{4}$. Dividing the interval into four equal parts gives us the following key points: $\left(\frac{\pi}{4},0\right)$, $\left(\frac{3\pi}{4},1\right)$, $\left(\frac{5\pi}{4},0\right)$, $\left(\frac{7\pi}{4},-1\right)$, $\left(\frac{9\pi}{4},0\right)$

Step 2: The vertical asymptotes of $y=\csc\left(x-\frac{\pi}{4}\right)$ are at the *x*-intercepts of $y=\sin\left(x-\frac{\pi}{4}\right)$, which are $x=\frac{\pi}{4}$, $x=\frac{5\pi}{4}$, and $x=\frac{9\pi}{4}$.

Step 3: Sketch the graph.

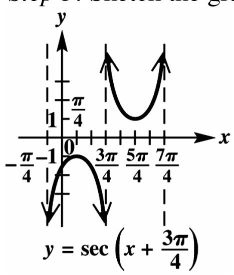


54. $y=\sec\left(x+\frac{3\pi}{4}\right)$

Step 1: Graph the corresponding reciprocal function $y=\cos\left(x+\frac{3\pi}{4}\right)$. The period is 2π and its amplitude is $|1|=1$. One period is in the interval $-\frac{3\pi}{4}\le x\le\frac{5\pi}{4}$.
Dividing the interval into four equal parts gives us the following key points: $\left(-\frac{3\pi}{4},1\right)$, $\left(-\frac{\pi}{4},0\right)$, $\left(\frac{\pi}{4},-1\right)$, $\left(\frac{3\pi}{4},0\right)$, $\left(\frac{5\pi}{4},1\right)$

Step 2: The vertical asymptotes of $y=\sec\left(x+\frac{3\pi}{4}\right)$ are at the x-intercepts of $y=\cos\left(x+\frac{3\pi}{4}\right)$, which are $x=-\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$. Continuing this pattern to the right, we also have a vertical asymptote of $x=\frac{7\pi}{4}$.

Step 3: Sketch the graph.

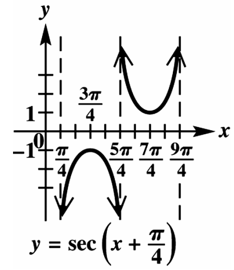


55. $y=\sec\left(x+\frac{\pi}{4}\right)$

Step 1: Graph the corresponding reciprocal function $y=\cos\left(x+\frac{\pi}{4}\right)$. The period is 2π and its amplitude is $|1|=1$. One period is in the interval $-\frac{\pi}{4}\le x\le\frac{7\pi}{4}$. Dividing the interval into four equal parts gives us the following key points: $\left(-\frac{\pi}{4},1\right)$, $\left(\frac{\pi}{4},0\right)$, $\left(\frac{3\pi}{4},-1\right)$, $\left(\frac{5\pi}{4},0\right)$, $\left(\frac{7\pi}{4},1\right)$

Step 2: The vertical asymptotes of $y=\sec\left(x+\frac{\pi}{4}\right)$ are at the x-intercepts of $y=\cos\left(x+\frac{\pi}{4}\right)$, which are $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$. Continuing this pattern to the right, we also have a vertical asymptote of $x=\frac{9\pi}{4}$.

Step 3: Sketch the graph.

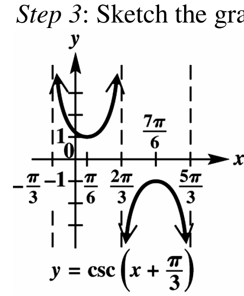


56. $y=\csc\left(x+\frac{\pi}{3}\right)$

Step 1: Graph the corresponding reciprocal function $y=\sin\left(x+\frac{\pi}{3}\right)$. The period is 2π and its amplitude is 1. One period is in the interval $-\frac{\pi}{3}\le x\le\frac{5\pi}{3}$. Dividing the interval into four equal parts gives us the following key points: $\left(-\frac{\pi}{3},0\right)$, $\left(\frac{\pi}{6},1\right)$, $\left(\frac{2\pi}{3},0\right)$, $\left(\frac{7\pi}{6},-1\right)$, $\left(\frac{5\pi}{3},0\right)$

Step 2: The vertical asymptotes of $y=\csc\left(x+\frac{\pi}{3}\right)$ are at the x-intercepts of $y=\sin\left(x+\frac{\pi}{3}\right)$, which are $x=-\frac{\pi}{3}$, $x=\frac{2\pi}{3}$, and $x=\frac{5\pi}{3}$.

Step 3: Sketch the graph.



57. $y=\sec\left(\frac{1}{2}x+\frac{\pi}{3}\right)=\sec\frac{1}{2}\left(x+\frac{2\pi}{3}\right)$

Step 1: Graph the corresponding reciprocal function $y=\cos\frac{1}{2}\left(x+\frac{2\pi}{3}\right)$.

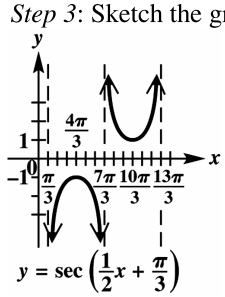
(*continued on next page*)

(*continued from page 593*)

The period is $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$ and its amplitude is $\left|\frac{1}{2}\right| = \frac{1}{2}$. One period is in the interval $-\frac{2\pi}{3} \le x \le \frac{10\pi}{3}$. Dividing the interval into four equal parts gives us the following key points: $\left(-\frac{2\pi}{3}, 1\right)$, $\left(\frac{\pi}{3}, 0\right)$, $\left(\frac{4\pi}{3}, -1\right)$, $\left(\frac{7\pi}{3}, 0\right)$, $\left(\frac{10\pi}{3}, 1\right)$

Step 2: The vertical asymptotes of $y = \sec\frac{1}{2}\left(x + \frac{2\pi}{3}\right)$ are at the x-intercepts of $y = \cos\frac{1}{2}\left(x + \frac{2\pi}{3}\right)$, which are $x = \frac{\pi}{3}$ and $x = \frac{7\pi}{3}$. Continuing this pattern to the right, we also have a vertical asymptote of $x = \frac{13\pi}{3}$.

Step 3: Sketch the graph.



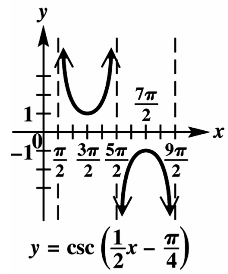
58. $y = \csc\left(\frac{1}{2}x - \frac{\pi}{4}\right) = \csc\frac{1}{2}\left(x - \frac{\pi}{2}\right)$

Step 1: Graph the corresponding reciprocal function $y = \sin\frac{1}{2}\left(x - \frac{\pi}{2}\right)$. The period is $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$ and its amplitude is $|1| = 1$.

One period is in the interval $\frac{\pi}{2} \le x \le \frac{9\pi}{2}$. Dividing the interval into four equal parts gives us the following key points: $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 1\right)$, $\left(\frac{5\pi}{2}, 0\right)$, $\left(\frac{7\pi}{2}, -1\right)$, $\left(\frac{9\pi}{2}, 0\right)$

Step 2: The vertical asymptotes of $y = \csc\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ are at the x-intercepts of $y = \sin\frac{1}{2}\left(x - \frac{\pi}{2}\right)$, which are $x = \frac{\pi}{2}$, $x = \frac{5\pi}{2}$, and $x = \frac{9\pi}{2}$.

Step 3: Sketch the graph.



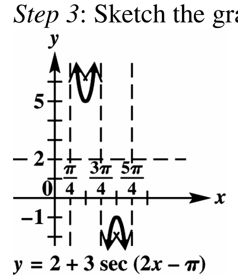
59. $y = 2 + 3\sec(2x - \pi) = 2 + 3\sec 2\left(x - \frac{\pi}{2}\right)$

Step 1: Graph the corresponding reciprocal function $y = 2 + 3\cos 2\left(x - \frac{\pi}{2}\right)$. The period is π and its amplitude is $|3| = 3$.

One period is in the interval $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. Dividing the interval into four equal parts gives us the following key points: $\left(\frac{\pi}{2}, 5\right)$, $\left(\frac{3\pi}{4}, 2\right)$, $(\pi, -1)$, $\left(\frac{5\pi}{4}, 2\right)$, $\left(\frac{3\pi}{2}, 5\right)$

Step 2: The vertical asymptotes of $y = 2 + 3\sec 2\left(x - \frac{\pi}{2}\right)$ are at the x-intercepts of $y = 3\cos 2\left(x - \frac{\pi}{2}\right)$, which are $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$. Continuing this pattern to the left, we also have a vertical asymptote of $x = \frac{\pi}{4}$.

Step 3: Sketch the graph.

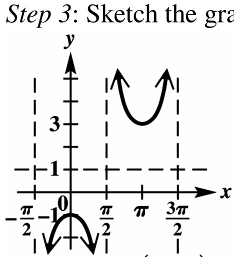


60. $y = 1 - 2\csc\left(x + \frac{\pi}{2}\right)$

Step 1: Graph the corresponding reciprocal function $y = 1 - 2\sin\left(x + \frac{\pi}{2}\right)$. The period is 2π and its amplitude is $|-2| = 2$. One period is in the interval $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. Dividing the interval into four equal parts gives us the following key points: $\left(-\frac{\pi}{2}, 1\right)$, $(0, -1)$, $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 3)$, $\left(\frac{3\pi}{2}, 1\right)$

Step 2: The vertical asymptotes of $y = 1 - 2\csc\left(x + \frac{\pi}{2}\right)$ are at the x-intercepts of $y = -2\sin\left(x + \frac{\pi}{2}\right)$, which are $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$.

Step 3: Sketch the graph.

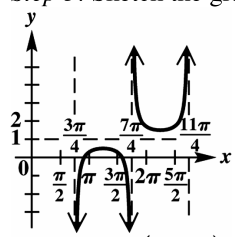


$y = 1 - 2\csc\left(x + \frac{\pi}{2}\right)$

61. *Step 1*: Graph the corresponding reciprocal function $y = 1 - \frac{1}{2}\sin\left(x - \frac{3\pi}{4}\right)$. The period is 2π and its amplitude is $\frac{1}{2}$. One period is in the interval $\frac{3\pi}{4} \le x \le \frac{11\pi}{4}$. Dividing the interval into four equal parts gives us the following key points: $\left(\frac{3\pi}{4}, 1\right)$, $\left(\frac{5\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{7\pi}{4}, 1\right)$, $\left(\frac{9\pi}{4}, \frac{3}{2}\right)$, $\left(\frac{11\pi}{4}, 1\right)$

Step 2: The vertical asymptotes of $y = 1 - \frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right)$ are at the x-intercepts of $y = -\frac{1}{2}\sin\left(x - \frac{3\pi}{4}\right)$, which are $x = \frac{3\pi}{4}$, $x = \frac{7\pi}{4}$, and $x = \frac{11\pi}{4}$.

Step 3: Sketch the graph.

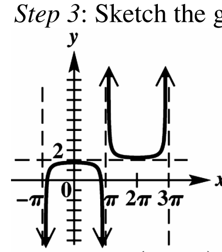


$y = 1 - \frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right)$

62. *Step 1*: Graph the corresponding reciprocal function $y = 2 + \frac{1}{4}\cos\frac{1}{2}(x - 2\pi)$. The period is $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$ and its amplitude is $\left|\frac{1}{4}\right| = \frac{1}{4}$. One period is in the interval $2\pi \le x \le 6\pi$.
Dividing the interval into four equal parts gives us the following key points: $\left(2\pi, \frac{9}{4}\right)$, $(3\pi, 2)$, $\left(4\pi, \frac{7}{4}\right)$, $(5\pi, 2)$, $\left(6\pi, \frac{9}{4}\right)$

Step 2: The vertical asymptotes of $y = 2 + \frac{1}{4}\sec\frac{1}{2}(x - 2\pi)$ are at the x-intercepts of $y = \frac{1}{4}\cos\frac{1}{2}(x - 2\pi)$, which are $x = 3\pi$ and $x = 5\pi$. Continuing this pattern to the left, we also have a vertical asymptote of $x = -\pi$.

Step 3: Sketch the graph.



$y = 2 + \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right)$

For exercises 63–66, other answers are possible.

63. Since the graph crosses the y-axis at (0, 1), this is a secant graph with $a = 1$. The period is $\left|-\frac{\pi}{4}-\frac{\pi}{4}\right| = \left|-\frac{\pi}{2}\right| = \frac{\pi}{2}$. Thus, $b = \frac{2\pi}{\frac{\pi}{2}} \Rightarrow b = 4$. The equation of the graph is $y = \sec 4x$.

64. Since the graph crosses the y-axis at (0, 1), this is a secant graph with $a = 1$. The period is $\left|-\frac{\pi}{2}-\frac{\pi}{2}\right| = \left|-\pi\right| = \pi$. Thus, $b = \frac{2\pi}{\pi} \Rightarrow b = 2$. The equation of the graph is $y = \sec 2x$.

65. This is the graph of $y = \csc x$ translated two units down. Thus, the equation of the graph is $y = -2 + \csc x$.

66. This is the graph of $y = \csc x$ translated one unit up. Thus, the equation of the graph is $y = 1 + \csc x$.

67. True; since $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$, the tangent and secant functions will be undefined at the same values.

68. False; secant values are undefined when $x = (2n+1)\frac{\pi}{2}$, while cosecant values are undefined when $x = n\pi$.

69. None; $\cos x \le 1$ for all x, so $\frac{1}{\cos x} \ge 1$ and $\sec x \ge 1$. Since $\sec x \ge 1$, $\sec x$ has no values in the interval $(-1,1)$.

70. The domain of the cosecant function, $y = \csc x$ is $\{x \mid x \ne n\pi, \text{ where } n \text{ is any integer}\}$, and the range is $(-\infty, -1] \cup [1, \infty)$. For the function $g(x) = -2\csc(4x+\pi) = -2\csc 4\left(x+\frac{\pi}{4}\right)$, the period is $\frac{\pi}{2}$. Therefore, the domain is $\left\{x \,\middle|\, x \ne \frac{n\pi}{4}, \text{ where } n \text{ is any integer}\right\}$. The range becomes $(-\infty, -2] \cup [2, \infty)$ since $a = -2$.

71. $\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos(x)} = \sec(x)$,

$\left\{x \,\middle|\, x \ne (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer}\right\}$.

72. $\csc(-x) = \frac{1}{\sin(-x)} = \frac{1}{-\sin x} = -\csc x$,

$\{x \mid x \ne n\pi, \text{ where } n \text{ is any integer}\}$

73. $a = 4\left|\sec 2\pi t\right|$

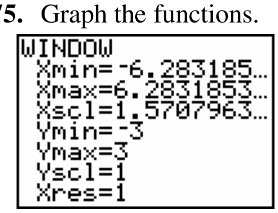
(a) $t = 0$
$a = 4\left|\sec 0\right| = 4\left|1\right| = 4(1) = 4$ m

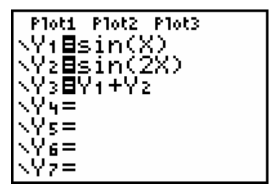
(b) $t = .86$
$a = 4\left|\sec 2\pi(.86)\right| \approx 4\left|1.5688\right|$
$= 4(1.5688) \approx 6.3$ m

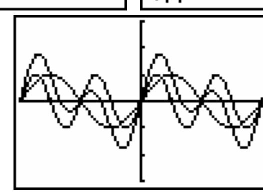
(c) $t = 1.24$
$a = 4\left|\sec 2\pi(1.24)\right| \approx 4\left|15.9260\right|$
$= 4(15.9260) \approx 63.7$ m

74. Answers will vary. No, these portions are not actually parabolas.

75. Graph the functions.

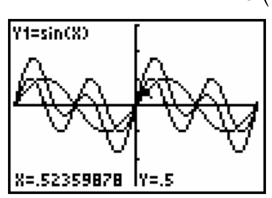


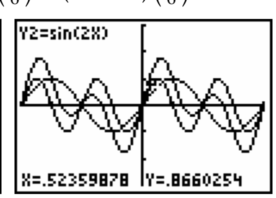


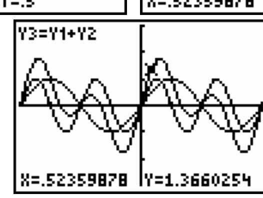


Notice that

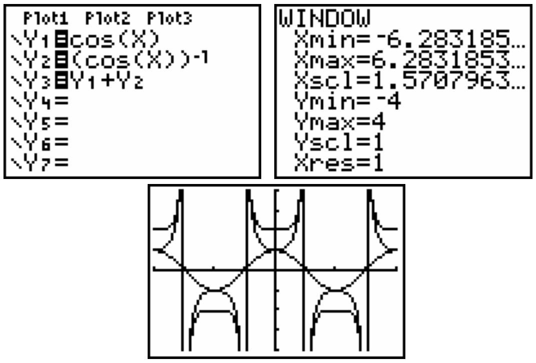
$Y_1\left(\frac{\pi}{6}\right) + Y_2\left(\frac{\pi}{6}\right) \approx .5 + .8660254 = 1.3660254$
$= Y_3\left(\frac{\pi}{6}\right) = (Y_1 + Y_2)\left(\frac{\pi}{6}\right)$





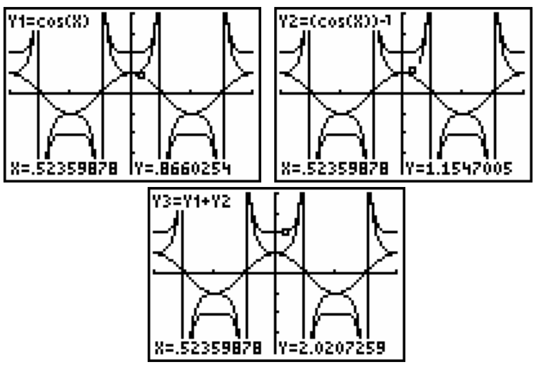


76. Graph the functions.



Notice that

$$\begin{aligned}Y_1\left(\tfrac{\pi}{6}\right)+Y_2\left(\tfrac{\pi}{6}\right) &\approx .8660254+1.1547005\\ &= 2.0207259\\ &= Y_3\left(\tfrac{\pi}{6}\right)=\left(Y_1+Y_2\right)\left(\tfrac{\pi}{6}\right)\end{aligned}$$



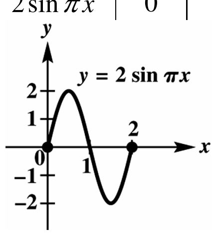
Summary Exercises on Graphing Circular Functions

1. $y=2\sin \pi x$

Period: $\dfrac{2\pi}{\pi}=2$ and amplitude: $|2|=2$

Divide the interval $[0,2]$ into four equal parts to get the *x*-values that will yield minimum and maximum points and *x*-intercepts. Then make a table.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \pi x$	0	1	0	-1	0
$2\sin \pi x$	0	2	0	-2	0

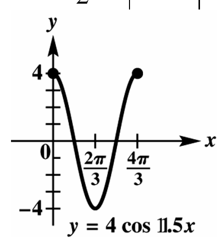


2. $y=4\cos(1.5x)=y=4\cos\left(\dfrac{3}{2}x\right)$

Period: $\dfrac{2\pi}{\frac{3}{2}}=2\pi\cdot\dfrac{2}{3}=\dfrac{4\pi}{3}$ and amplitude: $|4|=4$

Divide the interval $\left[0,\dfrac{4\pi}{3}\right]$ into four equal parts to get the *x*-values that will yield minimum and maximum points and *x*-intercepts. Then make a table.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$
$\frac{3}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{3}{2}x$	1	0	-1	0	1
$4\cos\frac{3}{2}x$	4	0	-4	0	4



3. $y=-2+.5\cos\dfrac{\pi}{4}x$

Step 1: Find the interval whose length is $\dfrac{2\pi}{b}$.

$0\le\dfrac{\pi}{4}x\le 2\pi\Rightarrow 0\le x\le 8$

Step 2: Divide the period into four equal parts to get the following *x*-values: $0,2,4,6,8$

Step 3: Evaluate the function for each of the five *x*-values.

x	0	2	4	6	8
$\frac{\pi}{4}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{\pi}{4}x$	1	0	-1	0	1
$\frac{1}{2}\cos\frac{\pi}{4}x$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
$-2+\frac{1}{2}\cos\frac{\pi}{4}x$	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-2	$-\frac{3}{2}$

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.

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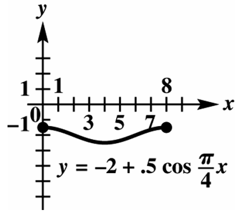


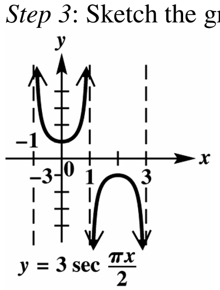
The amplitude is $|.5| = \left|\frac{1}{2}\right|$, which is $\frac{1}{2}$. The period is 8. The vertical translation is 2 units down. There is no phase shift.

4. $y = 3\sec\frac{\pi}{2}x$

Step 1: Graph the corresponding reciprocal function $y = 3\cos\frac{\pi}{2}x$ The period is $\frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$ and its amplitude is $|3| = 3.$ One period is in the interval $0 \le x \le 4$. Dividing the interval into four equal parts gives us the following key points: $(0,3)$, $(1,0)$, $(2,-3)$, $(3,0)$, $(4,3)$

Step 2: The vertical asymptotes of $y = 3\sec\frac{\pi}{2}x$ are at the x-intercepts of $y = 3\cos\frac{\pi}{2}x$ which are $x = 1$ and $x = 3.$ Continuing this pattern to the left, we also have a vertical asymptote of $x = -1.$

Step 3: Sketch the graph.



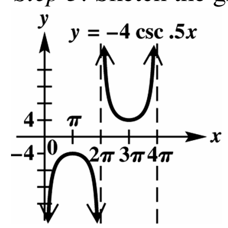
5. $y = -4\csc .5x = -4\csc\frac{1}{2}x$

Step 1: Graph the corresponding reciprocal function $y = -4\sin\frac{1}{2}x$ The period is $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$ and its amplitude is $|-4| = 4.$

One period is in the interval $0 \le x \le 4\pi$. Dividing the interval into four equal parts gives us the following key points: $(0,0)$, $(\pi,-4)$, $(2\pi,0)$, $(3\pi,4)$, $(4\pi,0)$

Step 2: The vertical asymptotes of $y = -4\csc\frac{1}{2}x$ are at the x-intercepts of $y = -4\sin\frac{1}{2}x$, which are $x = 0$, $x = 2\pi$, and $x = 4\pi$.

Step 3: Sketch the graph.



6. $y = 3\tan\left(\frac{\pi}{2}x + \pi\right) = 3\tan\frac{\pi}{2}(x+2)$

Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{\frac{\pi}{2}} = \pi \cdot \frac{2}{\pi} = 2$. Tangent has asymptotes of the form $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$. The asymptotes for $y = 3\tan\frac{\pi}{2}(x+2)$ are

$\frac{\pi}{2}(x+2) = -\frac{\pi}{2} \Rightarrow x + 2 = -1 \Rightarrow x = -3$ and

$\frac{\pi}{2}(x+2) = \frac{\pi}{2} \Rightarrow x + 2 = 1 \Rightarrow x = -1$

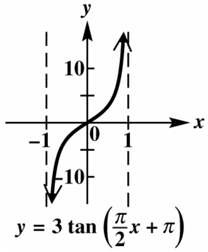
Continuing this pattern we see that $x = 1$ is also a vertical asymptote.

Step 2: Sketch the vertical asymptotes, $x = -1$ and $x = 1$.

Step 3: Divide the interval into four equal parts: $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(-\frac{1}{2},-3\right)$, $(0,0)$, $\left(\frac{1}{2},3\right)$

Step 5: Join the points with a smooth curve.

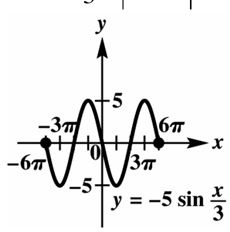


7. $y = -5\sin\frac{x}{3}$

Period: $\frac{2\pi}{\frac{1}{3}} = 2\pi \cdot \frac{3}{1} = 6\pi$ and amplitude: $|-5| = 5$

Divide the interval $[0, 6\pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table. Repeat this cycle for the interval $[-6\pi, 0]$.

x	0	$\frac{3\pi}{2}$	3π	$\frac{9\pi}{2}$	6π
$\frac{x}{3}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{x}{3}$	0	1	0	-1	0
$-5\sin\frac{x}{3}$	0	-5	0	5	0



8. $y = 10\cos\left(\frac{x}{4} + \frac{\pi}{2}\right) = 10\cos\frac{1}{4}(x + 2\pi)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

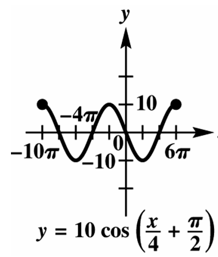
$0 \le \frac{1}{4}(x + 2\pi) \le 2\pi \Rightarrow 0 \le x + 2\pi \le 8\pi \Rightarrow$
$-2\pi \le x \le 6\pi$

Step 2: Divide the period into four equal parts to get the following x-values: -2π, 0, 2π, 4π, 6π

Step 3: Evaluate the function for each of the five x-values.

x	-2π	0	2π	4π	6π
$\frac{1}{4}(x + 2\pi)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{1}{4}(x + 2\pi)$	1	0	-1	0	1
$10\cos\frac{1}{4}(x + 2\pi)$	10	0	-10	0	10

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is 10. The period is 8π. There is no vertical translation. The phase shift is 2π units to the left.

9. $y = 3 - 4\sin(2.5x + \pi) = 3 - 4\sin\left(\frac{5}{2}x + \pi\right)$
$= 3 - 4\sin\frac{5}{2}\left(x + \frac{2\pi}{5}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

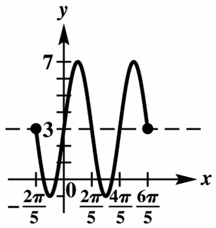
$0 \le \frac{5}{2}\left(x + \frac{2\pi}{5}\right) \le 2\pi \Rightarrow$
$0 \le x + \frac{2\pi}{5} \le \frac{4\pi}{5} \Rightarrow -\frac{2\pi}{5} \le x \le \frac{2\pi}{5}$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{2\pi}{5}$, $-\frac{\pi}{5}$, 0, $\frac{\pi}{5}$, $\frac{2\pi}{5}$

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{2\pi}{5}$	$-\frac{\pi}{5}$	0	$\frac{\pi}{5}$	$\frac{2\pi}{5}$
$\frac{5}{2}\left(x + \frac{2\pi}{5}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\frac{5}{2}\left(x + \frac{2\pi}{5}\right)$	0	1	0	-1	0
$-4\sin\frac{5}{2}\left(x + \frac{2\pi}{5}\right)$	0	-4	0	4	0
$3 - 4\sin\frac{5}{2}\left(x + \frac{2\pi}{5}\right)$	3	-1	3	7	3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the right, we obtain the following graph.



$y = 3 - 4\sin(2.5x + \pi)$

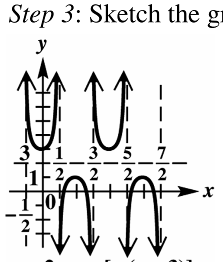
The amplitude is $|-4|$, which is 4. The period is $\frac{2\pi}{\frac{5}{2}}$, which is $\frac{4\pi}{5}$. The vertical translation is 3 units up. The phase shift is $\frac{2\pi}{5}$ units to the left.

10. $y = 2 - \sec\left[\pi(x-3)\right]$

Step 1: Graph the corresponding reciprocal function $y = 2 - \cos\left[\pi(x-3)\right]$. The period is 2 and its amplitude is $|-1| = 1$. One period is in the interval $\frac{3}{2} \le x \le \frac{7}{2}$. Dividing the interval into four equal parts gives us the following key points: $\left(\frac{3}{2}, 2\right)$, $(2,3)$, $\left(\frac{5}{2}, 2\right)$, $(3,1)$, $\left(\frac{7}{2}, 2\right)$

Step 2: The vertical asymptotes of $y = 2 - \sec\left[\pi(x-3)\right]$ are at the x-intercepts of $y = -\cos\left[\pi(x-3)\right]$, which are $x = \frac{3}{2}$, $x = \frac{5}{2}$, and $x = \frac{7}{2}$. Continuing this pattern to the left, we also have a vertical asymptote of $x = \frac{1}{2}$ and $x = -\frac{1}{2}$.

Step 3: Sketch the graph.



$y = 2 - \sec[\pi(x - 3)]$

Section 6.6: Harmonic Motion

1. $s(0) = 2$ in.; $P = .5$ sec

(a) Given $s(t) = a\cos\omega t$, the period is $\frac{2\pi}{\omega}$ and the amplitude is $|a|$.

$P = .5 \text{ sec} \Rightarrow .5 = \frac{2\pi}{\omega} \Rightarrow$

$\frac{1}{2} = \frac{2\pi}{\omega} \Rightarrow \omega = 4\pi$

$s(0) = 2 = a\cos\left[\omega(0)\right] \Rightarrow$

$2 = a\cos 0 \Rightarrow 2 = a(1) \Rightarrow a = 2$

Thus, $s(t) = 2\cos 4\pi t$.

(b) Since $s(1) = 2\cos\left[4\pi(1)\right] = 2\cos 4\pi = 2(1) = 2$, the weight is neither moving upward nor downward. At $t = 1$, the motion of the weight is changing from up to down.

2. $s(0) = 5$ in.; $P = 1.5$ sec

(a) Given $s(t) = a\cos\omega t$, the period is $\frac{2\pi}{\omega}$ and the amplitude is $|a|$.

$P = 1.5 \text{ sec} \Rightarrow 1.5 = \frac{2\pi}{\omega} \Rightarrow \frac{3}{2} = \frac{2\pi}{\omega} \Rightarrow$

$3\omega = 4\pi \Rightarrow \omega = \frac{4\pi}{3}$

$s(0) = 5 = a\cos\left[\omega(0)\right] \Rightarrow 5 = a\cos 0 \Rightarrow$

$5 = a(1) \Rightarrow a = 5$

Thus, $s(t) = 5\cos\frac{4\pi}{3}t$.

(b) Since $s(1) = 5\cos\left[\frac{4\pi}{3}(1)\right] = 5\cos\frac{4\pi}{3}$

$= 5\left(-\frac{1}{2}\right) = -\frac{5}{2} = -2.5$

the weight is moving upward.

3. $s(0) = -3$ in.; $P = .8$ sec

(a) Given $s(t) = a\cos\omega t$, the period is $\frac{2\pi}{\omega}$ and the amplitude is $|a|$.

$P = .8 \text{ sec} \Rightarrow .8 = \frac{2\pi}{\omega} \Rightarrow \frac{4}{5} = \frac{2\pi}{\omega} \Rightarrow$

$4\omega = 10\pi \Rightarrow \omega = \frac{10\pi}{4} = 2.5\pi$

$s(0) = -3 = a\cos[\omega(0)] \Rightarrow$
$-3 = a\cos 0 \Rightarrow -3 = a(1) \Rightarrow a = -3$

Thus, $s(t) = -3\cos 2.5\pi t.$

(b) Since $s(1) = -3\cos[2.5\pi(1)]$
$= -3\cos\frac{5\pi}{2} = -3(0) = 0$

the weight is moving upward.

4. $s(0) = -4$ in.; $P = 1.2$ sec

(a) Given $s(t) = a\cos\omega t,$ the period is $\frac{2\pi}{\omega}$ and the amplitude is $|a|$.

$P = 1.2 \text{ sec} \Rightarrow 1.2 = \frac{2\pi}{\omega} \Rightarrow \frac{6}{5} = \frac{2\pi}{\omega} \Rightarrow$
$6\omega = 10\pi \Rightarrow \omega = \frac{10\pi}{6} = \frac{5\pi}{3}$
$s(0) = -4 = a\cos[\omega(0)] \Rightarrow$
$-4 = a\cos 0 \Rightarrow -4 = a(1) \Rightarrow a = -4$

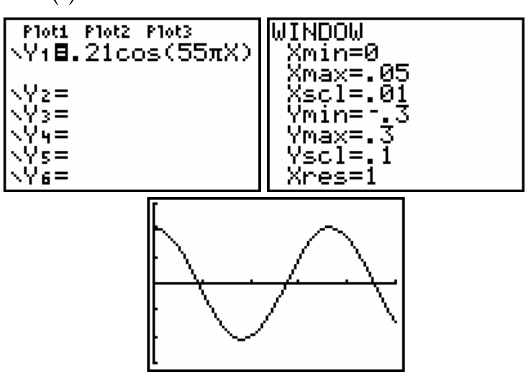
Thus, $s(t) = -4\cos\frac{5\pi}{3}t.$

(b) Since $s(1) = -4\cos\left[\frac{5\pi}{3}(1)\right]$
$= -4\cos\frac{5\pi}{3} = -4\left(\frac{1}{2}\right) = -2$

the weight is moving downward.

5. Since frequency is $\frac{\omega}{2\pi}$, we have

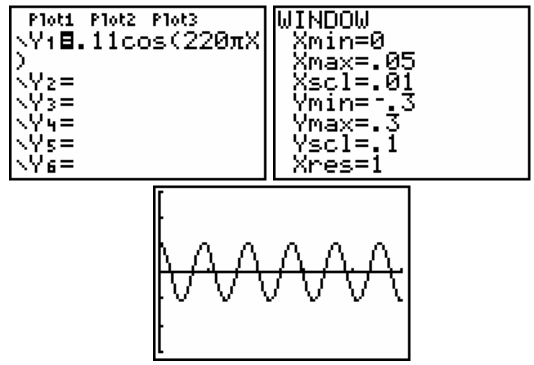
$27.5 = \frac{\omega}{2\pi} \Rightarrow \omega = 55\pi.$ Since $s(0) = .21,$
$.21 = a\cos[\omega(0)] \Rightarrow$
$.21 = a\cos 0 \Rightarrow .21 = a(1) \Rightarrow a = .21.$ Thus,
$s(t) = .21\cos 55\pi t.$



6. Since frequency is $\frac{\omega}{2\pi}$, we have

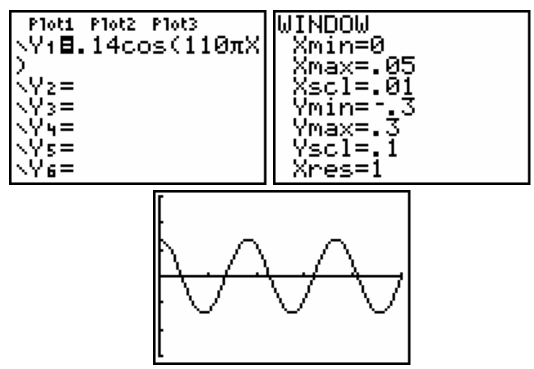
$110 = \frac{\omega}{2\pi} \Rightarrow \omega = 220\pi.$ Since $s(0) = .11,$
$.11 = a\cos[\omega(0)] \Rightarrow .11 = a\cos 0 \Rightarrow$
$.11 = a(1) \Rightarrow a = .11$

Thus, $s(t) = .11\cos 220\pi t.$



7. Since frequency is $\frac{\omega}{2\pi}$, we have

$55 = \frac{\omega}{2\pi} \Rightarrow \omega = 110\pi.$ Since $s(0) = .14,$
$.14 = a\cos[\omega(0)] \Rightarrow$
$.14 = a\cos 0 \Rightarrow .14 = a(1) \Rightarrow a = .14.$ Thus,
$s(t) = .14\cos 110\pi t.$



8. Since frequency is $\frac{\omega}{2\pi}$, we have

$220 = \frac{\omega}{2\pi} \Rightarrow \omega = 440\pi.$ Since $s(0) = .06,$
$.06 = a\cos[\omega(0)] \Rightarrow$
$.06 = a\cos 0 \Rightarrow .06 = a(1) \Rightarrow a = .06.$ Thus,
$s(t) = .06\cos 440\pi t.$

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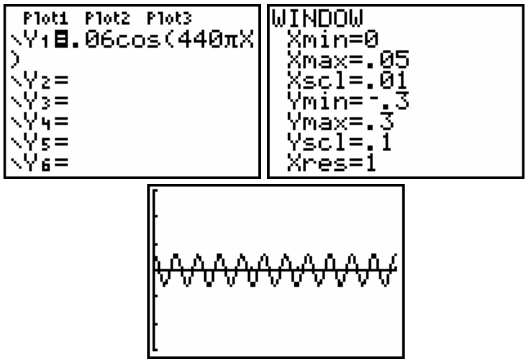


9. **(a)** Since the object is pulled down 4 units, $s(0)=-4.$ Thus, we have

$$s(0)=-4=a\cos\left[\omega(0)\right]\Rightarrow$$
$$-4=a\cos 0\Rightarrow -4=a(1)\Rightarrow a=-4$$

Since the time it takes to complete one oscillation is 3 sec, $P=3$ sec.

$$P=3\text{ sec}\Rightarrow 3=\frac{2\pi}{\omega}\Rightarrow 3=\frac{2\pi}{\omega}\Rightarrow$$
$$3\omega=2\pi\Rightarrow\omega=\frac{2\pi}{3}$$

Therefore, $s(t)=-4\cos\frac{2\pi}{3}t.$

(b) $$s(1)=-4\cos\left[\frac{2\pi}{3}(1.25)\right]$$
$$=-4\cos\left[\frac{2\pi}{3}\left(\frac{5}{4}\right)\right]=-4\cos\frac{5\pi}{6}$$
$$=-4\left(-\frac{\sqrt{3}}{2}\right)=2\sqrt{3}\approx 3.46\text{ units}$$

(rounded to three significant digits)

(c) The frequency is the reciprocal of the period, or $\frac{1}{3}$ oscillation per second.

10. **(a)** Since the object is pulled down 6 units, $s(0)=-6.$ Thus, we have

$$s(0)=-6=a\cos\left[\omega(0)\right]\Rightarrow$$
$$-6=a\cos 0\Rightarrow -6=a(1)\Rightarrow a=-6$$

Since the time it takes to complete one oscillation is 4 sec, $P=4$ sec.

$$P=4\text{ sec}\Rightarrow 4=\frac{2\pi}{\omega}\Rightarrow 4=\frac{2\pi}{\omega}\Rightarrow$$
$$4\omega=2\pi\Rightarrow\omega=\frac{2\pi}{4}=\frac{\pi}{2}$$

Therefore, $s(t)=-6\cos\frac{\pi}{2}t.$

(b) $$s(1.25)=-6\cos\left[\frac{\pi}{2}(1.25)\right]$$
$$=-6\cos\left[\frac{\pi}{2}\left(\frac{5}{4}\right)\right]$$
$$=-6\cos\frac{5\pi}{8}\approx 2.30\text{ units}$$

(rounded to three significant digits)

(c) The frequency is the reciprocal of the period, or $\frac{1}{4}$ oscillation per second.

11. **(a)** $a=2,\omega=2$

$s(t)=a\sin\omega t\Rightarrow s(t)=2\sin 2t$

amplitude $=|a|=|2|=2$; period $=\frac{2\pi}{\omega}=\frac{2\pi}{2}=\pi$; frequency $=\frac{\omega}{2\pi}=\frac{1}{\pi}$ rotation per second

(b) $a=2,\ \omega=4$

$s(t)=a\sin\omega t\Rightarrow s(t)=2\sin 4t$

amplitude $=|a|=|2|=2$; period $=\frac{2\pi}{\omega}=\frac{2\pi}{4}=\frac{\pi}{2}$; frequency $=\frac{\omega}{2\pi}=\frac{4}{2\pi}=\frac{2}{\pi}$ rotation per second

12. $P=2\pi\sqrt{\frac{L}{32}};L=\frac{1}{2}$ ft

The period is

$$P=2\pi\sqrt{\frac{L}{32}}\Rightarrow P=2\pi\sqrt{\frac{\frac{1}{2}}{32}}=2\pi\sqrt{\frac{1}{64}}$$
$$=2\pi\cdot\frac{1}{8}=\frac{\pi}{4}\text{ sec}$$

The frequency is the reciprocal of the period, or $\frac{4}{\pi}$ oscillations per second.

13. $$P=2\pi\sqrt{\frac{L}{32}}\Rightarrow 1=2\pi\sqrt{\frac{L}{32}}\Rightarrow$$
$$\frac{1}{2\pi}=\sqrt{\frac{L}{32}}\Rightarrow\frac{1}{(2\pi)^2}=\frac{L}{32}\Rightarrow\frac{1}{4\pi^2}=\frac{L}{32}\Rightarrow$$
$$\frac{32}{4\pi^2}=L\Rightarrow L=\frac{8}{\pi^2}\text{ ft}$$

14. $s(t) = a\sin\sqrt{\frac{k}{m}}t; k = 4; P = 1$ sec

A period of 1 sec is produced when $\frac{2\pi}{\sqrt{\frac{k}{m}}} = 1$

Since $k = 4$, we can solve

$$\frac{2\pi}{\sqrt{\frac{k}{m}}} = 1 \Rightarrow \frac{2\pi}{\sqrt{\frac{4}{m}}} = 1 \Rightarrow 2\pi = \sqrt{\frac{4}{m}} \Rightarrow$$

$$4\pi^2 = \frac{4}{m} \Rightarrow 4\pi^2 m = 4 \Rightarrow m = \frac{4}{4\pi^2} = \frac{1}{\pi^2}$$

15. $s(t) = -4\cos 8\pi t$

(a) The maximum of $s(t) = -4\cos 8\pi t$ is $|-4| = 4$ in.

(b) In order for $s(t) = -4\cos 8\pi t$ to reach its maximum, $y = \cos 8\pi t$ needs to be at a minimum. This occurs after $8\pi t = \pi \Rightarrow t = \frac{\pi}{8\pi} = \frac{1}{8}$ sec.

(c) Since $s(t) = -4\cos 8\pi t$ and $s(t) = a\cos\omega t$, $\omega = 8\pi$. Therefore, frequency $= \frac{\omega}{2\pi} = \frac{8\pi}{2\pi} = 4$ cycles per sec. The period is the reciprocal of the frequency, or $\frac{1}{4}$ sec.

16. $k = 2$, $m = 1$

(a) Since the spring is stretched $\frac{1}{2}$ ft, amplitude $= a = \frac{1}{2}$. From Exercise 14 we have, $s(t) = a\sin\sqrt{\frac{k}{m}}t$. Thus, $s(t) = a\sin\sqrt{\frac{2}{1}}t = a\sin\sqrt{2}t$. Since $s(t) = a\sin\omega t$, we have $\omega = \sqrt{2}$. This yields the following.

period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\,\pi$ and

frequency $= \frac{\omega}{2\pi} = \frac{\sqrt{2}}{2\pi}$ oscillatioin per second

(b) $s(t) = a\sin\omega t \Rightarrow s(t) = \frac{1}{2}\sin\sqrt{2}\,t$

17. $s(t) = -5\cos 4\pi t$, $a = |-5| = 5$, $\omega = 4\pi$

(a) maximum height = amplitude $= a = |-5| = 5$ in.

(b) frequency $= \frac{\omega}{2\pi} = \frac{4\pi}{2\pi} = 2$ cycles per sec; period $= \frac{2\pi}{\omega} = \frac{1}{2}$ sec

(c) $s(t) = -5\cos 4\pi t = 5 \Rightarrow \cos 4\pi t = -1 \Rightarrow$

$$4\pi t = \pi \Rightarrow t = \frac{1}{4}$$

The weight first reaches its maximum height after $\frac{1}{4}$ sec.

(d) Since $s(1.3) = -5\cos\left[4\pi(1.3)\right] = -5\cos 5.2\pi \approx 4$, after 1.3 sec, the weight is about 4 in. above the equilibrium position.

18. $s(t) = -4\cos 10t$, $a = -4$, $\omega = 10$

(a) maximum height = amplitude $= a = |-4| = 4$ in.

(b) frequency $= \frac{\omega}{2\pi} = \frac{10}{2\pi} = \frac{5}{\pi}$ cycles per sec; period $= \frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}$ sec

(c) $s(t) = -4\cos 10t = 4 \Rightarrow \cos 10t = -1 \Rightarrow$

$$10t = \pi \Rightarrow t = \frac{\pi}{10}$$

The weight first reaches its maximum height after $\frac{\pi}{10}$ sec.

(d) $s(1.466) = -4\cos(10 \cdot 1.466)$
$= -4\cos(14.66) \approx 2$

After 1.466 sec, the weight is about 2 in. above the equilibrium position.

19. $a = -3$

(a) We will use a model of the form $s(t) = a\cos\omega t$ with $a = -3$.

$$s(0) = -3\cos[\omega(0)] = -3\cos 0 = -3(1) = -3$$

Using a cosine function rather than a sine function will avoid the need for a phase shift. Since the frequency $= \frac{6}{\pi}$ cycles per sec, by definition,

$$\frac{\omega}{2\pi} = \frac{6}{\pi} \Rightarrow \omega\pi = 12\pi \Rightarrow \omega = 12.$$

Therefore, a model for the position of the weight at time t seconds is $s(t) = -3\cos 12t$.

(b) The period is the reciprocal of the frequency, or $\frac{\pi}{6}$ sec.

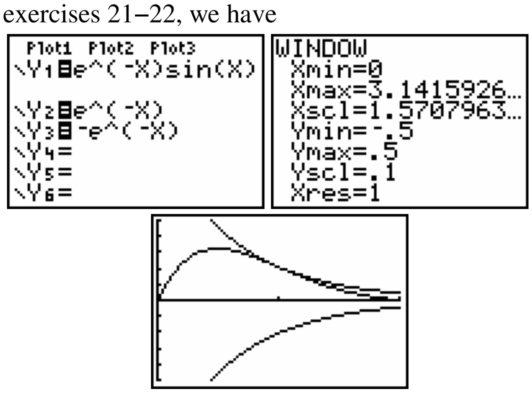
20. $a = -2$

(a) period: $\frac{2\pi}{\omega} = \frac{1}{3} \Rightarrow 6\pi = \omega$

$$s(t) = a\cos\omega t \Rightarrow s(t) = -2\cos 6\pi t$$

(b) frequency $= \frac{\omega}{2\pi} = \frac{6\pi}{2\pi} = 3$ cycles per sec

For exercises 21–22, we have



21. Since $e^{-t} \neq 0$, we have $e^{-t}\sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, \pi$. The x-intercepts of Y_1 are the same as these of $\sin x$.

22. Since $Y_1 = Y_2 \Rightarrow e^{-t}\sin t = e^{-t} \Rightarrow$ $\sin t = 1 \Rightarrow t = \frac{\pi}{2}$, the intersection occurs when $\sin t$ is at a maximum, that is, when $t = \frac{\pi}{2}$.

Thus, the point of intersection is $\left(\frac{\pi}{2}, e^{-\pi/2}\right)$.

$Y_1 = Y_3 \Rightarrow e^{-t}\sin t = -e^{-t} \Rightarrow$ $\sin t = -1 \Rightarrow x = \frac{3\pi}{2}$, so the intersection occurs when $\sin t$ is at a minimum but the minimum value of $\sin t$ does not occur in $[0, \pi]$. Because $\sin\frac{\pi}{2} = 1$,

$$e^{-\pi/2}\sin\frac{\pi}{2} = e^{-\pi/2}.$$

Chapter 6 Review Exercises

1. An angle of 1° is $\frac{1}{360}$ of the way around a circle. 1 radian is the measurement of an angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle. Thus, 1 radian is $\frac{1}{2\pi}$ of the way around the circle. Since $\frac{1}{360} < \frac{1}{2\pi}$, the angular measurement of 1 radian is larger than 1°.

2. (a) Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, we have $\frac{\pi}{2} < 3 < \pi$. Thus, the terminal side is in quadrant II.

(b) Since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, we have $\pi < 4 < \frac{3\pi}{2}$. Thus, the terminal side is in quadrant III.

(c) Since $-\frac{\pi}{2} \approx -1.57$ and $-\pi \approx -3.14$, we have $-\frac{\pi}{2} > -2 > -\pi$. Thus, the terminal side is in quadrant III.

(d) Since $2\pi \approx 6.28$ and $\frac{5\pi}{2} \approx 7.85$. we have $2\pi < 7 < \frac{5\pi}{2}$. Thus, the terminal side is in quadrant I.

3. To find a coterminal angle, add or subtract multiples of 2π. Three of the many possible answers are $1 + 2\pi$, $1 + 4\pi$, and $1 + 6\pi$.

4. To find a coterminal angle, add or subtract multiples of 2π. Since n represents any integer, the expression $\frac{\pi}{6}+2n\pi$ generates all coterminal angles with an angle of $\frac{\pi}{6}$ radians.

5. $45° = 45\left(\frac{\pi}{180}\text{radian}\right) = \frac{\pi}{4}\text{radians}$

6. $120° = 120\left(\frac{\pi}{180}\text{radian}\right) = \frac{2\pi}{3}\text{radians}$

7. $175° = 175\left(\frac{\pi}{180}\text{radian}\right) = \frac{35\pi}{36}\text{radians}$

8. $330° = 330\left(\frac{\pi}{180}\text{radian}\right) = \frac{11\pi}{6}\text{radians}$

9. $800° = 800\left(\frac{\pi}{180}\text{radian}\right) = \frac{40\pi}{9}\text{radians}$

10. $1020° = 1020\left(\frac{\pi}{180}\text{radian}\right) = \frac{17\pi}{3}\text{radians}$

11. $\frac{5\pi}{4} = \frac{5\pi}{4}\left(\frac{180°}{\pi}\right) = 225°$

12. $\frac{9\pi}{10} = \frac{9\pi}{10}\left(\frac{180°}{\pi}\right) = 162°$

13. $\frac{8\pi}{3} = \frac{8\pi}{3}\left(\frac{180°}{\pi}\right) = 480°$

14. $-\frac{6\pi}{5} = -\frac{6\pi}{5}\left(\frac{180°}{\pi}\right) = -216°$

15. $-\frac{11\pi}{18} = -\frac{11\pi}{18}\left(\frac{180°}{\pi}\right) = -110°$

16. $\frac{21\pi}{5} = \frac{21\pi}{5}\left(\frac{180°}{\pi}\right) = 756°$

17. Since $\frac{15}{60} = \frac{1}{4}$ rotation, we have

$\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}$. Thus,

$s = r\theta \Rightarrow s = 2\left(\frac{\pi}{2}\right) = \pi$ in.

18. Since $\frac{20}{60} = \frac{1}{3}$ rotation, we have

$\theta = \frac{1}{3}(2\pi) = \frac{2\pi}{3}$. Thus,

$s = r\theta \Rightarrow s = 2\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}$ in.

19. Since $\theta = 3(2\pi) = 6\pi$, we have

$s = r\theta \Rightarrow s = 2(6\pi) = 12\pi$ in.

20. Since $\theta = 10.5(2\pi) = 21\pi$, we have

$s = r\theta \Rightarrow s = 2(21\pi) = 42\pi$ in.

21. $r = 15.2$ cm, $\theta = \frac{3\pi}{4}$

$s = r\theta \Rightarrow s = 15.2\left(\frac{3\pi}{4}\right) = 11.4\pi \approx 35.8$ cm

22. $r = 11.4$ cm, $\theta = .769$

$s = r\theta \Rightarrow s = 11.4(.769) \approx 8.77$ cm

23. $r = 8.973$ cm, $\theta = 49.06°$

First convert $\theta = 49.06\frac{\pi}{180°}$ to radians:

$\theta = 49.06° = 49.06\left(\frac{\pi}{180}\right) = \frac{49.06\pi}{180}$

$s = r\theta \Rightarrow s = 8.973\left(\frac{49.06\pi}{180}\right) \approx 7.683$ cm

24. $r = 28.69, \theta = \frac{7\pi}{4}$

$A = \frac{1}{2}r^2\theta \Rightarrow$

$A = \frac{1}{2}(28.69)^2\left(\frac{7\pi}{4}\right)$

$= \frac{1}{2}(823.1161)\left(\frac{7\pi}{4}\right) \approx 2263 \text{ in.}^2$

25. $r = 38.0$ m, $\theta = 21°40'$

First convert $\theta = 21°40'$ to radians:

$\theta = 21°40' = \left(21+\frac{40}{60}\right)\left(\frac{\pi}{180}\right)$

$= \frac{65}{3}\left(\frac{\pi}{180}\right) = \frac{13\pi}{108}$

$A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(38.0)^2\left(\frac{13\pi}{108}\right) \approx 273 \text{ m}^2$

26. Because the central angle is very small, the arc length is approximately equal to the length of the inscribed chord. (See directions for Exercises 37–38 in Section 3.2)
Let h = height of tree, $r = 2000$, and $\theta = 1°10'$.
First, convert $\theta = 1°10'$ to radians:

$$\theta = 1°10' = \left(1+\tfrac{10}{60}\right)° = \left(1\tfrac{1}{6}\right)°$$
$$= \frac{7}{6}\left(\frac{\pi}{180}\text{radian}\right) = \frac{7\pi}{1080}\text{radian}$$

Thus, $h \approx r\theta \Rightarrow h \approx 2000\left(\frac{7\pi}{1080}\right) \approx 41$ yd

27. The cities are at 28ºN and 12ºS.
12ºS = −12ºN, so

$$\theta = 28° - (-12°) = 40° = 40\left(\frac{\pi}{180}\right) = \frac{2\pi}{9}$$

radians

$s = r\theta = 6400\left(\frac{2\pi}{9}\right) \approx 4500$ km (rounded to two significant digits)

28. The cities are at 72ºE and 35ºW.
35ºW = −35ºE

$$\theta = 72° - (-35°) = 107° = 107\left(\frac{\pi}{180}\right) = \frac{107\pi}{180}$$

radians

$s = r\theta = 6400\left(\frac{107\pi}{180}\right) \approx 12{,}000$ km (rounded to two significant digits)

29. $r = 2,\ s = 1.5$

$$s = r\theta \Rightarrow 1.5 = 2\theta \Rightarrow \theta = \frac{1.5}{2} = \frac{3}{4} \text{ radian}$$
$$A = \frac{1}{2}r^2\theta \Rightarrow$$
$$A = \frac{1}{2}(2)^2\left(\frac{3}{4}\right) = \frac{1}{2}(4)\left(\frac{3}{4}\right) = \frac{3}{2} = 1.5 \text{ sq units}$$

30. $s = 4,\ r = 8$

$$s = r\theta \Rightarrow 4 = 8\theta \Rightarrow \theta = \frac{4}{8} = \frac{1}{2} \text{ radian}$$
$$A = \frac{1}{2}r^2\theta \Rightarrow$$
$$A = \frac{1}{2}(8)^2\left(\frac{1}{2}\right) = \frac{1}{2}(64)\left(\frac{1}{2}\right) = 16 \text{ sq units}$$

31. $\tan\frac{\pi}{3}$

Converting $\frac{\pi}{3}$ to degrees, we have

$$\frac{\pi}{3} = \frac{1}{3}(180°) = 60° \Rightarrow \tan\frac{\pi}{3} = \tan 60° = \sqrt{3}$$

32. $\cos\frac{2\pi}{3}$

Since $\frac{2\pi}{3}$ is in quadrant II, the reference angle is $\pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}$. In quadrant II, the cosine is negative. Thus, $\cos\frac{2\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$. Converting $\frac{2\pi}{3}$ to degrees, we have $\frac{2\pi}{3} = \frac{2}{3}(180°) = 120°$. The reference angle is $180° - 120° = 60°$. Thus, $\cos\frac{2\pi}{3} = \cos 120° = -\cos 60° = -\frac{1}{2}$.

33. $\sin\left(-\frac{5\pi}{6}\right)$

$-\frac{5\pi}{6}$ is coterminal with $-\frac{5\pi}{6} + 2\pi = \frac{7\pi}{6}$.

Since $\frac{7\pi}{6}$ is in quadrant III, the reference angle is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$. In quadrant III, the sine is negative. Thus,

$$\sin\left(-\frac{5\pi}{6}\right) = \sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

Converting $\frac{5\pi}{6}$ to degrees, we have

$\frac{7\pi}{6}\left(\frac{180°}{\pi}\right) = 210°$. The reference angle is 210º − 180º = 30º. Thus,

$$\sin\left(-\frac{5\pi}{6}\right) = \sin\frac{7\pi}{6} = \sin 210°$$
$$= -\sin 30° = -\frac{1}{2}$$

34. $\tan\left(-\frac{7\pi}{3}\right)$

$-\frac{7\pi}{3}$ is coterminal with

$-\frac{7\pi}{3}+2(2\pi)=-\frac{7\pi}{3}+4\pi=-\frac{7\pi}{3}+\frac{12\pi}{3}=\frac{5\pi}{3}.$

Since $\frac{5\pi}{3}$ is in quadrant IV, the reference angle is $2\pi-\frac{5\pi}{3}=\frac{6\pi}{3}-\frac{5\pi}{3}=\frac{\pi}{3}$. In quadrant IV, the tangent is negative. Thus,

$\tan\left(-\frac{7\pi}{3}\right)=\tan\frac{5\pi}{3}=-\tan\frac{\pi}{3}=-\sqrt{3}.$

Converting $\frac{5\pi}{3}$ to degrees, we have

$\frac{5\pi}{3}=\frac{5}{3}(180°)=300°$. The reference angle is

$360°-300°=60°$. Thus,

$$\tan\left(-\frac{7\pi}{3}\right)=\tan\frac{5\pi}{3}=\tan 300°$$
$$=-\tan 60°=-\sqrt{3}$$

35. $\csc\left(-\frac{11\pi}{6}\right)$

$-\frac{11\pi}{6}$ is coterminal with

$-\frac{11\pi}{6}+2\pi=-\frac{11\pi}{6}+\frac{12\pi}{6}=\frac{\pi}{6}$. Since $\frac{\pi}{6}$ is in quadrant I, we have

$\csc\left(-\frac{11\pi}{6}\right)=\csc\frac{\pi}{6}=2$. Converting $\frac{\pi}{6}$ to degrees, we have $\frac{\pi}{6}=\frac{1}{6}(180°)=30°$. Thus,

$\csc\left(-\frac{11\pi}{6}\right)=\csc\frac{\pi}{6}=\csc 30°=2.$

36. $\cot\left(-\frac{17\pi}{3}\right)$

$-\frac{17\pi}{3}$ is coterminal with

$-\frac{17\pi}{3}+3(2\pi)=-\frac{17\pi}{3}+6\pi$

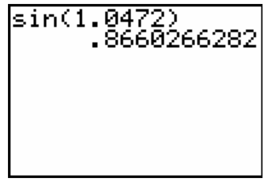
$=-\frac{17\pi}{3}+\frac{18\pi}{3}=\frac{\pi}{3}$. Since $\frac{\pi}{3}$ is in quadrant I, we have $\cot\left(-\frac{17\pi}{3}\right)=\cot\frac{\pi}{3}=\frac{\sqrt{3}}{3}$.

Converting $\frac{\pi}{3}$ to degrees, we have

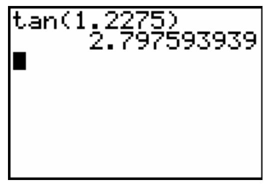
$\frac{\pi}{3}=\frac{1}{3}(180°)=60°$. Thus,

$\cot\left(-\frac{17\pi}{3}\right)=\cot\frac{\pi}{3}=\cot 60°=\frac{\sqrt{3}}{3}.$

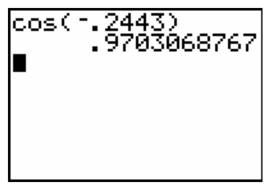
37. $\sin 1.0472\approx .8660$



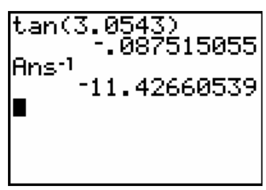
38. $\tan 1.2275\approx 2.7976$



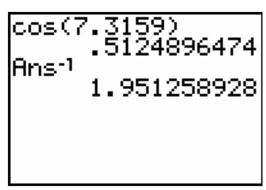
39. $\cos(-.2443)\approx .9703$



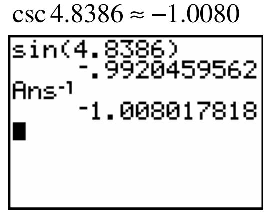
40. $\cot 3.0543\approx -11.4266$



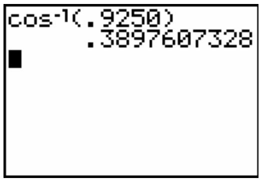
41. $\sec 7.3159\approx 1.9513$



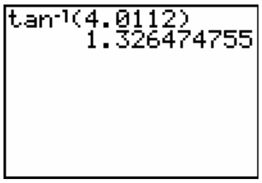
42. $\csc 4.8386\approx -1.0080$



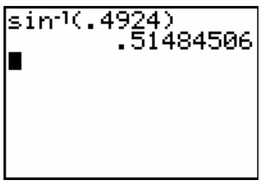
43. $\cos s = .9250 \Rightarrow s \approx .3898$



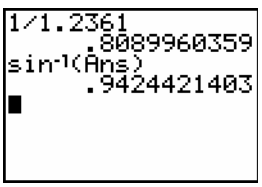
44. $\tan s \Rightarrow s \approx 1.3265$



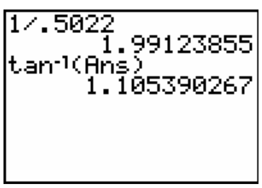
45. $\sin s = .4924 \Rightarrow s \approx .5148$



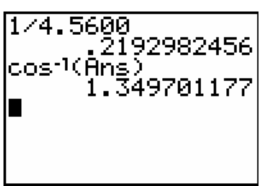
46. $\csc s = 1.2361 \Rightarrow s \approx .9424$



47. $\cot s = .5022 \Rightarrow s \approx 1.1054$



48. $\sec s = 4.5600 \Rightarrow s \approx 1.3497$



49. $\left[0, \frac{\pi}{2}\right], \cos s = \frac{\sqrt{2}}{2}$

Because $\cos s = \frac{\sqrt{2}}{2}$, the reference angle for s must be $\frac{\pi}{4}$ since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

For s to be in the interval $\left[0, \frac{\pi}{2}\right]$, s must be the reference angle. Therefore, $s = \frac{\pi}{4}$.

50. $\left[\frac{\pi}{2}, \pi\right]$, $\tan s = -\sqrt{3}$

Because $\tan s = -\sqrt{3}$, the reference angle for s must be $\frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$. For s to be in the interval $\left[\frac{\pi}{2}, \pi\right]$, we must subtract the reference angle from π. Therefore,

$s = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

51. $\left[\pi, \frac{3\pi}{2}\right]$, $\sec s = -\frac{2\sqrt{3}}{3}$

Because $\sec s = -\frac{2\sqrt{3}}{3}$, the reference angle for s must be $\frac{\pi}{6}$ since $\sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$. For s to be in the interval $\left[\pi, \frac{3\pi}{2}\right]$, we must add the reference angle to π. Therefore,

$s = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

52. $\left[\frac{3\pi}{2}, 2\pi\right], \sin s = -\frac{1}{2}$

Because $\sin s = -\frac{1}{2}$, the reference angle for must be $\frac{\pi}{6}$ since $\sin\frac{\pi}{6} = \frac{1}{2}$. For s to be in the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, we must subtract the reference angle from 2π. Therefore,

$s = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

53. $\theta = \frac{5\pi}{12}$, $\omega = \frac{8\pi}{9}$ radians per sec

$$\omega = \frac{\theta}{t} \Rightarrow \frac{8\pi}{9} = \frac{\frac{5\pi}{12}}{t} \Rightarrow \frac{8\pi}{9} = \frac{5\pi}{12t} \Rightarrow$$

$$96\pi t = 45\pi \Rightarrow t = \frac{45\pi}{96\pi} = \frac{15}{32} \text{ sec}$$

54. $t = 12$ sec, $\omega = 9$ radians per sec

$\omega = \frac{\theta}{t} \Rightarrow 9 = \frac{\theta}{12} \Rightarrow \theta = 108$ radians

55. $t = 8$ sec, $\theta = \frac{2\pi}{5}$ radians

$\omega = \frac{\theta}{t} \Rightarrow \omega = \frac{\frac{2\pi}{5}}{8} = \frac{2\pi}{5}\left(\frac{1}{8}\right) = \frac{\pi}{20}$ radians per sec

56. $s = \frac{12\pi}{25}$ ft, $r = \frac{3}{5}$ ft, $t = 15$ sec

$v = \frac{s}{t} \Rightarrow v = \frac{\frac{12\pi}{25}}{15} = \frac{12\pi}{25} \cdot \frac{1}{15} = \frac{12\pi}{375}$

$v = r\omega \Rightarrow \frac{12\pi}{375} = \frac{3}{5}\omega \Rightarrow \omega = \frac{12\pi}{375} \cdot \frac{5}{3} = \frac{4\pi}{75}$ radians per sec

57. $r = 11.46$ cm, $\omega = 4.283$ radians per sec, $t = 5.813$ sec

$s = r\theta = r\omega t = (11.46)(4.283)(5.813)$
≈ 285.3 cm

58. The flywheel is rotating 90 times per sec or $90(2\pi) = 180\pi$ radians per sec. Since $r = 7$ m, we have

$v = r\omega \Rightarrow v = 7(180\pi) = 1260\pi$ m per sec

59. Since $t = 30$ sec and $\theta = \frac{5\pi}{6}$, radians we have $\omega = \frac{\theta}{t} \Rightarrow \omega = \frac{\frac{5\pi}{6}}{30} = \frac{5\pi}{6} \cdot \frac{1}{30} = \frac{\pi}{36}$ radian per sec.

60. $F(t) = \frac{1}{2}(1 - \cos t)$

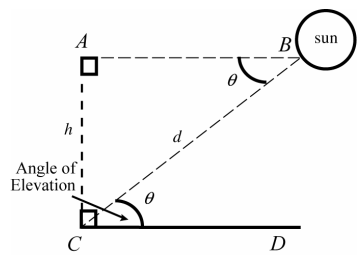
(a) $F(0) = \frac{1}{2}(1 - \cos 0) = \frac{1}{2}(1-1) = \frac{1}{2}(0) = 0;$
The face of the moon is not visible.

(b) $F\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(1 - \cos\frac{\pi}{2}\right) = \frac{1}{2}(1-0)$
$= \frac{1}{2}(1) = \frac{1}{2};$
Half the face of the moon is visible.

(c) $F(\pi) = \frac{1}{2}(1 - \cos\pi) = \frac{1}{2}[1-(-1)]$
$= \frac{1}{2}(2) = 1;$
The face of the moon is completely visible.

(d) $F\left(\frac{3\pi}{2}\right) = \frac{1}{2}\left(1 - \cos\frac{3\pi}{2}\right) = \frac{1}{2}(1-0)$
$= \frac{1}{2}(1) = \frac{1}{2};$
Half the face of the moon is visible.

61. (a) Because alternate interior angles of parallel lines with transversal have the same measure, the measure of angle *ABC* is equal to the measure of angle *BCD* (the angle of elevation). Moreover, triangle BAC is a right triangle and we can write the relation $\sin\theta = \frac{h}{d}$.



Solving for d we have,

$\sin\theta = \frac{h}{d} \Rightarrow d\sin\theta = h \Rightarrow$

$d = h \cdot \frac{1}{\sin\theta} \Rightarrow d = h\csc\theta$

(b) Since $d = h\csc\theta$, we substitute $2h$ for d and solve.

$2h = h\csc\theta \Rightarrow 2 = \csc\theta \Rightarrow$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

d is double h when the sun is $\frac{\pi}{6}$ radians (30°) above the horizon.

(c) $\csc\frac{\pi}{2} = 1$ and $\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \approx 1.15$

When the sun is lower in the sky $\left(\theta = \frac{\pi}{3}\right)$, sunlight is filtered by more atmosphere. There is less ultraviolet light reaching the earth's surface, and therefore, there is less likelihood of becoming sunburned. In this case, sunlight passes through 15% more atmosphere.

62. B; The amplitude is $|4| = 4$ and period is $\frac{2\pi}{2} = \pi$.

63. D; The amplitude is $|-3| = 3,$ but the period is $\frac{2\pi}{\frac{1}{2}} = 4\pi.$ All other statements are true.

64. The range of sine and cosine is [−1, 1] and the range of tangent and cotangent is $(-\infty, \infty)$, so those functions can have *y*-value $\frac{1}{2}$. Tangent, cotangent, secant, and cosecant can have *y*-value 2.

65. $y = 2 \sin x$
Amplitude: 2
Period: 2π
Vertical translation: none
Phase shift: none

66. $y = \tan 3x$
Amplitude: not applicable
Period: $\frac{\pi}{3}$
Vertical translation: none
Phase shift: none

67. $y = -\frac{1}{2}\cos 3x$
Amplitude: $\left|-\frac{1}{2}\right| = \frac{1}{2}$
Period: $\frac{2\pi}{3}$
Vertical translation: none
Phase shift: none

68. $y = 2 \sin 5x$
Amplitude: $|2| = 2$
Period: $\frac{2\pi}{5}$
Vertical translation: none
Phase shift: none

69. $y = 1 + 2\sin\frac{1}{4}x$
Amplitude: $|2| = 2$
Period: $\frac{2\pi}{\frac{1}{4}} = 8\pi$
Vertical translation: up 1 unit
Phase shift: none

70. $y = 3 - \frac{1}{4}\cos\frac{2}{3}x$
Amplitude: $\left|-\frac{1}{4}\right| = \frac{1}{4}$
Period: $\frac{2\pi}{\frac{2}{3}} = 3\pi$
Vertical translation: up 3 units
Phase shift: none

71. $y = 3\cos\left(x + \frac{\pi}{2}\right) = 3\cos\left[x - \left(-\frac{\pi}{2}\right)\right]$
Amplitude: $|3| = 3$
Period: 2π
Vertical translation: none
Phase shift: $\frac{\pi}{2}$ units to the left

72. $y = -\sin\left(x - \frac{3\pi}{4}\right)$
Amplitude: $|-1| = 1$
Period: 2π
Vertical translation: none
Phase shift: $\frac{3\pi}{4}$ units to the right

73. $y = \frac{1}{2}\csc\left(2x - \frac{\pi}{4}\right) = \frac{1}{2}\csc 2\left(x - \frac{\pi}{8}\right)$
Amplitude: not applicable
Period: $\frac{2\pi}{2} = \pi$
Vertical translation: none
Phase shift: $\frac{\pi}{8}$ unit to the right

74. $y = 2\sec(\pi x - 2\pi) = 2\sec\pi(x - 2)$
Amplitude: not applicable
Period: $\frac{2\pi}{\pi} = 2$
Vertical translation: none
Phase shift: 2 units to the right

75. $y = \frac{1}{3}\tan\left(3x - \frac{\pi}{3}\right) = \frac{1}{3}\tan 3\left(x - \frac{\pi}{9}\right)$
Amplitude: not applicable
Period: $\frac{\pi}{3}$
Vertical translation: none
Phase shift: $\frac{\pi}{9}$ unit to the right

76. $y = \cot\left(\frac{x}{2} + \frac{3\pi}{4}\right) = \cot\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$

Amplitude: not applicable

Period: $\frac{\pi}{\frac{1}{2}} = 2\pi$

Vertical translation: none

Phase shift: $\frac{3\pi}{2}$ unit to the left

77. The tangent function has a period of π and x-intercepts at integral multiples of π.

78. The sine function has a period of 2π and passes through the origin.

79. The cosine function has a period of 2π and has the value 0 when $x = \frac{\pi}{2}$.

80. The cosecant function has a period of 2π and is not defined at integral multiples of π.

81. The cotangent function has a period of π and decreases on the interval $(0, \pi)$.

82. The secant function has a period of 2π and vertical asymptotes at odd multiple of $\frac{\pi}{2}$, that is, at $x = (2n+1)\frac{\pi}{2}$, where n is an integer.

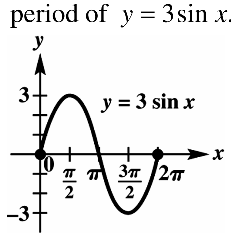
83. $y = 3\sin x$

Period: 2π and amplitude: $|3| = 3$

Divide the interval $[0, 2\pi]$ into four equal parts to get x-values that will yield minimum and maximum points and x-intercepts. Then make a table.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	−1	0
$3\sin x$	0	3	0	−3	0

This table gives five values for graphing one period of $y = 3\sin x$.



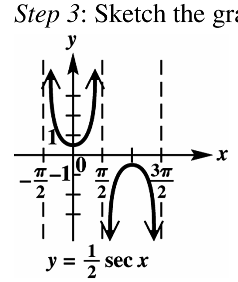
84. $y = \frac{1}{2}\sec x$

Step 1: Graph the corresponding reciprocal function $y = \frac{1}{2}\cos x$. The period is 2π and its amplitude is $\left|\frac{1}{2}\right| = \frac{1}{2}$. One period is in the interval $0 \le x \le 2\pi$. Dividing the interval into four equal parts gives the key points $\left(0, \frac{1}{2}\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\pi, -\frac{1}{2}\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(2\pi, \frac{1}{2}\right)$

Step 2: The vertical asymptotes of $y = \frac{1}{2}\sec x$ are at the x-intercepts of $y = \frac{1}{2}\cos x$, namely $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

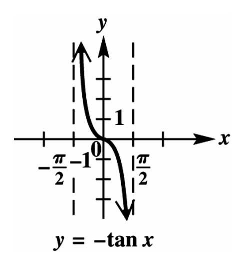
Continuing this pattern to the left, we also have a vertical asymptote at $x = -\frac{\pi}{2}$

Step 3: Sketch the graph



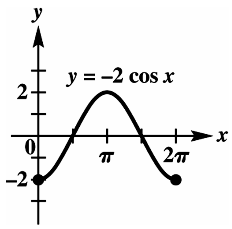
85. $y = -\tan x$

This is a reflection of the graph of $y = \tan x$ over the x-axis. The period is π and vertical asymptotes are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$



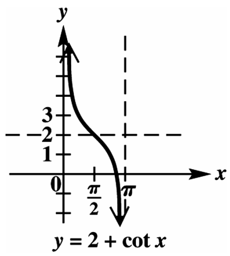
86. $y = -2\cos x$

This is a reflection of the graph of $y = \cos x$ over the x-axis. The amplitude is $|-2| = 2$, the period is 2π. Points on the graph are $(0, -2)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, 2)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, -2)$.



87. $y = 2 + \cot x$

This is the graph of $y = \cot x$ translated up 2 units. The period is π and the vertical asymptotes are $x = 0$ and $x = \pi$.



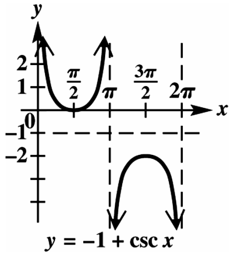
88. $y = -1 + \csc x$

This is the graph of $y = \csc x$ translated down 1 unit. The period is 2π and the vertical asymptotes are $x = 0$, $x = \pi$, and $x = 2\pi$.

Step 1: Graph the corresponding reciprocal function $y = -1 + \sin x$. The period is 2π, and its amplitude is $|1| = 1$. One period is in the interval $0 \le x \le 2\pi$. Dividing the interval into four equal parts gives the key points $(0, -1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, -2\right)$, $(2\pi, -1)$

Step 2: The vertical asymptotes of $y = -1 + \csc x$ are at the x-intercepts of $y = \sin x$, $x = 0$, $x = \pi$, and $x = 2\pi$.

Step 3: Sketch the graph.

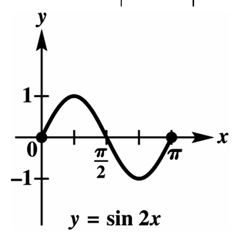


89. $y = \sin 2x$

Period: $\frac{2\pi}{2} = \pi$ and amplitude: $|1| = 1$

Divide the interval $[0, \pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2x$	0	1	0	–1	0



90. $y = \tan 3x$

Step 1: Find the period and locate the vertical asymptotes. The period of tangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{3}$. Tangent has asymptotes of the form $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$. The asymptotes for $y = \tan 3x$ are

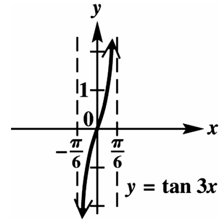
$3x = -\frac{\pi}{2} \Rightarrow x = -\frac{\pi}{6}$ and $3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$

Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $-\frac{\pi}{6}, -\frac{\pi}{12}, 0, \frac{\pi}{12}, \frac{\pi}{6}$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have $\left(-\frac{\pi}{12}, 1\right)$, $(0,0)$, $\left(\frac{\pi}{12}, 1\right)$

Step 5: Join the points with a smooth curve.

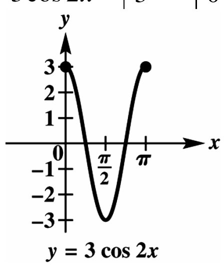


91. $y = 3\cos 2x$

Period: $\frac{2\pi}{2} = \pi$ and amplitude: $|3| = 3$

Divide the interval $[0, \pi]$ into four equal parts to get the x-values that will yield minimum and maximum points and x-intercepts. Then make a table.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 2x$	1	0	−1	0	1
$3\cos 2x$	3	0	−3	0	3



92. $y = \frac{1}{2}\cot 3x$

Step 1: Find the period and locate the vertical asymptotes. The period of cotangent is $\frac{\pi}{b}$, so the period for this function is $\frac{\pi}{3}$. Cotangent has asymptotes of the form $bx = 0$ and $bx = \pi$. The asymptotes for $y = \frac{1}{2}\cot 3x$ are $x = 0$ and $x = \frac{\pi}{3}$.

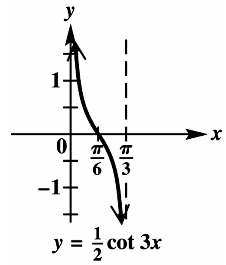
Step 2: Sketch the two vertical asymptotes found in Step 1.

Step 3: Divide the interval into four equal parts: $0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$

Step 4: Finding the first-quarter point, midpoint, and third-quarter point, we have: $\left(\frac{\pi}{12}, \frac{1}{2}\right)$, $\left(\frac{\pi}{6}, 0\right)$, $\left(\frac{\pi}{4}, -\frac{1}{2}\right)$

Step 5: Join the points with a smooth curve.

The graph is "shrunk" because $a = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$.



93. $y = \cos\left(x - \frac{\pi}{4}\right)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

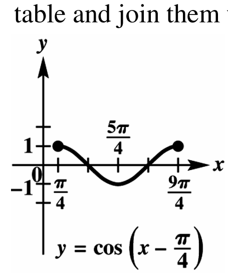
$$0 \le x - \frac{\pi}{4} \le 2\pi \Rightarrow 0 + \frac{\pi}{4} \le x \le 2\pi + \frac{\pi}{4} \Rightarrow$$
$$\frac{\pi}{4} \le x \le \frac{9\pi}{4}$$

Step 2: Divide the period into four equal parts to get the following x-values: $\frac{\pi}{4}$, π, $\frac{3\pi}{2}$, 2π, $\frac{9\pi}{4}$

Step 3: Evaluate the function for each of the five x-values.

x	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$
$x - \frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\left(x - \frac{\pi}{4}\right)$	1	0	−1	0	1

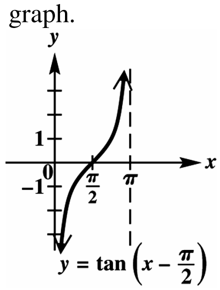
Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is 1. The period is 2π. There is no vertical translation. The phase shift is $\frac{\pi}{4}$ units to the right.

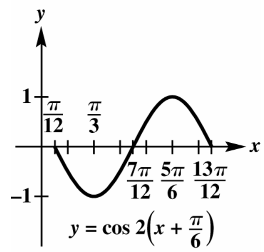
94. $y = \tan\left(x - \frac{\pi}{2}\right)$

Period: π
Vertical translation: none
Phase shift (horizontal translation): $\frac{\pi}{2}$ units to the right
Because the function is to be graphed over a one-period interval, locate two adjacent vertical asymptotes. Because asymptotes of the graph $y = \tan x$ occur at $-\frac{\pi}{2}$, and $\frac{\pi}{2}$, use the following equations to locate asymptotes: $x - \frac{\pi}{2} = -\frac{\pi}{2} \Rightarrow x = 0$ and $x - \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow x = \pi$
Divide the interval $(0, \pi)$ into four equal parts to obtain the following key x-values: first-quarter value: $\frac{\pi}{4}$; middle value: $\frac{\pi}{2}$; third-quarter value: $\frac{3\pi}{4}$. Evaluating the given function at these three key x-values gives the points $\left(\frac{\pi}{4}, -1\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{4}, 1\right)$
Connect these points with a smooth curve and continue to graph to approach the asymptote $x = 0$ and $x = \pi$ to complete one period of the graph.



95. $y = \sec\left(2x + \frac{\pi}{3}\right) = \sec 2\left(x + \frac{\pi}{6}\right)$

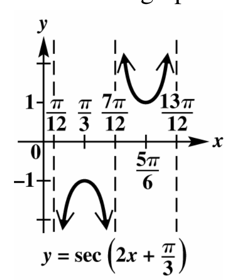
Step 1: Graph the corresponding reciprocal function $y = \cos 2\left(x + \frac{\pi}{6}\right)$.



The period is $\frac{2\pi}{2} = \pi$, and its amplitude is $|1| = 1$. One period is in the interval $\frac{\pi}{12} \le x \le \frac{13\pi}{12}$. Dividing the interval into four equal parts gives the key points $\left(\frac{\pi}{12}, 0\right), \left(\frac{\pi}{3}, -1\right)$, $\left(\frac{7\pi}{12}, 0\right), \left(\frac{5\pi}{6}, 1\right)$, and $\left(\frac{13\pi}{12}, 0\right)$.

Step 2: The vertical asymptotes of $y = \sec 2\left(x + \frac{\pi}{6}\right)$ are at the x-intercepts of $y = \cos 2\left(x + \frac{\pi}{6}\right)$, which are $x = \frac{\pi}{12}$, $x = \frac{7\pi}{12}$, and $x = \frac{13\pi}{12}$.

Step 3: Sketch the graph.



96. $y = \sin\left(3x + \frac{\pi}{2}\right) = \sin 3\left(x + \frac{\pi}{6}\right)$

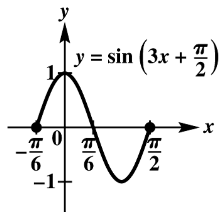
Step 1: Find the interval whose length is $\frac{2\pi}{b}$:
$0 \le 3\left(x + \frac{\pi}{6}\right) \le 2\pi \Rightarrow 0 \le x + \frac{\pi}{6} \le \frac{2\pi}{3} \Rightarrow -\frac{\pi}{6} \le x \le \frac{\pi}{2}$

Step 2: Divide the period into four equal parts to get the following x-values: $-\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

Step 3: Evaluate the function for each of the five x-values.

x	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$3\left(x+\frac{\pi}{6}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 3\left(x+\frac{\pi}{6}\right)$	0	1	0	-1	0

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



The amplitude is 1. The period is $\frac{2\pi}{3}$. There is no vertical translation. The phase shift is $\frac{\pi}{6}$ unit to the left.

97. $y = 1 + 2\cos 3x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

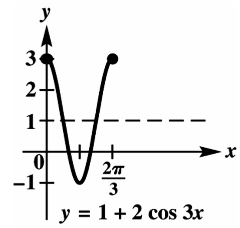
$0 \le 3x \le 2\pi \Rightarrow 0 \le x \le \frac{2\pi}{3}$

Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$

Step 3: Evaluate the function for each of the five x-values.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 3x$	1	0	-1	0	1
$2\cos 3x$	2	0	-2	0	2
$1+2\cos 3x$	3	1	-1	1	3

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve.



The period is $\frac{2\pi}{3}$. The amplitude is $|2|$, which is 2. The vertical translation is 1 unit up. There is no phase shift.

98. $y = -1 - 3\sin 2x$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

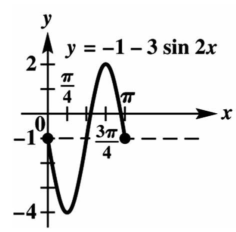
$0 \le 2x \le 2\pi \Rightarrow 0 \le x \le \pi$

Step 2: Divide the period into four equal parts to get the following x-values: 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π

Step 3: Evaluate the function for each of the five x-values.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2x$	0	1	0	-1	0
$-3\sin 2x$	0	-3	0	3	0
$-1-3\sin 2x$	-1	-4	-1	2	-1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. By graphing an additional period to the left, we obtain the following graph.



The amplitude is $|-3|$, which is 3. The period is $\frac{2\pi}{2} = \pi$. The vertical translation is 1 unit down. There is no phase shift.

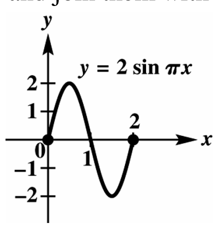
99. $y = 2\sin \pi x$

Period: $\frac{2\pi}{\pi} = 2$ and amplitude: $|1| = 1$

Divide the interval [0, 2] into four equal parts to get the *x*-values that will yield minimum and maximum points and *x*-intercepts. Then make a table.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \pi x$	0	1	0	−1	0
$2\sin \pi x$	0	2	0	−2	0

Steps 4 and 5: Plot the point found in the bale and join them with a sinusoidal curve.



100. $y = -\frac{1}{2}\cos(\pi x - \pi) = -\frac{1}{2}\cos \pi(x-1)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

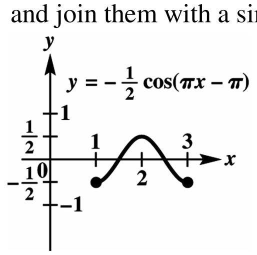
$0 \le \pi(x-1) \le 2\pi \Rightarrow 0 \le x-1 \le 2 \Rightarrow$
$1 \le x \le 3$

Step 2: Divide the period into four equal parts to get the *x*-values 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3.

Step 3: Evaluate the function for each of the five *x*-values.

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$\pi(x-1)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \pi(x-1)$	1	0	−1	0	1
$-\frac{1}{2}\cos \pi(x-1)$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$

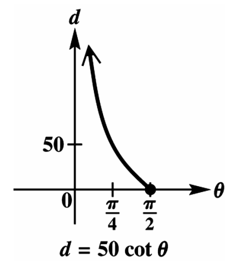
Steps 4 and 5: Plot the point found in the bale and join them with a sinusoidal curve.



101. **(a)** The shorter leg of the right triangle has length $h_2 - h_1$. Thus, we have

$$\cot\theta = \frac{d}{h_2 - h_1} \Rightarrow d = (h_2 - h_1)\cot\theta$$

(b) When $h_2 = 55$ and $h_1 = 5$,
$d = (55-5)\cot\theta = 50\cot\theta.$
The period is π, but the graph wanted is d for $0 < \theta < \frac{\pi}{2}$. The asymptote is the line $\theta = 0$. Also, when
$\theta = \frac{\pi}{4}, d = 50\cot\frac{\pi}{4} = 50(1) = 50.$



102. **(a)** Since $27 - 14.7 = 12.3$ and $14.7 - 2.4 = 12.3$, the time between high tides is 12.3 hr.

(b) Since $2.6 - 1.4 = 1.2$, the difference in water levels between high tide and low tide is 1.2 ft.

(c) Since $f(x) = .6\cos[.511(x-2.4)] + 2$, we have

$$\begin{aligned} f(10) &= .6\cos[.511(10-2.4)] + 2 \\ &= .6\cos(3.8836) + 2 \approx 1.56 \text{ ft} \end{aligned}$$

103. $t = 60 - 30\cos\frac{x\pi}{6}$

(a) For January, $x = 0$. Thus,

$$\begin{aligned} t &= 60 - 30\cos\frac{0\cdot\pi}{6} = 60 - 30\cos 0 \\ &= 60 - 30(1) = 60 - 30 = 30° \end{aligned}$$

(b) For April, $x = 3$. Thus,

$$\begin{aligned} t &= 60 - 30\cos\frac{3\pi}{6} = 60 - 30\cos\frac{\pi}{2} \\ &= 60 - 30(0) = 60 - 0 = 60° \end{aligned}$$

(c) For May, $x = 4$. Thus,

$$t = 60 - 30\cos\frac{4\pi}{6} = 60 - 30\cos\frac{2\pi}{3} = 60 - 30\left(-\frac{1}{2}\right) = 60 + 15 = 75°$$

(d) For June, $x = 5$. Thus,

$$t = 60 - 30\cos\frac{5\pi}{6} = 60 - 30\left(-\frac{\sqrt{3}}{2}\right) = 60 + 15\sqrt{3} \approx 86°$$

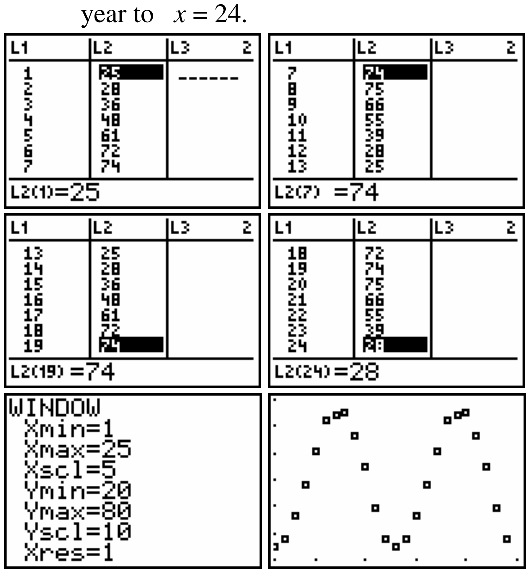
(e) For August, $x = 7$. Thus,

$$t = 60 - 30\cos\frac{7\pi}{6} = 60 - 30\left(-\frac{\sqrt{3}}{2}\right) = 60 + 15\sqrt{3} \approx 86°$$

(f) For October, $x = 9$. Thus,

$$t = 60 - 30\cos\frac{9\pi}{6} = 60 - 30\cos\frac{3\pi}{2} = 60 - 30(0) = 60 - 0 = 60°$$

104. (a) Let January correspond to $x = 1$, February to $x = 2$, … , and December of the second year to $x = 24$.



(b) The amplitude of the sine graph is approximately 25 since the average monthly high is 75, the average monthly low is 25, and $\frac{1}{2}(75 - 25) = \frac{1}{2}(50) = 25$. The period is 12 since the temperature cycles every twelve months.

Let $b = \frac{2\pi}{12} = \frac{\pi}{6}$. One way to determine the phase shift is to use the following technique. The minimum temperature occurs in January. Thus, when $x = 1$, $b(x - d)$ must equal $\left(-\frac{\pi}{2}\right) + 2\pi n$, where n is an integer, since the sine function is minimum at these values. Solving for d, we have

$$\frac{\pi}{6}(1 - d) = -\frac{\pi}{2} \Rightarrow 1 - d = \frac{6}{\pi}\left(-\frac{\pi}{2}\right) \Rightarrow$$
$$1 - d = -3 \Rightarrow -d = -4 \Rightarrow d = 4$$

This can be used as a first approximation. Let $f(x) = a \sin b(x - d) + c$. Since the amplitude is 25, let $a = 25$. The period is equal to 1 yr or 12 mo, so $b = \frac{\pi}{6}$. The average of the maximum and minimum temperatures is

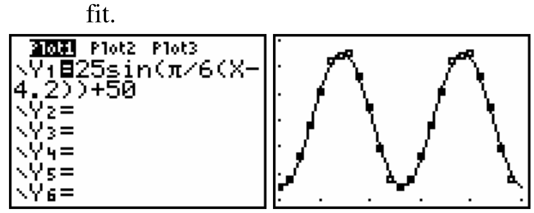
$$\frac{1}{2}(75 + 25) = \frac{1}{2}(100) = 50$$

Let the vertical translation be $c = 50$. Since the phase shift is approximately 4, it can be adjusted slightly to give a better visual fit. Try 4.2. Since the phase shift is 4.2, let $d = 4.2$. Thus,

$$f(x) = 25\sin\left[\frac{\pi}{6}(x - 4.2)\right] + 50.$$

(c) See part b.

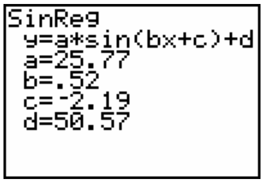
(d) Plotting the data with $f(x) = 25\sin\left[\frac{\pi}{6}(x - 4.2)\right] + 50$ on the same coordinate axes gives an excellent fit.



(e)

(*continued on next page*)

(*continue from page 617*)



From the sine regression we have
$y = 25.77\sin(.52x - 2.19) + 50.57$ or
$y = 25.77\sin\left[.52(x - 4.21)\right] + 50.57$

105. $P(x) = 7(1 - \cos 2\pi x)(x + 10) + 100e^{.2x}$

(a) January 1, base year $x = 0$
$$\begin{aligned} P(0) &= 7(1 - \cos 0)(10) + 100e^{0} \\ &= 7(1 - 1)(10) + 100(1) \\ &= 7(0)(10) + 100 = 0 + 100 = 100 \end{aligned}$$

(b) July 1, base year $x = .5$
$$\begin{aligned} P(.5) &= 7(1 - \cos \pi)(.5 + 10) + 100e^{.2(.5)} \\ &= 7\left[1 - (-1)\right](10.5) + 100e^{.1} \\ &= 7(2)(10.5) + 100e^{.1} \\ &= 147 + 100e^{.1} \approx 258 \end{aligned}$$

(c) January 1, following year $x = 1$
$$\begin{aligned} P(1) &= 7(1 - \cos 2\pi)(1 + 10) + 100e^{.2} \\ &= 7(1 - 1)(1 + 10) + 100e^{.2} \\ &= 7(0)(11) + 100e^{.2} = 0 + 100e^{.2} \\ &= 100e^{.2} \approx 122 \end{aligned}$$

(d) July 1, following year $x = 1.5$
$$\begin{aligned} P(1.5) &= 7(1 - \cos 3\pi)(1.5 + 10) + 100e^{.2(1.5)} \\ &= 7\left[1 - (-1)\right](11.5) + 100e^{.3} \\ &= 7(2)(11.5) + 100e^{.3} \\ &= 161 + +100e^{.3} \approx 296 \end{aligned}$$

106. **(a)** From the graph, one period is about 20 years.

(b) The population of hares fluctuates between a maximum of about 150,000 and a minimum of about 5000.

107. $s(t) = 4\sin \pi t$
$a = 4,\ \omega = \pi$
amplitude $= |a| = 4$; period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi} = 2$
frequency $= \dfrac{\omega}{2\pi} = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$

108. $s(t) = 3\cos 2t$
$a = 3,\ \omega = 2$
amplitude $= |a| = 3$
period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$
frequency $= \dfrac{\omega}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$

109. The frequency is the number of cycles in one unit of time.
$$\begin{aligned} s(1.5) &= 4\sin 1.5\pi = 4\sin\frac{3\pi}{2} = 4(-1) = -4 \\ s(2) &= 4\sin 2\pi = 4(0) = 0 \\ s(3.25) &= 4\sin 3.25\pi = 4\sin\frac{13\pi}{4} = 4\sin\frac{5\pi}{4} \\ &= 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} \end{aligned}$$

110. The period is the time to complete one cycle. The amplitude is the maximum distance (on either side) from the initial point.

Chapter 6 Test

1. $120° = 120\left(\dfrac{\pi}{180}\text{ radian}\right) = \dfrac{2\pi}{3}$ radians

2. $-45° = -45\left(\dfrac{\pi}{180}\text{ radian}\right) = -\dfrac{\pi}{4}$ radian

3. $5° = 5\left(\dfrac{\pi}{180}\text{ radian}\right) = \dfrac{\pi}{36} \approx .09$ radian

4. $\dfrac{3\pi}{4} = \dfrac{3\pi}{4}\left(\dfrac{180°}{\pi}\right) = 135°$

5. $-\dfrac{7\pi}{6} = -\dfrac{7\pi}{6}\left(\dfrac{180°}{\pi}\right) = -210°$

6. $4 = 4\left(\dfrac{180°}{\pi}\right) \approx 229.18°$

7. $r = 150$ cm, $s = 200$ cm

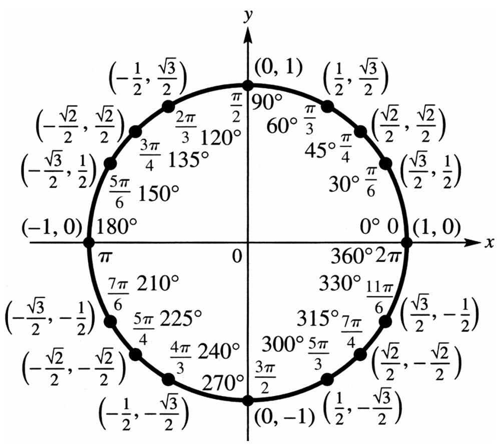
(a) $s = r\theta \Rightarrow 200 = 150\theta \Rightarrow \theta = \dfrac{200}{150} = \dfrac{4}{3}$

(b) $A = \dfrac{1}{2}r^2\theta$
$$\begin{aligned} A &= \frac{1}{2}(150)^2\left(\frac{4}{3}\right) = \frac{1}{2}(22{,}500)\left(\frac{4}{3}\right) \\ &= 15{,}000\text{ cm}^2 \end{aligned}$$

8. $r = \frac{1}{2}$ in., $s = 1$ in.

$s = r\theta \Rightarrow 1 = \frac{1}{2}\theta \Rightarrow \theta = 2$ radians

For Exercises 9–14, refer to Figure 12 on page 572 of the text.



9. $\sin \frac{3\pi}{4}$

Since $\frac{3\pi}{4}$ is in quadrant II, the reference angle is $\pi - \frac{3\pi}{4} = \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4}$. In quadrant II, the sine is positive. Thus, $\sin \frac{3\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Converting $\frac{3\pi}{4}$ to degrees, we have $\frac{3\pi}{4} = \frac{3}{4}(180°) = 135°$. The reference angle is $180° - 135° = 45°$. Thus, $\sin \frac{3\pi}{4} = \sin 135° = \sin 45° = \frac{\sqrt{2}}{2}$.

10. $\cos\left(-\frac{7\pi}{6}\right)$

$-\frac{7\pi}{6}$ is coterminal with $-\frac{7\pi}{6} + 2\pi = -\frac{7\pi}{6} + \frac{12\pi}{6} = \frac{5\pi}{6}$. Since $\frac{5\pi}{6}$ is in quadrant II, the reference angle is $\pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}$. In quadrant II, the cosine is negative. Thus,

$\cos\left(-\frac{7\pi}{6}\right) = \cos \frac{5\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$.

Converting $\frac{5\pi}{6}$ to degrees, we have $\frac{5\pi}{6} = \frac{5}{6}(180°) = 150°$. The reference angle is $180° - 150° = 30°$. Thus,

$$\cos\left(-\frac{7\pi}{6}\right) = \cos \frac{5\pi}{6} = \cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$$

11. $\tan \frac{3\pi}{2} = \tan 270°$ is undefined.

12. $\sec \frac{8\pi}{3}$

$\frac{8\pi}{3}$ is coterminal with $\frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$. Since $\frac{2\pi}{3}$ is in quadrant II, the reference angles is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$. In quadrant II, the secant is negative. Thus,

$\sec \frac{8\pi}{3} = \sec \frac{2\pi}{3} = -\sec \frac{\pi}{3} = -2$.

Converting $\frac{2\pi}{3}$ to degrees, we have $\frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{180°}{\pi} = 120°$. The reference angle is 180º − 120º = 60º. Thus,

$\sec \frac{8\pi}{3} = \sec \frac{2\pi}{3} = \sec 120° = -\sec 60° = -2$.

13. $\tan \pi = \tan 180° = 0$

14. $\cos \frac{3\pi}{2} = \cos 270° = 0$

15. $s = \frac{7\pi}{6}$

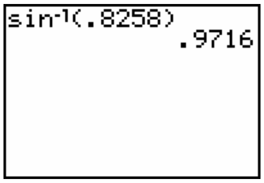
Since $\frac{7\pi}{6}$ is in quadrant III, the reference angle is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$. In quadrant III, the sine and cosine are negative.

$\sin \frac{7\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}$

$\cos \frac{7\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

$\tan \frac{7\pi}{6} = \sin \frac{\pi}{6} = \frac{\sqrt{3}}{3}$

16. **(a)** $\sin s = .8258 \Rightarrow s \approx .9716$



(b) Since $\cos\frac{\pi}{3} = \frac{1}{2}$ and $0 \le \frac{\pi}{3} \le \frac{\pi}{2}$,

$s = \frac{\pi}{3}.$

17. **(a)** The speed of ray *OP* is $\omega = \frac{\pi}{12}$ radian per sec. Since $\omega = \frac{\theta}{t}$, then in 8 sec,

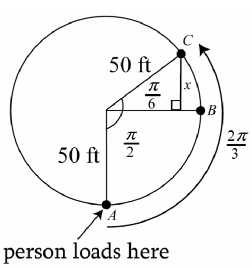
$\omega = \frac{\theta}{t} \Rightarrow \frac{\pi}{12} = \frac{\theta}{8} \Rightarrow \theta = \frac{8\pi}{12} = \frac{2\pi}{3}$
radians

(b) From part (a), *P* generates an angle of $\frac{2\pi}{3}$ radians in 8 sec. The distance traveled by *P* along the circle is

$s = r\theta \Rightarrow s = 60\left(\frac{2\pi}{3}\right) = 40\pi$ cm

(c) $v = \frac{s}{t} \Rightarrow \frac{40\pi}{8} = 5\pi$ cm per sec.

18. **(a)**



Suppose the person takes a seat at point *A*. When the person travels $\frac{\pi}{2}$ radians, the person is 50 ft above the ground. When the person travels $\frac{\pi}{6}$ more radians, we can let *x* be the additional vertical distance traveled:

$\sin\frac{\pi}{6} = \frac{x}{50} \Rightarrow x = 50\sin\frac{\pi}{6} = 50\left(\frac{1}{2}\right) = 25$

Thus, the person traveled an additional 25 ft above the ground, for a total of 75 ft above the ground.

(b) The Ferris wheel goes $\frac{2\pi}{3}$ radians per 30 sec or $\frac{2\pi}{90} = \frac{\pi}{45}$ radians per second.

19. **(a)** $y = \sec x$ **(b)** $y = \sin x$

(c) $y = \cos x$ **(d)** $y = \tan x$

(e) $y = \csc x$ **(e)** $y = \cot x$

20. $y = 3 - 6\sin\left(2x + \frac{\pi}{2}\right) = 3 - 6\sin 2\left(x + \frac{\pi}{4}\right)$
$= 3 - 6\sin 2\left[x - \left(-\frac{\pi}{4}\right)\right]$

(a) The period is $\frac{2\pi}{2} = \pi$.

(b) The amplitude is 6.

(c) The range is $[-3, 9]$.

(d) The *y*-intercept occurs when $x = 0$.

$$\begin{aligned} -6\sin\left(2\cdot 0 + \frac{\pi}{2}\right) + 3 &= -6\sin\left(0 + \frac{\pi}{2}\right) + 3 \\ &= -6\sin\left(\frac{\pi}{2}\right) + 3 \\ &= -6(1) + 3 \\ &= -6 + 3 = -3 \end{aligned}$$

(e) The phase shift is $\frac{\pi}{4}$ unit to the left $\left(\text{that is, } -\frac{\pi}{4}\right)$

21. $y = \sin(2x + \pi) = \sin 2\left(x + \frac{\pi}{2}\right)$
$= \sin 2\left[x - \left(-\frac{\pi}{2}\right)\right]$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

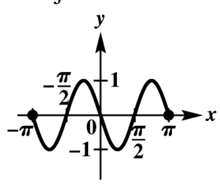
$0 \le 2\left(x + \frac{\pi}{2}\right) \le 2\pi \Rightarrow 0 \le x + \frac{\pi}{2} \le \frac{2\pi}{2} \Rightarrow$
$0 \le x + \frac{\pi}{2} \le \pi \Rightarrow -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Step 2:Divide the period into four equal parts to get the following *x*-values: $-\frac{\pi}{2}$, $-\frac{\pi}{4}$,0, $\frac{\pi}{4}$, $\frac{\pi}{2}$

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x+\frac{\pi}{2}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2\left(x+\frac{\pi}{2}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 2\left(x+\frac{\pi}{2}\right)$	0	1	0	-1	0

Steps 4 and 5:Plot the points found in the table and join them with a sinusoidal curve.

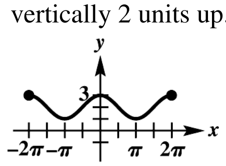


$y = \sin(2x + \pi)$

The period is π . There is no vertical translation. The phase shift is $\frac{\pi}{2}$ unit to the right.

22. $y = 2 + \cos x$

This is the graph of $y = \cos x$ translated vertically 2 units up.



$y = 2 + \cos x$

23. $y = -1 + 2\sin(x+\pi)$

Step 1: Find the interval whose length is $\frac{2\pi}{b}$.

$0 \le x+\pi \le 2\pi \Rightarrow\Rightarrow -\pi \le x \le \pi$

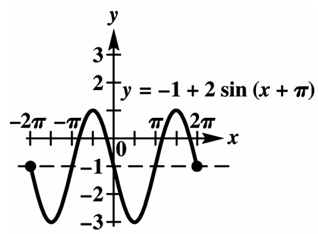
Step 2: Divide the period into four equal parts to get the following x-values: $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, π

Step 3: Evaluate the function for each of the five x-values

x	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$x+\pi$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x+\pi)$	0	1	0	-1	0
$2\sin(x+\pi)$	0	2	0	-2	0

x	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$-1+2\sin(x+\pi)$	-1	1	-1	-3	-1

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. Repeat this cycle for the interval $[-\pi, 0]$.



The amplitude is $|2|$, which is 2. The period is 2π. The vertical translation is 1 unit down. The phase shift is π units to the left.

24. $y = \tan\left(x - \frac{\pi}{2}\right)$

Period: π
Vertical translation: none
Phase shift (horizontal translation): $\frac{\pi}{2}$ units to the right
Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Because asymptotes of the graph $y = \tan x$ occur at $-\frac{\pi}{2}$, and $\frac{\pi}{2}$, use the following equations to locate asymptotes: $x - \frac{\pi}{2} = -\frac{\pi}{2} \Rightarrow x = 0$ and $x - \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow x = \pi.$ Divide the interval $(0, \pi)$ into four equal parts to obtain the key x-values: first-quarter value: $\frac{\pi}{4}$; middle value: $\frac{\pi}{2}$; third-quarter value: $\frac{3\pi}{4}$
Evaluating the given function at these three key x-values gives the points: $\left(\frac{\pi}{4}, -1\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{4}, 1\right)$. Connect these points with a smooth curve and continue to graph to approach the asymptote $x = 0$ and $x = \pi$ to complete one period of the graph. Repeat this cycle for the interval $[-\pi, 0]$.

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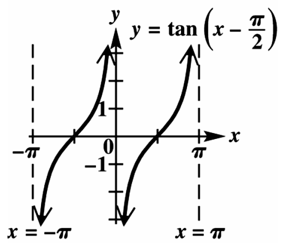
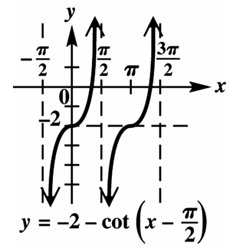


25. $y = -2 - \cot\left(x - \frac{\pi}{2}\right)$

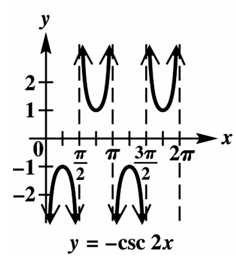
Period: $\frac{\pi}{b} = \frac{\pi}{1} = \pi$
Vertical translation: 2 units down
Phase shift (horizontal translation): $\frac{\pi}{2}$ units to the right
Because the function is to be graphed over a two-period interval, locate three adjacent vertical asymptotes. Because asymptotes of the graph $y = \cot x$ occur at multiples of π, use the following equations to locate asymptotes: $x - \frac{\pi}{2} = -\pi$, $x - \frac{\pi}{2} = 0$, and $x - \frac{\pi}{2} = \pi$. Solve each of these equations:

$x - \frac{\pi}{2} = -\pi \Rightarrow x = -\frac{\pi}{2}$; $x - \frac{\pi}{2} = 0 \Rightarrow x = \frac{\pi}{2}$;
$x - \frac{\pi}{2} = \pi \Rightarrow x = \frac{3\pi}{2}$

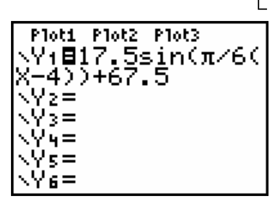
Divide the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ into four equal parts to obtain the following key *x*-values.
first-quarter value: $-\frac{\pi}{4}$ middle value: 0 ;
third-quarter value: $\frac{\pi}{4}$ Evaluating the given function at these three key *x*-values gives the points: $\left(-\frac{\pi}{4}, -3\right)$, $(0, -2)$, $\left(\frac{\pi}{4}, -1\right)$
Connect these points with a smooth curve and continue to graph to approach the asymptote $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ to complete one period of the graph. Sketch the identical curve between the asymptotes $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ to complete a second period of the graph.

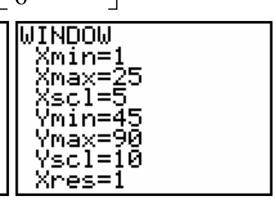


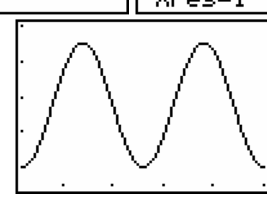
26. $y = -\csc 2x$
Step 1: Graph the corresponding reciprocal function $y = -\sin 2x$ The period is $\frac{2\pi}{2} = \pi$ and its amplitude is $|-1| = 1$. One period is in the interval $0 \le x \le \pi$. Dividing the interval into four equal parts gives us the key points:
$(0,0)$, $\left(\frac{\pi}{2}, -1\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{4}, 1\right)$, $(\pi, 0)$
Step 2: The vertical asymptotes of $y = -\csc 2x$ are at the x-intercepts of $y = -\sin 2x$, which are $x = 0, x = \frac{\pi}{2}$, and $x = \pi$. Continuing this pattern to the right, we also have a vertical asymptotes of $x = \frac{3\pi}{2}$ and $x = 2\pi$.
Step 3: Sketch the graph.



27. **(a)** $f(x) = 17.5 \sin\left[\frac{\pi}{6}(x - 4)\right] + 67.5$







(b) Amplitude: 17.5

Period: $\frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \frac{6}{\pi} = 12;$

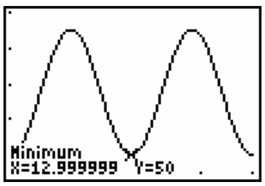
Phase shift: 4 units to the right

Vertical translation: 67.5 units up

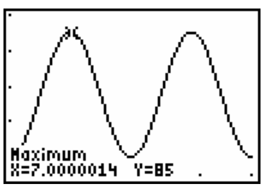
(c) For the month of December, $x = 12$.

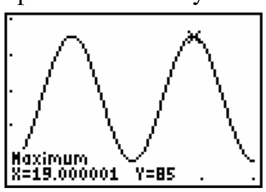
$$f(12) = 17.5\sin\left[\frac{\pi}{6}(12-4)\right] + 67.5$$
$$= 17.5\sin\left(\frac{4\pi}{3}\right) + 67.5$$
$$= 17.5\left(-\frac{\sqrt{3}}{2}\right) + 67.5 \approx 52°$$

(d) A minimum of 50 occurring at $x = 13 = 12 + 1$ implies 50°F in January.



A maximum of 85 occurring at $x = 7$ and $x = 19 = 12 + 7$ implies 85°F in July.





(e) Approximately 67.5° would be an average yearly temperature. This is the vertical translation.

28. $s(t) = -4\cos 8\pi t$, $a = |-4| = 4$, $\omega = 8\pi$

(a) maximum height = amplitude $= a = |-4| = 4$ in.

(b) $s(t) = -4\cos 8\pi t = 4 \Rightarrow \cos 8\pi t = -1 \Rightarrow$

$$8\pi t = \pi \Rightarrow t = \frac{1}{8}$$

The weight first reaches its maximum height after $\frac{1}{8}$ sec.

(c) frequency $= \frac{\omega}{2\pi} = \frac{8\pi}{2\pi} = 4$ cycles per sec;

period $= \frac{2\pi}{\omega} = \frac{2\pi}{8\pi} = \frac{1}{4}$ sec

Chapter 6: Quantitative Reasoning

1. $\Delta RQP \sim \Delta RMO$ because $m\angle R = m\angle R$ and $m\angle Q = m\angle M$.

2. $\frac{OM}{PR} = \frac{RM}{OR} \Rightarrow \frac{r}{c} = \frac{\frac{c}{2}}{b} \Rightarrow \frac{r}{c} = \frac{c}{2b} \Rightarrow r = \frac{c^2}{2b}$

3. From the Pythagorean theorem, $a^2 + b^2 = c^2$.

Thus, $r = \frac{a^2 + b^2}{2b}$

4. In the diagram, a = 1.4 in. and b = .2 in. Since $r = \frac{a^2 + b^2}{2b}$, we have

$$r = \frac{1.4^2 + .2^2}{2(.2)} = \frac{1.96 + .04}{.4} = \frac{2.00}{.4} = 5$$

Thus, the radius is 5 inches.

Chapter 7

TRIGONOMETRIC IDENTITIES AND EQUATIONS

Section 7.1: Fundamental Identities

1. By a negative-angle identity, $\tan(-\theta) = -\tan\theta$. Thus, if $\tan\theta = 2.6$, then $\tan(-\theta) = \underline{-2.6}$.

2. By a negative angle identity, $\cos(-\theta) = \cos\theta$. Thus, if $\cos\theta = -.65$, then $\cos(-\theta) = \underline{-.65}$.

3. By a reciprocal identity, $\cot\theta = \dfrac{1}{\tan\theta}$. Thus, if $\tan\theta = 1.6$, then $\cot\theta = \underline{.625}$.

4. By a quotient identity, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. By a negative angle identity $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. Thus, if $\cos x = .8$ and $\sin x = .6$, then

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = \frac{-.6}{.8} = \underline{-.75}.$$

5. By a negative angle identity $\sin(-\theta) = -\sin\theta$. Thus, $-\sin(-\theta) = -(-\sin\theta) = \sin\theta$. So, if $\sin\theta = \dfrac{2}{3}$, then $-\sin(-\theta) = \underline{\dfrac{2}{3}}$.

6. By a negative angle identity $\cos(-\theta) = \cos\theta$. Thus $-\cos(-\theta) = -\cos\theta$. So, if $\cos\theta = -\dfrac{1}{5}$, then $-\cos(-\theta) = \underline{\dfrac{1}{5}}$.

7. $\cos\theta = \dfrac{3}{4}$, θ is in quadrant I.

An identity that relates sine and cosine is $\sin^2 s + \cos^2 s = 1$.

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta + \left(\frac{3}{4}\right)^2 = 1 \Rightarrow$$

$$\sin^2\theta = 1 - \frac{9}{16} = \frac{7}{16} \Rightarrow \sin\theta = \pm\frac{\sqrt{7}}{4}$$

Since θ is in quadrant I, $\sin\theta = \dfrac{\sqrt{7}}{4}$.

8. $\cot\theta = -\dfrac{1}{3}$, θ in quadrant IV

Use the identity $1 + \cot^2\theta = \csc^2\theta$ since $\sin\theta = \dfrac{1}{\csc\theta}$.

$$1 + \cot^2\theta = \csc^2\theta \Rightarrow 1 + \left(-\frac{1}{3}\right)^2 = \csc^2\theta \Rightarrow$$

$$1 + \frac{1}{9} = \csc^2\theta \Rightarrow \frac{10}{9} = \csc^2\theta \Rightarrow$$

$$\csc\theta = \pm\frac{\sqrt{10}}{3}$$

Since θ is in quadrant IV, csc $\theta < 0$, so $\csc\theta = -\dfrac{\sqrt{10}}{3}$. Thus,

$$\sin\theta = \frac{1}{\csc\theta} = -\frac{3}{\sqrt{10}} = -\frac{3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

9. $\cos(-\theta) = \dfrac{\sqrt{5}}{5}$, $\tan\theta < 0$

Since $\cos(-\theta) = \dfrac{\sqrt{5}}{5}$, we have $\cos\theta = \dfrac{\sqrt{5}}{5}$ by a negative angle identity. An identity that relates sine and cosine is $\sin^2\theta + \cos^2\theta = 1$.

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta + \left(\frac{\sqrt{5}}{5}\right)^2 = 1 \Rightarrow$$

$$\sin^2\theta + \frac{5}{25} = 1 \Rightarrow \sin^2\theta + \frac{1}{5} = 1$$

$$\sin^2\theta = 1 - \frac{1}{5} = \frac{4}{5} \Rightarrow$$

$$\sin\theta = \pm\frac{2}{\sqrt{5}} = \pm\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$$

Since tan $\theta < 0$ and cos $\theta > 0$, θ is in quadrant IV, so sin $\theta < 0$. Thus, $\sin\theta = -\dfrac{2\sqrt{5}}{5}$.

10. $\tan\theta = -\dfrac{\sqrt{7}}{2}, \sec\theta > 0$

$$\tan^2\theta + 1 = \sec^2\theta \Rightarrow \left(-\frac{\sqrt{7}}{2}\right)^2 + 1 = \sec^2\theta \Rightarrow$$

$$\frac{11}{4} = \sec^2\theta \Rightarrow \sec\theta = \pm\frac{\sqrt{11}}{2}$$

Since $\sec\theta > 0$, $\sec\theta = \frac{\sqrt{11}}{2}$. Also,

since $\cos\theta = \frac{1}{\sec\theta}$, $\cos\theta = \frac{1}{\frac{\sqrt{11}}{2}} = \frac{2}{\sqrt{11}}$.

Now, use the identity $\sin^2\theta + \cos^2\theta = 1$:

$$\sin^2\theta + \left(\frac{2}{\sqrt{11}}\right)^2 = 1 \Rightarrow \sin^2\theta + \frac{4}{11} = 1 \Rightarrow$$
$$\sin^2\theta = 1 - \frac{4}{11} = \frac{7}{11} \Rightarrow$$
$$\sin\theta = \pm\sqrt{\frac{7}{11}} = \pm\frac{\sqrt{7}}{\sqrt{11}}$$
$$= \pm\frac{\sqrt{7}}{\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \pm\frac{\sqrt{77}}{11}$$

Since $\tan\theta < 0$ and $\sec\theta > 0$, θ is in quadrant IV, so $\sin\theta < 0$. Thus,

$$\sin\theta = -\sqrt{\frac{7}{11}} = -\frac{\sqrt{77}}{11}.$$

11. $\sec\theta = \frac{11}{4}, \tan\theta < 0$

Since $\cos\theta = \frac{1}{\sec\theta}, \cos\theta = \frac{1}{\frac{11}{4}} = \frac{4}{11}$. Use the identity $\sin^2\theta + \cos^2\theta = 1$, to obtain

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta + \left(\frac{4}{11}\right)^2 = 1 \Rightarrow$$
$$\sin^2\theta + \frac{16}{121} = 1 \Rightarrow \sin^2\theta = 1 - \frac{16}{121} \Rightarrow$$
$$\sin^2\theta = \frac{105}{121} \Rightarrow \sin\theta = \pm\frac{\sqrt{105}}{11}$$

Since $\tan\theta < 0$ and $\sec\theta > 0$, θ is in quadrant IV, so $\sin\theta < 0$. Thus,

$$\sin\theta = -\frac{\sqrt{105}}{11}.$$

12. $\csc\theta = -\frac{8}{5}$

$\sin\theta = \frac{1}{\csc\theta}$, so $\sin\theta = \frac{1}{-\frac{8}{5}} = -\frac{5}{8}$.

13. The quadrants are given so that one can determine which sign $(+ \text{ or } -)$ $\sin\theta$ will take.

Since $\sin\theta = \frac{1}{\csc\theta}$, the sign of $\sin\theta$ will be the same as $\csc\theta$.

14. The range of cosine is [−1, 1], thus there is no number or angle whose cosine is 3.

15. $\sin(-x) = \underline{-\sin x}$

16. Since $f(-x) = \sin(-x) = -\sin x = -f(x)$, $f(x) = \sin x$ is odd.

17. $\cos(-x) = \underline{\cos x}$

18. Since $f(-x) = \cos(-x) = \cos x = f(x)$, $f(x) = \cos x$ is even

19. $\tan(-x) = \underline{-\tan x}$

20. Since $f(-x) = \tan(-x) = -\tan x = -f(x)$, $f(x) = \tan x$ is odd.

21. This is the graph of $f(x) = \sec x$. It is symmetric about the y-axis. Moreover, since

$$f(-x) = \sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x = f(x),$$
$$f(-x) = f(x).$$

22. This is the graph of $f(x) = \csc x$. It is symmetric about the origin. Moreover, since

$$f(-x) = \csc(-x) = \frac{1}{\sin(-x)} = \frac{1}{-\sin x}$$
$$= -\csc x = -f(x),$$
$$f(-x) = -f(x).$$

23. This is the graph of $f(x) = \cot x$. It is symmetric about the origin. Moreover, since

$$f(-x) = \cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x}$$
$$= -\frac{\cos x}{\sin x} = -\cot x = -f(x)$$
$$f(-x) = -f(x).$$

24. This is the graph of $f(x) = \tan x$. It is symmetric about the origin. Moreover, since

$$f(-x) = \tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{\sin x}{-\cos x}$$
$$= -\frac{\sin x}{\cos x} = -\tan x = -f(x)$$
$$f(-x) = -f(x).$$

25. $\sin\theta = \frac{2}{3}, \theta$ in quadrant II

Since θ is in quadrant II, the sine and cosecant function values are positive. The cosine, tangent, cotangent, and secant function values are negative.

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow$

$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9} \Rightarrow$

$\cos\theta = -\frac{\sqrt{5}}{3}$, since $\cos\theta < 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2}{3}}{-\frac{\sqrt{5}}{3}} = -\frac{2}{\sqrt{5}}$

$= -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{2}{\sqrt{5}}} = -\frac{\sqrt{5}}{2}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{\sqrt{5}}{3}} = -\frac{3}{\sqrt{5}}$

$= -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

26. $\cos\theta = \frac{1}{5}, \theta$ in quadrant I

Since θ is in quadrant I, all the function values are positive.

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow$

$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{1}{5}\right)^2 = 1 - \frac{1}{25} = \frac{24}{25}$

$\sin\theta = \frac{\sqrt{24}}{25} = \frac{2\sqrt{6}}{5}$, since $\sin\theta > 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2\sqrt{6}}{5}}{\frac{1}{5}} = 2\sqrt{6}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{2\sqrt{6}}$

$= \frac{1}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{12}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{5}} = 5$

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{2\sqrt{6}}{5}} = \frac{5}{2\sqrt{6}}$

$= \frac{5}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{12}$

27. $\tan\theta = -\frac{1}{4}, \theta$ in quadrant IV

Since θ is in quadrant IV, the cosine and secant function values are positive. The sine, tangent, cotangent, and cosecant function values are negative.

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{1}{4}} = -4$

$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(-\frac{1}{4}\right)^2$

$= 1 + \frac{1}{16} = \frac{17}{16} \Rightarrow$

$\sec\theta = \frac{\sqrt{17}}{4}$, since $\sec\theta > 0$

$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{\sqrt{17}}{4}} = \frac{4}{\sqrt{17}}$

$= \frac{4}{\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow$

$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{4}{\sqrt{17}}\right)^2$

$\sin^2\theta = 1 - \frac{16}{17} = \frac{1}{17} \Rightarrow$

$\sin\theta = -\frac{1}{\sqrt{17}} = -\frac{1}{\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$,

since $\sin\theta < 0$

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{1}{\sqrt{17}}} = -\sqrt{17}$

28. $\csc\theta = -\frac{5}{2}, \theta$ in quadrant III

Since θ is in quadrant III, the tangent, and cotangent function values are positive. The sine, cosine, cosecant, and secant function values are negative.

$\sin\theta = \frac{1}{\csc\theta} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}$

$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(-\frac{2}{5}\right)^2 = 1 - \frac{4}{25} = \frac{21}{25} \Rightarrow$

$\cos\theta = -\frac{\sqrt{21}}{5}$, since $\cos\theta < 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{2}{5}}{-\frac{\sqrt{21}}{5}} = \frac{2}{\sqrt{21}}$

$= \frac{2}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{2}{\sqrt{21}}} = \frac{\sqrt{21}}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{\sqrt{21}}{5}} = -\frac{5}{\sqrt{21}}$$

$$= -\frac{5}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = -\frac{5\sqrt{21}}{21}$$

29. $\cot\theta = \frac{4}{3}, \sin\theta > 0$

Since $\cot\theta > 0$ and $\sin\theta > 0, \theta$ is in quadrant I, so all the function values are positive.

$$\tan = \frac{1}{\cot\theta} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow$$

$\sec\theta = \frac{5}{4}$, since $\sec\theta > 0$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow$$

$\sin\theta = \frac{3}{5}$, since $\sin\theta > 0$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

30. $\sin\theta = -\frac{4}{5}, \cos\theta < 0$

Since $\sin\theta < 0$ and $\cos\theta < 0$, θ is in quadrant III. Since θ is in quadrant III, the tangent, and cotangent function values are positive. The cosecant and secant function values are negative.

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(-\frac{4}{5}\right)^2$$

$$= 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow$$

$\cos\theta = -\frac{3}{5}$, since $\cos\theta < 0$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{3}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{4}{5}} = -\frac{5}{4}$$

31. $\sec\theta = \frac{4}{3}, \sin\theta < 0$

Since $\sec\theta > 0$ and $\sin\theta < 0, \theta$ is in quadrant IV. Since θ is in quadrant IV, the cosine function value is positive. The tangent, cotangent, and cosecant function values are negative.

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16} \Rightarrow$$

$\sin\theta = -\frac{\sqrt{7}}{4}$, since $\sin\theta < 0$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{7}}{4}}{\frac{3}{4}} = -\frac{\sqrt{7}}{3}$$

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{1}{-\frac{\sqrt{7}}{3}} = -\frac{3}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = -\frac{3\sqrt{7}}{7}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{\sqrt{7}}{4}} = -\frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = -\frac{4\sqrt{7}}{7}$$

32. $\cos\theta = -\frac{1}{4}, \sin\theta > 0$

Since $\cos\theta < 0$ and $\sin\theta > 0$, θ is in quadrant II. Since θ is in quadrant II, the cosecant function value is positive. The tangent, cotangent, and secant function values are negative.

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(-\frac{1}{4}\right)^2$$

$$= 1 - \frac{1}{16} = \frac{15}{16} \Rightarrow \sin\theta = \frac{\sqrt{15}}{4},$$

since $\sin\theta > 0$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{15}}{4}}{-\frac{1}{4}} = -\sqrt{15}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\sqrt{15}} = -\frac{1}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = -\frac{\sqrt{15}}{15}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{1}{4}} = -4$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{15}}{4}} = \frac{4}{\sqrt{15}}$$

$$= \frac{4}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{4\sqrt{15}}{15}$$

33. Since $\frac{\cos x}{\sin x} = \cot x$, choose expression B.

34. Since $\tan x = \dfrac{\sin x}{\cos x}$, choose expression D.

35. Since cos(–x) = cos x, choose expression E.

36. Since $\tan^2 x + 1 = \sec^2 x$, choose expression C.

37. Since $1 = \sin^2 x + \cos^2 x$, choose expression A.

38. $$-\tan x \cos x = -\frac{\sin x}{\cos x}\cdot \cos x = -\sin x = \sin(-x)$$
Choose expression C.

39. Since $\sec^2 x - 1 = \tan^2 x = \dfrac{\sin^2 x}{\cos^2 x}$, choose expression A.

40. Since $\dfrac{\sec x}{\csc x} = \dfrac{\frac{1}{\cos x}}{\frac{1}{\sin x}} = \dfrac{\sin x}{\cos x} = \tan x$, choose expression E.

41. Since $1 + \sin^2 x = \left(\csc^2 x - \cot^2 x\right) + \sin^2 x$, choose expression D.

42. Since $\cos^2 x = \dfrac{1}{\sec^2 x}$, choose expression B.

43. It is incorrect to state $1 + \cot^2 = \csc^2$. Cotangent and cosecant are functions of some variable such as $\theta, x, \text{or } t$. An acceptable statement would be $1 + \cot^2 \theta = \csc^2 \theta$.

44. In general, it is false that $\sqrt{x^2 + y^2} = x + y$. Stating $\sin^2 \theta + \cos^2 \theta = 1$ implies $\sin \theta + \cos \theta = 1$ is a false statement.

45. Find $\sin \theta$ if $\cos \theta = \dfrac{x}{x+1}$.

$\sin^2 \theta + \cos^2 \theta = 1$ and $\cos \theta = \dfrac{x}{x+1}$, so

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \left(\frac{x}{x+1}\right)^2 = 1 - \frac{x^2}{(x+1)^2} = \frac{(x+1)^2 - x^2}{(x+1)^2} = \frac{x^2 + 2x + 1 - x^2}{(x+1)^2} = \frac{2x+1}{(x+1)^2}$$

Thus, $\sin \theta = \dfrac{\pm\sqrt{2x+1}}{x+1}$.

46. Find $\tan \theta$ if $\sec \theta = \dfrac{p+4}{p}$.

$\tan^2 \theta + 1 = \sec^2 \theta$ and $\sec \theta = \dfrac{p+4}{p}$, so

$$\tan^2 \theta = \sec^2 \theta - 1 = \frac{(p+4)^2}{p^2} - 1 = \frac{p^2 + 8p + 16}{p^2} - \frac{p^2}{p^2} = \frac{8p+16}{p^2} = \frac{4(2p+4)}{p^2}$$

Thus, $\tan \theta = \dfrac{\pm 2\sqrt{2p+4}}{p}$.

47. $\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x \Rightarrow$
$\sin x = \pm\sqrt{1 - \cos^2 x}$

48. $\cot^2 x + 1 = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1 \Rightarrow$
$$\cot x = \pm\sqrt{\csc^2 x - 1} = \pm\sqrt{\frac{1}{\sin^2 x} - 1} = \frac{\pm\sqrt{1 - \sin^2 x}}{\sin x}$$

49. $\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1 \Rightarrow$
$\tan x = \pm\sqrt{\sec^2 x - 1}$

50. $\cot^2 x + 1 = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1 \Rightarrow$
$\cot x = \pm\sqrt{\csc^2 x - 1}$

51. $\csc x = \dfrac{1}{\sin x} \Rightarrow$
$$\csc x = \frac{1}{\pm\sqrt{1 - \cos^2 x}} = \frac{\pm 1}{\sqrt{1 - \cos^2 x}} \cdot \frac{\sqrt{1 - \cos^2 x}}{\sqrt{1 - \cos^2 x}} = \frac{\pm\sqrt{1 - \cos^2 x}}{1 - \cos^2 x}$$

52. $\sec x = \dfrac{1}{\cos x} \Rightarrow$
$$\sec x = \frac{1}{\pm\sqrt{1 - \sin^2 x}} = \frac{1}{\pm\sqrt{1 - \sin^2 x}} \cdot \frac{\sqrt{1 - \sin^2 x}}{\sqrt{1 - \sin^2 x}} = \frac{\pm\sqrt{1 - \sin^2 x}}{1 - \sin^2 x}$$

53. $\cot\theta\sin\theta = \dfrac{\cos\theta}{\sin\theta}\cdot\sin\theta = \cos\theta$

54. $\sec\theta\cot\theta\sin\theta = \dfrac{1}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\sin\theta}{1}$
$= \dfrac{\sin\theta\cos\theta}{\cos\theta\sin\theta} = 1$

55. $\cos\theta\csc\theta = \cos\theta\cdot\dfrac{1}{\sin\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

56. $\cot^2\theta\left(1+\tan^2\theta\right) = \dfrac{\cos^2\theta}{\sin^2\theta}\left(\sec^2\theta\right)$
$= \dfrac{\cos^2\theta}{\sin^2\theta}\left(\dfrac{1}{\cos^2\theta}\right)$
$= \dfrac{1}{\sin^2\theta} = \csc^2\theta$

57. $\sin^2\theta\left(\csc^2\theta - 1\right) = \sin^2\theta\left(\dfrac{1}{\sin^2\theta} - 1\right)$
$= \dfrac{\sin^2\theta}{\sin^2\theta} - \sin^2\theta$
$= 1 - \sin^2\theta = \cos^2\theta$

58. $(\sec\theta - 1)(\sec\theta + 1) = \sec^2\theta - 1 = \tan^2\theta$

59. $(1-\cos\theta)(1+\sec\theta)$
$= 1 + \sec\theta - \cos\theta - \cos\theta\sec\theta$
$= 1 + \sec\theta - \cos\theta - \cos\theta\left(\dfrac{1}{\cos\theta}\right)$
$= 1 + \sec\theta - \cos\theta - 1 = \sec\theta - \cos\theta$

60. $\dfrac{\cos\theta + \sin\theta}{\sin\theta} = \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\sin\theta} = \cot\theta + 1$

61. $\dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \dfrac{\cos^2\theta}{\sin\theta\cos\theta} - \dfrac{\sin^2\theta}{\sin\theta\cos\theta}$
$= \dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta - \tan\theta$

62. $\dfrac{1-\sin^2\theta}{1+\cot^2\theta} = \dfrac{\cos^2\theta}{\csc^2\theta} = \dfrac{\cos^2\theta}{\frac{1}{\sin^2\theta}} = \sin^2\theta\cos^2\theta$

63. $\sec\theta - \cos\theta = \dfrac{1}{\cos\theta} - \cos\theta = \dfrac{1}{\cos\theta} - \dfrac{\cos^2\theta}{\cos\theta}$
$= \dfrac{1-\cos^2\theta}{\cos\theta} = \dfrac{\sin^2\theta}{\cos\theta}$
$= \dfrac{\sin\theta}{\cos\theta}\cdot\sin\theta = \tan\theta\sin\theta$

64. $(\sec\theta + \csc\theta)(\cos\theta - \sin\theta)$
$= \left(\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}\right)(\cos\theta - \sin\theta)$
$= \dfrac{1}{\cos\theta}(\cos\theta) - \dfrac{1}{\cos\theta}(\sin\theta)$
$+ \dfrac{1}{\sin\theta}(\cos\theta) - \dfrac{1}{\sin\theta}(\sin\theta)$
$= 1 - \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} - 1 = -\tan\theta + \cot\theta$
$= \cot\theta - \tan\theta$

65. $\sin\theta(\csc\theta - \sin\theta) = \sin\theta\csc\theta - \sin^2\theta$
$= \sin\theta\cdot\dfrac{1}{\sin\theta} - \sin^2\theta$
$= 1 - \sin^2\theta = \cos^2\theta$

66. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \dfrac{\sec^2\theta}{\csc^2\theta} = \dfrac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}$
$= \dfrac{1}{\cos^2\theta}\cdot\dfrac{\sin^2\theta}{1} = \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

67. $\sin^2\theta + \tan^2\theta + \cos^2\theta$
$= \left(\sin^2\theta + \cos^2\theta\right) + \tan^2\theta$
$= 1 + \tan^2\theta = \sec^2\theta$

68. $\dfrac{\tan(-\theta)}{\sec\theta} = \dfrac{-\tan\theta}{\frac{1}{\cos\theta}} = -\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\cos\theta}{1} = -\sin\theta$

69. Since $\cos x = \dfrac{1}{5}$, which is positive, x is in quadrant I or quadrant IV.

$\sin x = \pm\sqrt{1-\cos^2 x} = \pm\sqrt{1-\left(\dfrac{1}{5}\right)^2} = \pm\sqrt{\dfrac{24}{25}}$
$= \pm\dfrac{\sqrt{24}}{5} = \pm\dfrac{2\sqrt{6}}{5}$

$\tan x = \dfrac{\sin x}{\cos x} = \dfrac{\pm\frac{2\sqrt{6}}{5}}{\frac{1}{5}} = \pm 2\sqrt{6}$

$\sec x = \dfrac{1}{\cos x} = \dfrac{1}{\frac{1}{5}} = 5$

Quadrant I:

$\dfrac{\sec x - \tan x}{\sin x} = \dfrac{5 - 2\sqrt{6}}{\frac{2\sqrt{6}}{5}} = \dfrac{25 - 10\sqrt{6}}{2\sqrt{6}}$
$= \dfrac{25 - 10\sqrt{6}}{2\sqrt{6}}\cdot\dfrac{\sqrt{6}}{\sqrt{6}} = \dfrac{25\sqrt{6} - 60}{12}$

(*continued on next page*)

(*continued from page 629*)

Quadrant IV:

$$\frac{\sec x - \tan x}{\sin x} = \frac{5-\left(-2\sqrt{6}\right)}{-\frac{2\sqrt{5}}{5}} = \frac{25+10\sqrt{6}}{-2\sqrt{6}}$$
$$= \frac{25+10\sqrt{6}}{-2\sqrt{6}} \cdot \frac{-\sqrt{6}}{-\sqrt{6}} = \frac{-25\sqrt{6}-60}{12}$$

70. Since $\csc x = -3$, which is negative, x is in quadrant III or quadrant IV.

Quadrant III:

$$\sin x = \frac{1}{\csc x} = -\frac{1}{3} \Rightarrow$$
$$\cos x = -\sqrt{1-\sin^2 x} = -\sqrt{1-\left(-\frac{1}{3}\right)^2}$$
$$= -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3} \Rightarrow$$
$$\sec x = \frac{1}{\cos x} = -\frac{3}{2\sqrt{2}}$$
$$\frac{\sin x + \cos x}{\sec x} = \frac{-\frac{1}{3} - \frac{2\sqrt{2}}{3}}{-\frac{3}{2\sqrt{2}}}$$
$$= \left(\frac{-1-2\sqrt{2}}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}+8}{9}$$

Quadrant IV:

$$\sin x = \frac{1}{\csc x} = -\frac{1}{3} \Rightarrow \cos x = \sqrt{1-\sin^2 x}$$
$$= \sqrt{1-\left(-\frac{1}{3}\right)^2} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$
$$\sec x = \frac{3}{2\sqrt{2}}$$
$$\frac{\sin x + \cos x}{\sec x} = \frac{-\frac{1}{3} + \frac{2\sqrt{2}}{3}}{\frac{3}{2\sqrt{2}}}$$
$$= \left(\frac{-1+2\sqrt{2}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{-2\sqrt{2}+8}{9}$$

71. $y = \sin(-2x) \Rightarrow y = -\sin(2x)$

72. It is the negative of $\sin(2x)$.

73. $y = \cos(-4x) \Rightarrow y = \cos(4x)$

74. It is the same function.

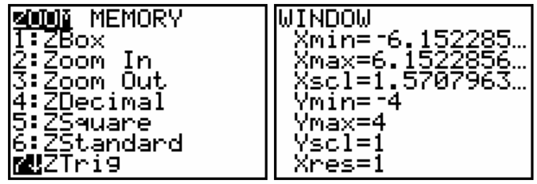
75. **(a)** $y = \sin(-4x) \Rightarrow y = -\sin(4x)$

(b) $y = \cos(-2x) \Rightarrow y = \cos(2x)$

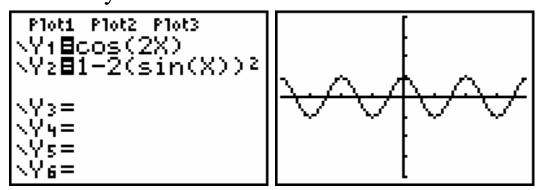
(c) $y = -5\sin(-3x) \Rightarrow y = -5\left[-\sin(3x)\right] \Rightarrow y = 5\sin(3x)$

76. Answers will vary.

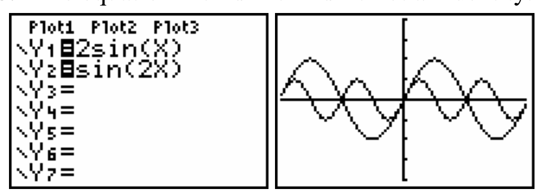
In Exercises 77−82, the functions are graphed in the following window.



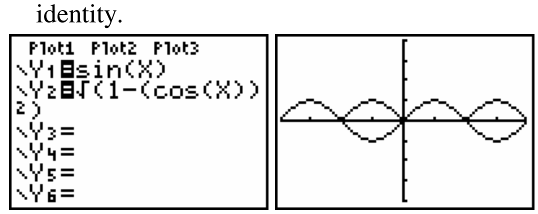
77. The equation $\cos 2x = 1 - 2\sin^2 x$ is an identity.



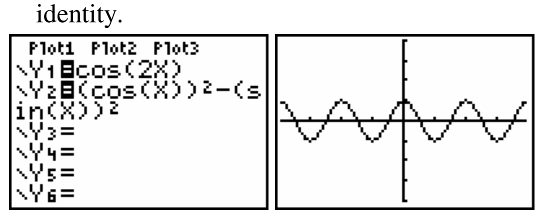
78. The equation $2\sin s = \sin 2s$ is not an identity.



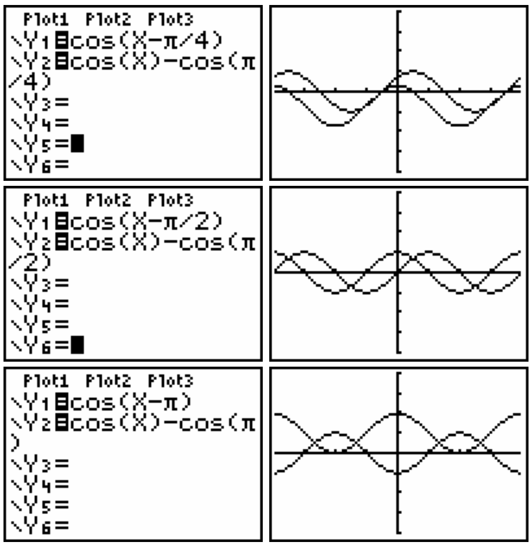
79. The equation $\sin x = \sqrt{1-\cos^2 x}$ is not an identity.



80. The equation $\cos 2x = \cos^2 x - \sin^2 x$ is an identity.

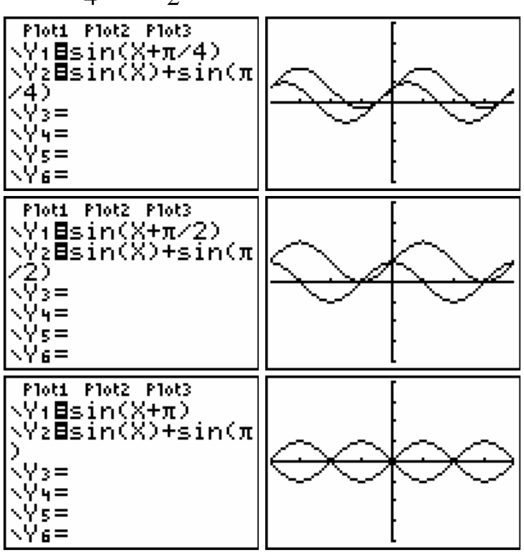


81. Does $\cos(x-y)=\cos x-\cos y$? If it does, then the graphs of $Y_1=\cos(x-y)$ and $Y_2=\cos x-\cos y$ will overlap for specific values of y. We will graph 3 cases: $y=\frac{\pi}{4}$, $y=\frac{\pi}{2}$, and $y=\pi$.



The equation $\cos(x-y)=\cos x-\cos y$ is not an identity.

82. Does $\sin(x+y)=\sin x+\sin y$? If it does, then the graphs of $Y_1=\sin(x+y)$ and $Y_2=\sin x+\sin y$ will overlap for specific values of y. We will graph 3 cases: $y=\frac{\pi}{4}$, $y=\frac{\pi}{2}$, and $y=\pi$.



The equation $\sin(x+y)=\sin x+\sin y$ is not an identity.

Section 7.2: Verifying Trigonometric Identities

1. $$\cot\theta+\frac{1}{\cot\theta}=\cot\theta+\tan\theta$$
$$=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}$$
$$=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{1}{\sin\theta\cos\theta}\text{ or }\csc\theta\sec\theta$$

2. $$\frac{\sec x}{\csc x}+\frac{\csc x}{\sec x}=\frac{\frac{1}{\cos x}}{\frac{1}{\sin x}}+\frac{\frac{1}{\sin x}}{\frac{1}{\cos x}}$$
$$=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}$$
$$=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}$$
$$=\frac{1}{\sin x\cos x}\text{ or }\csc x\sec x$$

3. $$\tan s(\cot s+\csc s)=\frac{\sin s}{\cos s}\left(\frac{\cos s}{\sin s}+\frac{1}{\sin s}\right)$$
$$=1+\frac{1}{\cos s}$$
$$=1+\sec s$$

4. $$\cos\beta(\sec\beta+\csc\beta)$$
$$=\cos\beta\left(\frac{1}{\cos\beta}+\frac{1}{\sin\beta}\right)$$
$$=1+\cot\beta$$

5. $$\frac{1}{\csc^2\theta}+\frac{1}{\sec^2\theta}=\sin^2\theta+\cos^2\theta=1$$

6. $$\frac{1}{\sin\alpha-1}-\frac{1}{\sin\alpha+1}$$
$$=\frac{\sin\alpha+1}{(\sin\alpha-1)(\sin\alpha+1)}-\frac{\sin\alpha-1}{(\sin\alpha-1)(\sin\alpha+1)}$$
$$=\frac{(\sin\alpha+1)-(\sin\alpha-1)}{(\sin\alpha-1)(\sin\alpha+1)}$$
$$=\frac{\sin\alpha+1-\sin\alpha+1}{(\sin\alpha-1)(\sin\alpha+1)}=\frac{2}{\sin^2\alpha-1}$$
$$=-\frac{2}{\cos^2\alpha}\text{ or }-2\sec^2\alpha$$

7. $\dfrac{\cos x}{\sec x}+\dfrac{\sin x}{\csc x}=\dfrac{\cos x}{\frac{1}{\cos x}}+\dfrac{\sin x}{\frac{1}{\sin x}}$
$=\cos x\left(\dfrac{\cos x}{1}\right)+\sin x\left(\dfrac{\sin x}{1}\right)$
$=\cos^2 x+\sin^2 x=1$

8. $\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}$
$=\dfrac{\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)}+\dfrac{\sin\theta\cdot\sin\theta}{\sin\theta(1+\cos\theta)}$
$=\dfrac{\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}+\dfrac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$
$=\dfrac{\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta(1+\cos\theta)}$
$=\dfrac{\cos\theta+\left(\cos^2\theta+\sin^2\theta\right)}{\sin\theta(1+\cos\theta)}=\dfrac{\cos\theta+1}{\sin\theta(1+\cos\theta)}$
$=\dfrac{1}{\sin\theta}$ or $\csc\theta$

9. $(1+\sin t)^2+\cos^2 t=1+2\sin t+\sin^2 t+\cos^2 t$
$=1+2\sin t+\left(\sin^2 t+\cos^2 t\right)$
$=1+2\sin t+1=2+2\sin t$

10. $(1+\tan s)^2-2\tan s=1+2\tan s+\tan^2 s-2\tan s$
$=1+\tan^2 s=\sec^2 s$

11. $\dfrac{1}{1+\cos x}-\dfrac{1}{1-\cos x}$
$=\dfrac{1-\cos x}{(1+\cos x)(1-\cos x)}-\dfrac{1+\cos x}{(1+\cos x)(1-\cos x)}$
$=\dfrac{(1-\cos x)-(1+\cos x)}{(1+\cos x)(1-\cos x)}$
$=\dfrac{1-\cos x-1-\cos x}{1-\cos^2 x}=-\dfrac{2\cos x}{\sin^2 x}$ or
$-\dfrac{2\cos x}{\sin^2 x}=-\dfrac{2\cos x}{\sin x\sin x}=-2\left(\dfrac{\cos x}{\sin x}\right)\left(\dfrac{1}{\sin x}\right)$
$=-2\cot x\csc x$

12. $(\sin\alpha-\cos\alpha)^2$
$=\sin^2\alpha-2\sin\alpha\cos\alpha+\cos^2\alpha$
$=\left(\sin^2\alpha+\cos^2\alpha\right)-2\sin\alpha\cos\alpha$
$=1-2\sin\alpha\cos\alpha$

13. $\sin^2\theta-1=(\sin\theta+1)(\sin\theta-1)$

14. $\sec^2\theta-1=(\sec\theta+1)(\sec\theta-1)$

15. $(\sin x+1)^2-(\sin x-1)^2$
$=\left[(\sin x+1)+(\sin x-1)\right]$
$\cdot\left[(\sin x+1)-(\sin x-1)\right]$
$=(\sin x+1+\sin x-1)(\sin x+1-\sin x+1)$
$=(2\sin x)(2)=4\sin x$

16. $(\tan x+\cot x)^2-(\tan x-\cot x)^2$
$=\left[(\tan x+\cot x)+(\tan x-\cot x)\right]$
$\cdot\left[(\tan x+\cot x)-(\tan x-\cot x)\right]$
$=(\tan x+\cot x+\tan x-\cot x)$
$\cdot(\tan x+\cot x-\tan x+\cot x)$
$=(2\tan x)(2\cot x)=4\cdot\dfrac{\sin x}{\cos x}\cdot\dfrac{\cos x}{\sin x}=4$

17. $2\sin^2 x+3\sin x+1$
Let $a=\sin x$.
$2\sin^2 x+3\sin x+1=2a^2+3a+1$
$=(2a+1)(a+1)$
$=(2\sin x+1)(\sin x+1)$

18. $4\tan^2\beta+\tan\beta-3$
Let $a=\tan\beta$.
$4\tan^2\beta+\tan\beta-3=4a^2+a-3$
$=(4a-3)(a+1)$
$=(4\tan\beta-3)(\tan\beta+1)$

19. $\cos^4 x+2\cos^2 x+1$
Let $\cos^2 x=a$.
$\cos^4 x+2\cos^2 x+1=a^2+2a+1$
$=(a+1)^2=\left(\cos^2 x+1\right)^2$

20. $\cot^4 x+3\cot^2 x+2$
Let $\cot^2 x=a$.
$\cot^4 x+3\cot^2 x+2=a^2+3a+2$
$=(a+2)(a+1)$
$=\left(\cot^2 x+2\right)\left(\cot^2 x+1\right)$
$=\left(\cot^2 x+2\right)\left(\csc^2 x\right)$
$=\csc^2 x\left(\cot^2 x+2\right)$

21. $\sin^3 x - \cos^3 x$

Let $\sin x = a$ and $\cos x = b$.

$$\begin{aligned}\sin^3 x - \cos^3 x &= a^3 - b^3 = (a-b)(a^2+ab+b^2)\\ &= (\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)\\ &= (\sin x - \cos x)\left[(\sin^2 x + \cos^2 x) + \sin x\cos x\right]\\ &= (\sin x - \cos x)(1 + \sin x\cos x)\end{aligned}$$

22. $\sin^3 \alpha + \cos^3 \alpha$

Let $\sin\alpha = a$ and $\cos\alpha = b$.

$$\begin{aligned}\sin^3 \alpha + \cos^3 \alpha &= a^3 + b^3 = (a+b)(a^2-ab+b^2)\\ &= (\sin\alpha + \cos\alpha)(\sin^2\alpha - \sin\alpha\cos\alpha + \cos^2\alpha)\\ &= (\sin\alpha + \cos\alpha)\left[(\sin^2\alpha + \cos^2\alpha) - \sin\alpha\cos\alpha\right]\\ &= (\sin\alpha + \cos\alpha)(1 - \sin\alpha\cos\alpha)\end{aligned}$$

23. $\tan\theta\cos\theta = \dfrac{\sin\theta}{\cos\theta}\cos\theta = \sin\theta$

24. $\cot\alpha\sin\alpha = \dfrac{\cos\alpha}{\sin\alpha}\cdot\sin\alpha = \cos\alpha$

25. $\sec r\cos r = \dfrac{1}{\cos r}\cdot\cos r = 1$

26. $\cot t\tan t = \dfrac{\cos t}{\sin t}\cdot\dfrac{\sin t}{\cos t} = 1$

27. $\dfrac{\sin\beta\tan\beta}{\cos\beta} = \tan\beta\tan\beta = \tan^2\beta$

28.
$$\begin{aligned}\frac{\csc\theta\sec\theta}{\cot\theta} &= \frac{\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}}\\ &= \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\\ &= \frac{1}{\cos^2\theta} = \sec^2\theta\end{aligned}$$

29.
$$\begin{aligned}\sec^2 x - 1 &= \frac{1}{\cos^2 x} - 1 = \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}\\ &= \frac{1-\cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x\end{aligned}$$

30. $\csc^2 t - 1 = \cot^2 t$

31.
$$\begin{aligned}\frac{\sin^2 x}{\cos^2 x} + \sin x\csc x &= \tan^2 x + \sin x\cdot\frac{1}{\sin x}\\ &= \tan^2 x + 1 = \sec^2 x\end{aligned}$$

32.
$$\begin{aligned}\frac{1}{\tan^2\alpha} + \cot\alpha\tan\alpha &= \cot^2\alpha + \frac{\cos\alpha}{\sin\alpha}\cdot\frac{\sin\alpha}{\cos\alpha}\\ &= \cot^2\alpha + 1 = \csc^2\alpha\end{aligned}$$

33. $1 - \dfrac{1}{\csc^2 x} = 1 - \sin^2 x = \cos^2 x$

34. $1 - \dfrac{1}{\sec^2 x} = 1 - \cos^2 x = \sin^2 x$

35. Verify $\dfrac{\cot\theta}{\csc\theta} = \cos\theta.$

$$\frac{\cot\theta}{\csc\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1} = \cos\theta$$

36. Verify $\dfrac{\tan\alpha}{\sec\alpha} = \sin\alpha.$

$$\frac{\tan\alpha}{\sec\alpha} = \frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha}} = \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha = \sin\alpha$$

37. Verify $\dfrac{1-\sin^2\beta}{\cos\beta} = \cos\beta.$

$$\frac{1-\sin^2\beta}{\cos\beta} = \frac{\cos^2\beta}{\cos\beta} = \cos\beta$$

38. Verify $\dfrac{\tan^2\alpha + 1}{\sec\alpha} = \sec\alpha$.

$$\frac{\tan^2\alpha + 1}{\sec\alpha} = \frac{\sec^2\alpha}{\sec\alpha} = \sec\alpha$$

39. Verify $\cos^2\theta(\tan^2\theta + 1) = 1.$

$$\begin{aligned}\cos^2\theta(\tan^2\theta + 1) &= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta} + 1\right)\\ &= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\right)\\ &= \cos^2\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}\right)\\ &= \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) = 1\end{aligned}$$

40. Verify $\sin^2\beta\left(1 + \cot^2\beta\right) = 1.$

$$\begin{aligned}\sin^2\beta(1 + \cot^2\beta) &= \sin^2\beta\csc^2\beta\\ &= \sin^2\beta\cdot\frac{1}{\sin^2\beta} = 1\end{aligned}$$

41. Verify $\cot s + \tan s = \sec s \csc s$.

$$\cot s + \tan s = \frac{\cos s}{\sin s} + \frac{\sin s}{\cos s} = \frac{\cos^2 s}{\sin s \cos s} + \frac{\sin^2 s}{\sin s \cos s} = \frac{\cos^2 s + \sin^2 s}{\cos s \sin s} = \frac{1}{\cos s \sin s} = \frac{1}{\cos s} \cdot \frac{1}{\sin s} = \sec s \csc s$$

42. Verify $\sin^2 \alpha + \tan^2 \alpha + \cos^2 \alpha = \sec^2 \alpha$.

$$\sin^2 \alpha + \tan^2 \alpha + \cos^2 \alpha = \left(\sin^2 \alpha + \cos^2 \alpha\right) + \tan^2 \alpha = 1 + \tan^2 \alpha = \sec^2 \alpha$$

43. Verify $\dfrac{\cos \alpha}{\sec \alpha} + \dfrac{\sin \alpha}{\csc \alpha} = \sec^2 \alpha - \tan^2 \alpha$.

Working with the left side, we have

$$\frac{\cos \alpha}{\sec \alpha} + \frac{\sin \alpha}{\csc \alpha} = \frac{\cos \alpha}{\frac{1}{\cos \alpha}} + \frac{\sin \alpha}{\frac{1}{\sin \alpha}} = \cos^2 \alpha + \sin^2 \alpha = 1$$

Working with the right side, we have $\sec^2 \alpha - \tan^2 \alpha = 1$.

Since $\dfrac{\cos \alpha}{\sec \alpha} + \dfrac{\sin \alpha}{\csc \alpha} = 1 = \sec^2 \alpha - \tan^2 \alpha$, the statement has been verified.

44. Verify $\dfrac{\sin^2 \theta}{\cos \theta} = \sec \theta - \cos \theta$.

$$\frac{\sin^2 \theta}{\cos \theta} = \frac{1 - \cos^2 \theta}{\cos \theta} = \frac{1}{\cos \theta} - \frac{\cos^2 \theta}{\cos \theta} = \sec \theta - \cos \theta$$

45. Verify $\sin^4 \theta - \cos^4 \theta = 2\sin^2 \theta - 1$.

$$\begin{aligned}\sin^4 \theta - \cos^4 \theta &= \left(\sin^2 \theta + \cos^2 \theta\right)\left(\sin^2 \theta - \cos^2 \theta\right) \\ &= 1 \cdot \left(\sin^2 \theta - \cos^2 \theta\right) = \sin^2 \theta - \cos^2 \theta \\ &= \sin^2 \theta - \left(1 - \sin^2 \theta\right) \\ &= \sin^2 \theta - 1 + \sin^2 \theta = 2\sin^2 \theta - 1\end{aligned}$$

46. Verify $\dfrac{\cos \theta}{\sin \theta \cot \theta} = 1$.

$$\frac{\cos \theta}{\sin \theta \cot \theta} = \frac{\cos \theta}{\sin \theta \cdot \frac{\cos \theta}{\sin \theta}} = \frac{\cos \theta}{\cos \theta} = 1$$

47. Verify $\dfrac{1 - \cos x}{1 + \cos x} = (\cot x - \csc x)^2$.

Work with the left side.

$$\frac{1 - \cos x}{1 + \cos x} = \frac{(1 - \cos x)(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} = \frac{1 - 2\cos x + \cos^2 x}{1 - \cos^2 x} = \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x}$$

Work with the right side.

$$(\cot x - \csc x)^2 = \left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right)^2 = \left(\frac{\cos x - 1}{\sin x}\right)^2 = \frac{\cos^2 x - 2\cos x + 1}{\sin^2 x}$$

$$\frac{1 - \cos x}{1 + \cos x} = \frac{\cos^2 x - 2\cos x + 1}{\sin^2 x} = (\cot x - \csc x)^2$$

Thus, the statement has been verified.

48. Verify $\sin^2 \theta\left(1 + \cot^2 \theta\right) - 1 = 0$

$$\sin^2 \theta\left(1 + \cot^2 \theta\right) - 1 = \sin^2 \theta\left(\csc^2 \theta\right) - 1 = \sin^2 \theta\left(\frac{1}{\sin^2 \theta}\right) - 1 = 1 - 1 = 0$$

49. Verify $\dfrac{\cos \theta + 1}{\tan^2 \theta} = \dfrac{\cos \theta}{\sec \theta - 1}$.

Work with the left side.

$$\begin{aligned}\frac{\cos \theta + 1}{\tan^2 \theta} &= \frac{\cos \theta + 1}{\sec^2 \theta - 1} = \frac{\cos \theta + 1}{\frac{1}{\cos^2 \theta} - 1} \\ &= \frac{(\cos \theta + 1)\cos^2 \theta}{\left(\frac{1}{\cos^2 \theta} - 1\right)\cos^2 \theta} \\ &= \frac{\cos^2 \theta(\cos \theta + 1)}{1 - \cos^2 \theta} \\ &= \frac{\cos^2 \theta(\cos \theta + 1)}{(1 + \cos \theta)(1 - \cos \theta)} = \frac{\cos^2 \theta}{1 - \cos \theta}\end{aligned}$$

Now work with the right side.

$$\frac{\cos \theta}{\sec \theta - 1} = \frac{\cos \theta}{\frac{1}{\cos \theta} - 1} = \frac{\cos \theta}{\frac{1}{\cos \theta} - 1} \cdot \frac{\cos \theta}{\cos \theta} = \frac{\cos^2 \theta}{1 - \cos \theta}$$

$$\frac{\cos \theta + 1}{\tan^2 \theta} = \frac{\cos^2 \theta}{1 - \cos \theta} = \frac{\cos \theta}{\sec \theta - 1}$$

Thus, the statement has been verified.

50. Verify $\dfrac{(\sec\theta-\tan\theta)^2+1}{\sec\theta\csc\theta-\tan\theta\csc\theta}=2\tan\theta.$

$$\begin{aligned}
&\frac{(\sec\theta-\tan\theta)^2+1}{\sec\theta\csc\theta-\tan\theta\csc\theta}\\
&=\frac{\sec^2\theta-2\sec\theta\tan\theta+\tan^2\theta+1}{\csc\theta(\sec\theta-\tan\theta)}\\
&=\frac{\sec^2\theta-2\sec\theta\tan\theta+(\tan^2\theta+1)}{\csc\theta(\sec\theta-\tan\theta)}\\
&=\frac{\sec^2\theta-2\sec\theta\tan\theta+\sec^2\theta}{\csc\theta(\sec\theta-\tan\theta)}\\
&=\frac{2\sec^2\theta-2\sec\theta\tan\theta}{\csc\theta(\sec\theta-\tan\theta)}\\
&=\frac{2\sec\theta(\sec\theta-\tan\theta)}{\csc\theta(\sec\theta-\tan\theta)}\\
&=\frac{2\sec\theta}{\csc\theta}=2\cdot\frac{\sin\theta}{\cos\theta}=2\tan\theta
\end{aligned}$$

51. Verify $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta.$

$$\begin{aligned}
&\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\\
&=\frac{1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}+\frac{1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}\\
&=\frac{(1+\sin\theta)+(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\\
&=\frac{1+\sin\theta+1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}\\
&=\frac{2}{1-\sin^2\theta}=\frac{2}{\cos^2\theta}=2\sec^2\theta
\end{aligned}$$

52. Verify $\dfrac{1}{\sec\alpha-\tan\alpha}=\sec\alpha+\tan\alpha.$

$$\begin{aligned}
\frac{1}{\sec\alpha-\tan\alpha}&=\frac{1}{\sec\alpha-\tan\alpha}\cdot\frac{\sec\alpha+\tan\alpha}{\sec\alpha+\tan\alpha}\\
&=\frac{\sec\alpha+\tan\alpha}{\sec^2\alpha-\tan^2\alpha}\\
&=\frac{\sec\alpha+\tan\alpha}{\frac{1}{\cos^2\alpha}-\frac{\sin^2\alpha}{\cos^2\alpha}}=\frac{\sec\alpha+\tan\alpha}{\frac{1-\sin^2\alpha}{\cos^2\alpha}}\\
&=\frac{\sec\alpha+\tan\alpha}{\frac{\cos^2\alpha}{\cos^2\alpha}}=\frac{\sec\alpha+\tan\alpha}{1}\\
&=\sec\alpha+\tan\alpha
\end{aligned}$$

53. Verify $\dfrac{\cot\alpha+1}{\cot\alpha-1}=\dfrac{1+\tan\alpha}{1-\tan\alpha}$.

$$\begin{aligned}
\frac{\cot\alpha+1}{\cot\alpha-1}&=\frac{\frac{\cos\alpha}{\sin\alpha}+1}{\frac{\cos\alpha}{\sin\alpha}-1}=\frac{\frac{\cos\alpha}{\sin\alpha}+1}{\frac{\cos\alpha}{\sin\alpha}-1}\cdot\frac{\sin\alpha}{\sin\alpha}\\
&=\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}\\
&=\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}\cdot\frac{\frac{1}{\cos\alpha}}{\frac{1}{\cos\alpha}}\\
&=\frac{\frac{\cos\alpha}{\cos\alpha}+\frac{\sin\alpha}{\cos\alpha}}{\frac{\cos\alpha}{\cos\alpha}-\frac{\sin\alpha}{\cos\alpha}}=\frac{1+\tan\alpha}{1-\tan\alpha}
\end{aligned}$$

54. Verify $\dfrac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\cot\theta\csc\theta.$

$$\begin{aligned}
\frac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}&=\frac{\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\sin\theta}\\
&=\frac{\frac{1+\cos\theta}{\sin\theta}}{\sin\theta\left(\frac{1}{\cos\theta}+1\right)}\cdot\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\
&=\frac{(1+\cos\theta)\cos\theta}{\sin^2\theta(1+\cos\theta)}=\frac{\cos\theta}{\sin^2\theta}\\
&=\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}=\csc\theta\cot\theta
\end{aligned}$$

55. Verify $\sec^4x-\sec^2x=\tan^4x+\tan^2x.$

Simplify the left side.

$$\begin{aligned}
\sec^4x-\sec^2x&=\sec^2x(\sec^2x-1)\\
&=\sec^2x\tan^2x=\tan^2x\sec^2x
\end{aligned}$$

Simplify the right side.

$$\begin{aligned}
\tan^4x+\tan^2x&=\tan^2x(\tan^2x+1)\\
&=\tan^2x\sec^2x
\end{aligned}$$

$\sec^4x-\sec^2x=\tan^2x\sec^2x=\tan^4x+\tan^2x$ Th us, the statement has been verified.

56. Verify $\left(\sec\alpha-\tan\alpha\right)^2=\dfrac{1-\sin\alpha}{1+\sin\alpha}$.

$$\begin{aligned}&\left(\sec\alpha-\tan\alpha\right)^2\\&=\sec^2\alpha-2\sec\alpha\tan\alpha+\tan^2\alpha\\&=\frac{1}{\cos^2\alpha}-2\cdot\frac{1}{\cos\alpha}\cdot\frac{\sin\alpha}{\cos\alpha}+\frac{\sin^2\alpha}{\cos^2\alpha}\\&=\frac{1-2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}=\frac{\left(1-\sin\alpha\right)^2}{1-\sin^2\alpha}\\&=\frac{\left(1-\sin\alpha\right)^2}{\left(1-\sin\alpha\right)\left(1+\sin\alpha\right)}=\frac{1-\sin\alpha}{1+\sin\alpha}\end{aligned}$$

57. Verify $\dfrac{\sec^4 s-\tan^4 s}{\sec^2 s+\tan^2 s}=\sec^2 s-\tan^2 s.$

$$\begin{aligned}&\frac{\sec^4 s-\tan^4 s}{\sec^2 s+\tan^2 s}\\&=\frac{\left(\sec^2 s+\tan^2 s\right)\left(\sec^2 s-\tan^2 s\right)}{\sec^2 s+\tan^2 s}\\&=\sec^2 s-\tan^2 s\end{aligned}$$

58. Verify $\dfrac{\cot^2 t-1}{1+\cot^2 t}=1-2\sin^2 t.$

$$\begin{aligned}\frac{\cot^2 t-1}{1+\cot^2 t}&=\frac{\frac{\cos^2 t}{\sin^2 t}-1}{1+\frac{\cos^2 t}{\sin^2 t}}=\frac{\frac{\cos^2 t}{\sin^2 t}-1}{1+\frac{\cos^2 t}{\sin^2 t}}\cdot\frac{\sin^2 t}{\sin^2 t}\\&=\frac{\cos^2 t-\sin^2 t}{\sin^2 t+\cos^2 t}=\frac{\cos^2 t-\sin^2 t}{1}\\&=\cos^2 t-\sin^2 t=\left(1-\sin^2 t\right)-\sin^2 t\\&=1-2\sin^2 t\end{aligned}$$

59. Verify $\dfrac{\tan^2 t-1}{\sec^2 t}=\dfrac{\tan t-\cot t}{\tan t+\cot t}.$

Simplify the right side

$$\begin{aligned}\frac{\tan t-\cot t}{\tan t+\cot t}&=\frac{\tan t-\frac{1}{\tan t}}{\tan t+\frac{1}{\tan t}}\\&=\frac{\tan t-\frac{1}{\tan t}}{\tan t+\frac{1}{\tan t}}\cdot\frac{\tan t}{\tan t}\\&=\frac{\tan^2 t-1}{\tan^2 t+1}=\frac{\tan^2 t-1}{\sec^2 t}\end{aligned}$$

60. Verify $\dfrac{\sin^4\alpha-\cos^4\alpha}{\sin^2\alpha-\cos^2\alpha}=1.$

$$\begin{aligned}&\frac{\sin^4\alpha-\cos^4\alpha}{\sin^2\alpha-\cos^2\alpha}\\&=\frac{\left(\sin^2\alpha+\cos^2\alpha\right)\left(\sin^2\alpha-\cos^2\alpha\right)}{\sin^2\alpha-\cos^2\alpha}\\&=\sin^2\alpha+\cos^2\alpha=1\end{aligned}$$

61. Verify

$\left(1-\cos^2\alpha\right)\left(1+\cos^2\alpha\right)=2\sin^2\alpha-\sin^4\alpha.$

$$\begin{aligned}\left(1-\cos^2\alpha\right)\left(1+\cos^2\alpha\right)&=\sin^2\alpha\left(1+\cos^2\alpha\right)\\&=\sin^2\alpha\left(2-\sin^2\alpha\right)\\&=2\sin^2\alpha-\sin^4\alpha\end{aligned}$$

62. Verify $\tan^2\alpha\sin^2\alpha=\tan^2\alpha+\cos^2\alpha-1.$

Work with the left side.

$$\begin{aligned}\tan^2\alpha\sin^2\alpha&=\tan^2\alpha\left(1-\cos^2\alpha\right)\\&=\tan^2\alpha-\tan^2\alpha\cos^2\alpha\\&=\tan^2\alpha-\sin^2\alpha\end{aligned}$$

Now work with the right side.

$$\begin{aligned}\tan^2\alpha+\cos^2\alpha-1&=\tan^2\alpha-\left(1-\cos^2\alpha\right)\\&=\tan^2\alpha-\sin^2\alpha\\\tan^2\alpha+\cos^2\alpha-1&=\tan^2\alpha-\sin^2\alpha\\&=\tan^2\alpha+\cos^2\alpha-1\end{aligned}$$

Thus, the statement has been verified.

63. Verify $\sin^2\alpha\sec^2\alpha+\sin^2\alpha\csc^2\alpha=\sec^2\alpha$.

$$\begin{aligned}&\sin^2\alpha\sec^2\alpha+\sin^2\alpha\csc^2\alpha\\&=\sin^2\alpha\cdot\frac{1}{\cos^2\alpha}+\sin^2\alpha\cdot\frac{1}{\sin^2\alpha}\\&=\frac{\sin^2\alpha}{\cos^2\alpha}+1=\tan^2\alpha+1=\sec^2\alpha\end{aligned}$$

64. Verify $\dfrac{-1}{\tan\alpha-\sec\alpha}+\dfrac{-1}{\tan\alpha+\sec\alpha}=2\tan\alpha.$

$$\begin{aligned}&\frac{-1}{\tan\alpha-\sec\alpha}+\frac{-1}{\tan\alpha+\sec\alpha}\\&=\frac{-\tan\alpha-\sec\alpha}{\left(\tan\alpha+\sec\alpha\right)\left(\tan\alpha-\sec\alpha\right)}\\&\qquad+\frac{-\tan\alpha+\sec\alpha}{\left(\tan\alpha+\sec\alpha\right)\left(\tan\alpha-\sec\alpha\right)}\\&=\frac{-\tan\alpha-\sec\alpha-\tan\alpha+\sec\alpha}{\left(\tan\alpha+\sec\alpha\right)\left(\tan\alpha-\sec\alpha\right)}\\&=\frac{-2\tan\alpha}{\tan^2\alpha-\sec^2\alpha}=\frac{-2\tan\alpha}{\tan^2\alpha-\left(\tan^2\alpha+1\right)}\\&=\frac{-2\tan\alpha}{\tan^2\alpha-\tan^2\alpha-1}=\frac{-2\tan\alpha}{-1}=2\tan\alpha\end{aligned}$$

65. Verify $\frac{\tan s}{1+\cos s}+\frac{\sin s}{1-\cos s}=\cot s+\sec s\csc s.$

$$\frac{\tan s}{1+\cos s}+\frac{\sin s}{1-\cos s}$$
$$=\frac{\tan s(1-\cos s)}{(1+\cos s)(1-\cos s)}+\frac{\sin s(1+\cos s)}{(1+\cos s)(1-\cos s)}$$
$$=\frac{\tan s(1-\cos s)+\sin s(1+\cos s)}{(1+\cos s)(1-\cos s)}$$
$$=\frac{\tan s-\sin s+\sin s+\sin s\cos s}{1-\cos^2 s}$$
$$=\frac{\tan s+\sin s\cos s}{\sin^2 s}=\frac{\tan s}{\sin^2 s}+\frac{\sin s\cos s}{\sin^2 s}$$
$$=\tan s\cdot\frac{1}{\sin^2 s}+\frac{\cos s}{\sin s}=\frac{\sin s}{\cos s}\cdot\frac{1}{\sin^2 s}+\cot s$$
$$=\frac{1}{\cos s}\cdot\frac{1}{\sin s}+\cot s=\sec s\csc s+\cot s$$

66. Verify
$$\frac{1-\cos x}{1+\cos x}=\csc^2 x-2\csc x\cot x+\cot^2 x$$
Work with the left side:
$$\frac{1-\cos x}{1+\cos x}=\frac{1-\cos x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}$$
$$=\frac{1-2\cos x+\cos^2 x}{1-\cos^2 x}$$
$$=\frac{1-2\cos x+\cos^2 x}{\sin^2 x}$$
Work with the right side:
$$\csc^2 x-2\csc x\cot x+\cot^2 x$$
$$=\frac{1}{\sin^2 x}-\frac{2\cos x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}$$
$$=\frac{1-2\cos x+\cos^2 x}{\sin^2 x}$$
$$\frac{1-\cos x}{1+\cos x}=\frac{1-2\cos x+\cos^2 x}{\sin^2 x}$$
$$=\csc^2 x-2\csc x\cot x+\cot^2 x$$
Thus, the statement has been verified.

67. Verify
$$\frac{1-\sin\theta}{1+\sin\theta}=\sec^2\theta-2\sec\theta\tan\theta+\tan^2\theta .$$
Simplify the right side
$$\sec^2\theta-2\sec\theta\tan\theta+\tan^2\theta$$
$$=\frac{1}{\cos^2\theta}-2\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}+\frac{\sin^2\theta}{\cos^2\theta}$$
$$=\frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}=\frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$
$$=\frac{(1-\sin\theta)^2}{(1+\sin\theta)(1-\sin\theta)}=\frac{1-\sin\theta}{1+\sin\theta}$$

68. Verify $\sin\theta+\cos\theta=\frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}$.

Simplify the right side.
$$\frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}$$
$$=\frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}\cdot\frac{\sin\theta}{\sin\theta}+\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}\cdot\frac{\cos\theta}{\cos\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{\cos\theta-\sin\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{-(\sin\theta-\cos\theta)}$$
$$=\frac{\sin^2\theta}{\sin\theta-\cos\theta}-\frac{\cos^2\theta}{\sin\theta-\cos\theta}=\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$$
$$=\frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta}=\sin\theta+\cos\theta$$

69. Verify $\frac{\sin\theta}{1-\cos\theta}-\frac{\sin\theta\cos\theta}{1+\cos\theta}=\csc\theta\left(1+\cos^2\theta\right)$.

$$\frac{\sin\theta}{1-\cos\theta}-\frac{\sin\theta\cos\theta}{1+\cos\theta}$$
$$=\frac{\sin\theta(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}-\frac{\sin\theta\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$
$$=\frac{\sin\theta(1+\cos\theta)-\sin\theta\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$
$$=\frac{\sin\theta+\sin\theta\cos\theta-\sin\theta\cos\theta+\sin\theta\cos^2\theta}{1-\cos^2\theta}$$
$$=\frac{\sin\theta+\sin\theta\cos^2\theta}{\sin^2\theta}=\frac{1+\cos^2\theta}{\sin\theta}$$
$$=\frac{1}{\sin\theta}\left(1+\cos^2\theta\right)=\csc\theta\left(1+\cos^2\theta\right)$$

70. Verify $(1+\sin x+\cos x)^2=2(1+\sin x)(1+\cos x)$.

Working with the right side, we have
$$2(1+\sin x)(1+\cos x)$$
$$=2(1+\sin x+\cos x+\sin x\cos x)$$
$$=2+2\sin x+2\cos x+2\sin x\cos x$$
Working with the left side, we have
$$(1+\sin x+\cos x)^2$$
$$=1+\sin x+\cos x+\sin x+\sin^2 x+\sin x\cos x$$
$$+\cos x+\sin x\cos x+\cos^2 x$$
$$=2+2\sin x+2\cos x+2\sin x\cos x$$

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$(1+\sin x+\cos x)^2$
$= 2+2\sin x+2\cos x+2\sin x\cos x$
$= 2(1+\sin x)(1+\cos x)$

Thus, the statement has been verified.

71. Verify $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$.

$$\begin{aligned}
&\frac{1+\cos x}{1-\cos x}-\frac{1-\cos x}{1+\cos x}\\
&=\frac{(1+\cos x)^2}{(1+\cos x)(1-\cos x)}-\frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)}\\
&=\frac{1+2\cos x+\cos^2 x}{(1+\cos x)(1-\cos x)}-\frac{1-2\cos x+\cos^2 x}{(1+\cos x)(1-\cos x)}\\
&=\frac{1+2\cos x+\cos^2 x-1+2\cos x-\cos^2 x}{(1+\cos x)(1-\cos x)}\\
&=\frac{4\cos x}{1-\cos^2 x}=\frac{4\cos x}{\sin^2 x}=4\cdot\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\\
&=4\cot x\csc x
\end{aligned}$$

72. Verify

$(\sec\alpha+\csc\alpha)(\cos\alpha-\sin\alpha)=\cot\alpha-\tan\alpha$.

$$\begin{aligned}
&(\sec\alpha+\csc\alpha)(\cos\alpha-\sin\alpha)\\
&=\left(\frac{1}{\cos\alpha}+\frac{1}{\sin\alpha}\right)(\cos\alpha-\sin\alpha)\\
&=\frac{\cos\alpha}{\cos\alpha}-\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}-\frac{\sin\alpha}{\sin\alpha}\\
&=1-\tan\alpha+\cot\alpha-1=\cot\alpha-\tan\alpha
\end{aligned}$$

73. Verify $\dfrac{1+\sin\theta}{1-\sin\theta}-\dfrac{1-\sin\theta}{1+\sin\theta}=4\tan\theta\sec\theta$

$$\begin{aligned}
&\frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta}\\
&=\frac{(1+\sin\theta)^2-(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}\\
&=\frac{\left(1+2\sin\theta+\sin^2\theta\right)-\left(1-2\sin\theta+\sin^2\theta\right)}{1-\sin^2\theta}\\
&=\frac{4\sin\theta}{\cos^2\theta}=4\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}=4\tan\theta\sec\theta
\end{aligned}$$

74. Verify $\dfrac{1-\cos\theta}{1+\cos\theta}=2\csc^2\theta-2\csc\theta\cot\theta-1$

Work with the left side:

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}=\frac{1-2\cos\theta+\cos^2\theta}{1-\cos^2\theta}=\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}$$

Work with the right side:

$$\begin{aligned}
2\csc^2\theta-2\csc x\cot\theta-1&=2\csc^2\theta-2\csc\theta\cot\theta-\left(\csc^2\theta-\cot^2\theta\right)=\csc^2\theta-2\csc\theta\cot\theta+\cot^2\theta\\
&=\frac{1}{\sin^2\theta}-\frac{2\cos\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}
\end{aligned}$$

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}=2\csc^2\theta-2\csc x\cot\theta-1$$

Thus, the statement has been verified.

75. Verify $(2\sin x+\cos y)^2+(2\cos x-\sin y)^2=5$

$$\begin{aligned}
(2\sin x+\cos x)^2+(2\cos x-\sin x)^2&=\left(4\sin^2 x+4\sin x\cos x+\cos^2 x\right)+\left(4\cos^2 x-4\sin x\cos x+\sin^2 x\right)\\
&=4\left(\sin^2 x+\cos^2 x\right)+\left(\cos^2 x+\sin^2 x\right)=4+1=5
\end{aligned}$$

76. Verify $\sin^2 x(1+\cot x)+\cos^2 x(1-\tan x)+\cot^2 x=\csc^2 x$

$$\begin{aligned}
\sin^2 x(1+\cot x)+\cos^2 x(1-\tan x)+\cot^2 x&=\sin^2 x+\sin^2 x\cot x+\cos^2 x-\cos^2\tan x+\cot^2 x\\
&=\left(\sin^2 x+\cos^2 x\right)+\sin^2 x\left(\frac{\cos x}{\sin x}\right)-\cos^2\left(\frac{\sin x}{\cos x}\right)+\cot^2 x\\
&=1+\sin x\cos x-\sin x\cos x+\cot^2 x=1+\cot^2 x=\csc^2 x
\end{aligned}$$

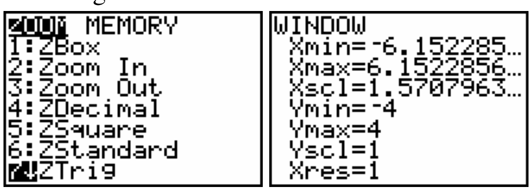
77. Verify $\sec x - \cos x + \csc x - \sin x - \sin x \tan x = \cos x \cot x$

$$\sec x - \cos x + \csc x - \sin x - \sin x \tan x = \frac{1}{\cos x} - \cos x + \frac{1}{\sin x} - \sin x - \sin x\left(\frac{\sin x}{\cos x}\right)$$
$$= \left(\frac{1}{\cos x} - \cos x\right) + \left(\frac{1}{\sin x} - \sin x\right) - \frac{\sin^2 x}{\cos x}$$
$$= \frac{1-\cos^2 x}{\cos x} + \frac{1-\sin^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}$$
$$= \left(\frac{1-\cos^2 x}{\cos x} - \frac{\sin^2 x}{\cos x}\right) + \frac{1-\sin^2 x}{\sin x} = \frac{1-\cos^2 x - \sin^2 x}{\cos x} + \frac{\cos^2 x}{\sin x}$$
$$= \frac{1-\left(\cos^2 x + \sin^2 x\right)}{\cos x} + \frac{\cos^2 x}{\sin x} = \frac{1-1}{\cos x} + \cos x \cdot \frac{\cos x}{\sin x} = \cos x \cot x$$

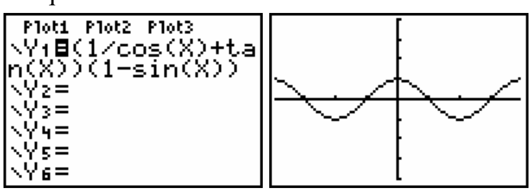
78. Verify $\sin^3\theta + \cos^3\theta = (\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta)$

$$\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)\left(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\right) = (\cos\theta + \sin\theta)(1 - \sin\theta\cos\theta)$$

In Exercises 79–86, the functions are graphed in the following window.

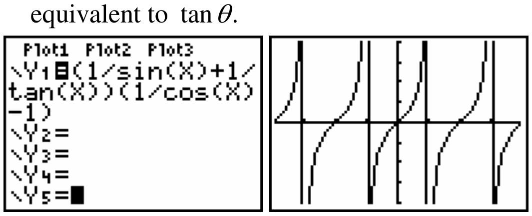


79. $(\sec\theta + \tan\theta)(1 - \sin\theta)$ appears to be equivalent to $\cos\theta$.



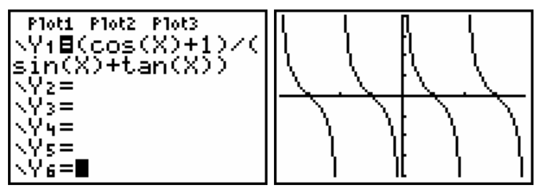
$$(\sec\theta + \tan\theta)(1 - \sin\theta)$$
$$= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)(1 - \sin\theta)$$
$$= \left(\frac{1+\sin\theta}{\cos\theta}\right)(1 - \sin\theta)$$
$$= \frac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta}$$
$$= \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta$$

80. $(\csc\theta + \cot\theta)(\sec\theta - 1)$ appears to be equivalent to $\tan\theta$.



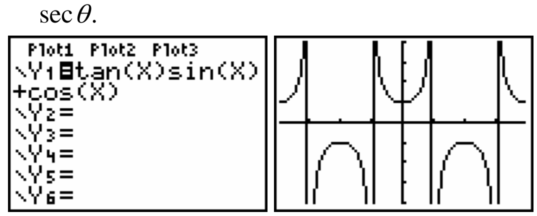
$$(\csc\theta + \cot\theta)(\sec\theta - 1)$$
$$= \left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\cos\theta} - 1\right)$$
$$= \left(\frac{1+\cos\theta}{\sin\theta}\right)\left(\frac{1-\cos\theta}{\cos\theta}\right)$$
$$= \frac{1-\cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

81. $\dfrac{\cos\theta + 1}{\sin\theta + \tan\theta}$ appears to be equivalent to $\cot\theta$.



$$\frac{\cos\theta + 1}{\sin\theta + \tan\theta} = \frac{1+\cos\theta}{\sin\theta + \frac{\sin\theta}{\cos\theta}} = \frac{1+\cos\theta}{\sin\theta\left(1 + \frac{1}{\cos\theta}\right)}$$
$$= \frac{1+\cos\theta}{\sin\theta\left(1 + \frac{1}{\cos\theta}\right)} \cdot \frac{\cos\theta}{\cos\theta}$$
$$= \frac{(1+\cos\theta)\cos\theta}{\sin\theta(\cos\theta + 1)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

82. $\tan\theta\sin\theta + \cos\theta$ appears to be equivalent to $\sec\theta$.



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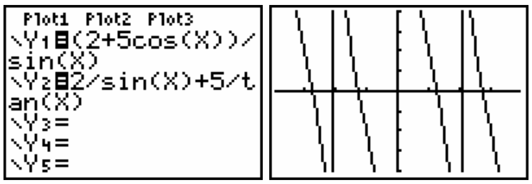
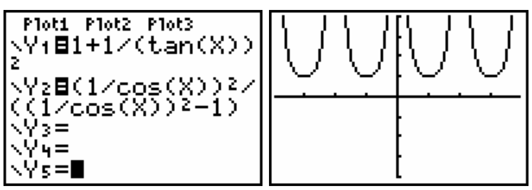
$$\tan\theta\sin\theta+\cos\theta=\frac{\sin\theta}{\cos\theta}\cdot\sin\theta+\cos\theta$$
$$=\frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\cos\theta}$$
$$=\frac{\sin^2\theta+\cos^2\theta}{\cos\theta}$$
$$=\frac{1}{\cos\theta}=\sec\theta$$

83. Is $\frac{2+5\cos x}{\sin x}=2\csc x+5\cot x$ an identity?



The graphs of $y=\frac{2+5\cos x}{\sin x}$ and $y=2\csc x+5\cot x$ appear to be the same. The given equation may be an identity. Since $\frac{2+5\cos x}{\sin x}=\frac{2}{\sin x}+\frac{5\cos x}{\sin x}=2\csc x+5\cot x,$ the given statement is an identity.

84. Is $1+\cot^2 x=\frac{\sec^2 x}{\sec^2 x-1}$ an identity?

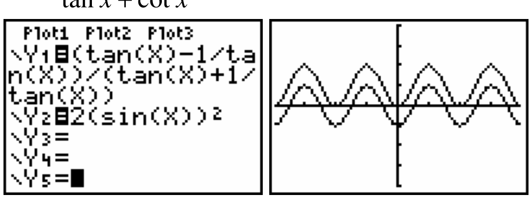


The graphs of $y=1+\cot^2 x$ and $y=\frac{\sec^2 x}{\sec^2 x-1}$ appear to be the same. The given equation may be an identity. Since

$$\frac{\sec^2 x}{\sec^2 x-1}=\frac{\sec^2 x}{\tan^2 x}=\frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}=\frac{1}{\sin^2 x}$$
$$=\csc^2 x=1+\cot^2 x$$

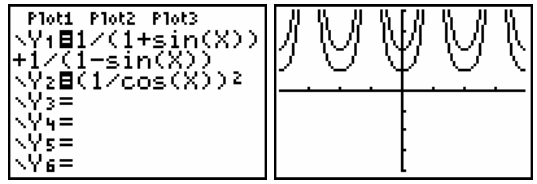
the given statement is an identity.

85. Is $\frac{\tan x-\cot x}{\tan x+\cot x}=2\sin^2 x$ an identity?



The graphs of $y=\frac{\tan x-\cot x}{\tan x+\cot x}$ and $y=2\sin^2 x$ are not the same. The given statement is not an identity.

86. Is $\frac{1}{1+\sin x}+\frac{1}{1-\sin x}=\sec^2 x$ an identity?



The graphs of $y=\frac{1}{1+\sin x}+\frac{1}{1-\sin x}$ and $y=\sec^2 x$ are not the same. The given statement is not an identity.

87. Show that $\sin(\csc s)=1$ is not an identity.

We need to find only one value for which the statement is false. Let $s = 2$. Use a calculator to find that sin (csc 2) ≈ .891094, which is not equal to 1. $\sin(\csc s)=1$ does not hold true for *all* real numbers *s*. Thus, it is not an identity.

88. Show that $\sqrt{\cos^2 s}=\cos s$ is not an identity.

Let $s=\frac{\pi}{3}$. We have $\cos\frac{\pi}{3}=\frac{1}{2}$ and $\sqrt{\cos^2\frac{\pi}{3}}=\sqrt{\left(\frac{1}{2}\right)^2}=\sqrt{\frac{1}{4}}=\frac{1}{2}.$

But let $s=\frac{2\pi}{3}$. We have, $\cos s=-\frac{1}{2}$ and $\sqrt{\cos^2\frac{2\pi}{3}}=\sqrt{\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{1}{4}}=\frac{1}{2}.$

$\sqrt{\cos^2 s}=\cos s$ does not hold true for *all* real numbers *s*. Thus, it is not an identity.

89. Show that $\csc t = \sqrt{1+\cot^2 t}$ is not an identity.

Let $t = \frac{\pi}{4}$. We have $\csc \frac{\pi}{4} = \sqrt{2}$ and $\sqrt{1+\cot^2 \frac{\pi}{4}} = \sqrt{1+1^2} = \sqrt{1+1} = \sqrt{2}$. But let $t = -\frac{\pi}{4}$. We have $\csc\left(-\frac{\pi}{4}\right) = -\sqrt{2}$ and $\sqrt{1+\cot^2\left(-\frac{\pi}{4}\right)} = \sqrt{1+(-1)^2} = \sqrt{1+1} = \sqrt{2}$.

$\csc t = \sqrt{1+\cot^2 t}$ does not hold true for *all* real numbers *s*. Thus, it is not an identity.

90. Show that $\cos t = \sqrt{1-\sin^2 t}$ is not an identity.

Let $t = \frac{\pi}{3}$. We have $\cos \frac{\pi}{3} = \frac{1}{2}$ and

$$\sqrt{1-\sin^2 \frac{\pi}{3}} = \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1-\frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

But let $t = \frac{2\pi}{3}$. We have $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and

$$\sqrt{1-\sin^2 \frac{2\pi}{3}} = \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1-\frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$\cos t = \sqrt{1-\sin^2 t}$ does not hold true for *all* real numbers *s*. Thus, it is not an identity.

91. $\sin x = \sqrt{1-\cos^2 x}$ is a true statement when $\sin x \ge 0$.

92. $\cos x = \sqrt{1-\sin^2 x}$ is a true statement when $\cos x \ge 0$.

93. **(a)** $I = k\cos^2\theta = k\left(1-\sin^2\theta\right)$

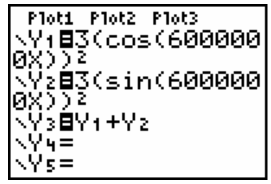
(b) For $\theta = 2\pi n$ for all integers *n*, $\cos^2\theta = 1$, its maximum value and *I* attains a maximum value of *k*.

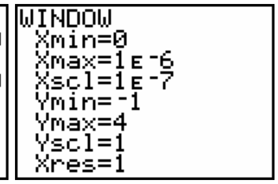
94. **(a)** $P = ky^2$ and $y = 4\cos(2\pi t)$

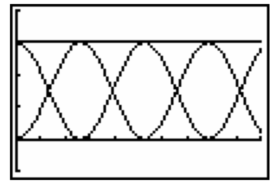
$$P = ky^2 = k\left[4\cos(2\pi t)\right]^2 = k\left[16\cos^2(2\pi t)\right] = 16k\cos^2(2\pi t)$$

(b) $P = 16k\cos^2(2\pi t) = 16k\left[1-\sin^2(2\pi t)\right]$

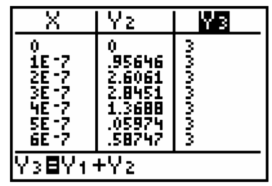
95. **(a)** The sum of *L* and *C* equals 3.







(b) Let $Y_1 = L(t), Y_2 = C(t)$, and $Y_3 = E(t)$. $Y_3 = 3$ for all inputs.



(c)
$$\begin{aligned} E(t) &= L(t)+C(t) \\ &= 3\cos^2(6{,}000{,}000t) + 3\sin^2(6{,}000{,}000t) \\ &= 3\left[\cos^2(6{,}000{,}000t) + \sin^2(6{,}000{,}000t)\right] \\ &= 3\cdot 1 = 3 \end{aligned}$$

Section 7.3: Sum and Difference Identities

1. Since $\cos(x+y) = \cos x\cos y - \sin x\sin y$, the correct choice is F.

2. Since $\cos(x-y) = \cos x\cos y + \sin x\sin y$, the correct choice is A.

3. Since $\cos(90° - x) = \sin x$, the correct choice is E.

4. Since $\sin(90° - x) = \cos x$, the correct choice is B.

5.
$$\begin{aligned} \cos 75° &= \cos(30° + 45°) \\ &= \cos 30°\cos 45° - \sin 30°\sin 45° \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

6. $\cos(-15°) = \cos(30° - 45°)$
$= \cos 30° \cos 45° + \sin 30° \sin 45°$
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

7. $\cos 105° = \cos(60° + 45°)$
$= \cos 60° \cos 45° - \sin 60° \sin 45°$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$

8. $\cos(-105°) = \cos[-60° + (-45°)]$
$= \cos(-60°)\cos(-45°) - \sin(-60°)\sin(-45°)$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)$
$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$

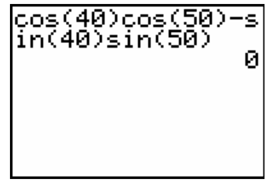
9. $\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$
$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$

10. $\cos\left(-\frac{\pi}{12}\right) = \cos\left(\frac{2\pi}{12} - \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$
$= \cos\frac{\pi}{6}\cos\frac{\pi}{4} + \sin\frac{\pi}{6}\sin\frac{\pi}{4}$
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

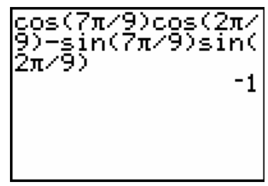
11. $\cos 40° \cos 50° - \sin 40° \sin 50°$
$= \cos(40° + 50°) = \cos 90° = 0$

12. $\cos\frac{7\pi}{9}\cos\frac{2\pi}{9} - \sin\frac{7\pi}{9}\sin\frac{2\pi}{9}$
$= \cos\left(\frac{7\pi}{9} + \frac{2\pi}{9}\right) = \cos\pi = -1$

13. The answer to exercise 11 is 0. Using a calculator to evaluate $\cos 40° \cos 50° - \sin 40° \sin 50°$ also gives a value of 0. (Make sure that your calculator is in DEGREE mode.)



14. The answer to exercise 12 is −1. Using a calculator to evaluate $\cos\frac{7\pi}{9}\cos\frac{2\pi}{9} - \sin\frac{7\pi}{9}\sin\frac{2\pi}{9}$ also gives a value of −1. (Make sure that your calculator is in RADIAN mode.)



15. Since $\frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{3}$, $\cot\frac{\pi}{3} = \tan\frac{\pi}{6}$.
$\cot\frac{\pi}{3} = \underline{\tan}\frac{\pi}{6}$

16. Since $-\frac{\pi}{6} = \frac{\pi}{2} - \frac{2\pi}{3}$,
$\sin\frac{2\pi}{3} = \cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \cos\left(-\frac{\pi}{6}\right)$.
$\sin\frac{2\pi}{3} = \underline{\cos}\left(-\frac{\pi}{6}\right)$

17. Since $90° - 57° = 33°$, $\sin 57° = \cos 33°$.
$\sin 57° = \underline{\cos} 33°$

18. Since 90° – 18° = 72°, cot 18° = tan 72°.
$\underline{\tan} 72° = \cot 18°$

19. Since 90º – 70º = 20º, and $\sin x = \frac{1}{\csc x}$,
$\cos 70° = \sin 20° = \frac{1}{\csc 20°}$.
$\cos 70° = \frac{1}{\underline{\csc} 20°}$

20. Since 90º – 24 º = 66º, and $\cot x = \frac{1}{\tan x}$,
$\tan 24° = \cot 66° = \frac{1}{\tan 66°}$.
$\tan 24° = \frac{1}{\underline{\tan} 66°}$

For exercises 21–26, other answers are possible.

21. $\tan\theta = \cot(45° + 2\theta)$

Since $\tan\theta = \cot(90° - \theta)$,

$$90° - \theta = 45° + 2\theta \Rightarrow 90° = 45° + 3\theta \Rightarrow$$
$$3\theta = 45° \Rightarrow \theta = 15°$$

22. $\sin\theta = \cos(2\theta - 10°)$

Since $\sin\theta = \cos(90° - \theta)$,

$$90° - \theta = 2\theta - 10° \Rightarrow 90° = 3\theta - 10° \Rightarrow$$
$$3\theta = 100° \Rightarrow \theta = \frac{100°}{3}$$

23. $\sec\theta = \csc\left(\frac{\theta}{2} + 20°\right)$

By a cofunction identity, $\sec\theta = \csc(90° - \theta)$.

Thus, $\csc\left(\frac{\theta}{2} + 20°\right) = \csc(90° - \theta)$. So,

$$\csc\left(\frac{\theta}{2} + 20°\right) = \csc(90° - \theta) \Rightarrow$$
$$\frac{\theta}{2} + 20° = 90° - \theta \Rightarrow \frac{3\theta}{2} + 20° = 90° \Rightarrow$$
$$\frac{3\theta}{2} = 70° \Rightarrow \theta = \frac{140°}{3}$$

24. $\cos\theta = \sin(3\theta + 10°)$

By a cofunction identity, $\cos\theta = \sin(90° - \theta)$.

Thus, $\sin(90° - \theta) = \sin(3\theta + 10°)$. So,

$$\sin(90° - \theta) = \sin(3\theta + 10°) \Rightarrow$$
$$90° - \theta = 3\theta + 10° \Rightarrow$$
$$80° = 4\theta \Rightarrow 20° = \theta$$

25. $\sin(3\theta - 15°) = \cos(\theta + 25°)$

Since $\sin\theta = \cos(90° - \theta)$, we have

$$\sin(3\theta - 15°) = \cos[90° - (3\theta - 15°)]$$
$$= \cos(90° - 3\theta + 15°)$$
$$= \cos(105° - 3\theta)$$

Solve $\cos(105° - 3\theta) = \cos(\theta + 25°)$.

$$\cos(105° - 3\theta) = \cos(\theta + 25°)$$
$$105° - 3\theta = \theta + 25°$$
$$105° = 4\theta + 25°$$
$$4\theta = 80° \Rightarrow \theta = 20°$$

26. $\cot(\theta - 10°) = \tan(2\theta + 20°)$

Since $\cot\theta = \tan(90° - \theta)$, we have

$$\cot(\theta - 10°) = \tan[90° - (\theta - 10°)]$$
$$= \tan(90° - \theta + 10°)$$
$$= \tan(100° - \theta)$$

Solve $\tan(100° - \theta) = \tan(2\theta + 20°)$.

$$\tan(100° - \theta) = \tan(2\theta + 20°)$$
$$100° - \theta = 2\theta + 20°$$
$$100° = 3\theta + 20°$$
$$3\theta = 80° \Rightarrow \theta = \frac{80°}{3}$$

27. $\cos(0° - \theta) = \cos 0° \cos\theta + \sin 0° \sin\theta$
$= (1)\cos\theta + (0)\sin\theta = \cos\theta$

28. $\cos(90° - \theta) = \cos 90° \cos\theta + \sin 90° \sin\theta$
$= (0)\cos\theta + (1)\sin\theta$
$= 0 + \sin\theta = \sin\theta$

29. $\cos(\theta - 180°) = \cos\theta\cos 180° + \sin\theta\sin 180°$
$= \cos\theta(-1) + \sin\theta(0)$
$= -\cos\theta + 0 = -\cos\theta$

30. $\cos(\theta - 270°) = \cos\theta\cos 270° + \sin\theta\sin 270°$
$= \cos\theta(0) + \sin\theta(-1)$
$= 0 - \sin\theta = -\sin\theta$

31. $\cos(0° + \theta) = \cos 0° \cos\theta - \sin 0° \sin\theta$
$= (1)\cos\theta - (0)\sin\theta = \cos\theta$

32. $\cos(90° + \theta) = \cos 90° \cos\theta - \sin 90° \sin\theta$
$= (0)\cos\theta - (1)\sin\theta$
$= 0 - \sin\theta = -\sin\theta$

33. $\cos(180° + \theta) = \cos 180° \cos\theta - \sin 180° \sin\theta$
$= (-1)\cos\theta - (0)\sin\theta$
$= -\cos\theta - 0 = -\cos\theta$

34. $\cos(270° + \theta) = \cos 270° \cos\theta - \sin 270° \sin\theta$
$= (0)\cos\theta - (-1)\sin\theta$
$= 0 + \sin\theta = \sin\theta$

35. $\cos s = -\frac{1}{5}$, $\sin t = \frac{3}{5}$, s and t are in quadrant II.

$\cos s = \frac{x}{r} \Rightarrow \cos s = -\frac{1}{5} = \frac{-1}{5} \Rightarrow x = -1, r = 5$.

Substituting into the Pythagorean theorem, we have $(-1)^2 + y^2 = 5^2 \Rightarrow y^2 = 24 \Rightarrow y = \sqrt{24}$, since sin $x > 0$. Thus, $\sin s = \frac{y}{r} = \frac{\sqrt{24}}{5}$.

We will use a Pythagorean identity to find the value of cos t.

$$\cos t = -\sqrt{1 - \sin^2 t} = -\sqrt{1 - \left(\frac{3}{5}\right)^2}$$
$$= -\sqrt{1 - \frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

(*continued on next page*)

(*continued from page 643*)

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$= \left(-\frac{1}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{\sqrt{24}}{5}\right)\left(\frac{3}{5}\right)$$
$$= \frac{4}{25} - \frac{3\sqrt{24}}{25} = \frac{4}{25} - \frac{6\sqrt{6}}{25} = \frac{4-6\sqrt{6}}{25}$$
$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$
$$= \left(-\frac{1}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{\sqrt{24}}{5}\right)\left(\frac{3}{5}\right)$$
$$= \frac{4}{25} + \frac{3\sqrt{24}}{25} = \frac{4}{25} + \frac{6\sqrt{6}}{25} = \frac{4+6\sqrt{6}}{25}$$

36. $\sin s = \frac{2}{3}$ and $\sin t = -\frac{1}{3}$, s is in quadrant II and t is in quadrant IV.

$\sin s = \frac{y}{r} = \frac{2}{3} \Rightarrow y = 2, r = 3$. Substituting into the Pythagorean theorem, we have $x^2 + 2^2 = 3^2 \Rightarrow x^2 = 5 \Rightarrow x = -\sqrt{5}$, since $\cos s < 0$. Thus, $\cos s = \frac{x}{r} = -\frac{\sqrt{5}}{3}$.

We will use a Pythagorean identity to find the value of $\cos t$.

$$\cos t = \sqrt{1-\left(-\frac{1}{3}\right)^2} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}}$$
$$= \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$
$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$= \left(-\frac{\sqrt{5}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$
$$= \frac{-2\sqrt{10}}{9} + \frac{2}{9} = \frac{-2\sqrt{10}+2}{9}$$
$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$
$$= \left(-\frac{\sqrt{5}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) + \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$
$$= \frac{-2\sqrt{10}}{9} - \frac{2}{9} = \frac{-2\sqrt{10}-2}{9}$$

37. $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, s is in quadrant I and t is in quadrant III.

$\sin s = \frac{y}{r} = \frac{3}{5} \Rightarrow y = 3, r = 5$. Substituting into the Pythagorean theorem, we have $x^2 + 3^2 = 5^2 \Rightarrow x^2 = 16 \Rightarrow x = 4$, since $\cos s > 0$. Thus, $\cos s = \frac{x}{r} = \frac{4}{5}$.

We will use a Pythagorean identity to find the value of $\cos t$.

$$\cos t = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\sqrt{1-\frac{144}{169}}$$
$$= -\sqrt{\frac{25}{169}} = -\frac{5}{13}$$
$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$
$$= -\frac{20}{65} + \frac{36}{65} = \frac{16}{65}$$
$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$
$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$
$$= -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$$

38. $\cos s = -\frac{8}{17}$ and $\cos t = -\frac{3}{5}$, s and t are in quadrant III

$\cos s = \frac{x}{r} = -\frac{8}{17} = \frac{-8}{17} \Rightarrow x = -8, r = 17$.

Substituting into the Pythagorean theorem, we have

$(-8)^2 + y^2 = 17^2 \Rightarrow y^2 = 225 \Rightarrow y = -15$, since $\sin s < 0$. Thus, $\sin s = \frac{y}{r} = -\frac{15}{17}$.

We will use a Pythagorean identity to find the value of $\sin t$.

$$\sin t = -\sqrt{1-\cos^2 t} = -\sqrt{1-\left(-\frac{3}{5}\right)^2}$$
$$= -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$
$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$= \left(-\frac{8}{17}\right)\left(-\frac{3}{5}\right) - \left(-\frac{15}{17}\right)\left(-\frac{4}{5}\right)$$
$$= \frac{24}{85} - \frac{60}{85} = -\frac{36}{85}$$
$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$
$$= \left(-\frac{8}{17}\right)\left(-\frac{3}{5}\right) + \left(-\frac{15}{17}\right)\left(-\frac{4}{5}\right)$$
$$= \frac{24}{85} + \frac{60}{85} = \frac{84}{85}$$

39. $\sin s = \frac{\sqrt{5}}{7}$ and $\sin t = \frac{\sqrt{6}}{8}$, s and t are in quadrant I.

$\sin s = \frac{y}{r} = \frac{\sqrt{5}}{7} \Rightarrow y = \sqrt{5}, r = 7$. Substituting into the Pythagorean theorem, we have $x^2 + \left(\sqrt{5}\right)^2 = 7^2 \Rightarrow x^2 = 44 \Rightarrow x = \sqrt{44},$ since $\cos s > 0$. Thus, $\cos s = \frac{x}{r} = \frac{\sqrt{44}}{7}$. We will use a Pythagorean identity to find the value of $\cos t$.

$$\cos t = \sqrt{1 - \left(\frac{\sqrt{6}}{8}\right)^2} = \sqrt{1 - \frac{6}{64}} = \sqrt{\frac{58}{64}} = \frac{\sqrt{58}}{8}$$

$$\begin{aligned}\cos(s+t) &= \cos s \cos t - \sin s \sin t \\ &= \left(\frac{\sqrt{44}}{7}\right)\left(\frac{\sqrt{58}}{8}\right) - \left(\frac{\sqrt{5}}{7}\right)\left(\frac{\sqrt{6}}{8}\right) \\ &= \frac{2\sqrt{638}}{56} - \frac{\sqrt{30}}{56} = \frac{2\sqrt{638} - \sqrt{30}}{56}\end{aligned}$$

$$\begin{aligned}\cos(s+t) &= \cos s \cos t - \sin s \sin t \\ &= \left(\frac{\sqrt{44}}{7}\right)\left(\frac{\sqrt{58}}{8}\right) + \left(\frac{\sqrt{5}}{7}\right)\left(\frac{\sqrt{6}}{8}\right) \\ &= \frac{2\sqrt{638}}{56} + \frac{\sqrt{30}}{56} = \frac{2\sqrt{638} + \sqrt{30}}{56}\end{aligned}$$

40. $\cos s = \frac{\sqrt{2}}{4}$ and $\sin t = -\frac{\sqrt{5}}{6}$, s and t are in quadrant IV.

$\cos s = \frac{x}{r} = \frac{\sqrt{2}}{4} \Rightarrow x = \sqrt{2}, r = 4$. Substituting into the Pythagorean theorem, we have $\left(\sqrt{2}\right)^2 + y^2 = 4^2 \Rightarrow y^2 = 14 \Rightarrow y = -\sqrt{14},$ since $\sin s < 0$. Thus, $\sin s = \frac{y}{r} = -\frac{\sqrt{14}}{4}$. We will use a Pythagorean identity to find the value of $\cos t$.

$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \left(-\frac{\sqrt{5}}{6}\right)^2} = \sqrt{1 - \frac{5}{36}} = \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{6}$$

$$\begin{aligned}\cos(s+t) &= \cos s \cos t - \sin s \sin t \\ &= \left(\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{31}}{6}\right) - \left(-\frac{\sqrt{14}}{4}\right)\left(-\frac{\sqrt{5}}{6}\right) \\ &= \frac{\sqrt{62}}{24} - \frac{\sqrt{70}}{24} = \frac{\sqrt{62} - \sqrt{70}}{24}\end{aligned}$$

$$\begin{aligned}\cos(s-t) &= \cos s \cos t + \sin s \sin t \\ &= \left(\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{31}}{6}\right) + \left(-\frac{\sqrt{14}}{4}\right)\left(-\frac{\sqrt{5}}{6}\right) \\ &= \frac{\sqrt{62}}{24} + \frac{\sqrt{70}}{24} = \frac{\sqrt{62} + \sqrt{70}}{24}\end{aligned}$$

41.
$$\begin{aligned}\cos 195° &= \cos(180° + 15°) \\ &= \cos 180° \cos 15° - \sin 180° \sin 15° \\ &= (-1)\cos 15° - (0)\sin 15° \\ &= -\cos 15° - 0 = -\cos 15°\end{aligned}$$

42.
$$\begin{aligned}-\cos 15° &= -\cos(45° - 30°) \\ &= -(\cos 45° \cos 30° + \sin 45° \sin 30°) \\ &= -\left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}\right) \\ &= -\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right) = \frac{-\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

43. $\cos 195° = -\cos 15° = \frac{-\sqrt{6} - \sqrt{2}}{4}$

44. **(a)**
$$\begin{aligned}\cos 255° &= \cos(180° + 75°) \\ &= \cos 180° \cos 75° - \sin 180° \sin 75° \\ &= (-1)\cos 75° - (0)\sin 75° \\ &= -\cos 75° = -\cos(45° + 30°) \\ &= -(\cos 45° \cos 30° - \sin 45° \sin 30°) \\ &= -\left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}\right) \\ &= -\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right) \\ &= -\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right) = \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

(b)
$$\begin{aligned}\cos \frac{11\pi}{12} &= \cos\left(\frac{12\pi}{12} - \frac{\pi}{12}\right) = \cos\left(\pi - \frac{\pi}{12}\right) \\ &= \cos \pi \cos \frac{\pi}{12} + \sin \pi \sin \frac{\pi}{12} \\ &= (-1)\cos \frac{\pi}{12} + (0)\sin \frac{\pi}{12} \\ &= -\cos \frac{\pi}{12} = -\cos\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) \\ &= -\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\end{aligned}$$

(*continued on next page*)

(*continued from page 645*)

$$= -\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)$$
$$= -\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right)$$
$$= -\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)$$
$$= -\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) = \frac{-\sqrt{6}-\sqrt{2}}{4}$$

45. $\sin 15° = \sin(45° - 30°)$
$$= \sin 45°\cos 30° - \cos 45°\sin 30°$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$
Thus, the correct choice is C.

46. $\sin 105° = \sin(60° + 45°)$
$$= \sin 60°\cos 45° + \cos 60°\sin 45°$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$
Thus, the correct choice is A.

47. $\tan 15° = \tan(60° - 45°)$
$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60°\tan 45°}$$
$$= \frac{\sqrt{3}-1}{1+\sqrt{3}(1)} = \frac{\sqrt{3}-1}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}}$$
$$= \frac{\sqrt{3}-3-1+\sqrt{3}}{1-3} = \frac{-4+2\sqrt{3}}{-2} = 2-\sqrt{3}$$
Thus, the correct choice is E.

48. $\tan 105° = \tan(60° + 45°)$
$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60°\tan 45°}$$
$$= \frac{\sqrt{3}+1}{1-\sqrt{3}(1)} = \frac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$= \frac{\sqrt{3}+3+1+\sqrt{3}}{1-3} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$
Thus, the correct choice is F.

49. $\sin(-105°) = \sin(45° - 150°)$
$$= \sin 45°\cos 150° - \cos 45°\sin 150°$$
$$= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$$
Thus, the correct choice is B.

50. $\tan(-105°) = -\tan 105° = -\tan(60° + 45°)$
$$= -\frac{\tan 60° + \tan 45°}{1 - \tan 60°\tan 45°}$$
$$= -\frac{\sqrt{3}+1}{1-\sqrt{3}(1)} = -\frac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$= -\frac{\sqrt{3}+3+1+\sqrt{3}}{1-3}$$
$$= -\frac{4+2\sqrt{3}}{-2} = 2+\sqrt{3}$$
Thus, the correct choice is D.

51. $\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$
$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

52. $\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \dfrac{\tan\frac{\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{6}\tan\frac{\pi}{4}}$
$$= \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}}\cdot\frac{3}{3} = \frac{\sqrt{3}+3}{3-\sqrt{3}}$$
$$= \frac{\sqrt{3}+3}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{\left(3+\sqrt{3}\right)^2}{3^2-\left(\sqrt{3}\right)^2}$$
$$= \frac{9+6\sqrt{3}+3}{9-3} = \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

53. $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \dfrac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} \cdot \frac{3}{3} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} = \frac{\left(3 - \sqrt{3}\right)^2}{3^2 - \left(\sqrt{3}\right)^2}$$

$$= \frac{9 - 6\sqrt{3} + 3}{9 - 3} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3}$$

54. $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

55. $\sin\left(-\frac{7\pi}{12}\right) = \sin\left(-\frac{\pi}{3} - \frac{\pi}{4}\right)$

$$= \sin\left(-\frac{\pi}{3}\right)\cos\frac{\pi}{4} - \cos\left(-\frac{\pi}{3}\right)\sin\frac{\pi}{4}$$

$$= -\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6} - \sqrt{2}}{4}$$

56. $\tan\left(-\frac{7\pi}{12}\right) = \tan\left[-\frac{\pi}{4} + \left(-\frac{\pi}{3}\right)\right]$

$$= \frac{\tan\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{3}\right)}{1 - \tan\left(-\frac{\pi}{4}\right)\tan\left(-\frac{\pi}{3}\right)}$$

$$= \frac{-1 + \left(-\sqrt{3}\right)}{1 - (-1)\left(-\sqrt{3}\right)} = \frac{-1 - \sqrt{3}}{1 - \sqrt{3}}$$

$$= \frac{-1 - \sqrt{3}}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{-1 - \sqrt{3} - \sqrt{3} - 3}{1^2 - \left(\sqrt{3}\right)^2} = \frac{-4 - 2\sqrt{3}}{1 - 3}$$

$$= \frac{-4 - 2\sqrt{3}}{-2} = 2 + \sqrt{3}$$

57. $\sin 76°\cos 31° - \cos 76°\sin 31° = \sin\left(76° - 31°\right)$

$$= \sin 45° = \frac{\sqrt{2}}{2}$$

58. $\sin 40°\cos 50° + \cos 40°\sin 50° = \sin\left(40° + 50°\right)$

$$= \sin 90° = 1$$

59. $\dfrac{\tan 80° + \tan 55°}{1 - \tan 80°\tan 55°} = \tan\left(80° + 55°\right)$

$$= \tan 135° = -1$$

60. $\dfrac{\tan 80° - \tan\left(-55°\right)}{1 + \tan 80°\tan\left(-55°\right)} = \tan\left[80° - \left(-55°\right)\right]$

$$= \tan 135° = -1$$

61. $\dfrac{\tan 100° + \tan 80°}{1 - \tan 100°\tan 80°} = \tan\left(100° + 80°\right)$

$$= \tan 180° = 0$$

62. $\sin 100°\cos 10° - \cos 100°\sin 10°$

$$= \sin\left(100° - 10°\right) = \sin 90° = 1$$

63. $\sin\frac{\pi}{5}\cos\frac{3\pi}{10} + \cos\frac{\pi}{5}\sin\frac{3\pi}{10}$

$$= \sin\left(\frac{\pi}{5} + \frac{3\pi}{10}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

64. $\dfrac{\tan\frac{5\pi}{12} + \tan\frac{\pi}{4}}{1 - \tan\frac{5\pi}{12}\tan\frac{\pi}{4}} = \tan\left(\frac{5\pi}{12} + \frac{\pi}{4}\right)$

$$= \tan\frac{2\pi}{3} = -\sqrt{3}$$

65. $\cos\left(30° + \theta\right) = \cos 30°\cos\theta - \sin 30°\sin\theta$

$$= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta$$

$$= \frac{1}{2}\left(\sqrt{3}\cos\theta - \sin\theta\right)$$

$$= \frac{\sqrt{3}\cos\theta - \sin\theta}{2}$$

66. $\cos\left(45° - \theta\right) = \cos 45°\cos\theta + \sin 45°\sin\theta$

$$= \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta$$

$$= \frac{\sqrt{2}}{2}\left(\cos\theta + \sin\theta\right)$$

$$= \frac{\sqrt{2}\left(\cos\theta + \sin\theta\right)}{2}$$

67. $\cos(60°+\theta)=\cos 60°\cos\theta-\sin 60°\sin\theta$
$$=\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta=\frac{1}{2}\left(\cos\theta-\sqrt{3}\sin\theta\right)=\frac{\cos\theta-\sqrt{3}\sin\theta}{2}$$

68. $\cos(\theta-30°)=\cos\theta\cos 30°+\sin\theta\sin 30°$
$$=\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta=\frac{1}{2}\left(\sqrt{3}\cos\theta+\sin\theta\right)=\frac{\sqrt{3}\cos\theta+\sin\theta}{2}$$

69. $\cos\left(\frac{3\pi}{4}-x\right)=\cos\frac{3\pi}{4}\cos x+\sin\frac{3\pi}{4}\sin x$
$$=\left(-\frac{\sqrt{2}}{2}\right)\cos x+\left(\frac{\sqrt{2}}{2}\right)\sin x=\frac{\sqrt{2}}{2}(-\cos x+\sin x)=\frac{\sqrt{2}(\sin x-\cos x)}{2}$$

70. $\sin(45°+\theta)=\sin 45°\cos\theta+\sin\theta\cos 45°$
$$=\frac{\sqrt{2}}{2}\sin\theta+\frac{\sqrt{2}}{2}\cos\theta=\frac{\sqrt{2}(\sin\theta+\cos\theta)}{2}$$

71. $\tan(\theta+30°)=\frac{\tan\theta+\tan 30°}{1-\tan\theta\tan 30°}$
$$=\frac{\tan\theta+\frac{1}{\sqrt{3}}}{1-\left(\frac{1}{\sqrt{3}}\right)\tan\theta}=\frac{\sqrt{3}\tan\theta+1}{\sqrt{3}-\tan\theta}$$

72. $\tan\left(\frac{\pi}{4}+x\right)=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}$
$$=\frac{1+\tan x}{1-1\cdot\tan x}=\frac{1+\tan x}{1-\tan x}$$

73. $\sin\left(\frac{\pi}{4}+x\right)=\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x$
$$=\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x=\frac{\sqrt{2}(\cos x+\sin x)}{2}$$

74. $\sin(180°-\theta)=\sin 180°\cos\theta-\cos 180°\sin\theta$
$$=(0)(\cos\theta)-(-1)(\sin\theta)=0+\sin\theta=\sin\theta$$

75. $\sin(270°-\theta)=\sin 270°\cos\theta-\cos 270°\sin\theta$
$$=(-1)(\cos\theta)-(0)(\sin\theta)=-\cos\theta$$

76. $\tan(180°+\theta)=\frac{\tan 180°+\tan\theta}{1-\tan 180°\tan\theta}$
$$=\frac{0+\tan\theta}{1-0\cdot\tan\theta}=\tan\theta$$

77. $\tan(360°-\theta)=\frac{\tan 360°-\tan\theta}{1+\tan 360°\tan\theta}$
$$=\frac{0-\tan\theta}{1+0\cdot\tan\theta}=-\tan\theta$$

78. $\sin(\pi+\theta)=\sin\pi\cos\theta+\cos\pi\sin\theta$
$$=0\cdot\cos\theta+(-1)\sin\theta=-\sin\theta$$

79. $\tan(\pi-\theta)=\frac{\tan\pi-\tan\theta}{1+\tan\pi\tan\theta}$
$$=\frac{0-\tan\theta}{1+0\cdot\tan\theta}=-\tan\theta$$

80. To follow the method of Example 2 to find $\tan(270°-\theta)$, we need to use the tangent of a difference formula:
$$\tan(270°-\theta)=\frac{\tan 270°-\tan\theta}{1+\tan 270°\tan\theta}$$
However, tan 270º is undefined.

81. Answers will vary.

82. If A, B, and C are angles of a triangle, then $A+B+C=180°$. Therefore, we have $\sin(A+B+C)=\sin 180°=0$.

83. $\cos s=\frac{3}{5}$, $\sin t=\frac{5}{13}$, and s and t are in quadrant I.
First find the values of sin s, tan s, cos t, and tan t. Because s and t are both in quadrant I, the values of sin s and cos t, tan s, and tan t will be positive.

$$\sin s = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos t = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

$$\tan s = \frac{\sin s}{\cos s} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

(a) $\sin(s+t) = \sin s\cos t + \cos s\sin t$

$$= \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

(b) $\tan(s+t) = \dfrac{\tan s + \tan t}{1 - \tan s\tan t} = \dfrac{\frac{4}{3}+\frac{5}{12}}{1-\left(\frac{4}{3}\right)\left(\frac{5}{12}\right)}$

$$= \frac{48+15}{36-20} = \frac{63}{16}$$

(c) From parts (a) and (b), $\sin(s+t) > 0$ and $\tan(s+t) > 0$. The only quadrant in which the values of both the sine and the tangent are positive is quadrant I, so $s+t$ is in quadrant I.

84. $\cos s = -\frac{1}{5}$, $\sin t = \frac{3}{5}$, and s and t are in quadrant II.
First find the values of sin s, tan s, cos t, and tan t. Because s and t are both in quadrant II, the values of sin s is positive. Cos t, tan s, and tan t are all negative

$$\sin s = \sqrt{1-\left(-\frac{1}{5}\right)^2} = \sqrt{1-\frac{1}{25}} = \sqrt{\frac{24}{25}} = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$$

$$\cos t = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

$$\tan s = \frac{\sin s}{\cos s} = \frac{\frac{2\sqrt{6}}{5}}{-\frac{1}{5}} = -2\sqrt{6}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$$

(a) $\sin(s+t) = \sin s\cos t + \cos s\sin t$

$$= \left(\frac{2\sqrt{6}}{5}\right)\left(-\frac{4}{5}\right) + \left(-\frac{1}{5}\right)\left(\frac{3}{5}\right) = \frac{-8\sqrt{6}}{25} - \frac{3}{25} = \frac{-8\sqrt{6}-3}{25}$$

(b) $\tan(s+t) = \dfrac{\tan s + \tan t}{1 - \tan s\tan t}$

$$= \frac{-2\sqrt{6}+\left(-\frac{3}{4}\right)}{1-\left(-2\sqrt{6}\right)\left(-\frac{3}{4}\right)} = \frac{\frac{-8\sqrt{6}-3}{4}}{\frac{4-6\sqrt{6}}{4}} = \frac{-8\sqrt{6}-3}{4-6\sqrt{6}}$$

Note that other forms of $\tan(s+t)$ will be obtained depending on whether tan s and tan t are written with rationalized deonominators.

(c) From parts (a) and (b), $\sin(s+t) < 0$ and $\tan(s+t) > 0$. The sine is negative in quadrants III and IV, while the tangent is positive in quadrants I and III. Therefore, $s+t$ is in quadrant III.

85. $\sin s = \frac{2}{3}$ and $\sin t = -\frac{1}{3}$, s is in quadrant II and t is in quadrant IV.
First find the values of cos s, cos t, tan s, and tan t. Because s is in quadrant II and t is in quadrant IV, the values of cos s, tan s, and tan t will be negative, while cos t will be positive.

$$\cos s = -\sqrt{1-\left(\frac{2}{3}\right)^2} = -\sqrt{1-\frac{4}{9}} = -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{3}$$

$$\cos t = \sqrt{1-\left(-\frac{1}{3}\right)^2} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

$$\tan s = \frac{\sin s}{\cos s} = \frac{\frac{2}{3}}{-\frac{\sqrt{5}}{3}} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

(a) $\sin(s+t) = \sin s\cos t + \cos s\sin t$

$$= \left(\frac{2}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) + \left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{1}{3}\right) = \frac{4\sqrt{2}}{9} + \frac{\sqrt{5}}{9} = \frac{4\sqrt{2}+\sqrt{5}}{9}$$

(b) Different forms of tan $(s + t)$ will be obtained depending on whether tan s and tan t are written with rationalized denominators.

$$\tan(s+t)=\frac{-\frac{2\sqrt{5}}{5}+\left(-\frac{\sqrt{2}}{4}\right)}{1-\left(-\frac{2\sqrt{5}}{5}\right)\left(-\frac{\sqrt{2}}{4}\right)}=\frac{-8\sqrt{5}-5\sqrt{2}}{20-2\sqrt{10}}$$

or
$$\tan(s+t)=\frac{-\frac{2}{\sqrt{5}}+\left(-\frac{1}{2\sqrt{2}}\right)}{1-\left(-\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{2\sqrt{2}}\right)}=\frac{-4\sqrt{2}-\sqrt{5}}{2\sqrt{10}-2}=\frac{4\sqrt{2}+\sqrt{5}}{2-2\sqrt{10}}$$

(c) To find the quadrant of $s + t$, notice from the preceding that $\sin(s+t)=\frac{4\sqrt{2}+\sqrt{5}}{9}>0$ and $\tan(s+t)=\frac{-8\sqrt{5}-5\sqrt{2}}{20-2\sqrt{10}}\approx -1.8<0$.
The only quadrant in which the values of sine are positive and tangent is negative is quadrant II. Therefore, $s + t$ is in quadrant II.

86. $\sin s=\frac{3}{5}$ and $\sin t=-\frac{12}{13}$, s is in quadrant I and t is in quadrant III.
First find the values of cos s, cos t, tan s, and tan t. Because s is in quadrant I and t is in quadrant III, the values of cos s, tan s, and tan t will be positive, while cos t will be negative.

$$\cos s=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\cos t=-\sqrt{1-\left(-\frac{12}{13}\right)^2}=-\sqrt{1-\frac{144}{169}}=-\sqrt{\frac{25}{169}}=-\frac{5}{13}$$

$$\tan s=\frac{\sin s}{\cos s}=\frac{\frac{3}{5}}{\frac{4}{5}}=\frac{3}{4}$$

$$\tan t=\frac{\sin t}{\cos t}=\frac{-\frac{12}{13}}{-\frac{5}{13}}=\frac{12}{5}$$

(a)
$$\sin(s+t)=\sin s\cos t+\cos s\sin t=\left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)=-\frac{15}{65}+\left(-\frac{48}{65}\right)=-\frac{63}{65}$$

(b)
$$\tan(s+t)=\frac{\frac{3}{4}+\frac{12}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{12}{5}\right)}=\frac{15+48}{20-36}=\frac{63}{-16}=-\frac{63}{16}$$

(c) From parts (a) and (b), sin $(s + t) < 0$ and tan $(s + t) < 0$. The only quadrant in which the values of sine and tangent are both negative is quadrant IV, so $s + t$ is in quadrant IV.

87. $\cos s=-\frac{8}{17}$ and $\cos t=-\frac{3}{5}$, s and t are in quadrant III
First find the values of sin s, sin t, tan s, and tan t. Because s and t are both in quadrant III, the values of sin s and sin t will be negative, while tan s and tan t will be positive.

$$\sin s=-\sqrt{1-\cos^2 s}=-\sqrt{1-\left(-\frac{8}{17}\right)^2}=-\sqrt{1-\frac{64}{289}}=-\sqrt{\frac{225}{289}}=-\frac{15}{17}$$

$$\sin t=-\sqrt{1-\cos^2 t}=-\sqrt{1-\left(-\frac{3}{5}\right)^2}=-\sqrt{1-\frac{9}{25}}=-\sqrt{\frac{16}{25}}=-\frac{4}{5}$$

$$\tan s=\frac{\sin s}{\cos s}=\frac{-\frac{15}{17}}{-\frac{8}{17}}=\frac{15}{8}$$

$$\tan t=\frac{\sin t}{\cos t}=\frac{-\frac{4}{5}}{-\frac{3}{5}}=\frac{4}{3}$$

(a)
$$\sin(s+t)=\sin s\cos t+\cos s\sin t=\left(-\frac{15}{17}\right)\left(-\frac{3}{5}\right)+\left(-\frac{8}{17}\right)\left(-\frac{4}{5}\right)=\frac{45}{85}+\frac{32}{85}=\frac{77}{85}$$

(b)
$$\tan(s+t)=\frac{\frac{15}{8}+\frac{4}{3}}{1-\left(\frac{15}{8}\right)\left(\frac{4}{3}\right)}=\frac{45+32}{24-60}=\frac{77}{-36}=-\frac{77}{36}$$

(c) From parts (a) and (b), sin $(s + t) > 0$ and tan $(s + t) < 0$. The only quadrant in which the value of the sine is positive and the value of the tangent is negative is quadrant II, so $s + t$ is in quadrant II.

88. $\cos s = -\frac{15}{17}$, $\sin t = \frac{4}{5}$, s is in quadrant II, and t is in quadrant I.
First find the values of sin s, cos t, tan s, and tan t. Because s is in quadrant II and t is in quadrant I, the values of sin s, cos t, and tan t will be positive, while tan s will be negative.

$$\sin s = \sqrt{1-\left(-\frac{15}{17}\right)^2} = \sqrt{1-\frac{225}{289}} = \sqrt{\frac{64}{289}} = \frac{8}{17}$$

$$\cos t = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

$$\tan s = \frac{\sin s}{\cos s} = \frac{\frac{8}{17}}{-\frac{15}{17}} = -\frac{8}{15}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

(a) $$\begin{aligned}\sin(s+t) &= \sin s\cos t + \cos s\sin t\\ &= \left(\frac{8}{17}\right)\left(\frac{3}{5}\right)+\left(-\frac{15}{17}\right)\left(\frac{4}{5}\right)\\ &= \frac{24}{85}-\frac{60}{85} = -\frac{36}{85}\end{aligned}$$

(b) $$\tan(s+t) = \frac{\left(-\frac{8}{15}\right)+\frac{4}{3}}{1-\left(-\frac{8}{15}\right)\left(\frac{4}{3}\right)} = \frac{-24+60}{45+32} = \frac{36}{77}$$

(c) To find the quadrant of $s + t$, notice from the preceding that $\sin(s+t) = -\frac{36}{85} < 0$ and $\tan(s+t) = \frac{36}{77} > 0$. The only quadrant in which the value of sine is negative and tangent is positive is quadrant III, so $s + t$ is in quadrant III.

89. $$\begin{aligned}\sin 165° &= \sin(180°-15°)\\ &= \sin 180°\cos 15° - \cos 180°\sin 15°\\ &= (0)\cos 15° - (-1)\sin 15° = 0 + \sin 15°\\ &= \sin 15°\end{aligned}$$

Now use a difference identity to find $\sin 15°$.

$$\begin{aligned}\sin 15° &= \sin(45°-30°)\\ &= \sin 45°\cos 30° - \cos 45°\sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

90. $$\begin{aligned}\tan 165° &= \tan(180°-15°)\\ &= \frac{\tan 180° - \tan 15°}{1+\tan 180°\tan 15°}\\ &= \frac{0-\tan 15°}{1+0\cdot\tan 15°} = -\tan 15°\end{aligned}$$

Now use a difference identity to find $\tan 15°$.

$$\begin{aligned}\tan 15° &= \tan(45°-30°) = \frac{\tan 45° - \tan 30°}{1+\tan 45°\tan 30°}\\ &= \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}}\\ &= \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{9-3\sqrt{3}-3\sqrt{3}+3}{9-3}\\ &= \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}\end{aligned}$$

Thus,
$\tan 165° = -\tan 15° = -\left(2-\sqrt{3}\right) = -2+\sqrt{3}.$

91. $$\begin{aligned}\sin 255° &= \sin(270°-15°)\\ &= \sin 270°\cos 15° - \cos 270°\sin 15°\\ &= (-1)\cos 15° - (0)\sin 15°\\ &= -\cos 15° - 0 = -\cos 15°\end{aligned}$$

Now use a difference identity to find $\cos 15°$.

$$\begin{aligned}\cos 15° &= \cos(45°-30°)\\ &= (\cos 45°\cos 30° + \sin 45°\sin 30°)\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Thus, $$\sin 255° = -\cos 15° = -\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) = \frac{-\sqrt{6}-\sqrt{2}}{4}$$

92. $$\begin{aligned}\tan 285° &= \tan(360°-75°)\\ &= \frac{\tan 360° - \tan 75°}{1+\tan 360°\tan 75°}\\ &= \frac{0-\tan 75°}{1+0\cdot\tan 75°} = \frac{-\tan 75°}{1+0}\\ &= -\tan 75°\end{aligned}$$

Now use a difference identity to find $\tan 75°$.

$$\begin{aligned}\tan 75° &= \tan(30°+45°) = \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}\cdot 1}\\ &= \frac{\sqrt{3}+3}{3-\sqrt{3}} = \frac{\sqrt{3}+3}{3-\sqrt{3}}\cdot\frac{\sqrt{3}+3}{3+\sqrt{3}}\end{aligned}$$

(*continued on next page*)

(*continued from page 651*)

$$= \frac{3+3\sqrt{3}+3\sqrt{3}+9}{9-3} = \frac{12+6\sqrt{3}}{6}$$
$$= 2+\sqrt{3}$$

Thus,

$$\tan 285° = -\tan 75° = -\left(2+\sqrt{3}\right) = -2-\sqrt{3}.$$

93. $\tan\frac{11\pi}{12} = \tan\left(\pi - \frac{\pi}{12}\right)$

$$= \frac{\tan\pi - \tan\frac{\pi}{12}}{1+\tan\pi\tan\frac{\pi}{12}} = -\tan\frac{\pi}{12}$$

Now use a difference identity to find $\tan\frac{\pi}{12}$.

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}}$$
$$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} = \frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}} = \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}$$
$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}$$
$$= \frac{9-6\sqrt{3}+3}{9-3} = \frac{12-6\sqrt{3}}{6}$$
$$= \frac{6\left(2-\sqrt{3}\right)}{6} = 2-\sqrt{3}$$

Thus,

$$\tan\frac{11\pi}{12} = -\tan\frac{\pi}{12} = -\left(2-\sqrt{3}\right) = -2+\sqrt{3}.$$

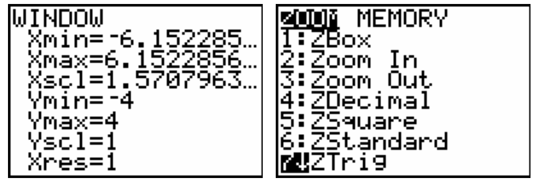
94. $\sin\left(-\frac{13\pi}{12}\right) = -\sin\left(\frac{13\pi}{12}\right) = -\sin\left(\pi+\frac{\pi}{12}\right)$

$$= -\left(\sin\pi\cos\frac{\pi}{12}+\cos\pi\sin\frac{\pi}{12}\right)$$
$$= -\left[(0)\cos\frac{\pi}{12}+(-1)\sin\frac{\pi}{12}\right]$$
$$= -\left(0-\sin\frac{\pi}{12}\right) = -\left(-\sin\frac{\pi}{12}\right)$$
$$= \sin\frac{\pi}{12}$$

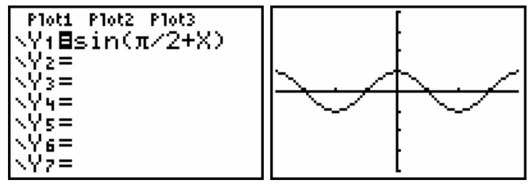
Now use a difference identity to find $\sin\frac{\pi}{12}$.

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

The graphs in exercises 95−98 are shown in the following window. This window can be obtained through the Zoom Trig feature.

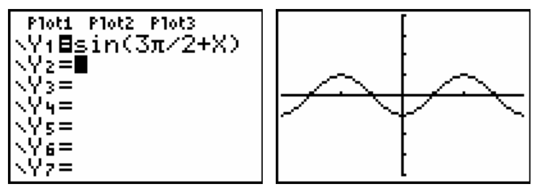


95. $\sin\left(\frac{\pi}{2}+\theta\right)$ appears to be equivalent to $\cos\theta$.



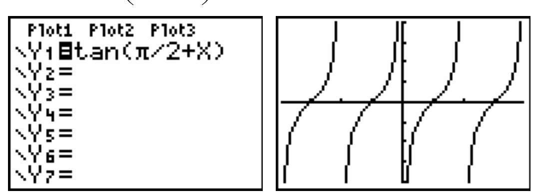
$$\sin\left(\frac{\pi}{2}+x\right) = \sin\frac{\pi}{2}\cos x+\sin x\cos\frac{\pi}{2}$$
$$= 1\cdot\cos x+\sin x\cdot 0$$
$$= \cos x+0 = \cos x$$

96. $\sin\left(\frac{3\pi}{2}+\theta\right)$ appears to be equivalent to $-\cos\theta$.



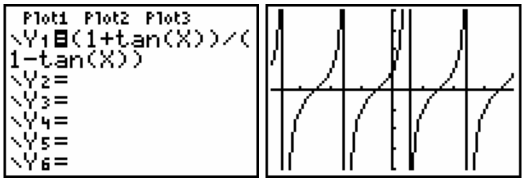
$$\sin\left(\frac{3\pi}{2}+\theta\right) = \sin\frac{3\pi}{2}\cos\theta+\cos\frac{3\pi}{2}\sin\theta$$
$$= (-1)\cos\theta+(0)\sin\theta = -\cos\theta$$

97. $\tan\left(\frac{\pi}{2}+\theta\right)$ appears to be equivalent to $-\cot\theta$.



$$\tan\left(\frac{\pi}{2}+\theta\right)$$
$$= \frac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)} = \frac{\sin\frac{\pi}{2}\cos\theta+\cos\frac{\pi}{2}\sin\theta}{\cos\frac{\pi}{2}\cos\theta-\sin\frac{\pi}{2}\sin\theta}$$
$$= \frac{1\cdot\cos\theta+0\cdot\sin\theta}{0\cdot\cos\theta-1\cdot\sin\theta} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

98. $\dfrac{1+\tan\theta}{1-\tan\theta}$ appears to be equivalent to $\tan\left(\dfrac{\pi}{4}+\theta\right)$.



Working with $\tan\left(\dfrac{\pi}{4}+\theta\right)$ we have

$$\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}=\frac{1+\tan\theta}{1-1\cdot\tan\theta}$$
$$=\frac{1+\tan\theta}{1-\tan\theta}$$

99. Verify $\sin 2x = 2\sin x\cos x$ is an identity.

$$\sin 2x = \sin(x+x) = \sin x\cos x+\cos x\sin x$$
$$= 2\sin x\cos x$$

100. Verify $\cos 2x = \cos^2 x - \sin^2 x$ is an identity.

$$\cos 2x = \cos(x+x)$$
$$= \cos x\cos x - \sin x\sin x$$
$$= \cos^2 x - \sin^2 x$$

101. Verify $1+\cos 2x - \cos^2 x = \cos^2 x$ is an identity

From exercise 65,

$\cos 2x = \cos^2 x - \sin^2 x$.

$$1+\cos 2x-\cos^2 x = 1+\cos^2 x - \sin^2 x - \cos^2 x$$
$$= 1-\sin^2 x = \cos^2 x$$

102. Verify $\sin(x+y)+\sin(x-y) = 2\sin x\cos y$ is an identity.

$$\sin(x+y)+\sin(x-y) = (\sin x\cos y+\cos x\sin y)+(\sin x\cos y-\cos x\sin y) = 2\sin x\cos y$$

103. Verify $\sin(210°+x)-\cos(210°+x)=0$ is an identity.

$$\sin(210°+x)-\cos(120°+x) = (\sin 210°\cos x+\cos 210°\sin x)-(\cos 120°\cos x - \sin 120°\sin x)$$
$$=\left(-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x\right)-\left(-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x\right)=0$$

104. Verify $\tan(x-y)-\tan(y-x)=\dfrac{2(\tan x-\tan y)}{1+\tan x\tan y}$ is an identity.

$$\tan(x-y)-\tan(y-x)=\frac{\tan x-\tan y}{1+\tan x\tan y}-\frac{\tan y-\tan x}{1+\tan y\tan x}=\frac{\tan x-\tan y-\tan y+\tan x}{1+\tan x\tan y}=\frac{2(\tan x-\tan y)}{1+\tan x\tan y}$$

105. Verify $\dfrac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}=\tan\alpha+\cot\beta$ is an identity.

$$\frac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\sin\beta}=\frac{\cos\alpha\cos\beta}{\cos\alpha\sin\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\sin\beta}=\frac{\cos\beta}{\sin\beta}+\frac{\sin\alpha}{\cos\alpha}=\cot\beta+\tan\alpha$$

106. Verify $\dfrac{\sin(s+t)}{\cos s\cos t}=\tan s+\tan t$ is an identity.

$$\frac{\sin(s+t)}{\cos s\cos t}=\frac{\sin s\cos t+\cos s\sin t}{\cos s\cos t}=\frac{\sin s\cos t}{\cos s\cos t}+\frac{\cos s\sin t}{\cos s\cos t}=\frac{\sin s}{\cos s}+\frac{\sin t}{\cos t}=\tan s+\tan t$$

107. Verify that $\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$ is an identity

$$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\sin x\cos y - \cos x\sin y}{\sin x\cos y + \cos x\sin y} = \frac{\frac{\sin x\cos y}{\cos x\cos y} - \frac{\cos x\sin y}{\cos x\cos y}}{\frac{\sin x\cos y}{\cos x\cos y} + \frac{\cos x\sin y}{\cos x\cos y}} = \frac{\frac{\sin x}{\cos x}\cdot\frac{\cos y}{\cos y} - \frac{\cos x}{\cos x}\cdot\frac{\sin y}{\cos y}}{\frac{\sin x}{\cos x}\cdot\frac{\cos y}{\cos y} + \frac{\cos x}{\cos x}\cdot\frac{\sin y}{\cos y}}$$

$$= \frac{\frac{\sin x}{\cos x}\cdot 1 - 1\cdot\frac{\sin y}{\cos y}}{\frac{\sin x}{\cos x}\cdot 1 + 1\cdot\frac{\sin y}{\cos y}} = \frac{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}} = \frac{\tan x - \tan y}{\tan x + \tan y}$$

108. Verify $\frac{\sin(x+y)}{\cos(x-y)} = \frac{\cot x + \cot y}{1 + \cot x\cot y}$ is an identity.

Working with the right side, we have

$$\frac{\cot x + \cot y}{1 + \cot x\cot y} = \frac{\frac{\cos x}{\sin x} + \frac{\cos y}{\sin y}}{1 + \frac{\cos x}{\sin x}\frac{\cos y}{\sin y}} = \frac{\frac{\cos x}{\sin x} + \frac{\cos y}{\sin y}}{1 + \frac{\cos x}{\sin x}\frac{\cos y}{\sin y}}\cdot\frac{\sin x\sin y}{\sin x\sin y} = \frac{\cos x\sin y + \sin x\cos y}{\sin x\sin y + \cos x\cos y} = \frac{\sin(x+y)}{\cos(x-y)}$$

109. Verify $\frac{\sin(s-t)}{\sin t} + \frac{\cos(s-t)}{\cos t} = \frac{\sin s}{\sin t\cos t}$ is an identity.

$$\frac{\sin(s-t)}{\sin t} + \frac{\cos(s-t)}{\cos t} = \frac{\sin s\cos t - \sin t\cos s}{\sin t} + \frac{\cos s\cos t + \sin t\sin s}{\cos t}$$

$$= \frac{\sin s\cos^2 t - \sin t\cos t\cos s}{\sin t\cos t} + \frac{\sin t\cos t\cos s + \sin^2 t\sin s}{\sin t\cos t} = \frac{\sin s\cos^2 t + \sin s\sin^2 t}{\sin t\cos t}$$

$$= \frac{\sin s\left(\cos^2 t + \sin^2 t\right)}{\sin t\cos t} = \frac{\sin s}{\sin t\cos t}$$

110. Verify $\frac{\tan(\alpha+\beta) - \tan\beta}{1 + \tan(\alpha+\beta)\tan\beta} = \tan\alpha$ is an identity.

$$\frac{\tan(\alpha+\beta) - \tan\beta}{1 + \tan(\alpha+\beta)\tan\beta} = \tan\left[(\alpha+\beta) - \beta\right]$$
$$= \tan\alpha$$

111. **(a)** $F = \frac{.6W\sin(\theta + 90°)}{\sin 12°}$
$= \frac{.6(170)\sin(30+90)°}{\sin 12°}$
$= \frac{102\sin 120°}{\sin 12°} \approx 425\text{ lb}$

(This is a good reason why people frequently have back problems.)

(b) $F = \frac{.6W\sin(\theta + 90°)}{\sin 12°}$
$= \frac{.6W(\sin\theta\cos 90° + \sin 90°\cos\theta)}{\sin 12°}$
$= \frac{.6W(\sin\theta\cdot 0 + 1\cdot\cos\theta)}{\sin 12°}$
$= \frac{.6W(0 + \cos\theta)}{\sin 12°}$
$= \frac{.6}{\sin 12°}W\cos\theta \approx 2.9W\cos\theta$

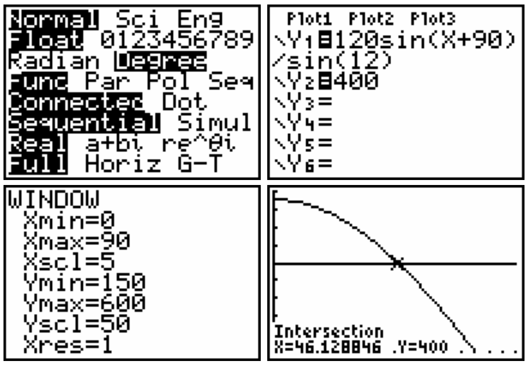
(c) F will be maximum when $\cos\theta = 1$ or $\theta = 0°$. ($\theta = 0°$ corresponds to the back being horizontal which gives a maximum force on the back muscles. This agrees with intuition since stress on the back increases as one bends farther until the back is parallel with the ground.)

112. **(a)** $F = \frac{.6W\sin(\theta + 90°)}{\sin 12°}, W = 200, \theta = 45°$
$F = \frac{.6(200)\sin(45+90)°}{\sin 12°}$
$= \frac{120\sin 135°}{\sin 12°} \approx 408\text{ lb}$

(b) The calculator should be in degree mode. Graph

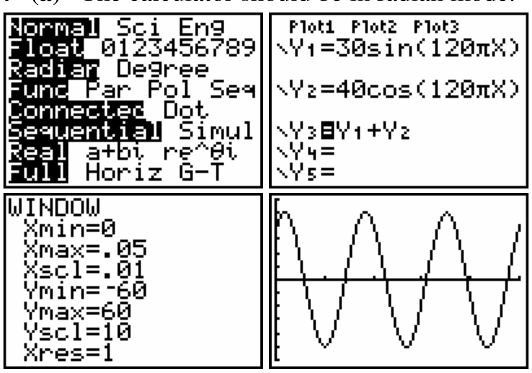
$$y=\frac{.6(200)\sin(x+90°)}{\sin 12°}=\frac{120\sin(x+90°)}{\sin 12°}$$

and $y = 400$ on the same screen to find the point of intersection.

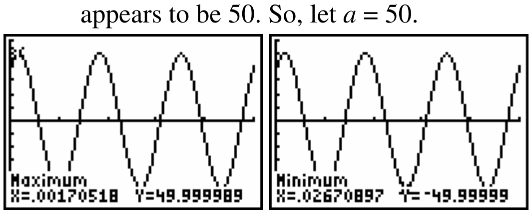


A force of 400 lb is exerted when $\theta \approx 46.1°$.

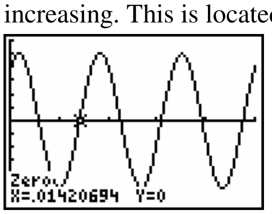
113. **(a)** The calculator should be in radian mode.



(b) Using the maximum and minimum features on your calculator, the amplitude appears to be 50. So, let $a = 50$.



Estimate the phase shift by approximating the first t-intercept where the graph of V is increasing. This is located at $t \approx .0142$.



$$\sin(120\pi t+\phi)=0 \Rightarrow$$
$$\sin[120\pi(.0142)+\phi]=0 \Rightarrow$$
$$120\pi(.0142)+\phi=0$$
$$\phi=-120\pi(.0142)$$
$$\approx -5.353$$

Thus, $V = 50 \sin(120\pi t - 5.353)$.

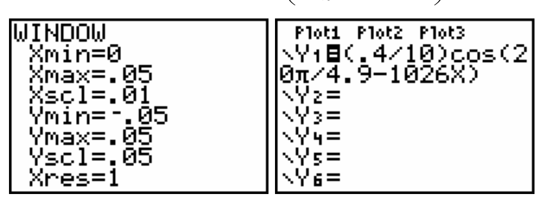
(c)
$$\begin{aligned}
&50\sin(120\pi t-5.353)\\
&=50\left[\begin{array}{l}(\sin 120\pi t)(\cos 5.353)\\ \quad -(\cos 120\pi t)(\sin 5.353)\end{array}\right]\\
&\approx 50\left[\begin{array}{l}(\sin 120\pi t)(.5977)\\ \quad -(\cos 120\pi t)(-.8017)\end{array}\right]\\
&\approx 29.89\sin 120\pi t+40.09\cos 120\pi t\\
&\approx 30\sin 120\pi t+40\cos 120\pi t
\end{aligned}$$

114.
$$\begin{aligned}
E&=20\sin\left(\frac{\pi t}{4}-\frac{\pi}{2}\right)\\
&=20\left(\sin\frac{\pi t}{4}\cos\frac{\pi}{2}-\cos\frac{\pi t}{4}\sin\frac{\pi}{2}\right)\\
&=20\left(\sin\frac{\pi t}{4}(0)-\cos\frac{\pi t}{4}(1)\right)\\
&=20\left(0-\cos\frac{\pi t}{4}\right)=-20\cos\frac{\pi t}{4}
\end{aligned}$$

115. **(a)** Since there are 60 cycles per sec, the number of cycles in .05 sec is given by (.05 sec)(60 cycles per sec) = 3 cycles

(b) Since $V = 163\sin\omega t$ and the maximum value of $\sin \omega t$ is 1, the maximum voltage is 163. Since $V = 163\sin\omega t$ and the minimum value of $\sin \omega t$ is –1, the minimum voltage is –163. Therefore, the voltage is not always equal to 115.

116. **(a)** Graph
$$\begin{aligned}
P&=\frac{a}{r}\cos\left[\frac{2\pi r}{\lambda}-ct\right]\\
&=\frac{.4}{10}\cos\left[\frac{2\pi(10)}{4.9}-1026t\right]\\
&=.04\cos\left(\frac{20\pi}{4.9}-1026t\right)
\end{aligned}$$



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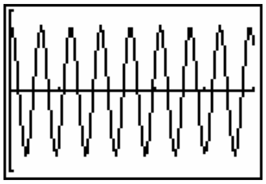
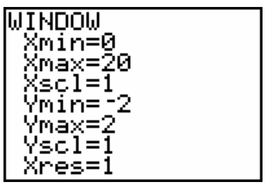


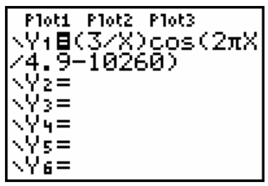
The pressure P is oscillating.

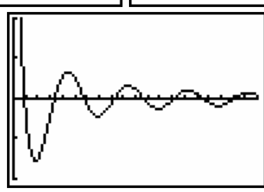
(b) Graph $P = \frac{a}{r}\cos\left[\frac{2\pi r}{\lambda} - ct\right]$

$$= \frac{3}{r}\cos\left[\frac{2\pi(r)}{4.9} - 1026(10)\right]$$

$$= \frac{3}{r}\cos\left(\frac{2\pi r}{4.9} - 10{,}260\right)$$







The pressure oscillates, and amplitude decreases as r increases.

(c) $P = \frac{a}{r}\cos\left[\frac{2\pi r}{\lambda} - ct\right]$

Let $r = n\lambda$.

$$P = \frac{a}{r}\cos\left[\frac{2\pi r}{\lambda} - ct\right]$$

$$= \frac{a}{n\lambda}\cos\left[\frac{2\pi n\lambda}{\lambda} - ct\right]$$

$$= \frac{a}{n\lambda}\cos\left[2\pi n - ct\right]$$

$$= \frac{a}{n\lambda}\left[\cos(2\pi n)\cos(ct) + \sin(2\pi n)\sin(ct)\right]$$

$$= \frac{a}{n\lambda}\left[(1)\cos(ct) + (0)\sin(ct)\right]$$

$$= \frac{a}{n\lambda}\left[\cos(ct) + 0\right] = \frac{a}{n\lambda}\cos(ct)$$

Chapter 7 Quiz
(Sections 7.1–7.3)

1. $\sin\theta = -\frac{7}{25}$, θ is in quadrant IV

In quadrant IV, the cosine and secant function values are positive. The tangent, cotangent, and cosecant function values are negative.

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(-\frac{7}{25}\right)^2}$$

$$= \sqrt{1-\frac{49}{625}} = \sqrt{\frac{576}{625}} = \frac{24}{25}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{7}{25}}{\frac{24}{25}} = -\frac{7}{24}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{7}{24}} = -\frac{24}{7}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{24}{25}} = \frac{25}{24}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{7}{25}} = -\frac{25}{7}$$

2. $\cot^2 x + \csc^2 x = \frac{\cos^2 x}{\sin^2 x} + \frac{1}{\sin^2 x} = \frac{1+\cos^2 x}{\sin^2 x}$

3. $\sin\left(-\frac{7\pi}{12}\right) = -\sin\left(\frac{7\pi}{12}\right) = -\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$

$$= -\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}\right)$$

$$= -\left[\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right) = -\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$= \frac{-\sqrt{6}-\sqrt{2}}{4}$$

4. $\cos(180° - \theta) = -\cos\theta$

5. $\cos A = \frac{3}{5}, \sin B = -\frac{5}{13}, 0 < A < \frac{\pi}{2}$, and $\pi < B < \frac{3\pi}{2}$

$\cos A = \frac{3}{5} = \frac{y}{r} \Rightarrow y = 3, r = 5$. Substituting into the Pythagorean theorem, we have $x^2 + 3^3 = 5^2 \Rightarrow x = 4$, since $\sin A > 0$. Thus, $\sin A = \frac{4}{5}$.

We will use a Pythagorean identity to find the value of cos *B*.

$$\cos B = -\sqrt{1-\left(-\frac{5}{13}\right)^2} = -\sqrt{1-\frac{25}{169}}$$
$$= -\sqrt{\frac{144}{169}} = -\frac{12}{13}$$

Note that cos *B* is negative because *B* is in quadrant III.

(a) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$$= \frac{3}{5}\left(-\frac{12}{13}\right) - \frac{4}{5}\left(-\frac{5}{13}\right)$$
$$= -\frac{36}{65} + \frac{20}{65} = -\frac{16}{65}$$

(b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$$= \frac{4}{5}\left(-\frac{12}{13}\right) + \frac{3}{5}\left(-\frac{5}{13}\right)$$
$$= -\frac{48}{65} - \frac{15}{65} = -\frac{63}{65}$$

(c) Both $\cos(A+B)$ and $\sin(A+B)$ are negative. Thus $(A+B)$ is in quadrant III.

6. $\tan\left(\frac{3\pi}{4}+x\right) = \frac{\tan\frac{3\pi}{4} + \tan x}{1-\tan\frac{3\pi}{4}\tan x}$
$$= \frac{-1+\tan x}{1-(-1)\tan x} = \frac{-1+\tan x}{1+\tan x}$$

7. Verify $\frac{1+\sin\theta}{\cot^2\theta} = \frac{\sin\theta}{\csc\theta - 1}$ is an identity.

Working with the right side, we have
$$\frac{\sin\theta}{\csc\theta-1} = \frac{\sin\theta}{\csc\theta-1}\cdot\frac{\csc\theta+1}{\csc\theta+1}$$
$$= \frac{\sin\theta\csc\theta+\sin\theta}{\csc^2\theta-1} = \frac{1+\sin\theta}{\cot^2\theta}$$

8. Verify $\frac{\sin^2\theta-\cos^2\theta}{\sin^4\theta-\cos^4\theta} = 1$ is an identity.
$$\frac{\sin^2\theta-\cos^2\theta}{\sin^4\theta-\cos^4\theta}$$
$$= \frac{\sin^2\theta-\cos^2\theta}{(\sin^2\theta-\cos^2\theta)(\sin^2\theta+\cos^2\theta)} = \frac{1}{1} = 1$$

9. Verify $\sin\left(\frac{\pi}{3}+\theta\right) - \sin\left(\frac{\pi}{3}-\theta\right) = \sin\theta$ is an identity.
$$\sin\left(\frac{\pi}{3}+\theta\right) - \sin\left(\frac{\pi}{3}-\theta\right) = \left(\sin\frac{\pi}{3}\cos\theta + \cos\frac{\pi}{3}\sin\theta\right) - \left(\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta\right)$$
$$= 2\cos\frac{\pi}{3}\sin\theta = 2\left(\frac{1}{2}\right)\sin\theta = \sin\theta$$

10. Verify $\frac{\cos(x+y)+\cos(x-y)}{\sin(x-y)+\sin(x-y)} = \cot x$ is an identity.
$$\frac{\cos(x+y)+\cos(x-y)}{\sin(x+y)+\sin(x-y)} = \frac{(\cos x\cos y - \sin x\sin y)+(\cos x\cos y+\sin x\sin y)}{(\sin x\cos y+\cos x\sin y)+(\sin x\cos y-\cos x\sin y)} = \frac{2\cos x\cos y}{2\sin x\cos y} = \frac{\cos x}{\sin x} = \cot x$$

Section 7.4: Double-Angle and Half-Angle Identities

1. C. $2\cos^2 15° - 1 = \cos(2\cdot 15°) = \cos 30° = \frac{\sqrt{3}}{2}$

2. E. $\frac{2\tan 15°}{1-\tan^2 15°} = \tan(2\cdot 15°) = \tan(30°) = \frac{\sqrt{3}}{3}$

3. B. $2\sin 22.5°\cos 22.5° = \sin(2\cdot 22.5°)$
$$= \sin 45° = \frac{\sqrt{2}}{2}$$

4. A. $\cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6} = \cos\left(2\cdot\frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

5. C. $2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \sin\left(2\cdot\frac{\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

6. D. $\frac{2\tan\frac{\pi}{3}}{1-\tan^2\frac{\pi}{3}} = \tan\left(2\cdot\frac{\pi}{3}\right) = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

7. $\cos 2\theta = \frac{3}{5}, \theta$ is in quadrant I.

$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \frac{3}{5} = 2\cos^2\theta - 1 \Rightarrow$

$2\cos^2\theta = \frac{3}{5} + 1 = \frac{8}{5} \Rightarrow \cos^2\theta = \frac{8}{10} = \frac{4}{5}$

Since θ is in quadrant I, $\cos\theta > 0$. Thus,

$\cos\theta = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$.

Since θ is in quadrant I, $\sin\theta > 0$.

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{2\sqrt{5}}{5}\right)^2} = \sqrt{1-\frac{20}{25}}$$
$$= \sqrt{\frac{5}{25}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

8. $\cos 2\theta = \frac{3}{4}, \theta$ is in quadrant III.

$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \frac{3}{4} = 2\cos^2\theta - 1 \Rightarrow$

$2\cos^2\theta = \frac{7}{4} \Rightarrow \cos^2\theta = \frac{7}{8}$

Since θ is in quadrant III, $\cos\theta < 0$.

$$\cos\theta = -\sqrt{\frac{7}{8}} = -\frac{\sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{7}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{14}}{4}$$

Since θ is in quadrant III, $\sin\theta < 0$.

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\frac{7}{8}} = -\sqrt{\frac{1}{8}}$$
$$= -\frac{1}{2\sqrt{2}} = -\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

9. $\cos 2\theta = -\frac{5}{12}, \frac{\pi}{2} < \theta < \pi$

$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow$

$2\cos^2\theta = \cos 2\theta + 1 = -\frac{5}{12} + 1 = \frac{7}{12} \Rightarrow$

$\cos^2\theta = \frac{7}{24}$

Since $\frac{\pi}{2} < \theta < \pi, \cos\theta < 0$. Thus,

$$\cos\theta = -\sqrt{\frac{7}{24}} = -\frac{\sqrt{7}}{\sqrt{24}} = -\frac{\sqrt{7}}{2\sqrt{6}}$$
$$= -\frac{\sqrt{7}}{2\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = -\frac{\sqrt{42}}{12}$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(-\sqrt{\frac{7}{24}}\right)^2$$
$$= 1-\frac{7}{24} = \frac{17}{24}$$

Since $\frac{\pi}{2} < \theta < \pi, \sin\theta > 0$. Thus,

$$\sin\theta = \sqrt{\frac{17}{24}} = \frac{\sqrt{17}}{\sqrt{24}} = \frac{\sqrt{17}}{2\sqrt{6}} = \frac{\sqrt{17}}{2\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{102}}{12}$$

10. $\cos 2x = \frac{2}{3}, \frac{\pi}{2} < x < \pi$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow \frac{2}{3} = 2\cos^2 x - 1 \Rightarrow$

$\cos^2 x = \frac{5}{6}$

Since x is in quadrant II, $\cos x < 0$.

$$\cos x = -\sqrt{\frac{5}{6}} = -\frac{\sqrt{5}}{\sqrt{6}} = -\frac{\sqrt{5}}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = -\frac{\sqrt{30}}{6}$$

Since x is in quadrant II, $\sin x > 0$.

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(-\sqrt{\frac{5}{6}}\right)^2} = \sqrt{1-\frac{5}{6}}$$
$$= \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

11. $\sin\theta = \frac{2}{5}, \cos\theta < 0$

$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow$

$\cos 2\theta = 1 - 2\left(\frac{2}{5}\right)^2 1 - 2\cdot\frac{4}{25} = 1 - \frac{8}{25} = \frac{17}{25}$

$\cos^2 2\theta + \sin^2 2\theta = 1 \Rightarrow$

$\sin^2 2\theta = 1 - \cos^2 2\theta \Rightarrow$

$\sin^2 2\theta = 1 - \left(\frac{17}{25}\right)^2 = 1 - \frac{289}{625} = \frac{336}{625}$

Since $\cos\theta < 0, \sin 2\theta < 0$ because $\sin 2\theta = 2\sin\theta\cos\theta < 0$ and $\sin\theta > 0$.

$$\sin 2\theta = -\sqrt{\frac{336}{625}} = -\frac{\sqrt{336}}{25} = -\frac{4\sqrt{21}}{25}$$

12. $\cos\theta = -\frac{12}{13}, \sin\theta > 0$

$\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow$

$$\sin\theta = \sqrt{1-\left(-\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = -\frac{120}{169}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{12}{13}\right)^2 - 1$$
$$= 2\cdot\frac{144}{169} - 1 = \frac{288}{169} - 1 = \frac{119}{169}$$

13. $\tan x = 2, \cos x > 0$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2(2)}{1-2^2} = \frac{4}{1-4} = -\frac{4}{3}$$

Since both tan x and cos x are positive, x must be in quadrant I. Since $0° < x < 90°$, then $0° < 2x < 180°$. Thus, $2x$ must be in either quadrant I or quadrant II. However, $\tan 2x < 0$, so $2x$ is in quadrant II, and $\sec 2x$ is negative.

$$\sec^2 2x = 1+\tan^2 2x = 1+\left(-\frac{4}{3}\right)^2 = 1+\frac{16}{9} = \frac{25}{9}$$

$$\sec 2x = -\frac{5}{3} \Rightarrow \cos 2x = \frac{1}{\sec 2x} = -\frac{3}{5}$$

$$\cos^2 2x + \sin^2 2x = 1 \Rightarrow \sin^2 2x = 1-\cos^2 2x \Rightarrow$$

$$\sin^2 2x = 1-\left(-\frac{3}{5}\right)^2 \Rightarrow \sin^2 2x = 1-\frac{9}{25} = \frac{16}{25}$$

In quadrants I and II, $\sin 2x > 0$. Thus, we have $\sin 2x = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

14. $\tan x = \frac{5}{3}, \sin x < 0$

$$\sec^2 x = 1+\tan^2 x = 1+\left(\frac{5}{3}\right)^2 = 1+\frac{25}{9} = \frac{34}{9}$$

Since $\sin x < 0$, and $\tan x > 0$, x is in quadrant III, so cos x and sec x are negative.

$$\sec x = -\frac{\sqrt{34}}{3}$$

$$\cos x = -\frac{3}{\sqrt{34}} = -\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34}$$

$$\sin x = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(-\frac{3\sqrt{34}}{34}\right)}$$

$$= -\sqrt{1-\frac{306}{1156}} = -\sqrt{\frac{850}{1156}} = -\frac{5\sqrt{34}}{34}$$

$$\cos 2x = 2\cos^2 x - 1 = 2\left(\frac{9}{34}\right)-1$$

$$= \frac{9}{17}-1 = -\frac{8}{17}$$

$$\sin 2x = 2\sin x\cos x$$

$$= 2\left(-\frac{5\sqrt{34}}{34}\right)\left(-\frac{3\sqrt{34}}{34}\right) = \frac{15}{17}$$

15. $\sin\theta = -\frac{\sqrt{5}}{7}, \cos\theta > 0$

$$\cos^2\theta = 1-\sin^2\theta = 1-\left(-\frac{\sqrt{5}}{7}\right)^2 = 1-\frac{5}{49} = \frac{44}{49}$$

Since $\cos\theta > 0$, $\cos\theta = \sqrt{\frac{44}{49}} = \frac{\sqrt{44}}{7} = \frac{2\sqrt{11}}{7}$.

$$\cos 2\theta = 1-2\sin^2\theta = 1-2\left(-\frac{\sqrt{5}}{7}\right)^2$$

$$= 1-2\cdot\frac{5}{49} = 1-\frac{10}{49} = \frac{39}{49}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{5}}{7}\right)\left(\frac{2\sqrt{11}}{7}\right)$$

$$= -\frac{4\sqrt{55}}{49}$$

16. $\cos\theta = \frac{\sqrt{3}}{5}, \sin\theta > 0$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{\sqrt{3}}{5}\right)^2} = \sqrt{1-\frac{3}{25}}$$

$$= \sqrt{\frac{22}{25}} = \frac{\sqrt{22}}{5}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{\sqrt{3}}{5}\right)^2 - 1 = 2\cdot\frac{3}{25}-1$$

$$= \frac{6}{25}-1 = -\frac{19}{25}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{22}}{5}\right)\left(\frac{\sqrt{3}}{5}\right) = \frac{2\sqrt{66}}{25}$$

17. $\cos^2 15° - \sin^2 15° = \cos\left[2(15°)\right]$

$$= \cos 30° = \frac{\sqrt{3}}{2}$$

18. $\frac{2\tan 15°}{1-\tan^2 15°} = \frac{\tan 15° + \tan 15°}{1-\tan^2 15°}$

$$= \tan 2(15°) = \tan 30° = \frac{\sqrt{3}}{3}$$

19. $1-2\sin^2 15° = \cos\left[2(15°)\right] = \cos 30° = \frac{\sqrt{3}}{2}$

20. $1-2\sin^2 22\frac{1}{2}° = \cos 2\left(22\frac{1}{2}°\right) = \cos 45° = \frac{\sqrt{2}}{2}$

21. $2\cos^2 67\frac{1}{2}° - 1 = \cos^2 67\frac{1}{2}° - \sin^2 67\frac{1}{2}°$

$$= \cos 2\left(67\frac{1}{2}°\right) = \cos 135°$$

$$= -\frac{\sqrt{2}}{2}$$

22. $\cos^2\frac{\pi}{8}-\frac{1}{2}=\frac{1}{2}\left(2\cos^2\frac{\pi}{8}-1\right)$
$=\frac{1}{2}\left[\cos\left(2\cdot\frac{\pi}{8}\right)\right]=\frac{1}{2}\cos\frac{\pi}{4}$
$=\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{4}$

23. $\frac{\tan 51°}{1-\tan^2 51°}$

Since $\frac{2\tan A}{1-\tan^2 A}=\tan 2A$, we have

$\frac{1}{2}\left(\frac{2\tan A}{1-\tan^2 A}\right)=\frac{1}{2}\tan 2A\Rightarrow$
$\frac{\tan A}{1-\tan^2 A}=\frac{1}{2}\tan 2A\Rightarrow$
$\frac{\tan 51°}{1-\tan^2 51°}=\frac{1}{2}\tan\left[2(51°)\right]=\frac{1}{2}\tan 102°$

24. $\frac{\tan 34°}{2\left(1-\tan^2 34°\right)}=\frac{1}{4}\left(\frac{2\tan 34°}{1-\tan^2 34°}\right)$
$=\frac{1}{4}\tan\left[2(34°)\right]=\frac{1}{4}\tan 68°$

25. $\frac{1}{4}-\frac{1}{2}\sin^2 47.1°=\frac{1}{4}\left(1-2\sin^2 47.1°\right)$
$=\frac{1}{4}\cos\left[2(47.1°)\right]$
$=\frac{1}{4}\cos 94.2°$

26. $\frac{1}{8}\sin 29.5°\cos 29.5°=\frac{1}{16}\left(2\sin 29.5°\cos 29.5°\right)$
$=\frac{1}{16}\sin\left[2(29.5°)\right]$
$=\frac{1}{16}\sin 59°$

27. $\sin^2\frac{2\pi}{5}-\cos^2\frac{2\pi}{5}=-\left(\cos^2\frac{2\pi}{5}-\sin^2\frac{2\pi}{5}\right)$

$\cos^2 A-\sin^2 A=\cos 2A\Rightarrow$
$-\left(\cos^2\frac{2\pi}{5}-\sin^2\frac{2\pi}{5}\right)=-\cos\left(2\cdot\frac{2\pi}{5}\right)$
$=-\cos\frac{4\pi}{5}$

28. $\cos^2 A-\sin^2 A=\cos 2A\Rightarrow$
$\cos^2 2x-\sin^2 2x=\cos(2\cdot 2x)=\cos 4x$

29. $\sin 4x=\sin\left[2(2x)\right]=2\sin 2x\cos 2x$
$=2(2\sin x\cos x)\left(\cos^2 x-\sin^2 x\right)$
$=4\sin x\cos^3 x-4\sin^3 x\cos x$

30. $\cos 3x=\cos(2x+x)$
$=\cos 2x\cos x-\sin 2x\sin x$
$=\left(1-2\sin^2 x\right)\cos x-(2\sin x\cos x)\sin x$
$=\cos x-2\sin^2 x\cos x-2\sin^2 x\cos x$
$=\cos x-4\sin^2 x\cos x$
$=\cos x\left(1-4\sin^2 x\right)$
$=\cos x\left[1-4\left(1-\cos^2 x\right)\right]$
$=\cos x\left(-3+4\cos^2 x\right)$
$=-3\cos x+4\cos^3 x$

31. $\tan 3x=\tan(2x+x)=\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$

$$=\frac{\frac{2\tan x}{1-\tan^2 x}+\tan x}{1-\frac{2\tan x}{1-\tan^2 x}\cdot\tan x}$$

$$=\frac{\frac{2\tan x}{1-\tan^2 x}+\frac{\left(1-\tan^2 x\right)\tan x}{1-\tan^2 x}}{\frac{1-\tan^2 x}{1-\tan^2 x}-\frac{2\tan^2 x}{1-\tan^2 x}}$$

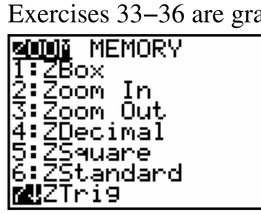
$$=\frac{\frac{2\tan x}{1-\tan^2 x}+\frac{\tan x-\tan^3 x}{1-\tan^2 x}}{\frac{1-\tan^2 x}{1-\tan^2 x}-\frac{2\tan^2 x}{1-\tan^2 x}}\cdot\frac{1-\tan^2 x}{1-\tan^2 x}$$

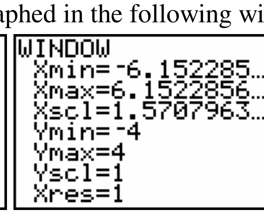
$$=\frac{2\tan x+\tan x-\tan^3 x}{1-\tan^2 x-2\tan^2 x}$$

$$=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$$

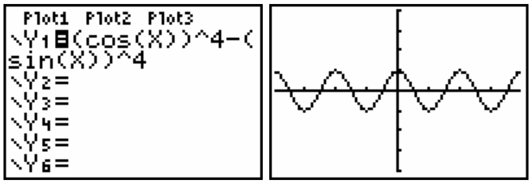
32. $\cos 4x=\cos\left[2(2x)\right]=2\cos^2 2x-1$
$=2\left(2\cos^2 x-1\right)^2-1$
$=2\left(4\cos^4 x-4\cos^2 x+1\right)-1$
$=8\cos^4 x-8\cos^2 x+2-1$
$=8\cos^4 x-8\cos^2 x+1$

Exercises 33–36 are graphed in the following window:



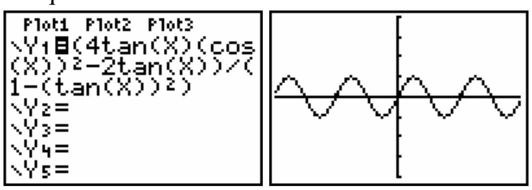


33. $\cos^4 x - \sin^4 x$ appears to be equivalent to $\cos 2x$.



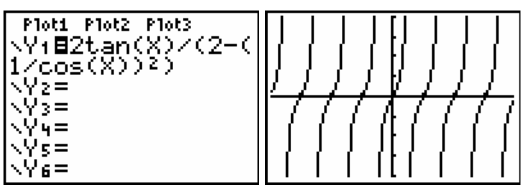
$$\begin{aligned}&\cos^4 x - \sin^4 x\\ &\quad= \left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right)\\ &\quad= 1\cdot \cos 2x = \cos 2x\end{aligned}$$

34. $\dfrac{4\tan x\cos^2 x - 2\tan x}{1-\tan^2 x}$ appears to be equivalent to $\sin 2x$.



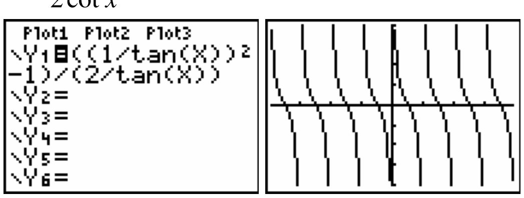
$$\begin{aligned}\frac{4\tan x\cos^2 x - 2\tan x}{1-\tan^2 x} &= \frac{2\tan x\left(2\cos^2 x - 1\right)}{1-\tan^2 x}\\ &= \frac{2\tan x}{1-\tan^2 x}\left(2\cos^2 x - 1\right)\\ &= \tan 2x\cos 2x\\ &= \frac{\sin 2x}{\cos 2x}\cdot\cos 2x\\ &= \sin 2x\end{aligned}$$

35. $\dfrac{2\tan x}{2-\sec^2 x}$ appears to be equivalent to $\tan 2x$.



$$\begin{aligned}\frac{2\tan x}{2-\sec^2 x} &= \frac{2\tan x}{1-\left(\sec^2 x - 1\right)}\\ &= \frac{2\tan x}{1-\tan^2 x} = \tan 2x\end{aligned}$$

36. $\dfrac{\cot^2 x - 1}{2\cot x}$ appears to be equivalent to $\cot 2x$.



$$\begin{aligned}\frac{\cot^2 x - 1}{2\cot x} &= \frac{\frac{1}{\tan^2 x} - 1}{\frac{2}{\tan x}}\\ &= \frac{\frac{1}{\tan^2 x} - 1}{\frac{2}{\tan x}}\cdot\frac{\tan^2 x}{\tan^2 x} = \frac{1-\tan^2 x}{2\tan x}\\ &= \frac{1}{\frac{2\tan x}{1-\tan^2 x}} = \frac{1}{\tan 2x} = \cot 2x\end{aligned}$$

37. Verify $\left(\sin x + \cos x\right)^2 = \sin 2x + 1$ is an identity.

$$\begin{aligned}\left(\sin x + \cos x\right)^2 &= \sin^2 x + 2\sin x\cos x + \cos^2 x\\ &= \left(\sin^2 x + \cos^2 x\right) + 2\sin x\cos x\\ &= 1 + \sin 2x\end{aligned}$$

38. Verify $\sec 2x = \dfrac{\sec^2 x + \sec^4 x}{2+\sec^2 x - \sec^4 x}$ is an identity.

Work with the right side.

$$\begin{aligned}&\frac{\sec^2 x + \sec^4 x}{2+\sec^2 x - \sec^4 x}\\ &= \frac{\frac{1}{\cos^2 x} + \frac{1}{\cos^4 x}}{2 + \frac{1}{\cos^2 x} - \frac{1}{\cos^4 x}}\\ &= \frac{\frac{1}{\cos^2 x} + \frac{1}{\cos^4 x}}{2 + \frac{1}{\cos^2 x} - \frac{1}{\cos^4 x}}\cdot\frac{\cos^4 x}{\cos^4 x}\\ &= \frac{\cos^2 x + 1}{2\cos^4 x + \cos^2 x - 1}\\ &= \frac{\cos^2 x + 1}{\left(2\cos^2 x - 1\right)\left(\cos^2 x + 1\right)}\\ &= \frac{1}{2\cos^2 x - 1} = \frac{1}{\cos 2x} = \sec 2x\end{aligned}$$

39. Verify $\tan 8\theta - \tan 8\theta\tan^2 4\theta = 2\tan 4\theta$ is an identity.

$$\begin{aligned}&\tan 8\theta - \tan 8\theta\tan^2 4\theta\\ &\quad= \tan 8\theta\left(1-\tan^2 4\theta\right)\\ &\quad= \frac{2\tan 4\theta}{1-\tan^2 4\theta}\left(1-\tan^2 4\theta\right) = 2\tan 4\theta\end{aligned}$$

40. Verify $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

Working with the right side, we have

$$\frac{2\tan x}{1+\tan^2 x} = \frac{2\cdot\frac{\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}} = \frac{2\cdot\frac{\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}}\cdot\frac{\cos^2 x}{\cos^2 x}$$
$$= \frac{2\sin x\cos x}{\cos^2 x+\sin^2 x} = 2\sin x\cos x$$
$$= \sin 2x$$

41. Verify $\cos 2\theta = \dfrac{2-\sec^2\theta}{\sec^2\theta}$ is an identity.

Working with the right side, we have

$$\frac{2-\sec^2\theta}{\sec^2\theta} = \frac{2-\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} = \frac{2-\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\cdot\frac{\cos^2\theta}{\cos^2\theta}$$
$$= \frac{2\cos^2\theta-1}{1} = \cos 2\theta$$

42. Verify $-\tan 2\theta = \dfrac{2\tan\theta}{\sec^2\theta-2}$ is an identity.

Working with the right side, we have

$$\frac{2\tan\theta}{\sec^2\theta-2} = \frac{2\tan\theta}{\left(1+\tan^2\theta\right)-2}$$
$$= \frac{2\tan\theta}{\tan^2\theta-1} = -\tan 2\theta$$

43. Verify that $\sin 4x = 4\sin x\cos x\cos 2x$ is an identity.

$$\sin 4x = \sin 2(2x) = 2\sin 2x\cos 2x$$
$$= 2(2\sin x\cos x)\cos 2x$$
$$= 4\sin x\cos x\cos 2x$$

44. Verify $\dfrac{1+\cos 2x}{\sin 2x} = \cot x$ is an identity.

$$\frac{1+\cos 2x}{\sin 2x} = \frac{1+\left(2\cos^2 x-1\right)}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x\cos x}$$
$$= \frac{\cos x}{\sin x} = \cot x$$

45. Verify $\dfrac{2\cos 2\theta}{\sin 2\theta} = \cot\theta-\tan\theta$ is an identity.

Work with the right side.

$$\cot\theta-\tan\theta = \frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}$$
$$= \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$$
$$= \frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta} = \frac{2\left(\cos^2\theta-\sin^2\theta\right)}{2\sin\theta\cos\theta} = \frac{2\cos 2\theta}{\sin 2\theta}$$

46. Verify $\cot 4\theta = \dfrac{1-\tan^2 2\theta}{2\tan 2\theta}$ is an identity.

$$\cot 4\theta = \frac{1}{\tan 4\theta} = \frac{1}{\tan 2(2\theta)} = \frac{1}{\frac{2\tan 2\theta}{1-\tan^2 2\theta}}$$
$$= \frac{1-\tan^2 2\theta}{2\tan 2\theta}$$

47. Verify $\tan x+\cot x = 2\csc 2x$ is an identity.

$$\tan x+\cot x = \frac{\sin x}{\cos x}\cdot\frac{\sin x}{\sin x}+\frac{\cos x}{\sin x}\cdot\frac{\cos x}{\cos x}$$
$$= \frac{\sin^2 x+\cos^2 x}{\cos x\sin x} = \frac{1}{\cos x\sin x}$$
$$= \frac{2}{2\cos x\sin x} = \frac{2}{\sin 2x} = 2\csc 2x$$

48. Verify $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ is an identity.

Work with the right side.

$$\frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}} = \frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}\cdot\frac{\cos^2 x}{\cos^2 x}$$
$$= \frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x} = \frac{\cos^2 x-\sin^2 x}{1}$$
$$= \cos^2 x-\sin^2 x = \cos 2x$$

49. Verify $1+\tan x\tan 2x = \sec 2x$ is an identity.

$1+\tan x\tan 2x$

$$= 1+\tan x\left(\frac{2\tan x}{1-\tan^2 x}\right) = 1+\frac{2\tan^2 x}{1-\tan^2 x}$$
$$= \frac{\left(1-\tan^2 x\right)+2\tan^2 x}{1-\tan^2 x} = \frac{1+\tan^2 x}{1-\tan^2 x}$$
$$= \frac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}} = \frac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}}\cdot\frac{\cos^2 x}{\cos^2 x}$$
$$= \frac{\cos^2 x+\sin^2 x}{\cos^2 x-\sin^2 x} = \frac{1}{\cos 2x} = \sec 2x$$

50. Verify $\frac{\cot A-\tan A}{\cot A+\tan A}=\cos 2A$ is an identity.

$$\frac{\cot A-\tan A}{\cot A+\tan A}=\frac{\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}}{\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}}=\frac{\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}}{\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}}\cdot\frac{\sin A\cos A}{\sin A\cos A}=\frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}=\frac{\cos^2 A-\sin^2 A}{1}$$
$$=\cos^2 A-\sin^2 A=\cos 2A$$

51. $2\sin 58°\cos 102°=2\left(\frac{1}{2}\left[\sin\left(58°+102°\right)+\sin\left(58°-102°\right)\right]\right)=\sin 160°+\sin\left(-44°\right)=\sin 160°-\sin 44°$

52. $2\cos 85°\sin 140°=2\left(\frac{1}{2}\left[\sin\left(85°+140°\right)-\sin\left(85°-140°\right)\right]\right)=\sin 225°-\sin\left(-55°\right)=\sin 225°+\sin 55°$

53. $2\sin\frac{\pi}{6}\cos\frac{\pi}{3}=2\cdot\frac{1}{2}\left[\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\right]=\sin\frac{\pi}{2}+\sin\left(-\frac{\pi}{6}\right)=\sin\frac{\pi}{2}-\sin\frac{\pi}{6}$

54. $5\cos 3x\cos 2x=5\left(\frac{1}{2}\left[\cos\left(3x+2x\right)+\cos\left(3x-2x\right)\right]\right)=\frac{5}{2}\left(\cos 5x+\cos x\right)=\frac{5}{2}\cos 5x+\frac{5}{2}\cos x$

55. $6\sin 4x\sin 5x=6\cdot\frac{1}{2}\left[\cos\left(4x-5x\right)-\cos\left(4x+5x\right)\right]=3\cos\left(-x\right)-3\cos 9x=3\cos x-3\cos 9x$

56. $8\sin 7x\sin 9x=8\cdot\frac{1}{2}\left[\cos\left(7x-9x\right)-\cos\left(7x+9x\right)\right]=4\left[\cos\left(-2x\right)-\cos 16x\right]$
$=4\left(\cos 2x-\cos 16x\right)=4\cos 2x-4\cos 16x$

57. $\cos 4x-\cos 2x=-2\sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)=-2\sin\frac{6x}{2}\sin\frac{2x}{2}=-2\sin 3x\sin x$

58. $\cos 5x+\cos 8x=2\cos\left(\frac{5x+8x}{2}\right)\cos\left(\frac{5x-8x}{2}\right)=2\cos\frac{13x}{2}\cos\frac{-3x}{2}=2\cos 6.5x\cos 1.5x$

59. $\sin 25°+\sin(-48°)=2\sin\left(\frac{25°+(-48°)}{2}\right)\cos\left(\frac{25°-(-48°)}{2}\right)=2\sin\frac{-23°}{2}\cos\frac{73°}{2}$
$=2\sin\left(-11.5°\right)\cos 36.5°=-2\sin 11.5°\cos 36.5°$

60. $\sin 102°-\sin 95°=2\cos\left(\frac{102°+95°}{2}\right)\sin\left(\frac{102°-95°}{2}\right)=2\cos\frac{197°}{2}\sin\frac{7°}{2}=2\cos 98.5°\sin 3.5°$

61. $\cos 4x+\cos 8x=2\cos\left(\frac{4x+8x}{2}\right)\cos\left(\frac{4x-8x}{2}\right)=2\cos\frac{12x}{2}\cos\frac{-4x}{2}=2\cos 6x\cos\left(-2x\right)=2\cos 6x\cos 2x$

62. $\sin 9x-\sin 3x=2\cos\left(\frac{9x+3x}{2}\right)\sin\left(\frac{9x-3x}{2}\right)=2\cos\frac{12x}{2}\sin\frac{6x}{2}=2\cos 6x\sin 3x$

63. C. $\sin 15°=\sin\frac{30°}{2}=\sqrt{\frac{1-\cos 30°}{2}}=\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$
$=\sqrt{\frac{2-\sqrt{3}}{2\cdot 2}}=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{\sqrt{2-\sqrt{3}}}{2}$

64. A. $\tan 15°=\tan\frac{30°}{2}=\frac{1-\cos 30°}{\sin 30°}=\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$
$=\frac{2-\sqrt{3}}{1}=2-\sqrt{3}$

65. D. $\cos\frac{\pi}{8} = \cos\frac{\frac{\pi}{4}}{2} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$
$= \sqrt{\frac{2+\sqrt{2}}{2\cdot 2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

66. E. $\tan\left(-\frac{\pi}{8}\right) = \tan\left(\frac{-\frac{\pi}{4}}{2}\right) = \frac{1-\cos\left(-\frac{\pi}{4}\right)}{\sin\left(-\frac{\pi}{4}\right)}$
$= \frac{1-\cos\frac{\pi}{4}}{-\sin\frac{\pi}{4}} = \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$
$= \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}\cdot\frac{2}{2} = \frac{2-\sqrt{2}}{-\sqrt{2}}$
$= \frac{2-\sqrt{2}}{-\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{-2} = 1-\sqrt{2}$

67. F. $\tan 67.5° = \tan\frac{135°}{2} = \frac{1-\cos 135°}{\sin 135°}$
$= \frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}} = \frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}}\cdot\frac{2}{2}$
$= \frac{2+\sqrt{2}}{\sqrt{2}} = \frac{2+\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{2\sqrt{2}+2}{2} = \sqrt{2}+1$

68. B. $\cos 67.5° = \cos\frac{135°}{2} = \sqrt{\frac{1+\cos 135°}{2}}$
$= \sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}\cdot\frac{2}{2}}$
$= \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

69. $\sin 67.5° = \sin\left(\frac{135°}{2}\right)$

Since 67.5° is in quadrant I, sin 67.5° > 0.

$\sin 67.5° = \sqrt{\frac{1-\cos 135°}{2}} = \sqrt{\frac{1-(-\cos 45°)}{2}}$
$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}\cdot\frac{2}{2}}$
$= \frac{\sqrt{2+\sqrt{2}}}{\sqrt{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

70. $\sin 195° = \sin\left(\frac{390°}{2}\right)$

Since 195° is in quadrant III, sin 195° < 0.

$\sin 195° = -\sqrt{\frac{1-\cos 390°}{2}} = -\sqrt{\frac{1-\cos 30°}{2}}$
$= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}}$
$= -\frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2}$

71. $\cos 195° = \cos\left(\frac{390°}{2}\right)$

Since 195° is in quadrant III, cos 195° < 0.

$\cos 195° = -\sqrt{\frac{1+\cos 390°}{2}} = -\sqrt{\frac{1+\cos 30°}{2}}$
$= -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}} = \frac{-\sqrt{2+\sqrt{3}}}{2}$

72. $\tan 195° = \tan\left(\frac{390°}{2}\right) = \frac{\sin 390°}{1+\cos 390°}$
$= \frac{\sin 30°}{1+\cos 30°} = \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}$
$= \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}\cdot\frac{2}{2} = \frac{1}{2+\sqrt{3}}$
$= \frac{1}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}$
$= \frac{2-\sqrt{3}}{4-3} = \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$

73. $\cos 165° = \cos\left(\frac{330°}{2}\right)$

Since 165° is in quadrant II, cos 165° < 0.

$\cos 165° = -\sqrt{\frac{1+\cos 330°}{2}} = -\sqrt{\frac{1+\cos 30°}{2}}$
$= -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}}$
$= \frac{-\sqrt{2+\sqrt{3}}}{2}$

74. $\sin 165° = \sin\left(\frac{330°}{2}\right)$

Since 165° is in quadrant II, sin 165° > 0.

$\sin 165° = \sqrt{\frac{1-\cos 330°}{2}} = \sqrt{\frac{1-\cos 30°}{2}}$
$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

75. Find $\cos\frac{x}{2}$, given $\cos x = \frac{1}{4}$, with $0 < x < \frac{\pi}{2}$.

Since $0 < x < \frac{\pi}{2} \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{4}$, $\cos\frac{x}{2} > 0$.

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1+\frac{1}{4}}{2}} = \sqrt{\frac{4+1}{8}} = \sqrt{\frac{5}{8}}$$
$$= \frac{\sqrt{5}}{\sqrt{8}} = \frac{\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{5}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{4}$$

76. Find $\sin\frac{x}{2}$ if $\cos x = -\frac{5}{8}$, with $\frac{\pi}{2} < x < \pi$.

Since $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$, $\sin\frac{x}{2} > 0$.

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\left(-\frac{5}{8}\right)}{2}} = \sqrt{\frac{1+\frac{5}{8}}{2}}$$
$$= \sqrt{\frac{8+5}{16}} = \sqrt{\frac{13}{16}} = \frac{\sqrt{13}}{4}$$

77. Find $\tan\frac{\theta}{2}$, given $\sin\theta = \frac{3}{5}$, with $90° < \theta < 180°$.

To find $\tan\frac{\theta}{2}$, we need the values of $\sin\theta$ and $\cos\theta$. We know $\sin\theta = \frac{3}{5}$.

$$\cos\theta = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\sqrt{1-\frac{9}{25}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

Since $90° < \theta < 180°$ (θ is in quadrant II), $\cos\theta < 0$. Thus, $\cos\theta = -\frac{4}{5}$.

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{\frac{3}{5}}{1+\left(-\frac{4}{5}\right)} = \frac{\frac{3}{5}}{1+\left(-\frac{4}{5}\right)} \cdot \frac{5}{5}$$
$$= \frac{3}{5-4} = \frac{3}{1} = 3$$

78. Find $\cos\frac{\theta}{2}$, if $\sin\theta = -\frac{4}{5}$, with $180° < \theta < 270°$.

Since $180° < \theta < 270°$, θ is in quadrant III, Thus, $\cos\theta < 0$.

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(-\frac{4}{5}\right)^2}$$
$$= -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

Now, find $\cos\frac{\theta}{2}$.

Since $180° < \theta < 270° \Rightarrow 90° < \frac{\theta}{2} < 135°$, $\frac{\theta}{2}$ is in quadrant II. Thus, $\cos\frac{\theta}{2} < 0$.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}$$
$$= -\sqrt{\frac{\frac{2}{5}}{2}} = -\sqrt{\frac{2}{10}} = -\sqrt{\frac{1}{5}}$$
$$= -\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

79. Find $\sin\frac{x}{2}$, given $\tan x = 2$, with $0 < x < \frac{\pi}{2}$.

Since x is in quadrant I, $\sec x > 0$.

$\sec^2 x = \tan^2 x + 1 \Rightarrow$
$\sec^2 x = 2^2 + 1 = 4 + 1 = 5 \Rightarrow \sec x = \sqrt{5}$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Since $0 < x < \frac{\pi}{2} \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{4}$, $\frac{x}{2}$ is in quadrant I. Thus, $\sin\frac{x}{2} > 0$.

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\frac{\sqrt{5}}{5}}{2}} = \frac{\sqrt{50-10\sqrt{5}}}{10}$$

80. Find $\cos\frac{x}{2}$ if $\cot x = -3$, with $\frac{\pi}{2} < x < \pi$.

Use identities to find $\cos x$. Since $\tan x = \frac{1}{\cot x} = -\frac{1}{3}$, we have

$\sec^2 x = \tan^2 x + 1 \Rightarrow \sec^2 x = \left(-\frac{1}{3}\right)^2 + 1 \Rightarrow$

$\sec^2 x = \frac{1}{9} + 1 = \frac{10}{9}$

Since $\frac{\pi}{2} < x < \pi$, $\sec x < 0$. Thus,

$\sec x = -\frac{\sqrt{10}}{3}$ and

$$\cos x = -\frac{3}{\sqrt{10}} = -\frac{3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}.$$

Since $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$, $\frac{x}{2}$ is in quadrant I. Thus, $\cos\frac{x}{2} > 0$.

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(*continued from page 665*)

$$\cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}}=\sqrt{\frac{1-\frac{3\sqrt{10}}{10}}{2}}$$
$$=\sqrt{\frac{10-3\sqrt{10}}{20}}=\frac{\sqrt{10-3\sqrt{10}}}{\sqrt{20}}$$
$$=\frac{\sqrt{10-3\sqrt{10}}}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{50-15\sqrt{10}}}{10}$$

81. Find $\tan\frac{\theta}{2}$, given $\tan\theta=\frac{\sqrt{7}}{3}$, with $180°<\theta<270°$.

$\sec^2\theta=\tan^2\theta+1\Rightarrow$

$$\sec^2\theta=\left(\frac{\sqrt{7}}{3}\right)^2+1=\frac{7}{9}+1=\frac{16}{9}$$

Since θ is in quadrant III, sec $\theta<0$ and $\sin\theta<0$.

$$\sec\theta=-\sqrt{\frac{16}{9}}=-\frac{4}{3}\text{ and}$$
$$\cos\theta=\frac{1}{\sec\theta}=\frac{1}{-\frac{4}{3}}=-\frac{3}{4}$$
$$\sin\theta=-\sqrt{1-\cos^2\theta}$$
$$=-\sqrt{1-\left(-\frac{3}{4}\right)^2}=-\sqrt{1-\frac{9}{16}}=-\frac{\sqrt{7}}{4}$$
$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\frac{-\frac{\sqrt{7}}{4}}{1+\left(-\frac{3}{4}\right)}=\frac{-\sqrt{7}}{4-3}=-\sqrt{7}$$

82. Find $\cot\frac{\theta}{2}$ if $\tan\theta=-\frac{\sqrt{5}}{2}$, with $90°<\theta<180°$.

Use identities to find sin θ and cos θ.

$\sec^2\theta=1+\tan^2\theta\Rightarrow$

$$\sec^2\theta=1+\left(-\frac{\sqrt{5}}{2}\right)^2=1+\frac{5}{4}=\frac{9}{4}$$

Since $90°<\theta<180°$, sec θ < 0. Thus,

$$\sec\theta=-\sqrt{\frac{9}{4}}=-\frac{3}{2}\Rightarrow\cos\theta=-\frac{2}{3}.$$

Since $90°<\theta<180°$, sin θ > 0. Thus,

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(-\frac{2}{3}\right)^2}$$
$$=\sqrt{1-\frac{4}{9}}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}$$

$$\cot\frac{\theta}{2}=\frac{1}{\tan\frac{\theta}{2}}=\frac{1}{\frac{\sin\theta}{1+\cos\theta}}=\frac{1+\cos\theta}{\sin\theta}$$
$$=\frac{1+\left(-\frac{2}{3}\right)}{\frac{\sqrt{5}}{3}}=\frac{3-2}{\sqrt{5}}=\frac{1}{\sqrt{5}}$$
$$=\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{5}$$

83. Find $\sin\theta$, given $\cos 2\theta=\frac{3}{5}$, θ is in quadrant I. Since θ is in quadrant I, $\sin\theta>0$.

$$\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}\Rightarrow$$
$$\sin\theta=\sqrt{\frac{1-\frac{3}{5}}{2}}=\sqrt{\frac{\frac{2}{5}}{2}}=\sqrt{\frac{2}{10}}=\sqrt{\frac{1}{5}}$$
$$=\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{5}$$

84. Find $\cos\theta$, given $\cos 2\theta=\frac{1}{2}$, θ is in quadrant II. Since θ is in quadrant II, $\cos\theta<0$.

$$\cos\theta=-\sqrt{\frac{1+\cos 2\theta}{2}}\Rightarrow$$
$$\cos\theta=-\sqrt{\frac{1+\frac{1}{2}}{2}}=-\sqrt{\frac{\frac{3}{2}}{2}}=-\sqrt{\frac{3}{4}}=-\frac{\sqrt{3}}{2}$$

85. Find cos x, given $\cos 2x=-\frac{5}{12},\frac{\pi}{2}<x<\pi$.

Since $\frac{\pi}{2}<x<\pi,\cos x<0.$

$$\cos x=-\sqrt{\frac{1+\cos 2x}{2}}\Rightarrow$$
$$\cos x=-\sqrt{\frac{1+\left(-\frac{5}{12}\right)}{2}}=-\sqrt{\frac{\frac{7}{12}}{2}}=-\sqrt{\frac{7}{24}}$$
$$=-\frac{\sqrt{7}}{2\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}=-\frac{\sqrt{42}}{12}$$

86. Find sin x, given $\cos 2x=\frac{2}{3},\pi<x<\frac{3\pi}{2}$.

Since x is in quadrant III, $\sin x<0$.

$$\sin x=-\sqrt{\frac{1-\cos 2\theta}{2}}\Rightarrow$$
$$\sin x=-\sqrt{\frac{1-\frac{2}{3}}{2}}=-\sqrt{\frac{\frac{1}{3}}{2}}=-\sqrt{\frac{1}{6}}$$
$$=-\frac{1}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}=-\frac{\sqrt{6}}{6}$$

87. $\sqrt{\dfrac{1-\cos 40°}{2}} = \sin\dfrac{40°}{2} = \sin 20°$

88. $\sqrt{\dfrac{1+\cos 76°}{2}} = \cos\dfrac{76°}{2} = \cos 38°$

89. $\sqrt{\dfrac{1-\cos 147°}{1+\cos 147°}} = \tan\dfrac{147°}{2} = \tan 73.5°$

90. $\sqrt{\dfrac{1+\cos 165°}{1-\cos 165°}} = \dfrac{1}{\tan 82.5°} = \cot 82.5°$

91. $\dfrac{1-\cos 59.74°}{\sin 59.74°} = \tan\dfrac{59.74°}{2} = \tan 29.87°$

92. $\dfrac{\sin 158.2°}{1+\cos 158.2°} = \tan\dfrac{158.2°}{2} = \tan 79.1°$

93. $\pm\sqrt{\dfrac{1+\cos 18x}{2}} = \cos\dfrac{18x}{2} = \cos 9x$

94. $\pm\sqrt{\dfrac{1+\cos 20\alpha}{2}} = \cos\dfrac{20\alpha}{2} = \cos 10\alpha$

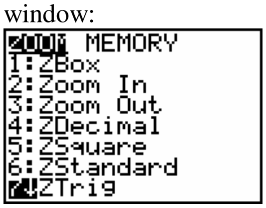
95. $\pm\sqrt{\dfrac{1-\cos 8\theta}{1+\cos 8\theta}} = \tan\dfrac{8\theta}{2} = \tan 4\theta$

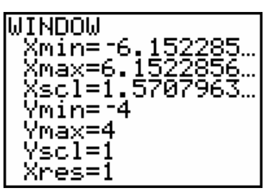
96. $\pm\sqrt{\dfrac{1-\cos 5A}{1+\cos 5A}} = \tan\dfrac{5A}{2}$

97. $\pm\sqrt{\dfrac{1+\cos\frac{x}{4}}{2}} = \cos\dfrac{\frac{x}{4}}{2} = \cos\dfrac{x}{8}$

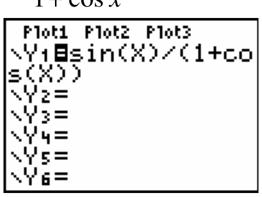
98. $\pm\sqrt{\dfrac{1-\cos\frac{3\theta}{5}}{2}} = \sin\dfrac{\frac{3\theta}{5}}{2} = \sin\dfrac{3\theta}{10}$

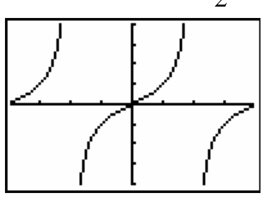
Exercises 99–102 are graphed in the following window:





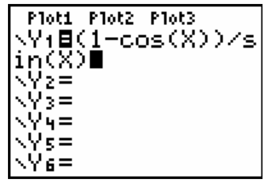
99. $\dfrac{\sin x}{1+\cos x}$ appears to be equivalent to $\tan\dfrac{x}{2}$.

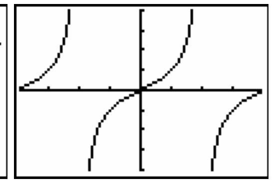




$$\frac{\sin x}{1+\cos x} = \frac{\sin 2\left(\frac{x}{2}\right)}{1+\cos 2\left(\frac{x}{2}\right)} = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{1+\left[2\cos^2\left(\frac{x}{2}\right)-1\right]}$$
$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \tan\left(\frac{x}{2}\right)$$

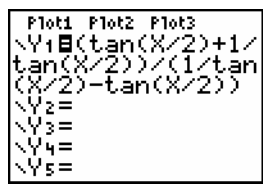
100. $\dfrac{1-\cos x}{\sin x}$ appears to be equivalent to $\tan\dfrac{x}{2}$.

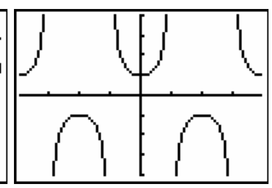




$$\frac{1-\cos x}{\sin x} = \frac{1-\cos 2\left(\frac{x}{2}\right)}{\sin 2\left(\frac{x}{2}\right)} = \frac{1-\left[1-2\sin^2\frac{x}{2}\right]}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}$$

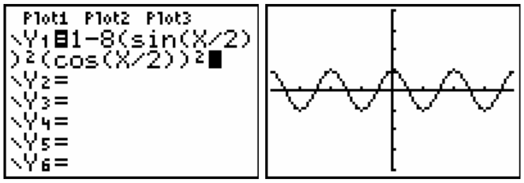
101. $\dfrac{\tan\frac{x}{2}+\cot\frac{x}{2}}{\cot\frac{x}{2}-\tan\frac{x}{2}}$ appears to be equivalent to $\sec x$.





$$\frac{\tan\frac{x}{2}+\cot\frac{x}{2}}{\cot\frac{x}{2}-\tan\frac{x}{2}} = \frac{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}$$
$$= \frac{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} \cdot \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{1}{\cos 2\left(\frac{x}{2}\right)}$$
$$= \frac{1}{\cos x} = \sec x$$

102. $1-8\sin^2\frac{x}{2}\cos^2\frac{x}{2}$ appears to be equivalent to $\cos 2x$.



$$1-8\sin^2\tfrac{x}{2}\cos^2\tfrac{x}{2}=1-2\left(4\sin^2\tfrac{x}{2}\cos^2\tfrac{x}{2}\right)$$
$$=1-2\left(2\sin\tfrac{x}{2}\cos\tfrac{x}{2}\right)^2$$
$$=1-2\left[\sin 2\left(\tfrac{x}{2}\right)\right]^2$$
$$=1-2\sin^2 x=\cos 2x$$

103. Verify $\sec^2\frac{x}{2}=\frac{2}{1+\cos x}$ is an identity.

$$\sec^2\frac{x}{2}=\frac{1}{\cos^2\frac{x}{2}}=\frac{1}{\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2}$$
$$=\frac{1}{\frac{1+\cos x}{2}}=\frac{2}{1+\cos x}$$

104. Verify $\cot^2\frac{x}{2}=\frac{(1+\cos x)^2}{\sin^2 x}$ is an identity.

$$\cot^2\frac{x}{2}=\left(\frac{1}{\tan\frac{x}{2}}\right)^2=\frac{1}{\left(\frac{\sin x}{1+\cos x}\right)^2}=\frac{(1+\cos x)^2}{\sin^2 x}$$

105. Verify $\sin^2\frac{x}{2}=\frac{\tan x-\sin x}{2\tan x}$ is an identity.

Work with the left side.

$$\sin^2\frac{x}{2}=\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2=\frac{1-\cos x}{2}$$

Work with the right side.

$$\frac{\tan x-\sin x}{2\tan x}=\frac{\frac{\sin x}{\cos x}-\sin x}{2\cdot\frac{\sin x}{\cos x}}$$
$$=\frac{\frac{\sin x}{\cos x}-\sin x}{2\cdot\frac{\sin x}{\cos x}}\cdot\frac{\cos x}{\cos x}$$
$$=\frac{\sin x-\cos x\sin x}{2\sin x}$$
$$=\frac{\sin x(1-\cos x)}{2\sin x}=\frac{1-\cos x}{2}$$

Since $\sin^2\frac{x}{2}=\frac{1-\cos x}{2}=\frac{\tan x-\sin x}{2\tan x}$, the statement has been verified.

106. Verify $\frac{\sin 2x}{2\sin x}=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$ is an identity.

Work with the left side.

$$\frac{\sin 2x}{2\sin x}=\frac{2\sin x\cos x}{2\sin x}=\cos x$$

Work with the right side.

$$\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1+\cos x}{2}-\frac{1-\cos x}{2}$$
$$=\frac{2\cos x}{2}=\cos x$$

Since $\frac{\sin 2x}{2\sin x}=\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$, the statement has been verified.

107. Verify $\frac{2}{1+\cos x}-\tan^2\frac{x}{2}=1$ is an identity.

$$\frac{2}{1+\cos x}-\tan^2\frac{x}{2}=\frac{2}{1+\cos x}-\left(\pm\sqrt{\frac{1-\cos x}{1+\cos x}}\right)^2$$
$$=\frac{2}{1+\cos x}-\frac{1-\cos x}{1+\cos x}$$
$$=\frac{2-1+\cos x}{1+\cos x}$$
$$=\frac{1+\cos x}{1+\cos x}=1$$

108. Verify $\tan\frac{\theta}{2}=\csc\theta-\cot\theta$ is an identity.

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\frac{\sin\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}$$
$$=\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}=\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$
$$=\frac{1-\cos\theta}{\sin\theta}=\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}$$
$$=\csc\theta-\cot\theta$$

109. Verify $1-\tan^2\frac{\theta}{2}=\frac{2\cos\theta}{1+\cos\theta}$ is an identity.

$$1-\tan^2\frac{\theta}{2}=1-\left(\frac{\sin\theta}{1+\cos\theta}\right)^2=1-\frac{\sin^2\theta}{(1+\cos\theta)^2}$$
$$=\frac{(1+\cos\theta)^2-\sin^2\theta}{(1+\cos\theta)^2}$$
$$=\frac{1+2\cos\theta+\cos^2\theta-\sin^2\theta}{(1+\cos\theta)^2}$$
$$=\frac{1+2\cos\theta+\cos^2\theta-\left(1-\cos^2\theta\right)}{(1+\cos\theta)^2}$$

$$= \frac{1 + 2\cos\theta + 2\cos^2\theta - 1}{(1+\cos\theta)^2}$$
$$= \frac{2\cos^2\theta + 2\cos\theta}{(1+\cos\theta)^2}$$
$$= \frac{2\cos\theta(1+\cos\theta)}{(1+\cos\theta)^2} = \frac{2\cos\theta}{1+\cos\theta}$$

110. Verify $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ is an identity.

Working with the right side, we have

$$\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-\frac{1-\cos x}{1+\cos x}}{1+\frac{1-\cos x}{1+\cos x}}$$
$$= \frac{1-\frac{1-\cos x}{1+\cos x}}{1+\frac{1-\cos x}{1+\cos x}} \cdot \frac{1+\cos x}{1+\cos x}$$
$$= \frac{(1+\cos x)-(1-\cos x)}{(1+\cos x)+(1-\cos x)}$$
$$= \frac{2\cos x}{2} = \cos x$$

111. $$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A} \cdot \frac{1-\cos A}{1-\cos A}$$
$$= \frac{\sin A(1-\cos A)}{1-\cos^2 A} = \frac{\sin A(1-\cos A)}{\sin^2 A}$$
$$= \frac{1-\cos A}{\sin A}$$

112. $\sin\frac{\theta}{2} = \frac{1}{m}, m = \frac{3}{2}$

Since $\sin\frac{\theta}{2} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$ and $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$,

we have $\left(\frac{2}{3}\right)^2 = \frac{1-\cos\theta}{2} \Rightarrow \frac{4}{9} = \frac{1-\cos\theta}{2} \Rightarrow$

$\frac{8}{9} = 1-\cos\theta \Rightarrow -\frac{1}{9} = -\cos\theta \Rightarrow \cos\theta = \frac{1}{9}$

Thus, we have $\theta = \cos^{-1}\frac{1}{9} \approx 84°$.

113. $\sin\frac{\theta}{2} = \frac{1}{m},\ \ m = \frac{5}{4}$

Since $\sin\frac{\theta}{2} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$ and $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$,

we have

$\left(\frac{4}{5}\right)^2 = \frac{1-\cos\theta}{2} \Rightarrow \frac{16}{25} = \frac{1-\cos\theta}{2} \Rightarrow$

$\frac{32}{25} = 1-\cos\theta \Rightarrow \frac{7}{25} = -\cos\theta \Rightarrow$

$\cos\theta = -\frac{7}{25}$

Thus, we have $\theta = \cos^{-1}\left(-\frac{7}{25}\right) \approx 106°$.

114. $\sin\frac{\theta}{2} = \frac{1}{m}, \theta = 30°$

$\sin\frac{30°}{2} = \frac{1}{m} \Rightarrow \sin 15° = \frac{1}{m}$

$m = \frac{1}{\sin 15°} \approx 3.9$

115. $\sin\frac{\theta}{2} = \frac{1}{m},\ \theta = 60°$

$\sin\frac{60°}{2} = \frac{1}{m} \Rightarrow \sin 30° = \frac{1}{m}$

$\frac{1}{2} = \frac{1}{m} \Rightarrow m = 2$

116. **(a)** $\cos\frac{\theta}{2} = \frac{R-b}{R}$

(b) $$\tan\frac{\theta}{4} = \frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \frac{1-\frac{R-b}{R}}{\frac{50}{R}}$$
$$= \frac{R-(R-b)}{50} = \frac{R-R+b}{50} = \frac{b}{50}$$

(c) If $\tan\frac{\theta}{4} = \frac{12}{50}$, then

$\frac{\theta}{4} = \tan^{-1}\frac{12}{50} \approx 13.4957 \Rightarrow$

$\theta \approx 53.9828 \approx 54°$

117. From Example 6, $W = \dfrac{(163\sin 120\pi t)^2}{15} \Rightarrow$

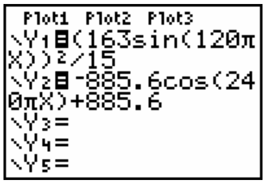
$W \approx 1771.3(\sin 120\pi t)^2$. Thus, we have

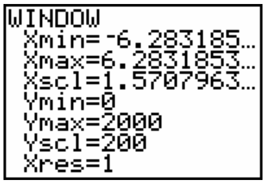
$$1771.3(\sin 120\pi t)^2$$
$$= 1771.3\sin 120\pi t \cdot \sin 120\pi t$$
$$= (1771.3)\left(\frac{1}{2}\right)\left[\cos(120\pi t - 120\pi t) - \cos(120\pi t + 120\pi t)\right]$$
$$= 885.6(\cos 0 - \cos 240\pi t)$$
$$= 885.6(1 - \cos 240\pi t)$$
$$= -885.6\cos 240\pi t + 885.6$$

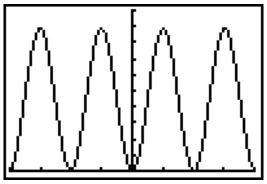
If we compare this to $W = a\cos(\omega t) + c$, then $a = -885.6$, $c = 885.6$, and $\omega = 240\pi$.

(*continued on next page*)

(*continued from page 669*)



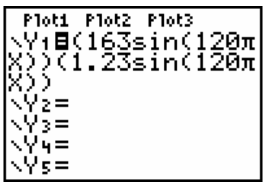


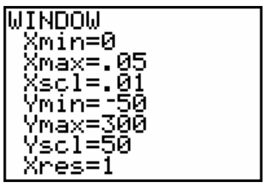


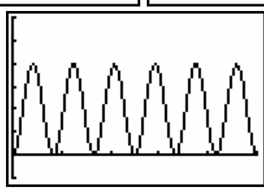
118. **(a)** Graph

$W = VI = \left[163\sin(120\pi t)\right]\left[1.23\sin(120\pi t)\right]$

over the interval $0 \le t \le .05$.







(b) The minimum wattage is 0 and the maximum wattage occurs whenever $\sin(120\pi t) = 1$. This would be 163(1.23) = 200.49 watts.

(c) $\left[163\sin(120\pi t)\right]\left[1.23\sin(120\pi t)\right]$

$= 200.49\sin^2(120\pi t)$

$= 200.49\left[\frac{1}{2}(1 - \cos 240\pi t)\right]$

$= -100.245\cos 240\pi t + 100.245$

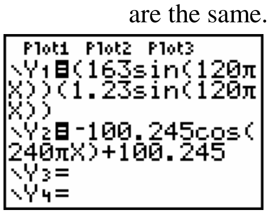
Then $a = -100.245$, $\omega = 240\pi$, and $c = 100.245$.

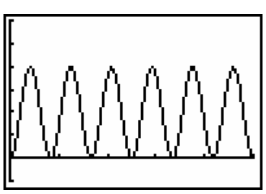
(d) The graphs of

$W = \left[163\sin(120\pi t)\right]\left[1.23\sin(120\pi t)\right]$

and $W = -100.245\cos 240\pi t + 100.245$

are the same.





(e) The graph of W is vertically centered about the line $y = 100.245$. An estimate for the average wattage consumed is 100.245 watts. (For sinusoidal current, the average wattage consumed by an electrical device will be equal to half of the peak wattage.) Thus, the light bulb is rated for 100 watts.

Summary Exercises on Verifying Trigonometric Identities

For the following exercises, other solutions are possible.

1. Verify $\tan\theta + \cot\theta = \sec\theta\csc\theta$ is an identity.

$$\begin{aligned}\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\\ &= \frac{1}{\cos\theta\sin\theta}\\ &= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ &= \sec\theta\csc\theta\end{aligned}$$

2. Verify $\csc\theta\cos^2\theta + \sin\theta = \csc\theta$ is an identity.

$$\begin{aligned}\csc\theta\cos^2\theta + \sin\theta &= \frac{1}{\sin\theta}\cdot\cos^2\theta + \sin\theta\\ &= \frac{\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta}{\sin\theta}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta}\\ &= \frac{1}{\sin\theta}\\ &= \csc\theta\end{aligned}$$

3. Verify $\tan\frac{x}{2} = \csc x - \cot x$ is an identity.

Starting on the right side, we have

$$\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1 - \cos x}{\sin x} = \tan\frac{x}{2}$$

4. Verify $\sec(\pi - x) = -\sec x$ is an identity.

$$\begin{aligned}\sec(\pi - x) &= \frac{1}{\cos(\pi - x)}\\ &= \frac{1}{\cos\pi\cos x + \sin\pi\sin x}\\ &= \frac{1}{(-1)\cos x + (0)\sin x}\\ &= \frac{1}{-\cos x + 0} = -\frac{1}{\cos x} = -\sec x\end{aligned}$$

5. Verify $\dfrac{\sin t}{1+\cos t} = \dfrac{1-\cos t}{\sin t}$ is an identity.

$$\begin{aligned}\frac{\sin t}{1+\cos t} &= \frac{\sin t}{1+\cos t}\cdot\frac{1-\cos t}{1-\cos t}\\ &= \frac{\sin t(1-\cos t)}{1-\cos^2 t} = \frac{\sin t(1-\cos t)}{\sin^2 t}\\ &= \frac{1-\cos t}{\sin t}\end{aligned}$$

6. Verify $\dfrac{1-\sin t}{\cos t} = \dfrac{1}{\sec t + \tan t}$ is an identity.

$$\begin{aligned}\frac{1}{\sec t + \tan t} &= \frac{1}{\frac{1}{\cos t} + \frac{\sin t}{\cos t}} = \frac{1}{\frac{1+\sin t}{\cos t}}\\ &= \frac{\cos t}{1+\sin t} = \frac{\cos t}{1+\sin t}\cdot\frac{1-\sin t}{1-\sin t}\\ &= \frac{\cos t(1-\sin t)}{1-\sin^2 t} = \frac{\cos t(1-\sin t)}{\cos^2 t}\\ &= \frac{1-\sin t}{\cos t}\end{aligned}$$

7. Verify $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ is an identity.

Starting on the right side, we have

$$\begin{aligned}\frac{2\tan\theta}{1+\tan^2\theta} &= \frac{2\tan\theta}{\sec^2\theta} = \frac{2\cdot\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\\ &= 2\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{1}\\ &= 2\sin\theta\cos\theta = \sin 2\theta\end{aligned}$$

8. Verify $\dfrac{2}{1+\cos x} - \tan^2\dfrac{x}{2} = 1$ is an identity.

$$\begin{aligned}\frac{2}{1+\cos x} - \tan^2\frac{x}{2} &= \frac{2}{1+\cos x} - \left(\frac{\sin x}{1+\cos x}\right)^2\\ &= \frac{2}{1+\cos x} - \frac{\sin^2 x}{(1+\cos x)^2}\\ &= \frac{2(1+\cos x)}{(1+\cos x)^2} - \frac{\sin^2 x}{(1+\cos x)^2}\\ &= \frac{2+2\cos x - \sin^2 x}{(1+\cos x)^2}\\ &= \frac{2+2\cos x - (1-\cos^2 x)}{(1+\cos x)^2}\\ &= \frac{2+2\cos x - 1 + \cos^2 x}{(1+\cos x)^2}\\ &= \frac{\cos^2 x + 2\cos x + 1}{(1+\cos x)^2}\\ &= \frac{(1+\cos x)^2}{(1+\cos x)^2} = 1\end{aligned}$$

9. Verify $\cot\theta - \tan\theta = \dfrac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$ is an identity.

$$\begin{aligned}\cot\theta - \tan\theta &= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\\ &= \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}\\ &= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\\ &= \frac{\cos^2\theta - (1-\cos^2\theta)}{\sin\theta\cos\theta}\\ &= \frac{\cos^2\theta - 1 + \cos^2\theta}{\sin\theta\cos\theta}\\ &= \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta}\end{aligned}$$

10. Verify $\dfrac{1}{\sec t-1}+\dfrac{1}{\sec t+1}=2\cot t\csc t$ is an identity.

$$\frac{1}{\sec t-1}+\frac{1}{\sec t+1}=\frac{1}{\frac{1}{\cos t}-1}+\frac{1}{\frac{1}{\cos t}+1}=\frac{1}{\frac{1}{\cos t}-1}\cdot\frac{\cos t}{\cos t}+\frac{1}{\frac{1}{\cos t}+1}\cdot\frac{\cos t}{\cos t}=\frac{\cos t}{1-\cos t}+\frac{\cos t}{1+\cos t}$$

$$=\frac{\cos t}{1-\cos t}\cdot\frac{1+\cos t}{1+\cos t}+\frac{\cos t}{1+\cos t}\cdot\frac{1-\cos t}{1-\cos t}=\frac{\cos t+\cos^2 t}{1-\cos^2 t}+\frac{\cos t-\cos^2 t}{1-\cos^2 t}$$

$$=\frac{\cos t+\cos^2 t+\cos t-\cos^2 t}{1-\cos^2 t}=\frac{2\cos t}{1-\cos^2 t}=\frac{2\cos t}{\sin^2 t}=2\cdot\frac{\cos t}{\sin t}\cdot\frac{1}{\sin t}=2\cot t\csc t$$

11. Verify $\dfrac{\sin(x+y)}{\cos(x-y)}=\dfrac{\cot x+\cot y}{1+\cot x\cot y}$ is an identity.

$$\frac{\sin(x+y)}{\cos(x-y)}=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y+\sin x\sin y}=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y+\sin x\sin y}\cdot\frac{\frac{1}{\cos x\cos y}}{\frac{1}{\cos x\cos y}}=\frac{\frac{\sin x\cos y}{\cos x\cos y}+\frac{\cos x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y}+\frac{\sin x\sin y}{\cos x\cos y}}$$

$$=\frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1+\frac{\sin x}{\cos x}\cdot\frac{\sin y}{\cos y}}=\frac{\cot x+\cot y}{1+\cot x\cot y}$$

12. Verify $1-\tan^2\dfrac{\theta}{2}=\dfrac{2\cos\theta}{1+\cos\theta}$ is an identity.

$$1-\tan^2\frac{\theta}{2}=1-\left(\frac{\sin\theta}{1+\cos\theta}\right)^2=1-\frac{\sin^2\theta}{(1+\cos\theta)^2}=\frac{(1+\cos\theta)^2}{(1+\cos\theta)^2}-\frac{\sin^2\theta}{(1+\cos\theta)^2}=\frac{1+2\cos\theta+\cos^2\theta}{(1+\cos\theta)^2}-\frac{\sin^2\theta}{(1+\cos\theta)^2}$$

$$=\frac{1+2\cos\theta+\cos^2\theta-\sin^2\theta}{(1+\cos\theta)^2}=\frac{2\cos\theta+\cos^2\theta+\left(1-\sin^2\theta\right)}{(1+\cos\theta)^2}=\frac{2\cos\theta+\cos^2\theta+\cos^2\theta}{(1+\cos\theta)^2}$$

$$=\frac{2\cos\theta+2\cos^2\theta}{(1+\cos\theta)^2}=\frac{2\cos\theta(1+\cos\theta)}{(1+\cos\theta)^2}=\frac{2\cos\theta}{1+\cos\theta}$$

13. Verify $\dfrac{\sin\theta+\tan\theta}{1+\cos\theta}=\tan\theta$ is an identity.

$$\frac{\sin\theta+\tan\theta}{1+\cos\theta}=\frac{\sin\theta+\frac{\sin\theta}{\cos\theta}}{1+\cos\theta}$$

$$=\frac{\sin\theta+\frac{\sin\theta}{\cos\theta}}{1+\cos\theta}\cdot\frac{\cos\theta}{\cos\theta}$$

$$=\frac{\sin\theta\cos\theta+\sin\theta}{\cos\theta(1+\cos\theta)}$$

$$=\frac{\sin\theta(\cos\theta+1)}{\cos\theta(1+\cos\theta)}=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

14. Verify $\csc^4 x-\cot^4 x=\dfrac{1+\cos^2 x}{1-\cos^2 x}$ is an identity.

$$\csc^4 x-\cot^4 x=\frac{1}{\sin^4 x}-\frac{\cos^4 x}{\sin^4 x}$$

$$=\frac{1-\cos^4 x}{\sin^4 x}$$

$$=\frac{\left(1+\cos^2 x\right)\left(1-\cos^2 x\right)}{\sin^4 x}$$

$$=\frac{\left(1+\cos^2 x\right)\left(\sin^2 x\right)}{\sin^4 x}$$

$$=\frac{1+\cos^2 x}{\sin^2 x}$$

$$=\frac{1+\cos^2 x}{1-\cos^2 x}$$

15. Verify $\cos x = \dfrac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$ is an identity.

$$\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{1-\left(\frac{1-\cos x}{\sin x}\right)^2}{1+\left(\frac{1-\cos x}{\sin x}\right)^2} = \frac{1-\frac{(1-\cos x)^2}{\sin^2 x}}{1+\frac{(1-\cos x)^2}{\sin^2 x}} = \frac{1-\frac{(1-\cos x)^2}{\sin^2 x}}{1+\frac{(1-\cos x)^2}{\sin^2 x}} \cdot \frac{\sin^2 x}{\sin^2 x} = \frac{\sin^2 x-(1-\cos x)^2}{\sin^2 x+(1-\cos x)^2}$$

$$= \frac{\sin^2 x-\left(1-2\cos x+\cos^2 x\right)}{\sin^2 x+\left(1-2\cos x+\cos^2 x\right)} = \frac{\sin^2 x-1+2\cos x-\cos^2 x}{\sin^2 x+1-2\cos x+\cos^2 x} = \frac{\left(1-\cos^2 x\right)-1+2\cos x-\cos^2 x}{\left(\sin^2 x+\cos^2 x\right)+1-2\cos x}$$

$$= \frac{1-\cos^2 x-1+2\cos x-\cos^2 x}{1+1-2\cos x} = \frac{2\cos x-2\cos^2 x}{2-2\cos x} = \frac{2\cos x(1-\cos x)}{2(1-\cos x)} = \cos x$$

16. Verify $\cos 2x = \dfrac{2-\sec^2 x}{\sec^2 x}$ is an identity. Starting on the right side, we have

$$\frac{2-\sec^2 x}{\sec^2 x} = \frac{2-\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} \cdot \frac{\cos^2 x}{\cos^2 x} = \frac{2\cos^2 x-1}{1} = 2\cos^2 x-1 = \cos 2x$$

17. Verify $\dfrac{\tan^2 t+1}{\tan t \csc^2 t} = \tan t$ is an identity.

$$\frac{\tan^2 t+1}{\tan t\csc^2 t} = \frac{\frac{\sin^2 t}{\cos^2 t}+1}{\frac{\sin t}{\cos t}\cdot\frac{1}{\sin^2 t}} = \frac{\frac{\sin^2 t}{\cos^2 t}+1}{\frac{1}{\cos t\sin t}} = \frac{\frac{\sin^2 t}{\cos^2 t}+1}{\frac{1}{\cos t\sin t}} \cdot \frac{\cos^2 t\sin t}{\cos^2 t\sin t} = \frac{\sin^3 t+\cos^2 t\sin t}{\cos t}$$

$$= \frac{\sin t\left(\sin^2 t+\cos^2 t\right)}{\cos t} = \frac{\sin t(1)}{\cos t} = \frac{\sin t}{\cos t} = \tan t$$

18. Verify $\dfrac{\sin s}{1+\cos s}+\dfrac{1+\cos s}{\sin s} = 2\csc s$ is an identity.

$$\frac{\sin s}{1+\cos s}+\frac{1+\cos s}{\sin s} = \frac{\sin^2 s}{\sin s(1+\cos s)}+\frac{(1+\cos s)^2}{\sin s(1+\cos s)} = \frac{\sin^2 s+(1+\cos s)^2}{\sin s(1+\cos s)} = \frac{\sin^2 s+\left(1+2\cos s+\cos^2 s\right)}{\sin s(1+\cos s)}$$

$$= \frac{\left(1-\cos^2 s\right)+1+2\cos s+\cos^2 s}{\sin s(1+\cos s)} = \frac{2+2\cos s}{\sin s(1+\cos s)} = \frac{2(1+\cos s)}{\sin s(1+\cos s)} = \frac{2}{\sin s} = 2\csc s$$

19. Verify $\tan 4\theta = \dfrac{2\tan 2\theta}{2-\sec^2 2\theta}$ is an identity.

$$\frac{2\tan 2\theta}{2-\sec^2 2\theta} = \frac{2\cdot\frac{\sin 2\theta}{\cos 2\theta}}{2-\frac{1}{\cos^2 2\theta}} = \frac{2\cdot\frac{\sin 2\theta}{\cos 2\theta}}{2-\frac{1}{\cos^2 2\theta}} \cdot \frac{\cos^2 2\theta}{\cos^2 2\theta} = \frac{2\sin 2\theta\cos 2\theta}{2\cos^2 2\theta-1} = \frac{\sin\left[2(2\theta)\right]}{\cos\left[2(2\theta)\right]} = \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta$$

20. Verify $\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)=\sec x+\tan x$ is an identity.

$$\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)=\frac{\tan\frac{x}{2}+\tan\frac{\pi}{4}}{1-\tan\frac{x}{2}\tan\frac{\pi}{4}}=\frac{\tan\frac{x}{2}+1}{1-\left(\tan\frac{x}{2}\right)(1)}=\frac{\tan\frac{x}{2}+1}{1-\tan\frac{x}{2}}=\frac{\frac{\sin x}{1+\cos x}+1}{1-\frac{\sin x}{1+\cos x}}=\frac{\frac{\sin x}{1+\cos x}+1}{1-\frac{\sin x}{1+\cos x}}\cdot\frac{1+\cos x}{1+\cos x}$$
$$=\frac{\sin x+(1+\cos x)}{(1+\cos x)-\sin x}=\frac{\sin x+1+\cos x}{1+\cos x-\sin x}=\frac{\sin x+1+\cos x}{1+\cos x-\sin x}\cdot\frac{\cos x}{\cos x}=\frac{\sin x\cos x+\cos x+\cos^2 x}{\cos x(1+\cos x-\sin x)}$$
$$=\frac{\cos x(1+\sin x)+(1+\sin x)(1-\sin x)}{\cos x(1+\cos x-\sin x)}=\frac{(1+\sin x)(\cos x+1-\sin x)}{\cos x(1+\cos x-\sin x)}$$
$$=\frac{1+\sin x}{\cos x}=\frac{1}{\cos x}+\frac{\sin x}{\cos x}=\sec x+\tan x$$

21. Verify $\frac{\cot s-\tan s}{\cos s+\sin s}=\frac{\cos s-\sin s}{\sin s\cos s}$ is an identity.

$$\frac{\cot s-\tan s}{\cos s+\sin s}=\frac{\frac{\cos s}{\sin s}-\frac{\sin s}{\cos s}}{\cos s+\sin s}=\frac{\frac{\cos s}{\sin s}-\frac{\sin s}{\cos s}}{\cos s+\sin s}\cdot\frac{\sin s\cos s}{\sin s\cos s}=\frac{\cos^2 s-\sin^2 s}{(\cos s+\sin s)\sin s\cos s}$$
$$=\frac{(\cos s+\sin s)(\cos s-\sin s)}{(\cos s+\sin s)\sin s\cos s}=\frac{\cos s-\sin s}{\sin s\cos s}$$

22. Verify $\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=1-2\cos^2\theta$ is an identity.

$$\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\frac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}=\frac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}\cdot\frac{\cos\theta\sin\theta}{\cos\theta\sin\theta}=\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta+\cos^2\theta}=\frac{\sin^2\theta-\cos^2\theta}{1}$$
$$=\sin^2\theta-\cos^2\theta=\left(1-\cos^2\theta\right)-\cos^2\theta=1-2\cos^2\theta$$

23. Verify $\frac{\tan(x+y)-\tan y}{1+\tan(x+y)\tan y}=\tan x$ is an identity.

$$\frac{\tan(x+y)-\tan y}{1+\tan(x+y)\tan y}=\frac{\frac{\tan x+\tan y}{1-\tan x\tan y}-\tan y}{1+\frac{\tan x+\tan y}{1-\tan x\tan y}\cdot\tan y}=\frac{\frac{\tan x+\tan y}{1-\tan x\tan y}-\tan y}{1+\frac{\tan x+\tan y}{1-\tan x\tan y}\cdot\tan y}\cdot\frac{1-\tan x\tan y}{1-\tan x\tan y}$$
$$=\frac{\tan x+\tan y-\tan y(1-\tan x\tan y)}{1-\tan x\tan y+(\tan x+\tan y)\tan y}=\frac{\tan x+\tan x\tan^2 y}{1-\tan x\tan y+\tan x\tan y+\tan^2 y}$$
$$=\frac{\tan x\left(1+\tan^2 y\right)}{1+\tan^2 y}=\tan x$$

24. Verify $2\cos^2\frac{x}{2}\tan x=\tan x+\sin x$ is an identity.

$$2\cos^2\frac{x}{2}\tan x=2\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2\cdot\frac{\sin x}{\cos x}=2\cdot\frac{1+\cos x}{2}\cdot\frac{\sin x}{\cos x}=(1+\cos x)\cdot\frac{\sin x}{\cos x}=\frac{\sin x}{\cos x}+\sin x=\tan x+\sin x$$

25. Verify $\dfrac{\cos^4 x - \sin^4 x}{\cos^2 x} = 1 - \tan^2 x$ is an identity.

$$\frac{\cos^4 x - \sin^4 x}{\cos^2 x} = \frac{\left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right)}{\cos^2 x} = \frac{(1)\left(\cos^2 x - \sin^2 x\right)}{\cos^2 x} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x}$$
$$= \frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = 1 - \tan^2 x$$

26. Verify $\dfrac{\csc t + 1}{\csc t - 1} = \left(\sec t + \tan t\right)^2$ is an identity.

$$\frac{\csc t + 1}{\csc t - 1} = \frac{\frac{1}{\sin t} + 1}{\frac{1}{\sin t} - 1} = \frac{\frac{1}{\sin t} + 1}{\frac{1}{\sin t} - 1} \cdot \frac{\sin t}{\sin t} = \frac{1 + \sin t}{1 - \sin t} = \frac{1 + \sin t}{1 - \sin t} \cdot \frac{1 + \sin t}{1 + \sin t}$$
$$= \frac{(1 + \sin t)^2}{1 - \sin^2 t} = \frac{(1 + \sin t)^2}{\cos^2 t} = \left(\frac{1 + \sin t}{\cos t}\right)^2 = \left(\frac{1}{\cos t} + \frac{\sin t}{\cos t}\right)^2 = \left(\sec t + \tan t\right)^2$$

27. Verify $\dfrac{2\left(\sin x - \sin^3 x\right)}{\cos x} = \sin 2x$ is an identity.

$$\frac{2\left(\sin x - \sin^3 x\right)}{\cos x} = \frac{2\sin x\left(1 - \sin^2 x\right)}{\cos x} = \frac{2\sin x \cos^2 x}{\cos x} = 2\sin x \cos x = \sin 2x$$

28. Verify $\dfrac{1}{2}\cot\dfrac{x}{2} - \dfrac{1}{2}\tan\dfrac{x}{2} = \cot x$ is an identity.

$$\frac{1}{2}\cot\frac{x}{2} - \frac{1}{2}\tan\frac{x}{2} = \frac{1}{2}\cdot\frac{1}{\tan\frac{x}{2}} - \frac{1}{2}\tan\frac{x}{2} = \frac{1}{2}\cdot\frac{1}{\frac{\sin x}{1 + \cos x}} - \frac{1}{2}\cdot\frac{1 - \cos x}{\sin x} = \frac{1 + \cos x}{2\sin x} - \frac{1 - \cos x}{2\sin x}$$
$$= \frac{1 + \cos x - (1 - \cos x)}{2\sin x} = \frac{1 + \cos x - 1 + \cos x}{2\sin x} = \frac{2\cos x}{2\sin x} = \frac{\cos x}{\sin x} = \cot x$$

29. Verify $\sin\left(60° + x\right) + \sin\left(60° - x\right) = \sqrt{3}\cos x$ is an identity.

$$\sin\left(60° + x\right) + \sin\left(60° - x\right) = \left(\sin 60° \cos x + \cos 60° \sin x\right) + \left(\sin 60° \cos x - \cos 60° \sin x\right)$$
$$= 2\sin 60° \cos x = 2\left(\frac{\sqrt{3}}{2}\right)\cos x = \sqrt{3}\cos x$$

30. Verify $\sin\left(60° - x\right) - \sin\left(60° + x\right) = -\sin x$ is an identity.

$$\sin\left(60° - x\right) - \sin\left(60° + x\right) = \left(\sin 60° \cos x - \cos 60° \sin x\right) - \left(\sin 60° \cos x + \cos 60° \sin x\right)$$
$$= -2\cos 60° \sin x = -2\left(\frac{1}{2}\right)\sin x = -\sin x$$

31. Verify $\dfrac{\cos(x + y) + \cos(y - x)}{\sin(x + y) - \sin(y - x)} = \cot x$ is an identity.

$$\frac{\cos(x + y) + \cos(y - x)}{\sin(x + y) - \sin(y - x)} = \frac{\left(\cos x \cos y - \sin x \sin y\right) + \left(\cos y \cos x + \sin y \sin x\right)}{\left(\sin x \cos y + \cos x \sin y\right) - \left(\sin y \cos x - \cos y \sin x\right)} = \frac{2\cos x \cos y}{2\cos y \sin x} = \frac{\cos x}{\sin x} = \cot x$$

32. Verify $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$ is an identity.

$$\begin{aligned}\sin x + \sin 3x + \sin 5x + \sin 7x &= (\sin x + \sin 3x) + (\sin 5x + \sin 7x)\\ &= 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right) + 2\sin\left(\frac{5x+7x}{2}\right)\cos\left(\frac{5x-7x}{2}\right)\\ &= 2\sin 2x\cos(-x) + 2\sin 6x\cos(-x)\\ &= 2\cos(-x)(\sin 2x + \sin 6x) = 2\cos x \sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right)\\ &= 2\cos x\sin 4x\cos(-2x) = 2\cos x\cos 2x\sin 4x\end{aligned}$$

33. Verify $\sin^3\theta + \cos^3\theta + \sin\theta\cos^2\theta + \sin^2\cos\theta = \sin\theta + \cos\theta$ is an identity.

$$\begin{aligned}\sin^3\theta + \cos^3\theta + \sin\theta\cos^2\theta + \sin^2\cos\theta &= \left(\sin^3\theta + \sin\theta\cos^2\theta\right) + \left(\cos^3\theta + \sin^2\cos\theta\right)\\ &= \sin\theta\left(\sin^2\theta + \cos^2\theta\right) + \cos\theta\left(\cos^2\theta + \sin^2\theta\right)\\ &= \left(\sin\theta + \cos\theta\right)\left(\sin^2\theta + \cos^2\theta\right) = \sin\theta + \cos\theta\end{aligned}$$

34. Verify $\dfrac{\cos x + \sin x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x$

$$\begin{aligned}\frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} &= \frac{(\cos x + \sin x)^2 - (\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)}\\ &= \frac{\left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right) - (\cos^2 x - 2\sin x\cos x + \sin^2 x)}{\cos^2 x - \sin^2 x}\\ &= \frac{4\sin x\cos x}{\cos^2 x - \sin^2 x} = \frac{2\sin 2x}{\cos 2x} = 2\tan 2x\end{aligned}$$

Section 7.5: Inverse Circular Functions

1. For a function to have an inverse, it must be one-to-one.

2. The domain of $y = \arcsin x$ equals the range of $y = \sin x$.

3. The range of $y = \cos^{-1} x$ equals the domain of $y = \cos x$.

4. The point $\left(\frac{\pi}{4}, 1\right)$ lies on the graph of $y = \tan x$. Therefore, the point $\left(1, \frac{\pi}{4}\right)$ lies on the graph of $y = \tan^{-1} x$.

5. If a function *f* has an inverse and $f(\pi) = -1$, then $f^{1}(-1) = \pi$.

6. Sketch the reflection of the graph of *f* across the line $y = x$.

7. **(a)** $[-1, 1]$

(b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) increasing

(d) -2 is not in the domain.

8. **(a)** $[-1, 1]$

(b) $[0, \pi]$

(c) decreasing

(d) $-\frac{4\pi}{3}$ is not in the range.

9. **(a)** $(-\infty, \infty)$

(b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) increasing

(d) no

10. **(a)** $(-\infty, -1] \cup [1, \infty)$; $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

(b) $(-\infty, -1] \cup [1, \infty)$; $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

(c) $(-\infty, \infty)$; $(0, \pi)$

11. $\cos^{-1}\frac{1}{a}$

12. Find $\tan^{-1}\frac{1}{a}+\pi\left(\text{ or }180°\right)$.

13. $y=\sin^{-1}0$

$\sin y=0,\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}$

Since $\sin 0=0,\quad y=0.$

14. $y=\tan^{-1}1$

$\tan y=1,\ -\frac{\pi}{2}<y<\frac{\pi}{2}$

Since $\tan\frac{\pi}{4}=1,\quad y=\frac{\pi}{4}.$

15. $y=\cos^{-1}(-1)$

$\cos y=-1,\ 0\le y\le\pi$

Since $\cos\pi=-1,\ y=\pi.$

16. $y=\arctan(-1)$

$\tan y=-1,-\frac{\pi}{2}<y<\frac{\pi}{2}$

Since $\tan\left(-\frac{\pi}{4}\right)=-1,\ y=-\frac{\pi}{4}.$

17. $y=\sin^{-1}(-1)$

$\sin y=-1,\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}$

Since $\sin\frac{\pi}{2}=-1,\ y=-\frac{\pi}{2}.$

18. $y=\cos^{-1}\frac{1}{2}$

$\cos y=\frac{1}{2},\ 0\le y\le\pi$

Since $\cos\frac{\pi}{3}=\frac{1}{2},\ y=\frac{\pi}{3}.$

19. $y=\arctan 0$

$\tan y=0,\ -\frac{\pi}{2}<y<\frac{\pi}{2}$

Since $\tan 0=0,\ y=0.$

20. $y=\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

$\sin y=-\frac{\sqrt{3}}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}$

Since $\sin\left(-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2},\ y=-\frac{\pi}{3}.$

21. $y=\arccos 0$

$\cos y=0,\ 0\le y\le\pi$

Since $\cos\frac{\pi}{2}=0,\ y=\frac{\pi}{2}.$

22. $y=\tan^{-1}(-1)$

$\tan y=-1,\ -\frac{\pi}{2}<y<\frac{\pi}{2}$

Since $\tan\left(-\frac{\pi}{4}\right)=-1,\ y=-\frac{\pi}{4}.$

23. $y=\sin^{-1}\frac{\sqrt{2}}{2}$

$\sin y=\frac{\sqrt{2}}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}$

Since $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2},\ y=\frac{\pi}{4}.$

24. $y=\cos^{-1}\left(-\frac{1}{2}\right)$

$\cos y=-\frac{1}{2},\ 0\le y\le\pi$

Since $\cos\frac{2\pi}{3}=-\frac{1}{2},\ y=\frac{2\pi}{3}.$

25. $y=\arccos\left(-\frac{\sqrt{3}}{2}\right)$

$\cos y=-\frac{\sqrt{3}}{2},\ 0\le y\le\pi$

Since $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2},\ y=\frac{5\pi}{6}.$

26. $y=\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

$\sin y=-\frac{\sqrt{2}}{2},-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

Since $\sin\left(-\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2},\ y=-\frac{\pi}{4}.$

27. $y=\cot^{-1}(-1)$

$\cot y=-1,\ 0<y<\pi$

y is in quadrant II. The reference angle is $\frac{\pi}{4}$.

Since $\cot\frac{3\pi}{4}=1,\ y=\frac{3\pi}{4}.$

28. $y = \sec^{-1}\left(-\sqrt{2}\right)$

$\sec y = -\sqrt{2},\ 0 \le y \le \pi,\ y \ne \frac{\pi}{2}$

y is in quadrant II. The reference angle is $\frac{\pi}{4}$.

Since $\sec\frac{3\pi}{4} = -\sqrt{2},\ y = \frac{3\pi}{4}$.

29. $y = \csc^{-1}(-2)$

$\csc y = -2,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

y is in quadrant IV. The reference angle is $\frac{\pi}{6}$.

Since $\csc\left(-\frac{\pi}{6}\right) = -2,\ y = -\frac{\pi}{6}$.

30. $y = \operatorname{arc}\cot\left(-\sqrt{3}\right)$

$\cot y = -\sqrt{3},\ 0 < y < \pi$

y is in quadrant II. The reference angle is $\frac{\pi}{6}$.

Since $\cot\frac{5\pi}{6} = -\sqrt{3},\ y = \frac{5\pi}{6}$.

31. $y = \operatorname{arc}\sec\left(\frac{2\sqrt{3}}{3}\right)$

$\sec y = \frac{2\sqrt{3}}{3},\ 0 \le y \le \pi,\ y \ne \frac{\pi}{2}$

Since $\sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3},\ y = \frac{\pi}{6}$.

32. $y = \csc^{-1}\sqrt{2}$

$\csc\ y = \sqrt{2},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

Since $\csc\frac{\pi}{4} = \sqrt{2},\ y = \frac{\pi}{4}$.

33. $y = \sec^{-1} 1$

$\sec\ y = 1,\ 0 \le y \le \pi,\ y \ne \frac{\pi}{2}$

Since $\sec 0 = 1,\ y = 0$.

34. No, because there is no angle θ such that $\sec\theta = 0$.

35. $\theta = \arctan(-1)$

$\tan\ \theta = -1,\ -90° < \theta < 90°$

θ is in quadrant IV. The reference angle is 45°. Thus, $\theta = -45°$.

36. $\theta = \arccos\left(-\frac{1}{2}\right)$

$\cos\theta = -\frac{1}{2},\ 0° \le \theta \le 180°$

θ is in quadrant II. The reference angle is 60°. Thus, $\theta = 180° - 60° = 120°$.

37. $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$

$\sin\theta = -\frac{\sqrt{3}}{2},\ -90° \le \theta \le 90°$

θ is in quadrant IV. The reference angle is 60°. $\theta = -60°$.

38. $\theta = \arcsin\left(-\frac{\sqrt{2}}{2}\right)$

$\sin\ \theta = -\frac{\sqrt{2}}{2},\ -90° \le \theta \le 90°$

θ is in quadrant IV. The reference angle is 45°. $\theta = -45°$

39. $\theta = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

$\cot\ \theta = -\frac{\sqrt{3}}{3},\ 0° < \theta < 180°$

θ is in quadrant II. The reference angle is 60°. $\theta = 180° - 60° = 120°$

40. $\theta = \csc^{-1}(-2)$

$\csc\ \theta = -2$ and $-90° < \theta < 90°,\ \theta \ne 0°$

θ is in quadrant IV. The reference angle is 30°. $\theta = -30°$

41. $\theta = \sec^{-1}(-2)$

$\sec\ \theta = -2,\ 0° \le \theta \le 180°, \theta \ne 90°$

θ is in quadrant II. The reference angle is 60°. $\theta = 180° - 60° = 120°$.

42. $\theta = \csc^{-1}(-1)$

$\csc\ \theta = -1,\ -90° \le \theta \le 90°,\ \theta \ne 0°$

Since the terminal side of θ lies on the y-axis, there is no reference angle. $\theta = -90°$.

43. $\theta = \tan^{-1}\sqrt{3}$

$\tan\theta = \sqrt{3}, -90° < \theta < 90°$

Since $\tan 60° = \sqrt{3}, \theta = 60°$.

44. $\theta = \cot^{-1}\frac{\sqrt{3}}{3}$

$\cot\theta = \frac{\sqrt{3}}{3}, -90° < \theta < 90°$

Since $\cot 60° = \frac{\sqrt{3}}{3}, \theta = 60°$.

45. $\theta = \sin^{-1} 2$

$\sin\theta = 2, 0° \le \theta \le 180°$

There is no angle θ such that $\sin\theta = 2$.

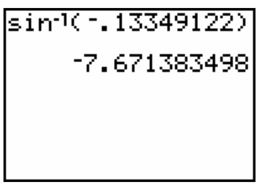
46. $\theta = \cos^{-1}(-2)$

$\cos\theta = -2, 0° \le \theta \le 180°$

There is no angle θ such that $\cos\theta = -2$.

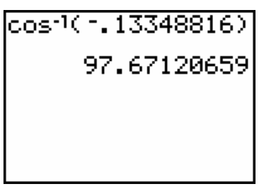
For Exercises 47–56, be sure that your calculator is in degree mode. Keystroke sequences may vary based on the type and/or model of calculator being used.

47. $\theta = \sin^{-1}(-.13349122)$



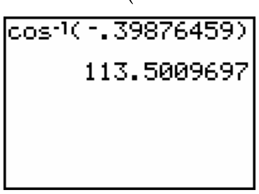
$\sin^{-1}(-.13349122) = -7.6713835°$

48. $\theta = \cos^{-1}(-.13348816)$



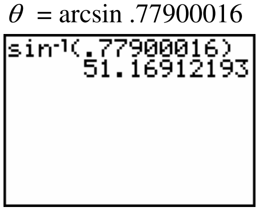
$\cos^{-1}(-.13348816) \approx 97.671207°$

49. $\theta = \arccos(-.39876459)$



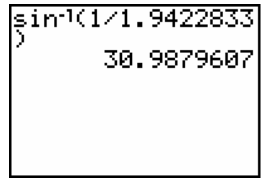
$\arccos(-.39876459) \approx 113.500970°$

50. $\theta = \arcsin .77900016$



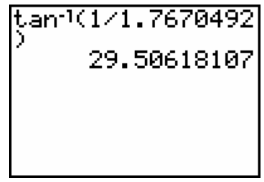
$\arcsin .77900016 \approx 51.1691219°$

51. $\theta = \csc^{-1} 1.9422833$



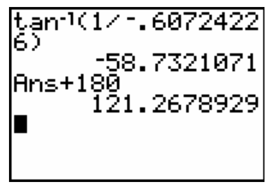
$\csc^{-1} 1.9422833 \approx 30.987961°$

52. $\theta = \cot^{-1} 1.7670492$



$\cot^{-1} 1.7670492 \approx 29.506181°$

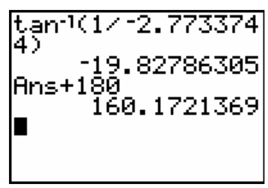
53. $\theta = \cot^{-1}(-.60724226)$



Note that θ is in quadrant II because $\cot^{-1}\theta$ is defined for $0° < \theta < 180°$.

$\cot^{-1}(-.60724226) \approx 121.267893°$

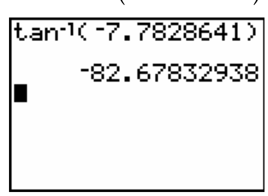
54. $\theta = \cot^{-1}(-2.7733744)$



Note that θ is in quadrant II because $\cot^{-1}\theta$ is defined for $0° < \theta < 180°$.

$\cot^{-1}(-2.77337744) \approx 160.172137°$

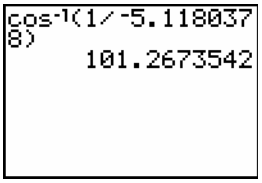
55. $\theta = \tan^{-1}(-7.7828641)$



Note that $\tan^{-1} y$ is defined for $-90° < y < 90°$.

$\tan^{-1}(-7.7828641) \approx -82.678329°$

56. $\theta = \sec^{-1}(-5.1180378)$

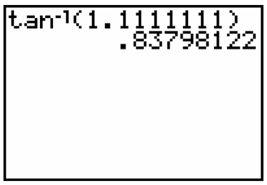


Note that $\sec^{-1}$ is defined for $0 < y < 180°$.

$\sec^{-1}(-5.1180378) \approx 101.267354°$

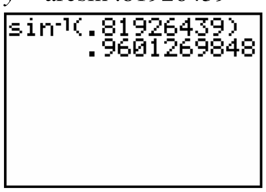
For Exercises 57–66, be sure that your calculator is in radian mode. Keystroke sequences may vary based on the type and/or model of calculator being used.

57. $y = \arctan 1.1111111$



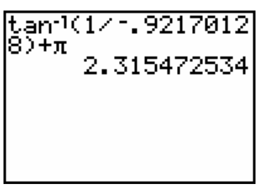
$\arctan 1.1111111 \approx .83798122$

58. $y = \arcsin .81926439$



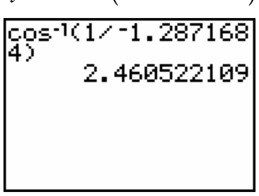
$\arcsin .81926439 \approx .96012698$

59. $y = \cot^{-1}(-.92170128)$



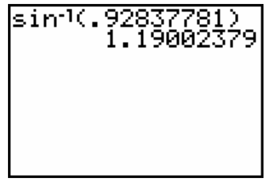
$\cot^{-1}(-.91270128) \approx 2.3154725$

60. $y = \sec^{-1}(-1.2871684)$



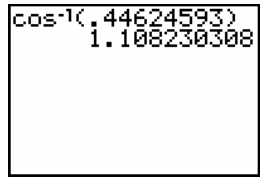
$\sec^{-1}(-1.2871684) \approx 2.4605221$

61. $y = \arcsin .92837781$



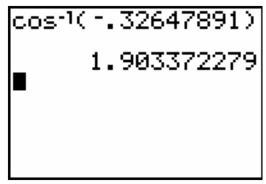
$\arcsin .92837781 \approx 1.1900238$

62. $y = \arccos .44624593$



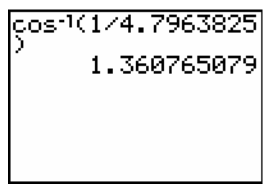
$\arccos .44624593 \approx 1.1082303$

63. $y = \cos^{-1}(-.32647891)$



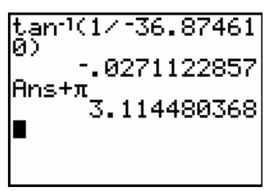
$\cos^{-1}(-.32647891) \approx 1.9033723$

64. $y = \sec^{-1}(4.7963825)$



$\sec^{-1}(4.7963825) \approx 1.3607651$

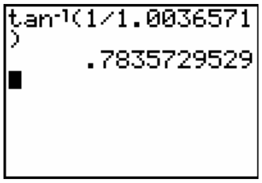
65. $y = \cot^{-1}(-36.874610)$



Note that θ is in quadrant II because $\cot^{-1}\theta$ is defined for $0 < \theta < \pi$.

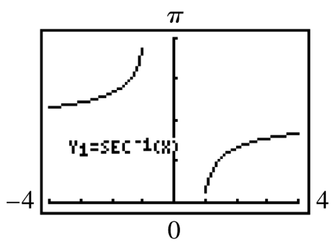
$\cot^{-1}(-36.874610) \approx 3.1144804$

66. $y = \cot^{-1}(1.0036571)$

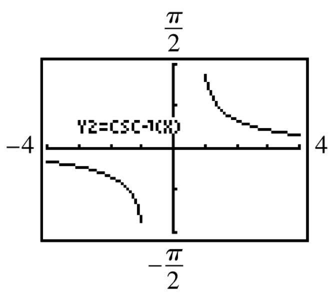


$\cot^{-1}(1.0036571) \approx .78357295$

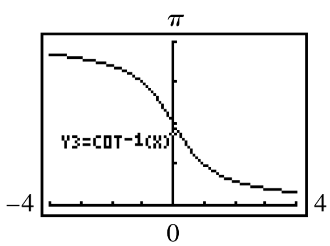
67.



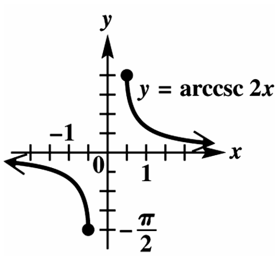
68.



69.



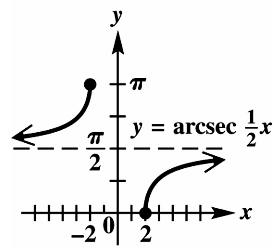
70.



Domain: $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

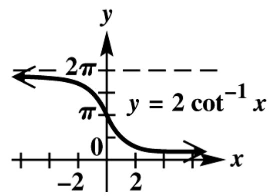
71.



Domain: $(-\infty, -2] \cup [2, \infty)$

Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

72.



Domain: $(-\infty, \infty)$

Range: $(0, 2\pi)$

73. 1.003 is not in the domain of $y = \sin^{-1} x$. (Alternatively, you could state that 1.003 is not in the range of $y = \sin x$.)

74. $f\left[f^{-1}(x)\right] = f\left[\frac{1}{3}x + \frac{2}{3}\right] = 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2$

$= x + 2 - 2 = x$

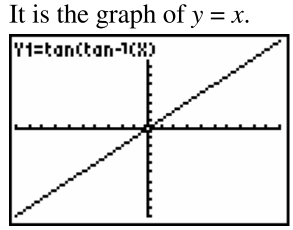
$f^{-1}\left[f(x)\right] = f^{-1}[3x - 2] = \frac{3x - 2}{3} + \frac{2}{3}$

$= \frac{3x - 2 + 2}{3} = x$

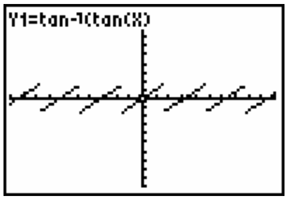
In each case the result is x. The graph is a straight line bisecting quadrants I and III (i.e., the line $y = x$).

The graphs for Exercises 75 and 76 are in the standard window. Exercise 76 is graphed in the dot mode to avoid the appearance of vertical line segments.

75. It is the graph of $y = x$.

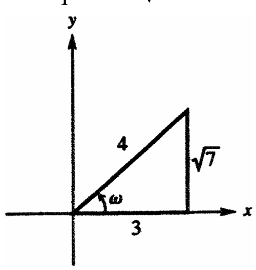


76. It does not agree because the range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, not $(-\infty, \infty)$, as was the case in Exercise 75.



77. $\tan\left(\arccos\frac{3}{4}\right)$

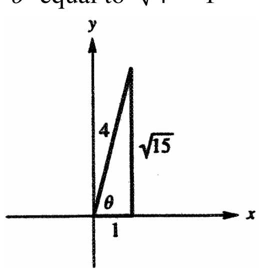
Let $\omega = \arccos\frac{3}{4}$, so that $\cos\omega = \frac{3}{4}$. Since arccos is defined only in quadrants I and II, and $\frac{3}{4}$ is positive, ω is in quadrant I. Sketch ω and label a triangle with the side opposite ω equal to $\sqrt{4^2-3^2} = \sqrt{16-9} = \sqrt{7}$.



$\tan\left(\arccos\frac{3}{4}\right) = \tan\omega = \frac{\sqrt{7}}{3}$

78. $\sin\left(\arccos\frac{1}{4}\right)$

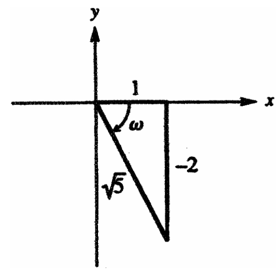
Let $\theta = \arccos\frac{1}{4}$, so that $\cos\theta = \frac{1}{4}$. Since arccos is defined only in quadrants I and II, and $\frac{1}{4}$ is positive, θ is in quadrant I. Sketch θ and label a triangle with the side opposite θ equal to $\sqrt{4^2-1^2} = \sqrt{16-1} = \sqrt{15}$.



$\sin\left(\arccos\frac{1}{4}\right) = \sin\theta = \frac{\sqrt{15}}{4}$

79. $\cos(\tan^{-1}(-2))$

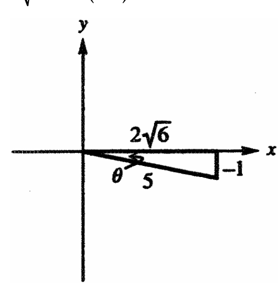
Let $\omega = \tan^{-1}(-2)$, so that $\tan\omega = -2$. Since $\tan^{-1}$ is defined only in quadrants I and IV, and -2 is negative, ω is in quadrant IV. Sketch ω and label a triangle with the hypotenuse equal to $\sqrt{(-2)^2+1} = \sqrt{4+1} = \sqrt{5}$.



$\cos(\tan^{-1}(-2)) = \cos\omega = \frac{\sqrt{5}}{5}$

80. $\sec\left(\sin^{-1}\left(-\frac{1}{5}\right)\right)$

Let $\theta = \sin^{-1}\left(-\frac{1}{5}\right)$, so that $\sin\theta = -\frac{1}{5}$. Since arcsin is defined only in quadrants I and IV, and $-\frac{1}{5}$ is negative, θ is in quadrant IV. Sketch θ and label a triangle with the side adjacent to θ equal to $\sqrt{5^2-(-1)^2} = \sqrt{25-1} = \sqrt{24} = 2\sqrt{6}$.



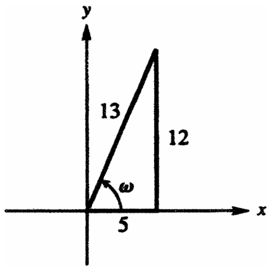
$\sec\left(\sin^{-1}\left(-\frac{1}{5}\right)\right) = \sec\theta = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12}$

81. $\sin\left(2\tan^{-1}\frac{12}{5}\right)$

Let $\omega = \tan^{-1}\frac{12}{5}$, so that $\tan\omega = \frac{12}{5}$. Since $\tan^{-1}\omega$ is defined only in quadrants I and IV, and $\frac{12}{5}$ is positive, ω is in quadrant I.

Sketch ω and label a right triangle with the hypotenuse equal to

$\sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13.$

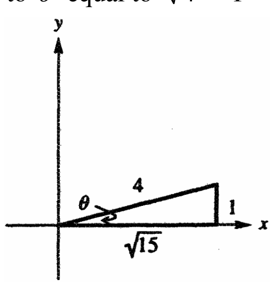


$\sin\omega = \frac{12}{13}; \cos\omega = \frac{5}{13}$

$$\sin\left(2\tan^{-1}\frac{12}{5}\right) = \sin(2\omega) = 2\sin\omega\cos\omega = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$$

82. $\cos\left(2\sin^{-1}\frac{1}{4}\right)$

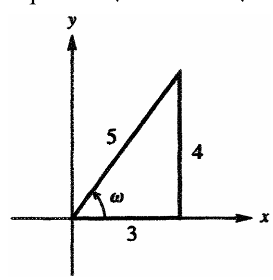
Let $\theta = \sin^{-1}\frac{1}{4}$, so that $\sin\theta = \frac{1}{4}$. Since $\sin^{-1}\theta$ is defined only in quadrants I and IV, and $\frac{1}{4}$ is positive, θ is in quadrant I. Sketch θ and label a triangle with the side adjacent to θ equal to $\sqrt{4^2-1^2} = \sqrt{16-1} = \sqrt{15}.$



$$\cos\left(2\sin^{-1}\frac{1}{4}\right) = \cos 2\theta = 1-2\sin^2\theta = 1-2\left(\frac{1}{4}\right)^2 = 1-2\left(\frac{1}{16}\right) = 1-\frac{1}{8} = \frac{7}{8}$$

83. $\cos\left(2\ \arctan\frac{4}{3}\right)$

Let $\omega = \arctan\frac{4}{3}$, so that $\tan\ \omega = \frac{4}{3}$. Since arctan is defined only in quadrants I and IV, and $\frac{4}{3}$ is positive, ω is in quadrant I. Sketch ω and label a triangle with the hypotenuse equal to $\sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5.$



$\cos\omega = \frac{3}{5}; \sin\omega = \frac{4}{5}$

$$\cos\left(2\arctan\frac{4}{3}\right) = \cos(2\omega) = \cos^2\omega - \sin^2\omega = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

84. $\tan\left(2\cos^{-1}\frac{1}{4}\right)$

Let $\theta = \cos^{-1}\frac{1}{4}$, so that $\cos\ \theta = \frac{1}{4}$. Since $\cos^{-1}\theta$ is defined only in quadrants I and II, and $\frac{1}{4}$ is positive, θ is in quadrant I. Sketch θ and label a triangle with the side opposite θ equal to $\sqrt{4^2-1^2} = \sqrt{16-1} = \sqrt{15}.$

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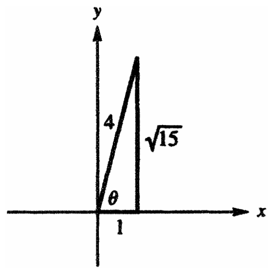
(*continued from page 683*)



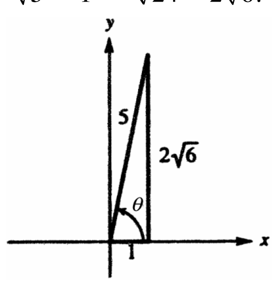
$$\tan\theta = \frac{\sqrt{15}}{1} = \sqrt{15}$$

$$\tan\left(2\cos^{-1}\frac{1}{4}\right) = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\sqrt{15}}{1-\left(\sqrt{15}\right)^2} = \frac{2\sqrt{15}}{-14} = -\frac{\sqrt{15}}{7}$$

85. $\sin\left(2\cos^{-1}\frac{1}{5}\right)$

Let $\theta = \cos^{-1}\frac{1}{5}$, so that $\cos\theta = \frac{1}{5}$. The inverse cosine function yields values only in quadrants I and II, and since $\frac{1}{5}$ is positive, θ is in quadrant I. Sketch θ and label the sides of the right triangle. By the Pythagorean theorem, the length opposite to θ will be $\sqrt{5^2-1^2} = \sqrt{24} = 2\sqrt{6}$.



From the figure,

$\sin\ \theta = \frac{2\sqrt{6}}{5}$. Then,

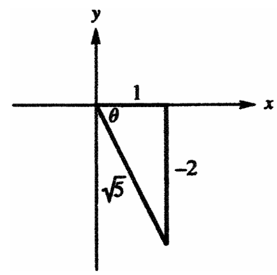
$$\sin\left(2\cos^{-1}\frac{1}{5}\right) = \sin 2\theta = 2\sin\theta\ \cos\theta$$

$$= 2\left(\frac{2\sqrt{6}}{5}\right)\left(\frac{1}{5}\right) = \frac{4\sqrt{6}}{25}$$

86. $\cos\left(2\tan^{-1}(-2)\right)$

Let θ = arctan (–2), so that tan θ = –2. Since arctan is defined only in quadrants I and IV, and –2 is negative, θ is in quadrant IV. Sketch θ and label a triangle with the hypotenuse equal to

$\sqrt{(-2)^2+1^2} = \sqrt{4+1} = \sqrt{5}$.



$$\cos\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos\left(2\tan^{-1}(-2)\right) = \cos 2\theta = 2\cos^2\theta - 1$$

$$= 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$$

87. $\sec(\sec^{-1} 2)$

Since secant and inverse secant are inverse functions, $\sec\left(\sec^{-1} 2\right) = 2$.

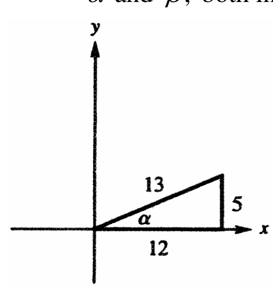
88. $\csc\left(\csc^{-1}\sqrt{2}\right) = \csc\frac{\pi}{4} = \sqrt{2}$

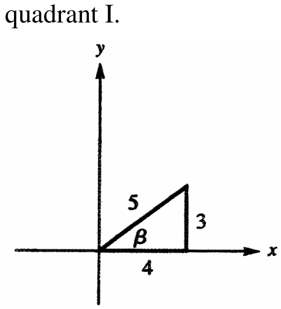
Since cosecant and inverse cosecant are inverse functions, $\csc\left(\csc^{-1}\sqrt{2}\right) = \sqrt{2}$.

89. $\cos\left(\tan^{-1}\frac{5}{12} - \cot^{-1}\frac{3}{4}\right)$

Let $\alpha = \tan^{-1}\frac{5}{12}$ and $\beta = \tan^{-1}\frac{4}{3}$. Then

$\tan\alpha = \frac{5}{12}$ and $\tan\beta = \frac{3}{4}$. Sketch angles α and β, both in quadrant I.



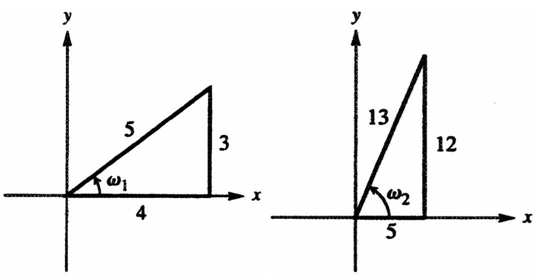


We have $\sin\alpha = \dfrac{5}{13}$, $\cos\alpha = \dfrac{12}{13}$, $\sin\beta = \dfrac{3}{5}$, and $\cos\beta = \dfrac{4}{5}$.

$$\begin{aligned}&\cos\left(\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{3}{4}\right)\\&= \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\\&= \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}\end{aligned}$$

90. $\cos\left(\sin^{-1}\dfrac{3}{5} + \cos^{-1}\dfrac{5}{13}\right)$

Let $\omega_1 = \sin^{-1}\dfrac{3}{5}$ and $\omega_2 = \cos^{-1}\dfrac{5}{13}$ Then $\sin\omega_1 = \dfrac{3}{5}$ and $\cos\omega_2 = \dfrac{5}{13}$. Sketch ω_1 and ω_2 in quadrant I.



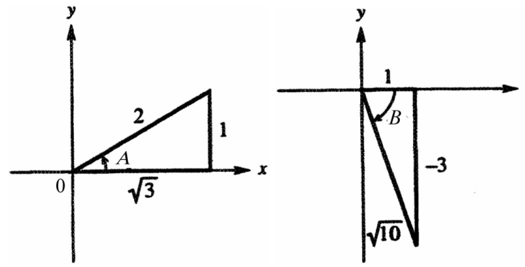
We have $\sin\omega_1 = \dfrac{3}{5}$, $\cos\omega_1 = \dfrac{4}{5}$, $\cos\omega_2 = \dfrac{5}{13}$, and $\sin\omega_2 = \dfrac{12}{13}$.

$$\begin{aligned}&\cos\left(\arcsin\frac{3}{5} + \arccos\frac{5}{13}\right)\\&= \cos(\omega_1 + \omega_2) = \cos\omega_1\cos\omega_2 - \sin\omega_1\sin\omega_2\\&= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}\end{aligned}$$

91. $\sin\left(\sin^{-1}\dfrac{1}{2} + \tan^{-1}(-3)\right)$

Let $\sin^{-1}\dfrac{1}{2} = A$ and $\tan^{-1}(-3) = B$.

Then $\sin A = \dfrac{1}{2}$ and $\tan B = -3$. Sketch angle A in quadrant I and angle B in quadrant IV.

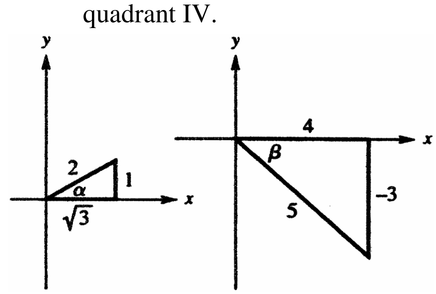


We have $\cos A = \dfrac{\sqrt{3}}{2}$, $\sin A = \dfrac{1}{2}$, $\cos B = \dfrac{1}{\sqrt{10}} = \dfrac{\sqrt{10}}{10}$, and $\sin B = \dfrac{-3}{\sqrt{10}} = -\dfrac{3\sqrt{10}}{10}$.

$$\begin{aligned}&\sin\left(\sin^{-1}\frac{1}{2} + \tan^{-1}(-3)\right)\\&= \sin(A + B) = \sin A\cos B + \cos A\sin B\\&= \frac{1}{2}\cdot\frac{1}{\sqrt{10}} + \frac{\sqrt{3}}{2}\cdot\frac{-3}{\sqrt{10}}\\&= \frac{1 - 3\sqrt{3}}{2\sqrt{10}} = \frac{\sqrt{10} - 3\sqrt{30}}{20}\end{aligned}$$

92. $\tan\left(\cos^{-1}\dfrac{\sqrt{3}}{2} - \sin^{-1}\left(-\dfrac{3}{5}\right)\right)$

Let $\alpha = \cos^{-1}\dfrac{\sqrt{3}}{2}$, $\beta = \sin^{-1}\left(-\dfrac{3}{5}\right)$. Sketch angle α in quadrant I and angle β in quadrant IV.



We have $\tan\alpha = \dfrac{1}{\sqrt{3}}$ and $\tan\beta = -\dfrac{3}{4}$.

$$\begin{aligned}&\tan\left(\cos^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\left(-\frac{3}{5}\right)\right)\\&= \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\ \tan\beta}\\&= \frac{\frac{1}{\sqrt{3}} - \left(-\frac{3}{4}\right)}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{3}{4}\right)} = \frac{\frac{4 + 3\sqrt{3}}{4\sqrt{3}}}{\frac{4\sqrt{3} - 3}{4\sqrt{3}}}\end{aligned}$$

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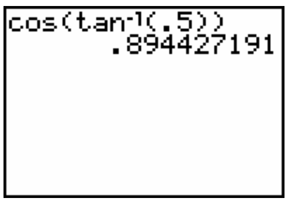
(*continued from page 685*)

$$=\frac{4+3\sqrt{3}}{4\sqrt{3}-3}=\frac{4+3\sqrt{3}}{-3+4\sqrt{3}}\cdot\frac{-3-4\sqrt{3}}{-3-4\sqrt{3}}$$
$$=\frac{-12-25\sqrt{3}-36}{9-48}=\frac{-48-25\sqrt{3}}{-39}$$
$$=\frac{48+25\sqrt{3}}{39}$$

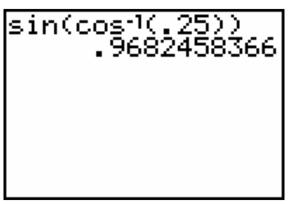
For Exercises 93–96, your calculator could be in either degree or radian mode. Keystroke sequences may vary based on the type and/or model of calculator being used.

93. $\cos\left(\tan^{-1}.5\right)$



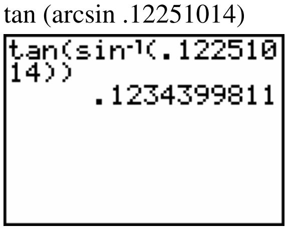
$\cos(\tan^{-1}.5)\approx .894427191$

94. $\sin\left(\cos^{-1}.25\right)$



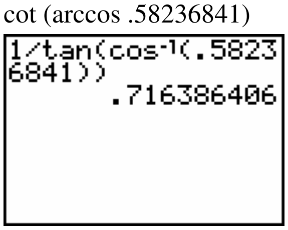
$\sin\left(\cos^{-1}.25\right)\approx .9682458366$

95. tan (arcsin .12251014)



tan (arcsin .12251014) $\approx$ .1234399811

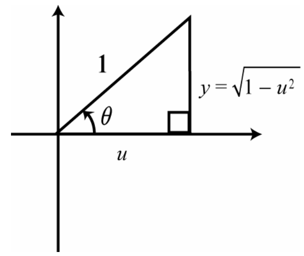
96. cot (arccos .58236841)



cot (arccos .58236841) $\approx$ .716386406

97. $\sin\left(\arccos u\right)$

Let $\theta=\arccos u$, so $\cos\theta=u=\frac{u}{1}$. Since $u>0$, $0<\theta<\frac{\pi}{2}$.

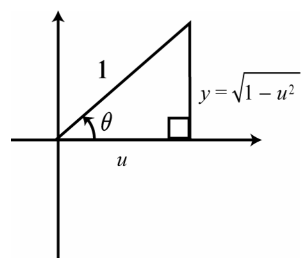


Since $y>0$, from the Pythagorean theorem, $y=\sqrt{1^2-u^2}=\sqrt{1-u^2}$.

Therefore, $\sin\theta=\frac{\sqrt{1-u^2}}{1}=\sqrt{1-u^2}$. Thus, $\sin\left(\arccos u\right)=\sqrt{1-u^2}$.

98. $\tan\left(\arccos u\right)$

Let $\theta=\arccos u$, so $\cos\theta=u=\frac{u}{1}$. Since $u>0$, $0<\theta<\frac{\pi}{2}$.

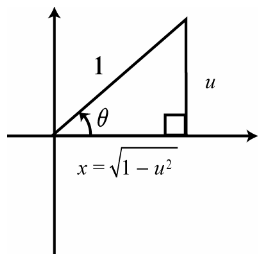


Since $y>0$, from the Pythagorean theorem, $y=\sqrt{1^2-u^2}=\sqrt{1-u^2}$.

Therefore, $\tan\theta=\frac{\sqrt{1-u^2}}{u}$. Thus, $\tan\left(\arccos u\right)=\frac{\sqrt{1-u^2}}{u}$.

99. $\cos\left(\arcsin u\right)$

Let $\theta=\arcsin u$, so $\sin\theta=u=\frac{u}{1}$. Since $u>0$, $0<\theta<\frac{\pi}{2}$.



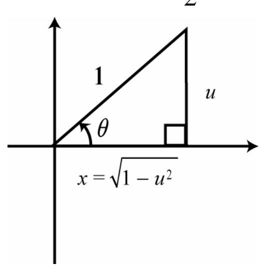
Since $x > 0$, from the Pythagorean theorem,
$x = \sqrt{1^2 - u^2} = \sqrt{1-u^2}$.

Therefore, $\cos\theta = \frac{\sqrt{1-u^2}}{1} = \sqrt{1-u^2}$. Thus,

$\cos(\arcsin u) = \sqrt{1-u^2}$

100. $\cot(\arcsin u)$

Let $\theta = \arcsin u$, so $\sin\theta = u = \frac{u}{1}$. Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.



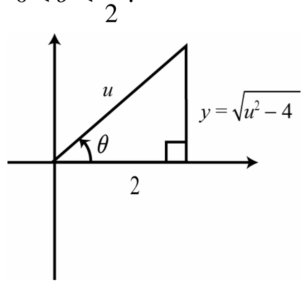
Since $x > 0$, from the Pythagorean theorem,
$x = \sqrt{1^2 - u^2} = \sqrt{1-u^2}$.

Therefore, $\cot\theta = \frac{\sqrt{1-u^2}}{u}$. Thus,

$\cot(\arcsin u) = \frac{\sqrt{1-u^2}}{u}$.

101. $\sin\left(2\sec^{-1}\frac{u}{2}\right)$

Let $\theta = \sec^{-1}\frac{u}{2}$, so $\sec\theta = \frac{u}{2}$. Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.



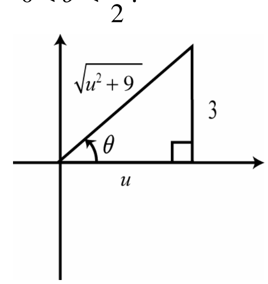
Since $y > 0$, from the Pythagorean theorem,
$y = \sqrt{u^2 - 2^2} = \sqrt{u^2 - 4}$.
Now find $\sin 2\theta$.

$\sin\theta = \frac{\sqrt{u^2-4}}{u}$ and $\cos\theta = \frac{2}{u}$ Thus,

$$\sin\left(2\sec^{-1}\frac{u}{2}\right) = \sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{\sqrt{u^2-4}}{u}\right)\left(\frac{2}{u}\right)$$
$$= \frac{4\sqrt{u^2-4}}{u^2}$$

102. $\cos\left(2\tan^{-1}\frac{3}{u}\right)$

Let $\theta = \tan^{-1}\frac{3}{u}$, so $\tan\theta = \frac{3}{u}$. Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.



From the Pythagorean theorem,
$r = \sqrt{u^2 + 3^2} = \sqrt{u^2+9}$. Now find $\cos 2\theta$

$\cos\theta = \frac{u}{\sqrt{u^2+9}}$. Thus,

$$\cos\left(2\tan^{-1}\frac{3}{u}\right) = \cos 2\theta = 2\cos^2\theta - 1$$
$$= 2\left(\frac{u}{\sqrt{u^2+9}}\right)^2 - 1$$
$$= \frac{2u^2}{u^2+9} - 1 = \frac{u^2-9}{u^2+9}$$

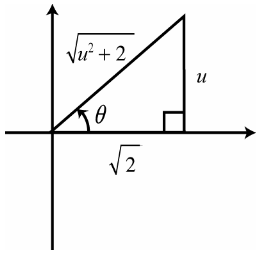
103. $\tan\left(\sin^{-1}\frac{u}{\sqrt{u^2+2}}\right)$

Let $\theta = \sin^{-1}\frac{u}{\sqrt{u^2+2}}$, so $\sin\theta = \frac{u}{\sqrt{u^2+2}}$.

Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.

(*continued on next page*)

(*continued from page 687*)



Since $x > 0$, from the Pythagorean theorem,

$$x = \sqrt{\left(\sqrt{u^2+2}\right)^2 - u^2} = \sqrt{u^2+2-u^2} = \sqrt{2}.$$

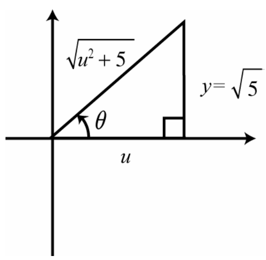
Therefore, $\tan\theta = \frac{u}{\sqrt{2}} = \frac{u\sqrt{2}}{2}$. Thus,

$$\tan\left(\sin^{-1}\frac{u}{\sqrt{u^2+2}}\right) = \frac{u\sqrt{2}}{2}.$$

104. $\sec\left(\cos^{-1}\frac{u}{\sqrt{u^2+5}}\right)$

Let $\theta = \cos^{-1}\frac{u}{\sqrt{u^2+5}}$, so $\cos\theta = \frac{u}{\sqrt{u^2+5}}$.

Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.



Since $y > 0$, from the Pythagorean theorem,

$$x = \sqrt{\left(\sqrt{u^2+5}\right)^2 - u^2} = \sqrt{u^2+5-u^2} = \sqrt{5}.$$

Therefore, $\sec\theta = \frac{\sqrt{u^2+5}}{u}$. Thus,

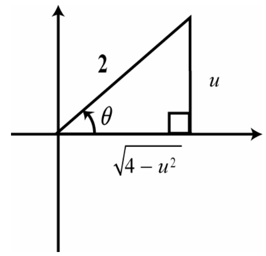
$\sec\left(\cos^{-1}\frac{u}{\sqrt{u^2+5}}\right) = \frac{\sqrt{u^2+5}}{u}$. Also note,

$$\sec\left(\cos^{-1}\frac{u}{\sqrt{u^2+5}}\right) = \frac{1}{\cos\left(\cos^{-1}\frac{u}{\sqrt{u^2+5}}\right)} = \frac{1}{\frac{u}{\sqrt{u^2+5}}} = \frac{\sqrt{u^2+5}}{u}.$$

105. $\sec\left(\operatorname{arc}\cot\frac{\sqrt{4-u^2}}{u}\right)$

Let $\theta = \operatorname{arc}\cot\frac{\sqrt{4-u^2}}{u}$, so $\cot\theta = \frac{\sqrt{4-u^2}}{u}$.

Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.



From the Pythagorean theorem,

$$r = \sqrt{\left(\sqrt{4-u^2}\right)^2 + u^2} = \sqrt{4-u^2+u^2} = \sqrt{4} = 2.$$

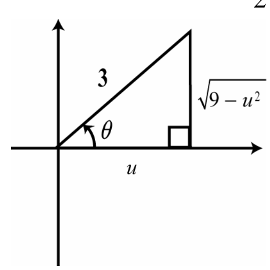
Therefore, $\sec\theta = \frac{2}{\sqrt{4-u^2}} = \frac{2\sqrt{4-u^2}}{4-u^2}$.

Thus, $\sec\left(\operatorname{arc}\cot\frac{\sqrt{4-u^2}}{u}\right) = \frac{2\sqrt{4-u^2}}{4-u^2}$.

106. $\csc\left(\arctan\frac{\sqrt{9-u^2}}{u}\right)$

Let $\theta = \arctan\frac{\sqrt{9-u^2}}{u}$, so $\tan\theta = \frac{\sqrt{9-u^2}}{u}$.

Since $u > 0$, $0 < \theta < \frac{\pi}{2}$.



From the Pythagorean theorem,

$$r = \sqrt{\left(\sqrt{9-u^2}\right)^2 + u^2} = \sqrt{9-u^2+u^2} = \sqrt{9} = 3.$$

Therefore, $\csc\theta = \frac{3}{\sqrt{9-u^2}} = \frac{3\sqrt{9-u^2}}{9-u^2}$. Thus,

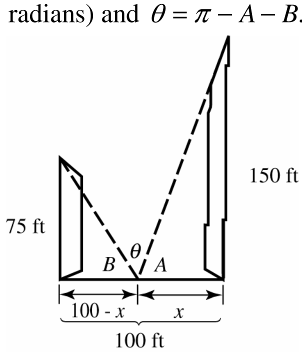
$$\csc\left(\arctan\frac{\sqrt{9-u^2}}{u}\right) = \frac{3\sqrt{9-u^2}}{9-u^2}.$$

107. **(a)** $\theta = \arcsin\sqrt{\dfrac{42^2}{2\left(42^2\right)+64(0)}}$
$= \arcsin\sqrt{\dfrac{42^2}{2\left(42^2\right)+0}} = \arcsin\sqrt{\dfrac{42^2}{2\left(42^2\right)}}$
$= \arcsin\sqrt{\dfrac{1}{2}} = \arcsin\dfrac{1}{\sqrt{2}}$
$= \arcsin\dfrac{\sqrt{2}}{2} = 45°$

(b) $\theta = \arcsin\sqrt{\dfrac{v^2}{2v^2+64(6)}}$
$= \arcsin\sqrt{\dfrac{v^2}{2v^2+384}}$

As v gets larger and larger,
$\sqrt{\dfrac{v^2}{2v^2+384}} \approx \sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2}$. Thus,
$\theta \approx \arcsin\dfrac{\sqrt{2}}{2} = 45°$.
The equation of the asymptote is $\theta = 45°$.

108. Let A be the triangle to the "right" of θ and let B be the angle to the "left" of θ. Then $A+\theta+B=\pi$ (since the angles sum to π radians) and $\theta = \pi - A - B$.



$\tan A = \dfrac{150}{x}$, so $A = \arctan\left(\dfrac{150}{x}\right)$ and
$\tan B = \dfrac{75}{100-x}$, so $B = \arctan\left(\dfrac{75}{100-x}\right)$.
As a result, $\theta = \pi - B - A \Rightarrow$
$\theta = \pi - \arctan\left(\dfrac{75}{100-x}\right) - \arctan\left(\dfrac{150}{x}\right)$.

109. $\theta = \tan^{-1}\left(\dfrac{x}{x^2+2}\right)$

(a) $x = 1$,
$\theta = \tan^{-1}\left(\dfrac{1}{1^2+2}\right) = \tan^{-1}\left(\dfrac{1}{3}\right) \approx 18°$

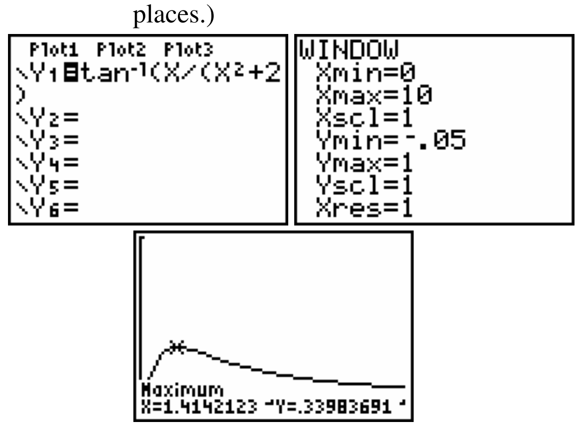
(b) $x = 2$,
$\theta = \tan^{-1}\left(\dfrac{2}{2^2+2}\right) = \tan^{-1}\dfrac{2}{6}$
$= \tan^{-1}\dfrac{1}{3} \approx 18°$

(c) $x = 3$,
$\theta = \tan^{-1}\left(\dfrac{3}{3^2+2}\right) = \tan^{-1}\dfrac{3}{11} \approx 15°$

(d) $\tan(\theta+\alpha) = \dfrac{1+1}{x} = \dfrac{2}{x}$ and $\tan\alpha = \dfrac{1}{x}$

$$\tan(\theta+\alpha) = \frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$
$$\frac{2}{x} = \frac{\tan\theta+\frac{1}{x}}{1-\tan\theta\left(\frac{1}{x}\right)}$$
$$\frac{2}{x} = \frac{x\tan\theta+1}{x-\tan\theta}$$
$$2(x-\tan\theta) = x(x\tan\theta+1)$$
$$2x-2\tan\theta = x^2\tan\theta+x$$
$$2x-x = x^2\tan\theta+2\tan\theta$$
$$x = \tan\theta\left(x^2+2\right)$$
$$\tan\theta = \frac{x}{x^2+2}$$
$$\theta = \tan^{-1}\left(\frac{x}{x^2+2}\right)$$

(e) If we graph $y_1 = \tan^{-1}\left(\dfrac{x}{x^2+2}\right)$ using a graphing calculator, the maximum value of the function occurs when x is 1.4142151 m. (Note: Due to the computational routine, there may be a discrepancy in the last few decimal places.)

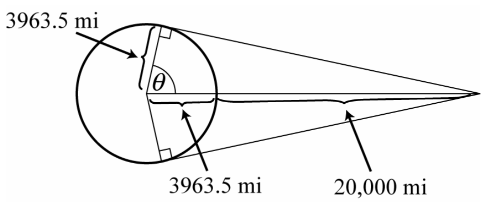


(f) $x = \sqrt{(1)(2)} = \sqrt{2}$

110. Since the diameter of the earth is 7927 miles at the equator, the radius of the earth is 3963.5 miles. Then

$\cos\theta = \dfrac{3963.5}{20,000 + 3963.5} = \dfrac{3963.5}{23,963.5}$ and

$\theta = \arccos\left(\dfrac{3963.5}{23,963.5}\right) \approx 80.48°.$



The percent of the equator that can be seen by the satellite is $\dfrac{2\theta}{360} \cdot 100 = \dfrac{2(80.48)}{360} \approx 44.7\%.$

Section 7.6: Trigonometric Equations

1. Solve the linear equation for $\cot x$.
2. Solve the linear equation for $\sin x$.
3. Solve the quadratic equation for $\sec x$ by factoring.
4. Solve the quadratic equation for $\cos x$ by the factoring.
5. Solve the quadratic equation for $\sin x$ using the quadratic formula.
6. Solve the quadratic for $\tan x$ using the quadratic formula.
7. Use the identity to rewrite as an equation with one trigonometric function.
8. Use an identity to rewrite as an equation with one trigonometric function.
9. $-30°$ is not in the interval $[0°, 360°)$.
10. To show that $\left\{0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}$ is not the correct solution set to the equation $\sin x = 1 - \cos x$, show that at least one element of the set is not a solution.

Check $x = 0$.

$$\begin{array}{ll} \sin x = 1 - \cos x & \\ \sin 0 = 1 - \cos 0 & ? \\ 0 = 1 - 1 & ? \\ 0 = 0 & \text{True} \end{array}$$

$x = 0$ is a solution.

Check $x = \dfrac{\pi}{2}$.

$$\begin{array}{ll} \sin x = 1 - \cos x & \\ \sin\frac{\pi}{2} = 1 - \cos\frac{\pi}{2} & ? \\ 1 = 1 - 0 & ? \\ 1 = 1 & \text{True} \end{array}$$

$x = \dfrac{\pi}{2}$ is a solution.

Check $x = \dfrac{3\pi}{2}$.

$$\begin{array}{ll} \sin x = 1 - \cos x & \\ \sin\frac{3\pi}{2} = 1 - \cos\frac{3\pi}{2} & ? \\ -1 = 1 - 0 & ? \\ -1 = 1 & \text{False} \end{array}$$

$x = \dfrac{3\pi}{2}$ is not a solution.

In general, when you square both sides of an equation or raise both sides of an equation to an even power, you must check all solutions in order to eliminate any extraneous solutions.

11. $2\cot x + 1 = -1 \Rightarrow 2\cot x = -2 \Rightarrow \cot x = -1$

Over the interval $[0, 2\pi)$, the equation $\cot x = -1$ has two solutions, the angles in quadrants II and IV that have a reference angle of $\dfrac{\pi}{4}$. These are $\dfrac{3\pi}{4}$ and $\dfrac{7\pi}{4}$.

Solution set: $\left\{\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right\}$

12. $\sin x + 2 = 3 \Rightarrow \sin x = 1$

Over the interval $[0, 2\pi)$, the equation $\sin x = 1$ has one solution. This solution is $\dfrac{\pi}{2}$.

Solution set: $\left\{\dfrac{\pi}{2}\right\}$

13. $2\sin x + 3 = 4 \Rightarrow 2\sin x = 1 \Rightarrow \sin x = \dfrac{1}{2}$

Over the interval $[0, 2\pi)$, the equation $\sin x = \dfrac{1}{2}$ has two solutions, the angles in quadrants I and II that have a reference angle of $\dfrac{\pi}{6}$. These are $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$.

Solution set: $\left\{\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right\}$

14. $2\sec x+1=\sec x+3 \Rightarrow \sec x=2$
Over the interval $[0,2\pi)$, the equation $\sec x=2$ has two solutions, the angles in quadrants I and IV that have a reference angle of $\frac{\pi}{3}$. These are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.
Solution set: $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

15. $\tan^2 x+3=0 \Rightarrow \tan^2 x=-3$
The square of a real number cannot be negative, so this equation has no solution.
Solution set: $\varnothing$

16. $\sec^2 x+2=-1 \Rightarrow \sec^2 x=-3$
The square of a real number cannot be negative, so this equation has no solution.
Solution set: $\varnothing$

17. $(\cot x-1)(\sqrt{3}\cot x+1)=0$
$\cot x-1=0 \Rightarrow \cot x=1$ or
$\sqrt{3}\cot x+1=0 \Rightarrow \sqrt{3}\cot x=-1 \Rightarrow$
$\cot x=-\frac{1}{\sqrt{3}} \Rightarrow \cot x=-\frac{\sqrt{3}}{3}$
Over the interval $[0,2\pi)$, the equation $\cot x=1$ has two solutions, the angles in quadrants I and III that have a reference angle of $\frac{\pi}{4}$. These are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. In the same interval, $\cot x=-\frac{\sqrt{3}}{3}$ also has two solutions. The angles in quadrants II and IV that have a reference angle of $\frac{\pi}{3}$ are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.
Solution set: $\left\{\frac{\pi}{4}, \frac{2\pi}{3}, \frac{5\pi}{4}, \frac{5\pi}{3}\right\}$

18. $(\csc x+2)(\csc x-\sqrt{2})=0$
$\csc x+2=0 \Rightarrow \csc x=-2$ or
$\csc x-\sqrt{2}=0 \Rightarrow \csc x=\sqrt{2}$
Over the interval $[0,2\pi)$, the equation $\csc x=-2$ has two solutions, the angles in quadrants III and IV that have a reference angle of $\frac{\pi}{6}$. These are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. In the same interval, $\csc x=\sqrt{2}$ also has two solutions, the angles in quadrants I and II that have a reference angle of $\frac{\pi}{4}$. These are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.
Solution set: $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

19. $\cos^2 x+2\cos x+1=0$
$\cos^2 x+2\cos x+1=0 \Rightarrow (\cos x+1)^2=0 \Rightarrow$
$\cos x+1=0 \Rightarrow \cos x=-1$
Over the interval $[0,2\pi)$, the equation $\cos x=-1$ has one solution. This solution is π. Solution set: $\{\pi\}$

20. $2\cos^2 x-\sqrt{3}\cos x=0$
$2\cos^2 x-\sqrt{3}\cos x=0$
$\cos x(2\cos x-\sqrt{3})=0$
$\cos x=0$ or
$2\cos x-\sqrt{3}=0 \Rightarrow 2\cos x=\sqrt{3} \Rightarrow \cos x=\frac{\sqrt{3}}{2}$
Over the interval $[0,2\pi)$, the equation $\cos x=0$ has two solutions. These solutions are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. In the same interval, $\cos x=\frac{\sqrt{3}}{2}$ also has two solutions. The angles in quadrants I and IV that have a reference angle of $\frac{\pi}{6}$ are $\frac{\pi}{6}$ and $\frac{11\pi}{6}$.
Solution set: $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

21. $-2\sin^2 x=3\sin x+1$
$-2\sin^2 x=3\sin x+1$
$2\sin^2 x+3\sin x+1=0$
$(2\sin x+1)(\sin x+1)=0$
$2\sin x+1=0 \Rightarrow \sin x=-\frac{1}{2}$ or
$\sin x+1=0 \Rightarrow \sin x=-1$
Over the interval $[0,2\pi)$, the equation $\sin x=-\frac{1}{2}$ has two solutions. The angles in quadrants III and IV that have a reference angle of $\frac{\pi}{6}$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. In the same interval, $\sin x=-1$ when the angle is $\frac{3\pi}{2}$.
Solution set: $\left\{\frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

22. $2\cos^2 x - \cos x = 1$

$$2\cos^2 x - \cos x = 1$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$

$2\cos x + 1 = 0 \Rightarrow 2\cos x = -1 \Rightarrow \cos x = -\frac{1}{2}$

or $\cos x - 1 = 0 \Rightarrow \cos x = 1$

Over the interval $[0, 2\pi)$, the equation $\cos x = -\frac{1}{2}$ has two solutions. The angles in quadrants II and III that have a reference angle of $\frac{\pi}{3}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. In the same interval, $\cos x = 1$ when the angle is 0.

Solution set: $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

23. $\left(\cot\theta - \sqrt{3}\right)\left(2\sin\theta + \sqrt{3}\right) = 0$

$\cot\theta - \sqrt{3} = 0 \Rightarrow \cot\theta = \sqrt{3}$ or

$2\sin\theta + \sqrt{3} = 0 \Rightarrow 2\sin\theta = -\sqrt{3} \Rightarrow$

$\sin\theta = -\frac{\sqrt{3}}{2}$

Over the interval $[0°, 360°)$, the equation $\cot\theta = \sqrt{3}$ has two solutions, the angles in quadrants I and III that have a reference angle of 30°. These are 30° and 210°. In the same interval, the equation $\sin\theta = -\frac{\sqrt{3}}{2}$ has two solutions, the angles in quadrants III and IV that have a reference angle of 60°. These are 240° and 300°.

Solution set: $\{30°, 210°, 240°, 300°\}$

24. $(\tan\theta - 1)(\cos\theta - 1) = 0$

$\tan\theta - 1 = 0 \Rightarrow \tan\theta = 1$ or

$\cos\theta - 1 = 0 \Rightarrow \cos\theta = 1$

Over the interval $[0°, 360°)$, the equation $\tan\theta = 1$ has two solutions, the angles in quadrants I and III that have a reference angle of 45°. These are 45° and 225°. In the same interval, the equation $\cos\theta = 1$ has one solution. The angle is 0°.

Solution set: $\{0°, 45°, 225°\}$

25. $2\sin\theta - 1 = \csc\theta$

$$2\sin\theta - 1 = \csc\theta$$
$$2\sin\theta - 1 = \frac{1}{\sin\theta}$$
$$2\sin^2\theta - \sin\theta = 1$$
$$2\sin^2\theta - \sin\theta - 1 = 0$$
$$(2\sin\theta + 1)(\sin\theta - 1) = 0$$

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$ or

$\sin\theta - 1 = 0 \Rightarrow \sin\theta = 1$

Over the interval $[0°, 360°)$, the equation $\sin\theta = -\frac{1}{2}$ has two solutions, the angles in quadrants III and IV that have a reference angle of 30°. These are 210° and 330°. In the same interval, the only angle θ for which $\sin\theta = 1$ is 90°.

Solution set: $\{90°, 210°, 330°\}$

26. $\tan\theta + 1 = \sqrt{3} + \sqrt{3}\cot\theta$

$$\tan\theta + 1 = \sqrt{3} + \sqrt{3}\cot\theta$$
$$\tan\theta + 1 = \sqrt{3} + \frac{\sqrt{3}}{\tan\theta}$$
$$\tan^2\theta + \tan\theta = \sqrt{3}\tan\theta + \sqrt{3}$$
$$\tan^2\theta + \left(1 - \sqrt{3}\right)\tan\theta - \sqrt{3} = 0$$
$$\left(\tan\theta - \sqrt{3}\right)(\tan\theta + 1) = 0$$
$$\tan\theta - \sqrt{3} = 0$$
$$\tan\theta = \sqrt{3} \text{ or}$$
$$\tan\theta + 1 = 0 \Rightarrow \tan\theta = -1$$

Over the interval $[0°, 360°)$, the equation $\tan\theta = \sqrt{3}$ has two solutions, the angles in quadrants I and III that have a reference angle of 60°. These are 60° and 240°. In the same interval, the equation $\tan\theta = -1$ has two solutions, the angles in quadrants II and IV that have a reference angle of 45°. These are 135° and 315°.

Solution set: $\{60°, 135°, 240°, 315°\}$

27. $\tan\theta - \cot\theta = 0$

$\tan\theta - \cot\theta = 0 \Rightarrow \tan\theta - \frac{1}{\tan\theta} = 0 \Rightarrow$

$\tan^2\theta - 1 = 0 \Rightarrow \tan^2\theta = 1 \Rightarrow \tan\theta = \pm 1$

Over the interval $[0°, 360°)$, the equation $\tan\theta = 1$ has two solutions, the angles in quadrants I and III that have a reference angle of 45°. These are 45° and 225°.

In the same interval, the equation $\tan\theta = -1$ has two solutions, the angles in quadrants II and IV that have a reference angle of $45°$. These are $135°$ and $315°$.
Solution set: $\{45°, 135°, 225°, 315°\}$

28. $\cos^2\theta = \sin^2\theta + 1$

$\cos^2\theta = \sin^2\theta + 1 \Rightarrow 1 - \sin^2\theta = \sin^2\theta + 1 \Rightarrow$
$1 = 2\sin^2\theta + 1 \Rightarrow 2\sin^2\theta = 0 \Rightarrow$
$\sin^2\theta = 0 \Rightarrow \sin\theta = 0$

Over the interval $[0°, 360°)$, the equation $\sin\theta = 0$ has two solutions. These are $0°$ and $180°$. Solution set: $\{0°, 180°\}$

29. $\csc^2\theta - 2\cot\theta = 0$

$$\csc^2\theta - 2\cot\theta = 0$$
$$\left(1 + \cot^2\theta\right) - 2\cot\theta = 0$$
$$\cot^2\theta - 2\cot\theta + 1 = 0$$
$$(\cot\theta - 1)^2 = 0$$
$$\cot\theta - 1 = 0 \Rightarrow \cot\theta = 1$$

Over the interval $[0°, 360°)$, the equation $\cot\theta = 1$ has two solutions, the angles in quadrants I and III that have a reference angle of $45°$. These are $45°$ and $225°$.
Solution set: $\{45°, 225°\}$

30. $\sin^2\theta\cos\theta = \cos\theta$

$$\sin^2\theta\cos\theta = \cos\theta$$
$$\sin^2\theta\cos\theta - \cos\theta = 0$$
$$\cos\theta\left(\sin^2\theta - 1\right) = 0$$
$$\cos\theta = 0 \text{ or}$$
$$\sin^2\theta - 1 = 0 \Rightarrow \sin^2\theta = 1 \Rightarrow$$
$$\sin\theta = \pm 1$$

Over the interval $[0°, 360°)$, the equation $\cos\theta = 0$ has two solutions. These are $90°$ and $270°$. In the same interval, the equation $\sin\theta = 1$ has one solution, namely $90°$. Finally, $\sin\theta = -1$ has one solution, namely $270°$. Solution set: $\{90°, 270°\}$

31. $2\tan^2\theta\sin\theta - \tan^2\theta = 0$

$$2\tan^2\theta\sin\theta - \tan^2\theta = 0$$
$$\tan^2\theta(2\sin\theta - 1) = 0$$
$$\tan^2\theta = 0$$
$$\tan\theta = 0 \text{ or } 2\sin\theta - 1 = 0 \Rightarrow$$
$$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$$

Over the interval $[0°, 360°)$, the equation $\tan\theta = 0$ has two solutions. These are $0°$ and $180°$. In the same interval, the equation $\sin\theta = \frac{1}{2}$ has two solutions, the angles in quadrants I and II that have a reference angle of $30°$. These are $30°$ and $150°$.
Solution set: $\{0°, 30°, 150°, 180°\}$

32. $\sin^2\theta\cos^2\theta = 0$

$\sin^2\theta\cos^2\theta = 0$
$\sin^2\theta = 0 \Rightarrow \sin\theta = 0$ or
$\cos^2\theta = 0 \Rightarrow \cos\theta = 0$

Over the interval $[0°, 360°)$, the equation $\sin\theta = 0$ has two solutions. These are $0°$ and $180°$. In the same interval, the equation $\cos\theta = 0$ has two solutions. These are $90°$ and $270°$.
Solution set: $\{0°, 90°, 180°, 270°\}$

33. $\sec^2\theta\tan\theta = 2\tan\theta$

$$\sec^2\theta\tan\theta = 2\tan\theta$$
$$\sec^2\theta\tan\theta - 2\tan\theta = 0$$
$$\tan\theta\left(\sec^2\theta - 2\right) = 0$$
$$\tan\theta = 0 \text{ or } \sec^2\theta - 2 = 0 \Rightarrow$$
$$\sec^2\theta = 2 \Rightarrow \sec\theta = \pm\sqrt{2}$$

Over the interval $[0°, 360°)$, the equation $\tan\theta = 0$ has two solutions. These are $0°$ and $180°$. In the same interval, the equation $\sec\theta = \sqrt{2}$ has two solutions, the angles in quadrants I and IV that have a reference angle of $45°$. These are $45°$ and $315°$. Finally, the equation $\sec\theta = -\sqrt{2}$ has two solutions, the angles in quadrants II and III that have a reference angle of $45°$. These are $135°$ and $225°$.
Solution set:
$\{0°, 45°, 135°, 180°, 225°, 315°\}$

34. $\cos^2\theta - \sin^2\theta = 0$

$$\cos^2\theta - \sin^2\theta = 0$$
$$\cos^2\theta - \left(1 - \cos^2\theta\right) = 0$$
$$2\cos^2\theta - 1 = 0 \Rightarrow 2\cos^2\theta = 1$$

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$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\sqrt{\frac{1}{2}} \Rightarrow \cos\theta = \pm\frac{\sqrt{2}}{2}$

Over the interval $[0°, 360°)$, the equation $\cos\theta = \frac{\sqrt{2}}{2}$ has two solutions, the angles in quadrants I and IV that have a reference angle of $45°$. These are $45°$ and $315°$. In the same interval, the equation $\cos\theta = -\frac{\sqrt{2}}{2}$ has two solutions, the angles in quadrants II and III that have a reference angle of $45°$. These are $135°$ and $225°$.

Solution set: $\{45°, 135°, 225°, 315°\}$

For Exercises 35–42, make sure your calculator is in degree mode.

35. $9\sin^2\theta - 6\sin\theta = 1$

$9\sin^2\theta - 6\sin\theta = 1 \Rightarrow 9\sin^2\theta - 6\sin\theta - 1 = 0$

We use the quadratic formula with $a = 9$, $b = -6$, and $c = -1$.

$$\sin\theta = \frac{6 \pm \sqrt{36 - 4(9)(-1)}}{2(9)} = \frac{6 \pm \sqrt{36+36}}{18} = \frac{6 \pm \sqrt{72}}{18} = \frac{6 \pm 6\sqrt{2}}{18} = \frac{1 \pm \sqrt{2}}{3}$$

Since $\sin\theta = \frac{1+\sqrt{2}}{3} > 0$ (and less than 1), we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant II. Using a calculator, if $\sin\theta = \frac{1+\sqrt{2}}{3} \approx .80473787$, the quadrant I angle will be approximately $53.6°$. The quadrant II angle will be approximately $180° - 53.6° = 126.4°$. Since $\sin\theta = \frac{1-\sqrt{2}}{3} < 0$ (and greater than -1), we will obtain two angles. One angle will be in quadrant III and the other will be in quadrant IV. Using a calculator, if $\sin\theta = \frac{1-\sqrt{2}}{3} \approx -.13807119$, then $\theta \approx -7.9°$. Since this solution is not in the interval [0°, 360°), we must use it as a reference angle to find angles in the interval. Our reference angle will be $7.9°$. The angle in quadrant III will be approximately $180° + 7.9° = 187.9°$. The angle in quadrant IV will be approximately $360° - 7.9° = 352.1°$.

Solution set: $\{53.6°, 126.4°, 187.9°, 352.1°\}$

36. $4\cos^2\theta + 4\cos\theta = 1$

$4\cos^2\theta + 4\cos\theta = 1 \Rightarrow 4\cos^2\theta + 4\cos\theta - 1 = 0$

We use the quadratic formula with $a = 4$, $b = 4$, and $c = -1$.

$$\cos\theta = \frac{-4 \pm \sqrt{4^2 - 4(4)(-1)}}{2(4)} = \frac{-4 \pm \sqrt{32}}{8} = \frac{-4 + 4\sqrt{2}}{8} = \frac{-1 \pm \sqrt{2}}{2}$$

$\frac{-1-\sqrt{2}}{2}$ is less than -1, which is an impossible value for the cosine function.

Since $\cos\theta = \frac{-1+\sqrt{2}}{2} > 0$ (and less than 1), we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant IV. Using a calculator, if $\cos\theta = \frac{-1+\sqrt{2}}{2} \approx .20710678$, the quadrant I angle will be approximately $78.0°$. The quadrant IV angle will be approximately $360° - 78.0° = 282.0°$.

Solution set: $\{78.0°, 282.0°\}$

37. $\tan^2\theta + 4\tan\theta + 2 = 0$

We use the quadratic formula with $a = 1$, $b = 4$, and $c = 2$.

$$\tan\theta = \frac{-4 \pm \sqrt{16 - 4(1)(2)}}{2(1)} = \frac{-4 \pm \sqrt{16-8}}{2} = \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

Since $\tan\theta = -2 + \sqrt{2} < 0$, we will obtain two angles. One angle will be in quadrant II and the other will be in quadrant IV. Using a calculator, if $\tan\theta = -2 + \sqrt{2} = -.5857864$, then $\theta \approx -30.4°$. Since this solution is not in the interval [0°, 360°), we must use it as a reference angle to find angles in the interval. Our reference angle will be $30.4°$. The angle in quadrant II will be approximately $180° - 30.4° = 149.6°$. The angle in quadrant IV will be approximately $360° - 30.4° = 329.6°$.

Since $\tan\theta = -2-\sqrt{2} < 0$, we will obtain two angles. One angle will be in quadrant II and the other will be in quadrant IV. Using a calculator, if $\tan\theta = -2-\sqrt{2} = -3.4142136$, then $\theta \approx -73.7°$. Since this solution is not in the interval [0°, 360°), we must use it as a reference angle to find angles in the interval. Our reference angle will be $73.7°$. The angle in quadrant II will be approximately $180° - 73.7° = 106.3°$. The angle in quadrant IV will be approximately $360° - 73.7° = 286.3°$.

Solution set: $\{106.3°, 149.6°, 286.3°, 329.6°\}$

38. $3\cot^2\theta - 3\cot\theta - 1 = 0$

We use the quadratic formula with $a = 3$, $b = -3$, and $c = -1$.

$$\cot\theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)} = \frac{3 \pm \sqrt{9+12}}{6} = \frac{3 \pm \sqrt{21}}{6}$$

Since $\cot\theta = \frac{3+\sqrt{21}}{6} > 0$, we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant III. Using a calculator, if $\cot\theta = \frac{3+\sqrt{21}}{6} \approx 1.2637626$, the quadrant I angle will be approximately $38.4°$. The quadrant III angle will be approximately $180° + 38.4° = 218.4°$.

Since $\cot\theta = \frac{3-\sqrt{21}}{6} < 0$, we will obtain two angles. One angle will be in quadrant II and the other will be in quadrant IV. Using a calculator, if $\cot\theta = \frac{3-\sqrt{21}}{6} \approx -.26376262$, the quadrant II angle will be approximately $104.8°$. (Note: You need to calculate

$$\tan^{-1}\left(\frac{1}{\frac{3-\sqrt{21}}{6}}\right) + 180$$

to obtain this angle.)

The reference angle is $180° - 104.8° = 75.2°$. Thus, the quadrant IV angle will be approximately $360° - 75.2° = 284.8°$.

Solution set: $\{38.4°, 104.8°, 218.4°, 284.8°\}$

39. $\sin^2\theta - 2\sin\theta + 3 = 0$

We use the quadratic formula with $a = 1$, $b = -2$, and $c = 3$.

$$\sin\theta = \frac{2 \pm \sqrt{4-(4)(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2i\sqrt{2}}{2} = 1 \pm i\sqrt{2}$$

Since $1 \pm i\sqrt{2}$ is not a real number, the equation has no real solutions.

Solution set: $\varnothing$

40. $2\cos^2\theta + 2\cos\theta - 1 = 0$

We use the quadratic formula with $a = 2$, $b = 2$, and $c = -1$.

$$\cos\theta = \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)} = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm \sqrt{12}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

$\frac{-1-\sqrt{3}}{2}$ is less than -1, which is an impossible value for the cosine function.

Since $\cos\theta = \frac{-1+\sqrt{3}}{2} > 0$ (and less than 1), we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant IV. Using a calculator, if $\cos\theta = \frac{-1+\sqrt{3}}{2} \approx .36602540$, the quadrant I angle will be approximately $68.5°$. The quadrant IV angle will be approximately $360° - 68.5° = 291.5°$.

Solution set: $\{68.5°, 291.5°\}$

41. $\cot\theta + 2\csc\theta = 3$

$$\cot\theta + 2\csc\theta = 3 \Rightarrow \frac{\cos\theta}{\sin\theta} + \frac{2}{\sin\theta} = 3$$

$$\cos\theta + 2 = 3\sin\theta$$

$$(\cos\theta + 2)^2 = (3\sin\theta)^2$$

$$\cos^2\theta + 4\cos\theta + 4 = 9\sin^2\theta$$

$$\cos^2\theta + 4\cos\theta + 4 = 9(1 - \cos^2\theta)$$

$$\cos^2\theta + 4\cos\theta + 4 = 9 - 9\cos^2\theta$$

$$10\cos^2\theta + 4\cos\theta - 5 = 0$$

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We use the quadratic formula with $a = 10$, $b = 4$, and $c = -5$.

$$\cos\theta = \frac{-4 \pm \sqrt{4^2 - 4(10)(-5)}}{2(10)} = \frac{-4 \pm \sqrt{16+200}}{20} = \frac{-4 \pm \sqrt{216}}{20} = \frac{-4 \pm 6\sqrt{6}}{20} = \frac{-2 \pm 3\sqrt{6}}{10}$$

Since $\cos\theta = \frac{-2+3\sqrt{6}}{10} > 0$ (and less than 1), we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant IV. Using a calculator, if $\cos\theta = \frac{-2+3\sqrt{6}}{10} \approx .53484692$, the quadrant I angle will be approximately $57.7°$. The quadrant IV angle will be approximately $360° - 57.7° = 302.3°$.

Since $\cos\theta = \frac{-2-3\sqrt{6}}{10} < 0$ (and greater than – 1), we will obtain two angles. One angle will be in quadrant II and the other will be in quadrant III. Using a calculator, if $\cos\theta = \frac{-2-3\sqrt{6}}{10} \approx -.93484692$, the quadrant II angle will be approximately $159.2°$. The reference angle is $180° - 159.2° = 20.8°$. Thus, the quadrant III angle will be approximately $180° + 20.8° = 200.8°$. Since the solution was found by squaring both sides of an equation, we must check that each proposed solution is a solution of the original equation. 302.3° and 200.8° do not satisfy our original equation. Thus, they are not elements of the solution set.

Solution set: $\{57.7°, 159.2°\}$

42. $2\sin\theta = 1 - 2\cos\theta$

$$2\sin\theta = 1 - 2\cos\theta$$
$$(2\sin\theta)^2 = (1 - 2\cos\theta)^2$$
$$4\sin^2\theta = 1 - 4\cos\theta + 4\cos^2\theta$$
$$4(1-\cos^2\theta) = 1 - 4\cos\theta + 4\cos^2\theta$$
$$4 - 4\cos^2\theta = 1 - 4\cos\theta + 4\cos^2\theta$$
$$0 = -3 - 4\cos\theta + 8\cos^2\theta$$

We use the quadratic formula with $a = 8$, $b = -4$, and $c = -3$.

$$\cos\theta = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(8)(-3)}}{2(8)} = \frac{4 \pm \sqrt{16+96}}{16} = \frac{4 \pm \sqrt{112}}{16} = \frac{4 \pm 4\sqrt{7}}{16} = \frac{1 \pm \sqrt{7}}{4}$$

Since $\cos\theta = \frac{1+\sqrt{7}}{4} > 0$ (and less than 1), we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant IV. Using a calculator, if $\cos\theta = \frac{1+\sqrt{7}}{4} \approx .91143783$, the quadrant I angle will be approximately $24.3°$. The quadrant IV angle will be approximately $360° - 24.3° = 335.7°$. Since $\cos\theta = \frac{1-\sqrt{7}}{4} < 0$ (and greater than – 1), we will obtain two angles. One angle will be in quadrant II and the other will be in quadrant III. Using a calculator, if $\cos\theta = \frac{1-\sqrt{7}}{4} \approx -.41143783$, the quadrant II angle will be approximately $114.3°$. The reference angle is $180° - 114.3° = 65.7°$. Thus, the quadrant III angle will be approximately $180° + 65.7° = 245.7°$. Since the solution was found by squaring both sides of an equation, we must check that each proposed solution is a solution of the original equation. 24.3° and 245.7° do not satisfy our original equation. Thus, they are not elements of the solution set.

Solution set: $\{114.3°, 335.7°\}$

In Exercises 43–46, if you are using a calculator, make sure it is in radian mode.

43. $3\sin^2 x - \sin x - 1 = 0$

We use the quadratic formula with $a = 3$, $b = -1$, and $c = -1$.

$$\sin x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-1)}}{2(3)} = \frac{1 \pm \sqrt{1+12}}{6} = \frac{1 \pm \sqrt{13}}{6}$$

Since $\sin x = \frac{1+\sqrt{13}}{6} > 0$ (and less than 1), we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant II. Using a calculator, if $\sin x = \frac{1+\sqrt{13}}{6} \approx .76759188,$ the quadrant I angle will be approximately $.8751.$ The quadrant II angle will be approximately $\pi - .88 \approx 2.2665.$ Since $\sin x = \frac{1-\sqrt{13}}{6} < 0$ (and greater than –1), we will obtain two angles. One angle will be in quadrant III and the other will be in quadrant IV. Using a calculator, if $\sin x = \frac{1-\sqrt{13}}{6} \approx -.43425855,$ then $x \approx -.4492.$ Since this solution is not in the interval $[0, 2\pi),$ we must use it as a reference angle to find angles in the interval. Our reference angle will be $.4492.$ The angle in quadrant III will be approximately $\pi + .4492 \approx 3.5908.$ The angle in quadrant IV will be approximately $2\pi - .4492 \approx 5.8340.$

Solution set:
$\{.8751 + 2n\pi,\ 2.2665 + 2n\pi,\ 3.5908 + 2n\pi,$ and $5.8340 + 2n\pi,$ where n is any integer.$\}$

44. $2\cos^2 x + \cos x = 1$

$$2\cos^2 x + \cos x = 1$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$

$2\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2}$ or
$\cos x + 1 = 0 \Rightarrow \cos x = -1$

Over the interval $[0, 2\pi),$ the equation $\cos x = \frac{1}{2}$ has two solutions. The angles in quadrants I and IV that have a reference angle of $\frac{\pi}{3}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. In the same interval, $\cos x = -1$ when the angle is π.

Solution set: $\left\{\frac{\pi}{3} + 2n\pi,\ \pi + 2n\pi,\right.$ and $\left.\frac{5\pi}{3} + 2n\pi,\right.$ where n is any integer$\left.\right\}$

45. $4\cos^2 x - 1 = 0$

$4\cos^2 x - 1 = 0 \Rightarrow \cos^2 x = \frac{1}{4} \Rightarrow \cos x = \pm\frac{1}{2}$

Over the interval $[0, 2\pi),$ the equation $\cos x = \frac{1}{2}$ has two solutions. The angles in quadrants I and IV that have a reference angle of $\frac{\pi}{3}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. In the same interval, $\cos x = -\frac{1}{2}$ has two solutions. The angles in quadrants II and III that have a reference angle of $\frac{\pi}{3}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Thus, the solutions are $\frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi$, and $\frac{5\pi}{3} + 2n\pi$, where n is any integer.

Solution set: $\left\{\frac{\pi}{3} + n\pi\right.$ and $\frac{2\pi}{3} + n\pi,$ where n is any integer$\}$

46. $2\cos^2 x + 5\cos x + 2 = 0$

$$2\cos^2 x + 5\cos x + 2 = 0$$
$$(2\cos x + 1)(\cos x + 2) = 0$$

$\cos x = -2$ is less than –1, which is an impossible value for the cosine function. Over the interval $[0, 2\pi),$ the equation $\cos x = -\frac{1}{2}$ has two solutions. The angles in quadrants II and III that have a reference angle of $\frac{\pi}{3}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Solution set: $\left\{\frac{2\pi}{3} + 2n\pi\right.$ and $\frac{4\pi}{3} + 2n\pi,$ where n is any integer$\}$

In Exercises 47–50, if you are using a calculator, make sure it is in degree mode.

47. $5\sec^2\theta = 6\sec\theta$

$5\sec^2\theta = 6\sec\theta \Rightarrow 5\sec^2\theta - 6\sec\theta = 0 \Rightarrow$
$\sec\theta(5\sec\theta - 6) = 0$

$\sec\theta = 0$ or $5\sec\theta - 6 = 0 \Rightarrow \sec\theta = \frac{6}{5}$

$\sec\theta = 0$ is an impossible values since the secant function must be either ≥ 1 or $\leq -1.$

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Since $\sec\theta = \frac{6}{5} > 1$, we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant IV. Using a calculator, if $\sec\theta = \frac{6}{5} = 1.2,$ the quadrant I angle will be approximately $33.6°.$ The quadrant IV angle will be approximately $360° - 33.6° = 326.4°.$
Solution set: $\{33.6° + 360°n$ and $326.4° + 360°n,$ where n is any integer$\}$

48. $3\sin^2\theta - \sin\theta = 2$

$$3\sin^2\theta - \sin\theta = 2$$
$$3\sin^2\theta - \sin\theta - 2 = 0$$
$$(3\sin\theta + 2)(\sin\theta - 1) = 0$$

$3\sin\theta + 2 = 0 \Rightarrow \sin\theta = -\frac{2}{3}$ or
$\sin\theta - 1 = 0 \Rightarrow \sin\theta = 1$
$\sin\theta = 1$ when $\theta = 90°.$

Since $\sin\theta = -\frac{2}{3} < 0$ (and greater than – 1), we will obtain two angles. One angle will be in quadrant III and the other will be in quadrant IV. Using a calculator, if $\sin\theta = -\frac{2}{3} \approx -.66666667,$ then $\theta \approx -41.8°.$
Since this solution is not in the interval [0°, 360°), we must use it as a reference angle to find angles in the interval. Our reference angle will be $41.8°.$ The angle in quadrant III will be approximately $180° + 41.8° = 221.8°.$ The angle in quadrant IV will be approximately $360° - 41.8° = 318.2°.$
Solution set: $\{90° + 360°n,\ 221.8° + 360°n,$ and $318.2° + 360°n,$ where n is any integer.$\}$

49. $\frac{2\tan\theta}{3 - \tan^2\theta} = 1$

$$\frac{2\tan\theta}{3 - \tan^2\theta} = 1$$
$$2\tan\theta = 3 - \tan^2\theta$$
$$\tan^2\theta + 2\tan\theta - 3 = 0$$
$$(\tan\theta - 1)(\tan\theta + 3) = 0$$

$\tan\theta - 1 = 0 \Rightarrow \tan\theta = 1$ or
$\tan\theta + 3 = 0 \Rightarrow \tan\theta = -3$

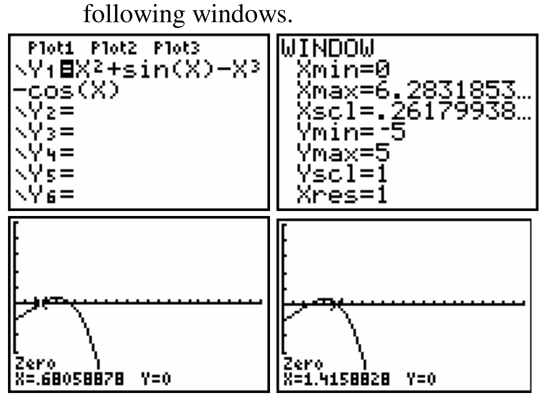
Over the interval $[0°, 360°)$, the equation $\tan\theta = 1$ has two solutions $45°$ and $225°.$ Over the same interval, the equation $\tan\theta = -3$ has two solutions that are approximately $-71.6° + 180° = 108.4°$ and $-71.6° + 360° = 288.4°.$ Thus, the solutions are $45° + 360°n,\ 108.4° + 360°n,\ 225° + 360°n$ and $288.4° + 360°n,$ where n is any integer. Since the period of the tangent function is $180°,$ the solutions can also be written as $45° + n \cdot 180°$ and $108.4° + n \cdot 180°,$ where n is any integer.

50. $\sec^2\theta = 2\tan\theta + 4$

$$\sec^2\theta = 2\tan\theta + 4$$
$$\tan^2\theta + 1 = 2\tan\theta + 4$$
$$\tan^2\theta - 2\tan\theta - 3 = 0$$
$$(\tan\theta - 3)(\tan\theta + 1) = 0$$
$$\tan\theta - 3 = 0$$
$$\tan\theta = 3 \text{ or}$$
$$\tan\theta + 1 = 0 \Rightarrow \tan\theta = -1$$

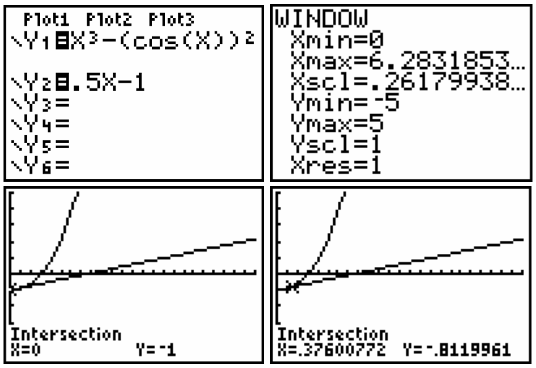
Over the interval $[0°, 360°)$, the equation $\tan\theta = -1$ has two solutions $135°$ and $315°.$ Over the same interval, the equation $\tan\theta = 3$ has two solutions that are approximately $71.6°$ and $180° + 71.6° = 251.6°.$ Thus, the solutions are $71.6° + 360°n,\ 135° + 360°n,$ $251.6° + 360°n$ and $315° + 360°n,$ where n is any integer. Since the period of the tangent function is $180°,$ the solutions can also be written as $71.6° + n \cdot 180°$ and $135° + n \cdot 180°,$ where n is any integer.

51. The x-intercept method is shown in the following windows.



Solution set: $\{.6806, 1.4159\}$

52. The intersection method is shown in the following screens.



Solution set: $\{0, .3760\}$

53. Since $2x = \frac{2\pi}{3}, 2\pi, \frac{8\pi}{3} \Rightarrow$
$x = \frac{2\pi}{6}, \frac{2\pi}{2}, \frac{8\pi}{6} \Rightarrow x = \frac{\pi}{3}, \pi, \frac{4\pi}{3}$, the solution set is $\left\{\frac{\pi}{3}, \pi, \frac{4\pi}{3}\right\}$.

54. Since $\frac{1}{2}x = \frac{\pi}{16}, \frac{5\pi}{12}, \frac{5\pi}{8} \Rightarrow$
$x = \frac{2\pi}{16}, \frac{2\cdot 5\pi}{12}, \frac{2\cdot 5\pi}{8} \Rightarrow x = \frac{\pi}{8}, \frac{5\pi}{6}, \frac{5\pi}{4}$,
the solution set is $\left\{\frac{\pi}{8}, \frac{5\pi}{6}, \frac{5\pi}{4}\right\}$.

55. Since $3\theta = 180°, 630°, 720°, 930° \Rightarrow$
$\theta = 60°, 210°, 240°, 310°$, the solution set is $\{60°, 210°, 240°, 310°\}$.

56. Since $\frac{1}{3}\theta = 45°, 60°, 75°, 90° \Rightarrow$
$\theta = 135°, 180°, 225°, 270°$, the solution set is $\{135°, 180°, 225°, 270°\}$.

57. $\cos 2x = \frac{\sqrt{3}}{2}$
Since $0 \le x < 2\pi$, $0 \le 2x < 4\pi$. Thus,
$2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6} \Rightarrow$
$x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$.
Solution set: $\left\{\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}\right\}$

58. $\cos 2x = -\frac{1}{2}$
Since $0 \le x < 2\pi$, $0 \le 2x < 4\pi$. Thus,
$2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3} \Rightarrow$
$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.
Solution set: $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

59. $\sin 3x = -1$
Since $0 \le x < 2\pi$, $0 \le 3x < 6\pi$. Thus,
$3x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2} \Rightarrow x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.
Solution set: $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

60. $\sin 3x = 0$
Since $0 \le x < 2\pi$, $0 \le 3x < 6\pi$. Thus,
$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$
$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$
Solution set: $\left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

61. $3\tan 3x = \sqrt{3} \Rightarrow \tan 3x = \frac{\sqrt{3}}{3}$
Since $0 \le x < 2\pi$, $0 \le 3x < 6\pi$.
Thus, $3x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}$
implies $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$.
Solution set:
$\left\{\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}\right\}$

62. $\cot 3x = \sqrt{3}$
Since $0 \le x < 2\pi$, $0 \le 3x < 6\pi$.
Thus, $3x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}$
implies $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$.
Solution set:
$\left\{\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}\right\}$

63. $\sqrt{2}\cos 2x = -1 \Rightarrow \cos 2x = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

Since $0 \le x < 2\pi,\ 0 \le 2x < 4\pi$. Thus,

$2x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4} \Rightarrow$

$x = \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

Solution set: $\left\{\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}\right\}$

64. $2\sqrt{3}\sin 2x = \sqrt{3} \Rightarrow \sin 2x = \frac{1}{2}$

Since $0 \le x < 2\pi,\ 0 \le 2x < 4\pi$. Thus,

$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \Rightarrow$

$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

Solution set: $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}$

65. $\sin\frac{x}{2} = \sqrt{2} - \sin\frac{x}{2}$

$\sin\frac{x}{2} = \sqrt{2} - \sin\frac{x}{2} \Rightarrow \sin\frac{x}{2} + \sin\frac{x}{2} = \sqrt{2} \Rightarrow$

$2\sin\frac{x}{2} = \sqrt{2} \Rightarrow \sin\frac{x}{2} = \frac{\sqrt{2}}{2}$

Since $0 \le x < 2\pi,\ 0 \le \frac{x}{2} < \pi$. Thus,

$\frac{x}{2} = \frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Solution set: $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

66. $\tan 4x = 0$

Since $0 \le x < 2\pi,\ 0 \le 4x < 8\pi$.

Thus, $4x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi$

implies $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

Solution set: $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\right\}$

67. $\sin x = \sin 2x$

$\sin x = \sin 2x \Rightarrow \sin x = 2\sin x\cos x \Rightarrow$

$\sin x - 2\sin x\cos x = 0 \Rightarrow \sin x(1 - 2\cos x) = 0$

Over the interval $[0, 2\pi)$, we have

$1 - 2\cos x = 0 \Rightarrow -2\cos x = -1 \Rightarrow$

$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$\sin x = 0 \Rightarrow x = 0$ or π

Solution set: $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

68. $\cos 2x - \cos x = 0$

We choose an identity for $\cos 2x$ that involves only the cosine function.

$$\cos 2x - \cos x = 0$$
$$\left(2\cos^2 x - 1\right) - \cos x = 0$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$2\cos x + 1 = 0 \text{ or } \cos x - 1 = 0$$

Over the interval $[0, 2\pi)$, we have

$2\cos x + 1 = 0 \Rightarrow 2\cos x = -1 \Rightarrow$

$\cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$\cos x - 1 = 0 \Rightarrow \cos x = 1 \Rightarrow x = 0$

Solution set: $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

69. $8\sec^2\frac{x}{2} = 4 \Rightarrow \sec^2\frac{x}{2} = \frac{1}{2} \Rightarrow \sec\frac{x}{2} = \pm\frac{\sqrt{2}}{2}$

Since $-\frac{\sqrt{2}}{2}$ is not in the interval $(-\infty, -1]$ and $\frac{\sqrt{2}}{2}$ is not in the interval $[1, \infty)$, this equation has no solution. Solution set: $\varnothing$

70. $\sin^2\frac{x}{2} - 2 = 0 \Rightarrow \sin^2\frac{x}{2} = 2 \Rightarrow \sin\frac{x}{2} = \pm\sqrt{2}$

Since neither $-\sqrt{2}$ nor $\sqrt{2}$ are in the interval $[-1, 1]$, this equation has no solution.

Solution set: $\varnothing$

71. $\sin\frac{x}{2} = \cos\frac{x}{2}$

$\sin\frac{x}{2} = \cos\frac{x}{2} \Rightarrow \sin^2\frac{x}{2} = \cos^2\frac{x}{2} \Rightarrow$

$\sin^2\frac{x}{2} = 1 - \sin^2\frac{x}{2} \Rightarrow 2\sin^2\frac{x}{2} = 1$

$\sin^2\frac{x}{2} = \frac{1}{2} \Rightarrow \sin\frac{x}{2} = \pm\sqrt{\frac{1}{2}} \Rightarrow \sin\frac{x}{2} = \pm\frac{\sqrt{2}}{2}$

Since $0 \le x < 2\pi,\ 0 \le \frac{x}{2} < \pi$.

If $\sin\frac{x}{2} = \frac{\sqrt{2}}{2},\ \frac{x}{2} = \frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$.

If $\sin\frac{x}{2} = -\frac{\sqrt{2}}{2}$, there are no solutions in the interval $[0, \pi)$. Because the solution was found by squaring an equation, the proposed solutions must be checked.

Check $x = \frac{\pi}{2}$

$$\sin\frac{x}{2} = \cos\frac{x}{2}$$
$$\sin\frac{\frac{\pi}{2}}{2} = \cos\frac{\frac{\pi}{2}}{2} \ ?$$
$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} \ ?$$
$$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \quad \text{True}$$

$\frac{\pi}{2}$ is a solution.

Check $x = \frac{3\pi}{2}$

$$\sin\frac{x}{2} = \cos\frac{x}{2}$$
$$\sin\frac{\frac{3\pi}{2}}{2} = \cos\frac{\frac{3\pi}{2}}{2} \ ?$$
$$\sin\frac{3\pi}{4} = \cos\frac{3\pi}{4} \ ?$$
$$\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} \quad \text{False}$$

$\frac{3\pi}{2}$ is not a solution.

Solution set: $\left\{\frac{\pi}{2}\right\}$

72. $\sec\frac{x}{2} = \cos\frac{x}{2}$

$$\sec\frac{x}{2} = \cos\frac{x}{2} \Rightarrow \frac{1}{\cos\frac{x}{2}} = \cos\frac{x}{2} \Rightarrow$$
$$\cos^2\frac{x}{2} = 1 \Rightarrow \cos\frac{x}{2} = \pm 1$$

Since $0 \le x < 2\pi$, $0 \le \frac{x}{2} < \pi$. Thus,

$\frac{x}{2} = 0 \Rightarrow x = 0$. Solution set: $\{0\}$

73. $\cos 2x + \cos x = 0$

We choose an identity for cos $2x$ that involves only the cosine function.

$$\cos 2x + \cos x = 0$$
$$\left(2\cos^2 x - 1\right) + \cos x = 0$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0 \Rightarrow$$
$$\cos x = \frac{1}{2} \text{ or } \cos x = -1$$

$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$\cos x = -1 \Rightarrow x = \pi$

Solution set: $\left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

74. $\sin x \cos x = \frac{1}{4}$

$$\sin x \cos x = \frac{1}{4} \Rightarrow 2\sin x \cos x = \frac{1}{2} \Rightarrow \sin 2x = \frac{1}{2}$$

Since $0 \le x < 2\pi, 0 \le 2x < 4\pi$. Thus,

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \Rightarrow$$
$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

Solution set: $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}$

75. $\sqrt{2}\sin 3\theta - 1 = 0$

$$\sqrt{2}\sin 3\theta - 1 = 0 \Rightarrow \sqrt{2}\sin 3\theta = 1 \Rightarrow$$
$$\sin 3\theta = \frac{1}{\sqrt{2}} \Rightarrow \sin 3\theta = \frac{\sqrt{2}}{2}$$

Since $0 \le \theta < 360°$, $0° \le 3\theta < 1080°$.

In quadrant I and II, sine is positive. Thus,

$$3\theta = 45°, 135°, 405°, 495°, 765°, 855° \Rightarrow$$
$$\theta = 15°, 45°, 135°, 165°, 255°, 285°$$

Solution set: $\{15°, 45°, 135°, 165°, 255°, 285°\}$

76. $-2\cos 2\theta = \sqrt{3}$

$$-2\cos 2\theta = \sqrt{3} \Rightarrow \cos 2\theta = -\frac{\sqrt{3}}{2}$$

Since $0 \le \theta < 360°$, $0° \le 2\theta < 720°$. In quadrant II and III, cosine is negative. Thus,

$$2\theta = 150°, 210°, 510°, 570°$$
$$\Rightarrow \theta = 75°, 105°, 255°, 285°.$$

Solution set: $\{75°, 105°, 255°, 285°\}$

77. $\cos\frac{\theta}{2} = 1$

Since $0 \le \theta < 360°$, $0° \le \frac{\theta}{2} < 180°$. Thus,

$\frac{\theta}{2} = 0° \Rightarrow \theta = 0°$. Solution set: $\{0°\}$

78. $\sin\frac{\theta}{2} = 1$

Since $0 \le \theta < 360°$, $0° \le \frac{\theta}{2} < 180°$. Thus,

$\frac{\theta}{2} = 90° \Rightarrow \theta = 180°$.

Solution set: $\{180°\}$

79. $2\sqrt{3}\sin\frac{\theta}{2}=3$

$2\sqrt{3}\sin\frac{\theta}{2}=3\Rightarrow\sin\frac{\theta}{2}=\frac{3}{2\sqrt{3}}\Rightarrow$

$\sin\frac{\theta}{2}=\frac{3\sqrt{3}}{6}\Rightarrow\sin\frac{\theta}{2}=\frac{\sqrt{3}}{2}$

Since $0\le\theta<360°$, $0°\le\frac{\theta}{2}<180°$. Thus,

$\frac{\theta}{2}=60°,120°\Rightarrow\theta=120°,240°$.

Solution set: $\{120°,240°\}$

80. $2\sqrt{3}\cos\frac{\theta}{2}=-3$

$2\sqrt{3}\cos\frac{\theta}{2}=-3\Rightarrow\cos\frac{\theta}{2}=\frac{-3}{2\sqrt{3}}\Rightarrow$

$\cos\frac{\theta}{2}=-\frac{\sqrt{3}}{2}$. Since $0\le\theta<360°$,

$0°\le\frac{\theta}{2}<180°$. Thus, $\frac{\theta}{2}=150°\Rightarrow\theta=300°$.

Solution set: $\{300°\}$

81. $2\sin\theta=2\cos 2\theta$

$2\sin\theta=2\cos 2\theta\Rightarrow\sin\theta=\cos 2\theta\Rightarrow$
$\sin\theta=1-2\sin^2\theta\Rightarrow 2\sin^2\theta+\sin\theta-1=0\Rightarrow$
$(2\sin\theta-1)(\sin\theta+1)=0\Rightarrow$
$2\sin\theta-1=0$ or $\sin\theta+1=0$

Over the interval $[0°,360°)$, we have

$2\sin\theta-1=0\Rightarrow 2\sin\theta=1\Rightarrow\sin\theta=\frac{1}{2}\Rightarrow$
$\theta=30°$ or $150°$
$\sin\theta+1=0\Rightarrow\sin\theta=-1\Rightarrow\theta=270°$

Solution set: $\{30°,150°,270°\}$

82. $\cos\theta-1=\cos 2\theta$

$\cos\theta-1=\cos 2\theta\Rightarrow\cos\theta-1=2\cos^2\theta-1\Rightarrow$
$2\cos^2\theta-\cos\theta=0\Rightarrow\cos\theta(2\cos\theta-1)=0$

Over the interval $[0°,360°)$, we have
$\cos\theta=0\Rightarrow\theta=90°$ or $270°$

$2\cos\theta-1=0\Rightarrow 2\cos\theta=1\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow$
$\theta=60°$ or $300°$

Solution set: $\{60°,90°,270°,300°\}$

In Exercises 83–90, we are to find all solutions.

83. $1-\sin\theta=\cos 2\theta$

$1-\sin\theta=\cos 2\theta\Rightarrow 1-\sin\theta=1-2\sin^2\theta\Rightarrow$
$2\sin^2\theta-\sin\theta=0\Rightarrow\sin\theta(2\sin\theta-1)=0$

Over the interval $[0°,360°)$, we have
$\sin\theta=0\Rightarrow\theta=0°$ or $180°$.

$2\sin\theta-1=0\Rightarrow\sin\theta=\frac{1}{2}\Rightarrow\theta=30°$ or $150°$

Solution set: $\{0°+360°n$, $30°+360°n$, $150°+360°n$, $180°+360°n$, where n is any integer$\}$ or $\{180°n$, $30°+360°n$, $150°+360°n$, where n is any integer$\}$

84. $\sin 2\theta=2\cos^2\theta$

$\sin 2\theta=2\cos^2\theta\Rightarrow 2\sin\theta\cos\theta=2\cos^2\theta\Rightarrow$
$\cos^2\theta-\sin\theta\cos\theta=0\Rightarrow$
$\cos\theta(\cos\theta-\sin\theta)=0$

Over the interval $[0°,360°)$, we have
$\cos\theta=0\Rightarrow\theta=90°$ or $270°$
$\cos\theta-\sin\theta=0\Rightarrow\cos\theta=\sin\theta\Rightarrow$
$\frac{\sin\theta}{\cos\theta}=1\Rightarrow\tan\theta=1\Rightarrow\theta=45°$ or $225°$

Solution set: $\{45°+360°n$, $90°+360°n$, $225°+360°n$, $270°+360°n$, where n is any integer$\}$ or $\{45°+180°n$, $90°+180°n$, where n is any integer$\}$

85. $\csc^2\frac{\theta}{2}=2\sec\theta$

$\csc^2\frac{\theta}{2}=2\sec\theta\Rightarrow\frac{1}{\sin^2\frac{\theta}{2}}=\frac{2}{\cos\theta}\Rightarrow$

$2\sin^2\frac{\theta}{2}=\cos\theta$

$2\left(\frac{1-\cos\theta}{2}\right)=\cos\theta\Rightarrow 1-\cos\theta=\cos\theta\Rightarrow$

$1=2\cos\theta\Rightarrow\cos\theta=\frac{1}{2}$

Over the interval $[0°,360°)$, we have

$\cos\theta=\frac{1}{2}\Rightarrow\theta=60°$ or $300°$

Solution set:
$\{60°+360°n$, $300°+360°n$, where n is any integer$\}$

86. $\cos\theta = \sin^2\frac{\theta}{2}$

$\cos\theta = \sin^2\frac{\theta}{2} \Rightarrow \cos\theta = \frac{1-\cos\theta}{2} \Rightarrow$

$2\cos\theta = 1-\cos\theta \Rightarrow 3\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{3}$

In quadrant I and IV, cosine is positive. Over the interval $[0°, 360°)$, we have

$\cos\theta = \frac{1}{3} \Rightarrow \theta \approx 70.5°$ or $289.5°$

Solution set:
$\{70.5° + 360°n,\ 289.5° + 360°n,$
where n is any integer$\}$

87. $2 - \sin 2\theta = 4\sin 2\theta$

$2 - \sin 2\theta = 4\sin 2\theta \Rightarrow 2 = 5\sin 2\theta \Rightarrow$

$\sin 2\theta = \frac{2}{5} \Rightarrow \sin 2\theta = .4$

Since $0 \le \theta < 360°$, $0° \le 2\theta < 720°$. In quadrant I and II, sine is positive.

$\sin 2\theta = .4 \Rightarrow 2\theta = 23.6°, 156.4°, 383.6°, 516.4°$

Thus, $\theta = 11.8°, 78.2°, 191.8°, 258.2°$.

Solution set:
$\{11.8° + 360°n,\ 78.2° + 360°n,\ 191.8° + 360°n,$
$258.2° + 360°n,$ where n is any integer$\}$ or
$\{11.8° + 180°n,\ 78.2° + 180°n,$
where n is any integer$\}$

88. $4\cos 2\theta = 8\sin\theta\cos\theta$

$4\cos 2\theta = 8\sin\theta\cos\theta$

$4\cos 2\theta = 4(2\sin\theta\cos\theta)$

$4\cos 2\theta = 4\sin 2\theta \Rightarrow \tan 2\theta = 1$

Since $0 \le \theta < 360°$, $0° \le 2\theta < 720°$. In quadrant I and III, tangent is positive. Thus,

$2\theta = 45°, 225°, 405°, 585° \Rightarrow$

$\theta = 22.5°, 112.5°, 202.5°, 292.5°$

Solution set: $\{22.5° + 360°n,\ 112.5° + 360°n,$
$202.5° + 360°n,\ 292.5° + 360°n,$ where n
is any integer$\}$ or
$\{22.5° + 180°n,\ 112.5° + 180°n,$
where n is any integer$\}$

89. $2\cos^2 2\theta = 1 - \cos 2\theta$

$$2\cos^2 2\theta = 1 - \cos 2\theta$$
$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$

Since $0 \le \theta < 360° \Rightarrow 0° \le 2\theta < 720°$, we have $2\cos 2\theta - 1 = 0 \Rightarrow 2\cos 2\theta = 1 \Rightarrow$

$\cos 2\theta = \frac{1}{2}$. Thus,

$2\theta = 60°, 300°, 420°, 660° \Rightarrow$

$\theta = 30°, 150°, 210°, 330°$ or

$\cos 2\theta + 1 = 0 \Rightarrow \cos 2\theta = -1$

$2\theta = 180°, 540° \Rightarrow \theta = 90°, 270°$

Solution set:
$\{30° + 360°n,\ 90° + 360°n,$
$150° + 360°n,\ 210° + 360°n,\ 270° + 360°n,$
$330° + 360°n,$ where n is any integer$\}$ or
$\{30° + 180°n,\ 90° + 180°n,\ 150° + 180°n,$
where n is any integer$\}$

90. $\sin\theta - \sin 2\theta = 0$

$\sin\theta - \sin 2\theta = 0 \Rightarrow \sin\theta - 2\sin\theta\cos\theta = 0 \Rightarrow$

$\sin\theta(1 - 2\cos\theta) = 0$

Over the interval $[0°, 360°)$, we have

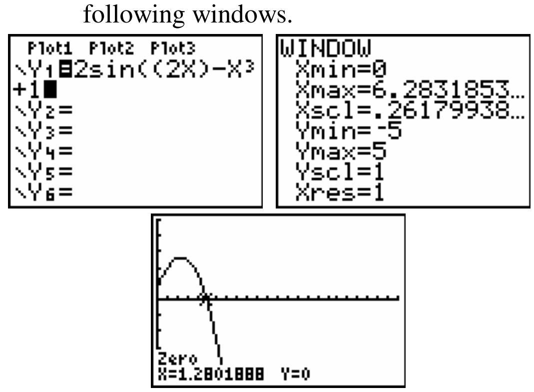
$\sin\theta = 0 \Rightarrow \theta = 0°$ or $180°$

$1 - 2\cos\theta = 0 \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow$

$\theta = 60°$ or $300°$

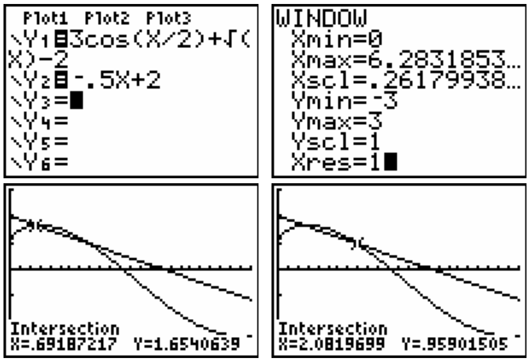
Solution set:
$\{0° + 360°n,\ 60° + 360°n,\ 180° + 360°n,$
$300° + 360°n,$ where n is any integer$\}$ or
$\{180°n,\ 60° + 360°n,\ 300° + 360°n,$
where n is any integer$\}$

91. The x-intercept method is shown in the following windows.



Solution set: {1.2802}

92. The intersection method is shown in the following screens.



Solution set: {.6919,2.0820}

93. $P = A\sin\left(2\pi ft + \phi\right)$

(a) $0 = .004\sin\left[2\pi\left(261.63\right)t + \frac{\pi}{7}\right]$

$0 = \sin\left(1643.87t + .45\right)$

Since $1643.87t + .45 = n\pi$, we have

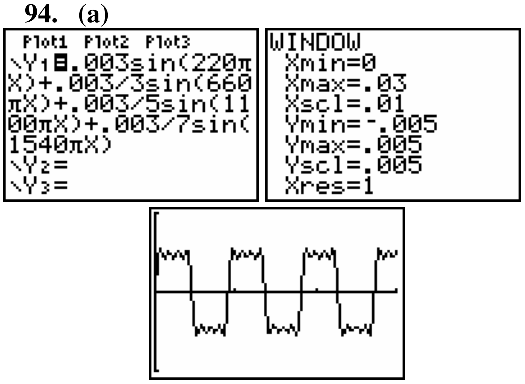
$t = \frac{n\pi - .45}{1643.87}$, where n is any integer.

If $n = 0$, then $t \approx .000274$. If $n = 1$, then $t \approx .00164$. If $n = 2$, then $t \approx .00355$. If $n = 3$, then $t \approx .00546$. The only solutions for t in the interval [0, .005] are .00164 and .00355.

(b) We must solve the trigonometric equation $P = 0$ to determine when $P \le 0$. From the graph we can estimate that $P \le 0$ on the interval [.00164, .00355].

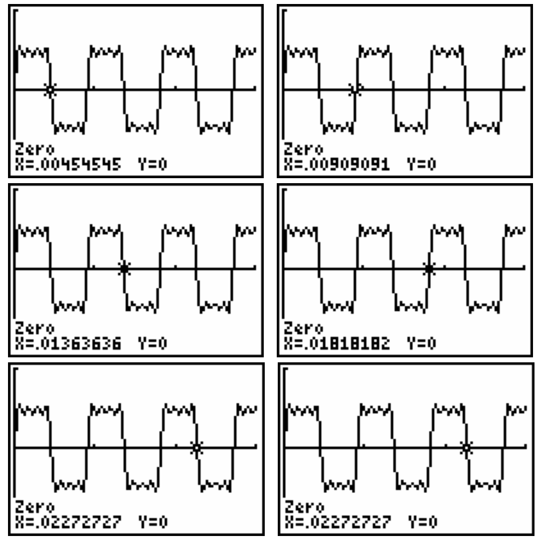
(c) $P < 0$ implies that there is a decrease in pressure so an eardrum would be vibrating outward.

94. **(a)**



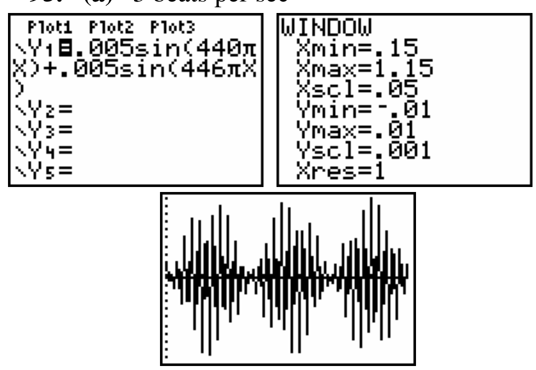
(b) The graph is periodic, and the wave has "jagged square" tops and bottoms.

(c) The eardrum is moving outward when $P < 0$.

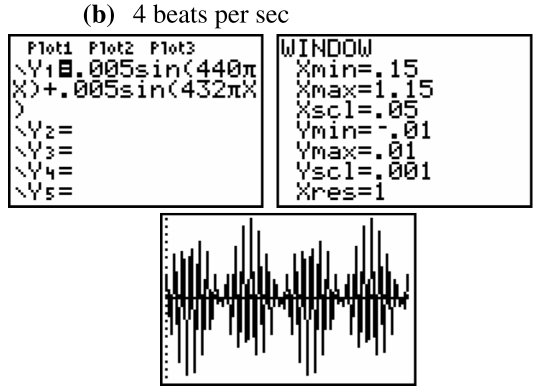


This occurs for the time intervals $(.0045, .0091), (.0136, .0182), (.0227, .0273)$.

95. **(a)** 3 beats per sec

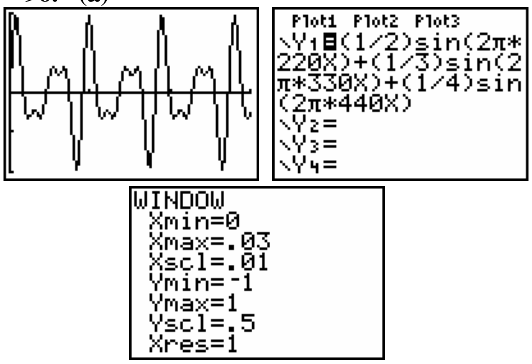


(b) 4 beats per sec

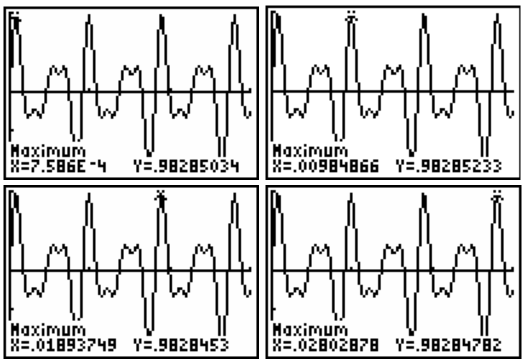


(c) The number of beats is equal to the absolute value of the difference in the frequencies of the two tones.

96. (a)

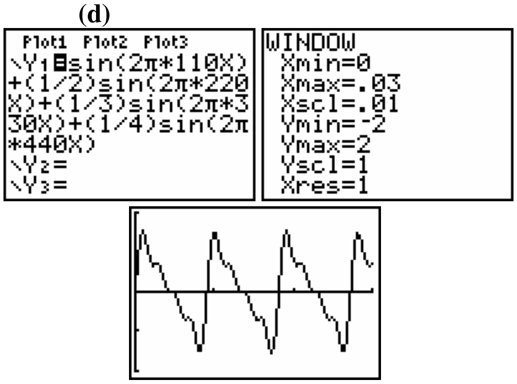


(b) .0007586, .009847, .01894, .02803



(c) 110 Hz

(d)



97. $i = I_{\max} \sin 2\pi ft$

Let $i = 40,\ I_{\max} = 100,\ f = 60.$

$40 = 100\sin\left[2\pi(60)t\right]$

$40 = 100\sin 120\pi t \Rightarrow .4 = \sin 120\pi t$

Using calculator,

$$120\pi t \approx .4115168 \Rightarrow t \approx \frac{.4115168}{120\pi} \Rightarrow$$

$t \approx .0010916 \Rightarrow t \approx .001\,\text{sec}$

98. $i = I_{\max} \sin 2\pi ft$

Let $i = 50,\ I_{\max} = 100,\ f = 120.$

$$50 = 100\sin\left[2\pi(120)t\right]$$

$$50 = 100\sin 240\pi t \Rightarrow \sin 240\pi t = \frac{1}{2}$$

$$240\pi t = \frac{\pi}{6} \Rightarrow t = \frac{1}{1440} \approx .0007 \text{ sec}$$

99. $i = I_{\max} \sin 2\pi ft$

Let $i = I_{\max},\ f = 60.$

$$I_{\max} = I_{\max}\sin\left[2\pi(60)t\right] \Rightarrow 1 = \sin 120\pi t \Rightarrow$$

$$120\pi t = \frac{\pi}{2} \Rightarrow 120t = \frac{1}{2} \Rightarrow t = \frac{1}{240} \approx .004\,\text{sec}$$

100. $i = I_{\max} \sin 2\pi ft$

Let $i = \frac{1}{2}I_{\max},\ f = 60.$

$$\frac{1}{2}I_{\max} = I_{\max}\sin\left[2\pi(60)t\right] \Rightarrow$$

$$\frac{1}{2} = \sin 120\pi t \Rightarrow 120\pi t = \frac{\pi}{6} \Rightarrow$$

$$t = \frac{1}{720} \approx .0014 \text{ sec}$$

101. $.342D\cos\theta + h\cos^2\theta = \dfrac{16D^2}{V_0^2}$

$V_0 = 60, D = 80, h = 2$

$$.342(80)\cos\theta + 2\cos^2\theta = \frac{16\cdot 80^2}{60^2}$$

$$2\cos^2\theta + 27.36\cos\theta - \frac{256}{9} = 0$$

$$\cos^2\theta + 13.68\cos\theta - \frac{128}{9} = 0$$

$b = 13.68$, and $c = -\dfrac{128}{9}$.

$$\cos\theta = \frac{-13.68 \pm \sqrt{13.68^2 - 4\left(-\frac{128}{9}\right)}}{2(1)}$$

$$= \frac{-13.68 \pm \sqrt{187.1424 + \frac{512}{9}}}{2}$$

$$\approx \frac{-13.68 \pm 15.6215}{2}$$

(*continued on next page*)

(*continued from page 705*)

$\frac{-13.68-15.6215}{2}$ is less than -1, which is an impossible value for the cosine function.
Since $\cos\theta = \frac{-13.68+15.6215}{2} > 0$ (and less than 1), we can obtain two angles. One angle will be in quadrant I and the other will be in quadrant IV. Using a calculator, if $\cos\theta \approx .97075$, the quadrant I angle will be approximately $14°$. The quadrant IV angle, however, is not meaningful in this application.

102. $V = \cos 2\pi t, 0 \le t \le \frac{1}{2}$

(a) $V = 0, \cos 2\pi t = 0 \Rightarrow 2\pi t = \cos^{-1} 0 \Rightarrow$

$2\pi t = \frac{\pi}{2} \Rightarrow t = \frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}\text{sec}$

(b) $V = .5, \cos 2\pi t = .5 \Rightarrow 2\pi t = \cos^{-1}(.5) \Rightarrow$

$2\pi t = \frac{\pi}{3} \Rightarrow t = \frac{\frac{\pi}{3}}{2\pi} = \frac{1}{6}$ sec

(c) $V = .25, \cos 2\pi t = .25 \Rightarrow$
$2\pi t = \cos^{-1}(.25) \Rightarrow 2\pi t \approx 1.3181161 \Rightarrow$
$t \approx \frac{1.3181161}{2\pi} \approx .21$ sec

103. $E = 20\sin\left(\frac{\pi t}{4} - \frac{\pi}{2}\right)$

(a) $E = 0 \Rightarrow 0 = 20\sin\left(\frac{\pi t}{4} - \frac{\pi}{2}\right)$
Since arcsin 0 = 0, solve
$\frac{\pi t}{4} - \frac{\pi}{2} = 0 \Rightarrow \frac{\pi t}{4} = \frac{\pi}{2} \Rightarrow t = 2$ sec

(b) $E = 10\sqrt{3} \Rightarrow 10\sqrt{3} = 20\sin\left(\frac{\pi t}{4} - \frac{\pi}{2}\right) \Rightarrow$
$\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi t}{4} - \frac{\pi}{2}\right)$
Since arcsin $\frac{\sqrt{3}}{2} = \frac{\pi}{3}$, solve
$\frac{\pi t}{4} - \frac{\pi}{2} = \frac{\pi}{3} \Rightarrow \frac{\pi t}{4} = \frac{10\pi}{12} \Rightarrow$
$t = \frac{10}{3}\text{sec} = 3\frac{1}{3}$ sec

104. $s(t) = \sin t + 2\cos t$

(a) $s(t) = \frac{2+\sqrt{3}}{2}$
$s(t) = \frac{2}{2} + \frac{\sqrt{3}}{2} = 2\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$
$= 2\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)$

One such value is $\frac{\pi}{3}$.

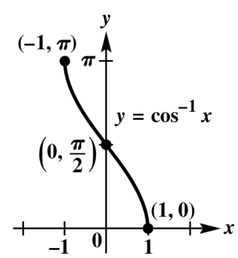
(b) $s(t) = \frac{3\sqrt{2}}{2}$
$s(t) = \frac{2\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}$
$= 2\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)$

One such value is $\frac{\pi}{4}$.

105. In the second line of the "solution", both sides of the equation were divided by $\sin x$. Instead of dividing by $\sin x$, one should have factored $\sin x$ from $\sin^2 x - \sin x$. In the process of dividing both sides by $\sin x$, the solutions of $x = 0$ and $x = \pi$ were eliminated.

Chapter 7 Quiz
(Sections 7.5–7.6)

1. Domain: [–1, 1]; range: $[0, \pi]$



2. (a) $y = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
$\sin y = -\frac{\sqrt{2}}{2}, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$
Since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $y = -\frac{\pi}{4}$.

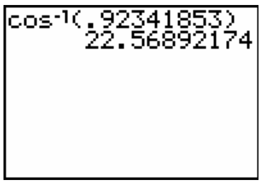
(b) $y = \tan^{-1}\sqrt{3}$
$\tan y = \sqrt{3}, -\frac{\pi}{2} < y < \frac{\pi}{2}$
Since $\tan\frac{\pi}{3} = \sqrt{3}$, $y = \frac{\pi}{3}$.

(c) $y = \sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

$\sec y = -\frac{2\sqrt{3}}{3},\ 0 \le y \le \pi,\ y \ne \frac{\pi}{2}$

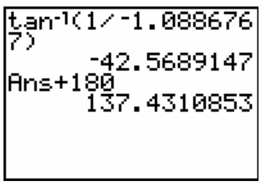
y is in quadrant II. The reference angle is $\frac{\pi}{6}$. Since $\sec\frac{5\pi}{6} = -\frac{2\sqrt{3}}{3},\ y = \frac{5\pi}{6}$.

3. (a)



$\theta = \arccos .92341853 \approx 22.568922°$

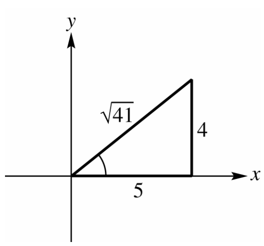
(b)



$\theta = \cot^{-1}(-1.08886767) \approx 137.431085°$

4. (a) $\cos\left(\tan^{-1}\frac{4}{5}\right)$

Let $\omega = \arctan\frac{4}{5}$, so that $\tan\omega = \frac{4}{5}$. Since arctan is defined only in quadrants I and IV, and $\frac{4}{5}$ is positive, ω is in quadrant I. Sketch ω and label a triangle with the hypotenuse equal to $\sqrt{4^2+5^2} = \sqrt{16+25} = \sqrt{41}$.

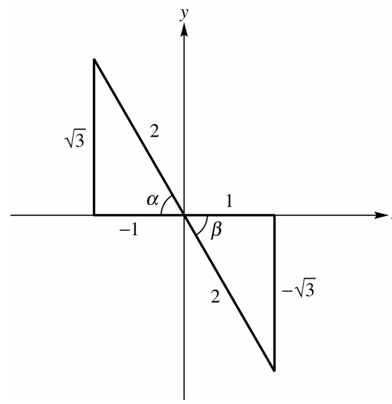


$\cos\omega = \frac{5}{\sqrt{41}} = \frac{5\sqrt{41}}{41}$

(b) $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\left(-\sqrt{3}\right)\right)$

Let $\alpha = \cos^{-1}\left(-\frac{1}{2}\right),\ \beta = \tan^{-1}\left(-\sqrt{3}\right)$.

Sketch angle α in quadrant II and angle β in quadrant IV.



We have $\sin\alpha = \frac{\sqrt{3}}{2}$, $\sin\beta = -\frac{\sqrt{3}}{2}$, and $\cos\beta = \frac{1}{2}$. Thus

$$\begin{aligned}&\sin\left(\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\left(-\sqrt{3}\right)\right)\\ &= \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}\end{aligned}$$

5. $2\sin\theta - \sqrt{3} = 0 \Rightarrow 2\sin\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

Over the interval $[0°, 360°)$, the equation $\sin\theta = \frac{\sqrt{3}}{2}$ has two solutions, the angles in quadrants I and II that have a reference angle of 60º. These are 60º and 120º.

Solution set: $\{60°, 120°\}$

6. $\cos\theta + 1 = 2\sin^2\theta$

$\cos\theta + 1 = 2\left(1-\cos^2\theta\right)$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

Over the interval $[0°, 360°)$, the equation $\cos\theta = \frac{1}{2}$ has two solutions, the angles in quadrants I and IV that have a reference angle of 60º. These are 60º and 300º. Over the interval $[0°, 360°)$, the equation $\cos\theta = -1$ has one solution, 180º.

Solution set: $\{60°, 180°, 300°\}$

7. $\tan^2 x - 5\tan x + 3 = 0$

We use the quadratic formula with $a = 1$, $b = -5$, and $c = 3$.

$$\tan x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{25-12}}{2} = \frac{5 \pm \sqrt{13}}{2}$$

Since $\tan x = \frac{5+\sqrt{13}}{2} > 0$, we will obtain two angles, one in quadrant I and the other in quadrant III. Using a calculator, we find $x \approx 1.3424$ and $x \approx 4.4840$.

Since $\tan x = \frac{5-\sqrt{13}}{2} > 0$, we will obtain two angles, one in quadrant I and the other in quadrant III. Using a calculator, we find $x \approx .6089$ and $x \approx 3.7505$.

Solution set: $\{.6089, 1.3424, 3.7505, 4.4840\}$

8. $3\cot 2x - \sqrt{3} = 0 \Rightarrow \cot 2x = \frac{\sqrt{3}}{3}$

Since $0 \le x < 2\pi$, $0 \le 2x < 4\pi$.

Thus, $2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$ implies

$x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$.

Solution set: $\left\{\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}\right\}$

9. $\cos\frac{x}{2} + \sqrt{3} = -\cos\frac{x}{2} \Rightarrow 2\cos\frac{x}{2} = -\sqrt{3} \Rightarrow$

$\cos\frac{x}{2} = -\frac{\sqrt{3}}{2}$

If $0 \le x < 2\pi$, then $0 \le \frac{x}{2} < \pi$. Over the interval $[0, \pi)$, $\frac{x}{2} = \frac{5\pi}{6}, \frac{7\pi}{6} \Rightarrow x = \frac{5\pi}{3}, \frac{7\pi}{3}$.

Since this is a cosine function with period 4π, the solution set is $\left\{\frac{5\pi}{3} + 4n\pi, \frac{7\pi}{3} + 4n\pi\right\}$.

10. $V = \cos 2\pi t, 0 \le t \le \frac{1}{2}$

(a) $V = 1, \cos 2\pi t = 1 \Rightarrow 2\pi t = \cos^{-1} 1 \Rightarrow$

$2\pi t = 0 \Rightarrow t = \frac{0}{2\pi} = 0\,\text{sec}$

(b) $V = .30, \cos 2\pi t = .30$

$2\pi t = \cos^{-1} .30 \Rightarrow$

$2\pi t \approx 1.266103673$

$t = \frac{1.266103673}{2\pi} = .20\,\text{sec}$

Section 7.7: Equations Involving Inverse Trigonometric Functions

1. Since arcsin 0 = 0, the correct choice is C.

2. Since $\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, the correct choice is A.

3. Since $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$, the correct choice is C.

4. Since $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, the correct choice is C.

5. $y = 5\cos x \Rightarrow \frac{y}{5} = \cos x \Rightarrow x = \arccos\frac{y}{5}$

6. $4y = \sin x \Rightarrow x = \arcsin 4y$

7. $2y = \cot 3x \Rightarrow 3x = \text{arccot}\, 2y \Rightarrow$

$x = \frac{1}{3}\text{arccot}\, 2y$

8. $6y = \frac{1}{2}\sec x \Rightarrow 12y = \sec x \Rightarrow x = \text{arcsec}\, 12y$

9. $y = 3\tan 2x \Rightarrow \frac{y}{3} = \tan 2x \Rightarrow 2x = \arctan \frac{y}{3} \Rightarrow$
$x = \frac{1}{2}\arctan \frac{y}{3}$

10. $y = 3\sin \frac{x}{2} \Rightarrow \frac{y}{3} = \sin \frac{x}{2} \Rightarrow \frac{x}{2} = \arcsin \frac{y}{3} \Rightarrow$
$x = 2\arcsin \frac{y}{3}$

11. $y = 6\cos \frac{x}{4} \Rightarrow \frac{y}{6} = \cos \frac{x}{4} \Rightarrow \frac{x}{4} = \arccos \frac{y}{6} \Rightarrow$
$x = 4\arccos \frac{y}{6}$

12. $y = -\sin \frac{x}{3} \Rightarrow \sin \frac{x}{3} = -y \Rightarrow$
$\frac{x}{3} = \arcsin(-y) \Rightarrow x = 3\arcsin(-y)$

13. $y = -2\cos 5x \Rightarrow -\frac{y}{2} = \cos 5x \Rightarrow$
$5x = \arccos\left(-\frac{y}{2}\right) \Rightarrow x = \frac{1}{5}\arccos\left(-\frac{y}{2}\right)$

14. $y = 3\cot 5x \Rightarrow \cot 5x = \frac{y}{3} \Rightarrow 5x = \text{arccot} \frac{y}{3} \Rightarrow$
$x = \frac{1}{5}\text{arccot} \frac{y}{3}$

15. $y = \cos(x+3) \Rightarrow x + 3 = \arccos y \Rightarrow$
$x = -3 + \arccos y$

16. $y = \tan(2x-1) \Rightarrow 2x - 1 = \arctan y \Rightarrow$
$2x = 1 + \arctan y \Rightarrow x = \frac{1}{2}(1 + \arctan y)$

17. $y = \sin x - 2 \Rightarrow y + 2 = \sin x \Rightarrow$
$x = \arcsin(y+2)$

18. $y = \cot x + 1 \Rightarrow \cot x = y - 1 \Rightarrow$
$x = \text{arccot}(y-1)$

19. $y = 2\sin x - 4 \Rightarrow y + 4 = 2\sin x \Rightarrow$
$\frac{y+4}{2} = \sin x \Rightarrow x = \arcsin\left(\frac{y+4}{2}\right)$

20. $y = 4 + 3\cos x \Rightarrow y - 4 = 3\cos x \Rightarrow$
$\frac{y-4}{3} = \cos x \Rightarrow x = \arccos\left(\frac{y-4}{3}\right)$

21. $y = \sqrt{2} + 3\sec 2x \Rightarrow y - \sqrt{2} = 3\sec 2x \Rightarrow$
$\frac{y-\sqrt{2}}{3} = \sec 2x \Rightarrow 2x = \sec^{-1}\left(\frac{y-\sqrt{2}}{3}\right) \Rightarrow$
$x = \frac{1}{2}\sec^{-1}\left(\frac{y-\sqrt{2}}{3}\right)$

22. $y = 2\csc \frac{x}{2} - \sqrt{3} \Rightarrow y + \sqrt{3} = 2\csc \frac{x}{2} \Rightarrow$
$\frac{y+\sqrt{3}}{2} = \csc \frac{x}{2} \Rightarrow \frac{x}{2} = \csc^{-1}\left(\frac{y+\sqrt{3}}{2}\right) \Rightarrow$
$x = 2\csc^{-1}\left(\frac{y+\sqrt{3}}{2}\right)$

23. First, $\sin x - 2 \neq \sin(x-2)$. If you think of the graph of $y = \sin x - 2$, this represents the graph of $f(x) = \sin x$, shifted 2 units down. If you think of the graph of $y = \sin(x-2)$, this represents the graph of $f(x) = \sin x$, shifted 2 units right.

24. $\cos^{-1} 2$ doesn't exist since there is no value x such that $\cos x = 2$.

25. $-4\arcsin x = \pi \Rightarrow \arcsin x = -\frac{\pi}{4} \Rightarrow$
$x = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
Solution set: $\left\{-\frac{\sqrt{2}}{2}\right\}$

26. $6\arccos x = 5\pi \Rightarrow \arccos x = \frac{5\pi}{6} \Rightarrow$
$x = \cos \frac{5\pi}{6} \Rightarrow x = -\frac{\sqrt{3}}{2}$
Solution set: $\left\{-\frac{\sqrt{3}}{2}\right\}$

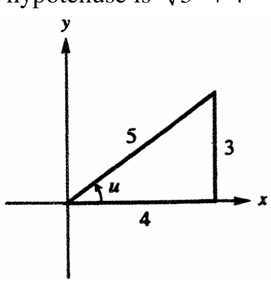
27. $\frac{4}{3}\cos^{-1} \frac{y}{4} = \pi \Rightarrow \cos^{-1} \frac{y}{4} = \frac{3\pi}{4} \Rightarrow$
$\frac{y}{4} = \cos \frac{3\pi}{4} \Rightarrow \frac{y}{4} = -\frac{\sqrt{2}}{2} \Rightarrow y = -2\sqrt{2}$
Solution set: $\left\{-2\sqrt{2}\right\}$

28. $4\pi + 4\tan^{-1} y = \pi \Rightarrow 4\tan^{-1} y = -3\pi \Rightarrow$
$\tan^{-1} y = -\frac{3\pi}{4}$
The range of $\tan^{-1} y$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Since $-\frac{3\pi}{4}$ is not in this interval, the equation has no solution. Solution set: $\varnothing$

29. $2\arccos\left(\frac{y-\pi}{3}\right) = 2\pi$
$2\arccos\left(\frac{y-\pi}{3}\right) = 2\pi \Rightarrow$
$\arccos\left(\frac{y-\pi}{3}\right) = \pi \Rightarrow \frac{y-\pi}{3} = \cos\pi$
$\frac{y-\pi}{3} = -1 \Rightarrow y - \pi = -3 \Rightarrow y = \pi - 3$
Solution set: $\{\pi - 3\}$

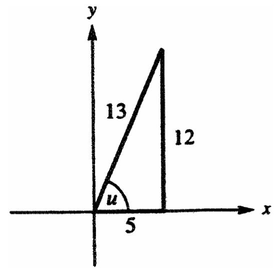
30. $\arccos\left(y - \frac{\pi}{3}\right) = \frac{\pi}{6} \Rightarrow y - \frac{\pi}{3} = \cos\frac{\pi}{6} \Rightarrow$
$y = \frac{\sqrt{3}}{2} + \frac{\pi}{3} \Rightarrow y = \frac{3\sqrt{3} + 2\pi}{6}$
Solution set: $\left\{\frac{3\sqrt{3} + 2\pi}{6}\right\}$

31. $\arcsin x = \arctan\frac{3}{4}$
Let $\arctan\frac{3}{4} = u$, so $\tan u = \frac{3}{4}$, u is in quadrant I. Sketch a triangle and label it. The hypotenuse is $\sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$.



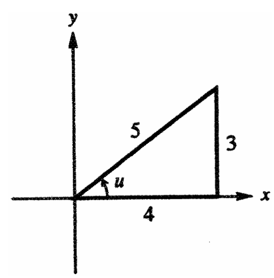
Therefore, $\sin u = \frac{3}{r} = \frac{3}{5}$. This equation becomes $\arcsin x = u$, or $x = \sin u$. Thus, $x = \frac{3}{5}$. Solution set: $\left\{\frac{3}{5}\right\}$

32. $\arctan x = \arccos\frac{5}{13}$
Let $\arccos\frac{5}{13} = u$, so $\cos\ u = \frac{5}{13}$. Sketch a triangle and label it. The side opposite angle u is $\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$.



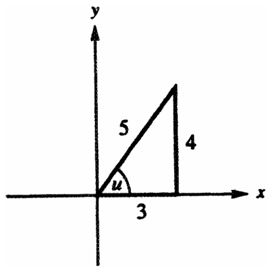
Therefore, $\tan u = \frac{12}{5}$. The equation becomes $\arctan x = u$, or $x = \tan u$. Thus, $x = \frac{12}{5}$.
Solution set: $\left\{\frac{12}{5}\right\}$

33. $\cos^{-1} x = \sin^{-1}\frac{3}{5}$
Let $\sin^{-1}\frac{3}{5} = u$, so $\sin u = \frac{3}{5}$. Sketch a triangle and label it. The hypotenuse is $\sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$.



Therefore, $\cos u = \frac{4}{5}$. The equation becomes $\cos^{-1} x = u$, or $x = \cos u$. Thus, $x = \frac{4}{5}$.
Solution set: $\left\{\frac{4}{5}\right\}$

34. $\cot^{-1} x = \tan^{-1}\frac{4}{3}$
Let $\tan^{-1}\frac{4}{3} = u$, so $\tan u = \frac{4}{3}$. Sketch a triangle and label it. The hypotenuse is $\sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$.



Therefore, $\cot u = \frac{3}{4}$. The equation becomes

$\cot^{-1} x = u$, or $x = \cot u$. Thus, $x = \frac{3}{4}$.

Solution set: $\left\{\frac{3}{4}\right\}$

35. $\sin^{-1} x - \tan^{-1} 1 = -\frac{\pi}{4}$

$$\sin^{-1} x - \tan^{-1} 1 = -\frac{\pi}{4} \Rightarrow$$
$$\sin^{-1} x = \tan^{-1} 1 - \frac{\pi}{4} \Rightarrow \sin^{-1} x = \frac{\pi}{4} - \frac{\pi}{4} \Rightarrow$$
$$\sin^{-1} x = 0 \Rightarrow \sin 0 = x \Rightarrow x = 0$$

Solution set: $\{0\}$

36. $\sin^{-1} x + \tan^{-1} \sqrt{3} = \frac{2\pi}{3}$

$$\sin^{-1} x + \tan^{-1} \sqrt{3} = \frac{2\pi}{3} \Rightarrow$$
$$\sin^{-1} x + \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow \sin^{-1} x = \frac{\pi}{3} \Rightarrow$$
$$x = \sin \frac{\pi}{3} \Rightarrow x = \frac{\sqrt{3}}{2}$$

Solution set: $\left\{\frac{\sqrt{3}}{2}\right\}$

37. $\arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \pi$

$$\arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \pi \Rightarrow$$
$$\arccos x = \pi - 2 \arcsin \frac{\sqrt{3}}{2} \Rightarrow$$
$$\arccos x = \pi - 2\left(\frac{\pi}{3}\right)$$
$$\arccos x = \pi - \frac{2\pi}{3} \Rightarrow \arccos x = \frac{\pi}{3} \Rightarrow$$
$$x = \cos \frac{\pi}{3} \Rightarrow x = \frac{1}{2}$$

Solution set: $\left\{\frac{1}{2}\right\}$

38. $\arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$$\arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3} \Rightarrow$$
$$\arccos x + 2\left(\frac{\pi}{3}\right) = \frac{\pi}{3} \Rightarrow \arccos x = -\frac{\pi}{3}$$

$-\frac{\pi}{3}$ is not in the range of arccos x. Therefore, the equation has no solution. Solution set: $\varnothing$

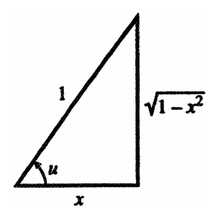
39. $\arcsin 2x + \arccos x = \frac{\pi}{6}$

$$\arcsin 2x + \arccos x = \frac{\pi}{6}$$
$$\arcsin 2x = \frac{\pi}{6} - \arccos x$$
$$2x = \sin\left(\frac{\pi}{6} - \arccos x\right)$$

Use the identity
$\sin(A - B) = \sin A \cos B - \cos A \sin B.$

$$2x = \sin \frac{\pi}{6} \cos(\arccos x) - \cos \frac{\pi}{6} \sin(\arccos x)$$

Let $u = \arccos x$. Thus, $\cos u = x = \frac{x}{1}$.



$\sin u = \sqrt{1 - x^2}$

$$2x = \sin \frac{\pi}{6} \cdot \cos u - \cos \frac{\pi}{6} \sin u \Rightarrow$$
$$2x = \frac{1}{2} x - \frac{\sqrt{3}}{2}\left(\sqrt{1 - x^2}\right) \Rightarrow$$
$$4x = x - \sqrt{3} \cdot \sqrt{1 - x^2}$$
$$3x = -\sqrt{3} \cdot \sqrt{1 - x^2}$$
$$(3x)^2 = \left(-\sqrt{3} \cdot \sqrt{1 - x^2}\right)^2 \Rightarrow 9x^2 = 3\left(1 - x^2\right)$$
$$9x^2 = 3 - 3x^2 \Rightarrow 12x^2 = 3$$
$$x^2 = \frac{3}{12} = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

Check these proposed solutions since they were found by squaring both side of an equation.

(*continued on next page*)

(*continued from page 711*)

Check $x = \frac{1}{2}$.

$$\arcsin 2x + \arccos x = \frac{\pi}{6}$$
$$\arcsin\left(2 \cdot \frac{1}{2}\right) + \arccos\left(\frac{1}{2}\right) = \frac{\pi}{6} \text{ ?}$$
$$\frac{\pi}{2} + \frac{\pi}{3} = \frac{\pi}{6} \text{ ?}$$
$$\frac{5\pi}{6} = \frac{\pi}{6} \text{ False}$$

$\frac{1}{2}$ is not a solution.

Check $x = -\frac{1}{2}$.

$$\arcsin 2x + \arccos x = \frac{\pi}{6}$$
$$\arcsin\left(2 \cdot -\frac{1}{2}\right) + \arccos\left(-\frac{1}{2}\right) = \frac{\pi}{6} \text{ ?}$$
$$-\frac{\pi}{2} + \frac{2\pi}{3} = \frac{\pi}{6} \text{ ?}$$
$$\frac{\pi}{6} = \frac{\pi}{6} \text{ True}$$

$-\frac{1}{2}$ is a solution.

Solution set: $\left\{-\frac{1}{2}\right\}$

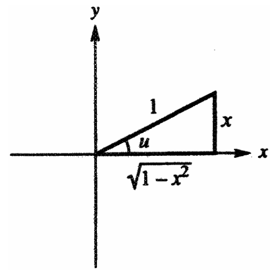
40. $\arcsin 2x + \arcsin x = \frac{\pi}{2}$

$$\arcsin 2x + \arcsin x = \frac{\pi}{2}$$
$$\arcsin 2x = \frac{\pi}{2} - \arcsin x$$
$$2x = \sin\left(\frac{\pi}{2} - \arcsin x\right)$$

Use the identity
$\sin (A - B) = \sin A \cos B - \cos A \sin B$.

$$2x = \sin\frac{\pi}{2}\cos(\arcsin x) - \cos\frac{\pi}{2}\sin(\arcsin x)$$
$$2x = 1 \cdot \cos(\arcsin x) - 0 \cdot \sin(\arcsin x)$$
$$2x = \cos(\arcsin x)$$

Let $\arcsin x = u$, so $\sin u = x = \frac{x}{1}$.



$\cos u = \sqrt{1-x^2}$

Substitute $\sqrt{1-x^2}$ for $\cos(\arcsin x)$ to obtain the following.

$2x = \sqrt{1-x^2} \Rightarrow 4x^2 = 1 - x^2 \Rightarrow$
$5x^2 = 1 \Rightarrow x^2 = \frac{1}{5} \Rightarrow x = \pm\frac{\sqrt{5}}{5}$

Check these proposed solutions since they were found by squaring an equation.

$x = \frac{\sqrt{5}}{5}$: Since

$\arcsin\left(2 \cdot \frac{\sqrt{5}}{5}\right) + \arcsin\left(\frac{\sqrt{5}}{5}\right) = \frac{\pi}{2}$, $\frac{\sqrt{5}}{5}$ is a solution.

$x = -\frac{\sqrt{5}}{5}$: Since

$\arcsin\left(2 \cdot \frac{-\sqrt{5}}{5}\right) + \arcsin\left(-\frac{\sqrt{5}}{5}\right) = -\frac{\pi}{2}$,

$\frac{-\sqrt{5}}{5}$ is not a solution

Solution set: $\left\{\frac{\sqrt{5}}{5}\right\}$

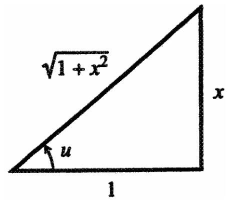
41. $\cos^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

$$\cos^{-1} x + \tan^{-1} x = \frac{\pi}{2}$$
$$\cos^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$
$$x = \cos\left(\frac{\pi}{2} - \tan^{-1} x\right)$$

Use the identity
$\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$x = \cos\frac{\pi}{2}\cos\left(\tan^{-1} x\right) + \sin\frac{\pi}{2}\sin\left(\tan^{-1} x\right)$$
$$x = 0 \cdot \cos\left(\tan^{-1} x\right) + 1 \cdot \sin\left(\tan^{-1} x\right)$$
$$x = \sin\left(\tan^{-1} x\right)$$

Let $u = \tan^{-1} x$. So, $\tan u = x$.



From the triangle, we find $\sin u = \frac{x}{\sqrt{1+x^2}}$, so the equation $x = \sin\left(\tan^{-1} x\right)$ becomes

$x = \frac{x}{\sqrt{1+x^2}}$. Solve this equation.

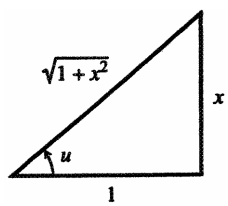
$$x = \frac{x}{\sqrt{1+x^2}} \Rightarrow x\sqrt{1+x^2} = x$$
$$x\sqrt{1+x^2} - x = 0 \Rightarrow x\left(\sqrt{1+x^2} - 1\right) = 0$$
$$x = 0 \text{ or } \sqrt{1+x^2} - 1 = 0 \Rightarrow \sqrt{1+x^2} = 1 \Rightarrow$$
$$1 + x^2 = 1 \Rightarrow x^2 = 0 \Rightarrow x = 0$$

Solution set: $\{0\}$

42. $\sin^{-1} x + \tan^{-1} x = 0$

$$\sin^{-1} x + \tan^{-1} x = 0 \Rightarrow \sin^{-1} x = -\tan^{-1} x \Rightarrow$$
$$x = \sin\left(-\tan^{-1} x\right) \Rightarrow x = -\sin\left(\tan^{-1} x\right)$$

Let $u = \tan^{-1} x$. So, $\tan u = x$.



From the triangle, we find $\sin u = \frac{x}{\sqrt{1+x^2}}$, so the equation $x = -\sin\left(\tan^{-1} x\right)$ becomes

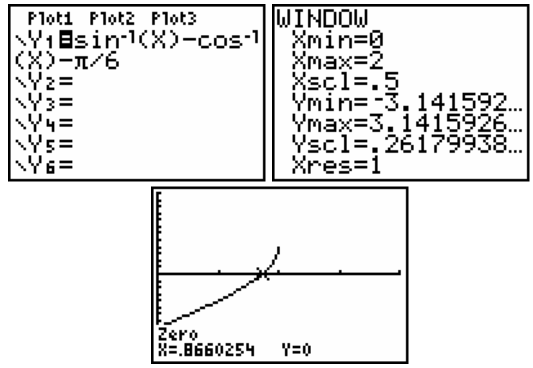
$x = -\frac{x}{\sqrt{1+x^2}}$. Solve this equation.

$$x = -\frac{x}{\sqrt{1+x^2}} \Rightarrow x\sqrt{1+x^2} = -x \Rightarrow$$
$$x\sqrt{1+x^2} + x = 0 \Rightarrow x\left(\sqrt{1+x^2} + 1\right) = 0 \Rightarrow$$
$$x = 0 \text{ or } \sqrt{1+x^2} + 1 = 0 \Rightarrow \sqrt{1+x^2} = -1,$$

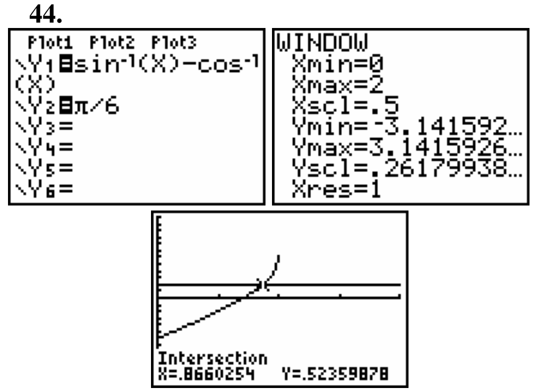
which is not real.

Solution set: $\{0\}$

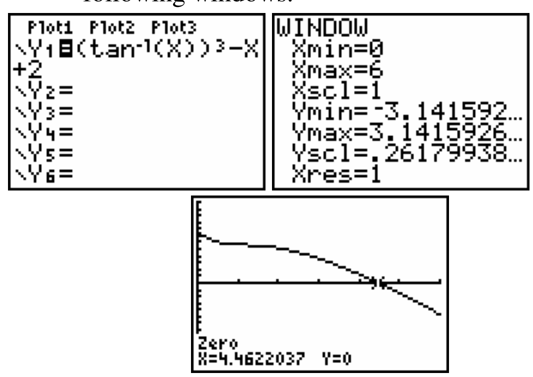
43.



44.

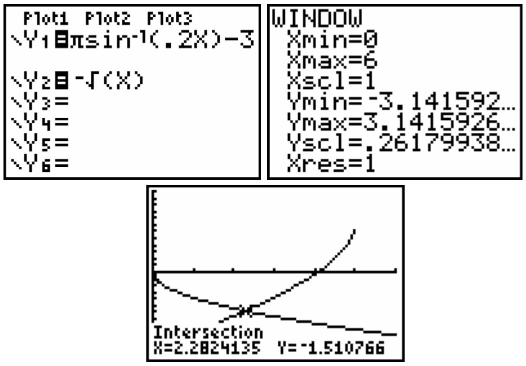


45. The *x*-intercept method is shown in the following windows.



Solution set: $\{4.4622\}$

46. The intersection method is shown in the following screens.



Solution set: $\{2.2824\}$

47. $A = \sqrt{\left(A_1 \cos\phi_1 + A_2 \cos\phi_2\right)^2 + \left(A_1 \sin\phi_1 + A_2 \sin\phi_2\right)^2}$ and

$\phi = \arctan\left(\dfrac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi + A_2 \cos\phi_2}\right)$

Make sure your calculator is in radian mode.

(a) Let $A_1 = .0012,\ \phi_1 = .052,\ A_2 = .004,$ and $\phi_2 = .61.$

$$A = \sqrt{(.0012\cos .052 + .004\cos .61)^2 + (.0012\sin .052 + .004\sin .61)^2}$$

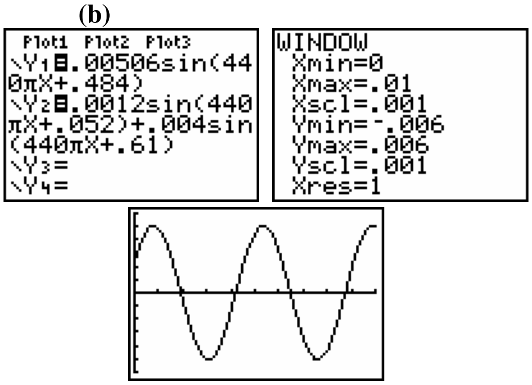
$\approx .00506$

$$\phi = \arctan\left(\frac{.0012\sin .052 + .004\sin .61}{.0012\cos .052 + .004\cos .61}\right)$$

$\approx .484$

If $f = 220,$ then $P = A\sin\left(2\pi ft + \phi\right)$ becomes $P = .00506\sin\left(440\pi t + .484\right).$

(b)



The two graphs are the same.

48. $A = \sqrt{\left(A_1 \cos\phi_1 + A_2 \cos\phi_2\right)^2 + \left(A_1 \sin\phi_1 + A_2 \sin\phi_2\right)^2}$ and

$\phi = \arctan\left(\dfrac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi + A_2 \cos\phi_2}\right)$

Make sure your calculator is in radian mode.

(a) Let $A_1 = .0025, \phi_1 = \dfrac{\pi}{7}, A_2 = .001, \phi_2 = \dfrac{\pi}{6},$ and $f = 300.$

$$A = \sqrt{\left[.0025\cos\left(\frac{\pi}{7}\right) + .001\cos\left(\frac{\pi}{6}\right)\right]^2 + \left[.0025\sin\left(\frac{\pi}{7}\right) + .001\sin\left(\frac{\pi}{6}\right)\right]^2}$$

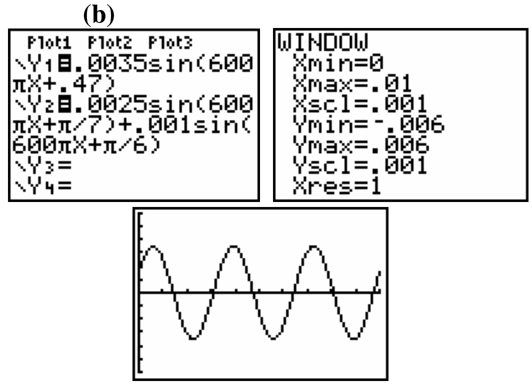
$\approx .0035$

$$\phi = \arctan\left[\frac{.0025\sin\left(\frac{\pi}{7}\right) + .001\sin\left(\frac{\pi}{6}\right)}{.0025\cos\left(\frac{\pi}{7}\right) + .001\cos\left(\frac{\pi}{6}\right)}\right]$$

$\approx .470$

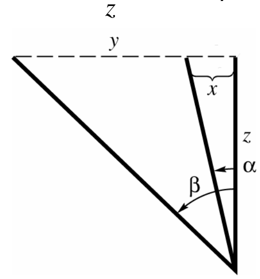
If $f = 300,$ then $P = A\sin\left(2\pi ft + \phi\right)$ becomes $P = .0035\sin\left(600\pi t + .47\right).$

(b)



The two graphs are the same.

49. **(a)** $\tan\alpha = \dfrac{x}{z}$ and $\tan\beta = \dfrac{x+y}{z}$



(b) Since

$$\tan\alpha = \frac{x}{z} \Rightarrow z\tan\alpha = x \Rightarrow z = \frac{x}{\tan\alpha} \text{ and}$$

$$\tan\beta = \frac{x+y}{z} \Rightarrow z\tan\beta = x+y \Rightarrow$$

$$z = \frac{x+y}{\tan\beta}, \text{ we have } \frac{x}{\tan\alpha} = \frac{x+y}{\tan\beta}$$

(c) $(x+y)\tan\alpha = x\tan\beta \Rightarrow$

$$\tan\alpha = \frac{x\tan\beta}{x+y} \Rightarrow \alpha = \arctan\left(\frac{x\tan\beta}{x+y}\right)$$

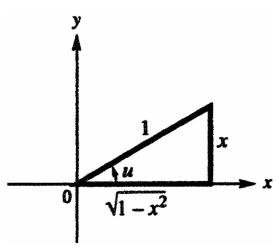
(d) $x\tan\beta = (x+y)\tan\alpha$

$$\tan\beta = \frac{(x+y)\tan\alpha}{x}$$

$$\beta = \arctan\left(\frac{(x+y)\tan\alpha}{x}\right)$$

50. **(a)** $u = \arcsin x \Rightarrow x = \sin u, -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

(b)



(c) $\tan u = \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{1-x^2}}{1-x^2}$

(d) $u = \arctan\left(\frac{x\sqrt{1-x^2}}{1-x^2}\right)$

51. **(a)** $E = E_{\max}\sin 2\pi ft \Rightarrow \frac{E}{E_{\max}} = \sin 2\pi ft \Rightarrow$

$$2\pi ft = \arcsin\frac{E}{E_{\max}} \Rightarrow$$

$$t = \frac{1}{2\pi f}\arcsin\frac{E}{E_{\max}}$$

(b) Let $E_{\max} = 12, E = 5,$ and $f = 100.$

$$t = \frac{1}{2\pi(100)}\arcsin\frac{5}{12}$$

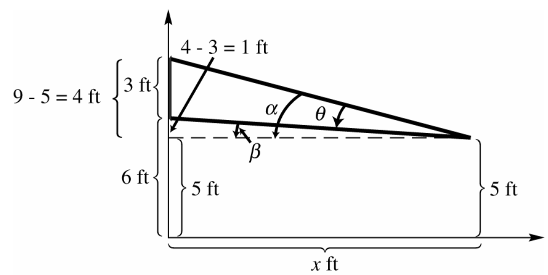
$$= \frac{1}{200\pi}\arcsin\frac{5}{12} \approx .00068\,\text{sec}$$

52. **(a)** $\theta = \alpha - \beta$

Since $\tan\alpha = \frac{4}{x} \Rightarrow \alpha = \tan^{-1}\left(\frac{4}{x}\right)$ and

$\tan\beta = \frac{1}{x} \Rightarrow \beta = \tan^{-1}\left(\frac{1}{x}\right)$, we have

$$\theta = \alpha - \beta \Rightarrow \theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right)$$



(b) $\theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right)$

(i) Let $\theta = \frac{\pi}{6}.$

$$\frac{\pi}{6} = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}$$

$$\tan\frac{\pi}{6} = \tan\left(\tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}\right)$$

$$\frac{\sqrt{3}}{3} = \frac{\frac{4}{x} - \frac{1}{x}}{1 + \frac{4}{x}\cdot\frac{1}{x}}$$

$$\frac{\sqrt{3}}{3} = \frac{\frac{3}{x}}{1+\frac{4}{x^2}} \Rightarrow \frac{\sqrt{3}}{3} = \frac{3x}{x^2+4}$$

$$\sqrt{3}x^2 + 4\sqrt{3} = 9x$$

$$\sqrt{3}x^2 - 9x + 4\sqrt{3} = 0$$

$$x^2 - 3\sqrt{3}x + 4 = 0$$

Using the quadratic formula with $a = 1$, $b = -3\sqrt{3}$, and $c = 4.$

$$x = \frac{-\left(-3\sqrt{3}\right) \pm \sqrt{\left(-3\sqrt{3}\right)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{3\sqrt{3} \pm \sqrt{27-16}}{2} = \frac{3\sqrt{3} \pm \sqrt{11}}{2}$$

$x \approx 4.26$ ft or .94 ft

(ii) Let $\theta = \dfrac{\pi}{8}.$

$$\frac{\pi}{8} = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}$$
$$\tan\frac{\pi}{8} = \tan\left(\tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}\right)$$

From part (a), we have

$$\tan\frac{\pi}{8} = \frac{3x}{x^2+4}$$
$$\left(\tan\frac{\pi}{8}\right)x^2 + 4\left(\tan\frac{\pi}{8}\right) = 3x$$
$$\left(\tan\frac{\pi}{8}\right)x^2 - 3x + 4\left(\tan\frac{\pi}{8}\right) = 0$$
$$x^2 - 3\left(\cot\frac{\pi}{8}\right)x + 4 = 0$$

Using the quadratic formula with

$a = 1,\ b = -3\left(\cot\dfrac{\pi}{8}\right),$ and $c = 4.$

$$x = \frac{-\left(-3\cot\frac{\pi}{8}\right) \pm \sqrt{\left(-3\cot\frac{\pi}{8}\right)^2 - 4(1)(4)}}{2(1)}$$

$x \approx 6.64$ ft or .60 ft

(c) $\theta = \tan^{-1}\left(\dfrac{4}{x}\right) - \tan^{-1}\left(\dfrac{1}{x}\right)$

(i) Let $x = 4.$

$$\theta = \tan^{-1}\frac{4}{4} - \tan^{-1}\frac{1}{4}$$
$$\theta = \tan^{-1}1 - \tan^{-1}\frac{1}{4}$$
$$\theta \approx \frac{\pi}{4} - .245 \approx .54$$

(ii) Let $x = 3.$

$$\theta = \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3} \Rightarrow \theta \approx .61$$

53. $y = \dfrac{1}{3}\sin\dfrac{4\pi t}{3}$

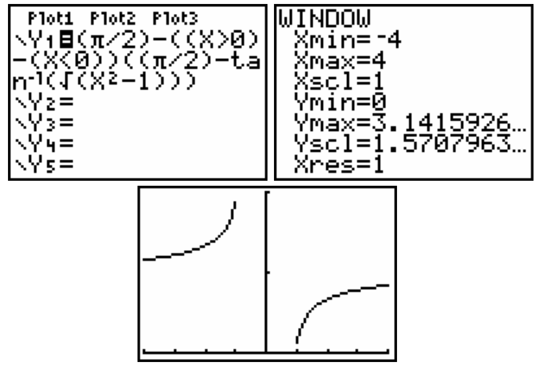
(a) $3y = \sin\dfrac{4\pi t}{3} \Rightarrow \dfrac{4\pi t}{3} = \arcsin 3y \Rightarrow$

$4\pi t = 3\arcsin 3y \Rightarrow t = \dfrac{3}{4\pi}\arcsin 3y$

(b) If $y = .3$ radian,

$t = \dfrac{3}{4\pi}\arcsin .9 \Rightarrow t \approx .27\,\text{sec}.$

54.



Chapter 7: Review Exercises

1. Since $\sec x = \dfrac{1}{\cos x},$ the correct choice is B.

2. Since $\csc x = \dfrac{1}{\sin x},$ the correct choice is A.

3. Since $\tan x = \dfrac{\sin x}{\cos x},$ the correct choice is C.

4. Since $\cot x = \dfrac{\cos x}{\sin x},$ the correct choice is F.

5. Since $\tan^2 x = \dfrac{1}{\cot^2 x},$ the correct choice is D.

6. Since $\sec^2 x = \dfrac{1}{\cos^2 x},$ the correct choice is E.

7. $$\sec^2\theta - \tan^2\theta = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} = 1$$

8. $$\frac{\cot\theta}{\sec\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{1} = \frac{\cos^2\theta}{\sin\theta}$$

9. $$\tan^2\theta\left(1+\cot^2\theta\right) = \frac{\sin^2\theta}{\cos^2\theta}\left(1+\frac{\cos^2\theta}{\sin^2\theta}\right) = \frac{\sin^2\theta}{\cos^2\theta}\left(\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}\right) = \frac{\sin^2\theta}{\cos^2\theta}\left(\frac{1}{\sin^2\theta}\right) = \frac{1}{\cos^2\theta}$$

10. $$\csc\theta + \cot\theta = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta}$$

11. $\tan\theta - \sec\theta\csc\theta = \frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$
$= \frac{\sin\theta}{\cos\theta} - \frac{1}{\sin\theta\cos\theta}$
$= \frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{1}{\sin\theta\cos\theta}$
$= \frac{\sin^2\theta - 1}{\sin\theta\cos\theta} = \frac{(1-\cos^2\theta)-1}{\sin\theta\cos\theta}$
$= \frac{-\cos^2\theta}{\sin\theta\cos\theta} = -\frac{\cos\theta}{\sin\theta}$

12. $\csc^2\theta + \sec^2\theta = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$
$= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}$
$= \frac{1}{\sin^2\theta\cos^2\theta}$

13. $\cos x = \frac{3}{5}$, x is in quadrant IV.

$\sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$

Since x is in quadrant IV, $\sin x < 0$.

$\sin x = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$

$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$

$\cot(-x) = -\cot x = \frac{1}{-\tan x} = \frac{1}{-\left(-\frac{4}{3}\right)} = \frac{3}{4}$

14. $\tan x = -\frac{5}{4}, \frac{\pi}{2} < x < \pi$

$\sec^2 x = \tan^2 x + 1 = \left(-\frac{5}{4}\right)^2 + 1 = \frac{25}{16} + 1 = \frac{41}{16}$

Since x is in quadrant II, $\sec x < 0$, so

$\sec x = -\frac{\sqrt{41}}{4}$.

$\cot x = \frac{1}{\tan x} = \frac{1}{-\frac{5}{4}} = -\frac{4}{5}$

$\cos x = \frac{1}{\sec x} = \frac{1}{-\frac{\sqrt{41}}{4}} = -\frac{4}{\sqrt{41}}\cdot\frac{\sqrt{41}}{\sqrt{41}} = -\frac{4\sqrt{41}}{41}$

Since x is in quadrant II, $\sin x > 0$.

$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\left(-\frac{4\sqrt{41}}{41}\right)^2}$
$= \sqrt{1-\frac{656}{1681}} = \sqrt{\frac{1025}{1681}} = \frac{5\sqrt{41}}{41}$

$\csc x = \frac{1}{\sin x} = \frac{1}{\frac{5\sqrt{41}}{41}} = \frac{41}{5\sqrt{41}}\cdot\frac{\sqrt{41}}{\sqrt{41}} = \frac{\sqrt{41}}{5}$

15. E. $\cos 210° = \cos(150° + 60°)$
$= \cos 150°\cos 60° - \sin 150°\sin 60°$

16. B. $\sin 35° = \cos(90° - 35°) = \cos 55°$

17. J. $\tan(-35°) = \cot[90° - (-35°)] = \cot 125°$

18. A. $-\sin 35° = \sin(-35°)$

19. I. $\cos 35° = \cos(-35°)$

20. C. $\cos 75° = \cos\frac{150°}{2} = \sqrt{\frac{1+\cos 150°}{2}}$

21. H. $\sin 75° = \sin(15° + 60°)$
$= \sin 15°\cos 60° + \cos 15°\sin 60°$

22. D. $\sin 300° = \sin 2(150°) = 2\sin 150°\cos 150°$

23. G. $\cos 300° = \cos 2(150°)$
$= \cos^2 150° - \sin^2 150°$

24. B. $\cos(-55°) = \cos 55°$

25. Find $\sin(x + y)$, $\cos(x - y)$, and $\tan(x + y)$, given $\sin x = -\frac{3}{5}$, $\cos y = -\frac{7}{25}$, x and y are in quadrant III. Since x and y are in quadrant III, $\cos x$ and $\sin y$ are negative.

$\cos x = -\sqrt{1-\sin^2 x} = -\sqrt{1-\left(-\frac{3}{5}\right)^2}$
$= -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$

$\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-\left(-\frac{7}{25}\right)^2}$
$= -\sqrt{1-\frac{49}{625}} = -\sqrt{\frac{596}{625}} = -\frac{24}{25}$

$\sin(x + y) = \sin x\cos y + \cos x\sin y$
$= \left(-\frac{3}{5}\right)\left(-\frac{7}{25}\right) + \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right)$
$= \frac{21}{125} + \frac{96}{125} = \frac{117}{125}$

(*continued on next page*)

(*continued from page 717*)

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$
$$= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right)+\left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right)$$
$$= \frac{28}{125}+\frac{72}{125}=\frac{100}{125}=\frac{4}{5}$$

To find $\tan(x+y)$, first find $\cos(x+y)$.

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right)-\left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right)$$
$$= \frac{28}{125}-\frac{72}{125}=-\frac{44}{125}$$

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\frac{117}{125}}{-\frac{44}{5}}\cdot\frac{125}{125}=-\frac{117}{44}$$

Note that using the formula

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}, \text{ we have}$$

$$\tan(x+y)=\frac{\frac{3}{4}+\frac{24}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{24}{7}\right)}=\frac{21+96}{28-72}=-\frac{117}{44}$$

To find the quadrant of $x + y$, notice that $\sin(x + y) > 0$, which implies $x + y$ is in quadrant I or II. Also $\tan(x + y) < 0$, which implies that $x + y$ is in quadrant II or IV. Therefore, $x + y$ is in quadrant II.

26. Find $\sin(x + y)$, $\cos(x - y)$, and $\tan(x + y)$, given $\sin x = \frac{3}{5}$, $\cos y = \frac{24}{25}$, x is in quadrant I, y is in quadrant IV.

Since x is in quadrant I, $\cos x > 0$ and

$$\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}$$
$$=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

Since y is in quadrant IV, $\sin y < 0$ and

$$\sin y=-\sqrt{1-\cos^2 y}=-\sqrt{1-\left(\frac{24}{25}\right)^2}$$
$$=-\sqrt{1-\frac{596}{625}}=-\sqrt{\frac{49}{625}}=-\frac{7}{25}$$

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$=\left(\frac{3}{5}\right)\left(\frac{24}{25}\right)+\frac{4}{5}\left(-\frac{7}{25}\right)$$
$$=\frac{72}{125}-\frac{28}{125}=\frac{44}{125}$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$
$$=\left(\frac{4}{5}\right)\left(\frac{24}{25}\right)+\left(\frac{3}{5}\right)\left(-\frac{7}{25}\right)$$
$$=\frac{96}{125}-\frac{21}{125}=\frac{75}{125}=\frac{3}{5}$$

To find $\tan(x+y)$, first find $\cos(x+y)$.

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$=\left(\frac{4}{5}\right)\left(\frac{24}{25}\right)-\left(\frac{3}{5}\right)\left(-\frac{7}{25}\right)$$
$$=\frac{96}{125}+\frac{21}{125}=\frac{117}{125}$$

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\frac{44}{125}}{\frac{117}{125}}=\frac{44}{117}$$

Note that using the formula

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}, \text{ we have}$$

$$\tan(x+y)=\frac{\frac{3}{4}+\left(-\frac{7}{24}\right)}{1-\left(\frac{3}{4}\right)\left(-\frac{7}{24}\right)}=\frac{72-28}{96+21}=\frac{44}{117}$$

To find the quadrant of $x + y$, notice that $\sin(x + y) > 0$, which implies $x + y$ is in quadrant I or II. Also $\tan(x + y) > 0$, which implies that $x + y$ is in quadrant I or III. Therefore, $x + y$ is in quadrant I.

27. Find $\sin(x + y)$, $\cos(x - y)$, and $\tan(x + y)$, given $\sin x = -\frac{1}{2}$, $\cos y = -\frac{2}{5}$, x and y are in quadrant III.

Since x and y are in quadrant III, $\cos x$ and $\sin y$ are negative.

$$\cos x=-\sqrt{1-\sin^2 x}=-\sqrt{1-\left(-\frac{1}{2}\right)^2}$$
$$=-\sqrt{1-\frac{1}{4}}=-\sqrt{\frac{3}{4}}=-\frac{\sqrt{3}}{2}$$

$$\sin y=-\sqrt{1-\cos^2 y}=-\sqrt{1-\left(-\frac{2}{5}\right)^2}$$
$$=-\sqrt{1-\frac{4}{25}}=-\sqrt{\frac{21}{25}}=-\frac{\sqrt{21}}{5}$$

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$=\left(-\frac{1}{2}\right)\left(-\frac{2}{5}\right)+\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{21}}{5}\right)$$
$$=\frac{2}{10}+\frac{\sqrt{63}}{10}=\frac{2+3\sqrt{7}}{10}$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$
$$=\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{2}{5}\right)+\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{21}}{5}\right)$$
$$=\frac{2\sqrt{3}}{10}+\frac{\sqrt{21}}{10}=\frac{2\sqrt{3}+\sqrt{21}}{10}$$

To find $\tan(x+y)$, first find $\cos(x+y)$.

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$=\left(-\frac{2}{5}\right)\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{21}}{5}\right)$$
$$=\frac{2\sqrt{3}}{10}-\frac{\sqrt{21}}{10}=\frac{2\sqrt{3}-\sqrt{21}}{10}$$

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\frac{2+3\sqrt{7}}{10}}{\frac{2\sqrt{3}-\sqrt{21}}{10}}$$
$$=\frac{2+3\sqrt{7}}{2\sqrt{3}-\sqrt{21}}$$

Using the formula $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$,

we have

$$\tan(x+y)=\frac{\frac{\sqrt{3}}{3}+\frac{\sqrt{21}}{2}}{1-\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{21}}{2}\right)}=\frac{2\sqrt{3}+3\sqrt{21}}{6-3\sqrt{7}}$$
$$=-\frac{75\sqrt{3}+24\sqrt{21}}{27}$$

The two forms of $\tan(x+y)$ are equal.

To find the quadrant of $x + y$, notice that $\sin(x + y) > 0$, which implies $x + y$ is in quadrant I or II. Also $\tan(x + y) < 0$, which implies that $x + y$ is in quadrant II or IV. Therefore, $x + y$ is in quadrant II.

28. Find $\sin(x + y)$, $\cos(x - y)$, and $\tan(x + y)$, given $\sin y = -\dfrac{2}{3}$, $\cos x = -\dfrac{1}{5}$, x is in quadrant II and y is in quadrant III.

Since x is in quadrant II, $\sin x$ is positive.

$$\sin x=\sqrt{1-\cos^2 x}=\sqrt{1-\left(-\frac{1}{5}\right)^2}$$
$$=\sqrt{1-\frac{1}{25}}=\sqrt{\frac{24}{25}}=\frac{2\sqrt{6}}{5}$$

Since y is in quadrant III, $\cos y$ is negative.

$$\cos y=-\sqrt{1-\sin^2 y}=-\sqrt{1-\left(-\frac{2}{3}\right)^2}$$
$$=-\sqrt{1-\frac{4}{9}}=-\sqrt{\frac{5}{9}}=-\frac{\sqrt{5}}{3}$$

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$=\left(\frac{2\sqrt{6}}{5}\right)\left(-\frac{\sqrt{5}}{3}\right)+\left(-\frac{1}{5}\right)\left(-\frac{2}{3}\right)$$
$$=-\frac{2\sqrt{30}}{15}+\frac{2}{15}=\frac{-2\sqrt{30}+2}{15}$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$
$$=\left(-\frac{1}{5}\right)\left(-\frac{\sqrt{5}}{3}\right)+\left(\frac{2\sqrt{6}}{5}\right)\left(-\frac{2}{3}\right)$$
$$=\frac{\sqrt{5}}{15}-\frac{4\sqrt{6}}{15}=\frac{\sqrt{5}-4\sqrt{6}}{15}$$

To find $\tan(x+y)$, first find $\cos(x+y)$.

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$=\left(-\frac{1}{5}\right)\left(-\frac{\sqrt{5}}{3}\right)-\left(\frac{2\sqrt{6}}{5}\right)\left(-\frac{2}{3}\right)$$
$$=\frac{\sqrt{5}}{15}+\frac{4\sqrt{6}}{15}=\frac{\sqrt{5}+4\sqrt{6}}{15}$$

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\frac{-2\sqrt{30}+2}{15}}{\frac{\sqrt{5}+4\sqrt{6}}{15}}$$
$$=\frac{-2\sqrt{30}+2}{\sqrt{5}+4\sqrt{6}}$$

To find $\tan(x+y)$ using the formula

$\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$, we have

$$\tan x=\frac{\sin x}{\cos x}=\frac{\frac{2\sqrt{6}}{5}}{-\frac{1}{5}}=-2\sqrt{6}$$

$$\tan y=\frac{\sin y}{\cos y}=\frac{-\frac{2}{3}}{-\frac{\sqrt{5}}{3}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{-2\sqrt{6}+\frac{2\sqrt{5}}{5}}{1-\left(-2\sqrt{6}\right)\left(\frac{2\sqrt{5}}{5}\right)}$$
$$=\frac{-2\sqrt{6}+\frac{2\sqrt{5}}{5}}{1-\left(-2\sqrt{6}\right)\left(\frac{2\sqrt{5}}{5}\right)}\cdot\frac{5}{5}$$
$$=\frac{-10\sqrt{6}+2\sqrt{5}}{5+4\sqrt{30}}$$

The two forms of $\tan(x+y)$ are equal.

To find the quadrant of $x + y$, notice that $\sin(x + y) < 0$, which implies $x + y$ is in quadrant III or IV. Also $\tan(x + y) < 0$, which implies that $x + y$ is in quadrant II or IV. Therefore, $x + y$ is in quadrant IV.

29. Find sin $(x + y)$, cos $(x - y)$, and tan $(x + y)$, given $\sin x = \frac{1}{10}$, $\cos y = \frac{4}{5}$, x is in quadrant I and y is in quadrant IV.
Since x is in quadrant I, cos x is positive.

$$\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{1}{10}\right)^2} = \sqrt{1-\frac{1}{100}} = \sqrt{\frac{99}{100}} = \frac{3\sqrt{11}}{10}$$

Since y is in quadrant IV, sin y is negative.

$$\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-\left(\frac{4}{5}\right)^2} = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y = \left(\frac{1}{10}\right)\left(\frac{4}{5}\right) + \left(\frac{3\sqrt{11}}{10}\right)\left(-\frac{3}{5}\right) = \frac{4}{50} - \frac{9\sqrt{11}}{50} = \frac{4-9\sqrt{11}}{50}$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y = \left(\frac{3\sqrt{11}}{10}\right)\left(\frac{4}{5}\right) + \left(\frac{1}{10}\right)\left(-\frac{3}{5}\right) = \frac{12\sqrt{11}}{50} - \frac{3}{50} = \frac{12\sqrt{11}-3}{50}$$

To find $\tan(x+y)$, first find $\cos(x+y)$.

$$\cos(x+y) = \cos x\cos y - \sin x\sin y = \left(\frac{3\sqrt{11}}{10}\right)\left(\frac{4}{5}\right) - \left(\frac{1}{10}\right)\left(-\frac{3}{5}\right) = \frac{12\sqrt{11}}{50} + \frac{3}{50} = \frac{12\sqrt{11}+3}{50}$$

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\frac{4-9\sqrt{11}}{50}}{\frac{12\sqrt{11}+3}{50}} = \frac{4-9\sqrt{11}}{12\sqrt{11}+3}$$

To find $\tan(x+y)$ using the formula $\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y}$, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{10}}{\frac{3\sqrt{11}}{10}} = \frac{1}{3\sqrt{11}}\cdot\frac{\sqrt{11}}{\sqrt{11}} = \frac{\sqrt{11}}{33}$$

$$\tan y = \frac{\sin y}{\cos y} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y} = \frac{\frac{\sqrt{11}}{33} + \left(-\frac{3}{4}\right)}{1-\left(\frac{\sqrt{11}}{33}\right)\left(-\frac{3}{4}\right)} = \frac{4\sqrt{11}-99}{132+3\sqrt{11}}$$

The two forms of $\tan(x+y)$ are equal.

To find the quadrant of $x + y$, notice that $\sin(x + y) < 0$, which implies $x + y$ is in quadrant III or IV. Also $\tan(x + y) < 0$, which implies that $x + y$ is in quadrant II or IV. Therefore, $x + y$ is in quadrant IV.

30. Find sin $(x + y)$, cos $(x - y)$, and tan $(x + y)$, given $\cos x = \frac{2}{9}$, $\sin y = -\frac{1}{2}$, x is in quadrant IV and y is in quadrant III.
Since x is in quadrant IV, sin x is negative.

$$\sin x = -\sqrt{1-\cos^2 x} = -\sqrt{1-\left(\frac{2}{9}\right)^2} = -\sqrt{1-\frac{4}{81}} = -\sqrt{\frac{77}{81}} = -\frac{\sqrt{77}}{9}$$

Since y is in quadrant III, cos y is negative.

$$\cos y = -\sqrt{1-\sin^2 y} = -\sqrt{1-\left(-\frac{1}{2}\right)^2} = -\sqrt{1-\frac{1}{4}} = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y = \left(-\frac{\sqrt{77}}{9}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{2}{9}\right)\left(-\frac{1}{2}\right) = \frac{\sqrt{231}}{18} - \frac{2}{18} = \frac{\sqrt{231}-2}{18}$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y = \left(\frac{2}{9}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{77}}{9}\right)\left(-\frac{1}{2}\right) = -\frac{2\sqrt{3}}{18} + \frac{\sqrt{77}}{18} = \frac{-2\sqrt{3}+\sqrt{77}}{18}$$

To find $\tan(x+y)$, first find $\cos(x+y)$.

$$\cos(x+y) = \cos x\cos y - \sin x\sin y = \left(\frac{2}{9}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{77}}{9}\right)\left(-\frac{1}{2}\right) = -\frac{2\sqrt{3}}{18} - \frac{\sqrt{77}}{18} = \frac{-2\sqrt{3}-\sqrt{77}}{18}$$

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\frac{\sqrt{231}-2}{18}}{\frac{-2\sqrt{3}-\sqrt{77}}{18}} = \frac{\sqrt{231}-2}{-2\sqrt{3}-\sqrt{77}}$$

To find $\tan(x+y)$ using the formula $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{\sqrt{77}}{9}}{\frac{2}{9}} = -\frac{\sqrt{77}}{2}$$

$$\tan y = \frac{\sin y}{\cos y} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{-\frac{\sqrt{77}}{2} + \frac{\sqrt{3}}{3}}{1 - \left(-\frac{\sqrt{77}}{2}\right)\left(\frac{\sqrt{3}}{3}\right)} = \frac{-3\sqrt{77}+2\sqrt{3}}{6+\sqrt{231}}$$

The two forms of $\tan(x+y)$ are equal.

To find the quadrant of $x + y$, notice that $\sin(x + y) > 0$, which implies $x + y$ is in quadrant I or II. Also $\tan(x + y) < 0$, which implies that $x + y$ is in quadrant II or IV. Therefore, $x + y$ is in quadrant II.

31. Find $\sin\theta$ and $\cos\theta$, given $\cos 2\theta = -\frac{3}{4}$, $90° < 2\theta < 180°$.

$90° < 2\theta < 180° \Rightarrow 45° < \theta < 90° \Rightarrow \theta$ is in quadrant I, so $\sin\theta$ and $\cos\theta$ are both positive.

$$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow -\frac{3}{4} = 1 - 2\sin^2\theta \Rightarrow$$

$$-\frac{7}{4} = -2\sin^2\theta \Rightarrow \frac{7}{8} = \sin^2\theta \Rightarrow$$

$$\sin\theta = \sqrt{\frac{7}{8}} = \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{14}}{4}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{7}{8}} = \sqrt{\frac{1}{8}} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

32. Find $\sin B$ and $\cos B$, given $\cos 2B = \frac{1}{8}$, B in quadrant IV.

Since B is in quadrant IV, $\sin B$ is negative and $\cos B$ is positive.

$$\sin B = -\sqrt{\frac{1-\cos 2B}{2}} = -\sqrt{\frac{1-\frac{1}{8}}{2}} = -\sqrt{\frac{8-1}{16}} = -\sqrt{\frac{7}{16}} = -\frac{\sqrt{7}}{4}$$

$$\cos B = \sqrt{\frac{1+\cos 2B}{2}} = \sqrt{\frac{1+\frac{1}{8}}{2}} = \sqrt{\frac{8+1}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

33. Find $\sin 2x$ and $\cos 2x$, given $\tan x = 3$, $\sin x < 0$.

Since $\tan x > 0$ and $\sin x < 0$, x is in quadrant III, and $2x$ is in quadrant I or II.

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2(3)}{1-3^2} = -\frac{6}{8} = -\frac{3}{4}$$

Since $\tan 2x < 0$, $2x$ is in quadrant II. Thus, $\sin 2x > 0$ and $\cos 2x < 0$.

$$\sec 2x = -\sqrt{1+\tan^2 x} = -\sqrt{1+\left(-\frac{3}{4}\right)^2} = -\sqrt{1+\frac{9}{16}} = -\sqrt{\frac{25}{16}} = -\frac{5}{4} \Rightarrow$$

$$\cos 2x = \frac{1}{-\frac{5}{4}} = -\frac{4}{5}$$

$$\sin 2x = \sqrt{1-\cos^2(2x)} = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

34. Find $\sin 2y$ and $\cos 2y$, given $\sec y = -\frac{5}{3}$, $\sin y > 0$

$$\cos y = \frac{1}{\sec y} = \frac{1}{-\frac{5}{3}} = -\frac{3}{5}$$

$$\sin y = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin 2y = 2\sin y\cos y = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

$$\cos 2y = \cos^2 y - \sin^2 y = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

35. Find $\cos\dfrac{\theta}{2}$, given $\cos\theta=-\dfrac{1}{2}$, $90°<\theta<180°$.

Since $90°<\theta<180°\Rightarrow 45°<\dfrac{\theta}{2}<90°$, $\dfrac{\theta}{2}$ is in quadrant I and $\cos\dfrac{\theta}{2}>0$.

$$\cos\frac{\theta}{2}=\sqrt{\frac{1+\left(-\frac{1}{2}\right)}{2}}=\sqrt{\frac{2-1}{4}}-\sqrt{\frac{1}{4}}=\frac{1}{2}$$

36. Find $\sin\dfrac{A}{2}$, given $\cos A=-\dfrac{3}{4}$, $90°<A<180°$.

Since $90°<A<180°\Rightarrow 45°<\dfrac{A}{2}<90°$, $\dfrac{A}{2}$ is in quadrant I and $\sin\dfrac{A}{2}>0$.

$$\sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}}=\sqrt{\frac{1-\left(-\frac{3}{4}\right)}{2}}=\sqrt{\frac{4+3}{8}}$$
$$=\sqrt{\frac{7}{8}}=\frac{\sqrt{7}}{2\sqrt{2}}=\frac{\sqrt{14}}{4}$$

37. Find tan x, given tan $2x = 2$, with $\pi<x<\dfrac{3\pi}{2}$.

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}\Rightarrow 2=\frac{2\tan x}{1-\tan^2 x}\Rightarrow$$
$$2\tan x=2\left(1-\tan^2 x\right),\text{ if }\tan x\neq\pm1$$

Thus, $2\left(\tan^2 x+2\tan x-1\right)=0\Rightarrow$

$\tan^2 x+2\tan x-1=0$, so we can use the quadratic formula to solve for tan x.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\Rightarrow$$
$$\tan x=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2}=\frac{-1\pm\sqrt{5}}{2}$$

Since x is in quadrant III, tan x >0, so

$$\tan x=\frac{-1+\sqrt{5}}{2}$$

38. Find $\sin y$, given $\cos 2y=-\dfrac{1}{3}$, with $\dfrac{\pi}{2}<y<\pi$.

$$\cos 2y=1-2\sin^2 y\Rightarrow-\frac{1}{3}=1-2\sin^2 y\Rightarrow$$
$$-2\sin^2 y=-\frac{4}{3}\Rightarrow\sin^2 y=\frac{2}{3}$$

Since $\dfrac{\pi}{2}<y<\pi$, y is in quadrant II and $\sin y>0$. Thus,

$$\sin y=\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{6}}{3}.$$

39. Find $\tan\dfrac{x}{2}$, given sin x = .8, with $0<x<\dfrac{\pi}{2}$.

$$\cos x=\pm\sqrt{1-\sin^2 x}=\pm\sqrt{1-.8^2}=\sqrt{1-.64}$$
$$=\pm\sqrt{.36}=\pm.6$$

Since x is in quadrant I, cos x > 0, so cos x = .6

$$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}=\frac{1-.6}{.8}=\frac{.4}{.8}=.5$$

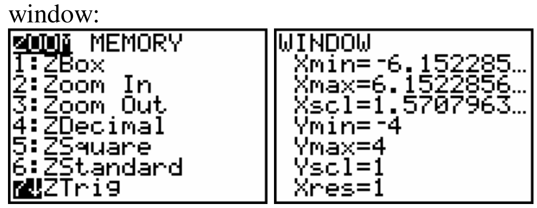
40. Find sin $2x$, given sin x = .6, with $\dfrac{\pi}{2}<x<\pi$.

$$\cos x=\pm\sqrt{1-\sin^2 x}=\pm\sqrt{1-.6^2}=\sqrt{1-.36}$$
$$=\pm\sqrt{.64}=\pm.8$$

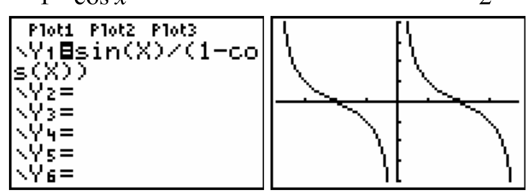
Since $\dfrac{\pi}{2}<x<\pi$, x is in quadrant II so cos x = −.8.

$$\sin 2x=2\sin x\cos x=2(.6)(-.8)=-.96$$

Exercises 41−44 are graphed in the following window:

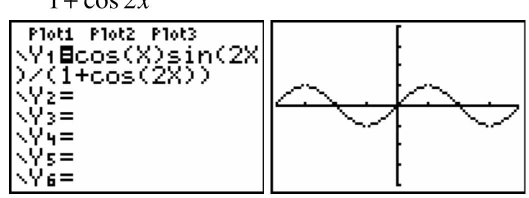


41. $\dfrac{\sin x}{1-\cos x}$ appears to be equivalent to $\cot\dfrac{x}{2}$.



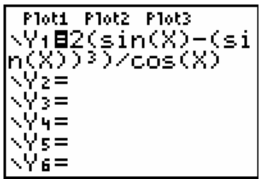
$$\frac{\sin x}{1-\cos x}=\frac{1}{\frac{1-\cos x}{\sin x}}=\frac{1}{\tan\frac{x}{2}}=\cot\frac{x}{2}$$

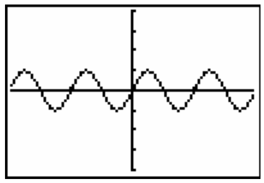
42. $\dfrac{\cos x\sin 2x}{1+\cos 2x}$ appears to be equivalent to sin x.



$$\frac{\cos x \sin 2x}{1+\cos 2x} = \frac{\cos x \cdot 2\sin x \cos}{1+2\cos^2 x - 1}$$
$$= \frac{2\sin x \cos^2 x}{2\cos^2 x} = \sin x$$

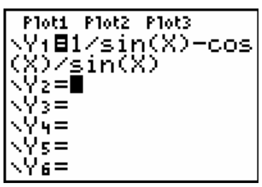
43. $\dfrac{2\left(\sin x - \sin^3 x\right)}{\cos x}$ appears to be equivalent to $\sin 2x$.

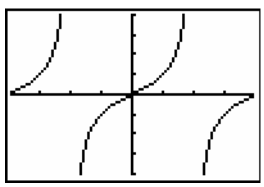




$$\frac{2\left(\sin x - \sin^3 x\right)}{\cos x} = \frac{2\sin x\left(1-\sin^2 x\right)}{\cos x}$$
$$= \frac{2\sin x \cos^2 x}{\cos x}$$
$$= 2\sin x \cos x = \sin 2x$$

44. $\csc x - \cot x$ appears to be equivalent to $\tan \dfrac{x}{2}$.





$$\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x} = \tan \frac{x}{2}$$

45. Verify $\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$ is an identity.

$$\sin^2 x - \sin^2 y = \left(1-\cos^2 x\right) - \left(1-\cos^2 y\right)$$
$$= 1 - \cos^2 x - 1 + \cos^2 y$$
$$= \cos^2 y - \cos^2 x$$

46. Verify $2\cos^3 x - \cos x = \dfrac{\cos^2 x - \sin^2 x}{\sec x}$ is an identity.

Work with the right side.

$$\frac{\cos^2 x - \sin^2 x}{\sec x} = \frac{\cos^2 x - \sin^2 x}{\frac{1}{\cos x}}$$
$$= \left(\cos^2 x - \sin^2 x\right) \cdot \cos x$$
$$= \cos^3 x - \sin^2 x \cos x$$
$$= \cos^3 x - \left(1 - \cos^2 x\right)\cos x$$
$$= \cos^3 x - \cos x + \cos^3 x$$
$$= 2\cos^3 x - \cos x$$

47. Verify $\dfrac{\sin^2 x}{2-2\cos x} = \cos^2 \dfrac{x}{2}$ is an identity.

$$\frac{\sin^2 x}{2-2\cos x} = \frac{1-\cos^2 x}{2\left(1-\cos x\right)}$$
$$= \frac{\left(1-\cos x\right)\left(1+\cos x\right)}{2\left(1-\cos x\right)}$$
$$= \frac{1+\cos x}{2} = \cos^2 \frac{x}{2}$$

48. Verify $\dfrac{\sin 2x}{\sin x} = \dfrac{2}{\sec x}$ is an identity.

$$\frac{\sin 2x}{\sin x} = \frac{2\sin x \cos x}{\sin x} = 2\cos x$$
$$= \frac{2}{\frac{1}{\cos x}} = \frac{2}{\sec x}$$

49. Verify $2\cos A - \sec A = \cos A - \dfrac{\tan A}{\csc A}$ is an identity.

Work with the right side.

$$\cos A - \frac{\tan A}{\csc A} = \cos A - \frac{\frac{\sin A}{\cos A}}{\frac{1}{\sin A}} = \cos A - \frac{\sin^2 A}{\cos A}$$
$$= \frac{\cos^2 A}{\cos A} - \frac{\sin^2 A}{\cos A}$$
$$= \frac{\cos^2 A - \sin^2 A}{\cos A}$$
$$= \frac{\cos^2 A - \left(1-\cos^2 A\right)}{\cos A}$$
$$= \frac{2\cos^2 A - 1}{\cos A} = 2\cos A - \frac{1}{\cos A}$$
$$= 2\cos A - \sec A$$

50. Verify $\dfrac{2\tan B}{\sin 2B} = \sec^2 B$ is an identity.

$$\frac{2\tan B}{\sin 2B} = \frac{2 \cdot \frac{\sin B}{\cos B}}{2\sin B \cos B} = \frac{2\sin B}{2\sin B \cos^2 B}$$
$$= \frac{1}{\cos^2 B} = \sec^2 B$$

51. Verify $1+\tan^2\alpha = 2\tan\alpha\csc 2\alpha$ is an identity.
Work with the right side.

$$2\tan\alpha\csc 2\alpha = \frac{2\tan\alpha}{\sin 2\alpha} = \frac{2\cdot\frac{\sin\alpha}{\cos\alpha}}{2\sin\alpha\cos\alpha} = \frac{2\sin\alpha}{2\sin\alpha\cos^2\alpha} = \frac{1}{\cos^2\alpha} = \sec^2\alpha = 1+\tan^2\alpha$$

52. Verify $\frac{2\cot x}{\tan 2x} = \csc^2 x - 2$ is an identity.

$$\frac{2\cot x}{\tan 2x} = \frac{2}{\tan x\left(\frac{2\tan x}{1-\tan^2 x}\right)} = \frac{2}{\tan x}\cdot\frac{1-\tan^2 x}{2\tan x}$$
$$= \frac{1-\tan^2 x}{\tan^2 x} = \frac{1-\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}\cdot\frac{\cos^2 x}{\cos^2 x}$$
$$= \frac{\cos^2 x - \sin^2 x}{\sin^2 x} = \frac{1-2\sin^2 x}{\sin^2 x}$$
$$= \csc^2 x - 2$$

53. Verify $\tan\theta\sin 2\theta = 2-2\cos^2\theta$ is an identity.

$$\tan\theta\sin 2\theta = \tan\theta(2\sin\theta\cos\theta) = \frac{\sin\theta}{\cos\theta}(2\sin\theta\cos\theta) = 2\sin^2\theta = 2(1-\cos^2\theta) = 2-2\cos^2\theta$$

54. Verify $\csc A\sin 2A - \sec A = \cos 2A\sec A$ is an identity.

$$\csc A\sin 2A - \sec A = \frac{1}{\sin A}(2\sin A\cos A) - \frac{1}{\cos A}$$
$$= 2\cos A - \frac{1}{\cos A} = \frac{2\cos^2 A}{\cos A} - \frac{1}{\cos A}$$
$$= \frac{2\cos^2 A - 1}{\cos A} = \frac{\cos 2A}{\cos A} = \cos 2A\sec A$$

55. Verify $2\tan x\csc 2x - \tan^2 x = 1$ is an identity.

$$2\tan x\csc 2x - \tan^2 x = 2\tan x\frac{1}{\sin 2x} - \tan^2 x$$
$$= 2\cdot\frac{\sin x}{\cos x}\cdot\frac{1}{2\sin x\cos x} - \frac{\sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = \frac{1-\sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} = 1$$

56. Verify $2\cos^2\theta - 1 = \frac{1-\tan^2\theta}{1+\tan^2\theta}$.
Work with the right side.

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}\cdot\frac{\cos^2\theta}{\cos^2\theta} = \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} = \frac{\cos^2\theta-\sin^2\theta}{1}$$
$$= \cos^2\theta - \sin^2\theta = \cos 2\theta = 2\cos^2\theta - 1$$

57. Verify $\tan\theta\cos^2\theta = \frac{2\tan\theta\cos^2\theta - \tan\theta}{1-\tan^2\theta}$ is an identity.
Work with the right side.

$$\frac{2\tan\theta\cos^2\theta-\tan\theta}{1-\tan^2\theta} = \frac{\tan\theta(2\cos^2\theta-1)}{1-\tan^2\theta} = \frac{\tan\theta(2\cos^2\theta-1)}{1-\frac{\sin^2\theta}{\cos^2\theta}}\cdot\frac{\cos^2\theta}{\cos^2\theta}$$
$$= \frac{\tan\theta\cos^2\theta(2\cos^2\theta-1)}{\cos^2\theta-\sin^2\theta} = \frac{\tan\theta\cos^2\theta(2\cos^2\theta-1)}{2\cos^2\theta-1} = \tan\theta\cos^2\theta$$

58. Verify $\sec^2\alpha - 1 = \frac{\sec 2\alpha - 1}{\sec 2\alpha + 1}$ is an identity.
Work with the right side.

$$\frac{\sec 2\alpha - 1}{\sec 2\alpha + 1} = \frac{\frac{1}{\cos 2\alpha}-1}{\frac{1}{\cos 2\alpha}+1} = \frac{\frac{1}{\cos^2\alpha-\sin^2\alpha}-1}{\frac{1}{\cos^2\alpha-\sin^2\alpha}+1}\cdot\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha-\sin^2\alpha}$$

$$= \frac{1-(\cos^2\alpha - \sin^2\alpha)}{\cos^2\alpha - \sin^2\alpha + 1}$$
$$= \frac{(1-\cos^2\alpha)+\sin^2\alpha}{\cos^2\alpha + (1-\sin^2\alpha)}$$
$$= \frac{\sin^2\alpha + \sin^2\alpha}{\cos^2\alpha + \cos^2\alpha} = \frac{2\sin^2\alpha}{2\cos^2\alpha}$$
$$= \tan^2\alpha = \sec^2\alpha - 1$$

59. Verify $\dfrac{\sin^2 x - \cos^2 x}{\csc x} = 2\sin^3 x - \sin x$ is an identity.

$$\frac{\sin^2 x - \cos^2 x}{\csc x} = \frac{\sin^2 x - (1-\sin^2 x)}{\frac{1}{\sin x}}$$
$$= \frac{2\sin^2 x - 1}{\frac{1}{\sin x}} \cdot \frac{\sin x}{\sin x}$$
$$= (2\sin^2 x - 1)\sin x$$
$$= 2\sin^3 x - \sin x$$

60. Verify $\sin^3\theta = \sin\theta - \cos^2\theta\sin\theta$ is an identity.
Work with the right side.

$$\sin\theta - \cos^2\theta\sin\theta = \sin\theta - (1-\sin^2\theta)\sin\theta$$
$$= \sin\theta - \sin\theta + \sin^3\theta$$
$$= \sin^3\theta$$

61. Verify $\tan 4\theta = \dfrac{2\tan 2\theta}{2-\sec^2 2\theta}$ is an identity.

$$\tan 4\theta = \tan[2(2\theta)] = \frac{2\tan 2\theta}{1-\tan^2 2\theta}$$
$$= \frac{2\tan 2\theta}{1-(\sec^2 2\theta - 1)} = \frac{2\tan 2\theta}{2-\sec^2 2\theta}$$

62. Verify $2\cos^2\dfrac{x}{2}\tan x = \tan x + \sin x$ is an identity. Work with the right side.

$$\tan x + \sin x = \frac{\sin x}{\cos x} + \sin x = \sin x\left(\frac{1}{\cos x} + 1\right)$$
$$= \sin x\left(\frac{1}{\cos\left[2\left(\frac{x}{2}\right)\right]} + 1\right)$$
$$= \sin x\left(\frac{1}{2\cos^2\left(\frac{x}{2}\right) - 1} + 1\right)$$
$$= \sin x\left(\frac{1 + \left(2\cos^2\left(\frac{x}{2}\right) - 1\right)}{2\cos^2\left(\frac{x}{2}\right) - 1}\right)$$
$$= \frac{2\sin x\cos^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right) - 1} = \frac{2\sin x\cos^2\left(\frac{x}{2}\right)}{\cos\left[2\left(\frac{x}{2}\right)\right]}$$
$$= \frac{2\sin x\cos^2\left(\frac{x}{2}\right)}{\cos x} = 2\cos^2\left(\frac{x}{2}\right)\tan x$$

63. Verify $\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \sec x + \tan x$ is an identity. Working with the left side,we have

$$\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{\tan\frac{x}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{x}{2}\tan\frac{\pi}{4}} = \frac{\tan\frac{x}{2} + 1}{1 - \tan\frac{x}{2}}$$

Working with the right side, we have

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1+\sin x}{\cos x}$$
$$= \frac{\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + \sin\left[2\left(\frac{x}{2}\right)\right]}{\cos\left[2\left(\frac{x}{2}\right)\right]}$$
$$= \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2 x - \sin^2 x}$$
$$= \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}$$
$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} \cdot \frac{\frac{1}{\cos\frac{x}{2}}}{\frac{1}{\cos\frac{x}{2}}}$$
$$= \frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

Since $\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \dfrac{\tan\frac{x}{2} + 1}{1 - \tan\frac{x}{2}} = \sec x + \tan x$, the statement is verified.

64. Verify $\dfrac{1}{2}\cot\dfrac{x}{2} - \dfrac{1}{2}\tan\dfrac{x}{2} = \cot x$ is an identity.

$$\frac{1}{2}\cot\frac{x}{2} - \frac{1}{2}\tan\frac{x}{2}$$
$$= \frac{1}{2}\left(\frac{1+\cos x}{\sin x}\right) - \frac{1}{2}\left(\frac{1-\cos x}{\sin x}\right)$$
$$= \frac{1+\cos x}{2\sin x} - \frac{1-\cos x}{2\sin x}$$
$$= \frac{2\cos x}{2\sin x} = \cot x$$

65. Verify $-\cot\frac{x}{2}=\frac{\sin 2x+\sin x}{\cos 2x-\cos x}$ is an identity.

Work with the right side.

$$\begin{aligned}\frac{\sin 2x+\sin x}{\cos 2x-\cos x}&=\frac{2\sin x\cos x+\sin x}{\left(2\cos^2 x-1\right)-\cos x}\\&=\frac{\sin x\left(2\cos x+1\right)}{2\cos^2 x-\cos x-1}\\&=\frac{\sin x\left(2\cos x+1\right)}{\left(2\cos x+1\right)\left(\cos x-1\right)}\\&=\frac{\sin x}{1-\cos x}=-\frac{\sin x}{\cos x-1}\\&=-\frac{1}{\frac{\sin x}{\cos x-1}}=-\frac{1}{\tan\frac{x}{2}}=-\cot\frac{x}{2}\end{aligned}$$

66. Verify $\frac{\sin 3t+\sin 2t}{\sin 3t-\sin 2t}=\frac{\tan\frac{5t}{2}}{\tan\frac{t}{2}}$ is an identity.

Using sum-to-product identities, we have

$$\begin{aligned}\frac{\sin 3t+\sin 2t}{\sin 3t-\sin 2t}&=\frac{2\sin\left(\frac{3t+2t}{2}\right)\cos\left(\frac{3t-2t}{2}\right)}{2\cos\left(\frac{3t+2t}{2}\right)\sin\left(\frac{3t-2t}{2}\right)}\\&=\frac{\sin\frac{5t}{2}\cos\frac{t}{2}}{\cos\frac{5t}{2}\sin\frac{t}{2}}\\&=\tan\frac{5t}{2}\cot\frac{t}{2}=\frac{\tan\frac{5t}{2}}{\tan\frac{t}{2}}\end{aligned}$$

67. $y=\sin^{-1}\frac{\sqrt{2}}{2}\Rightarrow\sin y=\frac{\sqrt{2}}{2}$

Since $-\frac{\pi}{2}\le y\le\frac{\pi}{2},\ y=\frac{\pi}{4}.$

68. $y=\arccos\left(-\frac{1}{2}\right)\Rightarrow\cos y=-\frac{1}{2}$

Since $0\le y\le\pi,\ y=\frac{2\pi}{3}.$

69. $y=\tan^{-1}\left(-\sqrt{3}\right)\Rightarrow\tan y=-\sqrt{3}$

Since $-\frac{\pi}{2}<y<\frac{\pi}{2},\ y=-\frac{\pi}{3}.$

70. $y=\arcsin\left(-1\right)\Rightarrow\sin y=-1$

Since $-\frac{\pi}{2}\le y\le\frac{\pi}{2},\ y=-\frac{\pi}{2}.$

71. $y=\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\Rightarrow\cos y=-\frac{\sqrt{2}}{2}$

Since $0\le y\le\pi,\ y=\frac{3\pi}{4}.$

72. $y=\arctan\frac{\sqrt{3}}{3}\Rightarrow\tan y=\frac{\sqrt{3}}{3}$

Since $-\frac{\pi}{2}<y<\frac{\pi}{2},\ y=\frac{\pi}{6}.$

73. $y=\sec^{-1}\left(-2\right)\Rightarrow\sec y=-2$

Since $0\le y\le\pi,\ y\ne\frac{\pi}{2}\Rightarrow y=\frac{2\pi}{3}.$

74. $y=\operatorname{arccsc}\frac{2\sqrt{3}}{3}\Rightarrow\csc y=\frac{2\sqrt{3}}{3}$

Since $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ and $y\ne 0,\ y=\frac{\pi}{3}.$

75. $y=\operatorname{arccot}\left(-1\right)\Rightarrow\cot y=-1$

Since $0<y<\pi,\ y=\frac{3\pi}{4}.$

76. $\theta=\arccos\frac{1}{2}\Rightarrow\cos\theta=\frac{1}{2}$

Since $0°\le\theta\le 180°,\ \theta=60°.$

77. $\theta=\arcsin\left(-\frac{\sqrt{3}}{2}\right)\Rightarrow\sin\theta=-\frac{\sqrt{3}}{2}$

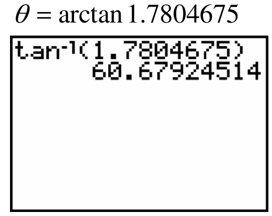
Since $-90°\le\theta\le 90°,\ \theta=-60°.$

78. $\theta=\tan^{-1}0\Rightarrow\tan\theta=0$

Since $-90°<\theta<90°,\ \theta=0°.$

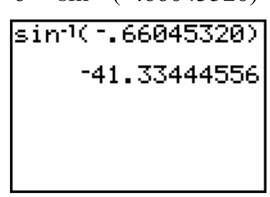
For Exercises79−84, be sure that your calculator is in degree mode. Keystroke sequences may vary based on the type and/or model of calculator being used.

79. $\theta=\arctan 1.7804675$



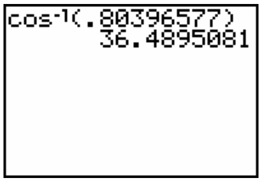
$\theta=60.67924514°$

80. $\theta=\sin^{-1}(-.66045320)$



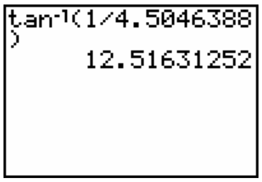
$\theta\approx-41.33444556°$

81. $\theta = \cos^{-1} .80396577$



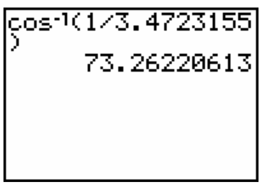
$\theta \approx 36.4895081°$

82. $\theta = \cot^{-1} 4.5046388$



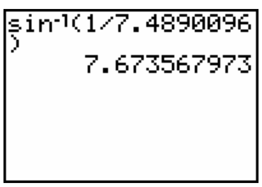
$\theta \approx 12.51631252°$

83. $\theta = \text{arc sec } 3.4723155$



$\theta \approx 73.26220613°$

84. $\theta = \csc^{-1} 7.4890096$



$\theta \approx 7.673567973°$

85. $\cos(\arccos(-1)) = \cos \pi = -1$ or
$\cos(\arccos(-1)) = \cos 180° = -1$

86. $\sin\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ or
$\sin\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right) = \sin(-60°) = -\frac{\sqrt{3}}{2}$

87. $\arccos\left(\cos\frac{3\pi}{4}\right) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

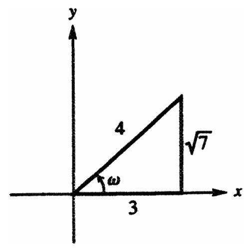
88. $\text{arcsec}(\sec \pi) = \text{arcsec}(-1) = \pi$

89. $\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \tan^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

90. $\cos^{-1}(\cos 0) = \cos^{-1} 1 = 0$

91. $\sin\left(\arccos\frac{3}{4}\right)$

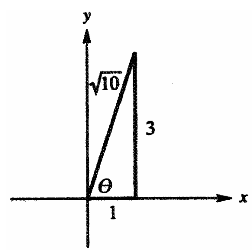
Let $\omega = \arccos\frac{3}{4}$, so that $\cos\omega = \frac{3}{4}$. Since arccos is defined only in quadrants I and II, and $\frac{3}{4}$ is positive, ω is in quadrant I. Sketch ω and label a triangle with the side opposite ω equal to $\sqrt{4^2 - 3^2} = \sqrt{16-9} = \sqrt{7}.$



$\sin\left(\arccos\frac{3}{4}\right) = \sin\omega = \frac{\sqrt{7}}{4}$

92. $\cos(\arctan 3)$

Let $\theta = \arctan 3$, so that $\tan\theta = 3 = \frac{3}{1}$. Since arctan is defined only in quadrants I and IV, and 3 is positive, θ is in quadrant I. Sketch θ and label a triangle with the hypotenuse equal to $\sqrt{3^2 + 1^2} = \sqrt{10}$.



$\cos(\arctan 3) = \cos\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

93. $\cos\left(\csc^{-1}(-2)\right)$

Let $\omega = \csc^{-1}(-2)$, so that $\csc\omega = -2$. Since $-\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$ and $\omega \ne 0$, and $\csc\omega = -2$ (negative), ω is in quadrant IV. Sketch ω and label a triangle with side adjacent to ω equal to $\sqrt{2^2 - (-1)^2} = \sqrt{4-1} = \sqrt{3}.$

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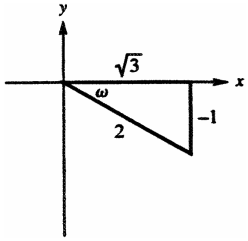
(*continued from page 727*)

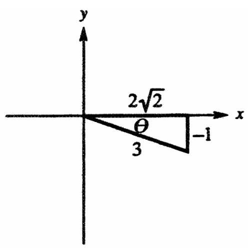


$$\cos\left(\csc^{-1}(-2)\right) = \cos\omega = \frac{\sqrt{3}}{2}$$

94. $\sec\left(2\sin^{-1}\left(-\frac{1}{3}\right)\right)$

Let $\theta = \sin^{-1}\left(-\frac{1}{3}\right)$, so $\sin\theta = -\frac{1}{3}$. Since arcsine is defined only in quadrants I and IV, and $-\frac{1}{3}$ is negative, θ is in quadrant IV. Sketch θ and label a triangle with the side adjacent to θ equal to

$$\theta = \sqrt{3^2 - (-1)^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}.$$



Thus, $\cos\theta = \frac{2\sqrt{2}}{3}$ and

$$\sec\left(2\sin^{-1}\left(-\frac{1}{3}\right)\right) = \sec 2\theta.$$

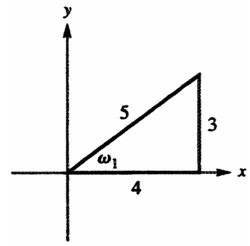
$$\sec 2\theta = \frac{1}{\cos 2\theta} = \frac{1}{2\cos^2\theta - 1} = \frac{1}{2\left(\frac{2\sqrt{2}}{3}\right)^2 - 1}$$

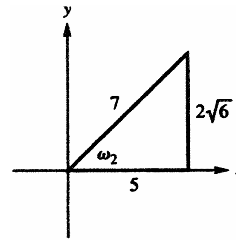
$$= \frac{1}{2\left(\frac{8}{9}\right) - 1} = \frac{1}{\frac{16}{9} - 1} = \frac{9}{16-9} = \frac{9}{7}$$

95. $\tan\left(\arcsin\frac{3}{5} + \arccos\frac{5}{7}\right)$

Let $\omega_1 = \arcsin\frac{3}{5}$, $\omega_2 = \arccos\frac{5}{7}$. Sketch angles ω_1 and ω_2 in quadrant I.

The side adjacent to ω_1 is $\sqrt{5^2 - 3^2} = \sqrt{25-9} = \sqrt{16} = 4$. The side opposite ω_2 is $\sqrt{7^2 - 5^2} = \sqrt{49-25} = \sqrt{24} = 2\sqrt{6}$.





We have $\tan\omega_1 = \frac{3}{4}$ and $\tan\omega_2 = \frac{2\sqrt{6}}{5}$.

$$\tan\left(\arcsin\frac{3}{5} + \arccos\frac{5}{7}\right)$$

$$= \tan(\omega_1 + \omega_2) = \frac{\tan\omega_1 + \tan\omega_2}{1 - \tan\omega_1 \tan\omega_2}$$

$$= \frac{\frac{3}{4} + \frac{2\sqrt{6}}{5}}{1 - \left(\frac{3}{4}\right)\left(\frac{2\sqrt{6}}{5}\right)} = \frac{\frac{15+8\sqrt{6}}{20}}{\frac{20-6\sqrt{6}}{20}}$$

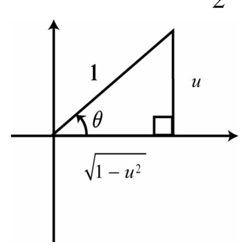
$$= \frac{15+8\sqrt{6}}{20-6\sqrt{6}} = \frac{15+8\sqrt{6}}{20-6\sqrt{6}} \cdot \frac{20+6\sqrt{6}}{20+6\sqrt{6}}$$

$$= \frac{588+250\sqrt{6}}{184} = \frac{294+125\sqrt{6}}{92}$$

96. $\cos\left(\arctan\frac{u}{\sqrt{1-u^2}}\right)$

Let $\theta = \arctan\frac{u}{\sqrt{1-u^2}}$, so $\tan\theta = \frac{u}{\sqrt{1-u^2}}$.

If $u > 0$, $0 < \theta < \frac{\pi}{2}$.



From the Pythagorean theorem,

$$r = \sqrt{\left(\sqrt{1-u^2}\right)^2 + u^2} = \sqrt{1-u^2+u^2} = \sqrt{1} = 1.$$

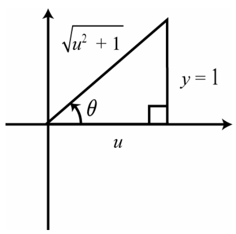
Therefore $\cos\theta = \frac{\sqrt{1-u^2}}{1} = \sqrt{1-u^2}$. Thus,

$$\cos\left(\arctan\frac{u}{\sqrt{1-u^2}}\right) = \sqrt{1-u^2}.$$

97. $\tan\left(\operatorname{arcsec}\frac{\sqrt{u^2+1}}{u}\right)$

Let $\theta = \operatorname{arcsec}\frac{\sqrt{u^2+1}}{u}$, so $\sec\theta = \frac{\sqrt{u^2+1}}{u}$.

If $u > 0$, $0 < \theta < \frac{\pi}{2}$.



From the Pythagorean theorem,

$$y = \sqrt{\left(\sqrt{u^2+1}\right)^2 - u^2} = \sqrt{u^2+1-u^2} = \sqrt{1} = 1.$$

Therefore $\tan\theta = \frac{1}{u}$. Thus,

$$\tan\left(\operatorname{arcsec}\frac{\sqrt{u^2+1}}{u}\right) = \frac{1}{u}.$$

98. $\sin^2 x = 1$

$\sin^2 x = 1 \Rightarrow \sin x = \pm 1$

Over the interval $[0, 2\pi)$, the equation $\sin x = 1$ has one solution. This solution is $\frac{\pi}{2}$. Over the same interval, the equation $\sin x = -1$ has one solution. This solution is $\frac{3\pi}{2}$. Solution set: $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

99. $2\tan x - 1 = 0$

$2\tan x - 1 = 0 \Rightarrow 2\tan x = 1 \Rightarrow \tan x = \frac{1}{2}$

Over the interval $[0, 2\pi)$, the equation $\tan x = \frac{1}{2}$ has two solutions. One solution is in quadrant I and the other is in quadrant III. Using a calculator, the quadrant I solution is approximately .4636. The quadrant III solution would be approximately $.4636 + \pi \approx 3.6052$.
Solution set: $\{.4636, 3.6052\}$

100. $3\sin^2 x - 5\sin x + 2 = 0$

$3\sin^2 x - 5\sin x + 2 = 0$

$(3\sin x - 2)(\sin x - 1) = 0$

$3\sin x - 2 \Rightarrow 3\sin x = 2 \Rightarrow \sin x = \frac{2}{3}$

$\sin x - 1 = 0 \Rightarrow \sin x = 1$

Over the interval $[0, 2\pi)$, the equation $\sin x = \frac{2}{3}$ has two solutions. One solution is in quadrant I and the other is in quadrant II. Using a calculator, the quadrant I solution is approximately .7297.
The quadrant II solution would be approximately $\pi - .7297 \approx 2.4119$. In the same interval, the equation $\sin x = 1$ has one solution. That solution is $\frac{\pi}{2}$.

Solution set: $\left\{.7297, \frac{\pi}{2}, 2.4119\right\}$

101. $\tan x = \cot x$

Use the identity $\cot x = \frac{1}{\tan x}$, $\tan x \neq 0$.

$\tan x = \cot x \Rightarrow \tan x = \frac{1}{\tan x} \Rightarrow$

$\tan^2 x = 1 \Rightarrow \tan x = \pm 1$

Over the interval $[0, 2\pi)$, the equation $\tan x = 1$ has two solutions. One solution is in quadrant I and the other is in quadrant III. These solutions are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. In the same interval, the equation $\tan x = -1$ has two solutions. One solution is in quadrant II and the other is in quadrant IV. These solutions are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$. Solution set: $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

102. $\sec^4 2x = 4$

$\sec^4 2x = 4 \Rightarrow \sec 2x = \pm\sqrt{2}$
Since $0 \le x < 2\pi, 0 \le 2x < 4\pi$. Thus,

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

implies

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}.$$

Solution set:

$$\left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$$

103. $\tan^2 2x - 1 = 0$

$\tan^2 2x - 1 = 0 \Rightarrow \tan^2 2x = 1 \Rightarrow \tan 2x = \pm 1$
Since $0 \le x < 2\pi, 0 \le 2x < 4\pi$. Thus,

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

implies

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}.$$

Solution set:

$$\left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$$

104. $\sec\frac{x}{2} = \cos\frac{x}{2}$

$$\sec\frac{x}{2} = \cos\frac{x}{2} \Rightarrow \frac{1}{\cos\frac{x}{2}} = \cos\frac{x}{2} \Rightarrow$$

$$\cos^2\frac{x}{2} = 1 \Rightarrow \cos\frac{x}{2} = \pm 1$$

Since $0 \le x < 2\pi, 0 \le \frac{x}{2} < \pi$. Thus, the only solution to $\cos\frac{x}{2} = \pm 1$ is $x = 0$.

Solution set:
$\{0 + 2n\pi, \text{where } n \text{ is any integer}\}$ or
$\{2n\pi, \text{where } n \text{ is any integer}\}$

105. $\cos 2x + \cos x = 0$

$\cos 2x + \cos x = 0 \Rightarrow 2\cos^2 x - 1 + \cos x = 0 \Rightarrow$
$2\cos^2 x + \cos x - 1 = 0 \Rightarrow$
$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0 \Rightarrow 2\cos x = 1 \Rightarrow \cos x = \frac{1}{2}$ or
$\cos x + 1 = 0 \Rightarrow \cos x = -1$

Over the interval $[0, 2\pi)$, the equation $\cos x = \frac{1}{2}$ has two solutions. The angles in quadrants I and IV that have a reference angle of $\frac{\pi}{3}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. In the same interval, $\cos x = -1$ when the angle is π.
Solution set:

$$\left\{\frac{\pi}{3} + 2n\pi, \pi + 2n\pi, \frac{5\pi}{3} + 2n\pi, \text{where } n \text{ is any integer}\right\}$$

106. $4\sin x\cos x = \sqrt{3}$

$$4\sin x\cos x = \sqrt{3} \Rightarrow 2\sin x\cos x = \frac{\sqrt{3}}{2} \Rightarrow$$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

Since $0 \le x < 2\pi, 0 \le 2x < 4\pi$. Thus,

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}.$$

Solution set:

$$\left\{\frac{\pi}{6} + 2n\pi, \frac{\pi}{3} + 2n\pi, \frac{7\pi}{6} + 2n\pi, \frac{4\pi}{3} + 2n\pi, \text{where } n \text{ is any integer}\right\}$$ or

$$\left\{\frac{\pi}{6} + n\pi, \frac{\pi}{3} + n\pi, \text{where } n \text{ is any integer}\right\}$$

45. $\sin^2\theta + 3\sin\theta + 2 = 0$

$\sin^2\theta + 3\sin\theta + 2 = 0$
$(\sin\theta + 2)(\sin\theta + 1) = 0$
In the interval $[0°, 360°)$, we have
$\sin\theta + 1 = 0 \Rightarrow \sin\theta = -1 \Rightarrow \theta = 270°$ and
$\sin\theta + 2 = 0 \Rightarrow \sin\theta = -2 < -1 \Rightarrow$ no solution
Solution set: $\{270°\}$

107. $2\tan^2\theta = \tan\theta + 1$

$2\tan^2\theta = \tan\theta + 1$
$2\tan^2\theta - \tan\theta - 1 = 0$
$(2\tan\theta + 1)(\tan\theta - 1) = 0$
In the interval $[0°, 360°)$, we have

$2\tan\theta + 1 = 0 \Rightarrow \tan\theta = -\frac{1}{2} \Rightarrow$
$\theta \approx 153.4°$ or $333.4°$ (using a calculator)
$\tan\theta - 1 = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°$ and $225°$
Solution set: $\{45°, 153.4°, 225°, 333.4°\}$

108. $\sin 2\theta = \cos 2\theta + 1$

$$\sin 2\theta = \cos 2\theta + 1$$
$$(\sin 2\theta)^2 = (\cos 2\theta + 1)^2$$
$$\sin^2 2\theta = \cos^2 2\theta + 2\cos 2\theta + 1$$
$$1 - \cos^2 2\theta = \cos^2 2\theta + 2\cos 2\theta + 1$$
$$2\cos^2 2\theta + 2\cos 2\theta = 0$$
$$\cos^2 2\theta + \cos 2\theta = 0$$
$$\cos 2\theta(\cos 2\theta + 1) = 0$$

Since $0° \le \theta < 360°,\ 0° \le 2\theta < 720°.$

$\cos 2\theta = 0 \Rightarrow 2\theta = 90°,\ 270°,\ 450°,\ 630° \Rightarrow$
$\theta = 45°,\ 135°,\ 225°,\ 315°$

$\cos 2\theta + 1 = 0 \Rightarrow \cos 2\theta = -1$
$2\theta = 180°,\ 540° \Rightarrow \theta = 90°,\ 270°$

Possible values for θ are
$\theta = 45°, 90°, 135°, 225°, 270°, 315°.$

All proposed solutions must be checked since the solutions were found by squaring an equation. A value for θ will be a solution if $\sin 2\theta - \cos 2\theta = 1.$

$\theta = 45°, 2\theta = 90° \Rightarrow$
$\sin 90° - \cos 90° = 1 - 0 = 1$
$\theta = 90°, 2\theta = 180° \Rightarrow$
$\sin 180° - \cos 180° = 0 - (-1) = 1$
$\theta = 135°, 2\theta = 270° \Rightarrow$
$\sin 270° - \cos 270° = -1 - 0 \ne 1$
$\theta = 225°, 2\theta = 450° \Rightarrow$
$\sin 450° - \cos 450° = 1 - 0 = 1$
$\theta = 270°, 2\theta = 540° \Rightarrow$
$\sin 540° - \cos 540° = 0 - (-1) = 1$
$\theta = 315°, 2\theta = 630° \Rightarrow$
$\sin 630° - \cos 630° = -1 - 0 \ne 1$

Thus, $\theta = 45°,\ 90°,\ 225°,\ 270°.$

Solution set: $\{45°,\ 90°,\ 225°,\ 270°\}$

109. $2\sin 2\theta = 1$

$2\sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2}$

Since $0° \le \theta < 360°,\ 0° \le 2\theta < 720°.$ Thus,
$2\theta = 30°, 150°, 390°, 510° \Rightarrow$
$\theta = 15°, 75°, 195°, 255°$

Solution set: $\{15°, 75°, 195°, 255°\}$

111. $3\cos^2\theta + 2\cos\theta - 1 = 0$
$(3\cos\theta - 1)(\cos\theta + 1) = 0$

In the interval $[0°, 360°),$ we have

$3\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{3} \Rightarrow$
$\theta \approx 70.5°$ and $289.5°$ (using a calculator)
$\cos\theta + 1 = 0 \Rightarrow \cos\theta = -1 \Rightarrow \theta = 180°$

Solution set: $\{70.5°,\ 180°,\ 289.5°\}$

112. $5\cot^2\theta - \cot\theta - 2 = 0$

We use the quadratic formula with $a = 5$, $b = -1$, and $c = -2$.

$$\cot\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(5)(-2)}}{2(5)} = \frac{1 \pm \sqrt{1+40}}{10} = \frac{1 \pm \sqrt{41}}{10}$$

Since $\cot\theta = \frac{1+\sqrt{41}}{10} > 0,$ we will obtain two angles. One angle will be in quadrant I and the other will be in quadrant III. Using a calculator, if $\cot\theta = \frac{1+\sqrt{41}}{10} \approx .7303124,$ the quadrant I angle will be approximately $53.5°.$ The quadrant III angle will be approximately $180° + 53.5° = 233.5°.$ Since

$\cot\theta = \frac{1-\sqrt{41}}{10} < 0,$ we will obtain two angles. One angle will be in quadrant II and the other will be in quadrant IV.

Using a calculator, if

$\cot\theta = \frac{1-\sqrt{41}}{10} \approx -.26376262,$ the quadrant II angle will be approximately $118.4°.$

(Note: You need to calculate

$\tan^{-1}\left(\dfrac{1}{\frac{1-\sqrt{41}}{10}}\right) + 180$ to obtain this angle.)

The reference angle is $180° - 118.4° = 61.6°.$ Thus, the quadrant IV angle will be approximately $360° - 61.675.2° = 298.4°.$

Solution set: $\{53.5°, 118.4°, 233.5°, 298.4°\}$

113. $4y = 2\sin x \Rightarrow 2y = \sin x \Rightarrow x = \arcsin 2y$

114. $y = 3\cos\frac{x}{2} \Rightarrow \frac{y}{3} = \cos\frac{x}{2} \Rightarrow \frac{x}{2} = \arccos\frac{y}{3} \Rightarrow$
$x = 2\arccos\frac{y}{3}$

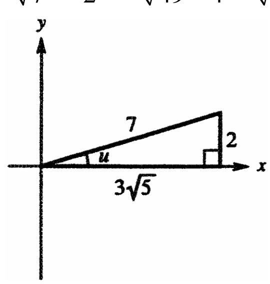
115. $2y = \tan(3x+2) \Rightarrow 3x + 2 = \arctan 2y \Rightarrow$
$3x = \arctan 2y - 2 \Rightarrow x = \left(\frac{1}{3}\arctan 2y\right) - \frac{2}{3}$

116. $5y = 4\sin x - 3 \Rightarrow \frac{5y+3}{4} = \sin x \Rightarrow$
$x = \arcsin\left(\frac{5y+3}{4}\right)$

117. $\frac{4}{3}\arctan\frac{x}{2} = \pi \Rightarrow \arctan\frac{x}{2} = \frac{3\pi}{4}$
But, by definition, the range of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, this equation has no solution.
Solution set: $\varnothing$

118. $\arccos x = \arcsin\frac{2}{7}$

Let $u = \arcsin\frac{2}{7}$. Then $\sin u = \frac{2}{7}$, u is in quadrant I since $\sin u = \frac{2}{7} > 0$. The side adjacent to u is
$\sqrt{7^2 - 2^2} = \sqrt{49-4} = \sqrt{45} = 3\sqrt{5}$.



Thus, $\cos u = \frac{3\sqrt{5}}{7}$. This equation becomes $\arccos x = u$, or $x = \cos u$. Thus, $x = \frac{3\sqrt{5}}{7}$.

Solution set: $\left\{\frac{3\sqrt{5}}{7}\right\}$

119. $\arccos x + \arctan 1 = \frac{11\pi}{12}$

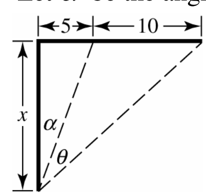
$$\arccos x + \arctan 1 = \frac{11\pi}{12}$$
$$\arccos x = \frac{11\pi}{12} - \arctan 1$$
$$\arccos x = \frac{11\pi}{12} - \frac{\pi}{4} = \frac{8\pi}{12} = \frac{2\pi}{3}$$
$$\cos\frac{2\pi}{3} = x \Rightarrow x = -\frac{1}{2}$$

Solution set: $\left\{-\frac{1}{2}\right\}$

120. $d = 550 + 450\cos\frac{\pi}{50}t$

$$d = 550 + 450\cos\frac{\pi}{50}t$$
$$450\cos\frac{\pi}{50}t = d - 550 \Rightarrow \cos\frac{\pi}{50}t = \frac{d-550}{450}$$
$$\frac{\pi}{50}t = \arccos\left(\frac{d-550}{450}\right)$$
$$t = \frac{50}{\pi}\arccos\left(\frac{d-550}{450}\right)$$

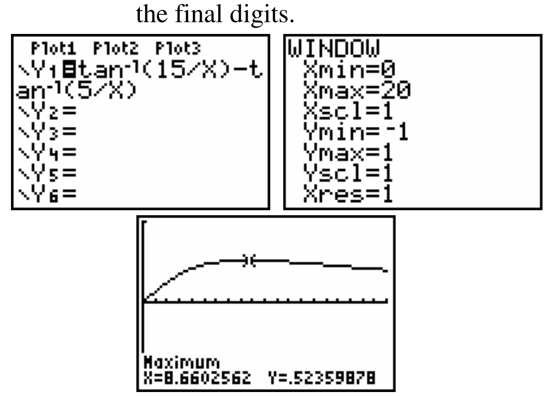
121. **(a)** Let α be the angle to the left of θ.



Thus, we have

$$\tan(\alpha + \theta) = \frac{5+10}{x}$$
$$\alpha + \theta = \arctan\left(\frac{15}{x}\right)$$
$$\theta = \arctan\left(\frac{15}{x}\right) - \alpha$$
$$\theta = \arctan\left(\frac{15}{x}\right) - \arctan\left(\frac{5}{x}\right)$$

(b) The maximum occurs at approximately 8.66026 ft. There may be a discrepancy in the final digits.



122. $\dfrac{c_1}{c_2} = \dfrac{\sin\theta_1}{\sin\theta_2}$

$\dfrac{c_1}{c_2} = \dfrac{\sin\theta_1}{\sin\theta_2} \Rightarrow .752 = \dfrac{\sin\theta_1}{\sin\theta_2}$

If $\theta_2 = 90°$, then $.752 = \dfrac{\sin\theta_1}{\sin 90°} = \dfrac{\sin\theta_1}{1}$

$= \sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}.752 \approx 48.8°.$

123. If $\theta_1 > 48.8°$, then $\theta_2 > 90°$ and the light beam is completely underwater.

124. $L = 6077 - 31\cos 2\theta$

(a) Solve the equation for θ.

$$L - 6077 = -31\cos 2\theta$$
$$\cos 2\theta = \frac{L-6077}{-31}$$
$$2\theta = \cos^{-1}\left(\frac{L-6077}{-31}\right)$$
$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{L-6077}{-31}\right)$$
$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{6074-6077}{-31}\right) \approx 42.2°$$

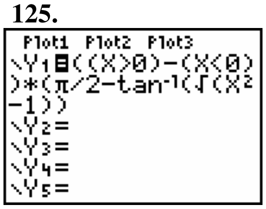
(b) Substitute $L = 6108$ in the equation from part (a).

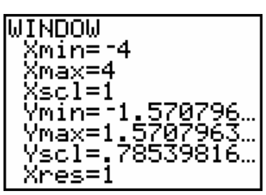
$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{6108-6077}{-31}\right)$$
$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{6108-6077}{-31}\right)$$
$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{31}{-31}\right) \Rightarrow \theta = \frac{1}{2}\cos^{-1}(-1)$$
$$\theta = \frac{1}{2}(180°) \Rightarrow \theta = 90°$$

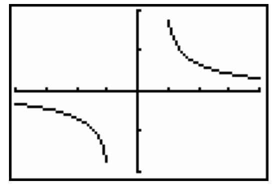
(c) Substitute $L = 6080.2$ in the equation from part (a).

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{6080.2-6077}{-31}\right)$$
$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{3.2}{-31}\right) \Rightarrow \theta \approx 48.0°$$

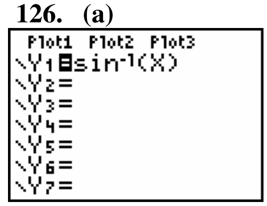
125.

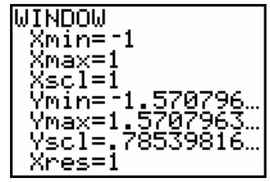


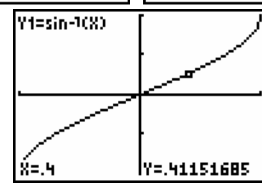




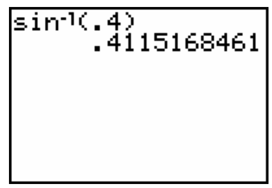
126. (a)



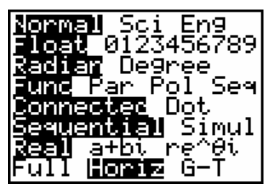


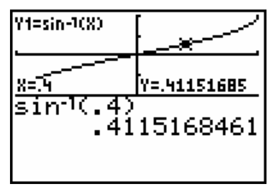


(b) In both cases, $\sin^{-1}.4 \approx .41151685$



To obtain the screen in the annotated instructor edition, change the mode of the calculator to Horizontal.





Chapter 7 Test

1. $\cos x = \dfrac{24}{25}$, x is in quadrant IV.

$$\sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{24}{25}\right)^2 = 1 - \frac{576}{625} = \frac{49}{625}$$

Since x is in quadrant IV, $\sin x < 0$.

$$\sin x = -\sqrt{\frac{49}{625}} = -\frac{7}{25}$$
$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{7}{25}}{\frac{24}{25}} = -\frac{7}{24}$$
$$\cot x = \frac{1}{\tan x} = \frac{1}{-\frac{7}{24}} = -\frac{24}{7}$$

(*continued on next page*)

(*continued from page 733*)

$$\sec x = \frac{1}{\cos x} = \frac{1}{\frac{24}{25}} = \frac{25}{24}$$
$$\csc x = \frac{1}{\sin x} = \frac{1}{-\frac{7}{25}} = -\frac{25}{7}$$

2. $\sec\theta - \sin\theta\tan\theta = \frac{1}{\cos\theta} - \sin\theta\cdot\frac{\sin\theta}{\cos\theta}$
$$= \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta$$

3. $\tan^2 x - \sec^2 x = \frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}$
$$= \frac{\sin^2 x - 1}{\cos^2 x} = -\frac{1-\sin^2 x}{\cos^2 x}$$
$$= -\frac{\cos^2 x}{\cos^2 x} = -1$$

4. $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$
$$= \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

5. **(a)** $\cos(270° - x)$
$$= \cos 270°\cos x + \sin 270°\sin x$$
$$= 0\cdot\cos x + (-1)\sin x = 0 - \sin x$$
$$= -\sin x$$

(b) $\tan(\pi + x) = \frac{\tan\pi + \tan x}{1 - \tan\pi\tan x} = \tan x$

6. $\sin(-22.5°) = \pm\sqrt{\frac{1-\cos(-45°)}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$
$$= \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$$

Since $-22.5°$ is in quadrant IV, $\sin(-22.5°)$ is negative. Thus, $\sin(-22.5°) = -\frac{\sqrt{2-\sqrt{2}}}{2}$.

7. Find sin $(A + B)$, cos $(A + B)$, and tan $(A - B)$, given $\sin A = \frac{5}{13}$, $\cos B = -\frac{3}{5}$, A is in quadrant I and B is in quadrant II. Thus, $\cos A > 0$ and $\sin B > 0$.

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\left(\frac{5}{13}\right)^2}$$
$$= \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$
$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\left(-\frac{3}{5}\right)^2}$$
$$= \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

(a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$$= \left(\frac{5}{13}\right)\left(-\frac{3}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{4}{5}\right)$$
$$= -\frac{15}{65} + \frac{48}{65} = \frac{33}{65}$$

(b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$$= \left(\frac{12}{13}\right)\left(-\frac{3}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$$
$$= -\frac{36}{65} - \frac{20}{65} = -\frac{56}{65}$$

(c) To use the formula $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$, first find $\tan A$ and $\tan B$:

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$
$$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$
$$\tan(A-B) = \frac{\frac{5}{12} - \left(-\frac{4}{3}\right)}{1 + \left(\frac{5}{12}\right)\left(-\frac{4}{3}\right)}$$
$$= \frac{15+48}{36-20} = \frac{63}{16}$$

(d) To find the quadrant of $A + B$, notice that $\sin(A + B) > 0$, which implies $x + y$ is in quadrant I or II. Also $\cos(A + B) < 0$, which implies that $A + B$ is in quadrant II or III. Therefore, $A + B$ is in quadrant II.

8. Given $\cos\theta = -\frac{3}{5}, \frac{\pi}{2} < \theta < \pi$

Since θ is in quadrant II, $\sin\theta > 0$, 2θ is in quadrant III or quadrant IV, and

$\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \frac{\theta}{2}$ is in quadrant I. Also

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

(a) $\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{3}{5}\right)^2 - 1 = -\frac{7}{25}$

Note that 2θ is in quadrant III because $\cos 2\theta < 0$.

(b) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$

(c) $$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{1-\frac{16}{9}} = \frac{-24}{9-16} = \frac{24}{7}$$

(d) $$\cos\tfrac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{5-3}{10}} = \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

(e) $$\tan\tfrac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta} = \frac{\frac{4}{5}}{1-\frac{3}{5}} = \frac{4}{5-3} = 2$$

9. Verify $\sec^2 B = \frac{1}{1-\sin^2 B}$ is an identity.

Work with the right side.

$$\frac{1}{1-\sin^2 B} = \frac{1}{\cos^2 B} = \sec^2 B$$

10. Verify $\tan^2 x - \sin^2 x = (\tan x \sin x)^2$ is an identity.

$$\begin{aligned}\tan^2 x - \sin^2 x &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x \\ &= \frac{\sin^2 x - \sin^2 x\cos^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x\left(1-\cos^2 x\right)}{\cos^2 x} = \frac{\sin^2 x\sin^2 x}{\cos^2 x} \\ &= \tan^2 x\sin^2 x = (\tan x\sin x)^2\end{aligned}$$

11. Verify $\frac{\tan x - \cot x}{\tan x + \cot x} = 2\sin^2 x - 1$ is an identity.

$$\begin{aligned}\frac{\tan x - \cot x}{\tan x + \cot x} &= \frac{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} \\ &= \frac{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} \cdot \frac{\cos x\sin x}{\cos x\sin x} \\ &= \frac{\sin^2 x - \cos^2 x}{\sin^2 + \cos^2 x} = \sin^2 x - \cos^2 x \\ &= \sin^2 x - \left(1-\sin^2 x\right) = 2\sin^2 x - 1\end{aligned}$$

12. Verify $\cos 2A = \frac{\cot A - \tan A}{\csc A\sec A}$ is an identity.

Work with the right side.

$$\begin{aligned}\frac{\cot A - \tan A}{\csc A\sec A} &= \frac{\frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}}{\left(\frac{1}{\sin A}\right)\left(\frac{1}{\cos A}\right)} \cdot \frac{\sin A\cos A}{\sin A\cos A} \\ &= \cos^2 A - \sin^2 A = \cos 2A\end{aligned}$$

13. **(a)** $V = 163\sin\omega t$. $\sin x = \cos\left(\frac{\pi}{2} - x\right) \Rightarrow$

$V = 163\cos\left(\frac{\pi}{2} - \omega t\right)$.

(b) $V = 163\sin\omega t = 163\sin 120\pi t$

$= 163\cos\left(\frac{\pi}{2} - 120\pi t\right) \Rightarrow$ the maximum voltage occurs when $\cos\left(\frac{\pi}{2} - 120\pi t\right) = 1$. Thus, the maximum voltage is V = 163 volts.

$\cos\left(\frac{\pi}{2} - 120\pi t\right) = 1$ when

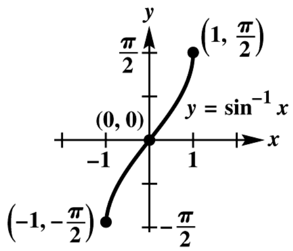
$\frac{\pi}{2} - 120\pi t = 2k\pi$, where k is any integer. The first maximum occurs when

$\frac{\pi}{2} - 120\pi t = 0 \Rightarrow$

$\frac{\pi}{2} = 120\pi t \Rightarrow \frac{1}{120\pi} \cdot \frac{\pi}{2} = t \Rightarrow t = \frac{1}{240}$

The maximum voltage will first occur at $\frac{1}{240}$ sec.

14.



Domain: $[-1,1]$; range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

15. **(a)** $y = \arccos\left(-\frac{1}{2}\right) \Rightarrow \cos y = -\frac{1}{2}$

Since $0 \le y \le \pi$, $y = \frac{2\pi}{3}$.

(b) $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \sin y = -\frac{\sqrt{3}}{2}$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y = -\frac{\pi}{3}$.

(c) $y = \tan^{-1} 0 \Rightarrow \tan y = 0$

Since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $y = 0$.

(d) $y = \text{arcsec}(-2) \Rightarrow \sec y = -2$

Since $0 \le y \le \pi$ and $y \ne \frac{\pi}{2}$, $y = \frac{2\pi}{3}$.

16. **(a)** $\theta = \arccos\frac{\sqrt{3}}{2} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

Since $0 \le y \le 180°$, $y = 30°$.

(b) $\theta = \tan^{-1}(-1) \Rightarrow \tan\theta = -1$

Since $-90° < y < 90°$, $y = -45°$.

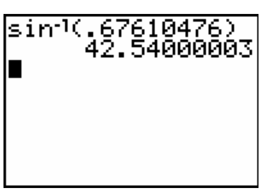
(c) $\theta = \cot^{-1}(-1) \Rightarrow \cot\theta = -1$

Since $0° < y < 180°$, $y = 135°$.

(d) $\theta = \csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right) \Rightarrow \csc\theta = -\frac{2\sqrt{3}}{3}$

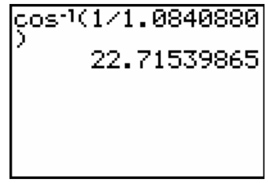
Since $-90 \le y \le 90°$, $\theta = -60°$.

17. **(a)**



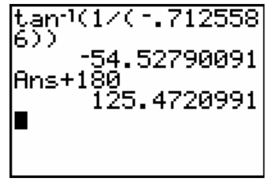
$\sin^{-1} .67610476 \approx 42.54°$

(b)



$\sec^{-1} 1.0840880 \approx 22.72°$

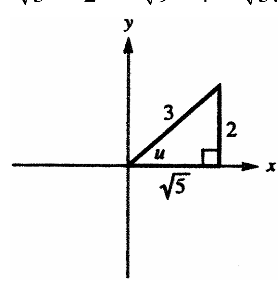
(c)



$\cot^{-1}(-.7125586) \approx 125.47°$

18. **(a)** $\cos\left(\arcsin\frac{2}{3}\right)$

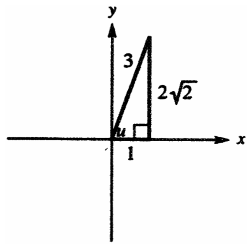
Let $\arcsin\frac{2}{3} = u$, so that $\sin u = \frac{2}{3}$. Since arcsine is defined only in quadrants I and IV, and $\frac{2}{3}$ is positive, u is in quadrant I. Sketch u and label a triangle with the side adjacent u equal to $\sqrt{3^2 - 2^2} = \sqrt{9-4} = \sqrt{5}$.



$\cos\left(\arcsin\frac{2}{3}\right) = \cos u = \frac{\sqrt{5}}{3}$

(b) $\sin\left(2\cos^{-1}\frac{1}{3}\right)$

Let $\theta = \cos^{-1}\frac{1}{3}$, so that $\cos\theta = \frac{1}{3}$. Since arccosine is defined only in quadrants I and II, and $\frac{1}{3}$ is positive, θ is in quadrant I. Sketch θ and label a triangle with the side opposite to θ equal to $\theta = \sqrt{3^2 - (-1)^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}$.



Thus, $\sin\theta = \frac{2\sqrt{2}}{3}$ and

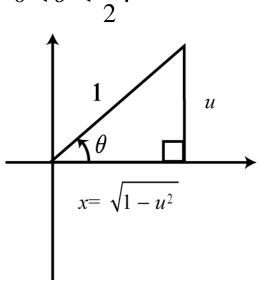
$\sin\left(2\cos^{-1}\frac{1}{3}\right) = \sin 2\theta$.

$$\sin\left(2\cos^{-1}\frac{1}{3}\right) = \sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right) = \frac{4\sqrt{2}}{9}$$

19. $\tan(\arcsin u)$

Let $\theta = \arcsin u$, so $\sin\theta = u = \frac{u}{1}$. If $u > 0$,

$0 < \theta < \frac{\pi}{2}$.



From the Pythagorean theorem,
$x = \sqrt{1^2 - u^2} = \sqrt{1-u^2}$. Therefore,

$\tan\theta = \frac{u}{\sqrt{1-u^2}} = \frac{u\sqrt{1-u^2}}{1-u^2}$. Thus,

$\tan(\arcsin u) = \frac{u\sqrt{1-u^2}}{1-u^2}$.

20. $-3\sec\theta + 2\sqrt{3} = 0 \Rightarrow \sec\theta = \frac{2\sqrt{3}}{3}$

Over the interval $[0, 360°)$, the equation

$\sec x = \frac{2\sqrt{3}}{3}$ has two solutions. One solution is in quadrant I and the other is in quadrant IV. These solutions are $30°$ and $330°$.

Solution set: $\{30°, 330°\}$

21.
$$\sin^2\theta = \cos^2\theta + 1$$
$$\sin^2\theta = 1 - \sin^2\theta + 1$$
$$2\sin^2\theta = 2 \Rightarrow \sin^2\theta = 1 \Rightarrow \sin\theta = \pm 1$$

Over the interval $[0, 360°)$, the equation $\sin\theta = 1$ has one solution, 90º. Over the interval $[0, 2\pi)$, the equation $\sin\theta = -1$ has one solution, 270º. Solution set: $\{90°, 270°\}$

22.
$$\csc^2\theta - 2\cot\theta = 4$$
$$1 + \cot^2\theta - 2\cot\theta = 4$$
$$\cot^2\theta - 2\cot\theta - 3 = 0$$
$$(\cot\theta - 3)(\cot\theta + 1) = 0 \Rightarrow$$
$$\cot\theta = 3 \text{ or } \cot\theta = -1$$

Over the interval $[0, 360°)$, the equation $\cot\theta = 3$ has two solutions, 18.4º and 198.4º (found using a calculator.) Over the interval $[0, 2\pi)$, the equation $\cot\theta = -1$ has two solutions, 135º and 315º.
Solution set: $\{18.4°, 135°, 198.4°, 315°\}$

23.
$$\cos x = \cos 2x$$
$$\cos x = 2\cos^2 x - 1$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0 \Rightarrow$$
$$\cos x = -\frac{1}{2} \text{ or } \cos x = 1$$

Over the interval $[0, 2\pi)$, the equation $\cos x = -\frac{1}{2}$ has two solutions, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.
Over the interval $[0, 2\pi)$, the equation $\cos x = 1$ has one solution, 0.

Solution set: $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

24. $\sqrt{2}\cos 3x - 1 = 0 \Rightarrow \cos 3x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Since $0 \le x < 2\pi, 0 \le 3x < 6\pi$. Thus

$$3x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4} \Rightarrow$$
$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

Solution set: $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}$

25. $\sin x \cos x = \frac{1}{3} \Rightarrow 2\sin x \cos x = \frac{2}{3} \Rightarrow$

$\sin 2x = \frac{2}{3}$

Since $0 \le x < 2\pi, 0 \le 2x < 4\pi$. Use a calculator to find $2x$:

$2x \approx .72672, 2.4118, 7.0129, 8.69505 \Rightarrow$

$x \approx .3649, 1.2059, 3.5605, 4.3475$

Solution set: $\{.3649, 1.2059, 3.5605, 4.3475\}$

26. $\sin^2 \theta = -\cos 2\theta$

$\sin^2 \theta = -\left(1 - 2\sin^2 \theta\right)$

$\sin^2 \theta = 1 \Rightarrow \sin \theta = \pm 1$

Over the interval $[0, 360°)$, the equation $\sin \theta = 1$ has one solution, 90º. Over the interval $[0, 360°)$, the equation $\sin \theta = -1$ has one solution, 270º. Since 270º = 90º + 180º, the solution set is {90º + 180ºn, where n is any integer}.

27. $2\sqrt{3}\sin \frac{x}{2} = 3 \Rightarrow \sin \frac{x}{2} = \frac{3}{2\sqrt{3}} \Rightarrow$

$\frac{x}{2} = \sin^{-1} \frac{3}{2\sqrt{3}} \Rightarrow \frac{x}{2} = \sin^{-1} \frac{\sqrt{3}}{2}$

Since $0 \le x < 2\pi \Rightarrow 0° \le \frac{x}{2} < \pi$, we have

$\frac{x}{2} = \frac{\pi}{3}, \frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Solution set:

$\left\{\frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n,\right.$

where n is any integer$\}$

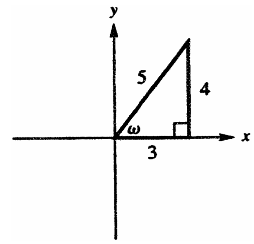
28. $y = \cos 3x \Rightarrow 3x = \arccos y \Rightarrow$

$x = \frac{1}{3}\arccos y$

29. $\arcsin x = \arctan \frac{4}{3}$

Let $\omega = \arctan \frac{4}{3}$. Then $\tan \omega = \frac{4}{3}$. Sketch ω in quadrant I and label a triangle with the hypotenuse equal to

$\sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5.$



Thus, we have

$\arcsin x = \arctan \frac{4}{3} \Rightarrow \arcsin x = \omega \Rightarrow$

$x = \sin \omega = \frac{4}{5}$

30. $y = \frac{\pi}{8}\cos\left[\pi\left(t - \frac{1}{3}\right)\right]$

Solve $0 = \frac{\pi}{8}\cos\left[\pi\left(t - \frac{1}{3}\right)\right]$.

$0 = \frac{\pi}{8}\cos\left[\pi\left(t - \frac{1}{3}\right)\right] \Rightarrow 0 = \cos\left[\pi\left(t - \frac{1}{3}\right)\right]$

$\pi\left(t - \frac{1}{3}\right) = \arccos 0 \Rightarrow \pi\left(t - \frac{1}{3}\right) = \frac{\pi}{2} + n\pi$

$t - \frac{1}{3} = \frac{1}{2} + n \Rightarrow t = \frac{1}{2} + \frac{1}{3} + n \Rightarrow t = \frac{5}{6} + n,$

where n is any integer

In the interval $[0, \pi)$, n = 0, 1, and 2 provide valid values for t. Thus, we have

$t = \frac{5}{6}, \frac{5}{6} + 1, \frac{5}{6} + 2 \Rightarrow t = \frac{5}{6}\text{sec}, \frac{11}{6}\text{sec}, \frac{17}{6}\text{sec}$

Chapter 7: Quantitative Reasoning

$$y' = r\cos(\theta + R) = r[\cos\theta\cos R - \sin\theta\sin R]$$
$$= (r\cos\theta)\cos R - (r\sin\theta)\sin R = y\cos R - z\sin R$$

$$z' = r\sin(\theta + R) = r\cos\left(\frac{\pi}{2} - (\theta + R)\right)$$
$$= r\cos\left(\left(\frac{\pi}{2} - \theta\right) - R\right)$$
$$= r\left[\cos\left(\frac{\pi}{2} - \theta\right)\cos R + \sin\left(\frac{\pi}{2} - \theta\right)\sin R\right]$$
$$= r[\sin\theta\cos R + \cos\theta\sin R]$$
$$= (r\sin\theta)\cos R + (r\cos\theta)\sin R = z\cos R + y\sin R$$

Chapter 8

APPLICATIONS OF TRIGONOMETRY

Section 8.1: The Law of Sines

1. A: $\frac{a}{b} = \frac{\sin A}{\sin B}$ can be rewritten as $\frac{a}{\sin A} = \frac{b}{\sin B}$. This is a valid proportion.

B: $\frac{a}{\sin A} = \frac{b}{\sin B}$ is a valid proportion.

C: $\frac{\sin A}{a} = \frac{b}{\sin B}$ cannot be rewritten as $\frac{a}{\sin A} = \frac{b}{\sin B}$. $\frac{\sin A}{a} = \frac{b}{\sin B}$ is not a valid proportion.

D: $\frac{\sin A}{a} = \frac{\sin B}{b}$ is a valid proportion.

2. A: With two angles and the side included between them, we could have

$$\underbrace{\frac{a}{\sin A}}=\underbrace{\frac{b}{\sin B}}=\frac{\overbrace{c}}{\sin C}.$$ If you know the measure of angles A and B, you can determine the measure of angle C. This provides enough information to solve the triangle.

B: With two angles and the side opposite one of them, we could have

$$\underbrace{\frac{a}{\sin A}}=\underbrace{\frac{\overbrace{b}}{\sin B}}=\frac{c}{\sin C}.$$ If you know the measure of angles A and B, you can determine the measure of angle C. This provides enough information to solve the triangle.

C: With two sides and the angle included between them, we could have

$$\underbrace{\frac{a}{\sin A}}=\frac{\overbrace{b}}{\sin B}=\frac{\overbrace{c}}{\sin C}.$$ This does not provide enough information to solve the triangle.

D: With three sides we have,

$$\frac{\overbrace{a}}{\sin A}=\frac{\overbrace{b}}{\sin B}=\frac{\overbrace{c}}{\sin C}.$$ This does not provide enough information to solve the triangle.

3. The measure of angle C is $180° - (60° + 75°) = 180° - 135° = 45°$.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 60°} = \frac{\sqrt{2}}{\sin 45°} \Rightarrow$$

$$a = \frac{\sqrt{2}\sin 60°}{\sin 45°} = \frac{\sqrt{2}\cdot\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \sqrt{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{2}} = \sqrt{3}$$

4. The measure of angle B is $180° - (45° + 105°) = 180° - 150° = 30°$.

$$\frac{a}{\sin A} = \frac{b}{\sin b} \Rightarrow \frac{a}{\sin 45°} = \frac{10}{\sin 30°} \Rightarrow$$

$$a = \frac{10\sin 45°}{\sin 30°} = \frac{10\cdot\frac{\sqrt{2}}{2}}{\frac{1}{2}}$$

$$= 10\cdot\frac{\sqrt{2}}{2}\cdot\frac{2}{1} = 10\sqrt{2}$$

5. $A = 37°,\ B = 48°,\ c = 18$ m

$C = 180° - A - B \Rightarrow$
$C = 180° - 37° - 48° = 95°$

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 48°} = \frac{18}{\sin 95°} \Rightarrow$$

$$b = \frac{18\sin 48°}{\sin 95°} \approx 13 \text{ m}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 37°} = \frac{18}{\sin 95°} \Rightarrow$$

$$a = \frac{18\sin 37°}{\sin 95°} \approx 11 \text{ m}$$

6. $B = 52°,\ C = 29°,\ a = 43$ cm

$A = 180° - B - C \Rightarrow$
$A = 180° - 52° - 29° = 99°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{43}{\sin 99°} = \frac{b}{\sin 52°} \Rightarrow$$
$$b = \frac{43 \sin 52°}{\sin 99°} \approx 34 \text{ cm}$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{43}{\sin 99°} = \frac{c}{\sin 29°} \Rightarrow$$
$$c = \frac{43 \sin 29°}{\sin 99°} \approx 21 \text{ cm}$$

7. $A = 27.2°,\ C = 115.5°,\ c = 76.0$ ft

$B = 180° - A - C \Rightarrow$
$B = 180° - 27.2° - 115.5° = 37.3°$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 27.2°} = \frac{76.0}{\sin 115.5°} \Rightarrow$$
$$a = \frac{76.0 \sin 27.2°}{\sin 115.5°} \approx 38.5 \text{ ft}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 37.3°} = \frac{76.0}{\sin 115.5°} \Rightarrow$$
$$b = \frac{76.0 \sin 37.3°}{\sin 115.5°} \approx 51.0 \text{ ft}$$

8. $C = 124.1°,\ B = 18.7°,\ b = 94.6$ m

$A = 180° - B - C \Rightarrow$
$A = 180° - 18.7° - 124.1° = 37.2°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 37.2°} = \frac{94.6}{\sin 18.7°} \Rightarrow$$
$$a = \frac{94.6 \sin 37.2°}{\sin 18.7°} \approx 178 \text{ m}$$
$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{c}{\sin 124.1°} = \frac{94.6}{\sin 18.7°} \Rightarrow$$
$$c = \frac{94.6 \sin 124.1°}{\sin 18.7°} \approx 244 \text{ m}$$

9. $A = 68.41°,\ B = 54.23°,\ a = 12.75$ ft

$C = 180° - A - B - C$
$= 180° - 68.41° - 54.23° = 57.36°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{12.75}{\sin 68.41°} = \frac{b}{\sin 54.23°} \Rightarrow$$
$$b = \frac{12.75 \sin 54.23°}{\sin 68.41°} \approx 11.13 \text{ ft}$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{12.75}{\sin 68.41°} = \frac{c}{\sin 57.36°} \Rightarrow$$
$$c = \frac{12.75 \sin 57.36°}{\sin 68.41°} \approx 11.55 \text{ ft}$$

10. $C = 74.08°,\ B = 69.38°,\ c = 45.38$ m

$A = 180° - B - C \Rightarrow$
$A = 180° - 69.38° - 74.08° = 36.54°$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 36.54°} = \frac{45.38}{\sin 74.08°} \Rightarrow$$
$$a = \frac{45.38 \sin 36.54°}{\sin 74.08°} \approx 28.10 \text{ m}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 69.38°} = \frac{45.38}{\sin 74.08°} \Rightarrow$$
$$b = \frac{45.38 \sin 69.38°}{\sin 74.08°} \approx 44.17 \text{ m}$$

11. $A = 87.2°, b = 75.9 \text{ yd}, C = 74.3°$

$B = 180° - A - C \Rightarrow$
$B = 180° - 87.2° - 74.3° = 18.5°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 87.2°} = \frac{75.9}{\sin 18.5°} \Rightarrow$$
$$a = \frac{75.9 \sin 87.2°}{\sin 18.5°} \approx 239 \text{ yd}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{75.9}{\sin 18.5°} = \frac{c}{\sin 74.3°} \Rightarrow$$
$$c = \frac{75.9 \sin 74.3°}{\sin 18.5°} \approx 230 \text{ yd}$$

12. $B = 38°40', a = 19.7 \text{ cm}, C = 91°40'$

$A = 180° - B - C \Rightarrow$
$A = 180° - 38°40' - 91°40' = 49°40'$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{19.7}{\sin 49°40'} = \frac{b}{\sin 38°40'} \Rightarrow$$
$$b = \frac{19.7 \sin 38°40'}{\sin 49°40'} \approx 16.1 \text{ cm}$$
$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 91°40'} = \frac{19.7}{\sin 49°40'} \Rightarrow$$
$$c = \frac{19.7 \sin 91°40'}{\sin 49°40'} \approx 25.8 \text{ cm}$$

13. $B = 20°50',\ AC = 132$ ft, $C = 103°10'$

$A = 180° - B - C$
$A = 180° - 20°50' - 103°10' \Rightarrow A = 56°00'$

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} \Rightarrow \frac{132}{\sin 20°50'} = \frac{AB}{\sin 103°10'} \Rightarrow$$
$$AB = \frac{132 \sin 103°10'}{\sin 20°50'} \approx 361 \text{ ft}$$
$$\frac{BC}{\sin A} = \frac{AC}{\sin B} \Rightarrow \frac{BC}{\sin 56°00'} = \frac{132}{\sin 20°50'} \Rightarrow$$
$$BC = \frac{132 \sin 56°00'}{\sin 20°50'} \approx 308 \text{ ft}$$

14. $A = 35.3°,\ B = 52.8°,\ AC = 675$ ft

$C = 180° - A - B \Rightarrow$
$C = 180° - 35.3° - 52.8° = 91.9°$

$$\frac{BC}{\sin A} = \frac{AC}{\sin B} \Rightarrow \frac{BC}{\sin 35.3°} = \frac{675}{\sin 52.8°} \Rightarrow$$
$$BC = \frac{675 \sin 35.3°}{\sin 52.8°} \approx 490 \text{ ft}$$
$$\frac{AB}{\sin C} = \frac{AC}{\sin B} \Rightarrow \frac{AB}{\sin 91.9°} = \frac{675}{\sin 52.8°} \Rightarrow$$
$$AB = \frac{675 \sin 91.9°}{\sin 52.8°} \approx 847 \text{ ft}$$

15. $A = 39.70°, C = 30.35°,\ b = 39.74$ m

$B = 180° - A - C \Rightarrow$
$B = 180° - 39.70° - 30.35° \Rightarrow$
$B = 109.95° \approx 110.0°$ (rounded)

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 39.70°} = \frac{39.74}{\sin 109.95°} \Rightarrow$$
$$a = \frac{39.74 \sin 39.70°}{\sin 110.0°} \approx 27.01 \text{ m}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{39.74}{\sin 109.95°} = \frac{c}{\sin 30.35°} \Rightarrow$$
$$c = \frac{39.74 \sin 30.35°}{\sin 110.0°} \approx 21.37 \text{ m}$$

16. $C = 71.83°,\ B = 42.57°,\ a = 2.614$ cm

$A = 180° - B - C \Rightarrow$
$A = 180° - 42.57° - 71.83° = 65.60°$

$$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{b}{\sin 42.57°} = \frac{2.614}{\sin 65.60°} \Rightarrow$$
$$b = \frac{2.614 \sin 42.57°}{\sin 65.60°} \approx 1.942 \text{ cm}$$
$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 71.83°} = \frac{2.614}{\sin 65.60°} \Rightarrow$$
$$c = \frac{2.614 \sin 71.83°}{\sin 65.60°} \approx 2.727 \text{ cm}$$

17. $B = 42.88°,\ C = 102.40°,\ b = 3974$ ft

$A = 180° - B - C \Rightarrow$
$A = 180° - 42.88° - 102.40° = 34.72°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 34.72°} = \frac{3974}{\sin 42.88°} \Rightarrow$$
$$a = \frac{3974 \sin 34.72°}{\sin 42.88°} \approx 3326 \text{ ft}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{3974}{\sin 42.88°} = \frac{c}{\sin 102.40°} \Rightarrow$$
$$c = \frac{3974 \sin 102.40°}{\sin 42.88°} \approx 5704 \text{ ft}$$

18. $A = 18.75°,\ B = 51.53°,\ c = 2798$ yd

$C = 180° - A - B \Rightarrow$
$C = 180° - 18.75° - 51.53° = 109.72°$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 18.75°} = \frac{2798}{\sin 109.72°} \Rightarrow$$
$$a = \frac{2798 \sin 18.75°}{\sin 109.72°} \approx 955.4 \text{ yd}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 51.53°} = \frac{2798}{\sin 109.72°} \Rightarrow$$
$$b = \frac{2798 \sin 51.53°}{\sin 109.72°} \approx 2327 \text{ yd}$$

19. $A = 39°54', a = 268.7 \text{ m}, B = 42°32'$

$C = 180° - A - B \Rightarrow$
$C = 180° - 39°54' - 42°32' = 97°34'$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{268.7}{\sin 39°54'} = \frac{b}{\sin 42°32'} \Rightarrow$$
$$b = \frac{268.7 \sin 42°32'}{\sin 39°54'} \approx 283.2 \text{ m}$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{268.7}{\sin 39°54'} = \frac{c}{\sin 97°54'} \Rightarrow$$
$$c = \frac{268.7 \sin 97°54'}{\sin 39°54'} \approx 415.2 \text{ m}$$

20. $C = 79°18', c = 39.81 \text{ mm}, A = 32°57'$

$B = 180° - A - C \Rightarrow$
$B = 180° - 32°57' - 79°18' = 67°45'$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 32°57'} = \frac{39.81}{\sin 79°18'} \Rightarrow$$
$$b = \frac{39.81 \sin 32°57'}{\sin 79°18'} \approx 22.04 \text{ mm}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 67°45'} = \frac{39.81}{\sin 79°18'} \Rightarrow$$
$$c = \frac{39.81 \sin 67°45'}{\sin 79°18'} \approx 37.50 \text{ mm}$$

21. Having three angles will not yield a unique triangle. The correct choice is A.

22. Having three angles will not yield a unique triangle. So, choices A and C cannot be correct. A triangle can be uniquely determined by three sides, assuming that triangle exists. In the case of choice B, no such triangle can be created with lengths 3, 5, and 20. The correct choice is D.

23. The vertical distance from the point (3, 4) to the x-axis is 4.

(a) If h is more than 4, two triangles can be drawn. But h must be less than 5 for both triangles to be on the positive x-axis. So, $4 < h < 5$.

(b) If $h = 4$, then exactly one triangle is possible. If $h > 5$, then only one triangle is possible on the positive *x*-axis.

(c) If $h < 4$, then no triangle is possible, since the side of length h would not reach the *x*-axis.

24. **(a)** Since the side must be drawn to the positive *x*-axis, no value of h would produce two triangles.

(b) Since the distance from the point (–3, 4) to the origin is 5, any value of h greater than 5 would produce exactly one triangle.

(c) Likewise, any value of h less than or equal to 5 would produce no triangle.

25. $a = 50, b = 26, A = 95°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 95°}{50} = \frac{\sin B}{26} \Rightarrow$$

$$\sin B = \frac{50 \sin 95°}{26} \approx .51802124 \Rightarrow B \approx 31.2°$$

Another possible value for B is $180° - 21.2° = 148.8°$, but this is too large to be in a triangle that also has a 95° angle. Therefore, only one triangle is possible.

26. $b = 60, a = 82, B = 100°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{82} = \frac{\sin 100°}{60} \Rightarrow$$

$$\sin A = \frac{82 \sin 100°}{60} \approx 1.34590390$$

Since $\sin A > 1$ is impossible, no triangle is possible for the given parts.

27. $a = 31, b = 26, B = 48°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{31} = \frac{\sin 48°}{26} \Rightarrow$$

$$\sin A = \frac{31 \sin 48°}{26} \approx .88605729 \Rightarrow A \approx 62.4°$$

Another possible value for A is $180° - 62.4° = 117.6°$. Therefore, two triangles are possible.

28. $a = 35, b = 30, A = 40°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 40°}{35} = \frac{\sin B}{30} \Rightarrow$$

$$\sin B = \frac{30 \sin 40°}{35} \approx .55096081 \Rightarrow B \approx 33.4°$$

Another possible value for B is $180° - 33.4° = 146.6°$, but this is too large to be in a triangle that also has a 40° angle. Therefore, only one triangle is possible.

29. $a = 50, b = 61, A = 58°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 58°}{50} = \frac{\sin B}{61} \Rightarrow$$

$$\sin B = \frac{61 \sin 58°}{50} \approx 1.03461868$$

Since $\sin B > 1$ is impossible, no triangle is possible for the given parts.

30. $B = 54°, c = 28, b = 23$

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin 54°}{23} = \frac{\sin C}{28} \Rightarrow$$

$$\sin C = \frac{28 \sin 54°}{23} \approx .98489025 \Rightarrow C \approx 80°$$

Another possible value for C is $180° - 80° = 100°$. Therefore, two triangles are possible.

31. $a = \sqrt{6}, b = 2, A = 60°$

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{2} = \frac{\sin 60°}{\sqrt{6}} \Rightarrow$$

$$\sin B = \frac{2 \sin 60°}{\sqrt{6}} = \frac{2 \cdot \frac{\sqrt{3}}{2}}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{6}} = \sqrt{\frac{3}{6}}$$

$$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{2}$$

$$\sin B = \frac{\sqrt{2}}{2} \Rightarrow B = 45°$$

There is another angle between $0°$ and $180°$ whose sine is $\frac{\sqrt{2}}{2}$: $180° - 45° = 135°$. However, this is too large because $A = 60°$ and $60° + 135° = 195° > 180°$, so there is only one solution, $B = 45°$.

32. $A = 45°,\ a = 3\sqrt{2},\ b = 3$

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{3} = \frac{\sin 45°}{3\sqrt{2}} \Rightarrow$$

$$\sin B = \frac{3 \sin 45°}{3\sqrt{2}} = \frac{3 \cdot \frac{\sqrt{2}}{2}}{3\sqrt{2}} = \frac{1}{2} \Rightarrow B = 30°$$

There is another angle between $0°$ and $180°$ whose sine is $\frac{1}{2}$: $180° - 30° = 150°$. However, this is too large because $A = 45°$ and $45° + 150° = 195° > 180°$, so there is only one solution, $B = 30°$.

33. $A = 29.7°$, $b = 41.5$ ft, $a = 27.2$ ft

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{41.5} = \frac{\sin 29.7°}{27.2} \Rightarrow$$

$$\sin B = \frac{41.5 \sin 29.7°}{27.2} \approx .75593878$$

There are two angles B between $0°$ and $180°$ that satisfy the condition. Since $\sin B \approx .75593878$, to the nearest tenth value of B is $B_1 = 49.1°$. Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 49.1° = 130.9°$. This is a valid angle measure for this triangle since $A + B_2 = 29.7° + 130.9° = 160.6° < 180°$.

Solving separately for angles C_1 and C_2 we have the following.

$$C_1 = 180° - A - B_1 = 180° - 29.7° - 49.1° = 101.2°$$

$$C_2 = 180° - A - B_2 = 180° - 29.7° - 130.9° = 19.4°$$

34. $B = 48.2°$, $a = 890$ cm, $b = 697$ cm

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{890} = \frac{\sin 48.2°}{697} \Rightarrow$$

$$\sin A = \frac{890 \sin 48.2°}{697} = .95189905$$

There are two angles A between $0°$ and $180°$ that satisfy the condition. Since $\sin A \approx .95189905$, to the nearest tenth value of A is $A_1 = 72.2°$. Supplementary angles have the same sine value, so another possible value of A is $A_2 = 180° - 72.2° = 107.8°$. This is a valid angle measure for this triangle since $B + A_2 = 48.2° + 107.8° = 156.0° < 180°$.

Solving separately for angles C_1 and C_2 we have the following.

$$C_1 = 180° - A_1 - B = 180° - 72.2° - 48.2° = 59.6°$$

$$C_2 = 180° - A_2 - B = 180° - 107.8° - 48.2° = 24.0°$$

35. $C = 41°20'$, $b = 25.9$ m, $c = 38.4$ m

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{25.9} = \frac{\sin 41°20'}{38.4} \Rightarrow$$

$$\sin B = \frac{25.9 \sin 41°20'}{38.4} \approx .44545209$$

There are two angles B between 0° and 180° that satisfy the condition. Since $\sin B \approx .44545209$, $B_1 \approx 26.5° = 26°30'$.

Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 26°30' = 153°30'$. This is not a valid angle measure for this triangle because $C + B_2 = 41°20' + 153°30' = 194°30' > 180°$.

Thus, $A = 180° - 26°30' - 41°20' = 112°10'$.

36. $B = 48°50'$, $a = 3850$ in., $b = 4730$ in.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{3850} = \frac{\sin 48°50'}{4730} \Rightarrow$$

$$\sin A = \frac{3850 \sin 48°50'}{4730} \approx .61274255$$

There are two angles A between 0° and 180° that satisfy the condition. Since $\sin A \approx .61274255$, $A_1 \approx 37.8° = 37°50'$.

Supplementary angles have the same sine value, so another possible value of A is $A_2 = 180° - 37°50' = 142°10'$. This is not a valid angle measure for this triangle because $B + A_2 = 48°50' + 142°10' = 191° > 180°$.

Thus, $C = 180° - 37°50' - 48°50' = 93°20'$.

37. $B = 74.3°$, $a = 859$ m, $b = 783$ m

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{859} = \frac{\sin 74.3°}{783} \Rightarrow$$

$$\sin A = \frac{859 \sin 74.3°}{783} \approx 1.0561331$$

Since $\sin A > 1$ is impossible, no such triangle exists.

38. $C = 82.2°$, $a = 10.9$ km, $c = 7.62$ km

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{10.9} = \frac{\sin 82.2°}{7.62} \Rightarrow$$

$$\sin A = \frac{10.9 \sin 82.2°}{7.62} \approx 1.4172115$$

Since $\sin A > 1$ is impossible, no such triangle exists.

39. $A = 142.13°$, $b = 5.432$ ft, $a = 7.297$ ft

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{b \sin A}{a} \Rightarrow$$
$$\sin B = \frac{5.432 \sin 142.13°}{7.297} \approx .45697580 \Rightarrow$$
$$B \approx 27.19°$$

Because angle A is obtuse, angle B must be acute, so this is the only possible value for B and there is one triangle with the given measurements.

$$C = 180° - A - B$$
$$= 180° - 142.13° - 27.19° = 10.68°$$

Thus, $B \approx 27.19°$ and $C \approx 10.68°$.

40. $B = 113.72°$, $a = 189.6$ yd, $b = 243.8$ yd

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{189.6} = \frac{\sin 113.72°}{243.8} \Rightarrow$$
$$\sin A = \frac{189.6 \sin 113.72°}{243.8} \approx .71198940$$

Because angle B is obtuse, angle A must be acute, so this is the only possible value for A and there is one triangle with the given measurements.

$$C = 180° - A - B$$
$$= 180° - 45.40° - 113.72° = 20.88°$$

Thus, $A = 45.40°$ and $C = 20.88°$.

41. $A = 42.5°$, $a = 15.6$ ft, $b = 8.14$ ft

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{8.14} = \frac{\sin 42.5°}{15.6} \Rightarrow$$
$$\sin B = \frac{8.14 \sin 42.5°}{15.6} \approx .35251951$$

There are two angles B between $0°$ and $180°$ that satisfy the condition. Since $\sin B \approx .35251951$, to the nearest tenth value of B is $B_1 = 20.6°$. Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 20.6° = 159.4°$. This is not a valid angle measure for this triangle since $A + B_2 = 42.5° + 159.4° = 201.9° > 180°$.

Thus, $C = 180° - 42.5° - 20.6° = 116.9°$.

Solving for c, we have the following.

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 116.9°} = \frac{15.6}{\sin 42.5°} \Rightarrow$$
$$c = \frac{15.6 \sin 116.9°}{\sin 42.5°} \approx 20.6 \text{ ft}$$

42. $C = 52.3°$, $a = 32.5$ yd, $c = 59.8$ yd

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{32.5} = \frac{\sin 52.3°}{59.8} \Rightarrow$$
$$\sin A = \frac{32.5 \sin 52.3°}{59.8} \approx .43001279$$

There are two angles A between $0°$ and $180°$ that satisfy the condition. Since $\sin A \approx .43001279$, to the nearest tenth value of A is $A_1 = 25.5°$. Supplementary angles have the same sine value, so another possible value of A is $A_2 = 180° - 25.5° = 154.5°$. This is not a valid angle measure for this triangle since $C + A_2 = 52.3° + 154.5° = 206.8° > 180°$.

Thus, $B = 180° - 52.3° - 25.5° = 102.2°$.

Solving for b, we have the following.

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 102.2°} = \frac{59.8}{\sin 52.3°} \Rightarrow$$
$$b = \frac{59.8 \sin 102.2°}{\sin 52.3°} \approx 73.9 \text{ yd}$$

43. $B = 72.2°$, $b = 78.3$ m, $c = 145$ m

$$\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \frac{\sin C}{145} = \frac{\sin 72.2°}{78.3} \Rightarrow$$
$$\sin C = \frac{145 \sin 72.2°}{78.3} \approx 1.7632026$$

Since $\sin C > 1$ is impossible, no such triangle exists.

44. $C = 68.5°$, $c = 258$ cm, $b = 386$ cm

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{386} = \frac{\sin 68.5°}{258} \Rightarrow$$
$$\sin B = \frac{386 \sin 68.5°}{258} \approx 1.39202008$$

Since $\sin B > 1$ is impossible, no such triangle exists.

45. $A = 38°40'$, $a = 9.72$ km, $b = 11.8$ km

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{11.8} = \frac{\sin 38°40'}{9.72} \Rightarrow$$
$$\sin B = \frac{11.8 \sin 38°40'}{9.72} \approx .75848811$$

There are two angles B between $0°$ and $180°$ that satisfy the condition. Since $\sin B \approx .75848811$, to the nearest tenth value of B is $B_1 = 49.3° \approx 49°20'$. Supplementary angles have the same sine value, so another possible value of B is

$$B_2 = 180° - 49°20'$$
$$= 179°60' - 49°20' = 130°40'$$

This is a valid angle measure for this triangle since

$$A + B_2 = 38°40' + 130°40' = 169°20' < 180°.$$

Solving separately for triangles AB_1C_1 and AB_2C_2 we have the following.

AB_1C_1:

$$C_1 = 180° - A - B_1 = 180° - 38°40' - 49°20' = 180° - 88°00' = 92°00'$$

$$\frac{c_1}{\sin C_1} = \frac{a}{\sin A} \Rightarrow \frac{c_1}{\sin 92°00'} = \frac{9.72}{\sin 38°40'} \Rightarrow$$

$$c_1 = \frac{9.72 \sin 92°00'}{\sin 38°40'} \approx 15.5 \text{ km}$$

AB_2C_2:

$$C_2 = 180° - A - B_2 = 180° - 38°40' - 130°40' = 10°40'$$

$$\frac{c_2}{\sin C_2} = \frac{a}{\sin A} \Rightarrow \frac{c_2}{\sin 10°40'} = \frac{9.72}{\sin 38°40'} \Rightarrow$$

$$c_2 = \frac{9.72 \sin 10°40'}{\sin 38°40'} \approx 2.88 \text{ km}$$

46. $C = 29°50'$, $a = 8.61$ m, $c = 5.21$ m

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{8.61} = \frac{\sin 29°50'}{5.21} \Rightarrow$$

$$\sin A = \frac{8.61 \sin 29°50'}{5.21} \approx .82212894$$

There are two angles A between $0°$ and $180°$ that satisfy the condition. Since $\sin A \approx .82212894$, to the nearest tenth value of A is $A_1 = 55.3° \approx 55°20'$. Supplementary angles have the same sine value, so another possible value of A is

$$A_2 = 180° - 55°20' = 179°60' - 55°20' = 124°40'$$

This is a valid angle measure for this triangle since

$$C + A_2 = 29°50' + 124°40' = 154°10' < 180°.$$

Solving separately for triangles A_1B_1C and A_2B_2C, we have the following.

A_1B_1C:

$$B_1 = 180° - C - A_1 = 180° - 55°20' - 29°50' = 179°60' - 85°10' = 94°50'$$

$$\frac{b_1}{\sin B_1} = \frac{c}{\sin C} \Rightarrow \frac{b_1}{\sin 94°50'} = \frac{5.21}{\sin 29°50'} \Rightarrow$$

$$b_1 = \frac{5.21 \sin 94°50'}{\sin 29°50'} \approx 10.4 \text{ m}$$

A_2B_2C:

$$B_2 = 180° - C - A_2 = 180° - 124°40' - 29°50' = 179°60' - 154°30' = 25°30'$$

$$\frac{b_2}{\sin B_2} = \frac{c}{\sin C} \Rightarrow \frac{b_2}{\sin 25°30'} = \frac{5.21}{\sin 29°50'} \Rightarrow$$

$$b_2 = \frac{5.21 \sin 25°30'}{\sin 29°50'} \approx 4.51 \text{ m}$$

47. $A = 96.80°$, $b = 3.589$ ft, $a = 5.818$ ft

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{3.589} = \frac{\sin 96.80°}{5.818} \Rightarrow$$

$$\sin B = \frac{3.589 \sin 96.80°}{5.818} \approx .61253922$$

There are two angles B between 0° and 180° that satisfy the condition. Since $\sin B \approx .61253922$, $B_1 \approx 37.77°$.

Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 37.77° = 142.23°$. This is not a valid angle measure for this triangle since $A + B_2 = 96.80° + 142.23° = 239.03° > 180°$.

Thus $C = 180° - 96.80° - 37.77° = 45.43°$.

Solving for c, we have the following.

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 45.43°} = \frac{5.8518}{\sin 96.80°} \Rightarrow$$

$$b = \frac{5.8518 \sin 45.43°}{\sin 96.80°} \approx 4.174 \text{ ft}$$

48. $C = 88.70°$, $b = 56.87$ yd, $c = 112.4$ yd

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{56.87} = \frac{\sin 88.70°}{112.4} \Rightarrow$$

$$\sin B = \frac{56.87 \sin 88.70°}{112.4} \approx .50583062$$

There are two angles B between 0° and 180° that satisfy the condition. Since $\sin B \approx .50583062$, $B_1 \approx 30.39°$.

Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 30.39° = 149.61°$. This is not a valid angle measure for this triangle since $C + B_2 = 96.80° + 149.61° = 238.31° > 180°$.

Thus $A = 180° - 88.70° - 30.39° = 60.91°$.

Solving for a, we have the following.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 60.91°} = \frac{112.4}{\sin 88.70°} \Rightarrow$$

$$a = \frac{112.4 \sin 60.91°}{\sin 88.70°} \approx 98.25 \text{ yd}$$

49. $B = 39.68°$, $a = 29.81$ m, $b = 23.76$ m

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{29.81} = \frac{\sin 39.68°}{23.76} \Rightarrow$$

$$\sin A = \frac{29.81 \sin 39.68°}{23.76} \approx .80108002$$

There are two angles A between $0°$ and $180°$ that satisfy the condition.

(*continued on next page*)

(*continued from page 745*)

Since $\sin A \approx .80108002,$ to the nearest hundredth value of A is $A_1 = 53.23°.$ Supplementary angles have the same sine value, so another possible value of A is $A_2 = 180° - 53.23° = 126.77°.$ This is a valid angle measure for this triangle since $B + A_2 = 39.68° + 126.77° = 166.45° < 180°.$ Solving separately for triangles A_1BC_1 and A_2BC_2 we have the following.

A_1BC_1:

$$C_1 = 180° - A_1 - B = 180° - 53.23° - 39.68° = 87.09°$$

$$\frac{c_1}{\sin C_1} = \frac{b}{\sin B} \Rightarrow \frac{c_1}{\sin 87.09°} = \frac{23.76}{\sin 39.68°} \Rightarrow$$

$$c_1 = \frac{23.76 \sin 87.09°}{\sin 39.68°} \approx 37.16 \text{ m}$$

A_2BC_2:

$$C_2 = 180° - A_2 - B = 180° - 126.77° - 39.68° = 13.55°$$

$$\frac{c_2}{\sin C_2} = \frac{b}{\sin B} \Rightarrow \frac{c_2}{\sin 13.55°} = \frac{23.76}{\sin 39.68°} \Rightarrow$$

$$c_1 = \frac{23.76 \sin 13.55°}{\sin 39.68°} \approx 8.719 \text{ m}$$

50. $A = 51.20°$, $c = 7986$ cm, $a = 7208$ cm

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin C}{7986} = \frac{\sin 51.20°}{7208} \Rightarrow$$

$$\sin C = \frac{7986 \sin 51.20°}{7208} \approx .86345630$$

There are two angles C between $0°$ and $180°$ that satisfy the condition. Since $\sin C \approx .86345630,$ to the nearest tenth value of C is $C_1 = 59.71°.$ Supplementary angles have the same sine value, so another possible value of C is $C_2 = 180° - 59.71° = 120.29°.$ This is a valid angle measure for this triangle since $A + C_2 = 51.20° + 120.29° = 171.49° < 180°.$ Solving separately for triangles AB_1C_1 and AB_2C_2 we have the following.

AB_1C_1:

$$B_1 = 180° - C_1 - A = 180° - 59.71° - 51.20° = 69.09°$$

$$\frac{b_1}{\sin B_1} = \frac{a}{\sin A} \Rightarrow \frac{b_1}{\sin 69.09°} = \frac{7208}{\sin 51.20°} \Rightarrow$$

$$b_1 = \frac{7208 \sin 69.09°}{\sin 51.20°} \approx 8640 \text{ cm}$$

AB_2C_2:

$$B_2 = 180° - C_2 - A = 180° - 120.29° - 51.20° = 8.51°$$

$$\frac{b_2}{\sin B_2} = \frac{a}{\sin A} \Rightarrow \frac{b_2}{\sin 8.51°} = \frac{7208}{\sin 51.20°} \Rightarrow$$

$$b_2 = \frac{7208 \sin 8.51°}{\sin 51.20°} \approx 1369 \text{ cm}$$

51. $a = \sqrt{5}, c = 2\sqrt{5},\ A = 30°$

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin C}{2\sqrt{5}} = \frac{\sin 30°}{\sqrt{5}} \Rightarrow$$

$$\sin C = \frac{2\sqrt{5}\sin 30°}{\sqrt{5}} = \frac{2\sqrt{5}\cdot\frac{1}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \Rightarrow$$

$C = 90°$

This is a right triangle.

52.–54. Answers will vary.

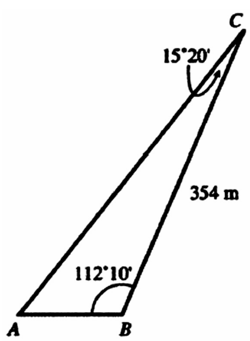
55. $B = 112°10'$; $C = 15°20'$; $BC = 354$ m

$$A = 180° - B - C$$
$$A = 180° - 112°10' - 15°20' = 179°60' - 127°30' = 52°30'$$

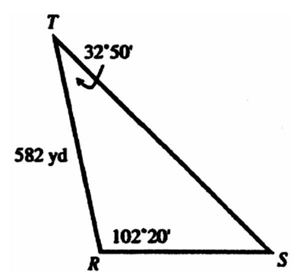
$$\frac{BC}{\sin A} = \frac{AB}{\sin C}$$

$$\frac{354}{\sin 52°30'} = \frac{AB}{\sin 15°20'}$$

$$AB = \frac{354 \sin 15°20'}{\sin 52°30'} \approx 118 \text{ m}$$



Exercise 55



Exercise 56

56. $T = 32°50'$; $R = 102°20'$; $TR = 582$ yd

$$S = 180° - 32°50' - 102°20' = 179°60' - 135°10' = 44°50'$$

$$\frac{RS}{\sin T} = \frac{TR}{\sin S}$$

$$\frac{RS}{\sin 32°50'} = \frac{582}{\sin 44°50'}$$

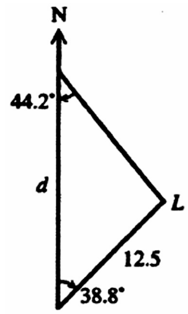
$$RS = \frac{582 \sin 32°50'}{\sin 44°50'} \approx 448 \text{ yd}$$

57. Let d = the distance the ship traveled between the two observations; L = the location of the lighthouse.

$L = 180° - 38.8° - 44.2° = 97.0°$

$$\frac{d}{\sin 97°} = \frac{12.5}{\sin 44.2°}$$

$$d = \frac{12.5 \sin 97°}{\sin 44.2°} \approx 17.8 \text{ km}$$



58. Let C = the transmitter. Since side AB is on an east-west line, the angle between it and any north-south line is 90°.

$A = 90° - 47.7° = 42.3°$

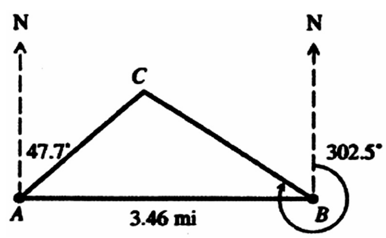
$B = 302.5° - 270° = 32.5°$

$C = 180° - A - B$

$\quad = 180° - 42.3° - 32.5° = 105.2°$

$$\frac{AC}{\sin 32.5°} = \frac{3.46}{\sin 105.2°}$$

$$AC = \frac{3.46 \sin 32.5°}{\sin 105.2°} \approx 1.93 \text{ mi}$$



59. $$\frac{x}{\sin 54.8°} = \frac{12.0}{\sin 70.4°}$$

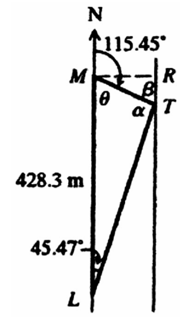
$$x = \frac{12.0 \sin 54.8°}{\sin 70.4°} \approx 10.4 \text{ in.}$$

60. Let M = Mark's location; L = Lisa's location; T = the tree's location; R = a point across the river from Mark so that MR is the distance across the river.

$\theta = 180° - 115.45° = 64.55°$

$\alpha = 180° - 45.47° - 64.55° = 69.98°$

$\beta = \theta = 64.55°$ (Alternate interior angles)



In triangle MTL,

$$\frac{MT}{\sin 45.47°} = \frac{428.3}{\sin 69.98°}$$

$$MT = \frac{428.3 \sin 45.47°}{\sin 69.98°} \approx 324.9645$$

In right triangle MTR,

$$\frac{MR}{MT} = \sin \beta \Rightarrow \frac{MR}{324.9645} = \sin 64.55°$$

$$MR = 324.9645 \sin 64.55° \approx 293.4 \text{ m}$$

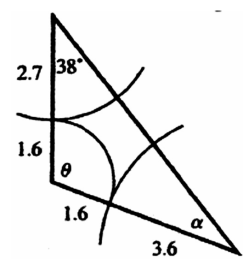
61. We cannot find θ directly because the length of the side opposite angle θ is not given. Redraw the triangle shown in the figure and label the third angle as α.

$$\frac{\sin \alpha}{1.6 + 2.7} = \frac{\sin 38°}{1.6 + 3.6}$$

$$\frac{\sin \alpha}{4.3} = \frac{\sin 38°}{5.2}$$

$$\sin \alpha = \frac{4.3 \sin 38°}{5.2} \approx .50910468$$

$$\alpha \approx \sin^{-1}(.50910468) \approx 31°$$



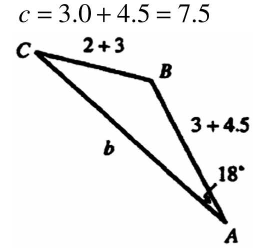
Thus,

$\theta = 180° - 38° - \alpha \approx 180° - 38° - 31° = 111°$

62. Label the centers of the atoms A, B, and C.

$a = 2.0 + 3.0 = 5.0$

$c = 3.0 + 4.5 = 7.5$



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$$\frac{\sin C}{c}=\frac{\sin A}{a}\Rightarrow\frac{\sin C}{7.5}=\frac{\sin 18°}{5}\Rightarrow$$

$$\sin C=\frac{7.5\sin 18°}{5}\Rightarrow \sin C\approx .46352549$$

$$C\approx \sin^{-1}(.46352549)\approx 28° \text{ and}$$

$$B\approx 180°-18°-28°=134°$$

$$\frac{b}{\sin B}=\frac{a}{\sin A}\Rightarrow\frac{b}{\sin 134°}=\frac{5}{\sin 18°}\Rightarrow$$

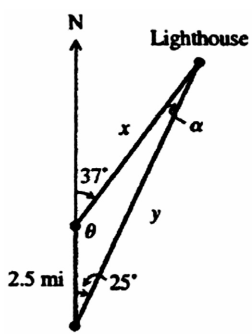
$$b=\frac{5.0\sin 134°}{\sin 18°}\approx 12$$

The distance between the centers of atoms A and C is 12.

63. Let x = the distance to the lighthouse at bearing N 37° E; y = the distance to the lighthouse at bearing N 25° E.

$$\theta=180°-37°=143°$$

$$\alpha=180°-\theta-25° =180°-143°-25°=12°$$

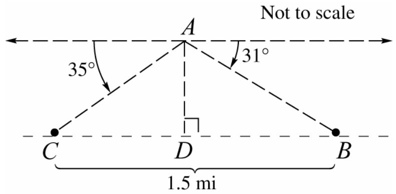


$$\frac{2.5}{\sin\alpha}=\frac{x}{\sin 25°}\Rightarrow x=\frac{2.5\sin 25°}{\sin 12°}\approx 5.1\text{ mi}$$

$$\frac{2.5}{\sin\alpha}=\frac{y}{\sin\theta}\Rightarrow\frac{2.5}{\sin 12°}=\frac{y}{\sin 143°}\Rightarrow$$

$$y=\frac{2.5\sin 143°}{\sin 12°}\approx 7.2\text{ mi}$$

64. Let A = the location of the balloon; B = the location of the farther town; C = the location of the closer town. Angle ABC = 31° and angle ACB = 35° because the angles of depression are alternate interior angles with the angles of the triangle.



Angle $BAC = 180° - 31° - 35° = 114°$

$$\frac{1.5}{\sin BAC}=\frac{AB}{\sin ACB}\Rightarrow\frac{1.5}{\sin 114°}=\frac{AB}{\sin 35°}\Rightarrow$$

$$AB=\frac{1.5\sin 35°}{\sin 114°}\approx .94178636$$

$$\sin ABC=\frac{AD}{AB}\Rightarrow \sin 31°=\frac{AD}{.94178636}\Rightarrow$$

$$AD=.94178636\cdot\sin 31°\approx .49$$

The balloon is .49 mi above the ground.

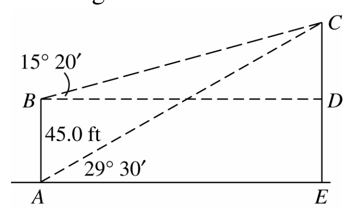
65. $Y = 43°30', Z = 95°30'$, $XY = 960$ m

We are looking for XZ.

$$\frac{XZ}{\sin Y}=\frac{XY}{\sin Z}\Rightarrow\frac{XZ}{\sin 43°30'}=\frac{960}{\sin 95°30'}\Rightarrow$$

$$XZ=\frac{960\sin 43°30'}{\sin 95°30'}\approx 664\text{ m}$$

66. The height of the tower = CE.



In triangle ABC,

$m\angle CAB = 90° - 29°30' = 60°30'$ and

$m\angle ABC = 90° + 15°20' = 105°20'$. Thus,

$m\angle ACB = 180° - 60°30' - 105°20' = 14°10'$.

This gives $\dfrac{AC}{\sin\angle CAB}=\dfrac{AB}{\sin\angle ACB}\Rightarrow$

$$\frac{AC}{\sin 105°20'}=\frac{45.0}{\sin 14°10'}\Rightarrow$$

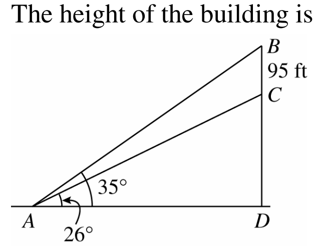
$$AC=\frac{45.0\sin 105°20'}{\sin 14°10'}\approx 177.3$$

In right triangle ACE,

$$\sin 29°30'=\frac{CE}{AC}\Rightarrow \sin 29°30'=\frac{CE}{177.3}\Rightarrow$$

$$CE=177.3\sin 29°30'\approx 87.3\text{ ft}$$

67. The height of the building is CD.



In right triangle *ABD,* we have $m\angle B = 90° - 35° = 55°$. In triangle *ABC*, we have $m\angle CAB = 35° - 26° = 9°$. This gives

$$\frac{BC}{\sin\angle CAB} = \frac{AC}{\sin\angle B} \Rightarrow \frac{95}{\sin 9°} = \frac{AC}{\sin 55°} \Rightarrow$$

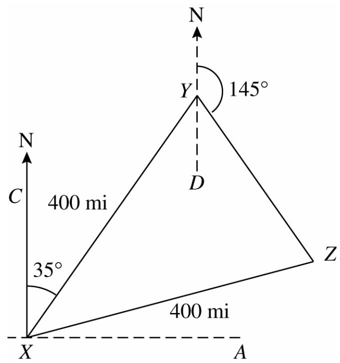
$$AC = \frac{95\sin 55°}{\sin 9°} \approx 497.5$$

In triangle *ACD*,

$$\sin\angle CAD = \frac{CD}{AC} \Rightarrow \sin 26° = \frac{CD}{497.5} \Rightarrow$$

$$CD = 497.5\sin 26° \approx 218 \text{ ft}$$

68. Recall that bearing is measured in a clockwise directed from due north. (See section 2.5)



$m\angle DYZ = 180° - 145° = 35°$ and $m\angle XYD = 35°$ (because $\angle YXC$ and $\angle XYD$ are alternate interior angles), so $m\angle XYZ = 35° + 35° = 70°$. Because $XY = XZ = 400$, $m\angle XYZ = m\angle Z \Rightarrow m\angle Z = 70°$. Thus, $m\angle CXZ = 180° - 70° - 70° = 40°$. So, the heading of *Z* from $X = m\angle CXZ = m\angle CXY + m\angle YXZ = 35° + 40° = 75°$.

$$\frac{YZ}{\sin\angle YXZ} = \frac{400}{\sin\angle Z} \Rightarrow YZ = \frac{400\sin\angle YXZ}{\sin\angle Z} \Rightarrow$$

$$YZ = \frac{400\sin\angle YXZ}{\sin\angle Z} = \frac{400\sin 40°}{\sin 70°} \approx 274 \text{ mi}$$

69. Angle *C* is equal to the difference between the angles of elevation.

$C = B - A = 52.7430° - 52.6997° = .0433°$

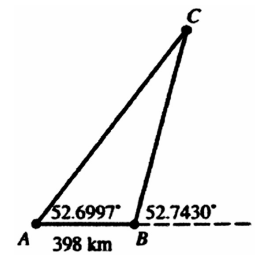
The distance *BC* to the moon can be determined using the law of sines.

$$\frac{BC}{\sin A} = \frac{AB}{\sin C}$$

$$\frac{BC}{\sin 52.6997°} = \frac{398}{\sin .0433°}$$

$$BC = \frac{398\sin 52.6997°}{\sin .0433°}$$

$$BC \approx 418{,}930 \text{ km}$$



If one finds distance *AC*, then we have

$$\frac{AC}{\sin B} = \frac{AB}{\sin C}$$

$$\frac{AC}{\sin(180° - 52.7430°)} = \frac{398}{\sin .0433°}$$

$$\frac{AC}{\sin 127.2570°} = \frac{398}{\sin .0433°}$$

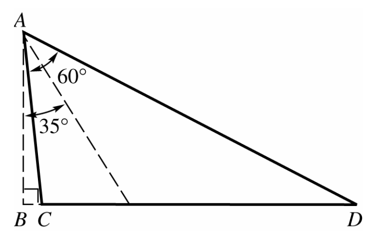
$$AC = \frac{398\sin 127.2570°}{\sin .0433°} \approx 419{,}171 \text{ km}$$

In either case the distance is approximately 419,000 km compared to the actual value of 406,000 km.

70. We must find the length of *CD*.

$m\angle BAC = 35° - 30° = 5°$ and $AB = 5000$ ft. Since triangle *ABC* is a right triangle,

$$\cos 5° = \frac{5000}{AC} \Rightarrow AC = \frac{5000}{\cos 5°} \approx 5019 \text{ ft}$$



The angular coverage of the lens is 60º, so $m\angle CAD = 60°$. From geometry, $m\angle ACB = 85°$, $m\angle ACD = 95°$, and $m\angle ADC = 25°$. We now know three angles and one side in triangle *ACD*, so we can use the law of sines to solve for *CD*.

$$\frac{CD}{\sin 60°} = \frac{AC}{\sin 25°} \Rightarrow \frac{CD}{\sin 60°} = \frac{5019}{\sin 25°} \Rightarrow$$

$$CD = \frac{5019\sin 60°}{\sin 25°} \approx 10{,}285 \text{ ft}$$

The photograph would cover a horizontal distance of approximately 10,285 ft.

71. To find the area of the triangle, use $\text{A} = \frac{1}{2}bh,$ with $b = 1$ and $h = \sqrt{3}.$ $\text{A} = \frac{1}{2}(1)\left(\sqrt{3}\right) = \frac{\sqrt{3}}{2}$

Now use $\text{A} = \frac{1}{2}ab\sin C,$ with $a = \sqrt{3},$ $b = 1,$ and $C = 90°$.

$$\text{A} = \frac{1}{2}\left(\sqrt{3}\right)(1)\sin 90° = \frac{1}{2}\left(\sqrt{3}\right)(1)(1)$$
$$= \frac{\sqrt{3}}{2} \text{ sq units}$$

72. To find the area of the triangle, use $\text{A} = \frac{1}{2}bh,$ with $b = 2$ and $h = \sqrt{3}.$ $\text{A} = \frac{1}{2}(2)\left(\sqrt{3}\right) = \sqrt{3}.$

Now use $\text{A} = \frac{1}{2}ab\sin C,$ with $a = 2,$ $b = 2,$ and $C = 60°$

$$\text{A} = \frac{1}{2}(2)(2)\sin 60° = \frac{1}{2}(2)(2)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \sqrt{3} \text{ sq units}$$

73. To find the area of the triangle, use $\text{A} = \frac{1}{2}bh,$ with $b = 1$ and $h = \sqrt{2}.$ $\text{A} = \frac{1}{2}(1)\left(\sqrt{2}\right) = \frac{\sqrt{2}}{2}$

Now use $\text{A} = \frac{1}{2}ab\sin C,$ with $a = 2,$ $b = 1,$ and $C = 45°$.

$$\text{A} = \frac{1}{2}(2)(1)\sin 45° = \frac{1}{2}(2)(1)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{2} \text{ sq unit}$$

74. To find the area of the triangle, use $\text{A} = \frac{1}{2}bh,$ with $b = 2$ and $h = 1.$ $\text{A} = \frac{1}{2}(2)(1) = 1$ Now use $\text{A} = \frac{1}{2}ab\sin C,$ with $a = 2,$ $b = \sqrt{2},$ and $C = 45°$.

$$\text{A} = \frac{1}{2}(2)\left(\sqrt{2}\right)\sin 45° = \frac{1}{2}(2)\left(\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= 1 \text{ sq unit}$$

75. $A = 42.5°$, $b = 13.6$ m, $c = 10.1$ m
Angle A is included between sides b and c. Thus, we have

$$\text{A} = \frac{1}{2}bc\sin A = \frac{1}{2}(13.6)(10.1)\sin 42.5°$$
$$\approx 46.4 \text{ m}^2$$

76. $C = 72.2°$, $b = 43.8$ ft, $a = 35.1$ ft
Angle C is included between sides a and b. Thus, we have

$$\text{A} = \frac{1}{2}ab\sin C = \frac{1}{2}(35.1)(43.8)\sin 72.2°$$
$$\approx 732 \text{ ft}^2$$

77. $B = 124.5°$, $a = 30.4$ cm, $c = 28.4$ cm
Angle B is included between sides a and c. Thus, we have

$$\text{A} = \frac{1}{2}ac\sin B = \frac{1}{2}(30.4)(28.4)\sin 124.5°$$
$$\approx 356 \text{ cm}^2$$

78. $C = 142.7°$, $a = 21.9$ km, $b = 24.6$ km
Angle C is included between sides a and b. Thus, we have

$$\text{A} = \frac{1}{2}ab\sin C = \frac{1}{2}(21.9)(24.6)\sin 142.7°$$
$$\approx 163 \text{ km}^2$$

79. $A = 56.80°$, $b = 32.67$ in., $c = 52.89$ in.
Angle A is included between sides b and c. Thus, we have

$$\text{A} = \frac{1}{2}bc\sin A = \frac{1}{2}(32.67)(52.89)\sin 56.80°$$
$$\approx 722.9 \text{ in.}^2$$

80. $A = 34.97°$, $b = 35.29$ m, $c = 28.67$ m
Angle A is included between sides b and c. Thus, we have

$$\text{A} = \frac{1}{2}bc\sin A = \frac{1}{2}(35.29)(28.67)\sin 34.97°$$
$$\approx 289.9 \text{ m}^2$$

81. $A = 30.50°,$ $b = 13.00$ cm, $C = 112.60°$
In order to use the area formula, we need to find either a or c.

$$B = 180° - A - C \Rightarrow$$
$$B = 180° - 30.50° - 112.60° = 36.90°$$

Finding a:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 30.5°} = \frac{13.00}{\sin 36.90°} \Rightarrow$$
$$a = \frac{13.00\sin 30.5°}{\sin 36.90°} \approx 10.9890\,\text{cm}$$

$$\text{A} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}(10.9890)(13.00)\sin 112.60°$$
$$\approx 65.94 \text{ cm}^2$$

Finding c:

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{13.00}{\sin 36.9°} = \frac{c}{\sin 112.6°} \Rightarrow$$
$$c = \frac{13.00\sin 112.6°}{\sin 36.9°} \approx 19.9889\,\text{cm}$$
$$\text{A} = \frac{1}{2}bc\sin A = \frac{1}{2}(19.9889)(13.00)\sin 30.5°$$
$$\approx 65.94 \text{ cm}^2$$

82. $A = 59.80°,\ b = 15.00 \text{ cm},\ C = 53.10°$

In order to use the area formula, we need to find either a or c.

$B = 180° - A - C \Rightarrow$

$B = 180° - 59.80° - 53.10° = 67.10°$

Finding a:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 59.80°} = \frac{15.00}{\sin 67.10°} \Rightarrow$$
$$a = \frac{15.00\sin 59.80°}{\sin 67.10°} \approx 14.0733\,\text{cm}$$
$$\text{A} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}(14.0733)(15.00)\sin 53.10° \approx 84.41 \text{ m}^2$$

Finding c:

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{15.00}{\sin 67.10°} = \frac{c}{\sin 53.10°} \Rightarrow$$
$$c = \frac{15.00\sin 53.10°}{\sin 67.10°} \approx 13.0216\,\text{m}$$
$$\text{A} = \frac{1}{2}bc\sin A = \frac{1}{2}(13.0216)(15.00)\sin 59.8°$$
$$\approx 84.41 \text{ m}^2$$

83. $$\text{A} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}(16.1)(15.2)\sin 125° \approx 100 \text{ m}^2$$

84. $$\text{A} = \frac{1}{2}(52.1)(21.3)\sin 42.2° \approx 373 \text{ m}^2$$

85. Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$ and $r = \frac{1}{2}$ (since the diameter is 1), we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\left(\frac{1}{2}\right) = 1$$

Then, $a = \sin A,\ b = \sin B,$ and $c = \sin C.$

86. Answers will vary.

87. Since triangles ACD and BCD are right triangles, we have $\tan\alpha = \frac{x}{d + BC}$ and $\tan\beta = \frac{x}{BC}$. Since $\tan\beta = \frac{x}{BC} \Rightarrow$ $BC = \frac{x}{\tan\beta}$, we can substitute into $\tan\alpha = \frac{x}{d + BC}$ and solve for x.

$$\tan\alpha = \frac{x}{d + BC} \Rightarrow \tan\alpha = \frac{x}{d + \frac{x}{\tan\beta}} \Rightarrow$$
$$\frac{\sin\alpha}{\cos\alpha} = \frac{x}{d + \frac{x\cos\beta}{\sin\beta}} \cdot \frac{\sin\beta}{\sin\beta}$$
$$\frac{\sin\alpha}{\cos\alpha} = \frac{x\sin\beta}{d\sin\beta + x\cos\beta}$$
$$\sin\alpha(d\sin\beta + x\cos\beta) = x\sin\beta(\cos\alpha)$$
$$d\sin\alpha\sin\beta + x\sin\alpha\cos\beta = x\cos\alpha\sin\beta$$
$$d\sin\alpha\sin\beta = x\cos\alpha\sin\beta - x\sin\alpha\cos\beta$$
$$= -x(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= -x\sin(a - \beta)$$
$$= x\sin[-(\alpha - \beta)]$$
$$d\sin\alpha\sin\beta = x\sin(\beta - \alpha)$$
$$\frac{d\sin\alpha\sin\beta}{\sin(\beta - \alpha)} = x$$

88. Prove that $\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B}$.

Start with the law of sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a = \frac{b\sin A}{\sin B}$$

Substitute for a in the expression $\frac{a+b}{b}$.

$$\frac{a+b}{b} = \frac{\frac{b\sin A}{\sin B} + b}{b} = \frac{\frac{b\sin A}{\sin B} + b}{b} \cdot \frac{\sin B}{\sin B}$$
$$= \frac{b\sin A + b\sin B}{b\sin B} = \frac{b(\sin A + \sin B)}{\sin B}$$
$$= \frac{\sin A + \sin B}{\sin B}$$

Section 8.2: The Law of Cosines

Connections (page 753)

1.
$$2s = a + b + c \quad (3)$$
$$2s - 2b = a - b + c \quad \text{Subtract } 2b \text{ from both sides.}$$
$$2(s - b) = a - b + c \quad (5)$$

(*continued on next page*)

(*continued from page 751*)

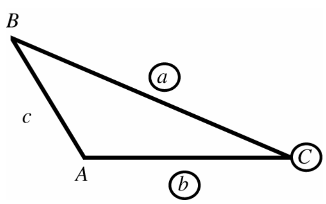
$$2s = a+b+c \quad (3)$$
$$2s-2c = a+b+c \quad \text{Subtract } 2c \text{ from both sides.}$$
$$2(s-c) = a+b-c \quad (6)$$

$$1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc} \quad \text{From equation (1)}$$
$$= \frac{2bc+b^2+c^2-a^2}{2bc}$$
$$= \frac{b^2+2bc+c^2-a^2}{2bc} \quad \text{Regroup}$$
$$= \frac{(b+c)^2-a^2}{2bc} \quad \text{Factor the perfect square trinomial}$$
$$= \frac{(b+c-a)(b+c+a)}{2bc} \quad \text{Factor the difference of squares}$$
$$= \frac{2(s-a)\cdot 2s}{2bc} \quad \text{From equations (4) and (3)}$$
$$1+\cos A = \frac{2s(s-a)}{bc} \quad (8)$$

2. Answers will vary.

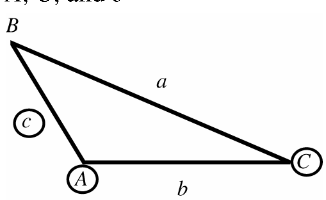
Exercises

1. a, b, and C



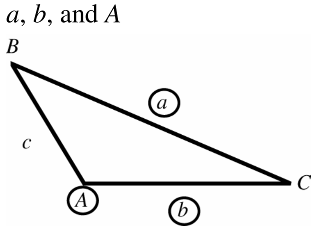
(a) SAS **(b)** law of cosines

2. A, C, and c



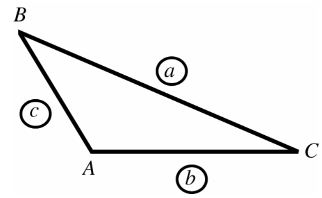
(a) SAA **(b)** law of sines

3. a, b, and A



(a) SSA **(b)** law of sines

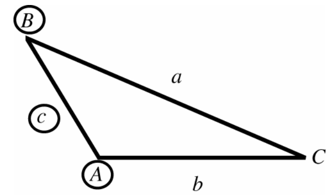
4. a, b, and c



(a) SSS

(b) law of cosines

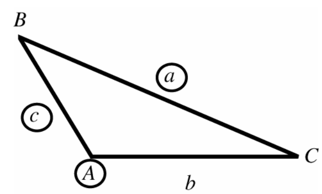
5. A, B, and c



(a) ASA

(b) law of sines

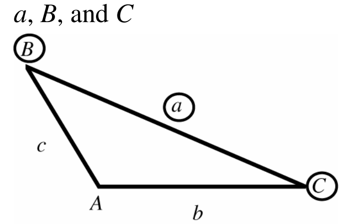
6. a, c, and A



(a) SSA

(b) law of sines

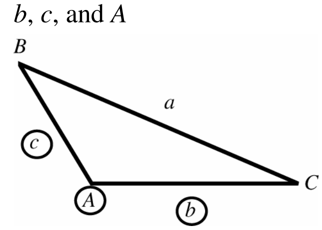
7. a, B, and C



(a) ASA

(b) law of sines

8. b, c, and A



(a) SAS

(b) law of cosines

9. $a^2 = 1^2 + \left(4\sqrt{2}\right)^2 - 2(1)\left(4\sqrt{2}\right)\cos 45°$
$= 1 + 32 - 8\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 33 - 8 = 25$
$a = \sqrt{25} = 5$

10. $a^2 = 3^2 + 8^2 - 2(3)(8)\cos 60°$
$= 9 + 64 - 48\left(\frac{1}{2}\right) = 73 - 24 = 49$
$a = \sqrt{49} = 7$

11. $\cos\theta = \frac{b^2 + c^2 - a^2}{2bc} = \frac{3^2 + 5^2 - 7^2}{2(3)(5)}$
$= \frac{9 + 25 - 49}{30} = -\frac{1}{2} \Rightarrow \theta = 120°$

12. $\cos\theta = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1^2 + \left(\sqrt{3}\right)^2 - 1^2}{2(1)\left(\sqrt{3}\right)}$
$= \frac{1 + 3 - 1}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{2}$
$\theta = 30°$

13. $A = 121°,\ b = 5,\ c = 3$
Start by finding a with the law of cosines.
$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$
$a^2 = 5^2 + 3^2 - 2(5)(3)\cos 121° \approx 49.5 \Rightarrow$
$a \approx 7.04 \approx 7.0$
Of the remaining angles B and C, C must be smaller since it is opposite the shorter of the two sides b and c. Therefore, C cannot be obtuse.

$\frac{\sin A}{a} = \frac{\sin C}{C} \Rightarrow \frac{\sin 121°}{7.04} = \frac{\sin C}{3} \Rightarrow$
$\sin C = \frac{3\sin 121°}{7.04} \approx .36527016 \Rightarrow C \approx 21.4°$
Thus, $B = 180° - 121° - 21.4° = 37.6°$.

14. $A = 61°,\ b = 4,\ c = 6$
Start by finding a with the law of cosines.
$a^2 = b^2 + c^2 - 2bc\cos A$
$a^2 = 4^2 + 6^2 - 2(4)(6)\cos 61° \approx 28.729$
$a \approx 5.36 \approx 5.4$
Of the remaining angles B and C, B must be smaller since it is opposite the shorter of the two sides b and c. Therefore, B cannot be obtuse.

$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 61°}{5.36} = \frac{\sin B}{4} \Rightarrow$
$\sin B = \frac{4\sin 61°}{5.36} \approx .65270127 \Rightarrow B \approx 40.7°$
Thus, $C = 180° - 61° - 40.7° = 78.3°$.

15. $a = 12,\ b = 10,\ c = 10$
We can use the law of cosines to solve for any angle of the triangle. Since b and c have the same measure, so do B and C since this would be an isosceles triangle. If we solve for B, we obtain
$b^2 = a^2 + c^2 - 2ac\cos B$
$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$
$\cos B = \frac{12^2 + 10^2 - 10^2}{2(12)(10)} = \frac{144}{240} = \frac{3}{5} \Rightarrow$
$B \approx 53.1°$
Therefore, $C = B \approx 53.1°$ and $A = 180° - 53.1° - 53.1° = 73.8°$. If we solve for A directly, however, we obtain
$a^2 = b^2 + c^2 - 2bc\cos A$
$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
$\cos A = \frac{10^2 + 10^2 - 12^2}{2(10)(10)} = \frac{56}{200} = \frac{7}{5}$
$A \approx 73.7°$
The angles may not sum to 180° due to rounding.

16. $a = 4,\ b = 10,\ c = 8$
We can use the law of cosines to solve for any angle of the triangle. We solve for B, the largest angle. We will know that B is obtuse if $\cos B < 0$.
$b^2 = a^2 + c^2 - 2ac\cos B$
$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$
$\cos B = \frac{4^2 + 8^2 - 10^2}{2(4)(8)} = \frac{-20}{64} = -\frac{5}{16} \Rightarrow$
$B \approx 108.2°$
We can now use the law of sines or the law of cosines to solve for either A or C. Using the law of cosines to solve for A, we have
$a^2 = b^2 + c^2 - 2bc\cos A$
$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
$\cos A = \frac{10^2 + 8^2 - 4^2}{2(10)(8)} = \frac{148}{160} = \frac{37}{40} \Rightarrow A \approx 22.3°$
Thus, $C = 180° - 108.2° - 22.3° = 49.5°$.

17. $B = 55°, a = 90, c = 100$

Start by finding b with the law of cosines.

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 90^2 + 100^2 - 2(90)(100)\cos 55° \approx 7775.6$

$b \approx 88.18$

(will be rounded as 88.2) Of the remaining angles A and C, A must be smaller since it is opposite the shorter of the two sides a and c. Therefore, A cannot be obtuse.

$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{90} = \frac{\sin 55°}{88.18} \Rightarrow$

$\sin A = \frac{90\sin 55°}{88.18} \approx .83605902 \Rightarrow A \approx 56.7°$

Thus, $C = 180° - 55° - 56.7° = 68.3°$.

18. $a = 5, b = 7, c = 9$

We can use the law of cosines to solve for any angle of the triangle. We solve for C, the largest angle. We will know that C is obtuse if $\cos C < 0$.

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \frac{5^2 + 7^2 - 9^2}{2(5)(7)} = \frac{-7}{70} = -\frac{1}{10} \Rightarrow C \approx 95.7°$

We can now use the law of sines or the law of cosines to solve for either A or B. Using the law of sines to solve for A, we have

$\frac{\sin A}{a} = \frac{\sin C}{C} \Rightarrow \frac{\sin A}{5} = \frac{\sin 95.7°}{9} \Rightarrow$

$\sin A = \frac{5\sin 95.7°}{9} \approx .55280865 \Rightarrow A \approx 33.6°$

Thus, $B = 180° - 95.7° - 33.6° = 50.7°$.

19. $A = 41.4°$, $b = 2.78$ yd, $c = 3.92$ yd

First find a.

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 2.78^2 + 3.92^2 - 2(2.78)(3.92)\cos 41.4°$

$\approx 6.7460 \Rightarrow a \approx 2.60$ yd

Find B next, since angle B is smaller than angle C (because $b < c$), and thus angle B must be acute.

$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{2.78} = \frac{\sin 41.4°}{2.597}$

$\sin B = \frac{2.78\sin 41.4°}{2.597} \approx .707091182 \Rightarrow$

$B \approx 45.1°$

Finally, $C = 180° - 41.4° - 45.1° = 93.5°$.

20. $C = 28.3°$, $b = 5.71$ in., $a = 4.21$ in.

First find c.

$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$

$c^2 = 4.21^2 + 5.71^2 - 2(4.21)(5.71)\cos 28.3°$

$\approx 7.9964 \Rightarrow c \approx 2.83$ in.

Find A next, since angle A is smaller than angle B (because $a < b$), and thus angle A must be acute.

$\frac{\sin A}{a} = \frac{\sin C}{C} \Rightarrow \frac{\sin A}{4.21} = \frac{\sin 28.3°}{2.828}$

$\sin A = \frac{4.21\sin 28.3°}{2.828} \approx .70576781 \Rightarrow$

$A \approx 44.9°$

Finally, $B = 180° - 28.3° - 44.9° = 106.8°$.

21. $C = 45.6°$, $b = 8.94$ m, $a = 7.23$ m

First find c.

$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$

$c^2 = 7.23^2 + 8.94^2 - 2(7.23)(8.94)\cos 45.6°$

$\approx 41.7493 \Rightarrow c \approx 6.46$ m

Find A next, since angle A is smaller than angle B (because $a < b$), and thus angle A must be acute.

$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{7.23} = \frac{\sin 45.6°}{6.461} \Rightarrow$

$\sin A = \frac{7.23\sin 45.6°}{6.461} \approx .79951052 \Rightarrow$

$A \approx 53.1°$

Finally, $B = 180° - 53.1° - 45.6° = 81.3°$.

22. $A = 67.3°$, $b = 37.9$ km, $c = 40.8$ km

First find a.

$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$

$a^2 = 37.9^2 + 40.8^2 - 2(37.9)(40.8)\cos 67.3°$

$\approx 1907.5815 \Rightarrow a \approx 43.7$ km

Find B next, since angle B is smaller than angle C (because $b < c$), and thus angle B must be acute.

$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{37.9} = \frac{\sin 67.3°}{43.68} \Rightarrow$

$\sin B = \frac{37.9\sin 67.3°}{43.68} \approx .80046231 \Rightarrow$

$B \approx 53.2°$

Finally, $C = 180° - 67.3° - 53.2° = 59.5°$.

23. $a = 9.3$ cm, $b = 5.7$ cm, $c = 8.2$ cm
We can use the law of cosines to solve for any of angle of the triangle. We solve for *A*, the largest angle. We will know that *A* is obtuse if $\cos A < 0$.

$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$$
$$\cos A = \frac{5.7^2 + 8.2^2 - 9.3^2}{2(5.7)(8.2)} \approx .14163457 \Rightarrow$$
$$A \approx 82^\circ$$

Find *B* next, since angle *B* is smaller than angle *C* (because $b < c$), and thus angle *B* must be acute.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{5.7} = \frac{\sin 82^\circ}{9.3} \Rightarrow$$
$$\sin B = \frac{5.7\sin 82^\circ}{9.3} \approx .60693849 \Rightarrow B \approx 37^\circ$$

Thus, $C = 180^\circ - 82^\circ - 37^\circ = 61^\circ$.

24. $a = 28$ ft, $b = 47$ ft, $c = 58$ ft
Angle *C* is the largest, so find it first.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$
$$\cos C = \frac{28^2 + 47^2 - 58^2}{2(28)(47)} \approx -.14095745 \Rightarrow$$
$$C \approx 98^\circ$$

Find *A* next, since angle *A* is smaller than angle *B* (because $a < b$), and thus angle *A* must be acute.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{28} = \frac{\sin 98^\circ}{58} \Rightarrow$$
$$\sin A = \frac{28\sin 98^\circ}{58} \approx .47806045 \Rightarrow A \approx 29^\circ$$

Thus, $B = 180^\circ - 29^\circ - 98^\circ = 53^\circ$.

25. $a = 42.9$ m, $b = 37.6$ m, $c = 62.7$ m
Angle *C* is the largest, so find it first.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$
$$\cos C = \frac{42.9^2 + 37.6^2 - 62.7^2}{2(42.9)(37.6)} \approx -.20988940 \Rightarrow$$
$$C \approx 102.1^\circ \approx 102^\circ 10'$$

Find *B* next, since angle *B* is smaller than angle *A* (because $b < a$), and thus angle *B* must be acute.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{37.6} = \frac{\sin 102.1^\circ}{62.7} \Rightarrow$$
$$\sin B = \frac{37.6\sin 102.1^\circ}{62.7} \approx .58635805 \Rightarrow$$
$$B \approx 35.9^\circ \approx 35^\circ 50'$$

Thus, $A = 180^\circ - 35^\circ 50' - 102^\circ 10'$
$= 180^\circ - 138^\circ = 42^\circ 00'$

26. $a = 189$ yd, $b = 214$ yd, $c = 325$ yd
Angle *C* is the largest, so find it first.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$
$$\cos C = \frac{189^2 + 214^2 - 325^2}{2(189)(214)} = -.29802700 \Rightarrow$$
$$C \approx 107.3^\circ \approx 107^\circ 20'$$

Find *B* next, since angle *B* is smaller than angle *A* (because $b < a$), and thus angle *B* must be acute.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{214} = \frac{\sin 107^\circ 20'}{325} \Rightarrow$$
$$\sin B = \frac{214\sin 107.3^\circ}{325} \approx .62867326 \Rightarrow$$
$$B \approx 39.0^\circ \approx 39^\circ 00'$$

Thus, $A = 180^\circ - 39^\circ 00' - 107^\circ 20'$
$= 179^\circ 60' - 146^\circ 20' = 33^\circ 40'$

27. $AB = 1240$ ft, $AC = 876$ ft, $BC = 965$ ft
Let $AB = c$, $AC = b$, $BC = a$
Angle *C* is the largest, so find it first.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$
$$\cos C = \frac{965^2 + 876^2 - 1240^2}{2(965)(876)} \approx .09522855 \Rightarrow$$
$$C \approx 84.5^\circ \text{ or } 84^\circ 30'$$

Find *B* next, since angle *B* is smaller than angle *A* (because $b < a$), and thus angle *B* must be acute.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{876} = \frac{\sin 84^\circ 30'}{1240} \Rightarrow$$
$$\sin B = \frac{876\sin 84^\circ 30'}{1240} \approx .70319925 \Rightarrow$$
$$B \approx 44.7^\circ \text{ or } 44^\circ 40'$$

Thus, $A = 180^\circ - 44^\circ 40' - 84^\circ 30'$
$= 179^\circ 60' - 129^\circ 10' = 50^\circ 50'$

28. $AB = 298$ m, $AC = 421$ m, $BC = 324$ m
Let $AB = c$, $AC = b$, $BC = a$.
Angle *B* is the largest, so find it first.

$$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow$$
$$\cos B = \frac{324^2 + 298^2 - 421^2}{2(324)(298)} \approx .08564815 \Rightarrow$$
$$B \approx 85.1^\circ \approx 85^\circ 10'$$

Find *C* next, since angle *C* is smaller than angle *A* (because $c < a$), and thus angle *B* must be acute.

$$\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \frac{\sin C}{298} = \frac{\sin 85.1^\circ}{421} \Rightarrow$$
$$\sin C = \frac{298\sin 85.1^\circ}{421} \approx .70525154 \Rightarrow$$
$$C \approx 44.8^\circ \approx 44^\circ 50'$$

(*continued on next page*)

(*continued from page 755*)

Thus, $A = 180° - 85°10' - 44°50'$
$= 180° - 130° = 50°00'$

29. $A = 80°40'$ $b = 143$ cm, $c = 89.6$ cm
First find a.
$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow$
$a^2 = 143^2 + 89.6^2 - 2(143)(89.6)\cos 80°40'$
$\approx 24,321.25 \Rightarrow a \approx 156$ cm
Find C next, since angle C is smaller than angle B (because $c < b$), and thus angle C must be acute.
$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin C}{89.6} = \frac{\sin 80°40'}{156.0} \Rightarrow$
$\sin C = \frac{89.6 \sin 80°40'}{156.0} \approx .56675534 \Rightarrow$
$C \approx 34.5° = 34°30'$
Finally, $B = 180° - 80°40' - 34°30'$
$= 179°60' - 115°10' = 64°50'$

30. $C = 72°40'$, $a = 327$ ft, $b = 251$ ft
First find c.
$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow$
$c^2 = 327^2 + 251^2 - 2(327)(251)\cos 72°40'$
$\approx 121,023.55 \Rightarrow c \approx 348$ ft
Find B next, since angle B is smaller than angle A (because $b < a$), and thus angle B must be acute.
$\frac{\sin B}{b} = \frac{\sin C}{C} \Rightarrow \frac{\sin B}{251} = \frac{\sin 72°40'}{347.9} \Rightarrow$
$\sin B = \frac{251 \sin 72°40'}{347.9} \approx .68870795 \Rightarrow$
$B \approx 43.5° = 43°30'$
Finally,
$A = 180° - 72°40' - 43°30'$
$= 179°60' - 116°10' = 63°50'$

31. $B = 74.80°$, $a = 8.919$ in., $c = 6.427$ in.
First find b.
$b^2 = a^2 + c^2 - 2ac \cos B \Rightarrow$
$b^2 = 8.919^2 + 6.427^2$
$- 2(8.919)(6.427)\cos 74.80°$
$\approx 90.7963 \Rightarrow b \approx 9.529$ in.
Find C next, since angle C is smaller than angle A (because $c < a$), and thus angle C must be acute.
$\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \frac{\sin C}{6.427} = \frac{\sin 74.80°}{9.5287} \Rightarrow$
$\sin C = \frac{6.427 \sin 74.80°}{9.5287} \approx .65089267 \Rightarrow$
$C \approx 40.61°$
Thus, $A = 180° - 74.80° - 40.61° = 64.59°$.

32. $C = 59.70°$, $a = 3.725$ mi, $b = 4.698$ mi
First find c.
$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow$
$c^2 = 3.725^2 + 4.698^2$
$- 2(3.725)(4.698)\cos 59.70°$
$\approx 18.28831 \Rightarrow c \approx 4.276$ mi
Find A next, since angle A is smaller than angle B (because $a < b$), and thus angle A must be acute.
$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{3.725} = \frac{\sin 59.70°}{4.27648} \Rightarrow$
$\sin A = \frac{3.725 \sin 59.70°}{4.27648} \approx .75205506 \Rightarrow$
$A \approx 48.77°$
Thus, $B = 180° - 48.77° - 59.70° = 71.53°$.

33. $A = 112.8°$, $b = 6.28$ m, $c = 12.2$ m
First find a.
$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow$
$a^2 = 6.28^2 + 12.2^2 - 2(6.28)(12.2)\cos 112.8°$
$\approx 247.658 \Rightarrow a \approx 15.7$ m
Find B next, since angle B is smaller than angle C (because $b < c$), and thus angle B must be acute.
$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{6.28} = \frac{\sin 112.8°}{15.74} \Rightarrow$
$\sin B = \frac{6.28 \sin 112.8°}{15.74} \approx .36780817 \Rightarrow$
$B \approx 21.6°$
Finally, $C = 180° - 112.8° - 21.6° = 45.6°$.

34. $B = 168.2°$, $a = 15.1$ cm, $c = 19.2$ cm
First find b.
$b^2 = a^2 + c^2 - 2ac \cos B \Rightarrow$
$b^2 = 15.1^2 + 19.2^2 - 2(15.1)(19.2)\cos 168.2°$
$\approx 1164.236 \Rightarrow b \approx 34.1$ cm
Find A next, since angle A is smaller than angle C (because $a < c$), and thus angle A must be acute.
$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{15.1} = \frac{\sin 168.2°}{34.12} \Rightarrow$
$\sin A = \frac{15.1 \sin 168.2°}{34.12} \approx .09050089 \Rightarrow A \approx 5.2°$
Thus, $C = 180° - 5.2° - 168.2° = 6.6°$.

35. $a = 3.0$ ft, $b = 5.0$ ft, $c = 6.0$ ft

Angle C is the largest, so find it first.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$

$$\cos C = \frac{3.0^2 + 5.0^2 - 6.0^2}{2(3.0)(5.0)} = -\frac{2}{30} = -\frac{1}{15}$$

$$\approx -.06666667 \Rightarrow C \approx 94°$$

Find A next, since angle A is smaller than angle B (because $a < b$), and thus angle A must be acute.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{3} = \frac{\sin 94°}{6} \Rightarrow$$

$$\sin A = \frac{3\sin 94°}{6} \approx .49878203 \Rightarrow A \approx 30°$$

Thus, $B = 180° - 30° - 94° = 56°$.

36. $a = 4.0$ ft, $b = 5.0$ ft, $c = 8.0$ ft

Angle C is the largest, so find it first.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$

$$\cos C = \frac{4.0^2 + 5.0^2 - 8.0^2}{2(4.0)(5.0)} = -\frac{23}{40}$$

$$\approx -.57500000 \Rightarrow C \approx 125°$$

Find A next, since angle A is smaller than angle B (because $a < b$), and thus angle A must be acute.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{4} = \frac{\sin 125°}{8} \Rightarrow$$

$$\sin A = \frac{4\sin 125°}{8} \approx .40957602 \Rightarrow A \approx 24°$$

Thus, $B = 180° - 24° - 125° = 31°$.

37. There are three ways to apply the law of cosines when $a = 3, b = 4$, and $c = 10$.

Solving for A:

$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$$

$$\cos A = \frac{4^2 + 10^2 - 3^2}{2(4)(10)} = \frac{107}{80} = 1.3375$$

Solving for B:

$$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow$$

$$\cos B = \frac{3^2 + 10^2 - 4^2}{2(3)(10)} = \frac{93}{60} = \frac{31}{20} = 1.55$$

Solving for C:

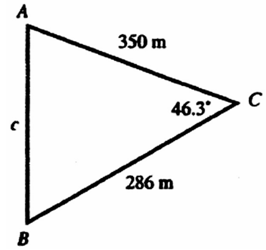
$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$

$$\cos C = \frac{3^2 + 4^2 - 10^2}{2(3)(4)} = \frac{-75}{24} = -\frac{25}{8} = -3.125$$

Since the cosine of any angle of a triangle must be between –1 and 1, a triangle cannot have sides 3, 4, and 10.

38. Answers will vary.

39. Find AB, or c, in the following triangle.



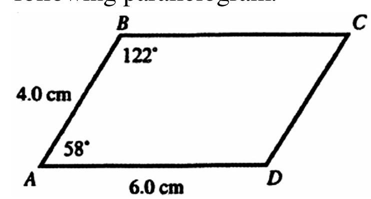
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 286^2 + 350^2 - 2(286)(350)\cos 46.3°$$

$$c^2 \approx 65{,}981.3 \Rightarrow c \approx 257$$

The length of AB is 257 m.

40. Find the diagonals, BD and AC, of the following parallelogram.



$$BD^2 = AB^2 + AD^2 - 2(AB)(AD)\cos A$$

$$BD^2 = 4^2 + 6^2 - 2(4)(6)\cos 58°$$

$$BD^2 \approx 26.563875 \Rightarrow BD \approx 5.2 \text{ cm}$$

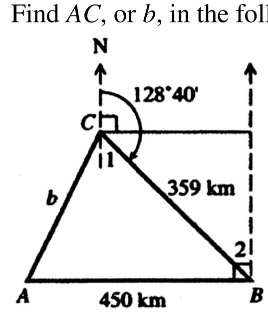
$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos B$$

$$AC^2 = 4^2 + 6^2 - 2(4)(6)\cos 122°$$

$$AC^2 \approx 77.436125 \Rightarrow AC \approx 8.8 \text{ cm}$$

The lengths of the diagonals are 5.2 cm and 8.8 cm.

41. Find AC, or b, in the following triangle.



$$m\angle 1 = 180° - 128°40' = 51°20'$$

Angles 1 and 2 are alternate interior angles formed when parallel lines (the north lines) are cut by a transversal, line BC, so $m\angle 2 = m\angle 1 = 51°20'$.

$$m\angle ABC = 90° - m\angle 2 = 90° - 51°20' = 38°40'$$

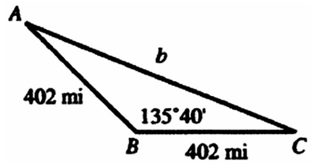
$$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow$$

$$b^2 = 359^2 + 450^2 - 2(359)(450)\cos 38°40'$$

$$\approx 79{,}106 \Rightarrow b \approx 281 \text{ km}$$

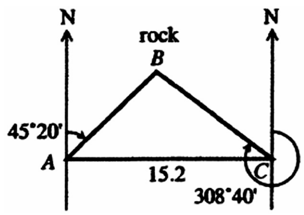
C is about 281 km from A.

42. Let B be the harbor, AB is the course of one ship, BC is the course of the other ship. Thus, b = the distance between the ships.



$$b^2 = 402^2 + 402^2 - 2(402)(402)\cos 135°40'$$
$$b^2 \approx 554394.25 \Rightarrow b = 745 \text{ mi}$$

43. Sketch a triangle showing the situation as follows.



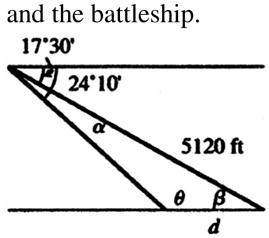
$$m\angle A = 90° - 45°20' = 44°40'$$
$$m\angle C = 308°40' - 270° = 38°40'$$
$$m\angle B = 180° - A - C$$
$$= 180° - 44°40' - 38°40' = 96°40'$$

Since we have only one side of a triangle, use the law of sines to find $BC = a$.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 44°\ 40'} = \frac{15.2}{\sin 96°\ 40'} \Rightarrow$$
$$a = \frac{15.2\sin 44°\ 40'}{\sin 96°\ 40'} \approx 10.8$$

The distance between the ship and the rock is about 10.8 miles.

44. Let d = the distance between the submarine and the battleship.



$\alpha = 24°\ 10' - 17°\ 30' = 6°\ 40'$

$\beta = 17°\ 30'$ since the angle of depression to the battleship equals the angle of elevation from the battleship (They are alternate interior angles.).

$$\theta = 180° - 6°\ 40' - 17°\ 30'$$
$$= 179°60' - 14°\ 10' = 155°\ 50'$$

Since we have only one side of a triangle, use the law of sines to find d.

$$\frac{d}{\sin 6°\ 40'} = \frac{5120}{\sin 155°\ 50'} \Rightarrow$$
$$d = \frac{5120\sin 6°\ 40'}{\sin 155°\ 50'} \approx 1451.9$$

The distance between the submarine and the battleship is 1450 ft. (rounded to three significant digits)

45. Use the law of cosines to find the angle, θ.

$$\cos\theta = \frac{20^2 + 16^2 - 13^2}{2(20)(16)} = \frac{487}{640} \approx .76093750 \Rightarrow$$
$$\theta \approx 40°$$

46. AB is the horizontal distance between points A and B. Using the laws of cosines, we have

$$AB^2 = 10^2 + 10^2 - 2(10)(10)\cos 128° \Rightarrow$$
$$AB^2 \approx 323.1 \Rightarrow AB \approx 18 \text{ ft}$$

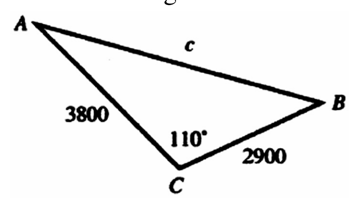
47. Let A = the angle between the beam and the 45-ft cable.

$$\cos A = \frac{45^2 + 90^2 - 60^2}{2(45)(90)} = \frac{6525}{8100} = \frac{29}{36}$$
$$\approx .80555556 \Rightarrow A \approx 36°$$

Let B = the angle between the beam and the 60-ft cable.

$$\cos B = \frac{90^2 + 60^2 - 45^2}{2(90)(60)} = \frac{9675}{10{,}800} = \frac{43}{48}$$
$$\approx .89583333 \Rightarrow B \approx 26°$$

48. Let c = the length of the tunnel.

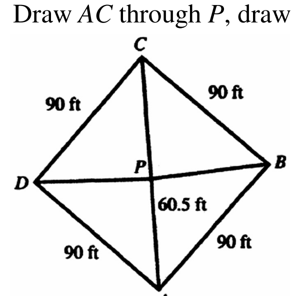


Use the law of cosines to find c.

$$c^2 = 3800^2 + 2900^2 - 2(3800)(2900)\cos 110°$$
$$\approx 30{,}388{,}124 \Rightarrow c \approx 5512.5$$

The tunnel is 5500 meters long. (rounded to two significant digits)

49. Let A = home plate; B = first base; C = second base; D = third base; P = pitcher's rubber. Draw AC through P, draw PB and PD.



In triangle *ABC*, angle $m\angle B = 90°$, and $m\angle A = m\angle C = 45°$.

$AC = \sqrt{90^2 + 90^2} = \sqrt{2 \cdot 90^2} = 90\sqrt{2}$ and

$PC = 90\sqrt{2} - 60.5 \approx 66.8$ ft

In triangle *APB*, $m\angle A = 45°$.

$PB^2 = AP^2 + AB^2 - 2(AP)(AB)\cos A$

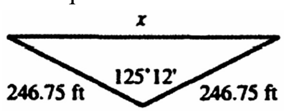
$PB^2 = 60.5^2 + 90^2 - 2(60.5)(90)\cos 45°$

$PB^2 \approx 4059.86 \Rightarrow PB \approx 63.7$ ft

Since triangles *APB* and *APD* are congruent, $PB = PD = 63.7$ ft.

The distance to second base is 66.8 ft and the distance to both first and third base is 63.7 ft.

50. Let *x* = the distance between the ends of the two equal sides.

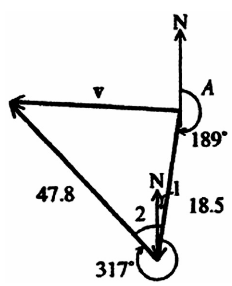


Use the law of cosines to find *x*.

$$x^2 = 246.75^2 + 246.75^2 - 2(246.75)(246.75)\cos 125°12' \approx 191{,}963.937 \Rightarrow x \approx 438.14$$

The distance between the ends of the two equal sides is 438.14 feet.

51. Find the distance of the ship from point *A*.



$m\angle 1 = 189° - 180° = 9°$

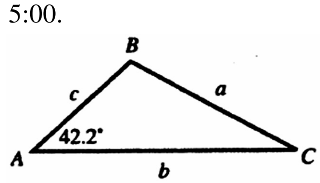
$m\angle 2 = 360° - 317° = 43°$

$m\angle 1 + m\angle 2 = 9° + 43° = 52°$

Use the law of cosines to find *v*.

$$v^2 = 47.8^2 + 18.5^2 - 2(47.8)(18.5)\cos 52° \approx 1538.23 \Rightarrow v \approx 39.2 \text{ km}$$

52. Let *A* = the man's location;
B = the factory whistle heard at 3 sec after 5:00;
C = the factory whistle heard at 6 sec after 5:00.



Since sound travels at 344 m per sec and the man hears the whistles in 3 sec and 6 sec, the factories are

$c = 3(344) = 1032$ m and $b = 6(344) = 2064$ m from the man.

Using the law of cosines we have

$$a^2 = 1032^2 + 2064^2 - 2(1032)(2064)\cos 42.2° \approx 2{,}169{,}221.3 \Rightarrow a \approx 1472.8$$

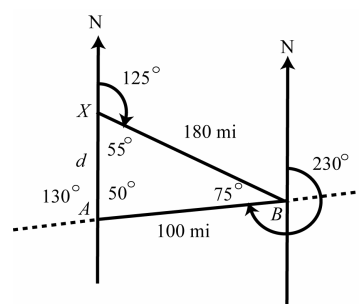
The factories are about 1473 m apart. (rounded to four significant digits)

53. $\cos A = \dfrac{17^2 + 21^2 - 9^2}{2(17)(21)} = \dfrac{649}{714} \approx .90896359 \Rightarrow$

$A \approx 25°$

Thus, the bearing of *B* from is $325° + 25° = 350°$.

54. Sketch a triangle showing the situation as follows.



The angle marked $130°$ is the corresponding angle that measures $360° - 230° = 130°$. The angle marked $55°$ is the supplement of the $125°$ angle. Finally, the $75°$ angle is marked as such because $180° - 55° - 50° = 75°$. We can use the law of cosines to solve for the side of the triangle marked *d*.

$$d^2 = 180^2 + 100^2 - 2(180)(100)\cos 75° \approx 33{,}082 \Rightarrow d \approx 181.9$$

The distance is approximately 180 mi. (rounded to two significant digits)

55. Let *c* = the length of the property line that cannot be directly measured.

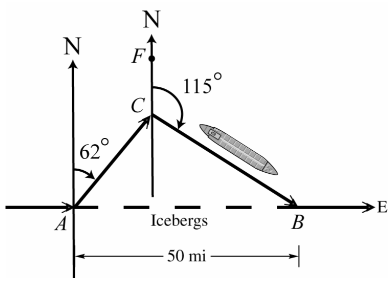
Using the law of cosines, we have

$$c^2 = 14.0^2 + 13.0^2 - 2(14.0)(13.0)\cos 70° \approx 240.5 \Rightarrow c \approx 15.5 \text{ ft}$$

(rounded to three significant digits)

The length of the property line is approximately 18.0 + 15.5 + 14.0 = 47.5 feet

56. Let A = the point where the ship changes to a bearing of 62°;
C = the point where it changes to a bearing of 115°.



$m\angle CAB = 90° - 62° = 28°$
$m\angle FCA = 180° - m\angle DAC = 180° - 62° = 118°$
$m\angle ACB = 360° - m\angle FCB - m\angle FCA = 360° - 115° - 118° = 127°$
$m\angle CBA = 180° - m\angle ACB - m\angle CAB = 180° - 127° - 28° = 25°$

Since we have only one side of a triangle, use the law of sines to find CB.

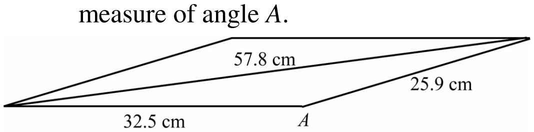
$$\frac{CB}{\sin 28°} = \frac{50}{\sin 127°} \Rightarrow CB = \frac{50 \sin 28°}{\sin 127°} \approx 29.4$$

and

$$\frac{AC}{\sin 25°} = \frac{50}{\sin 127°} \Rightarrow AC = \frac{50 \sin 25°}{\sin 127°} \approx 26.5$$

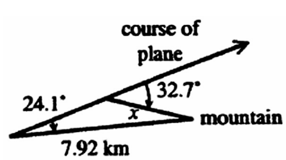
The ship traveled 26.5 + 29.4 = 55.9 mi. To avoid the iceberg, the ship had to travel 55.9 – 50 = 5.9 mi farther.

57. Using the law of cosines we can solve for the measure of angle A.



$$\cos A = \frac{25.9^2 + 32.5^2 - 57.8^2}{2(25.9)(32.5)} \approx -.95858628 \Rightarrow A \approx 163.5°$$

58. Let x = the distance from the plane to the mountain when the second bearing is taken.
$\theta = 180° - 32.7° = 147.3°$



Since we have only one side of a triangle, use the law of sines to find x.

$$\frac{x}{\sin 24.1°} = \frac{7.92}{\sin 147.3°}$$
$$x = \frac{7.92 \sin 24.1°}{\sin 147.3°} \approx 5.99$$

The plane is about 5.99 km from the mountain. (rounded to three significant digits)

59. Find x using the law of cosines.
$x^2 = 25^2 + 25^2 - 2(25)(25)\cos 52° \approx 480 \Rightarrow x \approx 22$ ft

60. To find the distance between the towns, d, use the law of cosines.
$d^2 = 3428^2 + 5631^2 - 2(3428)(5631)\cos 43.33° \approx 15{,}376{,}718 \Rightarrow d \approx 3921.3$
The distance between the two towns is about 3921 m (rounded to four significant digits.)

61. Let a be the length of the segment from (0, 0) to (6, 8). Use the distance formula.
$a = \sqrt{(6-0)^2 + (8-0)^2} = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$
Let b be the length of the segment from (0, 0) to (4, 3).
$b = \sqrt{(4-0)^2 + (3-0)^2} = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$
Let c be the length of the segment from (4, 3) to (6, 8).
$c = \sqrt{(6-4)^2 + (8-3)^2} = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow$$
$$\cos\theta = \frac{10^2 + 5^2 - \left(\sqrt{29}\right)^2}{2(10)(5)} = \frac{100 + 25 - 29}{100} = .96 \Rightarrow \theta \approx 16.26°$$

62. Let a be the length of the segment from (0, 0) to (8, 6). Use the distance formula.
$a = \sqrt{(8-0)^2 + (6-0)^2} = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$

Let b be the length of the segment from (0, 0) to (12, 5).
$b = \sqrt{(12-0)^2 + (5-0)^2} = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$

Let c be the length of the segment from (8, 6) to (12, 5).

$$c = \sqrt{(12-8)^2 + (5-6)^2} = \sqrt{4^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{10^2 + 13^2 - \left(\sqrt{17}\right)^2}{2(10)(13)} = \frac{100+169-17}{260} \approx .96923077 \Rightarrow$$

$$\theta \approx 14.25°$$

63. Using $A = \frac{1}{2}bh \Rightarrow A = \frac{1}{2}(16)\left(3\sqrt{3}\right) = 24\sqrt{3}.$

To use Heron's Formula, first find the semiperimeter,

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(6+14+16) = \frac{1}{2}\cdot 36 = 18.$$

Now find the area of the triangle.

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{18(18-6)(18-14)(18-16)} \\ &= \sqrt{18(12)(4)(2)} = \sqrt{1728} \\ &= 24\sqrt{3} \end{aligned}$$

Both formulas give the same area.

64. Using $A = \frac{1}{2}bh \Rightarrow A = \frac{1}{2}(10)\left(3\sqrt{3}\right) = 15\sqrt{3}.$

To use Heron's Formula, first find the semiperimeter,

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(10+6+14) = \frac{1}{2}\cdot 30 = 15.$$

Now find the area of the triangle.

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{15(15-10)(15-6)(15-14)} \\ &= \sqrt{15(5)(9)(1)} = \sqrt{675} = 15\sqrt{3} \end{aligned}$$

Both formulas give the same result.

65. $a = 12$ m, $b = 16$ m, $c = 25$ m

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(12+16+25) = \frac{1}{2}\cdot 53 = 26.5$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{26.5(26.5-12)(26.5-16)(26.5-25)} \\ &= \sqrt{26.5(14.5)(10.5)(1.5)} \approx 78 \text{ m}^2 \end{aligned}$$

(rounded to two significant digits)

66. $a = 22$ in., $b = 45$ in., $c = 31$ in.

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(22+45+31) = \frac{1}{2}\cdot 98 = 49$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{49(49-22)(49-45)(49-31)} \\ &= \sqrt{49(27)(4)(18)} \approx 310 \text{ in.}^2 \end{aligned}$$

(rounded to two significant digits)

67. $a = 154$ cm, $b = 179$ cm, $c = 183$ cm

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(154+179+183) = \frac{1}{2}\cdot 516 = 258$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{258(258-154)(258-179)(258-183)} \\ &= \sqrt{258(104)(79)(75)} \approx 12{,}600 \text{ cm}^2 \end{aligned}$$

(rounded to three significant digits)

68. $a = 25.4$ yd, $b = 38.2$ yd, $c = 19.8$ yd

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(25.4+38.2+19.8) = \frac{1}{2}\cdot 83.4 = 41.7$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{41.7(41.7-25.4)(41.7-38.2)\cdot(41.7-19.8)} \\ &= \sqrt{41.7(16.3)(3.5)(21.9)} \approx 228 \text{ yd}^2 \end{aligned}$$

(rounded to three significant digits)

69. $a = 76.3$ ft, $b = 109$ ft, $c = 98.8$ ft

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(76.3+109+98.8) = \frac{1}{2}\cdot 284.1 = 142.05$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{142.05(142.05-76.3)(142.05-109)\cdot(142.05-98.8)} \\ &= \sqrt{142.05(65.75)(33.05)(43.25)} \approx 3650 \text{ ft}^2 \end{aligned}$$

(rounded to three significant digits)

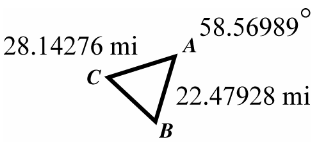
70. $a = 15.89$ in., $b = 21.74$ in., $c = 10.92$ in.

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(15.89+21.74+10.92) = \frac{1}{2}\cdot 48.55 = 24.275$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{24.275(24.275-15.89)\cdot(24.275-21.74)(24.275-10.92)} \\ &= \sqrt{24.275(8.385)(2.535)(13.355)} \\ &\approx 83.01 \text{ in.}^2 \end{aligned}$$

(rounded to four significant digits)

71. $AB = 22.47928$ mi, $AC = 28.14276$ mi, $A = 58.56989°$



This is SAS, so use the law of cosines.

$$BC^2 = AC^2 + AB^2 - 2(AC)(AB)\cos A$$

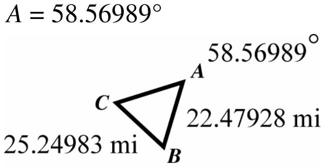
$$BC^2 = 28.14276^2 + 22.47928^2 - 2(28.14276)(22.47928)\cos 58.56989°$$

$$BC^2 \approx 637.55393$$

$$BC \approx 25.24983$$

BC is approximately 25.24983 mi. (rounded to seven significant digits)

72. $AB = 22.47928$ mi, $BC = 25.24983$ mi, $A = 58.56989°$



This is SSA, so use the law of sines.

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin C}{22.47928} = \frac{\sin 58.56989°}{25.24983} \Rightarrow$$

$$\sin C = \frac{22.47928 \sin 58.56989°}{25.24983} \approx .75965065$$

Thus, $C \approx 49.43341°$ and

$$\begin{aligned} B &= 180° - A - C \\ &= 180° - 58.56989° - 49.43341° \\ &= 71.99670°. \end{aligned}$$

73. Perimeter: 9 + 10 + 17 = 36 feet, so the semi-perimeter is $\frac{1}{2}\cdot 36 = 18$ feet.

Use Heron's Formula to find the area.

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{18(18-9)(18-10)(18-17)} \\ &= \sqrt{18(9)(8)(1)} = \sqrt{1296} = 36 \text{ ft} \end{aligned}$$

Since the perimeter and area both equal 36 feet, the triangle is a *perfect triangle.*

74. (a)

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(11+13+20) = \frac{1}{2}\cdot 44 = 22$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{22(22-11)(22-13)(22-20)} \\ &= \sqrt{22(11)(9)(2)} \\ &= \sqrt{4356} = 66, \text{ which is an integer} \end{aligned}$$

(b)

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(13+14+15) = \frac{1}{2}\cdot 42 = 21$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{21(21-13)(21-14)(21-15)} \\ &= \sqrt{21(8)(7)(6)} \\ &= \sqrt{7056} = 84, \text{ which is an integer} \end{aligned}$$

(c)

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(7+15+20) = \frac{1}{2}\cdot 42 = 21$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{21(21-7)(21-15)(21-20)} \\ &= \sqrt{21(14)(6)(1)} \\ &= \sqrt{1764} = 42, \text{ which is an integer} \end{aligned}$$

(d)

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(9+10+17) = \frac{1}{2}\cdot 36 = 18$$

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{18(18-9)(18-10)(18-17)} \\ &= \sqrt{18(9)(8)(1)} \\ &= \sqrt{1296} = 36, \text{ which is an integer} \end{aligned}$$

75. Find the area of the Bermuda Triangle using Heron's Formula.

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(850+925+1300) = \frac{1}{2}\cdot 3075 = 1537.5$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{1537.5(1537.5-850)\cdot(1537.5-925)(1537.5-1300)}$$
$$= \sqrt{1537.5(687.5)(612.5)(237.5)} \approx 392{,}128.82$$

The area of the Bermuda Triangle is about $390{,}000\ \text{mi}^2$.

76. Find the area of the region using Heron's Formula.

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(75+68+85) = \frac{1}{2}\cdot 228 = 114$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{114(114-75)(114-68)(114-85)}$$
$$= \sqrt{(114)(39)(46)(29)} \approx 2435.3571\ \text{m}^2$$

Number of cans needed

$$= \frac{(\text{area in m}^2)}{(\text{m}^2\text{per can})} = \frac{2435.3571}{75} = 32.471428 \text{ cans}$$

She will need to open 33 cans.

77. (a) Using the law of sines, we have

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin C}{15} = \frac{\sin 60°}{13} \Rightarrow$$
$$\sin C = \frac{15 \sin 60°}{13} = \frac{15}{13}\cdot\frac{\sqrt{3}}{2} \approx .99926008$$

There are two angles C between $0°$ and $180°$ that satisfy the condition. Since $\sin C \approx .99926008$, to the nearest tenth value of C is $C_1 = 87.8°$.

Supplementary angles have the same sine value, so another possible value of C is $B_2 = 180° - 87.8° = 92.2°$.

(b) By the law of cosines, we have

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos C = \frac{13^2+7^2-15^2}{2(13)(7)} = \frac{-7}{182}$$
$$= -\frac{1}{26} \approx -.03846154 \Rightarrow C \approx 92.2°$$

(c) With the law of cosines, we are required to find the inverse cosine of a negative number; therefore; we know angle C is greater than 90°.

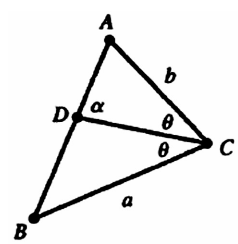
78. Using the law of cosines, we have

$$\cos B = \frac{a^2+c^2-b^2}{2ac} \Rightarrow$$
$$\cos B = \frac{6^2+5^2-4^2}{2(6)(5)} = \frac{36+25-16}{60} = \frac{3}{4}$$
$$\cos A = \frac{b^2+c^2-a^2}{2bc} \Rightarrow$$
$$\cos A = \frac{4^2+5^2-6^2}{2(4)(5)} = \frac{16+25-36}{40} = \frac{1}{8}$$

Since $2\cos^2 B - 1 = 2\left(\frac{3}{4}\right)^2 - 1 = 2\left(\frac{9}{16}\right) - \frac{16}{16}$

$= \frac{2}{16} = \frac{1}{8} = \cos A$, A is twice the size of B.

79. Given point D is on side $\overline{AB}$ of triangle ABC such that $\overline{CD}$ bisects $\angle C$, $m\angle ACD = m\angle DCB$. Show that $\frac{AD}{DB} = \frac{b}{a}$.



Let $\theta = m\angle ACD$, and $\alpha = m\angle ADC$

Then $\theta = m\angle DCB$ and $= m\angle BDC = 180 - \alpha$.

By the law of sines, we have

$$\frac{\sin\theta}{AD} = \frac{\sin\alpha}{b} \Rightarrow \sin\theta = \frac{AD\sin\alpha}{b} \text{ and}$$
$$\frac{\sin\theta}{DB} = \frac{\sin(180° - \alpha)}{a} \Rightarrow$$

(*continued on next page*)

(*continued from page 763*)

$$\sin\theta = \frac{DB\sin(180° - \alpha)}{a}$$

By substitution, we have

$$\frac{AD\sin\alpha}{b} = \frac{DB\sin(180° - \alpha)}{a}$$

Since $\sin\alpha = \sin(180° - \alpha), \frac{AD}{b} = \frac{DB}{a}.$

Multiplying both sides by $\frac{b}{DB}$, we have

$$\frac{AD}{b}\cdot\frac{b}{DB} = \frac{DB}{a}\cdot\frac{b}{DB} \Rightarrow \frac{AD}{DB} = \frac{b}{a}.$$

80. Let $a = 2,\ b = 2\sqrt{3}$, $A = 30°$, $B = 60°$.

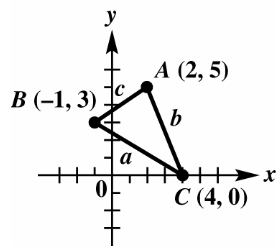
Verify $\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}$.

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(30° - 60°)}{\tan\frac{1}{2}(30° + 60°)} = \frac{\tan(-15°)}{\tan 45°} \approx -.26794919$$

$$\frac{a-b}{a+b} = \frac{2-2\sqrt{3}}{2+2\sqrt{3}}\cdot\frac{2-2\sqrt{3}}{2-2\sqrt{3}} = \frac{4-8\sqrt{3}+12}{4-12} = \frac{16-8\sqrt{3}}{-8} = -2+\sqrt{3} \approx -.26794919$$

Thus, $\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}$ using the given values of a, b, A, and B.

81.



82. $a = \sqrt{(-1-4)^2 + (3-0)^2} = \sqrt{25+9} = \sqrt{34}$

$b = \sqrt{(2-4)^2 + (5-0)^2} = \sqrt{4+25} = \sqrt{29}$

$c = \sqrt{(-1-2)^2 + (3-5)^2} = \sqrt{9+4} = \sqrt{13}$

83. Use the law of cosines to find the measure of $\angle A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\left(\sqrt{34}\right)^2 = \left(\sqrt{29}\right)^2 + \left(\sqrt{13}\right)^2 - 2\left(\sqrt{29}\right)\left(\sqrt{13}\right)\cos A$$

$$34 = 42 - 2\sqrt{377}\cos A$$

$$-8 = -2\sqrt{377}\cos A$$

$$\frac{4}{\sqrt{377}} = \cos A$$

$$\text{A} = \frac{1}{2}ab\sin C = \frac{1}{2}\left(\sqrt{29}\right)\left(\sqrt{13}\right)\sin C = \frac{\sqrt{377}}{2}\sin\left(\cos^{-1}\frac{4}{\sqrt{377}}\right) = 9.5 \text{ sq units}$$

84. First find the semiperimeter. $s = \frac{1}{2}(a+b+c) = \frac{1}{2}\left(\sqrt{34}+\sqrt{29}+\sqrt{13}\right)$

Using Heron's formula, we have

$$\text{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\frac{1}{2}\left(\sqrt{34}+\sqrt{29}+\sqrt{13}\right)\left(\frac{1}{2}\left(\sqrt{34}+\sqrt{29}+\sqrt{13}\right)-\sqrt{34}\right)\left(\frac{1}{2}\left(\sqrt{34}+\sqrt{29}+\sqrt{13}\right)-\sqrt{29}\right)\left(\frac{1}{2}\left(\sqrt{34}+\sqrt{29}+\sqrt{13}\right)-\sqrt{13}\right)}$$

= 9.5 sq units (found using a calculator)

85. $(x_1, y_1) = (2,5);\ (x_2, y_2) = (-1,3);\ (x_3, y_3) = (4,0)$

$$\text{A} = \frac{1}{2}\left|x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1\right|$$

$$= \frac{1}{2}\left|2(3) - 5(-1) + (-1)(0) - 3(4) + 4(5) - 0(2)\right|$$

= 9.5 sq units

Chapter 8 Quiz
(Sections 8.1–8.2)

1. Using the law of sines, we have

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin 30.6°}{7.42} = \frac{\sin C}{4.54} \Rightarrow$$
$$\sin C = \frac{4.54 \sin 30.6°}{7.42} \approx .311462 \Rightarrow$$
$$C \approx 18.1°$$
$$A = 180° - B - C$$
$$= 180° - 30.6° - 18.1° = 131.3°$$

2. Using the law of cosines, we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$a^2 = 75.0^2 + 135^2 - 2 \cdot 75.0 \cdot 135 \cos 144°$$
$$\approx 40,232.59 \Rightarrow$$

$a \approx 201$ m (rounded to three significant digits)

3. Using the law of cosines, we have

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$21.2^2 = 28.4^2 + 16.9^2 - 2 \cdot 28.4 \cdot 16.9 \cos C$$
$$-642.73 = -959.92 \cos C$$
$$\frac{642.73}{959.92} = \cos C \Rightarrow$$

$C \approx 48.0°$ (rounded to three significant digits)

4. $\text{A} = \frac{1}{2} ab \sin C = \frac{1}{2}(7)(9) \sin 150°$

$$= \frac{63}{4} = 15.75 \text{ sq units}$$

5. First find the semiperimeter:

$$s = \frac{1}{2}(19.5 + 21.0 + 22.5) = 31.5$$

Using Heron's formula, we have

$$\text{A} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{31.5(31.5-19.5)(31.5-21.0)(31.5-22.5)}$$
$$= \sqrt{31.5(12)(10.5)(9)}$$
$$= \sqrt{35,721} = 189 \text{ km}^2$$

6. Using the law of sines,we have

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{534} = \frac{\sin 25.4°}{345} \Rightarrow$$
$$\sin A = \frac{534 \sin 25.4°}{345} \approx .663917 \Rightarrow$$

$A \approx 41.6°$ or $A \approx 180° - 41.6° = 138.4°$

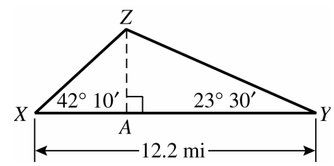
7. $\angle C = 180° - 111° - 41° = 28°$

Using the law of sines, we have

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 111°} = \frac{326}{\sin 28°} \Rightarrow$$
$$a = \frac{326 \sin 111°}{\sin 28°} \approx 648$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 41°} = \frac{326}{\sin 28°} \Rightarrow$$
$$b = \frac{326 \sin 41°}{\sin 28°} \approx 456$$

Note that both a and b have been rounded to three significant digits.

8. The height of the balloon is the length of the altitude of $\triangle XYZ$, AZ.



We will use the fact that $\sin 42°10' = \frac{AZ}{XZ}$ to find the length of AZ. First, we must find the length of XZ using the law of sines.

$$Z = 180° - 42°10' - 23°30' = 114°20'$$
$$\frac{XZ}{\sin Y} = \frac{XY}{\sin Z} \Rightarrow \frac{XZ}{\sin 23°30'} = \frac{12.2}{\sin 114°20'} \Rightarrow$$
$$XZ = \frac{12.2 \sin 23°30'}{\sin 114°20'} \approx 5.3390$$
$$\sin 42°10' = \frac{AZ}{5.3390}$$
$$AZ = 5.3390 \sin 42°10' \approx 3.6 \text{ mi}$$

Section 8.3: Vectors, Operations, and the Dot Product

1. Equal vectors have the same magnitude and direction. Equal vectors are **m** and **p**; **n** and **r**.

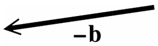
2. Opposite vectors have the same magnitude but opposite direction. Opposite vectors are **m** and **q**, **p** and **q**, **n** and **s**, **r** and **s**.

3. One vector is a positive scalar multiple of another if the two vectors point in the same direction; they may have different magnitudes.
m = 1**p**; **m** = 2**t**; **n** = 1**r**; **p** = 2**t** or

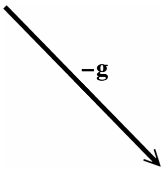
$$\mathbf{p} = 1\mathbf{m};\ \mathbf{t} = \frac{1}{2}\mathbf{m};\ \mathbf{r} = 1\mathbf{n};\ \mathbf{t} = \frac{1}{2}\mathbf{p}$$

4. One vector is a negative scalar multiple of another if the two vectors point in the opposite direction; they may have different magnitudes.
m = –1**q**; **p** = –1**q**; **r** = –1**s**; **q** = –2**t**; **n** = –1**s**

5.



6.



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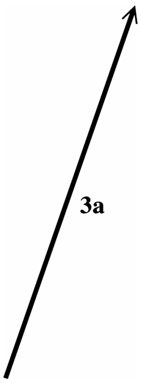

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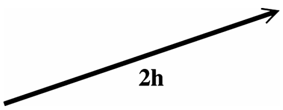

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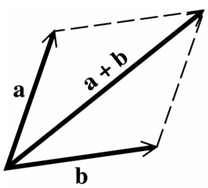

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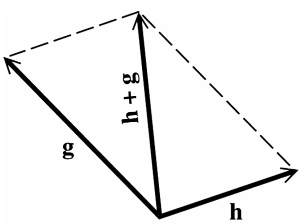

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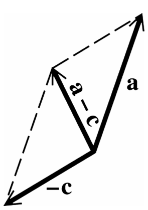

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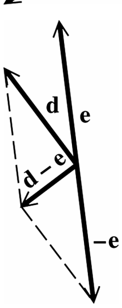
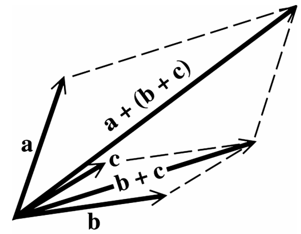
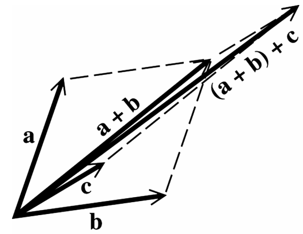


13.



14.



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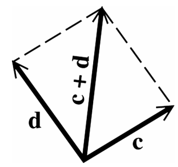
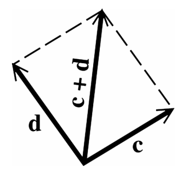
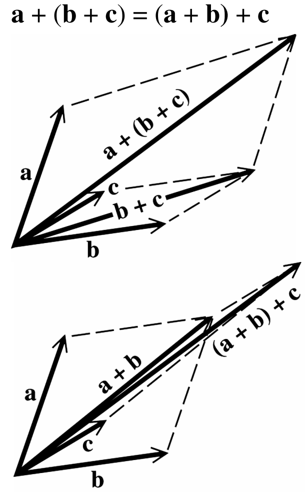


16.

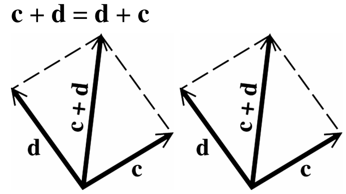


17. a + (b + c) = (a + b) + c



Yes, vector addition is associative.

18. c + d = d + c



Yes, vector addition is commutative.

19. Use the figure to find the components of **a** and **b**: $\mathbf{a} = \langle -8, 8\rangle$ and $\mathbf{b} = \langle 4, 8\rangle$.

(a) $\mathbf{a}+\mathbf{b} = \langle -8,8\rangle + \langle 4,8\rangle = \langle -8+4, 8+8\rangle = \langle -4,16\rangle$

(b) $\mathbf{a}-\mathbf{b} = \langle -8,8\rangle - \langle 4,8\rangle = \langle -8-4, 8-8\rangle = \langle -12,0\rangle$

(c) $-\mathbf{a} = -\langle -8,8\rangle = \langle 8,-8\rangle$

20. Use the figure to find the components of **a** and **b**: $\mathbf{a} = \langle 4,-4\rangle$ and $\mathbf{b} = \langle -8,-4\rangle$.

(a) $\mathbf{a}+\mathbf{b} = \langle 4,-4\rangle + \langle -8,-4\rangle = \langle 4-8, -4-4\rangle = \langle -4,-8\rangle$

(b) $\mathbf{a}-\mathbf{b} = \langle 4,-4\rangle - \langle -8,-4\rangle = \langle 4+8, -4+4\rangle = \langle 12,0\rangle$

(c) $-\mathbf{a} = -\langle 4,-4\rangle = \langle -4,4\rangle$

21. Use the figure to find the components of **a** and **b**: $\mathbf{a} = \langle 4,8\rangle$ and $\mathbf{b} = \langle 4,-8\rangle$.

(a) $\mathbf{a}+\mathbf{b} = \langle 4,8\rangle + \langle 4,-8\rangle = \langle 4+4, 8-8\rangle = \langle 8,0\rangle$

(b) $\mathbf{a}-\mathbf{b} = \langle 4,8\rangle - \langle 4,-8\rangle = \langle 4-4, 8-(-8)\rangle = \langle 0,16\rangle$

(c) $-\mathbf{a} = -\langle 4,8\rangle = \langle -4,-8\rangle$

22. Use the figure to find the components of **a** and **b**: $\mathbf{a} = \langle -4,-4\rangle$ and $\mathbf{b} = \langle 8,4\rangle$.

(a) $\mathbf{a}+\mathbf{b} = \langle -4,-4\rangle + \langle 8,4\rangle = \langle -4+8, -4+4\rangle = \langle 4,0\rangle$

(b) $\mathbf{a}-\mathbf{b} = \langle -4,-4\rangle - \langle 8,4\rangle = \langle -4-8, -4-4\rangle = \langle -12,-8\rangle$

(c) $-\mathbf{a} = -\langle -4,-4\rangle = \langle 4,4\rangle$

23. Use the figure to find the components of **a** and **b**: $\mathbf{a} = \langle -8,4\rangle$ and $\mathbf{b} = \langle 8,8\rangle$.

(a) $\mathbf{a}+\mathbf{b} = \langle -8,4\rangle + \langle 8,8\rangle = \langle -8+8, 4+8\rangle = \langle 0,12\rangle$

(b) $\mathbf{a}-\mathbf{b} = \langle -8,4\rangle - \langle 8,8\rangle = \langle -8-8, 4-8\rangle = \langle -16,-4\rangle$

(c) $-\mathbf{a} = -\langle -8,4\rangle = \langle 8,-4\rangle$

24. Use the figure to find the components of **a** and **b**: $\mathbf{a} = \langle 8,-4\rangle$ and $\mathbf{b} = \langle -4,8\rangle$.

(a) $\mathbf{a}+\mathbf{b} = \langle 8,-4\rangle + \langle -4,8\rangle = \langle 8-4, -4+8\rangle = \langle 4,4\rangle$

(b) $\mathbf{a}-\mathbf{b} = \langle 8,-4\rangle - \langle -4,8\rangle = \langle 8-(-4), -4-8\rangle = \langle 12,-12\rangle$

(c) $-\mathbf{a} = -\langle 8,-4\rangle = \langle -8,4\rangle$

25. **(a)** $2\mathbf{a} = 2(2\mathbf{i}) = 4\mathbf{i}$

(b) $2\mathbf{a}+3\mathbf{b} = 2(2\mathbf{i}) + 3(\mathbf{i}+\mathbf{j}) = 4\mathbf{i}+3\mathbf{i}+3\mathbf{j} = 7\mathbf{i}+3\mathbf{j}$

(c) $\mathbf{b}-3\mathbf{a} = \mathbf{i}+\mathbf{j}-3(2\mathbf{i}) = \mathbf{i}+\mathbf{j}-6\mathbf{i} = -5\mathbf{i}+\mathbf{j}$

26. **(a)** $2\mathbf{a} = 2(-\mathbf{i}+2\mathbf{j}) = -2\mathbf{i}+4\mathbf{j}$

(b) $2\mathbf{a}+3\mathbf{b} = 2(-\mathbf{i}+2\mathbf{j}) + 3(\mathbf{i}-\mathbf{j}) = -2\mathbf{i}+4\mathbf{j}+3\mathbf{i}-3\mathbf{j} = \mathbf{i}+\mathbf{j}$

(c) $\mathbf{b}-3\mathbf{a} = \mathbf{i}-\mathbf{j}-3(-\mathbf{i}+2\mathbf{j}) = \mathbf{i}-\mathbf{j}+3\mathbf{i}-6\mathbf{j} = 4\mathbf{i}-7\mathbf{j}$

27. **(a)** $2\mathbf{a} = 2\langle -1,2\rangle = \langle -2,4\rangle$

(b) $2\mathbf{a}+3\mathbf{b} = 2\langle -1,2\rangle + 3\langle 3,0\rangle = \langle -2,4\rangle + \langle 9,0\rangle = \langle -2+9, 4+0\rangle = \langle 7,4\rangle$

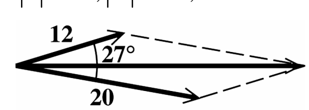
(c) $\mathbf{b}-3\mathbf{a} = \langle 3,0\rangle - 3\langle -1,2\rangle = \langle 3,0\rangle - \langle -3,6\rangle = \langle 3-(-3), 0-6\rangle = \langle 6,-6\rangle$

28. **(a)** $2\mathbf{a} = 2\langle -2,-1\rangle = \langle -4,-2\rangle$

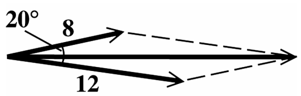
(b) $2\mathbf{a}+3\mathbf{b} = 2\langle -2,-1\rangle + 3\langle -3,2\rangle = \langle -4,-2\rangle + \langle -9,6\rangle = \langle -4-9, -2+6\rangle = \langle -13,4\rangle$

(c) $\mathbf{b}-3\mathbf{a} = \langle -3,2\rangle - 3\langle -2,-1\rangle = \langle -3,2\rangle - \langle -6,-3\rangle = \langle -3-(-6), 2-(-3)\rangle = \langle 3,5\rangle$

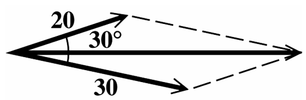
29. $|\boldsymbol{u}| = 12, |\boldsymbol{w}| = 20, \theta = 27°$



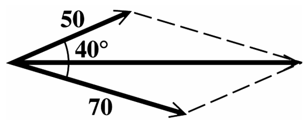
30. $|u| = 8, |w| = 12, \theta = 20°$



31. $|u| = 20, |w| = 30, \theta = 30°$



32. $|u| = 50, |w| = 70, \theta = 40°$



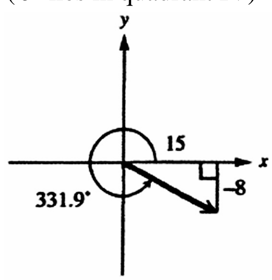
33. Magnitude:

$$\sqrt{15^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Angle: $\tan\theta' = \dfrac{b}{a} \Rightarrow \tan\theta' = \dfrac{-8}{15} \Rightarrow$

$$\theta' = \tan^{-1}\left(-\frac{8}{15}\right) \approx -28.1° \Rightarrow$$
$$\theta = -28.1° + 360° = 331.9°$$

(θ lies in quadrant IV)



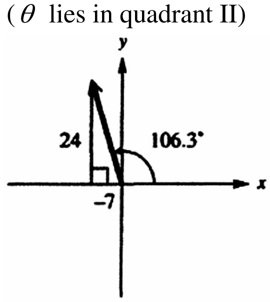
34. Magnitude:

$$\sqrt{(-7)^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$$

Angle: $\tan\theta' = \dfrac{b}{a} \Rightarrow \tan\theta' = \dfrac{24}{-7} \Rightarrow$

$$\theta' = \tan^{-1}\left(-\frac{24}{7}\right) \approx -73.7° \Rightarrow$$
$$\theta = -73.7° + 180° = 106.3°$$

(θ lies in quadrant II)



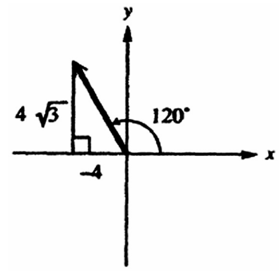
35. Magnitude:

$$\sqrt{(-4)^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16 + 48} = \sqrt{64} = 8$$

Angle: $\tan\theta' = \dfrac{b}{a} \Rightarrow \tan\theta' = \dfrac{4\sqrt{3}}{-4} \Rightarrow$

$$\theta' = \tan^{-1}\left(-\sqrt{3}\right) = -60° \Rightarrow$$
$$\theta = -60° + 180° = 120°$$

(θ lies in quadrant II)



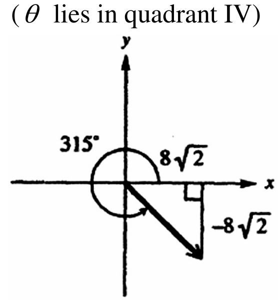
36. Magnitude:

$$\sqrt{\left(8\sqrt{2}\right)^2 + \left(-8\sqrt{2}\right)^2} = \sqrt{128 + 128} = \sqrt{256} = 16$$

Angle: $\tan\theta' = \dfrac{b}{a} \Rightarrow \tan\theta = \dfrac{-8\sqrt{2}}{8\sqrt{2}} \Rightarrow$

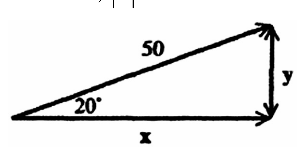
$$\theta' = \tan^{-1}(-1) \approx -45° \Rightarrow$$
$$\theta = -45° + 360° = 315°$$

(θ lies in quadrant IV)



In Exercises 37–42, **x** is the horizontal component of **v**, and **y** is the vertical component of **v**. Thus, $|\mathbf{x}|$ is the magnitude of **x** and $|\mathbf{y}|$ is the magnitude of **y**.

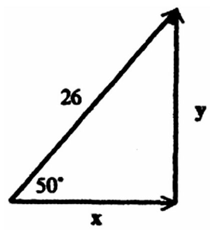
37. $\alpha = 20°, |\mathbf{v}| = 50$



$\mathbf{x} = 50\cos 20° \approx 47 \Rightarrow |\mathbf{x}| \approx 47$

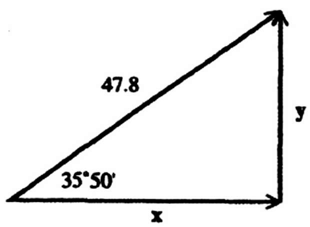
$\mathbf{y} = 50\sin 20° \approx 17 \Rightarrow |\mathbf{y}| \approx 17$

38. $\alpha = 50°$, $|\mathbf{v}| = 26$



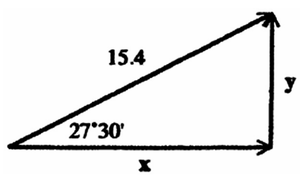
$\mathbf{x} = 26\cos 50° \approx 17 \Rightarrow |\mathbf{x}| \approx 17$
$\mathbf{y} = 26\sin 50° \approx 20 \Rightarrow |\mathbf{y}| \approx 20$

39. $\alpha = 35°50'$, $|\mathbf{v}| = 47.8$



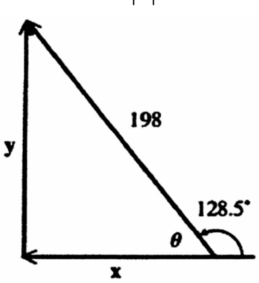
$\mathbf{x} = 47.8\cos 35°50' \approx 38.8 \Rightarrow |\mathbf{x}| \approx 38.8$
$\mathbf{y} = 47.8\sin 35°50' \approx 28.0 \Rightarrow |\mathbf{y}| \approx 28.0$

40. $\alpha = 27°30'$, $|\mathbf{v}| = 15.4$



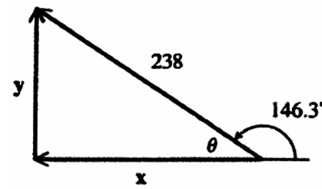
$\mathbf{x} = 15.4\cos 27°30' \approx 13.7 \Rightarrow |\mathbf{x}| \approx 13.7$
$\mathbf{y} = 15.4\sin 27°30' = 7.11 \Rightarrow |\mathbf{y}| \approx 7.11$

41. $\alpha = 128.5°$, $|\mathbf{v}| = 198$



$\mathbf{x} = 198\cos 128.5° \approx -123 \Rightarrow |\mathbf{x}| \approx 123$
$\mathbf{y} = 198\sin 128.5° \approx 155 \Rightarrow |\mathbf{y}| \approx 155$

42. $\alpha = 146.3°$, $|\mathbf{v}| = 238$



$\mathbf{x} = 238\cos 146.3° \approx -198 \Rightarrow |\mathbf{x}| \approx 198$
$\mathbf{y} = 238\sin 146.3° \approx 132 \Rightarrow |\mathbf{y}| \approx 132$

43. $\mathbf{u} = \langle a, b\rangle = \langle 5\cos(30°), 5\sin(30°)\rangle$
$= \left\langle \frac{5\sqrt{3}}{2}, \frac{5}{2} \right\rangle$

44. $\mathbf{u} = \langle a, b\rangle = \langle 8\cos(60°), 8\sin(60°)\rangle = \langle 4, 4\sqrt{3}\rangle$

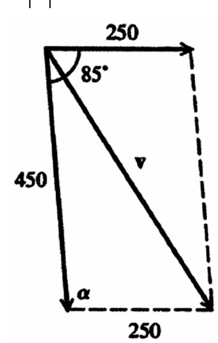
45. $\mathbf{v} = \langle a, b\rangle = \langle 4\cos(40°), 4\sin(40°)\rangle$
$\approx \langle 3.0642, 2.5712\rangle$

46. $\mathbf{v} = \langle a, b\rangle = \langle 3\cos(130°), 3\sin(130°)\rangle$
$\approx \langle -1.9284, 2.2981\rangle$

47. $\mathbf{v} = \langle a, b\rangle = \langle 5\cos(-35°), 5\sin(-35°)\rangle$
$\approx \langle 4.0958, -2.8679\rangle$

48. $\mathbf{v} = \langle a, b\rangle = \langle 2\cos(220°), 2\sin(220°)\rangle$
$\approx \langle -1.5321, -1.2856\rangle$

49. Forces of 250 newtons and 450 newtons, forming an angle of 85°
$\alpha = 180° - 85° = 95°$
$|\mathbf{v}|^2 = 250^2 + 450^2 - 2(250)(450)\cos 95°$
$|\mathbf{v}|^2 \approx 284{,}610.04$
$|\mathbf{v}| \approx 533.5$



The magnitude of the resulting force is 530 newtons. (rounded to two significant digits)

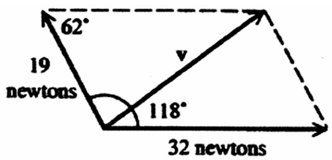
50. Forces of 19 newtons and 32 newtons, forming an angle of 118°

$\alpha = 180° - 118° = 62°$

$|\mathbf{v}|^2 = 19^2 + 32^2 - 2(19)(32)\cos 62°$

$|\mathbf{v}|^2 \approx 814.12257$

$|\mathbf{v}| \approx 28.53$ newtons



The magnitude of the resulting force is 29 newtons. (rounded to two significant digits)

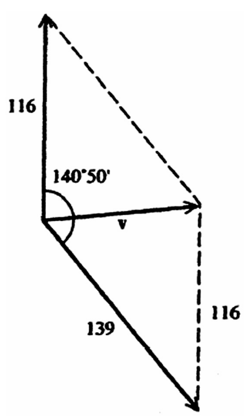
51. Forces of 116 lb and 139 lb, forming an angle of 140° 50′

$\alpha = 180° - 140°50'$
$= 179°60' - 140°50' = 39°10'$

$|\mathbf{v}|^2 = 139^2 + 116^2 - 2(139)(116)\cos 39°10'$

$|\mathbf{v}|^2 \approx 7774.7359$

$|\mathbf{v}| \approx 88.174$



The magnitude of the resulting force is 88.2 lb. (rounded to three significant digits)

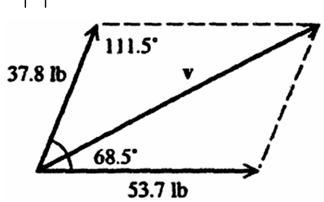
52. Forces of 37.8 lb and 53.7 lb, forming an angle of 68.5°

$\alpha = 180° - 68.5° = 111.5°$

$|\mathbf{v}|^2 = 37.8^2 + 53.7^2 - 2(37.8)(53.7)\cos 111.5°$

$|\mathbf{v}|^2 = 5800.4224$

$|\mathbf{v}| = 76.161$



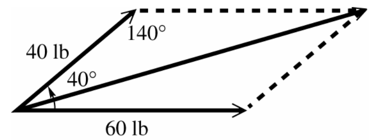
The magnitude of the resulting force is 76.2 lb. (rounded to three significant digits)

53. $\alpha = 180° - 40° = 140°$

$|\mathbf{v}|^2 = 40^2 + 60^2 - 2(40)(60)\cos 140°$

$|\mathbf{v}|^2 \approx 8877.0133$

$|\mathbf{v}| \approx 94.2$ lb

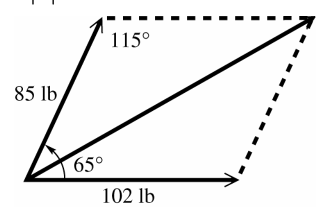


54. $\alpha = 180° - 65° = 115°$

$|\mathbf{v}|^2 = 85^2 + 102^2 - 2(85)(102)\cos 115°$

$|\mathbf{v}|^2 \approx 24,957.201$

$|\mathbf{v}| \approx 158.0$ lb

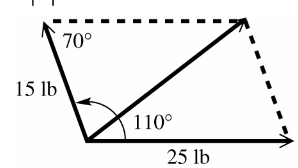


55. $\alpha = 180° - 110° = 70°$

$|\mathbf{v}|^2 = 15^2 + 25^2 - 2(15)(25)\cos 70°$

$|\mathbf{v}|^2 \approx 593.48489$

$|\mathbf{v}| \approx 24.4$ lb

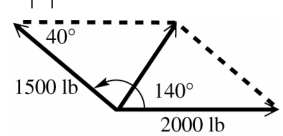


56. $\alpha = 180° - 140° = 40°$

$|\mathbf{v}|^2 = 1500^2 + 2000^2 - 2(1500)(2000)\cos 40°$

$|\mathbf{v}|^2 \approx 1,653,733.3$

$|\mathbf{v}| \approx 1286.0$ lb



57. If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $\boldsymbol{u} + \boldsymbol{v} = \langle a + c, b + d\rangle$.

58. With complex numbers, if $z_1 = a + bi$ and $z_2 = c + di$, then we have
$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$
Additional answers will vary.

For Exercises 59–66, $\mathbf{u} = \langle -2, 5\rangle$ and $\mathbf{v} = \langle 4, 3\rangle$.

59. $\mathbf{u}+\mathbf{v} = \langle -2, 5\rangle + \langle 4, 3\rangle$
$= \langle -2+4, 5+3\rangle = \langle 2, 8\rangle$

60. $\mathbf{u}-\mathbf{v} = \langle -2, 5\rangle - \langle 4, 3\rangle$
$= \langle -2-4, 5-3\rangle = \langle -6, 2\rangle$

61. $-4\mathbf{u} = -4\langle -2,5\rangle = \langle -4(-2), -4(5)\rangle = \langle 8, -20\rangle$

62. $-5\mathbf{v} = -5\langle 4, 3\rangle = \langle -5\cdot 4, -5\cdot 3\rangle = \langle -20, -15\rangle$

63. $3\mathbf{u}-6\mathbf{v} = 3\langle -2,5\rangle - 6\langle 4,3\rangle$
$= \langle -6,15\rangle - \langle 24,18\rangle$
$= \langle -6-24, 15-18\rangle = \langle -30,-3\rangle$

64. $-2\mathbf{u}+4\mathbf{v} = -2\langle -2,5\rangle + 4\langle 4,3\rangle$
$= \langle (-2)(-2), (-2)(5)\rangle + \langle 4\cdot 4, 4\cdot 3\rangle$
$= \langle 4,-10\rangle + \langle 16,12\rangle$
$= \langle 4+16, -10+12\rangle = \langle 20, 2\rangle$

65. $\mathbf{u}+\mathbf{v}-3\mathbf{u} = \langle -2,5\rangle + \langle 4,3\rangle - 3\langle -2,5\rangle$
$= \langle -2,5\rangle + \langle 4,3\rangle - \langle 3(-2), 3(5)\rangle$
$= \langle -2,5\rangle + \langle 4,3\rangle - \langle -6,15\rangle$
$= \langle -2+4, 5+3\rangle - \langle -6,15\rangle$
$= \langle 2,8\rangle - \langle -6,15\rangle$
$= \langle 2-(-6), 8-15\rangle = \langle 8,-7\rangle$

66. $2\mathbf{u}+\mathbf{v}-6\mathbf{v} = 2\langle -2,5\rangle + \langle 4,3\rangle - 6\langle 4,3\rangle$
$= \langle 2(-2), 2\cdot 5\rangle + \langle 4,3\rangle - \langle 6\cdot 4, 6\cdot 3\rangle$
$= \langle -4,10\rangle + \langle 4,3\rangle - \langle 24,18\rangle$
$= \langle -4+4-24, 10+3-18\rangle$
$= \langle -24,-5\rangle$

67. $\langle -5, 8\rangle = -5\mathbf{i}+8\mathbf{j}$

68. $\langle 6, -3\rangle = 6\mathbf{i}-3\mathbf{j}$

69. $\langle 2, 0\rangle = 2\mathbf{i}+0\mathbf{j} = 2\mathbf{i}$

70. $\langle 0, -4\rangle = 0\mathbf{i}-4\mathbf{j} = -4\mathbf{j}$

71. $\langle 6, -1\rangle\cdot\langle 2, 5\rangle = 6(2)+(-1)(5) = 12-5 = 7$

72. $\langle -3, 8\rangle\cdot\langle 7, -5\rangle = -3(7)+8(-5)$
$= -21-40 = -61$

73. $\langle 2, -3\rangle\cdot\langle 6, 5\rangle = 2(6)+(-3)(5) = 12-15 = -3$

74. $\langle 1, 2\rangle\cdot\langle 3, -1\rangle = 1(3)+2(-1) = 3-2 = 1$

75. $4\mathbf{i} = \langle 4,0\rangle; 5\mathbf{i}-9\mathbf{j} = \langle 5,-9\rangle$
$\langle 4,0\rangle\cdot\langle 5,-9\rangle = 4(5)+0(-9) = 20-0 = 20$

76. $2\mathbf{i}+4\mathbf{j} = \langle 2,4\rangle; -\mathbf{j} = \langle 0,-1\rangle$
$\langle 2,4\rangle\cdot\langle 0,-1\rangle = 2(0)+4(-1) = 0-4 = -4$

77. $\langle 2, 1\rangle\cdot\langle -3, 1\rangle$

$$\cos\theta = \frac{\langle 2, 1\rangle\cdot\langle -3, 1\rangle}{\sqrt{2^2+1^2}\cdot\sqrt{(-3)^2+1^2}} = \frac{-6+1}{\sqrt{5}\cdot\sqrt{10}}$$
$$= \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = 135°$$

78. $\langle 1, 7\rangle\cdot\langle 1, 1\rangle$

$$\cos\theta = \frac{\langle 1, 7\rangle\cdot\langle 1, 1\rangle}{\sqrt{1^2+7^2}\cdot\sqrt{1^2+1^2}} = \frac{1+7}{\sqrt{50}\cdot\sqrt{2}}$$
$$= \frac{8}{10} = .8 \Rightarrow \theta \approx 36.87°$$

79. $\langle 1, 2\rangle\cdot\langle -6, 3\rangle$

$$\cos\theta = \frac{\langle 1, 2\rangle\cdot\langle -6, 3\rangle}{\sqrt{1^2+2^2}\cdot\sqrt{(-6)^2+3^2}} \frac{-6+6}{\sqrt{5}\sqrt{45}}$$
$$= \frac{0}{15} = 0 \Rightarrow \theta = 90°$$

80. $\langle 4, 0\rangle\cdot\langle 2, 2\rangle$

$$\cos\theta = \frac{\langle 4, 0\rangle\cdot\langle 2, 2\rangle}{\sqrt{4^2+0^2}\cdot\sqrt{2^2+2^2}} = \frac{8+0}{\sqrt{16}\cdot\sqrt{8}}$$
$$= \frac{8}{8\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = 45°$$

81. First write the given vectors in component form: $3\mathbf{i}+4\mathbf{j} = \langle 3,4\rangle$ and $\mathbf{j} = \langle 0,1\rangle$

$$\cos\theta = \frac{\langle 3, 4\rangle\cdot\langle 0, 1\rangle}{|\langle 3, 4\rangle||\langle 0, 1\rangle|} = \frac{\langle 3, 4\rangle\cdot\langle 0, 1\rangle}{\sqrt{3^2+4^2}\cdot\sqrt{0^2+1^2}}$$
$$= \frac{0+4}{\sqrt{25}\cdot\sqrt{1}} = \frac{4}{5\cdot 1} = \frac{4}{5} = .8 \Rightarrow$$
$$\theta = \cos^{-1}.8 \approx 36.87°$$

82. First write the given vectors in component form: $-5\mathbf{i}+12\mathbf{j} = \langle -5,12\rangle$ and $3\mathbf{i}+2\mathbf{j} = \langle 3,2\rangle$

$$\cos\theta = \frac{\langle -5, 12\rangle\cdot\langle 3, 2\rangle}{\sqrt{(-5)^2+12^2}\cdot\sqrt{3^2+2^2}} = \frac{-15+24}{\sqrt{169}\sqrt{13}}$$
$$= \frac{9}{13\sqrt{13}} = \frac{9\sqrt{13}}{169} \Rightarrow \theta \approx 78.93°$$

For Exercises 83−86, $\mathbf{u} = \langle -2,1 \rangle$, $\mathbf{v} = \langle 3,4 \rangle$, and $w = \langle -5,12 \rangle$.

83. $(3\mathbf{u})\cdot\mathbf{v} = \left(3\langle -2, 1\rangle\right)\cdot\langle 3, 4\rangle$
$= \langle -6, 3\rangle\cdot\langle 3, 4\rangle = -18+12 = -6$

84. $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \langle -2,1\rangle\cdot\left(\langle 3,4\rangle - \langle -5,12\rangle\right)$
$= \langle -2,1\rangle\cdot\langle 3-(-5), 4-12\rangle$
$= \langle -2,1\rangle\cdot\langle 8,-8\rangle = -16-8 = -24$

85. $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} = \langle -2, 1\rangle\cdot\langle 3,4\rangle - \langle -2,1\rangle\cdot\langle -5, 12\rangle$
$= (-6+4)-(10+12)$
$= -2-22 = -24$

86. $\mathbf{u}\cdot(3\mathbf{v}) = \langle -2,1\rangle\cdot\left(3\langle 3,4\rangle\right) = \langle -2,1\rangle\cdot\langle 9,12\rangle$
$= -18+12 = -6$

87. Since $\langle 1,2\rangle\cdot\langle -6,3\rangle = -6+6 = 0$, the vectors are orthogonal.

88. Since $\langle 3,4\rangle\cdot\langle 6,8\rangle = 18+32 = 50 \neq 0$, the vectors are not orthogonal.

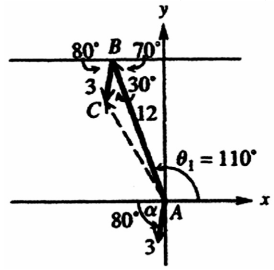
89. Since $\langle 1,0\rangle\cdot\langle \sqrt{2},0\rangle = \sqrt{2}+0 = \sqrt{2} \neq 0$, the vectors are not orthogonal

90. Since $\langle 1,1\rangle\cdot\langle 1,-1\rangle = 1-1 = 0$, the vectors are orthogonal.

91. $\sqrt{5}\mathbf{i} - 2\mathbf{j} = \langle \sqrt{5},-2\rangle; -5\mathbf{i}+2\sqrt{5}\mathbf{j} = \langle -5, 2\sqrt{5}\rangle$
Since $\langle \sqrt{5},-2\rangle\cdot\langle -5,2\sqrt{5}\rangle = -5\sqrt{5}-4\sqrt{5}$
$= -9\sqrt{5} \neq 0$, the vectors are not orthogonal.

92. $-4\mathbf{i}+3\mathbf{j} = \langle -4,3\rangle; 8\mathbf{i}-6\mathbf{j} = \langle 8,-6\rangle$
Since $\langle -4,3\rangle\cdot\langle 8,-6\rangle = -32-18 = -50 \neq 0$, the vectors are not orthogonal

93. Draw a line parallel to the *x*-axis and the vector **u + v** (shown as a dashed line)
Since $\theta_1 = 110°$, its supplementary angle is 70°. Further, since $\theta_2 = 260°$, the angle α is $260° - 180° = 80°$. Then the angle *CBA* becomes 180 – (80 + 70) = 180 – 150 = 30°.



Using the law of cosines, the magnitude of **u + v** is found as follows:

$$|\mathbf{u}+\mathbf{v}|^2 = a^2 + c^2 - 2ac\cos B$$
$$|\mathbf{u}+\mathbf{v}|^2 = 3^2 + 12^2 - 2(3)(12)\cos 30°$$
$$= 9+144-72\cdot\frac{\sqrt{3}}{2} = 153-36\sqrt{3}$$
$$\approx 90.646171$$

Thus, $|\mathbf{u}+\mathbf{v}| \approx 9.5208.$

Using the law of sines, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{3} = \frac{\sin 30°}{9.5208} \Rightarrow$$
$$\sin A = \frac{3\sin 30°}{9.5208} = \frac{3\cdot\frac{1}{2}}{9.5208} \approx .15754979 \Rightarrow$$
$A \approx 9.0647°$

The direction angle of **u + v** is $110° + 9.0647° = 119.0647°$.

94. Since $a = 12\cos 110° \approx -4.10424172$ and $b = 12\sin 110° \approx 11.27631145$,
$\langle a,b\rangle \approx \langle -4.1042, 11.2763\rangle.$

95. Since $c = 3\cos 260° \approx -.52094453$ and $d = 3\sin 260° \approx -2.95442326$,
$\langle c,d\rangle \approx \langle -.5209, -2.9544\rangle.$

96. If $\mathbf{u} = \langle a,b\rangle = \langle -4.10424, 11.27631\rangle$ and $\mathbf{v} = \langle c,d\rangle = \langle -.52094, -2.95442\rangle$, then
u + v
$$= \langle -4.10424172 + (-.52094453), 11.27631145 + (-2.95442326)\rangle$$
$$= \langle -4.62518625, 8.32188819\rangle$$
$$\approx \langle -4.6252, 8.3219\rangle$$

97. Magnitude:
$$\sqrt{(-4.62518625)^2 + 8.32188819^2} \approx 9.5208$$
Angle:
$$\tan\theta' = \frac{8.32188819}{-4.625186258} \Rightarrow \theta' \approx -60.9353° \Rightarrow$$
$\theta = -60.9353° + 180° = 119.0647°$
(θ lies in quadrant II)

98. They are the same. Preference of method is an individual choice.

Section 8.4: Applications of Vectors

1. Find the direction and magnitude of the equilibrant.
Since $A = 180° - 28.2° = 151.8°$, we can use the law of cosines to find the magnitude of the resultant, **v**.

$$|\mathbf{v}|^2 = 1240^2 + 1480^2 - 2(1240)(1480)\cos 151.8°$$
$$\approx 6962736.2 \Rightarrow |\mathbf{v}| \approx 2639 \text{ lb}$$

(will be rounded as 2640)
Use the law of sines to find α.

$$\frac{\sin\alpha}{1240} = \frac{\sin 151.8°}{2639}$$
$$\sin\alpha = \frac{1240\sin 151.8°}{2639} \approx .22203977$$
$$\alpha \approx 12.8°$$

Thus, we have 2640 lb at an angle of $\theta = 180° - 12.8° = 167.2°$ with the 1480-lb force.

2. Find the direction and magnitude of the equilibrant.
Since $A = 180° - 24.5° = 155.5°$, we can use the law of cosines to find the magnitude of the resultant, **v**.

$$|\mathbf{v}|^2 = 840^2 + 960^2 - 2(840)(960)\cos 155.5°$$
$$\approx 3094785.5 \Rightarrow |\mathbf{v}| \approx 1759 \text{ lb} \approx 1800 \text{ lb}$$

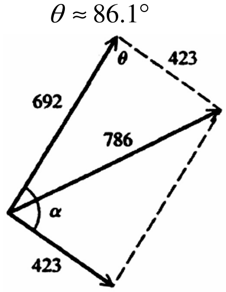
Use the law of sines to find α.

$$\frac{\sin\alpha}{960} = \frac{\sin 155.5°}{1759} \Rightarrow$$
$$\sin\alpha = \frac{960\sin 155.5°}{1759} \approx .22632491 \Rightarrow$$
$$\alpha \approx 13°$$

Thus, we have 1800 lb at an angle of $\theta = 180° - 13° = 167°$ with the 840-lb force.

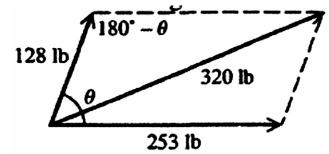
3. Let α = the angle between the forces.
To find α, use the law of cosines to find θ.

$$786^2 = 692^2 + 423^2 - 2(692)(423)\cos\theta$$
$$\cos\theta = \frac{692^2 + 423^2 - 786^2}{2(692)(423)} \approx .06832049$$
$$\theta \approx 86.1°$$



Thus, $\alpha = 180° - 86.1° = 93.9°$.

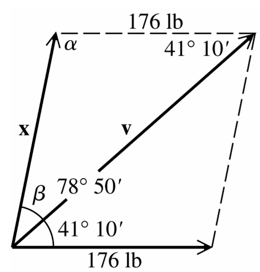
4. Let θ = the angle between the forces.



$$320^2 = 128^2 + 253^2 - 2(128)(253)\cos(180° - \theta)$$
$$\cos(180° - \theta) = \frac{128^2 + 253^2 - 320^2}{2(128)(253)}$$
$$\approx -.33978199$$
$$180° - \theta = 109.9°$$
$$\theta = 70.1°$$

5. Use the parallelogram rule. In the figure, **x** represents the second force and **v** is the resultant.

$$\alpha = 180° - 78°50'$$
$$= 179°60' - 78°50' = 101°10'$$
and
$$\beta = 78°50' - 41°10' = 37°40'$$

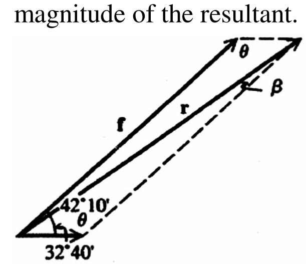


Using the law of sines, we have

$$\frac{|\mathbf{x}|}{\sin 41°10'} = \frac{176}{\sin 37°40'} \Rightarrow$$
$$|\mathbf{x}| = \frac{176\sin 41°10'}{\sin 37°40'} \approx 190$$
$$\frac{|\mathbf{v}|}{\sin\alpha} = \frac{176}{\sin 37°40'} \Rightarrow$$
$$|\mathbf{v}| = \frac{176\sin 101°10'}{\sin 37°40'} \approx 283$$

Thus, the magnitude of the second force is about 190 lb and the magnitude of the resultant is about 283 lb.

6. Let $|\mathbf{f}|$ = the second force and $|\mathbf{r}|$ = the magnitude of the resultant.



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Using the law of sines, we have

$$\frac{|\mathbf{r}|}{\sin\theta} = \frac{28.7}{\sin 9°30'} \Rightarrow$$

$$|\mathbf{r}| = \frac{28.7 \sin 9°30'}{\sin 37°40'} \approx 116.73 \text{ lb}$$

(will be rounded as 117)

$$\frac{|\mathbf{r}|}{\sin\theta} = \frac{28.7}{\sin 9°30'}$$

$$|\mathbf{r}| = \frac{28.7 \sin 137°50'}{\sin 9°30'} \approx 116.73 \text{ lb}$$

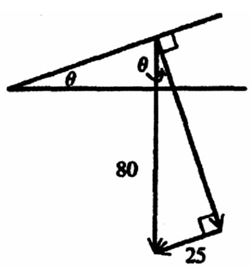
$$\frac{|\mathbf{f}|}{\sin 32°40'} = \frac{|\mathbf{r}|}{\sin 137°50'}$$

$$|\mathbf{f}| = \frac{116.73 \sin 32°40'}{\sin 137°50'} \approx 93.9 \text{ lb}$$

The magnitude of the resultant is 117 lb; the second force is 93.9 lb.

7. Let θ = the angle that the hill makes with the horizontal.
The 80-lb downward force has a 25-lb component parallel to the hill. The two right triangles are similar and have congruent angles.

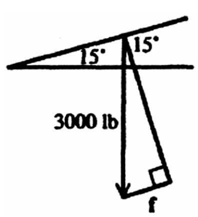
$$\sin\theta = \frac{25}{80} = \frac{5}{16} = .3125 \Rightarrow \theta \approx 18°$$



8. Let $|\mathbf{f}|$ = the force required to keep the car parked on hill.

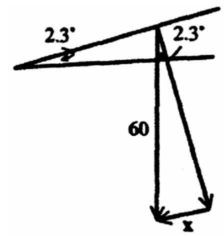
$$\frac{|\mathbf{f}|}{3000} = \sin 15° \Rightarrow |\mathbf{f}| = 3000 \sin 15° = 776.5 \text{ lb}$$

The force required to keep the car parked on the hill is approximately 800 lb.



9. Find the force needed to pull a 60-ton monolith along the causeway.
The force needed to pull 60 tons is equal to the magnitude of **x**, the component parallel to the causeway.

$$\sin 2.3° = \frac{|\mathbf{x}|}{60} \Rightarrow |\mathbf{x}| = 60 \sin 2.3° \approx 2.4 \text{ tons}$$

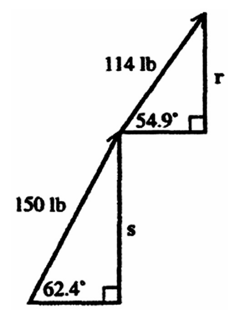


The force needed is 2.4 tons.

10. Let **r** = the vertical component of the person exerting a 114-lb force;
s = the vertical component of the person exerting a 150-lb force.
The weight of the box is the sum of the magnitudes of the vertical components of the two vectors representing the forces exerted by the two people.

$$|\mathbf{r}| = 114 \sin 54.9° \approx 93.27 \text{ and}$$

$$|\mathbf{s}| = 150 \sin 62.4° \approx 132.93$$



Thus, the weight of box is

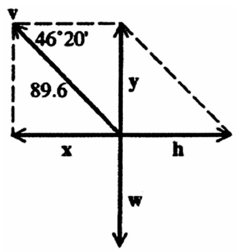
$$|\mathbf{r}| + |\mathbf{s}| \approx 93.27 + 132.93 = 226.2 \approx 226 \text{ lb.}$$

11. Like Example 3 on page 345 of the text, angle B equals angle θ, and here the magnitude of vector **BA** represents the weight of the stump grinder. The vector **AC** equals vector **BE**, which represents the force required to hold the stump grinder on the incline. Thus, we have

$$\sin B = \frac{18}{60} = \frac{3}{10} = .3 \Rightarrow B \approx 17.5°$$

12. Like Example 3 on page 345 of the text, angle B equals angle θ, and here the magnitude of vector **BA** represents the weight of the pressure washer. The vector **AC** equals vector **BE**, which represents the force required to hold the pressure washer on the incline. Thus we have $\sin B = \frac{30}{80} = \frac{3}{8} = .375 \Rightarrow B \approx 22.0°$

13. Find the weight of the crate and the tension on the horizontal rope.



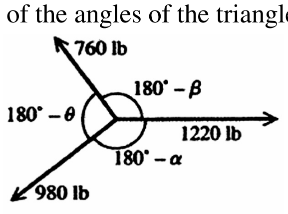
v has horizontal component **x** and vertical component **y**. The resultant **v + h** also has vertical component **y**. The resultant balances the weight of the crate, so its vertical component is the equilibrant of the crate's weight: $|\mathbf{w}| = |\mathbf{y}| = 89.6 \sin 46° 20' \approx 64.8$ lb

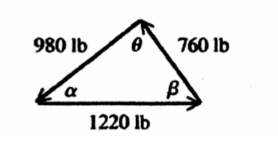
Since the crate is not moving side-to-side, **h**, the horizontal tension on the rope, is the opposite of **x**.

$|\mathbf{h}| = |\mathbf{x}| = 89.6 \cos 46° 20' \approx 61.9$ lb

The weight of the crate is 64.8 lb; the tension is 61.9 lb.

14. Draw a vector diagram showing the three forces acting at a point in equilibrium. Then, arrange the forces to form a triangle. The angles between the forces are the supplements of the angles of the triangle.





(1) $\cos\theta = \frac{980^2 + 760^2 - 1220^2}{2(980)(760)} = \frac{49,600}{1,489,600} = \frac{31}{931} \approx .03329753 \Rightarrow \theta \approx 88.1°$

The angle between the 760-lb and 980-lb forces is $180° - \theta = 180° - 88.1° = 91.9°$

(2) $\frac{\sin\alpha}{760} = \frac{\sin\theta}{1220} \Rightarrow$

$\sin\alpha = \frac{760 \sin 88.1°}{1220} \approx .62260833 \Rightarrow$

$\alpha \approx 38.5°$

The angle between the 1220-lb and 980-lb forces is

$180° - \alpha = 180° - 38.5° = 141.5°$

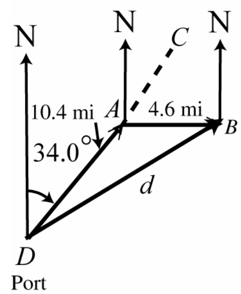
(3) $\beta = 180° - \alpha - \theta$

$= 180° - 38.5° - 88.1° = 53.4°$

The angle between the 760-lb and 1220-lb forces is

$180° - \beta = 180° - 53.4° = 126.6°$.

15. Refer to the diagram below. In order for the ship to turn due east, the ship must turn the measure of angle CAB, which is $90° - 34° = 56°$. Angle DAB is therefore $180° - 56° = 124°$.



Using the law of cosines, we can solve for the distance the ship is from port as follows.

$d^2 = 10.4^2 + 4.6^2 - 2(10.4)(4.6)\cos 124°$

$\approx 182.824 \Rightarrow d \approx 13.52$

hus, the distance the ship is from port is 13.5 mi. (rounded to three significant digits)

To find the bearing, we first seek the measure of angle ADB, which we will refer to as angle D. Using the law of cosines we have

$\cos D = \frac{13.52^2 + 10.4^2 - 4.6^2}{2(13.52)(10.4)} \approx .95937073 \Rightarrow$

$D \approx 16.4°$

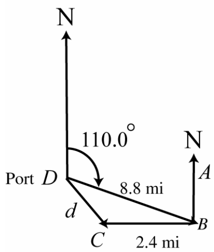
Thus, the bearing is

$34.0° + D = 34.0° + 16.4° = 50.4°$.

16. Refer to the diagram on the next page. In order for the luxury liner to turn due west, the measure of angle ABD must be $180° - 110° = 70°$. Thus, angle DBC is $90° - 70° = 20°$.

(*continued on next page*)

(*continued from page 775*)



Using the law of cosines, we can solve for the distance the luxury liner is from port as follows.

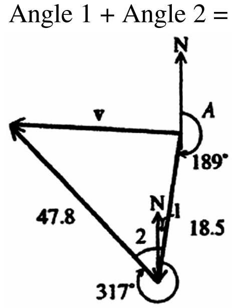
$$d^2 = 8.8^2 + 2.4^2 - 2(8.8)(2.4)\cos 20°$$
$$\approx 43.507 \Rightarrow d \approx 6.596$$

Thus, the distance the luxury liner is from port is 6.6 mi. (rounded to two significant digits) To find the bearing, we first seek the measure of angle *BDC*, which we will refer to as angle *D*. Using the law of cosines we have

$$\cos D = \frac{6.596^2 + 8.8^2 - 2.4^2}{2(6.596)(8.8)} \approx .99222683 \Rightarrow$$
$$D \approx 7.1°$$

Thus, the bearing is
$110.0° + D = 110.0° + 7.1° = 117.1°$.

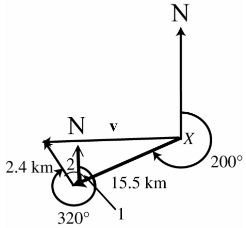
17. Find the distance of the ship from point *A*.
Angle 1 = 189° – 180° = 9°
Angle 2 = 360° – 317° = 43°
Angle 1 + Angle 2 = 9° + 43° = 52°



Use the law of cosines to find $|\mathbf{v}|$.

$$|\mathbf{v}|^2 = 47.8^2 + 18.5^2 - 2(47.8)(18.5)\cos 52°$$
$$\approx 1538.23 \Rightarrow |\mathbf{v}| \approx 39.2 \text{ km}$$

18. Find the distance of the ship from point *X*.
Angle 1 = 200° – 180° = 20°
Angle 2 = 360° – 320° = 40°
Angle 1 + Angle 2 = 20° + 40° = 60°

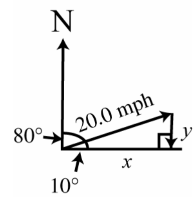


Use the law of cosines to find $|\mathbf{v}|$.

$$|\mathbf{v}|^2 = 2.4^2 + 15.5^2 - 2(2.4)(15.5)\cos 60°$$
$$= 208.81 \Rightarrow |\mathbf{v}| \approx 14.5 \text{ km}$$

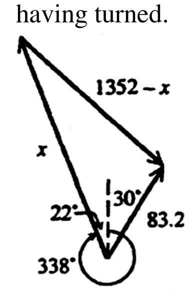
19. Let x = be the actual speed of the motorboat; y = the speed of the current.

$$\sin 10° = \frac{y}{20.0} \Rightarrow y = 20.0\sin 10° \approx 3.5$$
$$\cos 10° = \frac{x}{20.0} \Rightarrow x = 20.0\cos 10° \approx 19.7$$



The speed of the current is 3.5 mph and the actual speed of the motorboat is 19.7 mph.

20. At what time did the pilot turn? The plane will fly 2.6 hr before it runs out of fuel. In 2.6 hr, the carrier will travel (2.6)(32) = 83.2 mi and the plane will travel a total of (2.6)(520) = 1352 mi. Suppose it travels x mi on its initial bearing; then it travels 1352 – x mi after having turned.



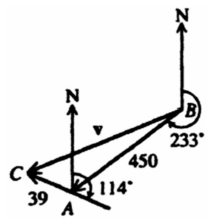
Use the law of cosines to get an equation in x and solve.

$$(1352 - x)^2 = x^2 + 83.2^2 - 2(x)(83.2)\cos 52°$$
$$1{,}827{,}904 - 2704x + x^2 = x^2 + 6922.24 - (166.4\cos 52°)x$$
$$1{,}827{,}904 - 2704x = 6922.24 - (166.4\cos 52°)x$$
$$1{,}820{,}982 - 2704x = -(166.4\cos 52°)x$$
$$1{,}820{,}982 = 2704x - (166.4\cos 52°)x$$
$$1{,}820{,}982 = 2(2704 - 166.4\cos 52°)x$$
$$x = \frac{1{,}820{,}982}{2704 - 166.4\cos 52°}$$
$$x \approx 700$$

To travel 700 mi at 520 mph requires $\frac{700}{520} = 1.35$ hr, or 1 hr and 21 min. Thus, the pilot turned at 1 hr 21 min after 2 P.M., or at 3:21 P.M.

21. Let **v** = the ground speed vector.
Find the bearing and ground speed of the plane.
Angle $A = 233° - 114° = 119°$
Use the law of cosines to find $|\mathbf{v}|$.

$$|\mathbf{v}|^2 = 39^2 + 450^2 - 2(39)(450)\cos 119°$$
$$|\mathbf{v}|^2 \approx 221{,}037.82$$
$$|\mathbf{v}| \approx 470.1$$



The ground speed is 470 mph. (rounded to two significant digits)
Use the law of sines to find angle B.

$$\frac{\sin B}{39} = \frac{\sin 119°}{470.1} \Rightarrow$$
$$\sin B = \frac{39 \sin 119°}{470.1} \approx .07255939 \Rightarrow B \approx 4°$$

Thus, the bearing is
$B + 233° = 4° + 233° = 237°$.

22. **(a)** Relative to the banks, the motorboat will be traveling at the following speed.

$$x^2 + 3.0^2 = 7.0^2 \Rightarrow x^2 + 9 = 49 \Rightarrow$$
$$x^2 = 40 \Rightarrow x = \sqrt{40} \approx 6.3 \text{ mph}$$

(b) At 6.3 mph, the motorboat travels the following rate.

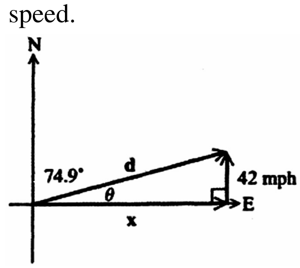
$$\frac{6.3 \text{ mi}}{\text{hr}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} = 9.24 \text{ ft/sec}$$

Crossing a 132-foot wide river would take

$$132 \text{ feet} \cdot \frac{1 \text{ second}}{9.24 \text{ feet}} \approx 14 \text{ seconds}$$

(c) $\sin\theta = \frac{3.0}{7.0} \approx .42857143 \Rightarrow \theta \approx 25°$

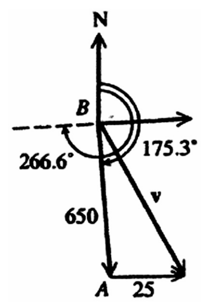
23. Let $|\mathbf{x}|$ = the airspeed and $|\mathbf{d}|$ = the ground speed.



$$\theta = 90° - 74.9° = 15.1°$$
$$\frac{|\mathbf{x}|}{42} = \cot 15.1° \Rightarrow$$
$$|\mathbf{x}| = 42 \cot 15.1° = \frac{42}{\tan 15.1°} \approx 156 \text{ mph}$$
$$\frac{|\mathbf{d}|}{42} = \csc 15.1° \Rightarrow$$
$$|\mathbf{d}| = 42 \csc 15.1° = \frac{42}{\sin 15.1°} \approx 161 \text{ mph}$$

24. Let **v** = the ground speed vector of the plane.
Find the actual bearing of the plane.
Angle $A = 266.6° - 175.3° = 91.3°$
Use the law of cosines to find $|\mathbf{v}|$.

$$|\mathbf{v}|^2 = 25^2 + 650^2 - 2(25)(650)\cos 91.3°$$
$$|\mathbf{v}|^2 = 423{,}862.33$$
$$|\mathbf{v}| = 651.0$$

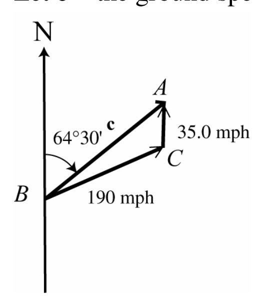


Use the law of sines to find, B the angle that **v** makes with the airspeed vector of the plane.

$$\frac{\sin B}{25} = \frac{\sin A}{651.0} \Rightarrow$$
$$\sin B = \frac{25 \sin 91.3°}{651.0} \approx .03839257 \Rightarrow B \approx 2.2°$$

Thus, the bearing is
$175.1° - B = 175.3° - 2.2° = 173.1°$.

25. Let **c** = the ground speed vector.



By alternate interior angles, angle $A = 64°30'$.
Use the law of sines to find B.

$$\frac{\sin B}{35.0} = \frac{\sin A}{190.0} \Rightarrow$$
$$\sin B = \frac{35.0 \sin 64°30'}{190.0} \approx .16626571 \Rightarrow$$
$$B \approx 9.57° \approx 9°30'$$

(*continued on next page*)

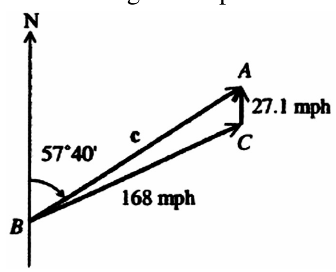
(*continued from page 777*)

Thus, the bearing is
$64°30' + B = 64°30' + 9°30' = 74°00'$.
Since $C = 180° - A - B$
$= 180° - 64.50° - 9.57° = 105.93°$, we use the law of sines to find the ground speed.

$$\frac{|\mathbf{c}|}{\sin C} = \frac{35.0}{\sin B} \Rightarrow$$
$$|\mathbf{c}| = \frac{35.0 \sin 105.93°}{\sin 9.57°} \approx 202 \text{ mph}$$

The bearing is $74°00'$; the ground speed is 202 mph.

26. Let **c** = the ground speed vector.



By alternate interior angles, angle $A = 57°40'$.
Use the law of sines to find B.

$$\frac{\sin B}{27.1} = \frac{\sin A}{168} \Rightarrow$$
$$\sin B = \frac{27.1 \sin 57°40'}{168} \approx .13629861 \Rightarrow$$
$$B \approx 7.83° \approx 7°50'$$

Thus, the bearing is
$57°40' + B = 57°40' + 7°50' = 65°30'$.
Since $C = 180° - A - B$
$= 180° - 57°40' - 7.83° \approx 114°30'$, we use the law of sine to find the ground speed.

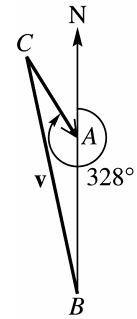
$$\frac{|\mathbf{c}|}{\sin C} = \frac{27.1}{\sin B} \Rightarrow$$
$$|\mathbf{c}| = \frac{27.1 \sin 114°30'}{\sin 7°50'} \approx 181 \text{ mph}$$

The bearing is $65°30'$; the ground speed is 181 mph.

27. Let **v** = the airspeed vector
The ground speed is $\dfrac{400 \text{ mi}}{2.5 \text{ hr}} = 160$ mph.
angle $BAC = 328° - 180° = 148°$
Using the law of cosines to find $|\mathbf{v}|$, we have

$$|\mathbf{v}|^2 = 11^2 + 160^2 - 2(11)(160)\cos 148°$$
$$|\mathbf{v}|^2 \approx 28{,}706.1$$
$$|\mathbf{v}| \approx 169.4$$

The airspeed must be 170 mph. (rounded to two significant digits)



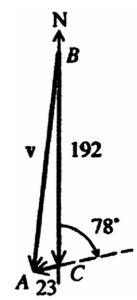
Use the law of sines to find B.

$$\frac{\sin B}{11} = \frac{\sin 148°}{169.4} \Rightarrow \sin B = \frac{11 \sin 148°}{169.4} \Rightarrow$$
$$\sin B \approx .03441034 \Rightarrow B \approx 2.0°$$

The bearing must be approximately
$360° - 2.0° = 358°$.

28. Let **v** = the ground speed vector.
angle $C = 180° - 78.0° = 102.0°$

$$|\mathbf{v}|^2 = 23.0^2 + 192^2 - 2(23)(192)\cos 102.0°$$
$$|\mathbf{v}|^2 \approx 39{,}229.28$$
$$|\mathbf{v}| \approx 198.1$$



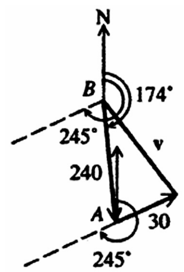
Thus, the ground speed is 198 mph. (rounded to three significant digits)

$$\frac{\sin B}{23} = \frac{\sin 102°}{198.1} \Rightarrow$$
$$\sin B = \frac{23 \sin 102°}{198.1} \approx .11356585 \Rightarrow$$
$$B \approx 6.5°$$

Thus, the bearing is
$180° + B = 180° + 6.5° = 186.5°$.

29. Find the ground speed and resulting bearing.
Angle $A = 245° - 174° = 71°$
Use the law of cosines to find $|\mathbf{v}|$.

$$|\mathbf{v}|^2 = 30^2 + 240^2 - 2(30)(240)\cos 71°$$
$$|\mathbf{v}|^2 \approx 53{,}811.8$$
$$|\mathbf{v}| \approx 232.1$$



The ground speed is 230 km per hr. (rounded to two significant digits)
Use the law of sines to find angle *B*.

$$\frac{\sin B}{30} = \frac{\sin 71°}{230} \Rightarrow$$
$$\sin B = \frac{30 \sin 71°}{230} \approx .12332851 \Rightarrow$$
$$B \approx 7°$$

Thus, the bearing is
$174° - B = 174° - 7° = 167°$.

30. **(a)** First change $10.34''$ to radians in order to use the length of arc formula.

$$10.34'' \cdot \frac{1°}{3600''} \cdot \frac{\pi}{180°} \approx 5.013 \times 10^{-5}$$

radian.
In one year Barnard's Star will move in the tangential direction the following distance.

$$s = r\theta = \left(35 \times 10^{12}\right)\left(5.013 \times 10^{-5}\right)$$
$$= 175.455 \times 10^{7} = 1,754,550,000 \text{ mi.}$$

In one second Barnard's Star moves

$$\frac{1,754,550,000}{60 \cdot 60 \cdot 24 \cdot 365} \approx 55.6 \text{ mi tangentially.}$$

Thus, $\mathbf{v}_t \approx 56$ mi/sec. (rounded to two significant digits)

(b) The magnitude of **v** is given by
$|\mathbf{v}|^2 = |\mathbf{v}_r|^2 + |\mathbf{v}_t|^2 - 2|\mathbf{v}_r||\mathbf{v}_t|\cos 90°$.
Since $|\mathbf{v}_r| = 67$ and $|\mathbf{v}_t| \approx 55.6$, we have

$$|\mathbf{v}|^2 = 67^2 + 55.6^2 - 2(67)(55.6)(0)$$
$$= 4489 + 3091.36 - 0 = 7580.36 \Rightarrow$$
$$|\mathbf{v}| = \sqrt{7580.36} \approx 87.1$$

Since the magnitude or length of **v** is 87, **v** represents a velocity of 87 mi/sec. (both rounded to two significant digits)

31. $\boldsymbol{R} = \boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{A} = .5\boldsymbol{i} + \boldsymbol{j}$

(a) Write the given vector in component form. $\boldsymbol{R} = \boldsymbol{i} - 2\boldsymbol{j} = \langle 1, \text{-}2 \rangle$ and
$\boldsymbol{A} = .5\boldsymbol{i} + \boldsymbol{j} = \langle .5, 1 \rangle$

$$|\boldsymbol{R}| = \sqrt{1^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5} \approx 2.2$$

and

$$|\boldsymbol{A}| = \sqrt{.5^2 + 1^2} = \sqrt{.25 + 1} = \sqrt{1.25} \approx 1.1$$

About 2.2 in. of rain fell. The area of the opening of the rain gauge is about 1.1 in.^{2}.

(b) $V = |\boldsymbol{R} \cdot \boldsymbol{A}| = |\langle 1, -2 \rangle \cdot \langle .5, 1 \rangle|$
$= |.5 + (-2)| = |-1.5| = 1.5$

The volume of rain was 1.5 in.^{3}.

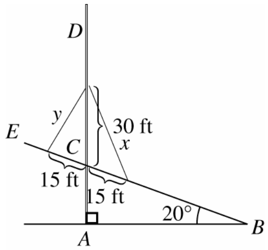
(c) **R** and **A** should be parallel and point in opposite directions.

32. $\boldsymbol{a} = \langle a_1, a_2 \rangle, \boldsymbol{b} = \langle b_1, b_2 \rangle$, and $\boldsymbol{a} - \boldsymbol{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$
$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1{}^2 + a_2{}^2 + b_1{}^2 + b_2{}^2 - 2|\text{a}||\text{b}|\cos$$
$$a_1{}^2 - 2a_1b_1 + b_1{}^2 + a_2{}^2 - 2a_2b_2 + b_2{}^2 = a_1{}^2 + a_2{}^2 + b_1{}^2 + b_2{}^2 - 2|\text{a}||\text{b}|\cos$$
$$-2a_1b_1 - 2a_2b_2 = -2|\mathbf{a}||\mathbf{b}|\cos\theta$$
$$a_1b_1 + a_2b_2 = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$\text{a} \cdot \text{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Summary Exercises on Applications of Trigonometry and Vectors

1. Consider the diagram below.



If we extend the flagpole, a right triangle *CAB* is formed. Thus, the measure of angle *BCA* is $90° - 20° = 70°.$ Since angle *DCB* and *BCA* are supplementary, the measure of angle *DCB* is $180° - 70° = 110°.$ We can now use the law of cosines to find the measure of the support wire on the right, *x*.

$$x^2 = 30^2 + 15^2 - 2(30)(15)\cos 110° \approx 1432.818 \approx 37.85 \approx 38 \text{ ft}$$

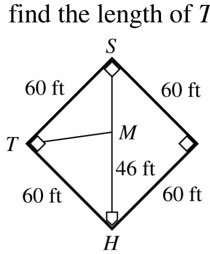
Now, to find the length of the support wire on the left, we have different ways to find it. One way would be to use the approximation for *x* and use the law of cosines. To avoid using the approximate value, we will find *y* with the same method as for *x*.

Since angle *DCB* and *DCE* are supplementary, the measure of angle *DCE* is $180° - 110° = 70°.$ We can now use the law of cosines to find the measure of the support wire on the left, *y*.

$$y^2 = 30^2 + 15^2 - 2(30)(15)\cos 70° \approx 817.182 \approx 28.59 \approx 29 \text{ ft}$$

The length of the two wires are about 29 ft and 38 ft.

2. We will find the measure of angle *THM* using right triangle *THS*, and then use that angle to find the length of *TM* in triangle *THM*.



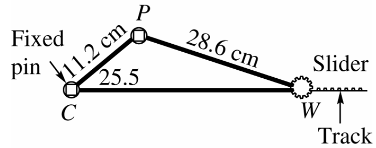
In ΔTHS, $m\angle THS = 45°$

In ΔTHM, we can use the law of cosines to find *TM*:

$$TM^2 = 60^2 + 46^2 - 2(60)(46)\cos 45° \approx 5716 - 3903.23 = 1812.77 \Rightarrow$$

$$TM \approx 43 \text{ ft}$$

3. Using the law of sines, we can find the measure of $\angle W$. Then find the measure of $\angle P$, and use the law of sines to find CW.



$$\frac{\sin W}{CP} = \frac{\sin C}{WP} \Rightarrow \frac{\sin W}{11.2} = \frac{\sin 25.5°}{28.6} \Rightarrow$$

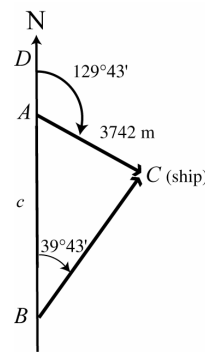
$$\sin W = \frac{11.2 \sin 25.5°}{28.6} \Rightarrow W \approx 9.7°$$

$$\angle P = 180° - 25.5° - 9.7° = 144.8°$$

$$\frac{CW}{\sin P} = \frac{PW}{\sin C} \Rightarrow \frac{CW}{\sin 144.8°} = \frac{28.6}{\sin 25.5°} \Rightarrow$$

$$CW = \frac{28.6 \sin 144.8°}{\sin 25.5°} \approx 38.3 \text{ cm}$$

4. Let *c* be the distance between the two lighthouses.



Since angle DAC and CAB form a line, angle CAB is the supplement of angle DAC. Thus, the measure of angle CAB is the following.

$$180° - 129°43' = 179°60' - 129°43' = 50°17'$$

Since the angles of a triangle must add up to 180°, the measure of angle ACB is the following.

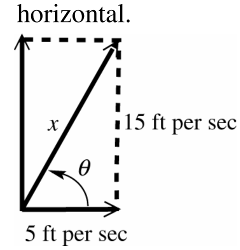
$$180° - 39°43' - 50°17' = 180° - 90° = 90°$$

Thus, we can solve the following.

$$\cos 50°17' = \frac{3742}{c} \Rightarrow c = \frac{3742}{\cos 50°17'} \approx 5856$$

The two lighthouses are 5856 m apart.

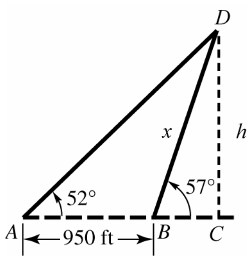
5. Let *x* be the new speed of the balloon. θ is the angle the balloon makes with horizontal.



To find x, use the Pythagorean theorem.
$x^2 = 15^2 + 5^2 \Rightarrow x^2 = 225 + 25 \Rightarrow x^2 = 250 \Rightarrow$
$x = \sqrt{250} = 5\sqrt{10} \approx 15.8$ ft per sec
To find θ, solve the following
$$\tan\theta = \frac{15}{5} \Rightarrow \tan\theta = 3 \Rightarrow \theta = \tan^{-1} 3 \approx 71.6°$$

6. Let **x** be the horizontal force.
$$\tan 40° = \frac{|\mathbf{x}|}{50} \Rightarrow |\mathbf{x}| = 50\tan 40° \approx 42 \text{ lb}$$

7. Let h be the height (vertical) of the airplane above the ground;
x is the distance between points B and D as labeled in the diagram.



Since angle ABD and DBC form a line, angle ABD is the supplement of angle DBC. Thus, the measure of angle ABD is $180° - 57° = 123°$. Since the angles of a triangle must add up to 180°, the measure of angle ADB is $180° - 123° - 52° = 5°$.
$$\frac{x}{\sin 52°} = \frac{950}{\sin 5°} \Rightarrow x = \frac{950\sin 52°}{\sin 5°} \approx 8589.34$$
$$\sin 57° = \frac{h}{8589.34} \Rightarrow$$
$h = 8589.34\sin 57° \approx 7203.6$
The airplane is approximately 7200 ft above the ground. (rounded to two significant digits)

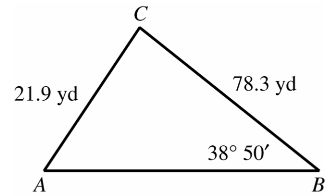
8. $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j} = \langle 6, 8\rangle$

(a) The speed of the wind is
$$|\mathbf{v}| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10 \text{ mph}$$

(b) $3\mathbf{v} = \langle 3\cdot 6, 3\cdot 8\rangle = \langle 18, 24\rangle = 18\mathbf{i} + 24\mathbf{j}$;
$$|3\mathbf{v}| = \sqrt{18^2 + 24^2} = \sqrt{324 + 576} = \sqrt{900} = 30$$
This represents a 30 mph wind in the direction of **v**.

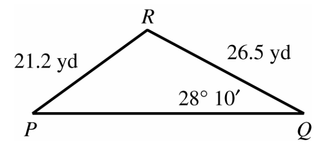
(c) **u** represents a southeast wind of
$$|\mathbf{u}| = \sqrt{(-8)^2 + 8^2} = \sqrt{64 + 64} = \sqrt{128} = 8\sqrt{2} \approx 11.3 \text{ mph}$$

9.



Using the law of sines, we have
$$\frac{\sin A}{BC} = \frac{\sin B}{AC} \Rightarrow \frac{\sin A}{78.3} = \frac{\sin 38°50'}{21.9} \Rightarrow$$
$$\sin A = \frac{78.3\sin 38°50'}{21.9} \approx 2.24$$
Since $-1 \le \sin A \le 1$, the triangle cannot exist.

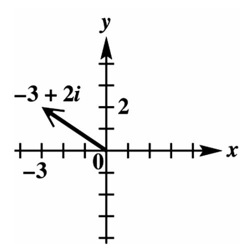
10.



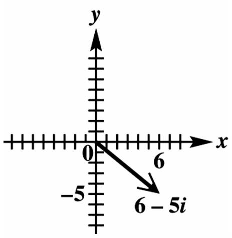
Using the law of sines, we have
$$\frac{\sin P}{QR} = \frac{\sin Q}{PR} \Rightarrow \frac{\sin P}{26.5} = \frac{\sin 28°10'}{21.2} \Rightarrow$$
$$\sin P = \frac{26.5\sin 28°10'}{21.2} \Rightarrow$$
$P \approx 36°10'$ or $P \approx 180° - 36°10' = 143°50'$
If $P = 36°10'$, $R = 180° - 36°10' - 28°10' = 115°40'$ and
$$\frac{PQ}{\sin R} = \frac{RP}{\sin Q} \Rightarrow \frac{PQ}{\sin 115°40'} = \frac{21.2}{\sin 28°10'} \Rightarrow$$
$$PQ = \frac{21.2\sin 115°40'}{\sin 28°10'} \approx 40.5 \text{ yd}$$
If $P = 143°50'$, $R = 180° - 143°50' - 28°10' = 8°00'$ and
$$\frac{PQ}{\sin R} = \frac{RP}{\sin Q} \Rightarrow \frac{PQ}{\sin 8°00'} = \frac{21.2}{\sin 28°10'} \Rightarrow$$
$$PQ = \frac{21.2\sin 8°00'}{\sin 28°10'} \approx 6.25 \text{ yd}$$
Thus, two cases are possible.

Section 8.5: Trigonometric (Polar) Form of Complex Numbers; Products and Quotients

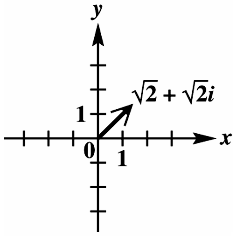
1.



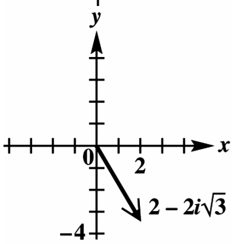
2.



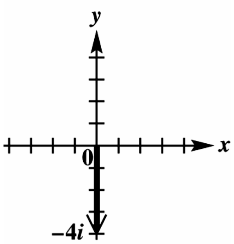
3.



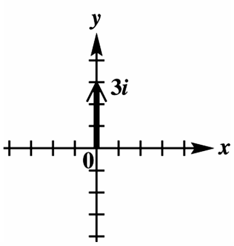
4.



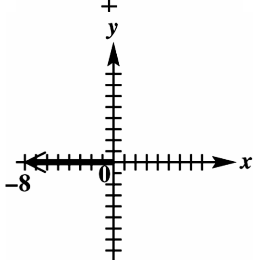
5.



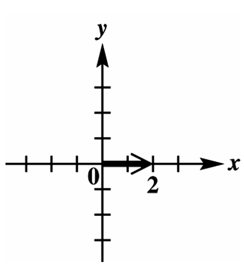
6.



7.



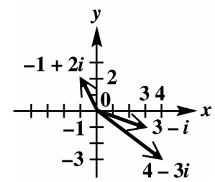
8.



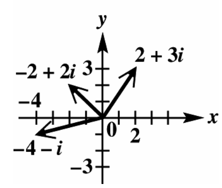
9. $1 - 4i$

10. $-4 - i$

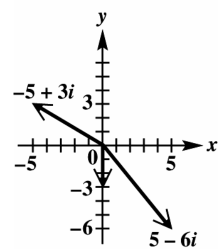
11. $(4-3i)+(-1+2i)=3-i$



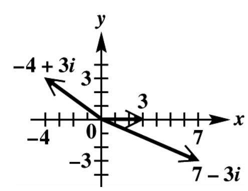
12. $(2+3i)+(-4-i)=-2+2i$



13. $(5-6i)+(-5+3i)=-3i$



14. $(7-3i)+(-4+3i)=3$



15. $-3+3i$

16. $6-2i$

17. $(2+6i)-2i=2+4i$

18. $(4-2i)+5=9-2i$

19. $(7+6i)+3i=7+9i$

20. $(-5-8i)-1=-6-8i$

21. $\left(\frac{1}{2}+\frac{2}{3}i\right)+\left(\frac{2}{3}+\frac{1}{2}i\right)=\frac{7}{6}+\frac{7}{6}i$

22. $\left(-\frac{1}{5}+\frac{2}{7}i\right)+\left(\frac{3}{7}-\frac{3}{4}i\right)=\frac{8}{35}-\frac{13}{28}i$

23. $2(\cos 45° + i\sin 45°) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$
$= \sqrt{2} + i\sqrt{2}$

24. $4(\cos 60° + i\sin 60°) = 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$
$= 2 + 2i\sqrt{3}$

25. $10(\cos 90° + i\ \sin 90°) = 10(0 + i)$
$= 0 + 10i = 10i$

26. $8(\cos 270° + i\sin 270°) = 8(0 - 1i) = -8i$

27. $4(\cos 240° + i\ \sin 240°) = 4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$
$= -2 - 2i\sqrt{3}$

28. $2(\cos 330° + i\sin\ 330°) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$
$= \sqrt{3} - i$

29. $3\text{ cis } 150° = 3(\cos\ 150° + i\ \sin 150°)$
$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

30. $2\text{ cis } 30° = 2(\cos 30° + i\sin 30°)$
$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$

31. $5\text{ cis } 300° = 5(\cos 300° + i\sin 300°)$
$= 5\left[\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)i\right] = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

32. $6\text{ cis } 135° = (\cos 135° + i\ \sin\ 135°)$
$= 6\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -3\sqrt{2} + 3i\sqrt{2}$

33. $\sqrt{2}\text{ cis } 225° = \sqrt{2}(\cos 225° + i\ \sin 225°)$
$= \sqrt{2}\left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}i\right)\right]$
$= -1 - i$

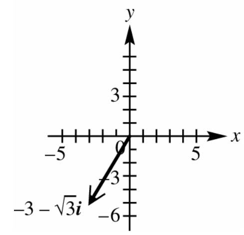
34. $\sqrt{3}\text{ cis } 315° = \sqrt{3}(\cos 315° + i\ \sin 315°)$
$= \sqrt{3}\left[\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$
$= \frac{\sqrt{6}}{2} - \frac{\sqrt{6}}{2}i$

35. $4(\cos(-30°) + i\sin(-30°))$
$= 4(\cos 30° - i\sin 30°) = 4\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$
$= 2\sqrt{3} - 2i$

36. $\sqrt{2}(\cos(-60°) + i\sin(-60°))$
$= \sqrt{2}(\cos 60° - i\sin 60°) = \sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$

37. $-3 - 3i\sqrt{3}$

Sketch a graph of $-3 - 3i\sqrt{3}$ in the complex plane.



Since $x = -3$ and $y = -3\sqrt{3}$,

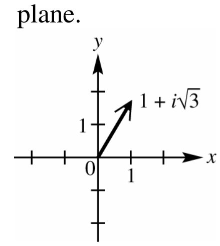
$r = \sqrt{(-3)^2 + (-3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6$

and $\tan\theta = \frac{-3\sqrt{3}}{-3} = \sqrt{3}.$ Thus, the reference angle for θ is 60º. The graph shows that θ is in quadrant III, so $\theta = 180° + 60° = 240°$. Therefore,

$-3 - 3i\sqrt{3} = 6(\cos 240° + i\sin 240°)$

38. $1 + i\sqrt{3}$

Sketch a graph of $1 + i\sqrt{3}$ in the complex plane.



Since $x = 1$ and $y = \sqrt{3}$,

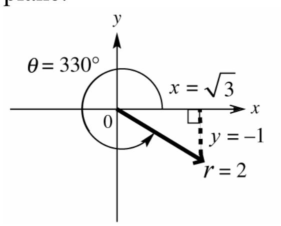
$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$ and

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}.$ Thus, the reference angle for θ is 60º. The graph shows that θ is in quadrant I, so $\theta = 60°$. Therefore,

$1 + i\sqrt{3} = 2(\cos 60° + i\sin 60°)$

39. $\sqrt{3} - i$

Sketch a graph of $\sqrt{3} - i$ in the complex plane.



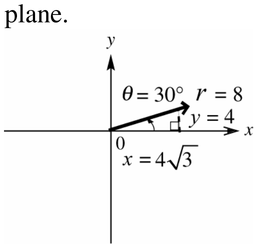
Since $x = \sqrt{3}$ and $y = -1$,

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$ and

$\tan\theta = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$. Since $\tan\theta = -\dfrac{\sqrt{3}}{3}$, the reference angle for θ is $30°$. The graph shows that θ is in quadrant IV, so $\theta = 360° - 30° = 330°$. Therefore, $\sqrt{3} - i = 2(\cos 330° + i \sin 330°)$.

40. $4\sqrt{3} + 4i$

Sketch a graph of $4\sqrt{3} + 4i$ in the complex plane.



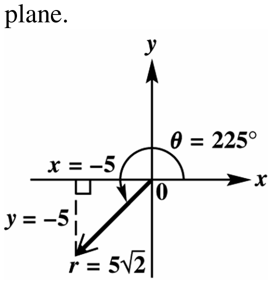
Since $x = 4\sqrt{3}$ and $y = 4$,

$r = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{48+16} = \sqrt{64} = 8$ and

$\tan\theta = \dfrac{4}{4\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. Since $\tan\theta = \dfrac{\sqrt{3}}{3}$, the reference angle for θ is $30°$. The graph shows that θ is in quadrant I, so $\theta = 30°$. Therefore, $4\sqrt{3} + 4i = 8(\cos 30° + i \sin 30°)$.

41. $-5 - 5i$

Sketch a graph of $-5 - 5i$ in the complex plane.



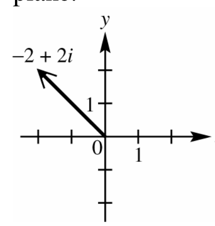
Since $x = -5$ and $y = -5$,

$r = \sqrt{(-5)^2 + (-5)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

and $\tan\theta = \dfrac{y}{x} = \dfrac{-5}{-5} = 1$. Since $\tan\theta = 1$, the reference angle for θ is $45°$. The graph shows that θ is in quadrant III, so $\theta = 180° + 45° = 225°$. Therefore, $-5 - 5i = 5\sqrt{2}(\cos 225° + i \sin 225°)$.

42. $-2 + 2i$

Sketch a graph of $-2 + 2i$ in the complex plane.



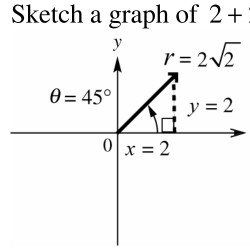
Since $x = -2$ and $y = 2$,

$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ and

$\tan\theta = \dfrac{2}{-2} = -1$. Since $\tan\theta = -1$, the reference angle for θ is $45°$. The graph shows that θ is in quadrant II, so $\theta = 180° - 45° = 135°$. Therefore, $-2 + 2i = 2\sqrt{2}(\cos 135° + i \sin 135°)$.

43. $2 + 2i$

Sketch a graph of $2 + 2i$ in the complex plane.



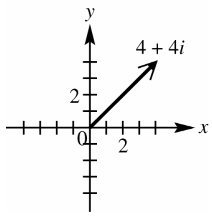
Since $x = 2$ and $y = 2$,

$r = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ and

$\tan\theta = \dfrac{2}{2} = 1$. Since $\tan\theta = 1$, the reference angle for θ is $45°$. The graph shows that θ is in quadrant I, so $\theta = 45°$. Therefore, $2 + 2i = 2\sqrt{2}(\cos 45° + i \sin 45°)$.

44. $4+4i$

Sketch a graph of $4+4i$ in the complex plane.



Since $x=4$ and $y=4$,

$r=\sqrt{4^2+4^2}=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$ and

$\tan\theta=\frac{4}{4}=1.$ Since $\tan\theta=1,$ the reference angle for θ is $45°.$ The graph shows that θ is in quadrant I, so $\theta=45°.$ Therefore,

$4+4i=4\sqrt{2}\left(\cos 45°+i\sin 45°\right).$

45. $5i=0+5i$

$0+5i$ is on the positive y-axis, so $\theta=90°$ and $x=0,\ y=5\Rightarrow$

$r=\sqrt{0^2+5^2}=\sqrt{0+25}=5$

Thus, $5i=5\left(\cos 90°+i\sin 90°\right).$

46. $-2i=0-2i$

$0-2i$ is on the negative y-axis, so $\theta=270°$ and $x=0,\ y=-2\Rightarrow$

$r=\sqrt{0^2+(-2)^2}=\sqrt{0+4}=2$

Thus, $-2i=2\left(\cos 270°+i\sin 270°\right).$

47. $-4=-4+0i$

$-4+0i$ is on the negative x-axis, so $\theta=180°$ and $x=-4,\ y=0\Rightarrow$

$r=\sqrt{(-4)^2+0^2}=\sqrt{16}=4$

Thus, $-4=4\left(\cos 180°+i\sin 180°\right).$

48. $7=7+0i$

$7+0i$ is on the positive x-axis, so $\theta=0°$ and

$x=7,\ y=0\Rightarrow r=\sqrt{7^2+0^2}=\sqrt{49}=7$

Thus, $7=7\left(\cos 0°+i\sin 0°\right).$

49. $2+3i$

$x=2, y=3\Rightarrow r=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}$

$\tan\theta=\frac{3}{2}$

$2+3i$ is in quadrant I, so

$\theta=56.31°.$

$2+3i=\sqrt{13}\left(\cos 56.31°+i\sin 56.31°\right)$

50. $\cos 35°+i\sin 35°=.8192+.5736i$

51. $3\left(\cos 250°+i\sin 250°\right)=-1.0261-2.8191i$

52. $-4+i$

$x=-4,\ y=1\Rightarrow$

$r=\sqrt{(-4)^2+1}=\sqrt{16+1}=\sqrt{17}$

$\tan\theta=\frac{1}{-4}=-\frac{1}{4}$

$-4+i$ is in the quadrant II, so $\theta=165.96°.$

$-4+i=\sqrt{17}\,(\cos 165.96°+i\sin 165.96°)$

53. $12i=0+12i$

$x=0,\ y=12\Rightarrow$

$r=\sqrt{0^2+12^2}=\sqrt{0+144}=\sqrt{144}=12$

$0+12i$ is on the positive y-axis, so $0=90°.$

$12i=12(\cos 90°+i\sin 90°)$

54. $3\text{ cis }180°=3\left(\cos 180°+i\sin 180°\right)$

$=3\left(-1+0i\right)=-3+0i$ or -3

55. $3+5i$

$x=3,\ y=5\Rightarrow r=\sqrt{3^2+5^2}=\sqrt{9+25}=\sqrt{34}$

$\tan\theta=\frac{5}{3}$

$3+5i$ is in the quadrant I, so $\theta=59.04°.$

$3+5i=\sqrt{34}\left(\cos 59.04°+i\sin 59.04°\right)$

56. $\text{cis }110.5°=\cos 110.5°+i\sin 110.5°$

$\approx -.3502+.9367i$

57. $z=-.2i$

$z^2-1=(-.2i)^2-1=.04i^2-1=.04(-1)-1$

$=-.04-1=-1.04$

The modulus is 1.04.

$(z^2-1)^2-1=(-1.04)^2-1=1.0816-1=.0816$

The modulus is .0816.

$\left[\left(z^2-1\right)^2-1\right]^2-1=(.0816)^2-1$

$=.0665856-1$

$=-.99334144$

The modulus is .99334144 .

The moduli do not exceed 2. Therefore, z is in the Julia set.

58. **(a)** Let $z_1=a+bi$ and its complex conjugate be $z_2=a-bi.$

$|z_1|=\sqrt{a^2+b^2}$ and

$|z_2|=\sqrt{a^2+(-b)^2}=\sqrt{a^2+b^2}=|z_1|.$

(b) Let $z_1 = a + bi$ and $z_2 = a - bi$.

$$z_1^2 - 1 = (a+bi)^2 - 1$$
$$= \left(a^2 + 2abi + b^2 i^2\right) - 1$$
$$= a^2 + 2abi + b^2(-1) - 1$$
$$= a^2 - b^2 + 2abi - 1$$
$$= \left(a^2 - b^2 - 1\right) + 2abi$$

$$z_2^2 - 1 = (a-bi)^2 - 1$$
$$= \left(a^2 - 2abi + b^2 i^2\right) - 1$$
$$= a^2 - 2abi + b^2(-1) - 1$$
$$= a^2 - b^2 - 2abi - 1$$
$$= \left(a^2 - b^2 - 1\right) - 2abi$$

(c)–(d) The results are again complex conjugates of each other. At each iteration, the resulting values from z_1 and z_2 will always be complex conjugates. Graphically, these represent points that are symmetric with respect to the *x*-axis, namely points such as (a, b) and $(a, -b)$.

(e) Answers will vary.

59. $\left[3(\cos 60° + i\sin 60°)\right]\left[2(\cos 90° + i\sin 90°)\right] = 3\cdot 2\left[\cos(60° + 90°) + i\sin(60° + 90°)\right]$
$$= 6(\cos 150° + i\sin 150°) = 6\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -3\sqrt{3} + 3i$$

60. $\left[4(\cos 30° + i\sin 30°)\right]\left[5(\cos 120° + i\sin 120°)\right] = 4\cdot 5\left[\cos(30° + 120°) + i\sin(30° + 120°)\right]$
$$= 20(\cos 150° + i\sin 150°) = 20\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -10\sqrt{3} + 10i$$

61. $\left[4(\cos 60° + i\sin 60°)\right]\cdot\left[6(\cos 330° + i\sin 330°)\right] = 4\cdot 6\left[(\cos(60° + 330°) + i\sin(60° + 330°)\right]$
$$= 24(\cos 390° + i\sin 390°) = 24(\cos 30° + i\sin 30°)$$
$$= 24\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 12\sqrt{3} + 12i$$

62. $\left[8(\cos 300° + i\sin 300°)\cdot[5(\cos 120° + i\sin 120°)\right] = 8\cdot 5\left[\cos(300° + 120°) + i\sin(300° + 120°)\right]$
$$= 40(\cos 420° + i\sin 420°) = 40(\cos 60° + i\sin 60°)$$
$$= 40\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 20 + 20i\sqrt{3}$$

63. $\left[2(\cos 45° + i\sin 45°)]\cdot[2(\cos 225° + i\sin 225°)\right] = 2\cdot 2\left[\cos(45° + 225°) + i\sin(45° + 225°)\right]$
$$= 4(\cos 270° + i\sin 270°) = 4(0 - i) = 0 - 4i \text{ or } -4i$$

64. $\left[8(\cos 210° + i\sin 210°)\right]\cdot\left[2(\cos 330° + i\sin 330°)\right] = 8\cdot 2\left[\cos(210° + 330°) + i\sin(210° + 330°)\right]$
$$= 16(\cos 540° + i\sin 540°)$$
$$= 16(\cos 180° + i\sin 180°)$$
$$= 16(-1 + 0\cdot i) = -16 + 0i \text{ or } -16$$

65. $\left[\sqrt{3}\text{ cis } 45°\right]\left[\sqrt{3}\text{ cis } 225°\right] = \sqrt{3}\cdot\sqrt{3}\left[\text{cis}(45° + 225°)\right]$
$$= 3\text{ cis } 270° = 3(\cos 270° + i\sin 270°) = 3(0 - i) = 0 - 3i \text{ or } -3i$$

66. $[6\text{ cis } 120°][5\text{ cis}(-30°)] = 6\cdot 5\left[\text{cis}(120° + (-30°))\right]$
$$= 30\text{ cis } 90° = 30(\cos 90° + i\sin 90°) = 30(0 + 1i) = 0 + 30i = 30i$$

67. $[5 \text{ cis } 90°][3 \text{ cis } 45°] = 5 \cdot 3\left[\text{cis}\left(90° + 45°\right)\right] = 15 \text{ cis } 135°$
$$= 15\left(\cos 135° + i \sin 135°\right) = 15\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = -\frac{15\sqrt{2}}{2} + \frac{15\sqrt{2}}{2} i$$

68. $\left[\sqrt{2} \text{ cis } 300°\right]\left[\sqrt{2} \text{ cis } 270°\right] = \sqrt{2} \cdot \sqrt{2}\left[\text{cis}\left(300° + 270°\right)\right] = 2 \text{ cis } 570°$
$$= 2 \text{ cis } 210° = 2\left(\cos 210° + i \sin 210°\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) = -\sqrt{3} - i$$

69. $\dfrac{4\left(\cos 120° + i \sin 120°\right)}{2\left(\cos 150° + i \sin 150°\right)} = \dfrac{4}{2}\left[\cos\left(120° - 150°\right) + i \sin\left(120° - 150°\right)\right]$
$$= 2\left(\cos\left(-30°\right) + i \sin\left(-30°\right)\right) = 2\left(\cos 30° - i \sin 30°\right)$$
$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) = \sqrt{3} - i$$

70. $\dfrac{24\left(\cos 150° + i \sin 150°\right)}{2\left(\cos 30° + i \sin 30°\right)} = \dfrac{24}{2}\left[\cos\left(150° - 30°\right) + i \sin\left(150° - 30°\right)\right]$
$$= 12\left(\cos 120° + i \sin 120°\right) = 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = -6 + 6i\sqrt{3}$$

71. $\dfrac{10\left(\cos 230° + i \sin 230°\right)}{5\left(\cos 50° + i \sin 50°\right)} = \dfrac{10}{5}\left[\cos\left(230° - 50°\right) + i \sin\left(230° - 50°\right)\right]$
$$= 2\left(\cos 180° + i \sin 180°\right) = 2\left(-1 + 0 \cdot i\right) = -2 + 0i \text{ or } -2$$

72. $\dfrac{12(\cos 293° + i \sin 293°)}{6(\cos 23° + i \sin 23°)} = \dfrac{12}{6}\left[\cos\left(293° - 23°\right) + i \sin\left(293° - 23°\right)\right]$
$$= 2\left(\cos 270° + i \sin 270°\right) = 2\left(0 - i\right) = -2i$$

73. $\dfrac{3 \text{ cis } 305°}{9 \text{ cis } 65°} = \dfrac{1}{3} \text{cis}\left(305° - 65°\right) = \dfrac{1}{3}\left(\text{cis } 240°\right)$
$$= \frac{1}{3}\left(\cos 240° + i \sin 240°\right)$$
$$= \frac{1}{3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -\frac{1}{6} - \frac{\sqrt{3}}{6} i$$

74. $\dfrac{16 \text{ cis } 310°}{8 \text{ cis } 70°} = 2 \text{ cis}\left(310° - 70°\right) = 2\left(\text{cis } 240°\right)$
$$= 2\left(\cos 240° + i \sin 240°\right)$$
$$= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -1 - i\sqrt{3}$$

75. $\dfrac{8}{\sqrt{3} + i}$

numerator: $8 = 8 + 0i$ and $r = \sqrt{8^2 + 0^2} = 8$
$\theta = 0°$ since $\cos 0° = 1$ and $\sin 0° = 0$, so
$8 = 8 \text{ cis } 0°$.
denominator: $\sqrt{3} + i$ and
$$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2$$
$$\tan \theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
Since x and y are both positive, θ is in quadrant I, so $\theta = 30°$. Thus $\sqrt{3} + i = 2 \text{ cis } 30°$.
$$\frac{8}{\sqrt{3} + i} = \frac{8 \text{ cis } 0°}{2 \text{ cis } 30°} = \frac{8}{2} \text{cis}\left(0 - 30°\right)$$
$$= 4\left[\cos\left(-30°\right) + i \sin\left(-30°\right)\right]$$
$$= 4\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) = 2\sqrt{3} - 2i$$

76. $\dfrac{2i}{-1-i\sqrt{3}}$

numerator: $2i = 0 + 2i$ and $r = \sqrt{0^2 + 2^2} = 2$
$\theta = 90°$ since $\cos 90° = 0$ and $\sin 90° = 1$, so $2i = 2\operatorname{cis} 90°$.

denominator: $-1-i\sqrt{3}$ and

$$r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

Since both x and y are negative, θ is in quadrant III, so $\theta = 60° + 180° = 240°$. Thus $-1-i\sqrt{3} = 2\operatorname{cis} 120°$.

$$\begin{aligned}\frac{2i}{-1-i\sqrt{3}} = \frac{2\operatorname{cis} 90°}{2\operatorname{cis} 240°} &= \operatorname{cis}(90 - 240°)\\ &= \cos(-150°) + i\sin(-150°)\\ &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i\end{aligned}$$

77. $\dfrac{-i}{1+i}$

numerator: $-i = 0 - i$ and

$$r = \sqrt{0^2 + (-1)^2} = \sqrt{0+1} = \sqrt{1} = 1$$

$\theta = 270°$ since $\cos 270° = 0$ and $\sin 270° = -1$, so $-i = 1\operatorname{cis} 270°$.

denominator: $1+i$

$r = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$ and

$$\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$$

Since x and y are both positive, θ is in quadrant I, so $\theta = 45°$. Thus,

$1 + i = \sqrt{2}\operatorname{cis} 45°$

$$\begin{aligned}\frac{-i}{1+i} = \frac{\operatorname{cis} 270°}{\sqrt{2}\operatorname{cis} 45°} &= \frac{1}{\sqrt{2}}\operatorname{cis}(270° - 45°)\\ &= \frac{\sqrt{2}}{2}\operatorname{cis} 225°\\ &= \frac{\sqrt{2}}{2}(\cos 225° + i\sin 225°)\\ &= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2} - i\cdot\frac{\sqrt{2}}{2}\right) = -\frac{1}{2} - \frac{1}{2}i\end{aligned}$$

78. $\dfrac{1}{2-2i}$

numerator: $1 = 1 + 0\cdot i$ and $r = 1$
$\theta = 0°$ since $\cos 0° = 1$ and $\sin 0° = 0$, so $1 = 1\operatorname{cis} 0°$.

denominator: $2-2i$ and

$$r = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{2} = -1$$

Since x is positive and y is negative, θ is in quadrant IV, so $\theta = -45°$. Thus, $2 - 2i = 2\sqrt{2}\operatorname{cis}(-45°)$.

$$\begin{aligned}\frac{1}{2-2i} = \frac{1\operatorname{cis} 0°}{2\sqrt{2}\operatorname{cis}(-45°)} &= \frac{1}{2\sqrt{2}}\operatorname{cis}\left[0 - (-45°)\right]\\ &= \frac{\sqrt{2}}{4}\operatorname{cis} 45° = \frac{\sqrt{2}}{4}(\cos 45° + i\sin 45°)\\ &= \frac{\sqrt{2}}{4}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{1}{4} + \frac{1}{4}i\end{aligned}$$

79. $\dfrac{2\sqrt{6} - 2i\sqrt{2}}{\sqrt{2} - i\sqrt{6}}$

numerator: $2\sqrt{6} - 2i\sqrt{2}$ and

$$\begin{aligned}r &= \sqrt{\left(2\sqrt{6}\right)^2 + \left(-2\sqrt{2}\right)^2} = \sqrt{24+8}\\ &= \sqrt{32} = 4\sqrt{2}\end{aligned}$$

$$\tan\theta = \frac{-2\sqrt{2}}{2\sqrt{6}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Since x is positive and y is negative, θ is in quadrant IV, so $\theta = -30°$. Thus, $2\sqrt{6} - 2i\sqrt{2} = 4\sqrt{2}\operatorname{cis}(-30°)$.

denominator: $\sqrt{2} - i\sqrt{6}$ and

$$r = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{6}\right)^2} = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{-\sqrt{6}}{\sqrt{2}} = -\sqrt{3}$$

Since x is positive and y is negative, θ is in quadrant IV, so $\theta = -60°$. Thus, $\sqrt{2} - i\sqrt{6} = 2\sqrt{2}\operatorname{cis}(-30°)$

$$\begin{aligned}\frac{2\sqrt{6} - 2i\sqrt{2}}{\sqrt{2} - i\sqrt{6}} &= \frac{4\sqrt{2}\operatorname{cis}(-30°)}{2\sqrt{2}\operatorname{cis}(-60°)}\\ &= \frac{4\sqrt{2}}{2\sqrt{2}}\operatorname{cis}\left[-30° - (-60°)\right]\\ &= 2\operatorname{cis} 30° = 2(\cos 30° + i\sin 30°)\\ &= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i\end{aligned}$$

80. $\dfrac{4+4i}{2-2i}$

numerator: $4+4i$ and

$r=\sqrt{4^2+4^2}=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$

$\tan\theta=\dfrac{4}{4}=1$

Since x and y are both positive, θ is in quadrant I, so $\theta=45°$. Thus,

$4+4i=4\sqrt{2}\text{ cis }45°$

denominator: $2-2i$ and

$r=\sqrt{2^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$

$\tan\theta=\dfrac{y}{x}=\dfrac{-2}{2}=-1$

Since x is positive and y is negative, θ is in quadrant IV, so $\theta=-45°$. Thus,

$2-2i=2\sqrt{2}\text{ cis}(-45°)$

$$\frac{4+4i}{2-2i}=\frac{4\sqrt{2}\text{ cis }45°}{2\sqrt{2}\text{ cis}(-45°)}$$
$$=\frac{4\sqrt{2}}{2\sqrt{2}}\text{cis}\left[45°-(-45°)\right]$$
$$=2\text{ cis }90°=2(\cos 90°+i\sin 90°)$$
$$=2(0+i)=0+2i=2i$$

81. $\left[2.5(\cos 35°+i\sin 35°)\right]\left[3.0(\cos 50°+i\sin 50°)\right]$
$$=2.5\cdot 3.0\left[\cos(35°+50°)+i\sin(35°+50°)\right]$$
$$=7.5(\cos 85°+i\sin 85°)\approx .6537+7.4715i$$

82. $\left[4.6(\cos 12°+i\sin 12°)\right]\left[2.0(\cos 13°+i\sin 13°)\right]$
$$=4.6\cdot 2.0\left[\cos(12°+13°)+i\sin(12°+13°)\right]$$
$$=9.2(\cos 25°+i\sin 25°)\approx 8.3380+3.8881i$$

83. $(12\text{ cis }18.5°)(3\text{ cis }12.5°)$
$$=12\cdot 3\text{ cis }(18.5°+12.5°)$$
$$=36\text{ cis }31°=36(\cos 31°+i\sin 31°)$$
$$\approx 30.8580+18.5414i$$

84. $(4\text{ cis }19.25°)(7\text{ cis }41.75°)$
$$=4\cdot 7\text{ cis }(19.25°+41.75°)$$
$$=28\text{ cis }61°=28(\cos 61°+i\sin 61°)$$
$$\approx 13.5747+24.4894i$$

85. $\dfrac{45(\cos 127°+i\sin 127°)}{22.5(\cos 43°+i\sin 43°)}$
$$=\frac{45}{22.5}\left[\cos(127°-43°)+i\sin(127°-43°)\right]$$
$$=2(\cos 84°+i\sin 84°)\approx .2091+1.9890i$$

86. $\dfrac{30(\cos 130°+i\sin 130°)}{10(\cos 21°+i\sin 21°)}$
$$=\frac{30}{10}\left[\cos(130°-21°)+i\sin(130°-21°)\right]$$
$$=3(\cos 109°+i\sin 109°)\approx -.9767+2.8366i$$

87. $\left[2\text{ cis}\dfrac{5\pi}{9}\right]^2=\left[2\text{ cis}\dfrac{5\pi}{9}\right]\left[2\text{ cis}\dfrac{5\pi}{9}\right]$
$$=2\cdot 2\text{ cis}\left(\frac{5\pi}{9}+\frac{5\pi}{9}\right)$$
$$=4\left(\cos\frac{10\pi}{9}+i\sin\frac{10\pi}{9}\right)$$
$$\approx -3.7588-1.3681i$$

88. $\left[24.3\text{ cis}\dfrac{7\pi}{12}\right]^2=\left[24.3\text{ cis}\dfrac{7\pi}{12}\right]\left[24.3\text{ cis}\dfrac{7\pi}{12}\right]$
$$=24.3\cdot 24.3\text{ cis}\left(\frac{7\pi}{12}+\frac{7\pi}{12}\right)$$
$$=590.49\text{ cis}\frac{7\pi}{6}$$
$$=590.49\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$$
$$\approx -511.3793-295.2450i$$

In Exercises 89–95, $w=-1+i$ and $z=-1-i$.

89. $w\cdot z=(-1+i)(-1-i)$
$$=-1(-1)+(-1)(-i)+i(-1)+i(-i)$$
$$=1+i-i-i^2=1-(-1)=2$$

90. $w=-1+i$:

$r=\sqrt{(-1)^2+1^2}=\sqrt{1+1}=\sqrt{2}$ and

$\tan\theta=\dfrac{1}{-1}=-1$

Since x is negative and y is positive, θ is in quadrant II, so $\theta=135°$. Thus,

$w=\sqrt{2}\text{ cis }135°$.

$z=-1-i$:

$r=\sqrt{(-1)^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}$ and

$\tan\theta=\dfrac{-1}{-1}=1$

Since x and y are negative, θ is in quadrant III, so $\theta=225°$. Thus, $z=\sqrt{2}\text{ cis }225°$.

91. $w\cdot z=\left(\sqrt{2}\text{ cis }135°\right)\left(\sqrt{2}\text{ cis }225°\right)$
$$=\sqrt{2}\cdot\sqrt{2}\left[\text{ cis }(135°+225°)\right]$$
$$=2\text{ cis }360°=2\text{ cis }0°$$

92. $2 \text{ cis } 0° = 2(\cos 0° + i \sin 0°)$
$= 2(1 + 0 \cdot i) = 2 \cdot 1 = 2$

It is the same.

93. $\frac{w}{z} = \frac{-1+i}{-1-i} = \frac{-1+i}{-1-i} \cdot \frac{-1+i}{-1+i} = \frac{1-i-i+i^2}{1-i^2}$
$= \frac{1-2i+(-1)}{1-(-1)} = \frac{-2i}{2} = -i$

94. $\frac{w}{z} = \frac{\sqrt{2} \text{ cis } 135°}{\sqrt{2} \text{ cis } 225°} = \frac{\sqrt{2}}{\sqrt{2}} \text{ cis } (135° - 225°)$
$= \text{ cis } (-90°)$

95. $\text{cis } (-90°) = \cos(-90°) + i \sin(-90°)$
$= 0 + i(-1) = 0 - i = -i$

It is the same.

96. Answers will vary.

97. $E = 8(\cos 20° + i \sin 20°), R = 6, X_L = 3,$
$I = \frac{E}{Z}, Z = R + X_L i$

Write $Z = 6 + 3i$ in trigonometric form.

$x = 6$, and $y = 3 \Rightarrow r = \sqrt{6^2 + 3^2} = \sqrt{36+9}$
$= \sqrt{45} = 3\sqrt{5}$.

$\tan\theta = \frac{3}{6} = \frac{1}{2}$, so $\theta \approx 26.6°$. Thus,

$Z = 3\sqrt{5} \text{ cis } 26.6°$.

$I = \frac{8 \text{ cis } 20°}{3\sqrt{5} \text{ cis } 26.6°} = \frac{8}{3\sqrt{5}} \text{ cis } (20° - 26.6°)$
$= \frac{8\sqrt{5}}{15} \text{ cis } (-6.6°)$
$= \frac{8\sqrt{5}}{15}[\cos(-6.6°) + i \sin(-6.6°)]$
$\approx 1.18 - .14i$

98. $E = 12(\cos 25° + i \sin 25°), R = 3, X_L = 4,$ and $X_c = 6$

$I = \frac{E}{R + (X_L - X_c)i} = \frac{12 \text{ cis } 25°}{3 + (4-6)i} = \frac{12 \text{ cis } 25°}{3 - 2i}$

Write $3 - 2i$ in trigonometric form: $x = 3$,
$y = -2$, so $r = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$.

$\tan\theta = -\frac{2}{3}$, so θ=326.31° since θ is in quadrant IV.

Continuing to find the current, we have

$I = \frac{12 \text{ cis } 25°}{\sqrt{13} \text{ cis } 326.31°} = \frac{12}{\sqrt{13}} \text{ cis } (25° - 326.3°)$
$= \frac{12\sqrt{13}}{13} \text{ cis } (-301.3°)$
$= \frac{12\sqrt{13}}{13}[\cos(-301.3°) + i \sin(-301.3°)]$
$\approx 1.7 + 2.8i$

99. Since $Z_1 = 50 + 25i$ and $Z_2 = 60 + 20i$, we have

$\frac{1}{Z_1} = \frac{1}{50+25i} \cdot \frac{50-25i}{50-25i} = \frac{50-25i}{50^2 - 25^2 i^2}$
$= \frac{50-25i}{2500 - 625(-1)} = \frac{50-25i}{2500+625}$
$= \frac{50-25i}{3125} = \frac{2}{125} - \frac{1}{125}i$ and

$\frac{1}{Z_2} = \frac{1}{60+20i} \cdot \frac{60-20i}{60-20i} = \frac{60-20i}{60^2 - 20^2 i^2}$
$= \frac{60-20i}{3600 - 400(-1)} = \frac{60-20i}{3600+400}$
$= \frac{60-20i}{4000} = \frac{3}{200} - \frac{1}{200}i$

$\frac{1}{Z_1} + \frac{1}{Z_2} = \left(\frac{2}{125} - \frac{1}{125}i\right) + \left(\frac{3}{200} - \frac{1}{200}i\right)$
$= \left(\frac{2}{125} + \frac{3}{200}\right) - \left(\frac{1}{125} + \frac{1}{200}\right)i$
$= \left(\frac{16}{1000} + \frac{15}{1000}\right) - \left(\frac{8}{1000} + \frac{5}{1000}\right)i$
$= \frac{31}{1000} - \frac{13}{1000}i$

$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{1}{\frac{31}{1000} - \frac{13}{1000}i}$
$= \frac{1000}{31-13i} \cdot \frac{31+13i}{31+13i} = \frac{1000(31+13i)}{31^2 - 13^2 i^2}$
$= \frac{31,000 + 13,000i}{961 - 169(-1)} = \frac{31,000 + 13,000i}{961+169}$
$= \frac{31,000 + 13,000i}{1130} = \frac{3100}{113} + \frac{1300}{113}i$
$\approx 27.43 + 11.5i$

100. $\tan\theta = \frac{11.5}{27.43} \Rightarrow \theta = \tan^{-1}\frac{11.5}{27.43} \approx 22.75°$ since θ is in quadrant I.

Section 8.6: DeMoivre's Theorem; Powers and Roots of Complex Numbers

1. $\left[3\left(\cos 30° + i\sin 30°\right)\right]^3$
$= 3^3\left[\cos\left(3\cdot 30°\right) + i\sin\left(3\cdot 30°\right)\right]$
$= 27\left(\cos 90° + i\sin 90°\right)$
$= 27\left(0 + 1\cdot i\right) = 0 + 27i$ or $27i$

2. $\left[2\left(\cos 135° + i\sin 135°\right)\right]^4$
$= 2^4\left[\cos\left(4\cdot 135°\right) + i\sin\left(4\cdot 135°\right)\right]$
$= 16\left(\cos 540° + i\sin 540°\right)$
$= 16\left(\cos 180° + i\sin 180°\right)$
$= 16(-1 + 0\cdot i) = -16 + 0i$ or -16

3. $\left(\cos 45° + i\sin 45°\right)^8$
$= \left[\cos\left(8\cdot 45°\right) + i\sin\left(8\cdot 45°\right)\right]$
$= \cos 360° + i\sin 360° = 1 + 0\cdot i$ or 1

4. $\left[2\left(\cos 120° + i\sin 120°\right)\right]^3$
$= 2^3\left[\cos\left(3\cdot 120°\right) + i\sin\left(3\cdot 120°\right)\right]$
$= 8\left(\cos 360° + i\sin 360°\right)$
$= 8\left(1 + 0\cdot i\right) = 8 + 0i$ or 8

5. $\left[3 \text{ cis } 100°\right]^3$
$= 3^3 \text{ cis } \left(3\cdot 100°\right)$
$= 27 \text{ cis } 300° = 27\left(\cos 300° + i\sin 300°\right)$
$= 27\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{27}{2} - \frac{27\sqrt{3}}{2}i$

6. $\left[3 \text{ cis } 40°\right]^3 = 3^3 \text{ cis } \left(3\cdot 40°\right) = 27 \text{ cis } 120°$
$= 27\left(\cos 120° + i\sin 120°\right)$
$= 27\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= -\frac{27}{2} + \frac{27\sqrt{3}}{2}i$

7. $\left(\sqrt{3} + i\right)^5$

First write $\sqrt{3} + i$ in trigonometric form.

$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$ and

$\tan\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Because x and y are both positive, θ is in quadrant I, so $\theta = 30°$.

$\sqrt{3} + i = 2\left(\cos 30° + i\sin 30°\right)$
$\left(\sqrt{3} + i\right)^5 = \left[2\left(\cos 30° + i\sin 30°\right)\right]^5$
$= 2^5\left[\cos\left(5\cdot 30°\right) + i\sin\left(5\cdot 30°\right)\right]$
$= 32\left(\cos 150° + i\sin 150°\right)$
$= 32\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -16\sqrt{3} + 16i$

8. $\left(2 - 2i\sqrt{3}\right)^4$

First write $2 - 2i\sqrt{3}$ in trigonometric form.

$r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$ and

$\tan\theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

Because x is positive and y is negative, θ is in quadrant IV, so $\theta = 300°$.

$2–2i\sqrt{3} = 4\left(\cos 300° + i\sin 300°\right)$
$\left(2–2i\sqrt{3}\right)^4 = \left[4\left(\cos 300° + i\sin 300°\right)\right]^4$
$= 4^4\left[\cos\left(4\cdot 300°\right) + i\sin\left(4\cdot 300°\right)\right]$
$= 256\left(\cos 1200° + i\sin 1200°\right)$
$= 256\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = -128 + 128i\sqrt{3}$

9. $\left(2\sqrt{2} - 2i\sqrt{2}\right)^6$

First write $2\sqrt{2} - 2i\sqrt{2}$ in trigonometric form.

$r = \sqrt{\left(2\sqrt{2}\right)^2 + \left(-2\sqrt{2}\right)^2} = \sqrt{8+8} = \sqrt{16} = 4$

and $\tan\theta = \frac{-2\sqrt{2}}{2\sqrt{2}} = -1$

Because x is positive and y is negative, θ is in quadrant IV, so $\theta = 315°$.

$2\sqrt{2} - 2i\sqrt{2} = 4\left(\cos 315° + i\sin 315°\right)^6$

$\left(2\sqrt{2} - 2i\sqrt{2}\right)^6$
$= \left[4\left(\cos 315° + i\sin 315°\right)\right]^6$
$= 4^6\left[\cos\left(6\cdot 315°\right) + i\sin\left(6\cdot 315°\right)\right]$
$= 4096\left[\cos 1890° + i\sin 1890°\right]$
$= 4096\left(\cos 90° + i\sin 90°\right)$
$= 4096\left(0 + 1\cdot i\right) = 0 + 4096i$ or $4096i$

10. $\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)^8$

First write $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$ in trigonometric form.

$$r=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2}=\sqrt{\frac{2}{4}+\frac{2}{4}}=\sqrt{1}=1$$

and $\tan\theta=\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=-1$

Because x is positive and y is negative, θ is in quadrant IV, so $\theta=315°$.

$$\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i=\cos 315°+i\sin 315°$$

$$\begin{aligned}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)^8&=(\cos 315°+i\sin 315°)^8\\&=\cos(8\cdot 315)°+i\sin(8\cdot 315)°\\&=\cos 2520°+i\sin 2520°\\&=\cos 0°+i\sin 0°=1+0i \text{ or } 1\end{aligned}$$

11. $(-2-2i)^5$

First write $-2-2i$ in trigonometric form.

$r=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$ and

$\tan\theta=\frac{-2}{-2}=1$

Because x and y are both negative, θ is in quadrant III, so $\theta=225°$.

$$-2-2i=2\sqrt{2}(\cos 225°+i\sin 225°)$$

$$\begin{aligned}(-2-2i)^5&=\left[2\sqrt{2}(\cos 225°+i\sin 225°)\right]^5\\&=\left(2\sqrt{2}\right)^5\left[\cos(5\cdot 225°)+i\sin(5\cdot 225°)\right]\\&=32\sqrt{32}(\cos 1125°+i\sin 1125°)\\&=128\sqrt{2}(\cos 45°+i\sin 45°)\\&=128\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=128+128i\end{aligned}$$

12. $(-1+i)^7$

First write $-1+i$ in trigonometric form.

$r=\sqrt{(-1)^2+1^2}=\sqrt{1+1}=\sqrt{2}=\sqrt{2}$ and

$\tan\theta=\frac{1}{-1}=-1$

Because x is negative and y is positive, θ is in quadrant II, so $\theta=135°$.

$$-1+i=\sqrt{2}(\cos 135°+i\sin 135°)$$

$$\begin{aligned}(-1+i)^7&=\left[\sqrt{2}(\cos 135°+i\sin 135°)\right]^7\\&=\left(\sqrt{2}\right)^7\left[\cos(7\cdot 135°)+i\sin(7\cdot 135°)\right]\\&=8\sqrt{2}(\cos 945°+i\sin 945°)\\&=8\sqrt{2}(\cos 225°+i\sin 225°)\\&=8\sqrt{2}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-8-8i\end{aligned}$$

13. **(a)** $\cos 0°+i\sin 0°=1(\cos 0°+i\sin 0°)$

We have $r=1$ and $\theta=0°$. Since $r^3(\cos 3\alpha+i\sin 3\alpha)=1(\cos 0°+i\sin 0°)$, then we have $r^3=1\Rightarrow r=1$ and

$$3\alpha=0°+360°\cdot k\Rightarrow\alpha=\frac{0°+360°\cdot k}{3}$$
$=0°+120°\cdot k=120°\cdot k$, k any integer.

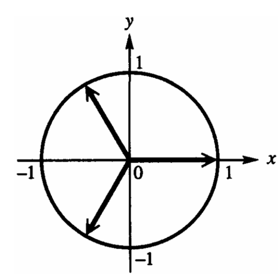
If $k=0$, then $\alpha=0°$.

If $k=1$, then $\alpha=120°$.

If $k=2$, then $\alpha=240°$.

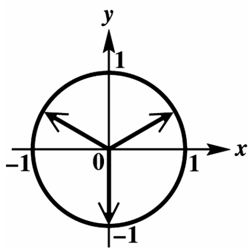
So, the cube roots are $\cos 0°+i\sin 0°$, $\cos 120°+i\sin 120°$, and $\cos 240°+i\sin 240°$.

(b)



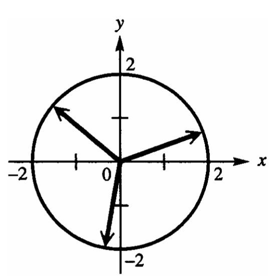
14. **(a)** Find the cube roots of
$\cos 90° + i\sin 90° = 1(\cos 90° + i\sin 90°)$.
We have $r = 1$ and $\theta = 90°$. Since
$r^3(\cos 3\alpha + i\sin 3\alpha) = 1(\cos 90° + i\sin 90°)$,
then we have $r^3 = 1 \Rightarrow r = 1$ and
$3\alpha = 90° + 360° \cdot k \Rightarrow \alpha = \dfrac{90° + 360° \cdot k}{3}$
$= 30° + 120° \cdot k, k$ any integer.
If $k = 0$, then $\alpha = 30° + 0° = 30°$.
If $k = 1$, then $\alpha = 30° + 120° = 150°$.
If $k = 2$, then $\alpha = 30° + 240° = 270°$.
So, the cube roots are
$\cos 30° + i\sin 30°$, $\cos 150° + i\sin 150°$, and $\cos 270° + i\sin 270°$.

(b)



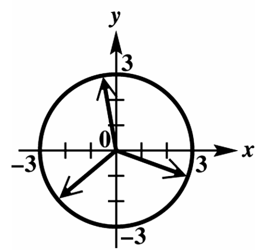
15. **(a)** Find the cube roots of 8 cis 60°
We have $r = 8$ and $\theta = 60°$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 8(\cos 60° + i\sin 60°)$, we have
$r^3 = 8 \Rightarrow r = 2$ and
$3\alpha = 60° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{60° + 360° \cdot k}{3} = 20° + 120° \cdot k$, k any integer.
If $k = 0$, then $\alpha = 20° + 0° = 20°$.
If $k = 1$, then $\alpha = 20° + 120° = 140°$.
If $k = 2$, then $\alpha = 20° + 240° = 260°$.
So, the cube roots are
$2 \text{ cis } 20°, 2 \text{ cis} 140°$, and $2 \text{ cis } 260°$.

(b)



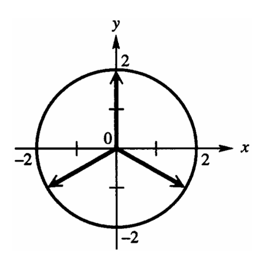
16. **(a)** Find the cube roots of 27 cis 300°. We have $r = 27$ and $\theta = 300°$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 27(\cos 300° + i\sin 300°)$, then we have
$r^3 = 27 \Rightarrow r = 3$ and
$3\alpha = 300° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{300° + 360° \cdot k}{3} = 100° + 120° \cdot k$, k any integer.
If $k = 0$, then $\alpha = 100° + 0° = 100°$.
If $k = 1$, then $\alpha = 100° + 120° = 220°$.
If $k = 2$, then $\alpha = 100° + 240° = 340°$.
So, the cube roots are
$3 \text{ cis} 100°, 3 \text{ cis } 220°$, and $3 \text{ cis } 340°$.

(b)



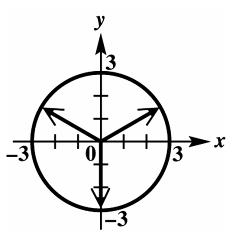
17. **(a)** Find the cube roots of
$-8i = 8(\cos 270° + i \sin 270°)$
We have $r = 8$ and $\theta = 270°$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 8(\cos 270° + i\sin 270°)$, then we have
$r^3 = 8 \Rightarrow r = 2$ and
$3\alpha = 270° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{270° + 360° \cdot k}{3} = 90° + 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 90° + 0° = 90°$.
If $k = 1$, then $\alpha = 90° + 120° = 210°$.
If $k = 2$, then $\alpha = 90° + 240° = 330°$.
So, the cube roots are
$2(\cos 90° + i\sin 90°)$,
$2(\cos 210° + i\sin 210°)$, and
$2(\cos 330° + i\sin 330°)$.

(b)



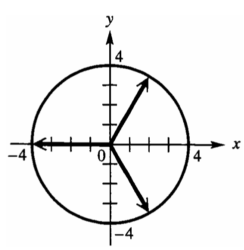
18. **(a)** Find the cube roots of
$27i = 27(\cos 90° + i \sin 90°)$.
We have $r = 27$ and $\theta = 90°$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 27(\cos 90° + i\sin 90°)$, then we have
$r^3 = 27 \Rightarrow r = 3$ and
$3\alpha = 90° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{90° + 360° \cdot k}{3} = 30° + 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 30° + 0° = 30°$.
If $k = 1$, then $\alpha = 30° + 120° = 150°$.
If $k = 2$, then $\alpha = 30° + 240° = 270°$.
So, the cube roots are
$3(\cos 30° + i\sin 30°)$,
$3(\cos 150° + i\sin 150°)$, and
$3(\cos 270° + i\sin 270°)$.

(b)



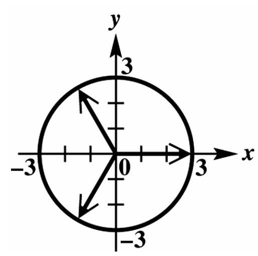
19. **(a)** Find the cube roots of
$-64 = 64(\cos 180° + i \sin 180°)$
We have $r = 64$ and $\theta = 180°$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 64(\cos 180° + i\sin 180°)$, then we have
$r^3 = 64 \Rightarrow r = 4$ and
$3\alpha = 180° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{180° + 360° \cdot k}{3} = 60° + 120° \cdot k$, k any integer.
If $k = 0$, then $\alpha = 60° + 0° = 60°$.
If $k = 1$, then $\alpha = 60° + 120° = 180°$.
If $k = 2$, then $\alpha = 60° + 240° = 300°$.
So, the cube roots are
$4(\cos 60° + i\sin 60°)$,
$4(\cos 180° + i\sin 180°)$, and
$4(\cos 300° + i\sin 300°)$.

(b)



20. **(a)** Find the cube roots of
$27 = 27(\cos 0° + i \sin 0°)$.
We have $r = 27$ and $\theta = 0°$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 27(\cos 0° + i\sin 0°)$, then we have
$r^3 = 27 \Rightarrow r = 3$ and $3\alpha = 0° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{0° + 360° \cdot k}{3} = 0° + 120° \cdot k = 120° \cdot k$,
k any integer. If $k = 0$, then $\alpha = 0°$.
If $k = 1$, then $\alpha = 120°$.
If $k = 2$, then $\alpha = 240°$. So, the cube roots are $3(\cos 0° + i\sin 0°)$,
$3(\cos 120° + i\sin 120°)$, and
$3(\cos 240° + i\sin 240°)$.

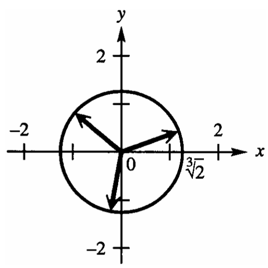
(b)



21. **(a)** Find the cube roots of $1 + i\sqrt{3}$.
We have
$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$ and
$\tan\theta = \dfrac{\sqrt{3}}{1} = \sqrt{3}$. Since θ is in quadrant I, $\theta = 60°$. Thus,
$1 + i\sqrt{3} = 2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$
$= 2(\cos 60° + i\sin 60°)$.
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 2(\cos 60° + i\sin 60°)$, then we have
$r^3 = 2 \Rightarrow r = \sqrt[3]{2}$ and
$3\alpha = 60° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{60° + 360° \cdot k}{3} = 20° + 120° \cdot k$, k any integer.
If $k = 0$, then $\alpha = 20° + 0° = 20°$.
If $k = 1$, then $\alpha = 20° + 120° = 140°$.
If $k = 2$, then $\alpha = 20° + 240° = 260°$.

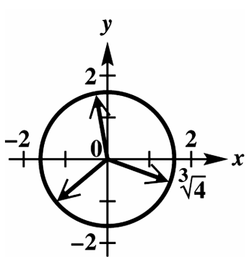
So, the cube roots are
$\sqrt[3]{2}\left(\cos 20° + i\sin 20°\right)$,
$\sqrt[3]{2}\left(\cos 140° + i\sin 140°\right)$, and
$\sqrt[3]{2}\left(\cos 260° + i\sin 260°\right)$.

(b)



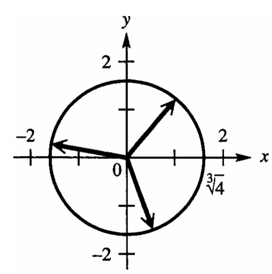
22. **(a)** Find the cube roots of $2 - 2i\sqrt{3}$.
We have
$$r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$$
and $\tan\theta = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3}$. Since θ is in quadrant IV, $\theta = 300°$. Thus,
$$2 - 2i\sqrt{3} = 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= 4\left(\cos 300° + i\sin 300°\right).$$
Since $r^3\left(\cos 3\alpha + i\sin 3\alpha\right)$
$= 4\left(\cos 300° + i\sin 300°\right)$, then we have
$r^3 = 4 \Rightarrow r = \sqrt[3]{4}$ and
$3\alpha = 300° + 360°\cdot k \Rightarrow$
$\alpha = \dfrac{300° + 360°\cdot k}{3} = 100° + 120°\cdot k$, k any integer.
If $k = 0$, then $\alpha = 100° + 0° = 100°$.
If $k = 1$, then $\alpha = 100° + 120° = 220°$.
If $k = 2$, then $\alpha = 100° + 240° = 340°$.
So, the cube roots are
$\sqrt[3]{4}\left(\cos 100° + i\sin 100°\right)$,
$\sqrt[3]{4}\left(\cos 220° + i\sin 220°\right)$, and
$\sqrt[3]{4}\left(\cos 340° + i\sin 340°\right)$.

(b)



23. **(a)** Find the cube roots of $-2\sqrt{3} + 2i$.
We have $r = \sqrt{\left(-2\sqrt{3}\right)^2 + 2^2} = \sqrt{12+4}$
$= \sqrt{16} = 4$ and $\tan\theta = \dfrac{2}{-2\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$.
Since θ is in quadrant II, $\theta = 150°$. Thus,
$$-2\sqrt{3} + 2i = 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= 4\left(\cos 150° + i\sin 150°\right).$$
Since $r^3\left(\cos 3\alpha + i\sin 3\alpha\right)$
$= 4\left(\cos 150° + i\sin 150°\right)$, then we have
$r^3 = 4 \Rightarrow r = \sqrt[3]{4}$ and
$3\alpha = 150° + 360°\cdot k \Rightarrow$
$\alpha = \dfrac{150° + 360°\cdot k}{3} = 50° + 120°\cdot k$, k any integer. If $k = 0$, then $\alpha = 50° + 0° = 50°$.
If $k = 1$, then $\alpha = 50° + 120° = 170°$.
If $k = 2$, then $\alpha = 50° + 240° = 290°$.
So, the cube roots are
$\sqrt[3]{4}\left(\cos 50° + i\sin 50°\right)$,
$\sqrt[3]{4}\left(\cos 170° + i\sin 170°\right)$, and
$\sqrt[3]{4}\left(\cos 290° + i\sin 290°\right)$.

(b)



24. **(a)** Find the cube roots of $\sqrt{3} - i$.
We have
$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$$
and $\tan\theta = -\dfrac{\sqrt{3}}{3}$. Since θ is in quadrant IV, $\theta = 330°$.

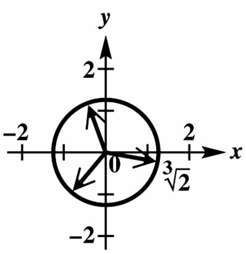
Thus, $\sqrt{3} - i = 2\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)$
$= 2\left(\cos 330° + i\sin 330°\right)$.

(*continued on next page*)

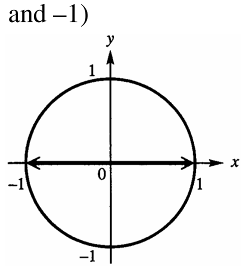
(*continued from page 795*)

Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 2(\cos 330° + i\sin 330°)$, then we have
$r^3 = 2 \Rightarrow r = \sqrt[3]{2}$ and
$3\alpha = 330° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{330° + 360° \cdot k}{3} = 110° + 120° \cdot k$, k any integer.
If $k = 0$, then $\alpha = 110° + 0° = 110°$.
If $k = 1$, then $\alpha = 110° + 120° = 230°$.
If $k = 2$, then $\alpha = 110° + 240° = 350°$.
So, the cube roots are
$\sqrt[3]{2}(\cos 110° + i\sin 110°)$,
$\sqrt[3]{2}(\cos 230° + i\sin 230°)$, and
$\sqrt[3]{2}(\cos 350° + i\sin 350°)$.

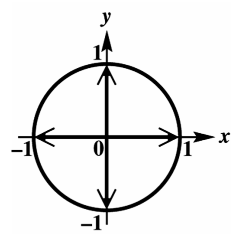
(b)



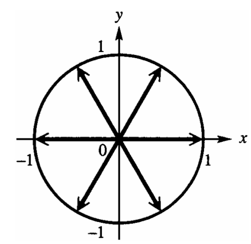
25. Find all the second (or square) roots of $1 = 1(\cos 0° + i\sin 0°)$.
Since $r^2(\cos 2\alpha + i\sin 2\alpha)$
$= 1(\cos 0° + i\sin 0°)$, then we have
$2\alpha = 0° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{0° + 360° \cdot k}{2} = 0° + 180° \cdot k = 180° \cdot k$, k any integer. If $k = 0$, then $\alpha = 0°$.
If $k = 1$, then $\alpha = 180°$. So, the second roots of 1 are
$\cos 0° + i\sin 0°$, and $\cos 180° + i\sin 180°$. (or 1 and −1)



26. Find all the fourth roots of $1 = 1(\cos 0° + i\sin 0°)$.
Since $r^4(\cos 4\alpha + i\sin 4\alpha)$
$= 1(\cos 0° + i\sin 0°)$, then we have
$r^4 = 1 \Rightarrow r = 1$ and $4\alpha = 0° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{0° + 360° \cdot k}{4} = 0° + 90° \cdot k = 90° \cdot k$, k any integer. If $k = 0$, then $\alpha = 0°$.
If $k = 1$, then $\alpha = 90°$.
If $k = 2$, then $\alpha = 180°$.
If $k = 3$, then $\alpha = 270°$. So, the fourth roots of 1 are $\cos 0° + i\sin 0°$, $\cos 90° + i\sin 90°$,
$\cos 180° + i\sin 180°$, and $\cos 270° + i\sin 270°$
(or 1, i, −1 and $-i$)



Exercise 26

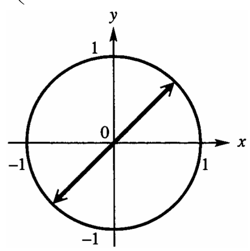


Exercise 27

27. Find all the sixth roots of $1 = 1(\cos 0° + i\sin 0°)$.
Since $r^6(\cos 6\alpha + i\sin 6\alpha)$
$= 1(\cos 0° + i\sin 0°)$, then we have
$r^6 = 1 \Rightarrow r = 1$ and $6\alpha = 0° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{0° + 360° \cdot k}{6} = 0° + 60° \cdot k = 60° \cdot k$, k any integer. If $k = 0$, then $\alpha = 0°$.
If $k = 1$, then $\alpha = 60°$.
If $k = 2$, then $\alpha = 120°$.
If $k = 3$, then $\alpha = 180°$.
If $k = 4$, then $\alpha = 240°$.
If $k = 5$, then $\alpha = 300°$. So, the sixth roots of 1 are $\cos 0° + i\sin 0°$, $\cos 60° + i\sin 60°$,
$\cos 120° + i\sin 120°$, $\cos 180° + i\sin 180°$,
$\cos 240° + i\sin 240°$, and $\cos 300° + i\sin 300°$.
$\left(\text{or } 1, \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i, -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i, -1, -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i, \text{ and } \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$

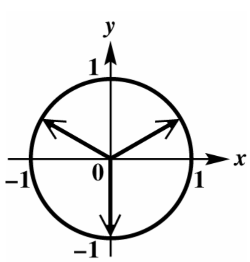
28. Find all the second (square) roots of $i = 1(\cos 90° + i\sin 90°)$.

Since $r^2(\cos 2\alpha + i\sin 2\alpha)$ $= 1(\cos 90° + i\sin 90°)$, then we have $r^2 = 1 \Rightarrow r = 1$ and $2\alpha = 90° + 360° \cdot k \Rightarrow$ $\alpha = \dfrac{90° + 360° \cdot k}{2} = 45° + 180° \cdot k$, k any integer. If $k = 0$, then $\alpha = 45° + 0° = 45°$. If $k = 1$, then $\alpha = 45° + 180° = 225°$. So, the second roots of i are $\cos 45° + i\sin 45°$ and $\cos 225° + i\sin 225°$ $\left(\text{or } \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i \text{ and } -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)$



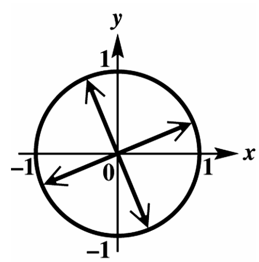
29. Find all the third (cube) roots of $i = 1(\cos 90° + i\sin 90°)$.

Since $r^3(\cos 3\alpha + i\sin 3\alpha) = 1(\cos 90° + i\sin 90°)$, then we have $r^3 = 1 \Rightarrow r = 1$ and $3\alpha = 90° + 360° \cdot k \Rightarrow$ $\alpha = \dfrac{90° + 360° \cdot k}{3} = 30° + 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 30° + 0° = 30°$. If $k = 1$, then $\alpha = 30° + 120° = 150°$. If $k = 2$, then $\alpha = 30° + 240° = 270°$. So, the third roots of i are $\cos 30° + i\sin 30°$, $\cos 150° + i\sin 150°$, and $\cos 270° + i\sin 270°$.



30. Find all the fourth roots of $i = 1(\cos 90° + i\sin 90°)$.

Since $r^4(\cos 4\alpha + i\sin 4\alpha) = 1(\cos 90° + i\sin 90°)$, then we have $r^4 = 1 \Rightarrow r = 1$ and $4\alpha = 90° + 360° \cdot k \Rightarrow$ $\alpha = \dfrac{90° + 360° \cdot k}{4} = 22.5° + 90° \cdot k$, k any integer. If $k = 0$, then $\alpha = 22.5° + 0° = 22.5°$. If $k = 1$, then $\alpha = 22.5° + 90° = 112.5°$. If $k = 2$, then $\alpha = 22.5° + 180° = 202.5°$. If $k = 3$, then $\alpha = 22.5° + 270° = 292.5°$. So, the fourth roots of i are $\cos 22.5° + i\sin 22.5°$, $\cos 112.5° + i\sin 112.5°$, $\cos 202.5° + i\sin 202.5°$, and $\cos 292.5° + i\sin 292.5°$.



31. $x^3 - 1 = 0 \Rightarrow x^3 = 1$

We have $r = 1$ and $\theta = 0°$.

$x^3 = 1 = 1 + 0i = 1(\cos 0° + i\sin 0°)$

Since $r^3(\cos 3\alpha + i\sin 3\alpha)$ $= 1(\cos 0° + i\sin 0°)$, then we have $r^3 = 1 \Rightarrow r = 1$ and $3\alpha = 0° + 360° \cdot k \Rightarrow$ $\alpha = \dfrac{0° + 360° \cdot k}{3} = 0° + 120° \cdot k = 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 0°$. If $k = 1$, then $\alpha = 120°$. If $k = 2$, then $\alpha = 240°$.

Solution set: $\{\cos 0° + i\sin 0°,\ \cos 120° + i\sin 120°,\ \cos 240° + i\sin 240°\}$ or $\left\{1,\ -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i,\ -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right\}$

32. $x^3+1=0 \Rightarrow x^3=-1$
We have $r=1$ and $\theta=180°$.
$x^3=-1=-1+0i=1(\cos 180°+i\sin 180°)$
Since $r^3(\cos 3\alpha+i\sin 3\alpha)$
$=1(\cos 180°+i\sin 180°)$, then we have
$r^3=1 \Rightarrow r=1$ and $3\alpha=180°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{180°+360°\cdot k}{3}=60°+120°\cdot k$, k any integer. If $k=0$, then $\alpha=60°+0°=60°$.
If $k=1$, then $\alpha=60°+120°=180°$.
If $k=2$, then $\alpha=60°+240°=300°$.
Solution set:
$\{\cos 60°+i\sin 60°,\ \cos 180°+i\sin 180°,$
$\cos 300°+i\sin 300°\}$ or
$\left\{\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i\right\}$

33. $x^3+i=0 \Rightarrow x^3=-i$
We have $r=1$ and $\theta=270°$.
$x^3=-i=0-i=1(\cos 270°+i\sin 270°)$
Since $r^3(\cos 3\alpha+i\sin 3\alpha)$
$=1(\cos 270°+i\sin 270°)$, then we have
$r^3=1 \Rightarrow r=1$ and $3\alpha=270°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{270°+360°\cdot k}{3}=90°+120°\cdot k$, k any integer. If $k=0$, then $\alpha=90°+0°=90°$.
If $k=1$, then $\alpha=90°+120°=210°$.
If $k=2$, then $\alpha=90°+240°=330°$.
Solution set:
$\{\cos 90°+i\sin 90°,\ \cos 210°+i\sin 210°,$
$\cos 330°+i\sin 330°\}$ or
$\left\{0,\ -\frac{\sqrt{3}}{2}-\frac{1}{2}i,\ \frac{\sqrt{3}}{2}-\frac{1}{2}i\right\}$

34. $x^4+i=0 \Rightarrow x^4=-i$
We have $r=1$ and $\theta=270°$.
$x^4=-i=0-i=1(\cos 270°+i\sin 270°)$
Since $r^4(\cos 4\alpha+i\sin 4\alpha)$
$=1(\cos 270°+i\sin 270°)$, then we have
$r^4=1 \Rightarrow r=1$ and $4\alpha=270°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{270°+360°\cdot k}{4}=67.5°+90°\cdot k$, k any integer. If $k=0$, then $\alpha=67.5°+0°=67.5°$.
If $k=1$, then $\alpha=67.5°+90°=157.5°$.
If $k=2$, then $\alpha=67.5°+180°=247.5°$.
If $k=3$, then $\alpha=67.5°+270°=337.5°$.
Solution set:
$\{\cos 67.5°+i\sin 67.5°, \cos 157.5°+i\sin 157.5°,$
$\cos 247.5°+i\sin 247.5°,$
$\cos 337.5°+i\sin 337.5°\}$

35. $x^3-8=0 \Rightarrow x^3=8$
We have $r=8$ and $\theta=0°$.
$x^3=8=8+0i=8(\cos 0°+i\sin 0°)$
Since $r^3(\cos 3\alpha+i\sin 3\alpha)$
$=8(\cos 0°+i\sin 0°)$, then we have
$r^3=8 \Rightarrow r=2$ and $3\alpha=0°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{0°+360°\cdot k}{3}=0°+120°\cdot k=120°\cdot k$, k any integer. If $k=0$, then $\alpha=0°$.
If $k=1$, then $\alpha=120°$. If $k=2$, then $\alpha=240°$.
Solution set:
$\{2(\cos 0°+i\sin 0°), 2(\cos 120°+i\sin 120°),$
$2(\cos 240°+i\sin 240°)\}$ or
$\{2,\ -1+\sqrt{3}i,\ -1-\sqrt{3}i\}$

36. $x^3+27=0 \Rightarrow x^3=-27$
We have $r=27$ and $\theta=180°$.
$x^3=-27=-27+0i=27(\cos 180°+i\sin 180°)$
Since $r^3(\cos 3\alpha+i\sin 3\alpha)$
$=27(\cos 180°+i\sin 180°)$, then we have
$r^3=27 \Rightarrow r=3$ and $3\alpha=180°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{180°+360°\cdot k}{3}=60°+120°\cdot k$, k any integer. If $k=0$, then $\alpha=60°+0°=60°$.
If $k=1$, then $\alpha=60°+120°=180°$.
If $k=2$, then $\alpha=60°+240°=300°$.
Solution set:
$\{3(\cos 60°+i\sin 60°), 3(\cos 180°+i\sin 180°),$
$3(\cos 300°+i\sin 300°)\}$ or
$\left\{\frac{3}{2}+\frac{3\sqrt{3}}{2}i,\ -3,\ \frac{3}{2}-\frac{3\sqrt{3}}{2}i\right\}$

37. $x^4+1=0 \Rightarrow x^4=-1$
We have $r=1$ and $\theta=180°$.
$x^4=-1=-1+0i=1(\cos 180°+i\sin 180°)$
Since $r^4(\cos 4\alpha+i\sin 4\alpha)$
$=1(\cos 180°+i\sin 180°)$, then we have
$r^4=1 \Rightarrow r=1$ and $4\alpha=180°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{180°+360°\cdot k}{4}=45°+90°\cdot k$, k any integer. If $k=0$, then $\alpha=45°+0°=45°$.
If $k=1$, then $\alpha=45°+90°=135°$.
If $k=2$, then $\alpha=45°+180°=225°$.
If $k=3$, then $\alpha=45°+270°=315°$.
Solution set:
$\{\cos 45°+i\sin 45°,\ \cos 135°+i\sin 135°,$
$\cos 225°+i\sin 225°,\ \cos 315°+i\sin 315°\}$ or
$\left\{\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i,\ -\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i,\ -\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i,\ \dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right\}$

38. $x^4+16=0 \Rightarrow x^4=-16$
We have $r=16$ and $\theta=180°$.
$x^4=-16=-16+0i=16(\cos 180°+i\sin 180°)$
Since $r^4(\cos 4\alpha+i\sin 4\alpha)$
$=16(\cos 180°+i\sin 180°)$, then we have
$r^4=16 \Rightarrow r=2$ and $4\alpha=180°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{180°+360°\cdot k}{4}=45°+90°\cdot k$, k any integer. If $k=0$, then $\alpha=45°+0°=45°$.
If $k=1$, then $\alpha=45°+90°=135°$.
If $k=2$, then $\alpha=45°+180°=225°$.
If $k=3$, then $\alpha=45°+270°=315°$.
Solution set:
$\{2(\cos 45°+i\sin 45°), 2(\cos 135°+i\sin 135°),$
$2(\cos 225°+i\sin 225°),$
$2(\cos 315°+i\sin 315°)\}$ or
$\{\sqrt{2}+i\sqrt{2},\ -\sqrt{2}+i\sqrt{2},\ -\sqrt{2}-i\sqrt{2},\ \sqrt{2}-i\sqrt{2}\}$

39. $x^4-i=0 \Rightarrow x^4=i$
We have $r=1$ and $\theta=90°$.
$x^4=i=0+i=1(\cos 90°+i\sin 90°)$
Since $r^4(\cos 4\alpha+i\sin 4\alpha)$
$=1(\cos 90°+i\sin 90°)$, then we have
$r^4=1 \Rightarrow r=1$ and $4\alpha=90°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{90°+360°\cdot k}{4}=22.5°+90°\cdot k$, k any integer. If $k=0$, then $\alpha=22.5°+0°=22.5°$.
If $k=1$, then $\alpha=22.5°+90°=112.5°$.
If $k=2$, then $\alpha=22.5°+180°=202.5°$.
If $k=3$, then $\alpha=22.5°+270°=292.5°$.
Solution set:
$\{\cos 22.5°+i\sin 22.5°, \cos 112.5°+i\sin 112.5°,$
$\cos 202.5°+i\sin 202.5°,$
$\cos 292.5°+i\sin 292.5°\}$

40. $x^5-i=0 \Rightarrow x^5=i$
We have $r=1$ and $\theta=90°$.
$x^5=i=0+i=1(\cos 90°+i\sin 90°)$
Since $r^5(\cos 5\alpha+i\sin 5\alpha)$
$=1(\cos 90°+i\sin 90°)$, then we have
$r^5=1 \Rightarrow r=1$ and $5\alpha=90°+360°\cdot k \Rightarrow$
$\alpha=\dfrac{90°+360°\cdot k}{5}=18°+72°\cdot k$, k any integer.
If $k=0$, then $\alpha=18°+0°=18°$.
If $k=1$, then $\alpha=18°+72°=90°$.
If $k=2$, then $\alpha=18°+144°=162°$.
If $k=3$, then $\alpha=18°+216°=234°$.
If $k=4$, then $\alpha=18°+288°=306°$.
Solution set:
$\{\cos 18°+i\sin 18°,\ \cos 90°+i\sin 90°(\text{or } 0),$
$\cos 162°+i\sin 162°, \cos 234°+i\sin 234°,$
$\cos 306°+i\sin 306°\}$

41. $x^3-(4+4i\sqrt{3})=0 \Rightarrow x^3=4+4i\sqrt{3}$
We have
$r=\sqrt{4^2+(4\sqrt{3})^2}=\sqrt{16+48}=\sqrt{64}=8$ and
$\tan\theta=\dfrac{4\sqrt{3}}{4}=\sqrt{3}$. Since θ is in quadrant I, $\theta=60°$.
$x^3=4+4i\sqrt{3}=8\left(\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)$
$=8(\cos 60°+i\sin 60°)$

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(*continued from page 799*)

Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 8(\cos 60° + i\sin 60°)$, then we have
$r^3 = 8 \Rightarrow r = 2$ and $3\alpha = 60° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{60° + 360° \cdot k}{3} = 20° + 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 20° + 0° = 20°$.
If $k = 1$, then $\alpha = 20° + 120° = 140°$.
If $k = 2$, then $\alpha = 20° + 240° = 260°$.
Solution set:
$\{2(\cos 20° + i\sin 20°), 2(\cos 140° + i\sin 140°), 2(\cos 260° + i\sin 260°)\}$

42. $x^4 - (8 + 8i\sqrt{3}) = 0 \Rightarrow x^4 = 8 + 8i\sqrt{3}$
We have
$r = \sqrt{8^2 + (8\sqrt{3})^2} = \sqrt{64 + 192} = \sqrt{256} = 16$
and $\tan\theta = \dfrac{8\sqrt{3}}{8} = \sqrt{3}$. Since θ is in quadrant I, $\theta = 60°$.
$x^4 = 8 + 8i\sqrt{3} = 16\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$
$= 16(\cos 60° + i\sin 60°)$
Since $r^4(\cos 4\alpha + i\sin 4\alpha)$
$= 16(\cos 60° + i\sin 60°)$, then we have
$r^4 = 16 \Rightarrow r = 2$ and $4\alpha = 60° + 360° \cdot k \Rightarrow$
$\alpha = \dfrac{60° + 360° \cdot k}{4} = 15° + 90° \cdot k$, k any integer. If $k = 0$, then $\alpha = 15° + 0° = 15°$.
If $k = 1$, then $\alpha = 15° + 90° = 105°$.
If $k = 2$, then $\alpha = 15° + 180° = 195°$.
If $k = 3$, then $\alpha = 15° + 270° = 285°$.
Solution set:
$\{2(\cos 15° + i\sin 15°), 2(\cos 105° + i\sin 105°), 2(\cos 195° + i\sin 195°), 2(\cos 285° + i\sin 285°)\}$

43. $x^3 - 1 = 0 \Rightarrow (x-1)(x^2 + x + 1) = 0$
Setting each factor equal to zero, we have
$x - 1 = 0 \Rightarrow x = 1$ and
$x^2 + x + 1 = 0 \Rightarrow$
$x = \dfrac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \dfrac{-1 \pm \sqrt{-3}}{2}$
$= \dfrac{-1 \pm i\sqrt{3}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$
Thus, $x = 1, -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i, -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i.$ We see that the solutions are the same as Exercise 31.

44. $x^3 + 27 = 0 \Rightarrow (x+3)(x^2 - 3x + 9) = 0$
Setting each factor equal to zero, we have
$x + 3 = 0 \Rightarrow x = -3$ and
$x^2 - 3x + 9 = 0 \Rightarrow$
$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1} = \dfrac{3 \pm \sqrt{-27}}{2}$
$= \dfrac{3 \pm 3i\sqrt{3}}{2} = \dfrac{3}{2} \pm \dfrac{3\sqrt{3}}{2}i$
Thus, $x = -3, \dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i, \dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i.$ We see that the solutions are the same as Exercise 36.

45. De Moivre's theorem states that
$(\cos\theta + i\sin\theta)^2 = 1^2(\cos 2\theta + i\sin 2\theta)$
$= \cos 2\theta + i\sin 2\theta$

46. $(\cos\theta + i\sin\theta)^2$
$= \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$
$= \cos^2\theta + 2i\sin\theta\cos\theta + (-1)\sin^2\theta$
$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$
$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$
$= \cos 2\theta + i\sin 2\theta$

47. Two complex numbers $a + bi$ and $c + di$ are equal only if $a = c$ and $b = d$. Thus, $a = c$ implies $\cos^2\theta - \sin^2\theta = \cos 2\theta.$

48. Two complex numbers $a + bi$ and $c + di$ are equal only if $a = c$ and $b = d$. Thus, $b = d$ implies $2\sin\theta\cos\theta = \sin 2\theta$.

49. **(a)** If $z = 0 + 0i$, then $z = 0$, $0^2 + 0 = 0$, $0^2 + 0 = 0$, and so on. The calculations repeat as 0, 0, 0, . . ., and will never exceed a modulus of 2. The point $(0, 0)$ is part of the Mandelbrot set. The pixel at the origin should be turned on.

(b) If $z = 1 - 1i$, then $(1-i)^2 + (1-i) = 1 - 3i.$ The modulus of $1 - 3i$ is $\sqrt{1^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$, which is greater than 2. Therefore, $1 - 1i$ is not part of the Mandelbrot set, and the pixel at $(1, -1)$ should be left off.

(c) If $z = -.5i$, then $(-.5i)^2 - .5i = -.25 - .5i$;

$(-.25 - .5i)^2 + (-.25 - .5i) = -.4375 - .25i$;

$(-.4375 - .25i)^2 + (-.4375 - .25i)$
$= -.308593 - .03125i$;

$(-.308593 - .03125i)^2$
$+ (-.308593 - .03125i)$
$= -.214339 - .0119629i$;

$(-.214339 - .0119629i)^2$
$+ (-.214339 - .0119629i)$
$= -.16854 - .00683466i$

This sequence appears to be approaching the origin, and no number has a modulus greater than 2. Thus, $-.5i$ is part of the Mandelbrot set, and the pixel at $(0, -.5)$ should be turned on.

50. (a) Let $f(z) = \dfrac{2z^3 + 1}{3z^2}$ and $z_1 = i$. Then,

$$z_2 = f(z_1) = \frac{2i^3 + 1}{3i^2} = \frac{2(-i) + 1}{3(-1)}$$
$$= \frac{-2i + 1}{-3} = -\frac{1}{3} + \frac{2}{3}i$$

Similarly, $z_3 = f(z_2) = \dfrac{2z_2^{\;3} + 1}{3z_2^{\;2}}$

$\approx -.58222 + .92444i$ and

$z_4 = f(z_3) = \dfrac{2z_3 + 1}{3z_3^{\;2}}$

$\approx -.50879 + .868165i$. The values of z seem to approach $w_2 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$. Color the pixel at $(0,1)$ blue.

(b) Let $f(z) = \dfrac{2z^3 + 1}{3z^2}$ and $z_1 = 2 + i$. Then,

$$z_2 = f(z_1) = \frac{2(2+i)^3 + 1}{3(2+i)^2}$$
$\approx 1.37333 + .61333i$

Similarly, $z_3 = f(z_2) = \dfrac{2z_2^{\;3} + 1}{3z_2^{\;2}}$

$\approx 1.01389 + .299161i$,

$z_4 = f(z_3) = \dfrac{2z_3^{\;3} + 1}{3z_3^{\;2}}$

$\approx .926439 + .0375086i$,

and $z_5 = f(z_4) = \dfrac{2z_4^{\;3} + 1}{3z_4^{\;2}}$

$\approx 1.00409 - .00633912i$.

The values of z seem to approach $w_1 = 1$. Color the pixel at $(2,1)$ red.

(c) Let $f(z) = \dfrac{2z^3 + 1}{3z^2}$ and $z_1 = -1 - i$. Then,

$$z_2 = f(z_1) = \frac{2(-1-i)^3 + 1}{3(-1-i)^2} = -\frac{2}{3} - \frac{5}{6}i.$$

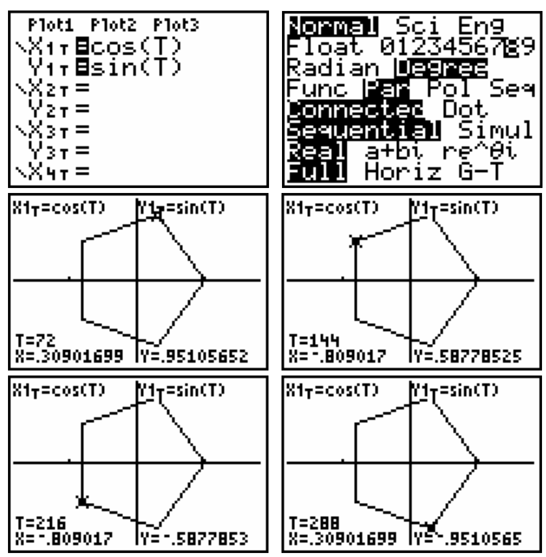
Similarly, $z_3 = f(z_2) = \dfrac{2z_2^{\;3} + 1}{3z_2^{\;2}}$

$\approx .508691 - .841099i$ and

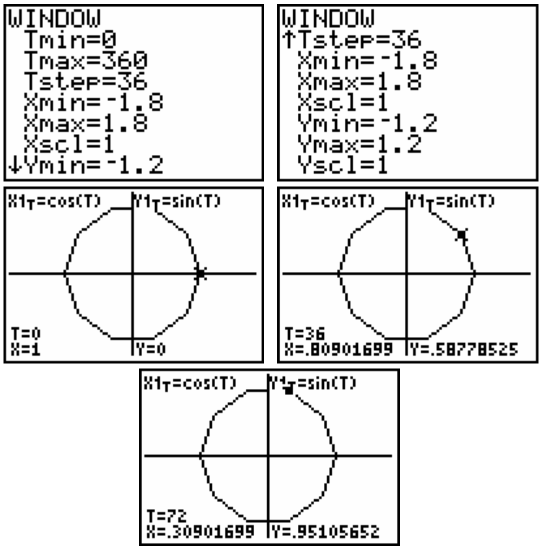
$z_4 = f(z_3) = \dfrac{2z_3^{\;3} + 1}{3z_3^{\;2}}$

$\approx -.499330 - .866269i$. The values of z seem to approach $w_3 = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$. Color the pixel at (–1, –1) yellow.

51. Using the trace function, we find that the other four fifth roots of 1 are:
$.30901699 + .95105652i$,
$-.809017 + .58778525i$,
$-.809017 - .5877853i$,
$.30901699 - .9510565i$.



52. Using the trace function, we find that the first three tenth roots of 1 are: 1, $.80901699 + .58778525i$, $.30901699 + .95105652i$.



53. $2+2i\sqrt{3}$ is one cube root of a complex number.

$$r=\sqrt{2^2+\left(2\sqrt{3}\right)^2}=\sqrt{4+12}=4 \text{ and}$$

$$\tan\theta=\frac{2\sqrt{3}}{2}=\sqrt{3}$$

Because x and y are both positive, θ is in quadrant I, so $\theta=60°$.

$$2+2i\sqrt{3}=4\left(\cos 60°+i\sin 60°\right)$$

$$\begin{aligned}\left(2+2i\sqrt{3}\right)^3&=\left[4\left(\cos 60°+i\sin 60°\right)\right]^3\\&=4^3\left[\cos\left(3\cdot 60°\right)+i\sin\left(3\cdot 60°\right)\right]\\&=64\left(\cos 180°+i\sin 180°\right)\\&=64\left(-1+0\cdot i\right)=-64+0\cdot i\end{aligned}$$

Since the graphs of the other roots of – 64 must be equally spaced around a circle and the graphs of these roots are all on a circle that has center at the origin and radius 4, the other roots are

$$\begin{aligned}&4\left[\cos\left(60°+120°\right)+i\sin\left(60°+120°\right)\right]\\&=4\left(\cos 180°+i\sin 180°\right)=4\left(-1+i\cdot 0\right)=-4\end{aligned}$$

$$\begin{aligned}&4\left[\cos\left(60°+240°\right)+i\sin\left(60°+240°\right)\right]\\&=4\left(\cos 300°+i\sin 300°\right)=4\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\\&=2-2i\sqrt{3}\end{aligned}$$

54. $x^2+2-i=0\Rightarrow x^2=-2+i$

$$r=\sqrt{(-2)^2+1^2}=\sqrt{5}\Rightarrow$$

$$r^{1/n}=r^{1/2}=\left(\sqrt{5}\right)^{1/2}\approx 1.49535 \text{ and}$$

$\tan\theta=-\frac{1}{2}$. Since θ is in quadrant II, $\theta\approx 153.4349°$ and

$$\alpha=\frac{153.4349°+360°\cdot k}{2}\approx 76.717°+180°\cdot k,$$

where k is an integer.

$$\begin{aligned}x\approx\ &1.49535\left(\cos 76.717°+i\sin 76.717°\right),\\&1.49535\left(\cos 256.717°+i\sin 256.717°\right)\end{aligned}$$

Solution set:
$\{.3436+1.4553i, -.3436-1.4553i\}$

55. $x^2-3+2i=0\Rightarrow x^2=3-2i$

$$r=\sqrt{3^2+(-2)^2}=\sqrt{9+4}=\sqrt{13}\Rightarrow$$

$$r^{1/n}=r^{1/2}=\left(\sqrt{13}\right)^{1/2}\approx 1.89883 \text{ and}$$

$\tan\theta=-\frac{2}{3}$. Since θ is in quadrant IV, $\theta\approx 326.3099°$ and

$$\alpha=\frac{326.3099°+360°\cdot k}{2}\approx 163.155°+180°\cdot k,$$

where k is an integer.

$$\begin{aligned}x\approx\ &1.89833\left(\cos 163.155°+i\sin 163.155°\right),\\&1.89833\left(\cos 343.155°+i\sin 343.155°\right)\end{aligned}$$

Solution set:
$\{-1.8174+.5503i, 1.8174-.5503i\}$

56. $x^3+4-5i=0\Rightarrow x^3=-4+5i$

$$r=\sqrt{(-4)^2+5^2}=\sqrt{16+25}=\sqrt{41}\Rightarrow$$

$$r^{1/n}=r^{1/3}=\left(\sqrt{41}\right)^{1/3}\approx 1.85694 \text{ and}$$

$$\tan\theta=-\frac{5}{4}=-1.25$$

Since θ is in quadrant II, $\theta\approx 128.6598°$ and

$$\alpha=\frac{128.6598°+360°\cdot k}{3}\approx 42.887°+120°\cdot k,$$

where k is an integer.

$$\begin{aligned}x\approx\ &1.85694\left(\cos 42.887°+i\sin 42.887°\right),\\&1.85694\left(\cos 162.887°+i\sin 162.887°\right),\\&1.85694\left(\cos 282.887°+i\sin 282.887°\right)\end{aligned}$$

Solution set:
$\{1.3606+1.2637i, -1.7747+.5464i, .4141-1.8102i\}$

57. $x^5 + 2 + 3i = 0 \Rightarrow x^5 = -2 - 3i$

$r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13} \Rightarrow$

$r^{1/n} = r^{1/5} = \left(\sqrt{13}\right)^{1/5} = 13^{1/10} \approx 1.2924$

and $\tan\theta = \frac{-3}{-2} = 1.5$

Since θ is in quadrant III, $\theta \approx 236.310°$ and

$\alpha = \frac{236.310° + 360° \cdot k}{5} = 47.262° + 72° \cdot k,$

where k is an integer.

$x \approx 1.29239\left(\cos 47.262° + i\sin 47.262°\right),$
$1.29239\left(\cos 119.262° + i\sin 119.2622°\right),$
$1.29239\left(\cos 191.262° + i\sin 191.2622°\right),$
$1.29239\left(\cos 263.262° + i\sin 263.2622°\right),$
$1.29239\left(\cos 335.262° + i\sin 335.2622°\right)$

Solution set:
$\{.87708 + .94922i, -.63173 + 1.1275i, -1.2675 - .25240i, -.15164 - 1.28347i, 1.1738 - .54083i\}$

58. The number 1 has 64 complex 64th roots. Two of them are real, 1 and –1, and 62 of them are not real.

59. The statement, "Every real number must have two real square roots," is false. Consider, for example, the real number –4. Its two square roots are 2i and –2i, which are not real.

60. The statement, "Some real numbers have three real cube roots," is false. Every real number has only one cube root that is also a real number. Then there are two cube roots, which are conjugates that are not real.

61. If z is an nth root of 1, then $z^n = 1$. Since $1 = \frac{1}{1} = \frac{1}{z^n} = \left(\frac{1}{z}\right)^n$, then $\frac{1}{z}$ is also an nth root of 1.

62. – 64. Answers will vary.

Chapter 8 Quiz
(Sections 8.3–8.6)

1. $\mathbf{a} = \langle -1,4\rangle$, $\mathbf{b} = \langle 5,2\rangle$

(a) $3\mathbf{a} = 3\langle -1,4\rangle = \langle -3,12\rangle$

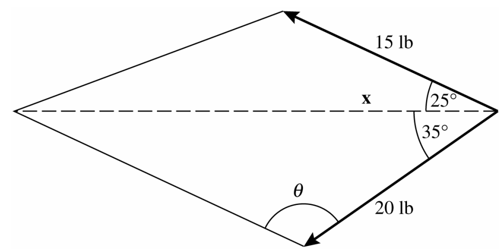
(b) $4\mathbf{a} - 2\mathbf{b} = 4\langle -1,4\rangle - 2\langle 5,2\rangle = \langle -4,16\rangle - \langle 10,4\rangle = \langle -14,12\rangle$

(c) $|\mathbf{a}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$

(d) $\mathbf{a}\cdot\mathbf{b} = (-1)(5) + (4)(2) = 3$

(e) $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(-1)(5)+(4)(2)}{\sqrt{(-1)^2+4^2}\cdot\sqrt{5^2+2^2}}$
$= \frac{3}{\sqrt{17}\cdot\sqrt{29}} \approx .1351132047 \Rightarrow$
$\theta \approx 82.23°$

2.



Let $|\mathbf{x}|$ be the equilibrant force.

$\theta = 180° - 35° - 25° = 120°$

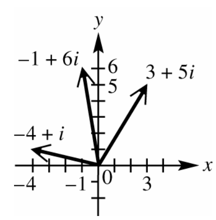
Using the law of cosines, we have

$|\mathbf{x}|^2 = 15^2 + 20^2 - 2\cdot 15\cdot 20\cos 120° \Rightarrow$

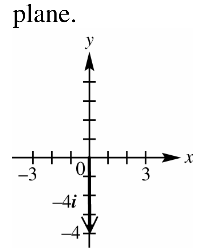
$|\mathbf{x}|^2 = 925 \Rightarrow \mathbf{x} \approx 30.4 \approx 30$ lb

3. $w = 3 + 5i, z = -4 + i$

$w + z = (3+5i) + (-4+i) = -1 + 6i$



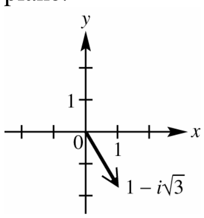
4. **(a)** Sketch a graph of $-4i$ in the complex plane.



Since $-4i = 0 - 4i$, we have $x = 0$ and $y = -4$, so $r = \sqrt{0^2 + (-4)^2} = 4$. We cannot find θ by using $\tan\theta = \frac{y}{x}$ because $x = 0$. From the graph, we see that $-4i$ is on the negative y-axis, so $\theta = 270°$. Thus, $-4i = 4\left(\cos 270° + i\sin 270°\right)$

(b) $1-i\sqrt{3}$

Sketch a graph of $1-i\sqrt{3}$ in the complex plane.



Since $x = 1$ and $y=-\sqrt{3}$,

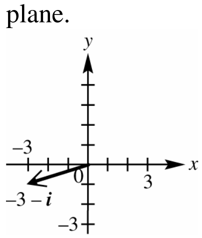
$r=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=\sqrt{1+3}=\sqrt{4}=2$ and

$\tan\theta=\frac{-\sqrt{3}}{1}=-\sqrt{3}$. The graph shows that θ is in quadrant IV, so $\theta=300°$. Therefore,

$1-i\sqrt{3}=2(\cos 300°+i\sin 300°)$

(c) $-3-i$

Sketch a graph of $-3-i$ in the complex plane.



Using a calculator, we find that the reference angle is 18.4º. The graph shows that θ is in quadrant III, so $\theta=180°+18.4°=198.4°$. Therefore,

$-3-i=\sqrt{10}\left(\cos 198.4°+i\sin 198.4°\right)$.

5. **(a)** $$4(\cos 60°+i\sin 60°)=4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2+2i\sqrt{3}$$

(b) $$\begin{aligned}5\operatorname{cis}130°&=5(\cos 130°+i\sin 130°)\\&=5\cos 130°+5i\sin 130°\\&=-3.2139+3.8302i\end{aligned}$$

(c) $$7(\cos 270°+i\sin 270°)=7(0+(-1)i)=0-7i$$

6. $w=12(\cos 80°+i\sin 80°)$,
$z=3(\cos 50°+i\sin 50°)$

(a) $$\begin{aligned}wz&=12(\cos 80°+i\sin 80°)\cdot 3(\cos 50°+i\sin 50°)\\&=12\cdot 3\left[\cos(80°+50°)+i\sin(80°+50°)\right]\\&=36(\cos 130°+i\sin 130°)\end{aligned}$$

(b) $$\begin{aligned}\frac{w}{z}&=\frac{12(\cos 80°+i\sin 80°)}{3(\cos 50°+i\sin 50°)}\\&=4\left[\cos(80°-50°)+i\sin(80°-50°)\right]\\&=4(\cos 30°+i\sin 30°)\\&=4\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\sqrt{3}+2i\end{aligned}$$

(c) $$\begin{aligned}z^3&=\left[3(\cos 50°+i\sin 50°)\right]^3\\&=3^3\left[\cos(3\cdot 50°)+i\sin(3\cdot 50°)\right]\\&=27(\cos 150°+i\sin 150°)\\&=27\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\frac{27\sqrt{3}}{2}+\frac{27}{2}i\end{aligned}$$

7. $(1-i)^3$

First write 1 − *i* in trigonometric form:

$r=\sqrt{1^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}$ and

$\tan\theta=\frac{-1}{1}=-1\Rightarrow\theta=315°$, since θ is in quadrant IV. Thus, $1-i=\sqrt{2}\operatorname{cis}315°$.

Using DeMoivre's theorem,

$$\begin{aligned}(1-i)^3&=\left(\sqrt{2}\right)^3\operatorname{cis}315°=2\sqrt{2}\operatorname{cis}315°\\&=2\sqrt{2}\cos 315°+2i\sqrt{2}\sin 315°\\&=2\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)+2i\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)=2-2i\end{aligned}$$

8. Find all the fourth roots of $-16=16(\cos 180°+i\sin 180°)$.

Since $r^4(\cos 4\alpha+i\sin 4\alpha)$

$=16(\cos 180°+i\sin 180°)$, then we have

$r^4=16\Rightarrow r=2$ and $4\alpha=180°+360°\cdot k\Rightarrow$

$\alpha=\frac{180°+360°\cdot k}{4}=45°+90°\cdot k$, k any integer.

If $k = 0,$ then $\alpha = 45° + 0° = 45°.$
If $k = 1,$ then $\alpha = 45° + 90° = 135°.$
If $k = 2,$ then $\alpha = 45° + 180° = 225°.$
If $k = 3,$ then $\alpha = 45° + 270° = 315°.$
Solution set:
$\{2(\cos 45° + i\sin 45°), 2(\cos 135° + i\sin 135°),$
$2(\cos 225° + i\sin 225°),$
$2(\cos 315° + i\sin 315°)\}$ or
$\{\sqrt{2} + i\sqrt{2},\ -\sqrt{2} + i\sqrt{2},\ -\sqrt{2} - i\sqrt{2},\ \sqrt{2} - i\sqrt{2}\}$

Section 8.7: Polar Equations and Graphs

1. **(a)** II (since $r > 0$ and $90° < \theta < 180°$)

(b) I (since $r > 0$ and $0° < \theta < 90°$)

(c) IV (since $r > 0$ and $-90° < \theta < 0°$)

(d) III (since $r > 0$ and $180° < \theta < 270°$)

2. **(a)** positive x-axis

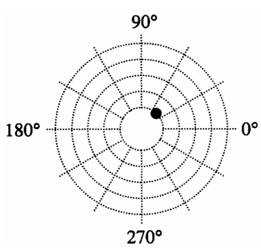
(b) negative x-axis

(c) negative y-axis

(d) positive y-axis (since $450° - 360° = 90°$)

For Exercises 3(b)–12(b), answers may vary.

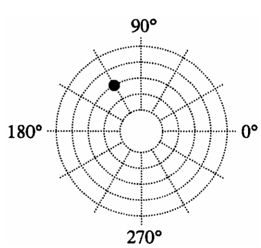
3. **(a)**



(b) Two other pairs of polar coordinates for (1, 45°) are (1, 405°) and (–1, 225°).

(c) Since $x = r\cos\theta \Rightarrow x = 1 \cdot \cos 45° = \frac{\sqrt{2}}{2}$ and $y = r\sin\theta \Rightarrow y = 1 \cdot \sin 45° = \frac{\sqrt{2}}{2},$ the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$

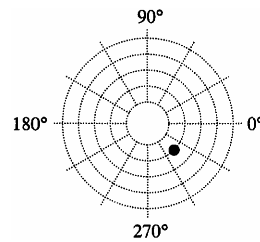
4. **(a)**



(b) Two other pairs of polar coordinates for (3, 120°) are (3, 480°) and (–3, 300°).

(c) Since $x = r\cos\theta \Rightarrow x = 3\cos 120 = -\frac{3}{2}$ and $y = r\sin\theta \Rightarrow y = 3\sin 120° = \frac{3\sqrt{3}}{2},$ the point is $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right).$

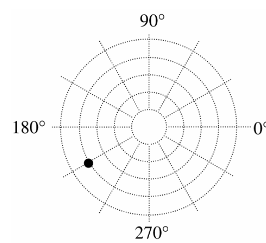
5. **(a)**



(b) Two other pairs of polar coordinates for (–2, 135°) are (–2, 495°) and (2, 315°).

(c) Since
$x = r\cos\theta \Rightarrow x = (-2)\cos 135° = \sqrt{2}$ and
$y = r\sin\theta \Rightarrow y = (-2)\sin 135° = \sqrt{2},$ the point is $(\sqrt{2}, -\sqrt{2}).$

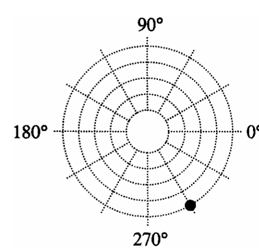
6. **(a)**



(b) Two other pairs of polar coordinates for (–4, 30°) are $(-4, 390°)$ and $(4, 210°).$

(c) Since
$x = r\cos\theta \Rightarrow x = (-4)\cos 30° = -2\sqrt{3}$ and
$y = r\sin\theta \Rightarrow y = (-4)\sin 30° = -2,$ the point is $(-2\sqrt{3}, -2).$

7. **(a)**



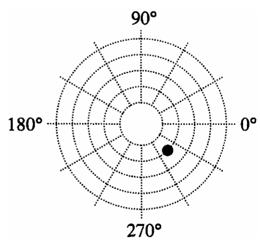
(b) Two other pairs of polar coordinates for (5, – 60°) are (5, 300°) and (–5, 120°).

(c) Since $x = r\cos\theta \Rightarrow x = 5\cos(-60°) = \dfrac{5}{2}$ and

$y = r\sin\theta \Rightarrow y = 5\sin(-60°) = -\dfrac{5\sqrt{3}}{2}$,

the point is $\left(\dfrac{5}{2}, -\dfrac{5\sqrt{3}}{2}\right)$.

8. **(a)**



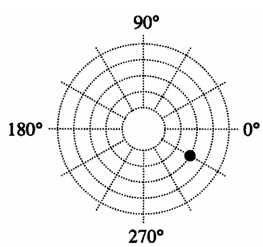
(b) Two other pairs of polar coordinates for (2, – 45°) are (2, 315°) and (–2, 135°).

(c) Since

$x = r\cos\theta \Rightarrow x = 2\cos(-45°) = \sqrt{2}$ and

$y = r\sin\theta \Rightarrow y = 2\sin(-45°) = -\sqrt{2}$, the point is $\left(\sqrt{2}, -\sqrt{2}\right)$.

9. **(a)**



(b) Two other pairs of polar coordinates for (–3, –210°) are (–3, 150°) and (3, –30°).

(c) Since

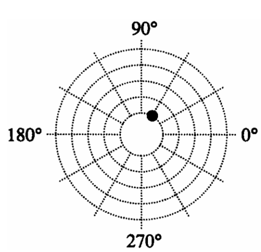
$x = r\cos\theta \Rightarrow x = (-3)\cos(-210°) = \dfrac{3\sqrt{3}}{2}$

and

$y = r\sin\theta \Rightarrow y = (-3)\sin(-210°) = -\dfrac{3}{2}$,

the point is $\left(\dfrac{3\sqrt{3}}{2}, -\dfrac{3}{2}\right)$.

10. **(a)**



(b) Two other pairs of polar coordinates for (–1, –120°) are (–1, 240°) and (1, 60°).

(c) Since

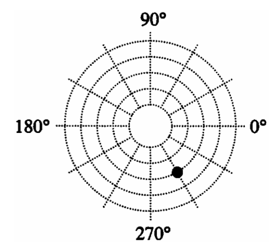
$x = r\cos\theta \Rightarrow x = (-1)\cos(-120°) = \dfrac{1}{2}$

and

$y = r\sin\theta \Rightarrow y = (-1)\sin(-120°) = \dfrac{\sqrt{3}}{2}$,

the point is $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

11. **(a)**

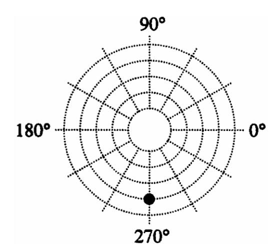


(b) Two other pairs of polar coordinates for $\left(3, \dfrac{5\pi}{3}\right)$ are $\left(3, \dfrac{11\pi}{3}\right)$ and $\left(-3, \dfrac{2\pi}{3}\right)$.

(c) Since $x = r\cos\theta \Rightarrow x = 3\cos\dfrac{5\pi}{3} = \dfrac{3}{2}$ and

$y = r\sin\theta \Rightarrow y = 3\sin\dfrac{5\pi}{3} = -\dfrac{3\sqrt{3}}{2}$, the point is $\left(\dfrac{3}{2}, -\dfrac{3\sqrt{3}}{2}\right)$.

12. **(a)**



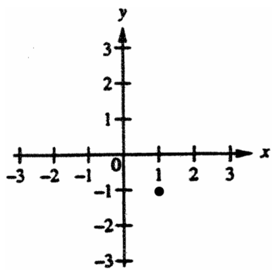
(b) Two other pairs of polar coordinates for $\left(4, \dfrac{3\pi}{2}\right)$ are $\left(4, -\dfrac{\pi}{2}\right)$ and $\left(-4, \dfrac{\pi}{2}\right)$.

(c) Since $x = r\cos\theta \Rightarrow x = 4\cos\dfrac{3\pi}{2} = 0$ and

$y = r\sin\theta \Rightarrow y = 4\sin\dfrac{3\pi}{2} = -4$, the point is $(0, -4)$.

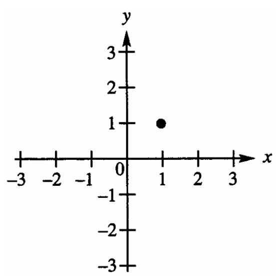
For Exercises 13(b)–22(b), answers may vary.

13. (a)



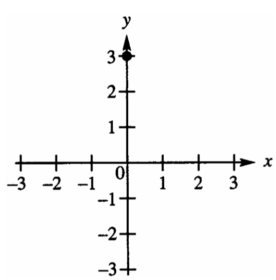
(b) $r=\sqrt{1^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}$ and $\theta=\tan^{-1}\left(\frac{-1}{1}\right)=\tan^{-1}(-1)=-45°$, since θ is in quadrant IV. Since $360°-45°=315°$, one possibility is $\left(\sqrt{2},315°\right)$. Alternatively, if $r=-\sqrt{2}$, then $\theta=315°-180°=135°$. Thus, a second possibility is $\left(-\sqrt{2},135°\right)$.

14. (a)



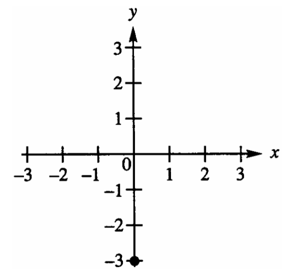
(b) $r=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$ and $\theta=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}1=45°$, since θ is in quadrant I. So, one possibility is $\left(\sqrt{2},45°\right)$. Alternatively, if $r=-\sqrt{2}$, then $\theta=45°+180°=225°$. Thus, a second possibility is $\left(-\sqrt{2},225°\right)$.

15. (a)



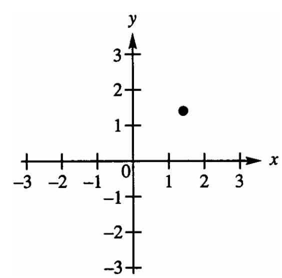
(b) $r=\sqrt{0^2+3^2}=\sqrt{0+9}=\sqrt{9}=3$ and $\theta=90°$, since $(0,3)$ is on the positive y-axis. So, one possibility is $(3,90°)$. Alternatively, if $r=-3$, then $\theta=90°+180°=270°$. Thus, a second possibility is $(-3,270°)$.

16. (a)



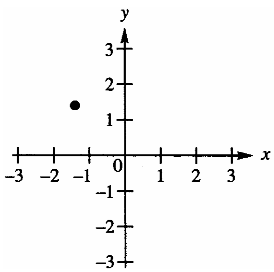
(b) $r=\sqrt{0^2+(-3)^2}=\sqrt{0+9}=\sqrt{9}=3$ and $\theta=270°$, since $(0,3)$ is on the negative y-axis. So, one possibility is $(3,270°)$. Alternatively, if $r=-3$, then $\theta=270°-180°=90°$. Thus, a second possibility is $(-3,90°)$.

17. (a)



(b) $r=\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}=\sqrt{2+2}=\sqrt{4}=2$ and $\theta=\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\tan^{-1}1=45°$, since θ is in quadrant I. So, one possibility is $(2,45°)$. Alternatively, if $r=-2$, then $\theta=45°+180°=225°$. Thus, a second possibility is $(-2,225°)$.

18. (a)

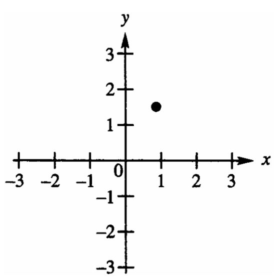


(b) $r = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2$

and $\theta = \tan^{-1}\left(\dfrac{\sqrt{2}}{-\sqrt{2}}\right) = \tan^{-1}(-1)$, since

θ is in quadrant II we have $\theta = 135°$.
So, one possibility is $(2, 135°)$.
Alternatively, if $r = -2$, then
$\theta = 135° + 180° = 315°$. Thus, a second possibility is $(-2, 315°)$.

19. (a)

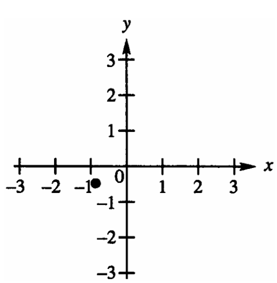


(b) $r = \sqrt{\left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(\dfrac{3}{2}\right)^2} = \sqrt{\dfrac{3}{4} + \dfrac{9}{4}}$

$= \sqrt{\dfrac{12}{4}} = \sqrt{3}$ and $\theta = \arctan\left(\dfrac{3}{2} \cdot \dfrac{2}{\sqrt{3}}\right)$

$= \tan^{-1}\left(\sqrt{3}\right) = 60°$, since θ is in quadrant I. So, one possibility is $\left(\sqrt{3}, 60°\right)$. Alternatively, if $r = -\sqrt{3}$, then $\theta = 60° + 180° = 240°$. Thus, a second possibility is $\left(-\sqrt{3}, 240°\right)$.

20. (a)



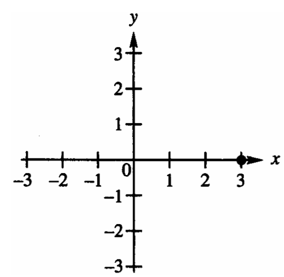
(b) $r = \sqrt{\left(-\dfrac{\sqrt{3}}{2}\right)^2 + \left(-\dfrac{1}{2}\right)^2} = \sqrt{\dfrac{3}{4} + \dfrac{1}{4}}$

$= \sqrt{\dfrac{4}{4}} = 1$ and $\theta = \arctan\left(\dfrac{-1}{2} \cdot \left(\dfrac{-2}{\sqrt{3}}\right)\right)$

$= \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \tan^{-1}\left(\dfrac{\sqrt{3}}{3}\right) = 210°$ since

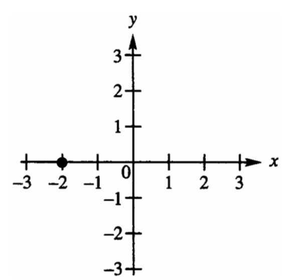
θ is in quadrant III. So, one possibility is $(1, 210°)$. Alternatively, if $r = -1$, then $\theta = 210° - 180° = 30°$. Thus, a second possibility is $(-1, 30°)$.

21. (a)



(b) $r = \sqrt{3^2 + 0^2} = \sqrt{9+0} = \sqrt{9} = 3$ and $\theta = 0°$, since $(3, 0)$ is on the positive x-axis. So, one possibility is $(3, 0°)$.
Alternatively, if $r = -3$, then
$\theta = 0° + 180° = 180°$. Thus, a second possibility is $(-3, 180°)$.

22. (a)

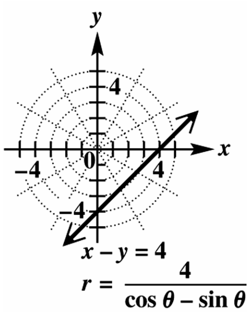


(b) $r = \sqrt{(-2)^2 + 0^2} = \sqrt{4+0} = \sqrt{4} = 2$ and $\theta = 180°$, since $(-2, 0)$ is on the negative x-axis. So, one possibility is $(2, 180°)$.
Alternatively, if $r = -2$, then
$\theta = 180° - 180° = 0°$. Thus, a second possibility is $(-2, 0°)$.

23. $x - y = 4$

Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 1, b = -1$, and $c = 4$, the polar equation is

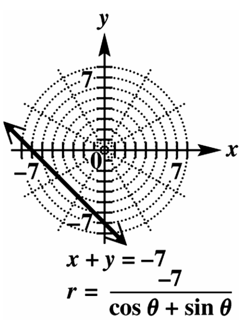
$r = \dfrac{4}{\cos\theta - \sin\theta}$.



24. $x + y = -7$

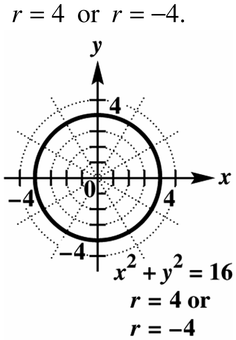
Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 1, b = 1$, and $c = -7$, the polar equation is

$r = \dfrac{-7}{\cos\theta + \sin\theta}$.



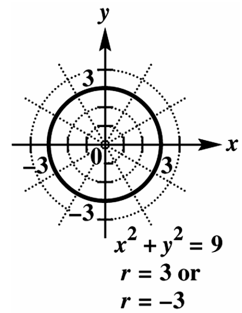
25. $x^2 + y^2 = 16 \Rightarrow r^2 = 16 \Rightarrow r = \pm 4$

The equation of the circle in polar form is $r = 4$ or $r = -4$.



26. $x^2 + y^2 = 9 \Rightarrow r^2 = 9 \Rightarrow r = \pm 3$

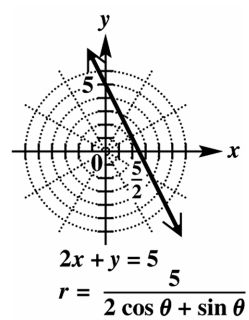
The equation of the circle in polar form is $r = 3$ or $r = -3$.



27. $2x + y = 5$

Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 2, b = 1$, and $c = 5$, the polar equation is

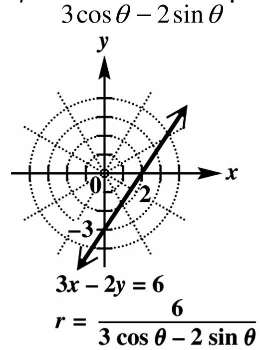
$r = \dfrac{5}{2\cos\theta + \sin\theta}$.



28. $3x - 2y = 6$

Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 3, b = -2$, and $c = 6$, the polar equation is

$r = \dfrac{6}{3\cos\theta - 2\sin\theta}$.

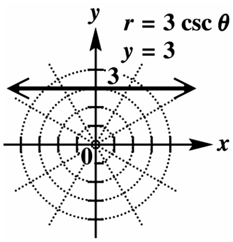


29. $r\sin\theta = k$

30. $r = \dfrac{k}{\sin\theta}$

31. $r = \dfrac{k}{\sin\theta} \Rightarrow r = k\csc\theta$

32. $y = 3$

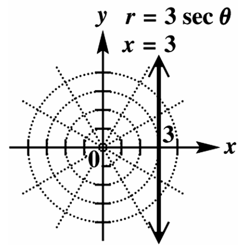


33. $r\cos\theta = k$

34. $r = \dfrac{k}{\cos\theta}$

35. $r = \dfrac{k}{\cos\theta} \Rightarrow r = k\sec\theta$

36. $x = 3$



37. $r = 3$ represents the set of all points 3 units from the pole. The correct choice is C.

38. $r = \cos 3\theta$ is a rose curve with 3 petals. The correct choice is D.

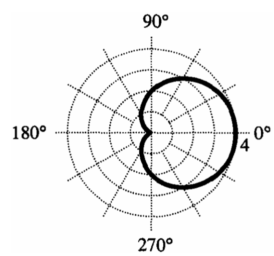
39. $r = \cos 2\theta$ is a rose curve with $2 \cdot 2 = 4$ petals. The correct choice is A

40. The general form for the polar equation of a line is $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, where the standard form of a line $ax + by = c$. $r = \dfrac{2}{\cos\theta + \sin\theta}$ is a line. The correct choice is B.

41. $r = 2 + 2\cos\theta$ (cardioid)

θ	0°	30°	60°	90°	120°	150°
$\cos\ \theta$	1	.9	.5	0	–.5	–.9
$r = 2 + 2\cos\theta$	4	3.8	3	2	1	.2

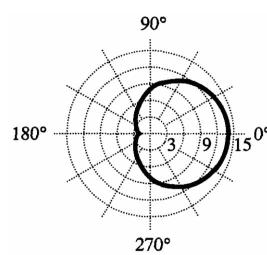
θ	180°	210°	240°	270°	300°	330°
$\cos\ \theta$	–1	–.9	–.5	0	.5	.9
$r = 2 + 2\cos\theta$	0	.3	1	2	3	3.7



42. $r = 8 + 6\cos\theta$ (limaçon)

θ	0°	30°	60°	90°	120°	150°
$\cos\ \theta$	1	.9	.5	0	–.5	–.9
$r = 8 + 6\cos\theta$	14	13.2	11	8	5	2.8

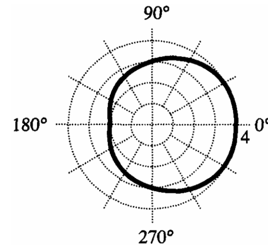
θ	180°	210°	240°	270°	300°	330°
$\cos\ \theta$	–1	–.9	–.5	0	.5	.9
$r = 8 + 6\cos\theta$	2	2.8	5	8	11	13.2



43. $r = 3 + \cos\theta$ (limaçon)

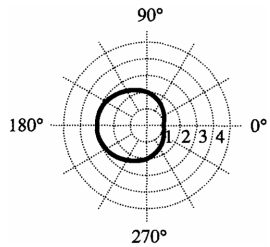
θ	0°	30°	60°	90°	120°	150°
$r = 3 + \cos\theta$	4	3.9	3.5	3	2.5	2.1

θ	180°	210°	240°	270°	300°	330°
$r = 3 + \cos\theta$	2	2.1	2.5	3	3.5	3.9



44. $r = 2 - \cos\theta$ (limaçon)

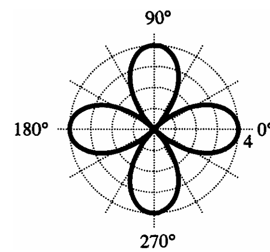
θ	0°	30°	60°	90°	135°	180°	225°	270°	315°
$r = 2 - \cos\theta$	1	1.1	1.5	2	2.7	3	2.7	2	1.3



45. $r = 4\cos 2\theta$ (four-leaved rose)

θ	0°	30°	45°	60°	90°	120°	135°	150°
$r = 4\cos 2\theta$	4	2	0	–2	–4	–2	0	2

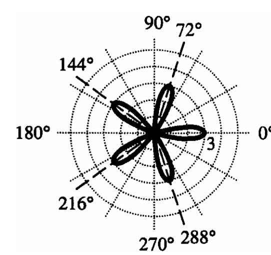
θ	180°	210°	225°	240°	270°	300°	315°	330°
$r = 4\cos 2\theta$	4	2	0	–2	–4	–2	0	2



46. $r = 3\cos 5\theta$ (five-leaved rose)

$r = 0$ when $\cos 5\theta = 0$, or $5\theta = 90° + 360° \cdot k = 18° + 72° \cdot k$, where k is an integer, or $\theta = 18°, 90°, 162°, 234°$.

θ	0°	18°	36°	54°	72°	90°	108°	162°
$r = 3\cos 5\theta$	3	0	–3	0	3	0	–3	0



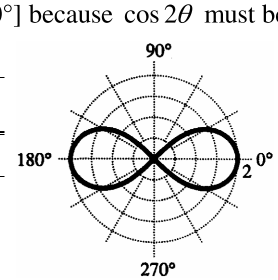
Pattern 3, 0, –3, 0, 3 continues for every 18°.

47. $r^2 = 4\cos 2\theta \Rightarrow r = \pm 2\sqrt{\cos 2\theta}$ (lemniscate)

Graph only exists for [0°, 45°], [135°, 225°], and [315°, 360°] because $\cos 2\theta$ must be positive.

θ	0°	30°	45°	135°	150°
$r = \pm 2\sqrt{\cos 2\theta}$	±2	±1.4	0	0	±1.4

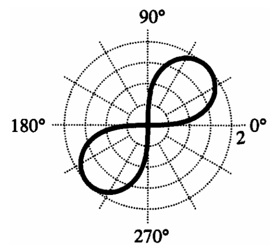
θ	180°	210°	225°	315°	330°
$r = \pm 2\sqrt{\cos 2\theta}$	±2	±1.4	0	0	±1.4



48. $r^2 = 4\sin 2\theta \Rightarrow r = \pm 2\sqrt{\sin 2\theta}$ (lemniscate)

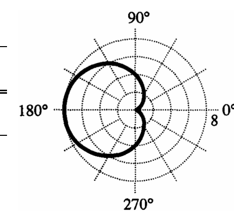
Graph only exists for [0°, 90°] and [180°, 270°] because sin θ must be positive.

θ	0°	30°	45°	60°	90°	180°	225°	270°
$r = \pm 2\sqrt{\sin 2\theta}$	0	±1.86	±2	±1.86	0	0	±2	0



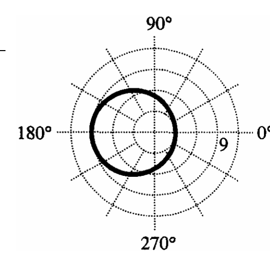
49. $r = 4 - 4\cos\theta$ (cardioid)

θ	0°	30°	60°	90°	120°	150°
$r = 4 - 4\cos\theta$	0	.5	2	4	6	7.5
θ	180°	210°	240°	270°	300°	330°
$r = 4 - 4\cos\theta$	8	7.5	6	4	2	.5



50. $r = 6 - 3\cos\theta$ (limaçon)

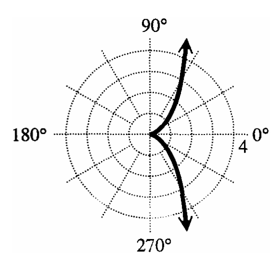
θ	0°	45°	90°	135°	180°	270°	360°
$r = 6 - 3\cos\theta$	3	3.9	6	8.1	9	6	3



51. $r = 2\sin\theta\tan\theta$ (cissoid)

r is undefined at $\theta = 90°$ and $\theta = 270°$.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
$r = 2\sin\theta\tan\theta$	0	.6	1.4	3	–	–3	–1.4	–.6	0

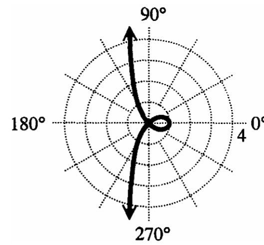


Notice that for [180°, 360°), the graph retraces the path traced for [0°, 180°).

52. $r = \dfrac{\cos 2\theta}{\cos\theta}$ (cissoid with a loop)

r is undefined at $\theta = 90°$ and $\theta = 270°$ and $r = 0$ at 45°, 135°, 225°, and 315°.

θ	0°	45°	60°	70°	80°
$r = \dfrac{\cos 2\theta}{\cos\theta}$	1	0	–1	–2.2	–5.4
θ	90°	100°	110°	135°	180°
$r = \dfrac{\cos 2\theta}{\cos\theta}$	–	5.4	2.2	0	–1



Notice that for [180°, 360°), the graph retraces the path traced for [0°, 180°).

53. $r = 2\sin\theta$

Multiply both sides by r to obtain

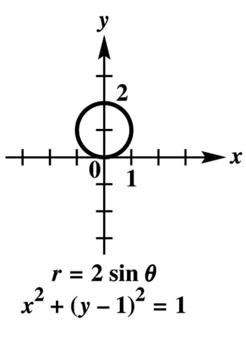
$r^2 = 2r\sin\theta$. Since

$r^2 = x^2 + y^2$ and $y = r\sin\theta$, $x^2 + y^2 = 2y$.

Complete the square on y to obtain

$x^2 + y^2 - 2y + 1 = 1 \Rightarrow x^2 + (y-1)^2 = 1.$

The graph is a circle with center at $(0,1)$ and radius 1.



54. $r = 2\cos\theta$

Multiply both sides by r to obtain

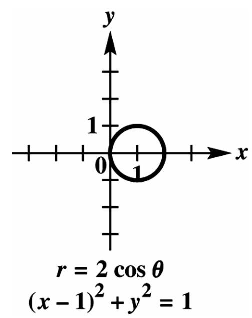
$r^2 = 2r\cos\theta.$

Since $r^2 = x^2 + y^2$ and $x = r\cos\theta$,

$x^2 + y^2 = 2x$. Complete the square on x to get the equation of a circle to obtain

$x^2 - 2x + y^2 = 0 \Rightarrow (x-1)^2 + y^2 = 1.$

The graph is a circle with center at $(1,0)$ and radius 1.



55. $r = \dfrac{2}{1-\cos\theta}$

Multiply both sides by $1 - \cos\theta$ to obtain

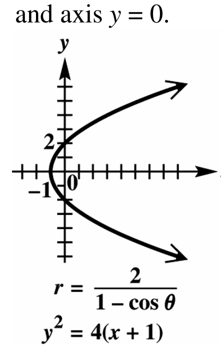
$r - r\cos\theta = 2$. Substitute $r = \sqrt{x^2 + y^2}$ to obtain

$\sqrt{x^2 + y^2} - x = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 + x \Rightarrow$

$x^2 + y^2 = (2+x)^2 \Rightarrow x^2 + y^2 = 4 + 4x + x^2 \Rightarrow$

$y^2 = 4(1+x)$

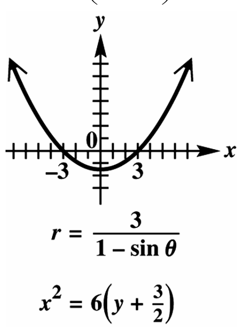
The graph is a parabola with vertex at $(-1,0)$ and axis $y = 0$.



56. $r = \dfrac{3}{1 - \sin\theta}$

$r = \dfrac{3}{1 - \sin\theta} \Rightarrow r - r\sin\theta = 3 \Rightarrow$
$r = r\sin\theta + 3 \Rightarrow \sqrt{x^2 + y^2} = y + 3$
$x^2 + y^2 = y^2 + 6y + 9 \Rightarrow x^2 = 6y + 9 \Rightarrow$
$x^2 = 6\left(y + \dfrac{3}{2}\right)$

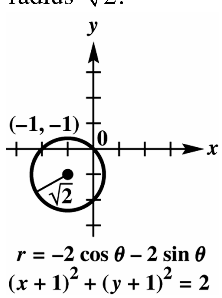
The graph is a parabola with axis $x = 0$ and vertex $\left(0, -\dfrac{3}{2}\right)$.



57. $r + 2\cos\theta = -2\sin\theta$

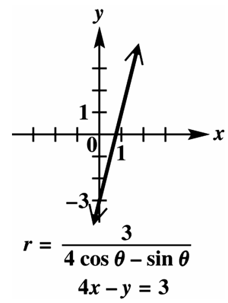
$r + 2\cos\theta = -2\sin\theta \Rightarrow$
$r^2 = -2r\sin\theta - 2r\cos\theta \Rightarrow x^2 + y^2 = -2y - 2x$
$x^2 + 2x + y^2 + 2y = 0 \Rightarrow$
$x^2 + 2x + 1 + y^2 + 2y + 1 = 2 \Rightarrow$
$(x+1)^2 + (y+1)^2 = 2$

The graph is a circle with center $(-1, -1)$ and radius $\sqrt{2}$.



58. $r = \dfrac{3}{4\cos\theta - \sin\theta}$

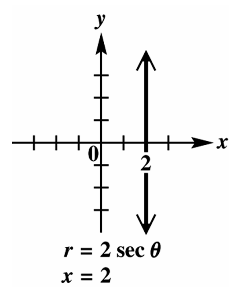
Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 4,\ b = -1,$ and $c = 3,$ we have $4x - y = 3.$ The graph is a line with intercepts $(0, -3)$ and $\left(\frac{3}{4}, 0\right)$.



59. $r = 2\sec\theta$

$r = 2\sec\theta \Rightarrow r = \dfrac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$

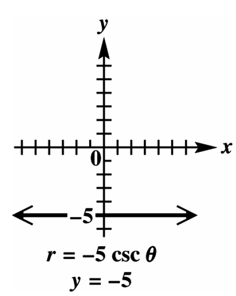
The graph is a vertical line, intercepting the x-axis at 2.



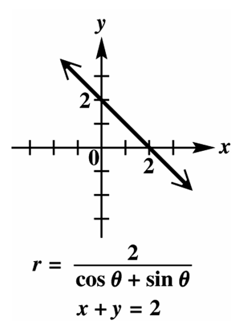
60. $r = -5\csc\theta$

$r = -5\csc\theta \Rightarrow r = -\dfrac{5}{\sin\theta} \Rightarrow$
$r\sin\theta = -5 \Rightarrow y = -5$

The graph is a horizontal line, intercepting the y-axis at -5.



Exercise 60



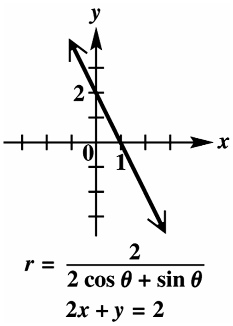
Exercise 61

61. $r = \dfrac{2}{\cos\theta + \sin\theta}$

Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 1,\ b = 1,$ and $c = 2,$ we have $x + y = 2.$ The graph is a line with intercepts $(0, 2)$ and $(2, 0)$.

62. $r = \dfrac{2}{2\cos\theta + \sin\theta}$

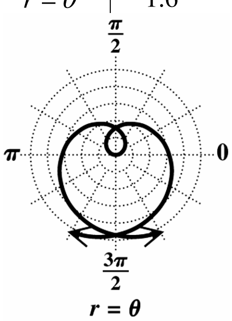
Using the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 2,\ b = 1,$ and $c = 2,$ we have $2x + y = 2.$ The graph is a line with intercepts $(0, 2)$ and $(1, 0)$.



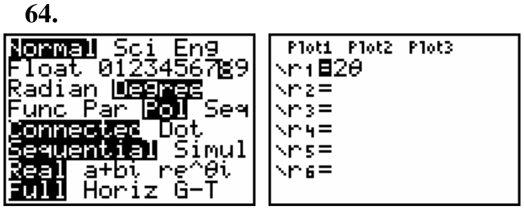
63. Graph $r = \theta$, a spiral of Archimedes.

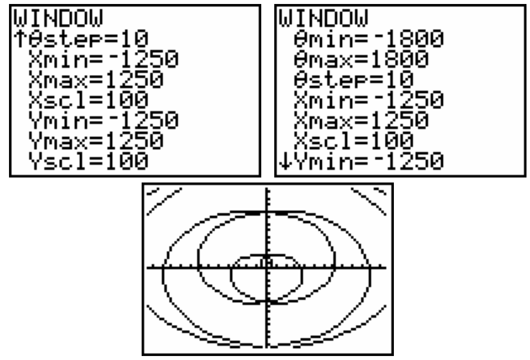
θ	–360°	–270°	–180°	–90°	0°
θ (radians)	–6.3	–4.7	–3.1	–1.6	0
$r = \theta$	–6.3	–4.7	–3.1	–1.6	0

θ	90°	180°	270°	360°
θ (radians)	1.6	3.1	4.7	6.3
$r = \theta$	1.6	3.1	4.7	6.3



64.





65. In rectangular coordinates, the line passes through $(1, 0)$ and $(0, 2)$. So

$m = \dfrac{2-0}{0-1} = \dfrac{2}{-1} = -2$ and

$(y - 0) = -2(x - 1) \Rightarrow y = -2x + 2 \Rightarrow$

$2x + y = 2.$ Converting to polar form

$r = \dfrac{c}{a\cos\theta + b\sin\theta}$, we have:

$r = \dfrac{2}{2\cos\theta + \sin\theta}$.

66. Answers will vary.

67. **(a)** $(r, -\theta)$

(b) $(r, \pi - \theta)$ or $(-r, -\theta)$

(c) $(r, \pi + \theta)$ or $(-r, \theta)$

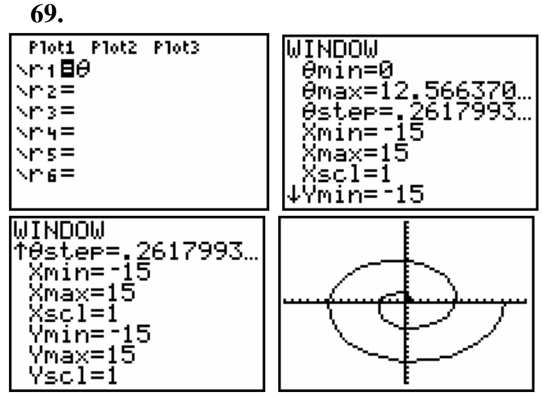
68. **(a)** $-\theta$ **(b)** $\pi - \theta$

(c) $-r; -\theta$ **(d)** $-r$

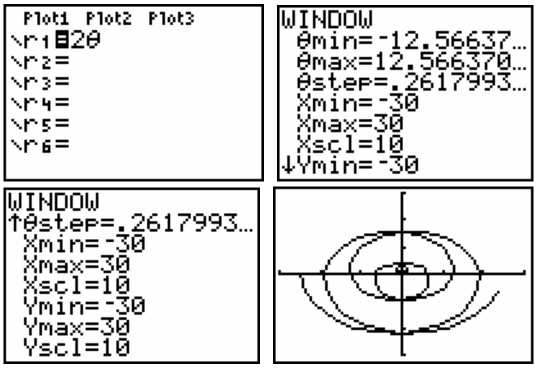
(e) $\pi + \theta$ **(f)** the polar axis

(g) the line $\theta = \dfrac{\pi}{2}$

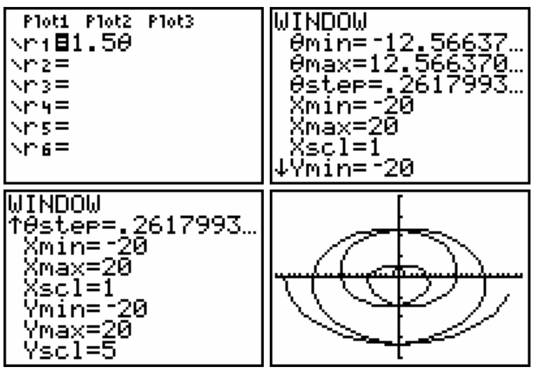
69.



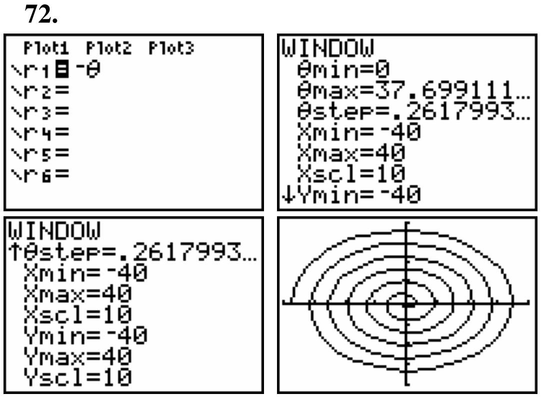
70.



71.



72.



73. $r = 4\sin\theta$, $r = 1 + 2\sin\theta$, $0 \le \theta < 2\pi$

$4\sin\theta = 1 + 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow$

$\sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

The points of intersection are

$\left(4\sin\dfrac{\pi}{6}, \dfrac{\pi}{6}\right) = \left(2, \dfrac{\pi}{6}\right)$ and

$\left(4\sin\dfrac{5\pi}{6}, \dfrac{5\pi}{6}\right) = \left(2, \dfrac{5\pi}{6}\right)$.

74. $r = 3$, $r = 2 + 2\cos\theta$; $0° \le \theta < 360°$

$3 = 2 + 2\cos\theta \Rightarrow 1 = 2\cos\theta \Rightarrow$

$\cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60°$ or $300°$

The points of intersection are

$(3, 60°), (3, 300°)$

75. $r = 2 + \sin\theta$, $r = 2 + \cos\theta$, $0 \le \theta < 2\pi$

$2 + \sin\theta = 2 + \cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow$

$\theta = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

$r = 2 + \sin\dfrac{\pi}{4} = 2 + \dfrac{\sqrt{2}}{2} = \dfrac{4+\sqrt{2}}{2}$ and

$r = 2 + \sin\dfrac{5\pi}{4} = 2 - \dfrac{\sqrt{2}}{2} = \dfrac{4-\sqrt{2}}{2}$

The points of intersection are

$\left(\dfrac{4+\sqrt{2}}{2}, \dfrac{\pi}{4}\right)$ and $\left(\dfrac{4-\sqrt{2}}{2}, \dfrac{5\pi}{4}\right)$.

76. $r = \sin 2\theta$, $r = \sqrt{2}\cos\theta$, $0 \le \theta < \pi$

$\sin 2\theta = \sqrt{2}\cos\theta \Rightarrow 2\sin\theta\cos\theta = \sqrt{2}\cos\theta \Rightarrow$

$2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0 \Rightarrow$

$\cos\theta\left(2\sin\theta - \sqrt{2}\right) = 0$

$\cos\theta = 0$ or $2\sin\theta - \sqrt{2} = 0 \Rightarrow$

$2\sin\theta = \sqrt{2} \Rightarrow \sin\theta = \dfrac{\sqrt{2}}{2}$

Thus, $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$. The points of intersection are $\left(\sin 2\cdot\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) = \left(0, \dfrac{\pi}{2}\right)$,

$\left(\sin 2\cdot\dfrac{\pi}{4}, \dfrac{\pi}{4}\right) = \left(1, \dfrac{\pi}{4}\right)$, and

$\left(\sin 2\cdot\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right) = \left(-1, \dfrac{3\pi}{4}\right)$.

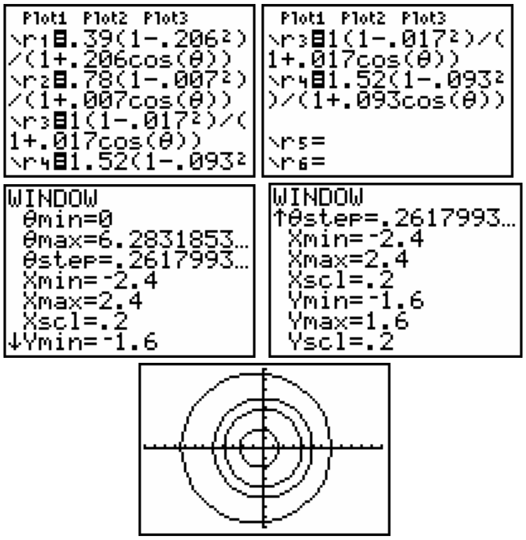
77. (a) Plot the following polar equations on the same polar axis in radian mode:

Mercury: $r = \dfrac{.39(1-.206^2)}{1+.206\cos\theta}$;

Venus: $r = \dfrac{.78(1-.007^2)}{1+.007\cos\theta}$;

Earth: $r = \dfrac{1(1-.017^2)}{1+.017\cos\theta}$;

Mars: $r = \dfrac{1.52(1-.093^2)}{1+.093\cos\theta}$.



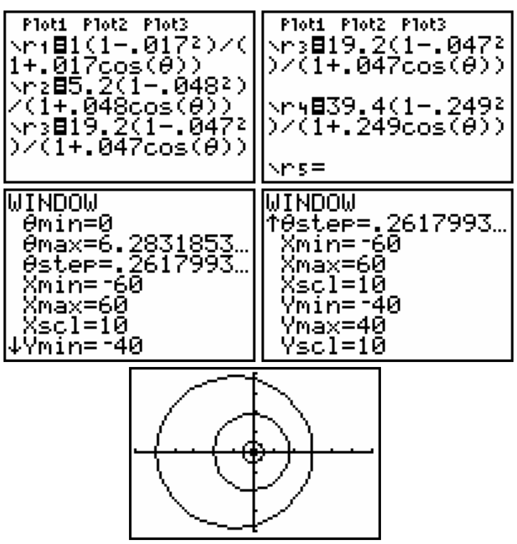
(b) Plot the following polar equations on the same polar axis:

Earth: $r = \dfrac{1(1-.017^2)}{1+.017\cos\theta}$;

Jupiter: $r = \dfrac{5.2(1-.048^2)}{1+.048\cos\theta}$;

Uranus: $r = \dfrac{19.2(1-.047^2)}{1+.047\cos\theta}$;

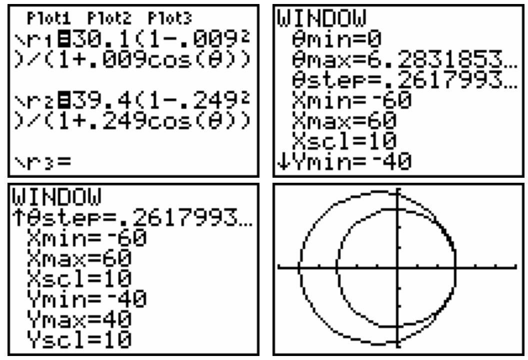
Pluto: $r = \dfrac{39.4(1-.249^2)}{1+.249\cos\theta}$.



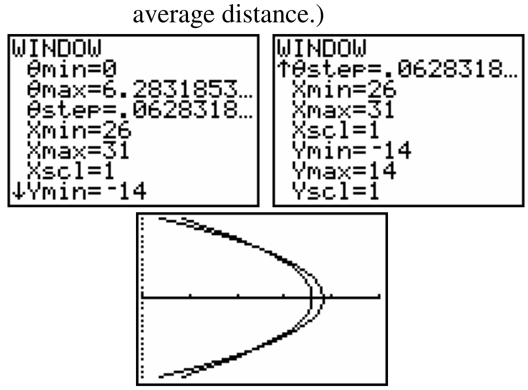
(c) We must determine if the orbit of Pluto is always outside the orbits of the other planets. Since Neptune is closest to Pluto, plot the orbits of Neptune and Pluto on the same polar axes.

Neptune: $r = \dfrac{30.1(1-.009^2)}{1+.009\cos\theta}$;

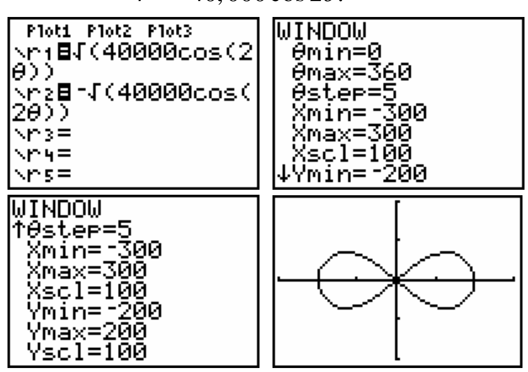
Pluto: $r = \dfrac{39.4(1-.249^2)}{1+.249\cos\theta}$



The graph shows that their orbits are very close near the polar axis. Use ZOOM or change your window to see that the orbit of Pluto does indeed pass inside the orbit of Neptune. Therefore, there are times when Neptune, not Pluto, is the farthest planet from the sun. (However, Pluto's average distance from the sun is considerably greater than Neptune's average distance.)



78. (a) In degree mode, graph $r^2 = 40{,}000\cos 2\theta$.

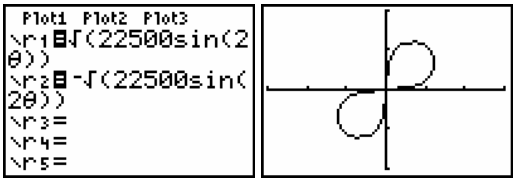


(*continued on next page*)

(*continued from page 817*)

Inside the “figure eight” the radio signal can be received. This region is generally in an east-west direction from the two radio towers with a maximum distance of 200 mi.

(b) In degree mode, graph $r^2 = 22{,}500\sin 2\theta$.



Inside the “figure eight” the radio signal can be received. This region is generally in a northeast-southwest direction from the two radio towers with a maximum distance of 150 mi.

Section 8.8: Parametric Equations, Graphs, and Applications

1. At $t = 2$, $x = 3(2) + 6 = 12$ and $y = -2(2) + 4 = 0$. The correct choice is C.

2. At $t = \frac{\pi}{4}$, $x = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

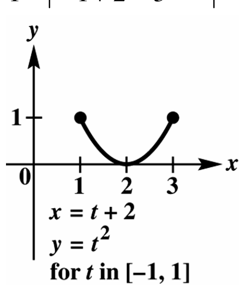
$y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. The correct choice is D.

3. At $t = 5$, $x = 5$ and $y = 5^2 = 25$. The correct choice is A.

4. At $t = 2$, $x = 2^2 + 3 = 7$ and $y = 2^2 - 2 = 2$. The correct choice is B.

5. **(a)** $x = t + 2$, $y = t^2$, for t in [–1, 1]

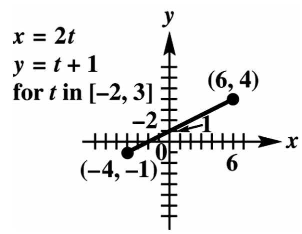
t	$x = t + 2$	$y = t^2$
-1	$-1 + 2 = 1$	$(-1)^2 = 1$
0	$0 + 2 = 2$	$0^2 = 0$
1	$1 + 2 = 3$	$1^2 = 1$



(b) $x - 2 = t$, therefore $y = (x - 2)^2$ or $y = x^2 - 4x + 4$. Since t is in [–1, 1], x is in [–1 + 2, 1 + 2] or [1, 3].

6. **(a)** $x = 2t$, $y = t + 1$, for t in [–2, 3]

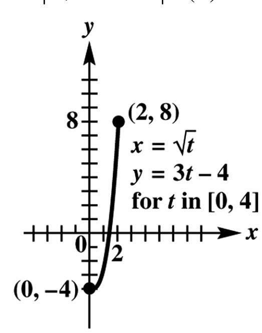
t	$x = 2t$	$y = t + 1$
-2	$2(-2) = -4$	$-2 + 1 = -1$
-1	$2(-1) = -2$	$-1 + 1 = 0$
0	$2(0) = 0$	$0 + 1 = 1$
1	$2(1) = 2$	$1 + 1 = 2$
2	$2(2) = 4$	$2 + 1 = 3$
3	$2(3) = 6$	$3 + 1 = 4$



(b) Since $x = 2t \Rightarrow \frac{x}{2} = t$, we have $y = \frac{x}{2} + 1$. Since t is in [–2, 3], x is in [2(–2), 2(3)] or [–4, 6].

7. **(a)** $x = \sqrt{t}$, $y = 3t - 4$, for t in [0, 4].

t	$x = \sqrt{t}$	$y = 3t - 4$
0	$\sqrt{0} = 0$	$3(0) - 4 = -4$
1	$\sqrt{1} = 1$	$3(1) - 4 = -1$
2	$\sqrt{2} = 1.4$	$3(2) - 4 = 2$
3	$\sqrt{3} = 1.7$	$3(3) - 4 = 5$
4	$\sqrt{4} = 2$	$3(4) - 4 = 8$

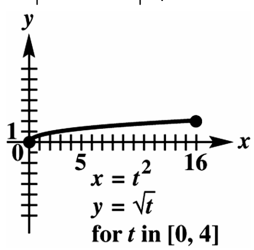


(b) $x=\sqrt{t}$, $y=3t-4$

Since $x=\sqrt{t} \Rightarrow x^2=t$, we have $y=3x^2-4$. Since t is in [0, 4], x is in $[\sqrt{0}, \sqrt{4}]$ or [0, 2].

8. (a) $x=t^2$, $y=\sqrt{t}$, for t in [0, 4]

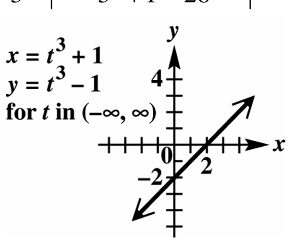
t	$x=t^2$	$y=\sqrt{t}$
0	$0^2=0$	$\sqrt{0}=0$
1	$1^2=1$	$\sqrt{1}=1$
2	$2^2=4$	$\sqrt{2}=1.414$
3	$3^2=9$	$\sqrt{3}=1.732$
4	$4^2=16$	$\sqrt{4}=2$



(b) Since $y=\sqrt{t} \Rightarrow y^2=t$, we have $x=t^2=\left(y^2\right)^2=y^4$ or $y=\sqrt[4]{x}$. Since t is in [0, 4], x is in $[0^2, 4^2]$, or [0, 16].

9. (a) $x=t^3+1$, $y=t^3-1$, for t in $(-\infty, \infty)$

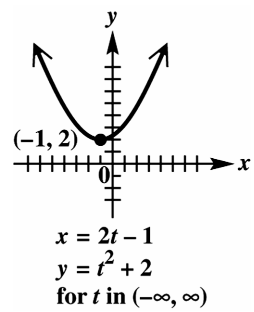
t	$x=t^3+1$	$y=t^3-1$
-2	$(-2)^3+1=-7$	$(-2)^3-1=-9$
-1	$(-1)^3+1=0$	$(-1)^3-1=-2$
0	$0^3+1=1$	$0^3-1=-1$
1	$1^3+1=2$	$1^3-1=0$
2	$2^3+1=9$	$2^3-1=7$
3	$3^3+1=28$	$3^3-1=26$



(b) Since $x=t^3+1$, we have $x-1=t^3$. Since $y=t^3-1$, we have $y=(x-1)-1=x-2$. Since t is in $(-\infty, \infty)$, x is in $(-\infty, \infty)$.

10. (a) $x=2t-1$, $y=t^2+2$, for t in $(-\infty, \infty)$

t	$x=2t-1$	$y=t^2+2$
-3	$2(-3)-1=-7$	$(-3)^2+2=11$
-2	$2(-2)-1=-5$	$(-2)^2+2=6$
-1	$2(-1)-1=-3$	$(-1)^2+2=3$
0	$2(0)-1=-1$	$0^2+2=2$
1	$2(1)-1=1$	$1^2+2=3$
2	$2(2)-1=3$	$2^2+2=6$
3	$2(3)-1=5$	$3^2+2=11$



(b) Since $x=2t-1 \Rightarrow x+1=2t$, we have $\dfrac{x+1}{2}=t$.

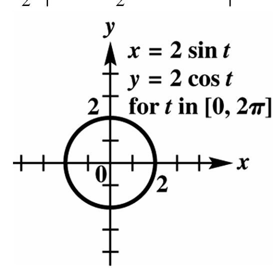
Since $y=t^2+2$, we have

$$y=\left(\frac{x+1}{2}\right)^2+2=\frac{1}{4}(x+1)^2+2.$$

Since t is in $(-\infty, \infty)$ and $x=2t-1$, x is in $(-\infty, \infty)$.

11. (a) $x=2\sin t$, $y=2\cos t$, for t in $[0, 2\pi]$

t	$x=2\sin t$	$y=2\cos t$
0	$2\sin 0=0$	$2\cos\theta=2$
$\frac{\pi}{6}$	$2\sin\frac{\pi}{6}=1$	$2\cos\frac{\pi}{6}=\sqrt{3}$
$\frac{\pi}{4}$	$2\sin\frac{\pi}{4}=\sqrt{2}$	$2\cos\frac{\pi}{4}=\sqrt{2}$
$\frac{\pi}{3}$	$2\sin\frac{\pi}{3}=\sqrt{3}$	$2\cos\frac{\pi}{3}=1$
$\frac{\pi}{2}$	$2\sin\frac{\pi}{2}=2$	$2\cos\frac{\pi}{2}=0$



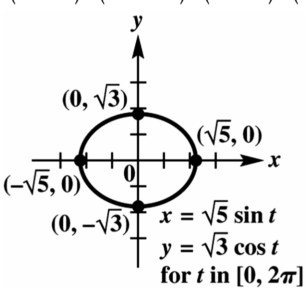
(b) Since $x = 2\sin t$ and $y = 2\cos t$, we have $\frac{x}{2} = \sin t$ and $\frac{y}{2} = \cos t$. Since $\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{4} = 1 \Rightarrow$$

$x^2 + y^2 = 4$. Since t is in $[0, 2\pi]$, x is in $[-2, 2]$ because the graph is a circle, centered at the origin, with radius 2.

12. **(a)** $x = \sqrt{5}\sin t,\ y = \sqrt{3}\cos t$, for t in $[0, 2\pi]$

Rewriting our parametric equations, we have $\frac{x}{\sqrt{5}} = \sin t$ and $\frac{y}{\sqrt{3}} = \cos t$. Since $-1 \le \sin t \le 1$, we have $-\sqrt{5} \le \sqrt{5}\sin t \le \sqrt{5}$. Therefore, x is in $[-\sqrt{5}, \sqrt{5}]$. The graph is an ellipse, centered at the origin, with vertices $\left(\sqrt{5}, 0\right), \left(-\sqrt{5}, 0\right), \left(0, \sqrt{3}\right), \left(0, -\sqrt{3}\right)$.

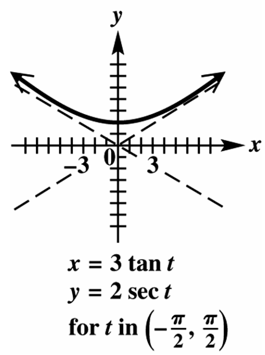


(b) Since $\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{x}{\sqrt{5}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1 \Rightarrow \frac{x^2}{5} + \frac{y^2}{3} = 1.$$

13. **(a)** $x = 3\tan t,\ y = 2\sec t$, for t in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

t	$x = 3\tan t$	$y = 2\sec t$
$-\frac{\pi}{3}$	$3\tan\left(-\frac{\pi}{3}\right) = -3\sqrt{3}$	$2\sec\left(-\frac{\pi}{3}\right) = 4$
$-\frac{\pi}{6}$	$3\tan\left(-\frac{\pi}{6}\right) = -\sqrt{3}$	$2\sec\left(-\frac{\pi}{6}\right) = \frac{4\sqrt{3}}{3}$
0	$3\tan 0 = 0$	$2\sec 0 = 2$
$\frac{\pi}{6}$	$3\tan\frac{\pi}{6} = \sqrt{3}$	$2\sec\frac{\pi}{6} = \frac{4\sqrt{3}}{3}$
$\frac{\pi}{3}$	$3\tan\frac{\pi}{3} = 3\sqrt{3}$	$2\sec\frac{\pi}{3} = 4$



(b) Since $\frac{x}{3} = \tan t$, $\frac{y}{2} = \sec t$, and $1 + \tan^2 t = \left(\frac{y}{2}\right)^2 = \sec^2 t$, we have

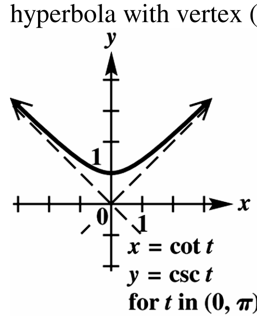
$$1 + \left(\frac{x}{3}\right)^2 = \left(\frac{y}{2}\right)^2 \Rightarrow 1 + \frac{x^2}{9} = \frac{y^2}{4} \Rightarrow$$

$$y^2 = 4\left(1 + \frac{x^2}{9}\right) \Rightarrow y = 2\sqrt{1 + \frac{x^2}{9}}$$

Since this graph is the top half of a hyperbola, x is in $(-\infty, \infty)$.

14. **(a)** $x = \cot t,\ y = \csc t$, for t in $(0, \pi)$

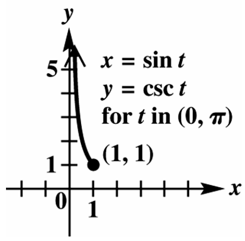
Since t is in $(0, \pi)$ and the value of the cotangent of a value close to 0 is very large and the value of the cotangent of a value close to π is very small, x is in $(-\infty, \infty)$. The graph is the top half of a hyperbola with vertex $(0, 1)$.



(b) Since $1 + \cot^2 t = \csc^2 t$, we have $1 + x^2 = y^2 \Rightarrow y = \sqrt{1 + x^2}$.

15. (a) $x = \sin t,\ y = \csc t$ for t in $(0, \pi)$

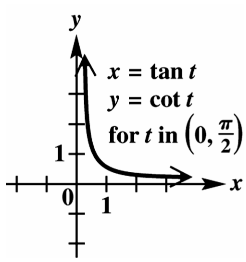
Since t is in $(0, \pi)$ and $x = \sin t$, x is in $(0, 1]$.



(b) Since $x = \sin t$ and $y = \csc t = \dfrac{1}{\sin t}$, we have $y = \dfrac{1}{x}$, where x is in $(0, 1]$.

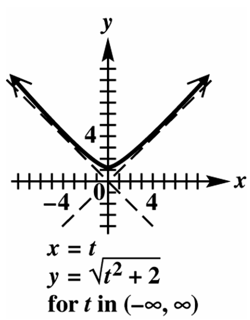
16. (a) $x = \tan t,\ y = \cot t,$ for t in $\left(0, \dfrac{\pi}{2}\right)$

Since t is in $\left(0, \dfrac{\pi}{2}\right)$, x is in $(0, \infty)$.



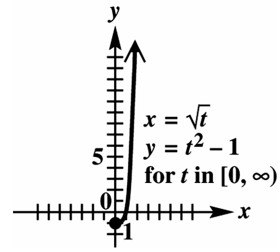
(b) Since $\cot t = \dfrac{1}{\tan t}$, $y = \dfrac{1}{x}$, where x is in $(0, \infty)$.

17. (a) $x = t,\ y = \sqrt{t^2 + 2},$ for t in $(-\infty, \infty)$



(b) Since $x = t$ and $y = \sqrt{t^2 + 2}$, $y = \sqrt{x^2 + 2}$. Since t is in $(-\infty\ \infty)$ and $x = t$, x is in $(-\infty, \infty)$.

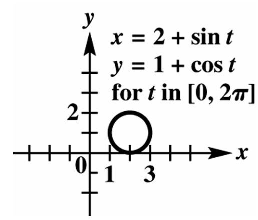
18. (a) $x = \sqrt{t},\ y = t^2 - 1,$ for t in $[0, \infty)$



(b) Since $x = \sqrt{t}$, we have $x^2 = t$. Therefore, $y = t^2 - 1 = (x^2)^2 - 1 = x^4 - 1$. Since t is in $[0, \infty)$, x is in $\left[\sqrt{0}, \sqrt{\infty}\right)$ or $[0, \infty)$.

19. (a) $x = 2 + \sin t,\ y = 1 + \cos t,$ for t in $[0, 2\pi]$

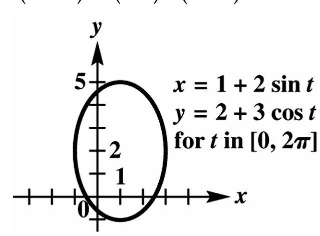
Since this is a circle centered at $(2,1)$ with radius 1, and t is in $[0, 2\pi]$, x is in $[1, 3]$.



(b) Since $x = 2 + \sin t$ and $y = 1 + \cos t$, $x - 2 = \sin t$ and $y - 1 = \cos t$. Since $\sin^2 t + \cos^2 t = 1$, we have $(x-2)^2 + (y-1)^2 = 1$.

20. (a) $x = 1 + 2\sin t,\ y = 2 + 3\cos t,$ for t in $[0, 2\pi]$

Since $x - 1 = 2\sin t \Rightarrow \dfrac{x-1}{2} = \sin t$ and $y - 2 = 3\cos t \Rightarrow \dfrac{y-2}{3} = \cos t$, x is in $[-1, 3]$. The graph is an ellipse with center $(1, 2)$ and axes endpoints $(3, 2)$, $(-1, 2)$, $(1, 5)$, $(1, -1)$.

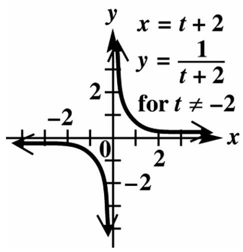


(b) Since $\sin^2 t + \cos^2 t = 1,$

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1 \Rightarrow$$
$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1.$$

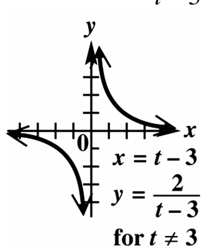
Also, $-1 \le \sin t \le 1, \Rightarrow -2 \le 2\sin t \le 2 \Rightarrow$
$-1 \le 1 + 2\sin t \le 3.$

21. (a) $x = t + 2,\ y = \dfrac{1}{t+2}$, for $t \ne 2$



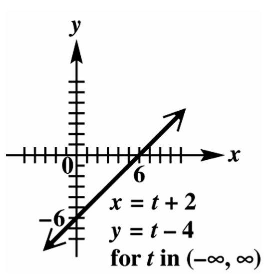
(b) Since $x = t + 2$ and $y = \dfrac{1}{t+2}$, we have $y = \dfrac{1}{x}$. Since $t \ne -2$, $x \ne -2 + 2$, $x \ne 0$. Therefore, x is in $(-\infty, 0) \cup (0, \infty)$.

22. (a) $x = t - 3,\ y = \dfrac{2}{t-3}$, for $t \ne 3$



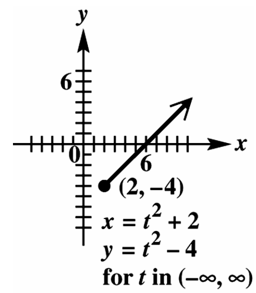
(b) Since $y = \dfrac{2}{t-3}$, we have $y = \dfrac{2}{x}$. Since $t \ne 3$, $x \ne 3 - 3 = 0$. Therefore, x is in $(-\infty, 0) \cup (0, \infty)$.

23. (a) $x = t + 2,\ y = t - 4,$ for t in $(-\infty, \infty)$



(b) Since $x = t + 2$, we have $t = x - 2$. Since $y = t - 4$, we have $y = (x - 2) - 4 = x - 6$. Since t is in $(-\infty, \infty)$, x is in $(-\infty, \infty)$.

24. (a) $x = t^2 + 2,\ y = t^2 - 4,$ for t in $(-\infty, \infty)$



(b) Since $x = t^2 + 2$, we have $t^2 = x - 2$. Since $y = t^2 - 4$, we have $y = (x - 2) - 4 = x - 6$. Since t is in $(-\infty, \infty)$, x is in $[2, \infty)$.

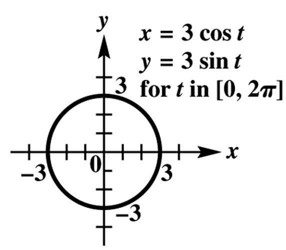
25. $x = 3\cos t,\ y = 3\sin t$

Since $x = 3\cos t \Rightarrow \cos t = \dfrac{x}{3}$,

$y = 3\sin t \Rightarrow \sin t = \dfrac{y}{3}$, and

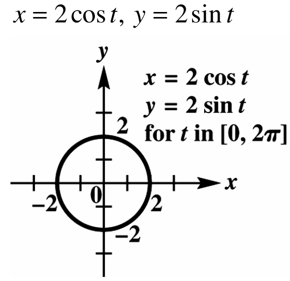
$\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \Rightarrow \frac{y^2}{9} + \frac{x^2}{9} = 1 \Rightarrow$$
$$x^2 + y^2 = 9$$



This is a circle centered at the origin with radius 3.

26. $x = 2\cos t,\ y = 2\sin t$



Since $x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$,

$y = 2\sin t \Rightarrow \sin t = \frac{y}{2}$, and $\sin^2 t + \cos^2 t = 1$,

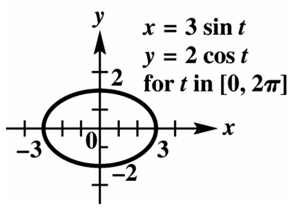
we have

$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 1 \Rightarrow$$

$$\frac{y^2}{4} + \frac{x^2}{4} = 1 \Rightarrow x^2 + y^2 = 4$$

This is a circle centered at the origin with radius 2.

27. $x = 3\sin t,\ y = 2\cos t$



Since $x = 3\sin t \Rightarrow \sin t = \frac{x}{3}$,

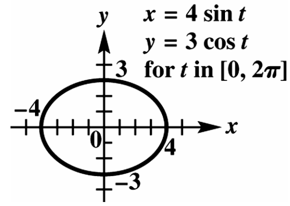
$y = 2\cos t \Rightarrow \cos t = \frac{y}{2}$, and

$\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

This is an ellipse centered at the origin with axes endpoints $(-3,0), (3,0), (0,-2), (0,2)$.

28. $x = 4\sin t,\ y = 3\cos t$



Since $x = 4\sin t \Rightarrow \sin t = \frac{x}{4}$,

$y = 3\cos t \Rightarrow \cos t = \frac{y}{3}$, and

$\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

This is an ellipse centered at the origin axes endpoints $(-4,0), (4,0), (0,-3), (0,3)$.

In Exercises 29 – 32, answers may vary.

29. $y = (x+3)^2 - 1$

$x = t,\ y = (t+3)^2 - 1$ for t in $(-\infty, \infty)$;

$x = t - 3,\ y = t^2 - 1$ for t in $(-\infty, \infty)$

30. $y = (x+4)^2 + 2$

$x = t,\ y = (t+4)^2 + 2$ for t in $(-\infty, \infty)$;

$x = t - 4,\ y = t^2 + 2$ for t in $(-\infty, \infty)$

31. $y = x^2 - 2x + 3 = (x-1)^2 + 2$

$x = t,\ y = (t-1)^2 + 2$ for t in $(-\infty, \infty)$;

$x = t + 1,\ y = t^2 + 2$ for t in $(-\infty, \infty)$

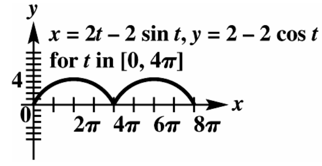
32. $y = x^2 - 4x + 6 = (x-2)^2 + 2$

$x = t,\ y = (t-2)^2 + 2$ for t in $(-\infty, \infty)$;

$x = t + 2,\ y = t^2 + 2$ for t in $(-\infty, \infty)$

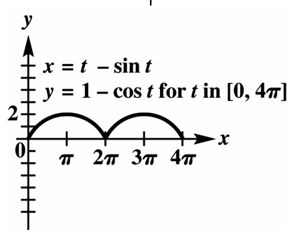
33. $x = 2t - 2\sin t, y = 2 - 2\cos t,$ for t in $[0, 4\pi]$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	3π	4π
$x = 2t - 2\sin t$	0	1.4	2π	11.4	4π	6π	8π
$y = 2 - 2\cos t$	0	2	4	2	0	4	0

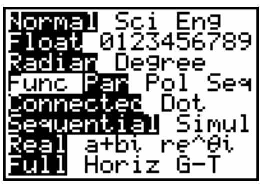


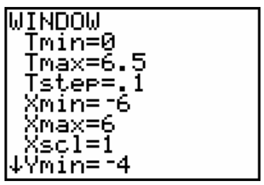
34. $x = t - \sin t$, $y = 1 - \cos t$, for t in $[0, 4\pi]$

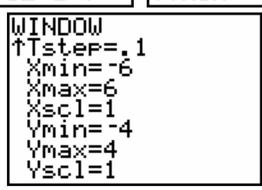
t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$x = t - \sin t$	0	.6	π	5.7	2π	6.8	3π	12.0	4π
$y = 1 - \cos t$	0	1	2	1	0	1	2	1	0



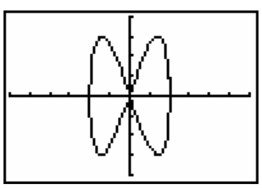
Exercises 35–38 are graphed in parametric mode in the following window.



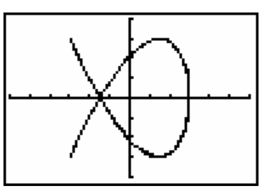




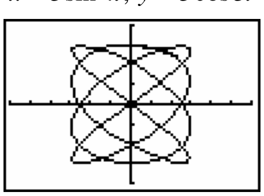
35. $x = 2\cos t$, $y = 3\sin 2t$



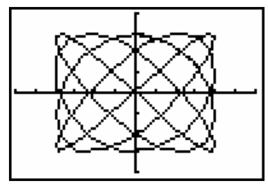
36. $x = 3\cos 2t$, $y = 3\sin 3t$



37. $x = 3\sin 4t$, $y = 3\cos 3t$



38. $x = 4\sin 4t$, $y = 3\sin 5t$



For Exercises 39–42, recall that the motion of a projectile (neglecting air resistance) can be modeled by: $x = (v_0 \cos\theta)t$, $y = (v_0 \sin\theta)t - 16t^2$ for t in $[0, k]$.

39. (a) $x = (v\cos\theta)t \Rightarrow$

$x = (48\cos 60°)t = 48\left(\frac{1}{2}\right)t = 24t$

$y = (v\sin\theta)t - 16t^2 \Rightarrow$

$y = (48\sin 60°)t - 16t^2 = 48 \cdot \frac{\sqrt{3}}{2}t - 16t^2$

$= -16t^2 + 24\sqrt{3}t$

(b) $t = \frac{x}{24}$, so $y = -16\left(\frac{x}{24}\right)^2 + 24\sqrt{3}\left(\frac{x}{24}\right)$

$= -\frac{x^2}{36} + \sqrt{3}x$

(c) $y = -16t^2 + 24\sqrt{3}t$

When the rocket is no longer in flight, $y = 0$. Solve $0 = -16t^2 + 24\sqrt{3}t \Rightarrow$

$0 = t\left(-16t + 24\sqrt{3}\right)$.

$t = 0$ or $-16t + 24\sqrt{3} = 0 \Rightarrow$

$-16t = -24\sqrt{3} \Rightarrow t = \frac{24\sqrt{3}}{16} \Rightarrow$

$t = \frac{3\sqrt{3}}{2} \approx 2.6$

The flight time is about 2.6 seconds. The horizontal distance at $t = \frac{3\sqrt{3}}{2}$ is

$$x = 24t = 24\left(\frac{3\sqrt{3}}{2}\right) \approx 62 \text{ ft}$$

40. **(a)** $x = (v\cos\theta)t \Rightarrow$

$$x = (150\cos 60°)t = 150\left(\frac{1}{2}\right)t = 75t$$

$$y = (v\sin\theta)t - 16t^2$$
$$= (150\sin 60°)t - 16t^2$$
$$= 150\frac{\sqrt{3}}{2}t - 16t^2 = -16t^2 + 75\sqrt{3}t$$

(b) $t = \frac{x}{75}$, so $y = -16\left(\frac{x}{75}\right)^2 + 75\sqrt{3}\left(\frac{x}{75}\right)$

$$= -\frac{16}{5625}x^2 + \sqrt{3}x$$

(c) $y = -16t^2 + 75\sqrt{3}t$

When the golf ball is no longer in flight, $y = 0$. Solve $0 = -16t^2 + 75\sqrt{3}t \Rightarrow$ $0 = t\left(-16t + 75\sqrt{3}\right)$.

$t = 0$ or $-16t + 75\sqrt{3} = 0 \Rightarrow$

$$-16t = -75\sqrt{3} \Rightarrow t = \frac{75\sqrt{3}}{16} \approx 8.1$$

The flight time is about 8.1 seconds. The horizontal distance at $t = \frac{75\sqrt{3}}{16}$ is

$$x = 75t = 75\left(\frac{75\sqrt{3}}{16}\right) \approx 609 \text{ ft}$$

41. **(a)** $x = (v\cos\theta)t \Rightarrow x = (88\cos 20°)t$

$$y = (v\sin\theta)t - 16t^2 + 2 \Rightarrow$$
$$y = (88\sin 20°)t - 16t^2 + 2$$

(b) Since $t = \frac{x}{88\cos 20°}$, we have

$$y = 88\sin 20°\left(\frac{x}{88\cos 20°}\right) - 16\left(\frac{x}{88\cos 20°}\right)^2 + 2$$

$$= (\tan 20°)x - \frac{x^2}{484\cos^2 20°} + 2$$

(c) Solving $0 = -16t^2 + (88\sin 20°)t + 2$ by the quadratic formula, we have

$$t = \frac{-88\sin 20° \pm \sqrt{(88\sin 20°)^2 - 4(-16)(2)}}{(2)(-16)}$$

$$= \frac{-30.098 \pm \sqrt{905.8759 + 128}}{-32} \Rightarrow$$

$t \approx -0.064$ or 1.9

Discard $t = -0.064$ since it is an unacceptable answer.

At $t \approx 1.9$ sec, $x = (88\cos 20°)t \approx 161$ ft.

The softball traveled 1.9 sec and 161 feet.

42. **(a)** $x = (v\cos\theta)t \Rightarrow x = (136\cos 29°)t$

$$y = (v\sin\theta)t - 16t^2 + 2.5 \Rightarrow$$
$$y = (136\sin 29°)t - 16t^2 + 2.5$$
$$= 2.5 - 16t^2 + (136\sin 29°)t$$

(b) Since $t = \frac{x}{136\cos 29°}$, we have

$$y = (136\sin 29°)\left(\frac{x}{136\cos 29°}\right) - 16\left(\frac{x}{136\cos 29°}\right)^2 + 2.5$$

$$= (\tan 29°)x - \frac{x^2}{1156\cos^2 29°} + 2.5$$

(c) Solving $0 = -16t^2 + (136\sin 29°)t + 2.5$ by the quadratic formula, we have

$$t = \frac{-136\sin 29° \pm \sqrt{(136\sin 29°)^2 - 4(-16)(2.5)}}{2(-16)}$$

$$= \frac{-65.934 \pm \sqrt{4347.3066 + 160}}{-32} \Rightarrow$$

$t \approx -.04,\ 4.2$

Discard $t = -.04$ since it gives an unacceptable answer. At $t \approx 4.2$ sec, $x = (136\cos 29°)t \approx 495$ ft. The baseball traveled 4.2 sec and 495 feet.

43. **(a)** $x = (v\cos\theta)t \Rightarrow$

$$x = (128\cos 60°)t = 128\left(\frac{1}{2}\right)t = 64t$$

$$y = (v\sin\theta)t - 16t^2 + 8 \Rightarrow$$
$$y = (128\sin 60°)t - 16t^2 + 8$$
$$= 128\left(\frac{\sqrt{3}}{2}\right)t - 16t^2 + 8$$
$$= 64\sqrt{3}t - 16t^2 + 8$$

(*continued on next page*)

(*continued from page 825*)

Since $t = \frac{x}{64}$,

$y = 64\sqrt{3}\left(\frac{x}{64}\right) - 16\left(\frac{x}{64}\right)^2 + 8 \Rightarrow$

$y = -\frac{1}{256}x^2 + \sqrt{3}x + 8.$

This is a parabolic path.

(b) Solving $0 = -16t^2 + 64\sqrt{3}t + 8$ by the quadratic formula, we have

$$t = \frac{-64\sqrt{3} \pm \sqrt{\left(64\sqrt{3}\right)^2 - 4(-16)(8)}}{2(-16)} = \frac{-64\sqrt{3} \pm \sqrt{12,800}}{-32} = \frac{-64\sqrt{3} \pm 80\sqrt{2}}{-32} \Rightarrow t \approx -.07,\ 7.0$$

Discard $t = -.07$ since it gives an unacceptable answer. At $t \approx 7.0$ sec, $x = 64t = 448$ ft. The rocket traveled approximately 7 sec and 448 feet.

44. $x = (v\cos\theta)t \Rightarrow$

$x = (88\cos 45°)t = 88\left(\frac{\sqrt{2}}{2}\right)t = 44\sqrt{2}t$

$y = (v\sin\theta)t - 2.66t^2 + h \Rightarrow$

$y = (88\sin 45°)t - 2.66t^2 + 0$

$= 88\left(\frac{\sqrt{2}}{2}\right)t - 2.66t^2 = 44\sqrt{2}t - 2.66t^2$

Solving $0 = 44\sqrt{2}t - 2.66t^2$ by factoring, we have

$0 = 44\sqrt{2}t - 2.66t^2 \Rightarrow 0 = \left(44\sqrt{2} - 2.66t\right)t \Rightarrow$

$t = 0$ or $44\sqrt{2} - 2.66t = 0$

$t = 0$ implies that the golf ball was initially on the ground, which is true. Solving $44\sqrt{2} - 2.66t = 0,$ we obtain

$44\sqrt{2} - 2.66t = 0 \Rightarrow 44\sqrt{2} = 2.66t \Rightarrow$

$t = \frac{44\sqrt{2}}{2.66} \approx 23.393$ sec

At $t \approx 23.393$ sec, $x = 44\sqrt{2}t \approx 1456$ ft. The golf ball traveled approximately 1456 feet.

45. (a) $x = (v\cos\theta)t \Rightarrow$

$x = (64\cos 60°)t = 64\left(\frac{1}{2}\right)t = 32t$

$y = (v\sin\theta)t - 16t^2 + 3 \Rightarrow$

$y = (64\sin 60°)t - 16t^2 + 3$

$= 64\left(\frac{\sqrt{3}}{2}\right)t - 16t^2 + 3$

$= 32\sqrt{3}t - 16t^2 + 3$

(b) Solving $0 = -16t^2 + 32\sqrt{3}t + 3$ by the quadratic formula, we have

$$t = \frac{-32\sqrt{3} \pm \sqrt{\left(32\sqrt{3}\right)^2 - 4(-16)(3)}}{2(-16)} = \frac{-32\sqrt{3} \pm \sqrt{3264}}{-32} = \frac{-32\sqrt{3} \pm 8\sqrt{51}}{-32} \Rightarrow$$

$t \approx -.05,\ 3.52$

Discard $t = -.07$ since it gives an unacceptable answer. At $t \approx 3.52$ sec, $x = 32t \approx 112.6$ ft. The ball traveled approximately 112.6 feet.

(c) To find the maximum height, find the vertex of $y = -16t^2 + 32\sqrt{3}t + 3.$

$y = -16t^2 + 32\sqrt{3}t + 3$

$= -16\left(t^2 - 2\sqrt{3}t\right) + 3$

$= -16\left(t^2 - 2\sqrt{3}t + 3 - 3\right) + 3$

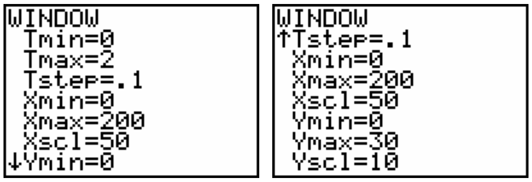
$y = -16\left(t - \sqrt{3}\right)^2 + 48 + 3$

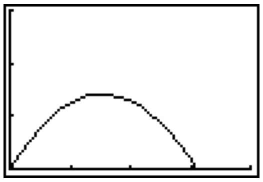
$= -16\left(t - \sqrt{3}\right)^2 + 51$

The maximum height of 51 ft is reached at $\sqrt{3} \approx 1.73$ sec. Since $x = 32t,$ the ball has traveled horizontally $32\sqrt{3} \approx 55.4$ ft.

(d) To determine if the ball would clear a 5-ft-high fence that is 100 ft from the batter, we need to first determine at what time is the ball 100 ft from the batter. Since $x = 32t,$ the time the ball is 100 ft from the batter is $t = \frac{100}{32} = 3.125$ sec. We next need to determine how high off the ground the ball is at this time. Since $y = 32\sqrt{3}t - 16t^2 + 3,$ the height of the ball is $y = 32\sqrt{3}(3.125) - 16(3.125)^2 + 3$ ≈ 20.0 ft . Since this height exceeds 5 ft, the ball will clear the fence.

46. (a)





(b) $x = 82.69265063t = v(\cos\theta)t \Rightarrow$
$82.69265063 = v\cos\theta$

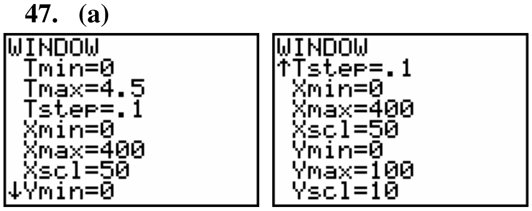
$$y = -16t^2 + 30.09777261t$$
$$= v(\sin\theta)t - 16t^2 \Rightarrow$$

$30.09777261 = v\sin\theta$

Thus, $\dfrac{30.09777261}{82.69265063} = \dfrac{v\sin\theta}{v\cos\theta} \Rightarrow$
$0.3697 = \tan\theta \Rightarrow \theta \approx 20.0°$

(c) $30.09777261 = v\sin 20.0° \Rightarrow v \approx 88.0$ ft/sec
Thus, the parametric equations are
$x = 88(\cos 20.0°)t,$
$y = -16t^2 + 88(\sin 20.0°)t.$

47. (a)



(b) $x = 56.56530965t = v(\cos\theta)t \Rightarrow$
$56.56530965 = v\cos\theta$

$$y = -16t^2 + 67.41191099t$$
$$= -16t^2 + v(\sin\theta)t \Rightarrow$$

$67.41191099 = v\sin\theta$

Thus, $\dfrac{67.41191099}{56.56530965} = \dfrac{v\sin\theta}{v\cos\theta} \Rightarrow$
$1.1918 = \tan\theta \Rightarrow \theta \approx 50.0°.$

(c) $67.41191099 = v\sin 50.0° \Rightarrow$
$v \approx 88.0$ ft/sec. Thus, the parametric equations are $x = 88(\cos 50.0°)t,$
$y = -16t^2 + 88(\sin 50.0°)t.$

For Exercises 48–51, many answers are possible.

48. The equation of a line with slope m through (x_1, y_1) is $y - y_1 = m(x - x_1)$.
To find two parametric representations, let $x = t$. We therefore have
$y - y_1 = m(t - x_1) \Rightarrow y = m(t - x_1) + y_1$. For another representation, let $x = t^2$. We therefore have
$y - y_1 = m(t^2 - x_1) \Rightarrow y = m(t^2 - x_1) + y_1$.

49. $y = a(x-h)^2 + k$
To find one parametric representation, let $x = t$. We therefore have, $y = a(t-h)^2 + k$.
For another representation, let $x = t + h$. We therefore have $y = a(t + h - h)^2 + k = at^2 + k$.

50. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
To find a parametric representation, let $x = a\sec\theta$. We therefore have

$$\frac{(a\sec\theta)^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow$$
$$\frac{a^2\sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow$$
$$\sec^2\theta - \frac{y^2}{b^2} = 1 \Rightarrow \sec^2\theta - 1 = \frac{y^2}{b^2}$$
$$\tan^2\theta = \frac{y^2}{b^2} \Rightarrow b^2\tan^2\theta = y^2 \Rightarrow$$
$$b\tan\theta = y$$

51. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
To find a parametric representation, let $x = a\sin t$. We therefore have

$$\frac{(a\sin t)^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{a^2\sin^2 t}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$$
$$\sin^2 t + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \sin^2 t$$
$$y^2 = b^2(1 - \sin^2 t) \Rightarrow y^2 = b^2\cos^2 t \Rightarrow$$
$$y = b\cos t$$

52. To show that $r = a\theta$ is given parametrically by $x = a\theta\cos\theta,\ y = a\theta\sin\theta,$ for θ in $(-\infty,\infty)$, we must show that the parametric equations yield $r = a\theta$, where $r^2 = x^2 + y^2$.

$r^2 = x^2 + y^2 \Rightarrow$
$r^2 = (a\theta\cos\theta)^2 + (a\theta\sin\theta)^2 \Rightarrow$
$r^2 = a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta$
$r^2 = a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta \Rightarrow$
$r^2 = a^2\theta^2\left(\cos^2\theta + \sin^2\theta\right) \Rightarrow r^2 = a^2\theta^2 \Rightarrow$
$r = \pm a\theta$ or just $r = a\theta$

This implies that the parametric equations satisfy $r = a\theta$.

53. To show that $r\theta = a$ is given parametrically by $x = \frac{a\cos\theta}{\theta},\ y = \frac{a\sin\theta}{\theta},$ for θ in $(-\infty,0)\cup(0,\infty)$, we must show that the parametric equations yield $r\theta = a$, where $r^2 = x^2 + y^2$.

$r^2 = x^2 + y^2 \Rightarrow$
$r^2 = \left(\frac{a\cos\theta}{\theta}\right)^2 + \left(\frac{a\sin\theta}{\theta}\right)^2 \Rightarrow$
$r^2 = \frac{a^2\cos^2\theta}{\theta^2} + \frac{a^2\sin^2\theta}{\theta^2}$
$r^2 = \frac{a^2}{\theta^2}\cos^2\theta + \frac{a^2}{\theta^2}\sin^2\theta \Rightarrow$
$r^2 = \frac{a^2}{\theta^2}\left(\cos^2\theta + \sin^2\theta\right) \Rightarrow r^2 = \frac{a^2}{\theta^2} \Rightarrow$
$r = \pm\frac{a}{\theta}$ or just $r = \frac{a}{\theta}$

This implies that the parametric equations satisfy $r\theta = a$.

54. The second set of equations $x = \cos t$, $y = -\sin t$, t in $[0, 2\pi]$ trace the circle out clockwise. A table of values confirms this.

t	$x = \cos t$	$y = -\sin t$
0	$\cos 0 = 1$	$-\sin 0 = 0$
$\frac{\pi}{6}$	$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$-\sin\frac{\pi}{6} = -\frac{1}{2}$
$\frac{\pi}{4}$	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$-\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\cos\frac{\pi}{3} = \frac{1}{2}$	$-\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$\cos\frac{\pi}{2} = 0$	$-\sin\frac{\pi}{2} = -1$

55. If $x = f(t)$ is replaced by $x = c + f(t)$, the graph will be translated c units horizontally.

56. If $y = g(t)$ is replaced by $y = d + g(t)$, the graph is translated vertically d units.

Chapter 8 Review Exercises

1. Find b, given $C = 74.2°$, $c = 96.3$ m, $B = 39.5°$.
Use the law of sines to find b.

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 39.5°} = \frac{96.3}{\sin 74.2°} \Rightarrow$$
$$b = \frac{96.3\sin 39.5°}{\sin 74.2°} \approx 63.7 \text{ m}$$

2. Find B, given $A = 129.7°$, $a = 127$ ft, $b = 69.8$ ft.
Use the law of sines to find B.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{69.8} = \frac{\sin 129.7°}{127} \Rightarrow$$
$$\sin B = \frac{69.8\sin 129.7°}{127} \approx .42286684$$

Since angle A is obtuse, angle B is acute. Thus, $B \approx 25.0°$.

3. Find B, given $C = 51.3°$, $c = 68.3$ m, $b = 58.2$ m.
Use the law of sines to find B.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{58.2} = \frac{\sin 51.3°}{68.3} \Rightarrow$$
$$\sin B = \frac{58.2\sin 51.3°}{68.3} \approx .66502269$$

There are two angles B between $0°$ and $180°$ that satisfy the condition. Since $\sin B \approx .66502269$, to the nearest tenth value of B is $B_1 = 41.7°$. Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 41.7° = 138.3°$. This is not a valid angle measure for this triangle since $C + B_2 = 51.3° + 138.3° = 189.6° > 180°$.
Thus, $B = 41.7°$.

4. Find b, given $a = 165$ m, $A = 100.2°$, $B = 25.0°$.
Use the law of sines to find b.

$$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{b}{\sin 25.0°} = \frac{165}{\sin 100.2°} \Rightarrow$$
$$b = \frac{165\sin 25.0°}{\sin 100.2°} \approx 70.9 \text{ m}$$

5. Find A, given $B = 39°50'$, $b = 268$ m, $a = 340$ m.
Use the law of sines to find A.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{340} = \frac{\sin 39°50'}{268} \Rightarrow$$
$$\sin A = \frac{340\sin 39°50'}{268} \approx .81264638$$

There are two angles A between $0°$ and $180°$ that satisfy the condition. Since $\sin A \approx .81264638,$ to the nearest tenth value of A is $A_1 = 54.4° \approx 54°20'$. Supplementary angles have the same sine value, so another possible value of A is $A_2 = 180° - 54°20'$ $= 179°60' - 54°20' = 125°40'$. This is a valid angle measure for this triangle since $B + A_2 = 39°50' + 125°40' = 165°30' < 180°$.
$A = 54°20'$ or $A = 125°40'$

6. Find A, given $C = 79°20'$, $c = 97.4$ mm, $a = 75.3$ mm.
Use the law of sines to find A.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{75.3} = \frac{\sin 79°20'}{97.4} \Rightarrow$$
$$\sin A = \frac{75.3 \sin 79°20'}{97.4} \approx .75974194$$

There are two angles A between $0°$ and $180°$ that satisfy the condition. Since $\sin A \approx .75974194,$ to the nearest hundredth value of A is $A_1 = 49.44° \approx 49°30'$.
Supplementary angles have the same sine value, so another possible value of A is
$A_2 = 180° - 49°30' = 179°60' - 49°30' = 130°30'$.
This is not a valid angle measure for this triangle since
$C + A_2 = 79°20' + 130°30' = 209°50' > 180°$.
Thus, $A = 49°30'$.

7. No; If you are given two angles of a triangle, then the third angle is known since the sum of the measures of the three angles is $180°$. Since you are also given one side, there will only be one triangle that will satisfy the conditions.

8. No; the sum of a and b do not exceed c.

9. $a = 10, B = 30°$

(a) The value of b that forms a right triangle would yield exactly one value for A. That is, $b = 10 \sin 30° = 5$. Also, any value of b greater than or equal to 10 would yield a unique value for A.

(b) Any value of b between 5 and 10, would yield two possible values for A.

(c) If b is less than 5, then no value for A is possible.

10. $A = 140°$, $a = 5$, and $b = 7$
With these conditions, we can try to solve the triangle with the law of sines.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{7} = \frac{\sin 140°}{5} \Rightarrow$$
$$\sin B = \frac{7 \sin 140°}{5} \approx .89990265 \Rightarrow B \approx 64°$$

Since $A + B = 140° + 64° = 204° > 180°,$ no such triangle exists.

11. Find A, given $a = 86.14$ in., $b = 253.2$ in., $c = 241.9$ in.
Use the law of cosines to find A.

$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{253.2^2 + 241.9^2 - 86.14^2}{2(253.2)(241.9)} \approx .94046923$$

Thus, $A \approx 19.87°$ or $19°\ 52'$.

12. Find b, given $B = 120.7°$, $a = 127$ ft, $c = 69.8$ ft.
Use the law of cosines to find b.

$$b^2 = a^2 + c^2 - 2ac \cos B \Rightarrow b^2 = 127^2 + 69.8^2 - 2(127)(69.8)\cos 120.7° \approx 30{,}052.6 \Rightarrow b \approx 173 \text{ ft}$$

13. Find a, given $A = 51°\ 20'$, $c = 68.3$ m, $b = 58.2$ m.
Use the law of cosines to find a.

$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow$$
$$a^2 = 58.2^2 + 68.3^2 - 2(58.2)(68.3)\cos 51°\ 20' \approx 3084.99 \Rightarrow a \approx 55.5 \text{ m}$$

14. Find B, given $a = 14.8$ m, $b = 19.7$ m, $c = 31.8$ m.
Use the law of cosines to find B.

$$b^2 = a^2 + c^2 - 2ac \cos B \Rightarrow$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{14.8^2 + 31.8^2 - 19.7^2}{2(14.8)(31.8)} \approx .89472845$$

Thus, $B \approx 26.5°$ or $26°\ 30'$.

15. Find a, given $A = 60°$, $b = 5$cm, $c = 21$ cm.
Use the law of cosines to find a.

$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow$$
$$a^2 = 5^2 + 21^2 - 2(5)(21)\cos 60° = 361 \Rightarrow$$
$$a = 19 \text{ cm}$$

16. Find A, given $a = 13$ ft, $b = 17$ ft, $c = 8$ ft.
Use the law of cosines to find A.

$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{17^2 + 8^2 - 13^2}{2(17)(8)}$$
$$= \frac{184}{272} = \frac{23}{34} \approx .67647059 \Rightarrow A \approx 47°$$

17. Solve the triangle, given $A = 25.2°$, $a = 6.92$ yd, $b = 4.82$ yd.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{b\sin A}{a} \Rightarrow$$
$$\sin B = \frac{4.82\sin 25.2°}{6.92} \approx .29656881$$

There are two angles B between $0°$ and $180°$ that satisfy the condition. Since $\sin B \approx .29656881,$ to the nearest tenth value of B is $B_1 = 17.3°$. Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 17.3° = 162.7°$. This is not a valid angle measure for this triangle since $A + B_2 = 25.2° + 162.7° = 187.9° > 180°$.

$$C = 180° - A - B \Rightarrow$$
$$C = 180° - 25.2° - 17.3° \Rightarrow C = 137.5°$$

Use the law of sines to find c.

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 137.5°} = \frac{6.92}{\sin 25.2°} \Rightarrow$$
$$c = \frac{6.92\sin 137.5°}{\sin 25.2°} \approx 11.0 \text{ yd}$$

18. Solve the triangle, given $A = 61.7°$, $a = 78.9$ m, $b = 86.4$ m.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{b\sin A}{a} \Rightarrow$$
$$\sin B = \frac{86.4\sin 61.7°}{78.9} \approx .96417292$$

There are two angles B between $0°$ and $180°$ that satisfy the condition. Since $\sin B \approx .96417292,$ to the nearest tenth value of B is $B_1 = 74.6°$. Supplementary angles have the same sine value, so another possible value of B is $B_2 = 180° - 74.6° = 105.4°$. This is a valid angle measure for this triangle since $A + B_2 = 61.7° + 105.4° = 167.1° < 180°$.

Solving separately for triangles AB_1C_1 and AB_2C_2 we have

AB_1C_1:

$$C_1 = 180° - A - B_1$$
$$= 180° - 61.7° - 74.6° = 43.7°$$

$$\frac{c_1}{\sin C_1} = \frac{a}{\sin A} \Rightarrow c_1 = \frac{a\sin C_1}{\sin A} \Rightarrow$$
$$c_1 = \frac{78.9\sin 43.7°}{\sin 61.7°} \approx 61.9 \text{ m}$$

AB_2C_2:

$$C_2 = 180° - A - B_2$$
$$= 180° - 61.7° - 105.4° = 12.9°$$

$$\frac{c_2}{\sin C_2} = \frac{a}{\sin A} \Rightarrow c_2 = \frac{a\sin C_2}{\sin A} \Rightarrow$$
$$c_2 = \frac{78.9\sin 12.9°}{\sin 61.7°} \approx 20.0 \text{ m}$$

19. Solve the triangle, given $a = 27.6$ cm, $b = 19.8$ cm, $C = 42°\ 30'$.
This is a SAS case, so using the law of cosines.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$$
$$c^2 = 27.6^2 + 19.8^2 - 2(27.6)(19.8)\cos 42°\ 30'$$
$$\approx 347.985 \Rightarrow c \approx 18.65 \text{ cm}$$

(will be rounded as 18.7)
Of the remaining angles A and B, B must be smaller since it is opposite the shorter of the two sides a and b. Therefore, B cannot be obtuse.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{19.8} = \frac{\sin 42°30'}{18.65} \Rightarrow$$
$$\sin B = \frac{19.8\sin 42°30'}{18.65} \approx .717124859 \Rightarrow$$
$$B \approx 45.8° \approx 45°50'$$

Thus,

$$A = 180° - B - C = 180° - 45°\ 50' - 42°\ 30'$$
$$= 179°60' - 88°20' = 91°\ 40'$$

20. Solve the triangle, given $a = 94.6$ yd, $b = 123$ yd, $c = 109$ yd.
We can use the law of cosines to solve for any angle of the triangle. We solve for B, the largest angle. We will know that B is obtuse if $\cos B < 0$.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow$$
$$\cos B = \frac{94.6^2 + 109^2 - 123^2}{2(94.6)(109)} = .27644937 \Rightarrow$$
$$B \approx 73.9°$$

Of the remaining angles A and C, A must be smaller since it is opposite the shorter of the two sides a and c. Therefore, A cannot be obtuse.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{94.6} = \frac{\sin 74.0°}{123} \Rightarrow$$
$$\sin A = \frac{94.6 \sin 74.0°}{123} \approx .73931184 \Rightarrow$$
$$A \approx 47.7°$$

Thus,
$C = 180° - A - B = 180° - 47.7° - 73.9° = 58.4°$.

21. Given $b = 840.6$ m, $c = 715.9$ m, $A = 149.3°$, find the area.
Angle A is included between sides b and c. Thus, we have
$$\mathcal{A} = \frac{1}{2} bc \sin A$$
$$= \frac{1}{2}(840.6)(715.9)\sin 149.3° \approx 153,600 \text{ m}^2$$
(rounded to four significant digits)

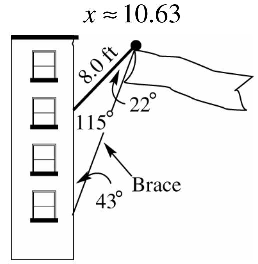
22. Given $a = 6.90$ ft, $b = 10.2$ ft, $C = 35°\ 10'$, find the area.
Angle C is included between sides a and b. Thus, we have
$$\mathcal{A} = \frac{1}{2} ab \sin C$$
$$= \frac{1}{2}(6.90)(10.2)\sin 35°\ 10' \approx 20.3 \text{ ft}^2$$
(rounded to three significant digits)

23. Given $a = .913$ km, $b = .816$ km, $c = .582$ km, find the area.
Use Heron's formula to find the area.
$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(.913 + .816 + .582)$$
$$= \frac{1}{2} \cdot 2.311 = 1.1555$$
$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{1.1555(1.1555-.913)\cdot (1.1555-.816)(1.1555-.582)}$$
$$= \sqrt{1.1555(.2425)(.3395)(.5735)}$$
$\approx .234 \text{ km}^2$ (rounded to three significant digits)

24. Given $a = 43$ m, $b = 32$ m, $c = 51$ m, find the area.
Use Heron's formula to find the area.
$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(43+32+51)$$
$$= \frac{1}{2} \cdot 126 = 63$$
$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{63(63-43)(63-32)(63-51)}$$
$$= \sqrt{63(20)(31)(12)} \approx 680 \text{ m}^2$$
(rounded to two significant digits)

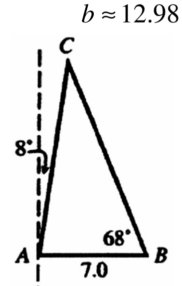
25. Since $B = 58.4°$ and $C = 27.9°$,
$A = 180° - B - C = 180° - 58.4° - 27.9° = 93.7°$.
Using the law of sines, we have
$$\frac{AB}{\sin C} = \frac{125}{\sin A} \Rightarrow \frac{AB}{\sin 27.9°} = \frac{125}{\sin 93.7°} \Rightarrow$$
$$AB = \frac{125 \sin 27.9°}{\sin 93.7°} \approx 58.61$$
The canyon is 58.6 feet across (rounded to three significant digits.)

26. The angle opposite the 8.0 ft flagpole is $180° - 115° - 22° = 43°$.
Using the law of sines, we have
$$\frac{8}{\sin 43°} = \frac{x}{\sin 115°} \Rightarrow x = \frac{8 \sin 115°}{\sin 43°} \Rightarrow$$
$$x \approx 10.63$$



The brace is 11 feet long. (rounded to two significant digits)

27. Let AC = the height of the tree.
Angle $A = 90° - 8.0° = 82°$
Angle $C = 180° - B - A = 30°$
Use the law of sines to find $AC = b$.
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 68°} = \frac{7.0}{\sin 30°}$$
$$b = \frac{7.0 \sin 68°}{\sin 30°}$$
$$b \approx 12.98$$



The tree is 13 meters tall. (rounded to two significant digits)

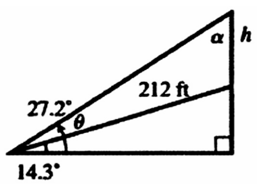
28. Let d = the distance between the ends of the wire.
This situation is SAS, so we should use the law of cosines.

$$d^2 = 15.0^2 + 12.2^2 - 2(15.0)(12.2)\cos 70.3°$$
$$\approx 250.463 \Rightarrow d \approx 15.83 \text{ ft}$$

The ends of the wire should be places 15.8 ft apart. (rounded to three significant digits)

29. Let h = the height of tree.

$$\theta = 27.2° - 14.3° = 12.9°$$
$$\alpha = 90° - 27.2° = 62.8°$$
$$\frac{h}{\sin\theta} = \frac{212}{\sin\alpha}$$
$$\frac{h}{\sin 12.9°} = \frac{212}{\sin 62.8°}$$
$$h = \frac{212 \sin 12.9°}{\sin 62.8°}$$
$$h \approx 53.21$$



The height of the tree is 53.2 ft. (rounded to three significant digits)

30. $AB = 150$ km, $AC = 102$ km, $BC = 135$ km
Use the law of cosines to find the measure of angle C.

$$(AB)^2 = (AC)^2 + (BC)^2 - 2(AC)(BC)\cos C \Rightarrow$$
$$150^2 = 102^2 + 135^2 - 2(102)(135)\cos C$$
$$\cos C = \frac{102^2 + 135^2 - 150^2}{2(102)(135)} \approx .22254902 \Rightarrow$$
$$C \approx 77.1°$$

31. Let x = the distance between the boats.
In 3 hours the first boat travels 3(36.2) = 108.6 km and the second travels 3(45.6) = 136.8 km.
Use the law of cosines to find x.

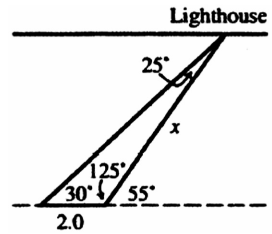
$$x^2 = 108.6^2 + 136.8^2 - 2(108.6)(136.8)\cos 54°10'$$
$$\approx 13{,}113.359 \Rightarrow x \approx 115 \text{ km}$$

They are 115 km apart.

32. To find the angles of the triangle formed by the ship's positions with the lighthouse, we find the supplementary angle to 55°:
180° – 55° = 125°. The third angle in the triangle is $180° - 125° - 30° = 25°$

Using the law of sines, we have

$$\frac{2}{\sin 25°} = \frac{x}{\sin 30°}$$
$$x = \frac{2\sin 30°}{\sin 25°}$$
$$\approx 2.4 \text{ mi}$$



The ship is 2.4 miles from the lighthouse.

33. Use the distance formula to find the distances between the points.
Distance between (–8, 6) and (0, 0):

$$\sqrt{(-8-0)^2 + (6-0)^2} = \sqrt{(-8)^2 + 6^2}$$
$$= \sqrt{64+36} = \sqrt{100} = 10$$

Distance between (–8, 6) and (3, 4):

$$\sqrt{(-8-3)^2 + (6-4)^2} = \sqrt{(-11)^2 + 2^2}$$
$$= \sqrt{121+4} = \sqrt{125}$$
$$= 5\sqrt{5} \approx 11.18$$

Distance between (3, 4) and (0, 0):

$$\sqrt{(3-0)^2 + (4-0)^2} = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9+16} = \sqrt{25} = 5$$

$$s \approx \frac{1}{2}(10 + 11.18 + 5) = \frac{1}{2} \cdot 26.18 = 13.09$$
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{13.09(13.09-10)\cdot(13.09-11.18)(13.09-5)}$$
$$= \sqrt{13.09(3.09)(1.91)(8.09)}$$

≈ 25 sq units (rounded to two significant digits)

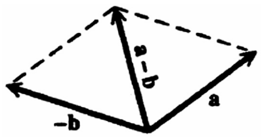
34. The sides of the triangle measure 5, 10, and $5\sqrt{5}$, which satisfy the converse of the Pythagorean theorem:

$$\left(5\sqrt{5}\right)^2 = 5^2 + 10^2 \Rightarrow 125 = 25 + 100 \Rightarrow$$
$$125 = 125$$

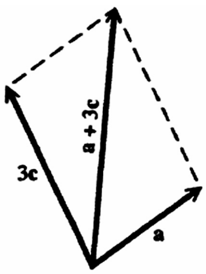
Thus, the triangle is a right triangle.
Let $\angle B$ be the angle with vertex (0, 0), the right angle. Then

$$A = \frac{1}{2}ac\sin B = \frac{1}{2} \cdot 5 \cdot 10 \cdot 1 = 25 \text{ sq units}$$

35. $\mathbf{a} - \mathbf{b}$



36. $\mathbf{a} + 3\mathbf{c}$



37. **(a)** true **(b)** false

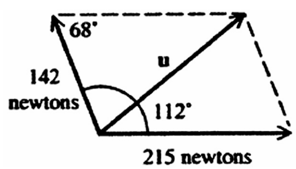
38. Forces of 142 newtons and 215 newtons, forming an angle of 112°

$180° - 112° = 68°$

$|\mathbf{u}|^2 = 215^2 + 142^2 - 2(215)(142)\cos 68°$

$|\mathbf{u}|^2 \approx 43515.5$

$|\mathbf{u}| \approx 209$ newtons

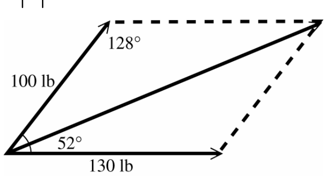


39. $\alpha = 180° - 52° = 128°$

$|\mathbf{v}|^2 = 100^2 + 130^2 - 2(100)(130)\cos 128°$

$|\mathbf{v}|^2 \approx 42907.2$

$|\mathbf{v}| \approx 207$ lb



40. $|\mathbf{v}| = 50, \theta = 45°$

horizontal:

$$x = |\mathbf{v}|\cos\theta = 50\cos 45° = \frac{50\cdot\sqrt{2}}{2} = 25\sqrt{2}$$

vertical:

$$y = |\mathbf{v}|\sin\theta = 50\sin 45° = \frac{50\cdot\sqrt{2}}{2} = 25\sqrt{2}$$

41. $|\mathbf{v}| = 964, \theta = 154°20'$

horizontal:

$x = |\mathbf{v}|\cos\theta = 964\cos 154°\ 20' \approx 869$

vertical: $y = |\mathbf{v}|\sin\theta = 964\sin 154°\ 20' \approx 418$

42. $\mathbf{u} = \langle 21, -20\rangle$

magnitude:

$$|\mathbf{u}| = \sqrt{21^2 + (-20)^2} = \sqrt{441 + 400} = \sqrt{841} = 29$$

Angle: $\tan\theta' = \frac{b}{a} \Rightarrow \tan\theta' = \frac{-20}{21} \Rightarrow$

$$\theta' = \tan^{-1}\left(-\frac{20}{21}\right) \approx -43.6° \Rightarrow$$

$$\theta = -43.6° + 360° = 316.4°$$

(θ lies in quadrant IV)

43. $\mathbf{u} = \langle -9, 12\rangle$

magnitude:

$$|\mathbf{u}| = \sqrt{(-9)^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$$

Angle: $\tan\theta' = \frac{b}{a} \Rightarrow \tan\theta' = \frac{12}{-9} \Rightarrow$

$$\theta' = \tan^{-1}\left(-\frac{4}{3}\right) \approx -53.1° \Rightarrow$$

$$\theta = -53.1° + 180° = 126.9°$$

(θ lies in quadrant II)

44. $\mathbf{v} = 2\mathbf{i} - \mathbf{j},\ \mathbf{u} = -3\mathbf{i} + 2\mathbf{j}$

First write the given vectors in component form.

$\mathbf{v} = 2\mathbf{i} - \mathbf{j} = \langle 2, -1\rangle$ and $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} = \langle -3, 2\rangle$

(a)
$$\begin{aligned} 2\mathbf{v} + \mathbf{u} &= 2\langle 2,-1\rangle + \langle -3,2\rangle \\ &= \langle 2\cdot 2, 2(-1)\rangle + \langle -3,2\rangle \\ &= \langle 4,-2\rangle + \langle -3,2\rangle \\ &= \langle 4+(-3), -2+2\rangle = \langle 1,0\rangle = \mathbf{i} \end{aligned}$$

(b)
$$\begin{aligned} 2\mathbf{v} &= 2\langle 2,-1\rangle = \langle 2\cdot 2, 2(-1)\rangle \\ &= \langle 4,-2\rangle = 4\mathbf{i} - 2\mathbf{j} \end{aligned}$$

(c)
$$\begin{aligned} \mathbf{v} - 3\mathbf{u} &= \langle 2,-1\rangle - 3\langle -3,2\rangle \\ &= \langle 2,-1\rangle - \langle 3(-3), 3\cdot 2\rangle \\ &= \langle 2,-1\rangle - \langle -9,6\rangle \\ &= \langle 2-(-9), -1-6\rangle \\ &= \langle 11,-7\rangle = 11\mathbf{i} - 7\mathbf{j} \end{aligned}$$

45. $\mathbf{a} = \langle 3,-2\rangle, \mathbf{b} = \langle -1,3\rangle$

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= \langle 3,-2\rangle\cdot\langle -1,3\rangle = 3(-1) + (-2)\cdot 3 \\ &= -3 - 6 = -9 \end{aligned}$$

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \Rightarrow \cos\theta = \frac{-9}{|\langle 3,-2\rangle||\langle -1,3\rangle|}$$

$$= \frac{-9}{\sqrt{3^2 + (-2)^2}\sqrt{(-1)^2 + 3^2}}$$

(*continued on next page*)

(*continued from page 833*)

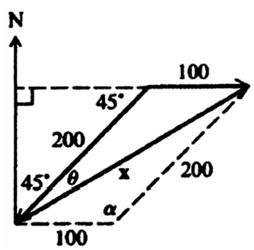
$$= -\frac{9}{\sqrt{9+4}\sqrt{1+9}} = -\frac{9}{\sqrt{13}\sqrt{10}}$$
$$= -\frac{9}{\sqrt{130}} \approx -.78935222$$

Thus, $\theta \approx 142.1°$.

46. $|\mathbf{u}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$
$$\mathbf{v} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle -4, 3\rangle}{5} = \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle$$

47. $|\mathbf{u}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$
$$\mathbf{v} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 5, 12\rangle}{13} = \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$$

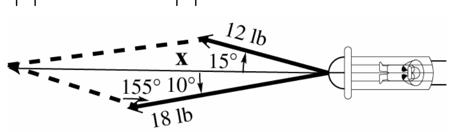
48. Let $\mathbf{x}$ = the resultant vector.
$$\alpha = 180° - 45° = 135°$$
$$|\mathbf{x}|^2 = 200^2 + 100^2 - 2(200)(100)\cos 135°$$
$$|\mathbf{x}|^2 = 40{,}000 + 10{,}000 - 40{,}000\left(-\frac{\sqrt{2}}{2}\right)$$
$$|\mathbf{x}|^2 = 50{,}000 + 20{,}000\sqrt{2}$$
$$|\mathbf{x}|^2 \approx 78{,}284.27 \Rightarrow |\mathbf{x}| \approx 279.8$$



Using the law of cosines again, we have
$$\cos\theta = \frac{100^2 + 279.8^2 - 200^2}{2(100)(279.8)} \approx .86290279 \Rightarrow$$
$$\theta \approx 30.4°$$

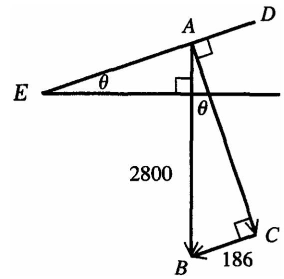
The force is 280 newtons (rounded) at an angle of 30.4° with the first rope.

49. Let $|\mathbf{x}|$ be the resultant force.
$$\theta = 180° - 15° - 10° = 155°$$
$$|\mathbf{x}|^2 = 12^2 + 18^2 - 2(12)(18)\cos 155°$$
$$|\mathbf{x}|^2 \approx 859.5 \Rightarrow |\mathbf{x}| \approx 29$$



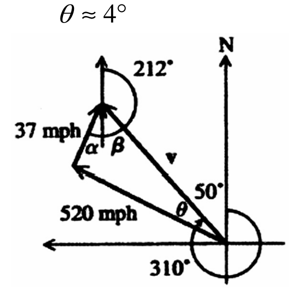
The magnitude of the resultant force on Jessie and the sled is 29 lb.

50. Let θ = the angle that the hill makes with the horizontal.
The downward force of 2800 lb has component *AC* perpendicular to the hill and component *CB* parallel to the hill. *AD* represents the force of 186 lb that keeps the car from rolling down the hill. Since vectors *AD* and *BC* are equal, $|BC| = 186$.
Angle B = angle *EAB* because they are alternate interior angles to the two right triangles are similar. Hence angle θ = angle *BAC*.



Since $\sin BAC = \frac{186}{2800} = \frac{93}{1400} \Rightarrow$
$BAC = \theta \approx 3.8° \approx 4°$, the angle that the hill makes with the horizontal is about 4º.

51. Let $\mathbf{v}$ = the ground speed vector.
$\alpha = 212° - 180° = 32°$ and $\beta = 50°$ because they are alternate interior angles. Angle opposite to 520 is $\alpha + \beta = 82°$.
Using the law of sines, we have
$$\frac{\sin\theta}{37} = \frac{\sin 82°}{520}$$
$$\sin\theta = \frac{37\sin 82°}{520}$$
$$\sin\theta \approx .07046138$$
$$\theta \approx 4°$$

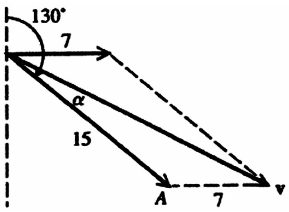


Thus, the bearing is 360° – 50° – θ = 306°.
The angle opposite $\mathbf{v}$ is 180° – 82° – 4° = 94°.
Using the laws of sines, we have
$$\frac{|\mathbf{v}|}{\sin 94°} = \frac{520}{\sin 82°} \Rightarrow$$
$$|\mathbf{v}| = \frac{520\sin 94°}{\sin 82°} \approx 524 \text{ mph}$$

The pilot should fly on a bearing of 306°. Her actual speed is 524 mph.

52. Let **v** = the resultant vector.
Angle $A = 180° - (130° - 90°) = 140°$
Use the law of cosines to find the magnitude of the resultant **v**.

$$|\mathbf{v}|^2 = 15^2 + 7^2 - 2(15)(7)\cos 140°$$
$$|\mathbf{v}|^2 \approx 434.87$$
$$|\mathbf{v}| \approx 20.9$$

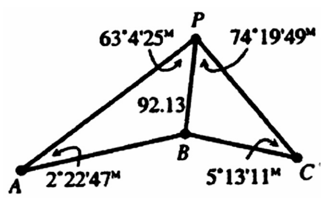


Use the law of sines to find α.

$$\frac{\sin\alpha}{7} = \frac{\sin 140°}{20.9} \Rightarrow$$
$$\sin\alpha = \frac{7\sin 140°}{20.9} \approx .21528772 \Rightarrow \alpha \approx 12°$$

The resulting speed is 21 km per hr (rounded) and bearing is 130° – 12° = 118°.

53. Refer to the diagram below. In each of the triangles *ABP* and *PBC*, we know two angles and one side. Solve each triangle using the law of sines.



$$AB = \frac{92.13\sin 63°\ 4'\ 25''}{\sin 2°\ 22'\ 47''} \approx 1978.28 \text{ ft and}$$
$$BC = \frac{92.13\sin 74°\ 19'\ 49''}{\sin 5°\ 13'\ 11''} \approx 975.05 \text{ ft}$$

54. $\left[5(\cos 90° + i\sin 90°)\right] \cdot \left[6(\cos 180° + i\sin 180°)\right]$

$$= 5\cdot 6\left[\cos(90°+180°) + i\sin(90°+180°)\right]$$
$$= 30(\cos 270° + i\sin 270°)$$
$$= 30(0 - i)$$
$$= 0 - 30i \text{ or } -30i$$

55. $[3\operatorname{cis}135°][2\operatorname{cis}105°]$

$$= 3\cdot 2\operatorname{cis}(135°+105°) = 6\operatorname{cis}240°$$
$$= 6(\cos 240° + i\sin 240°)$$
$$= 6\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -3 - 3\sqrt{3}i$$

56. $\dfrac{2(\cos 60° + i\sin 60°)}{8(\cos 300° + \sin 300°)}$

$$= \frac{2}{8}\left[\cos(60° - 300°) + i\sin(60° - 300°)\right]$$
$$= \frac{1}{4}\left[\cos(-240°) + i\sin(-240°)\right]$$
$$= \frac{1}{4}\left[\cos(240°) - i\sin(240°)\right]$$
$$= \frac{1}{4}\left[-\cos 60° + i\sin 60°\right]$$
$$= \frac{1}{4}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = -\frac{1}{8} + \frac{\sqrt{3}}{8}i$$

57. $\dfrac{4\operatorname{cis}270°}{2\operatorname{cis}90°} = \dfrac{4}{2}\operatorname{cis}(270° - 90°) = 2\operatorname{cis}180°$

$$= 2(\cos 180° + i\sin 180°)$$
$$= 2(-1 + 0i) = -2 + 0i \text{ or } -2$$

58. $\left(\sqrt{3} + i\right)^3$

$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$ and since

θ is in quadrant I, $\tan\theta = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} \Rightarrow$

$\theta = 30°$.

$$\left(\sqrt{3}+i\right)^3 = \left[2(\cos 30° + i\sin 30°)\right]^3$$
$$= 2^3\left[\cos(3\cdot 30°) + i\sin(3\cdot 30°)\right]$$
$$= 8\left[\cos 90° + i\sin 90°\right]$$
$$= 8(0 + i) = 0 + 8i = 8i$$

59. $(2 - 2i)^5$

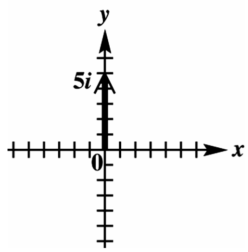
$r = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ and

$\tan\theta = \dfrac{-2}{2} = -1 \Rightarrow \theta = 315°$, since θ is in quadrant IV,

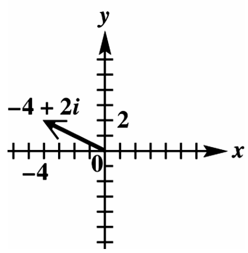
$$(2-2i)^5 = \left[2\sqrt{2}(\cos 315° + i\sin 315°)\right]^5$$
$$= \left(2\sqrt{2}\right)^5\left[\cos(5\cdot 315°) + i\sin(5\cdot 315°)\right]$$
$$= 128\sqrt{2}(\cos 1575° + i\sin 1575°)$$
$$= 128\sqrt{2}(\cos 135° + i\sin 135°)$$
$$= 128\sqrt{2}(-\cos 45° + i\sin 45°)$$
$$= 128\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -128 + 128i$$

60. $\left(\cos 100° + i\sin 100°\right)^6$
$= \cos\left(6\cdot 100°\right) + i\sin\left(6\cdot 100°\right)$
$= \cos 600° + i\sin 600° = \cos 240° + i\sin 240°$
$= -\cos 60° - i\sin 60° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

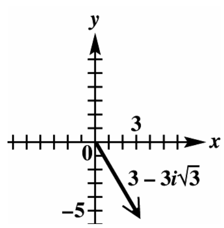
61.



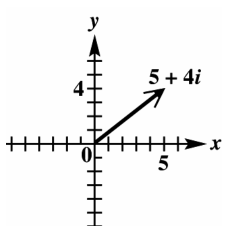
62.



63.



64. The resultant of 7 + 3i and −2 + i is $\left(7+3i\right)+\left(-2+i\right) = 5+4i$.



65. $-2 + 2i$
$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$
Since θ is in quadrant II,
$\tan\theta = \frac{2}{-2} = -1 \Rightarrow \theta = 135°$. Thus,
$-2+2i = 2\sqrt{2}\left(\cos 135° + i\sin 135°\right)$.

66. $3\left(\cos 90° + i\sin 90°\right) = 3\left(0+i\right) = 0+3i$ or $3i$

67. $2\left(\cos 225° + i\sin 225°\right)$
$= 2\left(-\cos 45° - i\sin 45°\right)$
$= 2\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$

68. $-4+4i\sqrt{3}$
$r = \sqrt{(-4)^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16+48} = 8$
Since θ is in quadrant II,
$\tan\theta = \frac{4\sqrt{3}}{-4} = -\sqrt{3} \Rightarrow \theta = 120°$.
Thus, $-4+4i\sqrt{3} = 8\left(\cos 120° + i\sin 120°\right)$.

69. $1-i$
$r = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$ and
$\tan\theta = \frac{-1}{1} = -1 \Rightarrow \theta = 315°$, since θ is in quadrant IV. Thus,
$1 - i = \sqrt{2}\left(\cos 315° + i\sin 315°\right)$.

70. $4\,\text{cis}\,240° = 4\left(\cos 240° + i\sin 240°\right)$
$= 4\left(-\cos 60° - i\sin 60°\right)$
$= 4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -2 - 2i\sqrt{3}$

71. $-4i$
Since $r = 4$ and the point $(0,-4)$ intersects the negative y-axis, $\theta = 270°$ and
$-4i = 4\left(\cos 270° + i\sin 270°\right)$.

72. Since the modulus of z is 2, the graph would be a circle, centered at the origin, with radius 2.

73. $z = x + yi$
Since the imaginary part of z is the negative of the real part of z, we are saying $y = -x$. This is a line.

74. Convert $-2 + 2i$ to polar form.
$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8}$ and
$\theta = \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}\left(-1\right) = 135°$, since θ is in quadrant II. Thus,
$-2+2i = \sqrt{8}\left(\cos 135° + i\sin 135°\right)$.

Since $r^5(\cos 5\alpha + i\sin 5\alpha)$
$= \sqrt{8}(\cos 135° + i\sin 135°)$, then we have
$r^5 = \sqrt{8} \Rightarrow r = \sqrt[10]{8}$ and
$5\alpha = 135° + 360° \cdot k \Rightarrow$
$\alpha = \frac{135° + 360° \cdot k}{5} = 27° + 72° \cdot k$, k any integer. If $k = 0$, then $\alpha = 27° + 0° = 27°$.
If $k = 1$, then $\alpha = 27° + 72° = 99°$.
If $k = 2$, then $\alpha = 27° + 144° = 171°$.
If $k = 3$, then $\alpha = 27° + 216° = 243°$.
If $k = 4$, then $\alpha = 27° + 288° = 315°$.
So, the fifth roots of –2 + 2*i* are
$\sqrt[10]{8}(\cos 27° + i\sin 27°)$,
$\sqrt[10]{8}(\cos 99° + i\sin 99°)$,
$\sqrt[10]{8}(\cos 171° + i\sin 171°)$,
$\sqrt[10]{8}(\cos 243° + i\sin 243°)$, and
$\sqrt[10]{8}(\cos 315° + i\sin 315°)$..

75. Convert 1 – *i* to polar form
$r = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$ and
$\tan\theta = \frac{-1}{1} = -1 \Rightarrow \theta = 315°$, since θ is in quadrant IV. Thus,
$1 - i = \sqrt{2}(\cos 315° + i\sin 315°)$. Since
$r^3(\cos 3\alpha + i\sin 3\alpha)$
$= \sqrt{2}(\cos 315° + i\sin 315°)$, then we have
$r^3 = \sqrt{2} \Rightarrow r = \sqrt[6]{2}$ and
$3\alpha = 315° + 360° \cdot k \Rightarrow$
$\alpha = \frac{315° + 360° \cdot k}{3} = 105° + 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 105° + 0° = 105°$.
If $k = 1$, then $\alpha = 105° + 120° = 225°$.
If $k = 2$, then $\alpha = 105° + 240° = 345°$.
So, the cube roots of 1 – *i* are
$\sqrt[6]{2}(\cos 105° + i\sin 105°)$,
$\sqrt[6]{2}(\cos 225° + i\sin 225°)$, and
$\sqrt[6]{2}(\cos 345° + i\sin 345°)$.

76. The real number –32 has one real fifth root. The one real fifth root is –2, and all other fifth roots are not real.

77. The number –64 has no real sixth roots because a real number raised to the sixth power will never be negative.

78. $x^3 + 125 = 0 \Rightarrow x^3 = -125$
We have $r = 125$ and $\theta = 180°$.
$x^3 = -125 = -125 + 0i$
$= 125(\cos 180° + i\sin 180°)$
Since $r^3(\cos 3\alpha + i\sin 3\alpha)$
$= 125(\cos 180° + i\sin 180°)$, then we have
$r^3 = 125 \Rightarrow r = 5$ and $3\alpha = 180° + 360° \cdot k \Rightarrow$
$\alpha = \frac{180° + 360° \cdot k}{3} = 60° + 120° \cdot k$, k any integer. If $k = 0$, then $\alpha = 60° + 0° = 60°$.
If $k = 1$, then $\alpha = 60° + 120° = 180°$.
If $k = 2$, then $\alpha = 60° + 240° = 300°$.
Solution set:
$\{5(\cos 60° + i\sin 60°), 5(\cos 180° + i\sin 180°), 5(\cos 300° + i\sin 300°)\}$

79. $x^4 + 16 = 0 \Rightarrow x^4 = -16$
We have, $r = 16$ and $\theta = 180°$.
$x^4 = -16 = -16 + 0i = 16(\cos 180° + i\sin 180°)$
Since $r^4(\cos 4\alpha + i\sin 4\alpha)$
$= 16(\cos 180° + i\sin 180°)$, then we have
$r^4 = 16 \Rightarrow r = 2$ and $4\alpha = 180° + 360° \cdot k \Rightarrow$
$\alpha = \frac{180° + 360° \cdot k}{4} = 45° + 90° \cdot k$, k any integer. If $k = 0$, then $\alpha = 45° + 0° = 45°$.
If $k = 1$, then $\alpha = 45° + 90° = 135°$.
If $k = 2$, then $\alpha = 45° + 180° = 225°$.
If $k = 3$, then $\alpha = 45° + 270° = 315°$.
Solution set:
$\{2(\cos 45° + i\sin 45°), 2(\cos 135° + i\sin 135°), 2(\cos 225° + i\sin 225°), 2(\cos 315° + i\sin 315°)\}$

80. $x^2 + i = 0 \Rightarrow x^2 = -i$
We have $r = 1$ and $\theta = 270°$.
$x^2 = -i = 0 - i = 1(\cos 270° + i\sin 270°)$
Since $r^2(\cos 2\alpha + i\sin 2\alpha)$
$= 1(\cos 270° + i\sin 270°)$, then we have
$r^2 = 1 \Rightarrow r = 1$ and $2\alpha = 270° + 360° \cdot k \Rightarrow$
$\alpha = \frac{270° + 360° \cdot k}{2} = 135° + 180° \cdot k$, k any integer. If $k = 0$, then $\alpha = 135° + 0° = 135°$.
If $k = 1$, then $\alpha = 135° + 180° = 315°$.
Solution set: {cos 135° + *i* sin 135°, cos 315° + *i* sin 315°}

81. $\left(-1, \sqrt{3}\right)$

$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$ and

$\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = 120°$, since

θ is in quadrant II. Thus, the polar coordinates are $\left(2, 120°\right)$.

82. $\left(5, 315°\right)$

$x = r\cos\theta \Rightarrow x = 5\cos 315° = 5\left(\frac{\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{2}$

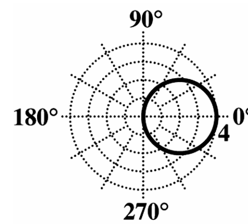
and $y = r\sin\theta \Rightarrow$

$y = 5\sin 315° = 5\left(-\frac{\sqrt{2}}{2}\right) = -\frac{5\sqrt{2}}{2}$. Thus, the

rectangular coordinates are $\left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$.

83. $r = 4\cos\theta$ is a circle.

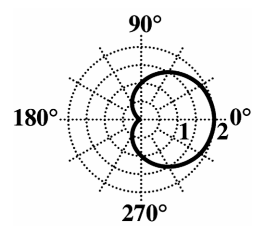
θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
$r = 4\cos\theta$	4	3.5	2.8	2	0	–2	–2.8	–3.5	–4



***r* = 4 cos θ**
Graph is retraced in the interval (180°, 360°).

84. $r = -1 + \cos\theta$ is a cardioid.

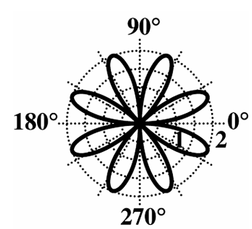
θ	0°	30°	45°	60°	90°	
$r = -1 + \cos\theta$	0	–.7	–.3	–.5	0	
θ	120°	135°	150°	180°	270°	315°
$r = -1 + \cos\theta$	–1.5	–1.7	–1.9	–2	–1	–.3



***r* = –1 + cos θ**

85. $r = 2\sin 4\theta$ is an eight-leaved rose.

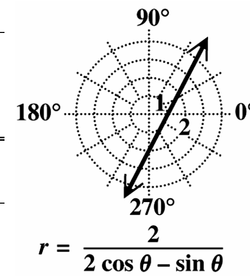
θ	0°	7.5°	15°	22.5°	0°	37.5°	45°
$r = 2\sin 4\theta$	0	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0
θ	52.5°	60°	67.5°	75°	82.5°	90°	52.5°
$r = 2\sin 4\theta$	–1	$-\sqrt{3}$	–2	$-\sqrt{3}$	–1	0	–1



***r* = 2 sin 4θ**
The graph continues to form eight petals for the interval [0°, 360°).

86. Since $r = \dfrac{2}{2\cos\theta - \sin\theta}$, we can use the general form for the polar equation of a line, $r = \dfrac{c}{a\cos\theta + b\sin\theta}$, with $a = 2,\ b = -1,$ and $c = 2,$ we have $2x - y = 2$. The graph is a line with intercepts $(0,-2)$ and $(1,0)$. Constructing a table of values, will also result in the graph.

θ	0°	30°	45°	60°	90°	
$r = \dfrac{2}{2\cos\theta - \sin\theta}$	1	1.6	2.8	15.9	−2	
θ	120°	135°	150°	180°	270°	315°
$r = \dfrac{2}{2\cos\theta - \sin\theta}$	−1.1	−.9	−1.9	−1	2	.9



87. $r = \dfrac{3}{1+\cos\theta}$

$r = \dfrac{3}{1+\cos\theta} \Rightarrow r(1+\cos\theta) = 3 \Rightarrow$

$r + r\cos\theta = 3 \Rightarrow \sqrt{x^2+y^2} + x = 3 \Rightarrow$

$\sqrt{x^2+y^2} = 3 - x$

$x^2 + y^2 = (3-x)^2 \Rightarrow x^2 + y^2 = 9 - 6x + x^2 \Rightarrow$

$y^2 = 9 - 6x \Rightarrow y^2 + 6x - 9 = 0 \Rightarrow y^2 = -6x + 9$

$y^2 = -6\left(x - \dfrac{3}{2}\right)$ or $y^2 + 6x - 9 = 0$

88. $r = \sin\theta + \cos\theta$

$r = \sin\theta + \cos\theta \Rightarrow r^2 = r\sin\theta + r\cos\theta \Rightarrow$

$x^2 + y^2 = x + y$

$x^2 + y^2 - x - y = 0 \Rightarrow (x^2 - x) + (y^2 - y) = 0 \Rightarrow$

$\left(x^2 - x + \dfrac{1}{4}\right) + \left(y^2 - y + \dfrac{1}{4}\right) = \dfrac{1}{4} + \dfrac{1}{4}$

$\left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{2}$ or $x^2 + y^2 - x - y = 0$

89. $r = 2 \Rightarrow \sqrt{x^2+y^2} = 2 \Rightarrow x^2 + y^2 = 4$

90. $y = x \Rightarrow r\sin\theta = r\cos\theta \Rightarrow$
$\sin\theta = \cos\theta$ or $\tan\theta = 1$

91. $y = x^2 \Rightarrow r\sin\theta = r^2\cos^2\theta \Rightarrow$

$\sin\theta = r\cos^2\theta \Rightarrow r = \dfrac{\sin\theta}{\cos^2\theta} \Rightarrow$

$r = \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\cos\theta} = \tan\theta\sec\theta$

$r = \tan\theta\sec\theta$ or $r = \dfrac{\tan\theta}{\cos\theta}$

92. $y = 2$

$y = r\sin\theta \Rightarrow r\sin\theta = 2 \Rightarrow$

$r = \dfrac{2}{\sin\theta}$ or $r = 2\csc\theta$

93. $x = 2$

$x = r\cos\theta \Rightarrow r\cos\theta = 2 \Rightarrow$

$r = \dfrac{2}{\cos\theta}$ or $r = 2\sec\theta$

94. $x^2 + y^2 = 4$

Since $x = r\cos\theta$ and $y = r\sin\theta,$ we have

$(r\cos\theta)^2 + (r\sin\theta)^2 = 4 \Rightarrow$

$r^2\cos\theta^2 + r^2\sin^2\theta = 4 \Rightarrow$

$r^2(\cos\theta^2 + \sin^2\theta) = 4 \Rightarrow r^2 = 4 \Rightarrow$

$r = -2$ or $r = -2$

95. $x + 2y = 4$

Since $x = r\cos\theta$ and $y = r\sin\theta,$ we have

$(r\cos\theta)^2 + (r\sin\theta)^2 = 4 \Rightarrow$

$r^2\cos\theta^2 + r^2\sin^2\theta = 4 \Rightarrow$

$r^2(\cos\theta^2 + \sin^2\theta) = 4 \Rightarrow r^2 = 4 \Rightarrow$

$r = -2$ or $r = -2$

96. To show that the distance between (r_1, θ_1) and (r_2, θ_2) is given by $d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$, we can convert the polar coordinates to rectangular coordinates, apply the distance formula:

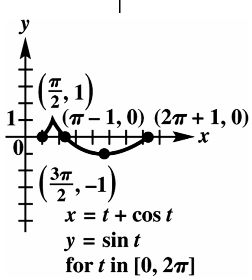
(r_1, θ_1) in rectangular coordinates is $(r_1\cos\theta_1, r_1\sin\theta_1)$.

(r_2, θ_2) in rectangular coordinates is $(r_2\cos\theta_2, r_2\sin\theta_2)$.

$$\begin{aligned}
d &= \sqrt{(r_1\cos\theta_1 - r_2\cos\theta)^2 + (r_1\sin\theta_1 - r_2\sin\theta)^2} \\
&= \sqrt{\left(r_1^2\cos^2\theta_1 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos^2\theta\right) + \left(r_1^2\sin^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin^2\theta\right)} \\
&= \sqrt{r_1^2\left(\cos^2\theta_1 + \sin^2\theta_1\right) - 2r_1r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + r_2^2\left(\cos^2\theta + \sin^2\theta\right)} \\
&= \sqrt{r_1^2\cdot 1 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2\cdot 1} \\
&= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}
\end{aligned}$$

97. $x = t + \cos t,\ y = \sin t$ for t in $[0, 2\pi]$

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$x = t + \cos t$	0	$\frac{\pi}{6} + \frac{\sqrt{3}}{2}$ ≈ 1.4	$\frac{\pi}{3} + \frac{1}{2}$ ≈ 1.5	$\frac{\pi}{2}$ ≈ 1.6	$\frac{3\pi}{4} - \frac{\sqrt{2}}{2}$ ≈ 1.6	$\pi - 1$ ≈ 2.1
$y = \sin t$	0	$\frac{1}{2} = .5$	$\frac{\sqrt{3}}{2} \approx 1.7$	1	$\frac{\sqrt{2}}{2} \approx .7$	0
t	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$x = t + \cos t$	$\frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ ≈ 2.8	$\frac{5\pi}{4} - \frac{\sqrt{2}}{2}$ ≈ 3.2	$\frac{4\pi}{3} - \frac{1}{2}$ ≈ 3.7	$\frac{3\pi}{2}$ ≈ 4.7	$\frac{7\pi}{4} + \frac{\sqrt{2}}{2}$ ≈ 6.2	$2\pi + 1$ ≈ 7.3
$y = \sin t$	$-\frac{1}{2} = -.5$	$-\frac{\sqrt{2}}{2} \approx -.7$	$-\frac{\sqrt{3}}{2} \approx -1.7$	-1	$-\frac{\sqrt{2}}{2} \approx -.7$	0



98. $x = 3t + 2,\ y = t - 1$, for t in $[-5, 5]$

Since $t = y + 1$, substitute $y + 1$ for t in the equation for x: $x = 3(y+1) + 2 \Rightarrow$

$x = 3y + 3 + 2 \Rightarrow x = 3y + 5 \Rightarrow x - 3y = 5$

Since t is in $[-5, 5]$, x is in $[3(-5) + 2, 3(5) + 2]$ or $[-13, 17]$.

99. $x = \sqrt{t-1},\ y = \sqrt{t}$, for t in $[1, \infty)$

Since $x = \sqrt{t-1} \Rightarrow x^2 = t - 1 \Rightarrow t = x^2 + 1$, substitute $x^2 + 1$ for t in the equation for y to obtain $y = \sqrt{x^2 + 1}$. Since t is in $[1, \infty)$, x is in $[\sqrt{1-1}, \infty)$ or $[0, \infty)$.

100. $x = t^2 + 5,\ y = \dfrac{1}{t^2+1}$, for t in $(-\infty, \infty)$

Since $t^2 = x - 5$, substitute $x - 5$ for t^2 in the equation for y: $y = \dfrac{1}{t^2+1} \Rightarrow y = \dfrac{1}{(x-5)+1} \Rightarrow$

$y = \dfrac{1}{x-4}$. Since $x = t^2 + 5$ and $t^2 \ge 0$, $x \ge 0 + 5 = 5$. Therefore, x is in $[5, \infty)$.

101. $x = 5 \tan t,\ y = 3 \sec t$, for t in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Since $\dfrac{x}{5} = \tan t,\ \dfrac{y}{3} = \sec t$, and $1 + \tan^2 t = \sec^2 t$, we have

$$1 + \left(\frac{x}{5}\right)^2 = \left(\frac{y}{3}\right)^2 \Rightarrow 1 + \frac{x^2}{25} = \frac{y^2}{9} \Rightarrow$$

$$9\left(1 + \frac{x^2}{25}\right) = y^2 \Rightarrow y = \sqrt{9\left(1 + \frac{x^2}{25}\right)} \Rightarrow$$

$$y = 3\sqrt{1 + \frac{x^2}{25}}.$$

y is positive since $y = 3 \sec t > 0$ for t in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Since t is in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $x = 5 \tan t$ is undefined at $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, x is in $(-\infty, \infty)$.

102. $x = \cos 2t,\ y = \sin t$ for t in $(-\pi, \pi)$

$\cos 2t = \cos^2 t - \sin^2 t$ (double angle formula)

Since $\cos^2 t + \sin^2 t = 1$, we have

$$\cos^2 t + \sin^2 t = 1 \Rightarrow$$

$$\left(\cos^2 t - \sin^2 t\right) + 2\sin^2 t = 1 \Rightarrow$$

$$x + 2y^2 = 1 \Rightarrow 2y^2 = -x + 1 \Rightarrow 2y^2 = -(x-1)$$

$$y^2 = -\frac{1}{2}(x-1) \text{ or } 2y^2 + x - 1 = 0$$

Since t is in $(-\pi, \pi)$ and $\cos 2t$ is in $[-1, 1]$, x is in $[-1, 1]$.

103. The radius of the circle that has center $(3, 4)$ and passes through the origin is

$$r = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16} = \sqrt{25} = 5$$

Thus, the equation of this circle is $(x-3)^2 + (y-4)^2 = 5^2$. Since

$$\cos^2 t + \sin^2 y = 1 \Rightarrow$$

$$25\cos^2 t + 25\sin^2 t = 25 \Rightarrow$$

$$(5\cos t)^2 + (5\sin t)^2 = 5^2,$$ we can have $5\cos t = x - 3$ and $5\sin t = y - 4$. Thus, a pair of parametric equations can be $x = 3 + 5\cos t$, $y = 4 + 5\sin t$, where t in $[0, 2\pi]$

104. **(a)** Let $z_1 = a + bi$ and its complex conjugate be $z_2 = a - bi$.

$|z_1| = \sqrt{a^2 + b^2}$ and

$$|z_2| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z_1|.$$

(b) Let $z_1 = a + bi$ and $z_2 = a - bi$.

$$\begin{aligned} z_1^2 + z_1 &= (a+bi)^2 + (a+bi) \\ &= \left(a^2 + 2abi + b^2 i^2\right) + (a+bi) \\ &= a^2 + 2abi + b^2(-1) + a + bi \\ &= a^2 - b^2 + 2abi + a + bi \\ &= \left(a^2 - b^2 + a\right) + (2ab + b)i = c + di \end{aligned}$$

$$\begin{aligned} z_2^2 + z_2 &= (a-bi)^2 + (a-bi) \\ &= \left(a^2 - 2abi + b^2 i^2\right) + (a-bi) \\ &= a^2 - 2abi + b^2(-1) + a - bi \\ &= a^2 - b^2 - 2abi + a - bi \\ &= \left(a^2 - b^2 + a\right) - (2ab + b)i = c - di \end{aligned}$$

(c) – (d) The results are again complex conjugates of each other. At each iteration, the resulting values from z_1 and z_2 will always be complex conjugates. Graphically, these represent points that are symmetric with respect to the x-axis, namely points such as (a, b) and $(a, -b)$.

105. **(a)** $x = (v\cos\theta)t \Rightarrow x = (118\cos 27°)t$ and

$$y = (v\sin\theta)t - 16t^2 + h \Rightarrow$$
$$y = (118\sin 27°)t - 16t^2 + 3.2$$

(b) Since $t = \dfrac{x}{118\cos 27°}$, we have

$$y = 118\sin 27° \cdot \frac{x}{118\cos 27°} - 16\left(\frac{x}{118\cos 27°}\right)^2 + 3.2$$
$$= 3.2 - \frac{4}{3481\cos^2 27°}x^2 + (\tan 27°)x$$

(c) Solving $0 = -16t^2 + (118\sin 27°)t + 3.2$ by the quadratic formula, we have

$$t = \frac{-118\sin 27° \pm \sqrt{(118\sin 27°)^2 - 4(-16)(3.2)}}{2(-16)} \Rightarrow$$

$t \approx -.06,\ 3.406$

Discard $t = -0.06$ sec since it is an unacceptable answer. At $t = 3.4$ sec, the baseball traveled

$x = (118\cos 27°)(3.406) \approx 358$ ft .

Chapter 8 Test

1. Find C, given $A = 25.2°$, $a = 6.92$ yd, $b = 4.82$ yd.
Use the law of sines to first find the measure of angle B.

$$\frac{\sin 25.2°}{6.92} = \frac{\sin B}{4.82} \Rightarrow$$
$$\sin B = \frac{4.82\sin 25.2°}{6.92} \approx .29656881 \Rightarrow$$
$$B \approx 17.3°$$

Use the fact that the angles of a triangle sum to 180° to find the measure of angle C.

$C = 180 - A - B = 180° - 25.2° - 17.3° = 137.5°$

Angle C measures 137.5°.

2. Find c, given $C = 118°$, $b = 131$ km, $a = 75$ km.
Using the law of cosines to find the length of c.

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c^2$$
$$= 75^2 + 131^2 - 2(75)(131)\cos 118°$$
$$\approx 32011.12 \Rightarrow c \approx 178.9 \text{ km}$$

c is approximately 179 km. (rounded to two significant digits)

3. Find B, given $a = 17.3$ ft, $b = 22.6$ ft, $c = 29.8$ ft.
Using the law of cosines, find the measure of angle B.

$$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{17.3^2 + 29.8^2 - 22.6^2}{2(17.3)(29.8)}$$
$$\approx .65617605 \Rightarrow B \approx 49.0°$$

B is approximately 49.0°.

4. $a = 14, b = 30, c = 40$
We can use Heron's formula to find the area.

$$s = \tfrac{1}{2}(a+b+c) = \tfrac{1}{2}(14+30+40) = 42$$
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{42(42-14)(42-30)(42-40)}$$
$$= \sqrt{42 \cdot 28 \cdot 12 \cdot 2}$$
$$= \sqrt{28{,}224} = 168 \text{ sq units}$$

5. This is SAS, so we can use the formula

$$A = \frac{1}{2}zy\sin X\ .$$
$$A = \frac{1}{2}\cdot 6 \cdot 12\sin 30° = \frac{1}{2}\cdot 6 \cdot 12 \cdot \frac{1}{2} = 18 \text{ sq units}$$

6. Since $B > 90°$, b must be the longest side of the triangle.

(a) $b > 10$

(b) none

(c) $b \le 10$

7. $A = 60°, b = 30 \text{ m}, c = 45 \text{ m}$
This is SAS, so use the law of cosines to find a: $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$

$$a^2 = 30^2 + 45^2 - 2 \cdot 30 \cdot 45\cos 60° = 1575 \Rightarrow$$
$$a = 15\sqrt{7} \approx 40 \text{ m}$$

Now use the law of sines to find B:

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{30} = \frac{\sin 60°}{15\sqrt{7}} \Rightarrow$$
$$\sin B = \frac{30\sin 60°}{15\sqrt{7}} \Rightarrow B \approx 41°$$
$$C = 180° - A - B = 180° - 60° - 41° = 79°$$

8. $b = 1075$ in., $c = 785$ in., $C = 38°30'$

We can use the law of sines.

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{1075} = \frac{\sin 38°30'}{785} \Rightarrow$$

$$\sin B = \frac{1075 \sin 38°30'}{785} \Rightarrow$$

$B_1 \approx 58.5° = 58°30'$ or

$B_2 = 180° - 58°30' = 121°30'$

Solving separately for triangles A_1B_1C and A_2B_2C, we have the following.

A_1B_1C:

$$A_1 = 180° - B_1 - C = 180° - 58°30' - 38°30' = 83°00'$$

$$\frac{a_1}{\sin A_1} = \frac{b}{\sin B_1} \Rightarrow \frac{a_1}{\sin 83°} = \frac{1075}{\sin 58°30'} \Rightarrow$$

$a_1 = \frac{1075 \sin 83°}{\sin 58°30'} \approx 1251 \approx 1250$ in. (rounded to three significant digits)

A_2B_2C:

$$A_2 = 180° - B_2 - C = 180° - 121°30' - 38°30' = 20°00'$$

$$\frac{a_2}{\sin A_2} = \frac{b}{\sin B_2} \Rightarrow \frac{a_2}{\sin 20°} = \frac{1075}{\sin 121°30'} \Rightarrow$$

$a_2 = \frac{1075 \sin 20°}{\sin 121°30'} \approx 431$ in. (rounded to three significant digits)

9. magnitude:

$$|\mathbf{v}| = \sqrt{(-6)^2 + 8^2} = \sqrt{36+64} = \sqrt{100} = 10$$

angle:

$$\tan\theta' = \frac{y}{x} \Rightarrow$$

$$\tan\theta' = \frac{8}{-6} = -\frac{4}{3} \approx -1.33333333 \Rightarrow$$

$$\theta' \approx -53.1° \Rightarrow \theta = -53.1° + 180° = 126.9°$$

(θ lies in quadrant II)

The magnitude $|\mathbf{v}|$ is 10 and $\theta = 126.9°$.

10. $\mathbf{u} = \langle -1, 3\rangle, \mathbf{v} = \langle 2, -6\rangle$

(a) $\mathbf{u} + \mathbf{v} = \langle -1, 3\rangle + \langle 2, -6\rangle = \langle -1+2, 3+(-6)\rangle = \langle 1, -3\rangle$

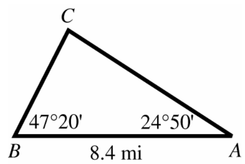
(b) $-3\mathbf{v} = -3\langle 2, -6\rangle = \langle -3\cdot 2, -3(-6)\rangle = \langle -6, 18\rangle$

(c) $\mathbf{u}\cdot\mathbf{v} = \langle -1, 3\rangle \cdot \langle 2, -6\rangle = -1(2) + 3(-6) = -2 - 18 = -20$

(d) $|\mathbf{u}| = \sqrt{(-1)^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$

11. Given $A = 24°\ 50', B = 47°\ 20'$ and $AB = 8.4$ mi, first find the measure of angle C.

$$C = 180° - 47°\ 20' - 24°\ 50' = 179°60' - 72°\ 10' = 107°\ 50'$$

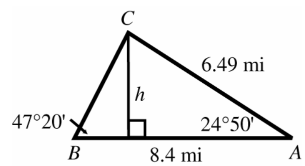


Use this information and the law of sines to find AC.

$$\frac{AC}{\sin 47°\ 20'} = \frac{8.4}{\sin 107°\ 50'} \Rightarrow$$

$$AC = \frac{8.4 \sin 47°\ 20'}{\sin 107°50'} \approx 6.49 \text{ mi}$$

Drop a perpendicular line from C to segment AB.



Thus, $\sin 24°50' = \frac{h}{6.49} \Rightarrow$

$h \approx 6.49 \sin 24°50' \approx 2.7$ mi.

The balloon is 2.7 miles off the ground.

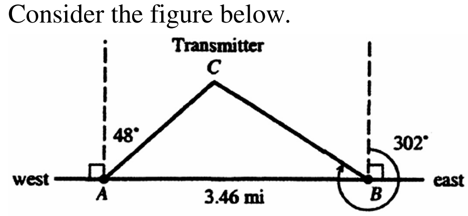
12. horizontal:

$x = |\mathbf{v}|\cos\theta = 569\cos 127.5° \approx -346$ and

vertical: $y = |\mathbf{v}|\sin\theta = 569\sin 127.5° \approx 451$

The vector is $\langle -346, 451\rangle$.

13. Consider the figure below.



Since the bearing is 48° from A, angle A in ABC must be 90° – 48° = 42°. Since the bearing is 302° from B, angle B in ABC must be 302° – 270° = 32°. The angles of a triangle sum to 180°, so

$$C = 180° - A - B = 180° - 42° - 32° = 106°.$$

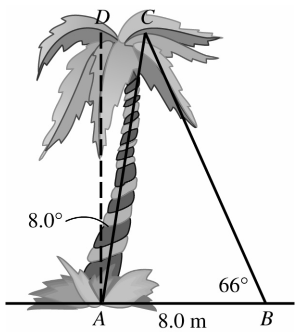
Using the law of sines, we have

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 32°} = \frac{3.46}{\sin 106°} \Rightarrow$$

$$b = \frac{3.46 \sin 32°}{\sin 106°} \approx 1.91 \text{ mi}$$

The distance from A to the transmitter is 1.91 miles. (rounded to two significant digits)

14.

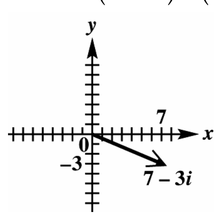


Since $m\angle DAC = 8.0°$, $m\angle CAD = 90° - 8.0° = 82.0°$. $m\angle B = 66°$, so $m\angle C = 180° - 82° - 66° = 32°$. Now use the law of sines to find AC:

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} \Rightarrow \frac{AC}{\sin 66°} = \frac{8.0}{\sin 32°} \Rightarrow$$
$$AC = \frac{8.0 \sin 66°}{\sin 32°} \approx 13.8 \approx 14 \text{ m}$$

15. $w = 2 - 4i, z = 5 + i$

$w + z = (2 - 4i) + (5 + i) = 7 - 3i$



16. **(a)** $3i$

$r = \sqrt{0^2 + 3^2} = \sqrt{0 + 9} = \sqrt{9} = 3$

The point $(0,3)$ is on the positive y-axis, so, $\theta = 90°$. Thus, $3i = 3(\cos 90° + i \sin 90°)$.

(b) $1 + 2i$

$r = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$

Since θ is in quadrant I,

$\theta = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1} 2 \approx 63.43°$. Thus,

$1 + 2i = \sqrt{5}(\cos 63.43° + i \sin 63.43°)$.

(c) $-1 - \sqrt{3}i$

$r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$

Since θ is in quadrant III,

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \tan^{-1}\sqrt{3} = 240°$.

Thus, $-1 - \sqrt{3}i = 2(\cos 240° + i \sin 240°)$

17. **(a)** $3(\cos 30° + i \sin 30°) = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

(b) $4 \text{ cis } 40° = 3.06 + 2.57i$

(c) $3(\cos 90° + i \sin 90°) = 3(0 + 1 \cdot i)$
$= 0 + 3i = 3i$

18. $w = 8(\cos 40° + i \sin 40°)$,
$z = 2(\cos 10° + i \sin 10°)$

(a) wz
$= 8 \cdot 2[\cos(40° + 10°) + i \sin(40° + 10°)]$
$= 16(\cos 50° + i \sin 50°)$

(b) $\frac{w}{z} = \frac{8}{2}[\cos(40° - 10°) + i \sin(40° - 10°)]$
$= 4(\cos 30° + i \sin 30°) = 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= 2\sqrt{3} + 2i$

(c) $z^3 = [2(\cos 10° + i \sin 10°)]^3$
$= 2^3(\cos 3 \cdot 10° + i \sin 3 \cdot 10°)$
$= 8(\cos 30° + i \sin 30°)$
$= 8\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 4\sqrt{3} + 4i$

19. Find all the fourth roots of $-16i = 16(\cos 270° + i \sin 270°)$.

Since $r^4(\cos 4\alpha + i \sin 4\alpha)$ $= 16(\cos 270° + i \sin 270°)$, then we have $r^4 = 16 \Rightarrow r = 2$ and $4\alpha = 270° + 360° \cdot k \Rightarrow$ $\alpha = \frac{270° + 360° \cdot k}{4} = 67.5° + 90° \cdot k$, k any integer. If $k = 0$, then $\alpha = 67.5°$.
If $k = 1$, then $\alpha = 157.5°$.
If $k = 2$, then $\alpha = 247.5°$.
If $k = 3$, then $\alpha = 337.5°$.

The fourth roots of $-16i$ are
$2(\cos 67.5° + \sin 67.5)$,
$2(\cos 157.5° + i \sin 157.5°)$,
$2(\cos 247.5° + i \sin 247.5°)$, and
$2(\cos 337.5° + i \sin 337.5°)$.

20. Answers may vary.

(a) $(0,5)$

$r=\sqrt{0^2+5^2}=\sqrt{0+25}=\sqrt{25}=5$

The point $(0,5)$ is on the positive y-axis. Thus, $\theta=90°$. One possibility is (5, 90°). Alternatively, if $\theta=90°-360°=-270°$, a second possibility is (5, –270°).

(b) $(-2,-2)$

$r=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$

Since θ is in quadrant III,

$\theta=\tan^{-1}\left(\frac{-2}{-2}\right)=\tan^{-1}1=225°$. One possibility is $\left(2\sqrt{2},225°\right)$. Alternatively, if $\theta=225°-360°=-135°$, a second possibility is $\left(2\sqrt{2},-135°\right)$.

21. **(a)** $(3,315°)$

$x=r\cos\theta\Rightarrow$

$x=3\cos 315°=3\cdot\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}$ and

$y=r\sin\theta\Rightarrow$

$y=3\sin 315°=3\left(-\frac{\sqrt{2}}{2}\right)=\frac{-3\sqrt{2}}{2}$

The rectangular coordinates are $\left(\frac{3\sqrt{2}}{2},\frac{-3\sqrt{2}}{2}\right)$.

(b) $(-4,90°)$

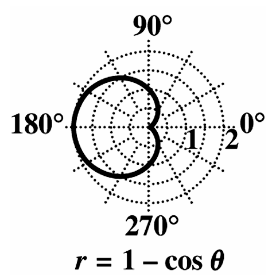
$x=r\cos\theta\Rightarrow x=-4\cos 90°=0$ and

$y=r\sin\theta\Rightarrow y=-4\sin 90°=-4$

The rectangular coordinates are $(0,-4)$.

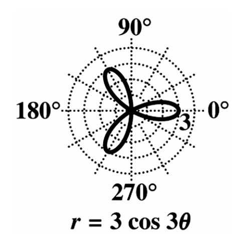
22. $r=1-\cos\theta$ is a cardioid.

θ	0°	30°	45°	60°	90°	135°
$r=1-\cos\theta$	0	.1	.3	.5	1	1.7
θ	180°	225°	270°	315°	360°	
$r=1-\cos\theta$	2	1.7	1	.3	0	



23. $r=3\cos 3\theta$ is a three-leaved rose.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
$r=3\cos 3\theta$	3	0	–2.1	–3	0	3	2.1	0	–3

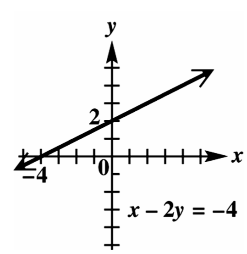


Graph is retraced in the interval (180°, 360°).

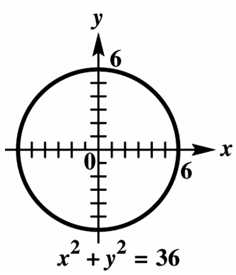
24. **(a)** Since

$r=\frac{4}{2\sin\theta-\cos\theta}=\frac{4}{-1\cdot\cos\theta+2\sin\theta}$,

we can use the general form for the polar equation of a line, $r=\frac{c}{a\cos\theta+b\sin\theta}$, with $a=-1$, $b=2$, and $c=4$, we have $-x+2y=4$ or $x-2y=-4$. The graph is a line with intercepts $(-4,0)$ and $(0,2)$.



(b) $r = 6$ represents the equation of a circle centered at the origin with radius 6, namely $x^2 + y^2 = 36$.

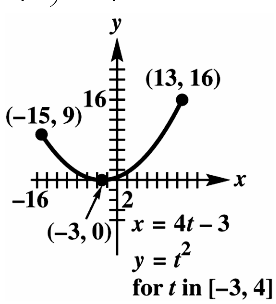


25. $x = 4t - 3,\ y = t^2$ for t in [–3, 4]

t	x	y
–3	–15	9
–1	–7	1
0	–3	0
1	1	1
2	5	4
4	13	16

Since $x = 4t - 3 \Rightarrow t = \dfrac{x+3}{4}$ and $y = t^2$, we have

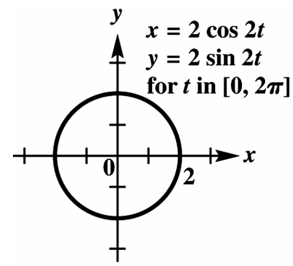
$y = \left(\dfrac{x+3}{4}\right)^2 = \dfrac{1}{4}(x+3)^2$, where x is in [–15, 13]



26. $x = 2\cos 2t,\ y = 2\sin 2t$ for t in $[0, 2\pi]$

t	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$
x	2	$\sqrt{2}$	0	$-\sqrt{2}$	–2	$-\sqrt{2}$
y	0	$\sqrt{2}$	2	$\sqrt{2}$	0	$-\sqrt{2}$

t	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	0	2	0	–2	0	2
y	–2	0	2	0	–2	0



Since $x = 2\cos 2t \Rightarrow \cos 2t = \dfrac{x}{2}$,

$y = 2\sin 2t \Rightarrow \sin 2t = \dfrac{y}{2}$, and

$\cos^2(2t) + \sin^2(2t) = 1$, we have

$\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{4} = 1 \Rightarrow x^2 + y^2 = 4$,

where x is in [–1, 1].

Chapter 8: Quantitative Reasoning

1. We can use the area formula $A = \frac{1}{2} rR\sin B$ for this triangle. By the law of sines, we have

$$\frac{r}{\sin A} = \frac{R}{\sin C} \Rightarrow r = \frac{R\sin A}{\sin C}$$

Since

$\sin C = \sin\left[180° - (A+B)\right] = \sin(A+B),$ we have $r = \frac{R\sin A}{\sin C} \Rightarrow r = \frac{R\sin A}{\sin(A+B)}$

By substituting into our area formula, we have

$$A = \frac{1}{2} rR\sin B \Rightarrow$$

$$A = \frac{1}{2}\left[\frac{R\sin A}{\sin(A+B)}\right] R\sin B \Rightarrow$$

$$A = \frac{1}{2} \cdot \frac{\sin A \sin B}{\sin(A+B)} R^2$$

Since there are a total of 10 stars, the total area covered by the stars is

$$A = 10\left[\frac{1}{2} \cdot \frac{\sin A \sin B}{\sin(A+B)} R^2\right] = \left[5\frac{\sin A \sin B}{\sin(A+B)}\right] R^2$$

2. If $A = 18°$ and $B = 36°$ we have

$$A = \left[\frac{5\sin 18° \sin 36°}{\sin(18+36)°}\right] R^2 \approx 1.12257 R^2$$

3. **(a)** $11.4 \text{ in.} \cdot \frac{10}{13} \text{ in.} \approx 8.77 \text{ in.}^2$

(b) $A = 50\left[5\frac{\sin 18° \sin 36°}{\sin(18° + 36°)}\right] \cdot .308^2 \approx 5.32 \text{ in.}^2$

(c) red

Chapter 9

SYSTEMS AND MATRICES

Section 9.1: Systems of Linear Equations

1. In 2005, the trend "Unadjusted for Weather" shows a higher ozone level than the trend "Adjusted for Weather."

2. The ozone level was approximately .056 ppm.

3. {(1998, .060), (2000, .059), (2001, .058), (2003, .059), (2005, .056)}

4. The graph is increasing from 1997 to 1998, 2000 to 2002, and 2004 to 2005. The graph is decreasing from 1998 to 2000 and 2002 to 2004.

5. t would represent time in years and y would represent the ozone level in ppm.

6. Answers will vary. One answer is that the graphs pass the vertical line test.

7. $4x + 3y = -13$ (1)
$-x + y = 5$ (2)

Solve equation (2) for y.

$-x + y = 5$ (2)
$y = x + 5$ (3)

Replace y with $x + 5$ in equation (1), and solve for x.

$$4x + 3y = -13 \Rightarrow 4x + 3(x+5) = -13 \Rightarrow$$
$$4x + 3x + 15 = -13 \Rightarrow 7x + 15 = -13 \Rightarrow$$
$$7x = -28 \Rightarrow x = -4$$

Replace x with -4 in equation (3) to obtain $y = -4 + 5 = 1$.

Check:

$4x + 3y = -13$ (1)	$-x + y = 5$ (2)
$4(-4) + 3(1) = -13$?	$-(-4) + 1 = 5$?
$-16 + 3 = -13$	$4 + 1 = 5$
$-13 = -13$ True	$5 = 5$ True

Solution set: $\{(-4, 1)\}$

8. $3x + 4y = 4$ (1)
$x - y = 13$ (2)

Solve equation (2) for x.

$x - y = 13$ (2)
$x = y + 13$ (3)

Replace x with $y + 13$ in equation (1), and solve for y.

$$3x + 4y = 4 \Rightarrow 3(y+13) + 4y = 4 \Rightarrow$$
$$3y + 39 + 4y = 4 \Rightarrow 7y + 39 = 4 \Rightarrow$$
$$7y = -35 \Rightarrow y = -5$$

Replace y with -5 in equation (3) to obtain $x = -5 + 13 = 8$.

Check:

$3x + 4y = 4$ (1)	$x - y = 13$ (2)
$3(8) + 4(-5) = 4$?	$8 - (-5) = 13$?
$24 + (-20) = 4$	$8 + 5 = 13$
$4 = 4$ True	$13 = 13$ True

Solution set: $\{(8, -5)\}$

9. $x - 5y = 8$ (1)
$x = 6y$ (2)

Replace x with $6y$ in equation (1), and solve for y: $x - 5y = 8 \Rightarrow 6y - 5y = 8 \Rightarrow y = 8$

Replace y with 8 in equation (2) to obtain $x = 6(8) = 48$.

Check:

$x - 5y = 8$ (1)	$x = 6y$ (2)
$48 - 5(8) = 8$?	$48 = 6(8)$?
$48 - 40 = 8$	$48 = 48$ True
$8 = 8$ True	

Solution set: $\{(48, 8)\}$

10. $6x - y = 5$ (1)
$y = 11x$ (2)

Replace y with $11x$ for y in equation (1).

$$6x - y = 5 \Rightarrow 6x - 11x = 5 \Rightarrow$$
$$-5x = 5 \Rightarrow x = -1$$

Replace x with -1 in equation (2) to obtain $y = 11(-1) = -11$.

Check:

$6x - y = 5$ (1)	$y = 11x$ (2)
$6(-1) - (-11) = 5$?	$-11 = 11(-1)$?
$-6 + 11 = 5$	$-11 = -11$ True
$5 = 5$ True	

Solution set: $\{(-1, -11)\}$

11. $8x - 10y = -22$ (1)
$3x + y = 6$ (2)

Solve equation (2) for y.

$3x + y = 6$ (2)
$y = -3x + 6$ (3)

Replace y with $-3x+6$ in equation (1), and solve for x.

$$\begin{aligned}8x-10y&=-22\\8x-10(-3x+6)&=-22\\8x+30x-60&=-22\\38x-60&=-22\\38x&=38\\x&=1\end{aligned}$$

Replace x with 1 in equation (3) to obtain $y=-3(1)+6=3$.

Check:

$8x-10y=-22$ (1)	$3x+y=6$ (2)
$8(1)-10(3)=-22$?	$3(1)+3=6$?
$8-30=-22$	$3+3=6$
$-22=-22$ True	$6=6$ True

Solution set: $\{(1,3)\}$

12. $4x-5y=-11$ (1)
$2x+y=5$ (2)

Solve equation (2) for y.

$2x+y=5$ (2)
$y=-2x+5$ (3)

Replace y with $-2x+5$ in equation (1), and solve for x.

$$\begin{aligned}4x-5y&=-11\\4x-5(-2x+5)&=-11\\4x+10x-25&=-11\\14x-25&=-11\Rightarrow 14x=14\Rightarrow x=1\end{aligned}$$

Replace x with 1 in equation (3) to obtain $y=-2(1)+5=3$.

Check:

$4x-5y=-11$ (1)	$2x+y=5$ (2)
$4(1)-5(3)=-11$?	$2(1)+3=5$?
$4-15=-11$	$2+3=5$
$-11=-11$ True	$5=5$ True

Solution set: $\{(1,3)\}$

13. $7x-y=-10$ (1)
$3y-x=10$ (2)

Solve equation (1) for y.

$7x-y=-10$ (1)
$y=7x+10$ (3)

Replace y with $7x+10$ in equation (2), and solve for x.

$$\begin{aligned}3y-x=10\Rightarrow 3(7x+10)-x=10\Rightarrow\\21x+30-x=10\Rightarrow 20x+30=10\Rightarrow\\20x=-20\Rightarrow x=-1\end{aligned}$$

Replace x with -1 in equation (3) to obtain $y=7(-1)+10=3$.

Check:

$7x-y=-10$ (1)	$3y-x=10$ (2)
$7(-1)-3=-10$?	$3(3)-(-1)=10$?
$-7-3=-10$	$9+1=10$
$-10=-10$ True	$10=10$ True

Solution set: $\{(-1,3)\}$

14. $4x+5y=7$ (1)
$9y=31+2x$ (2)

Solve equation (2) for y.

$9y=31+2x$ (2)
$y=\frac{31}{9}+\frac{2}{9}x$ (3)

Replace y with $\frac{31}{9}+\frac{2}{9}x$ in equation (1), and solve for x.

$$\begin{aligned}4x+5y&=7\\4x+5\left(\tfrac{31}{9}+\tfrac{2}{9}x\right)&=7\Rightarrow 4x+\tfrac{155}{9}+\tfrac{10}{9}x=7\\9\cdot\left(4x+\tfrac{155}{9}+\tfrac{10}{9}x\right)&=9\cdot 7\\36x+155+10x&=63\\46x&=-92\Rightarrow x=-2\end{aligned}$$

Replace x with -2 in equation (3) to obtain $y=\frac{31}{9}+\frac{2}{9}(-2)=\frac{31-4}{9}=\frac{27}{9}=3$.

Check:

$4x+5y=7$ (1)	$9y=31+2x$ (2)
$4(-2)+5(3)=7$?	$9(3)=31+2(-2)$?
$-8+15=7$	$27=31-4$
$7=7$ True	$27=27$ True

Solution set: $\{(-2,3)\}$

15. $-2x=6y+18$ (1)
$-29=5y-3x$ (2)

Solve equation (1) for x.

$-2x=6y+18$ (1)
$x=-3y-9$ (3)

Replace x with $-3y-9$ in equation (2), and solve for y.

$$\begin{aligned}&-29=5y-3(-3y-9)\\&-29=5y+9y+27\Rightarrow -29=14y+27\Rightarrow\\&-56=14y\Rightarrow -4=y\end{aligned}$$

Replace y with -4 in equation (3) to obtain $x=-3(-4)-9=12-9=3$.

Check: $-2x=6y+18$ (1)

$$\begin{aligned}-2(3)&=6(-4)+18\ ?\\-6&=-24+18\Rightarrow -6=-6\ \text{True}\end{aligned}$$

$-29=5y-3x$ (2)

$$\begin{aligned}-29&=5(-4)-3(3)\ ?\\-29&=-20-9\Rightarrow -29=-29\ \text{True}\end{aligned}$$

Solution set: $\{(3,-4)\}$

16. $3x - 7y = 15\,(1)$
$3x + 7y = 15\,(2)$

Solve equation (1) for *x*.

$3x - 7y = 15 \quad (1)$
$3x = 7y + 15$
$x = \frac{7}{3}y + 5 \quad (3)$

Replace *x* with $\frac{7}{3}y + 5$ in equation (2), and solve for *y*.

$3x + 7y = 15 \Rightarrow 3\left(\frac{7}{3}y + 5\right) + 7y = 15 \Rightarrow$
$7y + 15 + 7y = 15 \Rightarrow 14y + 15 = 15 \Rightarrow$
$14y = 0 \Rightarrow y = 0$

Replace *y* with 0 in equation (3) to obtain $x = \frac{7}{3}(0) + 5 = 5.$

Check:

(1)	(2)
$3x - 7y = 15$ (1)	$3x + 7y = 15$ (2)
$3(5) - 7(0) = 15$?	$3(5) + 7(0) = 15$?
$15 - 0 = 15$	$15 + 0 = 15$
$15 = 15$ True	$15 = 15$ True

Solution set: $\{(5, 0)\}$

17. $3y = 5x + 6 \quad (1)$
$x + y = 2 \quad (2)$

Solve equation (2) for *y*. (You could solve equation (2) for *x* just as easily.)

$x + y = 2 \quad (2)$
$y = 2 - x \quad (3)$

Replace *y* with $2 - x$ in equation (1), and solve for *x*.

$3y = 5x + 6 \Rightarrow 3(2 - x) = 5x + 6 \Rightarrow$
$6 - 3x = 5x + 6 \Rightarrow 6 = 8x + 6 \Rightarrow$
$0 = 8x \Rightarrow 0 = x$

Replace *x* with 0 in equation (3) to obtain $y = 2 - 0 = 2.$

Check:

(1)	(2)
$3y = 5x + 6$ (1)	$x + y = 2$ (2)
$3(2) = 5(0) + 6$?	$0 + 2 = 2$?
$6 = 0 + 6$	$2 = 2$ True
$6 = 6$ True	

Solution set: $\{(0, 2)\}$

18. $4y = 2x - 4 \quad (1)$
$x - y = 4 \quad (2)$

Solve equation (2) for *x*.

$x - y = 4 \quad (2)$
$x = y + 4 \quad (3)$

Replace *x* with $y + 4$ in equation (1), and solve for *y*.

$4y = 2x - 4 \Rightarrow 4y = 2(y + 4) - 4 \Rightarrow$
$4y = 2y + 8 - 4 \Rightarrow 4y = 2y + 4 \Rightarrow$
$2y = 4 \Rightarrow y = 2$

Replace *y* with 2 in equation (3) to obtain $x = 2 + 4 = 6.$

Check:

(1)	(2)
$4y = 2x - 4$ (1)	$x - y = 4$ (2)
$4(2) = 2(6) - 4$?	$6 - 2 = 4$?
$8 = 12 - 4$	$4 = 4$ True
$8 = 8$ True	

Solution set: $\{(6, 2)\}$

19. $3x - y = -4 \quad (1)$
$x + 3y = 12 \quad (2)$

Multiply equation (2) by –3 and add the result to equation (1).

$$\begin{array}{rcrcr} 3x & - & y & = & -4 \\ -3x & - & 9y & = & -36 \\ \hline & & -10y & = & -40 \Rightarrow y = 4 \end{array}$$

Substitute 4 for *y* in equation (2) and solve for *x*: $x + 3(4) = 12 \Rightarrow x + 12 = 12 \Rightarrow x = 0$

Check:

(1)	(2)
$3x - y = -4$ (1)	$x + 3y = 12$ (2)
$3(0) - 4 = -4$?	$0 + 3(4) = 12$?
$0 - 4 = -4$	$0 + 12 = 12$
$-4 = -4$ True	$12 = 12$ True

Solution set: $\{(0, 4)\}$

20. $4x + y = -23 \quad (1)$
$x - 2y = -17 \quad (2)$

Multiply equation (1) by 2 and add the result to equation (2).

$$\begin{array}{rcrcr} 8x & + & 2y & = & -46 \\ x & - & 2y & = & -17 \\ \hline 9x & & & = & -63 \Rightarrow x = -7 \end{array}$$

Substitute -7 for *x* in equation (2) and solve for *x*.

$-7 - 2y = -17 \Rightarrow -2y = -10 \Rightarrow y = 5$

Check:

(1)	(2)
$4x + y = -23$ (1)	$x - 2y = -17$ (2)
$4(-7) + 5 = -23$?	$-7 - 2(5) = -17$?
$-28 + 5 = -23$	$-7 - 10 = -17$
$-23 = -23$ True	$-17 = -17$ True

Solution set: $\{(-7, 5)\}$

21. $2x-3y=-7$ (1)
$5x+4y=17$ (2)
Multiply equation (1) by 4 and equation (2) by 3 and then add the resulting equations.
$$\begin{array}{r} 8x-12y=-28 \\ \underline{15x+12y=\ \ 51} \\ 23x \qquad = \ \ 23 \Rightarrow x=1 \end{array}$$
Substitute 1 for *x* in equation (2) and solve for *y*.
$$5(1)+4y=17 \Rightarrow 5+4y=17 \Rightarrow$$
$$4y=12 \Rightarrow y=3$$
Check:

$2x-3y=-7$ (1)	$5x+4y=17$ (2)
$2(1)-3(3)=-7$?	$5(1)+4(3)=17$?
$2-9=-7$	$5+12=17$
$-7=-7$ True	$17=17$ True

Solution set: $\{(1,3)\}$

22. $4x+3y=-1$ (1)
$2x+5y=3$ (2)
Multiply equation (2) by –2 and add to equation (1).
$$\begin{array}{r} 4x+\ \ 3y=-1 \\ \underline{-4x-10y=-6} \\ -7y=-7 \Rightarrow y=1 \end{array}$$
Substitute 1 for *y* in equation (2) and solve for *x*.
$$2x+5(1)=3 \Rightarrow 2x+5=3 \Rightarrow$$
$$2x=-2 \Rightarrow x=-1$$
Check:

$4x+3y=-1$ (1)	$2x+5y=3$ (2)
$4(-1)+3(1)=-1$?	$2(-1)+5(1)=3$?
$-4+3=-1$	$-2+5=3$
$-1=-1$ True	$3=3$ True

Solution set: $\{(-1,1)\}$

23. $5x+7y=\ 6$ (1)
$10x-3y=46$ (2)
Multiply equation (1) by –2 and add to equation (2).
$$\begin{array}{r} -10x-14y=-12 \\ \underline{10x-\ \ 3y=\ \ 46} \\ -17y=\ \ 34 \Rightarrow y=-2 \end{array}$$
Substitute -2 for *y* in equation (2) and solve for *x*.
$$10x-3(-2)=46 \Rightarrow 10x+6=46$$
$$10x=40 \Rightarrow x=4$$
Check: $5x+7y=6$ (1)
$$5(4)+7(-2)=6 \quad ?$$
$$20-14=6$$
$$6=6 \text{ True}$$
$$10x-3y=46 \quad (2)$$
$$10(4)-3(-2)=46 \quad ?$$
$$40+6=46$$
$$46=46 \text{ True}$$
Solution set: $\{(4,-2)\}$

24. $12x-5y=9$ (1)
$3x-8y=-18$ (2)
Multiply equation (2) by –4 and add to equation (1).
$$\begin{array}{r} 12x-\ \ 5y=\ \ 9 \\ \underline{-12x+32y=72} \\ 27y=81 \Rightarrow y=3 \end{array}$$
Substitute 3 for *y* in equation (2) and solve for *x*.
$$3x-8(3)=-18 \Rightarrow 3x-24=-18$$
$$3x=6 \Rightarrow x=2$$
Check:

$12x-5y=9$ (1)	$3x-8y=-18$ (2)
$12(2)-5(3)=9$?	$3(2)-8(3)=-18$?
$24-15=9$	$6-24=-18$
$9=9$ True	$-18=-18$ True

Solution set: $\{(2,3)\}$

25. $6x+7y+\ 2=0$ (1)
$7x-6y-26=0$ (2)
Multiply equation (1) by 6 and equation (2) by 7 and then add the resulting equations.
$$\begin{array}{r} 36x+42y+\ \ 12=0 \\ \underline{49x-42y-182=0} \\ 85x \qquad -170=0 \Rightarrow 85x=170 \Rightarrow x=2 \end{array}$$
Substitute 2 for *x* in equation (1).
$$6(2)+7y+2=0$$
$$12+7y+2=0$$
$$7y+14=0$$
$$7y=-14 \Rightarrow y=-2$$
Check: $6x+7y+\ 2=0$ (1)
$$6(2)+7(-2)+2=0 \quad ?$$
$$12-14+2=0 \Rightarrow 0=0 \text{ True}$$
$$7x-6y-26=0 \quad (2)$$
$$7(2)-6(-2)-26=0 \quad ?$$
$$14+12-26=0 \Rightarrow 0=0 \text{ True}$$
Solution set: $\{(2,-2)\}$

26. $5x+4y+\ 2=0$ (1)
$4x-5y-23=0$ (2)

Multiply equation (1) by 5 and equation (2) by 4 and then add the resulting equations.

$$\begin{aligned} 25x+20y+10&=0\\ 16x-20y-92&=0\\ \hline 41x\qquad -82&=0\Rightarrow 41x=82\Rightarrow x=2 \end{aligned}$$

Substitute 2 for x in equation (1).

$$\begin{aligned} 5(2)+4y+2&=0\\ 10+4y+2&=0\\ 4y+12&=0\\ 4y&=-12\Rightarrow y=-3 \end{aligned}$$

Check:
$$\begin{aligned} 5x+4y+2&=0 \quad (1)\\ 5(2)+4(-3)+2&=0 \quad ?\\ 10-12+2&=0\\ 0&=0 \text{ True} \end{aligned}$$

$$\begin{aligned} 4x-5y-23&=0 \quad (2)\\ 4(2)-5(-3)-23&=0 \quad ?\\ 8+15-23&=0\\ 0&=0 \text{ True} \end{aligned}$$

Solution set: $\{(2,-3)\}$

27. $\frac{x}{2}+\frac{y}{3}=4$ (1)
$\frac{3x}{2}+\frac{3y}{2}=15$ (2)

To clear denominators, multiply equation (1) by 6 and equation (2) by 2.

$$6\left(\frac{x}{2}+\frac{y}{3}\right)=6(4)$$
$$2\left(\frac{3x}{2}+\frac{3y}{2}\right)=2(15)$$

This gives the system
$3x+2y=24$ (3)
$3x+3y=30$ (4).

Multiply equation (3) by -1 and add to equation (4).

$$\begin{aligned} -3x-2y&=-24\\ 3x+3y&=\ 30\\ \hline y&=\ 6 \end{aligned}$$

Substitute 6 for y in equation (1).

$$\frac{x}{2}+\frac{6}{3}=4\Rightarrow\frac{x}{2}+2=4\Rightarrow\frac{x}{2}=2\Rightarrow x=4$$

Check:
$$\begin{aligned} \frac{x}{2}+\frac{y}{3}&=4 \quad (1)\\ \frac{4}{2}+\frac{6}{3}&=4 \quad ?\\ 2+2&=4\\ 4&=4 \text{ True} \end{aligned}$$

$$\begin{aligned} \frac{3x}{2}+\frac{3y}{2}&=15 \quad (2)\\ \frac{3(4)}{2}+\frac{3(6)}{2}&=15 \quad ?\\ 6+9&=15\\ 15&=15 \text{ True} \end{aligned}$$

Solution set: $\{(4,6)\}$

28. $\frac{3x}{2}+\frac{y}{2}=-2$ (1)
$\frac{x}{2}+\frac{y}{2}=\ 0$ (2)

To clear denominators, multiply equations (1) and (2) by 2.

$$2\left(\frac{3x}{2}+\frac{y}{2}\right)=2(-2)$$
$$2\left(\frac{x}{2}+\frac{y}{2}\right)=2(0)$$

This gives the system
$3x+y=-4$ (3)
$x+y=\ 0$ (4).

Multiply equation (4) by -1 and add to equation (3).

$$\begin{aligned} 3x+y&=-4\\ -x-y&=\ 0\\ \hline 2x\qquad&=-4\Rightarrow x=-2 \end{aligned}$$

Substitute –2 for x into equation (4) and solve for y.

$$-2+y=0\Rightarrow y=2$$

Check:
$$\begin{aligned} \frac{3x}{2}+\frac{y}{2}&=-2 \quad (1)\\ \frac{3(-2)}{2}+\frac{2}{2}&=-2 \quad ?\\ -3+1&=-2\\ -2&=-2 \text{ True} \end{aligned}$$

$$\begin{aligned} \frac{x}{2}+\frac{y}{2}&=0 \quad (2)\\ \frac{-2}{2}+\frac{2}{2}&=0 \quad ?\\ -1+1&=0\\ 0&=0 \text{ True} \end{aligned}$$

Solution set: $\{(-2,2)\}$

29. $\frac{2x-1}{3}+\frac{y+2}{4}=4$ (1)
$\frac{x+3}{2}-\frac{x-y}{3}=3$ (2)

Multiply equation (1) by 12 and equation (2) by 6 to clear denominators. Also, remove parentheses and combine like terms.

$$\begin{aligned} 12\left(\frac{2x-1}{3}+\frac{y+2}{4}\right)&=12(4)\\ 4(2x-1)+3(y+2)&=48\\ 8x-4+3y+6\ &=48\Rightarrow 8x+3y=46 \quad (3)\\ 6\left(\frac{x+3}{2}-\frac{x-y}{3}\right)&=6(3)\\ 3(x+3)-2(x-y)\ &=18\\ 3x+9-2x+2y&=18\Rightarrow x+2y=9 \quad (4) \end{aligned}$$

Multiply equation (4) by –8 and then add the result to equation (3).

$$\begin{aligned} 8x+\ 3y&=\ 46\\ -8x-16y&=-72\\ \hline -13y&=-26\Rightarrow y=2 \end{aligned}$$

Substitute 2 for y into equation (4) and solve for x: $x+2(2)=9\Rightarrow x+4=9\Rightarrow x=5$

Check:

$\frac{2x-1}{3}+\frac{y+2}{4}=4 \quad (1)$

$\frac{2(5)-1}{3}+\frac{2+2}{4}=4 \quad ?$

$\frac{10-1}{3}+\frac{4}{4}=4$

$\frac{9}{3}+1=4$

$3+1=4$

$4=4$ True

$\frac{x+3}{2}-\frac{x-y}{3}=3 \quad (2)$

$\frac{5+3}{2}-\frac{5-2}{3}=3 \quad ?$

$\frac{8}{2}-\frac{3}{3}=3$

$4-1=3$

$3=3$ True

Solution set: $\{(5,2)\}$

30. $\frac{x+6}{5}+\frac{2y-x}{10}=1 \quad (1)$

$\frac{x+2}{4}+\frac{3y+2}{5}=-3 \quad (2)$

Multiply equation (1) by 10 and equation (2) by 20 to clear denominators. Also, remove parentheses and combine like terms.

$$10\left(\frac{x+6}{5}+\frac{2y-x}{10}\right)=10(1)$$
$$2(x+6)+(2y-x)=10$$
$$2x+12+2y-x=10 \Rightarrow x+2y=-2 \quad (3)$$
$$20\left(\frac{x+2}{4}+\frac{3y+2}{5}\right)=20(-3)$$
$$5(x+2)+4(3y+2)=-60$$
$$5x+10+12y+8=-60 \Rightarrow$$
$$5x+12y=-78 \quad (4)$$

Multiply equation (3) by –5 and then add the result to equation (4).

$$\begin{array}{r} -5x-10y=\ \ 10 \\ 5x+12y=-78 \\ \hline 2y=-68 \Rightarrow y=-34 \end{array}$$

Substitute –34 for y in equation (3) and solve for x.

$x+2(-34)=-2 \Rightarrow x-68=-2 \Rightarrow x=66$

Check: $\frac{x+6}{5}+\frac{2y-x}{10}=1 \quad (1)$

$\frac{66+6}{5}+\frac{2(-34)-66}{10}=1 \quad ?$

$\frac{72}{5}+\frac{-68-66}{10}=1$

$\frac{144}{10}+\frac{-134}{10}=1$

$\frac{10}{10}=1$

$1=1$ True

$\frac{x+2}{4}+\frac{3y+2}{5}=-3 \quad (2)$

$\frac{66+2}{4}+\frac{3(-34)+2}{5}=-3 \quad ?$

$\frac{68}{4}+\frac{-102+2}{5}=-3$

$17+\frac{-100}{5}=-3$

$17-20=-3$

$-3=-3$ True

Solution set: $\{(66,-34)\}$

31. $9x-5y=1 \ (1)$

$-18x+10y=1 \ (2)$

Multiply equation (1) by 2 and add the result to equation (2).

$$\begin{array}{r} 18x-10y=2 \\ -18x+10y=1 \\ \hline 0=3 \end{array}$$

This is a false statement. The solution set is $\varnothing$, and the system is inconsistent.

32. $3x+2y=5 \ (1)$

$6x+4y=8 \ (2)$

Multiply equation (1) by –2 and add the resulting equations.

$$\begin{array}{r} -6x-4y=-10 \\ 6x+4y=\ \ 8 \\ \hline 0=\ -2 \end{array}$$

This is a false statement. The solution set is $\varnothing$, and the system is inconsistent.

33. $4x-y=9 \ (1)$

$-8x+2y=-18 \ (2)$

Multiply equation (1) by 2 and add the result to equation (2).

$$\begin{array}{r} 8x-2y=\ \ 18 \\ -8x+2y=-18 \\ \hline 0=\ \ 0 \end{array}$$

This is a true statement. There are infinitely many solutions. We will now express the solution set with y as the arbitrary variable. Solve equation (1) for x.

$4x-y=9 \Rightarrow 4x=y+9 \Rightarrow x=\frac{y+9}{4}$

Solution set: $\left\{\left(\frac{y+9}{4},\ y\right)\right\}$

34. $3x+5y=-2 \ (1)$

$9x+15y=-6 \ (2)$

Multiply equation (1) by –3 and add the result to equation (2).

$$\begin{array}{r} -9x-15y=\ \ 6 \\ 9x+15y=-6 \\ \hline 0=\ \ 0 \end{array}$$

This is a true statement. There are infinitely many solutions. We will now express the solution set with y as the arbitrary variable. Solve equation (1) for x.

$3x+5y=-2 \Rightarrow 3x=-5y-2 \Rightarrow x=\frac{-5y-2}{3}$

Solution set: $\left\{\left(\frac{-5y-2}{3},\ y\right)\right\}$

35. $5x - 5y - 3 = 0$ (1)
$x - y - 12 = 0$ (2)

Multiply equation (2) by –5 and then add the resulting equations.

$$\begin{array}{r} 5x - 5y - 3 = 0 \\ \underline{-5x + 5y + 60 = 0} \\ 57 = 0 \end{array}$$

This is a false statement. The solution set is $\varnothing$, and the system is inconsistent.

36. $2x - 3y - 7 = 0$ (1)
$-4x + 6y - 14 = 0$ (2)

Multiply equation (1) by 2 and then add the resulting equations.

$$\begin{array}{r} 4x - 6y - 14 = 0 \\ \underline{-4x + 6y - 14 = 0} \\ -28 = 0 \end{array}$$

This is a false statement. The solution set is $\varnothing$, and the system is inconsistent.

37. $7x + 2y = 6$ (1)
$14x + 4y = 12$ (2)

Multiply equation (1) by –2 and add the result to equation (2).

$$\begin{array}{r} -14x - 4y = -12 \\ \underline{14x + 4y = 12} \\ 0 = 0 \end{array}$$

This is a true statement. There are infinitely many solutions. We will express the solution set with *y* as the arbitrary variable. Solve equation (1) for *x*.

$7x + 2y = 6 \Rightarrow 7x = 6 - 2y \Rightarrow x = \frac{6-2y}{7}$

Solution set: $\left\{\left(\frac{6-2y}{7},\ y\right)\right\}$

38. $2x - 8y = 4$ (1)
$x - 4y = 2$ (2)

Multiply equation (2) by –2 and then add the result to equation (1).

$$\begin{array}{r} 2x - 8y = 4 \\ \underline{-2x + 8y = -4} \\ 0 = 0 \end{array}$$

This is a true statement. There are infinitely many solutions. We will now express the solution set with *y* as the arbitrary variable. Solve equation (2) for *x*.

$x - 4y = 2 \Rightarrow x = 2 + 4y$

Solution set: $\{(2 + 4y,\ y)\}$

39. $4x - 5y = -11$
$-5y = -4x - 11$
$y = \frac{4}{5}x + \frac{11}{5}$
$2x + y = 5$
$y = -2x + 5$

Screen A is the correct choice.

40. Since $y = ax + b$ and one line passes through (0, 3) and (3, 0), we have the equations $3 = a(0) + b$ and $0 = a(3) + b.$

This becomes the following system.

$b = 3$ (1)
$3a + b = 0$ (2)

Substitute $b = 3$ into equation (2) to solve for a: $3a + 3 = 0 \Rightarrow 3a = -3 \Rightarrow a = -1$

The equation is $y = -x + 3 \Rightarrow x + y = 3$

The other line passes through (0, –2) and (2, 0), so we have the equations $-2 = a(0) + b$ and $0 = a(2) + b.$

This becomes the following system.

$b = -2$ (3)
$2a + b = 0$ (4)

Substitute $b = -2$ into equation (4) to solve for a: $2a - 2 = 0 \Rightarrow 2a = 2 \Rightarrow a = 1$

Thus, the equation is $y = x - 2 \Rightarrow x - y = 2$

We can verify that $\left(\frac{5}{2}, \frac{1}{2}\right)$ satisfies the equations by finding the solution of the system

$x + y = 3$ (5)
$x - y = 2$ (6)

Add the two equations and solve for *x*:

$$\begin{array}{l} x + y = 3 \\ \underline{x - y = 2} \\ 2x = 5 \Rightarrow x = \frac{5}{2} \end{array}$$

Substitute $x = \frac{5}{2}$ into equation (5) and solve for *y*: $\frac{5}{2} + y = 3 \Rightarrow y = x = \frac{1}{2}$.

Therefore, the equations of the two lines are $x + y = 3$ and $x - y = 2$

41. Since $y = ax + b$ and the line passes through (2, 0) and (0, 3), we have the equations $0 = a(2) + b$ and $3 = a(0) + b$

These becomes the following system.

$2a + b = 0$ (1)
$b = 3$ (2)

Substitute $b = 3$ into equation (1) to solve for a: $2a + 3 = 0 \Rightarrow 2a = -3 \Rightarrow a = -\frac{3}{2}$. Thus, the equation is $y = -\frac{3}{2}x + 3 \Rightarrow 3x + 2y = 6$.

The other line goes through the points (0, 1) and (−3, 0), so we have the equations $1 = a(0) + b$ and $0 = a(-3) + b$. These become the following system:

$$b = 1 \quad (3)$$
$$-3a + b = 0 \quad (4)$$

Substitute $b = 1$ into equation (4), then solve for a:

$$-3a + 1 = 0 \Rightarrow a = \tfrac{1}{3}$$

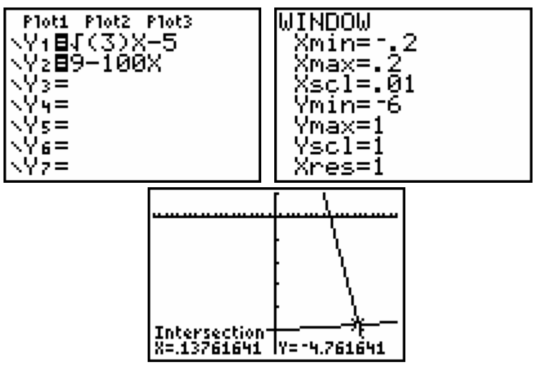
Thus, the equation of this line is

$$y = \tfrac{1}{3}x + 1 \Rightarrow 3y = x + 3 \Rightarrow x - 3y = -3$$

42. Solve each equation for y

$$\sqrt{3}x - y = 5 \Rightarrow -y = -\sqrt{3}x + 5 \Rightarrow y = \sqrt{3}x - 5$$

and $100x + y = 9 \Rightarrow y = 9 - 100x$

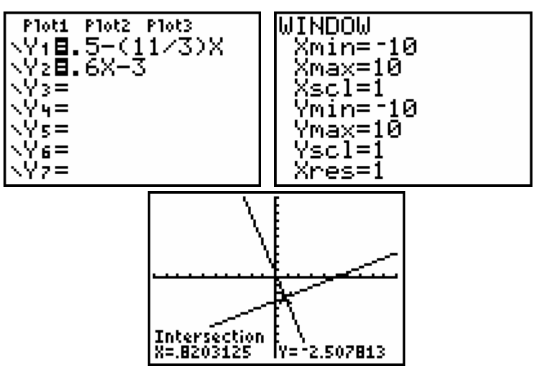


Solution set: $\{(.138, -4.762)\}$

43. Solve each equation for y.

$\frac{11}{3}x + y = .5 \Rightarrow y = .5 - \frac{11}{3}x$ and

$.6x - y = 3 \Rightarrow -y = 3 - .6x \Rightarrow y = .6x - 3$



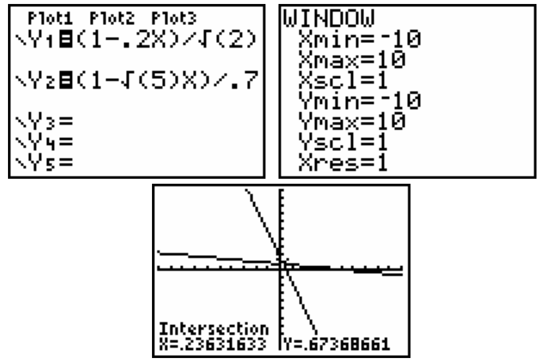
Solution set: $\{(.820, -2.508)\}$

44. Solve each equation for y.

$$.2x + \sqrt{2}y = 1 \Rightarrow \sqrt{2}y = 1 - .2x \Rightarrow y = \tfrac{1-.2x}{\sqrt{2}}$$

and

$$\sqrt{5}x + .7y = 1 \Rightarrow .7y = 1 - \sqrt{5}x \Rightarrow y = \tfrac{1-\sqrt{5}x}{.7}$$

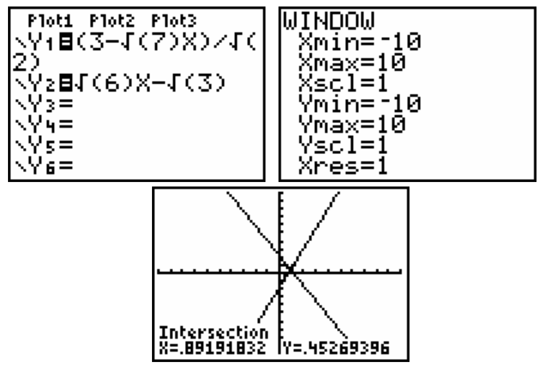


Solution set: $\{(.236, .674)\}$

45. Solve each equation for y.

$$\sqrt{7}x + \sqrt{2}y - 3 = 0 \Rightarrow \sqrt{2}y = 3 - \sqrt{7}x \Rightarrow y = \tfrac{3-\sqrt{7}x}{\sqrt{2}}$$

and $\sqrt{6}x - y - \sqrt{3} = 0 \Rightarrow y = \sqrt{6}x - \sqrt{3}$



Solution set: $\{(.892, .453)\}$

46.
$$x - 2y = 3 \quad (1)$$
$$-2x + 4y = k \quad (2)$$

Multiply equation (1) by 2 and add the result to equation (2).

$$\begin{array}{r} 2x - 4y = 6 \\ \underline{-2x + 4y = k} \\ 0 = 6 + k \end{array}$$

The system will have no solution when $6 + k \neq 0 \Rightarrow k \neq -6.$

The system will have infinitely many solutions when $6 + k = 0 \Rightarrow k = -6.$

47. $x+y+z=2$ (1)
$2x+y-z=5$ (2)
$x-y+z=-2$ (3)

Eliminate z by adding equations (1) and (2) to get $3x+2y=7$ (4).

Eliminate z by adding equations (2) and (3) to get $3x=3 \Rightarrow x=1$ (5).

Using $x=1$, find y from equation (4) by substitution.

$3(1)+2y=7 \Rightarrow 3+2y=7 \Rightarrow 2y=4 \Rightarrow y=2$

Substitute 1 for x and 2 for y in equation (1) to find z.

$1+2+z=2 \Rightarrow 3+z=2 \Rightarrow z=-1$

Verify that the ordered triple $(1,2,-1)$ satisfies all three equations.

Check:

$x+y+z=2$ (1)
$1+2+(-1)=2$?
$2=2$ True

$2x+y-z=5$ (2)
$2(1)+2-(-1)=5$?
$2+2+1=5$
$5=5$ True

$x-y+z=-2$ (3)
$1-2+(-1)=-2$?
$-1-1=-2$
$-2=-2$ True

Solution set: $\{(1,2,-1)\}$

48. $2x+y+z=9$ (1)
$-x-y+z=1$ (2)
$3x-y+z=9$ (3)

Eliminate y by adding equations (1) and (2) to get $x+2z=10$ (4).

Eliminate y by adding equations (1) and (3) to get $5x+2z=18$ (5).

Use the system formed by equations (4) and (5) to find the value for x. Multiply equation (5) by -1 and add to equation (4).

$$\begin{array}{l} x+2z=\ \ 10 \\ \underline{-5x-2z=-18} \\ -4x\qquad = -8 \Rightarrow x=2 \end{array}$$

Using $x=2$, find z from equation (4) by substitution.

$2+2z=10 \Rightarrow 2z=8 \Rightarrow z=4$

Substitute 2 for x and 4 for z in equation (1) to find y.

$2(2)+y+4=9 \Rightarrow 4+y+4=9 \Rightarrow$
$y+8=9 \Rightarrow y=1$

Verify that the ordered triple $(2,1,4)$ satisfies all three equations.

Check:

$2x+y+z=9$ (1)
$2(2)+1+4=9$?
$4+1+4=9$
$9=9$ True

$-x-y+z=1$ (2)
$-2-1+4=1$?
$1=1$ True

$3x-y+z=9$ (3)
$3(2)-1+4=9$?
$6-1+4=9$
$9=9$ True

Solution set: $\{(2,1,4)\}$

49. $x+3y+4z=14$ (1)
$2x-3y+2z=10$ (2)
$3x-y+z=9$ (3)

Eliminate y by adding equations (1) and (2) to get $3x+6z=24$ (4).

Multiply equation (3) by 3 and add the result to equation (1).

$$\begin{array}{l} x+3y+4z=14 \\ \underline{9x-3y+3z=27} \\ 10x+\qquad 7z=41 \ (5) \end{array}$$

Multiply equation (4) by 10 and equation (5) by –3 and add in order to eliminate y.

$$\begin{array}{l} 30x+60z=\ \ 240 \\ \underline{-30x-21z=-123} \\ \qquad 39z=\ \ 117 \Rightarrow z=3 \end{array}$$

Using $z=3$, find x from equation (5) by substitution.

$10x+7(3)=41 \Rightarrow 10x+21=41 \Rightarrow$
$10x=20 \Rightarrow x=2$

Substitute 2 for x and 3 for z in equation (1) to find y.

$2+3y+4(3)=14 \Rightarrow 3y=0 \Rightarrow y=0$

Verify that the ordered triple $(2,0,3)$ satisfies all three equations.

Check:

$x+3y+4z=14$ (1)
$2+3(0)+4(3)=14$?
$2+0+12=14$
$14=14$ True

$2x-3y+2z=10$ (2)
$2(2)-3(0)+2(3)=10$?
$4-0+6=10$
$10=10$ True

$3x-y+z=9$ (3)
$3(2)-0+3=9$?
$6-0+3=9$
$9=9$ True

Solution set: $\{(2,0,3)\}$

50. $4x - y + 3z = -2$ (1)
$3x + 5y - z = 15$ (2)
$-2x + y + 4z = 14$ (3)

Eliminate y by adding equations (1) and (3) to get $2x + 7z = 12$ (4).

Multiply equation (1) by 5 and add the result to equation (2).

$$\begin{array}{l} 20x - 5y + 15z = -10 \\ \underline{3x + 5y - z = 15} \\ 23x + 14z = 5 \quad (5) \end{array}$$

Multiply equation (4) by -2 and then add the result to equation (5) in order to eliminate y.

$$\begin{array}{l} -4x - 14z = -24 \\ \underline{23x + 14z = 5} \\ 19x = -19 \Rightarrow x = -1 \end{array}$$

Using $x = -1$, find z from equation (4) by substitution.

$2(-1) + 7z = 12 \Rightarrow 7z = 14 \Rightarrow z = 2$

Substitute -1 for x and 2 for z in equation (3) to find y.

$-2(-1) + y + 4(2) = 14 \Rightarrow 2 + y + 8 = 14 \Rightarrow$
$10 + y = 14 \Rightarrow y = 4$

Verify that the ordered triple $(-1, 4, 2)$ satisfies all three equations.

Check:
$$\begin{array}{r} 4x - y + 3z = -2 \quad (1) \\ 4(-1) - 4 + 3(2) = -2 \quad ? \\ -4 - 4 + 6 = -2 \\ -2 = -2 \text{ True} \end{array}$$

$$\begin{array}{r} 3x + 5y - z = 15 \quad (2) \\ 3(-1) + 5(4) - 2 = 15 \quad ? \\ -3 + 20 - 2 = 15 \\ 15 = 15 \text{ True} \end{array}$$

$$\begin{array}{r} -2x + y + 4z = 14 \quad (3) \\ -2(-1) + 4 + 4(2) = 14 \quad ? \\ 2 + 4 + 8 = 14 \\ 14 = 14 \text{ True} \end{array}$$

Solution set: $\{(-1, 4, 2)\}$

51. $x + 4y - z = 6$ (1)
$2x - y + z = 3$ (2)
$3x + 2y + 3z = 16$ (3)

Eliminate z by adding equations (1) and (2) to get $3x + 3y = 9$ or $x + y = 3$ (4).

Multiply equation (1) by 3 and add the result to equation (3).

$$\begin{array}{l} 3x + 12y - 3z = 18 \\ \underline{3x + 2y + 3z = 16} \\ 6x + 14y = 34 \quad (5) \end{array}$$

Multiply equation (4) by -6 and then add the result to equation (5) in order to eliminate x.

$$\begin{array}{l} -6x - 6y = -18 \\ \underline{6x + 14y = 34} \\ 8y = 16 \Rightarrow y = 2 \end{array}$$

Using $y = 2$, find x from equation (4) by substitution: $x + 2 = 3 \Rightarrow x = 1$

Substitute 1 for x and 2 for y in equation (1) to find z.

$1 + 4(2) - z = 6 \Rightarrow 1 + 8 - z = 6 \Rightarrow$
$9 - z = 6 \Rightarrow -z = -3 \Rightarrow z = 3$

Verify that the ordered triple $(1, 2, 3)$ satisfies all three equations.

Check:

$x + 4y - z = 6$ (1)	$2x - y + z = 3$ (2)
$1 + 4(2) - 3 = 6$?	$2(1) - 2 + 3 = 3$?
$1 + 8 - 3 = 6$	$2 - 2 + 3 = 3$
$6 = 6$ True	$3 = 3$ True

$$\begin{array}{r} 3x + 2y + 3z = 16 \quad (3) \\ 3(1) + 2(2) + 3(3) = 16 \quad ? \\ 3 + 4 + 9 = 16 \\ 16 = 16 \text{ True} \end{array}$$

Solution set: $\{(1, 2, 3)\}$

52. $4x - 3y + z = 9$ (1)
$3x + 2y - 2z = 4$ (2)
$x - y + 3z = 5$ (3)

Eliminate x by multiplying equation (3) by -4 and add to equation (1).

$$\begin{array}{l} 4x - 3y + z = 9 \\ \underline{-4x + 4y - 12z = -20} \\ y - 11z = -11 \quad (4) \end{array}$$

Eliminate x by multiplying equation (3) by -3 and add to equation (2).

$$\begin{array}{l} 3x + 2y - 2z = 4 \\ \underline{-3x + 3y - 9z = -15} \\ 5y - 11z = -11 \quad (5) \end{array}$$

Multiply equation (4) by -1 and add to equation (5) in order to eliminate z.

$$\begin{array}{l} -y + 11z = 11 \\ \underline{5y - 11z = -11} \\ 4y = 0 \Rightarrow y = 0 \end{array}$$

Using $y = 0$, find z from equation (4) by substitution.

$0 - 11z = -11 \Rightarrow -11z = -11 \Rightarrow z = 1$

Substitute 0 for y and 1 for z in equation (3) to find x.

$x - 0 + 3(1) = 5 \Rightarrow x + 3 = 5 \Rightarrow x = 2$

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Verify that the ordered triple $(2,0,1)$ satisfies all three equations.

Check: $4x-3y+z=9$ (1)

$$4(2)-3(0)+1=9 \quad ?$$
$$8-0+1=9$$
$$9=9 \text{ True}$$

$$3x+2y-2z=4 \quad (2)$$
$$3(2)+2(0)-2(1)=4 \quad ?$$
$$6+0-2=4$$
$$4=4 \text{ True}$$

$$x-y+3z=5 \quad (3)$$
$$2-0+3(1)=5 \quad ?$$
$$2-0+3=5$$
$$5=5 \text{ True}$$

Solution set: $\{(2,0,1)\}$

53. $x-3y-2z=-3$ (1)
$3x+2y-z=12$ (2)
$-x-y+4z=3$ (3)

Eliminate x by adding equations (1) and (3) to get $-4y+2z=0$ (4).

Eliminate x by multiplying equation (3) by 3 and add to equation (2).

$$\begin{array}{r} -3x-3y+12z=9 \\ 3x+2y-z=12 \\ \hline -y+11z=21 \quad (5) \end{array}$$

Multiply equation (5) by -4 and add to equation (4) in order to eliminate y.

$$\begin{array}{r} -4y+2z=0 \\ 4y-44z=-84 \\ \hline -42z=-84 \Rightarrow z=2 \end{array}$$

Using $z=2$, find y from equation (4) by substitution.

$$-4y+2(2)=0 \Rightarrow -4y+4=0 \Rightarrow$$
$$-4y=-4 \Rightarrow y=1$$

Substitute 1 for y and 2 for z in equation (1) to find x.

$$x-3(1)-2(2)=-3 \Rightarrow x-3-4=-3 \Rightarrow$$
$$x-7=-3 \Rightarrow x=4$$

Verify that the ordered triple $(4,1,2)$ satisfies all three equations.

Check: $x-3y-2z=-3$ (1)

$$4-3(1)-2(2)=-3 \quad ?$$
$$4-3-4=-3$$
$$-3=-3 \text{ True}$$

$$3x+2y-z=12 \quad (2)$$
$$3(4)+2(1)-2=12 \quad ?$$
$$12+2-2=12$$
$$12=12 \text{ True}$$

$$-x-y+4z=3 \quad (3)$$
$$-4-1+4(2)=3 \quad ?$$
$$-4-1+8=3$$
$$3=3 \text{ True}$$

Solution set: $\{(4,1,2)\}$

54. $x+y+z=3$ (1)
$3x-3y-4z=-1$ (2)
$x+y+3z=11$ (3)

Eliminate x by multiplying equation (1) by -3 and add to equation (2).

$$\begin{array}{r} -3x-3y-3z=-9 \\ 3x-3y-4z=-1 \\ \hline -6y-7z=-10 \quad (4) \end{array}$$

Eliminate x by multiplying equation (3) by -1 and add to equation (1).

$$\begin{array}{r} x+y+z=3 \\ -x-y-3z=-11 \\ \hline -2z=-8 \Rightarrow z=4 \end{array}$$

Using $z=4$, find y from equation (4) by substitution.

$$-6y-7(4)=-10 \Rightarrow -6y-28=-10 \Rightarrow$$
$$-6y=18 \Rightarrow y=-3$$

Substitute -3 for y and 4 for z in equation (1) to find x.

$$x+(-3)+4=3 \Rightarrow x+1=3 \Rightarrow x=2$$

Verify that the ordered triple $(2,-3,4)$ satisfies all three equations.

Check: $x+y+z=3$ (1)

$$2+(-3)+4=3 \quad ?$$
$$3=3 \text{ True}$$

$$3x-3y-4z=-1 \quad (2)$$
$$3(2)-3(-3)-4(4)=-1 \quad ?$$
$$6+9-16=-1$$
$$-1=-1 \text{ True}$$

$$x+y+3z=11 \quad (3)$$
$$2+(-3)+3(4)=11 \quad ?$$
$$2-3+12=11$$
$$11=11 \text{ True}$$

Solution set: $\{(2,-3,4)\}$

55. $2x + 6y - z = 6$ (1)
$4x - 3y + 5z = -5$ (2)
$6x + 9y - 2z = 11$ (3)

Eliminate y by multiplying equation (2) by 2 and add to equation (1).

$$\begin{array}{rl} 2x + 6y - z = & 6 \\ 8x - 6y + 10z = & -10 \\ \hline 10x + 9z = & -4 \quad (4) \end{array}$$

Eliminate y by multiplying equation (2) by 3 and add to equation (3).

$$\begin{array}{rl} 6x + 9y - 2z = & 11 \\ 12x - 9y + 15z = & -15 \\ \hline 18x + 13z = & -4 \quad (5) \end{array}$$

Multiply equation (4) by 9 and equation (5) by -5 and add in order to eliminate x.

$$\begin{array}{rl} 90x + 81z = & -36 \\ -90x - 65z = & 20 \\ \hline 16z = & -16 \Rightarrow z = -1 \end{array}$$

Using $z = -1$, find x from equation (4) by substitution.

$$10x + 9(-1) = -4 \Rightarrow 10x - 9 = -4$$
$$10x = 5 \Rightarrow x = \frac{1}{2}$$

Substitute $\frac{1}{2}$ for x and -1 for z in equation (1) to find y.

$$2\left(\frac{1}{2}\right) + 6y - (-1) = 6 \Rightarrow 1 + 6y + 1 = 6 \Rightarrow$$
$$6y + 2 = 6 \Rightarrow 6y = 4 \Rightarrow y = \frac{2}{3}$$

Verify that the ordered triple $\left(\frac{1}{2}, \frac{2}{3}, -1\right)$ satisfies all three equations.

Check:

$$2x + 6y - z = 6 \quad (1)$$
$$2\left(\frac{1}{2}\right) + 6\left(\frac{2}{3}\right) - (-1) = 6 \quad ?$$
$$1 + 4 + 1 = 6$$
$$6 = 6 \text{ True}$$

$$4x - 3y + 5z = -5 \quad (2)$$
$$4\left(\frac{1}{2}\right) - 3\left(\frac{2}{3}\right) + 5(-1) = -5 \quad ?$$
$$2 - 2 - 5 = -5$$
$$-5 = -5 \text{ True}$$

$$6x + 9y - 2z = 11 \quad (3)$$
$$6\left(\frac{1}{2}\right) + 9\left(\frac{2}{3}\right) - 2(-1) = 11 \quad ?$$
$$3 + 6 + 2 = 11$$
$$11 = 11 \text{ True}$$

Solution set: $\left\{\left(\frac{1}{2}, \frac{2}{3}, -1\right)\right\}$

56. $8x - 3y + 6z = -2$ (1)
$4x + 9y + 4z = 18$ (2)
$12x - 3y + 8z = -2$ (3)

Eliminate y by multiplying equation (1) by 3 and add to equation (2).

$$\begin{array}{rl} 24x - 9y + 18z = & -6 \\ 4x + 9y + 4z = & 18 \\ \hline 28x + 22z = & 12 \Rightarrow 14x + 11z = 6 \quad (4) \end{array}$$

Eliminate y by multiplying equation (3) by -1 and add to equation (1).

$$\begin{array}{rl} 8x - 3y + 6z = & -2 \\ -12x + 3y - 8z = & 2 \\ \hline -4x - 2z = & 0 \Rightarrow -2x - z = 0 \quad (5) \end{array}$$

Multiply equation (5) by 7 and add to equation (4) in order to eliminate x.

$$\begin{array}{rl} 14x + 11z = & 6 \\ -14x - 7z = & 0 \\ \hline 4z = & 6 \Rightarrow z = \frac{3}{2} \end{array}$$

Using $z = \frac{3}{2}$, find x from equation (5) by substitution.

$$-2x - \frac{3}{2} = 0 \Rightarrow -2x = \frac{3}{2} \Rightarrow x = -\frac{3}{4}$$

Substitute $-\frac{3}{4}$ for x and $\frac{3}{2}$ for z in equation (1) to find y.

$$8\left(-\frac{3}{4}\right) - 3y + 6\left(\frac{3}{2}\right) = -2 \Rightarrow -6 - 3y + 9 = -2 \Rightarrow$$
$$3 - 3y = -2 \Rightarrow -3y = -5 \Rightarrow y = \frac{5}{3}$$

Verify that the ordered triple $\left(-\frac{3}{4}, \frac{5}{3}, \frac{3}{2}\right)$ satisfies all three equations.

Check:

$$8x - 3y + 6z = -2 \quad (1)$$
$$8\left(-\frac{3}{4}\right) - 3\left(\frac{5}{3}\right) + 6\left(\frac{3}{2}\right) = -2 \quad ?$$
$$-6 - 5 + 9 = -2$$
$$-2 = -2 \text{ True}$$

$$4x + 9y + 4z = 18 \quad (2)$$
$$4\left(-\frac{3}{4}\right) + 9\left(\frac{5}{3}\right) + 4\left(\frac{3}{2}\right) = 18 \quad ?$$
$$-3 + 15 + 6 = 18$$
$$18 = 18 \text{ True}$$

$$12x - 3y + 8z = -2 \quad (3)$$
$$12\left(-\frac{3}{4}\right) - 3\left(\frac{5}{3}\right) + 8\left(\frac{3}{2}\right) = -2 \quad ?$$
$$-9 - 5 + 12 = -2$$
$$-2 = -2 \text{ True}$$

Solution set: $\left\{\left(-\frac{3}{4}, \frac{5}{3}, \frac{3}{2}\right)\right\}$

57. $2x - 3y + 2z - 3 = 0$ (1)
$4x + 8y + z - 2 = 0$ (2)
$-x - 7y + 3z - 14 = 0$ (3)

Eliminate x by multiplying equation (3) by 2 and add to equation (1).

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$$\begin{array}{r} 2x-3y+2z-3=0 \\ -2x-14y+6z-28=0 \\ \hline -17y+8z-31=0 \quad (4) \end{array}$$

Eliminate x by multiplying equation (3) by 4 and add to equation (2).

$$\begin{array}{r} 4x+8y+z-2=0 \\ -4x-28y+12z-56=0 \\ \hline -20y+13z-58=0 \quad (5) \end{array}$$

Multiply equation (5) by 17, equation (4) by -20, and add in order to eliminate y.

$$\begin{array}{r} 340y-160z+620=0 \\ -340y+221z-986=0 \\ \hline 61z-366=0 \Rightarrow z=6 \end{array}$$

Using $z=6$, find y from equation (5) by substitution.

$-20y+13(6)-58=0 \Rightarrow -20y+78-58=0$
$-20y+20=0 \Rightarrow -20y=-20 \Rightarrow y=1$

Substitute 1 for y and 6 for z in equation (1) to find x.

$2x-3(1)+2(6)-3=0 \Rightarrow 2x-3+12-3=0 \Rightarrow$
$2x+6=0 \Rightarrow 2x=-6 \Rightarrow x=-3$

Verify that the ordered triple $(-3,1,6)$ satisfies all three equations.

Check:
$$\begin{aligned} 2x-3y+2z-3&=0 \quad (1) \\ 2(-3)-3(1)+2(6)-3&=0 \quad ? \\ -6-3+12-3&=0 \\ 0&=0 \text{ True} \end{aligned}$$

$$\begin{aligned} 4x+8y+z-2&=0 \quad (2) \\ 4(-3)+8(1)+6-2&=0 \\ -12+8+6-2&=0 \quad ? \\ 0&=0 \text{ True} \end{aligned}$$

$$\begin{aligned} -x-7y+3z-14&=0 \quad (3) \\ -(-3)-7(1)+3(6)-14&=0 \quad ? \\ 3-7+18-14&=0 \\ 0&=0 \text{ True} \end{aligned}$$

Solution set: $\{(-3,1,6)\}$

58. $-x+2y-z-1=0$ (1)
$-x-y-z+2=0$ (2)
$x-y+2z-2=0$ (3)

Eliminate x by adding equations (1) and (3).

$$\begin{array}{r} -x+2y-z-1=0 \\ x-y+2z-2=0 \\ \hline y+z-3=0 \quad (4) \end{array}$$

Eliminate x by adding equations (2) and (3).

$$\begin{array}{r} -x-y-z+2=0 \\ x-y+2z-2=0 \\ \hline -2y+z=0 \quad (5) \end{array}$$

Multiply equation (5) by -1 and add to equation (4) order to eliminate z.

$$\begin{array}{r} y+z-3=0 \\ 2y-z=0 \\ \hline 3y-3=0 \Rightarrow y=1 \end{array}$$

Using $y=1$, find z from equation (5) by substitution.

$-2(1)+z=0 \Rightarrow -2+z=0 \Rightarrow z=2$

Substitute 1 for y and 2 for z in equation (1) to find x.

$-x+2(1)-2-1=0 \Rightarrow -x+2-2-1=0 \Rightarrow$
$-x-1=0 \Rightarrow -1=x$

Verify that the ordered triple $(-1,1,2)$ satisfies all three equations.

Check:
$$\begin{aligned} -x+2y-z-1&=0 \quad (1) \\ -(-1)+2(1)-2-1&=0 \quad ? \\ 1+2-2-1&=0 \\ 0&=0 \text{ True} \end{aligned}$$

$$\begin{aligned} -x-y-z+2&=0 \quad (2) \\ -(-1)-1-2+2&=0 \\ 1-1-2+2&=0 \quad ? \\ 0&=0 \text{ True} \end{aligned}$$

$$\begin{aligned} x-y+2z-2&=0 \quad (3) \\ -1-1+2(2)-2&=0 \quad ? \\ -1-1+4-2&=0 \\ 0&=0 \text{ True} \end{aligned}$$

Solution set: $\{(-1,1,2)\}$

59. $x-2y+3z=6$ (1)
$2x-y+2z=5$ (2)

Multiply equation (2) by -2 and add to equation (1) in order to eliminate y.

$$\begin{array}{r} x-2y+3z=6 \\ -4x+2y-4z=-10 \\ \hline -3x-z=-4 \quad (3) \end{array}$$

Solve equation (3) for z.

$-3x-z=-4 \Rightarrow -z=3x-4 \Rightarrow z=-3x+4$

Express y in terms of x by solving equation (1) for y and substituting $-3x+4$ for z.

$$x-2y+3z=6 \Rightarrow -2y=-x-3z+6 \Rightarrow$$
$$y=\frac{-x-3z+6}{-2}=\frac{x+3z-6}{2} \Rightarrow$$
$$y=\frac{x+3(-3x+4)-6}{2}$$
$$=\frac{x-9x+12-6}{2}=\frac{-8x+6}{2}=-4x+3$$

With x arbitrary, the solution set is of the form $\{(x,-4x+3,-3x+4)\}$.

60. $3x-2y+z=15$ (1)
$x+4y-z=11$ (2)

Multiply equation (1) by 2 and add to equation (2) in order to eliminate y.

$$\begin{array}{l} 6x-4y+2z=30 \\ \underline{\;\;x+4y-\;\;z=11} \\ 7x \qquad\;\; +\;z=41\ (3) \end{array}$$

Solve equation (3) for z.
$7x+z=41 \Rightarrow z=-7x+41$

Express y in terms of x by solving equation (1) for y and substituting $-7x+41$ for z.

$$3x-2y+z=15 \Rightarrow -2y=-3x-z+15 \Rightarrow$$
$$y=\tfrac{-3x-z+15}{-2}=\tfrac{3x+z-15}{2}$$
$$=\tfrac{3x+(-7x+41)-15}{2}=\tfrac{-4x+26}{2}$$
$$=-2x+13$$

With x arbitrary, the solution set is of the form $\{(x,-2x+13,-7x+41)\}$.

61. $5x-4y+z=\ 9$ (1)
$x+\ y\qquad=15$ (2)

Multiply equation (2) by 4 and add to equation (1) in order to eliminate y.

$$\begin{array}{l} 5x-4y+z=\ 9 \\ \underline{4x+4y\qquad=60} \\ 9x \qquad\; +z=69\ (3) \end{array}$$

Solve equation (3) for z.
$9x+z=69 \Rightarrow z=-9x+69$

Express y in terms of x by solving equation (1) for y and substituting $-9x+69$ for z.

$$5x-4y+z=9 \Rightarrow -4y=-5x-z+9 \Rightarrow$$
$$y=\tfrac{-5x-z+9}{-4}=\tfrac{5x+z-9}{4}$$
$$=\tfrac{5x+(-9x+69)-9}{4}=\tfrac{-4x+60}{4}=-x+15$$

With x arbitrary, the solution set is of the form $\{(x,-x+15,-9x+69)\}$.

62. $3x-5y-4z=\ -7$ (1)
$y-\ z=-13$ (2)

Multiply equation (2) by 5 and add to equation (1) in order to eliminate y.

$$\begin{array}{l} 3x-5y-4z=\ -7 \\ \underline{\qquad\;\; 5y-5z=-65} \\ 3x \qquad\; -9z=-72\ (3) \end{array}$$

Solve equation (3) for z.
$3x-9z=-72 \Rightarrow -9z=-72-3x \Rightarrow$
$$z=\tfrac{-72-3x}{-9}=\tfrac{24+x}{3}$$

Express y in terms of x by solving equation (1) for y and substituting $\frac{24+x}{3}$ for z.

$$3x-5y-4z=-7 \Rightarrow -5y=-3x+4z-7 \Rightarrow$$
$$y=\tfrac{-3x+4z-7}{-5}=\tfrac{3x-4z+7}{5}$$
$$y=\tfrac{3x-4\left(\frac{24+x}{3}\right)+7}{5}=\tfrac{9x-4(24+x)+21}{15}$$
$$=\tfrac{9x-96-4x+21}{15}=\tfrac{-75+5x}{15}=\tfrac{-15+x}{3}$$

With x arbitrary, the solution set is of the form $\left\{\left(x,\frac{-15+x}{3},\frac{24+x}{3}\right)\right\}$.

63. $3x+4y-z=13$ (1)
$x+\ y+2z=15$ (2)

Multiply equation (2) by -4 and add to equation (1) in order to eliminate y.

$$\begin{array}{l} 3x+4y-\ z=\ 13 \\ \underline{-4x-4y-8z=-60} \\ -x \qquad -9z=-47\ (3) \end{array}$$

Solve equation (3) for z.
$-x-9z=-47 \Rightarrow -9z=-47+x \Rightarrow$
$$z=\tfrac{-47+x}{-9}=\tfrac{47-x}{9}$$

Express y in terms of x by solving equation (2) for y and substituting $\frac{47-x}{9}$ for z.

$$x+y+2z=15 \Rightarrow y=-x-2z+15 \Rightarrow$$
$$y=-x-2\left(\tfrac{47-x}{9}\right)+15$$
$$y=\tfrac{-9x-2(47-x)+135}{9}$$
$$=\tfrac{-9x-94+2x+135}{9}=\tfrac{41-7x}{9}$$

With x arbitrary, the solution set is of the form $\left\{\left(x,\frac{41-7x}{9},\frac{47-x}{9}\right)\right\}$.

64. $x-y+z=-6$ (1)
$4x+y+z=\ 7$ (2)

Add equations (1) and (2) in order to eliminate y.

$$\begin{array}{l} x-y+z=-6 \\ \underline{4x+y+z=\ 7} \\ 5x \quad +2z=\ 1\ (3) \end{array}$$

Solve equation (3) for z.
$5x+2z=1 \Rightarrow 2z=-5x+1 \Rightarrow z=\frac{-5x+1}{2}$

Express y in terms of x by solving equation (1) for y and substituting $\frac{-5x+1}{2}$ for z.

$$x-y+z=-6 \Rightarrow -y=-x-z-6 \Rightarrow$$
$$y=x+z+6$$
$$y=x+\tfrac{-5x+1}{2}+6=\tfrac{2x-5x+1+12}{2}=\tfrac{-3x+13}{2}$$

With x arbitrary, the solution set is of the form $\left\{\left(x,\frac{-3x+13}{2},\frac{-5x+1}{2}\right)\right\}$.

65.
$$\begin{aligned} 3x + 5y - z &= -2 \ (1) \\ 4x - y + 2z &= 1 \ (2) \\ -6x - 10y + 2z &= 0 \ (3) \end{aligned}$$

Multiply equation (1) by 2 and add the result to equation (3).

$$\begin{aligned} 6x + 10y - 2z &= -4 \\ -6x - 10y + 2z &= 0 \\ \hline 0 &= -4 \end{aligned}$$

We obtain a false statement. The solution set is $\varnothing$, and the system is inconsistent.

66.
$$\begin{aligned} 3x + y + 3z &= 1 \ (1) \\ x + 2y - z &= 2 \ (2) \\ 2x - y + 4z &= 4 \ (3) \end{aligned}$$

Multiply equation (2) by 3 and add the result to equation (1) in order to eliminate *z*.

$$\begin{aligned} 3x + y + 3z &= 1 \\ 3x + 6y - 3z &= 6 \\ \hline 6x + 7y &= 7 \ (4) \end{aligned}$$

Multiply equation (2) by 4 and add the result to equation (3) in order to eliminate *z*.

$$\begin{aligned} 4x + 8y - 4z &= 8 \\ 2x - y + 4z &= 4 \\ \hline 6x + 7y &= 12 \ (5) \end{aligned}$$

Multiply equation (4) by –1 and add the result to equation (5).

$$\begin{aligned} -6x - 7y &= -7 \\ 6x + 7y &= 12 \\ \hline 0 &= 5 \end{aligned}$$

This is a false statement. The solution set is $\varnothing$, and the system is inconsistent.

67.
$$\begin{aligned} 5x - 4y + z &= 0 \ (1) \\ x + y &= 0 \ (2) \\ -10x + 8y - 2z &= 0 \ (3) \end{aligned}$$

Multiply equation (2) by 4 and add the result to equation (1) in order to eliminate *y*.

$$\begin{aligned} 5x - 4y + z &= 0 \\ 4x + 4y &= 0 \\ \hline 9x + z &= 0 \ (4) \end{aligned}$$

Multiply equation (2) by -8 and add the result to equation (3) in order to eliminate *y*.

$$\begin{aligned} -8x - 8y &= 0 \\ -10x + 8y - 2z &= 0 \\ \hline -18x - 2z &= 0 \ (5) \end{aligned}$$

Multiply equation (4) by 2 and add the result to equation (5).

$$\begin{aligned} 18x + 2z &= 0 \\ -18x - 2z &= 0 \\ \hline 0 &= 0 \end{aligned}$$

This is a true statement and the system has infinitely many solutions.
Solve equation (4) for *x*.

$$9x + z = 0 \Rightarrow 9x = -z \Rightarrow x = -\tfrac{z}{9}$$

Express *y* in terms of *z* by solving equation (1) for *y* and substituting $-\frac{z}{9}$ for *x*.

$$5x - 4y + z = 0 \Rightarrow -4y = -5x - z \Rightarrow$$
$$y = \tfrac{-5x-z}{-4} = \tfrac{5x+z}{4} \Rightarrow$$
$$y = \tfrac{5\left(-\frac{z}{9}\right)+z}{4} = \tfrac{-5z+9z}{36} = \tfrac{4z}{36} = \tfrac{z}{9}$$

With *z* arbitrary, the solution set is of the form $\left\{\left(-\frac{z}{9}, \frac{z}{9}, z\right)\right\}$.

68.
$$\begin{aligned} 2x + y - 3z &= 0 \ (1) \\ 4x + 2y - 6z &= 0 \ (2) \\ x - y + z &= 0 \ (3) \end{aligned}$$

Multiply equation (1) by -2 and add the result to equation (2) in order to eliminate *y*.

$$\begin{aligned} -4x - 2y + 6z &= 0 \\ 4x + 2y - 6z &= 0 \\ \hline 0 &= 0 \end{aligned}$$

This is a true statement and the system has infinitely many solutions.
We now need to solve for one of the variables in terms of another variable by eliminating a third variable. Multiply equation (3) by 2 and add the result to equation (2) in order to eliminate *y*.

$$\begin{aligned} 4x + 2y - 6z &= 0 \\ 2x - 2y + 2z &= 0 \\ \hline 6x - 4z &= 0 \ (4) \end{aligned}$$

Solve equation (4) for *x*.

$$6x - 4z = 0 \Rightarrow 6x = 4z \Rightarrow x = \tfrac{2z}{3}$$

Express *y* in terms of *z* by solving equation (3) for *y* and substituting $\frac{2z}{3}$ for *x*.

$$x - y + z = 0 \Rightarrow y = x + z \Rightarrow$$
$$y = \tfrac{2z}{3} + z = \tfrac{2z+3z}{3} = \tfrac{5z}{3}$$

With *z* arbitrary, the solution set is of the form $\left\{\left(\frac{2z}{3}, \frac{5z}{3}, z\right)\right\}$.

69. $\frac{2}{x}+\frac{1}{y}=\frac{3}{2}$ (1)
$\frac{3}{x}-\frac{1}{y}=1$ (2)

Let $\frac{1}{x}=t$ and $\frac{1}{y}=u$. With these substitutions, the system becomes
$2t+u=\frac{3}{2}$ (3)
$3t-u=1$ (4)

Add these equations, eliminate u, and solve for t.
$5t=\frac{5}{2}\Rightarrow t=\frac{1}{2}$

Substitute $\frac{1}{2}$ for t in equation (3) and solve for u.
$2\left(\frac{1}{2}\right)+u=\frac{3}{2}\Rightarrow 1+u=\frac{3}{2}\Rightarrow u=\frac{1}{2}$

Now find the values of x and y, the variables in the original system. So, $\frac{1}{x}=t$, $tx=1$, and $x=\frac{1}{t}$. Likewise $y=\frac{1}{u}$.
$x=\frac{1}{t}=\frac{1}{\frac{1}{2}}=2$ and $y=\frac{1}{u}=\frac{1}{\frac{1}{2}}=2$

Solution set: $\{(2,2)\}$

70. $\frac{1}{x}+\frac{3}{y}=\frac{16}{5}$ (1)
$\frac{5}{x}+\frac{4}{y}=5$ (2)

Let $\frac{1}{x}=t$ and $\frac{1}{y}=u$. With these substitutions, the system becomes
$t+3u=\frac{16}{5}$ (3)
$5t+4u=5$ (4)

Multiply equation (3) by –5 and add to equation (4), eliminating t, and solve for u

$$\begin{array}{r} -5t-15u=-16 \\ 5t+\ \ 4u=\ \ \ 5 \\ \hline -11u=-11\Rightarrow u=1 \end{array}$$

Substitute 1 for u in equation (3) and solve for t: $t+3(1)=\frac{16}{5}\Rightarrow t+3=\frac{16}{5}\Rightarrow t=\frac{16}{5}-\frac{15}{5}=\frac{1}{5}$

Now find the values of x and y, the variables in the original system. So, $x=\frac{1}{t}=\frac{1}{\frac{1}{5}}=5$ and $y=\frac{1}{u}=\frac{1}{1}=1.$

Solution set: $\{(5,1)\}$

71. $\frac{2}{x}+\frac{1}{y}=11$ (1)
$\frac{3}{x}-\frac{5}{y}=10$ (2)

Let $\frac{1}{x}=t$ and $\frac{1}{y}=u$. With these substitutions, the system becomes
$2t+\ u=11$ (3)
$3t-5u=10$ (4)

Multiply equation (3) by 5 and add to equation (4), eliminating u, and solve for t.

$$\begin{array}{l} 10t+5u=55 \\ \ \ 3t-5u=10 \\ \hline 13t\qquad\ =65\Rightarrow t=5 \end{array}$$

Substitute 5 for t in equation (3) and solve for u: $2(5)+u=11\Rightarrow 10+u=11\Rightarrow u=1$

Now find the values of x and y, the variables in the original system. So, $x=\frac{1}{t}=\frac{1}{5}$ and $y=\frac{1}{u}=\frac{1}{1}=1.$

Solution set: $\left\{\left(\frac{1}{5},1\right)\right\}$

72. $\frac{2}{x}+\frac{3}{y}=18$ (1)
$\frac{4}{x}-\frac{5}{y}=-8$ (2)

Let $\frac{1}{x}=t$ and $\frac{1}{y}=u$. With these substitutions, the system becomes
$2t+3u=18$ (3)
$4t-5u=-8.$ (4)

Multiply equation (3) by –2 and add to equation (4), eliminating t, and solve for u.

$$\begin{array}{r} -4t-6u=-36 \\ 4t-5u=\ \ -8 \\ \hline -11u=-44\Rightarrow u=4 \end{array}$$

Substitute 4 for u in equation (3) and solve for t.
$2t+3(4)=18\Rightarrow 2t+12=18$
$2t=6\Rightarrow t=3$

Now find the values of x and y, the variables in the original system. So, $x=\frac{1}{t}=\frac{1}{3}$ and $y=\frac{1}{u}=\frac{1}{4}.$

Solution set: $\left\{\left(\frac{1}{3},\frac{1}{4}\right)\right\}$

73. $\frac{2}{x}+\frac{3}{y}-\frac{2}{z}=-1$ (1)
$\frac{8}{x}-\frac{12}{y}+\frac{5}{z}=5$ (2)
$\frac{6}{x}+\frac{3}{y}-\frac{1}{z}=1$ (3)

Let $\frac{1}{x}=t,\ \frac{1}{y}=u,$ and $\frac{1}{z}=v.$ With these substitutions, the system becomes
$2t+3u-2v=-1$ (4)
$8t-12u+5v=5$ (5)
$6t+3u-v=1$ (6)

Eliminate v by multiplying equation (6) by 5 and add to equation (5).

$$\begin{array}{r} 8t-12u+5v=5 \\ \underline{30t+15u-5v=5} \\ 38t+3u\qquad=10 \end{array}\ (7)$$

Eliminate v by multiplying equation (6) by -2 and add to equation (4).

$$\begin{array}{r} 2t+3u-2v=-1 \\ \underline{-12t-6u+2v=-2} \\ -10t-3u\qquad=-3 \end{array}\ (8)$$

Add equations (7) and (8) in order to eliminate u.

$$\begin{array}{r} 38t+3u=10 \\ \underline{-10t-3u=-3} \\ 28t\qquad=7 \end{array} \Rightarrow t=\frac{1}{4}$$

Using $t=\frac{1}{4}$, find u from equation (7) by substitution.

$$38\left(\frac{1}{4}\right)+3u=10\Rightarrow\frac{19}{2}+3u=10\Rightarrow$$
$$3u=\frac{20}{2}-\frac{19}{2}=\frac{1}{2}\Rightarrow u=\frac{1}{6}$$

Substitute $\frac{1}{4}$ for t and $\frac{1}{6}$ for u in equation (4) to find v.

$$2\left(\frac{1}{4}\right)+3\left(\frac{1}{6}\right)-2v=-1\Rightarrow\frac{1}{2}+\frac{1}{2}-2v=-1\Rightarrow$$
$$1-2v=-1\Rightarrow-2v=-2\Rightarrow v=1$$

Now find the values of x, y, and z, the variables in the original system. So,
$x=\frac{1}{t}=\frac{1}{\frac{1}{4}}=4,\quad y=\frac{1}{u}=\frac{1}{\frac{1}{6}}=6,$ and
$z=\frac{1}{v}=\frac{1}{1}=1.$

Solution set: $\{(4,6,1)\}$

74. $-\frac{5}{x}+\frac{4}{y}+\frac{3}{z}=2$ (1)
$\frac{10}{x}+\frac{3}{y}-\frac{6}{z}=7$ (2)
$\frac{5}{x}+\frac{2}{y}-\frac{9}{z}=6$ (3)

Let $\frac{1}{x}=t,\ \frac{1}{y}=u,$ and $\frac{1}{z}=v.$

With these substitutions, the system becomes
$-5t+4u+3v=2$ (4)
$10t+3u-6v=7$ (5)
$5t+2u-9v=6$ (6)

Eliminate t by adding equations (4) and (6) to obtain $6u-6v=8$ or $3u-3v=4$ (7).

Eliminate t by multiplying equation (4) by 2 and add to equation (5).

$$\begin{array}{r} -10t+8u+6v=4 \\ \underline{10t+3u-6v=7} \\ 11u\qquad=11 \end{array} \Rightarrow u=1$$

Using $u=1$, find v from equation (7) by substitution.

$$3(1)-3v=4\Rightarrow3-3v=4\Rightarrow-3v=1\Rightarrow v=-\frac{1}{3}$$

Substitute 1 for u and $-\frac{1}{3}$ for v in equation (4) to find t.

$$-5t+4(1)+3\left(-\frac{1}{3}\right)=2\Rightarrow-5t+4-1=2\Rightarrow$$
$$-5t+3=2\Rightarrow-5t=-1\Rightarrow t=\frac{1}{5}$$

Now find the values of x, y, and z, the variables in the original system. Since
$x=\frac{1}{t}=\frac{1}{\frac{1}{5}}=5,\quad y=\frac{1}{u}=\frac{1}{1}=1,$ and
$z=\frac{1}{v}=\frac{1}{-\frac{1}{3}}=-3.$

Solution set: $\{(5,1,-3)\}$

75. Answers will vary.

(a) $x+y+z=4$
$x+2y+z=5$
$2x-y+3z=4$

This system has exactly one solution, namely (4, 1, –1). (There are other equations that would do the same.)

(b) $x+y+z=4$
$x+y+z=5$
$2x-y+3z=4$

This system has no solution, since no ordered triple can satisfy the first two equations simultaneously. (There are other equations that would do the same.)

(c) $x+y+z=4$
$2x+2y+2z=8$
$2x-y+3z=4$

This system has infinitely many solutions, since all the ordered triples that satisfy the first equation will also satisfy the second equation. (There are other equations that would do the same.)

76. **(a)** One example is the ceiling and two perpendicular walls of a standard room meet at a single point.

(b) One example is three pages of this book (or any book) intersect in a line (represented by the spine of the book).

77. Since $y = ax^2 + bx + c$ and the parabola passes through the points (2, 3), (–1, 0), and (–2, 2), we have the following equations.

$$3 = a(2)^2 + b(2) + c \quad \Rightarrow 4a + 2b + c = 3 \ (1)$$
$$0 = a(-1)^2 + b(-1) + c \Rightarrow \quad a - \ b + c = 0 \ (2)$$
$$2 = a(-2)^2 + b(-2) + c \Rightarrow 4a - 2b + c = 2 \ (3)$$

Multiply equation (2) by –1 and add the result to equation (1) in order to eliminate *c*.

$$\begin{aligned} 4a + 2b + c &= 3 \\ -a + \ b - c &= 0 \\ \hline 3a + 3b \quad &= 3 \quad (4) \end{aligned}$$

Multiplying equation (4) by –1 and then adding the result to equation (5).

$$\begin{aligned} -3a - 3b &= -3 \\ 3a - \ b &= \ 2 \\ \hline -4b &= -1 \Rightarrow b = \tfrac{1}{4} \end{aligned}$$

Multiply equation (2) by –1 and add the result to equation (3) in order to eliminate *c*.

$$\begin{aligned} -a + \ b - c &= 0 \\ 4a - 2b + c &= 2 \\ \hline 3a - \ b \quad &= 2 \quad (5) \end{aligned}$$

Substitute this value into equation (5).

$$3a - \tfrac{1}{4} = 2 \Rightarrow 3a = \tfrac{8}{4} + \tfrac{1}{4} = \tfrac{9}{4} \Rightarrow a = \tfrac{3}{4}$$

Substitute $a = \frac{3}{4}$ and $b = \frac{1}{4}$ into equation (1) in order to solve for *c*.

$$4\left(\tfrac{3}{4}\right) + 2\left(\tfrac{1}{4}\right) + c = 3$$
$$3 + \tfrac{1}{2} + c = 3 \Rightarrow c = -\tfrac{1}{2}$$

The equation of the parabola is $y = \frac{3}{4}x^2 + \frac{1}{4}x - \frac{1}{2}$.

78. Since $y = ax + b$ and the line passes through (–2, 1) and (–1, –2), we have the equations $1 = a(-2) + b$ and $-2 = a(-1) + b.$

This becomes the following system.

$$\begin{aligned} -2a + b &= \ \ 1 \quad (1) \\ -a + b &= -2 \quad (2) \end{aligned}$$

Multiply equation (1) by –1 and add the result to equation (2) in order to eliminate *b*.

$$\begin{aligned} 2a - b &= -1 \\ -a + b &= -2 \\ \hline a \quad &= -3 \end{aligned}$$

Substitute this value into equation (1).

$$-2(-3) + b = 1 \Rightarrow 6 + b = 1 \Rightarrow b = -5$$

The equation is $y = -3x - 5$.

79. Since $y = ax + b$ and the line passes through the points $(2,5)$ and $(-1,-4)$, we have the equations $5 = a(2) + b$ and $-4 = a(-1) + b.$

This becomes the following system.

$$\begin{aligned} 2a + b &= \ \ 5 \ \ (1) \\ -a + b &= -4. \ (2) \end{aligned}$$

Multiply equation (2) by –1 and add the result to equation (1) in order to eliminate *b*.

$$\begin{aligned} 2a + b &= 5 \\ a - b &= 4 \\ \hline 3a \quad &= 9 \Rightarrow a = 3 \end{aligned}$$

Substitute this value into equation (1).

$$2(3) + b = 5 \Rightarrow 6 + b = 5 \Rightarrow b = -1$$

The equation is $y = 3x - 1$.

80. Since $y = ax^2 + b + c$ and the parabola passes through the points (2, 9), (–2, 1), and (–3, 4), we have the following equations.

$$9 = a(2)^2 + b(2) + c \quad \Rightarrow 4a + 2b + c = 9 \ (1)$$
$$1 = a(-2)^2 + b(-2) + c \Rightarrow 4a - 2b + c = 1 \ (2)$$
$$4 = a(-3)^2 + b(-3) + c \Rightarrow 9a - 3b + c = 4 \ (3)$$

Multiply equation (2) by –1 and add the result to equation (1) in order to eliminate *c*.

$$\begin{aligned} 4a + 2b + c &= \ \ 9 \\ -4a + 2b - c &= -1 \\ \hline 4b \quad &= \ \ 8 \Rightarrow b = 2 \end{aligned}$$

Substitute $b = 2$ into equation (5) in order to solve for *a*.

$$5a - 2 = 3 \Rightarrow 5a = 5 \Rightarrow a = 1$$

Multiply equation (2) by –1 and add the result to equation (3).

$$\begin{aligned} -4a + 2b - c &= -1 \\ 9a - 3b + c &= \ \ 4 \\ \hline 5a - \ b \quad &= \ \ 3 \ \ (5) \end{aligned}$$

Substitute $a = 1$ and $b = 2$ into equation (1) in order to solve for *c*.

$$4(1) + 2(2) + c = 9$$
$$4 + 4 + c = 9 \Rightarrow 8 + c = 9 \Rightarrow c = 1$$

The equation of the parabola is $y = x^2 + 2x + 1$.

81. Since $y = ax^2 + bx + c$ and the parabola passes through the points $(-2,-3.75)$, $(4,-3.75)$, and $(-1,-1.25)$, we have the following equations.

$$-3.75 = a(-2)^2 + b(-2) + c$$
$$4a - 2b + c = -3.75 \quad (1)$$
$$-3.75 = a(4)^2 + b(4) + c$$
$$16a + 4b + c = -3.75 \quad (2)$$
$$-1.25 = a(-1)^2 + b(-1) + c$$
$$a - b + c = -1.25 \quad (3)$$

Multiply equation (2) by –1 and add the result to equation (1) in order to eliminate c.

$$\begin{array}{rl} 4a - 2b + c = & -3.75 \\ -16a - 4b - c = & 3.75 \\ \hline -12a - 6b \quad = & 0 \text{ or } 2a + b = 0 \ (4) \end{array}$$

Multiply equation (2) by –1 and add the result to equation (3) in order to eliminate c.

$$\begin{array}{rl} -16a - 4b - c = & 3.75 \\ a - b + c = & -1.25 \\ \hline -15a - 5b \quad = & 2.50 \text{ or } -3a - b = .5 \ (5) \end{array}$$

Add equations (4) and (5) in order to solve for b.

$$\begin{array}{rl} 2a + b = & 0 \\ -3a - b = & .5 \\ \hline -a \quad = & .5 \Rightarrow a = -.5 \end{array}$$

Substitute $a = -.5$ into equation (4) in order to solve for b.

$$2(-.5) + b = 0 \Rightarrow -1 + b = 0 \Rightarrow b = 1$$

Substitute $a = -.5$ and $b = 1$ into equation (3) in order to solve for c.

$$-.5 - 1 + c = -1.25$$
$$-1.5 + c = -1.25 \Rightarrow c = .25$$

The equation of the parabola is $y = -.5x^2 + x + .25$ or $y = -\frac{1}{2}x^2 + x + \frac{1}{4}$.

82. Since $y = ax^2 + bx + c$ and the parabola passes through the points (–2, 2.9), (0, .56), and (1, .8), we have the equations.

$$2.9 = a(-2)^2 + b(-2) + c$$
$$4a - 2b + c = 2.9 \quad (1)$$
$$.56 = a(0)^2 + b(0) + c$$
$$c = .56 \quad (2)$$
$$.8 = a(1)^2 + b(1) + c$$
$$a + b + c = .8 \quad (3)$$

Since $c = .56$, we substitute this value into equation (1) and (2) to obtain the following.

$$4a - 2b + .56 = 2.9 \ (4)$$
$$a + b + .56 = .8 \quad (5)$$

$$\begin{array}{rl} 4a - 2b = & 2.34 \\ 2a + 2b = & .48 \\ \hline 6a \quad = & 2.82 \Rightarrow a = .47 \end{array}$$

This becomes the following system.

$$4a - 2b = 2.34 \ (6)$$
$$a + b = .24 \quad (7)$$

Multiply equation (7) by 2 and add the result to equation (6) in order to eliminate b. Substitute the value for a into equation (7).

$$.47 + b = .24 \Rightarrow b = -.23$$

The equation of the parabola is $Y_1 = .47X^2 - .23X + .56$.

83. Since $x^2 + y^2 + ax + by + c = 0$ and the circle passes through the points $(-1,3)$, $(6,2)$, and $(-2,-4)$, we have the following equations.

$$(-1)^2 + 3^2 + a(-1) + b(3) + c = 0$$
$$-a + 3b + c = -10 \quad (1)$$
$$6^2 + 2^2 + a(6) + b(2) + c = 0 \Rightarrow$$
$$6a + 2b + c = -40 \quad (2)$$
$$(-2)^2 + (-4)^2 + a(-2) + b(-4) + c = 0$$
$$-2a - 4b + c = -20 \quad (3)$$

Multiply equation (1) by –2 and adding the result to equation (3) in order to eliminate a.

$$\begin{array}{rl} 2a - 6b - 2c = & 20 \\ -2a - 4b + c = & -20 \\ \hline -10b - c = & 0 \ (4) \end{array}$$

Multiply equation (1) by 6 and adding the result to equation (2) in order to eliminate a.

$$\begin{array}{rl} -6a + 18b + 6c = & -60 \\ 6a + 2b + c = & -40 \\ \hline 20b + 7c = & -100 \ (5) \end{array}$$

Multiply equation (4) by 7 and adding the result to equation (5) in order to eliminate c.

$$\begin{array}{rl} -70b - 7c = & 0 \\ 20b + 7c = & -100 \\ \hline -50b \quad = & -100 \Rightarrow b = 2 \end{array}$$

We substitute this value into equation (4) in order to solve for c.

$$-10(2) - c = 0 \Rightarrow -20 - c = 0 \Rightarrow -20 = c$$

Substitute $b = 2$ and $c = -20$ into equation (1).

$$-a + 3(2) - 20 = -10 \Rightarrow -a + 6 - 20 = -10 \Rightarrow$$
$$-a - 14 = -10 \Rightarrow -a = 4 \Rightarrow a = -4$$

The equation of the circle is $x^2 + y^2 - 4x + 2y - 20 = 0$.

84. Since $x^2 + y^2 + ax + by + c = 0$ and the circle passes through the points $(-1,5)$, $(6,6)$, and $(7,-1)$, we have the following equations.

$$(-1)^2 + 5^2 + a(-1) + b(5) + c = 0$$
$$-a + 5b + c = -26 \quad (1)$$
$$6^2 + 6^2 + a(6) + b(6) + c = 0$$
$$6a + 6b + c = -72 \quad (2)$$
$$7^2 + (-1)^2 + a(7) + b(-1) + c = 0$$
$$7a - b + c = -50 \quad (3)$$

Multiply equation (1) by -1 and add the result to equation (2) in order to eliminate c.

$$\begin{aligned} a - 5b - c &= 26 \\ 6a + 6b + c &= -72 \\ \hline 7a + b &= -46 \quad (4) \end{aligned}$$

Multiply equation (1) by -1 and add the result to equation (3) in order to eliminate c.

$$\begin{aligned} a - 5b - c &= 26 \\ 7a - b + c &= -50 \\ \hline 8a - 6b &= -24 \quad (5) \end{aligned}$$

Multiply equation (4) by 6 and add the result to equation (5) in order to eliminate b.

$$\begin{aligned} 42a + 6b &= -276 \\ 8a - 6b &= -24 \\ \hline 50a &= -300 \Rightarrow a = -6 \end{aligned}$$

We substitute this value into equation (4) in order to solve for b.

$$7(-6) + b = -46$$
$$-42 + b = -46$$
$$b = -4$$

Substitute $a = -6$ and $b = -4$ into equation (1).

$$-(-6) + 5(-4) + c = -26 \Rightarrow$$
$$6 - 20 + c = -26 \Rightarrow$$
$$-14 + c = -26 \Rightarrow c = -12$$

The equation of the circle is $x^2 + y^2 - 6x - 4y - 12 = 0.$

85. Since $x^2 + y^2 + ax + by + c = 0$ and the circle passes through the points (2, 1), (–1, 0), and (3, 3), we have the following equations.

$$2^2 + 1^2 + a(2) + b(1) + c = 0$$
$$2a + b + c = -5 \quad (1)$$
$$(-1)^2 + 0^2 + a(-1) + b(0) + c = 0$$
$$-a + c = -1 \quad (2)$$
$$3^2 + 3^2 + a(3) + b(3) + c = 0$$
$$3a + 3b + c = -18 \quad (3)$$

Multiply equation (1) by –3 and add the result to equation (3) in order to eliminate b.

$$\begin{aligned} -6a - 3b - 3c &= 15 \\ 3a + 3b + c &= -18 \\ \hline -3a - 2c &= -3 \quad (4) \end{aligned}$$

Multiply equation (2) by 2 and add the result to equation (4) in order to eliminate c.

$$\begin{aligned} -2a + 2c &= -2 \\ -3a - 2c &= -3 \\ \hline -5a &= -5 \Rightarrow a = 1 \end{aligned}$$

We substitute this value into equation (2) in order to solve for c.

$$-1 + c = -1 \Rightarrow c = 0$$

Substitute $a = 1$ and $c = 0$ into equation (1).

$$2(1) + b + 0 = -5 \Rightarrow 2 + b = -5 \Rightarrow b = -7$$

The equation of the circle is $x^2 + y^2 + x - 7y = 0$.

86. Since $x^2 + y^2 + ax + by + c = 0$ and the circle passes through points (0, 3), (4, 2), and (–5, –2), we have the following equations.

$$0^2 + 3^2 + a(0) + b(3) + c = 0$$
$$3b + c = -9 \quad (1)$$
$$4^2 + 2^2 + a(4) + b(2) + c = 0$$
$$4a + 2b + c = -20 \quad (2)$$
$$(-5)^2 + (-2)^2 + a(-5) + b(-2) + c = 0$$
$$-5a - 2b + c = -29 \quad (3)$$

Multiply equation (2) by 5 and multiply equation (3) by 4, and then add the results in order to eliminate a.

$$\begin{aligned} 20a + 10b + 5c &= -100 \\ -20a - 8b + 4c &= -116 \\ \hline 2b + 9c &= -216 \quad (4) \end{aligned}$$

Multiply equation (1) by –9 and add the result to equation (4) in order to eliminate c.

$$\begin{aligned} -27b - 9c &= 81 \\ 2b + 9c &= -216 \\ \hline -25b &= -135 \Rightarrow b = \tfrac{27}{5} \end{aligned}$$

Substitute $b = \frac{27}{5}$ into equation (1) in order to solve for c.

$$3\left(\tfrac{27}{5}\right) + c = -9 \Rightarrow \tfrac{81}{5} + c = -9 \Rightarrow$$
$$c = -\tfrac{81}{5} - \tfrac{45}{5} = -\tfrac{126}{5}$$

Substitute $b = \frac{27}{5}$ and $c = -\frac{126}{5}$ into equation (2).

$$4a+2\left(\frac{27}{5}\right)+\left(-\frac{126}{5}\right)=-20 \Rightarrow$$
$$4a+\frac{54}{5}-\frac{126}{5}=-20$$
$$4a-\frac{72}{5}=-20 \Rightarrow 4a=\frac{72}{5}-\frac{100}{5} \Rightarrow$$
$$4a=-\frac{28}{5} \Rightarrow a=-\frac{7}{5}$$

The equation of the circle is
$x^2+y^2-\frac{7}{5}x+\frac{27}{5}y-\frac{126}{5}=0.$

87. (a) Since $C=at^2+bt+c$ and we have the ordered pairs (0, 318), (20, 341), and (40, 371), we have the following equations.

$$318=a(0)^2+b(0)+c$$
$$c=318 \quad (1)$$
$$341=a(20)^2+b(20)+c$$
$$400a+20b+c=341 \quad (2)$$
$$371=a(40)^2+b(40)+c$$
$$1600a+40b+c=371 \quad (3)$$

Since $c = 318$, we substitute this value into equations (2) and (3) to obtain the following system.

$$400a+20b+318=341$$
$$400a+20b=23 \quad (4)$$
$$1600a+40b+318=371$$
$$1600a+40b=53 \quad (5)$$

Multiply equation (4) by –2 and add the result to equation (5) in order to eliminate b.

$$-800a-40b=-46$$
$$1600a+40b=53$$
$$800a=7 \Rightarrow a=\frac{7}{800}$$

Substitute $a=\frac{7}{800}$ into equation (5) in order to solve for b.

$$1600\left(\frac{7}{800}\right)+40b=53$$
$$14+40b=53$$
$$40b=39 \Rightarrow b=\frac{39}{40}$$

The constants are $a=\frac{7}{800}$, $b=\frac{39}{40}$, and $c=318$ and the relationship is $C=\frac{7}{800}t^2+\frac{39}{40}t+318.$

(b) Since $t = 0$ corresponds to 1962, the amount of carbon dioxide will be double its 1962 level when $\frac{7}{800}t^2+\frac{39}{40}t+318=2(318).$

Solve this equation.

$$\frac{7}{800}t^2+\frac{39}{40}t+318=2(318)$$
$$\frac{7}{800}t^2+\frac{39}{40}t-318=0$$
$$7t^2+780t-254,400=0$$

Use the quadratic formula, where $a=7, b=780,$ and $c=-254,400.$

$$t=\frac{-780\pm\sqrt{780^2-4(7)(-254,400)}}{2(7)}$$
$$=\frac{-780\pm\sqrt{608,400+7,123,200}}{14}$$
$$=\frac{-780\pm\sqrt{7,731,600}}{14}$$
$$=\frac{-780-\sqrt{7,731,600}}{14}\approx-254.3 \text{ or}$$
$$t=\frac{-780+\sqrt{7,731,600}}{14}\approx 142.9$$

We reject the first proposed solution because time cannot be negative. If t is approximately 142.9 then the year is $1962+142.9=2104.9$ (near the end of 2104).

88. (a) Since $C=aS^2+bS+c$ and we have the ordered pairs (320, 33), (600, 40), and (1283, 50), we obtain the following system.

$$33=a(320)^2+b(320)+c$$
$$102,400a+320b+c=33 \quad (1)$$
$$40=a(600)^2+b(600)+c$$
$$360,000a+600b+c=40 \quad (2)$$
$$50=a(1283)^2+b(1283)+c$$
$$1,646,089a+1283b+c=50 \quad (3)$$

Multiply equation (1) by -1 and add the result to equation (2) in order to eliminate c.

$$-102,400a-320b-c=-33$$
$$360,000a+600b+c=40$$
$$257,600a+280b=7 \text{ or}$$
$$36,800a+40b=1 \quad (4)$$

Multiply equation (1) by -1 and add the result to equation (3) in order to eliminate c.

$$-102,400a-320b-c=-33$$
$$1,646,089a+1283b+c=50$$
$$1,543,689a+963b=17 \quad (5)$$

Multiply equation (4) by -963 and equation (5) by 40 and add the results in order to eliminate b.

$$-35,438,400a-38,520b=-963$$
$$61,747,560a+38,520b=680$$
$$26,309,160a=-283 \Rightarrow$$
$$a=\frac{-283}{26,309,160}\approx-.0000107567$$

We substitute this value into equation (4) in order to solve for b.

$$36,800\left(\frac{-283}{26,309,160}\right)+40b=1 \Rightarrow -\frac{10,414,400}{26,309,160}+40b=1 \Rightarrow 40b=1+\frac{10,414,400}{26,309,160} \Rightarrow$$
$$40b=\frac{918,089}{657,729} \Rightarrow b=\frac{918,089}{26,309,160} \approx .034896$$

Substitute $a=\frac{-283}{26,309,160}$ and $b=\frac{918,089}{26,309,160}$ into equation (1).

$$102,400\left(\frac{-283}{26,309,160}\right)+320\left(\frac{918,089}{26,309,160}\right)+c=33 \Rightarrow -\frac{724,480}{657,729}+\frac{7,344,712}{657,729}+c=33$$
$$\frac{2,206,744}{657,729}+c=33 \Rightarrow c=33-\frac{2,206,744}{657,729} \approx 22.9347$$

The relationship is $C=-.0000107567S^2+.034896S+22.9347$

(b) We need to solve the equation $-.0000107567S^2+.034896S$
$+22.9347=45$

$$-.0000107567S^2+.034896S+22.9347=45 \Rightarrow .0000107567S^2-.034896S+22.0653=0$$

Use the quadratic formula, where $a=.0000107567, b=-.034896,$ and $c=22.0653.$

$$S=\frac{-(-.034896)\pm\sqrt{(-.034896)^2-4(.0000107567)(22.0653)}}{2(.0000107567)} \Rightarrow S\approx 2383 \text{ or } S\approx 861$$

The top speed of 2383 knots is not reasonable. We use the top speed of 861 knots.

89. Let x = one number; let y = the other number We have the following system of equations.
$x+y=47$ (1)
$x-y=1$ (2)
Add the two equations to eliminate y.
$2x=48 \Rightarrow x=24$. Substitute this value into equation (2) and solve for y:
$24+y=47 \Rightarrow y=23$.
The two numbers are 23 and 24.

90. Let x = the cost of a goat; let y = the cost of a sheep. We have the following system of equations:
$6x+5y=305$ (1)
$2x+9y=285$ (2)
Multiply equation (1) by 9, and multiply equation (2) by −5, then add the resulting equations to eliminate y:
$54x+45y=2745$ (3)
$-10x-45y=-1425$ (4)
$44x = 1320 \Rightarrow x=30$
Substitute this value into equation (1) and solve for y:
$6(30)+5y=305 \Rightarrow 5y=125 \Rightarrow y=25$
Verify that (30, 25) satisfies both equations in the original system.
Goats cost \$30 each and sheep cost \$25 each.

91. Let x = the FCI for football; let y = the FCI for baseball. We have the following system of equations.
$\frac{x+y}{2}=250.51 \Rightarrow x+y=501.02$ (1)
$x-y=158.62$ (2)
Add the two equations to eliminate y.
$2x=659.64 \Rightarrow x=329.82$
Substitute this value into equation (2) and solve for y:
$329.82-y=158.62 \Rightarrow y=171.20$
Verify that (329.82, 171.20) satisfies both equations in the original system.
The FCI for football was \$329.82, and the FCI for baseball was \$171.20.

92. Let x = the number of ones; let y = the number of fives; let z = the number of twenties. We have the following system of equations.
$x+y+z=30$ (1)
$z-x=9$ (2)
$x+5y+20z=351$ (3)
Solve equation (2) for z and substitute this value for z into equations (1) and (3):
$z=x+9$
$x+y+(x+9)=30 \Rightarrow$
$2x+y=21$ (4)
$x+5y+20(x+9)=351 \Rightarrow$
$21x+5y=171$ (5)

(*continued on next page*)

(*continued from page 869*)

Solve equation (4) for *y*, then substitute this value into equation (5) to solve for *x*:

$$2x + y = 21 \Rightarrow y = -2x + 21$$
$$21x + 5(-2x + 21) = 171 \Rightarrow 11x + 105 = 171 \Rightarrow$$
$$11x = 66 \Rightarrow x = 6$$

Substitute *x* = 6 into equation (2) to solve for *z*: $z - 6 = 9 \Rightarrow z = 15$

Substitute *x* = 6 and *z* = 15 into equation (1) to solve for *y*: $6 + y + 15 = 30 \Rightarrow y = 9$

Verify that (6, 9, 15) satisfies all three equations in the original system.
The cashier had 6 ones, 9 fives, and 15 twenties.

93. Let *x* = the number of \$3.00 gallons;
y = the number of \$4.50 gallons;
z = the number of \$9.00 gallons.
We have the following equations.

$$x + y + z = 300 \quad (1)$$
$$2x = y \Rightarrow$$
$$2x - y = 0 \quad (2)$$
$$3.00x + 4.50y + 9.00z = 6.00(300) \Rightarrow$$
$$3x + 4.5y + 9z = 1800 \quad (3)$$

Multiply equation (1) by –9 and add to equation (3) in order to eliminate *z*.

$$\begin{array}{r} -9x - 9y - 9z = -2700 \\ 3x + 4.5y + 9z = 1800 \\ \hline -6x - 4.5y = -900 \text{ (4)} \end{array}$$

Multiply equation (2) by 3 and add to equation (4) in order to eliminate *x*.

$$\begin{array}{r} 6x - 3y = 0 \\ -6x - 4.5y = -900 \\ \hline -7.5y = -900 \Rightarrow y = 120 \end{array}$$

Substitute this value into equation (2) to solve for *x*.

$$2x - 120 = 0 \Rightarrow 2x = 120 \Rightarrow x = 60$$

Substitute $x = 60$ and $y = 120$ into equation (1) and solve for *y*.

$$60 + 120 + z = 300 \Rightarrow 180 + z = 300 \Rightarrow z = 120$$

She should use 60 gal of the \$3.00 water, 120 gal of the \$4.50 water, and 120 gal of the \$9.00 water.

94. Let *x* = the number of barrels of \$150 glue;
y = the number of barrels of \$190 glue.
There are a total of $x + y + 150$ barrels that will be produced which the company wants to sell for \$120 per barrel. One equation will be $x = y$. (1) The other equation will be $150x + 190y + 100(150) = 120(x + y + 150)$.
This simplifies as follows.

$$150x + 190y + 100(150) = 120(x + y + 150)$$
$$150x + 190y + 15{,}000 = 120x + 120y + 18{,}000$$
$$30x + 70y = 3000 \text{ (2)}$$

Replace *y* with *x* in equation (2), and solve for *x*.

$$30x + 70y = 3000 \Rightarrow 30x + 70x = 3000 \Rightarrow$$
$$100x = 3000 \Rightarrow x = 30$$

30 barrels each of \$150 and \$190 glue are needed. A total of $30 + 30 + 150 = 210$ barrels of \$120 glue will be produced.

95. Let *x* = the length of the shortest side;
y = the length of the medium side;
z = the length of the longest side.
We have the following equations.

$$x + y + z = 59 \quad (1)$$
$$z = y + 11 \Rightarrow \quad -y + z = 11 \quad (2)$$
$$y = x + 3 \Rightarrow -x + y = 3 \quad (3)$$

Add equations (1) and (3) together in order to eliminate *x*.

$$\begin{array}{r} x + y + z = 59 \\ -x + y = 3 \\ \hline 2y + z = 62 \text{ (4)} \end{array}$$

Multiply equation (2) by 2 and then add the result to equation (4) to solve for *z*.

$$\begin{array}{r} -2y + 2z = 22 \\ 2y + z = 62 \\ \hline 3z = 84 \Rightarrow z = 28 \end{array}$$

Substitute this value into equation (2) in order to solve for *y*.

$$-y + 28 = 11 \Rightarrow -y = -17 \Rightarrow y = 17$$

Substitute $y = 17$ into equation (3) in order to solve for *x*: $-x + 17 = 3 \Rightarrow -x = -14 \Rightarrow x = 14$

The lengths of the sides of the triangle are 14 inches, 17 inches, and 28 inches.
Note: $14 + 17 + 28 = 59$

96. Let *x* = the measure of the largest angle;
y = the measure of the medium angle;
z = the measure of the smallest angle.
We have the following equations.

$$x + y + z = 180 \quad (1)$$
$$x = 2y - 55 \Rightarrow x - 2y = -55 \quad (2)$$
$$z = y - 25 \Rightarrow -y + z = -25 \quad (3)$$

Multiply equation (3) by -1 and add to equation (1) to solve for *z*.

$$\begin{array}{r} y - z = 25 \\ x + y + z = 180 \\ \hline x + 2y = 205 \text{ (4)} \end{array}$$

Add equations (2) and (4) in order to solve for *x*.

$$\begin{aligned} x - 2y &= -55 \\ x + 2y &= 205 \\ \hline 2x \quad &= 150 \Rightarrow x = 75 \end{aligned}$$

Substitute this value into equation (2) in order to solve for *y*.

$75 - 2y = -55 \Rightarrow -2y = -130 \Rightarrow y = 65$

Substitute this value into equation (3) in order to solve for *z*: $-65 + z = -25 \Rightarrow z = 40$

The angle measures are 75°, 65°, and 40°.

Note: $75 + 65 + 40 = 180$

97. Let *x* = the amount invested in real estate (at 3%); *y* = the amount invested in a money market account (at 2.5%); *z* = the amount invested in CDs (at 1.5%).

Completing the table we have the following.

	Amount Invested	**Rate (as a decimal)**	**Annual Interest**
Real Estate	x	.03	$.03x$
Money Market	y	.025	$.025y$
CDs	z	.015	$.015z$

We have the following equations.

$$\begin{aligned} x + y + z &= 200{,}000 \qquad (1) \\ z = x + y - 80{,}000 &\Rightarrow \\ x + y - z &= 80{,}000 \qquad (2) \\ .03x + .025y + .015z = 4900 &\Rightarrow \\ 30x + 25y + 15z &= 4{,}900{,}000 \qquad (3) \end{aligned}$$

Add equations (1) and (2) in order to eliminate *z*.

$$\begin{aligned} x + y + z &= 200{,}000 \\ x + y - z &= 80{,}000 \\ \hline 2x + 2y \quad &= 280{,}000 \text{ or } x + y = 140{,}000 \ (4) \end{aligned}$$

Multiply equation (2) by 15 and add the result to equation (3) in order to eliminate *z*.

$$\begin{aligned} 15x + 15y - 15z &= 1{,}200{,}000 \\ 30x + 25y + 15z &= 4{,}900{,}000 \\ \hline 45x + 40y \quad &= 6{,}100{,}000 \text{ or} \\ 9x + 8y &= 1{,}220{,}000 \qquad (5) \end{aligned}$$

Multiply equation (4) by -8 and add the result to equation (5).

$$\begin{aligned} -8x - 8y &= -1{,}120{,}000 \\ 9x + 8y &= 1{,}220{,}000 \\ \hline x \quad &= 100{,}000 \end{aligned}$$

Substitute this value into equation (4) in order to solve for *y*.

$100{,}000 + y = 140{,}000 \Rightarrow y = 40{,}000$

Substitute $x = 100{,}000$ and $y = 40{,}000$ into equation (1) in order to solve for *z*.

$$\begin{aligned} 100{,}000 + 40{,}000 + z &= 200{,}000 \\ 140{,}000 + z &= 200{,}000 \Rightarrow z = 60{,}000 \end{aligned}$$

The amounts invested were \$100,000 at 3% (real estate), \$40,000 at 2.5% (money market), and \$60,000 at 1.5% (CDs).

98. Let x = the amount invested in real estate (at 2%); *y* = the amount invested in a money market account (at 2.5%); *z* = the amount invested in CDs (at 1.25%).

Completing the table we have the following.

	Amount Invested	**Rate (as a decimal)**	**Annual Interest**
Mutual Funds	x	.02	$.02x$
Government Bonds	y	.025	$.025y$
Savings Account	z	.0125	$.0125z$

We have the following equations.

$$\begin{aligned} x + y + z &= 40{,}000 \qquad (1) \\ y = 2x &\Rightarrow \\ 2x - y &= 0 \qquad (2) \\ .02x + .025y + .0125z = 825 &\Rightarrow \\ 200x + 250y + 125z &= 8{,}250{,}000 \qquad (3) \end{aligned}$$

Multiply equation (1) by -125 and add the result to equation (3) in order to eliminate *z*.

$$\begin{aligned} -125x - 125y - 125z &= -5{,}000{,}000 \\ 200x + 250y + 125z &= 8{,}250{,}000 \\ \hline 75x + 125y \quad &= 3{,}250{,}000 \text{ or} \\ 3x + 5y &= 130{,}000 \qquad (4) \end{aligned}$$

Multiply equation (2) by 5 and add the result to equation (4).

$$\begin{aligned} 10x - 5y &= 0 \\ 3x + 5y &= 130{,}000 \\ \hline 13x \quad &= 130{,}000 \Rightarrow x = 10{,}000 \end{aligned}$$

Substitute this value into equation (2) in order to solve for *y*.

$$\begin{aligned} 2(10{,}000) - y = 0 \Rightarrow 20{,}000 - y = 0 &\Rightarrow \\ 20{,}000 &= y \end{aligned}$$

Substitute $x = 10{,}000$ and $y = 20{,}000$ into equation (1) in order to solve for *z*.

$$\begin{aligned} 10{,}000 + 20{,}000 + z = 40{,}000 &\Rightarrow \\ 30{,}000 + z = 40{,}000 &\Rightarrow z = 10{,}000 \end{aligned}$$

The amounts invested were \$10,000 at 2% (mutual funds), \$20,000 at 2.5% (government bonds), and \$10,000 at 1.25% (savings account).

99. $p = 16 - \frac{5}{4}q$

(a) $p = 16 - \frac{5}{4}(0) = 16 - 0 = 16$
The price is \$16.

(b) $p = 16 - \frac{5}{4}(4) = 16 - 5 = 16 - 5 = 11$
The price is \$11.

(c) $p = 16 - \frac{5}{4}(8) = 16 - 10 = 6$
The price is \$6.

100. $p = 16 - \frac{5}{4}q$

(a) $6 = 16 - \frac{5}{4}q$
$-10 = -\frac{5}{4}q \Rightarrow -40 = -5q \Rightarrow 8 = q$
The demand is 8 units.

(b) $11 = 16 - \frac{5}{4}q$
$-5 = -\frac{5}{4}q \Rightarrow -20 = -5q \Rightarrow 4 = q$
The demand is 4 units.

(c) $16 = 16 - \frac{5}{4}q \Rightarrow 0 = -\frac{5}{4}q \Rightarrow 0 = q$
The demand is 0 units.

101. See answer to Exercise 103.

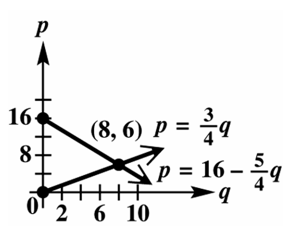
102. $p = \frac{3}{4}q$

(a) $0 = \frac{3}{4}q \Rightarrow 0 = q$

(b) $10 = \frac{3}{4}q \Rightarrow 40 = 3q \Rightarrow q = \frac{40}{3}$

(c) $20 = \frac{3}{4}q \Rightarrow 80 = 3q \Rightarrow q = \frac{80}{3}$

103.



104. To find the equilibrium supply, solve the following system.
$p = 16 - \frac{5}{4}q$ (1)
$p = \frac{3}{4}q.$ (2)

Replace p with $16 - \frac{5}{4}q$ in equation (2), and solve for q. The value of q will give the equilibrium demand.
$16 - \frac{5}{4}q = \frac{3}{4}q$
$4\left(\frac{3}{4}q\right) = 4\left(16 - \frac{5}{4}q\right)$
$3q = 64 - 5q \Rightarrow 8q = 64 \Rightarrow q = 8$

The equilibrium demand is 8.
To find p, substitute $q = 8$ into equation (2).
$p = \frac{3}{4}(8) = 6$
The equilibrium price is \$6.

105.
$25x + 40y + 20z = 2200$ (4)
$4x + 2y + 3z = 280$ (5)
$3x + 2y + z = 180$ (6)

Eliminate z by multiplying equation (6) by -3 and add to equation (5).
$$\begin{array}{l} -9x - 6y - 3z = -540 \\ \underline{4x + 2y + 3z = 280} \\ -5x - 4y = -260 \quad (7) \end{array}$$

Eliminate z by multiplying equation (6) by -20 and add to equation (4).
$$\begin{array}{l} -60x - 40y - 20z = -3600 \\ \underline{25x + 40y + 20z = 2200} \\ -35x = -1400 \Rightarrow x = 40 \end{array}$$

Substitute this value in equation (7) in order to solve for y.
$-5(40) - 4y = -260 \Rightarrow -200 - 4y = -260 \Rightarrow$
$-4y = -60 \Rightarrow y = 15$

Substitute $x = 40$ and $y = 15$ into equation (6) to solve for z.
$3(40) + 2(15) + z = 180 \Rightarrow$
$120 + 30 + z = 180 \Rightarrow$
$150 + z = 180 \Rightarrow z = 30$

Solution set: $\{(40, 15, 30)\}$

106. *Check*:
$25x + 40y + 20z = 2200$ (4)
$25(40) + 40(15) + 20(30) = 2200$?
$1000 + 600 + 600 = 2200$
$2200 = 2200$ True

$4x + 2y + 3z = 280$ (5)
$4(40) + 2(15) + 3(30) = 280$?
$160 + 30 + 90 = 280$
$280 = 280$ True

$3x + 2y + z = 180$ (6)
$3(40) + 2(15) + 30 = 180$?
$120 + 30 + 30 = 180$
$180 = 180$ True

$(40, 15, 30)$ satisfies all three equations.

107. Let x = the number of pounds of Arabian Mocha Sanai; y = the number of pounds of Organic Shade-Grown Mexico; z = the number of pounds of Guatemala Antigua. Completing the table we have the following.

	Number of Pounds	Cost per Pound	Total Cost
Arabian Mocha	x	15.99	$15.99x$
Organic Mexico	y	12.99	$12.99y$
Guatemala Antigua	z	10.19	10.19z
Total	50	12.37	50(12.37) = 618.50

We have the following equations.

$$x + y + z = 50 \qquad (1)$$
$$z = 2x \qquad (2)$$
$$15.99x + 12.99y + 10.19z = 618.50 \qquad (3)$$

Substitute $z = 2x$ into equations (1) and (3):

$$x + y + 2x = 50 \Rightarrow$$
$$3x + y = 50 \qquad (4)$$
$$15.99x + 12.99y + 10.19(2x) = 618.50 \Rightarrow$$
$$36.37x + 12.99y = 618.50 \qquad (5)$$

Solve equation (4) for y, then substitute that value into equation (5) and solve for x:

$3x + y = 50 \Rightarrow y = -3x + 50$

$$36.37x + 12.99(-3x + 50) = 618.50 \Rightarrow$$
$$36.37x - 38.97x + 649.50 = 618.50 \Rightarrow$$
$$-2.6x = -31 \Rightarrow$$
$$x \approx 11.9231 \approx 11.92$$

Substitute $x = 11.9231$ into equation (2) to solve for z: $z \approx 2(11.9231) \approx 23.85$

Substitute the values for x and z into equation (1) to solve for y:

$11.92 + y + 23.85 = 50 \Rightarrow y \approx 14.23$

11.92 pounds of Arabian Mocha Sanani, 14.23 pounds of Organic Shade-Grown Mexico, and 23.85 pounds of Guatemala Antigua are needed. (Answers are approximations.)

108. Let x = the number of pounds of Arabian Mocha Sanai; y = the number of pounds of Organic Shade-Grown Mexico; z = the number of pounds of Guatemala Antigua. Completing the table we have the following.

	Number of Pounds	Cost per Pound	Total Cost
Arabian Mocha	x	15.99	$15.99x$
Organic Mexico	y	12.99	$12.99y$
Guatemala Antigua	z	12.49	12.49z
Total	50	12.37	50(12.37) = 618.50

We have the following equations.

$$x + y + z = 50 \qquad (1)$$
$$z = 2x \qquad (2)$$
$$15.99x + 12.99y + 12.49z = 618.50 \qquad (3)$$

Substitute $z = 2x$ into equations (1) and (3):

$$x + y + 2x = 50 \Rightarrow$$
$$3x + y = 50 \qquad (4)$$
$$15.99x + 12.99y + 12.49(2x) = 618.50 \Rightarrow$$
$$40.97x + 12.99y = 618.50 \qquad (5)$$

Solve equation (4) for y, then substitute that value into equation (5) and solve for x:

$3x + y = 50 \Rightarrow y = -3x + 50$

$$40.97x + 12.99(-3x + 50) = 618.50 \Rightarrow$$
$$40.97x - 38.97x + 649.50 = 618.50 \Rightarrow$$
$$2x = -31 \Rightarrow x = -15.5$$

Substitute $x = -15.5$ into equation (2) to solve for z: $z = 2(-15.5) = -31$

Substitute the values for x and z into equation (1) to solve for y:

$-15.5 + y - 31 = 50 \Rightarrow y = 96.5$

−15.5 pounds of Arabian Mocha Sanai, 96.5 pounds of Organic Shade-Grown Mexico, and −31 pounds of Guatemala Antigua are needed. The answer is not reasonable, since the mixture cannot include negative numbers of ingredients.

Section 9.2: Matrix Solution of Linear Systems

Connections (page 861)

1. $T(n)=\frac{2}{3}n^3+\frac{3}{2}n^2-\frac{7}{6}n$

Continuing in this manner we have the following.

n	T
3	28
6	191
10	805
29	17,487
100	681,550
200	5,393,100
400	42,906,200
1000	668,165,500
5000	8.3×10^{10}
10,000	6.7×10^{11}
100,000	6.7×10^{14}

2. Using the tables he would have to do 17,487 operations. Yes, this is too many to do by hand.

3. If the number of variables doubles, the number of operations increases by a factor of 8. 100 variables require 681,550 operations and 200 variables requires 5,393,100 operations. The ratio $\frac{5{,}393{,}100}{681{,}550}\approx 7.91\approx 8$.

4. A system of 100,000 variables has 6.7×10^{14} operations. If a Cray-T90 does 60 billion $=6\times10^{10}$ operations per second, then the system would take $\frac{6.7\times10^{14}}{6\times10^{10}}\approx 11{,}166.67$ sec. Since there are 3600 seconds in 1 hr, the system would take $\frac{11{,}166.67}{3600}\approx 3.1$ hr.

Exercises

1. $\begin{bmatrix}2&4\\4&7\end{bmatrix}$; -2 times row 1 added to row 2

$$\begin{bmatrix}2&4\\4+(-2)(2)&7+(-2)(4)\end{bmatrix}=\begin{bmatrix}2&4\\0&-1\end{bmatrix}$$

2. $\begin{bmatrix}-1&4\\7&0\end{bmatrix}$; 7 times row 1 added to row 2

$$\begin{bmatrix}-1&4\\7+7(-1)&0+7(4)\end{bmatrix}=\begin{bmatrix}-1&4\\0&28\end{bmatrix}$$

3. $\begin{bmatrix}1&5&6\\-2&3&-1\\4&7&0\end{bmatrix}$; 2 times row 1 added to row 2

$$\begin{bmatrix}1&5&6\\-2+2(1)&3+2(5)&-1+2(6)\\4&7&0\end{bmatrix}=\begin{bmatrix}1&5&6\\0&13&11\\4&7&0\end{bmatrix}$$

4. $\begin{bmatrix}2&5&6\\4&-1&2\\3&7&1\end{bmatrix}$; -6 times row 3 added to row 1

$$\begin{bmatrix}2+(-6)(3)&5+(-6)(7)&6+(-6)(1)\\4&-1&2\\3&7&1\end{bmatrix}=\begin{bmatrix}-16&-37&0\\4&-1&2\\3&7&1\end{bmatrix}$$

5. $2x+3y=11$
$x+2y=8$

The augmented matrix is $\left[\begin{array}{cc|c}2&3&11\\1&2&8\end{array}\right]$. The size is 2×3.

6. $3x+5y=-13$
$2x+3y=-9$

The augmented matrix is $\left[\begin{array}{cc|c}3&5&-13\\2&3&-9\end{array}\right]$. The size is 2×3.

7. $2x+y+z-3=0$
$3x-4y+2z+7=0$
$x+y+z-2=0$

Each equation of a linear system must have the constant term isolated on one side of the equal sign. Rewriting this system, we have
$2x+y+z=3$
$3x-4y+2z=-7$
$x+y+z=2$

The augmented matrix is $\left[\begin{array}{ccc|c}2&1&1&3\\3&-4&2&-7\\1&1&1&2\end{array}\right]$.

The size is 3×4.

8. $4x-2y+3z-4=0$
$3x+5y+\ \ z-7=0$
$5x-\ \ y+4z-7=0$

Each equation of a linear system must have the constant term isolated on one side of the equal sign. Rewriting this system we have

$4x-2y+3z=4$
$3x+5y+\ \ z=7$
$5x-\ \ y+4z=7.$

The augmented matrix is $\left[\begin{array}{ccc|c}4&-2&3&4\\3&5&1&7\\5&-1&4&7\end{array}\right]$.

The size is 3×4.

9. $\left[\begin{array}{ccc|c}3&2&1&1\\0&2&4&22\\-1&-2&3&15\end{array}\right]$ is associated with the system $\begin{aligned}3x+2y+z&=1\\2y+4z&=22\\-x-2y+3z&=15.\end{aligned}$

10. $\left[\begin{array}{ccc|c}2&1&3&12\\4&-3&0&10\\5&0&-4&-11\end{array}\right]$ is associated with the system $\begin{aligned}2x+y+3z&=12\\4x-3y&=10\\5x-4z&=-11.\end{aligned}$

11. $\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&3\\0&0&1&-2\end{array}\right]$ is associated with the system

$x=\ \ 2$
$y=\ \ 3$
$z=-2.$

12. $\left[\begin{array}{ccc|c}1&0&0&4\\0&1&0&2\\0&0&1&3\end{array}\right]$ is associated with the system

$x=4$
$y=2$
$z=3.$

13. $\left[\begin{array}{ccc|c}1&1&0&3\\0&2&1&-4\\1&0&-1&5\end{array}\right]$ is associated with the system $\begin{aligned}x+y&=3\\2y+z&=-4\\x-z&=5.\end{aligned}$

14. $\left[\begin{array}{ccc|c}2&0&1&9\\0&-1&-1&5\\3&1&0&8\end{array}\right]$ is associated with the system

$2x+z=9$
$-y-z=5$
$3x+y=8.$

15. $x+y=5$
$x-y=-1$

This system has the augmented matrix

$\left[\begin{array}{cc|c}1&1&5\\1&-1&-1\end{array}\right]$.

$\left[\begin{array}{cc|c}1&1&5\\0&-2&-6\end{array}\right]\begin{matrix}\\-1\text{R1}+\text{R2}\end{matrix}\Rightarrow$

$\left[\begin{array}{cc|c}1&1&5\\0&1&3\end{array}\right]\begin{matrix}\\-\frac{1}{2}\text{R2}\end{matrix}\Rightarrow\left[\begin{array}{cc|c}1&0&2\\0&1&3\end{array}\right]\begin{matrix}-1\text{R2}+\text{R1}\\ \\ \end{matrix}$

Solution set: $\{(2,3)\}$

16. $x+2y=5$
$2x+\ \ y=-2$

This system has the augmented matrix

$\left[\begin{array}{cc|c}1&2&5\\2&1&-2\end{array}\right]$.

$\left[\begin{array}{cc|c}1&2&5\\0&-3&-12\end{array}\right]\begin{matrix}\\-2\text{R1}+\text{R2}\end{matrix}\Rightarrow$

$\left[\begin{array}{cc|c}1&2&5\\0&1&4\end{array}\right]\begin{matrix}\\-\frac{1}{3}\text{R2}\end{matrix}\Rightarrow\left[\begin{array}{cc|c}1&0&-3\\0&1&4\end{array}\right]\begin{matrix}-2\text{R2}+\text{R1}\\ \\ \end{matrix}$

Solution set: $\{(-3,4)\}$

17. $3x+2y=-9$
$2x-5y=-6$

This system has the augmented matrix

$\left[\begin{array}{cc|c}3&2&-9\\2&-5&-6\end{array}\right]$.

$\left[\begin{array}{cc|c}1&7&-3\\2&-5&-6\end{array}\right]\begin{matrix}-1\text{R2}+\text{R1}\\ \\ \end{matrix}\Rightarrow$

$\left[\begin{array}{cc|c}1&7&-3\\0&-19&0\end{array}\right]\begin{matrix}\\-2\text{R1}+\text{R2}\end{matrix}\Rightarrow$

$\left[\begin{array}{cc|c}1&7&-3\\0&1&0\end{array}\right]\begin{matrix}\\-\frac{1}{19}\text{R2}\end{matrix}\Rightarrow$

$\left[\begin{array}{cc|c}1&0&-3\\0&1&0\end{array}\right]\begin{matrix}-7\text{R2}+\text{R1}\\ \\ \end{matrix}$

Solution set: $\{(-3,0)\}$

18. $2x-3y=10$
$2x+2y=5$

This system has the augmented matrix

$$\left[\begin{array}{cc|c}2 & -3 & 10\\ 2 & 2 & 5\end{array}\right].$$

$$\left[\begin{array}{cc|c}1 & -\frac{3}{2} & 5\\ 2 & 2 & 5\end{array}\right]\tfrac{1}{2}\text{R1} \Rightarrow \left[\begin{array}{cc|c}1 & -\frac{3}{2} & 5\\ 0 & 5 & -5\end{array}\right]-2\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & -\frac{3}{2} & 5\\ 0 & 1 & -1\end{array}\right]\tfrac{1}{5}\text{R2} \Rightarrow \left[\begin{array}{cc|c}1 & 0 & \frac{7}{2}\\ 0 & 1 & -1\end{array}\right]\tfrac{3}{2}\text{R2+R1}$$

Solution set: $\left\{\left(\frac{7}{2},-1\right)\right\}$

19. $6x+y-5=0$
$5x+y-3=0$

Rewrite the system as $\begin{array}{l}6x+y=5\\ 5x+y=3\end{array}$

The augmented matrix is $\left[\begin{array}{cc|c}6 & 1 & 5\\ 5 & 1 & 3\end{array}\right]$.

$$\left[\begin{array}{cc|c}1 & \frac{1}{6} & \frac{5}{6}\\ 5 & 1 & 3\end{array}\right]\tfrac{1}{6}\text{R1} \Rightarrow \left[\begin{array}{cc|c}1 & \frac{1}{6} & \frac{5}{6}\\ 0 & \frac{1}{6} & -\frac{7}{6}\end{array}\right]-5\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & \frac{1}{6} & \frac{5}{6}\\ 0 & 1 & -7\end{array}\right]6\text{R2} \Rightarrow \left[\begin{array}{cc|c}1 & 0 & 2\\ 0 & 1 & -7\end{array}\right]-\tfrac{1}{6}\text{R2}+\text{R1}$$

or

$$\left[\begin{array}{cc|c}1 & 0 & 2\\ 5 & 1 & 3\end{array}\right]-1\text{R2+R1} \Rightarrow \left[\begin{array}{cc|c}1 & 0 & 2\\ 0 & 1 & -7\end{array}\right]-5\text{R1+R2}$$

Solution set: $\{(2,-7)\}$

20. $2x-5y-10=0$
$3x+y-15=0$

Rewrite the system as $\begin{array}{l}2x-5y=10\\ 3x+y=15.\end{array}$

The augmented matrix is $\left[\begin{array}{cc|c}2 & -5 & 10\\ 3 & 1 & 15\end{array}\right]$.

$$\left[\begin{array}{cc|c}1 & -\frac{5}{2} & 5\\ 3 & 1 & 15\end{array}\right]\tfrac{1}{2}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & -\frac{5}{2} & 5\\ 0 & \frac{17}{2} & 0\end{array}\right]-3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & -\frac{5}{2} & 5\\ 0 & 1 & 0\end{array}\right]\tfrac{2}{17}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & 0 & 5\\ 0 & 1 & 0\end{array}\right]\tfrac{5}{2}\text{R2}+\text{R1}$$

or

$$\left[\begin{array}{cc|c}-1 & -6 & -5\\ 3 & 1 & 15\end{array}\right]-1\text{R2}+\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & 6 & 5\\ 3 & 1 & 15\end{array}\right]-1\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & 6 & 5\\ 0 & -17 & 0\end{array}\right]-3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & 6 & 5\\ 0 & 1 & 0\end{array}\right]-\tfrac{1}{17}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & 0 & 5\\ 0 & 1 & 0\end{array}\right]-6\text{R2}+\text{R1}$$

Solution set: $\{(5,0)\}$

21. $2x-y=6$
$4x-2y=0$

This system has the augmented matrix

$$\left[\begin{array}{cc|c}2 & -1 & 6\\ 4 & -2 & 0\end{array}\right].$$

$$\left[\begin{array}{cc|c}1 & -\frac{1}{2} & 3\\ 4 & -2 & 0\end{array}\right]\tfrac{1}{2}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & -\frac{1}{2} & 3\\ 0 & 0 & -12\end{array}\right]-4\text{R1+R2}$$

The second row of the augmented matrix corresponds to the equation $0x+0y=-12$, which has no solution. Thus, the solution set is $\varnothing$.

22. $3x-2y=1$
$6x-4y=-1$

This system has the augmented matrix

$$\left[\begin{array}{cc|c}3 & -2 & 1\\ 6 & -4 & -1\end{array}\right].$$

$$\left[\begin{array}{cc|c}1 & -\frac{2}{3} & \frac{1}{3}\\ 6 & -4 & -1\end{array}\right]\tfrac{1}{3}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c}1 & -\frac{2}{3} & \frac{1}{3}\\ 0 & 0 & -3\end{array}\right]-6\text{R1+R2}$$

The second row of the augmented matrix corresponds to the equation $0x+0y=-3$, which has no solution. Thus, the solution set is $\varnothing$.

23. $3x-4y=7$
$-6x+8y=-14$

This system has the augmented matrix
$$\left[\begin{array}{cc|c}3&-4&7\\-6&8&-14\end{array}\right]$$
$$\left[\begin{array}{cc|c}1&-\frac{4}{3}&\frac{7}{3}\\-6&8&-14\end{array}\right]\tfrac{1}{3}\text{R1}\Rightarrow$$
$$\left[\begin{array}{cc|c}1&-\frac{4}{3}&\frac{7}{3}\\0&0&0\end{array}\right]6R1+R2$$
It is impossible to go further. The equation that corresponds to the first row in the final matrix is $x-\frac{4}{3}y=\frac{7}{3}\Rightarrow x=\frac{4}{3}y+\frac{7}{3}$

Solution set: $\left\{\frac{4}{3}y+\frac{7}{3},y\right\}$

24. $\frac{1}{2}x+\frac{3}{5}y=\frac{1}{4}$
$10x+12y=5$

This system has the augmented matrix
$$\left[\begin{array}{cc|c}\frac{1}{2}&\frac{3}{5}&\frac{1}{4}\\10&12&5\end{array}\right]$$
$$\left[\begin{array}{cc|c}1&\frac{6}{5}&\frac{1}{2}\\10&12&5\end{array}\right]2\text{R1}\Rightarrow$$
$$\left[\begin{array}{cc|c}1&\frac{6}{5}&\frac{1}{2}\\0&0&0\end{array}\right]-10R1+R2$$
It is impossible to go further. The equation that corresponds to the first row in the final matrix is $x+\frac{6}{5}y=\frac{1}{2}\Rightarrow x=-\frac{6}{5}y+\frac{1}{2}$

Solution set: $\left\{-\frac{6}{5}y+\frac{1}{2},y\right\}$

25. $\begin{aligned}x+\ \ y-5z&=-18\\3x-3y+\ \ z&=6\\x+3y-2z&=-13\end{aligned}$; This system has the augmented matrix $\left[\begin{array}{ccc|c}1&1&-5&-18\\3&-3&1&6\\1&3&-2&-13\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&1&-5&-18\\0&-6&16&60\\1&3&-2&-13\end{array}\right]-3\text{R1}+\text{R2}\Rightarrow\left[\begin{array}{ccc|c}1&1&-5&-18\\0&-6&16&60\\0&2&3&5\end{array}\right]-1\text{R1}+\text{R3}\Rightarrow\left[\begin{array}{ccc|c}1&1&-5&-18\\0&1&-\frac{8}{3}&-10\\0&2&3&5\end{array}\right]-\tfrac{1}{6}\text{R2}$$
$$\left[\begin{array}{ccc|c}1&0&-\frac{7}{3}&-8\\0&1&-\frac{8}{3}&-10\\0&2&3&5\end{array}\right]-1\text{R2+R1}\Rightarrow\left[\begin{array}{ccc|c}1&0&-\frac{7}{3}&-8\\0&1&-\frac{8}{3}&-10\\0&0&\frac{25}{3}&25\end{array}\right]-2\text{R2+R3}\Rightarrow\left[\begin{array}{ccc|c}1&0&-\frac{7}{3}&-8\\0&1&-\frac{8}{3}&-10\\0&0&1&3\end{array}\right]\tfrac{3}{25}\text{R3}$$
$$\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&-\frac{8}{3}&-10\\0&0&1&3\end{array}\right]\tfrac{7}{3}\text{R3+R1}\Rightarrow\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&-2\\0&0&1&3\end{array}\right]\tfrac{8}{3}\text{R3+R2}$$
Solution set: $\{(-1,-2,3)\}$

26. $\begin{aligned}-x+2y+6z&=2\\3x+2y+6z&=6\\x+4y-3z&=1\end{aligned}$; This system has the augmented matrix $\left[\begin{array}{ccc|c}-1&2&6&2\\3&2&6&6\\1&4&-3&1\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&-2&-6&-2\\3&2&6&6\\1&4&-3&1\end{array}\right]-1\text{R1}\Rightarrow\left[\begin{array}{ccc|c}1&-2&-6&-2\\0&8&24&12\\1&4&-3&1\end{array}\right]-3\text{R1}+\text{R2}\Rightarrow\left[\begin{array}{ccc|c}1&-2&-6&-2\\0&8&24&12\\0&6&3&3\end{array}\right]-1\text{R1}+\text{R3}$$
$$\left[\begin{array}{ccc|c}1&-2&-6&-2\\0&1&3&\frac{3}{2}\\0&6&3&3\end{array}\right]\tfrac{1}{8}\text{R2}\Rightarrow\left[\begin{array}{ccc|c}1&0&0&1\\0&1&3&\frac{3}{2}\\0&6&3&3\end{array}\right]2\text{R2}+\text{R1}\Rightarrow\left[\begin{array}{ccc|c}1&0&0&1\\0&1&3&\frac{3}{2}\\0&0&-15&-6\end{array}\right]-6\text{R2}+\text{R3}$$
$$\left[\begin{array}{ccc|c}1&0&0&1\\0&1&3&\frac{3}{2}\\0&0&1&\frac{2}{5}\end{array}\right]-\tfrac{1}{15}\text{R3}\Rightarrow\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&\frac{3}{10}\\0&0&1&\frac{2}{5}\end{array}\right]-3\text{R3+R2}$$
Solution set: $\left\{\left(1,\frac{3}{10},\frac{2}{5}\right)\right\}$

27. $\begin{aligned} x + \ \ y - \ \ z &= \ \ 6 \\ 2x - \ \ y + \ \ z &= -9 \\ x - 2y + 3z &= \ \ 1 \end{aligned}$; This system has the augmented matrix $\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 2 & -1 & 1 & -9 \\ 1 & -2 & 3 & 1 \end{array}\right]$.

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & -3 & 3 & -21 \\ 1 & -2 & 3 & 1 \end{array}\right] \begin{array}{l} \\ -2\text{R}1 + \text{R}2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & -3 & 3 & -21 \\ 0 & -3 & 4 & -5 \end{array}\right] \begin{array}{l} \\ \\ -1\text{R}1 + \text{R}3 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 7 \\ 0 & -3 & 4 & -5 \end{array}\right] \begin{array}{l} \\ -\frac{1}{3}\text{R}2 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 7 \\ 0 & -3 & 4 & -5 \end{array}\right] \begin{array}{l} -1\text{R}2 + \text{R}1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & 16 \end{array}\right] \begin{array}{l} \\ \\ 3\text{R}2 + \text{R}3 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & 16 \end{array}\right] \begin{array}{l} \\ \text{R}3 + \text{R}2 \\ \\ \end{array}$$

Solution set: $\{(-1, 23, 16)\}$

28. $\begin{aligned} x + 3y - 6z &= \ \ 7 \\ 2x - \ \ y + \ \ z &= \ \ 1 \\ x + 2y + 2x &= -1 \end{aligned}$; This system has the augmented matrix $\left[\begin{array}{rrr|r} 1 & 3 & -6 & 7 \\ 2 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \end{array}\right]$.

$$\left[\begin{array}{rrr|r} 1 & 3 & -6 & 7 \\ 0 & -7 & 13 & -13 \\ 1 & 2 & 2 & -1 \end{array}\right] \begin{array}{l} \\ -2\text{R}1 + \text{R}2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 3 & -6 & 7 \\ 0 & -7 & 13 & -13 \\ 0 & -1 & 8 & -8 \end{array}\right] \begin{array}{l} \\ \\ -1\text{R}1 + \text{R}3 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 3 & -6 & 7 \\ 0 & -1 & 8 & -8 \\ 0 & -7 & 13 & -13 \end{array}\right] \begin{array}{l} \\ \text{R}3 \leftrightarrow \text{R}2 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 3 & -6 & 7 \\ 0 & 1 & -8 & 8 \\ 0 & -7 & 13 & -13 \end{array}\right] \begin{array}{l} \\ -1\text{R}2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 18 & -17 \\ 0 & 1 & -8 & 8 \\ 0 & -7 & 13 & -13 \end{array}\right] \begin{array}{l} -3\text{R}2 + \text{R}1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 18 & -17 \\ 0 & 1 & -8 & 8 \\ 0 & 0 & -43 & 43 \end{array}\right] \begin{array}{l} \\ \\ 7\text{R}2 + \text{R}3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 18 & -17 \\ 0 & 1 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{43}\text{R}3 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} -18\text{R}3 + \text{R}1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \\ 8\text{R}3 + \text{R}2 \\ \\ \end{array}$$

Solution set: $\{(1, 0, -1)\}$

29. $\begin{aligned} x - z &= -3 \\ y + z &= 9 \\ x + z &= 7 \end{aligned}$; This system has the augmented matrix $\left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 1 & 0 & 1 & 7 \end{array}\right]$.

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 10 \end{array}\right] \begin{array}{l} \\ \\ -1\text{R}1 + \text{R}3 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}\text{R}3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \text{R}3 + \text{R}1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \\ -1\text{R}3 + \text{R}2 \\ \\ \end{array}$$

Solution set: $\{(2, 4, 5)\}$

30. $\begin{aligned}-x+y&=-1\\ y-z&=\ \ 6\\ x+z&=-1\end{aligned}$; This system has the augmented matrix $\left[\begin{array}{ccc|c}-1&1&0&-1\\0&1&-1&6\\1&0&1&-1\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&-1&0&1\\0&1&-1&6\\1&0&1&-1\end{array}\right]-1\text{R1} \Rightarrow \left[\begin{array}{ccc|c}1&-1&0&1\\0&1&-1&6\\0&1&1&-2\end{array}\right]-1\text{R1}+\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&-1&7\\0&1&-1&6\\0&1&1&-2\end{array}\right]\text{R2}+\text{R1}$$

$$\left[\begin{array}{ccc|c}1&0&-1&7\\0&1&-1&6\\0&0&2&-8\end{array}\right]-1\text{R2}+\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&-1&7\\0&1&-1&6\\0&0&1&-4\end{array}\right]\tfrac{1}{2}\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&0&3\\0&1&-1&6\\0&0&1&-4\end{array}\right]\text{R3}+\text{R1}$$

$$\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&2\\0&0&1&-4\end{array}\right]\text{R3}+\text{R2}$$

Solution set: $\{(3,2,-4)\}$

31. $\begin{aligned}y&=-2x-2z+1\\ x&=-2y-z+2\\ z&=x-y\end{aligned}$; Rewrite the system as $\begin{aligned}2x+y+2z&=1\\ x+2y+z&=2\\ x-y-z&=0.\end{aligned}$ The augmented matrix is $\left[\begin{array}{ccc|c}2&1&2&1\\1&2&1&2\\1&-1&-1&0\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&2&1&2\\2&1&2&1\\1&-1&-1&0\end{array}\right]\text{R1}\leftrightarrow\text{R2} \Rightarrow \left[\begin{array}{ccc|c}1&2&1&2\\0&-3&0&-3\\1&-1&-1&0\end{array}\right]-2\text{R1}+\text{R2} \Rightarrow \left[\begin{array}{ccc|c}1&2&1&2\\0&-3&0&-3\\0&-3&-2&-2\end{array}\right]-1\text{R1}+\text{R3}$$

$$\left[\begin{array}{ccc|c}1&2&1&2\\0&1&0&1\\0&-3&-2&-2\end{array}\right]-\tfrac{1}{3}\text{R2} \Rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&1\\0&-3&-2&-2\end{array}\right]-2\text{R2}+\text{R1} \Rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&1\\0&0&-2&1\end{array}\right]3\text{R2}+\text{R3}$$

$$\left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&1\\0&0&1&-\frac{1}{2}\end{array}\right]-\tfrac{1}{2}\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&0&\frac{1}{2}\\0&1&0&1\\0&0&1&-\frac{1}{2}\end{array}\right]-1\text{R3}+\text{R1}$$

Solution set: $\left\{\left(\frac{1}{2},1,-\frac{1}{2}\right)\right\}$

32. $\begin{aligned}x+y&=\ \ 1\\ 2x-z&=\ \ 0\\ y+2z&=-2\end{aligned}$; This system has the augmented matrix $\left[\begin{array}{ccc|c}1&1&0&1\\2&0&-1&0\\0&1&2&-2\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&1&0&1\\0&-2&-1&-2\\0&1&2&-2\end{array}\right]-2\text{R1}+\text{R2} \Rightarrow \left[\begin{array}{ccc|c}1&1&0&1\\0&1&2&-2\\0&-2&-1&-2\end{array}\right]\text{R2}\leftrightarrow\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&-2&3\\0&1&2&-2\\0&-2&-1&-2\end{array}\right]-1\text{R2}+\text{R1}$$

$$\left[\begin{array}{ccc|c}1&0&-2&3\\0&1&2&-2\\0&0&3&-6\end{array}\right]2\text{R2}+\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&-2&3\\0&1&2&-2\\0&0&1&-2\end{array}\right]\tfrac{1}{3}\text{R3} \Rightarrow \left[\begin{array}{ccc|c}1&0&0&-1\\0&1&2&-2\\0&0&1&-2\end{array}\right]2\text{R3}+\text{R1}$$

$$\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&2\\0&0&1&-2\end{array}\right]-2\text{R3}+\text{R2}$$

Solution set: $\{(-1,2,-2)\}$

33. $\begin{array}{l} 2x-y+3z=0 \\ x+2y-z=5 \\ 2y+z=1 \end{array}$; This system has the augmented matrix $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 1 & 2 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} R1 \leftrightarrow R2 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] -2R1+R2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{array}\right] -\tfrac{1}{5}R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} -2R2+R1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & -3 \end{array}\right] \begin{array}{l} \\ \\ -2R2+R3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \\ \\ \tfrac{1}{3}R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} -1R3+R1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] R3+R2$$

Solution set: $\{(2,1,-1)\}$

34. $\begin{array}{l} 4x+2y-3z=6 \\ x-4y+z=-4 \\ -x+2z=2 \end{array}$; This system has the augmented matrix $\left[\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 1 & -4 & 1 & -4 \\ -1 & 0 & 2 & 2 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & -4 \\ 4 & 2 & -3 & 6 \\ -1 & 0 & 2 & 2 \end{array}\right] \begin{array}{l} R1 \leftrightarrow R2 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 1 & -4 \\ 0 & 18 & -7 & 22 \\ -1 & 0 & 2 & 2 \end{array}\right] -4R1+R2 \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 1 & -4 \\ 0 & 18 & -7 & 22 \\ 0 & -4 & 3 & -2 \end{array}\right] \begin{array}{l} \\ \\ R1+R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & -4 \\ 0 & 1 & -\frac{7}{18} & \frac{11}{9} \\ 0 & -4 & 3 & -2 \end{array}\right] \tfrac{1}{18}R2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{9} & \frac{8}{9} \\ 0 & 1 & -\frac{7}{18} & \frac{11}{9} \\ 0 & -4 & 3 & -2 \end{array}\right] \begin{array}{l} 4R2+R1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{9} & \frac{8}{9} \\ 0 & 1 & -\frac{7}{18} & \frac{11}{9} \\ 0 & 0 & \frac{13}{9} & \frac{26}{9} \end{array}\right] \begin{array}{l} \\ \\ 4R2+R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{9} & \frac{8}{9} \\ 0 & 1 & -\frac{7}{18} & \frac{11}{9} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \\ \tfrac{9}{13}R3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -\frac{7}{18} & \frac{11}{9} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \tfrac{5}{9}R3+R1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \tfrac{7}{18}R3+R2$$

Solution set: $\{(2,2,2)\}$

35. $\begin{array}{l} 3x+5y-z+2=0 \\ 4x-y+2z-1=0 \\ -6x-10y+2z=0 \end{array}$; Rewrite the system as $\begin{array}{l} 3x+5y-z=-2 \\ 4x-y+2z=1 \\ -6x-10y+2z=0. \end{array}$

The augmented matrix is $\left[\begin{array}{ccc|c} 3 & 5 & -1 & -2 \\ 4 & -1 & 2 & 1 \\ -6 & -10 & 2 & 0 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 4 & -1 & 2 & 1 \\ -6 & -10 & 2 & 0 \end{array}\right] \begin{array}{l} \tfrac{1}{3}R1 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{23}{3} & \frac{10}{3} & \frac{11}{3} \\ -6 & -10 & 2 & 0 \end{array}\right] -4R1+R2 \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{23}{3} & \frac{10}{3} & \frac{11}{3} \\ 0 & 0 & 0 & -4 \end{array}\right] \begin{array}{l} \\ \\ 6R1+R3 \end{array}$$

The last row indicates that there is no solution. The solution set is $\varnothing$.

36. $\begin{aligned}3x+\ y+3z&=1\\ x+2y-\ z&=2\\ 2x-\ y+4z&=4\end{aligned}$; This system has the augmented matrix $\left[\begin{array}{ccc|c}3&1&3&1\\1&2&-1&2\\2&-1&4&4\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&2&-1&2\\3&1&3&1\\2&-1&4&4\end{array}\right]\text{R1}\leftrightarrow\text{R2}\Rightarrow\left[\begin{array}{ccc|c}1&2&-1&2\\0&-5&6&-5\\2&-1&4&4\end{array}\right]-3\text{R1}+\text{R2}\Rightarrow\left[\begin{array}{ccc|c}1&2&-1&2\\0&-5&6&-5\\0&-5&6&0\end{array}\right]-2\text{R1}+\text{R3}$$

$$\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&-\frac{6}{5}&1\\0&-5&6&0\end{array}\right]-\tfrac{1}{5}\text{R2}\Rightarrow\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&-\frac{6}{5}&1\\0&0&0&5\end{array}\right]5\text{R2}+\text{R3}$$

The last row indicates that there is no solution. The solution set is $\varnothing$.

37. $\begin{aligned}x-8y+\ z&=4\\ 3x-\ y+2z&=-1\end{aligned}$

This system has the augmented matrix
$\left[\begin{array}{ccc|c}1&-8&1&4\\3&-1&2&-1\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&-8&1&4\\0&23&-1&-13\end{array}\right]-3\text{R1}+\text{R2}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&-8&1&4\\0&1&-\frac{1}{23}&-\frac{13}{23}\end{array}\right]\tfrac{1}{23}\text{R2}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&0&\frac{15}{23}&-\frac{12}{23}\\0&1&-\frac{1}{23}&-\frac{13}{23}\end{array}\right]8\text{R2}+\text{R1}$$

The equations that correspond to the final matrix are $x+\frac{15}{23}z=-\frac{12}{23}$ and $y-\frac{1}{23}z=-\frac{13}{23}$. This system has infinitely many solutions. We will express the solution set with z as the arbitrary variable. Therefore,
$x=-\frac{15}{23}z-\frac{12}{23}$ and $y=\frac{1}{23}z-\frac{13}{23}$.

Solution set: $\left\{\left(-\frac{15}{23}z-\frac{12}{23},\frac{1}{23}z-\frac{13}{23},z\right)\right\}$

38. $\begin{aligned}5x-3y+z&=1\\ 2x+y-z&=4\end{aligned}$

This system has the augmented matrix
$\left[\begin{array}{ccc|c}5&-3&1&1\\2&1&-1&4\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1&-\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\2&1&-1&4\end{array}\right]\tfrac{1}{5}\text{R1}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&-\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\0&\frac{11}{5}&-\frac{7}{5}&\frac{18}{5}\end{array}\right]-2\text{R1}+\text{R2}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&-\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\0&1&-\frac{7}{11}&\frac{18}{11}\end{array}\right]\tfrac{5}{11}\text{R2}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&0&-\frac{2}{11}&\frac{13}{11}\\0&1&-\frac{7}{11}&\frac{18}{11}\end{array}\right]\tfrac{3}{5}\text{R2}+\text{R1}\Rightarrow$$

The equations that correspond to the final matrix are $x-\frac{2}{11}z=\frac{13}{11}$ and $y-\frac{7}{11}z=\frac{18}{11}$. This system has infinitely many solutions. We will express the solution set with z as the arbitrary variable. Therefore,
$x=\frac{2}{11}z+\frac{13}{11}$ and $y=\frac{7}{11}z+\frac{18}{11}$.

Solution set: $\left\{\left(\frac{2}{11}z+\frac{13}{11},\frac{7}{11}z+\frac{18}{11},z\right)\right\}$

39. $\begin{aligned}x-y+2z+w&=4\\ y+z&=3\\ z-w&=2\\ x-y&=0\end{aligned}$

This system has the augmented matrix
$\left[\begin{array}{cccc|c}1&-1&2&1&4\\0&1&1&0&3\\0&0&1&-1&2\\1&-1&0&0&0\end{array}\right]$

$$\left[\begin{array}{cccc|c}1&-1&2&1&4\\0&1&1&0&3\\0&0&1&-1&2\\0&0&-2&-1&-4\end{array}\right]-1\text{R1}+\text{R4}\Rightarrow$$

$$\left[\begin{array}{cccc|c}1&0&3&1&7\\0&1&1&0&3\\0&0&1&-1&2\\0&0&-2&-1&-4\end{array}\right]\text{R2}+\text{R1}\Rightarrow$$

$$\left[\begin{array}{cccc|c}1&0&0&4&1\\0&1&1&0&3\\0&0&1&-1&2\\0&0&-2&-1&-4\end{array}\right]-3\text{R3}+\text{R1}\Rightarrow$$

$$\left[\begin{array}{cccc|c}1&0&0&4&1\\0&1&0&1&1\\0&0&1&-1&2\\0&0&-2&-1&-4\end{array}\right]-1\text{R3}+\text{R2}\Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right]\begin{array}{l} \\ \\ \\ 2R3+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} \\ \\ \\ -\frac{1}{3}R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} -4R4+R1 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} \\ -1R4+R2 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} \\ \\ R4+R3 \\ \\ \end{array}$$

Solution set: $\{(1,1,2,0)\}$

40. $\begin{array}{l} x+2y+z-3w=7 \\ y+z=0 \\ x-w=4 \\ -x+y=-3 \end{array}$; This system has the augmented matrix $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 4 \\ -1 & 1 & 0 & 0 & -3 \end{array}\right]$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 2 & -3 \\ -1 & 1 & 0 & 0 & -3 \end{array}\right]\begin{array}{l} \\ \\ -1R1+R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 2 & -3 \\ 0 & 3 & 1 & -3 & 4 \end{array}\right]\begin{array}{l} \\ \\ \\ R1+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 2 & -3 \\ 0 & 3 & 1 & -3 & 4 \end{array}\right]\begin{array}{l} -2R2+R1 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 3 & 1 & -3 & 4 \end{array}\right]\begin{array}{l} \\ \\ 2R2+R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -2 & -3 & 4 \end{array}\right]\begin{array}{l} \\ \\ \\ -3R2+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -2 & -3 & 4 \end{array}\right]\begin{array}{l} R3+R1 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -2 & -3 & 4 \end{array}\right]\begin{array}{l} \\ -1R3+R2 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} \\ \\ \\ 2R3+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} R4+R1 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} \\ 2R4+R2 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} \\ \\ -2R4+R3 \\ \\ \end{array}$$

Solution set: $\{(2,-1,1,-2)\}$

41. $\begin{aligned} x+3y-2z-\ \ w&=9\\ 4x+\ \ y+\ \ z+2w&=2\\ -3x-\ \ y+\ \ z-\ \ w&=-5\\ x-\ \ y-3z-2w&=2 \end{aligned}$; This system has the augmented matrix $\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 4 & 1 & 1 & 2 & 2\\ -3 & -1 & 1 & -1 & -5\\ 1 & -1 & -3 & -2 & 2 \end{array}\right]$.

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 0 & -11 & 9 & 6 & -34\\ -3 & -1 & 1 & -1 & -5\\ 1 & -1 & -3 & -2 & 2 \end{array}\right]\begin{array}{l} \\ -4R1+R2\\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 0 & -11 & 9 & 6 & -34\\ 0 & 8 & -5 & -4 & 22\\ 1 & -1 & -3 & -2 & 2 \end{array}\right]\begin{array}{l} \\ \\ 3R1+R3\\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 0 & -11 & 9 & 6 & -34\\ 0 & 8 & -5 & -4 & 22\\ 0 & -4 & -1 & -1 & -7 \end{array}\right]\begin{array}{l} \\ \\ \\ -1R1+R4 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 0 & 1 & -\frac{9}{11} & -\frac{6}{11} & \frac{34}{11}\\ 0 & 8 & -5 & -4 & 22\\ 0 & -4 & -1 & -1 & -7 \end{array}\right]\begin{array}{l} \\ -\frac{1}{11}R2\\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & \frac{5}{11} & \frac{7}{11} & -\frac{3}{11}\\ 0 & 1 & -\frac{9}{11} & -\frac{6}{11} & \frac{34}{11}\\ 0 & 8 & -5 & -4 & 22\\ 0 & -4 & -1 & -1 & -7 \end{array}\right]\begin{array}{l} -3R2+R1\\ \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & \frac{5}{11} & \frac{7}{11} & -\frac{3}{11}\\ 0 & 1 & -\frac{9}{11} & -\frac{6}{11} & \frac{34}{11}\\ 0 & 0 & \frac{17}{11} & \frac{4}{11} & -\frac{30}{11}\\ 0 & -4 & -1 & -1 & -7 \end{array}\right]\begin{array}{l} \\ \\ -8R2+R3\\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & \frac{5}{11} & \frac{7}{11} & -\frac{3}{11}\\ 0 & 1 & -\frac{9}{11} & -\frac{6}{11} & \frac{34}{11}\\ 0 & 0 & \frac{17}{11} & \frac{4}{11} & -\frac{30}{11}\\ 0 & 0 & -\frac{47}{11} & -\frac{35}{11} & \frac{59}{11} \end{array}\right]\begin{array}{l} \\ \\ \\ 4R2+R4 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & \frac{5}{11} & \frac{7}{11} & -\frac{3}{11}\\ 0 & 1 & -\frac{9}{11} & -\frac{6}{11} & \frac{34}{11}\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & -\frac{47}{11} & -\frac{35}{11} & \frac{59}{11} \end{array}\right]\begin{array}{l} \\ \\ \frac{11}{17}R3\\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & \frac{9}{17} & \frac{9}{17}\\ 0 & 1 & -\frac{9}{11} & -\frac{6}{11} & \frac{34}{11}\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & -\frac{47}{11} & -\frac{35}{11} & \frac{59}{11} \end{array}\right]\begin{array}{l} -\frac{5}{11}R3+R1\\ \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & \frac{9}{17} & \frac{9}{17}\\ 0 & 1 & 0 & -\frac{6}{17} & \frac{28}{71}\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & -\frac{47}{11} & -\frac{35}{11} & \frac{59}{11} \end{array}\right]\begin{array}{l} \\ \frac{9}{11}R3+R2\\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & \frac{9}{17} & \frac{9}{17}\\ 0 & 1 & 0 & -\frac{6}{17} & \frac{28}{17}\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & 0 & -\frac{37}{17} & -\frac{37}{17} \end{array}\right]\begin{array}{l} \\ \\ \\ \frac{47}{11}R3+R4 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & \frac{9}{17} & \frac{9}{17}\\ 0 & 1 & 0 & -\frac{6}{17} & \frac{28}{17}\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l} \\ \\ \\ -\frac{17}{37}R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -\frac{6}{17} & \frac{28}{17}\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l} -\frac{9}{17}R4+R1\\ \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 1 & \frac{4}{17} & -\frac{30}{17}\\ 0 & 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l} \\ \frac{6}{17}R4+R2\\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l} \\ \\ -\frac{4}{17}R4+R3\\ \\ \end{array}$$

Solution set: $\{(0,2,-2,1)\}$

42. $\begin{aligned} 2x+y-z+3w&=0\\ 3x-2y+z-4w&=-24\\ x+y-z+w&=2\\ x-y+2z-5w&=-16 \end{aligned}$; This system has the augmented matrix $\left[\begin{array}{rrrr|r} 2 & 1 & -1 & 3 & 0\\ 3 & -2 & 1 & -4 & -24\\ 1 & 1 & -1 & 1 & 2\\ 1 & -1 & 2 & -5 & -16 \end{array}\right]$.

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 1 & 2\\ 3 & -2 & 1 & -4 & -24\\ 2 & 1 & -1 & 3 & 0\\ 1 & -1 & 2 & -5 & -16 \end{array}\right] \text{R1}\leftrightarrow\text{R3} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 1 & 2\\ 0 & -5 & 4 & -7 & -30\\ 2 & 1 & -1 & 3 & 0\\ 1 & -1 & 2 & -5 & -16 \end{array}\right] -3\text{R1}+\text{R2}$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 1 & 2\\ 0 & -5 & 4 & -7 & -30\\ 0 & -1 & 1 & 1 & -4\\ 1 & -1 & 2 & -5 & -16 \end{array}\right] -2\text{R1}+\text{R3} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 1 & 2\\ 0 & -5 & 4 & -7 & -30\\ 0 & -1 & 1 & 1 & -4\\ 0 & -2 & 3 & -6 & -18 \end{array}\right] -1\text{R1}+\text{R4}$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 1 & 2\\ 0 & -1 & 1 & 1 & -4\\ 0 & -5 & 4 & -7 & -30\\ 0 & -2 & 3 & -6 & -18 \end{array}\right] \text{R2}\leftrightarrow\text{R3} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 1 & 2\\ 0 & 1 & -1 & -1 & 4\\ 0 & -5 & 4 & -7 & -30\\ 0 & -2 & 3 & -6 & -18 \end{array}\right] -1\text{R2}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & -1 & -1 & 4\\ 0 & -5 & 4 & -7 & -30\\ 0 & -2 & 3 & -6 & -18 \end{array}\right] -1\text{R2}+\text{R1} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & -1 & -1 & 4\\ 0 & 0 & -1 & -12 & -10\\ 0 & -2 & 3 & -6 & -18 \end{array}\right] 5\text{R2}+\text{R3}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & -1 & -1 & 4\\ 0 & 0 & -1 & -12 & -10\\ 0 & 0 & 1 & -8 & -10 \end{array}\right] 2\text{R2}+\text{R4} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & -1 & -1 & 4\\ 0 & 0 & 1 & 12 & 10\\ 0 & 0 & 1 & -8 & -10 \end{array}\right] -1\text{R3}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & 0 & 11 & 14\\ 0 & 0 & 1 & 12 & 10\\ 0 & 0 & 1 & -8 & -10 \end{array}\right] \text{R3}+\text{R2} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & 0 & 11 & 14\\ 0 & 0 & 1 & 12 & 10\\ 0 & 0 & 0 & -20 & -20 \end{array}\right] -1\text{R3}+\text{R4}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -2\\ 0 & 1 & 0 & 11 & 14\\ 0 & 0 & 1 & 12 & 10\\ 0 & 0 & 0 & 1 & 1 \end{array}\right] -\tfrac{1}{20}\text{R4} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -4\\ 0 & 1 & 0 & 11 & 14\\ 0 & 0 & 1 & 12 & 10\\ 0 & 0 & 0 & 1 & 1 \end{array}\right] -2\text{R4}+\text{R1}$$

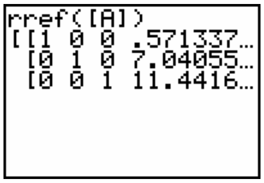
$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -4\\ 0 & 1 & 0 & 0 & 3\\ 0 & 0 & 1 & 12 & 10\\ 0 & 0 & 0 & 1 & 1 \end{array}\right] -11\text{R4}+\text{R2} \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -4\\ 0 & 1 & 0 & 0 & 3\\ 0 & 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 1 & 1 \end{array}\right] -12\text{R4}+\text{R3}$$

Solution set: $\{(-4,3,-2,1)\}$

43. $.3x + 2.7y - \sqrt{2}z = 3$
$\sqrt{7}x - 20y + 12z = -2$
$4x + \sqrt{3}y - 1.2z = \frac{3}{4}$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} .3 & 2.7 & -\sqrt{2} & 3 \\ \sqrt{7} & -20 & 12 & -2 \\ 4 & \sqrt{3} & -1.2 & \frac{3}{4} \end{array}\right].$$

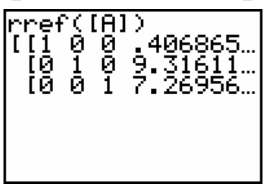


Using a graphing calculator. we obtain the approximate solution set.
Solution set: $\{(.571, 7.041, 11.442)\}$.

44. $\sqrt{5}x - 1.2y + z = -3$
$\frac{1}{2}x - 3y + 4z = \frac{4}{3}$
$4x + 7y - 9z = \sqrt{2}$

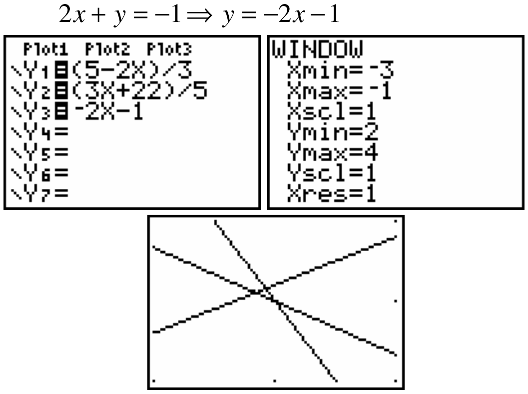
This system has the augmented matrix

$$\left[\begin{array}{ccc|c} \sqrt{5} & -1.2 & 1 & -3 \\ \frac{1}{2} & -3 & 4 & \frac{4}{3} \\ 4 & 7 & -9 & \sqrt{2} \end{array}\right].$$



Using a graphing calculator, we obtain the approximate solution set:
Solution set: $\{(.407, 9.316, 7.270)\}$

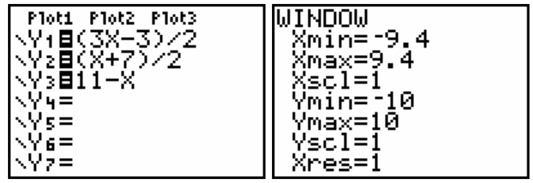
45. $2x + 3y = 5 \Rightarrow 3y = 5 - 2x \Rightarrow y = \frac{5-2x}{3}$
$-3x + 5y = 22 \Rightarrow 5y = 3x + 22 \Rightarrow y = \frac{3x+22}{5}$
$2x + y = -1 \Rightarrow y = -2x - 1$



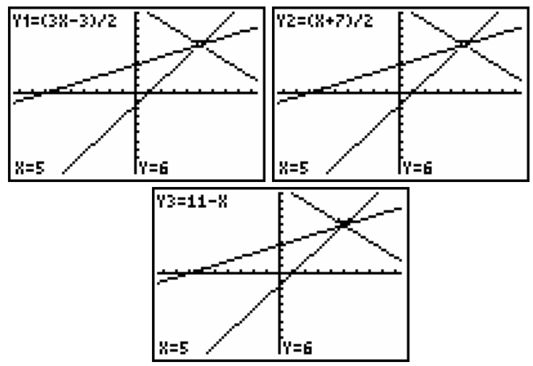
There are no solutions since the three lines do not intersect at one point.

46. $3x - 2y = 3 \Rightarrow -2y = -3x + 3 \Rightarrow$
$y = \frac{-3x+3}{-2} \Rightarrow y = \frac{3x-3}{2}$
$-2x + 4y = 14 \Rightarrow 4y = 2x + 14 \Rightarrow$
$y = \frac{2x+14}{4} \Rightarrow y = \frac{x+7}{2}$
$x + y = 11 \Rightarrow y = 11 - x$

If you graph these lines on the same axes, they do appear to intersect at one point.



If you trace on the first line to the point (5,6) and then toggle between the graphs, you will see that the point lies on all three lines. The solution set is $\{(5, 6)\}$.



Confirm answer by solving the system with the Gauss-Jordan method.

$$\left[\begin{array}{cc|c} 3 & -2 & 3 \\ -2 & 4 & 14 \\ 1 & 1 & 11 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 11 \\ -2 & 4 & 14 \\ 3 & -2 & 3 \end{array}\right] \begin{matrix} R1 \leftrightarrow R3 \\ \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & 6 & 36 \\ 3 & -2 & 3 \end{array}\right] 2R1 + R2 \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & 6 & 36 \\ 0 & -5 & -30 \end{array}\right] \begin{matrix} \\ \\ -3R1 + R3 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & 1 & 6 \\ 0 & -5 & -30 \end{array}\right] \tfrac{1}{6}R2 \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & -5 & -30 \end{array}\right] \begin{matrix} -1R2 + R1 \\ \\ \\ \end{matrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ \\ 5R2 + R3 \end{matrix}$$

This confirms the solution set, $\{(5, 6)\}$. Note: The last equation does not provide any additional information it implies $0x + 0y = 0$, which is a true statement for all (x, y).

47. $\dfrac{1}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$

If we add the rational expression on the right, we get the following.

$$\frac{1}{(x-1)(x+1)}=\frac{A(x+1)}{(x-1)(x+1)}+\frac{B(x-1)}{(x-1)(x+1)}$$
$$\frac{1}{(x-1)(x+1)}=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

Since the denominators are equal, the numerators must be equal. Thus we have the equation $1=A(x+1)+B(x-1)$.

$1=A(x+1)+B(x-1)\Rightarrow$
$1=Ax+A+Bx-B$
$1=(A-B)+(Ax+Bx)\Rightarrow$
$1=(A-B)+(A+B)x$

Equating the coefficients of like powers of x gives the following system of equations.

$A+B=0$
$A-B=1$

Solve this system by the Gauss-Jordan method.

$$\left[\begin{array}{cc|c}1&1&0\\1&-1&1\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1&1&0\\0&-2&1\end{array}\right]-\text{R1}+\text{R2}\Rightarrow$$
$$\left[\begin{array}{cc|c}1&1&0\\0&1&-\frac{1}{2}\end{array}\right]-\tfrac{1}{2}\text{R2}\Rightarrow$$
$$\left[\begin{array}{cc|c}1&0&\frac{1}{2}\\0&1&-\frac{1}{2}\end{array}\right]-1\text{R2}+\text{R1}$$

Thus, $A=\frac{1}{2}$ and $B=-\frac{1}{2}$.

48. $\dfrac{x+4}{x^2}=\dfrac{A}{x}+\dfrac{B}{x^2}$

If we add the rational expression on the right, we get the following.

$$\frac{x+4}{x^2}=\frac{Ax}{x^2}+\frac{B}{x^2}$$
$$\frac{x+4}{x^2}=\frac{Ax+B}{x^2}$$

Since the denominators are equal, the numerators must be equal, so we have $x+4=Ax+B$. Thus, $A=1$ and $B=4$.

49. $\dfrac{x}{(x-a)(x+a)}=\dfrac{A}{x-a}+\dfrac{B}{x+a}$

If we add the rational expression on the right, we get the following.

$$\frac{x}{(x-a)(x+a)}=\frac{A(x+a)}{(x-a)(x+a)}+\frac{B(x-a)}{(x-a)(x+a)}$$
$$\frac{x}{(x-a)(x+a)}=\frac{A(x+a)+B(x-a)}{(x-a)(x+a)}$$

Since the denominators are equal, the numerators must be equal. Thus we have the equation $x=A(x+a)+B(x-a)$.

$x=A(x+a)+B(x-a)\Rightarrow$
$x=Ax+Aa+Bx-Ba\Rightarrow$
$x=(A+B)x+(A-B)a$

Equating the coefficients of like powers of x gives the following system of equations.

$A+B=1$
$A-B=0$

Solve this system by the Gauss-Jordan method.

$$\left[\begin{array}{cc|c}1&1&1\\1&-1&0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1&1&1\\0&-2&-1\end{array}\right]-\text{R1}+\text{R2}\Rightarrow$$
$$\left[\begin{array}{cc|c}1&1&1\\0&1&\frac{1}{2}\end{array}\right]-\tfrac{1}{2}\text{R2}\Rightarrow\left[\begin{array}{cc|c}1&0&\frac{1}{2}\\0&1&\frac{1}{2}\end{array}\right]-1\text{R2}+\text{R1}$$

Thus, $A=\frac{1}{2}$ and $B=\frac{1}{2}$.

50. $\dfrac{2x}{(x+2)(x-1)}=\dfrac{A}{x+2}+\dfrac{B}{x-1}$

If we add the rational expression on the right, we get the following.

$$\frac{2x}{(x+2)(x-1)}=\frac{A(x-1)}{(x+2)(x-1)}+\frac{B(x+2)}{(x-1)(x+2)}$$
$$\frac{2x}{(x+2)(x-1)}=\frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Since the denominators are equal, the numerators must be equal. Thus we have the equation $2x=A(x-1)+B(x+2)$.

$2x=A(x-1)+B(x+2)\Rightarrow$
$2x=Ax-A+Bx+2B\Rightarrow$
$2x=(A+B)x+(-A+2B)$

Equating the coefficients of like powers of x gives the following system of equations.

$A+B=2$
$-A+2B=0$

Solve this system by the Gauss-Jordan method.

$$\left[\begin{array}{cc|c}1&1&2\\-1&2&0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1&1&2\\0&3&2\end{array}\right]\text{R1}+\text{R2}\Rightarrow$$
$$\left[\begin{array}{cc|c}1&1&2\\0&1&\frac{2}{3}\end{array}\right]\tfrac{1}{3}\text{R2}\Rightarrow\left[\begin{array}{cc|c}1&0&\frac{4}{3}\\0&1&\frac{2}{3}\end{array}\right]-1\text{R2}+\text{R1}$$

Thus, $A=\frac{4}{3}$ and $B=\frac{2}{3}$.

51. Let x = the daily wage for a day laborer
y = the daily wage for a concrete finisher
From the information, we can write the system

$$\begin{aligned} 7x + 2y &= 1384 \\ x + 5y &= 952 \end{aligned}$$

Solve this system by the Gauss-Jordan method:

$$\left[\begin{array}{cc|c} 7 & 2 & 1384 \\ 1 & 5 & 952 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 5 & 952 \\ 7 & 2 & 1384 \end{array}\right] \text{R1} \leftrightarrow \text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 5 & 952 \\ 0 & -33 & -5280 \end{array}\right] -7\text{R1} + \text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 5 & 952 \\ 0 & 1 & 160 \end{array}\right] -\tfrac{1}{33}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 152 \\ 0 & 1 & 160 \end{array}\right] -5\text{R2} + \text{R1}$$

From the final matrix, $x = 152$ and $y = 160$, so the day laborers earn \$152 per day and the concreter finishers earn \$160 per day.

52. Let x = the cost of a pound of peanuts
y = the cost of a pound of cashews
From the information, we can write the system

$$\begin{aligned} 5x + 6y &= 33.60 \\ 3x + 7y &= 32.40 \end{aligned}$$

Solve this system by the Gauss-Jordan method:

$$\left[\begin{array}{cc|c} 5 & 6 & 33.60 \\ 3 & 7 & 32.40 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & \frac{6}{5} & 6.72 \\ 3 & 7 & 32.40 \end{array}\right] \tfrac{1}{5}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & \frac{6}{5} & 6.72 \\ 0 & \frac{17}{5} & 12.24 \end{array}\right] -3\text{R1} + \text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & \frac{6}{5} & 6.72 \\ 0 & 1 & 3.60 \end{array}\right] \tfrac{5}{17}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2.40 \\ 0 & 1 & 3.60 \end{array}\right] -\tfrac{6}{5}\text{R2} + \text{R1}$$

From the final matrix, $x = 2.40$ and $y = 3.60$, so a pound of peanuts costs \$2.40 and a pound of cashews costs \$3.60.

53. Let x = the first number; y = the second number; z = the third number.
From the information, we can write the system

$$\begin{aligned} & & x + y + z &= 20 \\ x = 3(y - z) &\Rightarrow x = 3y - 3z \Rightarrow & x - 3y + 3z &= 0 \\ y = 2 + 2z &\Rightarrow & y - 2z &= 2 \end{aligned}$$

Solve this system by the Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 1 & -3 & 3 & 0 \\ 0 & 1 & -2 & 2 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 1 & -\frac{1}{2} & 5 \\ 0 & 1 & -2 & 2 \end{array}\right] \tfrac{1}{4}(\text{R1} - \text{R2}) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 1 & -\frac{1}{2} & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \tfrac{2}{3}(\text{R2} - \text{R3}) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{R2} + \tfrac{1}{2}\text{R3} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{R1} - \text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{R1} - \text{R3}$$

From the final matrix, $x = 12$, $y = 6$, and $z = 2$. Thus, the numbers are 12, 6, and 2.

54. Let x = the number of small cars; y = the number of medium cars; z = the number of large cars.
From the information, we can write the system

$$\begin{aligned} & & x + y + z &= 24 \\ x = 3 + y &\Rightarrow & x - y \quad &= 3 \\ y = z &\Rightarrow & y - z &= 0 \end{aligned}$$

Solve this system by the Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 0 & -2 & -1 & -21 \\ 0 & 1 & -1 & 0 \end{array}\right] -\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 0 & -2 & -1 & -21 \\ 0 & 0 & 1 & 7 \end{array}\right] -\tfrac{1}{3}(\text{R2} + 2\text{R3}) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 7 \end{array}\right] -\tfrac{1}{2}(\text{R2} + \text{R3}) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 17 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} R1-R3 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ R1-R2 \\ \\ \end{array}$$

From the final matrix, $x = 10$, $y = 7$, and $z = 7$, so she must sell 10 small cars, 7 medium cars, and 7 large cars.

55. Let x = number of cubic centimeters of the 2% solution; y = number of cubic centimeters of the 7% solution.
From the information we can write the system

$$\begin{array}{l} x + y = 40 \Rightarrow x + y = 40 \\ .02x + .07y = .032(40) \Rightarrow .02x + .07y = 1.28 \end{array}$$

Solve this system by the Gauss-Jordan method.

$$\left[\begin{array}{cc|c} 1 & 1 & 40 \\ .02 & .07 & 1.28 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 40 \\ 0 & .05 & .48 \end{array}\right] \begin{array}{l} \\ -.02R1+R2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 40 \\ 0 & 1 & 9.6 \end{array}\right] \begin{array}{l} \\ \frac{1}{.05}R2 \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 30.4 \\ 0 & 1 & 9.6 \end{array}\right] \begin{array}{l} -1R2+R1 \\ \\ \end{array}$$

From the final matrix, $x = 30.4$ and $y = 9.6$, so the chemist should mix 30.4 cm^3 of the 2% solution with 9.6 cm^3 of the 7% solution.

56. Let x = the amount borrowed at 8%; y = the amount borrowed at 10%; z = the amount borrowed at 9%.
Completing the table we have the following.

Amount Invested	Rate (in %)	Annual Interest
x	8	$.08x$
y	10	$.10y$
z	9	$.09z$
		2220

From the information given in the exercise, we can write the system.

$$\begin{cases} .08x + .010y + .09z = 2200 \\ y = \frac{1}{2}x + 2000 \end{cases} \Rightarrow$$

$$\begin{array}{l} x + y + z = 25,000 \\ 8x + 10y + 9z = 222,000 \\ -x + 2y = 4,000 \end{array}$$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 8 & 10 & 9 & 222,000 \\ -1 & 2 & 0 & 4000 \end{array}\right].$$

Solve by the Gauss-Jordan method.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 0 & 2 & 1 & 22,000 \\ -1 & 2 & 0 & 4000 \end{array}\right] \begin{array}{l} \\ -8R1+R2 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 0 & 2 & 1 & 22,000 \\ 0 & 3 & 1 & 29,000 \end{array}\right] \begin{array}{l} \\ \\ R1+R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 0 & 1 & \frac{1}{2} & 11,000 \\ 0 & 3 & 1 & 29,000 \end{array}\right] \begin{array}{l} \\ \frac{1}{2}R2 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 14,000 \\ 0 & 1 & \frac{1}{2} & 11,000 \\ 0 & 3 & 1 & 29,000 \end{array}\right] \begin{array}{l} -1R2+R1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 14,000 \\ 0 & 1 & \frac{1}{2} & 11,000 \\ 0 & 0 & -\frac{1}{2} & -4000 \end{array}\right] \begin{array}{l} \\ \\ -3R2+R3 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 14,000 \\ 0 & 1 & \frac{1}{2} & 11,000 \\ 0 & 0 & 1 & 8000 \end{array}\right] \begin{array}{l} \\ \\ -2R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10,000 \\ 0 & 1 & \frac{1}{2} & 11,000 \\ 0 & 0 & 1 & 8000 \end{array}\right] \begin{array}{l} -\frac{1}{2}R3+R1 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10,000 \\ 0 & 1 & 0 & 7000 \\ 0 & 0 & 1 & 8000 \end{array}\right] \begin{array}{l} \\ -\frac{1}{2}R3+R2 \\ \\ \end{array}$$

From the final matrix, we have $x = 10,000$, $y = 7000$, and $z = 8000$. The company borrowed \$10,000 at 8%, \$7000 at 10%, and \$8000 at 9%.

57. Answers will vary.
If the condition that correlates to the third row in the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 8 & 10 & 9 & 222,000 \\ -1 & 2 & 0 & 4000 \end{array}\right],$$ was dropped, then that row would be dropped. We would have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 8 & 10 & 9 & 222,000 \end{array}\right].$$

Solve by the Gauss-Jordan method.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 8 & 10 & 9 & 222,000 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25,000 \\ 0 & 2 & 1 & 22,000 \end{array}\right] \begin{array}{l} \\ -8R1+R2 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & \frac{1}{2} & 11{,}000 \end{array}\right] \tfrac{1}{2}\text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 14{,}000 \\ 0 & 1 & \frac{1}{2} & 11{,}000 \end{array}\right] -1\text{R2}+\text{R1}$$

We can see it is not possible to find a unique solution.

58. Answers will vary.
Given the conditions in Exercise 56, the only solution that satisfies all three conditions is borrowing \$10,000 at 8%, \$7000 at 10%, and \$8000 at 9%. If a fourth condition is included, that is that the company can borrow only \$6000 at 9%, then all conditions cannot be met.

59. Let x = number of grams of food A;
y = number of grams of food B;
z = number of grams of food C.
Completing the table we have the following.

				Total
Food Group	A	B	C	
Grams/Meal	x	y	z	400

From the information given in the exercise, we can write the system.

$$\begin{aligned} x + y + z &= 400 \\ x = \tfrac{1}{3}y \Rightarrow 3x - y &= 0 \\ x + z = 2y \Rightarrow x - 2y + z &= 0 \end{aligned}$$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 400 \\ 3 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right]$$

Solve by the Gauss-Jordan method.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 400 \\ 0 & -4 & -3 & -1200 \\ 1 & -2 & 1 & 0 \end{array}\right] -3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 400 \\ 0 & -4 & -3 & -1200 \\ 0 & -3 & 0 & -400 \end{array}\right] -1\text{R1}+\text{R3} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 400 \\ 0 & 1 & \frac{3}{4} & 300 \\ 0 & -3 & 0 & -400 \end{array}\right] -\tfrac{1}{4}\text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 100 \\ 0 & 1 & \frac{3}{4} & 300 \\ 0 & -3 & 0 & -400 \end{array}\right] -1\text{R2}+\text{R1} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 100 \\ 0 & 1 & \frac{3}{4} & 300 \\ 0 & 0 & \frac{9}{4} & 500 \end{array}\right] 3\text{R2}+\text{R3} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 100 \\ 0 & 1 & \frac{3}{4} & 300 \\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right] \tfrac{4}{9}\text{R3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{400}{9} \\ 0 & 1 & \frac{3}{4} & 300 \\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right] -\tfrac{1}{4}\text{R3}+\text{R1} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{400}{9} \\ 0 & 1 & 0 & \frac{400}{3} \\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right] -\tfrac{3}{4}\text{R3}+\text{R2}$$

From the final matrix, we have $x = \frac{400}{9} \approx 44.4$, $y = \frac{400}{3} \approx 133.3$, and $z = \frac{2000}{9} \approx 222.2$. The diet should include 44.4 g of food A, 133.3 g of food B, and 222.2 g of food C.

60. Answers will vary.
The condition that foods A and B cost 2¢ per gram and food C costs 3¢ per gram, and that a meal must cost \$8 correlates to the equation $.02x + .02y + .03z = 8$. If the solution found in Exercise 53 also satisfies this condition, then a solution is possible. Substituting the values found in Exercise 53 we have the following.

$$\begin{aligned} .02x + .02y + .03z &= 8 \\ 2x + 2y + 3z &= 800 \\ 2\left(\tfrac{400}{9}\right) + 2\left(\tfrac{400}{3}\right) + 3\left(\tfrac{2000}{9}\right) &= 800\ ? \\ \tfrac{9200}{9} &= 800 \text{ False} \end{aligned}$$

Since a false statement is obtained, all four conditions cannot be met simultaneously.

61. (a) For the 65 or older group:
With $x = 0$ representing 2005 and $x = 45$ representing 2050, we have information that correlates to the points (0, .124) and (45, .207).

$$m = \frac{.207 - .124}{45 - 0} = \frac{.083}{45} = .00184$$

Since we have the point (0, .124), the equation in slope-intercept form is $y = .00184x + .124$

For the 25–34 group:
With $x = 0$ representing 2005 and $x = 45$ representing 2050, we have information that correlates to the points (0, .134) and (50, .126).

$m = \frac{.126 - .134}{45 - 0} = \frac{-.008}{45} = -.000178$
(three significant figures)
Since we have the point (0, .134), the equation in slope-intercept form is
$y = -.000178x + .134$

(b) We need to solve the following system.

$\begin{cases} y = .00184x + .124 \\ y = -.000178x + .134 \end{cases} \Rightarrow \begin{cases} -.00184x + y = .124 \\ .000178x + y = .134 \end{cases} \Rightarrow$

$-1840x + 1{,}000{,}000y = 124{,}000$
$178x + 1{,}000{,}000y = 134{,}000$

This system has the augmented matrix

$$\left[\begin{array}{cc|c} -1840 & 1{,}000{,}000 & 124{,}000 \\ 178 & 1{,}000{,}000 & 134{,}000 \end{array}\right]$$

Solve by the Gauss-Jordan method.

$$\left[\begin{array}{cc|c} 1 & -\frac{1{,}000{,}000}{1840} & -\frac{124{,}000}{1840} \\ 1 & \frac{1{,}000{,}000}{178} & \frac{134{,}000}{178} \end{array}\right] \begin{array}{l} -\frac{1}{1840}\text{R1} \\ \frac{1}{178}\text{R2} \end{array}$$

Reducing the fractions, we have

$$\left[\begin{array}{cc|c} 1 & -\frac{12{,}500}{23} & -\frac{1550}{23} \\ 1 & \frac{500000}{89} & \frac{67{,}000}{89} \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & -\frac{12{,}500}{23} & -\frac{1550}{23} \\ 0 & -\frac{12{,}612{,}500}{2047} & -\frac{1{,}678{,}950}{2047} \end{array}\right] \text{R1} - \text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & -\frac{12{,}500}{23} & -\frac{1550}{23} \\ 0 & 1 & \frac{1{,}678{,}950}{12{,}612{,}500} \end{array}\right] -\frac{2047}{12{,}612{,}500}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{5000}{1009} \\ 0 & 1 & \frac{33{,}579}{252{,}250} \end{array}\right] \frac{12{,}500}{23}\text{R2} + \text{R1}$$

The solution set is

$\left\{\left(\frac{5000}{1009}, \frac{33{,}579}{252{,}250}\right)\right\} \approx \{(4.9554, .1331)\}$

$x = \frac{5000}{1009} \approx 4.9554$ represents the year 2009 and $y = \frac{33{,}579}{252{,}250} \approx .1331 \approx 13.31\%$.

The two groups will have the same percent of the population in 2009, about 13.3%

62. (a) For the 35–44 group:
With $x = 0$ representing 2005 and $x = 45$ represent 2050, we have information that correlates to the points (0, .148) and (50, .124).

$m = \frac{.124 - .148}{45 - 0} = \frac{-.024}{45} = -.000533$
(three significant figures)

Since we have the point (0, .148), the equation in slope-intercept form is
$y = -.000533x + .148$

(b) We need to solve the following system.

$\begin{cases} y = -.000533x + .148 \\ y = \ \ .00184\ \ x + .124 \end{cases} \Rightarrow \begin{cases} .000533x + y = .148 \\ -.00184x + y = .124 \end{cases} \Rightarrow$

$533x + 1{,}000{,}000y = 148{,}000$
$-1840x + 1{,}000{,}000y = 124{,}000$

This system has the augmented matrix

$$\left[\begin{array}{cc|c} 533 & 1{,}000{,}000 & 148{,}000 \\ -1840 & 1{,}000{,}000 & 124{,}000 \end{array}\right]$$

Solve by the Gauss-Jordan method.

$$\left[\begin{array}{cc|c} 1 & \frac{1{,}000{,}000}{533} & \frac{148{,}000}{533} \\ 1 & -\frac{1{,}000{,}000}{1840} & -\frac{124{,}000}{1840} \end{array}\right] \begin{array}{l} \frac{1}{533}\text{R1} \\ -\frac{1}{1840}\text{R2} \end{array}$$

Reducing the fractions, we have

$$\left[\begin{array}{cc|c} 1 & \frac{1{,}000{,}000}{533} & \frac{148{,}000}{533} \\ 1 & -\frac{12{,}500}{23} & -\frac{1550}{23} \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & \frac{1{,}000{,}000}{533} & \frac{148{,}000}{533} \\ 0 & 1 & \frac{28{,}201}{197{,}750} \end{array}\right] \frac{12{,}259}{29{,}662{,}500}(\text{R1} - \text{R2}) \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & \frac{1{,}000{,}000}{533} & \frac{148{,}000}{533} \\ 0 & 1 & \frac{28{,}201}{197{,}750} \end{array}\right] -\frac{1{,}000{,}000}{533}\text{R2} + \text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{8000}{791} \\ 0 & 1 & \frac{28{,}201}{197{,}750} \end{array}\right]$$

The solution set is

$\left\{\left(\frac{8000}{791}, \frac{28{,}201}{197{,}750}\right)\right\} \approx \{(10.11, .1426)\}$.

$x = \frac{8000}{791} \approx 10.11$ represents the year 2015 and $y = \frac{28{,}201}{197{,}750} \approx .1426 \approx 14.3\%$. The two groups will have the same percent of the population in 2015, about 14.3%

63. (a) A height of 6′11″ is 83″.
If $W = 7.46\text{H} - 374$, then
$W = 7.46(83) - 374 = 245.18$.
Using the first equation, the predicted weight is approximately 245 pounds.
If $W = 7.93H - 405$, then
$W = 7.93(83) - 405 = 253.19$
Using the second equation, the predicted weight is approximately 253 pounds.

(b) For the first model $W = 7.46H - 374$, a 1-inch increase in height results in a 7.46-pound increase in weight.
For the second model $W = 7.93H - 405$, a 1-inch increase in height results in a 7.93-pound increase in weight. In each case, the change is given by the slope of the line that is the graph of the given equation.

(c) $W - 7.46H = -374$
$W - 7.93H = -405$
This system has the augmented matrix
$$\left[\begin{array}{cc|c} 1 & -7.46 & -374 \\ 1 & -7.93 & -405 \end{array}\right].$$
Solve this system by the Gauss-Jordan method.
$$\left[\begin{array}{cc|c} 1 & -7.46 & -374 \\ 0 & -.47 & -31 \end{array}\right] \begin{array}{l} \\ -1\text{R1} + \text{R2} \end{array} \Rightarrow$$
$$\left[\begin{array}{cc|c} 1 & -7.46 & -374 \\ 0 & 1 & 65.957 \end{array}\right] \begin{array}{l} \\ -\frac{1}{.47}\text{R2} \end{array} \Rightarrow$$
$$\left[\begin{array}{cc|c} 1 & 0 & 118.043 \\ 0 & 1 & 65.957 \end{array}\right] \begin{array}{l} 7.46\text{R2} + \text{R1} \\ \\ \end{array}$$
From the last matrix, we have $W \approx 118$ and $H \approx 66$. The two models agree at a height of 66 inches and a weight of 118 pounds.

64. (a) At intersection D, x_3 cars enter on N Street and x_4 enter on 10th street. There are 400 cars leaving D on 10th and 200 leaving on N. Therefore,
$x_3 + x_4 = 400 + 200$ or $x_3 + x_4 = 600$.

(b) We have the system
$$\begin{aligned} x_1 + x_4 &= 1000 \\ x_1 + x_2 &= 1100 \\ x_2 + x_3 &= 700 \\ x_3 + x_4 &= 600 \end{aligned}.$$
This system has the augmented matrix
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 1 & 1 & 0 & 0 & 1100 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right].$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right] \begin{array}{l} \\ -1\text{R1} + \text{R2} \\ \\ \\ \end{array} \Rightarrow$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right] \begin{array}{l} \\ \\ -1\text{R2} + \text{R3} \\ \\ \end{array} \Rightarrow$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ -1\text{R3} + \text{R4} \end{array}$$

(c) $x_1 + x_4 = 1000 \Rightarrow x_4 = 1000 - x_1$
$x_2 - x_4 = 100 \Rightarrow x_4 = x_2 - 100$
$x_3 + x_4 = 600 \Rightarrow x_4 = 600 - x_3$

(d) Since $x_4 = 1000 - x_1$, the largest possible value for x_1 so that x_4 is not negative is $x_1 = 1000$.

(e) Since $x_4 = x_2 - 100$, the smallest possible value for x_2 so that x_4 is not negative is $x_2 = 100$.

(f) Since $x_3 + x_4 = 600$, the largest possible value of x_3 and x_4 so that neither variable is negative is $x_3 = 600$ or $x_4 = 600$.

(g) A solution of the problem in which all equations are satisfied and all variables are nonnegative is $x_1 = 1000, x_2 = 100,$ $x_3 = 600,$ and $x_4 = 0.$ The solution is not unique. Another possible solution is $x_1 = 1000, x_2 = 100,$ $x_3 = 0,$ and $x_4 = 600.$ There are many other possibilities.

65. $F = a + bA + cP + dW$
Substituting the values, we have the following system of equations.
$a + 871b + 11.5c + 3d = 239$
$a + 847b + 12.2c + 2d = 234$
$a + 685b + 10.6c + 5d = 192$
$a + 969b + 14.2c + 1d = 343.$

66. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 871 & 11.5 & 3 & 239 \\ 1 & 847 & 12.2 & 2 & 234 \\ 1 & 685 & 10.6 & 5 & 192 \\ 1 & 969 & 14.2 & 1 & 343 \end{array}\right].$$

$$\left[\begin{array}{cccc|c} 1 & 871 & 11.5 & 3 & 239 \\ 1 & 847 & 12.2 & 2 & 234 \\ 1 & 685 & 10.6 & 5 & 192 \\ 1 & 969 & 14.2 & 1 & 343 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 871 & 11.5 & 3 & 239 \\ 0 & -24 & .7 & -1 & -5 \\ 1 & 685 & 10.6 & 5 & 192 \\ 1 & 969 & 14.2 & 1 & 343 \end{array}\right]\begin{array}{l} \\ -1R1+R2 \\ -1R1+R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 871 & 11.5 & 3 & 239 \\ 0 & -24 & .7 & -1 & -5 \\ 0 & -186 & -.9 & 2 & -47 \\ 1 & 969 & 14.2 & 1 & 343 \end{array}\right]\begin{array}{l} \\ \\ -1R1+R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 871 & 11.5 & 3 & 239 \\ 0 & -24 & .7 & -1 & -5 \\ 0 & -186 & -.9 & 2 & -47 \\ 0 & 98 & 2.7 & -2 & 104 \end{array}\right]\begin{array}{l} \\ \\ \\ -1R1+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 871 & 11.5 & 3 & 239 \\ 0 & 1 & -\frac{7}{240} & \frac{1}{24} & \frac{5}{24} \\ 0 & -186 & -.9 & 2 & -47 \\ 0 & 98 & 2.7 & -2 & 104 \end{array}\right]\begin{array}{l} \\ -\frac{1}{24}R2 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{8857}{240} & -\frac{799}{24} & \frac{1381}{24} \\ 0 & 1 & -\frac{7}{240} & \frac{1}{24} & \frac{5}{24} \\ 0 & -186 & -.9 & 2 & -47 \\ 0 & 98 & 2.7 & -2 & 104 \end{array}\right]\begin{array}{l} -871R2+R1 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{8857}{240} & -\frac{799}{24} & \frac{1381}{24} \\ 0 & 1 & -\frac{7}{240} & \frac{1}{24} & \frac{5}{24} \\ 0 & 0 & -\frac{253}{40} & \frac{39}{4} & -\frac{33}{4} \\ 0 & 98 & 2.7 & -2 & 104 \end{array}\right]\begin{array}{l} \\ \\ 186R2+R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{8857}{240} & -\frac{799}{24} & \frac{1381}{24} \\ 0 & 1 & -\frac{7}{240} & \frac{1}{24} & \frac{5}{24} \\ 0 & 0 & -\frac{253}{40} & \frac{39}{4} & -\frac{33}{4} \\ 0 & 0 & \frac{667}{120} & -\frac{73}{12} & \frac{1003}{12} \end{array}\right]\begin{array}{l} \\ \\ \\ -98R2+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{8857}{240} & -\frac{799}{24} & \frac{1381}{24} \\ 0 & 1 & -\frac{7}{240} & \frac{1}{24} & \frac{5}{24} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & \frac{667}{120} & -\frac{73}{12} & \frac{1003}{12} \end{array}\right]\begin{array}{l} \\ \\ -\frac{40}{253}R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{35{,}819}{1518} & \frac{649}{69} \\ 0 & 1 & -\frac{7}{240} & \frac{1}{24} & \frac{5}{24} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & \frac{667}{120} & -\frac{73}{12} & \frac{1003}{12} \end{array}\right]\begin{array}{l} -\frac{8857}{240}R3+R1 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{35{,}819}{1518} & \frac{649}{69} \\ 0 & 1 & 0 & -\frac{5}{1518} & \frac{17}{69} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & \frac{667}{120} & -\frac{73}{12} & \frac{1003}{12} \end{array}\right]\begin{array}{l} \\ \frac{7}{240}R3+R2 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{35{,}819}{1518} & \frac{649}{69} \\ 0 & 1 & 0 & -\frac{5}{1518} & \frac{17}{69} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & 0 & \frac{82}{33} & \frac{229}{3} \end{array}\right]\begin{array}{l} \\ \\ \\ -\frac{667}{120}R3+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{35{,}819}{1518} & \frac{649}{69} \\ 0 & 1 & 0 & -\frac{5}{1518} & \frac{17}{69} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & 0 & 1 & \frac{2519}{82} \end{array}\right]\begin{array}{l} \\ \\ \\ \frac{33}{82}R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{117{,}335}{164} \\ 0 & 1 & 0 & -\frac{5}{1518} & \frac{17}{69} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & 0 & 1 & \frac{2519}{82} \end{array}\right]\begin{array}{l} -\frac{35819}{1518}R4+R1 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{117{,}335}{164} \\ 0 & 1 & 0 & 0 & \frac{57}{164} \\ 0 & 0 & 1 & -\frac{390}{253} & \frac{30}{23} \\ 0 & 0 & 0 & 1 & \frac{2519}{82} \end{array}\right]\begin{array}{l} \\ \frac{5}{1518}R4+R2 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{117{,}335}{164} \\ 0 & 1 & 0 & 0 & \frac{57}{164} \\ 0 & 0 & 1 & 0 & \frac{1995}{41} \\ 0 & 0 & 0 & 1 & \frac{2519}{82} \end{array}\right]\begin{array}{l} \\ \\ \frac{390}{253}R4+R3 \\ \\ \end{array}$$

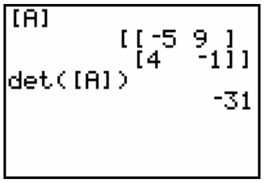
We obtain
$a = -\frac{117{,}335}{164} \approx -715.457$, $b = \frac{57}{164} \approx .34756$,
$c = \frac{1995}{41} \approx 48.6585$, and $d = \frac{2519}{82} \approx 30.71951$

67. Using these values,
$F = -714.457 + .34756A + 48.6585P + 30.71951W$

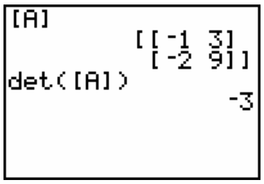
68. Using $A = 960$, $P = 12.6$, and $W = 3$ we obtain the following.
$F = -715.457 + .34756(960) + 48.6585(12.6) + 30.71951(3) = 323.45623 \approx 323$
Therefore, the predicted fawn count is approximately 323, which is just slightly higher than the actual value of 320.

Section 9.3: Determinant Solution of Linear Systems

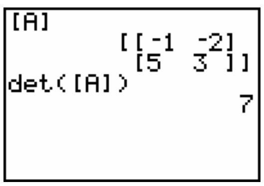
1. $\begin{vmatrix} -5 & 9 \\ 4 & -1 \end{vmatrix} = -5(-1) - 4\cdot 9 = 5 - 36 = -31$



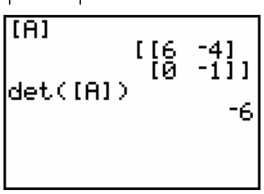
2. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -1\cdot 9 - (-2)\cdot 3 = -9 - (-6) = -3$



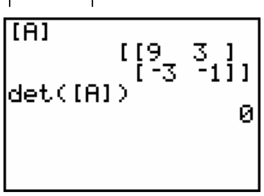
3. $\begin{vmatrix} -1 & -2 \\ 5 & 3 \end{vmatrix} = -1\cdot 3 - 5(-2) = -3 - (-10) = 7$



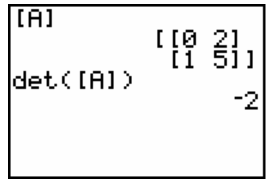
4. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = 6(-1) - 0(-4) = -6 - 0 = -6$



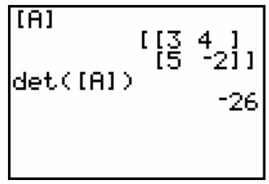
5. $\begin{vmatrix} 9 & 3 \\ -3 & -1 \end{vmatrix} = 9(-1) - (-3)\cdot 3 = -9 - (-9) = 0$



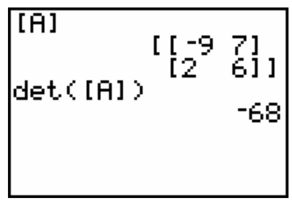
6. $\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 0\cdot 5 - 1\cdot 2 = 0 - 2 = -2$



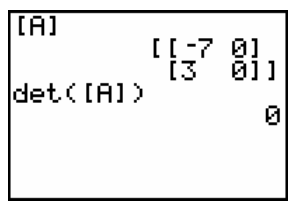
7. $\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = 3(-2) - 5\cdot 4 = -6 - 20 = -26$



8. $\begin{vmatrix} -9 & 7 \\ 2 & 6 \end{vmatrix} = -9(6) - 2\cdot 7 = -54 - 14 = -68$



9. $\begin{vmatrix} -7 & 0 \\ 3 & 0 \end{vmatrix} = -7\cdot 0 - 3\cdot 0 = 0 - 0 = 0$



10. The determinate value is 0.

11. $\begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix}$

To find the cofactor of 1, we have $i = 2$, $j = 1$,

$M_{21} = \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 0\cdot 1 - 2\cdot 1 = 0 - 2 = -2$

Thus, the cofactor is

$(-1)^{2+1}(-2) = (-1)^3(-2) = (-1)(-2) = 2.$

(*continued on next page*)

(*continued from page 893*)

To find the cofactor of 2, we have $i = 2,\ j = 2,$

$$M_{22} = \begin{vmatrix} -2 & 1 \\ 4 & 1 \end{vmatrix} = -2\cdot 1 - 4\cdot 1 = -2 - 4 = -6$$

Thus, the cofactor is

$(-1)^{2+2}(-6) = (-1)^4(-6) = 1(-6) = -6.$

To find the cofactor of 0, we have $i = 2,\ j = 3,$

$$M_{23} = \begin{vmatrix} -2 & 0 \\ 4 & 2 \end{vmatrix} = -2\cdot 2 - 4\cdot 0 = -4 - 0 = -4$$

Thus, the cofactor is

$(-1)^{2+3}(-4) = (-1)^5(-4) = (-1)(-4) = 4.$

12. $\begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 0 & -3 & 1 \end{vmatrix}$

To find the cofactor of 1, we have $i = 2,\ j = 1,$

$$M_{21} = \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = -1\cdot 1 - (-3)\cdot 2 = -1 - (-6) = 5$$

Thus, the cofactor is

$(-1)^{2+1}(5) = (-1)^3(5) = (-1)(5) = -5.$

To find the cofactor of 0, we have

$i = 2,\ j = 2, M_{22} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 2 = 1 - 0 = 1$

Thus, the cofactor is

$(-1)^{2+2}(1) = (-1)^4(1) = 1(1) = 1.$

To find the cofactor of 2, we have $i = 2,\ j = 3,$

$$M_{23} = \begin{vmatrix} 1 & -1 \\ 0 & -3 \end{vmatrix} = 1(-3) - 0(-1) = -3 - 0 = -3$$

Thus, the cofactor is

$(-1)^{2+3}(-3) = (-1)^5(-3) = (-1)(-3) = 3.$

13. $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & 4 & 1 \end{vmatrix}$

To find the cofactor of 2, we have $i = 2,\ j = 1,$

$$M_{21} = \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 2\cdot 1 - 4(-1) = 2 - (-4) = 6$$

Thus, the cofactor is

$(-1)^{2+1}(6) = (-1)^3(6) = (-1)(6) = -6.$

To find the cofactor of 3, we have $i = 2,\ j = 2,$

$$M_{22} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1\cdot 1 - (-1)(-1) = 1 - 1 = 0$$

Thus, the cofactor is

$(-1)^{2+2}(0) = (-1)^4(0) = 1(0) = 0.$

To find the cofactor of -2, we have $i = 2,\ j = 3,$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 1\cdot 4 - (-1)\cdot 2 = 4 - (-2) = 6$$

Thus, the cofactor is

$(-1)^{2+3}(6) = (-1)^5(6) = (-1)(6) = -6.$

14. $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \\ -2 & 1 & 4 \end{vmatrix}$

To find the cofactor of 3, we have $i = 2,\ j = 1,$

$$M_{21} = \begin{vmatrix} -1 & 4 \\ 1 & 4 \end{vmatrix} = -1\cdot 4 - 1\cdot 4 = -4 - 4 = -8.$$

Thus, the cofactor is

$(-1)^{2+1}(-8) = (-1)^3(-8) = (-1)(-8) = 8.$

To find the cofactor of 0, we have $i = 2,\ j = 2,$

$$M_{22} = \begin{vmatrix} 2 & 4 \\ -2 & 4 \end{vmatrix} = 2\cdot 4 - (-2)\cdot 4 = 8 - (-8) = 16$$

Thus, the cofactor is

$(-1)^{2+2}(16) = (-1)^4(16) = 1(16) = 16.$

To find the cofactor of 1, we have $i = 2,\ j = 3,$

$$M_{23} = \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = 2\cdot 1 - (-2)(-1) = 2 - 2 = 0$$

Thus, the cofactor is

$(-1)^{2+3}(0) = (-1)^5(0) = (-1)(0) = 0.$

For Exercises 15–28, an answer can be arrived at by expanding on any row or column as noted on page 870 of your text. In the solutions, we will expand on a row or column that allows a minimum number of calculations. Any row or column containing zero, will reduce the number of calculations as noted in the same paragraph.

15. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

If we expand by the third row, we will need to find $a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$. However, we do not need to calculate A_{33}, since $a_{33} = 0$.

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -7 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^4(-7\cdot 3 - 1\cdot 8)$$
$$= 1(-21 - 8) = 1(-29) = -29$$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 8 \\ 2 & 3 \end{vmatrix} = (-1)^5(4\cdot 3 - 2\cdot 8)$
$= -1(12-16) = -1(-4) = 4$

$a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$
$= a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + 0\cdot A_{33}$
$= -6(-29) + 3(4) + 0 = 174 + 12 = 186$

16. $\begin{vmatrix} 8 & -2 & -4 \\ 7 & 0 & 3 \\ 5 & -1 & 2 \end{vmatrix}$

If we expand by the second row, we will need to find $a_{21}\cdot A_{21} + a_{22}\cdot A_{22} + a_{23}\cdot A_{23}$. However, we do not need to calculate A_{22}, since $a_{22} = 0$.

$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & -4 \\ -1 & 2 \end{vmatrix}$
$= (-1)^3\left[-2\cdot 2 - (-1)\cdot(-4)\right]$
$= -1[-4-4] = -1(-8) = 8$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 8 & -2 \\ 5 & -1 \end{vmatrix} = (-1)^5\left[8(-1) - 5(-2)\right]$
$= -1\left[-8-(-10)\right] = -1(2) = -2$

$a_{21}\cdot A_{21} + a_{22}\cdot A_{22} + a_{23}\cdot A_{23}$
$= a_{21}\cdot A_{21} + 0\cdot A_{22} + a_{23}\cdot A_{23}$
$= 7(8) + 0 + 3(-2) = 56 + (-6) = 50$

17. $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 4 \end{vmatrix}$

If we expand by the third row, we will need to find $a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$. However, we do not need to calculate A_{31}, since $a_{31} = 0$.

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}$
$= (-1)^5\left[1(-1) - (-1)\cdot 0\right]$
$= -1(-1-0) = -1(-1) = 1$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$
$= (-1)^6\left[1\cdot 2 - (-1)\cdot 2\right]$
$= 1\left[2-(-2)\right] = 1(4) = 4$

$a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$
$= 0\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$
$= 0 + 1(1) + 4(4) = 1 + 16 = 17$

18. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

If we expand by the third row, we will need to find $a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$. However, we do not need to calculate A_{33}, since $a_{33} = 0$.

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & -1 \\ 7 & -2 \end{vmatrix} = (-1)^4\left[1(-2) - 7(-1)\right]$
$= 1\left[-2-(-7)\right] = 1(5) = 5$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = (-1)^5\left[2(-2) - 4(-1)\right]$
$= -1\left[-4-(-4)\right] = -1(0) = 0$

$a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + a_{33}\cdot A_{33}$
$= a_{31}\cdot A_{31} + a_{32}\cdot A_{32} + 0\cdot A_{33}$
$= 2(5) + 4(0) + 0 = 10 + 0 = 10$

For Exercises 19–28, we will use the sign checkerboard as described in the margin of page 526 in calculations.

19. $\begin{vmatrix} 10 & 2 & 1 \\ -1 & 4 & 3 \\ -3 & 8 & 10 \end{vmatrix}$

If we expand by the first row, we will need to find $a_{11}\cdot M_{11} - a_{12}\cdot M_{12} + a_{13}\cdot M_{13}$.

$M_{11} = \begin{vmatrix} 4 & 3 \\ 8 & 10 \end{vmatrix} = 4\cdot 10 - 8\cdot 3 = 40 - 24 = 16,$

$M_{12} = \begin{vmatrix} -1 & 3 \\ -3 & 10 \end{vmatrix} = -1\cdot 10 - (-3)\cdot 3$
$= -10 - (-9) = -1,$ and

$M_{13} = \begin{vmatrix} -1 & 4 \\ -3 & 8 \end{vmatrix} = -1\cdot 8 - (-3)\cdot 4 = -8 - (-12) = 4$

$a_{11}\cdot M_{11} - a_{12}\cdot M_{12} + a_{13}\cdot M_{13}$
$= 10\cdot 16 - 2(-1) + 1\cdot 4$
$= 160 - (-2) + 4 = 166$

20. $\begin{vmatrix} 7 & -1 & 1 \\ 1 & -7 & 2 \\ -2 & 1 & 1 \end{vmatrix}$

If we expand by the first row, we will need to find $a_{11}\cdot M_{11} - a_{12}\cdot M_{12} + a_{13}\cdot M_{13}$.

$M_{11} = \begin{vmatrix} -7 & 2 \\ 1 & 1 \end{vmatrix} = -7 \cdot 1 - 1 \cdot 2 = -7 - 2 = -9,$

$M_{12} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 \cdot 1 - (-2) \cdot 2 = 1 - (-4) = 5,$

and

$M_{13} = \begin{vmatrix} 1 & -7 \\ -2 & 1 \end{vmatrix} = 1 \cdot 1 - (-2)(-7) = 1 - 14 = -13$

$$\begin{aligned} &a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13} \\ &= 7(-9) - (-1) \cdot 5 + 1(-13) \\ &= -63 - (-5) + (-13) = -71 \end{aligned}$$

21. $\begin{vmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 1 & 10 & -12 \end{vmatrix}$

If we expand by the second row, we will need to find $-a_{21} \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23}$. Since $a_{21} = a_{22} = a_{23} = 0,$ the result will be zero.

22. $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 9 & 0 \\ -1 & -2 & 0 \end{vmatrix}$

If we expand by the third column, we will need to find $a_{13} \cdot M_{13} - a_{23} \cdot M_{23} + a_{33} \cdot M_{33}$. Since $a_{13} = a_{23} = a_{33} = 0,$ the result will be zero.

23. $\begin{vmatrix} 3 & 3 & -1 \\ 2 & 6 & 0 \\ -6 & -6 & 2 \end{vmatrix}$

If we expand by the second row, we will need to find $-a_{21} \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23}$. However, we do not need to calculate M_{23}, since $a_{23} = 0$.

$M_{21} = \begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 3 \cdot 2 - (-6)(-1) = 6 - 6 = 0$

and

$M_{22} = \begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 3 \cdot 2 - (-6)(-1) = 6 - 6 = 0$

$$\begin{aligned} &a_{21} \cdot M_{21} - a_{22} \cdot M_{22} + a_{23} \cdot M_{23} \\ &= a_{21} \cdot M_{21} - a_{22} \cdot M_{22} + 0 \cdot M_{23} \\ &= 2 \cdot 0 - 6 \cdot 0 + 0 = 0 - 0 = 0 \end{aligned}$$

24. $\begin{vmatrix} 5 & -3 & 2 \\ -5 & 3 & -2 \\ 1 & 0 & 1 \end{vmatrix}$

If we expand by the third row, we will need to find $a_{31} \cdot M_{31} - a_{32} \cdot M_{32} + a_{33} \cdot M_{33}$. However, we do not need to calculate M_{32}, since $a_{32} = 0$.

$M_{31} = \begin{vmatrix} -3 & 2 \\ 3 & -2 \end{vmatrix} = -3(-2) - 3 \cdot 2 = 6 - 6 = 0$ and

$M_{33} = \begin{vmatrix} 5 & -3 \\ -5 & 3 \end{vmatrix} = 5 \cdot 3 - (-5)(-3) = 15 - 15 = 0$

$$\begin{aligned} &a_{31} \cdot M_{31} - a_{32} \cdot M_{32} + a_{33} \cdot M_{33} \\ &= a_{31} \cdot M_{31} - 0 \cdot M_{32} + a_{33} \cdot M_{33} \\ &= 1 \cdot 0 - 0 + 1 \cdot 0 = 0 + 0 = 0 \end{aligned}$$

25. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

If we expand by the first row, we will need to find $a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13}$. However, we do not need to calculate M_{12} or M_{13}, since $a_{12} = a_{13} = 0$.

$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1 - 0 = 1$

$$\begin{aligned} &a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13} \\ &= a_{11} \cdot M_{11} - 0 \cdot M_{12} + 0 \cdot M_{13} \\ &= 1 \cdot 1 - 0 + 0 = 1 \end{aligned}$$

26. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

If we expand by the first row, we will need to find $a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13}$. However, we do not need to calculate M_{12} or M_{13}, since $a_{12} = a_{13} = 0$.

$M_{11} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1(1) - 0 \cdot 0 = -1 - 0 = -1$

$$\begin{aligned} &a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13} \\ &= a_{11} \cdot M_{11} - 0 \cdot M_{12} + 0 \cdot M_{13} \\ &= 1(-1) - 0 + 0 = -1 \end{aligned}$$

27. $\begin{vmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$

If we expand by the first column, we will need to find $a_{11} \cdot M_{11} - a_{21} \cdot M_{21} + a_{31} \cdot M_{31}$. However, we do not need to calculate M_{21} or M_{31}, since $a_{21} = a_{31} = 0$.

$$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = 1(-1) - 0 \cdot 0 = -1 - 0 = -1$$

$$\begin{aligned} &a_{11} \cdot M_{11} - a_{21} \cdot M_{21} + a_{31} \cdot M_{31} \\ &= a_{11} \cdot M_{11} - 0 \cdot M_{21} + 0 \cdot M_{31} \\ &= -2(-1) - 0 + 0 = 2 \end{aligned}$$

28. $\begin{vmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$

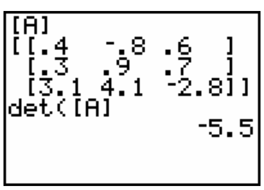
If we expand by the first column, we will need to find $a_{11} \cdot M_{11} - a_{21} \cdot M_{21} + a_{31} \cdot M_{31}$. However, we do not need to calculate M_{11} or M_{31}, since $a_{11} = a_{31} = 0$.

$$M_{21} = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = 0 \cdot 0 - (-1)(-1) = 0 - 1 = -1$$

$$\begin{aligned} &a_{11} \cdot M_{11} - a_{21} \cdot M_{21} + a_{31} \cdot M_{31} \\ &= 0 \cdot M_{11} - a_{21} \cdot M_{21} + 0 \cdot M_{31} \\ &= 0 - (-1)(-1) + 0 = -1 \end{aligned}$$

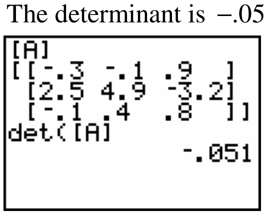
29. $\begin{vmatrix} .4 & -.8 & .6 \\ .3 & .9 & .7 \\ 3.1 & 4.1 & -2.8 \end{vmatrix}$

The determinant is -5.5.



30. $\begin{vmatrix} -.3 & -.1 & .9 \\ 2.5 & 4.9 & -3.2 \\ -.1 & .4 & .8 \end{vmatrix}$

The determinant is $-.051$.



31. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

$d_1 = a_{11}a_{22}a_{33}; d_2 = a_{12}a_{23}a_{31}; d_3 = a_{13}a_{21}a_{32}$
$d_4 = a_{13}a_{22}a_{31}; d_5 = a_{11}a_{23}a_{32}; d_6 = a_{12}a_{21}a_{33}$

$$\begin{aligned} &(d_1 + d_2 + d_3) - (d_4 + d_5 + d_6) \\ &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\quad - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}) \\ &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\quad - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}) \end{aligned}$$

32. $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & 6 \\ 7 & 1 & 5 \end{vmatrix} \begin{matrix} 1 & 3 \\ 0 & 2 \\ 7 & 1 \end{matrix}$

$d_1 = 1 \cdot 2 \cdot 5 = 10; d_2 = 3 \cdot 6 \cdot 7 = 126;$
$d_3 = 2 \cdot 0 \cdot 1 = 0$
$d_4 = 2 \cdot 2 \cdot 7 = 28; d_5 = 1 \cdot 6 \cdot 1 = 6;$
$d_6 = 3 \cdot 0 \cdot 5 = 0$
$(10 + 126 + 0) - (28 + 6 + 0) = 102$

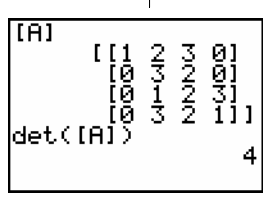
33. $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & 6 \\ 7 & 1 & 5 \end{vmatrix}$

If we expand by the first column, we will need to find $a_{11} \cdot M_{11} - a_{21} \cdot M_{21} + a_{31} \cdot M_{31}$.

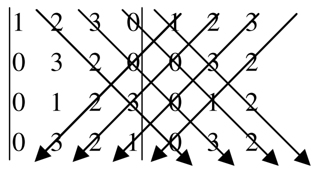
$$\begin{aligned} &(-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix} - (-1)^{1+2} \cdot 0 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} \\ &\qquad + (-1)^{1+3} \cdot 7 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 6 \end{vmatrix} \\ &= 1(10 - 6) + 0 + 7(18 - 4) = 4 + 98 = 102 \end{aligned}$$

Both methods give the same determinant.

34. No. For example, using a graphing calculator, we find that $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{vmatrix} = 4$



Using diagonals, we have



$d_1 = 1\cdot3\cdot2\cdot1 = 6, d_2 = 2\cdot2\cdot3\cdot0 = 0,$
$d_3 = 3\cdot0\cdot0\cdot3 = 0, d_4 = 0\cdot0\cdot1\cdot2 = 0,$
$d_5 = 3\cdot3\cdot0\cdot1 = 0, d_6 = 2\cdot0\cdot3\cdot2 = 0,$
$d_7 = 1\cdot0\cdot2\cdot3 = 0, d_8 = 0\cdot2\cdot1\cdot0 = 0,$

$$(d_1+d_2+d_3+d_4)-(d_5+d_6+d_7+d_8) = (6+0+0+0)-(0+0+0+0) = 6$$

35. To solve the equation $\begin{vmatrix} 5 & x \\ -3 & 2 \end{vmatrix} = 6,$ we need to solve $5\cdot2-(-3)\cdot x = 6.$

$$5\cdot2-(-3)\cdot x = 6 \Rightarrow 10+3x = 6 \Rightarrow 3x = -4 \Rightarrow x = -\frac{4}{3}$$

Verifying $\begin{vmatrix} 5 & -\frac{4}{3} \\ -3 & 2 \end{vmatrix} = 6,$ we have

$5\cdot2-(-3)\left(-\frac{4}{3}\right) = 10-4 = 6.$

Solution set: $\left\{-\frac{4}{3}\right\}$

36. To solve the equation $\begin{vmatrix} -.5 & 2 \\ x & x \end{vmatrix} = 0,$ we need to solve $-.5x-2x = 0.$

$-.5x-2x = 0 \Rightarrow -2.5x = 0 \Rightarrow x = 0$

Verifying $\begin{vmatrix} -.5 & 2 \\ 0 & 0 \end{vmatrix} = 0,$ we have

$-.5\cdot0-0\cdot2 = 0-0 = 0.$

Solution set: $\{0\}$

37. To solve the equation $\begin{vmatrix} x & 3 \\ x & x \end{vmatrix} = 4,$ we need to solve

$$x\cdot x-3x = 4 \Rightarrow x^2-3x = 4 \Rightarrow x^2-3x-4 = 0 \Rightarrow (x+1)(x-4) = 0$$

$x+1 = 0 \Rightarrow x = -1$ or $x-4 = 0 \Rightarrow x = 4$

Verify $x = -1.$

$$\begin{vmatrix} -1 & 3 \\ -1 & -1 \end{vmatrix} = -1(-1)-(-1)\cdot3 = 1-(-3) = 4$$

Verify $x = 4.$

$$\begin{vmatrix} 4 & 3 \\ 4 & 4 \end{vmatrix} = 4\cdot4-4\cdot3 = 16-12 = 4$$

Solution set: $\{-1, 4\}$

38. To solve the equation $\begin{vmatrix} 2x & x \\ 11 & x \end{vmatrix} = 6,$ we need to solve $2x\cdot x-11x = 6.$

$$2x\cdot x-11x = 6 \Rightarrow 2x^2-11x = 6 \Rightarrow 2x^2-11x-6 = 0 \Rightarrow (2x+1)(x-6) = 0$$

$2x+1 = 0 \Rightarrow x = -\frac{1}{2}$ or $x-6 = 0 \Rightarrow x = 6$

Verify $x = -\frac{1}{2}.$

$$\begin{vmatrix} 2\left(-\frac{1}{2}\right) & -\frac{1}{2} \\ 11 & -\frac{1}{2} \end{vmatrix} = \begin{vmatrix} -1 & -\frac{1}{2} \\ 11 & -\frac{1}{2} \end{vmatrix} = -1\left(-\frac{1}{2}\right)-11\left(-\frac{1}{2}\right) = \frac{1}{2}-\left(-\frac{11}{2}\right) = \frac{12}{2} = 6$$

Verify $x = 6.$

$$\begin{vmatrix} 2(6) & 6 \\ 11 & 6 \end{vmatrix} = \begin{vmatrix} 12 & 6 \\ 11 & 6 \end{vmatrix} = 12\cdot6-11\cdot6 = 72-66 = 6$$

Solution set: $\left\{-\frac{1}{2}, 6\right\}$

39. To solve the equation $\begin{vmatrix} -2 & 0 & 1 \\ -1 & 3 & x \\ 5 & -2 & 0 \end{vmatrix} = 3,$ expand by the first row. In order to do this, we will need to find $a_{11}\cdot M_{11}-a_{12}\cdot M_{12}+a_{13}\cdot M_{13}.$ However, we do not need to calculate $M_{12},$ since $a_{12} = 0.$

$$M_{11} = \begin{vmatrix} 3 & x \\ -2 & 0 \end{vmatrix} = 3\cdot0-(-2)\cdot x = 0-(-2x) = 2x$$

$$M_{13} = \begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix} = (-1)(-2)-5\cdot3 = 2-15 = -13$$

$$\begin{aligned} a_{11}\cdot M_{11}-a_{12}\cdot M_{12}+a_{13}\cdot M_{13} &= a_{11}\cdot M_{11}-0\cdot M_{12}+a_{13}\cdot M_{13} \\ &= -2(2x)-0+1(-13) \\ &= -4x+(-13) = -4x-13 \end{aligned}$$

Set this equal to 3 and solve to get $-4x-13 = 3 \Rightarrow -4x = 16 \Rightarrow x = -4.$

Verify $\begin{vmatrix} -2 & 0 & 1 \\ -1 & 3 & -4 \\ 5 & -2 & 0 \end{vmatrix} = 3.$

Since

$M_{11} = \begin{vmatrix} 3 & -4 \\ -2 & 0 \end{vmatrix} = 3 \cdot 0 - (-2)(-4) = 0 - 8 = -8$

and $M_{13} = \begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix} = -13,$ we have

$$\begin{aligned} &a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13} \\ &= a_{11} \cdot M_{11} - 0 \cdot M_{12} + a_{13} \cdot M_{13} \\ &= -2(-8) - 0 + 1(-13) = 16 + (-13) = 3 \end{aligned}$$

Solution set: $\{-4\}$

40. To solve the equation $\begin{vmatrix} 4 & 3 & 0 \\ 2 & 0 & 1 \\ -3 & x & -1 \end{vmatrix} = 5,$ expand by second column. We will need to find $-a_{12} \cdot M_{12} + a_{22} \cdot M_{22} - a_{32} \cdot M_{32}.$ However, we do not need to calculate $M_{22},$ since $a_{22} = 0.$

$M_{12} = \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = 2(-1) - (-3) \cdot 1 = -2 - (-3) = 1$

and $M_{32} = \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} = 4 \cdot 1 - 2 \cdot 0 = 4 - 0 = 4$

$$\begin{aligned} &-a_{12} \cdot M_{12} + a_{22} \cdot M_{22} - a_{32} \cdot M_{32} \\ &= -a_{12} \cdot M_{12} + 0 \cdot M_{22} - a_{32} \cdot M_{32} \\ &= -3 \cdot 1 - x \cdot 4 = -3 - 4x \end{aligned}$$

Set this equal to 5 and solve to get $-3 - 4x = 5 \Rightarrow -4x = 8 \Rightarrow x = -2.$

Verify $\begin{vmatrix} 4 & 3 & 0 \\ 2 & 0 & 1 \\ -3 & -2 & -1 \end{vmatrix} = 5.$

Since $M_{12} = \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = 1$ and

$M_{32} = \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} = 4,$ we have the following.

$$\begin{aligned} &-a_{12} \cdot M_{12} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23} \\ &= -a_{12} \cdot M_{12} + 0 \cdot M_{22} - a_{23} \cdot M_{23} \\ &= -3 \cdot 1 + 0 - (-2) \cdot 4 = -3 - (-8) = 5 \end{aligned}$$

Solution set: $\{-2\}$

41. To solve the equation $\begin{vmatrix} 5 & 3x & -3 \\ 0 & 2 & -1 \\ 4 & -1 & x \end{vmatrix} = -7,$ expand by the second row. We will need to find $-a_{21} \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23}.$ However, we do not need to calculate $M_{21},$ since $a_{21} = 0.$

$M_{22} = \begin{vmatrix} 5 & -3 \\ 4 & x \end{vmatrix} = 5x - 4(-3) = 5x - (-12) = 5x + 12$

and $M_{23} = \begin{vmatrix} 5 & 3x \\ 4 & -1 \end{vmatrix} = 5(-1) - 4 \cdot 3x = -5 - 12x$

$$\begin{aligned} &-a_{21} \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23} \\ &= -0 \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23} \\ &= 0 + 2(5x + 12) - (-1)(-5 - 12x) \\ &= 2(5x + 12) + (-5 - 12x) \\ &= 10x + 24 - 5 - 12x = 19 - 2x \end{aligned}$$

Set this equal to -7 and solve to get $19 - 2x = -7 \Rightarrow -2x = -26 \Rightarrow x = 13.$

Verify $\begin{vmatrix} 5 & 3(13) & -3 \\ 0 & 2 & -1 \\ 4 & -1 & 13 \end{vmatrix} = \begin{vmatrix} 5 & 39 & -3 \\ 0 & 2 & -1 \\ 4 & -1 & 13 \end{vmatrix} = -7.$

Since

$M_{22} = \begin{vmatrix} 5 & -3 \\ 4 & 13 \end{vmatrix} = 5 \cdot 13 - 4(-3) = 65 - (-12) = 77$

and

$M_{23} = \begin{vmatrix} 5 & 39 \\ 4 & -1 \end{vmatrix} = 5(-1) - 4 \cdot 39 = -5 - 156 = -161,$

we have the following.

$$\begin{aligned} &-a_{21} \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23} \\ &= -0 \cdot M_{21} + a_{22} \cdot M_{22} - a_{23} \cdot M_{23} \\ &= 0 + 2 \cdot 77 - (-1)(-161) = 154 - 161 = -7 \end{aligned}$$

Solution set: $\{13\}$

42. To solve the equation $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} = x,$ expand by the first column. We will need to find $a_{11} \cdot M_{11} - a_{21} \cdot M_{21} + a_{31} \cdot M_{31}.$ However, we do not need to calculate $M_{21},$ since $a_{21} = 0.$

$M_{11} = \begin{vmatrix} 4 & x \\ 0 & 2 \end{vmatrix} = 4 \cdot 2 - 0 \cdot x = 8 - 0 = 8$ and

$M_{31} = \begin{vmatrix} 1 & -1 \\ 4 & x \end{vmatrix} = 1 \cdot x - 4(-1) = x - (-4) = x + 4$

(*continued on next page*)

(*continued from page 899*)

$a_{11}\cdot M_{11}-a_{21}\cdot M_{21}+a_{31}\cdot M_{31}$
$=a_{11}\cdot M_{11}-0\cdot M_{21}+a_{31}\cdot M_{31}$
$=2x(8)-0+3(x+4)$
$=16x+3x+12=19x+12$

Set this equal to x and solve to get
$19x+12=x\Rightarrow 12=-18x\Rightarrow \frac{12}{-18}=x\Rightarrow x=-\frac{2}{3}.$

Verify $\begin{vmatrix}2\left(-\frac{2}{3}\right) & 1 & -1\\ 0 & 4 & -\frac{2}{3}\\ 3 & 0 & 2\end{vmatrix}=\begin{vmatrix}-\frac{4}{3} & 1 & -1\\ 0 & 4 & -\frac{2}{3}\\ 3 & 0 & 2\end{vmatrix}=-\frac{2}{3}.$

Since

$M_{11}=\begin{vmatrix}4 & -\frac{2}{3}\\ 0 & 2\end{vmatrix}=4\cdot 2-0\left(-\frac{2}{3}\right)=8-0=8$ and

$M_{31}=\begin{vmatrix}1 & -1\\ 4 & -\frac{2}{3}\end{vmatrix}=1\left(-\frac{2}{3}\right)-4(-1)$
$=-\frac{2}{3}-(-4)=\frac{10}{3}$

$a_{11}\cdot M_{11}-a_{21}\cdot M_{21}+a_{31}\cdot M_{31}$
$=a_{11}\cdot M_{11}-0\cdot M_{21}+a_{31}\cdot M_{31}$
$=-\frac{4}{3}(8)-0+3\cdot\frac{10}{3}=-\frac{32}{3}+\frac{30}{3}=-\frac{2}{3}$

Solution set: $\left\{-\frac{2}{3}\right\}$

43. $P(0,0), Q(0,2), R(1,4)$

Find $D=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$, where

$P=(x_1, y_1)=(0,0), Q(x_2, y_2)=(0,2)$, and $R=(x_3, y_3)=(1,4)$

Expanding by the first row, we have the following

$$D=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ 0 & 2 & 1\\ 1 & 4 & 1\end{vmatrix}=\frac{1}{2}\left[0\begin{vmatrix}2 & 1\\ 4 & 1\end{vmatrix}-0\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}+1\begin{vmatrix}0 & 2\\ 1 & 4\end{vmatrix}\right]$$
$=\frac{1}{2}[0(2-4)-0(0-1)+1(0-2)]$
$=\frac{1}{2}[0(-2)-0(-1)+1(-2)]$
$=\frac{1}{2}[0-0+(-2)]=\frac{1}{2}(-2)=-1$

Area of triangle $=|D|=|-1|=1.$

44. $P(0,1), Q(2,0), R(1,5)$

Find $D=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$, where

$P=(x_1, y_1)=(0,1), Q(x_2, y_2)=(2,0)$, and $R=(x_3, y_3)=(1,5)$.

Expanding by the first row, we have the following.

$$D=\frac{1}{2}\begin{vmatrix}0 & 1 & 1\\ 2 & 0 & 1\\ 1 & 5 & 1\end{vmatrix}=\frac{1}{2}\left[0\begin{vmatrix}0 & 1\\ 5 & 1\end{vmatrix}-1\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}+1\begin{vmatrix}2 & 0\\ 1 & 5\end{vmatrix}\right]$$
$=\frac{1}{2}[0(0-5)-1(2-1)+1(10-0)]$
$=\frac{1}{2}[0(-5)-1(1)+1(10)]=\frac{1}{2}(0-1+10)$
$=\frac{1}{2}(9)=\frac{9}{2}=4.5$

Area of triangle $=|D|=|4.5|=4.5.$

45. $P(2,5), Q(-1,3), R(4,0)$

Find $D=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$, where

$P=(x_1, y_1)=(2,5), Q(x_2, y_2)=(-1,3)$, and $R=(x_3, y_3)=(4,0)$.

Expanding by the third row, we have the following.

$$D=\frac{1}{2}\begin{vmatrix}2 & 5 & 1\\ -1 & 3 & 1\\ 4 & 0 & 1\end{vmatrix}$$
$$=\frac{1}{2}\left[4\begin{vmatrix}5 & 1\\ 3 & 1\end{vmatrix}-0\begin{vmatrix}2 & 1\\ -1 & 1\end{vmatrix}+1\begin{vmatrix}2 & 5\\ -1 & 3\end{vmatrix}\right]$$
$=\frac{1}{2}[4(5-3)-0(2+1)+1(6+5)]$
$=\frac{1}{2}[4(2)-0(3)+1(11)]=\frac{1}{2}(8-0+11)$
$=\frac{1}{2}(19)=\frac{19}{2}=9.5$

Area of triangle $=|D|=|9.5|=9.5.$

46. $P(2,-2), Q(0,0), R(-3,-4)$

Find $D=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$, where

$P=(x_1, y_1)=(2,-2), Q(x_2, y_2)=(0,0)$, and $R=(x_3, y_3)=(-3,-4)$.

Expanding by the second row, we have the following.

$$D=\frac{1}{2}\begin{vmatrix}2 & -2 & 1\\ 0 & 0 & 1\\ -3 & -4 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[-0\begin{vmatrix}-2 & 1\\ -4 & 1\end{vmatrix}+0\begin{vmatrix}2 & 1\\ -3 & 1\end{vmatrix}-1\begin{vmatrix}2 & -2\\ -3 & -4\end{vmatrix}\right]$$

$$=\tfrac{1}{2}\left[0(-2+4)+0(2+3)-1(-8-6)\right]$$

$$=\tfrac{1}{2}\left[0(2)+0(5)-1(-14)\right]$$

$$=\tfrac{1}{2}(0+0+14)=\tfrac{1}{2}(14)=7$$

Area of triangle $=|D|=|7|=7.$

47. $(101.3,52.7),(117.2,253.9),(313.1,301.6)$

Label the points as follows.

$P=(x_1,\,y_1)=(101.3,52.7)$,

$Q(x_2,\,y_2)=(117.2,253.9)$, and

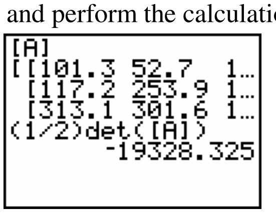
$R=(x_3,\,y_3)=(313.1,301.6)$

Since

$$D=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}101.3 & 52.7 & 1\\ 117.2 & 253.9 & 1\\ 313.1 & 301.6 & 1\end{vmatrix},$$

we will enter the 3×3 as $\begin{bmatrix}101.3 & 52.7 & 1\\ 117.2 & 253.9 & 1\\ 313.1 & 301.6 & 1\end{bmatrix}$

and perform the calculations as shown below.



Area of triangular lot is $|-19,328.325|$ ft^2 or approximately 19,328.3 ft^2.

48. $A=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$

$$=a_{31}\begin{vmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}-a_{32}\begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}+a_{33}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$

$$=a_{31}\left(a_{12}a_{23}-a_{22}a_{13}\right)-a_{32}\left(a_{11}a_{23}-a_{21}a_{13}\right)+a_{33}\left(a_{11}a_{22}-a_{21}a_{12}\right)$$

$$=a_{31}a_{12}a_{23}-a_{31}a_{22}a_{13}-a_{32}a_{11}a_{23}+a_{32}a_{21}a_{13}+a_{33}a_{11}a_{22}-a_{33}a_{21}a_{12}$$

$$=a_{31}a_{12}a_{23}+a_{32}a_{21}a_{13}+a_{33}a_{11}a_{22}-a_{31}a_{22}a_{13}-a_{32}a_{11}a_{23}-a_{33}a_{21}a_{12}$$

$$=\left(a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}+a_{11}a_{22}a_{33}\right)-\left(a_{31}a_{22}a_{13}+a_{32}a_{23}a_{11}+a_{33}a_{21}a_{12}\right)$$

$$=\left(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}\right)-\left(a_{31}a_{22}a_{13}+a_{32}a_{23}a_{11}+a_{33}a_{21}a_{12}\right)$$

49. $\begin{vmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 3 & 0 & 0\end{vmatrix}$

By Theorem 1, the determinant is 0 since every entry in column 2 is 0.

50. $\begin{vmatrix}-1 & 2 & 4\\ 4 & -8 & -16\\ 3 & 0 & 5\end{vmatrix}$

Row 1 of the matrix equals row 2 multiplied by $-\frac{1}{4}$, so, by Theorem 4, we have the following.

$$\begin{vmatrix}-1 & 2 & 4\\ 4 & -8 & -16\\ 3 & 0 & 5\end{vmatrix}=-\frac{1}{4}\begin{vmatrix}4 & -8 & -16\\ 4 & -8 & -16\\ 3 & 0 & 5\end{vmatrix}$$

By Theorem 5, the determinant of a matrix with two identical rows equals 0, so the value of the original determinant is $-\frac{1}{4}\cdot 0=0.$

51. $\begin{vmatrix}6 & 8 & -12\\ -1 & 0 & 2\\ 4 & 0 & -8\end{vmatrix}$

Given $\begin{bmatrix}6 & 8 & -12\\ -1 & 0 & 2\\ 4 & 0 & -8\end{bmatrix}$, add 2 times column 1 to column 3 to obtain $\begin{bmatrix}6 & 8 & 0\\ -1 & 0 & 0\\ 4 & 0 & 0\end{bmatrix}$. By Theorem 6 the following statement is true.

$$\begin{vmatrix}6 & 8 & -12\\ -1 & 0 & 2\\ 4 & 0 & -8\end{vmatrix}=\begin{vmatrix}6 & 8 & 0\\ -1 & 0 & 0\\ 4 & 0 & 0\end{vmatrix}$$

By Theorem 1, the determinant is 0 since every entry in column 3 is 0.

52. $\begin{vmatrix} 4 & 8 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & 3 \end{vmatrix}$

Given $\begin{bmatrix} 4 & 8 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$, add -2 times column 1 to column 2 to obtain $\begin{bmatrix} 4 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. By Theorem 6 the following statement is true.

$$\begin{vmatrix} 4 & 8 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix}$$

By Theorem 1, the determinant is 0 since every entry in column 2 is 0.

53. $\begin{vmatrix} -4 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix}$

Since $\begin{bmatrix} -4 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 6 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{bmatrix} \begin{matrix} 2R2+R1 \\ \\ \\ \end{matrix}$,

by Theorem 6 we have the following.

$$\begin{vmatrix} -4 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 6 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix}$$

Expanding by the first column, we have the following.

$$\begin{vmatrix} 0 & 1 & 6 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix} = 0\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} + 0\begin{vmatrix} 1 & 6 \\ 0 & 1 \end{vmatrix}$$
$$= 0 - 2(4-12) + 0 = -2(-8) = 16$$

54. $\begin{vmatrix} 6 & 3 & 2 \\ 1 & 0 & 2 \\ 5 & 7 & 3 \end{vmatrix}$

Given $\begin{bmatrix} 6 & 3 & 2 \\ 1 & 0 & 2 \\ 5 & 7 & 3 \end{bmatrix}$, add -2 times column 1 to column 3 to obtain $\begin{bmatrix} 6 & 3 & -10 \\ 1 & 0 & 0 \\ 5 & 7 & -7 \end{bmatrix}$. By Theorem 6 the following statement is true.

$$\begin{vmatrix} 6 & 3 & 2 \\ 1 & 0 & 2 \\ 5 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 3 & -10 \\ 1 & 0 & 0 \\ 5 & 7 & -7 \end{vmatrix}$$

Expanding by the second row we have the following.

$$\begin{vmatrix} 6 & 3 & -10 \\ 1 & 0 & 0 \\ 5 & 7 & -7 \end{vmatrix} = -1\begin{vmatrix} 3 & -10 \\ 7 & -7 \end{vmatrix} + 0\begin{vmatrix} 6 & -10 \\ 5 & -7 \end{vmatrix} - 0\begin{vmatrix} 6 & 3 \\ 5 & 7 \end{vmatrix}$$
$$= -1(-21+70) + 0 - 0$$
$$= -1(49) = -49$$

As noted on page 870 of the text, the array of signs can be extended for determinants of 4×4 matrices. For Exercises 55–58, use the following array of signs.

$$\underline{\textit{For } 4\times 4 \textit{ matrices}}$$
$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

Like previous exercises, you can arrive at a solution by expanding on any row or column. We will also use determinant theorems to reduce the number of calculations.

55. $\begin{vmatrix} 3 & -6 & 5 & -1 \\ 0 & 2 & -1 & 3 \\ -6 & 4 & 2 & 0 \\ -7 & 3 & 1 & 1 \end{vmatrix}$

Add 3 times row 1 to row 2 and add row 1 to row 4 to obtain

$$\begin{vmatrix} 3 & -6 & 5 & -1 \\ 0 & 2 & -1 & 3 \\ -6 & 4 & 2 & 0 \\ -7 & 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -6 & 5 & -1 \\ 9 & -16 & 14 & 0 \\ -6 & 4 & 2 & 0 \\ -4 & -3 & 6 & 0 \end{vmatrix}.$$

Expanding by the fourth column, we have the following.

$$\begin{vmatrix} 3 & -6 & 5 & -1 \\ 9 & -16 & 14 & 0 \\ -6 & 4 & 2 & 0 \\ -4 & -3 & 6 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} 9 & -16 & 14 \\ -6 & 4 & 2 \\ -4 & -3 & 6 \end{vmatrix}$$
$$= \begin{vmatrix} 9 & -16 & 14 \\ -6 & 4 & 2 \\ -4 & -3 & 6 \end{vmatrix}$$

Adding -7 times row 2 to row 1 and -3 times row 2 added to row 3 we have the following.

$$\begin{vmatrix} 9 & -16 & 14 \\ -6 & 4 & 2 \\ -4 & -3 & 6 \end{vmatrix} = \begin{vmatrix} 51 & -44 & 0 \\ -6 & 4 & 2 \\ 14 & -15 & 0 \end{vmatrix}$$

Expanding by the third column, we have

$$-2\begin{vmatrix} 51 & -44 \\ 14 & -15 \end{vmatrix} = -2(-765+616)$$
$$= -2(-149) = 298$$

56. $$\begin{vmatrix} 4 & 5 & -1 & -1 \\ 2 & -3 & 1 & 0 \\ -5 & 1 & 3 & 9 \\ 0 & -2 & 1 & 5 \end{vmatrix}$$

Add 9 times row 1 to row 3 and add 5 times row 1 to row 4 to obtain the following.

$$\begin{vmatrix} 4 & 5 & -1 & -1 \\ 2 & -3 & 1 & 0 \\ -5 & 1 & 3 & 9 \\ 0 & -2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 5 & -1 & -1 \\ 2 & -3 & 1 & 0 \\ 31 & 46 & -6 & 0 \\ 20 & 23 & -4 & 0 \end{vmatrix}$$

Expanding by the fourth column, we have the following.

$$\begin{vmatrix} 4 & 5 & -1 & -1 \\ 2 & -3 & 1 & 0 \\ 31 & 46 & -6 & 0 \\ 20 & 23 & -4 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} 2 & -3 & 1 \\ 31 & 46 & -6 \\ 20 & 23 & -4 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & -3 & 1 \\ 31 & 46 & -6 \\ 20 & 23 & -4 \end{vmatrix}$$

Adding 6 times row 1 to row 2 and 4 times row 1 added to row 3, we have the following.

$$\begin{vmatrix} 2 & -3 & 1 \\ 31 & 46 & -6 \\ 20 & 23 & -4 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 1 \\ 43 & 28 & 0 \\ 28 & 11 & 0 \end{vmatrix}$$

Expanding by the third column, we have

$$1\begin{vmatrix} 43 & 28 \\ 28 & 11 \end{vmatrix} = 473 - 784 = -311.$$

57. $$\begin{vmatrix} 4 & 0 & 0 & 2 \\ -1 & 0 & 3 & 0 \\ 2 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

Expanding by the second column, we have

$$\begin{vmatrix} 4 & 0 & 0 & 2 \\ -1 & 0 & 3 & 0 \\ 2 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = -4\begin{vmatrix} 4 & 0 & 2 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

Using the definition of the determinant in the text, we have the following.

$$-4\begin{vmatrix} 4 & 0 & 2 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$
$$= -4\left(\begin{array}{l}[4(3)(2)+0(0)(0)+2(-1)(1)] \\ \quad -[0(3)(2)+1(0)(4)+2(-1)(0)]\end{array}\right)$$
$$= -4[(24+0-2)-(0+0+0)]$$
$$= -4(22-0) = -4(22) = -88$$

58. $$\begin{vmatrix} -2 & 0 & 4 & 2 \\ 3 & 6 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 9 & 0 & 2 & -1 \end{vmatrix}$$

Expanding by the third row, we have the following.

$$\begin{vmatrix} -2 & 0 & 4 & 2 \\ 3 & 6 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 9 & 0 & 2 & -1 \end{vmatrix} = -3\begin{vmatrix} -2 & 0 & 4 \\ 3 & 6 & 0 \\ 9 & 0 & 2 \end{vmatrix}$$

Using the definition of the determinant in the text, we have the following.

$$-3\begin{vmatrix} -2 & 0 & 4 \\ 3 & 6 & 0 \\ 9 & 0 & 2 \end{vmatrix}$$
$$= -3\left(\begin{array}{l}[-2(6)(2)+0(0)(9)+4(3)(0)] \\ \quad -[9(6)(4)+0(0)(-2)+2(3)(0)]\end{array}\right)$$
$$= -3[(-24+0+0)-(216+0+0)]$$
$$= -3(-24-216) = -3(-240) = 720$$

59. For $\begin{array}{l} 4x+3y-2z=1 \\ 7x-4y+3z=2 \\ -2x+y-8z=0 \end{array}$, we have the following.

$$D = \begin{vmatrix} 4 & 3 & -2 \\ 7 & -4 & 3 \\ -2 & 1 & -8 \end{vmatrix}, D_x = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -4 & 3 \\ 0 & 1 & -8 \end{vmatrix},$$
$$D_y = \begin{vmatrix} 4 & 1 & -2 \\ 7 & 2 & 3 \\ -2 & 0 & -8 \end{vmatrix}, \text{ and } D_z = \begin{vmatrix} 4 & 3 & 1 \\ 7 & -4 & 2 \\ -2 & 1 & 0 \end{vmatrix}$$

(a) D **(b)** A

(c) C **(d)** B

60. $x = \frac{D_x}{D} = \frac{-43}{-43} = 1,\ y = \frac{D_y}{D} = \frac{0}{-43} = 0,$ and

$z = \frac{D_z}{D} = \frac{43}{-43} = -1$

Solution set: $\{(1, 0, -1)\}$

61. $x + y = 4$
$2x - y = 2$

$D = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2(1) = -1 - 2 = -3,$

$D_x = \begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = 4(-1) - 2(1) = -4 - 2 = -6,$

$D_y = \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} = 1(2) - 2(4) = 2 - 8 = -6 \Rightarrow$

$x = \frac{D_x}{D} = \frac{-6}{-3} = 2$ and $y = \frac{D_y}{D} = \frac{-6}{-3} = 2.$

Solution set: $\{(2, 2)\}$

62. $3x + 2y = -4$
$2x - y = -5$

$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = 3(-1) - 2(2) = -3 - 4 = -7,$

$D_x = \begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix} = -4(-1) - (-5)(2) = 4 + 10 = 14,$

$D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = 3(-5) - 2(-4) = -15 + 8 = -7 \Rightarrow$

$x = \frac{D_x}{D} = \frac{14}{-7} = -2$ and $y = \frac{D_y}{D} = \frac{-7}{-7} = 1.$

Solution set: $\{(-2, 1)\}$

63. $4x + 3y = -7$
$2x + 3y = -11$

$D = \begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix} = 4(3) - 2(3) = 12 - 6 = 6,$

$D_x = \begin{vmatrix} -7 & 3 \\ -11 & 3 \end{vmatrix} = -7(3) - (-11)(3)$
$= -21 + 33 = 12,$

$D_y = \begin{vmatrix} 4 & -7 \\ 2 & -11 \end{vmatrix} = 4(-11) - 2(-7)$
$= -44 + 14 = -30 \Rightarrow$

$x = \frac{D_x}{D} = \frac{12}{6} = 2$ and $y = \frac{D_y}{D} = \frac{-30}{6} = -5.$

Solution set: $\{(2, -5)\}$

64. $4x - y = 0$
$2x + 3y = 14$

$D = \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 4(3) - 2(-1) = 12 + 2 = 14,$

$D_x = \begin{vmatrix} 0 & -1 \\ 14 & 3 \end{vmatrix} = 0(3) - 14(-1) = 0 + 14 = 14,$

$D_y = \begin{vmatrix} 4 & 0 \\ 2 & 14 \end{vmatrix} = 4(14) - 2(0) = 56 - 0 = 56 \Rightarrow$

$x = \frac{D_x}{D} = \frac{14}{14} = 1$ and $y = \frac{D_y}{D} = \frac{56}{14} = 4.$

Solution set: $\{(1, 4)\}$

65. $5x + 4y = 10$
$3x - 7y = 6$

$D = \begin{vmatrix} 5 & 4 \\ 3 & -7 \end{vmatrix} = 5(-7) - 3(4) = -35 - 12 = -47,$

$D_x = \begin{vmatrix} 10 & 4 \\ 6 & -7 \end{vmatrix} = 10(-7) - 6(4) = -70 - 24 = -94,$

$D_y = \begin{vmatrix} 5 & 10 \\ 3 & 6 \end{vmatrix} = 5(6) - 3(10) = 30 - 30 = 0 \Rightarrow$

$x = \frac{D_x}{D} = \frac{-94}{-47} = 2$ and $y = \frac{D_y}{D} = \frac{0}{-47} = 0.$

Solution set: $\{(2, 0)\}$

66. $3x + 2y = -4$
$5x - y = 2$

$D = \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} = 3(-1) - 5(2) = -3 - 10 = -13,$

$D_x = \begin{vmatrix} -4 & 2 \\ 2 & -1 \end{vmatrix} = -4(-1) - 2(2) = 4 - 4 = 0,$

$D_y = \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix} = 3(2) - 5(-4) = 6 + 20 = 26 \Rightarrow$

$x = \frac{D_x}{D} = \frac{0}{-13} = 0$ and $y = \frac{D_y}{D} = \frac{26}{-13} = -2.$

Solution set: $\{(0, -2)\}$

67. $1.5x + 3y = 5$ (1)
$2x + 4y = 3$ (2)

$D = \begin{vmatrix} 1.5 & 3 \\ 2 & 4 \end{vmatrix} = (1.5)(4) - 2(3) = 6 - 6 = 0$

Because $D = 0$, Cramer's rule does not apply. To determine whether the system is inconsistent or has infinitely many solutions, use the elimination method.

$$\begin{aligned} 6x+12y&=20 && \text{Multiply equation (1) by 4.}\\ -6x-12y&=-9 && \text{Multiply equation (2) by } -3.\\ \hline 0&=11 && \text{False}\end{aligned}$$

The system is inconsistent.
Solution set: $\varnothing$

68. $12x+8y=3$ (1)
$15x+10y=9$ (2)

$$D=\begin{vmatrix}12 & 8\\ 15 & 10\end{vmatrix}=12(10)-15(8)=120-120=0$$

Because $D = 0$, Cramer's rule does not apply. To determine whether the system is inconsistent or has infinitely many solutions, use the elimination method.

$$\begin{aligned} 60x+40y&=15 && \text{Multiply equation (1) by 5.}\\ -60x-40y&=-36 && \text{Multiply equation (2) by } -4.\\ \hline 0&=-21 && \text{False}\end{aligned}$$

The system is inconsistent.
Solution set: $\varnothing$

69. $3x+2y=4$ (1)
$6x+4y=8$ (2)

$$D=\begin{vmatrix}3 & 2\\ 6 & 4\end{vmatrix}=3(4)-6(2)=12-12=0$$

Because $D = 0$, Cramer's rule does not apply. To determine whether the system is inconsistent or has infinitely many solutions, use the elimination method.

$$\begin{aligned} -6x-4y&=-8 && \text{Multiply equation (1) by } -2.\\ 6x+4y&=8\\ \hline 0&=0 && \text{True}\end{aligned}$$

This shows that equations (1) and (2) are dependent. To write the solution set with y as the arbitrary variable, solve equation (1) for x in terms of y.

$$3x+2y=4 \Rightarrow 3x=4-2y \Rightarrow x=\tfrac{4-2y}{3}$$

Solution set: $\left\{\left(\frac{4-2y}{3}, y\right)\right\}$

70. $4x+3y=9$
$12x+9y=27$

$$D=\begin{vmatrix}4 & 3\\ 12 & 9\end{vmatrix}=4(9)-12(3)=36-36=0$$

Because $D = 0$, Cramer's rule does not apply. To determine whether the system is inconsistent or has infinitely many solutions, use the elimination method.

$$\begin{aligned} -12x-9y&=-27 && \text{Multiply equation (1) by } -3.\\ 12x+9y&=27\\ \hline 0&=0 && \text{True}\end{aligned}$$

This shows that equations (1) and (2) are dependent. To write the solution set with y as the arbitrary variable, solve equation (1) for x in terms of y.

$$4x+3y=9 \Rightarrow 4x=9-3y \Rightarrow x=\tfrac{9-3y}{4}$$

Solution set: $\left\{\left(\frac{9-3y}{4}, y\right)\right\}$

71. $\frac{1}{2}x+\frac{1}{3}y=2$
$\frac{3}{2}x-\frac{1}{2}y=-12$

$$D=\begin{vmatrix}\frac{1}{2} & \frac{1}{3}\\ \frac{3}{2} & -\frac{1}{2}\end{vmatrix}=\tfrac{1}{2}\left(-\tfrac{1}{2}\right)-\tfrac{3}{2}\left(\tfrac{1}{3}\right)=-\tfrac{1}{4}-\tfrac{1}{2}=-\tfrac{1}{4}-\tfrac{2}{4}=-\tfrac{3}{4}$$

$$D_x=\begin{vmatrix}2 & \frac{1}{3}\\ -12 & -\frac{1}{2}\end{vmatrix}=2\left(-\tfrac{1}{2}\right)-(-12)\left(\tfrac{1}{3}\right) \text{ and } =-1+4=3$$

$$D_y=\begin{vmatrix}\frac{1}{2} & 2\\ \frac{3}{2} & -12\end{vmatrix}=\tfrac{1}{2}(-12)-\tfrac{3}{2}(2)=-6-3=-9$$

Thus, we have $x=\frac{D_x}{D}=\frac{3}{-\frac{3}{4}}=3\left(-\frac{4}{3}\right)=-4$ and

$$y=\frac{D_y}{D}=\frac{-9}{-\frac{3}{4}}=-9\left(-\tfrac{4}{3}\right)=12$$

Solution set: $\{(-4,12)\}$

72. $-\frac{3}{4}x+\frac{2}{3}y=16$
$\frac{5}{2}x+\frac{1}{2}y=-37$

$$D=\begin{vmatrix}-\frac{3}{4} & \frac{2}{3}\\ \frac{5}{2} & \frac{1}{2}\end{vmatrix}=-\tfrac{3}{4}\left(\tfrac{1}{2}\right)-\tfrac{5}{2}\left(\tfrac{2}{3}\right)=-\tfrac{3}{8}-\tfrac{5}{3}=-\tfrac{9}{24}-\tfrac{40}{24}=-\tfrac{49}{24}$$

$$D_x=\begin{vmatrix}16 & \frac{2}{3}\\ -37 & \frac{1}{2}\end{vmatrix}=16\left(\tfrac{1}{2}\right)-(-37)\left(\tfrac{2}{3}\right)=8+\tfrac{74}{3}=\tfrac{24}{3}+\tfrac{74}{3}=\tfrac{98}{3}$$

$$D_y=\begin{vmatrix}-\frac{3}{4} & 16\\ \frac{5}{2} & -37\end{vmatrix}=-\tfrac{3}{4}(-37)-\tfrac{5}{2}(16)=\tfrac{111}{4}-40=\tfrac{111}{4}-\tfrac{160}{4}=-\tfrac{49}{4}$$

Thus,we have $x=\frac{D_x}{D}=\frac{\frac{98}{3}}{-\frac{49}{24}}=\frac{98}{3}\left(-\frac{24}{49}\right)=-16$

and $y=\frac{D_y}{D}=\frac{-\frac{49}{4}}{-\frac{49}{24}}=-\frac{49}{4}\left(-\frac{24}{49}\right)=6$

Solution set: $\{(-16,6)\}$

In Exercises 73–84, we will be using the Determinant Theorems on page 876 of the text to reduce the number of calculations necessary in simplifying determinants.

73. $\begin{aligned}2x-y+4z+2&=0\\3x+2y-z+3&=0\\x+4y+2z-17&=0\end{aligned} \Rightarrow \begin{aligned}2x-y+4z&=-2\\3x+2y-z&=-3\\x+4y+2z&=17\end{aligned}$

Adding 2 times row 1 to row 2 and 4 times row 1 to row 3 we have

$$D=\begin{vmatrix}2&-1&4\\3&2&-1\\1&4&2\end{vmatrix}=\begin{vmatrix}2&-1&4\\7&0&7\\9&0&18\end{vmatrix}.$$

Expanding by column two, we have

$$D=-(-1)\begin{vmatrix}7&7\\9&18\end{vmatrix}=126-63=63.$$

Adding 2 times row 1 to row 2 and 4 times row 1 to row 3, we have

$$D_x=\begin{vmatrix}-2&-1&4\\-3&2&-1\\17&4&2\end{vmatrix}=\begin{vmatrix}-2&-1&4\\-7&0&7\\9&0&18\end{vmatrix}.$$

Expanding about column two, we have

$$D_x=-(-1)\begin{vmatrix}-7&7\\9&18\end{vmatrix}=-126-63=-189.$$

Adding column 2 to column 1, we have

$$D_y=\begin{vmatrix}2&-2&4\\3&-3&-1\\1&17&2\end{vmatrix}=\begin{vmatrix}0&-2&4\\0&-3&-1\\18&17&2\end{vmatrix}.$$

Expanding by column one, we have

$$D_y=18\begin{vmatrix}-2&4\\-3&-1\end{vmatrix}=18(2+12)=18(14)=252.$$

Adding column 3 to column 1, we have

$$D_z=\begin{vmatrix}2&-1&-2\\3&2&-3\\1&4&17\end{vmatrix}=\begin{vmatrix}0&-1&-2\\0&2&-3\\18&4&17\end{vmatrix}.$$

Expanding by column one we have

$$D_z=18\begin{vmatrix}-1&-2\\2&-3\end{vmatrix}=18(3+4)=18(7)=126.$$

Thus, we have $x=\frac{D_x}{D}=\frac{-189}{63}=-3$,

$y=\frac{D_y}{D}=\frac{252}{63}=4$, and $z=\frac{D_z}{D}=\frac{126}{63}=2$.

Solution set: $\{(-3,4,2)\}$

74. $\begin{aligned}x+y+z-4&=0\\2x-y+3z-4&=0\\4x+2y-z+15&=0\end{aligned} \Rightarrow \begin{aligned}x+y+z&=4\\2x-y+3z&=4\\4x+2y-z&=-15\end{aligned}$

Adding row 1 to row 2 and -2 times row 1 to row 3, we have $D=\begin{vmatrix}1&1&1\\2&-1&3\\4&2&-1\end{vmatrix}=\begin{vmatrix}1&1&1\\3&0&4\\2&0&-3\end{vmatrix}$.

Expanding by column two we have

$$D=-1\begin{vmatrix}3&4\\2&-3\end{vmatrix}=-1(-9-8)=-1(-17)=17.$$

Adding row 1 to row 2 and -2 times row 1 to row 3, we have

$$D_x=\begin{vmatrix}4&1&1\\4&-1&3\\-15&2&-1\end{vmatrix}=\begin{vmatrix}4&1&1\\8&0&4\\-23&0&-3\end{vmatrix}.$$

Expanding about column two, we have

$$D_x=-1\begin{vmatrix}8&4\\-23&-3\end{vmatrix}=-1(-24+92)$$
$$=-1(68)=-68$$

Adding -3 times row 1 to row 2 and row 1 to row 3, we have

$$D_y=\begin{vmatrix}1&4&1\\2&4&3\\4&-15&-1\end{vmatrix}=\begin{vmatrix}1&4&1\\-1&-8&0\\5&-11&0\end{vmatrix}.$$

Expanding by column three, we have

$$D_y=1\begin{vmatrix}-1&-8\\5&-11\end{vmatrix}=11+40=51.$$

Adding row 1 to row 2 and -2 times row 1 to row 3, we have

$$D_z=\begin{vmatrix}1&1&4\\2&-1&4\\4&2&-15\end{vmatrix}=\begin{vmatrix}1&1&4\\3&0&8\\2&0&-23\end{vmatrix}.$$

Expanding by column two, we have

$$D_z=-1\begin{vmatrix}3&8\\2&-23\end{vmatrix}=-1(-69-16)=-1(-85)=85$$

Thus, we have $x=\frac{D_x}{D}=\frac{-68}{17}=-4$,

$y=\frac{D_y}{D}=\frac{51}{17}=3$, and $z=\frac{D_z}{D}=\frac{85}{17}=5$.

Solution set: $\{(-4,3,5)\}$

75. $\begin{aligned}4x-3y+z&=-1\\5x+7y+2z&=-2\\3x-5y-z&=1\end{aligned}$

Adding row 3 to row 1 and 2 times row 3 to row 2, we have

$$D=\begin{vmatrix}4&-3&1\\5&7&2\\3&-5&-1\end{vmatrix}=\begin{vmatrix}7&-8&0\\11&-3&0\\3&-5&-1\end{vmatrix}.$$

Expanding by column three, we have

$$D = -1\begin{vmatrix} 7 & -8 \\ 11 & -3 \end{vmatrix} = -1(-21+88) = -1(67) = -67$$

Adding column 1 to column 3 we have

$$D_x = \begin{vmatrix} -1 & -3 & 1 \\ -2 & 7 & 2 \\ 1 & -5 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 0 \\ -2 & 7 & 0 \\ 1 & -5 & 0 \end{vmatrix}.$$

Since we have a column of zeros, $D_x = 0$.

Adding column 2 to column 3, we have

$$D_y = \begin{vmatrix} 4 & -1 & 1 \\ 5 & -2 & 2 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 0 \\ 5 & -2 & 0 \\ 3 & 1 & 0 \end{vmatrix}.$$

Since we have a column of zeros, $D_y = 0$.

Adding row 3 to row 1 and 2 times row 3 to row 2, we have

$$D_z = \begin{vmatrix} 4 & -3 & -1 \\ 5 & 7 & -2 \\ 3 & -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -8 & 0 \\ 11 & -3 & 0 \\ 3 & -5 & 1 \end{vmatrix}.$$

Expanding by the third column, we have

$$D_z = 1\begin{vmatrix} 7 & -8 \\ 11 & -3 \end{vmatrix} = -21+88 = 67.$$

Thus, we have $x = \frac{D_x}{D} = \frac{0}{-67} = 0$,

$y = \frac{D_y}{D} = \frac{0}{-67} = 0$, and $z = \frac{D_z}{D} = \frac{67}{-67} = -1$.

Solution set: $\{(0,0,-1)\}$

76. $2x - 3y + z = 8$
$-x - 5y + z = -4$
$3x - 5y + 2z = 12$

Adding 2 times row 2 to row 1 and 3 times row 2 to row 3, we have

$$D = \begin{vmatrix} 2 & -3 & 1 \\ -1 & -5 & 1 \\ 3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -13 & 3 \\ -1 & -5 & 1 \\ 0 & -20 & 5 \end{vmatrix}.$$

Expanding by column one, we have

$$D = -(-1)\begin{vmatrix} -13 & 3 \\ -20 & 5 \end{vmatrix} = -65+60 = -5.$$

Given that $D_x = \begin{vmatrix} 8 & -3 & 1 \\ -4 & -5 & 1 \\ 12 & -5 & 2 \end{vmatrix}$, we add -1 times row 1 to row 2 and -2 times row 1 to row 3 to obtain $D_x = \begin{vmatrix} 8 & -3 & 1 \\ -12 & -2 & 0 \\ -4 & 1 & 0 \end{vmatrix}$.

Expanding about column three, we have

$$D_x = 1\begin{vmatrix} -12 & -2 \\ -4 & 1 \end{vmatrix} = -12 - 8 = -20.$$

Adding 2 times row 2 to row 1, we have

$$D_y = \begin{vmatrix} 2 & 8 & 1 \\ -1 & -4 & 1 \\ 3 & 12 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 3 \\ -1 & -4 & 1 \\ 3 & 12 & 2 \end{vmatrix}.$$

Expanding by row one, we have

$$D_y = 3\begin{vmatrix} -1 & -4 \\ 3 & 12 \end{vmatrix} = 3(-12+12) = 3(0) = 0.$$

Adding -4 times column 1 to column 3, we have $D_z = \begin{vmatrix} 2 & -3 & 8 \\ -1 & -5 & -4 \\ 3 & -5 & 12 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 0 \\ -1 & -5 & 0 \\ 3 & -5 & 0 \end{vmatrix}$.

Since we have a column of zeros, $D_z = 0$.

Thus, we have $x = \frac{D_x}{D} = \frac{-20}{-5} = 4$,

$y = \frac{D_y}{D} = \frac{0}{-5} = 0$, and $z = \frac{D_z}{D} = \frac{0}{-5} = 0$.

Solution set: $\{(4,0,0)\}$

77. $x + 2y + 3z = 4$ (1)
$4x + 3y + 2z = 1$ (2)
$-x - 2y - 3z = 0$ (3)

Adding row 1 to row 3, we have

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ -1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 0 & 0 & 0 \end{vmatrix}.$$

Since we have a row of zeros, $D = 0$ and we cannot use Cramer's rule.
Using the elimination method, we can add equations (1) and (3).

$$\begin{array}{r} x + 2y + 3z = 4 \\ -x - 2y - 3z = 0 \\ \hline 0 = 4 \end{array} \quad \text{False}$$

The system is inconsistent.
Solution set: $\varnothing$

78. $2x - y + 3z = 1$ (1)
$-2x + y - 3z = 2$ (2)
$5x - y + z = 2$ (3)

Adding row 1 to row 2, we have

$$D = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 1 & -3 \\ 5 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 5 & -1 & 1 \end{vmatrix}.$$

Since we have a row of zeros, $D = 0$ and we cannot use Cramer's rule.Using the elimination method, we can add equations (1) and (2).

$$\begin{aligned} 2x - y + 3z &= 1 \\ -2x + y - 3z &= 2 \\ \hline 0 &= 3 \quad \text{False} \end{aligned}$$

The system is inconsistent. Solution set: $\varnothing$

79. $-2x - 2y + 3z = 4$ (1)
$5x + 7y - z = 2$ (2)
$2x + 2y - 3z = -4$ (3)

Adding row 1 to row 3, we have

$$D = \begin{vmatrix} -2 & -2 & 3 \\ 5 & 7 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 3 \\ 5 & 7 & -1 \\ 0 & 0 & 0 \end{vmatrix}.$$

Since we have a row of zeros, $D = 0$ and we cannot use Cramer's rule. Using the elimination method, we can add equations (1) and (3).

$$\begin{aligned} -2x - 2y + 3z &= 4 \\ 2x + 2y - 3z &= -4 \\ \hline 0 &= 0 \quad \text{True} \end{aligned}$$

This system will have infinitely many solutions.
Solve the system made up of equations (2) and (3) in terms of the arbitrary variable z. To eliminate x, multiply equation (2) by –2 and equation (3) by 5 and add the results

$$\begin{aligned} -10x - 14y + 2z &= -4 \\ 10x + 10y - 15z &= -20 \\ \hline -4y - 13z &= -24 \end{aligned}$$

Solve for y in terms of z.

$-4y - 13z = -24 \Rightarrow -4y = -24 + 13z \Rightarrow$
$y = \frac{-24+13z}{-4} \Rightarrow y = \frac{24-13z}{4}$

Express x also in terms of z by solving equation (3) for x and substituting $\frac{24-13z}{4}$ for y.

$2x + 2y - 3z = -4 \Rightarrow 2x = -2y + 3z - 4 \Rightarrow$
$x = \frac{-2y+3z-4}{2}$

$x = \frac{-2\left(\frac{24-13z}{4}\right)+3z-4}{2} = \frac{\frac{-24+13z}{2}+3z-4}{2}$
$= \frac{-24+13z+6z-8}{4} = \frac{-32+19z}{4}$

Solution set (with z arbitrary):
$\left\{\left(\frac{-32+19z}{4}, \frac{24-13z}{4}, z\right)\right\}$

80. $3x - 2y + 4z = 1$
$4x + y - 5z = 2$
$-6x + 4y - 8z = -2$

Adding 2 times row 1 to row 3, we have

$$D = \begin{vmatrix} 3 & -2 & 4 \\ 4 & 1 & -5 \\ -6 & 4 & -8 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 4 \\ 4 & 1 & -5 \\ 0 & 0 & 0 \end{vmatrix}.$$

Since we have a row of zeros, $D = 0$ and we cannot use Cramer's rule.Using the Gauss-Jordan method, the system can be expressed as the following augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & 1 \\ 4 & 1 & -5 & 2 \\ -6 & 4 & -8 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \\ 4 & 1 & -5 & 2 \\ -6 & 4 & -8 & -2 \end{array}\right] \frac{1}{3}\text{R1} \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & \frac{11}{3} & -\frac{31}{3} & \frac{2}{3} \\ -6 & 4 & -8 & -2 \end{array}\right] -4\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & \frac{11}{3} & -\frac{31}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] 6\text{R1}+\text{R3} \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{31}{11} & \frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right] \frac{3}{11}\text{R2} \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{6}{11} & \frac{5}{11} \\ 0 & 1 & -\frac{31}{11} & \frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right] \frac{2}{3}\text{R2}+\text{R1}$$

The third row of the last matrix shows that the system has infinitely many solutions. We will give the solution set with z as the arbitrary variable. From the first two rows of the final matrix, we obtain the equations
$x - \frac{6}{11}z = \frac{5}{11}$ and $y - \frac{31}{11}z = \frac{2}{11}$.

Solving the first equation for x and the second for y, we obtain $x = \frac{6}{11}z + \frac{5}{11} = \frac{6z+5}{11}$ and $y = \frac{31}{11}z + \frac{2}{11} = \frac{31z+2}{11}$.

Solution set (with z arbitrary): $\left\{\left(\frac{6z+5}{11}, \frac{31z+2}{11}, z\right)\right\}$

81. $5x - y = -4$
$3x + 2z = 4$
$4y + 3z = 22$

Adding 4 times row 1 to row 3, we have
$$D = \begin{vmatrix} 5 & -1 & 0 \\ 3 & 0 & 2 \\ 0 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 5 & -1 & 0 \\ 3 & 0 & 2 \\ 20 & 0 & 3 \end{vmatrix}.$$
Expanding by column two, we have
$$D = -(-1)\begin{vmatrix} 3 & 2 \\ 20 & 3 \end{vmatrix} = 9 - 40 = -31.$$
Adding 4 times row 1 to row 3, we have
$$D_x = \begin{vmatrix} -4 & -1 & 0 \\ 4 & 0 & 2 \\ 22 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -4 & -1 & 0 \\ 4 & 0 & 2 \\ 6 & 0 & 3 \end{vmatrix}.$$
Expand by column two, we have
$$D_x = -(-1)\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 12 - 12 = 0.$$
Adding column 2 to column 1, we have
$$D_y = \begin{vmatrix} 5 & -4 & 0 \\ 3 & 4 & 2 \\ 0 & 22 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 0 \\ 7 & 4 & 2 \\ 22 & 22 & 3 \end{vmatrix}.$$
Adding 4 times column 1 to column 2, we now have $D_y = \begin{vmatrix} 1 & 0 & 0 \\ 7 & 32 & 2 \\ 22 & 110 & 3 \end{vmatrix}$.

Expanding by row one, we have
$$D_y = 1\begin{vmatrix} 32 & 2 \\ 110 & 3 \end{vmatrix} = 96 - 220 = -124.$$
Adding 4 times row 1 to row 3, we have
$$D_z = \begin{vmatrix} 5 & -1 & -4 \\ 3 & 0 & 4 \\ 0 & 4 & 22 \end{vmatrix} = \begin{vmatrix} 5 & -1 & -4 \\ 3 & 0 & 4 \\ 20 & 0 & 6 \end{vmatrix}.$$
Expanding by column two ,we have
$$D_z = -(-1)\begin{vmatrix} 3 & 4 \\ 20 & 6 \end{vmatrix} = 18 - 80 = -62.$$
Thus, we have
$x = \frac{D_x}{D} = \frac{0}{-31} = 0$, $y = \frac{D_y}{D} = \frac{-124}{-31} = 4$, and $z = \frac{D_z}{D} = \frac{-62}{-31} = 2$.

Solution set: $\{(0, 4, 2)\}$

82. $3x + 5y = -7$
$2x + 7z = 2$
$4y + 3z = -8$

Adding -1 times row 2 to row 1, we have
$$D = \begin{vmatrix} 3 & 5 & 0 \\ 2 & 0 & 7 \\ 0 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -7 \\ 2 & 0 & 7 \\ 0 & 4 & 3 \end{vmatrix}.$$
Adding -2 times row 1 to row 2, we now have $D = \begin{vmatrix} 1 & 5 & -7 \\ 0 & -10 & 21 \\ 0 & 4 & 3 \end{vmatrix}$.

Expanding by column one, we have
$$D = 1\begin{vmatrix} -10 & 21 \\ 4 & 3 \end{vmatrix} = -30 - 84 = -114.$$
Adding -1 times row 3 to row 1, we have
$$D_x = \begin{vmatrix} -7 & 5 & 0 \\ 2 & 0 & 7 \\ -8 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 0 & 7 \\ -8 & 4 & 3 \end{vmatrix}.$$
Adding -4 times row 1 to row 3, we have
$$D_x = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 0 & 7 \\ -12 & 0 & 15 \end{vmatrix}.$$
Expanding by column two, we have
$$D_x = -1\begin{vmatrix} 2 & 7 \\ -12 & 15 \end{vmatrix} = -1(30 + 84)$$
$$= (-1)114 = -114$$
Adding -1 times row 2 to row 1, we have
$$D_y = \begin{vmatrix} 3 & -7 & 0 \\ 2 & 2 & 7 \\ 0 & -8 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -9 & -7 \\ 2 & 2 & 7 \\ 0 & -8 & 3 \end{vmatrix}.$$
Adding -2 times row 1 to row 2, we now have $D_y = \begin{vmatrix} 1 & -9 & -7 \\ 0 & 20 & 21 \\ 0 & -8 & 3 \end{vmatrix}$.

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Expanding by column one, we have

$$D_y = 1\begin{vmatrix} 20 & 21 \\ -8 & 3 \end{vmatrix} = 60 + 168 = 228.$$

Adding -1 times row 2 to row 1, we have

$$D_z = \begin{vmatrix} 3 & 5 & -7 \\ 2 & 0 & 2 \\ 0 & 4 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -9 \\ 2 & 0 & 2 \\ 0 & 4 & -8 \end{vmatrix}.$$

Adding -2 times row 1 to row 2, we now have $D_z = \begin{vmatrix} 1 & 5 & -9 \\ 0 & -10 & 20 \\ 0 & 4 & -8 \end{vmatrix}.$

Expanding by column one, we have

$$D_z = 1\begin{vmatrix} -10 & 20 \\ 4 & -8 \end{vmatrix} = 80 - 80 = 0.$$

Thus, we have

$x = \frac{D_x}{D} = \frac{-114}{-114} = 1,\ y = \frac{D_y}{D} = \frac{228}{-114} = -2,$ and

$z = \frac{D_z}{D} = \frac{0}{-114} = 0.$

Solution set: $\{(1, -2, 0)\}$

83. $x + 2y = 10$
$3x + 4z = 7$
$-y - z = 1$

Adding -3 times row 1 to row 2, we have

$$D = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -6 & 4 \\ 0 & -1 & -1 \end{vmatrix}.$$

Expanding by column one, we have

$$D = 1\begin{vmatrix} -6 & 4 \\ -1 & -1 \end{vmatrix} = 6 + 4 = 10.$$

Adding column 1 to column 2 and column 1 to column 3, we have

$$D_x = \begin{vmatrix} 10 & 2 & 0 \\ 7 & 0 & 4 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 10 & 12 & 10 \\ 7 & 7 & 11 \\ 1 & 0 & 0 \end{vmatrix}.$$

Expanding by row three, we have

$$D_x = 1\begin{vmatrix} 12 & 10 \\ 7 & 11 \end{vmatrix} = 132 - 70 = 62.$$

Adding column 2 to column 3, we have

$$D_y = \begin{vmatrix} 1 & 10 & 0 \\ 3 & 7 & 4 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 10 & 10 \\ 3 & 7 & 11 \\ 0 & 1 & 0 \end{vmatrix}.$$

Expanding by row three, we have

$$D_y = -1\begin{vmatrix} 1 & 10 \\ 3 & 11 \end{vmatrix} = -1(11 - 30) = -1(-19) = 19.$$

Adding column 3 to column 2, we have

$$D_z = \begin{vmatrix} 1 & 2 & 10 \\ 3 & 0 & 7 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 12 & 10 \\ 3 & 7 & 7 \\ 0 & 0 & 1 \end{vmatrix}.$$

Expanding by row three, we have

$$D_z = 1\begin{vmatrix} 1 & 12 \\ 3 & 7 \end{vmatrix} = 7 - 36 = -29.$$

Thus, we have $x = \frac{D_x}{D} = \frac{62}{10} = \frac{31}{5},\ y = \frac{D_y}{D} = \frac{19}{10},$ and $z = \frac{D_z}{D} = \frac{-29}{10} = -\frac{29}{10}.$

Solution set: $\left\{\left(\frac{31}{5}, \frac{19}{10}, -\frac{29}{10}\right)\right\}$

84. $5x - 2y = 3$
$4y + z = 8$
$x + 2z = 4$

Adding -5 times row 3 to row 1, we have

$$D = \begin{vmatrix} 5 & -2 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -10 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{vmatrix}.$$

Expanding by column one, we have

$$D = 1\begin{vmatrix} -2 & -10 \\ 4 & 1 \end{vmatrix} = -2 + 40 = 38.$$

Adding 2 times row 1 to row 2, we have

$$D_x = \begin{vmatrix} 3 & -2 & 0 \\ 8 & 4 & 1 \\ 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 0 \\ 14 & 0 & 1 \\ 4 & 0 & 2 \end{vmatrix}.$$

Expanding by column two, we have

$$D_x = -(-2)\begin{vmatrix} 14 & 1 \\ 4 & 2 \end{vmatrix} = 2(28 - 4) = 2(24) = 48.$$

Adding -5 row 3 to row 1, we have

$$D_y = \begin{vmatrix} 5 & 3 & 0 \\ 0 & 8 & 1 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -17 & -10 \\ 0 & 8 & 1 \\ 1 & 4 & 2 \end{vmatrix} = .$$

Expanding by column one, we have

$$D_y = 1\begin{vmatrix} -17 & -10 \\ 8 & 1 \end{vmatrix} = -17 + 80 = 63.$$

Adding -5 row 3 to row 1, we have

$$D_z = \begin{vmatrix} 5 & -2 & 3 \\ 0 & 4 & 8 \\ 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -17 \\ 0 & 4 & 8 \\ 1 & 0 & 4 \end{vmatrix}.$$

Expanding by row three, we have

$$D_z = 1\begin{vmatrix} -2 & -17 \\ 4 & 8 \end{vmatrix} = -16 + 68 = 52.$$

Thus, we have $x = \frac{D_x}{D} = \frac{48}{38} = \frac{24}{19}$, $y = \frac{D_y}{D} = \frac{63}{38}$,

and $z = \frac{D_z}{D} = \frac{52}{38} = \frac{26}{19}$.

Solution set: $\left\{\left(\frac{24}{19}, \frac{63}{38}, \frac{26}{19}\right)\right\}$

85. $\frac{\sqrt{3}}{2}(W_1 + W_2) = 100$
$W_1 - W_2 = 0$

Using the distributive property, we have the following system.

$\frac{\sqrt{3}}{2}W_1 + \frac{\sqrt{3}}{2}W_2 = 100$
$W_1 - W_2 = 0$

Using Cramer's rule, we have

$$D = \begin{vmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 1 & -1 \end{vmatrix} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3},$$

$$D_{W_1} = \begin{vmatrix} 100 & \frac{\sqrt{3}}{2} \\ 0 & -1 \end{vmatrix} = -100 - 0 = -100, \text{ and}$$

$$D_{W_2} = \begin{vmatrix} \frac{\sqrt{3}}{2} & 100 \\ 1 & 0 \end{vmatrix} = 0 - 100 = -100. \text{ This yields}$$

the following solution.

$W_1 = \frac{D_{W_1}}{D} = \frac{-100}{-\sqrt{3}} = \frac{100}{\sqrt{3}} = \frac{100\sqrt{3}}{3} \approx 58$ and

$W_2 = \frac{D_{W_2}}{D} = \frac{-100}{-\sqrt{3}} \approx 58$

Both W_1 and W_2 are approximately 58 lb.

86. $W_1 + \sqrt{2}W_2 = 300$
$\sqrt{3}W_1 - \sqrt{2}W_2 = 0$

Using Cramer's rule, we have the following.

$$D = \begin{vmatrix} 1 & \sqrt{2} \\ \sqrt{3} & -\sqrt{2} \end{vmatrix} = -\sqrt{2} - \sqrt{6}$$

$$D_{W_1} = \begin{vmatrix} 300 & \sqrt{2} \\ 0 & -\sqrt{2} \end{vmatrix} = -300\sqrt{2} - 0 = -300\sqrt{2}$$

and $D_{W_2} = \begin{vmatrix} 1 & 300 \\ \sqrt{3} & 0 \end{vmatrix} = 0 - 300\sqrt{3} = -300\sqrt{3}$

This yields the following solution.

$W_1 = \frac{D_{W_1}}{D} = \frac{-300\sqrt{2}}{-\sqrt{2}-\sqrt{6}} = \frac{-\sqrt{2}(300)}{-\sqrt{2}(1+\sqrt{3})} = \frac{300}{1+\sqrt{3}} \approx 110$

and $W_2 = \frac{-300\sqrt{3}}{-\sqrt{2}-\sqrt{6}} = \frac{300\sqrt{3}}{\sqrt{2}+\sqrt{6}} \approx 134$

W_1 is approximately 110 lb and W_2 is approximately 134 lb.

87. $bx + y = a^2$
$ax + y = b^2$

Using Cramer's rule, we have

$$D = \begin{vmatrix} b & 1 \\ a & 1 \end{vmatrix} = b - a, \quad D_x = \begin{vmatrix} a^2 & 1 \\ b^2 & 1 \end{vmatrix} = a^2 - b^2,$$

and $D_y = \begin{vmatrix} b & a^2 \\ a & b^2 \end{vmatrix} = b^3 - a^3.$

$$x = \frac{D_x}{D} = \frac{a^2 - b^2}{b - a} = \frac{(a+b)(a-b)}{b-a}$$
$$= \frac{(a+b)(a-b)}{-(a-b)} = -(a+b) = -a - b$$

$$y = \frac{D_y}{D} = \frac{b^3 - a^3}{b - a} = \frac{(b-a)(b^2 + ab + a^2)}{b-a}$$
$$= b^2 + ab + a^2$$

Solution set: $\left\{\left(-a - b, a^2 + ab + b^2\right)\right\}$

88. $ax + by = \frac{b}{a}$
$x + y = \frac{1}{b}$

Using Cramer's rule, we have

$$D = \begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix} = a - b, \quad D_x = \begin{vmatrix} \frac{b}{a} & b \\ \frac{1}{b} & 1 \end{vmatrix} = \frac{b}{a} - 1, \text{ and}$$

$$D_y = \begin{vmatrix} a & \frac{b}{a} \\ 1 & \frac{1}{b} \end{vmatrix} = \frac{a}{b} - \frac{b}{a}.$$

$$x = \frac{D_x}{D} = \frac{\frac{b}{a} - 1}{a - b} = \frac{b - a}{a(a-b)} = \frac{-(a-b)}{a(a-b)} = \frac{-1}{a}$$

$$y = \frac{D_y}{D} = \frac{\frac{a}{b} - \frac{b}{a}}{a - b} = \frac{a^2 - b^2}{ab(a-b)}$$
$$= \frac{(a+b)(a-b)}{ab(a-b)} = \frac{a+b}{ab}$$

Note: In order for $D \neq 0$, we also assumed $a \neq b$.

Solution set: $\left\{\left(-\frac{1}{a}, \frac{a+b}{ab}\right)\right\}$

89. $b^2x + a^2y = b^2$
$ax + by = a$

Using Cramer's rule, we have

$$D = \begin{vmatrix} b^2 & a^2 \\ a & b \end{vmatrix} = b^3 - a^3,$$

$$D_x = \begin{vmatrix} b^2 & a^2 \\ a & b \end{vmatrix} = b^3 - a^3, \text{ and}$$

$$D_y = \begin{vmatrix} b^2 & b^2 \\ a & a \end{vmatrix} = ab^2 - ab^2 = 0.$$

$$x = \frac{D_x}{D} = \frac{b^3 - a^3}{b^3 - a^3} = 1$$

$$y = \frac{D_y}{D} = \frac{0}{b^3 - a^3} = 0$$

Note: In order for $D \neq 0$, we also assumed $a \neq b$.

Solution set: $\{(1,0)\}$

90. $x + by = b$
$ax + y = a$

Using Cramer's rule, we have

$$D = \begin{vmatrix} 1 & b \\ a & 1 \end{vmatrix} = 1 - ab, \quad D_x = \begin{vmatrix} b & b \\ a & 1 \end{vmatrix} = b - ab, \text{ and}$$

$$D_y = \begin{vmatrix} 1 & b \\ a & a \end{vmatrix} = a - ab.$$

$$x = \frac{D_x}{D} = \frac{b - ab}{1 - ab}$$

$$y = \frac{D_y}{D} = \frac{a - ab}{1 - ab}$$

Note: In order for $D \neq 0$, we also assumed $ab \neq 1$.

Solution set: $\left\{\left(\frac{b-ab}{1-ab}, \frac{a-ab}{1-ab}\right)\right\}$

Section 9.4: Partial Fractions

1. $\dfrac{5}{3x(2x+1)} = \dfrac{A}{3x} + \dfrac{B}{2x+1}$

Multiply both sides by $3x(2x+1)$ to get

$5 = A(2x+1) + B(3x)$. (1)

First substitute 0 for x to get

$5 = A(2\cdot 0 + 1) + B(3\cdot 0) \Rightarrow A = 5.$

Replace A with 5 in equation (1) and substitute $-\frac{1}{2}$ for x to get the following.

$$5 = 5\left[2\left(-\tfrac{1}{2}\right)+1\right] + B\left[3\left(-\tfrac{1}{2}\right)\right]$$
$$= 5(-1+1) - \tfrac{3}{2}B = 5(0) - \tfrac{3}{2}B \Rightarrow$$
$$5 = -\tfrac{3}{2}B \Rightarrow -\tfrac{10}{3} = B$$

Thus, we have $\dfrac{5}{3x(2x+1)} = \dfrac{5}{3x} + \dfrac{-\frac{10}{3}}{2x+1} \Rightarrow$

$$\frac{5}{3x(2x+1)} = \frac{5}{3x} + \frac{-10}{3(2x+1)}$$

2. $\dfrac{3x-1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

Multiply both sides by $x(x+1)$ to get

$3x - 1 = A(x+1) + Bx$. (1)

First substitute 0 for x to get

$$3(0) - 1 = A(0+1) + B(0) \Rightarrow$$
$$0 - 1 = A + 0 \Rightarrow A = -1$$

Replace A with -1 in equation (1) and substitute -1 for x to get the following.

$$3(-1) - 1 = -1(-1+1) + B(-1) \Rightarrow$$
$$-3 - 1 = -1(0) - B \Rightarrow -4 = -B \Rightarrow B = 4$$

Thus, we have $\dfrac{3x-1}{x(x+1)} = \dfrac{-1}{x} + \dfrac{4}{x+1}.$

3. $\dfrac{4x+2}{(x+2)(2x-1)} = \dfrac{A}{x+2} + \dfrac{B}{2x-1}$

Multiply both sides by $(x+2)(2x-1)$ to get

$4x + 2 = A(2x-1) + B(x+2)$. (1)

First substitute -2 for x to get the following.

$$4(-2) + 2 = A[2(-2) - 1] + B(-2+2)$$
$$-8 + 2 = A(-4-1) + B(0) = -6 = -5A$$
$$A = \tfrac{6}{5}$$

Replace A with $\frac{6}{5}$ in equation (1) and substitute $\frac{1}{2}$ for x to get the following.

$$4\cdot\tfrac{1}{2} + 2 = \tfrac{6}{5}\left(2\cdot\tfrac{1}{2} - 1\right) + B\left(\tfrac{1}{2} + 2\right)$$
$$2 + 2 = \tfrac{6}{5}(1-1) + B\cdot\tfrac{5}{2} \Rightarrow 4 = \tfrac{6}{5}(0) + \tfrac{5}{2}B \Rightarrow$$
$$4 = \tfrac{5}{2}B \Rightarrow B = \tfrac{8}{5}$$

Thus, we have

$$\frac{4x+2}{(x+2)(2x-1)} = \frac{\frac{6}{5}}{(x+2)} + \frac{\frac{8}{5}}{(2x-1)} \Rightarrow$$
$$\frac{4x+2}{(x+2)(2x-1)} = \frac{6}{5(x+2)} + \frac{8}{5(2x-1)}$$

4. $\dfrac{x+2}{(x+1)(x-1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$

Multiply both sides by $(x+1)(x-1)$ to get

$x+2=A(x-1)+B(x+1).$ (1)

First substitute -1 for x to get the following.

$-1+2=A(-1-1)+B(-1+1)\Rightarrow$

$1=A(-2)+B(0)\Rightarrow 1=-2A\Rightarrow A=-\frac{1}{2}$

Replace A with $-\frac{1}{2}$ in equation (1) and substitute 1 for x to get the following.

$1+2=-\frac{1}{2}(1-1)+B(1+1)\Rightarrow$

$3=-\frac{1}{2}(0)+B(2)\Rightarrow 3=2B\Rightarrow B=\frac{3}{2}$

Thus, we have

$$\frac{x+2}{(x+1)(x-1)}=\frac{-1}{2(x+1)}+\frac{3}{2(x-1)}.$$

5. $\dfrac{x}{x^2+4x-5}=\dfrac{x}{(x+5)(x-1)}=\dfrac{A}{x+5}+\dfrac{B}{x-1}$

Multiply both sides by $(x+5)(x-1)$ to get

$x=A(x-1)+B(x+5).$ (1)

First substitute -5 for x to get

$-5=A(-5-1)+B(-5+5)\Rightarrow$

$-5=A(-6)+B(0)\Rightarrow -5=-6A\Rightarrow A=\frac{5}{6}$

Replace A with $\frac{5}{6}$ in equation (1) and substitute 1 for x to get

$1=\frac{5}{6}(1-1)+B(1+5)\Rightarrow 1=\frac{5}{6}(0)+B(6)\Rightarrow$

$1=6B\Rightarrow B=\frac{1}{6}$

Thus, we have

$$\frac{x}{(x+5)(x-1)}=\frac{\frac{5}{6}}{x+5}+\frac{\frac{1}{6}}{x-1}=\frac{5}{6(x+5)}+\frac{1}{6(x-1)}$$

6. $\dfrac{5x-3}{(x+1)(x-3)}=\dfrac{A}{x+1}+\dfrac{B}{x-3}$

Multiply both sides by $(x+1)(x-3)$ to get

$5x-3=A(x-3)+B(x+1).$ (1)

First substitute -1 for x to get the following.

$5(-1)-3=A(-1-3)+B(-1+1)$

$-5-3=A(-4)+B(0)$

$-8=-4A\Rightarrow A=2$

Replace A with 2 in equation (1) and substitute 3 for x to get the following.

$5(3)-3=2(3-3)+B(3+1)\Rightarrow$

$15-3=2(0)+B(4)\Rightarrow 12=4B\Rightarrow B=3$

Thus, we have $\dfrac{5x-3}{(x+1)(x-3)}=\dfrac{2}{x+1}+\dfrac{3}{x-3}.$

7. $\dfrac{2x}{(x+1)(x+2)^2}=\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

Multiply both sides by $(x+1)(x+2)^2$ to get

$2x=A(x+2)^2+B(x+1)(x+2)+C(x+1)$ (1)

First substitute -1 for x to get the following.

$2(-1)=A(-1+2)^2+B(-1+1)(-1+2)+C(-1+1)$

$-2=A(1)^2+B(0)(1)+C(0)\Rightarrow -2=A$

Replace A with -2 in equation (1) and substitute -2 for x to get the following.

$2(-2)=-2(-2+2)^2+B(-2+1)(-2+2)+C(-2+1)$

$-4=0+0-C\Rightarrow C=4$

$2x=-2(x+2)^2+B(x+1)(x+2)+4(x+1)$ (2)

Now substitute 0 (arbitrary choice) for x in equation (2) to get the following.

$2(0)=-2(0+2)^2+B(0+1)(0+2)+4(0+1)$

$0=-2(2)^2+B(1)(2)+4(1)$

$0=-2(4)+2B+4\Rightarrow 0=-8+2B+4$

$0=-4+2B\Rightarrow 4=2B\Rightarrow B=2$

Thus we have

$$\frac{2x}{(x+1)(x+2)^2}=\frac{-2}{x+1}+\frac{2}{x+2}+\frac{4}{(x+2)^2}.$$

8. $\dfrac{2}{x^2(x+3)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+3}$

Multiply both sides by $x^2(x+3)$ to get

$2=Ax(x+3)+B(x+3)+Cx^2.$ (1)

First substitute 0 for x to get the following.

$2=A(0)(0+3)+B(0+3)+C(0)^2$

$2=0+3B+0\Rightarrow 2=3B\Rightarrow B=\frac{2}{3}$

Replace B with $\frac{2}{3}$ in equation (1) and substitute -3 for x to get the following.

$2=A(-3)(-3+3)+\frac{2}{3}(-3+3)+C(-3)^2$

$2=0+0+9C\Rightarrow C=\frac{2}{9}$

$2=Ax(x+3)+\frac{2}{3}(x+3)+\frac{2}{9}x^2$ (2)

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Now substitute 1 (arbitrary choice) for x in equation (2) to get the following.

$$2 = A(1)(1+3) + \tfrac{2}{3}(1+3) + \tfrac{2}{9}(1)^2$$
$$2 = 4A + \tfrac{2}{3}(4) + \tfrac{2}{9}$$
$$2 = 4A + \tfrac{8}{3} + \tfrac{2}{9} \Rightarrow 2 = 4A + \tfrac{26}{9}$$
$$\tfrac{18}{9} - \tfrac{26}{9} = 4A \Rightarrow -\tfrac{8}{9} = 4A \Rightarrow A = -\tfrac{2}{9}$$

Thus, we have

$$\frac{2}{x^2(x+3)} = \frac{-\frac{2}{9}}{x} + \frac{\frac{2}{3}}{x^2} + \frac{\frac{2}{9}}{x+3} = \frac{-2}{9x} + \frac{2}{3x^2} + \frac{2}{9(x+3)}$$

9. $\dfrac{4}{x(1-x)} = \dfrac{A}{x} + \dfrac{B}{1-x}$

Multiply both sides by $x(1-x)$ to get

$4 = A(1-x) + Bx.$ (1)

First substitute 0 for x to get the following.

$4 = A(1-0) + B(0) \Rightarrow 4 = A + 0 \Rightarrow A = 4$

Replace A with 4 in equation (1) and substitute 1 for x to get the following.

$4 = 4(1-1) + B(1) \Rightarrow 4 = 0 + B \Rightarrow B = 4$

Thus, we have $\dfrac{4}{x(1-x)} = \dfrac{4}{x} + \dfrac{4}{1-x}.$

10. $\dfrac{4x^2 - 4x^3}{x^2(1-x)}$

Since the numerator and denominator have the same degree, first find the quotient. Since there is a common factor in the terms of the numerator, the expression simplifies to be

$$\frac{4x^2(1-x)}{x^2(1-x)} = 4.$$

11. $\dfrac{2x+1}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$

Multiply both sides by $(x+2)^3$ to get

$2x+1 = A(x+2)^2 + B(x+2) + C.$ (1)

First substitute -2 for x to get the following.

$$2(-2)+1 = A(-2+2)^2 + B(-2+2) + C$$
$$-4+1 = 0+0+C \Rightarrow C = -3$$

Replace C with -3 in equation (1) and substitute 0 (arbitrary choice) for x to get the following.

$$2(0)+1 = A(0+2)^2 + B(0+2) - 3$$
$$1 = 4A + 2B - 3 \Rightarrow 4A + 2B = 4$$
$$2A + B = 2 \quad (3)$$

Replace C with -3 in equation (1) and substitute -1 (arbitrary choice) for x to get the following.

$$2(-1)+1 = A(-1+2)^2 + B(-1+2) - 3$$
$$-1 = A + B - 3 \Rightarrow A + B = 2 \quad (4)$$

Solve the system of equations by multiplying equation (4) by -1 and adding to equation 3.

$$\begin{aligned} 2A + B &= 2 \\ \underline{-A - B} &\underline{= -2} \\ A \qquad &= 0 \end{aligned}$$

Substituting 0 for A in equation (4) we obtain $0 + B = 2 \Rightarrow B = 2.$

Thus, we have

$$\frac{2x+1}{(x+2)^3} = \frac{0}{x+2} + \frac{2}{(x+2)^2} + \frac{-3}{(x+2)^3} = \frac{2}{(x+2)^2} + \frac{-3}{(x+2)^3}$$

12. $\dfrac{4x^2 - x - 15}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

Multiply both sides by $x(x+1)(x-1)$ to get

$$4x^2 - x - 15 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \quad (1)$$

First substitute 0 for x to get the following.

$$4(0)^2 - 0 - 15 = A(0+1)(0-1) + B(0)(0-1) + C(0)(0+1)$$
$$0 - 0 - 15 = -A + 0 + 0 \Rightarrow -15 = -A \Rightarrow A = 15$$

Replace A with 15 in equation (1) and substitute -1 for x to get the following.

$$4(-1)^2 - (-1) - 15 = 15(-1+1)(-1-1) + B(-1)(-1-1) + C(-1)(-1+1)$$
$$4 + 1 - 15 = 15(0)(-2) + B(-1)(-2) + C(-1)(0)$$
$$-10 = 0 + 2B + 0 \Rightarrow -10 = 2B \Rightarrow B = -5$$
$$4x^2 - x - 15 = 15(x+1)(x-1) - 5x(x-1) + Cx(x+1) \quad (2)$$

Now substitute 1 for x in equation (2) to get the following.

$$4(1)^2 - 1 - 15 = 15(1+1)(1-1) - 5(1)(1-1) + C(1)(1+1)$$
$$4 - 1 - 15 = 15(2)(0) - 5(1)(0) + C(1)(2)$$
$$-12 = 0 - 0 + 2C$$
$$-12 = 2C \Rightarrow C = -6$$

Thus, we have

$$\frac{4x^2 - x - 15}{x(x+1)(x-1)} = \frac{15}{x} + \frac{-5}{x+1} + \frac{-6}{x-1}.$$

13. $\dfrac{x^2}{x^2 + 2x + 1}$

This is not a proper fraction; the numerator has degree greater than or equal to that of the denominator. Divide the numerator by the denominator.

$$\begin{array}{r|l} & 1 \\ x^2 + 2x + 1 & x^2 \\ & \underline{x^2 + 2x + 1} \\ & \quad -2x - 1 \end{array}$$

Find the partial fraction decomposition for $\dfrac{-2x-1}{x^2+2x+1} = \dfrac{-2x-1}{(x+1)^2}$.

$$\frac{-2x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Multiply both sides by $(x+1)^2$ to get

$-2x - 1 = A(x+1) + B.$ (1)

First substitute -1 for x to get the following.

$$-2(-1) - 1 = A(-1+1) + B$$
$$2 - 1 = 0 + B \Rightarrow 1 = B$$

Replace B with 1 in equation (1) and substitute 2 (arbitrary choice) for x to get the following.

$$-2(2) - 1 = A(2+1) + 1 \Rightarrow -5 = 3A + 1$$
$$-6 = 3A \Rightarrow A = -2$$

Thus, we have

$$\frac{x^2}{x^2+2x+1} = 1 + \frac{-2}{x+1} + \frac{1}{(x+1)^2}.$$

14. $\dfrac{3}{x^2+4x+3} = \dfrac{3}{(x+3)(x+1)} = \dfrac{A}{x+3} + \dfrac{B}{x+1}$

Multiply both sides by $(x+3)(x+1)$ to get

$3 = A(x+1) + B(x+3).$ (1)

First substitute -3 for x to get the following.

$$3 = A(-3+1) + B(-3+3)$$
$$3 = A(-2) + B(0)$$
$$3 = -2A \Rightarrow A = -\tfrac{3}{2}$$

Replace A with $-\frac{3}{2}$ in equation (1) and substitute -1 for x to get the following.

$$3 = -\tfrac{3}{2}(-1+1) + B(-1+3)$$
$$3 = -\tfrac{3}{2}(0) + B(2) \Rightarrow 3 = 2B \Rightarrow B = \tfrac{3}{2}$$

Thus, we have

$$\frac{3}{x^2+4x+3} = \frac{-\frac{3}{2}}{x+3} + \frac{\frac{3}{2}}{x+1} = \frac{-3}{2(x+3)} + \frac{3}{2(x+1)}$$

15. $\dfrac{2x^5 + 3x^4 - 3x^3 - 2x^2 + x}{2x^2 + 5x + 2}$

The degree of the numerator is greater than the degree of the denominator, so first find the quotient.

$$\begin{array}{r|l} & x^3 - x^2 \\ 2x^2 + 5x + 2 & 2x^5 + 3x^4 - 3x^3 - 2x^2 + x \\ & \underline{2x^5 + 5x^4 + 2x^3} \\ & \quad -2x^4 - 5x^3 - 2x^2 + x \\ & \quad \underline{-2x^4 - 5x^3 - 2x^2} \\ & \qquad\qquad\qquad\qquad x \end{array}$$

Find the partial fraction decomposition for $\dfrac{x}{2x^2+5x+2} = \dfrac{x}{(2x+1)(x+2)}$.

$$\frac{x}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

Multiply both sides by $(2x+1)(x+2)$ to get

$x = A(x+2) + B(2x+1).$ (1)

First substitute $-\frac{1}{2}$ for x to get the following.

$$-\tfrac{1}{2} = A\left(-\tfrac{1}{2} + 2\right) + B\left[2\left(-\tfrac{1}{2}\right) + 1\right]$$
$$-\tfrac{1}{2} = A\left(\tfrac{3}{2}\right) + B(0) \Rightarrow -\tfrac{1}{2} = \tfrac{3}{2}A \Rightarrow A = -\tfrac{1}{3}$$

Replace A with $-\frac{1}{3}$ in equation (1) and substitute -2 for x to get the following.

$$-2 = -\tfrac{1}{3}(-2+2) + B\left[2(-2) + 1\right]$$
$$-2 = -\tfrac{1}{3}(0) + B(-3) \Rightarrow -2 = -3B \Rightarrow B = \tfrac{2}{3}$$

Thus, we have

$$\frac{2x^5 + 3x^4 - 3x^3 - 2x^2 + x}{2x^2 + 5x + 2} = x^3 - x^2 + \frac{-1}{3(2x+1)} + \frac{2}{3(x+2)}$$

16. $\dfrac{6x^5+7x^4-x^2+2x}{3x^2+2x-1}$

The degree of the numerator is greater than the degree of the denominator, so first find the quotient.

$$\begin{array}{r} 2x^3+\ x^2 \qquad\qquad\qquad \\ 3x^2+2x-1\overline{\big)\,6x^5+7x^4+0x^3-x^2+2x} \\ \underline{6x^5+4x^4-2x^3\qquad\qquad} \\ 3x^4+2x^3-x^2+2x \\ \underline{3x^4+2x^3-x^2\quad\ \ } \\ 2x \end{array}$$

Find the partial fraction decomposition for $\dfrac{2x}{3x^2+2x-1}=\dfrac{2x}{(3x-1)(x+1)}$.

$$\frac{2x}{(3x-1)(x+1)}=\frac{A}{3x-1}+\frac{B}{x+1}$$

Multiply both sides by $(3x-1)(x+1)$ to get $2x=A(x+1)+B(3x-1)$. (1)

First substitute $\frac{1}{3}$ for x to get the following.

$$2\left(\tfrac{1}{3}\right)=A\left(\tfrac{1}{3}+1\right)+B\left[3\left(\tfrac{1}{3}\right)-1\right]$$
$$\tfrac{2}{3}=A\left(\tfrac{4}{3}\right)+B(0)\Rightarrow\tfrac{2}{3}=\tfrac{4}{3}A\Rightarrow A=\tfrac{1}{2}$$

Replace A with $\frac{1}{2}$ in equation (1) and substitute -1 for x to get the following.

$$2(-1)=\tfrac{1}{2}(-1+1)+B\left[3(-1)-1\right]$$
$$-2=\tfrac{1}{2}(0)+B(-4)\Rightarrow-2=-4B\Rightarrow B=\tfrac{1}{2}$$

Thus, we have

$$\frac{6x^5+7x^4-x^2+2x}{3x^2+2x-1}=2x^3+x^2+\frac{1}{2(3x-1)}+\frac{1}{2(x+1)}$$

17. $\dfrac{x^3+4}{9x^3-4x}$

Find the quotient since the degrees of the numerator and denominator are the same.

$$\begin{array}{r} \tfrac{1}{9}\qquad\qquad\qquad\quad \\ 9x^3-4x\overline{\big)\,x^3+0x^2+0x+4} \\ \underline{x^3\qquad\ -\tfrac{4}{9}x\quad\ \ } \\ \tfrac{4}{9}x+4 \end{array}$$

Find the partial fraction decomposition for $\dfrac{\frac{4}{9}x+4}{9x^3-4x}$.

$$\frac{\frac{4}{9}x+4}{9x^3-4x}=\frac{\frac{4}{9}x+4}{x\left(9x^2-4\right)}=\frac{\frac{4}{9}x+4}{x(3x+2)(3x-2)}=\frac{A}{x}+\frac{B}{3x+2}+\frac{C}{3x-2}$$

Multiply both sides of $\dfrac{\frac{4}{9}x+4}{9x^3-4x}=\dfrac{A}{x}+\dfrac{B}{3x+2}+\dfrac{C}{3x-2}$ by $x(3x+2)(3x-2)$ to get the following.

$$\tfrac{4}{9}x+4=A(3x+2)(3x-2)+Bx(3x-2)+Cx(3x+2)\quad(1)$$

First substitute 0 for x to get the following.

$$\tfrac{4}{9}(0)+4=A\left[3(0)+2\right]\left[3(0)-2\right]+B(0)\left[3(0)-2\right]+C(0)\left[3(0)+2\right]$$

$$0+4=A(2)(-2)+0+0\Rightarrow4=-4A\Rightarrow A=-1$$

Replace A with -1 in equation (1) and substitute $-\frac{2}{3}$ for x to get the following.

$$\tfrac{4}{9}\left(-\tfrac{2}{3}\right)+4=-\left[3\left(-\tfrac{2}{3}\right)+2\right]\left[3\left(-\tfrac{2}{3}\right)-2\right]+B\left(-\tfrac{2}{3}\right)\left[3\left(-\tfrac{2}{3}\right)-2\right]+C\left(-\tfrac{2}{3}\right)\left[3\left(-\tfrac{2}{3}\right)+2\right]$$

$$-\tfrac{8}{27}+4=0+B\left(-\tfrac{2}{3}\right)(-4)+0$$

$$\tfrac{100}{27}=\tfrac{8}{3}B\Rightarrow B=\tfrac{25}{18}$$

$$\tfrac{4}{9}x+4=-(3x+2)(3x-2)+\tfrac{25}{18}x(3x-2)+Cx(3x+2)\quad(2)$$

Substitute $\frac{2}{3}$ in equation (2) for x to get the following.

$$\tfrac{4}{9}\left(\tfrac{2}{3}\right)+4=-\left[3\left(\tfrac{2}{3}\right)+2\right]\left[3\left(\tfrac{2}{3}\right)-2\right]+\tfrac{25}{18}\left(\tfrac{2}{3}\right)\left[3\left(\tfrac{2}{3}\right)-2\right]+C\left(\tfrac{2}{3}\right)\left[3\left(\tfrac{2}{3}\right)+2\right]$$

$$\tfrac{8}{27}+4=0+0+C\left(\tfrac{2}{3}\right)(4)$$

$$\tfrac{116}{27}=\tfrac{8}{3}C\Rightarrow C=\tfrac{29}{18}$$

Thus, we have

$$\frac{x^3+4}{9x^3-4x}=\frac{1}{9}+\frac{-1}{x}+\frac{25}{18(3x+2)}+\frac{29}{18(3x-2)}.$$

18. $\dfrac{x^3+2}{x^3-3x^2+2x}$

Find the quotient since the degrees of the numerator and denominator are the same.

$$\begin{array}{r} 1 \\ x^3-3x^2+2x\overline{\smash{\big)}\, x^3+0x^2+0x+2} \\ \underline{x^3-3x^2+2x} \\ 3x^2-2x+2 \end{array}$$

Find the partial fraction decomposition for $\dfrac{3x^2-2x+2}{x^3-3x^2+2x}$.

$$\frac{3x^2-2x+2}{x^3-3x^2+2x}=\frac{3x^2-2x+2}{x\left(x^2-3x+2\right)}=\frac{3x^2-2x+2}{x(x-2)(x-1)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-1}$$

Multiply both sides of $\dfrac{3x^2-2x+2}{x(x-2)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x-1}$ by $x(x-2)(x-1)$ to get the following.

$$3x^2-2x+2=A(x-2)(x-1)+Bx(x-1)+Cx(x-2)\quad(1)$$

First substitute 0 for x to get

$$3(0)^2-2(0)+2=A(0-2)(0-1)+B(0)(0-1)+C(0)(0-2)$$
$$0-0+2=A(-2)(-1)+0+0$$
$$2=2A\Rightarrow A=1$$

Replace A with 1 in equation (1) and substitute 2 for x to get the following.

$$3(2)^2-2(2)+2=(2-2)(2-1)+B(2)(2-1)+C(2)(2-2)$$
$$12-4+2=0+B(2)(1)+0$$
$$10=2B\Rightarrow B=5$$
$$3x^2-2x+2=(x-2)(x-1)+5x(x-1)+Cx(x-2)\quad(2)$$

Substitute 1 in equation (2) for x to get the following.

$$3(1)^2-2(1)+2=(1-2)(1-1)+5(1)(1-1)+C(1)(1-2)$$
$$3-2+2=0+0+C(1)(-1)$$
$$3=-C\Rightarrow C=-3$$

Thus, we have

$$\frac{x^3+2}{x^3-3x^2+2x}=1+\frac{1}{x}+\frac{5}{x-2}+\frac{-3}{x-1}.$$

19. $\dfrac{-3}{x^2\left(x^2+5\right)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+5}$ Multiply both sides by $x^2\left(x^2+5\right)$ to get the following.

$$-3=Ax\left(x^2+5\right)+B\left(x^2+5\right)+(Cx+D)x^2$$

Distributing and combining like terms on the right side of the equation, we have the following.

$$-3=Ax\left(x^2+5\right)+B\left(x^2+5\right)+(Cx+D)x^2$$
$$-3=Ax^3+5Ax+Bx^2+5B+Cx^3+Dx^2$$
$$-3=(A+C)x^3+(B+D)x^2+(5A)x+5B$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^3- term, we have $0=A+C$.

For the x^2- term, we have $0=B+D$.

For the x-term, we have $0=5A\Rightarrow A=0$.

For the constant term, we have $-3=5B\Rightarrow B=-\frac{3}{5}$.

Since $A=0$ and $0=A+C$, we have $C=0$.

Since $B=-\frac{3}{5}$ and $0=B+D$, we have $D=\frac{3}{5}$.

Thus, we have $\dfrac{-3}{x^2\left(x^2+5\right)}=\dfrac{-3}{5x^2}+\dfrac{3}{5\left(x^2+5\right)}$.

20. $\dfrac{2x+1}{(x+1)\left(x^2+2\right)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+2}$

Multiply both sides by $(x+1)\left(x^2+2\right)$ to get the following.

$$2x+1=A\left(x^2+2\right)+(Bx+C)(x+1)\quad(1)$$

First substitute -1 for x to get the following.

$$2(-1)+1=A\left[(-1)^2+2\right]+\left[B(-1)+C\right](-1+1)$$
$$-2+1=A(1+2)+0$$
$$-1=3A\Rightarrow A=-\tfrac{1}{3}$$

Replace A with $-\frac{1}{3}$ in equation (1) and substitute 0 for x to get the following.

$$2(0)+1=-\tfrac{1}{3}\left(0^2+2\right)+\left[B(0)+C\right](0+1)$$
$$0+1=-\tfrac{1}{3}(2)+C\Rightarrow 1=-\tfrac{2}{3}+C\Rightarrow C=\tfrac{5}{3}$$
$$2x+1=-\tfrac{1}{3}\left(x^2+2\right)+\left(Bx+\tfrac{5}{3}\right)(x+1)\quad(2)$$

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Substitute 1 (arbitrary value) in equation (2) for x to get the following.

$$2(1)+1=-\tfrac{1}{3}(1^2+2)+\left[B(1)+\tfrac{5}{3}\right](1+1)$$
$$2+1=-\tfrac{1}{3}(3)+2\left(B+\tfrac{5}{3}\right)$$
$$3=-1+2B+\tfrac{10}{3}\Rightarrow 3=2B+\tfrac{7}{3}$$
$$\tfrac{2}{3}=2B\Rightarrow B=\tfrac{1}{3}$$

Thus, we have

$$\frac{2x+1}{(x+1)(x^2+2)}=\frac{-\frac{1}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{5}{3}}{x^2+2}=\frac{-1}{3(x+1)}+\frac{x+5}{3(x^2+2)}$$

21. $$\frac{3x-2}{(x+4)(3x^2+1)}=\frac{A}{x+4}+\frac{Bx+C}{3x^2+1}$$

Multiply both sides by $(x+4)(3x^2+1)$ to get the following.

$$3x-2=A(3x^2+1)+(Bx+C)(x+4)\quad(1)$$

First substitute -4 for x to get the following.

$$3(-4)-2=A\left[3(-4)^2+1\right]+\left[B(-4)+C\right](-4+4)$$
$$-12-2=A(48+1)+0$$
$$-14=49A\Rightarrow -\tfrac{14}{49}=A\Rightarrow A=-\tfrac{2}{7}$$

Replace A with $-\frac{2}{7}$ in equation (1) and substitute 0 for x to get the following.

$$3(0)-2=-\tfrac{2}{7}\left[3(0)^2+1\right]+\left[B(0)+C\right](0+4)$$
$$-2=-\tfrac{2}{7}+4C\Rightarrow -\tfrac{12}{7}=4C\Rightarrow C=-\tfrac{3}{7}$$
$$3x-2=-\tfrac{2}{7}(3x^2+1)+\left(Bx-\tfrac{3}{7}\right)(x+4)\quad(2)$$

Substitute 1 (arbitrary value) in equation (2) for x to get the following.

$$3(1)-2=-\tfrac{2}{7}\left[3(1)^2+1\right]+\left[B(1)-\tfrac{3}{7}\right](1+4)$$
$$3-2=-\tfrac{2}{7}(4)+\left[B-\tfrac{3}{7}\right](5)$$
$$1=-\tfrac{8}{7}+5B-\tfrac{15}{7}$$
$$1=5B-\tfrac{23}{7}\Rightarrow \tfrac{30}{7}=5B\Rightarrow B=\tfrac{6}{7}$$

Thus, we have

$$\frac{3x-2}{(x+4)(3x^2+1)}=\frac{-\frac{2}{7}}{x+4}+\frac{\frac{6}{7}x+\left(-\frac{3}{7}\right)}{3x^2+1}=\frac{-2}{7(x+4)}+\frac{6x-3}{7(3x^2+1)}$$

22. $$\frac{3}{x(x+1)(x^2+1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

Multiply both sides by $x(x+1)(x^2+1)$ to get the following.

$$3=A(x+1)(x^2+1)+Bx(x^2+1)+(Cx+D)(x)(x+1)$$

Expanding and combining like terms on the right side of the equation, we have the following.

$$3=A(x+1)(x^2+1)+Bx(x^2+1)+(Cx+D)(x)(x+1)$$
$$3=A(x^3+x^2+x+1)+B(x^3+x)+C(x^3+x^2)+D(x^2+x)$$
$$3=Ax^3+Ax^2+Ax+A+Bx^3+Bx+Cx^3+Cx^2+Dx^2+Dx$$
$$3=(A+B+C)x^3+(A+C+D)x^2+(A+B+D)x+A$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^3- term, we have $0=A+B+C.$

For the x^2- term, we have $0=A+C+D.$

For the x-term, we have $0=A+B+D.$

For the constant term, we have $3=A.$

If we use the Gauss-Jordan method, we begin with the following augmented matrix.

$$\left[\begin{array}{cccc|c}1&1&1&0&0\\1&0&1&1&0\\1&1&0&1&0\\1&0&0&0&3\end{array}\right]$$

We will start using a notation which combines row operation steps in a single matrix.

$$\left[\begin{array}{cccc|c}1&1&1&0&0\\0&-1&0&1&0\\0&0&-1&1&0\\0&-1&-1&0&3\end{array}\right]\begin{array}{l}\\-1\text{R1}+\text{R2}\\-1\text{R1}+\text{R3}\\-1\text{R1}+\text{R4}\end{array}\Rightarrow$$

$$\left[\begin{array}{cccc|c}1&1&1&0&0\\0&1&0&-1&0\\0&0&1&-1&0\\0&-1&-1&0&3\end{array}\right]\begin{array}{l}\\-1\text{R2}\\-1\text{R3}\\ \end{array}\Rightarrow$$

$$\left[\begin{array}{cccc|c}1&0&1&1&0\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&-1&-1&3\end{array}\right]\begin{array}{l}-\text{R2}+\text{R1}\\ \\ \\ \text{R2}+\text{R4}\end{array}\Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 & 3 \end{array}\right]\begin{array}{l} -R3+R1 \\ \\ \\ R3+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]\begin{array}{l} \\ \\ \\ -\frac{1}{2}R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]\begin{array}{l} -2R4+R1 \\ R4+R2 \\ R4+R3 \\ \end{array}$$

Thus, we have

$$\frac{3}{x(x+1)(x^2+1)} = \frac{3}{x} + \frac{-\frac{3}{2}}{x+1} + \frac{-\frac{3}{2}x + \left(-\frac{3}{2}\right)}{x^2+1}$$

$$= \frac{3}{x} + \frac{-3}{2(x+1)} + \frac{-3(x+1)}{2(x^2+1)}$$

23. $$\frac{1}{x(2x+1)(3x^2+4)} = \frac{A}{x} + \frac{B}{2x+1} + \frac{Cx+D}{3x^2+4}$$

Multiply both sides by $x(2x+1)(3x^2+4)$ to get the following.

$$1 = A(2x+1)(3x^2+4) + Bx(3x^2+4) + (Cx+D)(x)(2x+1)$$

Expanding and combining like terms on the right side of the equation, we have the following.

$$1 = A(2x+1)(3x^2+4) + Bx(3x^2+4) + (Cx+D)(x)(2x+1)$$

$$1 = A(6x^3+3x^2+8x+4) + B(3x^3+4x) + C(2x^3+x^2) + D(2x^2+x)$$

$$1 = 6Ax^3 + 3Ax^2 + 8Ax + 4A + 3Bx^3 + 4Bx + 2Cx^3 + Cx^2 + 2Dx^2 + Dx$$

$$1 = (6A+3B+2C)x^3 + (3A+C+2D)x^2 + (8A+4B+D)x + 4A$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^3- term, we have $0 = 6A+3B+2C$.
For the x^2- term, we have $0 = 3A+C+2D$.
For the x-term, we have $0 = 8A+4B+D$.
For the constant term, we have $1 = 4A$.

If we use the Gauss-Jordan method, we begin with the following augmented matrix.

$$\left[\begin{array}{cccc|c} 6 & 3 & 2 & 0 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 8 & 4 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 6 & 3 & 2 & 0 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 8 & 4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right]\begin{array}{l} \\ \\ \\ \frac{1}{4}R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 3 & 0 & 1 & 2 & 0 \\ 8 & 4 & 0 & 1 & 0 \\ 6 & 3 & 2 & 0 & 0 \end{array}\right]\begin{array}{l} R1 \leftrightarrow R4 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 4 & 0 & 1 & -2 \\ 0 & 3 & 2 & 0 & -\frac{3}{2} \end{array}\right]\begin{array}{l} \\ -3R1+R2 \\ -8R1+R3 \\ -6R1+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 3 & 2 & 0 & -\frac{3}{2} \end{array}\right]\begin{array}{l} \\ \\ \frac{1}{4}R3 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 3 & 2 & 0 & -\frac{3}{2} \end{array}\right]\begin{array}{l} \\ R2 \leftrightarrow R3 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 0 & 2 & -\frac{3}{4} & 0 \end{array}\right]\begin{array}{l} \\ \\ \\ -3R2+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 0 & 0 & -\frac{19}{4} & \frac{3}{2} \end{array}\right]\begin{array}{l} \\ \\ \\ -2R3+R4 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 & -\frac{6}{19} \end{array}\right]\begin{array}{l} \\ \\ \\ -\frac{4}{19}R4 \end{array} \Rightarrow$$

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$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & -\frac{8}{19} \\ 0 & 0 & 1 & 0 & -\frac{9}{76} \\ 0 & 0 & 0 & 1 & -\frac{6}{19} \end{array}\right] \begin{array}{l} \\ -\frac{1}{4}R4+R2 \\ -2R4+R3 \\ \\ \end{array}$$

Thus, we have

$$\frac{1}{x(2x+1)(3x^2+4)} = \frac{\frac{1}{4}}{x} + \frac{-\frac{8}{19}}{2x+1} + \frac{-\frac{9}{76}x + \left(-\frac{6}{19}\right)}{3x^2+4}$$
$$= \frac{\frac{1}{4}}{x} + \frac{-\frac{8}{19}}{2x+1} + \frac{-\frac{9}{76}x - \frac{24}{76}}{3x^2+4}$$
$$= \frac{1}{4x} + \frac{-8}{19(2x+1)} + \frac{-9x-24}{76(3x^2+4)}$$

24. $\dfrac{x^4+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

Multiply both sides by $x(x^2+1)^2$ to get the following.

$$x^4+1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

Expanding and combining like terms on the right side of the equation, we have the following.

$$x^4+1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$
$$x^4+1 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$
$$x^4+1 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$
$$x^4+1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

Equate the coefficients of like powers of *x* on the two sides of the equation.

For the x^4- term, we have $1 = A+B$.

For the x^3- term, we have $0 = C$.

For the x^2- term, we have $0 = 2A+B+D$.

For the *x*-term, we have $0 = C+E$.

For the constant term, we have $1 = A$.

Since $A = 1$ and $1 = A+B$, we have $B = 0$. Since $A = 1$, $B = 0$, and $0 = 2A+B+D$, we have $D = -2$. Since $C = 0$ and $0 = C+E$, we have $E = 0$.

Thus, we have

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{0\cdot x+0}{x^2+1} + \frac{-2x+0}{(x^2+1)^2}$$
$$= \frac{1}{x} + \frac{-2x}{(x^2+1)^2}$$

25. $\dfrac{3x-1}{x(2x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{2x^2+1} + \dfrac{Dx+E}{(2x^2+1)^2}$

Multiply both sides by $x(2x^2+1)^2$ to get the following.

$$3x-1 = A(2x^2+1)^2 + (Bx+C)(x)(2x^2+1) + (Dx+E)x$$

Expanding and combining like terms on the right side of the equation, we have the following.

$$3x-1 = A(2x^2+1)^2 + (Bx+C)(x)(2x^2+1) + (Dx+E)x$$
$$= A(4x^4+4x^2+1) + B(2x^4+x^2) + C(2x^3+x) + Dx^2 + Ex$$
$$= 4Ax^4 + 4Ax^2 + A + 2Bx^4 + Bx^2 + 2Cx^3 + Cx + Dx^2 + Ex$$
$$3x-1 = (4A+2B)x^4 + 2Cx^3 + (4A+B+D)x^2 + (C+E)x + A$$

Equate the coefficients of like powers of *x* on the two sides of the equation.

For the x^4- term, we have $0 = 4A+2B$.

For the x^3- term, we have $0 = 2C \Rightarrow C = 0$.

For the x^2- term, we have $0 = 4A+B+D$.

For the *x*-term, we have $3 = C+E$.

For the constant term, we have $-1 = A$.

Since $A = -1$ and $0 = 4A+2B$, we have $B = 2$. Since $A = -1$, $B = 2$ and $0 = 4A+B+D$, we have $D = 2$. Since $C = 0$ and $3 = C+E$, we have $E = 3$.

Thus, we have

$$\frac{3x-1}{x(2x^2+1)^2} = \frac{-1}{x} + \frac{2x+0}{2x^2+1} + \frac{2x+3}{(2x^2+1)^2}$$
$$= \frac{-1}{x} + \frac{2x}{2x^2+1} + \frac{2x+3}{(2x^2+1)^2}$$

26. $$\frac{3x^4+x^3+5x^2-x+4}{(x-1)\left(x^2+1\right)^2} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

Multiply both sides by $(x-1)\left(x^2+1\right)^2$ to get the following.

$$\begin{aligned}3x^4+x^3+5x^2-x+4 &= A\left(x^2+1\right)^2+(Bx+C)(x-1)\left(x^2+1\right)\\ &\quad+(Dx+E)(x-1)\end{aligned}$$

Expanding and combining like terms on the right side of the equation, we have

$$\begin{aligned}3x^4+x^3+5x^2-x+4 &= A\left(x^2+1\right)^2+(Bx+C)(x-1)\left(x^2+1\right)\\ &\quad+(Dx+E)(x-1)\\ &= A\left(x^4+2x^2+1\right)+(Bx+C)\left(x^3-x^2+x-1\right)\\ &\quad+D\left(x^2-x\right)+E(x-1)\\ &= Ax^4+2Ax^2+A+Bx^4-Bx^3+Bx^2-Bx\\ &\quad+Cx^3-Cx^2+Cx-C\\ &\quad+Dx^2-Dx+Ex-E\end{aligned}$$

$$\begin{aligned}3x^4+x^3+5x^2-x+4 &= (A+B)x^4+(-B+C)x^3+(2A+B-C+D)x^2\\ &\quad+(-B+C-D+E)x+(A-C-E)\end{aligned}$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^4- term, we have $3=A+B.$

For the x^3- term, we have $1=-B+C.$

For the x^2- term, we have $5=2A+B-C+D.$

For the x-term, we have $-1=-B+C-D+E.$

For the constant term, we have $4=A-C-E.$

If we use the Gauss-Jordan method, we begin with the following augmented matrix.

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 0 & 0 & 3\\ 0 & -1 & 1 & 0 & 0 & 1\\ 2 & 1 & -1 & 1 & 0 & 5\\ 0 & -1 & 1 & -1 & 1 & -1\\ 1 & 0 & -1 & 0 & -1 & 4 \end{array}\right]$$

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 0 & 0 & 3\\ 0 & -1 & 1 & 0 & 0 & 1\\ 0 & -1 & -1 & 1 & 0 & -1\\ 0 & -1 & 1 & -1 & 1 & -1\\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right]\begin{array}{l} \\ \\ -2R1+R3 \\ \\ -1R1+R5 \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 0 & 0 & 3\\ 0 & 1 & -1 & 0 & 0 & -1\\ 0 & -1 & -1 & 1 & 0 & -1\\ 0 & -1 & 1 & -1 & 1 & -1\\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right]\begin{array}{l} \\ -1R2 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 0 & 4\\ 0 & 1 & -1 & 0 & 0 & -1\\ 0 & 0 & -2 & 1 & 0 & -2\\ 0 & 0 & 0 & -1 & 1 & -2\\ 0 & 0 & -2 & 0 & -1 & 0 \end{array}\right]\begin{array}{l} -1R2+R1 \\ \\ R2+R3 \\ R2+R4 \\ R2+R5 \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 0 & 4\\ 0 & 1 & -1 & 0 & 0 & -1\\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1\\ 0 & 0 & 0 & -1 & 1 & -2\\ 0 & 0 & -2 & 0 & -1 & 0 \end{array}\right]\begin{array}{l} \\ \\ -\frac{1}{2}R3 \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & \frac{1}{2} & 0 & 3\\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1\\ 0 & 0 & 0 & -1 & 1 & -2\\ 0 & 0 & 0 & -1 & -1 & 2 \end{array}\right]\begin{array}{l} -1R3+R1 \\ R3+R2 \\ \\ \\ 2R3+R5 \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & \frac{1}{2} & 0 & 3\\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1\\ 0 & 0 & 0 & 1 & -1 & 2\\ 0 & 0 & 0 & -1 & -1 & 2 \end{array}\right]\begin{array}{l} \\ \\ \\ -1R4 \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & \frac{1}{2} & 2\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 1\\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 2\\ 0 & 0 & 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0 & -2 & 4 \end{array}\right]\begin{array}{l} -\frac{1}{2}R4+R1 \\ \frac{1}{2}R4+R2 \\ \frac{1}{2}R4+R3 \\ \\ R4+R5 \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & \frac{1}{2} & 2\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 1\\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 2\\ 0 & 0 & 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} \\ \\ \\ \\ -\frac{1}{2}R5 \end{array} \Rightarrow$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 0 & 3\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right]\begin{array}{l} -\frac{1}{2}R5+R1 \\ \frac{1}{2}R5+R2 \\ \frac{1}{2}R5+R3 \\ R5+R4 \\ \\ \end{array}$$

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Thus, we have

$$\frac{3x^4+x^3+5x^2-x+4}{(x-1)(x^2+1)^2} = \frac{3}{x-1}+\frac{0\cdot x+1}{x^2+1}+\frac{0\cdot x+(-2)}{(x^2+1)^2} = \frac{3}{x-1}+\frac{1}{x^2+1}+\frac{-2}{(x^2+1)^2}$$

27. $$\frac{-x^4-8x^2+3x-10}{(x+2)(x^2+4)^2} = \frac{A}{x+2}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

Multiply both sides by $(x+2)(x^2+4)^2$ to get the following.

$$-x^4-8x^2+3x-10 = A(x^2+4)^2+(Bx+C)(x+2)(x^2+4) + (Dx+E)(x+2) \quad (1)$$

Substitute then expand and combine like terms on the right side of the equation.
Substitute -2 in equation (1) for x to get the following.

$$-(-2)^4-8(-2)^2+3(-2)-10 = A\left[(-2)^2+4\right]^2 + \left[B(-2)+C\right](-2+2)\left[(-2)^2+4\right] + \left[D(-2)+E\right](-2+2)$$

$$-16-32+(-6)-10 = A(4+4)^2+0+0$$

$$-64 = 64A \Rightarrow A=-1$$

Substituting -1 for A in equation (1) and expanding and combining like terms, we have the following.

$$\begin{aligned}-x^4-8x^2+3x-10 &= -1(x^2+4)^2+(Bx+C)(x+2)(x^2+4) + (Dx+E)(x+2)\\ &= -1(x^4+8x^2+16) + (Bx+C)(x^3+2x^2+4x+8) + D(x^2+2x)+E(x+2)\\ &= -x^4-8x^2-16+Bx^4+2Bx^3+4Bx^2+8Bx + Cx^3+2Cx^2+4Cx+8C+Dx^2+2Dx + Ex+2E\\ &= (-1+B)x^4+(2B+C)x^3 + (-8+4B+2C+D)x^2 + (8B+4C+2D+E)x + (-16+8C+2E)\end{aligned}$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^4- term, we have
$-1=-1+B \Rightarrow B=0.$

For the x^3- term, we have $0=2B+C.$

For the x^2- term, we have
$-8=-8+4B+2C+D \Rightarrow 0=4B+2C+D.$

For the x-term, we have
$3=8B+4C+2D+E.$

For the constant term, we have
$-10=-16+8C+2E \Rightarrow 6=8C+2E.$

Since $B=0$ and $0=2B+C,$ we have $C=0.$ Since $B=0,$ $C=0,$ and $0=4B+2C+D,$ we have $D=0.$ Since $C=0$ and $6=8C+2E,$ we have $E=3.$

Thus we have,

$$\frac{-x^4-8x^2+3x-10}{(x+2)(x^2+4)^2} = \frac{-1}{x+2}+\frac{0\cdot x+0}{x^2+4}+\frac{0\cdot x+3}{(x^2+4)^2} = \frac{-1}{x+2}+\frac{3}{(x^2+4)^2}$$

28. $$\frac{x^2}{x^4-1} = \frac{x^2}{(x^2+1)(x^2-1)} = \frac{x^2}{(x^2+1)(x+1)(x-1)} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}$$

Multiply both sides by $(x^2+1)(x+1)(x-1)$ to get the following.

$$x^2 = A(x^2+1)(x-1)+B(x^2+1)(x+1) + (Cx+D)(x+1)(x-1) \quad (1)$$

Substitute then expand and combine like terms on the right side of the equation.
Substitute -1 in equation (1) for x to get the following.

$$(-1)^2 = A\left[(-1)^2+1\right](-1-1) + B\left[(-1)^2+1\right](-1+1) + \left[C(-1)+D\right](-1+1)(-1-1)$$
$$1 = A(1+1)(-1-1)+0+0$$
$$1 = -4A \Rightarrow A = -\tfrac{1}{4}$$

Substituting $-\frac{1}{4}$ for A and 1 for x in equation (1), we have:

$$1^2 = -\tfrac{1}{4}\left(1^2+1\right)(1-1) + B\left(1^2+1\right)(1+1) + \left[C(1)+D\right](1+1)(1-1)$$
$$1 = 0+0+4B \Rightarrow 1 = 4B \Rightarrow B = \tfrac{1}{4}$$
$$x^2 = -\tfrac{1}{4}\left(x^2+1\right)(x-1)+\tfrac{1}{4}\left(x^2+1\right)(x+1) + (Cx+D)(x+1)(x-1) \qquad (2)$$

Expanding and combining like terms, we have the following.

$$\begin{aligned} x^2 &= -\tfrac{1}{4}\left(x^2+1\right)(x-1)+\tfrac{1}{4}\left(x^2+1\right)(x+1) + (Cx+D)(x+1)(x-1) \\ &= -\tfrac{1}{4}\left(x^3-x^2+x-1\right)+\tfrac{1}{4}\left(x^3+x^2+x+1\right) + (Cx+D)\left(x^2-1\right) \\ &= \tfrac{1}{2}x^2+\tfrac{1}{2}+Cx^3-Cx+Dx^2-D \end{aligned}$$
$$x^2 = Cx^3+\left(D+\tfrac{1}{2}\right)x^2-Cx+\left(\tfrac{1}{2}-D\right)$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^3- term, we have $0 = C$.

For the x^2- term, we have

$1 = D+\frac{1}{2} \Rightarrow D = \frac{1}{2}$.

For the x-term, we again have $C = 0$.

For the constant term, we have

$0 = \frac{1}{2} - D \Rightarrow D = \frac{1}{2}$

Thus, we have

$$\frac{x^2}{x^4-1} = \frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}+\frac{0\cdot x+\frac{1}{2}}{x^2+1} = \frac{-1}{4(x+1)}+\frac{1}{4(x-1)}+\frac{1}{2\left(x^2+1\right)}$$

29. $\dfrac{5x^5+10x^4-15x^3+4x^2+13x-9}{x^3+2x^2-3x}$

Since the degree of the numerator is higher than the degree of the denominator, first find the quotient.

$$\begin{array}{r} 5x^2 \qquad\qquad\qquad\qquad\qquad\qquad \\ x^3+2x^2-3x \overline{\big)\, 5x^5+10x^4-15x^3+4x^2+13x-9} \\ \underline{5x^5+10x^4-15x^3 \qquad\qquad\qquad\quad} \\ 4x^2+13x-9 \end{array}$$

Find the partial fraction decomposition of $\dfrac{4x^2+13x-9}{x^3+2x^2-3x}$.

$$\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{4x^2+13x-9}{x\left(x^2+2x-3\right)} = \frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$$

Multiply by $x(x+3)(x-1)$ to obtain

$$4x^2+13x-9 = A(x+3)(x-1)+Bx(x-1)+Cx(x+3) \quad (1)$$

First substitute 0 for x in equation (1) to get the following.

$$4(0)^2+13(0)-9 = A(0+3)(0-1)+B(0)(0-1)+C(0)(0+3)$$
$$-9 = -3A \Rightarrow A = 3$$

Replace A with 3 in equation (1) and substitute -3 for x to get the following.

$$4(-3)^2+13(-3)-9 = 3(-3+3)(-3-1)+B(-3)(-3-1)+C(-3)(-3+3)$$
$$-12 = 12B \Rightarrow B = -1$$
$$4x^2+13x-9 = 3(x+3)(x-1)-x(x-1)+Cx(x+3) \quad (2)$$

Now substitute 1 (arbitrary choice) for x in equation (2) to get the following.

$$4(1)^2+13(1)-9 = 3(1+3)(1-1)-1(1-1)+C(1)(1+3)$$
$$8 = 4C \Rightarrow C = 2$$

Thus, we have

$$\frac{5x^5+10x^4-15x^3+4x^2+13x-9}{x^3+2x^2-3x} = 5x^2+\frac{3}{x}+\frac{-1}{x+3}+\frac{2}{x-1}$$

30. $\dfrac{3x^6+3x^4+3x}{x^4+x^2}$

Since the degree of the numerator is greater than the degree of the denominator, first find the quotient.

$$\begin{array}{r|l} & 3x^2 \\ x^4+x^2 & 3x^6+0x^5+3x^4+0x^3+0x^2+3x+0 \\ & \underline{3x^6 \qquad\quad +3x^4 \qquad\qquad\qquad\qquad\quad} \\ & \qquad\qquad\qquad\qquad\qquad\quad 3x \end{array}$$

Now find the partial fraction decomposition of

$$\frac{3x}{x^4+x^2}=\frac{3x}{x^2\left(x^2+1\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}.$$

Multiply by $x^2\left(x^2+1\right)$ to obtain

$$3x=Ax\left(x^2+1\right)+B\left(x^2+1\right)+\left(Cx+D\right)x^2. \ (1)$$

First substitute 0 for x in equation (1) to get

$$3(0)=A(0)\left(0^2+1\right)+B\left(0^2+1\right)+\left(C(0)+D\right)(0)^2$$

$$0=B$$

$$3x=Ax\left(x^2+1\right)+\left(Cx+D\right)x^2 \ (2)$$

Expanding and combining like terms we have the following.

$$3x=Ax\left(x^2+1\right)+\left(Cx+D\right)x^2$$

$$3x=Ax^3+Ax+Cx^3+Dx^2$$

$$3x=\left(A+C\right)x^3+Dx^2+Ax$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^3- term, we have $0=A+C$.

For the x^2- term, we have $D=0$.

For the x-term, we have again have $A=3$.

Since $A=3$ and $0=A+C$, we have $C=-3$.

Thus we have,

$$\frac{3x^6+3x^4+3x}{x^4+x^2}=3x^2+\frac{3}{x}+\frac{0}{x^2}+\frac{-3x+0}{x^2+1}=3x^2+\frac{3}{x}+\frac{-3x}{x^2+1}$$

Alternatively we could have found the partial fraction decomposition of

$$\frac{3x}{x^4+x^2}=\frac{3}{x^3+x}=\frac{3}{x\left(x^2+1\right)}.$$

Since $\dfrac{3}{x\left(x^2+1\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$, we multiply by $x\left(x^2+1\right)$ to obtain

$$3=A\left(x^2+1\right)+\left(Bx+C\right)x. \ (1)$$

First substitute 0 for x in equation (1) to get

$$3=A\left(0^2+1\right)+\left(B(0)+C\right)(0)\Rightarrow A=3.$$

$$3=3\left(x^2+1\right)+\left(Bx+C\right)x \ (2)$$

Expanding and combining like terms we have the following.

$$3=3\left(x^2+1\right)+\left(Bx+C\right)x$$

$$3=3x^2+3+Bx^2+Cx$$

$$3=\left(3+B\right)x^2+Cx+3$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^2- term, $0=3+B\Rightarrow B=-3.$

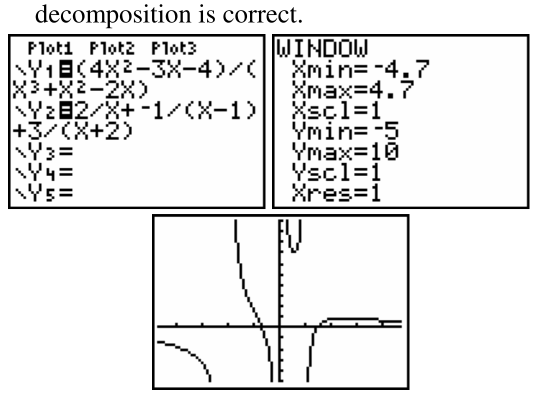
For the x-term, $C=0.$

Thus, we have

$$\frac{3x^6+3x^4+3x}{x^4+x^2}=3x^2+\frac{3}{x}+\frac{-3x+0}{x^2+1}=3x^2+\frac{3}{x}+\frac{-3x}{x^2+1}$$

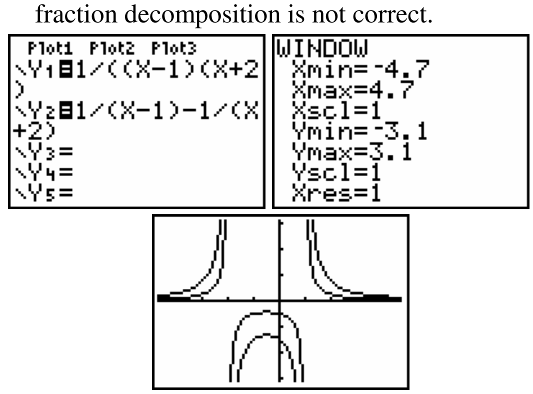
31. $\dfrac{4x^2-3x-4}{x^3+x^2-2x}=\dfrac{2}{x}+\dfrac{-1}{x-1}+\dfrac{3}{x+2}$

The graphs coincide. The partial fraction decomposition is correct.



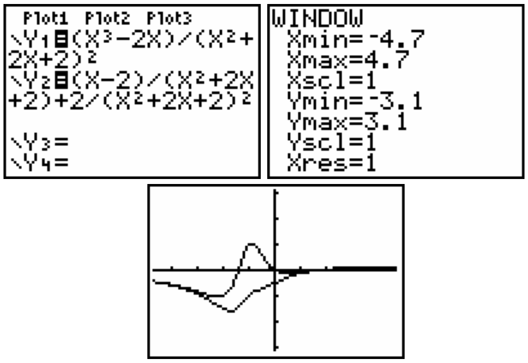
32. $\dfrac{1}{(x-1)(x+2)}=\dfrac{1}{x-1}-\dfrac{1}{x+2}$

The graphs do not coincide. The partial fraction decomposition is not correct.



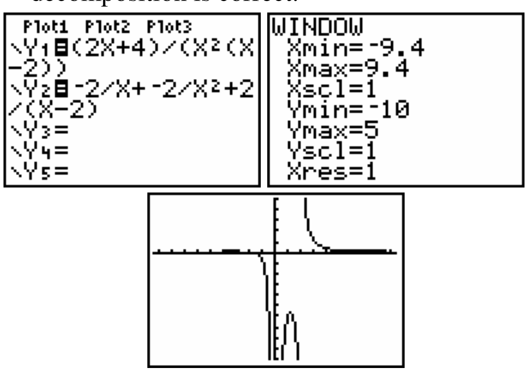
33. $\dfrac{x^3-2x}{\left(x^2+2x+2\right)^2}=\dfrac{x-2}{x^2+2x+2}+\dfrac{2}{\left(x^2+2x+2\right)^2}$

The graphs do not coincide. The partial fraction decomposition is not correct.



34. $\dfrac{2x+4}{x^2\left(x-2\right)}=\dfrac{-2}{x}+\dfrac{-2}{x^2}+\dfrac{2}{x-2}$

The graphs coincide. The partial fraction decomposition is correct.



Chapter 9 Quiz
(Sections 9.1–9.4)

1. $2x+y=-4$ (1)
$-x+2y=2$ (2)

Solve equation (1) for y:
$2x+y=-4\Rightarrow y=-2x-4$ (3)

Replace y in equation (2) with $-2x-4$ and solve for x:
$-x+2y=2\Rightarrow -x+2(-2x-4)=2$
$-5x-8=2\Rightarrow x=-2$

Substitute -2 for x in equation (3) and solve for y: $y=-2(-2)-4=0$

Verify that the ordered pair $(-2,0)$ satisfies both equations.

Check:

$2x+y=-4$ (1)	$-x+2y=2$ (2)
$2(-2)+0=-4$?	$-(-2)+2(0)=2$?
$-4=-4$ True	$2=2$ True

Solution set: $\{(-2,0)\}$

2. $5x+10y=10$ (1)
$x+2y=2$ (2)

Solve equation (2) for x:
$x+2y=2\Rightarrow x=-2y+2$

Substitute $-2y+2$ for x in equation (1):
$5\left(-2y+2\right)+10y=10$
$-10y+10+10y=10\Rightarrow 10=10$

This is a true statement. There are infinitely many solutions. Express the solution set with y as the arbitrary variable. Solve equation (1) for x: $5x+10y=10\Rightarrow 5x=-10y+10\Rightarrow$
$x=-2y+2$

Alternatively, express the solution set with x as the arbitrary variable by solution equation (1) for y:

$5x+10y=10\Rightarrow 10y=-5x+10\Rightarrow$
$y=-\frac{1}{2}x+1=\frac{2-x}{2}$

Solution set: $\left\{\left(x,\frac{2-x}{2}\right)\right\}$ or $\left\{\left(-2y+2,y\right)\right\}$

3. $x-y=6$ (1)
$x-y=4$ (2)

Multiply equation (2) by -1, then add the result to equation (1):

$$\begin{array}{r} x-y=6 \\ \underline{-x+y=-4} \\ 0=2 \end{array}$$

This is a false statement. The system is inconsistent and the solution set is $\varnothing$.

4. $2x-3y=18$ (1)
$5x+2y=7$ (2)

Multiply equation (1) by 2 and equation (2) by 3 and then add the resulting equations.

$$\begin{array}{r} 4x-6y=36 \\ \underline{15x+6y=21} \\ 19x=57\Rightarrow x=3 \end{array}$$

Substitute 3 for x in equation 1 and solve for y:
$2(3)-3y=18\Rightarrow -3y=12\Rightarrow y=-4$

Verify that the ordered pair $(3,-4)$ satisfies both equations.

Check:

$2x-3y=18$ (1)
$2(3)-3(-4)=18$?
$6+12=18$
$18=18$ True

$5x+2y=7$ (2)
$5(3)+2(-4)=7$?
$15-8=7$
$7=7$ True

Solution set: $\{(3,-4)\}$

5. $3x+5y=-5$
$-2x+3y=16$

This system has the augmented matrix
$\left[\begin{array}{rr|r} 3 & 5 & -5 \\ -2 & 3 & 16 \end{array}\right]$.

$\left[\begin{array}{rr|r} 1 & 8 & 11 \\ -2 & 3 & 16 \end{array}\right] \text{R1}+\text{R2} \Rightarrow$

$\left[\begin{array}{rr|r} 1 & 8 & 11 \\ 0 & 19 & 38 \end{array}\right] 2\text{R1}+\text{R2} \Rightarrow$

$\left[\begin{array}{rr|r} 1 & 8 & 11 \\ 0 & 1 & 2 \end{array}\right] \tfrac{1}{19}\text{R2} \Rightarrow$

$\left[\begin{array}{rr|r} 1 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right] \text{R1}-8\text{R2}$

Verify that the ordered pair $(-5,2)$ satisfies both equations.

Check:

$3x+5y=-5$ (1)
$3(-5)+5(2)=-5$?
$-15+10=-5$
$-5=-5$ True

$-2x+3y=16$ (2)
$-2(-5)+3(2)=16$?
$10+6=16$
$16=16$ True

Solution set: {(−5, 2)}

6. $5x+2y=-3$
$4x-3y=-30$

$D=\begin{vmatrix} 5 & 2 \\ 4 & -3 \end{vmatrix}=5(-3)-4(2)=-23$

$D_x=\begin{vmatrix} -3 & 2 \\ -30 & -3 \end{vmatrix}=(-3)(-3)-(-30)(2)=69$

$D_y=\begin{vmatrix} 5 & -3 \\ 4 & -30 \end{vmatrix}=5(-30)-4(-3)=-138$

$x=\frac{D_x}{D}=\frac{69}{-23}=-3$ and $y=\frac{-138}{-23}=6$

Verify that the ordered pair $(-3,6)$ satisfies both equations.

Check: $5x+2y=-3$ (1)
$5(-3)+2(6)=-3$?
$-15+12=-3$
$-3=-3$ True

$5x+2y=7$ (2)
$5(3)+2(-4)=7$?
$15-8=7$
$7=7$ True

Solution set: {(−3, 6)}

7. $x+y+z=1$ (1)
$-x+y+z=5$ (2)
$y+2z=5$ (3)

Eliminate *x* by adding equations (1) and (2) to get $2y+2z=6 \Rightarrow y+z=3$ (4).

Now solve the system consisting of equations (3) and (4) by subtracting equation (4) from equation (3):

$$\begin{array}{rl} y+2z=5 & (3) \\ y+z=3 & (4) \\ \hline z=2 & \end{array}$$

Substitute the value for *z* into equation (4) to solve for *y*: $y+2=3 \Rightarrow y=1$

Now, substitute the values for *y* and *z* into equation (1) to solve for *x*:
$x+1+2=1 \Rightarrow x=-2$

Verify that the ordered triple $(-2,1,2)$ satisfies all three equations.

Check:

$x+y+z=1$ (1)	$-x+y+z=5$ (2)
$-2+1+2=1$?	$-(-2)+1+2=5$?
$1=1$ True	$5=5$ True

$y+2z=5$ (3)
$1+2(2)=5$?
$5=5$ True

Solution set: $\{(-2,1,2)\}$

8. $2x+4y+4z=4$
$x+3y+z=4$
$-x+3y+2z=-1$

This system has the augmented matrix

$\left[\begin{array}{rrr|r} 2 & 4 & 4 & 4 \\ 1 & 3 & 1 & 4 \\ -1 & 3 & 2 & -1 \end{array}\right]$

$\left[\begin{array}{rrr|r} 1 & 2 & 2 & 2 \\ 1 & 3 & 1 & 4 \\ -1 & 3 & 2 & -1 \end{array}\right] \begin{array}{l} \tfrac{1}{2}\text{R1} \\ \\ \\ \end{array} \Rightarrow$

$\left[\begin{array}{rrr|r} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right] \begin{array}{l} \\ -\text{R1}+\text{R2} \\ \text{R1}+\text{R3} \end{array} \Rightarrow$

$\left[\begin{array}{rrr|r} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \\ \\ \tfrac{1}{9}(-5\text{R2}+\text{R3}) \end{array} \Rightarrow$

$$\begin{bmatrix} 1 & 2 & 2 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} R2+R3 \quad \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} R1+-2R2 \quad \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} R1+-2R3$$

Verify that the ordered triple $(2,1,-1)$ satisfies all three equations.

Check: $2x+4y+4z=4$ (1)

$$2(2)+4(1)+4(-1)=4 \ ?$$
$$4+4-4=4$$
$$4=4 \text{ True}$$

$$x+3y+z=4 \quad (2)$$
$$2+3(1)+(-1)=4 \ ?$$
$$2+3-1=4$$
$$4=4 \quad \text{True}$$

$$-x+3y+2z=-1 \quad (3)$$
$$-2+3(1)+2(-1)=-1 \ ?$$
$$-2+3-2=-1$$
$$-1=-1 \quad \text{True}$$

Solution set: $\{(-2,1,2)\}$

9. $7x+y-z=4$ (1)
$2x-3y+z=2$ (2)
$-6x+9y-3z=-6$ (3)

$$D=\begin{vmatrix} 7 & 1 & -1 \\ 2 & -3 & 1 \\ -6 & 9 & -3 \end{vmatrix}$$

Adding 3 times row 2 to row 3, we have

$$D=\begin{vmatrix} 7 & 1 & -1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

Since we have a row of zeros, $D=0$ and we cannot use Cramer's rule.
Using the elimination method, we can add 3 times equation (2) to equation (3).

$$\begin{array}{rl} 6x-9y+3z=6 & (2) \\ -6x+9y-3z=-6 & (3) \\ \hline 0=0 & \text{True} \end{array}$$

This system will have infinitely many solutions.
Solve the system made up of equations (1) and (3) in terms of the arbitrary variable y. To eliminate z, multiply equation (1) by −3 and add the result to equation (3):

$$\begin{array}{r} -21x-3y+3z=-12 \\ -6x+9y-3z=-6 \\ \hline -27x+6y=-18 \end{array} \Rightarrow -27x=-6y-18 \Rightarrow$$
$$x=\frac{6y+18}{27}=\frac{2y+6}{9}$$

Now, express z also in terms of y by solving equation (1) for y and substituting $\frac{2y+6}{9}$ for x in the result.

$$7x+y-z=4 \Rightarrow z=7x+y-4 \Rightarrow$$
$$z=7\left(\frac{2y+6}{9}\right)+y-4=\frac{14y+42}{9}+y-4=\frac{23y+6}{9}$$

Solution set: $\left\{\left(\frac{2y+6}{9}, y, \frac{23y+6}{9}\right)\right\}$

10. Let x = number of stereo tvs; y = number of non-stereo tvs
The information in the problem gives the system $x+y=32{,}000{,}000$ (1)

$$\frac{x}{y}=\frac{10}{19} \Rightarrow 19x=10y \quad (2)$$

Solve equation (1) for y, then substitute into equation (2):

$$x+y=32{,}000{,}000 \Rightarrow y=32{,}000{,}000-x$$
$$19x=10(32{,}000{,}000-x)$$
$$19x=320{,}000{,}000-10x$$
$$29x=320{,}000{,}000$$
$$x \approx 11034483 \approx 11.03 \text{ million}$$

Substitute this value into equation (1) to solve for y: $11.03 \text{ million} + y = 32$ million

$$y \approx 21 \text{ million}$$

About 11.03 million tvs with stereo sound and about 21 million tvs without stereo sound were sold.

11. Let x = the amount invested at 8%; y = the amount invested at 11%; z = the amount invested at 14%
The information in the problem gives the system:

$$\begin{cases} x+y+z=5000 \\ z=x+y \Rightarrow \\ .08x+.11y+.14z=595 \end{cases}$$

$$\begin{cases} x+y+z=5000 & (1) \\ -x-y+z=0 & (2) \\ .08x+.11y+.14z=595 & (3) \end{cases}$$

This system has the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & | & 5000 \\ -1 & -1 & 1 & | & 0 \\ .08 & .11 & .14 & | & 595 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 5000 \\ 0 & 0 & 1 & | & 2500 \\ .08 & .11 & .14 & | & 595 \end{bmatrix} \tfrac{1}{2}(R1+R2) \quad \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 5000 \\ .08 & .11 & .14 & 595 \\ 0 & 0 & 1 & 2500 \end{bmatrix} \begin{matrix} \\ \\ R2 \leftrightarrow R3 \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 5000 \\ 8 & 11 & 14 & 59,500 \\ 0 & 0 & 1 & 2500 \end{bmatrix} \begin{matrix} \\ 100R2 \\ \\ \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 5000 \\ 0 & 1 & 2 & 6500 \\ 0 & 0 & 1 & 2500 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}(-8R1+R2) \\ \\ \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 5000 \\ 0 & 1 & 0 & 1500 \\ 0 & 0 & 1 & 2500 \end{bmatrix} \begin{matrix} \\ R2-2R3 \\ \\ \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & 3500 \\ 0 & 1 & 0 & 1500 \\ 0 & 0 & 1 & 2500 \end{bmatrix} \begin{matrix} R1-R2 \\ \\ \\ \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 1000 \\ 0 & 1 & 0 & 1500 \\ 0 & 0 & 1 & 2500 \end{bmatrix} \begin{matrix} R1-R3 \\ \\ \\ \end{matrix} \Rightarrow$$

\$1000 was invested at 8%, \$1500 was invested at 11%, and \$2500 was invested at 14%.

12. $A = \begin{bmatrix} -5 & 4 \\ 2 & -1 \end{bmatrix}$

$|A| = (-5)(-1) - (2)(4) = 5 - 8 = -3$

13. $\begin{vmatrix} -1 & 2 & 4 \\ -3 & -2 & -3 \\ 2 & -1 & 5 \end{vmatrix}$

To expand by the first row, we will need to find $a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13}$.

$$M_{11} = \begin{vmatrix} -2 & -3 \\ -1 & 5 \end{vmatrix} = (-2)(5) - (-1)(-3) = -13$$

$$M_{12} = \begin{vmatrix} -3 & -3 \\ 2 & 5 \end{vmatrix} = (-3)(5) - (2)(-3) = -9$$

$$M_{13} = \begin{vmatrix} -3 & -2 \\ 2 & -1 \end{vmatrix} = (-3)(-1) - (2)(-2) = 7$$

$$a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13} = (-1)(-13) - (2)(-9) + (4)(7) = 59$$

14. $\dfrac{10x+13}{x^2-x-20} = \dfrac{10x+13}{(x-5)(x+4)} = \dfrac{A}{x-5} + \dfrac{B}{x+4}$

Multiply both sides by $(x-5)(x+4)$ to get

$10x+13 = A(x+4) + B(x-5) \quad (1)$.

Substitute 5 for x to get

$10(5)+13 = A(5+4) + B(5-5)$

$63 = 9A \Rightarrow 7 = A$

In equation (1), replace A with 7 and substitute −4 for x to get

$10(-4)+13 = 7((-4)+4) + B((-4)-5)$

$27 = -9B \Rightarrow -3 = B$

Thus, we have $\dfrac{10x+13}{x^2-x-20} = \dfrac{7}{x-5} - \dfrac{3}{x+4}$

15. $\dfrac{2x^2-15x-32}{(x-1)(x^2+6x+8)} = \dfrac{2x^2-15x-32}{(x-1)(x+2)(x+4)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+4}$

Multiply both sides by $(x-1)(x+2)(x+4)$:

$$2x^2-15x-32 = A(x+2)(x+4) + B(x-1)(x+4) + C(x-1)(x+2)$$

Expanding and combining like terms on the right side of the equation, we have

$$\begin{aligned} 2x^2-15x-32 &= A(x+2)(x+4) + B(x-1)(x+4) + C(x-1)(x+2) \\ &= Ax^2 + 6Ax + 8A + Bx^2 + 3Bx - 4B + Cx^2 + Cx - 2C \\ &= x^2(A+B+C) + x(6A+3B+C) + (8A-4B-2C) \end{aligned}$$

Equate the coefficients of like powers of x on the two sides of the equation.

For the x^2-term, $2 = A+B+C$

For the x-term, $-15 = 6A+3B+C$

For the constant term, $-32 = 8A-4B-2C$

Using the Gauss-Jordan method, we have

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 6 & 3 & 1 & -15 \\ 8 & -4 & -2 & -32 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -3 & -5 & -27 \\ 0 & -12 & -10 & -48 \end{bmatrix} \begin{matrix} \\ -6R1+R2 \\ -8R1+R2 \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -3 & -5 & -27 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{10}(-4R2+R3) \end{matrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{matrix} \\ -\frac{1}{3}(5R3+R2) \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{array}\right] \text{R1} - \text{R2} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{array}\right] \text{R1} - \text{R3}$$

$A = -3, B = -1, C = 6.$

Thus

$$\frac{2x^2 - 15x - 32}{(x-1)(x^2 + 6x + 8)} = \frac{-3}{x-1} + \frac{-1}{x+2} + \frac{6}{x+4}$$

Section 9.5: Nonlinear Systems of Equations

1. The system is $\begin{aligned} x^2 &= y - 1 \\ y &= 3x + 5 \end{aligned}$

The proposed solution set is $\{(-1, 2), (4, 17)\}$.

Check $(-1, 2)$.

$$\begin{aligned} x^2 &= y - 1 \\ (-1)^2 &= 2 - 1 \ ? \\ 1 &= 1 \end{aligned} \qquad \begin{aligned} y &= 3x + 5 \\ 2 &= 3(-1) + 5 \ ? \\ 2 &= -3 + 5 \\ 2 &= 2 \end{aligned}$$

Check $(4, 17)$.

$$\begin{aligned} x^2 &= y - 1 \\ 4^2 &= 17 - 1 \ ? \\ 16 &= 16 \end{aligned} \qquad \begin{aligned} y &= 3x + 5 \\ 17 &= 3(4) + 5 \ ? \\ 17 &= 12 + 5 \\ 17 &= 17 \end{aligned}$$

Both solutions are valid.

2. The system is $\begin{aligned} 2x^2 &= 3y + 23 \\ y &= 2x - 5 \end{aligned}$

The proposed solution set is $\{(-1, -7), (4, 3)\}$.

Check $(-1, -7)$.

$$\begin{aligned} 2x^2 &= 3y + 23 \\ 2(-1)^2 &= 3(-7) + 23 \ ? \\ 2(1) &= -21 + 23 \\ 2 &= 2 \end{aligned} \qquad \begin{aligned} y &= 2x - 5 \\ -7 &= 2(-1) - 5 \ ? \\ -7 &= -2 - 5 \\ -7 &= -7 \end{aligned}$$

Check $(4, 3)$.

$$\begin{aligned} 2x^2 &= 3y + 23 \\ 2(4)^2 &= 3(3) + 23 \ ? \\ 2(16) &= 9 + 23 \\ 32 &= 32 \end{aligned} \qquad \begin{aligned} y &= 2x - 5 \\ 3 &= 2(4) - 5 \ ? \\ 3 &= 8 - 5 \\ 3 &= 3 \end{aligned}$$

Both solutions are valid.

3. The system is $\begin{aligned} x^2 + y^2 &= 5 \\ -3x + 4y &= 2 \end{aligned}$

The proposed solution set is $\left\{(-2, -1), \left(\frac{38}{25}, \frac{41}{25}\right)\right\}$.

Check $(-2, -1)$.

$$\begin{aligned} x^2 + y^2 &= 5 \\ (-2)^2 + (-1)^2 &= 5 \ ? \\ 4 + 1 &= 5 \\ 5 &= 5 \end{aligned} \qquad \begin{aligned} -3x + 4y &= 2 \\ -3(-2) + 4(-1) &= 2 \ ? \\ 6 + (-4) &= 2 \\ 2 &= 2 \end{aligned}$$

Check $\left(\frac{38}{25}, \frac{41}{25}\right)$.

$$\begin{aligned} x^2 + y^2 &= 5 \\ \left(\tfrac{38}{25}\right)^2 + \left(\tfrac{41}{25}\right)^2 &= 5 \ ? \\ \tfrac{1444}{625} + \tfrac{1681}{625} &= 5 \\ \tfrac{3125}{625} &= 5 \\ 5 &= 5 \end{aligned} \qquad \begin{aligned} -3x + 4y &= 2 \\ -3\left(\tfrac{38}{25}\right) + 4\left(\tfrac{41}{25}\right) &= 2 \ ? \\ -\tfrac{114}{25} + \tfrac{164}{25} &= 2 \\ \tfrac{50}{25} &= 2 \\ 2 &= 2 \end{aligned}$$

Both solutions are valid.

4. The system is $\begin{aligned} x + y &= -3 \\ x^2 + y^2 &= 45 \end{aligned}$

The proposed solution set is $\{(-6, 3), (3, -6)\}$.

Check $(-6, 3)$.

$$\begin{aligned} x + y &= -3 \\ -6 + 3 &= -3 \ ? \\ -3 &= -3 \end{aligned} \qquad \begin{aligned} x^2 + y^2 &= 45 \\ (-6)^2 + 3^2 &= 45 \ ? \\ 36 + 9 &= 45 \\ 45 &= 45 \end{aligned}$$

Check $(3, -6)$.

$$\begin{aligned} x + y &= -3 \\ 3 + (-6) &= -3 \ ? \\ -3 &= -3 \end{aligned} \qquad \begin{aligned} x^2 + y^2 &= 45 \\ 3^2 + (-6)^2 &= 45 \ ? \\ 9 + 36 &= 45 \\ 45 &= 45 \end{aligned}$$

Both solutions are valid.

5. The system is $\begin{aligned} y &= \log x \\ x^2 - y^2 &= 4 \end{aligned}$

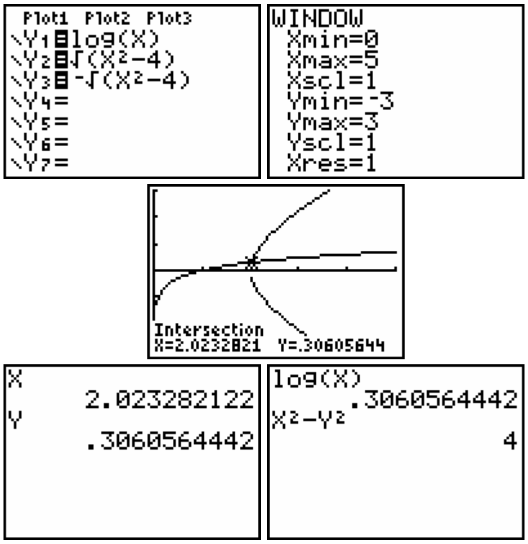
The proposed approximate solution set is $\{(2.0232821, .30605644)\}$.

To show this, we will recreate the solution with the graphing calculator. In order to get $x^2 - y^2 = 4$ to be displayed, solve for y and enter the relation as two functions.

$$x^2 - y^2 = 4 \Rightarrow y^2 = x^2 - 4 \Rightarrow y = \pm\sqrt{x^2 - 4}$$

(*continued on next page*)

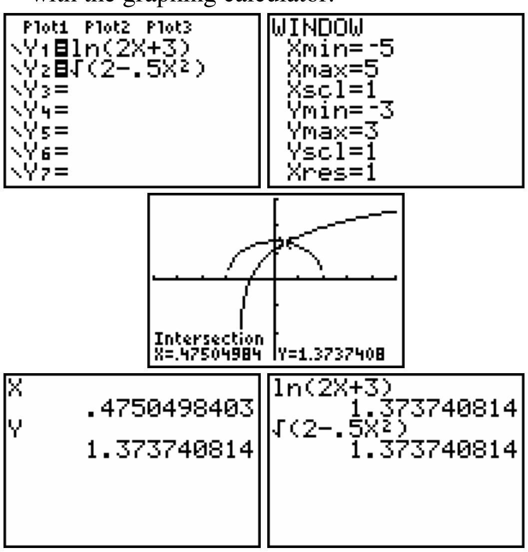
(*continued from page 929*)



6. The system is $\begin{array}{l} y = \ln(2x+3) \\ y = \sqrt{2 - .5x^2} \end{array}$

The proposed approximate solution set is $\{(.47504984, 1.3737408)\}$.

To show this, we will recreate the solution with the graphing calculator.



7. The system $\begin{array}{l} x^2 - y = 4 \\ x + y = -2 \end{array}$ cannot have more than two solutions because a parabola and a line cannot intersect in more than two points.

8. No. complex solutions do not appear as intersection points of the graphs.

In the solutions to Exercises 9–14, we will include both the algebraic solution and graphing calculator solution (like Example 1). For Exercises 15–42, we will include just the algebraic solution. Also, in the solutions to Exercises 9–42, we will omit the checking of solutions. Recall, though, when checking elements of the solution set, substitute the ordered pair(s) into both equations of the system.

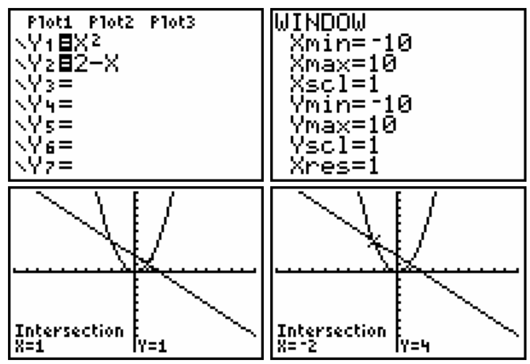
9. $x^2 - y = 0 \quad (1)$
$x + y = 2 \quad (2)$

Algebraic Solution:
Solving equation (2) for y, we have $y = 2 - x$.
Substitute this result into equation (1).

$$2 - x = x^2 \Rightarrow x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0 \Rightarrow x = -2 \text{ or } x = 1$$

If $x = -2$, then $y = 2 - (-2) = 4$. If $x = 1$, then $y = 2 - 1 = 1$.

Graphing Calculator Solution
$x^2 - y = 0 \Rightarrow y = x^2$ and $x + y = 2 \Rightarrow y = 2 - x$



Solution set: $\{(1,1), (-2,4)\}$

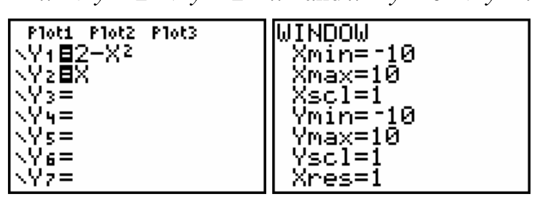
10. $x^2 + y = 2 \quad (1)$
$x - y = 0 \quad (2)$

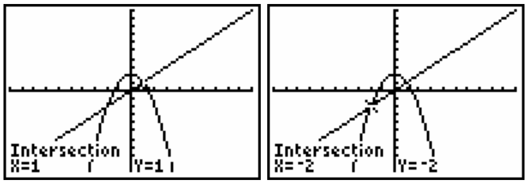
Algebraic Solution:
Solving equation (2) for y, we have $y = x$.
Substitute this result into equation (1).

$$x^2 + x = 2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow$$
$$(x+2)(x-1) = 0 \Rightarrow x = -2 \text{ or } x = 1$$

If $x = -2$, then $y = -2$. If $x = 1$, then $y = 1$.

Graphing Calculator Solution
$x^2 + y = 2 \Rightarrow y = 2 - x^2$ and $x - y = 0 \Rightarrow y = x$





Solution set: $\{(1,1),(-2,-2)\}$

11. $y = x^2 - 2x + 1$ (1)
$x - 3y = -1$ (2)

Algebraic Solution:

Solving equation (2) for *y*, we have $y = \frac{x+1}{3}$.

Substitute this result into equation (1).

$\frac{x+1}{3} = x^2 - 2x + 1 \Rightarrow x + 1 = 3x^2 - 6x + 3 \Rightarrow$

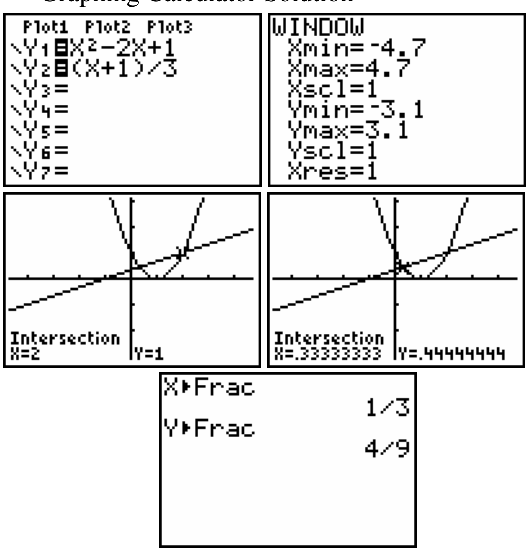
$3x^2 - 7x + 2 = 0 \Rightarrow (3x-1)(x-2) = 0 \Rightarrow$

$x = \frac{1}{3}$ or $x = 2$

If $x = 2$, then $y = \frac{2+1}{3} = 1$. If $x = \frac{1}{3}$, then

$y = \frac{\frac{1}{3}+1}{3} = \frac{1+3}{9} = \frac{4}{9}$.

Graphing Calculator Solution



Solution set: $\left\{(2,1),\left(\frac{1}{3},\frac{4}{9}\right)\right\}$

12. $y = x^2 + 6x + 9$ (1)
$x + 2y = -2$ (2)

Algebraic Solution:

Solving equation (2) for *y*, we have $y = \frac{-x-2}{2}$.

Substitute this result into equation (1).

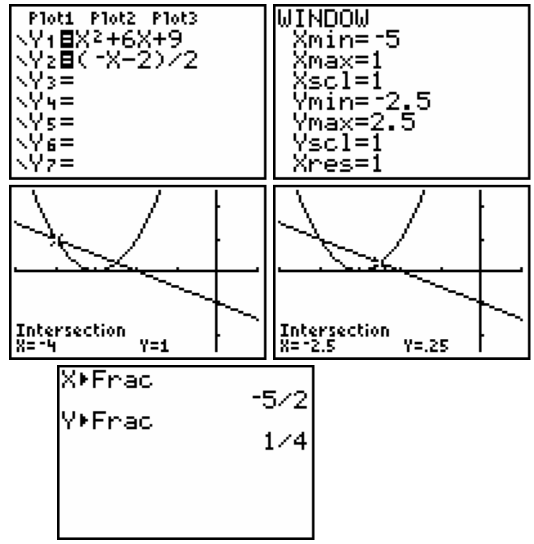
$$\frac{-x-2}{2} = x^2 + 6x + 9$$
$$-x - 2 = 2x^2 + 12x + 18$$
$$0 = 2x^2 + 13x + 20$$

$(x+4)(2x+5) = 0 \Rightarrow x = -4$ or $x = -\frac{5}{2}$

If $x = -4$, then $y = \frac{-(-4)-2}{2} = \frac{4-2}{2} = 1$.

If $x = -\frac{5}{2}$, then $y = \frac{-\left(-\frac{5}{2}\right)-2}{2} = \frac{5-4}{4} = \frac{1}{4}$.

Graphing Calculator Solution



Solution set: $\left\{(-4,1),\left(-\frac{5}{2},\frac{1}{4}\right)\right\}$

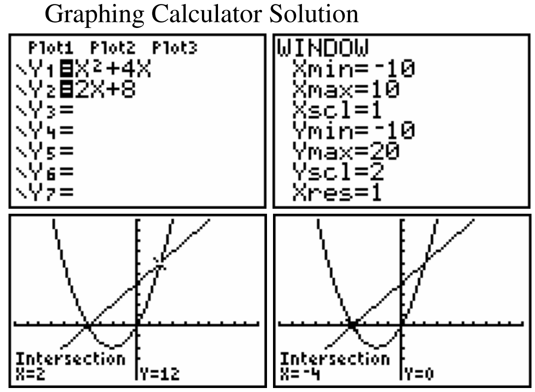
13. $y = x^2 + 4x$ (1)
$2x - y = -8$ (2)

Algebraic Solution:

Solving equation (2) for *y*, we have $y = 2x + 8$. Substitute this result into equation (1): $2x + 8 = x^2 + 4x \Rightarrow x^2 + 2x - 8 = 0 \Rightarrow$ $(x+4)(x-2) = 0 \Rightarrow x = -4$ or $x = 2$

If $x = -4$, then $y = 2(-4) + 8 = 0$. If $x = 2$, then $y = 2(2) + 8 = 12$.

Graphing Calculator Solution



Solution set: $\{(2,12),(-4,0)\}$

14. $y = 6x + x^2$ (1)
$3x - 2y = 10$ (2)

Algebraic Solution:

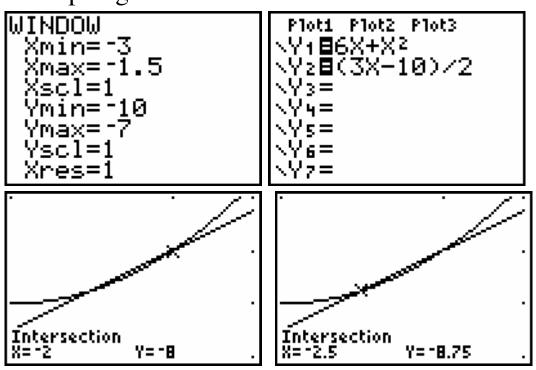
Solving equation (2) for y, we have $y = \frac{3x-10}{2}$.

Substitute this result into equation (1).

$\frac{3x-10}{2} = 6x + x^2 \Rightarrow 3x - 10 = 12x + 2x^2 \Rightarrow$
$0 = 2x^2 + 9x + 10 \Rightarrow (x+2)(2x+5) = 0 \Rightarrow$
$x = -2$ or $x = -\frac{5}{2}$

If $x = -2$, then $y = \frac{3(-2)-10}{2} = \frac{-6-10}{2} = -8$. If $x = -\frac{5}{2}$, then $y = \frac{3\left(-\frac{5}{2}\right)-10}{2} = \frac{3(-5)-20}{4} = -\frac{35}{4}$.

Graphing Calculator Solution



Solution set: $\left\{(-2,-8), \left(-\frac{5}{2}, -\frac{35}{4}\right)\right\}$

15. $3x^2 + 2y^2 = 5$ (1)
$x - y = -2$ (2)

Solving equation (2) for y, we have $y = x + 2$.

Substitute this result into equation (1).

$$3x^2 + 2(x+2)^2 = 5$$
$$3x^2 + 2\left(x^2 + 4x + 4\right) = 5$$
$$3x^2 + 2x^2 + 8x + 8 = 5$$
$$5x^2 + 8x + 3 = 0$$
$$(5x+3)(x+1) = 0 \Rightarrow x = -\tfrac{3}{5} \text{ or } x = -1$$

If $x = -\frac{3}{5}$, then $y = -\frac{3}{5} + 2 = -\frac{3}{5} + \frac{10}{5} = \frac{7}{5}$. If $x = -1$, then $y = -1 + 2 = 1$.

Solution set: $\left\{\left(-\frac{3}{5}, \frac{7}{5}\right), (-1, 1)\right\}$

16. $x^2 + y^2 = 5$ (1)
$-3x + 4y = 2$ (2)

Solving equation (2) for y, we have $y = \frac{3x+2}{4}$.

Substitute this result into equation (1).

$$x^2 + \left(\tfrac{3x+2}{4}\right)^2 = 5 \Rightarrow x^2 + \tfrac{(3x+2)^2}{16} = 5 \Rightarrow$$
$$16x^2 + (3x+2)^2 = 80$$
$$16x^2 + 9x^2 + 12x + 4 = 80$$
$$25x^2 + 12x - 76 = 0$$
$$(x+2)(25x-38) = 0 \Rightarrow x = -2 \text{ or } x = \tfrac{38}{25}$$

If $x = -2$, then $y = \frac{3(-2)+2}{4} = \frac{-6+2}{4} = -1$. If $x = \frac{38}{25}$, then $y = \frac{3\left(\frac{38}{25}\right)+2}{4} = \frac{3(38)+50}{100} = \frac{164}{100} = \frac{41}{25}$.

Solution set: $\left\{(-2,-1), \left(\frac{38}{25}, \frac{41}{25}\right)\right\}$

17. $x^2 + y^2 = 8$ (1)
$x^2 - y^2 = 0$ (2)

Using the elimination method, we add equations (1) and (2).

$$\begin{aligned} x^2 + y^2 &= 8 \\ x^2 - y^2 &= 0 \\ \hline 2x^2 \quad &= 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \end{aligned}$$

Find y by substituting back into equation (2).

If $x = 2$, then $2^2 - y^2 = 0 \Rightarrow 4 - y^2 = 0 \Rightarrow$
$y^2 = 4 \Rightarrow y = \pm 2$. If $x = -2$, then
$(-2)^2 - y^2 = 0 \Rightarrow 4 - y^2 = 0 \Rightarrow y^2 = 4 \Rightarrow$
$y = \pm 2$.

Solution set: $\{(2,2), (2,-2), (-2,2), (-2,-2)\}$

18. $x^2 + y^2 = 10$ (1)
$2x^2 - y^2 = 17$ (2)

Using the elimination method, we add equations (1) and (2).

$$\begin{aligned} x^2 + y^2 &= 10 \\ 2x^2 - y^2 &= 17 \\ \hline 3x^2 \quad &= 27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3 \end{aligned}$$

Find y by substituting back into equation (1).

If $x = 3$, then
$3^2 + y^2 = 10 \Rightarrow 9 + y^2 = 10 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$.
If $x = -3$, then
$(-3)^2 + y^2 = 10 \Rightarrow 9 + y^2 = 10 \Rightarrow y^2 = 1 \Rightarrow$
$y = \pm 1$.

Solution set: $\{(3,1), (3,-1), (-3,1), (-3,-1)\}$

19. $5x^2 - y^2 = 0$ (1)
$3x^2 + 4y^2 = 0$ (2)
Using the elimination method, we multiply equation (1) by 4 and add to equation (2).

$$\begin{array}{l} 20x^2 - 4y^2 = 0 \\ \underline{\ \ 3x^2 + 4y^2 = 0} \\ 23x^2 \qquad\quad = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \end{array}$$

Find y by substituting back into equation (1).
If $x = 0$, then
$5(0)^2 - y^2 = 0 \Rightarrow 0 - y^2 = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0.$
Solution set: $\{(0,0)\}$

20. $x^2 + y^2 = 4$ (1)
$2x^2 - 3y^2 = -12$ (2)
Using the elimination method, we multiply equation (1) by 3 and add to equation (2).

$$\begin{array}{l} 3x^2 + 3y^2 = 12 \\ \underline{2x^2 - 3y^2 = -12} \\ 5x^2 \qquad\quad = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \end{array}$$

Find y by substituting back into equation (1). If $x = 0$, then $0^2 + y^2 = 4 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$
Solution set: $\{(0,2),(0,-2)\}$

21. $3x^2 + y^2 = 3$ (1)
$4x^2 + 5y^2 = 26$ (2)
Using the elimination method, we multiply equation (1) by -5 and add to equation (2).

$$\begin{array}{l} -15x^2 - 5y^2 = -15 \\ \underline{\quad\ 4x^2 + 5y^2 = 26} \\ -11x^2 \qquad\quad = 11 \Rightarrow x^2 = -1 \Rightarrow x = \pm i \end{array}$$

Find y by substituting back into equation (1).
If $x = i$, then $3(i)^2 + y^2 = 3 \Rightarrow$
$3(-1) + y^2 = 3 \Rightarrow -3 + y^2 = 3 \Rightarrow y^2 = 6 \Rightarrow$
$y = \pm\sqrt{6}$. If $x = -i$, then $3(-i)^2 + y^2 = 3 \Rightarrow$
$3(-1) + y^2 = 3 \Rightarrow -3 + y^2 = 3 \Rightarrow y^2 = 6 \Rightarrow$
$y = \pm\sqrt{6}.$
Solution set:
$\{(i,\sqrt{6}),(-i,\sqrt{6}),(i,-\sqrt{6}),(-i,-\sqrt{6})\}$

22. $x^2 + 2y^2 = 9$ (1)
$3x^2 - 4y^2 = 27$ (2)
Using the elimination method, we multiply equation (1) by 2 and add to equation (2).

$$\begin{array}{l} 2x^2 + 4y^2 = 18 \\ \underline{3x^2 - 4y^2 = 27} \\ 5x^2 \qquad\quad = 45 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3 \end{array}$$

Find y by substituting back into equation (1).
If $x = 3$, then $3^2 + 2y^2 = 9 \Rightarrow 9 + 2y^2 = 9 \Rightarrow$
$2y^2 = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0.$ If $x = -3$, then
$(-3)^2 + 2y^2 = 9 \Rightarrow 9 + 2y^2 = 9 \Rightarrow 2y^2 = 0 \Rightarrow$
$y^2 = 0 \Rightarrow y = 0.$
Solution set: $\{(-3,0),(3,0)\}$

23. $2x^2 + 3y^2 = 5$ (1)
$3x^2 - 4y^2 = -1$ (2)
Using the elimination method, we multiply equation (1) by 4 and equation (2) by 3 and then add the resulting equations.

$$\begin{array}{l} 8x^2 + 12y^2 = 20 \\ \underline{9x^2 - 12y^2 = -3} \\ 17x^2 \qquad\quad = 17 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \end{array}$$

Find y by substituting back into equation (1).
If $x = 1$, then $2(1)^2 + 3y^2 = 5 \Rightarrow$
$2 + 3y^2 = 5 \Rightarrow 3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$

If $x = -1$, then $2(-1)^2 + 3y^2 = 5 \Rightarrow$
$2 + 3y^2 = 5 \Rightarrow 3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$
Solution set: $\{(1,-1),(-1,1),(1,1),(-1,-1)\}$

24. $3x^2 + 5y^2 = 17$ (1)
$2x^2 - 3y^2 = 5$ (2)
Using the elimination method, we multiply equation (1) by 3 and equation (2) by 5 and then add the resulting equations.

$$\begin{array}{l} 9x^2 + 15y^2 = 51 \\ \underline{10x^2 - 15y^2 = 25} \\ 19x^2 \qquad\quad = 76 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \end{array}$$

Find y by substituting back into equation (1).
If $x = 2$, then $3(2)^2 + 5y^2 = 17 \Rightarrow$
$12 + 5y^2 = 17 \Rightarrow 5y^2 = 5 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$
If $x = -2$, then $3(-2)^2 + 5y^2 = 17 \Rightarrow$
$12 + 5y^2 = 17 \Rightarrow 5y^2 = 5 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$
Solution set: $\{(2,1),(-2,1),(2,-1),(-2,-1)\}$

25. $2x^2+2y^2=20$ (1)
$4x^2+4y^2=30$ (2)
Using the elimination method, we multiply equation (1) by -2 and add to equation (2).

$$\begin{aligned} -4x^2-4y^2&=-40\\ 4x^2+4y^2&=\ \ 30\\ \hline 0&=-10 \end{aligned}$$

This is a false statement.
Solution set: $\varnothing$

26. $x^2+y^2=4$ (1)
$5x^2+5y^2=28$ (2)
Using the elimination method, we multiply equation (1) by -5 and add to equation (2).

$$\begin{aligned} -5x^2-5y^2&=-20\\ 5x^2+5y^2&=\ \ 28\\ \hline 0&=\ \ 8 \end{aligned}$$

This is a false statement.
Solution set: $\varnothing$

27. $2x^2-3y^2=8$ (1)
$6x^2+5y^2=24$ (2)
Using the elimination method, we multiply equation (1) by -3 and add to equation (2).

$$\begin{aligned} -6x^2+9y^2&=-24\\ 6x^2+5y^2&=\ \ 24\\ \hline 14y^2&=\ \ 0\Rightarrow y^2=0\Rightarrow y=0 \end{aligned}$$

Find x by substituting back into equation (2).
If $y=0$, then $6x^2+5(0)^2=24\Rightarrow$
$6x^2+0=24\Rightarrow x^2=4\Rightarrow x=\pm 2.$
Solution set: $\{(2,0),(-2,0)\}$

28. $5x^2-2y^2=25$ (1)
$10x^2+\ y^2=50$ (2)
Using the elimination method, we multiply equation (2) by 2 and add to equation (1).

$$\begin{aligned} 5x^2-2y^2&=25\\ 20x^2+2y^2&=100\\ \hline 25x^2\qquad&=125\Rightarrow x^2=5\Rightarrow x=\pm\sqrt{5} \end{aligned}$$

Find y by substituting back into equation (2).
If $x=\sqrt{5}$, then $10(\sqrt{5})^2+y^2=50\Rightarrow$
$50+y^2=50\Rightarrow y^2=0\Rightarrow y=0.$
If $x=-\sqrt{5}$, then
$10(-\sqrt{5})^2+y^2=50\Rightarrow 50+y^2=50\Rightarrow$
$y^2=0\Rightarrow y=0.$
Solution set: $\{(-\sqrt{5},0),(\sqrt{5},0)\}$

29. $xy=-15$ (1)
$4x+3y=3$ (2)
Solving equation (1) for y, we have $y=-\frac{15}{x}$.
Substitute this result into equation (2).
$4x+3\left(-\frac{15}{x}\right)=3\Rightarrow 4x^2-45=3x\Rightarrow$
$4x^2-3x-45=0\Rightarrow(x+3)(4x-15)=0\Rightarrow$
$x=-3$ or $x=\frac{15}{4}$
If $x=-3$, then $y=-\frac{15}{-3}=5$. If $x=\frac{15}{4}$, then
$y=-\frac{15}{\frac{15}{4}}=-4.$
Solution set: $\left\{(-3,5),\left(\frac{15}{4},-4\right)\right\}$

30. $xy=8$ (1)
$3x+2y=-16$ (2)
Solving equation (1) for y, we have $y=\frac{8}{x}$.
Substitute this result into equation (2).
$3x+2\left(\frac{8}{x}\right)=-16\Rightarrow 3x^2+16=-16x\Rightarrow$
$3x^2+16x+16=0\Rightarrow(x+4)(3x+4)=0\Rightarrow$
$x=-4$ or $x=-\frac{4}{3}$
If $x=-4$, then $y=\frac{8}{-4}=-2$. If $x=-\frac{4}{3}$, then
$y=\frac{8}{-\frac{4}{3}}=-6.$
Solution set: $\left\{(-4,-2),\left(-\frac{4}{3},-6\right)\right\}$

31. $2xy+1=0$ (1)
$x+16y=2$ (2)
Solving equation (2) for x, we have $x=2-16y$.
Substitute this result into equation (1).
$2(-16y+2)y+1=0\Rightarrow-32y^2+4y+1=0\Rightarrow$
$32y^2-4y-1=0\Rightarrow(8y+1)(4y-1)=0\Rightarrow$
$y=-\frac{1}{8}$ or $y=\frac{1}{4}$
If $y=-\frac{1}{8}$, then $x=2-16\left(-\frac{1}{8}\right)=2+2=4$. If $y=\frac{1}{4}$, then $x=2-16\left(\frac{1}{4}\right)=2-4=-2.$
Solution set: $\left\{\left(4,-\frac{1}{8}\right),\left(-2,\frac{1}{4}\right)\right\}$

32. $-5xy+2=0\,(1)$
$x-15y=5\,(2)$
Solving equation (2) for *x*, we have $x=5+15y$.
Substitute this result into equation (1).
$-5(5+15y)y+2=0$
$-75y^2-25y+2=0 \Rightarrow 75y^2+25y-2=0 \Rightarrow$
$(15y-1)(5y+2)=0 \Rightarrow y=\frac{1}{15}$ or $y=-\frac{2}{5}$
If $y=\frac{1}{15}$, then $x=5+15\left(\frac{1}{15}\right)=5+1=6$. If $y=-\frac{2}{5}$, then $x=5+15\left(-\frac{2}{5}\right)=5+(-6)=-1$.
Solution set: $\left\{\left(6,\frac{1}{15}\right),\left(-1,-\frac{2}{5}\right)\right\}$

33. $x^2+4y^2=25\,(1)$
$xy=6 \quad (2)$
Solving equation (2) for *x*, we have $x=\frac{6}{y}$.
Substitute this result into equation (1).
$\left(\frac{6}{y}\right)^2+4y^2=25 \Rightarrow \frac{36}{y^2}+4y^2=25 \Rightarrow$
$36+4y^4=25y^2$
$4y^4-25y^2+36=0 \Rightarrow (y^2-4)(4y^2-9)=0$
$y^2=4 \Rightarrow y=\pm 2$ or $y^2=\frac{9}{4} \Rightarrow y=\pm\frac{3}{2}$
If $y=2$, then $x=\frac{6}{2}=3$. If $y=-2$, then $x=\frac{6}{-2}=-3$. If $y=\frac{3}{2}$, then $x=\frac{6}{\frac{3}{2}}=4$. If $y=-\frac{3}{2}$, then $x=\frac{6}{-\frac{3}{2}}=-4$.
Solution set:
$\left\{(3,2),(-3,-2),\left(4,\frac{3}{2}\right),\left(-4,-\frac{3}{2}\right)\right\}$

34. $5x^2-2y^2=6\;(1)$
$xy=2 \quad (2)$
Solving equation (2) for *y*, we have $y=\frac{2}{x}$.
Substitute this result into equation (1).
$5x^2-2\left(\frac{2}{x}\right)^2=6 \Rightarrow 5x^2-\frac{8}{x^2}=6$
$5x^4-8=6x^2$
$5x^4-6x^2-8=0 \Rightarrow (5x^2+4)(x^2-2)=0$
$x^2=-\frac{4}{5}$ or $x^2=2$
$x=\pm\frac{2}{\sqrt{5}}i=\pm\frac{2\sqrt{5}}{5}i \qquad x=\pm\sqrt{2}$
If $x=\frac{2\sqrt{5}}{5}i$, then $y=\frac{2}{\frac{2\sqrt{5}}{5}i}=\frac{5}{i\sqrt{5}}=\frac{\sqrt{5}}{i}=-i\sqrt{5}$.
If $x=-\frac{2\sqrt{5}}{5}i$, then
$y=\frac{2}{-\frac{2\sqrt{5}}{5}i}=-\frac{5}{i\sqrt{5}}=-\frac{\sqrt{5}}{i}=i\sqrt{5}$.
If $x=\sqrt{2}$, then $y=\frac{2}{\sqrt{2}}=\sqrt{2}$. If $x=-\sqrt{2}$, then $y=\frac{2}{-\sqrt{2}}=-\sqrt{2}$.
Solution set:
$\left\{\left(\frac{2\sqrt{5}}{5}i,-i\sqrt{5}\right),\left(-\frac{2\sqrt{5}}{5}i,i\sqrt{5}\right),\right.$
$\left.\left(\sqrt{2},\sqrt{2}\right),\left(-\sqrt{2},-\sqrt{2}\right)\right\}$

35. $x^2-xy+y^2=5 \quad (1)$
$2x^2+xy-y^2=10\,(2)$
Using the elimination method, we add equations (1) and (2).
$$\begin{array}{l} x^2-xy+y^2=5 \\ \underline{2x^2+xy-y^2=10} \\ 3x^2 \qquad\qquad =15 \Rightarrow x^2=5 \Rightarrow x=\pm\sqrt{5} \end{array}$$
Find *y* by substituting back into equation (1).
If $x=\sqrt{5}$, then
$\left(\sqrt{5}\right)^2-\sqrt{5}y+y^2=5 \Rightarrow 5-\sqrt{5}y+y^2=5 \Rightarrow$
$y^2-\sqrt{5}y=0 \Rightarrow y\left(y-\sqrt{5}\right)=0$
Thus, we have $y=0$ or $y=\sqrt{5}$. If $x=-\sqrt{5}$, then
$\left(-\sqrt{5}\right)^2-\left(-\sqrt{5}\right)y+y^2=5$
$5+\sqrt{5}y+y^2=5 \Rightarrow y^2+\sqrt{5}y=0 \Rightarrow$
$y\left(y+\sqrt{5}\right)=0 \Rightarrow y=0$ or $y=-\sqrt{5}$
Thus, we have $y=0$ or $y=-\sqrt{5}$.
Solution set:
$\left\{\left(\sqrt{5},0\right),\left(-\sqrt{5},0\right),\left(\sqrt{5},\sqrt{5}\right),\left(-\sqrt{5},-\sqrt{5}\right)\right\}$

36. $3x^2+xy+3y^2=7\;(1)$
$x^2+y^2=2\,(2)$
This system can be solved using a combination of the elimination and substitution methods. Using the elimination method, multiply equation (2) by -3 and add to equation (1).
$$\begin{array}{l} 3x^2+xy+3y^2=\ \ 7 \\ \underline{-3x^2 \qquad -3y^2=-6} \\ \qquad\quad xy \qquad\ \ = \ \ 1\ (3) \end{array}$$
Solve equation (3) for *y*. $xy=1 \Rightarrow y=\frac{1}{x}$
Find *x* by substituting back into equation (2).
$x^2+\left(\frac{1}{x}\right)^2=2 \Rightarrow x^2+\frac{1}{x^2}=2 \Rightarrow$
$x^4+1=2x^2 \Rightarrow x^4-2x^2+1=0 \Rightarrow$
$\left(x^2-1\right)^2=0 \Rightarrow x^2-1=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

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(*continued from page 935*)

If $x = 1,$, then $y = \frac{1}{1} = 1$. If $x = -1,$, then $y = \frac{1}{-1} = -1$.

Solution set: $\{(1,1),(-1,-1)\}$

37. $x^2 + 2xy - y^2 = 14 \quad (1)$
$x^2 - y^2 = -16 \quad (2)$

This system can be solved using a combination of the elimination and substitution methods. Using the elimination method, multiply equation (2) by -1 and add to equation (1).

$$\begin{aligned} x^2 + 2xy - y^2 &= 14 \\ -x^2 \qquad + y^2 &= 16 \\ \hline 2xy \qquad &= 30 \Rightarrow xy = 15\,(3) \end{aligned}$$

Solve equation (3) for *y*.

$xy = 15 \Rightarrow y = \frac{15}{x}\ (4)$

Find *x* by substituting equation (4) into equation (2).

$$x^2 - \left(\frac{15}{x}\right)^2 = -16 \Rightarrow x^2 - \frac{225}{x^2} = -16 \Rightarrow$$
$$x^4 - 225 = -16x^2 \Rightarrow$$
$$x^4 + 16x^2 - 225 = 0 \Rightarrow \left(x^2 - 9\right)\left(x^2 + 25\right) = 0$$

$x^2 = 9$ or $x^2 = -25$
$x = \pm 3$ $\quad x = \pm 5i$

If $x = 3$, then $y = \frac{15}{3} = 5$. If $x = -3$, then $y = \frac{15}{-3} = -5$. If $x = 5i$, then $y = \frac{15}{5i} = \frac{3}{i} = -3i$. If $x = -5i$, then $y = \frac{15}{-5i} = -\frac{3}{i} = 3i$.

Solution set: $\{(3,5),(-3,-5),(5i,-3i),(-5i,3i)\}$

38. $3x^2 + 2xy - y^2 = 9\ (1)$
$x^2 - xy + y^2 = 9\ (2)$

This system can be solved using a combination of the addition and substitution methods. Using the elimination method, we add equations (1) and (2).

$$\begin{aligned} 3x^2 + 2xy - y^2 &= 9 \\ x^2 - \ xy + y^2 &= 9 \\ \hline 4x^2 + \ xy \qquad &= 18\ (3) \end{aligned}$$

Solving equation (3) for *y* we have

$4x^2 + xy = 18 \Rightarrow xy = 18 - 4x^2 \Rightarrow y = \frac{18-4x^2}{x}$.

Find *x* by substituting equation (4) into equation (2).

$$x^2 - x\left(\frac{18-4x^2}{x}\right) + \left(\frac{18-4x^2}{x}\right)^2 = 9$$
$$x^2 - 18 + 4x^2 + \frac{\left(18-4x^2\right)^2}{x^2} = 9$$
$$x^4 - 18x^2 + 4x^4 + \left(18 - 4x^2\right)^2 = 9x^2$$
$$x^4 - 18x^2 + 4x^4 + 324 - 144x^2 + 16x^4 = 9x^2$$
$$21x^4 - 171x^2 + 324 = 0$$
$$7x^4 - 57x^2 + 108 = 0$$
$$(7x^2 - 36)(x^2 - 3) = 0$$

$x^2 = \frac{36}{7}$ or $x^2 = 3$
$x = \pm\frac{6}{\sqrt{7}} = \pm\frac{6\sqrt{7}}{7}$ $\quad x = \pm\sqrt{3}$

If $x = \frac{6\sqrt{7}}{7}$, then

$$y = \frac{18-4\left(\frac{6\sqrt{7}}{7}\right)^2}{\frac{6\sqrt{7}}{7}} = \left[18 - 4\left(\frac{36\cdot 7}{49}\right)\right]\cdot\frac{7}{6\sqrt{7}}$$
$$= \left[18 - 4\left(\frac{36}{7}\right)\right]\cdot\frac{7}{6\sqrt{7}} = \left(\frac{126}{7} - \frac{144}{7}\right)\cdot\frac{\sqrt{7}}{6}$$
$$= \frac{-18}{7}\cdot\frac{\sqrt{7}}{6} = -\frac{3\sqrt{7}}{7}$$

If $x = -\frac{6\sqrt{7}}{7}$, then

$$y = \frac{18-4\left(-\frac{6\sqrt{7}}{7}\right)^2}{-\frac{6\sqrt{7}}{7}} = \left[18 - 4\left(\frac{36\cdot 7}{49}\right)\right]\cdot\frac{-7}{6\sqrt{7}}$$
$$= \left[18 - 4\left(\frac{36}{7}\right)\right]\cdot\frac{-7}{6\sqrt{7}} = \left(\frac{126}{7} - \frac{144}{7}\right)\cdot\frac{-\sqrt{7}}{6}$$
$$= \frac{-18}{7}\cdot\frac{-\sqrt{7}}{6} = \frac{3\sqrt{7}}{7}$$

If $x = \sqrt{3}$, then

$$y = \frac{18-4\left(\sqrt{3}\right)^2}{\sqrt{3}} = \frac{18-4(3)}{\sqrt{3}} = \frac{18-12}{\sqrt{3}} = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

If $x = -\sqrt{3}$, then

$$y = \frac{18-4\left(-\sqrt{3}\right)^2}{-\sqrt{3}} = \frac{18-4(3)}{-\sqrt{3}} = \frac{18-12}{-\sqrt{3}} = \frac{6}{-\sqrt{3}}$$
$$= -\frac{6\sqrt{3}}{3} = -2\sqrt{3}$$

Solution set: $\left\{\left(\sqrt{3}, 2\sqrt{3}\right),\left(-\sqrt{3}, -2\sqrt{3}\right),\left(\frac{6\sqrt{7}}{7}, -\frac{3\sqrt{7}}{7}\right),\left(-\frac{6\sqrt{7}}{7}, \frac{3\sqrt{7}}{7}\right)\right\}$

39. $x = |y|$ (1)
$x^2 + y^2 = 18$ (2)

If $x = |y|$, then $x^2 = y^2$ since $|y|^2 = y^2$.

Substitute $x^2 = y^2$ into equation (2).

$y^2 + y^2 = 18 \Rightarrow 2y^2 = 18 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

If $y = 3$, then $x = |y| = |3| = 3$. If $y = -3$, then $x = |y| = |-3| = 3$.

Solution set: $\{(3,-3),(3,3)\}$

40. $2x + |y| = 4$ (1)
$x^2 + y^2 = 5$ (2)

Solve equation (1) for $|y|$ we have $|y| = 4 - 2x$. Since $|y|^2 = y^2$, we substitute $|y|^2 = y^2 = (4-2x)^2$ into equation (2) and solve for x.

$$x^2 + (4-2x)^2 = 5$$
$$x^2 + 16 - 16x + 4x^2 = 5 \Rightarrow$$
$$5x^2 - 16x + 11 = 0 \Rightarrow (5x - 11)(x - 1) = 0$$

$x = \frac{11}{5}$ or $x = 1$

If $x = \frac{11}{5}$, then

$|y| = 4 - 2\left(\frac{11}{5}\right) \Rightarrow |y| = \frac{20}{5} - \frac{22}{5} \Rightarrow |y| = -\frac{2}{5}$.

Since the absolute value of a number never results in a negative value, there is no corresponding value of y for $x = \frac{11}{5}$.

If $x = 1$, then $|y| = 4 - 2(1) \Rightarrow$
$|y| = 4 - 2 \Rightarrow |y| = 2 \Rightarrow y = \pm 2$.

Solution set: $\{(1,2),(1,-2)\}$

41. $2x^2 - y^2 = 4$ (1)
$|x| = |y|$ (2)

Since $|x|^2 = x^2$ and $|y|^2 = y^2$, we substitute $y^2 = x^2$ into equation (1) and solve for x.

$2x^2 - x^2 = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

By substituting these values back into equation (2) we have the following.

If $x = 2$, then $|2| = |y| \Rightarrow 2 = |y| \Rightarrow y = \pm 2$. If $x = -2$, then $|-2| = |y| \Rightarrow 2 = |y| \Rightarrow y = \pm 2$.

Solution set: $\{(2,2),(-2,-2),(2,-2),(-2,2)\}$

42. $x^2 + y^2 = 9$ (1)
$|x| = |y|$ (2)

Since $|x|^2 = x^2$ and $|y|^2 = y^2$, we substitute $y^2 = x^2$ into equation (1) and solve for x.

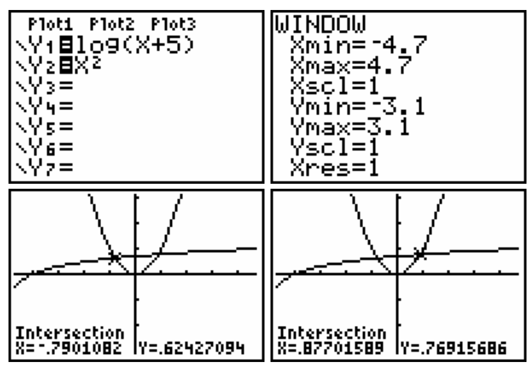
$x^2 + x^2 = 9 \Rightarrow 2x^2 = 9 \Rightarrow x^2 = \frac{9}{2} \Rightarrow$
$x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$

By substituting these values back into equation (2) we have the following.

If $x = \frac{3\sqrt{2}}{2}$, then $\left|\frac{3\sqrt{2}}{2}\right| = |y| \Rightarrow \frac{3\sqrt{2}}{2} = |y| \Rightarrow y = \pm\frac{3\sqrt{2}}{2}$. If $x = -\frac{3\sqrt{2}}{2}$, then $\left|-\frac{3\sqrt{2}}{2}\right| = |y| \Rightarrow \frac{3\sqrt{2}}{2} = |y| \Rightarrow y = \pm\frac{3\sqrt{2}}{2}$.

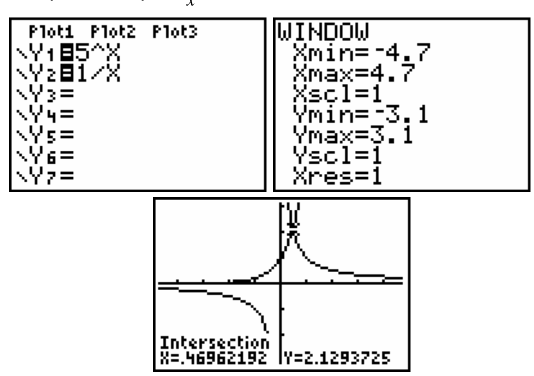
Solution set: $\left\{\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right), \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right), \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)\right\}$

43. $y = \log(x+5)$
$y = x^2$



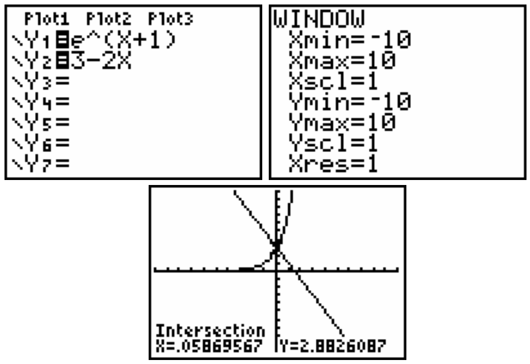
Solution set: $\{(-.79,.62),(.88,.77)\}$

44. $y = 5^x$
$xy = 1 \Rightarrow y = \frac{1}{x}$



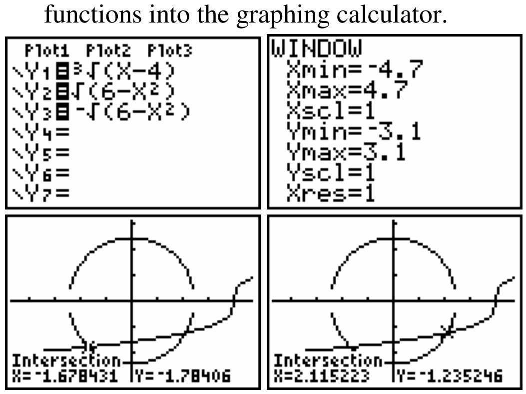
Solution set: $\{(.47, 2.13)\}$

45. $y = e^{x+1}$
$2x + y = 3 \Rightarrow y = 3 - 2x$



Solution set: $\{(.06, 2.88)\}$

46. $y = \sqrt[3]{x-4}$
$x^2 + y^2 = 6$
Since
$x^2 + y^2 = 6 \Rightarrow y^2 = 6 - x^2 \Rightarrow y = \pm\sqrt{6 - x^2}$,
we need to enter the second relation as two functions into the graphing calculator.



Solution set: $\{(-1.68, -1.78), (2.12, -1.24)\}$

47. Let x and y represent the numbers.
We obtain the following system.
$x + y = 17$ (1)
$xy = 42$ (2)
Solving equation (1) for y we have $y = 17 - x$. (3). Substituting this into equation (2) we have $x(17 - x) = 42$. Solving this equation for x we have the following.

$$x(17 - x) = 42 \Rightarrow 17x - x^2 = 42 \Rightarrow$$
$$0 = x^2 - 17x + 42$$
$$(x - 3)(x - 14) = 0 \Rightarrow x = 3 \text{ or } x = 14$$

Using equation (3), if $x = 3$ then
$3 + y = 17 \Rightarrow y = 14$. If $x = 14$, then
$14 + y = 17 \Rightarrow y = 3$.
The two numbers are 14 and 3.

48. Let x and y represent the numbers.
We obtain the following system.
$x + y = 10$ (1)
$x^2 - y^2 = 20$ (2)
Solving equation (1) for y we have $y = 10 - x$. (3). Substituting this into equation (2) we have $x^2 - (10 - x)^2 = 20$. Solving this equation for x we have the following.

$$x^2 - (10 - x)^2 = 20$$
$$x^2 - (100 - 20x + x^2) = 20$$
$$x^2 - 100 + 20x - x^2 = 20$$
$$-100 + 20x = 20$$
$$20x = 120 \Rightarrow x = 6$$

Using equation (3), if $x = 6$ then
$y = 10 - 6 = 4$.
The two numbers are 6 and 4.

49. Let x and y represent the numbers.
We obtain the following system.
$x^2 + y^2 = 100$ (1)
$x^2 - y^2 = 28$ (2)
Adding equations (1) and (2) in order to eliminate y^2, we have $2x^2 = 128$.
Solving this equation for x we have
$2x^2 = 128 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$
Using equation (1), if $x = 8$ then
$8^2 + y^2 = 100 \Rightarrow 64 + y^2 = 100 \Rightarrow$
$y^2 = 36 \Rightarrow y = \pm 6$
If $x = -8$ then
$(-8)^2 + y^2 = 100 \Rightarrow 64 + y^2 = 100$
$y^2 = 36 \Rightarrow y = \pm 6$
The two numbers are 8 and 6, or 8 and −6, or −8 and 6, or −8 and −6.

50. Let x and y represent the numbers.
We obtain the following system.
$x^2 + y^2 = 194$ (1)
$x^2 - y^2 = 144$ (2)
Adding equations (1) and (2) in order to eliminate y^2, we have $2x^2 = 338$. Solving this equation for x we have
$2x^2 = 338 \Rightarrow x^2 = 169 \Rightarrow x = \pm 13$
Using equation (1), if $x = 13$ then
$13^2 + y^2 = 194 \Rightarrow 169 + y^2 = 194 \Rightarrow$
$y^2 = 25 \Rightarrow y = \pm 5$

If $x=-13$ then $(-13)^2+y^2=194\Rightarrow$
$169+y^2=194\Rightarrow y^2=25\Rightarrow y=\pm5.$
The two numbers are 5 and 13, 5 and −13 or −5 and 13, or −5 and −13.

51. Let x and y represent the numbers.
We obtain the following system.
$\frac{x}{y}=\frac{9}{2}$ (1)
$xy=162$ (2)
Solve equation (1) for x.
$y\left(\frac{x}{y}\right)=y\left(\frac{9}{2}\right)\Rightarrow x=\frac{9}{2}y$
Substitute $\frac{9}{2}y$ for x in equation (2) and solve for y.
$\left(\frac{9}{2}y\right)y=162\Rightarrow \frac{9}{2}y^2=162\Rightarrow$
$\frac{2}{9}\left(\frac{9}{2}y^2\right)=\frac{2}{9}(162)\Rightarrow y^2=36\Rightarrow y=\pm6$
If $y=6$, $x=\frac{9}{2}(6)=27$. If $y=-6$,
$x=\frac{9}{2}(-6)=-27.$
The two numbers are either 27 and 6, or –27 and –6.

52. Let x and y represent the two numbers.
We obtain the following system.
$\frac{x}{y}=\frac{4}{3}$ (1)
$x^2+y^2=100$ (2)
Solve equation (1) for x.
$y\left(\frac{x}{y}\right)=y\left(\frac{4}{3}\right)\Rightarrow x=\frac{4}{3}y$
Substitute $\frac{4}{3}y$ for x in equation (2) and solve for y.
$\left(\frac{4}{3}y\right)^2+y^2=100\Rightarrow \frac{16}{9}y^2+y^2=100\Rightarrow$
$\frac{25}{9}y^2=100\Rightarrow \frac{9}{25}\left(\frac{25}{9}y^2\right)=\frac{9}{25}(100)\Rightarrow$
$y^2=36\Rightarrow y=\pm6$
If $y=6$, $x=\frac{4}{3}(6)=8$. If $y=-6$,
$x=\frac{4}{3}(-6)=-8.$.
The two numbers are either 8 and 6, or –8 and –6.

53. Let x = the length of the second side and y = the length of the third side.
We obtain the following system.
$x^2+y^2=13^2=169$ (1)
$x=y+7$ (2)
Substitute $y+7$ for x in equation (1) and solve for y.
$$(y+7)^2+y^2=169$$
$$y^2+14y+49+y^2=169$$
$$2y^2+14y-120=0$$
$$y^2+7y-60=0$$
$$(y+12)(y-5)=0\Rightarrow y=-12 \text{ or } y=5$$
We disregard the negative solution since a length cannot be negative. If y = 5, then $x=5+7=12.$
The lengths of the two shorter sides are 5 m and 12 m.

54. Let x = the length of the second side and y = the length of the third side.
We obtain the following system.
$x^2+y^2=29^2=841$ (1)
$x=y+1$ (2)
Substitute $y+1$ for x in equation (1) and solve for y.
$$(y+1)^2+y^2=841$$
$$y^2+2y+1+y^2=841$$
$$2y^2+2y-840=0\Rightarrow y^2+y-420=0$$
$$(y+21)(y-20)=0\Rightarrow y=-21 \text{ or } y=20$$
We disregard the negative solution since a length cannot be negative. If y = 20, then $x=20+1=21.$
The lengths of the two shorter sides are 20 ft and 21 ft.

55. If the system formed by the following equations has a solution, the line and the circle intersect.
$3x-2y=9$ (1)
$x^2+y^2=25$ (2)
Solving equation (1) for x we have
$3x=9+2y\Rightarrow x=\frac{9+2y}{3}.$ (3)
Substituting into equation (3) into equation (2) we have $\left(\frac{9+2y}{3}\right)^2+y^2=25.$
Solving this equation for y we have the following.
$$\left(\frac{9+2y}{3}\right)^2+y^2=25\Rightarrow \frac{(9+2y)^2}{9}+y^2=25$$
$$(9+2y)^2+9y^2=225$$
$$81+36y+4y^2+9y^2=225$$
$$13y^2+36y+81=225$$
$$13y^2+36y-144=0$$
Using the quadratic formula where $a=13, b=36,$ and $c=-144$, we have the following.

(*continued on next page*)

(*continued from page 939*)

$$y = \frac{-36 \pm \sqrt{(36)^2 - 4(13)(-144)}}{2(13)}$$
$$= \frac{-36 \pm \sqrt{1296 + 7488}}{26} = \frac{-36 \pm \sqrt{8784}}{26}$$

$y = \frac{-36-\sqrt{8784}}{26} \approx -4.989$ and

$y = \frac{-36+\sqrt{8784}}{26} \approx 2.220$

If $y = \frac{-36+\sqrt{8784}}{26}$, then

$$x = \frac{9+2\left(\frac{-36+\sqrt{8784}}{26}\right)}{3} = \frac{234+2\left(-36+\sqrt{8784}\right)}{78}$$
$$= \frac{162+2\sqrt{8784}}{78} \approx 4.480$$

If $y = \frac{-36-\sqrt{8784}}{26}$, then

$$x = \frac{9+2\left(\frac{-36-\sqrt{8784}}{26}\right)}{3} = \frac{234+2\left(-36-\sqrt{8784}\right)}{78}$$
$$= \frac{162-2\sqrt{8784}}{78} \approx -.326$$

Thus, the circle and line do intersect, in fact twice, at approximately (4.48, 2.22) and (–.326, –4.99).

56. $x + 2y = b$ (1)
$x^2 + y^2 = 9$ (2)

Solving equation (1) for x we have $x = b - 2y$. Substitute $b - 2y$ for x in equation (2) to obtain $(b-2y)^2 + y^2 = 9.$

Expanding and collecting like terms (in terms of y), we have the following.

$$(b-2y)^2 + y^2 = 9$$
$$b^2 - 4by + 4y^2 + y^2 = 9$$
$$b^2 - 4by + 5y^2 - 9 = 0$$
$$5y^2 - 4by + (b^2 - 9) = 0$$

This is a quadratic equation in terms of y and will have a unique solution when the discriminant is 0.

$$(-4b)^2 - 4(5)(b^2 - 9) = 0$$
$$16b^2 - 20b^2 + 180 = 0 \Rightarrow -4b^2 = -180$$
$$b^2 = 45 \Rightarrow b = \pm 3\sqrt{5}$$

Thus, the line $x + 2y = b$ will touch the circle $x^2 + y^2 = 9$ in only one point if $b = \pm 3\sqrt{5}$.

57. We must first find the solution to the following system.

$y = x^2$ (1)
$x^2 + y^2 = 90$ (2)

Substitute y for x^2 in equation (2) and solve for y.

$$y + y^2 = 90 \Rightarrow y^2 + y - 90 = 0$$
$$(y+10)(y-9) = 0 \Rightarrow y = -10 \text{ or } y = 9$$

If $y = -10$, then $-10 = x^2 \Rightarrow x = \pm i\sqrt{10}.$ Since we are seeking points of intersection, we reject these nonreal solutions. If $y = 9$, then $9 = x^2 \Rightarrow x = \pm 3.$ The two points of intersection of the two graphs are (3, 9) and (–3, 9). The slope of the line through these two points is $m = \frac{9-9}{-3-3} = 0,$ so this is the horizontal line with equation $y = 9.$

58. Answers will vary.

59. Let x represent the length and width of the square base, and let y represent the height. Using the formula for the volume of a box, $V = LWH$, we have the equation $x^2 y = 360.$ (1). Since the height is 4 ft greater than both the length and width we have the equation $y = x + 4.$ (2) Since the surface area consists of a square base and four rectangular sides we have the equation $x^2 + 4xy = 276.$ (3) Substituting equation (2) into equation (3) we have $x^2 + 4x(x+4) = 276.$ Solving this equation for x we have the following.

$$x^2 + 4x(x+4) = 276 \Rightarrow x^2 + 4x^2 + 16x = 276$$
$$5x^2 + 16x = 276 \Rightarrow 5x^2 + 16x - 276 = 0$$
$$(5x+46)(x-6) = 0 \Rightarrow x = -\tfrac{46}{5} \text{ or } x = 6$$

We disregard the negative solution since a length (or width) cannot be negative. If $x = 6$, then $y = 6 + 4 = 10.$ Thus, the length, which is equal to the width, is 6 ft and the height is 10 ft. Note: Since $6^2(10) = 360,$ the conditions for the volume are satisfied.

60. Let r represent the radius and let h represent the height.
Using the formula for the volume of a cylinder, $V = \pi r^2 h,$ we have the equation $\pi r^2 h = 50.$ (1) Since the lateral area is represented by a rectangle, the formula for the lateral area is $A = 2\pi rh.$ Thus, we have the equation $2\pi rh = 65.$ (2) Solving equation (2) for h, we have $h = \frac{65}{2\pi r}.$ (3) Substituting equation (3) into equation (1), we have $\pi r^2\left(\frac{65}{2\pi r}\right) = 50.$ Solving this equation for r we have the following.

$$\pi r^2\left(\frac{65}{2\pi r}\right) = 50 \Rightarrow \frac{65}{2}r = 50 \Rightarrow 65r = 100 \Rightarrow r = \frac{20}{13} \approx 1.538$$

If $r = \frac{20}{13},$ then $\frac{65}{2\pi\left(\frac{20}{13}\right)} = \frac{65\cdot 13}{40\pi} = \frac{845}{40\pi} \approx 6.724.$

Thus, the radius is approximately 1.538 in. and the height is approximately 6.724 in.

61. The system is $\begin{aligned} x^2 y &= 75 \\ x^2 + 4xy &= 85 \end{aligned}$

The first possible solution is $x = 5$ and $y = 3.$

$$\begin{aligned} x^2 y &= 75 & x^2 + 4xy &= 85 \\ 5^2\cdot 3 &= 75 \ ? & 5^2 + 4\cdot 5\cdot 3 &= 85\ ? \\ 25\cdot 3 &= 75 & 25 + 60 &= 85 \\ 75 &= 75 \text{ True} & 85 &= 85 \text{ True} \end{aligned}$$

Thus, the solution length = width = 5 in. and height = 3 in. is valid.
Ideally we should check the exact values in the system in order to determine if the approximate dimensions are valid. From Example 6 we have

$$x = \frac{-5+\sqrt{5^2-4(1)(-60)}}{2(1)} = \frac{-5+\sqrt{25+240}}{2} = \frac{-5+\sqrt{265}}{2}.$$

The corresponding y-value would be

$$y = \frac{75}{x^2} = \frac{75}{\left(\frac{-5+\sqrt{265}}{2}\right)^2} = \frac{75}{\frac{\left(-5+\sqrt{265}\right)^2}{4}} = \frac{300}{\left(-5+\sqrt{265}\right)^2}.$$

The second possible solution is $x = \frac{-5+\sqrt{265}}{2}$ and $y = \frac{300}{\left(-5+\sqrt{265}\right)^2}.$

$$\begin{aligned} x^2 y &= 75 \\ \left(\frac{-5+\sqrt{265}}{2}\right)^2\cdot\frac{300}{\left(-5+\sqrt{265}\right)^2} &= 75 \ ? \\ \frac{\left(-5+\sqrt{265}\right)^2}{4}\cdot\frac{300}{\left(-5+\sqrt{265}\right)^2} &= 75 \\ 75 &= 75 \text{ True} \end{aligned}$$

$$\begin{aligned} x^2 + 4xy &= 85 \\ \frac{\left(-5+\sqrt{265}\right)^2}{4} + \frac{600}{-5+\sqrt{265}} &= 85\ ? \\ \frac{\left(-5+\sqrt{265}\right)^3 + 2400}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{\left(-5+\sqrt{265}\right)^2\left(-5+\sqrt{265}\right)+2400}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{\left(25-10\sqrt{265}+265\right)\left(-5+\sqrt{265}\right)+2400}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{\left(290-10\sqrt{265}\right)\left(-5+\sqrt{265}\right)+2400}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{\left(-1450+340\sqrt{265}-2650\right)+2400}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{-4100+340\sqrt{265}+2400}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{-1700+340\sqrt{265}}{4\left(-5+\sqrt{265}\right)} &= 85 \\ \frac{340\left(-5+\sqrt{265}\right)}{4\left(-5+\sqrt{265}\right)} &= 85 \\ 85 &= 85 \text{ True} \end{aligned}$$

62. supply: $p = \frac{2000}{2000-q}$
demand: $p = \frac{7000-3q}{2q}$

(a) Equilibrium occurs when supply equals demand, so solve the system formed by the supply and demand equations. This system can be solved by substitution. Substitute $\frac{2000}{2000-q}$ for p in the demand equation and solve the resulting equation for q.

$$\begin{aligned} \frac{2000}{2000-q} &= \frac{7000-3q}{2q} \\ 2q(2000-q)\left(\frac{2000}{2000-q}\right) &= 2q(2000-q)\left(\frac{7000-3q}{2q}\right) \\ 2q(2000) &= (2000-q)(7000-3q) \\ 4000q &= 14{,}000{,}000 - 13{,}000q + 3q^2 \\ 0 &= 3q^2 - 17{,}000q + 14{,}000{,}000 \Rightarrow \\ (3q-14{,}000)(q-1000) &= 0 \Rightarrow \end{aligned}$$

$q = \frac{14{,}000}{3} \approx 4667$ or $q = 1000$

If $q = \frac{14{,}000}{3},$ then

$$p = \frac{2000}{2000-\frac{14{,}000}{3}} = \frac{6000}{6000-14{,}000} = \frac{6000}{-8000} = -\frac{3}{4}.$$

Since this is a negative result, we reject this solution. If $q = 1000,$ then $p = \frac{2000}{2000-1000} = \frac{2000}{1000} = 2.$ Thus, equilibrium demand is 1000 units.

(b) As calculated in part b, the equilibrium price, *p,* is \$2.

63. supply: $p = \sqrt{.1q + 9} - 2$
demand: $p = \sqrt{25 - .1q}$

(a) Equilibrium occurs when supply equals demand, so solve the system formed by the supply and demand equations. This system can be solved by substitution. Substitute $\sqrt{.1q+9} - 2$ for *p* in the demand equation and solve the resulting equation for *q*.

$$\begin{aligned}\sqrt{.1q+9} - 2 &= \sqrt{25-.1q}\\ \left(\sqrt{.1q+9}-2\right)^2 &= \left(\sqrt{25-.1q}\right)^2\\ .1q+9-4\sqrt{.1q+9}+4 &= 25-.1q\\ .2q-12 &= 4\sqrt{.1q+9}\\ (.2q-12)^2 &= \left(4\sqrt{.1q+9}\right)^2\\ .04q^2-4.8q+144 &= 16(.1q+9)\\ .04q^2-4.8q+144 &= 1.6q+144\\ .04q^2-6.4q &= 0\\ .04q(q-160) &= 0 \Rightarrow\\ q = 0 &\text{ or } q = 160\end{aligned}$$

Disregard an equilibrium demand of 0. The equilibrium demand is 160 units.

(b) Substitute 160 for *q* in either equation and solve for *p*.

$$\begin{aligned}p &= \sqrt{.1(160)+9} - 2 = \sqrt{16+9} - 2\\ &= \sqrt{25} - 2 = 5 - 2 = 3\end{aligned}$$

The equilibrium price is \$3.

64. $G = \frac{Bt}{R+R_t}$; $S = \frac{BR}{(R+R_t)^2}$

Find values of *R* and R_t when $B = 3.7$, $t = 90$, $G = .4$ and $S = .001$.

$$.4 = \frac{3.7(90)}{R+R_t} = \frac{333}{R+R_t} \quad (1)$$
$$.001 = \frac{3.7R}{(R+R_t)^2} \quad (2)$$

By clearing fractions in equations (1) and (2) we have the following.

$$.4 = \frac{333}{R+R_t} \Rightarrow R+R_t = \frac{333}{.4} = 832.5$$
$$.001 = \frac{3.7R}{(R+R_t)^2} \Rightarrow (R+R_t)^2 = \frac{3.7R}{.001} = 3700R \ (3)$$
$$R + R_t = 832.5 \Rightarrow (R+R_t)^2 = 693,056.25 \ (4)$$

Substituting equation (3) into equation (4) we have $3700R = 693,056.25 \Rightarrow R = 187.3125$. If $R = 187.3125$, then $187.3125 + R_t = 832.5 \Rightarrow R_t = 645.1875$. Thus, $R \approx 187$ and $R_t \approx 645$.

65. (a) The emission of carbon is increasing with time. The carbon emissions from the former USSR and Eastern Europe have surpassed the emissions of Western Europe.

(b) They were equal in 1963 when the levels were approximately 400 million metric tons.

(c)
$$W = E$$
$$375(1.008)^{(t-1950)} = 260(1.038)^{(t-1950)}$$
$$\ln\left[375(1.008)^{(t-1950)}\right] = \ln\left[260(1.038)^{(t-1950)}\right]$$
$$\ln 375 + (t-1950)\ln 1.008 = \ln 260 + (t-1950)\ln 1.038$$
$$\ln 375 - \ln 260 = (t-1950)\ln 1.038 - (t-1950)\ln 1.008$$
$$\ln 375 - \ln 260 = (\ln 1.038 - \ln 1.008)\cdot(t-1950)$$
$$\frac{\ln 375 - \ln 260}{\ln 1.038 - \ln 1.008} = t - 1950$$
$$t = 1950 + \frac{\ln 375 - \ln 260}{\ln 1.038 - \ln 1.008} \approx 1962.49$$

If *t* is approximately 1962.49, $W = 375(1.008)^{(1962.49-1950)} \approx 414.24$.

In 1962, the emission levels were equal and were approximately 414 million metric tons.

66. Shift the graph of $y = |x|$ one unit to the right to obtain the graph of $y = |x-1|$.

67. Shift the graph of $y = x^2$ four units down to obtain the graph of $y = x^2 - 4$.

68. If $x - 1 \geq 0 \ (x \geq 1)$, $|x-1| = x-1$. If $x - 1 < 0 \ (x < 1)$, $|x-1| = -(x-1) = 1-x$.

Therefore, $y = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1. \end{cases}$

69. $x^2 - 4 = x - 1$ if $x \geq 1$ and $x^2 - 4 = 1 - x$ if $x < 1$

70. $x^2 - 4 = x - 1$ if $x \geq 1$
$x^2 - x - 3 = 0$
Solve by the quadratic formula where $a = 1$, $b = -1$, and $c = -3$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

If $\frac{1+\sqrt{13}}{2} \approx 2.3 \geq 1$, but $\frac{1-\sqrt{13}}{2} \approx -1.3 < 1$.

Therefore, $x = \frac{1+\sqrt{13}}{2}$.

$x^2 - 4 = 1 - x$ if $x < 1$

$x^2 + x - 5 = 0$

Solve by the quadratic formula where $a = 1, b = 1$, and $c = -5$.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-5)}}{2(1)} = \frac{-1 \pm \sqrt{1+20}}{2} = \frac{-1 \pm \sqrt{21}}{2}$$

If $\frac{-1-\sqrt{21}}{2} \approx -2.8 < 1$, but $\frac{-1+\sqrt{21}}{2} \approx 1.8 > 1$,

Therefore, $x = \frac{-1-\sqrt{21}}{2}$.

71. If $y = |x-1|$ and $x = \frac{1+\sqrt{13}}{2}$, then

$$y = \left|\frac{1+\sqrt{13}}{2} - 1\right| = \left|\frac{1+\sqrt{13}}{2} - \frac{2}{2}\right| = \left|\frac{-1+\sqrt{13}}{2}\right| = \frac{-1+\sqrt{13}}{2}.$$

If $y = 1 - x$ and $x = \frac{-1-\sqrt{21}}{2}$, then

$$y = 1 - \left(\frac{-1-\sqrt{21}}{2}\right) = \frac{2}{2} + \frac{1+\sqrt{21}}{2} = \frac{3+\sqrt{21}}{2}.$$

Solution set:

$$\left\{\left(\frac{1+\sqrt{13}}{2}, \frac{-1+\sqrt{13}}{2}\right), \left(\frac{-1-\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}\right)\right\}$$

Summary Exercises on Systems of Equations

As noted in the text, different methods of solving equations have been introduced. In the solutions to these exercises, we will present the solution using one method (or combination of methods). In general, the nonlinear systems cannot use the matrix, Gauss-Jordan, Cramer's rule methods. Only the methods of substitution or elimination should be considered for such systems.

1. $2x + 5y = 4$
$3x - 2y = -13$

$$D = \begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix} = 2(-2) - 3(5) = -4 - 15 = -19,$$

$$D_x = \begin{vmatrix} 4 & 5 \\ -13 & -2 \end{vmatrix} = 4(-2) - (-13)(5)$$
$$= -8 - (-65) = 57$$

$$D_y = \begin{vmatrix} 2 & 4 \\ 3 & -13 \end{vmatrix} = 2(-13) - 3(4)$$
$$= -26 - 12 = -38 \Rightarrow$$

$$x = \frac{D_x}{D} = \frac{57}{-19} = -3 \text{ and } y = \frac{D_y}{D} = \frac{-38}{-19} = 2.$$

Solution set: $\{(-3, 2)\}$

2. $x - 3y = 7$ (1)
$-3x + 4y = -1$ (2)

Multiply equation (1) by 3 and add the result to equation (2).

$$\begin{aligned} 3x - 9y &= 21 \\ -3x + 4y &= -1 \\ \hline -5y &= 20 \Rightarrow y = -4 \end{aligned}$$

If $y = -4$, then $x - 3(-4) = 7 \Rightarrow$
$x - (-12) = 7 \Rightarrow x + 12 = 7 \Rightarrow x = -5$

Solution set: $\{(-5, -4)\}$

3. $2x^2 + y^2 = 5$ (1)
$3x^2 + 2y^2 = 10$ (2)

Using the elimination method, we add -2 times equation (1) to equation (2).

$$\begin{aligned} -4x^2 - 2y^2 &= -10 \\ 3x^2 + 2y^2 &= 10 \\ \hline -x^2 &= 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \end{aligned}$$

If $x = 0$, then $2(0)^2 + y^2 = 5 \Rightarrow$
$0 + y^2 = 5 \Rightarrow y^2 = 5 \Rightarrow y = \pm\sqrt{5}$.

Solution set: $\{(0, \sqrt{5}), (0, -\sqrt{5})\}$

4. $2x - 3y = -2$
$x + y = -16$
$3x - 2y + z = 7$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & -2 \\ 1 & 1 & 0 & -16 \\ 3 & -2 & 1 & 7 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -16 \\ 2 & -3 & 0 & -2 \\ 3 & -2 & 1 & 7 \end{array}\right] \begin{matrix} R1 \leftrightarrow R2 \\ \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -16 \\ 0 & -5 & 0 & 30 \\ 0 & -5 & 1 & 55 \end{array}\right] \begin{matrix} \\ -2R1 + R2 \\ -3R1 + R3 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -16 \\ 0 & 1 & 0 & -6 \\ 0 & -5 & 1 & 55 \end{array}\right] \begin{matrix} \\ -\frac{1}{5}R2 \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 25 \end{array}\right] \begin{matrix} -1R2 + R1 \\ \\ 5R2 + R3 \end{matrix} \Rightarrow$$

Solution set: $\{(-10, -6, 25)\}$

5. $6x - y = 5$ (1)
$xy = 4$ (2)

Solving equation (2) for y, we have $y = \frac{4}{x}$.
Substitute this result into equation (1).

$$6x - \frac{4}{x} = 5 \Rightarrow 6x^2 - 4 = 5x \Rightarrow$$
$$6x^2 - 5x - 4 = 0 \Rightarrow (2x+1)(3x-4) = 0 \Rightarrow \text{ If } x = -\tfrac{1}{2} \text{ or } x = \tfrac{4}{3}$$

$x = \frac{4}{3}$, then $y = \frac{4}{\frac{4}{3}} = 3$. If $x = -\frac{1}{2}$, then

$y = \frac{4}{-\frac{1}{2}} = -8$.

Solution set: $\left\{\left(\frac{4}{3}, 3\right), \left(-\frac{1}{2}, -8\right)\right\}$

6. $4x + 2z = -12$ (1)
$x + y - 3z = 13$ (2)
$-3x + y - 2z = 13$ (3)

Eliminate y by adding -1 times equations (3) to equation (2).

$$\begin{array}{l} x + y - 3z = 13 \\ \underline{3x - y + 2z = -13} \\ 4x \quad - z = \quad 0 \ (4) \end{array}$$

Eliminate z by adding 2 times equation (4) to equation (1).

$$\begin{array}{l} 8x - 2z = 0 \\ \underline{4x + 2z = -12} \\ 12x \quad = -12 \Rightarrow x = -1 \end{array}$$

Using $x = -1$, find z from equation (4) by substitution.

$4(-1) - z = 0 \Rightarrow -4 - z = 0 \Rightarrow z = -4$

Substitute -1 for x and -4 for z in equation (2) to find y.

$$-1 + y - 3(-4) = 13 \Rightarrow -1 + y + 12 = 13 \Rightarrow$$
$$11 + y = 13 \Rightarrow y = 2$$

Solution set: $\{(-1, 2, -4)\}$

7. $x + 2y + z = 5$ (1)
$y + 3z = 9$ (2)

This system has infinitely many solutions. We will express the solution set with z as the arbitrary variable. Solving equation (2) for y, we have $y = 9 - 3z$. Substituting y (in terms of z) into equation (1) and solving for x we have the following.

$$x + 2(9 - 3z) + z = 5 \Rightarrow x + 18 - 6z + z = 5 \Rightarrow$$
$$x + 18 - 5z = 5 \Rightarrow x = -13 + 5z$$

Solution set: $\{(-13 + 5z, 9 - 3z, z)\}$

8. $x - 4 = 3y$ (1)
$x^2 + y^2 = 8$ (2)

Solving equation (1) for x, we have $x = 3y + 4$.
Substitute this result into equation (2).

$$(3y+4)^2 + y^2 = 8 \Rightarrow 9y^2 + 24y + 16 + y^2 = 8$$
$$10y^2 + 24y + 16 = 8 \Rightarrow 10y^2 + 24y + 8 = 0 \Rightarrow$$
$$5y^2 + 12y + 4 = 0 \Rightarrow (y+2)(5y+2) = 0$$
$$y = -2 \text{ or } y = -\tfrac{2}{5}$$

If $y = -2$, then $x = 3(-2) + 4 = -6 + 4 = -2$.

If $y = -\frac{2}{5}$, then $x = 3\left(-\frac{2}{5}\right) + 4$
$= -\frac{6}{5} + \frac{20}{5} = \frac{14}{5}$.

Solution set: $\left\{(-2, -2), \left(\frac{14}{5}, -\frac{2}{5}\right)\right\}$

9. $3x + 6y - 9z = 1$
$2x + 4y - 6z = 1$
$3x + 4y + 5z = 0$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 6 & -9 & 1 \\ 2 & 4 & -6 & 1 \\ 3 & 4 & 5 & 0 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 1 \\ 3 & 6 & -9 & 1 \\ 3 & 4 & 5 & 0 \end{array}\right] \begin{array}{l} \text{R1} \leftrightarrow \text{R2} \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & \frac{1}{2} \\ 3 & 6 & -9 & 1 \\ 3 & 4 & 5 & 0 \end{array}\right] \begin{array}{l} \frac{1}{2}\text{R1} \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -2 & 14 & -\frac{3}{2} \end{array}\right] \begin{array}{l} \\ -3\text{R1+R2} \\ -3\text{R1+R3} \end{array}$$

The second row of the augmented matrix corresponds to the statement $0 = -\frac{1}{2}$. This is a false statement. Thus the solution set is $\varnothing$.

10. $x + 2y + z = 0$ (1)
$x + 2y - z = 6$ (2)
$2x - y = -9$ (3)

Add equations (1) and (2) in order to eliminate z to obtain $2x + 4y = 6$ or $x + 2y = 3$. (4)

Multiply equation (3) by 2 and add the result to equation (4) in order to eliminate y.

$$\begin{aligned} x + 2y &= 3 \\ 4x - 2y &= -18 \\ \hline 5x &= -15 \Rightarrow x = -3 \end{aligned}$$

Substitute $x = -3$ into equation (3) to solve for y.

$$2(-3) - y = -9 \Rightarrow -6 - y = -9 \Rightarrow -y = -3 \Rightarrow y = 3$$

Substitute $x = -3$ and $y = 3$ into equation (1) to solve for z.

$$-3 + 2(3) + z = 0 \Rightarrow -3 + 6 + z = 0 \Rightarrow 3 + z = 0 \Rightarrow z = -3$$

Solution set: $\{(-3, 3, -3)\}$

11. $x^2 + y^2 = 4$ (1)
$y = x + 6$ (2)

Substitute equation (1) into equation (2).

$$\begin{aligned} x^2 + (x+6)^2 &= 4 \\ x^2 + x^2 + 12x + 36 &= 4 \\ 2x^2 + 12x + 36 &= 4 \\ 2x^2 + 12x + 32 = 0 &\Rightarrow x^2 + 6x + 16 = 0 \end{aligned}$$

Using the quadratic formula where $a = 1, b = 6,$ and $c = 16,$ we have the following.

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(16)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 64}}{2} = \frac{-6 \pm \sqrt{-28}}{2} = \frac{-6 \pm 2i\sqrt{7}}{2} = -3 \pm i\sqrt{7}$$

If $x = -3 + i\sqrt{7},$ then $y = -3 + i\sqrt{7} + 6 = 3 + i\sqrt{7}.$ If $x = -3 - i\sqrt{7},$ then $y = -3 - i\sqrt{7} + 6 = 3 - i\sqrt{7}.$

Solution set: $\left\{\left(-3 + i\sqrt{7}, 3 + i\sqrt{7}\right), \left(-3 - i\sqrt{7}, 3 - i\sqrt{7}\right)\right\}$

12. $x + 5y = -23$
$4y - 3z = -29$
$2x + 5z = 19$

Adding -2 times row 1 to row 3, we have

$$D = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 4 & -3 \\ 2 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 4 & -3 \\ 0 & -10 & 5 \end{vmatrix}.$$

Expanding by column one, we have

$$D = 1 \cdot \begin{vmatrix} 4 & -3 \\ -10 & 5 \end{vmatrix} = 20 - 30 = -10.$$

Expanding about column two, we have

$$D_x = \begin{vmatrix} -23 & 5 & 0 \\ -29 & 4 & -3 \\ 19 & 0 & 5 \end{vmatrix} \text{ as follows.}$$

$$-1(5)\begin{vmatrix} -29 & -3 \\ 19 & 5 \end{vmatrix} + 4\begin{vmatrix} -23 & 0 \\ 19 & 5 \end{vmatrix}$$
$$= -5(-145 + 57) + 4(-115 - 0)$$
$$= -5(-88) + 4(-115) = 440 + (-460) = -20$$

Adding -2 times row 1 to row 3, we have

$$D_y = \begin{vmatrix} 1 & -23 & 0 \\ 0 & -29 & -3 \\ 2 & 19 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -23 & 0 \\ 0 & -29 & -3 \\ 0 & 65 & 5 \end{vmatrix}.$$

Expanding by column one, we have

$$D_y = 1 \cdot \begin{vmatrix} -29 & -3 \\ 65 & 5 \end{vmatrix} = -145 + 195 = 50.$$

Adding -2 times row 1 to row 3, we have

$$D_z = \begin{vmatrix} 1 & 5 & -23 \\ 0 & 4 & -29 \\ 2 & 0 & 19 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -23 \\ 0 & 4 & -29 \\ 0 & -10 & 65 \end{vmatrix}.$$

Expanding by column one, we have

$$D_z = 1 \cdot \begin{vmatrix} 4 & -29 \\ -10 & 65 \end{vmatrix} = 260 - 290 = -30.$$

Thus, we have

$x = \frac{D_x}{D} = \frac{-20}{-10} = 2,$ $y = \frac{D_y}{D} = \frac{50}{-10} = -5,$ and $z = \frac{D_z}{D} = \frac{30}{-10} = 3.$

Solution set: $\{(2, -5, 3)\}$

13. $y + 1 = x^2 + 2x$ (1)
$y + 2x = 4$ (2)

Solving equation (2) for y, we have $y = 4 - 2x$. Substitute this result into equation (1).

$$4 - 2x + 1 = x^2 + 2x \Rightarrow 5 - 2x = x^2 + 2x \Rightarrow 0 = x^2 + 4x - 5 \Rightarrow (x+5)(x-1) = 0 \Rightarrow x = -5 \text{ or } x = 1$$

If $x = -5,$ then $y = 4 - 2(-5) = 4 + 10 = 14.$ If $x = 1,$ then $y = 4 - 2(1) = 4 - 2 = 2.$

Solution set: $\{(1, 2), (-5, 14)\}$

14. $3x+6z=-3$ (1)
$y-\ z=\ 3$ (2)
$2x+4z=-1$ (3)

Add 2 times equation (1) to -3 times equation (3) in order to eliminate *x*.

$$\begin{array}{r} 6x+12z=-6 \\ \underline{-6x-12z=\ 3} \\ 0=-3 \end{array}$$

This results in a false statement. The solution set is $\varnothing$.

15. $2x+3y+4z=3$
$-4x+2y-6z=2$
$4x+3z=0$

This system has the augmented matrix

$$\left[\begin{array}{rrr|r} 2 & 3 & 4 & 3 \\ -4 & 2 & -6 & 2 \\ 4 & 0 & 3 & 0 \end{array}\right].$$

$$\left[\begin{array}{rrr|r} 1 & \frac{3}{2} & 2 & \frac{3}{2} \\ -4 & 2 & -6 & 2 \\ 4 & 0 & 3 & 0 \end{array}\right]\begin{array}{l}\frac{1}{2}\text{R1}\\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & \frac{3}{2} & 2 & \frac{3}{2} \\ 0 & 8 & 2 & 8 \\ 0 & -6 & -5 & -6 \end{array}\right]\begin{array}{l} \\ 4\text{R1+R2} \\ -4\text{R1+R3}\end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & \frac{3}{2} & 2 & \frac{3}{2} \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & -6 & -5 & -6 \end{array}\right]\begin{array}{l} \\ \frac{1}{8}\text{R2} \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & \frac{13}{8} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -\frac{7}{2} & 0 \end{array}\right]\begin{array}{l} -\frac{3}{2}\text{R2+R1} \\ \\ 6\text{R2+R3}\end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & \frac{13}{8} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} \\ \\ -\frac{2}{7}\text{R3}\end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} -\frac{13}{8}\text{R3+R1} \\ -\frac{1}{4}\text{R3+R2} \\ \\ \end{array}$$

Solution set: $\{(0,1,0)\}$

16. $\frac{3}{x}+\frac{4}{y}=4$ (1)
$\frac{1}{x}+\frac{2}{y}=\frac{2}{3}$ (2)

Let $\frac{1}{x}=t$ and $\frac{1}{y}=u$. With these substitutions,

the system becomes $\begin{array}{r} 3t+4u=4\ (3) \\ t+2u=\frac{2}{3}.\ (4)\end{array}$

Add -3 times equation (4) to equation (3) and solve for *u*.

$$\begin{array}{r} 3t+4u=\ 4 \\ \underline{-3t-6u=-2} \\ -2u=\ 2\Rightarrow u=-1 \end{array}$$

Substitute -1 for *u* in equation (3) and solve for *t*.

$3t+4(-1)=4\Rightarrow 3t+(-4)=4\Rightarrow 3t=8\Rightarrow t=\frac{8}{3}$

Now find the values of *x* and *y*, the variables in the original system. Since $\frac{1}{x}=t$, $tx=1$, and $x=\frac{1}{t}=\frac{1}{\frac{8}{3}}=\frac{3}{8}$. Likewise $y=\frac{1}{u}=\frac{1}{-1}=-1$.

Solution set: $\left\{\left(\frac{3}{8},-1\right)\right\}$

17. $-5x+2y+z=5$ (1)
$-3x-2y-z=3$ (2)
$-x+6y\quad =1$ (3)

Add equations (1) and (2) in order to eliminate *z* to obtain $-8x=8$ or $x=-1$.
Substitute $x=-1$ into equation (3) to solve for *y*.

$-(-1)+6y=1\Rightarrow 1+6y=1\Rightarrow 6y=0\Rightarrow y=0$

Substitute $x=-1$ and $y=0$ into equation (1) to solve for *z*.

$-5(-1)+2(0)+z=5\Rightarrow 5+0+z=5\Rightarrow$
$5+z=5\Rightarrow z=0$

Solution set: $\{(-1,0,0)\}$

18. $x+5y+3z=9$
$2x+9y+7z=5$

This system has the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 5 & 3 & 9 \\ 2 & 9 & 7 & 5 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 5 & 3 & 9 \\ 0 & -1 & 1 & -13 \end{array}\right]\begin{array}{l} \\ -2\text{R1}+\text{R2}\end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 5 & 3 & 9 \\ 0 & 1 & -1 & 13 \end{array}\right]\begin{array}{l} \\ -1\text{R2}\end{array} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 8 & -56 \\ 0 & 1 & -1 & 13 \end{array}\right]\begin{array}{l} -5\text{R2}+\text{R1} \\ \\ \end{array}$$

The equations that correspond to the final matrix are $x+8z=-56$ and $y-z=13$.
Solving these equations for *x* and *y*, respectively we have

$x+8z=-56\Rightarrow x=-56-8z$ and
$y-z=13\Rightarrow y=13+z$.

With *z* arbitrary, the solution set is of the form $\{(-56-8z,13+z,z)\}$.

19. $2x^2 + y^2 = 9$ (1)
$3x - 2y = -6$ (2)
Solving equation (2) for x, we have
$3x = 2y - 6 \Rightarrow x = \frac{2y-6}{3}$.
Substitute this result into equation (1).

$$\begin{aligned} 2\left(\tfrac{2y-6}{3}\right)^2 + y^2 &= 9 \\ 2 \cdot \tfrac{(2y-6)^2}{9} + y^2 &= 9 \\ 2 \cdot (2y-6)^2 + 9y^2 &= 81 \\ 2(4y^2 - 24y + 36) + 9y^2 &= 81 \\ 8y^2 - 48y + 72 + 9y^2 &= 81 \\ 17y^2 - 48y + 72 &= 81 \\ 17y^2 - 48y - 9 &= 0 \\ (y-3)(17y+3) = 0 &\Rightarrow y - 3 \text{ or } y = -\tfrac{3}{17} \end{aligned}$$

If $y = 3$, then $x = \frac{2(3)-6}{3} = \frac{6-6}{3} = \frac{0}{3} = 0$. If $y = -\frac{3}{17}$, then $x = \frac{2\left(-\frac{3}{17}\right)-6}{3} = \frac{2(-3)-102}{51}$
$= \frac{-6-102}{51} = \frac{-108}{51} = -\frac{36}{17}$.
Solution set: $\left\{(0,3), \left(-\frac{36}{17}, -\frac{3}{17}\right)\right\}$

20. $2x - 4y - 6 = 0$ (1)
$-x + 2y + 3 = 0$ (2)
Solving equation (2) for x, we obtain
$-x + 2y + 3 = 0 \Rightarrow -x = -3 - 2y \Rightarrow$
$x = 3 + 2y$ (3)
Substitute equation (3) into equation (1) and solve for y.
$2(3 + 2y) - 4y - 6 = 0$
$6 + 4y - 4y - 6 = 0 \Rightarrow 0 = 0$
This is a true statement. This system has infinitely many solutions. We will express the solution set with y as the arbitrary variable.
Solution set: $\{(3 + 2y, y)\}$

21. $x + y - z = 0$ (1)
$2y - z = 1$ (2)
$2x + 3y - 4z = -4$ (3)
Add –2 times equation (1) to equation (3) in order to eliminate x.

$$\begin{aligned} -2x - 2y + 2z &= 0 \\ 2x + 3y - 4z &= -4 \\ \hline y - 2z &= -4 \ (4) \end{aligned}$$

Add –2 times equation (4) to equation (2) in order to eliminate y and solve for z.

$$\begin{aligned} 2y - z &= 1 \\ -2y + 4z &= 8 \\ \hline 3z = 9 &\Rightarrow z = 3 \end{aligned}$$

Substitute 3 for z in equation (4) to obtain
$y - 2(3) = -4 \Rightarrow y - 6 = -4 \Rightarrow y = 2$.
Substitute 3 for z and 2 for y in equation (1) to obtain $x + 2 - 3 = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$.
Solution set: $\{(1, 2, 3)\}$

22. $2y = 3x - x^2$ (1)
$x + 2y = 12$ (2)
Solve equation (2) for y to obtain
$x + 2y = 12 \Rightarrow 2y = 12 - x \Rightarrow y = \frac{12-x}{2}$. (3)
Substitute equation (3) into equation (1) and solve for x.
$2\left(\frac{12-x}{2}\right) = 3x - x^2 \Rightarrow 12 - x = 3x - x^2$
$x^2 - 4x + 12 = 0$
Using the quadratic formula where $a = 1, b = -4$, and $c = 12$, we have

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(12)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16-48}}{2} = \frac{4 \pm \sqrt{-32}}{2} \\ &= \frac{4 \pm 4i\sqrt{2}}{2} = 2 \pm 2i\sqrt{2} \end{aligned}$$

If $x = 2 + 2i\sqrt{2}$, then
$y = \frac{12-\left(2+2i\sqrt{2}\right)}{2} = \frac{12-2-2i\sqrt{2}}{2} = \frac{10-2i\sqrt{2}}{2} = 5 - i\sqrt{2}$
If $x = 2 - 2i\sqrt{2}$, then
$y = \frac{12-\left(2-2i\sqrt{2}\right)}{2} = \frac{12-2+2i\sqrt{2}}{2} = \frac{10+2i\sqrt{2}}{2} = 5 + i\sqrt{2}$
Solution set:
$\left\{\left(2 + 2i\sqrt{2}, 5 - i\sqrt{2}\right), \left(2 - 2i\sqrt{2}, 5 + i\sqrt{2}\right)\right\}$

23. $4x - z = -6$ (1)
$\frac{3}{5}y + \frac{1}{2}z = 0$ (2)
$\frac{1}{3}x + \frac{2}{3}z = -5$ (3)

Before applying one of the methods, it would be helpful to clear the fractions in equations (2) and (3) by multiplying equation (2) by 10 and equation (3) by 3 to obtain the following.
$6y + 5z = 0$ (4)
$x + 2z = -15$ (5)
The system formed by equations (1), (4), and (5) can be represented by the following augmented matrix.

$$\left[\begin{array}{ccc|c} 4 & 0 & -1 & -6 \\ 0 & 6 & 5 & 0 \\ 1 & 0 & 2 & -15 \end{array}\right]$$

(*continued on next page*)

(*continued from page 947*)

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -15 \\ 0 & 6 & 5 & 0 \\ 4 & 0 & -1 & -6 \end{array}\right] \begin{matrix} R1 \leftrightarrow R3 \\ \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -15 \\ 0 & 6 & 5 & 0 \\ 0 & 0 & -9 & 54 \end{array}\right] \begin{matrix} \\ \\ -4R1+R3 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -15 \\ 0 & 1 & \frac{5}{6} & 0 \\ 0 & 0 & -9 & 54 \end{array}\right] \begin{matrix} \\ \frac{1}{6}R2 \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -15 \\ 0 & 1 & \frac{5}{6} & 0 \\ 0 & 0 & 1 & -6 \end{array}\right] \begin{matrix} \\ \\ -\frac{1}{9}R3 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -6 \end{array}\right] \begin{matrix} -2R3+R1 \\ -\frac{5}{6}R3+R2 \\ \\ \end{matrix}$$

Solution set: $\{(-3,5,-6)\}$

24. $x - y + 3z = 3$
$-2x + 3y - 11z = -4$
$x - 2y + 8z = 6$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ -2 & 3 & -11 & -4 \\ 1 & -2 & 8 & 6 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -5 & 2 \\ 0 & -1 & 5 & 3 \end{array}\right] \begin{matrix} \\ 2R1+R2 \\ -1R1+R3 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right] \begin{matrix} R2+R1 \\ \\ R2+R3 \end{matrix}$$

The third row of the augmented matrix corresponds to the statement $0 = 5$. This is a false statement. Thus, the solution set is $\varnothing$.

25. $x^2 + 3y^2 = 28$ (1)
$y - x = -2$ (2)

Solving equation (2) for *y*, we have $y = x - 2$. Substitute this result into equation (1) and solve for *y*

$$x^2 + 3(x-2)^2 = 28$$
$$x^2 + 3(x^2 - 4x + 4) = 28$$
$$x^2 + 3x^2 - 12x + 12 = 28$$
$$4x^2 - 12x + 12 = 28$$
$$4x^2 - 12x - 16 = 0$$
$$x^2 - 3x - 4 = 0$$
$$(x+1)(x-4) = 0 \Rightarrow x = -1 \text{ or } x = 4$$

If $x = -1$, then $y = -1 - 2 = -3$. If $x = 4$, then $y = 4 - 2 = 2$.

Solution set: $\{(4,2),(-1,-3)\}$

26. $5x - 2z = 8$ (1)
$4y + 3z = -9$ (2)
$\frac{1}{2}x + \frac{2}{3}y = -1$ (3)

Add 3 times equation (1) to 2 times equation (2) to eliminate *z*.

$$\begin{array}{rcl} 15x \quad - 6z &=& 24 \\ 8y + 6z &=& -18 \\ \hline 15x + 8y \qquad &=& 6 \ (4) \end{array}$$

Multiply equation (3) by –30 and add to equation (4).

$$\begin{array}{rcl} -15x - 20y &=& 30 \\ 15x + 8y &=& 6 \\ \hline -12y &=& 36 \Rightarrow y = -3 \end{array}$$

Substitute –3 for *y* in equation (4) to solve for *x*. $15x + 8(-3) = 6 \Rightarrow 15x + (-24) = 6 \Rightarrow 15x = 30 \Rightarrow x = 2$

Substitute 2 for *x* in equation (1) to solve for *z*. $5(2) - 2z = 8 \Rightarrow 10 - 2z = 8 \Rightarrow -2z = -2 \Rightarrow z = 1$

Solution set: $\{(2,-3,1)\}$

27. $2x^2 + 3y^2 = 20$ (1)
$x^2 + 4y^2 = 5$ (2)

Using the elimination method, we add −2 times equation (2) to equation (1).

$$\begin{array}{rcl} 2x^2 + 3y^2 &=& 20 \\ -2x^2 - 8y^2 &=& -10 \\ \hline -5y^2 &=& 10 \Rightarrow y^2 = -2 \Rightarrow y = \pm i\sqrt{2} \end{array}$$

Find *x* by substituting back into equation (2).

If $y = i\sqrt{2}$, then $x^2 + 4(i\sqrt{2})^2 = 5 \Rightarrow x^2 + 4(-1)(2) = 5 \Rightarrow x^2 - 8 = 5 \Rightarrow x^2 = 13 \Rightarrow x = \pm\sqrt{13}$. If $y = -i\sqrt{2}$, then $x^2 + 4(-i\sqrt{2})^2 = 5 \Rightarrow x^2 + 4(-1)(2) = 5 \Rightarrow x^2 - 8 = 5 \Rightarrow x^2 = 13 \Rightarrow x = \pm\sqrt{13}$

Solution set: $\{(\sqrt{13}, i\sqrt{2}), (-\sqrt{13}, i\sqrt{2}), (\sqrt{13}, -i\sqrt{2}), (-\sqrt{13}, -i\sqrt{2})\}$

28. $x + y + z = -1$ (1)
$2x + 3y + 2z = 3$ (2)
$2x + y + 2z = -7$ (3)

Add -2 times equation (1) to equation (2) in order to eliminate z.

$$\begin{array}{r} -2x - 2y - 2z = 2 \\ 2x + 3y + 2z = 3 \\ \hline y \quad = 5 \end{array}$$

Since $y = 5$, substitute this value into equations (1) and (3) to obtain the following.

$$x + 5 + z = -1 \Rightarrow x + z = -6$$
$$2x + 5 + 2z = -7 \Rightarrow 2x + 2z = -12 \Rightarrow x + z = -6$$

If we also substitute $y = 5$ into equation (2), we obtain the following.

$$2x + 3(5) + 2z = 3 \Rightarrow 2x + 15 + 2z = 3 \Rightarrow 2x + 2z = -12 \Rightarrow x + z = -6$$

After these substitutions, if we multiply any of these three identical equations by –1 and add to another of these equations, we will obtain the statement $0 = 0$. This is a true statement and the system has infinitely many solutions. Solving $x + z = -6$ for x, we have $x = -6 - z$. With z arbitrary, the solution set is of the form $\{(-6 - z, 5, z)\}$.

29. $x + 2z = 9$
$y + z = 1$
$3x - 2y = 9$

Adding -3 times row 1 to row 3, we have

$$D = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & -6 \end{vmatrix}.$$

Expanding by column one, we have

$$D = 1 \cdot \begin{vmatrix} 1 & 1 \\ -2 & -6 \end{vmatrix} = -6 + 2 = -4.$$

Adding 2 times row 2 to row 3, we have

$$D_x = \begin{vmatrix} 9 & 0 & 2 \\ 1 & 1 & 1 \\ 9 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 9 & 0 & 2 \\ 1 & 1 & 1 \\ 11 & 0 & 2 \end{vmatrix}.$$

Expanding by column two, we have

$$D_x = 1 \cdot \begin{vmatrix} 9 & 2 \\ 11 & 2 \end{vmatrix} = 18 - 22 = -4.$$

Adding -3 times row 1 to row 3 we have

$$D_y = \begin{vmatrix} 1 & 9 & 2 \\ 0 & 1 & 1 \\ 3 & 9 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 9 & 2 \\ 0 & 1 & 1 \\ 0 & -18 & -6 \end{vmatrix}.$$

Expanding by column one, we have

$$D_y = 1 \cdot \begin{vmatrix} 1 & 1 \\ -18 & -6 \end{vmatrix} = -6 + 18 = 12.$$

Adding -3 times row 1 to row 3, we have

$$D_z = \begin{vmatrix} 1 & 0 & 9 \\ 0 & 1 & 1 \\ 3 & -2 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 9 \\ 0 & 1 & 1 \\ 0 & -2 & -18 \end{vmatrix}.$$

Expanding by column one, we have

$$D_z = 1 \cdot \begin{vmatrix} 1 & 1 \\ -2 & -18 \end{vmatrix} = -18 + 2 = -16.$$

Thus, we have

$x = \frac{D_x}{D} = \frac{-4}{-4} = 1,\ y = \frac{D_y}{D} = \frac{12}{-4} = -3,$ and $z = \frac{D_z}{D} = \frac{-16}{-4} = 4.$

Solution set: $\{(1, -3, 4)\}$

30. $x^2 - y^2 = 15$ (1)
$x - 2y = 2$ (2)

Solving equation (2) for x, we have $x = 2y + 2$. Substitute this result into equation (1).

$$(2y + 2)^2 - y^2 = 15 \Rightarrow$$
$$4y^2 + 8y + 4 - y^2 = 15 \Rightarrow 3y^2 + 8y + 4 = 15$$
$$3y^2 + 8y - 11 = 0 \Rightarrow (3y + 11)(y - 1) = 0 \Rightarrow$$
$$y = -\tfrac{11}{3} \text{ or } y = 1$$

If $y = -\frac{11}{3}$, then $x = 2\left(-\frac{11}{3}\right) + 2 = -\frac{22}{3} + \frac{6}{3} = -\frac{16}{3}$. If $y = 1$, then $x = 2(1) + 2 = 2 + 2 = 4$.

Solution set: $\left\{\left(-\frac{16}{3}, -\frac{11}{3}\right), (4, 1)\right\}$

31. $-x + y = -1$
$x + z = 4$
$6x - 3y + 2z = 10$

This system has the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 4 \\ 6 & -3 & 2 & 10 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 4 \\ 6 & -3 & 2 & 10 \end{array}\right] \begin{matrix} -1\text{R1} \\ \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 2 & 4 \end{array}\right] \begin{matrix} \\ -1\text{R1}+\text{R2} \\ -6\text{R1}+\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right] \begin{array}{l} \text{R2+R1} \\ \\ -3\text{R2+R3} \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \\ \\ -1\text{R3} \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} -1\text{R3+R1} \\ -1\text{R3+R2} \\ \\ \end{array}$$

Solution set: $\{(-1,-2,5)\}$

32. $2x - y - 2z = -1$ (1)
$-x + 2y + 13z = 12$ (2)
$3x + 9z = 6$ (3)

Add 2 times equation (1) to equation (2) in order to eliminate *y*.

$$\begin{array}{l} 4x - 2y - 4z = -2 \\ -x + 2y + 13z = 12 \\ \hline 3x + 9y = 10 \ (4) \end{array}$$

Add –1 times equation (3) to equation (4).

$$\begin{array}{l} -3x - 9z = -6 \\ 3x + 9z = 10 \\ \hline 0 = 4 \end{array}$$

This is a false statement. The solution set is $\emptyset$.

33. $xy = -3$ (1)
$x + y = -2$ (2)

Solving equation (1) for *y*, we have $y = -\frac{3}{x}$.

Substitute this result into equation (2).

$$x + \left(-\tfrac{3}{x}\right) = -2 \Rightarrow x^2 - 3 = -2x \Rightarrow$$
$$x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0 \Rightarrow$$
$$x = -3 \text{ or } x = 1$$

If $x = -3$, then $y = -\frac{3}{-3} = 1$. If $x = 1$, then $y = -\frac{3}{1} = -3$.

Solution set: $\{(1,-3),(-3,1)\}$

34. $-3x + 2z = 1$ (1)
$4x + y - 2z = -6$ (2)
$x + y + 4z = 3$ (3)

Add –1 times equation (3) to equation (3) in order to eliminate *y*.

$$\begin{array}{l} 4x + y - 2z = -6 \\ -x - y - 4z = -3 \\ \hline 3x - 6z = -9 \Rightarrow x - 2z = -3 \ (4) \end{array}$$

Add equation (1) to equation (4) to obtain $-2x = -2 \Rightarrow x = 1$.

Substitute 1 for *x* in equation (4) to obtain $1 - 2z = -3 \Rightarrow -2z = -4 \Rightarrow z = 2$.

Substitute 1 for *x* and 2 for *z* in equation (3) to obtain $1 + y + 4(2) = 3 \Rightarrow y + 9 = 3 \Rightarrow y = -6$.

Solution set: $\{(1,-6,2)\}$

35. $y = x^2 + 6x + 9$ (1)
$x + y = 3$ (2)

Solve equation (2) for *y* to obtain $x + y = 3 \Rightarrow y = 3 - x$. (3)

Substitute equation (3) into equation (1) and solve for *x*.

$$3 - x = x^2 + 6x + 9$$
$$0 = x^2 + 7x + 6$$
$$(x+6)(x+1) = 0 \Rightarrow x = -6 \text{ or } x = -1$$

If $x = -6$, then $y = 3 - (-6) = 9$. If $x = -1$, then $y = 3 - (-1) = 4$.

Solution set: $\{(-6,9),(-1,4)\}$

36. $x^2 + y^2 = 9$ (1)
$2x - y = 3$ (2)

Solving equation (2) for *y*, we have $y = 2x - 3$.

Substitute this result into equation (1).

$$x^2 + (2x-3)^2 = 9 \Rightarrow x^2 + 4x^2 - 12x + 9 = 9 \Rightarrow$$
$$5x^2 - 12x + 9 = 9 \Rightarrow 5x^2 - 12x = 0 \Rightarrow$$
$$x(5x - 12) = 0 \Rightarrow x = 0 \text{ or } x = \tfrac{12}{5}$$

If $x = 0$, then $y = 2(0) - 3 = -3$. If $x = \frac{12}{5}$, then $y = 2\left(\frac{12}{5}\right) - 3 = \frac{24}{5} - \frac{15}{5} = \frac{9}{5}$.

Solution set: $\left\{(0,-3),\left(\frac{12}{5},\frac{9}{5}\right)\right\}$

37. $$\begin{aligned} 2x-3y \qquad &= -2 \quad (1)\\ x+y-4z &= -16 \quad (2)\\ 3x-2y+z &= 7 \quad (3)\end{aligned}$$

Add 4 times equation (3) to equation (2) in order to eliminate z.

$$\begin{array}{l} x+y-4z=-16\\ \underline{12x-8y+4z=28}\\ 13x-7y \qquad = 12 \quad (4)\end{array}$$

Add 3 times equation (4) to -7 times equation (1) in order to eliminate y and solve for x.

$$\begin{array}{l} -14x+21y=14\\ \underline{39x-21y=36}\\ 25x \qquad = 50 \Rightarrow x=2\end{array}$$

Substitute 2 for x in equation (4) to obtain

$13(2)-7y=12 \Rightarrow 26-7y=12 \Rightarrow$
$-7y=-14 \Rightarrow y=2$

Substitute 2 for x and 2 for y in equation (2) to obtain the following.

$2+2-4z=-16 \Rightarrow 4-4z=-16 \Rightarrow$
$-4z=-20 \Rightarrow z=5$

Solution set: $\{(2,2,5)\}$

38. $$\begin{aligned} 3x-y &= -2\\ y+5z &= -4\\ -2x+3y-z &= -8\end{aligned}$$

Add -5 times column 2 to column 3:

$$D=\begin{vmatrix} 3 & -1 & 0\\ 0 & 1 & 5\\ -2 & 3 & -1\end{vmatrix}=\begin{vmatrix} 3 & -1 & 5\\ 0 & 1 & 0\\ -2 & 3 & -16\end{vmatrix}.$$

Expand by row two:

$$D=1\cdot\begin{vmatrix} 3 & 5\\ -2 & -16\end{vmatrix}=-48+10=-38.$$

Add –2 times column 2 to column 1:

$$D_x=\begin{vmatrix} -2 & -1 & 0\\ -4 & 1 & 5\\ -8 & 3 & -1\end{vmatrix}=\begin{vmatrix} 0 & -1 & 0\\ -6 & 1 & 5\\ -14 & 3 & -1\end{vmatrix}.$$

Expand by row one:

$$D_x=(-1)(-1)\begin{vmatrix} -6 & 5\\ -14 & -1\end{vmatrix}=6+70=76.$$

Add 5 times row 3 to row 2:

$$D_y=\begin{vmatrix} 3 & -2 & 0\\ 0 & -4 & 5\\ -2 & -8 & -1\end{vmatrix}=\begin{vmatrix} 3 & -2 & 0\\ -10 & -44 & 0\\ -2 & -8 & -1\end{vmatrix}.$$

Expanding by column three, we have

$$D_y=-1\cdot\begin{vmatrix} 3 & -2\\ -10 & -44\end{vmatrix}=-(-132-20)=152.$$

Add 4 times column 2 to column 3:

$$D_z=\begin{vmatrix} 3 & -1 & -2\\ 0 & 1 & -4\\ -2 & 3 & -8\end{vmatrix}=\begin{vmatrix} 3 & -1 & -6\\ 0 & 1 & 0\\ -2 & 3 & 4\end{vmatrix}.$$

Expand by row two:

$$D_z=1\cdot\begin{vmatrix} 3 & -6\\ -2 & 4\end{vmatrix}=12-12=0.$$

Thus, we have

$x=\frac{D_x}{D}=\frac{76}{-38}=-2,\ y=\frac{D_y}{D}=\frac{152}{-38}=-4,$
and $z=\frac{D_z}{D}=\frac{0}{-38}=0.$

Solution set: $\{(-2,-4,0)\}$

39. $$\begin{aligned} y &= (x-1)^2+2 \quad (1)\\ y &= 2x-1 \quad (2)\end{aligned}$$

Substitute $2x - 1$ for y in equation (1) and solve for x:

$$\begin{aligned} 2x-1 &= (x-1)^2+2\\ 2x-1 &= x^2-2x+1+2\end{aligned}$$
$$x^2-4x+4=0 \Rightarrow (x-2)^2=0 \Rightarrow x=2$$

Substitute 2 for x in equation (2) and solve for y: $y=2(2)-1=3$

Solution set: $\{(2, 3)\}$

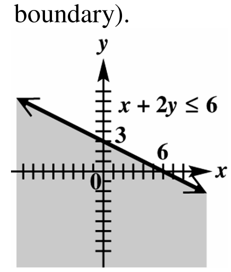
Section 9.6: Systems of Inequalities and Linear Programming

1. $x+2y\le 6$

The boundary is the line $x+2y=6$, which can be graphed using the x-intercept 6 and y-intercept 3. The boundary is included in the graph, so draw a solid line. Solving for y, we have the following.

$$x+2y\le 6 \Rightarrow 2y\le -x+6 \Rightarrow y\le -\tfrac{1}{2}x+3$$

Since y is *less than* or equal to $-\frac{1}{2}x+3$, the graph of the solution set is the line and the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $0+2(0)\le 6 \Rightarrow 0\le 6$ is a true statement, shade line and the side of the graph containing the test point (0, 0) (the half-plane below the boundary).

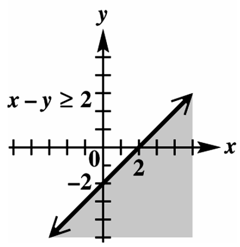


2. $x - y \geq 2$

The boundary is the line $x - y = 2$, which can be graphed using the x-intercept 2 and y-intercept –2. The boundary is included in the graph, so draw a solid line. Solving for y, we have the following.

$x - y \geq 2 \Rightarrow -y \geq -x + 2 \Rightarrow y \leq x - 2$

Since y is *less than* or equal to $x - 2$, the graph of the solution set is the line and the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $0 - 0 \geq 2 \Rightarrow 0 \geq 2$ is a false statement, shade the line and the side of the graph not containing the test point $(0, 0)$ (the half-plane below the boundary).

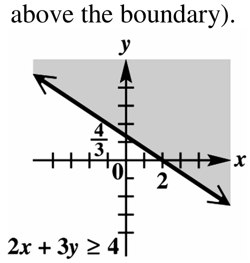


3. $2x + 3y \geq 4$

The boundary is the line $2x + 3y = 4$, which can be graphed using the x-intercept 2 and y-intercept $\frac{4}{3}$. The boundary is included in the graph, so draw a solid line. Solving for y, we have the following.

$2x + 3y \geq 4 \Rightarrow 3y \geq -2x + 4 \Rightarrow y \geq -\frac{2}{3}x + \frac{4}{3}$

Since y is *greater than* or equal to $-\frac{2}{3}x + \frac{4}{3}$, the graph of the solution set is the line and the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $2(0) + 3(0) \geq 4 \Rightarrow 0 \geq 4$ is a false statement, shade the line and the side of the graph not containing the test point (0,0) (the half-plane above the boundary).

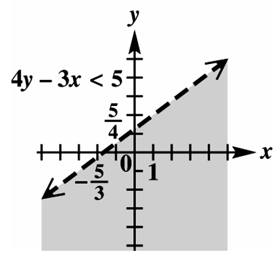


4. $4y - 3x < 5$

The boundary is the line $4y - 3x = 5$, which can be graphed using the x-intercept $-\frac{5}{3}$ and y-intercept $\frac{5}{4}$. The boundary is not included in the graph, so draw a dashed line. Solving for y, we have the following.

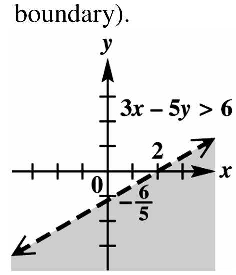
$4y - 3x < 5 \Rightarrow 4y < 3x + 5 \Rightarrow y < \frac{3}{4}x + \frac{5}{4}$

Since y is *less than* $-\frac{2}{3}x + \frac{4}{3}$, the graph of the solution set is the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $4(0) - 3(0) < 5 \Rightarrow 0 < 5$ is a true statement, shade the side of the graph containing the test point (0, 0) (the half-plane below the boundary).



5. $3x - 5y > 6$

The boundary is the line $3x - 5y = 6$, which can be graphed using the x-intercept 2 and y-intercept $-\frac{6}{5}$. The boundary is not included in the graph, so draw a dashed line. Solving for y, we have the following Since y is *less than* $\frac{3}{5}x - \frac{6}{5}$, the graph of the solution set is the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $3(0) - 5(0) > 6 \Rightarrow 0 > 6$ is a false statement, shade the side of the graph not containing the test point (0, 0) (the half-plane below the boundary).

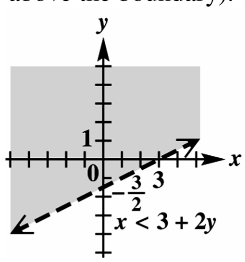


6. $x < 3 + 2y$

The boundary is the line $x = 3 + 2y$, which can be graphed using the x-intercept 3 and y-intercept $-\frac{3}{2}$. The boundary is not included in the graph, so draw a dashed line. Solving for y, we have the following.

$x < 3 + 2y \Rightarrow -2y < -x + 3 \Rightarrow y > \frac{1}{2}x - \frac{3}{2}$

Since y is *greater than* $\frac{1}{2}x - \frac{3}{2}$, the graph of the solution set is the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $0 < 3 + 2(0) \Rightarrow 0 < 3$ is a true statement, shade the side of the graph containing the test point (0, 0) (the half-plane above the boundary).

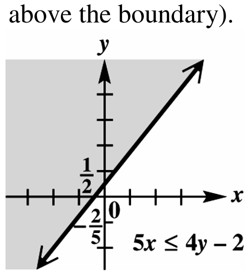


7. $5x \le 4y - 2$

The boundary is the line $5x = 4y - 2$, which can be graphed using the x-intercept $-\frac{2}{5}$ and y-intercept $\frac{1}{2}$. The boundary is included in the graph, so draw a solid line. Solving for y, we have the following.

$5x \le 4y - 2 \Rightarrow -4y \le -5x - 2 \Rightarrow y \ge \frac{5}{4}x + \frac{1}{2}$

Since y is *greater than* or equal to $\frac{5}{4}x + \frac{1}{2}$, the graph of the solution set is the line and the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $5(0) \le 4(0) - 2 \Rightarrow 0 \le -2$ is a false statement, shade the line and the side of the graph not containing the test point (0, 0) (the half-plane above the boundary).

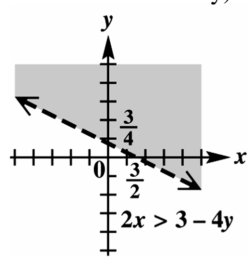


8. $2x > 3 - 4y$

The boundary is the line $2x = 3 - 4y$, which can be graphed using the x-intercept $\frac{3}{2}$ and y-intercept $\frac{3}{4}$. The boundary is not included in the graph, so draw a dashed line. Solving for y, we have the following.

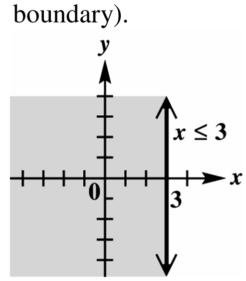
$2x > 3 - 4y \Rightarrow 4y > -2x + 3 \Rightarrow y > -\frac{1}{2}x + \frac{3}{4}$

Since y is *greater than* $-\frac{1}{2}x + \frac{3}{4}$, the graph of the solution set is the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $2(0) > 3 - 4(0) \Rightarrow 0 > 3$ is a false statement, shade the side of the graph not containing the test point (0, 0) (the half-plane above the boundary).



9. $x \le 3$

The boundary is the vertical line $x = 3$, which intersects the x-axis at 3. The boundary is included in the graph, so draw a solid line. Since $x \le 3$, it can easily be determined that we should shade to the left of the boundary. We also can use (0, 0) as a test point. Since $0 \le 3$ is a true statement, shade the line and the side of the graph containing the test point (0, 0) (the half-plane to the left of the boundary).



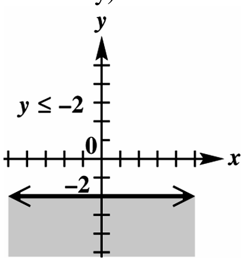
10. $y \le -2$

The boundary is the horizontal line $y = -2$, which intersects the y-axis at -2. The boundary is included in the graph, so draw a solid line. Since $y \le -2$, it can easily be determined that we should shade below the boundary.

(*continued on next page*)

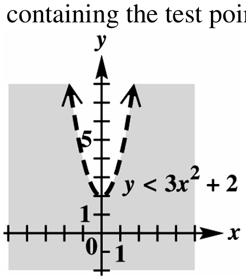
(*continued from page 953*)

We also can use (0, 0) as a test point. Since $0 \le -2$ is a false statement, shade the line and the side of the graph not containing the test point (0, 0) (the half-plane below the boundary).

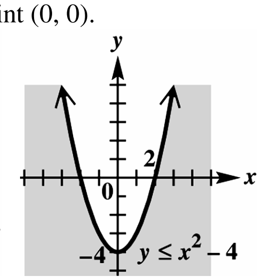


11. $y < 3x^2 + 2$

The boundary is the parabola $y = 3x^2 + 2$, which opens upwards. It has vertex $(0, 2)$, y-intercept 2, and no x-intercepts. Since the inequality symbol is <, draw a dashed curve. Since y is *less than* $3x^2 + 2$, the graph of the solution set is the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $0 < 3(0)^2 + 2 \Rightarrow 0 < 2$ is a true statement, shade the region of the graph containing the test point (0, 0).



Exercise 11



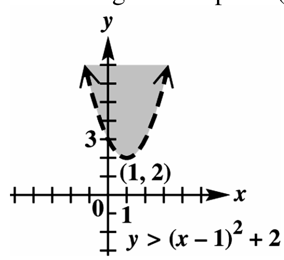
Exercise 12

12. $y \le x^2 - 4$

The boundary is the parabola $y = x^2 - 4$, which opens upwards. It has vertex $(0, -4)$, y-intercept –4, and x-intercepts of ± 2. Since the inequality symbol is $\le$, draw a solid curve. Since y is *less than* $x^2 - 4$, the graph of the solution set is the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $0 \le 0^2 - 4 \Rightarrow 0 < -4$ is a false statement, shade the region of the graph not containing the test point (0, 0).

13. $y > (x-1)^2 + 2$

The boundary is the parabola $y = (x-1)^2 + 2$, which opens upwards. It has vertex $(1, 2)$, y-intercept 3, and no x-intercepts. Since the inequality symbol is >, draw a dashed curve. Since y is *greater than* $(x-1)^2 + 2$, the graph of the solution set is the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $0 > (0-1)^2 + 2 \Rightarrow 0 > 3$ is a false statement, shade the region of the graph not containing the test point (0, 0).

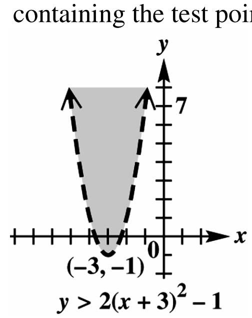


14. $y > 2(x+3)^2 - 1$

The boundary is the parabola $y = 2(x+3)^2 - 1$, which opens upwards. It has vertex $(-3, -1)$, y-intercept $2(0+3)^2 - 1 = 2(9) - 1 = 18 - 1 = 17$, and x-intercepts when $0 = 2(x+3)^2 - 1$.

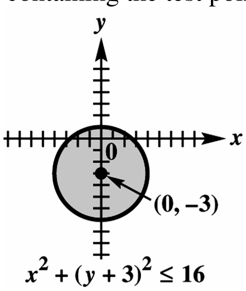
$$0 = 2(x+3)^2 - 1 \Rightarrow 1 = 2(x+3)^2 \Rightarrow$$
$$\tfrac{1}{2} = (x+3)^2 \Rightarrow \pm\tfrac{\sqrt{2}}{2} = x + 3 \Rightarrow x = -3 \pm \tfrac{\sqrt{2}}{2}$$
$$x = -3 - \tfrac{\sqrt{2}}{2} \approx -3.7 \text{ or } x = -3 + \tfrac{\sqrt{2}}{2} \approx -2.3$$

Since the inequality symbol is >, draw a dashed curve. Since y is *greater than* $2(x+3)^2 - 1$, the graph of the solution set is the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $0 > 2(0+3)^2 - 1 \Rightarrow 0 > 17$ is a false statement, shade the region of the graph not containing the test point (0, 0).



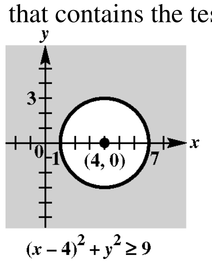
15. $x^2 + (y+3)^2 \le 16$

The boundary is a circle with center (0, –3) and radius 4. Draw a solid circle to show that the boundary is included in the graph. Since $x^2 + (y+3)^2 \le 16,$ it can easily be determined that points in the interior or on the boundary would satisfy this relation. We also can use (0, 0) as a test point. Since we have $0^2 + (0+3)^2 \le 16 \Rightarrow 9 \le 16$ is a true statement, shade the region of the graph containing the test point (0, 0).

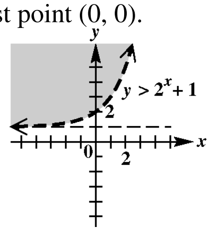


16. $(x-4)^2 + y^2 \ge 9$

The boundary is a circle with center (4, 0) and radius 3. Draw a solid circle to show that the boundary is included in the graph. Since $(x-4)^2 + y^2 \ge 9,$ it can easily be determined that points on the boundary and exterior to the circle satisfy this relation. We can use (0, 0) as a test point. Since $(0-4)^2 + 0^2 \ge 9 \Rightarrow 16 + 9 \ge 9 \Rightarrow 25 \ge 9$ is a true statement, shade the region of the graph that contains the test point (0, 0).



Exercise 16



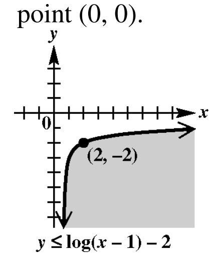
Exercise 17

17. $y > 2^x + 1$

The boundary is an exponential function with y-intercept 2.The inequality symbol is >, wo draw a dashed curve. Since y is *greater than* $2^x + 1,$ the graph of the solution set is the half-plane *above* the boundary. We also can use (0, 0) as a test point. Since $0 > 2^0 + 1 \Rightarrow 0 > 2$ is a false statement, shade the region of the graph not containing the test point (0, 0).

18. $y \le \log(x-1) - 2$

The boundary is a logarithmic function that passes through (2, −2) and has asymptotes $y = -1$ and $x = 1$.The boundary is included in the graph, so draw a solid line. Since $y \le \log(x-1) - 2$, the graph of the solution set is the half-plane *below* the boundary. We also can use (0, 0) as a test point. Since $0 \le \log(0-1) - 2 \Rightarrow 0 \le \log(-1) - 2$ is a false statement ($\log(-1)$ is not defined), shade the region of the graph not containing the test point (0, 0).



19. If an inequality is a “strict” inequality $(<\text{ or }>)$, then the boundary is not included and is represented by a dashed line. If an inequality is a “nonstrict” or “weak” inequality $(\le\text{ or }\ge)$, then the boundary is included and is represented by a solid line.

20. One will shade below the line $y = 3x - 6$. Additional responses will vary.

21. $Ax + By \ge C, B > 0$

Solving for y we have

$Ax + By \ge C \Rightarrow By \ge -Ax + C \Rightarrow y \ge -\frac{A}{B}x + \frac{C}{B}$

Since $B > 0$, the inequality symbol was not reversed when both sides are divided by B. Since $y \ge -\frac{A}{B}x + \frac{C}{B},$ you would shade above the line.

22. $Ax + By \ge C, B < 0$

Solving for y we have

$Ax + By \ge C \Rightarrow By \ge -Ax + C \Rightarrow y \le -\frac{A}{B}x + \frac{C}{B}$

Since $B < 0$, the inequality symbol was reversed when both sides are divided by B. Since $y \le -\frac{A}{B}x + \frac{C}{B},$ you would shade below the line.

23. The graph of $(x-5)^2 + (y-2)^2 = 4$ is a circle with center (5, 2) and radius $r = \sqrt{4} = 2$. The graph of $(x-5)^2 + (y-2)^2 < 4$ is the region in the interior of this circle. The correct response is B.

24. The graph of $y \geq 3x+5$ consists of the line $y = 3x + 5$ and the region above it. The graph of $y < 3x+5$ is the region below the line $y \geq 3x+5,$ and does not include the boundary line. Therefore, the graph of $y < 3x+5$ does not intersect the graph $y \geq 3x+5.$ (Many other answers are possible. The graphs of $y < 3x+b$ and $y \leq 3x+b$ do not intersect the graph of $y \geq 3x+5$ if $b < 5.$)

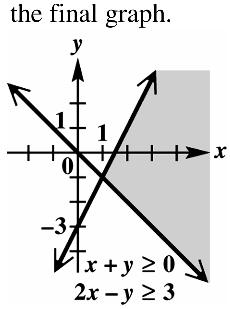
25. The graph of $y \leq 3x-6$ is the region below the line with slope 3 and *y*-intercept – 6. This is graph C.

26. The graph of $y \geq 3x-6$ is the region above the line with a slope of 3 and a *y*-intercept of – 6. This is graph B.

27. The graph of $y \leq -3x-6$ is the region below the line with slope –3 and *y*-intercept – 6. This is graph A.

28. The graph of $y \geq -3x-6$ is the region above the line with a slope of –3 and a *y*-intercept of –6. This is graph D.

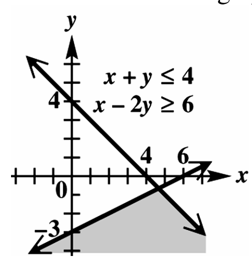
29. $x + y \geq 0$
$2x - y \geq 3$
Graph $x + y = 0$ as a solid line through the origin with a slope of –1. Shade the region above this line. Graph $2x - y = 3$ as a solid line with *x*-intercept $\frac{3}{2}$ and *y*-intercept –3. Shade the region below this line. To find where the boundaries of the two lines intersect, solve the system $\begin{array}{l} x + y = 0 \ (1) \\ 2x - y = 3 \ (2) \end{array}$ by adding the two equations together to obtain $3x = 3 \Rightarrow x = 1.$ Substituting this value into equation (1), we obtain $1 + y = 0 \Rightarrow y = -1.$ Thus, the boundaries intersect at $(1, -1).$ The solution set is the common region, which is shaded in the final graph.



30. $x + y \leq 4$
$x - 2y \geq 6$
Graph $x + y = 4$ as a solid line with *y*-intercept 4 and *x*-intercept 4. Shade the region below this line. Graph $x - 2y = 6$ as a solid line with *y*-intercept –3 and *x*-intercept 6. Shade the region below this line. To find where the boundaries of the two lines intersect, solve the system $\begin{array}{l} x + y = 4 \ (1) \\ x - 2y = 6 \ (2) \end{array}$. Add –1 times equation (1) to equation (2) in order to eliminate *x* and solve for *y*.

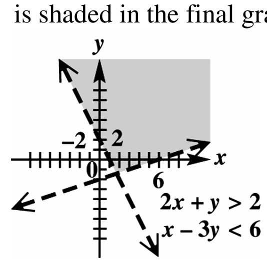
$$\begin{array}{r} -x - y = -4 \\ x - 2y = 6 \\ \hline -3y = 2 \Rightarrow y = -\frac{2}{3} \end{array}$$

Substitute this value into equation (1) to obtain $x + \left(-\frac{2}{3}\right) = 4 \Rightarrow x = 4 + \frac{2}{3} = \frac{14}{3}.$ Thus, the boundaries intersect at $\left(\frac{14}{3}, -\frac{2}{3}\right).$ The solution set is the common region, which is shaded in the final graph.



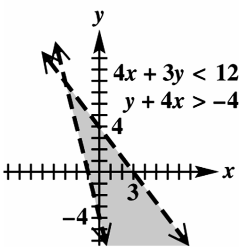
For Exercises 31–62, we will omit finding the points in which the boundaries intersect.

31. $2x + y > 2$
$x - 3y < 6$
Graph $2x + y = 2$ as a dashed line with *y*-intercept 2 and *x*-intercept 1. Shade the region above this line. Graph $x - 3y = 6$ as a dashed line with *y*-intercept –2 and *x*-intercept 6. Shade the region above this line. The solution set is the common region, which is shaded in the final graph.



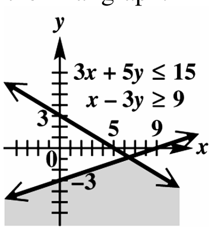
32. $4x + 3y < 12$
$y + 4x > -4$

Graph $4x + 3y = 12$ as a dashed line with y-intercept 4 and x-intercept 3. Shade the region below the line. Graph $y + 4x = -4$ as a dashed line with y-intercept –4 and x-intercept –1. Shade the region above the line. The solution set is the common region, which is shaded in the final graph.



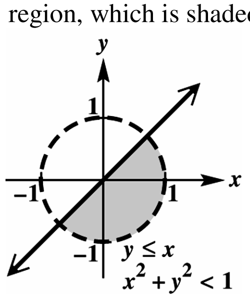
33. $3x + 5y \le 15$
$x - 3y \ge 9$

Graph $3x + 5y = 15$ as a solid line with y-intercept 3 and x-intercept 5. Shade the region below this line. Graph $x - 3y = 9$ as a solid line with y-intercept –3 and x-intercept 9. Shade the region below this line. The solution set is the common region, which is shaded in the final graph.



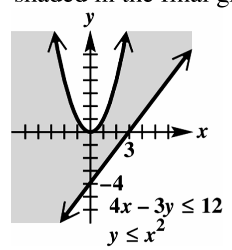
34. $y \le x$
$x^2 + y^2 < 1$

Graph $y = x$ as a solid line through (0, 0) and (1, 1). Shade the region below the line. Graph $x^2 + y^2 = 1$ as a dashed circle centered at $(0,0)$ and radius 1. Shade the region inside the circle. The solution set is the common region, which is shaded in the final graph.



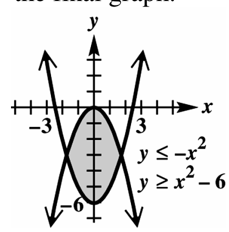
35. $4x - 3y \le 12$
$y \le x^2$

Graph $4x - 3y = 12$ as a solid line with y-intercept –4 and x-intercept 3. Shade the region above this line. Graph the solid parabola $y = x^2$. Shade the region outside of this parabola. The solution set is the intersection of these two regions, which is shaded in the final graph.



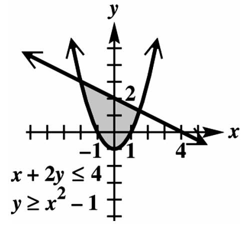
36. $y \le -x^2$
$y \ge x^2 - 6$

Graph $y = -x^2$ as a solid parabola. Shade the region below the curve. Graph $y = x^2 - 6$ using a solid parabola. Shade the region above the curve. The solution set is the region between the two curves, which is shaded in the final graph.



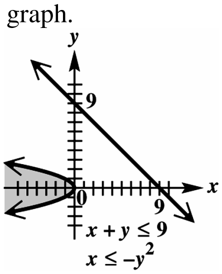
37. $x + 2y \le 4$
$y \ge x^2 - 1$

Graph $x + 2y = 4$ as a solid line with y-intercept 2 and x-intercept 4. Shade the region below the line. Graph the solid parabola $y = x^2 - 1$. Shade the region inside of the parabola. The solution set is the common region, which is shaded in the final graph.



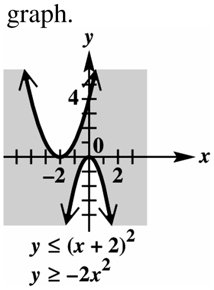
38. $x + y \le 9$
$x \le -y^2$

Graph $x + y = 9$ as a solid line with y-intercept 9 and x-intercept 9. Shade the region below this line. Graph the solid horizontal parabola $x = -y^2$. Shade the region inside of this parabola. The solution set is the intersection of these two regions, which is shaded in the final graph.



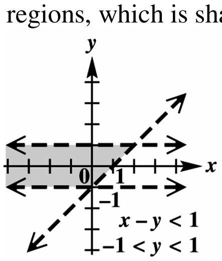
39. $y \le (x+2)^2$
$y \ge -2x^2$

Graph $y = (x+2)^2$ as a solid parabola opening up with a vertex at (–2, 0). Shade the region below the parabola. Graph $y = -2x^2$ as a solid parabola opening down with a vertex at the origin. Shade the region above the parabola. The solution set is the intersection of these two regions, which is shaded in the final graph.



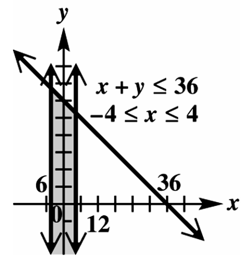
40. $x - y < 1$
$-1 < y < 1$

Graph $x - y = 1$ as a dashed line with y-intercept –1 and x-intercept 1. Shade the region above the line. Graph the horizontal lines $y = -1$ and $y = 1$ as dashed lines. Shade the region between these two lines. The solution set is the intersection of these two regions, which is shaded in the final graph.



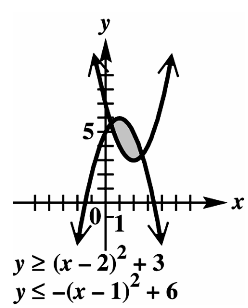
41. $x + y \le 36$
$-4 \le x \le 4$

Graph $x + y = 36$ as a solid line with y-intercept 36 and x-intercept 36. Shade the region below this line. Graph the vertical lines $x = -4$ and $x = 4$ as solid lines. Shade the region between these lines. The solution set is the intersection of these two regions, which is shaded in the final graph.

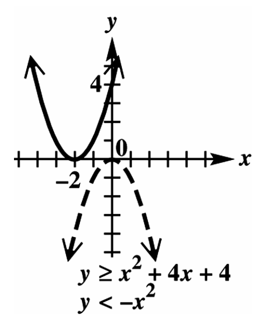


42. $y \ge (x-2)^2 + 3$
$y \le -(x-1)^2 + 6$

Graph $y = (x-2)^2 + 3$ as a solid parabola opening up with a vertex at (2, 3). Shade the region above the parabola. Graph $y = -(x-1)^2 + 6$ as a solid parabola opening down with a vertex at (1, 6). Shade the region below the parabola. The solution set is the intersection of these two regions, which is shaded in the final graph.



Exercise 42



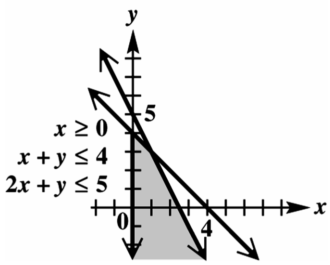
Exercise 43

43. $y \ge x^2 + 4x + 4$
$y < -x^2$

Graph the solid parabola $y = x^2 + 4x + 4$ or $y = (x+2)^2$, which has vertex (–2, 0) and opens upward. Shade the region inside of this parabola. Graph the dashed parabola $y = -x^2$, which has vertex (0, 0) and opens downward. Shade the region inside this parabola. These two regions have no points in common, so the system has no solution.

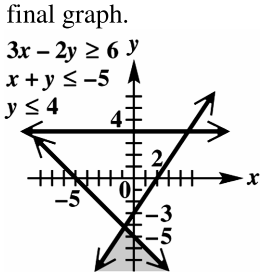
44. $x > 0$
$x + y \le 4$
$2x + y \le 5$

Graph $x + y = 4$ as a solid line through (0, 4) and (4, 0). Shade the region below the line. Graph $2x + y = 5$ as a solid line through $(0, 5)$ and $\left(\frac{5}{2}, 0\right)$. Shade the region below the line. The solution set is the intersection of these two regions and the region to the right of the x-axis, as shown as the final graph.



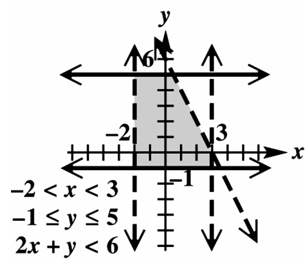
45. $3x - 2y \ge 6$
$x + y \le -5$
$y \le 4$

Graph $3x - 2y = 6$ as a solid line and shade the region below it. Graph $x + y = -5$ as a solid line and shade the region below it. Graph $y = 4$ as a solid horizontal line and shade the region below it. The solution set is the intersection of these three regions, which is shaded in the final graph.



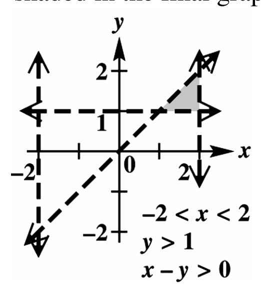
46. $-2 < x < 3$
$-1 \le y \le 5$
$2x + y < 6$

Graph $2x + y = 6$ as a dashed line through (0, 6) and (3, 0). Shade the region below this line. Shade the region between the two solid horizontal lines $y = 5$ and $y = -1$. Shade the region between the two vertical dashed lines $x = -2$ and $x = 3$. The solution set is the common region, which is shaded in the final graph.



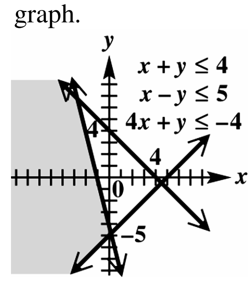
47. $-2 < x < 2$
$y > 1$
$x - y > 0$

Graph the vertical lines $x = -2$ and $x = 2$ as a dashed line. Shade the region between the two lines. Graph the horizontal line $y = 1$ as a dashed line. Shade the region above the line. Graph the line $x - y = 0$ as a dashed line through the origin with a slope of 1. Shade the region below this line. The solution set is the intersection of these three regions, which is shaded in the final graph.



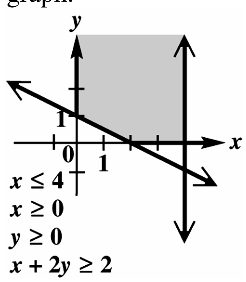
48. $x + y \le 4$
$x - y \le 5$
$4x + y \le -4$

Graph $x + y = 4$ as a solid line and shade the region below it. Graph $x - y = 5$ as a solid line and shade the region above it. Graph $4x + y = -4$ as a solid line and shade the region below it. The solution set is the common region, which is shaded in the final graph.



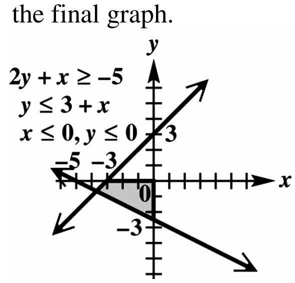
49. $x \le 4$
$x \ge 0$
$y \ge 0$
$x + 2y \ge 2$

Graph $x = 4$ as a solid vertical line. Shade the region to the left of this line. Graph $x = 0$ as a solid vertical line. (This is the y-axis.) Shade the region to the right of this line. Graph $y = 0$ as a solid horizontal line. (This is the x-axis.) Shade the region above the line. Graph $x + 2y = 2$ as a solid line with x-intercept 2 and y-intercept 1. Shade the region above the line. The solution set is the intersection of these four regions, which is shaded in the final graph.



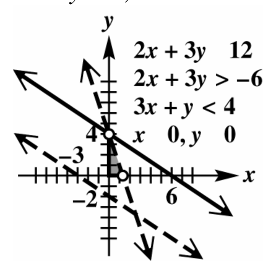
50. $2y + x \ge -5$
$y \le 3 + x$
$x \le 0$
$y \le 0$

Graph $2y + x = -5$ as a solid line through $\left(0, -\frac{5}{2}\right)$ and (–5, 0). Shade the region above the line. Graph $y = 3 + x$ as a solid line through (0, 3) and (–3, 0). Shade the region below this line. The solution is the common region, which is also below the x-axis and to the left of the y-axis. This region is shaded in the final graph.



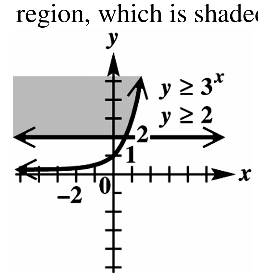
51. $2x + 3y \le 12$
$2x + 3y > -6$
$3x + y < 4$
$x \ge 0$
$y \ge 0$

Graph $2x + 3y = 12$ as a solid line and shade the region below it. Graph $2x + 3y = 6$ as a dashed line and shade the region above it. Graph $3x + y = 4$ as a dashed line and shade the region below it. $x = 0$ is the y-axis. Shade the region to the right of it. $y = 0$ is the x-axis. Shade the region above it. The solution set is the intersection of these five regions, which is shaded in the final graph. The open circles at (0, 4) and $\left(\frac{4}{3}, 0\right)$ indicate that those points are not included in the solution (due to the fact that the boundary line on which they lie, $3x + y = 4$, is not included).



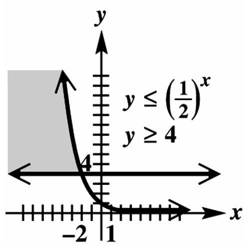
52. $y \ge 3^x$
$y \ge 2$

Graph the exponential function $y = 3^x$ using a solid curve passing through the points $\left(-2, \frac{1}{9}\right)$, $\left(-1, \frac{1}{3}\right)$, (0, 1), (1, 3), and (2, 9); shade the region above it. Graph the solid horizontal line $y = 2$ and shade the region above it. The solution set is the common region, which is shaded in the final graph.



53. $y \le \left(\frac{1}{2}\right)^x$
$y \ge 4$

Graph $y = \left(\frac{1}{2}\right)^x$ using a solid curve passing through the points (–2, 4), (–1, 2), (0, 1), $\left(1, \frac{1}{2}\right)$, and $\left(2, \frac{1}{4}\right)$. Shade the region below this curve. Graph the solid horizontal line $y = 4$ and shade the region above it. The solution set consists of the intersection of these two regions, which is shaded in the final graph.



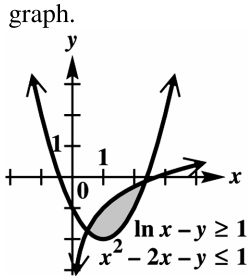
54. $\ln x - y \ge 1$
$x^2 - 2x - y \le 1$

In order to graph the curves, we must solve each relation for y.

$$\ln x - y \ge 1 \Rightarrow -y \ge -\ln x + 1 \Rightarrow y \le \ln x - 1 \quad (1)$$

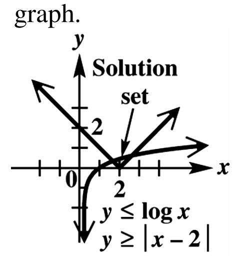
$$x^2 - 2x - y \le 1 \Rightarrow -y \le -x^2 + 2x + 1 \Rightarrow y \ge x^2 - 2x - 1 \Rightarrow y \ge (x-1)^2 - 2 \quad (2)$$

The boundary for inequality (1) is the graph of $y = \ln x - 1$, which is the curve $y = \ln x$ translated down 1 unit. Shade the region below this solid curve. The boundary for inequality (2) is the graph of $y = (x-1)^2 - 2$, which is the parabola whose vertex is (1, –2) and whose y-intercept is –1. Shade the region above this solid curve. The solution set is the common region, which is shaded in the final graph.



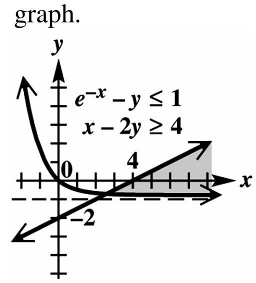
55. $y \le \log x$
$y \ge |x-2|$

Graph $y = \log x$ using a solid curve because $y \le \log x$ is a nonstrict inequality. (Recall that "log x" means $\log_{10} x$.) This graph contains the points $(.1, -1)$, $(1, 0)$, and $(10, 1)$. Use a calculator to approximate other points on the graph, such as $(2, .30)$ and $(4, .60)$. Because the symbol is $\le$, shade the region *below* the curve. Now graph $y = |x-2|$. Make this boundary solid because $y \ge |x-2|$ is also a nonstrict inequality. This graph can be obtained by translating the graph of $y = |x|$ to the right 2 units. It contains points $(0, 2)$, $(2, 0)$, and $(4, 2)$. Because the symbol is $\ge$, shade the region *above* the absolute value graph. The solution set is the intersection of the two regions, which is shaded in the final graph.



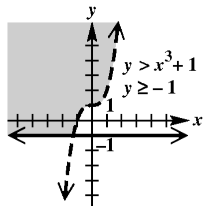
56. $e^{-x} - y \le 1$
$x - 2y \ge 4$

Graph $e^{-x} - y = 1 \Rightarrow y = e^{-x} - 1$ as the solid curve which is the graph of $y = e^{-x}$, translated down 1 unit. Shade the region above this curve. Graph $x - 2y = 4$ as the solid line through (0, –2) and (4, 0). Shade the region below this line. The solution set is the common region, which is shaded in the final graph.



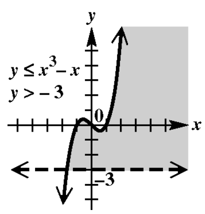
57. $y > x^3 + 1$
$y \geq -1$

Graph $y > x^3 + 1$ as a dashed curve which is the graph of $y = x^3$ translated up 1 unit. Shade the region above the curve. Graph $y = -1$ as a solid horizontal line through (0, −1). Shade the area above the line. The solution set is the common region, which is shaded in the final graph.



58. $y \leq x^3 - x$
$y > -3$

Graph $y = x^3 - x$ as a solid curve passing through the points (−2, −36), (−1, 0), (0, 0), (1, 0), and (2, 36). Shade the region below the curve. Graph $y = -3$ as a solid horizontal line through (0, −3). Shade the region above the curve. The solution set is the common region, which is shaded in the final graph.



59. The upper line passes through (0, 2) and (4, 0), so $m = \frac{0-2}{4-0} = -\frac{1}{2}$, and the equation is $y = -\frac{1}{2}x + 2 \Rightarrow x + 2y = 4$. The line is solid, so it is included in the inequality. The lower line passes through (0, −3) and (4, 0), so $m = \frac{0-(-3)}{4-0} = \frac{3}{4}$ and the equation is $y = \frac{3}{4}x - 3 \Rightarrow 3x - 4y = 12$. The line is solid, so it is included in the inequality. The shaded region contains the point (0, 0), so test (0, 0) in each equation to determine the direction of the inequalities:
$0 + 2(0) = 0 < 4 \Rightarrow x + 2y \leq 4$,
$3(0) + 4(0) = 0 < 12 \Rightarrow 3x - 4y \leq 12$
The system is $\begin{array}{c} x + 2y \leq 4 \\ 3x - 4y \leq 12 \end{array}$.

60. The dashed line with positive slope passes through (−1, 0) and (0, 3), so $m = \frac{0-3}{-1-0} = 3$, and the equation of the line is $y = 3x + 3 \Rightarrow 3x - y = -3$. The line is dashed so it is not included in the inequality. The dashed line with negative slope passes through (−2, 0) and (0, −3), so $m = \frac{-3-0}{0-(-2)} = -\frac{3}{2}$, and the equation of the line is $y = -\frac{3}{2}x - 3 \Rightarrow 3x + 2y = -6$
The line is dashed so it is not included in the inequality. The shaded region contains the point (−3, −1), so test (−3, −1) in each equation to determine the direction of the inequalities:
$3(-3) - (-1) = 10 > -3 \Rightarrow 3x - y > -3$,
$3(-3) + 2(-1) = -11 < -6 \Rightarrow 3x + 2y < -6$.
The system is $\begin{array}{c} 3x - y > -3 \\ 3x + 2y < -6 \end{array}$.

61. The circle has center (0, 0) and radius 4, so its equation is $x^2 + y^2 = 16$. The curve is solid, so it is included in the inequality. The horizontal line passes through (0, 2), so its equation is $y = 2$. It is solid, so it is included in the inequality. The shaded region includes the point (−1, 3), so test (−1, 3) in each equation to determine the direction of the inequalities:
$(-1)^2 + 3^2 = 10 < 16 \Rightarrow x^2 + y^2 \leq 16$,
$3 > 2 \Rightarrow y > 2$. The system is $\begin{array}{c} x^2 + y^2 \leq 16 \\ y \geq 2 \end{array}$.

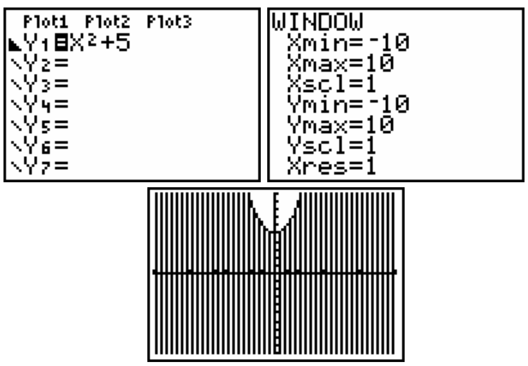
62. The parabola passes through (−1, 0), (0, −1), and (1, 0) and appears to be the graph of $y = x^2$ transformed one unit down, that is $y = x^2 - 1$. Check each point in the equation to verify: $0 = (-1)^2 - 1, -1 = 0^2 - 1, 0 = 1^2 - 1$.
The curve is solid, so it is included in the inequality. The dashed line passes through (−4, 0) and (0, 2), so $m = \frac{2-0}{0-(-4)} = \frac{1}{2}$, and the equation is $y = \frac{1}{2}x + 2 \Rightarrow x - 2y = -4$. The line is dashed so it is not included in the inequality. The shaded region includes (0, 0), so test (0, 0) in each equation to determine the direction of the inequalities:
$0 > 0^2 - 1 = -1 \Rightarrow y \geq x^2 - 1$,
$0 - 2(0) = 0 > -4 \Rightarrow y - 2x > -4$
The system is $\begin{array}{c} y \geq x^2 - 1 \\ y - 2x > -4 \end{array}$.

63. The graph is in the first quadrant, so $x > 0$ and $y > 0$. A circle with radius 2 centered at the origin has the equation $x^2 + y^2 = 4$. Since the region includes and is inside the circle, the inequality is $x^2 + y^2 \le 4$. A line that passes through (0, −1) and (2, 2) has slope $m = \frac{2-(-1)}{2-0} = \frac{3}{2}$ and equation $y = \frac{3}{2}x - 1$. Since the region is above the line, but does not include the line, the inequality is $y > \frac{3}{2}x - 1$.

The system is
$$\begin{aligned} x &> 0 \\ y &> 0 \\ x^2 + y^2 &\le 4 \\ y &> \tfrac{3}{2}x - 1 \end{aligned}.$$

64. $y \le x^2 + 5$

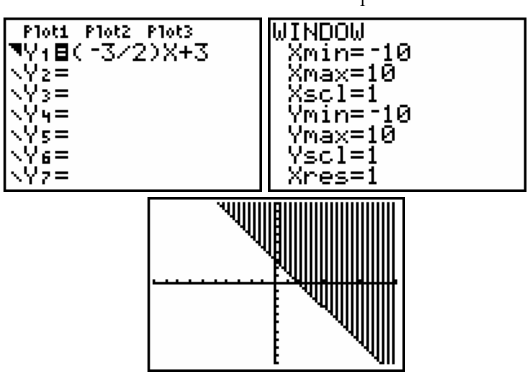
Enter $Y_1 = x^2 + 5$ and use a graphing calculator to shade the region below the parabola. Notice the icon to the left of Y_1.



65. $3x + 2y \ge 6$

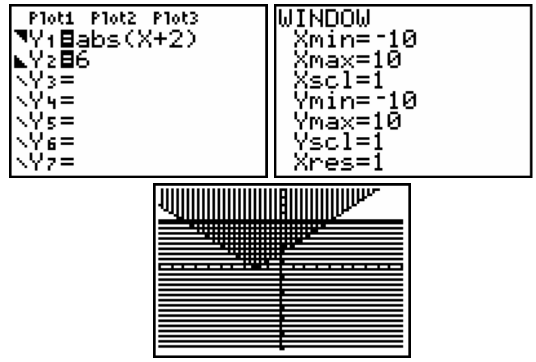
Solving the inequality for y we have $3x + 2y \ge 6 \Rightarrow 2y \ge -3x + 6 \Rightarrow y \ge -\frac{3}{2}x + 3.$

Enter $Y_1 = (-3/2)x + 3$ and use a graphing calculator to shade the region above the line. Notice the icon to the left of Y_1.



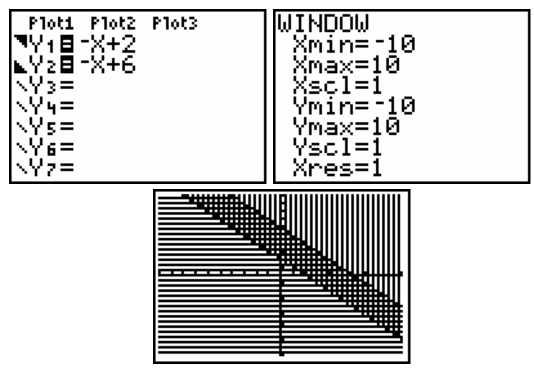
66. $y \ge |x + 2|;\ y \le 6$

Enter $Y_1 = \text{abs}(x + 2)$ and $Y_2 = 6$. Shade the region above the graph of $Y_1 = \text{abs}(x + 2)$ and below the graph of $Y_2 = 6$.



67. $x + y \ge 2;\ x + y \le 6$

Solving each inequality for y, we have $y = -x + 2$ and $y = -x + 6$. Enter $Y_1 = -x + 2$ and $Y_2 = -x + 6$ and use the graphing calculator to shade the region above the graph of $y_1 = -x + 2$ and below the graph of $y_2 = -x + 6$.



68. **(a)** $c = 10x + 20y \Rightarrow 20y = -10x + c \Rightarrow y = -\frac{1}{2}x + \frac{c}{20}$

(b) As c increases, the line of constant cost moves up.

(c) C gives the optimal solution.

69.

Point	Value of $3x + 5y$	
$(1,1)$	$3(1)+5(1)=8$	← Minimum
$(2,7)$	$3(2)+5(7)=41$	
$(5,10)$	$3(5)+5(10)=65$	← Maximum
$(6,3)$	$3(6)+5(3)=33$	

The maximum value is 65 at $(5,10)$. The minimum value is 8 at (1, 1).

70.

Point	Value of $6x + y$	
$(1, 2)$	$6(1) + 2 = 8$	← Minimum
$(1, 5)$	$6(1) + 5 = 11$	
$(6, 8)$	$6(6) + 8 = 44$	
$(9, 1)$	$6(9) + 1 = 55$	← Maximum

The maximum value is 55 at $(9, 1)$. The minimum value is 8 at $(1, 2)$.

71.

Point	Value of $3x + 5y$	
$(1, 0)$	$3(1) + 5(0) = 3$	← Minimum
$(1, 10)$	$3(1) + 5(10) = 53$	
$(7, 9)$	$3(7) + 5(9) = 66$	← Maximum
$(7, 6)$	$3(7) + 5(6) = 51$	

The maximum value is 66 at $(7, 9)$. The minimum value is 3 at $(1, 0)$.

72.

Point	Value of $5x + 5y$	
$(1, 0)$	$5(1) + 5(0) = 5$	← Minimum
$(1, 10)$	$5(1) + 5(10) = 55$	
$(7, 9)$	$5(7) + 5(9) = 80$	← Maximum
$(7, 6)$	$5(7) + 5(6) = 65$	

The maximum value is 80 at $(7, 9)$. The minimum value is 5 at $(1, 0)$.

73.

Point	Value of $10y$	
$(1, 0)$	$10(0) = 0$	← Minimum
$(1, 10)$	$10(10) = 100$	← Maximum
$(7, 9)$	$10(9) = 90$	
$(7, 6)$	$10(6) = 60$	

The maximum value is 100 at $(1, 10)$. The minimum value is 0 at $(1, 0)$.

74.

Point	Value of $3x - y$	
$(1, 0)$	$3(1) - 0 = 3$	
$(1, 10)$	$3(1) - 10 = -7$	← Minimum
$(7, 9)$	$3(7) - 9 = 12$	
$(7, 6)$	$3(7) - 6 = 15$	← Maximum

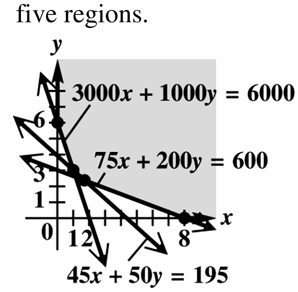
The maximum value is 15 at $(7, 6)$. The minimum value is –7 at $(1, 10)$.

75. Let x = the number of Brand X pills; y = the number of Brand Y pills. The following table is helpful in organizing the information.

	Number of Brand X pills (x)	Number of Brand Y pills (y)	Restrictions
Vitamin A	3000	1000	At least 6000
Vitamin C	45	50	At least 195
Vitamin D	75	200	At least 600

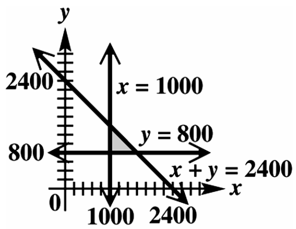
We have
$$3000x + 1000y \ge 6000$$
$$45x + 50y \ge 195$$
$$75x + 200y \ge 600$$
$$x \ge 0,\ y \ge 0.$$

Graph $3000x + 1000y = 6000$ as a solid line with x-intercept 2 and y-intercept 6. Shade the region above the line. Graph $45x + 50y = 195$ as a solid line with x-intercept $4.\overline{3}$ and y-intercept 3.9. Shade the region above the line. Graph $75x + 200y = 600$ as a solid line with x-intercept 8 and y-intercept 3. Shade the region above the line. Graph $x = 0$ (the y-axis) as a solid line and shade the region to the right of it. Graph $y = 0$ (the x-axis) as a solid line and shade the region above it. The region of feasible solutions is the intersection of these five regions.



76. Let x = number of boxes to Des Moines, and y = number of boxes to San Antonio.

We have
$$x \geq 1000$$
$$y \geq 800$$
$$x + y \leq 2400.$$



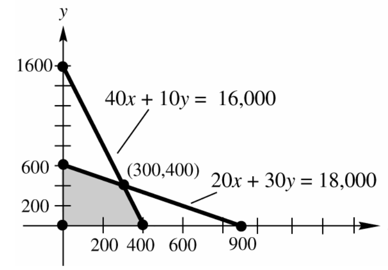
77. **(a)** Let x = number of cartons of food; y = number of cartons of clothing. The following table is helpful in organizing the information.

	Number of cartons of food (x)	Number of cartons of clothing (y)	Restrictions
Weight	40	10	Cannot Exceed 16,000
Volume	20	30	No more than 18,000

We have
$$40x + 10y \leq 16,000$$
$$20x + 30y \leq 18,000$$
$$x \geq 0,\ y \geq 0.$$

Maximize objective function, number of people $= 10x + 8y$.

Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $40x + 10y \leq 16,000,$ draw the line with x-intercept 400 and y-intercept 1600 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $20x + 30y \leq 18,000,$ draw the line with x-intercept 900 and y-intercept 600 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. The constraints $x \geq 0$ and $y \geq 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three are the vertices are $(0,0), (0,600),$ and $(400,0)$. The fourth vertex is the intersection point of the lines $40x + 10y = 16,000$ and $20x + 30y = 18,000.$ To find this point, solve the system
$$40x + 10y = 16,000$$
$$20x + 30y = 18,000.$$

The first equation can be written as $4x + y = 1600 \Rightarrow y = 1600 - 4x.$

Substituting this equation into $20x + 30y = 18,000,$ we have
$$20x + 30(1600 - 4x) = 18,000$$
$$20x + 48,000 - 120x = 18,000$$
$$48,000 - 100x = 18,000$$
$$-100x = -30,000$$
$$x = 300$$

Substituting $x = 300$ into $y = 1600 - 4x,$ we have
$$y = 1600 - 4(300) = 1600 - 1200 = 400.$$

Thus, the fourth vertex is $(300, 400)$.

Next, evaluate the objective function at each vertex.

Point	Number of people $= 10x + 8y$	
$(0,0)$	$10(0) + 8(0) = 0$	
$(0,600)$	$10(0) + 8(600) = 4800$	
$(300,400)$	$10(300) + 8(400) = 6200$	$\leftarrow$ Maximum
$(400,0)$	$10(400) + 8(0) = 4000$	

The maximum value of $10x + 8y$ occurs at $(300, 400)$, so they should send 300 cartons of food and 400 cartons of clothes to maximize the number of people helped.

(b) The maximum number of people helped is 6200.

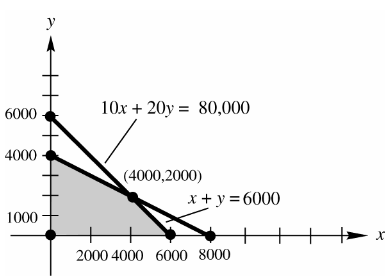
78. (a) Let x = number of medical kits, y = number of containers of water. The following table is helpful in organizing the information.

	Number of medical kits (x)	Number of containers of water (y)	Restrictions
Weight	10	20	No more than 80,000
Volume	1	1	No more than 6000

We have
$$10x + 20y \le 80,000$$
$$x + y \le 6000$$
$$x \ge 0,\ y \ge 0.$$

Maximize objective function, number of people $= 4x + 10y$.

Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $10x + 20y \le 80,000,$ draw the line with x-intercept 8000 and y-intercept 4000 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $x + y \le 6000,$ draw the line with x-intercept 6000 and y-intercept 6000 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. The constraints $x \ge 0$ and $y \ge 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three vertices are $(0,0), (0,4000),$ and $(6000,0)$. The fourth vertex is the intersection point of the lines $10x + 20y = 80,000$ and $x + y = 6000.$

To find this point, solve the system
$$10x + 20y = 80,000$$
$$x + y = 6000.$$

The second equation can be written as $y = 6000 - x.$ Substituting this equation into $10x + 20y = 80,000,$ we have
$$10x + 20(6000 - x) = 80,000$$
$$10x + 120,000 - 20x = 80,000$$
$$-10x = -40,000$$
$$x = 4000$$

Substituting $x = 4000$ into $y = 6000 - x,$ we have $y = 6000 - 4000 = 2000.$ Thus, the fourth vertex is $(4000, 2000)$. Next, evaluate the objective function at each vertex.

Point	Number of people $= 4x + 10y$	
$(0,0)$	$4(0) + 10(0) = 0$	
$(0,4000)$	$4(0) + 10(4000) = 40,000$	← Maximum
$(4000,2000)$	$4(4000) + 10(2000) = 36,000$	
$(6000,0)$	$4(6000) + 10(0) = 24,000$	

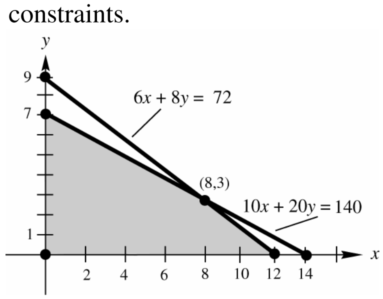
The maximum value of $4x + 10y$ occurs at $(0,4000)$, so they should ship no medical kits and 4000 containers of water to maximize the number of people helped.

(b) Maximize $6x + 10y$ by evaluating at each vertex.

Point	Number of people $= 6x + 10y$	
$(0,0)$	$6(0) + 10(0) = 0$	
$(0,4000)$	$6(0) + 10(4000) = 40,000$	
$(4000,2000)$	$6(4000) + 10(2000) = 44,000$	← Maximum
$(6000,0)$	$6(6000) + 10(0) = 36,000$	

Ship 4000 medical kits and 2000 containers of water to maximize the number of people helped.

79. Let x = number of cabinet A; y = number of cabinet B.
The cost constraint is $10x + 20y \le 140$. The space constraint is $6x + 8y \le 72$. Since the numbers of cabinets cannot be negative, we also have $x \ge 0$ and $y \ge 0$. We want to maximize the objective function, storage capacity $= 8x + 12y$. Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $10x + 20y \le 140$, draw the line with x-intercept 14 and y-intercept 7 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region below the line. To graph $6x + 8y \le 72$, draw the line with x-intercept 12 and y-intercept 9 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region below the line. The constraints $x \ge 0$ and $y \ge 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three vertices are $(0,0), (0,7)$, and $(12,0)$. The fourth vertex is the intersection point of the lines $10x + 20y = 140$ and $6x + 8y = 72$. To find this point, solve the system

$$\begin{aligned} 10x + 20y &= 140 \\ 6x + 8y &= 72. \end{aligned}$$

The first equation can be written as $x + 2y = 14 \Rightarrow x = 14 - 2y$. Substituting this equation into $6x + 8y = 72$, we have

$$6(14 - 2y) + 8y = 72 \Rightarrow 84 - 12y + 8y = 72 \Rightarrow$$
$$84 - 4y = 72 \Rightarrow -4y = -12 \Rightarrow y = 3$$

Substituting $y = 3$ into $x = 14 - 2y$, we have $x = 14 - 2(3) = 14 - 6 = 8$. Thus, the fourth vertex is $(8,3)$. Next, evaluate the objective function at each vertex.

Point	Storage Capacity $= 8x + 12y$	
$(0,0)$	$8(0) + 12(0) = 0$	
$(0,7)$	$8(0) + 12(7) = 84$	
$(8,3)$	$8(8) + 12(3) = 100$	← Maximum
$(12,0)$	$8(12) + 12(0) = 96$	

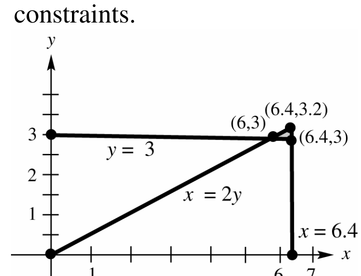
The maximum value of $8x + 12y$ occurs at $(8,3)$, so the office manager should buy 8 of cabinet A and 3 of cabinet B for a total storage capacity of 100 ft^3.

80. Let x = number of millions of gallons of gasoline; y = number of millions of gallons of fuel oil.
Constraints are

$$\begin{aligned} x &\ge 2y \\ y &\ge 3 \\ x &\le 6.4 \\ x &\ge 0. \end{aligned}$$

We want to maximize the objective function, revenue $= 1.90x + 1.50y$.
Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $x \ge 2y$, draw the line that passes through the origin and another point such as $(6,3)$. Since $x \ge 2y \Rightarrow y \le \frac{1}{2}x$, we shade the region *below* the line. To graph $y \ge 3$, we shade *above* the solid line $y = 3$. To graph $x \le 6.4$, we shade *below* the solid line $x = 6.4$.

The constraints $x \ge 0$ and $y \ge 3$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that the three vertices are $(6.4,3)$, $(6,3)$, and $(6.4,3.2)$. Next, evaluate the objective function at each vertex.

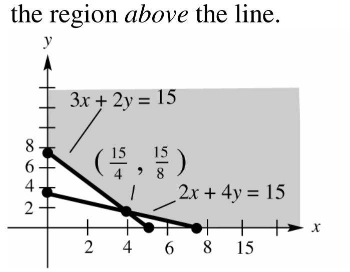
(*continued on next page*)

(*continued from page 967*)

Point	Storage Capacity $= 1.90x + 1.50y$	
$(6.4, 3)$	$1.90(6.4) + 1.50(3) = 16.66$	
$(6, 3)$	$1.90(6) + 1.50(3) = 15.9$	
$(6.4, 3.2)$	$1.90(6.4) + 1.50(3.2) = 16.96$	← Maximum

The maximum value of $1.90x + 1.50y$ occurs at $(6.4, 3.2)$, so the revenue will be maximized when 6.4 million gallons of gasoline and 3.2 million gallons of fuel oil are produced for a maximum revenue of \$16,960,000.

81. Let x = number of servings of product A; y = number of servings of product B.
The Supplement I constraint is $3x + 2y \ge 15$.
The Supplement II constraint is $2x + 4y \ge 15$.
Since the numbers of servings cannot be negative, we also have $x \ge 0$ and $y \ge 0$. We want to minimize the objective function, cost $= .25x + .40y$. Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $3x + 2y \ge 15$, draw the line with x-intercept 5 and y-intercept $\frac{15}{2} = 7\frac{1}{2}$ as a solid line. Because the test point $(0, 0)$ does not satisfy this inequality, shade the region *above* the line. To graph $2x + 4y \ge 15$, draw the line with x-intercept $\frac{15}{2} = 7\frac{1}{2}$ and y-intercept $\frac{15}{4} = 3\frac{3}{4}$ as a solid line. Because the test point $(0, 0)$ does not satisfy this inequality, shade the region *above* the line.



The constraints $x \ge 0$ and $y \ge 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.

From the graph, observe that two vertices are $\left(0, \frac{15}{2}\right)$ and $\left(\frac{15}{2}, 0\right)$. The third vertex is the intersection point of the lines $3x + 2y = 15$ and $2x + 4y = 15$. To find this point, solve the system
$$3x + 2y = 15$$
$$2x + 4y = 15.$$
Multiply the first equation by –2 and add it to the second equation.
$$-6x - 4y = -30$$
$$2x + 4y = 15$$
$$-4x = -15 \Rightarrow x = \frac{15}{4}$$
Substituting this equation into $2x + 4y = 15$, we have $2\left(\frac{15}{4}\right) + 4y = 15 \Rightarrow \frac{15}{2} + 4y = 15 \Rightarrow 15 + 8y = 30 \Rightarrow 8y = 15 \Rightarrow y = \frac{15}{8}$. Thus, the third vertex is $\left(\frac{15}{4}, \frac{15}{8}\right)$. Next, evaluate the objective function at each vertex.

Point	Cost $= .25x + .40y$	
$\left(0, \frac{15}{2}\right)$	$.25(0) + .40\left(\frac{15}{2}\right) = 3.00$	
$\left(\frac{15}{4}, \frac{15}{8}\right)$	$.25\left(\frac{15}{4}\right) + .40\left(\frac{15}{8}\right) = 1.69$	← Minimum
$\left(\frac{15}{2}, 0\right)$	$.25\left(\frac{15}{2}\right) + .40(0) = 1.88$	

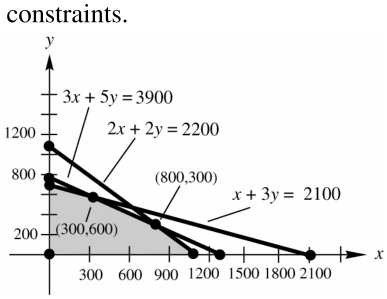
The minimum cost is \$1.69 for $\frac{15}{4} = 3\frac{3}{4}$ servings of *A* and $\frac{15}{8} = 1\frac{7}{8}$ servings of *B*.

82. Let x = the number of bargain sets; y = the number of deluxe sets.
The following table is helpful in organizing the information.

	Number of bargain sets (x)	Number of deluxe sets (y)	Restrictions
Assemby	3	5	No more than 3900
Cabinet	1	3	No more than 2100
Testing and Packaging	2	2	No more than 2200

We have
$$3x + 5y \le 3900$$
$$x + 3y \le 2100$$
$$2x + 2y \le 2200$$
$$x \ge 0,\ y \ge 0.$$
Maximize profit function, number of people $= 100x + 150y$.
Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints.

To graph $3x+5y \le 3900$, draw the line with x-intercept 1300 and y-intercept 780 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $x+3y \le 2100$, draw the line with x-intercept 2100 and y-intercept 700 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $2x+2y \le 2200$, draw the line with x-intercept 1100 and y-intercept 1100 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. The constraints $x \ge 0$ and $y \ge 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three vertices are $(0,0), (0,700)$, and $(1100,0)$. A fourth vertex is the intersection point of the lines $3x+5y=3900$ and $2x+2y=2200$. To find this point, solve the system

$$\begin{aligned} 3x+5y &= 3900 \\ 2x+2y &= 2200. \end{aligned}$$

The second equation can be written as $x+y=1100 \Rightarrow y=1100-x$. Substituting this equation into $3x+5y=3900$, we have

$$\begin{aligned} 3x+5(1100-x) &= 3900 \\ 3x+5500-5x &= 3900 \\ -2x &= -1600 \Rightarrow x = 800 \end{aligned}$$

Substituting $x=800$ into $y=1100-x$, we have $y=1100-600=300$. Thus, the fourth vertex is $(800,300)$. The fifth vertex is the intersection point of the lines $3x+5y=3900$ and $x+3y=2100$. To find this point, solve the system

$$\begin{aligned} 3x+5y &= 3900 \\ x+3y &= 2100. \end{aligned}$$

The second equation can be written as $x=2100-3y$. Substituting this equation into $3x+5y=3900$, we have

$$\begin{aligned} 3(2100-3y)+5y &= 3900 \\ 6300-9y+5y &= 3900 \\ -4y &= -2400 \Rightarrow y = 600 \end{aligned}$$

Substituting $y=600$ into $x=2100-3y$, we have $x=2100-3(600)=2100-1800=300$. Thus, the fifth vertex is $(300,600)$. Next, evaluate the objective function at each vertex.

Point	Number of people $=100x+150y$	
$(0,0)$	$100(0)+150(0)=0$	
$(0,700)$	$100(0)+150(700)=105,000$	
$(300,600)$	$100(300)+150(600)=120,000$	
$(800,300)$	$100(800)+150(300)=125,000$	← Maximum
$(1100,0)$	$100(1100)+150(0)=110,000$	

The company should produce 800 bargain and 300 deluxe sets for a maximum profit of \$125,000.

Section 9.7: Properties of Matrices

1. $\begin{bmatrix} -3 & a \\ b & 5 \end{bmatrix} = \begin{bmatrix} c & 0 \\ 4 & d \end{bmatrix}$

Since corresponding elements are equal, we have the following.
$a=0, b=4, c=-3, d=5$

2. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Since corresponding elements are equal, we have the following.
$w=9, x=17, y=8, z=-12$

3. $\begin{bmatrix} x+2 & y-6 \\ z-3 & w+5 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 0 & 3 \end{bmatrix}$

Since corresponding elements are equal, we have the following.
$x+2=-2 \Rightarrow x=-4$
$y-6=8 \Rightarrow y=14$
$z-3=0 \Rightarrow z=3$
$w+5=3 \Rightarrow w=-2$

4. $\begin{bmatrix} 6 & a+3 \\ b+2 & 9 \end{bmatrix} = \begin{bmatrix} c-3 & 4 \\ -2 & d-4 \end{bmatrix}$

Since corresponding elements are equal, we have the following.

$a+3=4 \Rightarrow a=1$
$b+2=-2 \Rightarrow b=-4$
$6=c-3 \Rightarrow c=9$
$9=d-4 \Rightarrow d=13$

5. $\begin{bmatrix} 0 & 5 & x \\ -1 & 3 & y+2 \\ 4 & 1 & z \end{bmatrix} = \begin{bmatrix} 0 & w+3 & 6 \\ -1 & 3 & 0 \\ 4 & 1 & 8 \end{bmatrix}$

Since corresponding elements are equal, we have the following.

$x=6$
$y+2=0 \Rightarrow y=-2$
$z=8$
$w+3=5 \Rightarrow w=2$

6. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Since corresponding elements are equal, we have the following.

$x-4=2 \Rightarrow x=6$
$5=y+3 \Rightarrow y=2$
$2=z+4 \Rightarrow z=-2$
$w=5$

7. Since $\begin{bmatrix} x & y & z \end{bmatrix}$ is a 1×3 matrix and $\begin{bmatrix} 21 & 6 \end{bmatrix}$ is a 1×2 matrix, the statement cannot be true, hence we cannot find values of *x*, *y*, and *z*.

8. Since $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ is a 3×1 matrix and $\begin{bmatrix} 3 \\ -9 \end{bmatrix}$ is a 2×1 matrix, the statement cannot be true, hence we cannot find values of *p*, *q*, and *r*.

9. $\begin{bmatrix} -7+z & 4r & 8s \\ 6p & 2 & 5 \end{bmatrix} + \begin{bmatrix} -9 & 8r & 3 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 36 & 27 \\ 20 & 7 & 12a \end{bmatrix} \Rightarrow$

$\begin{bmatrix} -16+z & 12r & 8s+3 \\ 6p+2 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 36 & 27 \\ 20 & 7 & 12a \end{bmatrix}$

Since corresponding elements are equal, we have

$-16+z=2 \Rightarrow z=18,\ 12r=36 \Rightarrow r=3,$
$8s+3=27 \Rightarrow s=3,\ 6p+2=20 \Rightarrow p=3,$
$9=12a \Rightarrow a=\frac{3}{4}$

Thus, $z = 18$, $r = 3$, $s = 3$, $p = 3$, and $a=\frac{3}{4}$.

10. $\begin{bmatrix} a+2 & 3z+1 & 5m \\ 8k & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2z & 5m \\ 2k & 5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 4a+2 & 5z+1 & 10m \\ 10k & 5 & 9 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$

Since corresponding elements are equal, we have $4a+2=10 \Rightarrow 4a=8 \Rightarrow a=2,$
$5z+1=-14 \Rightarrow 5z=-15 \Rightarrow z=-3,$
$10m=80 \Rightarrow m=8,\ 10k=10 \Rightarrow k=1$

Thus, $a = 2$, $z = -3$, $m = 8$, and $k = 1$.

11. Two matrices are equal if they have the same <u>size</u> and if corresponding elements are <u>equal</u>.

12. In order to add two matrices, they must be the same <u>size</u>.

13. $\begin{bmatrix} -4 & 8 \\ 2 & 3 \end{bmatrix}$

This matrix has 2 rows and 2 columns, so it is a 2×2 square matrix.

14. $\begin{bmatrix} -9 & 6 & 2 \\ 4 & 1 & 8 \end{bmatrix}$

This matrix has 2 rows and 3 columns, so it is a 2×3 matrix.

15. $\begin{bmatrix} -6 & 8 & 0 & 0 \\ 4 & 1 & 9 & 2 \\ 3 & -5 & 7 & 1 \end{bmatrix}$

This matrix has 3 rows and 4 columns, so it is a 3×4 matrix.

16. $\begin{bmatrix} 8 & -2 & 4 & 6 & 3 \end{bmatrix}$

This matrix has 1 row and 5 columns, so it is a 1×5 row matrix.

17. $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

This matrix has 2 rows and 1 column, so it is a 2×1 column matrix.

18. [–9]
This matrix has 1 row and 1 column, so it is a 1×1 square matrix. It is also a row matrix since it has only one row, and a column matrix because it has only one column.

19.–20. Answers will vary.

21. $\begin{bmatrix}-4 & 3\\ 12 & -6\end{bmatrix}+\begin{bmatrix}2 & -8\\ 5 & 10\end{bmatrix}=\begin{bmatrix}-4+2 & 3+(-8)\\ 12+5 & -6+10\end{bmatrix}=\begin{bmatrix}-2 & -5\\ 17 & 4\end{bmatrix}$

22. $\begin{bmatrix}9 & 4\\ -8 & 2\end{bmatrix}+\begin{bmatrix}-3 & 2\\ -4 & 7\end{bmatrix}=\begin{bmatrix}9+(-3) & 4+2\\ -8+(-4) & 2+7\end{bmatrix}=\begin{bmatrix}6 & 6\\ -12 & 9\end{bmatrix}$

23. $\begin{bmatrix}6 & -9 & 2\\ 4 & 1 & 3\end{bmatrix}+\begin{bmatrix}-8 & 2 & 5\\ 6 & -3 & 4\end{bmatrix}=\begin{bmatrix}6+(-8) & -9+2 & 2+5\\ 4+6 & 1+(-3) & 3+4\end{bmatrix}=\begin{bmatrix}-2 & -7 & 7\\ 10 & -2 & 7\end{bmatrix}$

24. $\begin{bmatrix}4 & -3\\ 7 & 2\\ -6 & 8\end{bmatrix}+\begin{bmatrix}9 & -10\\ 0 & 5\\ -1 & 6\end{bmatrix}=\begin{bmatrix}4+9 & -3+(-10)\\ 7+0 & 2+5\\ -6+(-1) & 8+6\end{bmatrix}=\begin{bmatrix}13 & -13\\ 7 & 7\\ -7 & 14\end{bmatrix}$

25. Since $\begin{bmatrix}2 & 4 & 6\end{bmatrix}$ is a 1×3 matrix and $\begin{bmatrix}-2\\ -4\\ -6\end{bmatrix}$ is a 3×1 matrix, $\begin{bmatrix}2 & 4 & 6\end{bmatrix}+\begin{bmatrix}-2\\ -4\\ -6\end{bmatrix}$ cannot be added.

26. Since $\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}$ is a 3×1 matrix and $\begin{bmatrix}2\\ -6\end{bmatrix}$ is a 2×1 matrix, $\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}+\begin{bmatrix}2\\ -6\end{bmatrix}$ cannot be added.

27. $\begin{bmatrix}-6 & 8\\ 0 & 0\end{bmatrix}-\begin{bmatrix}0 & 0\\ -4 & -2\end{bmatrix}=\begin{bmatrix}-6-0 & 8-0\\ 0-(-4) & 0-(-2)\end{bmatrix}=\begin{bmatrix}-6 & 8\\ 4 & 2\end{bmatrix}$

28. $\begin{bmatrix}11 & 0\\ -4 & 0\end{bmatrix}-\begin{bmatrix}0 & 12\\ 0 & -14\end{bmatrix}=\begin{bmatrix}11-0 & 0-12\\ -4-0 & 0-(-14)\end{bmatrix}=\begin{bmatrix}11 & -12\\ -4 & 14\end{bmatrix}$

29. $\begin{bmatrix}12\\ -1\\ 3\end{bmatrix}-\begin{bmatrix}8\\ 4\\ -1\end{bmatrix}=\begin{bmatrix}12-8\\ -1-4\\ 3-(-1)\end{bmatrix}=\begin{bmatrix}4\\ -5\\ 4\end{bmatrix}$

30. $\begin{bmatrix}10 & -4 & 6\end{bmatrix}-\begin{bmatrix}-2 & 5 & 3\end{bmatrix}=\begin{bmatrix}10-(-2) & -4-5 & 6-3\end{bmatrix}=\begin{bmatrix}12 & -9 & 3\end{bmatrix}$

31. Since $\begin{bmatrix}-4 & 3\end{bmatrix}$ is a 1×2 matrix and $\begin{bmatrix}5 & 8 & 2\end{bmatrix}$ is a 1×3 matrix, $\begin{bmatrix}-4 & 3\end{bmatrix}-\begin{bmatrix}5 & 8 & 2\end{bmatrix}$ cannot be subtracted.

32. Since $\begin{bmatrix}4 & 6\end{bmatrix}$ is a 1×2 matrix and $\begin{bmatrix}2\\ 3\end{bmatrix}$ is a 2×1 matrix, $\begin{bmatrix}4 & 6\end{bmatrix}-\begin{bmatrix}2\\ 3\end{bmatrix}$ cannot be subtracted.

33. $\begin{bmatrix}\sqrt{3} & -4\\ 2 & -\sqrt{5}\\ -8 & \sqrt{8}\end{bmatrix}-\begin{bmatrix}2\sqrt{3} & 9\\ -2 & \sqrt{5}\\ -7 & 3\sqrt{2}\end{bmatrix}=\begin{bmatrix}\sqrt{3}-2\sqrt{3} & -4-9\\ 2-(-2) & -\sqrt{5}-\sqrt{5}\\ -8-(-7) & \sqrt{8}-3\sqrt{2}\end{bmatrix}=\begin{bmatrix}-\sqrt{3} & -13\\ 4 & -2\sqrt{5}\\ -1 & -\sqrt{2}\end{bmatrix}$

34. $\begin{bmatrix}2 & \sqrt{7}\\ 3\sqrt{28} & -6\end{bmatrix}-\begin{bmatrix}-1 & 5\sqrt{7}\\ 2\sqrt{7} & 2\end{bmatrix}=\begin{bmatrix}2 & \sqrt{7}\\ 6\sqrt{7} & -6\end{bmatrix}-\begin{bmatrix}-1 & 5\sqrt{7}\\ 2\sqrt{7} & 2\end{bmatrix}=\begin{bmatrix}2-(-1) & \sqrt{7}-5\sqrt{7}\\ 6\sqrt{7}-2\sqrt{7} & -6-2\end{bmatrix}=\begin{bmatrix}3 & -4\sqrt{7}\\ 4\sqrt{7} & -8\end{bmatrix}$

35. $$\begin{bmatrix} 3x+y & x-2y & 2x \\ 5x & 3y & x+y \end{bmatrix} + \begin{bmatrix} 2x & 3y & 5x+y \\ 3x+2y & x & 2x \end{bmatrix}$$
$$= \begin{bmatrix} (3x+y)+2x & (x-2y)+3y & 2x+(5x+y) \\ 5x+(3x+2y) & 3y+x & (x+y)+2x \end{bmatrix}$$
$$= \begin{bmatrix} 5x+y & x+y & 7x+y \\ 8x+2y & x+3y & 3x+y \end{bmatrix}$$

36. $$\begin{bmatrix} 4k-8y \\ 6z-3x \\ 2k+5a \\ -4m+2n \end{bmatrix} - \begin{bmatrix} 5k+6y \\ 2z+5x \\ 4k+6a \\ 4m-2n \end{bmatrix}$$
$$= \begin{bmatrix} (4k-8y)-(5k+6y) \\ (6z-3x)-(2z+5x) \\ (2k+5a)-(4k+6a) \\ (-4m+2n)-(4m-2n) \end{bmatrix}$$
$$= \begin{bmatrix} 4k-8y-5k-6y \\ 6z-3x-2z-5x \\ 2k+5a-4k-6a \\ -4m+2n-4m+2n \end{bmatrix} = \begin{bmatrix} -k-14y \\ 4z-8x \\ -2k-a \\ -8m+4n \end{bmatrix}$$

In Exercises 37–40, $A = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$.

37. $$2A = 2\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2(-2) & 2(4) \\ 2(0) & 2(3) \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 0 & 6 \end{bmatrix}$$

38. $$-3B = -3\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -3(-6) & -3(2) \\ -3(4) & -3(0) \end{bmatrix} = \begin{bmatrix} 18 & -6 \\ -12 & 0 \end{bmatrix}$$

39. $$\tfrac{3}{2}B = \tfrac{3}{2}\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}(-6) & \frac{3}{2}(2) \\ \frac{3}{2}(4) & \frac{3}{2}(0) \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ 6 & 0 \end{bmatrix}$$

40. $$-1.5A = 2\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1.5(-2) & -1.5(4) \\ -1.5(0) & -1.5(3) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & -4.5 \end{bmatrix}$$

41. $$2A - B = 2\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -4-(-6) & 8-2 \\ 0-4 & 6-0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 6 \end{bmatrix}$$

42. $$-2A + 4B = -2\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 0 & -6 \end{bmatrix} + \begin{bmatrix} -24 & 8 \\ 16 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 4+(-24) & -8+8 \\ 0+16 & -6+0 \end{bmatrix} = \begin{bmatrix} -20 & 0 \\ 16 & -6 \end{bmatrix}$$

43. $$-A + \tfrac{1}{2}B = -\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2+(-3) & -4+1 \\ 0+2 & -3+0 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -3 \end{bmatrix}$$

44. $$\tfrac{3}{4}A - B = \tfrac{3}{4}\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 3 \\ 0 & \frac{9}{4} \end{bmatrix} - \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{3}{2}-(-6) & 3-2 \\ 0-4 & \frac{9}{4}-0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}+\frac{12}{2} & 1 \\ -4 & \frac{9}{4} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & 1 \\ -4 & \frac{9}{4} \end{bmatrix}$$

45. *AB* can be calculated and the result will be a 2×5 matrix.

Matrix *A* Matrix *B*
2×3 3×5
matches
size of *AB*
2×5

46. *CA* can be calculated and the result will be a 5×3 matrix.

Matrix *C* Matrix *A*
5×2 2×3
matches
size of *CA*
5×3

47. *BA* cannot be calculated.

Matrix *B* Matrix *A*

$\underbrace{3\times 5 \quad 2}_{\text{different}}\times 3$

48. *AC* cannot be calculated.

Matrix *A* Matrix *C*

$\underbrace{2\times 3 \quad 5}_{\text{different}}\times 2$

49. *BC* can be calculated and the result will be a 3×2 matrix.

Matrix *B* Matrix *C*

$\underbrace{3\times \underbrace{5 \quad 5}_{\text{matches}}\times 2}_{\text{size of } CA \; 3\times 2}$

50. *CB* cannot be calculated.

Matrix *C* Matrix *B*

$\underbrace{5\times 2 \quad 3}_{\text{different}}\times 5$

51. A 2×2 matrix multiplied by a 2×1 matrix results in a 2×1 matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1(-1)+2(7) \\ 3(-1)+4(7) \end{bmatrix} = \begin{bmatrix} -1+14 \\ -3+28 \end{bmatrix} = \begin{bmatrix} 13 \\ 25 \end{bmatrix}$$

52. A 2×2 matrix multiplied by a 2×1 matrix results in a 2×1 matrix.

$$\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1(6)+5(2) \\ 7(6)+0(2) \end{bmatrix} = \begin{bmatrix} -6+10 \\ 42+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 42 \end{bmatrix}$$

53. A 2×3 matrix multiplied by a 3×1 matrix results in a 2×1 matrix.

$$\begin{bmatrix} 3 & -4 & 1 \\ 5 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(-1)+(-4)(4)+1(2) \\ 5(-1)+0(4)+2(2) \end{bmatrix} = \begin{bmatrix} -3+(-16)+2 \\ -5+0+4 \end{bmatrix} = \begin{bmatrix} -17 \\ -1 \end{bmatrix}$$

54. A 2×3 matrix multiplied by a 3×1 matrix results in a 2×1 matrix.

$$\begin{bmatrix} -6 & 3 & 5 \\ 2 & 9 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -6(-2)+3(0)+5(3) \\ 2(-2)+9(0)+1(3) \end{bmatrix} = \begin{bmatrix} 12+0+15 \\ -4+0+3 \end{bmatrix} = \begin{bmatrix} 27 \\ -1 \end{bmatrix}$$

55. A 2×3 matrix multiplied by a 3×2 matrix results in a 2×2 matrix.

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & -3\sqrt{2} \\ \sqrt{3} & 3\sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8\sqrt{2}+9\sqrt{2} - 3\sqrt{2}(0) & -10\sqrt{2}+12\sqrt{2} - 3\sqrt{2}(2) \\ 8\sqrt{3}+3\sqrt{3}(9) + 0(0) & -10\sqrt{3}+3\sqrt{3}(12) + 0(2) \end{bmatrix} = \begin{bmatrix} 17\sqrt{2} & -4\sqrt{2} \\ 35\sqrt{3} & 26\sqrt{3} \end{bmatrix}$$

56. A 2×3 matrix multiplied by a 3×1 matrix results in a 2×1 matrix.

$$\begin{bmatrix} -9 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{5} \\ \sqrt{20} \\ -2\sqrt{5} \end{bmatrix} = \begin{bmatrix} -9 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{5} \\ 2\sqrt{5} \\ -2\sqrt{5} \end{bmatrix} = \begin{bmatrix} -9\sqrt{5}+2\left(2\sqrt{5}\right)+1\left(-2\sqrt{5}\right) \\ 3\sqrt{5}+0\left(2\sqrt{5}\right)+0\left(-2\sqrt{5}\right) \end{bmatrix} = \begin{bmatrix} -7\sqrt{5} \\ 3\sqrt{5} \end{bmatrix}$$

57. A 2×2 matrix multiplied by a 2×2 matrix results in a 2×2 matrix.

$$\begin{bmatrix}\sqrt{3} & 1\\ 2\sqrt{5} & 3\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{3} & -\sqrt{6}\\ 4\sqrt{3} & 0\end{bmatrix}=\begin{bmatrix}\sqrt{3}\left(\sqrt{3}\right)+1\left(4\sqrt{3}\right) & \sqrt{3}\left(-\sqrt{6}\right)+1(0)\\ 2\sqrt{5}\left(\sqrt{3}\right)+3\sqrt{2}\left(4\sqrt{3}\right) & 2\sqrt{5}\left(-\sqrt{6}\right)+3\sqrt{2}\,(0)\end{bmatrix}$$
$$=\begin{bmatrix}3+4\sqrt{3} & -\sqrt{18}\\ 2\sqrt{15}+12\sqrt{6} & -2\sqrt{30}\end{bmatrix}=\begin{bmatrix}3+4\sqrt{3} & -3\sqrt{2}\\ 2\sqrt{15}+12\sqrt{6} & -2\sqrt{30}\end{bmatrix}$$

58. A 2×2 matrix multiplied by a 2×2 matrix results in a 2×2 matrix.

$$\begin{bmatrix}\sqrt{7} & 0\\ 2 & \sqrt{28}\end{bmatrix}\begin{bmatrix}2\sqrt{3} & -\sqrt{7}\\ 0 & -6\end{bmatrix}=\begin{bmatrix}\sqrt{7}\left(2\sqrt{3}\right)+0(0) & \sqrt{7}\left(-\sqrt{7}\right)+0(-6)\\ 2\left(2\sqrt{3}\right)+\sqrt{28}\,(0) & 2\left(-\sqrt{7}\right)+\sqrt{28}\,(-6)\end{bmatrix}$$
$$=\begin{bmatrix}2\sqrt{21} & -7\\ 4\sqrt{3} & -2\sqrt{7}-6\sqrt{28}\end{bmatrix}=\begin{bmatrix}2\sqrt{21} & -7\\ 4\sqrt{3} & -2\sqrt{7}-12\sqrt{7}\end{bmatrix}=\begin{bmatrix}2\sqrt{21} & -7\\ 4\sqrt{3} & -14\sqrt{7}\end{bmatrix}$$

59. $\begin{bmatrix}-3 & 0 & 2 & 1\\ 4 & 0 & 2 & 6\end{bmatrix}\begin{bmatrix}-4 & 2\\ 0 & 1\end{bmatrix}$

A 2×4 matrix cannot be multiplied by a 2×2 matrix because the number of columns of the first matrix (four) is not equal to the number of rows of the second matrix (two).

60. $\begin{bmatrix}-1 & 2 & 4 & 1\\ 0 & 2 & -3 & 5\end{bmatrix}\begin{bmatrix}1 & 2 & 4\\ -2 & 5 & 1\end{bmatrix}$

A 2×4 matrix cannot be multiplied by a 2×3 matrix because the number of columns of the first matrix (four) is not equal to the number of rows of the second matrix (two).

61. A 1×3 matrix multiplied by a 3×3 matrix results in a 1×3 matrix.

$$\begin{bmatrix}-2 & 4 & 1\end{bmatrix}\begin{bmatrix}3 & -2 & 4\\ 2 & 1 & 0\\ 0 & -1 & 4\end{bmatrix}=\begin{bmatrix}-2(3)+4(2)+1(0) & -2(-2)+4(1)+1(-1) & -2(4)+4(0)+1(4)\end{bmatrix}$$
$$=\begin{bmatrix}-6+8+0 & 4+4+(-1) & -8+0+4\end{bmatrix}=\begin{bmatrix}2 & 7 & -4\end{bmatrix}$$

62. A 1×3 matrix multiplied by a 3×3 matrix results in a 1×3 matrix.

$$\begin{bmatrix}0 & 3 & -4\end{bmatrix}\begin{bmatrix}-2 & 6 & 3\\ 0 & 4 & 2\\ -1 & 1 & 4\end{bmatrix}=\begin{bmatrix}0(-2)+3(0)+(-4)(-1) & 0(6)+3(4)+(-4)(1) & 0(3)+3(2)+(-4)(4)\end{bmatrix}$$
$$=\begin{bmatrix}0+0+4 & 0+12+(-4) & 0+6+(-16)\end{bmatrix}=\begin{bmatrix}4 & 8 & -10\end{bmatrix}$$

63. A 3×3 matrix multiplied by a 3×3 matrix results in a 3×3 matrix.

$$\begin{bmatrix}-2 & -3 & -4\\ 2 & -1 & 0\\ 4 & -2 & 3\end{bmatrix}\begin{bmatrix}0 & 1 & 4\\ 1 & 2 & -1\\ 3 & 2 & -2\end{bmatrix}$$
$$=\begin{bmatrix}-2(0)+(-3)(1)+(-4)(3) & -2(1)+(-3)(2)+(-4)(2) & -2(4)+(-3)(-1)+(-4)(-2)\\ 2(0)+(-1)(1)+0(3) & 2(1)+(-1)(2)+0(2) & 2(4)+(-1)(-1)+0(-2)\\ 4(0)+(-2)(1)+3(3) & 4(1)+(-2)(2)+3(2) & 4(4)+(-2)(-1)+3(-2)\end{bmatrix}$$
$$=\begin{bmatrix}0+(-3)+(-12) & -2+(-6)+(-8) & -8+3+8\\ 0+(-1)+0 & 2+(-2)+0 & 8+1+0\\ 0+(-2)+9 & 4+(-4)+6 & 16+2+(-6)\end{bmatrix}=\begin{bmatrix}-15 & -16 & 3\\ -1 & 0 & 9\\ 7 & 6 & 12\end{bmatrix}$$

64. A 3×3 matrix multiplied by a 3×3 matrix results in a 3×3 matrix.

$$\begin{bmatrix}-1&2&0\\0&3&2\\0&1&4\end{bmatrix}\begin{bmatrix}2&-1&2\\0&2&1\\3&0&-1\end{bmatrix}$$
$$=\begin{bmatrix}(-1)(2)+2(0)+0(3)&(-1)(-1)+2(2)+0(0)&(-1)(2)+2(1)+0(-1)\\0(2)+3(0)+2(3)&0(-1)+3(2)+2(0)&0(2)+3(1)+2(-1)\\0(2)+1(0)+4(3)&0(-1)+1(2)+4(0)&0(2)+1(1)+4(-1)\end{bmatrix}$$
$$=\begin{bmatrix}-2+0+0&1+4+0&-2+2+0\\0+0+6&0+6+0&0+3+(-2)\\0+0+12&0+2+0&0+1+(-4)\end{bmatrix}=\begin{bmatrix}-2&5&0\\6&6&1\\12&2&-3\end{bmatrix}$$

In Exercises 65–72, $A=\begin{bmatrix}4&-2\\3&1\end{bmatrix}$, $B=\begin{bmatrix}5&1\\0&-2\\3&7\end{bmatrix}$ and $C=\begin{bmatrix}-5&4&1\\0&3&6\end{bmatrix}$.

65. A 3×2 matrix multiplied by a 2×2 matrix results in a 3×2 matrix.

$$BA=\begin{bmatrix}5&1\\0&-2\\3&7\end{bmatrix}\begin{bmatrix}4&-2\\3&1\end{bmatrix}=\begin{bmatrix}5(4)+1(3)&5(-2)+1(1)\\0(4)+(-2)(3)&0(-2)+(-2)(1)\\3(4)+7(3)&3(-2)+7(1)\end{bmatrix}=\begin{bmatrix}20+3&-10+1\\0+(-6)&0+(-2)\\12+21&-6+7\end{bmatrix}=\begin{bmatrix}23&-9\\-6&-2\\33&1\end{bmatrix}$$

66. A 2×2 matrix multiplied by a 2×3 matrix results in a 2×3 matrix.

$$AC=\begin{bmatrix}4&-2\\3&1\end{bmatrix}\begin{bmatrix}-5&4&1\\0&3&6\end{bmatrix}=\begin{bmatrix}4(-5)+(-2)(0)&4(4)+(-2)(3)&4(1)+(-2)(6)\\3(-5)+1(0)&3(4)+1(3)&3(1)+1(6)\end{bmatrix}$$
$$=\begin{bmatrix}-20+0&16+(-6)&4+(-12)\\-15+0&12+3&3+6\end{bmatrix}=\begin{bmatrix}-20&10&-8\\-15&15&9\end{bmatrix}$$

67. A 3×2 matrix multiplied by a 2×3 matrix results in a 3×3 matrix.

$$BC=\begin{bmatrix}5&1\\0&-2\\3&7\end{bmatrix}\begin{bmatrix}-5&4&1\\0&3&6\end{bmatrix}=\begin{bmatrix}5(-5)+1(0)&5(4)+1(3)&5(1)+1(6)\\0(-5)+(-2)(0)&0(4)+(-2)(3)&0(1)+(-2)(6)\\3(-5)+7(0)&3(4)+7(3)&3(1)+7(6)\end{bmatrix}$$
$$=\begin{bmatrix}-25+0&20+3&5+6\\0+0&0+(-6)&0+(-12)\\-15+0&12+21&3+42\end{bmatrix}=\begin{bmatrix}-25&23&11\\0&-6&-12\\-15&33&45\end{bmatrix}$$

68. A 2×3 matrix multiplied by a 3×2 matrix results in a 2×2 matrix.

$$CB=\begin{bmatrix}-5&4&1\\0&3&6\end{bmatrix}\begin{bmatrix}5&1\\0&-2\\3&7\end{bmatrix}=\begin{bmatrix}-5(5)+4(0)+1(3)&-5(1)+4(-2)+1(7)\\0(5)+3(0)+6(3)&0(1)+3(-2)+6(7)\end{bmatrix}$$
$$=\begin{bmatrix}-25+0+3&-5+(-8)+7\\0+0+18&0+(-6)+42\end{bmatrix}=\begin{bmatrix}-22&-6\\18&36\end{bmatrix}$$

69. $AB=\begin{bmatrix}4&-2\\3&1\end{bmatrix}\begin{bmatrix}5&1\\0&-2\\3&7\end{bmatrix}$ is the product of a 2×2 matrix multiplied by a 3×2 matrix, which is not possible.

70. $CA=\begin{bmatrix}-5&4&1\\0&3&6\end{bmatrix}\begin{bmatrix}4&-2\\3&1\end{bmatrix}$ is the product of a 2×3 matrix multiplied by a 2×2 matrix, which is not possible.

71. Since $A^2 = AA,$ we are finding the product of two 2×2 matrices, which results in a 2×2 matrix.

$$A^2 = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4(4)+(-2)(3) & 4(-2)+(-2)(1) \\ 3(4)+1(3) & 3(-2)+1(1) \end{bmatrix} = \begin{bmatrix} 16+(-6) & -8+(-2) \\ 12+3 & -6+1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ 15 & -5 \end{bmatrix}$$

72. $A^2 = \begin{bmatrix} 10 & -10 \\ 15 & -5 \end{bmatrix}$ from Exercise 71. Since $A^3 = A^2 \cdot A,$ we are now finding the product of a 2×2 matrix multiplied by another 2×2 matrix, which results in a 2×2 matrix.

$$A^3 = A^2A = \begin{bmatrix} 10 & -10 \\ 15 & -5 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10(4)+(-10)(3) & 10(-2)+(-10)(1) \\ 15(4)+(-5)(3) & 15(-2)+(-5)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 40+(-30) & -20+(-10) \\ 60+(-15) & -30+(-5) \end{bmatrix} = \begin{bmatrix} 10 & -30 \\ 45 & -35 \end{bmatrix}$$

73. Since the answers to 65 and 69 are not equal, $BA \neq AB.$
Since the answers to 67 and 68 are not equal, $BC \neq CB.$
Since the answers to 66 and 70 are not equal, $AC \neq CA.$
No, matrix multiplication is not commutative.

74. Answers will vary

75. $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 0 \\ 5 & -2 \end{bmatrix}$

(a) $AB = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 3(6)+4(5) & 3(0)+4(-2) \\ -2(6)+1(5) & -2(0)+1(-2) \end{bmatrix} = \begin{bmatrix} 18+20 & 0+(-8) \\ -12+5 & 0+(-2) \end{bmatrix} = \begin{bmatrix} 38 & -8 \\ -7 & -2 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 6 & 0 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6(3)+0(-2) & 6(4)+0(1) \\ 5(3)+(-2)(-2) & 5(4)+(-2)(1) \end{bmatrix} = \begin{bmatrix} 18+0 & 24+0 \\ 15+4 & 20+(-2) \end{bmatrix} = \begin{bmatrix} 18 & 24 \\ 19 & 18 \end{bmatrix}$

Note: $AB \neq BA$

76. $A = \begin{bmatrix} 0 & -5 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix}$

(a) $AB = \begin{bmatrix} 0 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 0(3)+(-5)(-5) & 0(-1)+(-5)(4) \\ -4(3)+2(-5) & -4(-1)+2(4) \end{bmatrix} = \begin{bmatrix} 0+25 & 0+(-20) \\ -12+(-10) & 4+8 \end{bmatrix} = \begin{bmatrix} 25 & -20 \\ -22 & 12 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 0 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3(0)+(-1)(-4) & 3(-5)+(-1)(2) \\ -5(0)+4(-4) & -5(-5)+4(2) \end{bmatrix} = \begin{bmatrix} 0+4 & -15+(-2) \\ 0+(-16) & 25+8 \end{bmatrix} = \begin{bmatrix} 4 & -17 \\ -16 & 33 \end{bmatrix}$

Note: $AB \neq BA$

77. $A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) $AB = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0(1)+1(0)+(-1)(0) & 0(0)+1(1)+(-1)(0) & 0(0)+1(0)+(-1)(1) \\ 0(1)+1(0)+0(0) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ 0(1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$

$$= \begin{bmatrix} 0+0+0 & 0+1+0 & 0+0+(-1) \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $BA=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&-1\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1(0)+0(0)+0(0)&1(1)+0(1)+0(0)&1(-1)+0(0)+0(1)\\0(0)+1(0)+0(0)&0(1)+1(1)+0(0)&0(-1)+1(0)+0(1)\\0(0)+0(0)+1(0)&0(1)+0(1)+1(0)&0(-1)+0(0)+1(1)\end{bmatrix}$

$=\begin{bmatrix}0+0+0&1+0+0&-1+0+0\\0+0+0&0+1+0&0+0+0\\0+0+0&0+0+0&0+0+1\end{bmatrix}=\begin{bmatrix}0&1&-1\\0&1&0\\0&0&1\end{bmatrix}$

Note: $AB=BA=A$

78. $A=\begin{bmatrix}-1&0&1\\0&1&1\\-1&-1&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

(a) $AB=\begin{bmatrix}-1&0&1\\0&1&1\\-1&-1&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}-1(0)+0(0)+1(1)&-1(0)+0(1)+1(0)&-1(1)+0(0)+1(0)\\0(0)+1(0)+1(1)&0(0)+1(1)+1(0)&0(1)+1(0)+1(0)\\-1(0)+(-1)(0)+0(1)&-1(0)+(-1)(1)+0(0)&-1(1)+(-1)(0)+0(0)\end{bmatrix}$

$=\begin{bmatrix}0+0+1&0+0+0&-1+0+0\\0+0+1&0+1+0&0+0+0\\0+0+0&0+(-1)+0&-1+0+0\end{bmatrix}=\begin{bmatrix}1&0&-1\\1&1&0\\0&-1&-1\end{bmatrix}$

(b) $BA=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}-1&0&1\\0&1&1\\-1&-1&0\end{bmatrix}=\begin{bmatrix}0(-1)+0(0)+1(-1)&0(0)+0(1)+1(-1)&0(1)+0(1)+1(0)\\0(-1)+1(0)+0(-1)&0(0)+1(1)+0(-1)&0(1)+1(1)+0(0)\\1(-1)+0(0)+0(-1)&1(0)+0(1)+0(-1)&1(1)+0(1)+0(0)\end{bmatrix}$

$=\begin{bmatrix}0+0+(-1)&0+0+(-1)&0+0+0\\0+0+0&0+1+0&0+1+0\\-1+0+0&0+0+0&1+0+0\end{bmatrix}=\begin{bmatrix}-1&-1&0\\0&1&1\\-1&0&1\end{bmatrix}$

Note: Although $AB\neq BA$, there is a relation between the matrix A and the product. In the case of AB, the product is matrix A with columns 1 and 3 swapped. In the case of BA, the product is matrix A with rows 1 and 3 swapped. Matrix B is known as a "permutation" matrix.

79. In Exercise 77, $AB=A$ and $BA=A$. For this pair of matrices, B acts in the same way for multiplication as the number 1 acts for multiplication of real numbers.

80. $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$AB=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$=\begin{bmatrix}a(1)+b(0)&a(0)+b(1)\\c(1)+d(0)&c(0)+d(1)\end{bmatrix}$

$=\begin{bmatrix}a+0&0+b\\c+0&0+d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$BA=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$=\begin{bmatrix}1(a)+0(c)&1(b)+0(d)\\0(a)+1(c)&0(b)+1(d)\end{bmatrix}$

$=\begin{bmatrix}a+0&b+0\\0+c&0+d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

Matrix B acts as the multiplicative identity element for 2×2 square matrices.

81. **(a)** The sales figure information may be written as the 3×3 matrix where column 1 represents nonfat, column 2 represents regular, and column 3 represents supercreamy.

$\begin{matrix}\text{Location I}\\\text{Location II}\\\text{Location III}\end{matrix}\begin{bmatrix}50&100&30\\10&90&50\\60&120&40\end{bmatrix}.$

(b) The income per gallon information may be written as the 3×1 matrix $\begin{bmatrix} 12 \\ 10 \\ 15 \end{bmatrix}$.

Note: If the matrix in part (a) had been written with its rows and columns interchanged, then this income per gallon information would be written instead as a 1×3 matrix.

(c) $$\begin{bmatrix} 50 & 100 & 30 \\ 10 & 90 & 50 \\ 60 & 120 & 40 \end{bmatrix}\begin{bmatrix} 12 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 2050 \\ 1770 \\ 2520 \end{bmatrix}$$

Note: This result may be written as a 1×3 matrix instead.

(d) 2050 + 1770 + 2520 = 6340; The total daily income from the three locations is \$6340.

82. **(a)** A 4×5 matrix multiplied by a 5×2 matrix results in a 4×2 matrix.

$$\begin{bmatrix} 1 & 4 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & 3 & 0 & \frac{1}{4} & 0 \\ 4 & 3 & 2 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} 5 & 5 \\ 8 & 10 \\ 10 & 12 \\ 12 & 15 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1(5)+4(8)+\frac{1}{4}(10)+\frac{1}{4}(12)+1(5) & 1(5)+4(10)+\frac{1}{4}(12)+\frac{1}{4}(15)+1(6) \\ 0(5)+3(8)+0(10)+\frac{1}{4}(12)+0(5) & 0(5)+3(10)+0(12)+\frac{1}{4}(15)+0(6) \\ 4(5)+3(8)+2(10)+1(12)+1(5) & 4(5)+3(10)+2(12)+1(15)+1(6) \\ 0(5)+1(8)+0(10)+\frac{1}{3}(12)+0(5) & 0(5)+1(10)+0(12)+\frac{1}{3}(15)+0(6) \end{bmatrix}$$

$$= \begin{bmatrix} 5+32+2.5+3+5 & 5+40+3+3.75+6 \\ 0+24+0+3+0 & 0+30+0+3.75+0 \\ 20+24+20+12+5 & 20+30+24+15+6 \\ 0+8+0+4+0 & 0+10+0+5+0 \end{bmatrix} = \begin{bmatrix} 47.5 & 57.75 \\ 27 & 33.75 \\ 81 & 95 \\ 12 & 15 \end{bmatrix}$$

(b) $\begin{bmatrix} 20 & 200 & 50 & 60 \end{bmatrix}$

A 1×4 matrix multiplied by a 4×5 matrix results in a 1×5 matrix.

$$\begin{bmatrix} 20 & 200 & 50 & 60 \end{bmatrix}\begin{bmatrix} 1 & 4 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & 3 & 0 & \frac{1}{4} & 0 \\ 4 & 3 & 2 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$= [20(1)+200(0)+50(4)+60(0) \quad 20(4)+200(3)+50(3)+60(1)$ continued

$20\left(\frac{1}{4}\right)+200(0)+50(2)+60(0) \quad 20\left(\frac{1}{4}\right)+200\left(\frac{1}{4}\right)+50(1)+60\left(\frac{1}{3}\right)$ continued

$20(1)+200(0)+50(1)+60(0)]$

$$= \begin{bmatrix} 20+0+200+0 & 80+600+150+60 & 5+0+100+0 & 5+50+50+20 & 20+0+50+0 \end{bmatrix}$$
$$= \begin{bmatrix} 220 & 890 & 105 & 125 & 70 \end{bmatrix}$$

(c) A 1×4 matrix multiplied by a 4×2 matrix results in a 1×2 matrix.

$$\begin{bmatrix} 20 & 200 & 50 & 60 \end{bmatrix}\begin{bmatrix} 47.5 & 57.75 \\ 27 & 33.75 \\ 81 & 95 \\ 12 & 15 \end{bmatrix}$$
$$= \begin{bmatrix} 20(47.5)+200(27)+50(81)+60(12) & 20(57.75)+200(33.75)+50(95)+60(15) \end{bmatrix}$$
$$= \begin{bmatrix} 950+5400+4050+720 & 1155+6750+4750+900 \end{bmatrix} = \begin{bmatrix} 11{,}120 & 13{,}555 \end{bmatrix}$$

83. Answers will vary depending on when numbers are rounded.

(a) $j_1 = 690, s_p = 210, a_1 = 2100$

1st year:

$$\begin{aligned} j_2 &= .33a_1 = .33(2100) = 693 \\ s_2 &= .18j_1 = .18(690) \approx 124.2 \\ &\approx 124 \\ a_2 &= .71s_1 + .94a_1 \\ &= .71(210) + .94(2100) \\ &\approx 2123.1 \approx 2123 \end{aligned}$$

$j_2 + s_2 + a_2 = 693 + 124 + 2123 = 2940$

2nd year:

$$\begin{aligned} j_3 &= .33a_2 = .33(2123) \approx 700.6 \\ s_3 &= .18j_2 = .18(693) \approx 124.7 \\ a_3 &= .71s_2 + .94a_2 \\ &= .71(124) + .94(2123) \approx 2083.7 \end{aligned}$$

$j_3 + s_3 + a_3 = 700.6 + 124.7 + 2083.7 = 2909$

3rd year:

$$\begin{aligned} j_4 &= .33a_3 = .33(2084) \approx 687 \\ s_4 &= .18j_3 = .18(700) = 126 \\ a_4 &= .72s_3 + .94a_3 \\ &= .71(125) + .94(2084) = 2048 \end{aligned}$$

$j_3 + s_3 + a_3 = 687 + 126 + 2048 = 2861$

4th year:

$$\begin{aligned} j_5 &= .33a_4 = .33(2048) \approx 676 \\ s_5 &= .18j_4 = .18(687) \approx 124 \\ a_5 &= .72s_4 + .94a_4 \\ &= .71(126) + .94(2048) = 2014 \end{aligned}$$

$j_5 + s_5 + a_5 = 686 + 124 + 2048 = 2814$

5th year:

$$\begin{aligned} j_6 &= .33a_5 = .33(2014) \approx 664 \\ s_6 &= .18j_5 = .18(676) \approx 122 \\ a_6 &= .72s_5 + .94a_5 \\ &= .71(124) + .94(2014) \approx 1981 \end{aligned}$$

$j_6 + s_6 + a_6 = 664 + 122 + 1981 = 2767$

(b) The northern spotted owl will become extinct.

(c) $j_1 = 690, s_1 = 210, a_1 = 2100$

1st year:

$$\begin{aligned} j_2 &= .33a_1 = .33(2100) = 693 \\ s_2 &= .3j_1 = .3(690) = 207 \\ a_2 &= .71s_1 + .94a_1 \\ &= .71(210) + .94(2100) \approx 2123 \end{aligned}$$

$j_2 + s_2 + a_2 = 693 + 207 + 2123 = 3023$

2nd year:

$$\begin{aligned} j_3 &= .33a_2 = .33(2123) \approx 701 \\ s_3 &= .3j_2 = .3(693) \approx 208 \\ a_3 &= .71s_2 + .94a_2 \\ &= .71(207) = .94(2123) \approx 2143 \end{aligned}$$

$j_3 + s_3 + a_3 = 701 + 208 + 2143 = 3052$

3rd year:

$$\begin{aligned} j_4 &= .33a_3 = .33(2143) \approx 707 \\ s_4 &= .3j_3 = .3(701) = 210 \\ a_4 &= .71s_3 + .94a_3 \\ &= .71(208) + .94(2143) \approx 2162 \end{aligned}$$

$j_4 + s_4 + a_4 = 707 + 210 + 2162 = 3079$

4th year:

$$\begin{aligned} j_5 &= .33a_4 = .33(2162) \approx 714 \\ s_5 &= .3j_4 = .3(707) \approx 212 \\ a_5 &= .72s_4 + .94a_4 \\ &= .71(210) + .94(2162) \approx 2181 \end{aligned}$$

$j_5 + s_5 + a_5 = 714 + 212 + 2181 = 3107$

5th year:

$$\begin{aligned} j_6 &= .71s_5 = .33a_5 \\ &= .33(2181) \approx 720 \\ s_6 &= .3j_5 = .3(714) \approx 214 \\ a_6 &= .71s_5 + .94a_5 \\ &= .71(212) + .94(2181) = 2201 \end{aligned}$$

$j_6 + s_6 + a_6 = 720 + 214 + 2201 = 3135$

84. **(a)** $\begin{bmatrix} m_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} .51 & .4 \\ -.05 & 1.05 \end{bmatrix} \begin{bmatrix} m_n \\ d_n \end{bmatrix}$

$d_{n+1} = -.05m_n + 1.05d_n$

The rate the deer population will grow from year to year if there are no mountain lions is 1.05.

(b) $m_1 = 2000, d_1 = 5000$

$$\begin{aligned} m_2 &= .51m_1 + .4d_1 \\ &= .51(2000) + .4(5000) = 3020 \end{aligned}$$

$$\begin{aligned} d_2 &= -.05m_1 + 1.05d_1 \\ &= -.05(2000) + 1.05(5000) = 5150 \end{aligned}$$

After 1 year, there will be 3020 mountain lions and 5150 hundred (515,000) deer.

$$\begin{aligned} m_3 &= .51m_2 + .4d_2 \\ &= .51(3020) + .4(5150) \approx 3600 \end{aligned}$$

$$\begin{aligned} d_3 &= -.05m_2 + 1.05d_2 \\ &= -.05(3020) + 1.05(5150) \approx 5256.5 \end{aligned}$$

After 2 years, there will be 3600 mountain lions and 5256.5 hundred (525,650) deer.

(c) $m_1 = 4000, d_1 = 5000$

$m_2 = .51m_1 + .4d_1$
$= .51(4000) + .4(5000) = 4040$

$d_2 = -.05m_1 + 1.05d_1$
$= -.05(4000) + 1.05(5000) = 5050$

$m_2 = 1.01m_1 = 1.01(4000) = 4040$
$d_2 = 1.01d_1 = 1.01(5000) = 5050$

$m_3 = .51m_2 + .4d_2$
$= .51(4040) + .4(5050) \approx 4080$

$d_3 = -.05m_2 + 1.05d_2$
$= -.05(4040) + 1.05(5050) = 5101$

$m_3 = 1.01m_2 = 1.01(4040) \approx 4080$
$d_3 = 1.01d_2 = 1.01(5050) \approx 5101$

The populations each grow at a steady annual rate of 1.01.

85. Expanding $\begin{vmatrix} -x & 0 & .33 \\ .18 & -x & 0 \\ 0 & .71 & .94-x \end{vmatrix}$ by row one, we have the following.

$$-x\begin{vmatrix} -x & 0 \\ .71 & .94-x \end{vmatrix} - 0 + .33\begin{vmatrix} .18 & -x \\ 0 & .71 \end{vmatrix} = -x\left[-x(.94-x) - .71(0)\right] + .33\left[(.18)(.71) - 0(-x)\right]$$
$$= -x\left(-.94x + x^2 - 0\right) + .33(.1278 - 0) = .94x^2 - x^3 + .042174$$

Thus, the polynomial is $-x^3 + .94x^2 + .042174$. Evaluating this polynomial with $x = .98359$, we have $-(.98359)^3 + .94(.98359)^2 + .042174 \approx .0000029$. Thus $.98359$ is an approximate zero.

86. Since $\begin{bmatrix} .51-x & .4 \\ -.05 & 1.05-x \end{bmatrix} = (.51-x)(1.05-x) - (-.05)(.4)$. This simplifies as follows.

$.5355 - .51x - 1.05x + x^2 + .02 = x^2 - 1.56x + .5555$

Evaluating this polynomial with $x = 1.01$, we have the following.

$(1.01)^2 - 1.56(1.01) + .5555 = 1.0201 - 1.5756 + .5555 = 0$

Thus, 1.01 is a zero.

In Exercises 87–94, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$.

87. $A + B = B + A$

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix} = B + A$$

88. $$A + (B + C) = A + \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} \end{bmatrix}$$
$$= \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$
$$= \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) + C = (A + B) + C$$

89. $(AB)C = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)C = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

$$= \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21})c_{11}+(a_{11}b_{12}+a_{12}b_{22})c_{21} & (a_{11}b_{11}+a_{12}b_{21})c_{12}+(a_{11}b_{12}+a_{12}b_{22})c_{22} \\ (a_{21}b_{11}+a_{22}b_{21})c_{11}+(a_{21}b_{12}+a_{22}b_{22})c_{21} & (a_{21}b_{11}+a_{22}b_{21})c_{12}+(a_{21}b_{12}+a_{22}b_{22})c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{12}b_{21}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{22}b_{21}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{22}b_{21}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22} \end{bmatrix}$$

$A(BC) = A\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21})+a_{12}(b_{21}c_{11}+b_{22}c_{21}) & a_{11}(b_{11}c_{12}+b_{12}c_{22})+a_{12}(b_{21}c_{12}+b_{22}c_{22}) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21})+a_{22}(b_{21}c_{11}+b_{22}c_{21}) & a_{21}(b_{11}c_{12}+b_{12}c_{22})+a_{22}(b_{21}c_{12}+b_{22}c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{21}c_{11}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{21}c_{12}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{21}c_{11}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{21}c_{12}+a_{22}b_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{12}b_{21}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{22}b_{21}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{22}b_{21}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22} \end{bmatrix}$$

Since the final matrix for $(AB)C$ and $A(BC)$ are the same, we have obtained the desired results.

90. $A(B+C) = A\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}+\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12})+a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11})+a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12})+a_{22}(b_{22}+c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{11}c_{11}+a_{12}b_{21}+a_{12}c_{21} & a_{11}b_{12}+a_{11}c_{12}+a_{12}b_{22}+a_{12}c_{22} \\ a_{21}b_{11}+a_{21}c_{11}+a_{22}b_{21}+a_{22}c_{21} & a_{21}b_{12}+a_{21}c_{12}+a_{22}b_{22}+a_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{11}c_{11}+a_{12}c_{21} & a_{11}b_{12}+a_{12}b_{22}+a_{11}c_{12}+a_{12}c_{22} \\ a_{21}b_{11}+a_{22}b_{21}+a_{21}c_{11}+a_{22}c_{21} & a_{21}b_{12}+a_{22}b_{22}+a_{21}c_{12}+a_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21})+(a_{11}c_{11}+a_{12}c_{21}) & (a_{11}b_{12}+a_{12}b_{22})+(a_{11}c_{12}+a_{12}c_{22}) \\ (a_{21}b_{11}+a_{22}b_{21})+(a_{21}c_{11}+a_{22}c_{21}) & (a_{21}b_{12}+a_{22}b_{22})+(a_{21}c_{12}+a_{22}c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}+\begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}+\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = AB+AC$$

91. $c(A+B) = cA+cB,$ for any real number c.

$c(A+B) = c\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = c\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} c(a_{11}+b_{11}) & c(a_{12}+b_{12}) \\ c(a_{21}+b_{21}) & c(a_{22}+b_{22}) \end{bmatrix}$

$$= \begin{bmatrix} c\cdot a_{11}+c\cdot b_{11} & c\cdot a_{12}+c\cdot b_{12} \\ c\cdot a_{21}+c\cdot b_{21} & c\cdot a_{22}+c\cdot b_{22} \end{bmatrix} = \begin{bmatrix} c\cdot a_{11} & c\cdot a_{12} \\ c\cdot a_{21} & c\cdot a_{22} \end{bmatrix}+\begin{bmatrix} c\cdot b_{11} & c\cdot b_{12} \\ c\cdot b_{21} & c\cdot b_{22} \end{bmatrix}$$

$$= c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+c\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = cA+cB$$

92. $(c+d)A = cA + dA$, for any real numbers c and d.

$$(c+d)A = (c+d)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (c+d)\cdot a_{11} & (c+d)\cdot a_{12} \\ (c+d)\cdot a_{21} & (c+d)\cdot a_{22} \end{bmatrix} = \begin{bmatrix} c\cdot a_{11} + d\cdot a_{11} & c\cdot a_{12} + d\cdot a_{12} \\ c\cdot a_{21} + d\cdot a_{21} & c\cdot a_{22} + d\cdot a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} c\cdot a_{11} & c\cdot a_{12} \\ c\cdot a_{21} & c\cdot a_{22} \end{bmatrix} + \begin{bmatrix} d\cdot a_{11} & d\cdot a_{12} \\ d\cdot a_{21} & d\cdot a_{22} \end{bmatrix} = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + d\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = cA + dA$$

93. $c(A)d = (cd)A$, for any real numbers c and d.

$$c(A)d = \left(c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)d = \begin{bmatrix} c\cdot a_{11} & c\cdot a_{12} \\ c\cdot a_{21} & c\cdot a_{22} \end{bmatrix}d = \begin{bmatrix} c\cdot a_{11}\cdot d & c\cdot a_{12}\cdot d \\ c\cdot a_{21}\cdot d & c\cdot a_{22}\cdot d \end{bmatrix} = \begin{bmatrix} c\cdot d\cdot a_{11} & c\cdot d\cdot a_{12} \\ c\cdot d\cdot a_{21} & c\cdot d\cdot a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (cd)\cdot a_{11} & (cd)\cdot a_{12} \\ (cd)\cdot a_{21} & (cd)\cdot a_{22} \end{bmatrix} = (cd)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = (cd)A$$

94. $(cd)A = c(dA)$, for any real numbers c and d.

$$(cd)A = (cd)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (cd)\cdot a_{11} & (cd)\cdot a_{12} \\ (cd)\cdot a_{21} & (cd)\cdot a_{22} \end{bmatrix} = \begin{bmatrix} c\cdot(d\cdot a_{11}) & c\cdot(d\cdot a_{12}) \\ c\cdot(d\cdot a_{21}) & c\cdot(d\cdot a_{22}) \end{bmatrix} = c\begin{bmatrix} d\cdot a_{11} & d\cdot a_{12} \\ d\cdot a_{21} & d\cdot a_{22} \end{bmatrix}$$

$$= c\left(d\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = c(dA)$$

Section 9.8: Matrix Inverses

1. $A = \begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix} = \begin{bmatrix} 1(-2)+0(3)+0(0) & 1(4)+0(5)+0(8) & 1(0)+0(9)+0(-6) \\ 0(-2)+1(3)+0(0) & 0(4)+1(5)+0(8) & 0(0)+1(9)+0(-6) \\ 0(-2)+0(3)+1(0) & 0(4)+0(5)+1(8) & 0(0)+0(9)+1(-6) \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+0 & 4+0+0 & 0+0+0 \\ 0+3+0 & 0+5+0 & 0+9+0 \\ 0+0+0 & 0+0+8 & 0+0+(-6) \end{bmatrix} = \begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix}$$

2. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AI_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a(1)+b(0) & a(0)+b(1) \\ c(1)+d(0) & c(0)+d(1) \end{bmatrix} = \begin{bmatrix} a+0 & 0+b \\ c+0 & 0+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1(a)+0(c) & 1(b)+0(d) \\ 0(a)+1(c) & 0(b)+1(d) \end{bmatrix} = \begin{bmatrix} a+0 & b+0 \\ 0+c & 0+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus, $AI_2 = I_2A = A$.

3.
$$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5(3)+7(-2) & 5(-7)+7(5) \\ 2(3)+3(-2) & 2(-7)+3(5) \end{bmatrix} = \begin{bmatrix} 15+(-14) & -35+35 \\ 6+(-6) & (-14)+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3(5)+(-7)(2) & 3(7)+(-7)(3) \\ (-2)(5)+5(2) & (-2)(7)+5(3) \end{bmatrix} = \begin{bmatrix} 15+(-14) & 21+(-21) \\ (-10)+10 & (-14)+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the products obtained by multiplying the matrices in either order are both the 2×2 identity matrix, the given matrices are inverses of each other.

4. $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2(-1)+3(1) & 2(3)+3(-2) \\ 1(-1)+1(1) & 1(3)+1(-2) \end{bmatrix} = \begin{bmatrix} -2+3 & 6+(-6) \\ -1+1 & 3+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1(2)+3(1) & -1(3)+3(1) \\ 1(2)+(-2)(1) & 1(3)+(-2)(1) \end{bmatrix} = \begin{bmatrix} -2+3 & -3+3 \\ 2+(-2) & 3+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since the products obtained by multiplying the matrices in either order are both the 2×2 identity matrix, the given matrices are inverses of each other.

5. $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -1(-5)+2(-3) & -1(-2)+2(-1) \\ 3(-5)+(-5)(-3) & 3(-2)+(-5)(-1) \end{bmatrix} = \begin{bmatrix} 5+(-6) & 2+(-2) \\ -15+15 & -6+5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Since this product is not the 2×2 identity matrix, the given matrices are not inverses of each other.

6. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2(2)+1(-3) & 2(1)+1(2) \\ 3(2)+2(-3) & 3(1)+2(2) \end{bmatrix} = \begin{bmatrix} 4+(-3) & 2+2 \\ 6+(-6) & 3+4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix}$

Since this product is not the 2×2 identity matrix, the given matrices are not inverses of each other.

7. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0(1)+1(1)+0(0) & 0(0)+1(0)+0(-1) & 0(1)+1(0)+0(0) \\ 0(1)+0(1)+(-2)(0) & 0(0)+0(0)+(-2)(-1) & 0(1)+0(0)+(-2)(0) \\ 1(1)+(-1)(1)+0(0) & 1(0)+(-1)(0)+0(-1) & 1(1)+(-1)(0)+0(0) \end{bmatrix}$

$= \begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+2 & 0+0+0 \\ 1+(-1)+0 & 0+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since this product is not the 3×3 identity matrix, the given matrices are not inverses of each other.

8. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+2(0)+0(0) & 1(-2)+2(1)+0(-1) & 1(0)+2(0)+0(1) \\ 0(1)+1(0)+0(0) & 0(-2)+1(1)+0(-1) & 0(0)+1(0)+0(1) \\ 0(1)+1(0)+0(0) & 0(-2)+1(1)+0(-1) & 0(0)+1(0)+0(1) \end{bmatrix}$

$= \begin{bmatrix} 1+0+0 & -2+2+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Since this product is not the 3×3 identity matrix, the given matrices are not inverses of each other.

9. $\begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1(15)+(-1)(-12)+(-1)(-4) & -1(4)+(-1)(-3)+(-1)(-1) & -1(-5)+(-1)(4)+(-1)(1) \\ 4(15)+5(-12)+0(-4) & 4(4)+5(-3)+0(-1) & 4(-5)+5(4)+0(1) \\ 0(15)+1(-12)+(-3)(-4) & 0(4)+1(-3)+(-3)(-1) & 0(-5)+1(4)+(-3)(1) \end{bmatrix}$

$= \begin{bmatrix} -15+12+4 & -4+3+1 & 5+(-4)+(-1) \\ 60+(-60)+0 & 16+(-15)+0 & -20+20+0 \\ 0+(-12)+12 & 0+(-3)+3 & 0+4+(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$$\begin{bmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 15(-1)+4(4)+(-5)(0) & 15(-1)+4(5)+(-5)(1) & 15(-1)+4(0)+(-5)(-3) \\ -12(-1)+(-3)(4)+4(0) & -12(-1)+(-3)(5)+4(1) & -12(-1)+(-3)(0)+4(-3) \\ -4(-1)+(-1)(4)+1(0) & -4(-1)+(-1)(5)+1(1) & -4(-1)+(-1)(0)+1(-3) \end{bmatrix}$$
$$= \begin{bmatrix} -15+16+0 & -15+20+(-5) & -15+0+15 \\ 12+(-12)+0 & 12+(-15)+4 & 12+0+(-12) \\ 4+(-4)+0 & 4+(-5)+1 & 0+4+(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

The given matrices are inverses of each other.

10. $$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1(7)+3(-1)+3(-1) & 1(-3)+3(1)+3(0) & 1(-3)+3(0)+3(1) \\ 1(7)+4(-1)+3(-1) & 1(-3)+4(1)+3(0) & 1(-3)+4(0)+3(1) \\ 1(7)+3(-1)+4(-1) & 1(-3)+3(1)+4(0) & 1(-3)+3(0)+4(1) \end{bmatrix}$$
$$= \begin{bmatrix} 7+(-3)+(-3) & -3+3+0 & -3+0+3 \\ 7+(-4)+(-3) & -3+4+0 & -3+0+3 \\ 7+(-3)+(-4) & -3+3+0 & -3+0+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 7(1)+(-3)(1)+(-3)(1) & 7(3)+(-3)(4)+(-3)(3) & 7(3)+(-3)(3)+(-3)(4) \\ -1(1)+1(1)+0(1) & -1(3)+1(4)+0(3) & -1(3)+1(3)+0(4) \\ -1(1)+0(1)+1(1) & -1(3)+0(4)+1(3) & -1(3)+0(3)+1(4) \end{bmatrix}$$
$$= \begin{bmatrix} 7+(-3)+(-3) & 21+(-12)+(-9) & 21+(-9)+(-12) \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

The given matrices are inverses of each other.

11. Find the inverse of $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$, if it exists. Since $[A \mid I_2] = \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{array}\right]$, we have

$$\left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ -2 & -1 & 0 & 1 \end{array}\right] -1\text{R1} \Rightarrow \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right] 2\text{R1}+\text{R2} \Rightarrow$$
$$\left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right] -\tfrac{1}{5}\text{R2} \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right] 2\text{R2}+\text{R1}$$

Thus, $A^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$.

12. Find the inverse of $A=\begin{bmatrix}1 & -1\\ 2 & 0\end{bmatrix}$, if it exists. Since $\left[A \mid I_2\right]=\left[\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 2 & 0 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 0 & 2 & -2 & 1\end{array}\right]-2\text{R1}+\text{R2} \Rightarrow \left[\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 0 & 1 & -1 & \frac{1}{2}\end{array}\right]\tfrac{1}{2}\text{R2} \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & -1 & \frac{1}{2}\end{array}\right]\text{R2}+\text{R1}.$$

Thus, $A^{-1}=\begin{bmatrix}0 & \frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}$.

13. Find the inverse of $A=\begin{bmatrix}-1 & -2\\ 3 & 4\end{bmatrix}$, if it exists. Since $\left[A \mid I_2\right]=\left[\begin{array}{cc|cc}-1 & -2 & 1 & 0\\ 3 & 4 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{cc|cc}-1 & -2 & 1 & 0\\ 0 & -2 & 3 & 1\end{array}\right]3\text{R1}+\text{R2} \Rightarrow \left[\begin{array}{cc|cc}1 & 2 & -1 & 0\\ 0 & -2 & 3 & 1\end{array}\right]-1\text{R1} \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 2 & 1\\ 0 & -2 & 3 & 1\end{array}\right]\text{R2}+\text{R1} \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 2 & 1\\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2}\end{array}\right]-\tfrac{1}{2}\text{R2}$$

Thus, $A^{-1}=\begin{bmatrix}2 & 1\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix}$.

14. Find the inverse of $A=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$, if it exists. Since $\left[A \mid I_2\right]=\left[\begin{array}{cc|cc}3 & -1 & 1 & 0\\ -5 & 2 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{cc|cc}1 & -\frac{1}{3} & \frac{1}{3} & 0\\ -5 & 2 & 0 & 1\end{array}\right]\tfrac{1}{3}\text{R1} \Rightarrow \left[\begin{array}{cc|cc}1 & -\frac{1}{3} & \frac{1}{3} & 0\\ 0 & \frac{1}{3} & \frac{5}{3} & 1\end{array}\right]5\text{R1}+\text{R2} \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 2 & 1\\ 0 & \frac{1}{3} & \frac{5}{3} & 1\end{array}\right]\text{R2}+\text{R1} \Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 2 & 1\\ 0 & 1 & 5 & 3\end{array}\right]3\text{R2}$$

Thus, $A^{-1}=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$.

15. Find the inverse of $\begin{bmatrix}5 & 10\\ -3 & -6\end{bmatrix}$, if it exists. Since $\left[A \middle| I_2\right]=\left[\begin{array}{cc|cc}5 & 10 & 1 & 0\\ -3 & -6 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{cc|cc}1 & 2 & \frac{1}{5} & 0\\ -3 & -6 & 0 & 1\end{array}\right]\tfrac{1}{5}\text{R1} \Rightarrow \left[\begin{array}{cc|cc}1 & 2 & \frac{1}{5} & 0\\ 0 & 0 & \frac{3}{5} & 1\end{array}\right]3\text{R1}+\text{R2}$$

At this point, the matrix should be changed so that the second-row, second-column element will be 1. Since that element is now 0, the desired transformation cannot be completed. Therefore, the inverse of the given matrix does not exist.

16. Find the inverse of $A=\begin{bmatrix}-6 & 4\\ -3 & 2\end{bmatrix}$, if it exists. Since $\left[A \middle| I_2\right]=\left[\begin{array}{cc|cc}-6 & 4 & 1 & 0\\ -3 & 2 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{cc|cc}1 & -\frac{2}{3} & -\frac{1}{6} & 0\\ -3 & 2 & 0 & 1\end{array}\right]-\tfrac{1}{6}\text{R1} \Rightarrow \left[\begin{array}{cc|cc}1 & -\frac{2}{3} & -\frac{1}{6} & 0\\ 0 & 0 & -\frac{1}{2} & 1\end{array}\right]3\text{R1}+\text{R2}.$$

At this point, the matrix should be changed so that the second-row, second-column element will be 1. Since that element is now 0, the desired transformation cannot be completed. Therefore, the inverse of the given matrix does not exist.

17. Find the inverse of $A=\begin{bmatrix}1 & 0 & 1\\ 0 & -1 & 0\\ 2 & 1 & 1\end{bmatrix}$, if it exists. Since $\left[A\,|\,I_3\right]=\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 2 & 1 & 1 & 0 & 0 & 1\end{array}\right]$ we have

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 1 & -1 & -2 & 0 & 1\end{array}\right]\begin{matrix}\\ \\ -2\text{R}1+\text{R}3\end{matrix} \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & -1 & -2 & 1 & 1\end{array}\right]\begin{matrix}\\ -1\text{R}2\\ \text{R}2+\text{R}3\end{matrix}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 2 & -1 & -1\end{array}\right]\begin{matrix}\\ \\ -1\text{R}3\end{matrix} \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 1 & 1\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 2 & -1 & -1\end{array}\right]\begin{matrix}-1\text{R}3+\text{R}1\\ \\ \\ \end{matrix}.$$

Thus, $A^{-1}=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 0\\ 2 & -1 & -1\end{bmatrix}$.

18. Find the inverse of $A=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 1 & 0 & 1\end{bmatrix}$, if it exists. Since $\left[A\,|\,I_3\right]=\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]\begin{matrix}\\ \\ -1\text{R}1+\text{R}3\end{matrix} \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]\begin{matrix}\\ -1\text{R}2.\\ \\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ -1 & 0 & 1\end{bmatrix}$.

19. Find the inverse of $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$, if it exists. Since $\left[A\,|\,I_3\right]=\left[\begin{array}{ccc|ccc}1 & 3 & 3 & 1 & 0 & 0\\ 1 & 4 & 3 & 0 & 1 & 0\\ 1 & 3 & 4 & 0 & 0 & 1\end{array}\right]$, we have

$$\left[\begin{array}{ccc|ccc}1 & 3 & 3 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]\begin{matrix}\\ -1\text{R}1+\text{R}2\\ -1\text{R}1+\text{R}3\end{matrix} \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 4 & -3 & 0\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]\begin{matrix}-3\text{R}2+\text{R}1\\ \\ \\ \end{matrix} \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 7 & -3 & -3\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]\begin{matrix}-3\text{R}3+\text{R}1\\ \\ \\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$.

20. Find the inverse of $A=\begin{bmatrix}-2&2&4\\-3&4&5\\1&0&2\end{bmatrix}$, if it exists. Since $\left[A\middle|I_3\right]=\left[\begin{array}{ccc|ccc}-2&2&4&1&0&0\\-3&4&5&0&1&0\\1&0&2&0&0&1\end{array}\right]$, we have

$$\left[\begin{array}{ccc|ccc}1&0&2&0&0&1\\-3&4&5&0&1&0\\-2&2&4&1&0&0\end{array}\right]\begin{array}{l}R1\leftrightarrow R3\\ \\ \\\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&2&0&0&1\\0&4&11&0&1&3\\0&2&8&1&0&2\end{array}\right]\begin{array}{l}\\3R1+R2\\2R1+R3\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&0&0&1\\0&1&\frac{11}{4}&0&\frac{1}{4}&\frac{3}{4}\\0&2&8&1&0&2\end{array}\right]\begin{array}{l}\\\frac{1}{4}R2\\ \\\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&2&0&0&1\\0&1&\frac{11}{4}&0&\frac{1}{4}&\frac{3}{4}\\0&0&\frac{5}{2}&1&-\frac{1}{2}&\frac{1}{2}\end{array}\right]\begin{array}{l}\\ \\-2R2+R3\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&0&0&1\\0&1&\frac{11}{4}&0&\frac{1}{4}&\frac{3}{4}\\0&0&1&\frac{2}{5}&-\frac{1}{5}&\frac{1}{5}\end{array}\right]\begin{array}{l}\\ \\\frac{2}{5}R3\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&-\frac{4}{5}&\frac{2}{5}&\frac{3}{5}\\0&1&0&-\frac{11}{10}&\frac{4}{5}&\frac{1}{5}\\0&0&1&\frac{2}{5}&-\frac{1}{5}&\frac{1}{5}\end{array}\right]\begin{array}{l}-2R3+R1\\-\frac{11}{4}R3+R2.\\ \\\end{array}$$

Thus, $A^{-1}=\begin{bmatrix}-\frac{4}{5}&\frac{2}{5}&\frac{3}{5}\\-\frac{11}{10}&\frac{4}{5}&\frac{1}{5}\\\frac{2}{5}&-\frac{1}{5}&\frac{1}{5}\end{bmatrix}$.

21. Find the inverse of $A=\begin{bmatrix}2&2&-4\\2&6&0\\-3&-3&5\end{bmatrix}$, if it exists. Since $\left[A\middle|I_3\right]=\left[\begin{array}{ccc|ccc}2&2&-4&1&0&0\\2&6&0&0&1&0\\-3&-3&5&0&0&1\end{array}\right]$,

$$\left[\begin{array}{ccc|ccc}1&1&-2&\frac{1}{2}&0&0\\2&6&0&0&1&0\\-3&-3&5&0&0&1\end{array}\right]\begin{array}{l}\frac{1}{2}R1\\ \\ \\\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&1&-2&\frac{1}{2}&0&0\\0&4&4&-1&1&0\\0&0&-1&\frac{3}{2}&0&1\end{array}\right]\begin{array}{l}\\-2R1+R2\\3R1+R3\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&1&-2&\frac{1}{2}&0&0\\0&1&1&-\frac{1}{4}&\frac{1}{4}&0\\0&0&-1&\frac{3}{2}&0&1\end{array}\right]\begin{array}{l}\\\frac{1}{4}R2\\ \\\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&-3&\frac{3}{4}&-\frac{1}{4}&0\\0&1&1&-\frac{1}{4}&\frac{1}{4}&0\\0&0&-1&\frac{3}{2}&0&1\end{array}\right]\begin{array}{l}-1R2+R1\\ \\ \\\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&-3&\frac{3}{4}&-\frac{1}{4}&0\\0&1&1&-\frac{1}{4}&\frac{1}{4}&0\\0&0&1&-\frac{3}{2}&0&-1\end{array}\right]\begin{array}{l}\\ \\-1R3\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&-\frac{15}{4}&-\frac{1}{4}&-3\\0&1&0&\frac{5}{4}&\frac{1}{4}&1\\0&0&1&-\frac{3}{2}&0&-1\end{array}\right]\begin{array}{l}3R3+R1\\-1R3+R2.\\ \\\end{array}$$

Thus, $A^{-1}=\begin{bmatrix}-\frac{15}{4}&-\frac{1}{4}&-3\\\frac{5}{4}&\frac{1}{4}&1\\-\frac{3}{2}&0&-1\end{bmatrix}$.

22. Find the inverse of $A=\begin{bmatrix}2&4&6\\-1&-4&-3\\0&1&-1\end{bmatrix}$, if it exists. Since $\left[A\,|\,I_3\right]=\left[\begin{array}{ccc|ccc}2&4&6&1&0&0\\-1&-4&-3&0&1&0\\0&1&-1&0&0&1\end{array}\right]$,

$$\left[\begin{array}{ccc|ccc}1&2&3&\frac{1}{2}&0&0\\-1&-4&-3&0&1&0\\0&1&-1&0&0&1\end{array}\right]\begin{matrix}\frac{1}{2}\text{R1}\\ \\ \\ \end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&2&3&\frac{1}{2}&0&0\\0&-2&0&\frac{1}{2}&1&0\\0&1&-1&0&0&1\end{array}\right]\begin{matrix} \\ \text{R1}+\text{R2}\\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc}1&2&3&\frac{1}{2}&0&0\\0&1&0&-\frac{1}{4}&-\frac{1}{2}&0\\0&1&-1&0&0&1\end{array}\right]\begin{matrix} \\ -\frac{1}{2}\text{R2}\\ \end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&3&1&1&0\\0&1&0&-\frac{1}{4}&-\frac{1}{2}&0\\0&0&-1&\frac{1}{4}&\frac{1}{2}&1\end{array}\right]\begin{matrix}-2\text{R2}+\text{R1}\\ \\ -\text{R2}+\text{R3}\end{matrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&3&1&1&0\\0&1&0&-\frac{1}{4}&-\frac{1}{2}&0\\0&0&1&-\frac{1}{4}&-\frac{1}{2}&-1\end{array}\right]\begin{matrix} \\ \\ -1\text{R3}\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&\frac{7}{4}&\frac{5}{2}&3\\0&1&0&-\frac{1}{4}&-\frac{1}{2}&0\\0&0&1&-\frac{1}{4}&-\frac{1}{2}&-1\end{array}\right]\begin{matrix}-3\text{R3}+\text{R1}\\ \\ \\ \end{matrix}.$$

Thus, $A^{-1}=\begin{bmatrix}\frac{7}{4}&\frac{5}{2}&3\\-\frac{1}{4}&-\frac{1}{2}&0\\-\frac{1}{4}&-\frac{1}{2}&-1\end{bmatrix}$.

23. Find the inverse of $A=\begin{bmatrix}1&1&0&2\\2&-1&1&-1\\3&3&2&-2\\1&2&1&0\end{bmatrix}$, if it exists.

$$\left[A\,|\,I_4\right]=\left[\begin{array}{cccc|cccc}1&1&0&2&1&0&0&0\\2&-1&1&-1&0&1&0&0\\3&3&2&-2&0&0&1&0\\1&2&1&0&0&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&1&0&2&1&0&0&0\\0&-3&1&-5&-2&1&0&0\\0&0&2&-8&-3&0&1&0\\0&1&1&-2&-1&0&0&1\end{array}\right]\begin{matrix} \\ -2\text{R1}+\text{R2}\\ -3\text{R1}+\text{R3}\\ -1\text{R1}+\text{R4}\end{matrix}$$

$$\Rightarrow\left[\begin{array}{cccc|cccc}1&1&0&2&1&0&0&0\\0&1&-\frac{1}{3}&\frac{5}{3}&\frac{2}{3}&-\frac{1}{3}&0&0\\0&0&2&-8&-3&0&1&0\\0&1&1&-2&-1&0&0&1\end{array}\right]\begin{matrix} \\ -\frac{1}{3}\text{R2}\\ \\ \\ \end{matrix}\Rightarrow\left[\begin{array}{cccc|cccc}1&0&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&-\frac{1}{3}&\frac{5}{3}&\frac{2}{3}&-\frac{1}{3}&0&0\\0&0&2&-8&-3&0&1&0\\0&0&\frac{4}{3}&-\frac{11}{3}&-\frac{5}{3}&\frac{1}{3}&0&1\end{array}\right]\begin{matrix}-1\text{R2}+\text{R1}\\ \\ \\ -1\text{R2}+\text{R4}\end{matrix}$$

$$\Rightarrow\left[\begin{array}{cccc|cccc}1&0&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&-\frac{1}{3}&\frac{5}{3}&\frac{2}{3}&-\frac{1}{3}&0&0\\0&0&1&-4&-\frac{3}{2}&0&\frac{1}{2}&0\\0&0&\frac{4}{3}&-\frac{11}{3}&-\frac{5}{3}&\frac{1}{3}&0&1\end{array}\right]\begin{matrix} \\ \\ \frac{1}{2}\text{R3}\\ \\ \end{matrix}\Rightarrow\left[\begin{array}{cccc|cccc}1&0&0&\frac{5}{3}&\frac{5}{6}&\frac{1}{3}&-\frac{1}{6}&0\\0&1&0&\frac{1}{3}&\frac{1}{6}&-\frac{1}{3}&\frac{1}{6}&0\\0&0&1&-4&-\frac{3}{2}&0&\frac{1}{2}&0\\0&0&0&\frac{5}{3}&\frac{1}{3}&\frac{1}{3}&-\frac{2}{3}&1\end{array}\right]\begin{matrix}-\frac{1}{3}\text{R3}+\text{R1}\\ \frac{1}{3}\text{R3}+\text{R2}\\ \\ -\frac{4}{3}\text{R3}+\text{R4}\end{matrix}$$

$$\Rightarrow\left[\begin{array}{cccc|cccc}1&0&0&\frac{5}{3}&\frac{5}{6}&\frac{1}{3}&-\frac{1}{6}&0\\0&1&0&\frac{1}{3}&\frac{1}{6}&-\frac{1}{3}&\frac{1}{6}&0\\0&0&1&-4&-\frac{3}{2}&0&\frac{1}{2}&0\\0&0&0&1&\frac{1}{5}&\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\end{array}\right]\begin{matrix} \\ \\ \\ \frac{3}{5}\text{R4}\end{matrix}\Rightarrow\left[\begin{array}{cccc|cccc}1&0&0&0&\frac{1}{2}&0&\frac{1}{2}&-1\\0&1&0&0&\frac{1}{10}&-\frac{2}{5}&\frac{3}{10}&-\frac{1}{5}\\0&0&1&0&-\frac{7}{10}&\frac{4}{5}&-\frac{11}{10}&\frac{12}{5}\\0&0&0&1&\frac{1}{5}&\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\end{array}\right]\begin{matrix}-\frac{5}{3}\text{R4}+\text{R1}\\ -\frac{1}{3}\text{R4}+\text{R2}\\ 4\text{R4}+\text{R3}\\ \\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}&-1\\\frac{1}{10}&-\frac{2}{5}&\frac{3}{10}&-\frac{1}{5}\\-\frac{7}{10}&\frac{4}{5}&-\frac{11}{10}&\frac{12}{5}\\\frac{1}{5}&\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\end{bmatrix}$.

24. Find the inverse of $A=\begin{bmatrix}1&-2&3&0\\0&1&-1&1\\-2&2&-2&4\\0&2&-3&1\end{bmatrix}$, if it exists.

$$\left[A\middle|I_4\right]=\left[\begin{array}{rrrr|rrrr}1&-2&3&0&1&0&0&0\\0&1&-1&1&0&1&0&0\\-2&2&-2&4&0&0&1&0\\0&2&-3&1&0&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{rrrr|rrrr}1&-2&3&0&1&0&0&0\\0&1&-1&1&0&1&0&0\\0&-2&4&4&2&0&1&0\\0&2&-3&1&0&0&0&1\end{array}\right]\begin{array}{l}\\ \\ 2R1+R3\\ \\ \end{array}$$

$$\left[\begin{array}{rrrr|rrrr}1&0&1&2&1&2&0&0\\0&1&-1&1&0&1&0&0\\0&0&2&6&2&2&1&0\\0&0&-1&-1&0&-2&0&1\end{array}\right]\begin{array}{l}2R2+R1\\ \\ 2R2+R3\\ -2R2+R4\end{array}\Rightarrow\left[\begin{array}{rrrr|rrrr}1&0&1&2&1&2&0&0\\0&1&-1&1&0&1&0&0\\0&0&1&3&1&1&\frac{1}{2}&0\\0&0&-1&-1&0&-2&0&1\end{array}\right]\begin{array}{l}\\ \\ \frac{1}{2}R3\\ \\ \end{array}$$

$$\left[\begin{array}{rrrr|rrrr}1&0&0&-1&0&1&-\frac{1}{2}&0\\0&1&0&4&1&2&\frac{1}{2}&0\\0&0&1&3&1&1&\frac{1}{2}&0\\0&0&0&2&1&-1&\frac{1}{2}&1\end{array}\right]\begin{array}{l}-1R3+R1\\ R3+R2\\ \\ R3+R4\end{array}\Rightarrow\left[\begin{array}{rrrr|rrrr}1&0&0&-1&0&1&-\frac{1}{2}&0\\0&1&0&4&1&2&\frac{1}{2}&0\\0&0&1&3&1&1&\frac{1}{2}&0\\0&0&0&1&\frac{1}{2}&-\frac{1}{2}&\frac{1}{4}&\frac{1}{2}\end{array}\right]\begin{array}{l}\\ \\ \\ \frac{1}{2}R4\end{array}$$

$$\left[\begin{array}{rrrr|rrrr}1&0&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{4}&\frac{1}{2}\\0&1&0&0&-1&4&-\frac{1}{2}&-2\\0&0&1&0&-\frac{1}{2}&\frac{5}{2}&-\frac{1}{4}&-\frac{3}{2}\\0&0&0&1&\frac{1}{2}&-\frac{1}{2}&\frac{1}{4}&\frac{1}{2}\end{array}\right]\begin{array}{l}R4+R1\\ -4R4+R2\\ -3R4+R3\\ \\ \end{array}\quad\text{Thus, } A^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&-\frac{1}{4}&\frac{1}{2}\\-1&4&-\frac{1}{2}&-2\\-\frac{1}{2}&\frac{5}{2}&-\frac{1}{4}&-\frac{3}{2}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{4}&\frac{1}{2}\end{bmatrix}.$$

25. Since $A^{-1}=\begin{bmatrix}5&-9\\-1&2\end{bmatrix}$, we need to find A, where $A=\left(A^{-1}\right)^{-1}$.

$$\left[A^{-1}\middle|I_2\right]=\left[\begin{array}{rr|rr}5&-9&1&0\\-1&2&0&1\end{array}\right]\Rightarrow\left[\begin{array}{rr|rr}1&-\frac{9}{5}&\frac{1}{5}&0\\-1&2&0&1\end{array}\right]\begin{array}{l}\frac{1}{5}R1\\ \\ \end{array}\Rightarrow\left[\begin{array}{rr|rr}1&-\frac{9}{5}&\frac{1}{5}&0\\0&\frac{1}{5}&\frac{1}{5}&1\end{array}\right]\begin{array}{l}\\ 1R1+R2\end{array}$$

$$\left[\begin{array}{rr|rr}1&-\frac{9}{5}&\frac{1}{5}&0\\0&1&1&5\end{array}\right]\begin{array}{l}\\ 5R2\end{array}\Rightarrow\left[\begin{array}{rr|rr}1&0&2&9\\0&1&1&5\end{array}\right]\begin{array}{l}\frac{9}{5}R2+R1\\ \\ \end{array}$$

Thus, $A=\begin{bmatrix}2&9\\1&5\end{bmatrix}$.

26. Since $A^{-1}=\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}&0\\\frac{1}{3}&-\frac{5}{3}&1\\\frac{1}{3}&\frac{1}{3}&0\end{bmatrix}$, we need to find A, where $A=\left(A^{-1}\right)^{-1}$.

$$\left[A^{-1}\middle|I_3\right]=\left[\begin{array}{rrr|rrr}\frac{2}{3}&-\frac{1}{3}&0&1&0&0\\\frac{1}{3}&-\frac{5}{3}&1&0&1&0\\\frac{1}{3}&\frac{1}{3}&0&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{rrr|rrr}1&-\frac{1}{2}&0&\frac{3}{2}&0&0\\\frac{1}{3}&-\frac{5}{3}&1&0&1&0\\\frac{1}{3}&\frac{1}{3}&0&0&0&1\end{array}\right]\begin{array}{l}\frac{3}{2}R1\\ \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|rrr}1&-\frac{1}{2}&0&\frac{3}{2}&0&0\\0&-\frac{3}{2}&1&-\frac{1}{2}&1&0\\0&\frac{1}{2}&0&-\frac{1}{2}&0&1\end{array}\right]\begin{array}{l}\\ -\frac{1}{3}R1+R2\\ -\frac{1}{3}R1+R3\end{array}\Rightarrow\left[\begin{array}{rrr|rrr}1&-\frac{1}{2}&0&\frac{3}{2}&0&0\\0&1&-\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}&0\\0&\frac{1}{2}&0&-\frac{1}{2}&0&1\end{array}\right]\begin{array}{l}\\ -\frac{2}{3}R2\\ \\ \end{array}$$

(*continued on next page*)

(*continued from page 989*)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}\text{R2}+\text{R1} \\ \\ -\frac{1}{2}\text{R2}+\text{R3} \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right] 3\text{R3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{array}\right] \begin{array}{l} \frac{1}{3}\text{R3}+\text{R1} \\ \frac{2}{3}\text{R3}+\text{R2} \\ \\ \end{array}$$

Thus, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$.

27. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $ad - bc = ad - cb$ is the determinant of matrix A.

28. If $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$, then

$$A^{-1} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}.$$

29. $A^{-1} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix} = \begin{bmatrix} \frac{1}{|A|}d & \frac{1}{|A|}(-b) \\ \frac{1}{|A|}(-c) & \frac{1}{|A|}a \end{bmatrix}$

$$= \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

30. Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, switch entries a_{11} and a_{22} (a and d, respectively). Then negate entries a_{12} and a_{21} (b and c, respectively). Finally, multiply the matrix by $\frac{1}{|A|}$ to obtain

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

31. Given $A = \begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}$, $|A| = 12 - 14 = -2$. Thus

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 3 & -2 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{7}{2} & -2 \end{bmatrix}.$$

32. The inverse of a 2×2 matrix A does not exist if the determinant of A has the value <u>zero</u>.

33. $-x + y = 1$
$2x - y = 1$

Given $A = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find A^{-1}. Since $\left[A \mid I_2\right] = \left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right]$,

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -1\text{R1} \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \begin{array}{l} \\ -2\text{R1}+\text{R2} \end{array} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] \begin{array}{l} \text{R2}+\text{R1} \\ \\ \end{array}$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

$$X = A^{-1}B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1)+1(1) \\ 2(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solution set: $\{(2,3)\}$

34. $x+y=5$
$x-y=-1$

Given $A=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, and $B=\begin{bmatrix}5\\ -1\end{bmatrix}$,

find A^{-1}. Since $\left[A|I_2\right]=\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 1 & -1 & 0 & 1\end{array}\right]$,

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & -2 & -1 & 1\end{array}\right]-1\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & 1 & \frac{1}{2} & -\frac{1}{2}\end{array}\right]-\tfrac{1}{2}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \frac{1}{2} & -\frac{1}{2}\end{array}\right]-\text{R2}+\text{R1}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}$.

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}5\\ -1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}(5)+\frac{1}{2}(-1)\\ \frac{1}{2}(5)+\left(-\frac{1}{2}\right)(-1)\end{bmatrix}=\begin{bmatrix}\frac{5}{2}+\left(-\frac{1}{2}\right)\\ \frac{5}{2}+\frac{1}{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{4}{2}\\ \frac{6}{2}\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$$

Solution set: $\{(2,3)\}$

35. $2x-y=-8$
$3x+y=-2$

Given $A=\begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, and $B=\begin{bmatrix}-8\\ -2\end{bmatrix}$,

find A^{-1}. We will use the relation given on page 588 to calculate A^{-1} in the solution to this exercise.

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}$.

Thus we have:

$$A^{-1}=\begin{bmatrix}\frac{1}{2(1)-(-1)(3)} & \frac{1}{2(1)-(-1)(3)}\\ \frac{-3}{2(1)-(-1)(3)} & \frac{2}{2(1)-(-1)(3)}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2-(-3)} & \frac{1}{2-(-3)}\\ \frac{-3}{2-(-3)} & \frac{2}{2-(-3)}\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ -\frac{3}{5} & \frac{2}{5}\end{bmatrix}$$

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ -\frac{3}{5} & \frac{2}{5}\end{bmatrix}\begin{bmatrix}-8\\ -2\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{5}(-8)+\frac{1}{5}(-2)\\ -\frac{3}{5}(-8)+\frac{2}{5}(-2)\end{bmatrix}=\begin{bmatrix}-\frac{8}{5}+\left(-\frac{2}{5}\right)\\ \frac{24}{5}+\left(-\frac{4}{5}\right)\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{10}{5}\\ \frac{20}{5}\end{bmatrix}=\begin{bmatrix}-2\\ 4\end{bmatrix}$$

Solution set: $\{(-2,4)\}$

36. $x+3y=-12$
$2x-y=11$

Given $A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, and $B=\begin{bmatrix}-12\\ 11\end{bmatrix}$,

find A^{-1}. We will use the relation given on page 588 to calculate A^{-1} in the solution to this exercise.

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}$.

Thus we have the following.

$$A^{-1}=\begin{bmatrix}\frac{-1}{1(-1)-3(2)} & \frac{-3}{1(-1)-3(2)}\\ \frac{-2}{1(-1)-3(2)} & \frac{1}{1(-1)-3(2)}\end{bmatrix}=\begin{bmatrix}\frac{-1}{-1-6} & \frac{-3}{-1-6}\\ \frac{-2}{-1-6} & \frac{1}{-1-6}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{7} & \frac{3}{7}\\ \frac{2}{7} & -\frac{1}{7}\end{bmatrix}$$

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{7} & \frac{3}{7}\\ \frac{2}{7} & -\frac{1}{7}\end{bmatrix}\begin{bmatrix}-12\\ 11\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{7}(-12)+\frac{3}{7}(11)\\ \frac{2}{7}(-12)+\left(-\frac{1}{7}\right)(11)\end{bmatrix}=\begin{bmatrix}-\frac{12}{7}+\frac{33}{7}\\ -\frac{24}{7}+\left(-\frac{11}{7}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{21}{7}\\ -\frac{35}{7}\end{bmatrix}=\begin{bmatrix}3\\ -5\end{bmatrix}$$

Solution set: $\{(3,-5)\}$

37. $2x+3y=-10$
$3x+4y=-12$

Given $A=\begin{bmatrix}2&3\\3&4\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$, and $B=\begin{bmatrix}-10\\-12\end{bmatrix}$,

find A^{-1}. Since $\left[A|I_2\right]=\left[\begin{array}{cc|cc}2&3&1&0\\3&4&0&1\end{array}\right]$,

$$\left[\begin{array}{cc|cc}1&\frac{3}{2}&\frac{1}{2}&0\\3&4&0&1\end{array}\right]\tfrac{1}{2}\text{R1} \Rightarrow$$
$$\left[\begin{array}{cc|cc}1&\frac{3}{2}&\frac{1}{2}&0\\0&-\frac{1}{2}&-\frac{3}{2}&1\end{array}\right]-3\text{R1}+\text{R2} \Rightarrow$$
$$\left[\begin{array}{cc|cc}1&\frac{3}{2}&\frac{1}{2}&0\\0&1&3&-2\end{array}\right]-2\text{R2} \Rightarrow$$
$$\left[\begin{array}{cc|cc}1&0&-4&3\\0&1&3&-2\end{array}\right]-\tfrac{3}{2}\text{R2}+\text{R1}$$

Thus, $A^{-1}=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}$.

$$X=A^{-1}B=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}\begin{bmatrix}-10\\-12\end{bmatrix}=\begin{bmatrix}-4(-10)+3(-12)\\3(-10)+(-2)(-12)\end{bmatrix}=\begin{bmatrix}40+(-36)\\-30+24\end{bmatrix}=\begin{bmatrix}4\\-6\end{bmatrix}$$

Solution set: $\{(4,-6)\}$

38. $2x-3y=10$
$2x+2y=5$

Given $A=\begin{bmatrix}2&-3\\2&2\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$, and $B=\begin{bmatrix}10\\5\end{bmatrix}$,

find A^{-1}. Since $\left[A|I_2\right]=\left[\begin{array}{cc|cc}2&-3&1&0\\2&2&0&1\end{array}\right]$,

$$\left[\begin{array}{cc|cc}1&-\frac{3}{2}&\frac{1}{2}&0\\2&2&0&1\end{array}\right]\tfrac{1}{2}\text{R1} \Rightarrow$$
$$\left[\begin{array}{cc|cc}1&-\frac{3}{2}&\frac{1}{2}&0\\0&5&-1&1\end{array}\right]-2\text{R1}+\text{R2} \Rightarrow$$
$$\left[\begin{array}{cc|cc}1&-\frac{3}{2}&\frac{1}{2}&0\\0&1&-\frac{1}{5}&\frac{1}{5}\end{array}\right]\tfrac{1}{5}\text{R2} \Rightarrow$$
$$\left[\begin{array}{cc|cc}1&0&\frac{1}{5}&\frac{3}{10}\\0&1&-\frac{1}{5}&\frac{1}{5}\end{array}\right]\tfrac{3}{2}\text{R2}+\text{R1}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{1}{5}&\frac{3}{10}\\-\frac{1}{5}&\frac{1}{5}\end{bmatrix}$.

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{5}&\frac{3}{10}\\-\frac{1}{5}&\frac{1}{5}\end{bmatrix}\begin{bmatrix}10\\5\end{bmatrix}=\begin{bmatrix}\frac{1}{5}(10)+\frac{3}{10}(5)\\-\frac{1}{5}(10)+\frac{1}{5}(5)\end{bmatrix}=\begin{bmatrix}2+\frac{3}{2}\\-2+1\end{bmatrix}=\begin{bmatrix}\frac{4}{2}+\frac{3}{2}\\-1\end{bmatrix}=\begin{bmatrix}\frac{7}{2}\\-1\end{bmatrix}$$

Solution set: $\left\{\left(\frac{7}{2},-1\right)\right\}$

39. $6x+9y=3$
$-8x+3y=6$

Given $A=\begin{bmatrix}6&9\\-8&3\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$, and $B=\begin{bmatrix}3\\6\end{bmatrix}$,

find A^{-1}. We will use the results of the Relating Concepts exercises to calculate A^{-1} in the solution to this exercise.

If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$.

Thus we have the following.

$$A^{-1}=\frac{1}{6(3)-9(-8)}\begin{bmatrix}3&-9\\8&6\end{bmatrix}=\frac{1}{18-(-72)}\begin{bmatrix}3&-9\\8&6\end{bmatrix}=\frac{1}{90}\begin{bmatrix}3&-9\\8&6\end{bmatrix}=\begin{bmatrix}\frac{3}{90}&\frac{-9}{90}\\\frac{8}{90}&\frac{6}{90}\end{bmatrix}=\begin{bmatrix}\frac{1}{30}&-\frac{1}{10}\\\frac{4}{45}&\frac{1}{15}\end{bmatrix}$$

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{30}&-\frac{1}{10}\\\frac{4}{45}&\frac{1}{15}\end{bmatrix}\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}\frac{1}{30}(3)+\left(-\frac{1}{10}\right)(6)\\\frac{4}{45}(3)+\frac{1}{15}(6)\end{bmatrix}=\begin{bmatrix}\frac{1}{10}+\left(-\frac{6}{10}\right)\\\frac{4}{15}+\frac{2}{5}\end{bmatrix}=\begin{bmatrix}-\frac{5}{10}\\\frac{4}{15}+\frac{6}{15}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\\frac{10}{15}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\\frac{2}{3}\end{bmatrix}$$

Solution set: $\left\{\left(-\frac{1}{2},\frac{2}{3}\right)\right\}$

40. $5x-3y=0$
$10x+6y=-4$

Given $A=\begin{bmatrix}5&-3\\10&6\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$, and $B=\begin{bmatrix}0\\-4\end{bmatrix}$,

find A^{-1}. We will use the results of the Relating Concepts exercises to calculate A^{-1} in the solution to this exercise.

If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$.

Thus we have

$$A^{-1} = \frac{1}{5(6)-(-3)(10)}\begin{bmatrix} 6 & 3 \\ -10 & 5 \end{bmatrix} = \frac{1}{30-(-30)}\begin{bmatrix} 6 & 3 \\ -10 & 5 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} 6 & 3 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} \frac{6}{60} & \frac{3}{60} \\ -\frac{10}{60} & \frac{5}{60} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{20} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} \frac{1}{10} & \frac{1}{20} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10}(0) + \frac{1}{20}(-4) \\ -\frac{1}{6}(0) + \frac{1}{12}(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 0 + \left(-\frac{1}{5}\right) \\ 0 + \left(-\frac{1}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{3} \end{bmatrix}$$

Solution set: $\left\{\left(-\frac{1}{5}, -\frac{1}{3}\right)\right\}$

41. $.2x + .3y = -1.9$
$.7x - .2y = 4.6$

Given

$$A = \begin{bmatrix} .2 & .3 \\ .7 & -.2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} -1.9 \\ 4.6 \end{bmatrix},$$

find A^{-1}. Since $\left[A \mid I_2\right] = \left[\begin{array}{cc|cc} .2 & .3 & 1 & 0 \\ .7 & -.2 & 0 & 1 \end{array}\right]$,

$$\left[\begin{array}{cc|cc} 1 & 1.5 & 5 & 0 \\ .7 & -.2 & 0 & 1 \end{array}\right] \begin{matrix} 5\text{R}1 \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 1.5 & 5 & 0 \\ 0 & -1.25 & -3.5 & 1 \end{array}\right] \begin{matrix} \\ -.7\text{R}1 + \text{R}2 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 1.5 & 5 & 0 \\ 0 & 1 & 2.8 & -.8 \end{array}\right] \begin{matrix} \\ \frac{1}{-1.25}\text{R}2 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & .8 & 1.2 \\ 0 & 1 & 2.8 & -.8 \end{array}\right] \begin{matrix} -1.5\text{R}2 + \text{R}1 \\ \end{matrix}$$

Thus, $A^{-1} = \begin{bmatrix} .8 & 1.2 \\ 2.8 & -.8 \end{bmatrix}$.

$$X = A^{-1}B = \begin{bmatrix} .8 & 1.2 \\ 2.8 & -.8 \end{bmatrix}\begin{bmatrix} -1.9 \\ 4.6 \end{bmatrix}$$

$$= \begin{bmatrix} .8(-1.9) + 1.2(4.6) \\ 2.8(-1.9) + (-.8)(4.6) \end{bmatrix}$$

$$= \begin{bmatrix} -1.52 + 5.52 \\ -5.32 + (-3.68) \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$$

Solution set: $\{(4, -9)\}$

42. (a) In order to solve $\begin{matrix} 7x - 2y = 3 \\ 14x - 4y = 1 \end{matrix}$ by the matrix inverse method, we must first find A^{-1} given that $A = \begin{bmatrix} 7 & -2 \\ 14 & -4 \end{bmatrix}$. Since $\left[A \middle| I_2\right] = \left[\begin{array}{cc|cc} 7 & -2 & 1 & 0 \\ 14 & -4 & 0 & 1 \end{array}\right]$, we have the following.

$$\left[\begin{array}{cc|cc} 1 & -\frac{2}{7} & \frac{1}{7} & 0 \\ 14 & -4 & 0 & 1 \end{array}\right] \begin{matrix} \frac{1}{7}\text{R}1 \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \begin{matrix} \\ -14\text{R}1 + \text{R}2 \end{matrix}$$

Since it is not possible to get a 1 in the second entry of the second column, the inverse of A does not exist, hence the matrix inverse method cannot be used to solve the system.

(b) In order to solve $\begin{matrix} x - 2y + 3z = 4 \\ 2x - 4y + 6z = 8 \\ 3x - 6y + 9z = 14 \end{matrix}$ by the matrix inverse method, we must first find A^{-1} given that $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}$. Since

$$\left[A \middle| I_3\right] = \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & -4 & 6 & 0 & 1 & 0 \\ 3 & -6 & 9 & 0 & 0 & 1 \end{array}\right],$$ we have the following.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \end{array}\right] \begin{matrix} \\ -2\text{R}1 + \text{R}2 \\ -3\text{R}1 + \text{R}3 \end{matrix}$$

Since it is not possible to get a 1 in the second entry of the second column (nor the third entry of the third column), the inverse of A does not exist, hence the matrix inverse method cannot be used to solve the system.

43. $x+\ \ y+z=6$
$2x+3y-z=7$
$3x-\ \ y-z=6$

Given $A=\begin{bmatrix}1&1&1\\2&3&-1\\3&-1&-1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, and $B=\begin{bmatrix}6\\7\\6\end{bmatrix}$, find A^{-1}. Since $\left[A\middle|I_3\right]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&3&-1&0&1&0\\3&-1&-1&0&0&1\end{array}\right]$,

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&-3&-2&1&0\\0&-4&-4&-3&0&1\end{array}\right]\begin{matrix}\\-2R1+R2\\-3R1+R3\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&4&3&-1&0\\0&1&-3&-2&1&0\\0&0&-16&-11&4&1\end{array}\right]\begin{matrix}-R2+R1\\ \\4R2+R3\end{matrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&4&3&-1&0\\0&1&-3&-2&1&0\\0&0&1&\frac{11}{16}&-\frac{1}{4}&-\frac{1}{16}\end{array}\right]\begin{matrix}\\ \\-\frac{1}{16}R3\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{4}&0&\frac{1}{4}\\0&1&0&\frac{1}{16}&\frac{1}{4}&-\frac{3}{16}\\0&0&1&\frac{11}{16}&-\frac{1}{4}&-\frac{1}{16}\end{array}\right]\begin{matrix}-4R3+R1\\3R3+R2\end{matrix}.\text{ Thus, }A^{-1}=\begin{bmatrix}\frac{1}{4}&0&\frac{1}{4}\\\frac{1}{16}&\frac{1}{4}&-\frac{3}{16}\\\frac{11}{16}&-\frac{1}{4}&-\frac{1}{16}\end{bmatrix}.$$

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{4}&0&\frac{1}{4}\\\frac{1}{16}&\frac{1}{4}&-\frac{3}{16}\\\frac{11}{16}&-\frac{1}{4}&-\frac{1}{16}\end{bmatrix}\begin{bmatrix}6\\7\\6\end{bmatrix}=\begin{bmatrix}\frac{1}{4}(6)+0(7)+\frac{1}{4}(6)\\\frac{1}{16}(6)+\frac{1}{4}(7)+\left(-\frac{3}{16}\right)(6)\\\frac{11}{16}(6)+\left(-\frac{1}{4}\right)(7)+\left(-\frac{1}{16}\right)(6)\end{bmatrix}=\begin{bmatrix}\frac{3}{2}+0+\frac{3}{2}\\\frac{3}{8}+\frac{7}{4}+\left(-\frac{9}{8}\right)\\\frac{33}{8}+\left(-\frac{7}{4}\right)+\left(-\frac{3}{8}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{6}{2}\\\frac{7}{4}+\left(-\frac{6}{8}\right)\\\frac{30}{8}+\left(-\frac{7}{4}\right)\end{bmatrix}=\begin{bmatrix}3\\\frac{7}{4}+\left(-\frac{3}{4}\right)\\\frac{15}{4}+\left(-\frac{7}{4}\right)\end{bmatrix}=\begin{bmatrix}3\\\frac{4}{4}\\\frac{8}{4}\end{bmatrix}=\begin{bmatrix}3\\1\\2\end{bmatrix}$$

Solution set: $\{(3,1,2)\}$

44. $2x+5y+2z=9$
$4x-7y-3z=7$
$3x-8y-2z=9$

Given $A=\begin{bmatrix}2&5&2\\4&-7&-3\\3&-8&-2\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, and $B=\begin{bmatrix}9\\7\\9\end{bmatrix}$, find A^{-1}. Since $\left[A\middle|I_3\right]=\left[\begin{array}{ccc|ccc}2&5&2&1&0&0\\4&-7&-3&0&1&0\\3&-8&-2&0&0&1\end{array}\right]$,

$$\left[\begin{array}{ccc|ccc}1&\frac{5}{2}&1&\frac{1}{2}&0&0\\4&-7&-3&0&1&0\\3&-8&-2&0&0&1\end{array}\right]\begin{matrix}\frac{1}{2}R1\\ \\ \end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&\frac{5}{2}&1&\frac{1}{2}&0&0\\0&-17&-7&-2&1&0\\0&-\frac{31}{2}&-5&-\frac{3}{2}&0&1\end{array}\right]\begin{matrix}\\-4R1+R2\\-3R1+R3\end{matrix}$$

$$\left[\begin{array}{ccc|ccc}1&\frac{5}{2}&1&\frac{1}{2}&0&0\\0&1&\frac{7}{17}&\frac{2}{17}&-\frac{1}{17}&0\\0&-\frac{31}{2}&-5&-\frac{3}{2}&0&1\end{array}\right]-\frac{1}{17}R2\Rightarrow\left[\begin{array}{ccc|ccc}1&0&-\frac{1}{34}&\frac{7}{34}&\frac{5}{34}&0\\0&1&\frac{7}{17}&\frac{2}{17}&-\frac{1}{17}&0\\0&0&\frac{47}{34}&\frac{11}{34}&-\frac{31}{34}&1\end{array}\right]\begin{matrix}-\frac{5}{2}R2+R1\\ \\\frac{31}{2}R2+R3\end{matrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&-\frac{1}{34}&\frac{7}{34}&\frac{5}{34}&0\\0&1&\frac{7}{17}&\frac{2}{17}&-\frac{1}{17}&0\\0&0&1&\frac{11}{47}&-\frac{31}{47}&\frac{34}{47}\end{array}\right]\begin{matrix}\\ \\\frac{34}{47}R3\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&\frac{10}{47}&\frac{6}{47}&\frac{1}{47}\\0&1&0&\frac{1}{47}&\frac{10}{47}&-\frac{14}{47}\\0&0&1&\frac{11}{47}&-\frac{31}{47}&\frac{34}{47}\end{array}\right]\begin{matrix}\frac{1}{34}R3+R1\\-\frac{7}{17}R3+R2.\\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{10}{47}&\frac{6}{47}&\frac{1}{47}\\\frac{1}{47}&\frac{10}{47}&-\frac{14}{47}\\\frac{11}{47}&-\frac{31}{47}&\frac{34}{47}\end{bmatrix}$.

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$$X = A^{-1}B = \begin{bmatrix} \frac{10}{47} & \frac{6}{47} & \frac{1}{47} \\ \frac{1}{47} & \frac{10}{47} & -\frac{14}{47} \\ \frac{11}{47} & -\frac{31}{47} & \frac{34}{47} \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{10}{47}(9)+\frac{6}{47}(7)+\frac{1}{47}(9) \\ \frac{1}{47}(9)+\frac{10}{47}(7)+\left(-\frac{14}{47}\right)(9) \\ \frac{11}{47}(9)+\left(-\frac{31}{47}\right)(7)+\frac{34}{47}(9) \end{bmatrix} = \begin{bmatrix} \frac{90}{47}+\frac{42}{47}+\frac{9}{47} \\ \frac{9}{47}+\frac{70}{47}+\left(-\frac{126}{47}\right) \\ \frac{99}{47}+\left(-\frac{217}{47}\right)+\frac{306}{47} \end{bmatrix} = \begin{bmatrix} \frac{141}{47} \\ \frac{-47}{47} \\ \frac{188}{47} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

Solution set: $\{(3,-1,4)\}$

45. $x+3y+3z=1$
$x+4y+3z=0$
$x+3y+4z=-1$

We have $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (from Exercise 19).

$$X = A^{-1}B = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7(1)+(-3)(0)+(-3)(-1) \\ -1(1)+1(0)+0(-1) \\ -1(1)+0(0)+1(-1) \end{bmatrix} = \begin{bmatrix} 7+0+3 \\ -1+0+0 \\ -1+0+(-1) \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ -2 \end{bmatrix}$$

Solution set: $\{(10,-1,-2)\}$

46. $-2x+2y+4z=3$
$-3x+4y+5z=1$
$x+2z=2$

We have $A = \begin{bmatrix} -2 & 2 & 4 \\ -3 & 4 & 5 \\ 1 & 0 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} -\frac{4}{5} & \frac{2}{5} & \frac{3}{5} \\ -\frac{11}{10} & \frac{4}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$ (from Exercise 20).

$$X = A^{-1}B = \begin{bmatrix} -\frac{4}{5} & \frac{2}{5} & \frac{3}{5} \\ -\frac{11}{10} & \frac{4}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5}(3)+\frac{2}{5}(1)+\frac{3}{5}(2) \\ -\frac{11}{10}(3)+\frac{4}{5}(1)+\frac{1}{5}(2) \\ \frac{2}{5}(3)+\left(-\frac{1}{5}\right)(1)+\frac{1}{5}(2) \end{bmatrix} = \begin{bmatrix} -\frac{12}{5}+\frac{2}{5}+\frac{6}{5} \\ -\frac{33}{10}+\frac{4}{5}+\frac{2}{5} \\ \frac{6}{5}+\left(-\frac{1}{5}\right)+\frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{33}{10}+\frac{6}{5} \\ \frac{7}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{33}{10}+\frac{12}{10} \\ \frac{7}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{21}{10} \\ \frac{7}{5} \end{bmatrix}$$

Solution set: $\left\{\left(-\frac{4}{5},-\frac{21}{10},\frac{7}{5}\right)\right\}$

47. $2x+2y-4z=12$
$2x+6y=16$
$-3x-3y+5z=-20$

We have $A = \begin{bmatrix} 2 & 2 & -4 \\ 2 & 6 & 0 \\ -3 & -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ 16 \\ -20 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} -\frac{15}{4} & -\frac{1}{4} & -3 \\ \frac{5}{4} & \frac{1}{4} & 1 \\ -\frac{3}{2} & 0 & -1 \end{bmatrix}$ (from Exercise 21).

$$X = A^{-1}B = \begin{bmatrix} -\frac{15}{4} & -\frac{1}{4} & -3 \\ \frac{5}{4} & \frac{1}{4} & 1 \\ -\frac{3}{2} & 0 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 16 \\ -20 \end{bmatrix} = \begin{bmatrix} -\frac{15}{4}(12)+\left(-\frac{1}{4}\right)(16)+(-3)(-20) \\ \frac{5}{4}(12)+\frac{1}{4}(16)+1(-20) \\ -\frac{3}{2}(12)+0(16)+(-1)(-20) \end{bmatrix} = \begin{bmatrix} -45+(-4)+60 \\ 15+4+(-20) \\ -18+0+20 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ 2 \end{bmatrix}$$

Solution set: $\{(11,-1,2)\}$

48. $\begin{aligned} 2x+4y+6z&=4\\ -x-4y-3z&=8\\ y-z&=-4\end{aligned}$

We have $A=\begin{bmatrix}2&4&6\\-1&-4&-3\\0&1&-1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}4\\8\\-4\end{bmatrix}$, and $A^{-1}=\begin{bmatrix}\frac{7}{4}&\frac{5}{2}&3\\-\frac{1}{4}&-\frac{1}{2}&0\\-\frac{1}{4}&-\frac{1}{2}&-1\end{bmatrix}$ (from Exercise 22).

$$X=A^{-1}B=\begin{bmatrix}\frac{7}{4}&\frac{5}{2}&3\\-\frac{1}{4}&-\frac{1}{2}&0\\-\frac{1}{4}&-\frac{1}{2}&-1\end{bmatrix}\begin{bmatrix}4\\8\\-4\end{bmatrix}=\begin{bmatrix}\frac{7}{4}(4)+\frac{5}{2}(8)+3(-4)\\-\frac{1}{4}(4)+\left(-\frac{1}{2}\right)(8)+0(-4)\\-\frac{1}{4}(4)+\left(-\frac{1}{2}\right)(8)+(-1)(-4)\end{bmatrix}=\begin{bmatrix}7+20+(-12)\\-1+(-4)+0\\-1+(-4)+4\end{bmatrix}=\begin{bmatrix}15\\-5\\-1\end{bmatrix}$$

Solution set: $\{(15,-5,-1)\}$

49. $\begin{aligned} x+y+2w&=3\\ 2x-y+z-w&=3\\ 3x+3y+2z-2w&=5\\ x+2y+z&=3\end{aligned}$

We have $A=\begin{bmatrix}1&1&0&2\\2&-1&1&-1\\3&3&2&-2\\1&2&1&0\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$, $B=\begin{bmatrix}3\\3\\5\\3\end{bmatrix}$, and $A^{-1}=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}&-1\\\frac{1}{10}&-\frac{2}{5}&\frac{3}{10}&-\frac{1}{5}\\-\frac{7}{10}&\frac{4}{5}&-\frac{11}{10}&\frac{12}{5}\\\frac{1}{5}&\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\end{bmatrix}$ (from Exercise 23).

$$X=A^{-1}B=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}&-1\\\frac{1}{10}&-\frac{2}{5}&\frac{3}{10}&-\frac{1}{5}\\-\frac{7}{10}&\frac{4}{5}&-\frac{11}{10}&\frac{12}{5}\\\frac{1}{5}&\frac{1}{5}&-\frac{2}{5}&\frac{3}{5}\end{bmatrix}\begin{bmatrix}3\\3\\5\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(3)+0(3)+\frac{1}{2}(5)+(-1)(3)\\\frac{1}{10}(3)+\left(-\frac{2}{5}\right)(3)+\frac{3}{10}(5)+\left(-\frac{1}{5}\right)(3)\\-\frac{7}{10}(3)+\frac{4}{5}(3)+\left(-\frac{11}{10}\right)(5)+\frac{12}{5}(3)\\\frac{1}{5}(3)+\frac{1}{5}(3)+\left(-\frac{2}{5}\right)(5)+\frac{3}{5}(3)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{2}+0+\frac{5}{2}+(-3)\\\frac{3}{10}+\left(-\frac{6}{5}\right)+\frac{3}{2}+\left(-\frac{3}{5}\right)\\-\frac{21}{10}+\frac{12}{5}+\left(-\frac{11}{2}\right)+\frac{36}{5}\\\frac{3}{5}+\frac{3}{5}+(-2)+\frac{9}{5}\end{bmatrix}=\begin{bmatrix}\frac{8}{2}+(-3)\\\frac{3}{10}+\left(-\frac{9}{5}\right)+\frac{3}{2}\\-\frac{21}{10}+\left(-\frac{11}{2}\right)+\frac{48}{5}\\(-2)+\frac{15}{5}\end{bmatrix}=\begin{bmatrix}4+(-3)\\\frac{3}{10}+\left(-\frac{18}{10}\right)+\frac{15}{10}\\-\frac{21}{10}+\left(-\frac{55}{10}\right)+\frac{96}{10}\\(-2)+3\end{bmatrix}=\begin{bmatrix}1\\\frac{0}{10}\\\frac{20}{10}\\1\end{bmatrix}=\begin{bmatrix}1\\0\\2\\1\end{bmatrix}$$

Solution set: $\{(1,0,2,1)\}$

50. $\begin{aligned} x-2y+3z&=1\\ y-z+w&=-1\\ -2x+2y-2z+4w&=2\\ 2y-3z+w&=-3\end{aligned}$

We have $A=\begin{bmatrix}1&-2&3&0\\0&1&-1&1\\-2&2&-2&4\\0&2&-3&1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$, $B=\begin{bmatrix}1\\-1\\2\\-3\end{bmatrix}$, and $A^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&-\frac{1}{4}&\frac{1}{2}\\-1&4&-\frac{1}{2}&-2\\-\frac{1}{2}&\frac{5}{2}&-\frac{1}{4}&-\frac{3}{2}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{4}&\frac{1}{2}\end{bmatrix}$ (from Exercise 24).

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(*continued from page 996*)

$$X = A^{-1}B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ -1 & 4 & -\frac{1}{2} & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{4} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1)+\frac{1}{2}(-1)+\left(-\frac{1}{4}\right)(2)+\frac{1}{2}(-3) \\ -1(1)+4(-1)+\left(-\frac{1}{2}\right)(2)+(-2)(-3) \\ -\frac{1}{2}(1)+\frac{5}{2}(-1)+\left(-\frac{1}{4}\right)(2)+\left(-\frac{3}{2}\right)(-3) \\ \frac{1}{2}(1)+\left(-\frac{1}{2}\right)(-1)+\frac{1}{4}(2)+\frac{1}{2}(-3) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)+\left(-\frac{3}{2}\right) \\ -1+(-4)+(-1)+6 \\ -\frac{1}{2}+\left(-\frac{5}{2}\right)+\left(-\frac{1}{2}\right)+\frac{9}{2} \\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\left(-\frac{3}{2}\right) \end{bmatrix} = \begin{bmatrix} -\frac{4}{2} \\ 0 \\ \frac{2}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

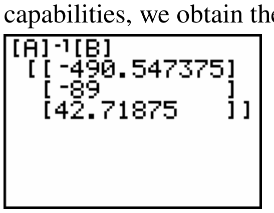
Solution set: $\{(-2, 0, 1, 0)\}$

51. **(a)** $602.7 = a + 5.543b + 37.14c$
$656.7 = a + 6.933b + 41.30c$
$778.5 = a + 7.638b + 45.62c$

(b) $A = \begin{bmatrix} 1 & 5.543 & 37.14 \\ 1 & 6.933 & 41.30 \\ 1 & 7.638 & 45.62 \end{bmatrix}, X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

and $B = \begin{bmatrix} 602.7 \\ 656.7 \\ 778.5 \end{bmatrix}$

Using a graphing calculator with matrix capabilities, we obtain the following.



$X = A^{-1}B$ or $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -490.547375 \\ -89 \\ 42.71875 \end{bmatrix}$.

Thus, $a \approx -490.547$, $b = -89$, $c = 42.71875$.

(c) $S = -490.547 - 89A + 42.71875B$

(d) If $A = 7.752$ and $B = 47.38$, the predicted value of S is given by the following.

$$S = -490.547 - 89(7.752) + 42.71875(47.38) = 843.539375 \approx 843.5$$

The predicted value is approximately 843.5.

(e) If $A = 8.9$ and $B = 66.25$, the predicted value of S is given by the following.

$$S = -490.547 - 89(8.9) + 42.71875(66.25) = 1547.470188 \approx 1547.5$$

The predicted value is approximately 1547.5.

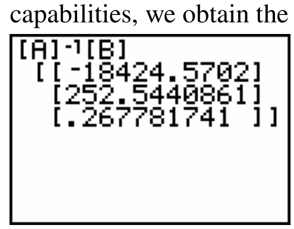
Using only three consecutive years to forecast six years into the future, it is probably not very accurate.

52. **(a)** $10{,}170 = a + 112.9b + 307.5c$
$15{,}305 = a + 132.9b + 621.63c$
$21{,}289 = a + 155.2b + 1937.13c$

(b) $A = \begin{bmatrix} 1 & 112.9 & 307.5 \\ 1 & 132.9 & 621.63 \\ 1 & 155.2 & 1937.13 \end{bmatrix}, X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

and $B = \begin{bmatrix} 10{,}170 \\ 15{,}305 \\ 21{,}289 \end{bmatrix}$

Using a graphing calculator with matrix capabilities, we obtain the following.



$X = A^{-1}B$ or $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -18{,}424.5702 \\ 252.5440861 \\ .267781741 \end{bmatrix}$.

Thus, $a \approx -18{,}425$, $b \approx 252.54$, and $c \approx .26778$.

(c) Using these values for the coefficients, we obtain the following equation.
$S = -18{,}425 + 252.54R + .26778I$

(d) If $R = 117.6$ and $I = 310.73$, the predicted value of S is given by the following.
$S = -18{,}425 + 252.54(117.6) + .26778(310.73) \approx 11{,}357$

(e) If $R = 143.8$ and $I = 829.06$, the predicted value of S is given by the following.
$S = -18{,}425 + 252.54(143.8) + .26778(829.06) \approx 18{,}112$

53. Answers will vary.

54. Given $A = \begin{bmatrix} \sqrt{2} & .5 \\ -17 & \frac{1}{2} \end{bmatrix}$, we obtain

$A^{-1} = \begin{bmatrix} .0543058761 & -.0543058761 \\ 1.846399787 & .153600213 \end{bmatrix}$ using a graphing calculator.

55. Given $A = \begin{bmatrix} \frac{2}{3} & .7 \\ 22 & \sqrt{3} \end{bmatrix}$, we obtain

$A^{-1} \approx \begin{bmatrix} -.1215875322 & .0491390161 \\ 1.544369078 & -.046799063 \end{bmatrix}$ using a graphing calculator.

56. Given $A = \begin{bmatrix} 1.4 & .5 & .59 \\ .84 & 1.36 & .62 \\ .56 & .47 & 1.3 \end{bmatrix}$, we obtain

$$A^{-1} = \begin{bmatrix} .9987635516 & -.252092087 & -.3330564627 \\ -.5037783375 & 1.007556675 & -.2518891688 \\ -.2481013617 & -.2556769758 & 1.003768868 \end{bmatrix}$$

using a graphing calculator.

57. Given $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$, we obtain

$A^{-1} = \begin{bmatrix} 2 & -2 & 0 \\ -4 & 0 & 4 \\ 3 & 3 & -3 \end{bmatrix}$

using a graphing calculator.

58. $x - \sqrt{2}y = 2.6$
$.75x + y = -7$
Given
$A = \begin{bmatrix} 1 & -\sqrt{2} \\ .75 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 2.6 \\ -7 \end{bmatrix}$,
using a graphing calculator, we have
$X = A^{-1}B = \begin{bmatrix} -3.542308934 \\ -4.343268299 \end{bmatrix}$. Thus, the solution set is
$\{(-3.542308934, -4.343268299)\}$.

59. $2.1x + y = \sqrt{5}$
$\sqrt{2}x - 2y = 5$
Given
$A = \begin{bmatrix} 2.1 & 1 \\ \sqrt{2} & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} \sqrt{5} \\ 5 \end{bmatrix}$,
using a graphing calculator, we have
$X = A^{-1}B = \begin{bmatrix} 1.68717058 \\ -1.306990242 \end{bmatrix}$. Thus, the solution set is
$\{(1.68717058, -1.306990242)\}$.

60. $\pi x + ey + \sqrt{2}z = 1$
$ex + \pi y + \sqrt{2}z = 2$
$\sqrt{2}x + ey + \pi z = 3$
Given
$A = \begin{bmatrix} \pi & e & \sqrt{2} \\ e & \pi & \sqrt{2} \\ \sqrt{2} & e & \pi \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,
using a graphing calculator, we have
$X = A^{-1}B = \begin{bmatrix} -.9704156959 \\ 1.391914631 \\ .1874077432 \end{bmatrix}$.
Solution set:
$\{(-.9704156959, 1.391914631, .1874077432)\}$

61. $(\log 2)x + (\ln 3)y + (\ln 4)z = 1$
$(\ln 3)x + (\log 2)y + (\ln 8)z = 5$
$(\log 12)x + (\ln 4)y + (\ln 8)z = 9$

Given $A = \begin{bmatrix} \log 2 & \ln 3 & \ln 4 \\ \ln 3 & \log 2 & \ln 8 \\ \log 12 & \ln 4 & \ln 8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and $B = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$, using a graphing calculator, we

have $X = A^{-1}B = \begin{bmatrix} 13.58736702 \\ 3.929011993 \\ -5.342780076 \end{bmatrix}$.

Solution set:
$\{(13.58736702, 3.929011993, -5.342780076)\}$

62. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A \cdot O = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a(0)+b(0) & a(0)+b(0) \\ c(0)+d(0) & c(0)+d(0) \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$O \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0(a)+0(c) & 0(b)+0(d) \\ 0(a)+0(c) & 0(b)+0(d) \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Thus, $A \cdot O = O \cdot A = O$.

63.–64. Answers will vary.

65. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (Other answers are possible.)

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+0(0) & 1(1)+0(1) \\ 1(1)+1(0) & 1(1)+1(1) \end{bmatrix} = \begin{bmatrix} 1+0 & 1+0 \\ 1+0 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Find $(AB)^{-1}$. Since $[AB|I_2] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$,

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{matrix} \\ -R1+R2 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{matrix} -R2+R1 \\ \end{matrix}$$

Thus, $(AB)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Now find A^{-1}. Since

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{matrix} \\ -R1+R2 \end{matrix}$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Now find B^{-1}. Since

$$[B|I_2] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{matrix} -R2+R1 \\ \end{matrix}$$

Thus, $B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Now find $A^{-1}B^{-1}$.

$$A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+0(0) & 1(-1)+0(1) \\ -1(1)+1(0) & -1(-1)+1(1) \end{bmatrix} = \begin{bmatrix} 1+0 & -1+0 \\ -1+0 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Thus $(AB)^{-1} \neq A^{-1}B^{-1}$.

66. Suppose A and B are matrices, where $A^{-1}, B^{-1},$ and AB all exist. Show that $(AB)^{-1} = B^{-1}A^{-1}$. Since $A^{-1}, B^{-1},$ and AB all exist, we can assume that A and B are $n \times n$ (square) matrices.

$$(AB)(AB)^{-1} = I_n \Rightarrow A^{-1}\left[(AB)(AB)^{-1}\right] = A^{-1}I_n \Rightarrow \left[A^{-1}(AB)\right](AB)^{-1} = A^{-1} \Rightarrow$$
$$\left[(A^{-1}A)B\right](AB)^{-1} = A^{-1} \Rightarrow (I_nB)(AB)^{-1} = A^{-1} \Rightarrow B(AB)^{-1} = A^{-1} \Rightarrow B^{-1}\left[B(AB)^{-1}\right] = B^{-1}A^{-1} \Rightarrow$$
$$(B^{-1}B)(AB)^{-1} = B^{-1}A^{-1} \Rightarrow I_n(AB)^{-1} = B^{-1}A^{-1} \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

67. Given $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, and since a, b, and c are all nonzero, $\frac{1}{a}, \frac{1}{b},$ and $\frac{1}{c}$ all exist.

Thus, $\left[A|I\right] = \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{c} \end{array}\right] \begin{matrix} \frac{1}{a}\text{R1} \\ \frac{1}{b}\text{R2} \\ \frac{1}{c}\text{R3} \end{matrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

68. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1)+0(0)+0(0) & 1(0)+0(0)+0(1) & 1(0)+0(-1)+0(-1) \\ 0(1)+0(0)+(-1)(0) & 0(0)+0(0)+(-1)(1) & 0(0)+0(-1)+(-1)(-1) \\ 0(1)+1(0)+(-1)(0) & 0(0)+1(0)+(-1)(1) & 0(0)+1(-1)+(-1)(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+(-1) & 0+0+1 \\ 0+0+0 & 0+0+(-1) & 0+(-1)+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1)+0(0)+0(0) & 1(0)+0(-1)+0(-1) & 1(0)+0(1)+0(0) \\ 0(1)+0(0)+(-1)(0) & 0(0)+0(-1)+(-1)(-1) & 0(0)+0(1)+(-1)(0) \\ 0(1)+1(0)+(-1)(0) & 0(0)+1(-1)+(-1)(-1) & 0(0)+1(1)+(-1)(0) \end{bmatrix}$$
$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ 0+0+0 & 0+(-1)+1 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AA^2 = I, A^2 = A^{-1}$. Therefore, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

69. Find the inverses of I_n, $-A$, and kA.

I_n is its own inverse, since $I_n \cdot I_n = I_n$.

The inverse of $-A$ is $-\left(A^{-1}\right)$, since

$(-A)\left[-\left(A^{-1}\right)\right] = (-1)(-1)\left(A \cdot A^{-1}\right) = I_n.$

The inverse of kA (k a scalar) is $\frac{1}{k}A^{-1}$, since

$(kA)\left(\frac{1}{k}A^{-1}\right) = \left(k \cdot \frac{1}{k}\right)\left(A \cdot A^{-1}\right) = I_n.$

70. Answers will vary.

Chapter 9: Review Exercises

1. $2x + 6y = 6 \quad (1)$
$5x + 9y = 9 \quad (2)$

Solving equation (1) for x, we have
$2x + 6y = 6 \Rightarrow 2x = 6 - 6y \Rightarrow x = 3 - 3y \ (3).$

Substituting equation (3) into equation (2) and solving for y, we have the following.

$5(3 - 3y) + 9y = 9 \Rightarrow 15 - 15y + 9y = 9 \Rightarrow$
$15 - 6y = 9 \Rightarrow -6y = -6 \Rightarrow y = 1$

Substituting 1 for y in equation (3), we have
$x = 3 - 3(1) = 3 - 3 = 0.$

Solution set: $\{(0,1)\}$

2. $3x - 5y = 7 \quad (1)$
$2x + 3y = 30 \ (2)$

Multiply equation (1) by 3 and multiply equation (2) by 5. Add the resulting equations.

$$\begin{array}{r} 9x - 15y = 21 \\ \underline{10x + 15y = 150} \\ 19x \qquad = 171 \end{array} \Rightarrow x = \frac{171}{19} = 9$$

Substitute this value into equation (2) to obtain
$2(9) + 3y = 30 \Rightarrow 18 + 3y = 30 \Rightarrow$
$3y = 12 \Rightarrow y = 4$

Solution set: $\{(9,4)\}$

3. $x + 5y = 9 \quad (1)$
$2x + 10y = 18 \ (2)$

Multiply equation (1) by -2 and add the resulting equation to equation (2).

$$\begin{array}{r} -2x - 10y = -18 \\ \underline{2x + 10y = \ \ 18} \\ 0 = 0 \end{array}$$

This is a true statement and implies that the system has infinitely many solutions. Solving equation (1) for x we have $x = 9 - 5y.$ Given y as an arbitrary value, the solution set is $\{(9 - 5y, y)\}$.

4. $\frac{1}{6}x + \frac{1}{3}y = 8 \quad (1)$
$\frac{1}{4}x + \frac{1}{2}y = 12 \ (2)$

To clear fractions, multiply equation (1) by 6 and equation (2) by 4 to obtain
$x + 2y = 48 \ (3)$
$x + 2y = 48 \ (4)$.

Since equations (3) and (4) represent the same equation, there are infinitely many solutions. Solving equation (3) (or (4)) for x we have $x = 48 - 2y.$ Given y as an arbitrary value, the solution set is $\{(48 - 2y, y)\}$.

5. $y = -x + 3 \quad (1)$
$2x + 2y = 1 \ (2)$

Substituting equation (1) into equation (2), we have $2x + 2(-x + 3) = 1.$ Solving for x we have $2x + 2(-x + 3) = 1 \Rightarrow$
$2x + (-2x) + 6 = 1 \Rightarrow 6 = 1.$ This is a false statement and the solution is inconsistent. Thus, the solution set is $\emptyset$.

6. $.2x + .5y = 6 \ (1)$
$.4x + \quad y = 9 \ (2)$

Multiply equation (1) by –2 and add the result to equation (2).

$$\begin{array}{r} -.4x - y = -12 \\ \underline{.4x + y = \quad 9} \\ 0 = \ -3 \end{array}$$

This is a false statement and the solution is inconsistent. Thus, the solution set is $\emptyset$.

7. $3x - 2y = 0 \ (1)$
$9x + 8y = 7 \ (2)$

Multiply equation (1) by –3 and add the result to equation (2).

$$\begin{array}{r} -9x + 6y = 0 \\ \underline{9x + 8y = 7} \\ 14y = 7 \end{array} \Rightarrow y = \frac{1}{2}$$

Substituting $\frac{1}{2}$ for y in equation (1) and solving for x, we have the following.

$3x - 2\left(\frac{1}{2}\right) = 0 \Rightarrow 3x - 1 = 0$
$3x = 1 \Rightarrow x = \frac{1}{3}$

Solution set: $\left\{\left(\frac{1}{3}, \frac{1}{2}\right)\right\}$

8. $6x + 10y = -11$ (1)
$9x + 6y = -3$ (2)

Add –4 times equation (2) to 6 times equation (1) in order to eliminate *x*.

$$\begin{array}{r} 36x + 60y = -66 \\ -36x - 24y = 12 \\ \hline 36y = -54 \Rightarrow y = \frac{-54}{36} = -\frac{3}{2} \end{array}$$

Substituting $-\frac{3}{2}$ for *y* in equation (2) and solving for *x*, we have the following.

$9x + 6\left(-\frac{3}{2}\right) = -3 \Rightarrow 9x + (-9) = -3 \Rightarrow$
$9x = 6 \Rightarrow x = \frac{6}{9} = \frac{2}{3}$

Solution set: $\left\{\left(\frac{2}{3}, -\frac{3}{2}\right)\right\}$

9. $2x - 5y + 3z = -1$ (1)
$x + 4y - 2z = 9$ (2)
$-x + 2y + 4z = 5$ (3)

First, we eliminate *x*. Multiply equation (2) by –2 and add the result to equation (1).

$$\begin{array}{r} 2x - 5y + 3z = -1 \\ -2x - 8y + 4z = -18 \\ \hline -13y + 7z = -19 \end{array} \quad (4)$$

Next, add equations (2) and (3) to obtain $6y + 2z = 14$ (5). Now, we solve the system

$-13y + 7z = -19$ (4)
$6y + 2z = 14$ (5).

Multiply equation (4) by 2, multiply equation (5) by –7, and add the resulting equations.

$$\begin{array}{r} -26y + 14z = -38 \\ -42y - 14z = -98 \\ \hline -68y \qquad = -136 \Rightarrow y = 2 \end{array}$$

Substituting this value into equation (4), we have $-13(2) + 7z = -19 \Rightarrow$
$-26 + 7z = -19 \Rightarrow 7z = 7 \Rightarrow z = 1.$

Substituting 2 for *y* and 1 for *z* in equation (2), we have $x + 4(2) - 2(1) = 9 \Rightarrow$
$x + 8 - 2 = 9 \Rightarrow x = 3.$

Solution set: $\{(3, 2, 1)\}$

10. $4x + 3y + z = -8$ (1)
$3x + y - z = -6$ (2)
$x + y + 2z = -2$ (3)

First, we eliminate *y*. Multiply equation (2) by –3 and add the result to equation (1).

$$\begin{array}{r} 4x + 3y + z = -8 \\ -9x - 3y + 3z = 18 \\ \hline -5x \qquad + 4z = 10 \end{array} \quad (4)$$

Multiply equation (2) by –1 and add the result to equation (3).

$$\begin{array}{r} -3x - y + z = 6 \\ x + y + 2z = -2 \\ \hline -2x \qquad + 3z = 4 \end{array} \quad (5)$$

Now, we solve the system $\begin{array}{l} -5x + 4z = 10 \ (4) \\ -2x + 3z = 4 \ (5) \end{array}$.

Multiply equation (4) by 3, multiply equation (5) by –4, and add the resulting equations.

$$\begin{array}{r} -15x + 12z = 30 \\ 8x - 12z = -16 \\ \hline -7x \qquad = 14 \Rightarrow x = -2 \end{array}$$

Substituting this value into equation (4), we have $-5(-2) + 4z = 10 \Rightarrow 10 + 4z = 10 \Rightarrow$
$4z = 0 \Rightarrow z = 0.$

Substituting –2 for *x* and 0 for *z* in equation (2), we have
$3(-2) + y - 0 = -6 \Rightarrow -6 + y = -6 \Rightarrow y = 0.$

Solution set: $\{(-2, 0, 0)\}$

11. $5x - y = 26$ (1)
$4y + 3z = -4$ (2)
$3x + 3z = 15$ (3)

Multiply equation (2) by –1 and add the result to equation (3) in order to eliminate *z*.

$$\begin{array}{r} -4y - 3z = 4 \\ 3x \qquad + 3z = 15 \\ \hline 3x - 4y \qquad = 19 \end{array} \quad (4)$$

Multiply equation (1) by –4 and add the result to equation (4).

$$\begin{array}{r} -20x + 4y = -104 \\ 3x - 4y = 19 \\ \hline -17x \qquad = -85 \Rightarrow x = 5 \end{array}$$

Substitute 5 for *x* in equation (1) and solve for *y* to obtain
$5(5) - y = 26 \Rightarrow 25 - y = 26 \Rightarrow y = -1.$

Substitute 5 for *x* equation (3) and solve for *z*:
$3(5) + 3z = 15 \Rightarrow 15 + 3z = 15 \Rightarrow$
$3z = 0 \Rightarrow z = 0$

Solution set: $\{(5, -1, 0)\}$

12. $x + \quad z = 2 \quad (1)$

$2y - \quad z = 2 \quad (2)$

$-x + 2y = -4 \ (3)$

Add equations (1) and (2) to obtain $x + 2y = 4$ (4). Add equations (3) and (4) together to obtain $4y = 0 \Rightarrow y = 0.$ Substituting this value into equation (2), we have $2(0) - z = 2 \Rightarrow 0 - z = 2 \Rightarrow z = -2.$ Substitute –2 for *z* in equation (1) and solve for *x* to obtain $x + (-2) = 2 \Rightarrow x = 4.$

Solution set: $\{(4, 0, -2)\}$

13. One possible answer is $\begin{array}{l} x + y = 2 \\ x + y = 3. \end{array}$

14. The line with positive slope passes through (0, –2) and (2, 0), so $m = \frac{0-(-2)}{2-0} = 1$, and the equation of the line is $y = x - 2 \Rightarrow x - y = 2$. The line with negative slope passes through (3, 0) and (0, 3), so $m = \frac{3-0}{0-3} = -1$, and the equation of the line is $y = -x + 3 \Rightarrow x + y = 3$

The system is $\begin{array}{l} x - y = 2 \\ x + y = 3 \end{array}$.

15. Let *x* = amount of rice; *y* = amount of soybeans.
We get the following system of equations. The first equation relates protein and the second relates calories.

$15x + 22.5y = 9.5 \quad (1)$

$810x + 270y = 324 \ (2)$

Multiply equation (1) by –12 and add the result to equation (2).

$$\begin{array}{rl} -180x - 270y &= -114 \\ 810x + 270y &= \ \ 324 \\ \hline 630x \qquad\quad &= \ \ 210 \Rightarrow x = \frac{1}{3} \end{array}$$

Substitute $\frac{1}{3}$ for x in equation (1) and solve for *y*.

$$15\left(\tfrac{1}{3}\right) + 22.5y = 9.5 \Rightarrow 5 + 22.5y = 9.5 \Rightarrow$$
$$22.5y = 4.5 \Rightarrow y = .20 = \tfrac{1}{5}$$

$\frac{1}{3}$ cup of rice and $\frac{1}{5}$ cup of soybeans should be used.

16. Let *x* = number of recordable CDs;
y = number of play-only CDs.
We get the following system of equations. The first equation relates number of CDs and the second relates revenue from the order.

$x + \quad y = 100 \ (1)$

$.80x + .60y = \ 76 \ (2)$

Solving equation (1) for *y*, we obtain $y = 100 - x$ (3). Substitute equation (3) into equation (2) and solve for *x*.

$$\begin{aligned} .80x + .60(100 - x) &= 76 \\ .80x + 60 - .60x &= 76 \\ .20x + 60 &= 76 \\ .20x &= 16 \Rightarrow x = 80 \end{aligned}$$

Substituting 80 for x in equation (3), we obtain $y = 100 - 80 = 20.$

The company should send 80 recordable CDs and 20 play-only CDs.

17. Let *x* = the number of blankets, *y* = the number of rugs, *z* = the number of skirts.
The following table is helpful in organizing the information.

	Number of blankets (*x*)	Number of rugs (*y*)	Number of skirts (*z*)	Available
Spinning yarn	24	30	12	306
Dying	4	5	3	59
Weaving	15	18	9	201

We have the following system of equations.

$24x + 30y + 12z = 306 \Rightarrow 4x + 5y + 2z = 51 \ (1)$

$4x + \ 5y + \ 3z = 59 \quad (2)$

$15x + 18y + \ 9z = 201 \Rightarrow 5x + 6y + 3z = 67 \ (3)$

Multiply equation (1) by –1 and add the result to equation (2).

$$\begin{array}{rl} -4x - 5y - 2z &= -51 \\ 4x + 5y + 3z &= \ \ 59 \\ \hline z &= \ \ \ 8 \end{array}$$

Substitute 8 for z in equations (1) and (3) and simplify.

$$4x + 5y + 2(8) = 51 \Rightarrow 4x + 5y + 16 = 51 \Rightarrow$$
$$4x + 5y = 35 \quad (4)$$
$$5x + 6y + 3(8) = 67 \Rightarrow 5x + 6y + 24 = 67 \Rightarrow$$
$$5x + 6y = 43 \quad (5)$$

Multiply equation (4) by 5 and equation (5) by –4 and add the results.

$$\begin{array}{rl} 20x + 25y &= \ \ 175 \\ -20x - 24y &= -172 \\ \hline y &= \ \ \ \ 3 \end{array}$$

Substituting 3 for y in equation (4) and solving for *x*, we obtain $4x + 5(3) = 35 \Rightarrow$

$4x + 15 = 35 \Rightarrow 4x = 20 \Rightarrow x = 5$

5 blankets, 3 rugs, and 8 skirts can be made.

18. $y = .0515x + 12.3$ (1)
$y = .255x + 9.01$ (2)

(a) Substitute equation (1) into equation (2) to obtain $.0515x + 12.3 = .255x + 9.01.$

$$.0515x + 12.3 = .255x + 9.01$$
$$12.3 = .2035x + 9.01$$
$$3.29 = .2035x \Rightarrow x \approx 16.17$$

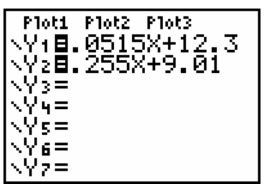
Since $x = 0$ corresponds to the year 1990, the populations will be equal during 2006.

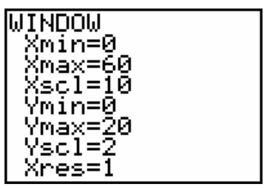
(b) Using $x \approx 16.17,$ we evaluate either of the two linear functions to find the percent of U.S. resident population that will be black or Hispanic.

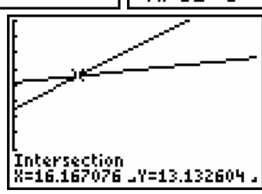
$$y = .255(16.17) + 9.01$$
$$= 4.12335 + 9.01 = 13.13335$$

The amount will be approximately 13.1%.

(c) The graphing calculator supports our algebraic solution. Note: An approximation for *x* was used when determining the corresponding value of *y* in the algebraic solution. This accounts for the difference in the values.





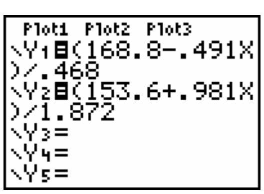


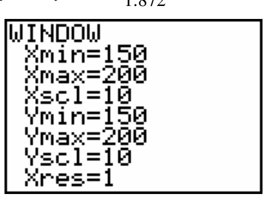
(d) Since $.255 > .0515,$ the Hispanic population is increasing more rapidly.

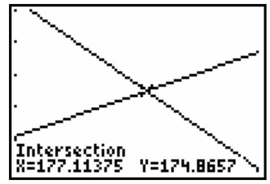
19. $H = .491x + .468y + 11.2$
$H = -.981x + 1.872y + 26.4$

Substitute $H = 180$ and solve each equation for y so they can be entered into the graphing calculator.

$$180 = .491x + .468y + 11.2$$
$$168.8 - .491x = .468y \Rightarrow y = \frac{168.8 - .491x}{.468}$$
$$180 = -.981x + 1.872y + 26.4$$
$$153.6 + .981x = 1.872y \Rightarrow y = \frac{153.6 + .981x}{1.872}$$





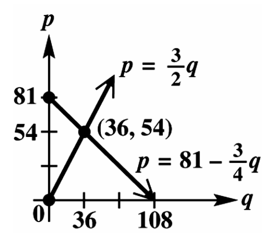


Thus, we have $x \approx 177.1$ and $y \approx 174.9.$

If an athlete's maximum heart rate is 180 beats per minute, then it will be about 177 beats per minute 5 seconds after stopping and 175 beats per second after stopping.

20. $p = \frac{3}{2}q$ (1)
$p = 81 - \frac{3}{4}q$ (2)

(a)



(b) To find the equilibrium demand, replace *p* with $\frac{3}{2}q$ in equation (2), and solve for *q*. The value of *q* will give the equilibrium demand.

$$\tfrac{3}{2}q = 81 - \tfrac{3}{4}q \Rightarrow 4\left(\tfrac{3}{2}q\right) = 4\left(81 - \tfrac{3}{4}q\right) \Rightarrow$$
$$6q = 324 - 3q \Rightarrow 9q = 324 \Rightarrow q = 36$$

The equilibrium demand is 36.

(c) To find *p*, substitute *q* = 36 into equation (1) to obtain $p = \frac{3}{2}(36) = 54.$

The equilibrium price is \$54.

21. Since $y = ax^2 + bx + c$ and the points $(1, -2.3), (2, -1.3),$ and $(3, 4.5)$ are on the parabola, we have the following system of equations.

$$-2.3 = a(1)^2 + b(1) + c \Rightarrow a + b + c = -2.3 \text{ (1)}$$
$$-1.3 = a(2)^2 + b(2) + c \Rightarrow 4a + 2b + c = -1.3 \text{ (2)}$$
$$4.5 = a(3)^2 + b(3) + c \Rightarrow 9a + 3b + c = 4.5 \text{ (3)}$$

First, eliminate *c* by multiplying equation (1) by –1 and adding the result to equation (2).

$$\begin{array}{l} -a - b - c = 2.3 \\ \underline{4a + 2b + c = -1.3} \\ 3a + b \phantom{{}+c} = 1 \text{ (4)} \end{array}$$

Next, multiply equation (2) by –1 and add to equation (3).

$$\begin{array}{r} -4a-2b-c=1.3 \\ 9a+3b+c=4.5 \\ \hline 5a+b\quad\ \ =5.8\ \ (5) \end{array}$$

Next, eliminate b by multiplying equation (4) by –1 and adding the result to equation (5).

$$\begin{array}{r} -3a-b=-1 \\ 5a+b=5.8 \\ \hline 2a\quad\ =4.8 \end{array} \Rightarrow a=2.4$$

Substitute this value into equation (4) to obtain $3(2.4)+b=1 \Rightarrow 7.2+b=1 \Rightarrow b=-6.2.$

Substitute 2.4 for a and –6.2 for b into equation (1) in order to solve for c.

$$(2.4)+(-6.2)+c=-2.3$$
$$-3.8+c=-2.3 \Rightarrow c=1.5$$

The equation of the parabola is $y=2.4x^2-6.2x+1.5$ or $Y_1=2.4X^2-6.2X+1.5.$

22. $2x-6y+4z=5\ (1)$
$5x+y-3z=1\ (2)$

Eliminate y by multiplying equation (2) by 6 and adding to equation (1). Then, solve for x.

$$\begin{array}{r} 2x-\ 6y+\ 4z=5 \\ 30x+6y-18z=6 \\ \hline 32x\qquad -14z=11 \end{array} \Rightarrow$$

$$32x=11+14z \Rightarrow x=\tfrac{11+14z}{32}=\tfrac{11}{32}+\tfrac{7}{16}z$$

Substitute $x=\frac{11}{32}+\frac{7}{16}z$ into equation (2) and solve for y.

$$5\left(\tfrac{11}{32}+\tfrac{7}{16}z\right)+y-3z=1$$
$$\tfrac{55}{32}+\tfrac{35}{16}z+y-3z=1$$
$$y-\tfrac{13}{16}z=\tfrac{32}{32}-\tfrac{55}{32}$$
$$y-\tfrac{13}{16}z=-\tfrac{23}{32} \Rightarrow y=-\tfrac{23}{32}+\tfrac{13}{16}z$$

Solution set: $\left\{\left(\frac{11}{32}+\frac{7}{16}z, -\frac{23}{32}+\frac{13}{16}z, z\right)\right\}$

23. $3x-4y+z=2\ \ (1)$
$2x+y=1\ \ (2)$

Solving equation (2) for y, we obtain $y = 1 - 2x$. Substitute $1 - 2x$ for y in equation (1) and solve for z.

$$3x-4(1-2x)+z=2 \Rightarrow 3x-4+8x+z=2 \Rightarrow$$
$$11x+z=6 \Rightarrow z=6-11x$$

Solution set: $\{(x, 1-2x, 6-11x)\}$

24. Writing $\begin{array}{r} 2x+3y=10 \\ -3x+y=18 \end{array}$ as an augmented matrix we have $\left[\begin{array}{rr|r} 2 & 3 & 10 \\ -3 & 1 & 18 \end{array}\right]$.

$$\left[\begin{array}{rr|r} 1 & \frac{3}{2} & 5 \\ -3 & 1 & 18 \end{array}\right] \tfrac{1}{2}\text{R1} \Rightarrow \left[\begin{array}{rr|r} 1 & \frac{3}{2} & 5 \\ 0 & \frac{11}{2} & 33 \end{array}\right] 3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & \frac{3}{2} & 5 \\ 0 & 1 & 6 \end{array}\right] \tfrac{2}{11}\text{R2} \Rightarrow \left[\begin{array}{rr|r} 1 & 0 & -4 \\ 0 & 1 & 6 \end{array}\right] -\tfrac{3}{2}\text{R2}+\text{R1}$$

Solution set: $\{(-4,6)\}$

25. Writing $\begin{array}{r} 5x+2y=-10 \\ 3x-5y=-6 \end{array}$ as an augmented matrix we have $\left[\begin{array}{rr|r} 5 & 2 & -10 \\ 3 & -5 & -6 \end{array}\right]$.

$$\left[\begin{array}{rr|r} 1 & \frac{2}{5} & -2 \\ 3 & -5 & -6 \end{array}\right] \tfrac{1}{5}\text{R1} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & \frac{2}{5} & -2 \\ 0 & -\frac{31}{5} & 0 \end{array}\right] -3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & \frac{2}{5} & -2 \\ 0 & 1 & 0 \end{array}\right] -\tfrac{5}{31}\text{R2} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & 0 & -2 \\ 0 & 1 & 0 \end{array}\right] -\tfrac{2}{5}\text{R2}+\text{R1}$$

Solution set: $\{(-2,0)\}$

26. Writing $\begin{array}{r} 3x+y=-7 \\ x-y=-5 \end{array}$ as an augmented matrix we have $\left[\begin{array}{rr|r} 3 & 1 & -7 \\ 1 & -1 & -5 \end{array}\right]$.

$$\left[\begin{array}{rr|r} 1 & -1 & -5 \\ 3 & 1 & -7 \end{array}\right] \text{R1} \leftrightarrow \text{R2} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & -1 & -5 \\ 0 & 4 & 8 \end{array}\right] -3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & -1 & -5 \\ 0 & 1 & 2 \end{array}\right] \tfrac{1}{4}\text{R2} \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right] \text{R2}+\text{R1}$$

Solution set: $\{(-3,2)\}$

27. Writing $\begin{aligned} 2x - y + 4z &= -1 \\ -3x + 5y - z &= 5 \\ 2x + 3y + 2z &= 3 \end{aligned}$ as an augmented matrix we have $\left[\begin{array}{rrr|r} 2 & -1 & 4 & -1 \\ -3 & 5 & -1 & 5 \\ 2 & 3 & 2 & 3 \end{array}\right]$.

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ -3 & 5 & -1 & 5 \\ 2 & 3 & 2 & 3 \end{array}\right] \begin{matrix} \frac{1}{2}\text{R1} \\ \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & \frac{7}{2} & 5 & \frac{7}{2} \\ 0 & 4 & -2 & 4 \end{array}\right] \begin{matrix} \\ 3\text{R1}+\text{R2} \\ -2\text{R1}+\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 1 & \frac{10}{7} & 1 \\ 0 & 4 & -2 & 4 \end{array}\right] \begin{matrix} \\ \frac{2}{7}\text{R2} \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & \frac{19}{7} & 0 \\ 0 & 1 & \frac{10}{7} & 1 \\ 0 & 0 & -\frac{54}{7} & 0 \end{array}\right] \begin{matrix} \frac{1}{2}\text{R2}+\text{R1} \\ \\ -4\text{R2}+\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & \frac{19}{7} & 0 \\ 0 & 1 & \frac{10}{7} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} \\ \\ -\frac{7}{54}\text{R2} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} -\frac{19}{7}\text{R3}+\text{R1} \\ -\frac{10}{7}\text{R3}+\text{R2} \\ \\ \end{matrix}$$

Solution set: $\{(0,1,0)\}$

28. Writing $\begin{aligned} x - z &= -3 \\ y + z &= 6 \\ 2x - 3z &= -9 \end{aligned}$ as an augmented matrix we have $\left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 6 \\ 2 & 0 & -3 & -9 \end{array}\right]$.

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -1 & -3 \end{array}\right] \begin{matrix} \\ \\ -2\text{R1}+\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{matrix} \\ \\ -1\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{matrix} \\ -1\text{R3}+\text{R2} \Rightarrow \\ \\ \end{matrix}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{matrix} \text{R3}+\text{R1} \\ \\ \\ \end{matrix}$$

Solution set: $\{(0,3,3)\}$

29. Let x = number of pounds of \$4.60 tea;
y = the number of pounds of \$5.75 tea;
z = the number of pounds of \$6.50 tea.
From the information in the exercise, we obtain the system

$$\begin{aligned} x + y + z &= 20 \\ 4.6x + 5.75y + 6.5z &= 20(5.25) \\ x &= y + z \end{aligned}.$$

Rewriting the system, we have

$$\begin{aligned} x + y + z &= 20 \\ 4.6x + 5.75y + 6.5z &= 105 \\ x - y - z &= 0 \end{aligned}.$$

With an augmented matrix of

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 20 \\ 4.6 & 5.75 & 6.5 & 105 \\ 1 & -1 & -1 & 0 \end{array}\right],$$

we solve by the Gauss-Jordan method.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 20 \\ 0 & 1.15 & 1.9 & 13 \\ 0 & -2 & -2 & -20 \end{array}\right] \begin{matrix} \\ -4.6\text{R1}+\text{R2} \\ -1\text{R1}+\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 20 \\ 0 & 1 & \frac{38}{23} & \frac{260}{23} \\ 0 & -2 & -2 & -20 \end{array}\right] \begin{matrix} \\ \frac{20}{23}\text{R2} \\ \\ \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{15}{23} & \frac{200}{23} \\ 0 & 1 & \frac{38}{23} & \frac{260}{23} \\ 0 & 0 & \frac{30}{23} & \frac{60}{23} \end{array}\right] \begin{matrix} -1\text{R2}+\text{R1} \\ \\ 2\text{R2}+\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{15}{23} & \frac{200}{23} \\ 0 & 1 & \frac{38}{23} & \frac{260}{23} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{matrix} \\ \\ \frac{23}{30}\text{R3} \end{matrix} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{matrix} \frac{15}{23}\text{R3}+\text{R1} \\ -\frac{38}{23}\text{R3}+\text{R2} \\ \\ \end{matrix}$$

From the final matrix, $x = 10$, $y = 8$, and $z = 2$. Therefore, 10 lb of \$4.60 tea, 8 lb of \$5.75 tea, and 2 lb of \$6.50 tea should be used.

30. Let x = the amount of 5% solution ;
y = the amount of 15% solution;
z = the amount of 10% solution.
From the information in the exercise, we obtain the system

$$\begin{aligned} x+y+z&=20\\ .05x+.15y+.10z&=.08(20)\\ x&=y+z+2\end{aligned}$$

Rewriting the system, we have

$$\begin{aligned} x+\quad y+\quad z&=\ 20\\ 5x+15y+10z&=160\\ x-\quad y-\quad z&=\ \ 2\end{aligned}.$$

With an augmented matrix of

$$\left[\begin{array}{ccc|c}1&1&1&20\\5&15&10&160\\1&-1&-1&2\end{array}\right],$$

we solve by the Gauss-Jordan method.

$$\left[\begin{array}{ccc|c}1&1&1&20\\0&10&5&60\\0&-2&-2&-18\end{array}\right]\begin{array}{l}\\-5\text{R1}+\text{R2}\\-1\text{R1}+\text{R3}\end{array}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&1&1&20\\0&-2&-2&-18\\0&10&5&60\end{array}\right]\text{R2}\leftrightarrow\text{R3}\quad\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&1&1&20\\0&1&1&9\\0&10&5&60\end{array}\right]-\tfrac{1}{2}\text{R2}\quad\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&0&0&11\\0&1&1&9\\0&0&-5&-30\end{array}\right]\begin{array}{l}-1\text{R2}+\text{R1}\\ \\-10\text{R2}+\text{R3}\end{array}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&0&0&11\\0&1&1&9\\0&0&1&6\end{array}\right]\begin{array}{l}\\ \\-\tfrac{1}{5}\text{R3}\end{array}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&0&0&11\\0&1&0&3\\0&0&1&6\end{array}\right]-1\text{R3}+\text{R2}$$

11 ml of 5% solution, 3 ml of 15% solution, and 6 ml of 10% solution should be used.

31. $y=.11x+2.9$ (1)
$y=.19x+1.4$ (2)
The given system can be solved by substitution. Substitute equation (1) into equation (2) to obtain $.11x+2.9=.19x+1.4.$
Solving this equation, we have
$1.5=.08x\Rightarrow x=18.75.$
Since $x=0$ corresponds 1960, $x=18.75$ implies during 1979 the male and female enrollments were the same.

The enrollment of males in this year is $y=.11(18.75)+2.9=4.9625$ million. The enrollment of females in this year (which is the same as the males) is $y=.19(18.75)+1.4=4.9625$ million. The total enrollment was therefore approximately 10 million.

32. $\begin{vmatrix}-2&4\\0&3\end{vmatrix}=-2(3)-(0)(4)=-6-0=-6$

33. $\begin{vmatrix}-1&8\\2&9\end{vmatrix}=-1(9)-2(8)=-9-16=-25$

34. $\begin{vmatrix}x&4x\\2x&8x\end{vmatrix}=x(8x)-2x(4x)=8x^2-8x^2=0$

35. Expanding $\begin{vmatrix}-1&2&3\\4&0&3\\5&-1&2\end{vmatrix}$ by the second column, we have

$$\begin{aligned}&-2\begin{vmatrix}4&3\\5&2\end{vmatrix}+0\begin{vmatrix}-1&3\\5&2\end{vmatrix}-(-1)\begin{vmatrix}-1&3\\4&3\end{vmatrix}\\&=-2[4(2)-5(3)]+0+[(-1)(3)-(4)(3)]\\&=-2(8-15)+(-3-12)=-2(-7)+(-15)\\&=14-15=-1\end{aligned}$$

36. Expanding $\begin{vmatrix}-2&4&1\\3&0&2\\-1&0&3\end{vmatrix}$ by the second column, we have

$$\begin{aligned}&(-1)(4)\begin{vmatrix}3&2\\-1&3\end{vmatrix}+0\begin{vmatrix}-2&1\\-1&3\end{vmatrix}-0\begin{vmatrix}-2&1\\3&2\end{vmatrix}\\&=-4(9+2)+0-0=-4(11)=-44\end{aligned}$$

37. $\begin{vmatrix}3x&7\\-x&4\end{vmatrix}=8\Rightarrow$

$12x-(-7x)=8\Rightarrow 19x=8\Rightarrow x=\frac{8}{19}$

Solution set: $\left\{\frac{8}{19}\right\}$

38. If we expand the left side of $\begin{vmatrix} 6x & 2 & 0 \\ 1 & 5 & 3 \\ x & 2 & -1 \end{vmatrix} = 2x$ by the first row and solve, we obtain the following.

$$6x\begin{vmatrix} 5 & 3 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ x & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ x & 2 \end{vmatrix} = 2x$$

$$\begin{aligned} 6x(-5-6) - 2(-1-3x) + 0 &= 2x \\ 6x(-11) + 2 + 6x &= 2x \\ -66x + 2 + 6x &= 2x \\ -60x + 2 &= 2x \\ 2 &= 62x \\ x &= \tfrac{2}{62} = \tfrac{1}{31} \end{aligned}$$

Solution set: $\left\{\frac{1}{31}\right\}$

39. $3x + 7y = 2$
$5x - y = -22$

$$D = \begin{vmatrix} 3 & 7 \\ 5 & -1 \end{vmatrix} = 3(-1) - 5(7) = -3 - 35 = -38$$

$$D_x = \begin{vmatrix} 2 & 7 \\ -22 & -1 \end{vmatrix} = 2(-1) - (-22)(7) = -2 - (-154) = 152$$

$$D_y = \begin{vmatrix} 3 & 2 \\ 5 & -22 \end{vmatrix} = 3(-22) - 5(2) = -66 - 10 = -76$$

$x = \frac{D_x}{D} = \frac{152}{-38} = -4$ and $y = \frac{D_y}{D} = \frac{-76}{-38} = 2.$

Solution set: $\{(-4, 2)\}$

40. $3x + y = -1$
$5x + 4y = 10$

$$D = \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 3(4) - 5(1) = 12 - 5 = 7$$

$$D_x = \begin{vmatrix} -1 & 1 \\ 10 & 4 \end{vmatrix} = (-1)(4) - (10)(1) = -4 - 10 = -14$$

$$D_y = \begin{vmatrix} 3 & -1 \\ 5 & 10 \end{vmatrix} = 3(10) - 5(-1) = 30 - (-5) = 35$$

$x = \frac{D_x}{D} = \frac{-14}{7} = -2$ and $y = \frac{D_y}{D} = \frac{35}{7} = 5.$

Solution set: $\{(-2, 5)\}$

41. Given $\begin{array}{ll} 6x + y = -3 & (1) \\ 12x + 2y = 1 & (2) \end{array}$, we have

$$D = \begin{vmatrix} 6 & 1 \\ 12 & 2 \end{vmatrix} = 6(2) - 12(1) = 12 - 12 = 0.$$

Because $D = 0$, Cramer's rule does not apply. To determine whether the system is inconsistent or has infinitely many solutions, use the elimination method.

$$\begin{array}{rl} -12x - 2y = 6 & \text{Multiply equation (1) by} -2. \\ 12x + 2y = 1 & \\ \hline 0 = 7 & \text{False} \end{array}$$

The system is inconsistent. Thus the solution set is $\varnothing$.

42. $3x + 2y + z = 2$
$4x - y + 3z = -16$
$x + 3y - z = 12$

Adding 2 times row 2 to row 1 and 3 times row 2 to row 3, we have

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & -1 & 3 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 11 & 0 & 7 \\ 4 & -1 & 3 \\ 13 & 0 & 8 \end{vmatrix}$$

Expanding by column two, we have

$$D = -1\begin{vmatrix} 11 & 7 \\ 13 & 8 \end{vmatrix} = -(88 - 91) = 3$$

Adding 2 times row 2 to row 1 and 3 times row 2 to row 3, we have

$$D_x = \begin{vmatrix} 2 & 2 & 1 \\ -16 & -1 & 3 \\ 12 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -30 & 0 & 7 \\ 4 & -1 & 3 \\ -36 & 0 & 8 \end{vmatrix}$$

Expanding by column two, we have

$$D_x = -1\begin{vmatrix} -30 & 7 \\ -36 & 8 \end{vmatrix} = -[-240 - (-252)] = -12$$

Adding 3 times row 3 to row 2 and row 3 to row 1, we have

$$D_y = \begin{vmatrix} 3 & 2 & 1 \\ 4 & -16 & 3 \\ 1 & 12 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 14 & 0 \\ 7 & 20 & 0 \\ 1 & 12 & -1 \end{vmatrix}$$

Expanding by column three, we have

$$D_y = -1\begin{vmatrix} 4 & 14 \\ 7 & 20 \end{vmatrix} = -1(80 - 98) = -(-18) = 18$$

Adding 2 times row 2 to row 1 and 3 times row 2 to row 3, we have

$$D_z = \begin{vmatrix} 3 & 2 & 2 \\ 4 & -1 & -16 \\ 1 & 3 & 12 \end{vmatrix} = \begin{vmatrix} 11 & 0 & -30 \\ 4 & -1 & -16 \\ 13 & 0 & -36 \end{vmatrix}$$

Expanding by column two, we have

$$D_z = -1\begin{vmatrix} 11 & -30 \\ 13 & -36 \end{vmatrix} = -1(-396 + 390) = -(-6) = 6$$

$x = \frac{D_x}{D} = \frac{-12}{3} = -4,\ y = \frac{D_y}{D} = \frac{18}{3} = 6,$
$z = \frac{D_z}{D} = \frac{6}{3} = 2$

Solution set: $\{(-4, 6, 2)\}$

43. $x + y = -1$
$2y + z = 5$
$3x - 2z = -28$

Adding –3 times row 1 to row 3, we have

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -3 & -2 \end{vmatrix}$$

Expanding by column one, we have

$$D = \begin{vmatrix} 2 & 1 \\ -3 & -2 \end{vmatrix} = -4 - (-3) = -1$$

Adding –2 times row 1 to row 2, we have

$$D_x = \begin{vmatrix} -1 & 1 & 0 \\ 5 & 2 & 1 \\ -28 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 0 \\ 7 & 0 & 1 \\ -28 & 0 & -2 \end{vmatrix}$$

Expanding about column two, we have

$$D_x = -(1)\begin{vmatrix} 7 & 1 \\ -28 & -2 \end{vmatrix} = -[-14 - (-28)] = -14$$

Adding 2 times row 2 to row 3, we have

$$D_y = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 5 & 1 \\ 3 & -28 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 5 & 1 \\ 3 & -18 & 0 \end{vmatrix}$$

Expanding by column three, we have

$$D_y = -(1)\begin{vmatrix} 1 & -1 \\ 3 & -18 \end{vmatrix}$$
$$= -1[-18 - (-3)] = -(-15) = 15$$

Adding –3 times row 1 to row 3, we have

$$D_z = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 5 \\ 3 & 0 & -28 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & -3 & -25 \end{vmatrix}$$

Expanding by column one, we have

$$D_z = 1\begin{vmatrix} 2 & 5 \\ -3 & -25 \end{vmatrix} = -50 - (-15) = -35.$$

Thus, we have

$x = \frac{D_x}{D} = \frac{-14}{-1} = 14,\ y = \frac{D_y}{D} = \frac{15}{-1} = -15,$
$z = \frac{D_z}{D} = \frac{-35}{-1} = 35$

Solution set: $\{(14, -15, 35)\}$

44. $5x - 2y - z = 8$ (1)
$-5x + 2y + z = -8$ (2)
$x - 4y - 2z = 0$ (3)

Adding row 1 to row 2, we have

$$D = \begin{vmatrix} 5 & -2 & -1 \\ -5 & 2 & 1 \\ 1 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 5 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & -4 & -2 \end{vmatrix} = 0$$

Since $D = 0$, Cramer's rule cannot be used. Adding the first and second equations results in the equality $0 = 0$. This implies that there are infinitely many solutions. We can use the elimination method to complete the solution. We will write the solution set with z as the arbitrary variable. Multiply equation (1) by –2 and add the result to equation (3) to obtain the following.

$$\begin{aligned} -10x + 4y + 2z &= -16 \\ x - 4y - 2z &= 0 \\ \hline -9x \qquad\qquad &= -16 \Rightarrow x = \tfrac{16}{9} \end{aligned}$$

Substitute this value of x in equation (1). Solve the resulting equation for y in terms of z.

$$5\left(\tfrac{16}{9}\right) - 2y - z = 8 \Rightarrow \tfrac{80}{9} - 2y - z = 8$$
$$-2y - z = \tfrac{72}{9} - \tfrac{80}{9} \Rightarrow -2y - z = -\tfrac{8}{9}$$
$$-2y = -\tfrac{8}{9} + z \Rightarrow -2y = \tfrac{-8+9z}{9}$$
$$y = \tfrac{8-9z}{18}$$

Solution set: $\left\{\left(\frac{16}{9}, \frac{8-9z}{18}, z\right)\right\}$

45. $\dfrac{2}{3x^2 - 5x + 2} = \dfrac{2}{(x-1)(3x-2)} = \dfrac{A}{x-1} + \dfrac{B}{3x-2}$

Multiply both sides by $(x-1)(3x-2)$ to get

$2 = A(3x-2) + B(x-1)$. (1)

First substitute 1 for x to get

$2 = A[3(1) - 2] + B(1-1)$
$2 = A(1) + B(0) \Rightarrow A = 2$

Replace A with 2 in equation (1) and substitute $\frac{2}{3}$ for x to get

$2 = 2\left[3\left(\frac{2}{3}\right) - 2\right] + B\left(\frac{2}{3} - 1\right)$
$2 = 2(0) + B\left(-\frac{1}{3}\right) \Rightarrow 2 = -\frac{1}{3}B \Rightarrow B = -6$

Thus, we have

$$\frac{2}{3x^2 - 5x + 2} = \frac{2}{x-1} + \frac{-6}{3x-2} \text{ or } \frac{2}{x-1} - \frac{6}{3x-2}$$

46. $\dfrac{11-2x}{x^2-8x+16}=\dfrac{11-2x}{(x-4)^2}=\dfrac{A}{(x-4)^2}+\dfrac{B}{x-4}$

Multiply both sides by $(x-4)^2$ to get

$11-2x=A+B(x-4).$ (1)

First substitute 4 for x to get the following.

$11-2(4)=A+B(4-4)$

$11-8=A+0\Rightarrow A=3$

Replace A with 3 in equation (1) and substitute 0 (arbitrary choice) for x to get the following.

$11-2(0)=3+B(0-4)$

$11-0=3+(-4B)\Rightarrow 8=-4B\Rightarrow B=-2$

Thus, we have

$\dfrac{11-2x}{x^2-8x+16}=\dfrac{3}{(x-4)^2}+\dfrac{-2}{x-4}$ or

$\dfrac{3}{(x-4)^2}-\dfrac{2}{x-4}$

47. $\dfrac{5-2x}{(x^2+2)(x-1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+2}$

Multiply both sides by $(x^2+2)(x-1)$ to get the following.

$5-2x=A(x^2+2)+(Bx+C)(x-1)$ (1)

First substitute 1 for x to get the following.

$5-2(1)=A(1^2+2)+[B(1)+C](1-1)$

$5-2=A(1+2)+0\Rightarrow 3=3A\Rightarrow A=1$

Replace A with 1 in equation (1) and substitute 0 for x to get the following.

$5-2(0)=(0^2+2)+[B(0)+C](0-1)$

$5-0=(0+2)+C(-1)\Rightarrow 5=2-C\Rightarrow$

$3=-C\Rightarrow C=-3$

$5-2x=(x^2+2)+(Bx-3)(x-1)$ (2)

Substitute 2 (arbitrary value) in equation (2) for x to get the following.

$5-2(2)=(2^2+2)+[B(2)-3](2-1)$

$5-4=(4+2)+(2B-3)(1)$

$1=6+2B-3\Rightarrow 1=3+2B$

$-2=2B\Rightarrow B=-1$

Thus, we have

$\dfrac{5-2x}{(x^2+2)(x-1)}=\dfrac{1}{x-1}+\dfrac{(-1)x+(-3)}{x^2+2}$ or

$\dfrac{1}{x-1}-\dfrac{x+3}{x^2+2}$

48. $\dfrac{x^3+2x^2-3}{x^4-4x^2+4}=\dfrac{x^3+2x^2-3}{(x^2-2)^2}$

$=\dfrac{Ax+B}{x^2-2}+\dfrac{Cx+D}{(x^2-2)^2}$

Multiply both sides by $(x^2-2)^2$ to get

$x^3+2x^2-3=(Ax+B)(x^2-2)+(Cx+D)$ (1)

Distributing and combining like terms on the right side of the equation, we have the following.

$x^3+2x^2-3=Ax^3+Bx^2-2Ax-2B+Cx+D$

$x^3+2x^2-3=Ax^3+Bx^2+(C-2A)x+(D-2B)$

Equate the coefficients of like powers of x on the two sides of the equation. For the x^3- term, we have $1=A.$ For the x^2- term, we have $2=B.$ For the x- term, we have $0=C-2A.$ For the constant term, we have $-3=D-2B.$ Since $A=1$ and $0=C-2A,$ we have $C=2.$ Since $B=2$ and $-3=D-2B,$ we have $D=1.$ Thus, we have

$\dfrac{x^3+2x^2-3}{x^4-4x^2+4}=\dfrac{x+2}{x^2-2}+\dfrac{2x+1}{(x^2-2)^2}.$

Note: The denominators of the partial fractions are irreducible over the rational numbers.

49. $y=2x+10$ (1)

$x^2+y=13$ (2)

Substituting equation (1) into equation (2), we have $x^2+(2x+10)=13.$ Solving for x we get the following.

$x^2+(2x+10)=13$

$x^2+2x-3=0$

$(x+3)(x-1)=0\Rightarrow x=-3$ or $x=1$

For each value of x, use equation (1) to find the corresponding value of y.

If $x=-3,\ y=2(-3)+10=-6+10=4.$

If $x=1,\ y=2(1)+10=2+10=12.$

Solution set: $\{(-3,4),(1,12)\}$

50. $x^2 = 2y - 3$ (1)
$x + y = 3$ (2)
Solving equation (2) for y we have $y = 3 - x$ (3). Substituting this expression into equation (1) we have $x^2 = 2(3 - x) - 3$. Solving for x we have

$$x^2 = 2(3 - x) - 3 \Rightarrow x^2 = 6 - 2x - 3$$
$$x^2 = -2x + 3 \Rightarrow x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0 \Rightarrow x = -3 \text{ or } x = 1$$

For each value of x, use equation (3) to find the corresponding value of y. If $x = 1$, $y = 3 - 1 = 2$. If $x = -3$, $y = 3 - (-3) = 6$.
Solution set: $\{(1,2),(-3,6)\}$

51. $x^2 + y^2 = 17$ (1)
$2x^2 - y^2 = 31$ (2)
Add equations (1) and (2) to obtain
$3x^2 = 48 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$
For each value of x, use equation (1) to find the corresponding value of y.
If $x = -4$, $(-4)^2 + y^2 = 17 \Rightarrow 16 + y^2 = 17 \Rightarrow$ $y^2 = 1 \Rightarrow y = \pm 1$. If $x = 4$, $4^2 + y^2 = 17 \Rightarrow$ $16 + y^2 = 17 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$.
Solution set: $\{(-4,1),(-4,-1),(4,-1),(4,1)\}$

52. $2x^2 + 3y^2 = 30$ (1)
$x^2 + y^2 = 13$ (2)
Multiply equation (2) by –2 and add the result to equation (1).

$$\begin{array}{r} 2x^2 + 3y^2 = 30 \\ -2x^2 - 2y^2 = -26 \\ \hline y^2 = 4 \end{array} \Rightarrow y = \pm 2$$

To find the corresponding values of x, substitute back into equation (2). If $y = 2$, then $x^2 + 2^2 = 13 \Rightarrow x^2 + 4 = 13 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.
If $y = -2$, then $x^2 + (-2)^2 = 13 \Rightarrow$ $x^2 + 4 = 13 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.
Solution set: $\{(-3,2),(-3,-2),(3,-2),(3,2)\}$

53. $xy = -10$ (1)
$x + 2y = 1$ (2)
Solving equation (2) for x gives $x = 1 - 2y$ (3). Substituting this expression into equation (1), we obtain $(1-2y)y = -10$. Solving for y we have

$$(1-2y)y = -10 \Rightarrow y - 2y^2 = -10$$
$$0 = 2y^2 - y - 10$$
$$(y+2)(2y-5) = 0 \Rightarrow y = -2 \text{ or } y = \tfrac{5}{2}$$

For each value of y, use equation (3) to find the corresponding value of x.
If $y = -2$, $x = 1 - 2(-2) = 1 - (-4) = 5$.
If $y = \frac{5}{2}$, $x = 1 - 2\left(\frac{5}{2}\right) = 1 - 5 = -4$.
Solution set: $\left\{(5,-2),\left(-4,\frac{5}{2}\right)\right\}$

54. $xy + 2 = 0$ (1)
$y - x = 3$ (2)
Solving equation (2) for y, we have $y = x + 3$ (3). Substituting this expression into equation (1), we obtain $x(x+3) + 2 = 0$. Solving for x we have the following.

$$x(x+3) + 2 = 0 \Rightarrow x^2 + 3x + 2 = 0$$
$$(x+1)(x+2) = 0 \Rightarrow x = -1 \text{ or } x = -2$$

For each value of x, use equation (3) to find the corresponding value of y. If $x = -1$, $y = (-1) + 3 = 2$. If $x = -2$, $y = (-2) + 3 = 1$.
Solution set: $\{(-1,2),(-2,1)\}$

55. $x^2 + 2xy + y^2 = 4$ (1)
$x - 3y = -2$ (2)
Solve equation (2) for x: $x = 3y - 2$ (3).
Substituting this expression into equation (1), we obtain $(3y-2)^2 + 2(3y-2)y + y^2 = 4$.
Solving for x we have the following.

$$(3y-2)^2 + 2(3y-2)y + y^2 = 4$$
$$9y^2 - 12y + 4 + 6y^2 - 4y + y^2 = 4$$
$$16y^2 - 16y + 4 = 4$$
$$16y^2 - 16y = 0$$
$$y^2 - y = 0$$
$$y(y-1) = 0$$
$$y = 0 \text{ or } y = 1$$

For each value of y, use equation (3) to find the corresponding value of x.
If $y = 0$, $x = 3(0) - 2 = 0 - 2 = -2$.
If $y = 1$, $x = 3(1) - 2 = 3 - 2 = 1$.
Solution set: $\{(-2,0),(1,1)\}$

56. $x^2 + 2xy = 15 + 2x$ (1)
$xy - 3x + 3 = 0$ (2)

Multiply equation (2) by –2 and add to equation (1) in order to eliminate the *xy*-term.

$$\begin{aligned} x^2 + 2xy &= 15 + 2x \\ -2xy + 6x - 6 &= 0 \\ \hline x^2 + 6x - 6 &= 15 + 2x \quad (3) \end{aligned}$$

Solve equation (3) for *x*.

$$x^2 + 6x - 6 = 15 + 2x$$
$$x^2 + 4x - 21 = 0 \Rightarrow (x-3)(x+7) = 0$$
$$x = 3 \text{ or } x = -7$$

For each value of *y*, use equation (2) to find the corresponding value of *y*.

If $x = 3$, $3y - 3(3) + 3 = 0 \Rightarrow 3y - 9 + 3 = 0 \Rightarrow$
$3y - 6 = 0 \Rightarrow 3y = 6 \Rightarrow y = 2$

If $x = -7$, $-7y - 3(-7) + 3 = 0 \Rightarrow$
$-7y - (-21) + 3 = 0 \Rightarrow -7y + 24 = 0 \Rightarrow$
$-7y = -24 \Rightarrow y = \frac{24}{7}$

Solution set: $\left\{(3,2), \left(-7, \frac{24}{7}\right)\right\}$

57. $3x - y = b$ (1)
$x^2 + y^2 = 25$ (2)

Solving equation (1) for *y* we have $y = 3x - b$. Substitute this expression into equation (2).

$$x^2 + (3x-b)^2 = 25$$
$$x^2 + 9x^2 - 6bx + b^2 = 25$$
$$10x^2 - 6bx + (b^2 - 25) = 0$$

Recall that there is only one solution when the discriminant equals 0:

$$b^2 - 4ac = 0 \Rightarrow$$
$$(-6b)^2 - 4(10)(b^2 - 25) = 0$$
$$36b^2 - 40b^2 + 1000 = 0$$
$$-4b^2 = -1000$$
$$b^2 = 250 \Rightarrow b = \pm\sqrt{250}$$
$$b = \pm 5\sqrt{10}$$

58. $x^2 + y^2 = 144$ (1)
$x + 2y = 8$ (2)

Solving equation (2) for *x*, we have $x = 8 - 2y$ (3). Substitute $-2y + 8$ for *x* in equation (1) to obtain

$$(8 - 2y)^2 + y^2 = 144$$
$$64 - 32y + 4y^2 + y^2 = 144$$
$$5y^2 - 32y - 80 = 0$$

We solve this quadratic equation for *x* by using the quadratic formula, with $a = 5, b = -32,$ and $c = -80$.

$$y = \frac{-(-32) \pm \sqrt{(-32)^2 - 4(5)(-80)}}{2(5)}$$
$$= \frac{32 \pm \sqrt{1,024 + 1,600}}{10} = \frac{32 + \sqrt{2,624}}{10}$$
$$= \frac{32 \pm 8\sqrt{41}}{10} = \frac{16 \pm 4\sqrt{41}}{5}$$

For each value of *y*, use equation (3) to find the corresponding value of *x*.

If $y = \frac{16+4\sqrt{41}}{5}$

$$x = 8 - 2\left(\frac{16+4\sqrt{41}}{5}\right) = \frac{40}{5} + \frac{-32-8\sqrt{41}}{5} = \frac{8-8\sqrt{41}}{5}$$

If $y = \frac{16-4\sqrt{41}}{5}$,

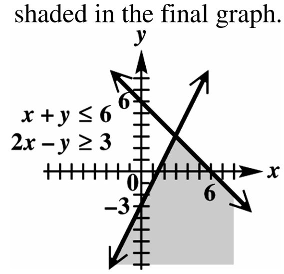
$$x = 8 - 2\left(\frac{16-4\sqrt{41}}{5}\right) = \frac{40}{5} + \frac{-32+8\sqrt{41}}{5} = \frac{8+8\sqrt{41}}{5}$$

Yes, the circle and the line have two points in common,

$\left(\frac{8-8\sqrt{41}}{5}, \frac{16+4\sqrt{41}}{5}\right)$ and $\left(\frac{8+8\sqrt{41}}{5}, \frac{16-4\sqrt{41}}{5}\right)$.

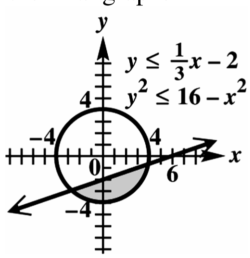
59. $x + y \le 6$
$2x - y \ge 3$

Graph the solid line $x + y = 6$, which has *x*-intercept 6 and *y*-intercept 6. Since $x + y \le 6 \Rightarrow y \le -x + 6$, shade the region below this line. Graph the solid line $2x - y = 3$, which has *x*-intercept $\frac{3}{2}$ and *y*-intercept –3. Since $2x - y \ge 3 \Rightarrow -y \ge -2x + 3 \Rightarrow y \le 2x - 3$, shade the region below this line. The solution set is the intersection of these two regions, which is shaded in the final graph.



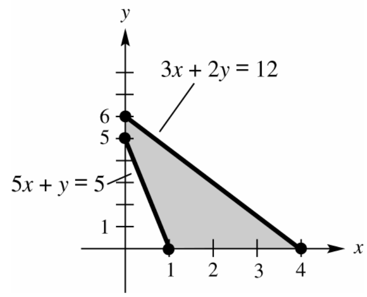
60. $y \le \frac{1}{3}x - 2$
$y^2 \le 16 - x^2$

Graph the line $y = \frac{1}{3}x - 2$ as a solid line. Since $y \le \frac{1}{3}x - 2$, shade the region below the line. Since, $y^2 \le 16 - x^2 \Rightarrow x^2 + y^2 \le 16$, graph the circle $x^2 + y^2 = 16$ as a solid curve with center $(0,0)$ and radius 4. Shade the interior of the circle. The solution set is the intersection of two regions, which is shaded in the final graph.



61. Find $x \ge 0$ and $y \ge 0$ such that $\begin{array}{l} 3x + 2y \le 12 \\ 5x + y \ge 5 \end{array}$ and $2x + 4y$ is maximized. To graph $3x + 2y \le 12$, draw the line with x-intercept 4 and y-intercept 6 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $5x + y \ge 5$, draw the line with x-intercept 1 and y-intercept 5 as a solid line. Because the test point $(0,0)$ does not satisfy this inequality, shade the region *above* the line. The constraints $x \ge 0$ and $y \ge 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints. The four vertices are $(1,0), (4,0), (0,5),$ and $(0,6)$.

Point	Value of $2x + 4y$	
$(1,0)$	$2(1) + 4(0) = 2$	
$(4,0)$	$2(4)+4(0)=8$	
$(0,5)$	$2(0)+4(5)=20$	
$(0,6)$	$2(0)+4(6)=24$	← Maximum



The maximum value is 24, which occurs at $(0,6)$.

62. The parabola passes through (−1, 0), (0, 1), and (1, 0), so those points satisfy the equation $y = ax^2 + bx + c$. Substituting each ordered pair into the equation gives the system

$$0 = a(-1)^2 + b(-1) + c$$
$$1 = a(0^2) + b(0) + c$$
$$0 = a(1^2) + b(1) + c$$

which simplifies to

$$0 = a - b + c \quad (1)$$
$$1 = \quad c \quad (2)$$
$$0 = a + b + c \quad (3)$$

Substitute $c = 1$ from equation (2) into equations (1) and (3), then solve the system consisting of those two equations:

$$0 = a - b + 1 \Rightarrow -1 = a - b \quad (4)$$
$$0 = a + b + 1 \Rightarrow -1 = a + b \quad (5)$$

Add equations (4) and (5), solve for a, then solve for b:

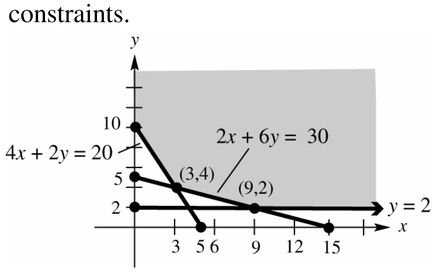
$$-1 = a - b$$
$$-1 = a + b$$
$$-2 = 2a \quad \Rightarrow a = -1$$
$$-1 = -1 + b \Rightarrow b = 0$$

Thus the equation is $y = -x^2 + 1$. The curve is solid, so it is included in the inequality. The dashed line passes through (4, 0) and (0, −2), so $m = \frac{-2-0}{0-4} = \frac{1}{2}$, and the equation is $y = \frac{1}{2}x + 2 \Rightarrow x - 2y = 4$. The line is dashed so it is not included in the inequality.
The shaded region includes (0, 0), so test (0, 0) in each equation to determine the direction of the inequalities:

$$0 < -(0^2) + 1 = 1 \Rightarrow y \le -x^2 + 1,$$
$$0 - 2(0) = 0 < 4 \Rightarrow y - 2x < 4$$

The system is $\begin{array}{l} x^2 + y \le 1 \\ x - 2y < 4 \end{array}$.

63. Let x = number of servings of food A; y = number of servings of food B.
The protein constraint is $2x+6y \geq 30$. The fat constraint is $4x+2y \geq 20$. Also, $y \geq 2$ and since the numbers of servings cannot be negative, we also have $x \geq 0$. We want to minimize the objective function, cost $= .18x + .12y$. Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $2x+6y \geq 30$, draw the line with x-intercept 15 and y-intercept 5 as a solid line. Because the test point $(0,0)$ does not satisfy this inequality, shade the region above the line. To graph $4x+2y \geq 20$, draw the line with x-intercept 5 and y-intercept 10 as a solid line. Because the test point $(0,0)$ does not satisfy this inequality, shade the region above the line. To graph $y \geq 2$, draw the horizontal line with y-intercept 2 as a solid line and shade the region above the line. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that one vertex is $(0,10)$. The second occurs when y = 2 and
$2x+6y=30 \Rightarrow 2x+6(2)=30 \Rightarrow 2x=18 \Rightarrow x=9.$
Thus, $(9,2)$ is a vertex. The third vertex is the intersection point of the lines $2x+6y=30$ and $4x+2y=20$. To find this point, solve the system

$$\begin{aligned} 2x+6y&=30 \\ 4x+2y&=20. \end{aligned}$$

Multiply the first equation by –2 and add it to the second equation.

$$\begin{aligned} -4x-12y&=-60 \\ 4x+2y&=20 \\ \hline -10y&=-40 \Rightarrow y=4 \end{aligned}$$

Substituting 4 for y into $2x+6y=30$, we have the following.

$$2x+6(4)=30 \Rightarrow 2x+24=30 \Rightarrow 2x=6 \Rightarrow x=3$$

Thus, the third vertex is $(3,4)$. Next, evaluate the objective function at each vertex.

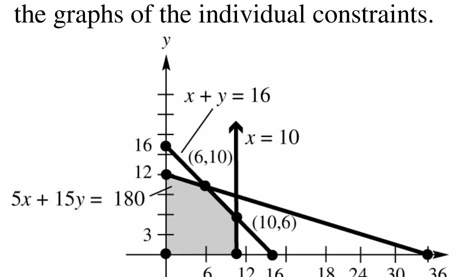
Point	Cost $= .18x + .12y$	
$(0,10)$	$.18(0)+.12(10)=1.20$	
$(3,4)$	$.18(3)+.12(4)=1.02$	← Minimum
$(9,2)$	$.18(9)+.12(2)=1.86$	

The minimum cost of \$1.02 per serving will be produced by 3 units of food A and 4 units of food B.

64. Let x = the number of geese; y = the number of pigs.
The constraints are

$$\begin{aligned} 5x+15y &\leq 180 \\ x+y &\leq 16 \\ x &\leq 10 \\ x \geq 0,\ y &\geq 0. \end{aligned}$$

Maximize profit function, number of people $= 6x+20y$.
Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $5x+15y \leq 180$, draw the line with x-intercept 36 and y-intercept 12 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $x+y \leq 16$, draw the line with x-intercept 16 and y-intercept 16 as a solid line. Because the test point $(0,0)$ satisfies this inequality, shade the region *below* the line. To graph $x \leq 10$, draw the vertical line with x-intercept 10 and shade the region to the *left* of the line. The constraints $x \geq 0$ and $y \geq 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three vertices are $(0,0), (10,0)$, and $(0,12)$. A fourth vertex is the intersection point of the lines $x=10$ and $x+y=16$. This point is $(10,6)$. The fifth vertex is the intersection point of the lines $5x+15y=180$ and $x+y=16$.

To find this point, solve the system $\begin{aligned}5x+15y&=180\\x+\quad y&=16.\end{aligned}$ The second equation can be written as $x=16-y$. Substituting this equation into $5x+15y=180$, we have the following.

$5(16-y)+15y=180 \Rightarrow 80-5y+15y=180$
$80+10y=180 \Rightarrow 10y=100 \Rightarrow y=10$

Substituting $y=10$ into $x=16-y$, we have $x=16-10=6$. Thus, the fifth vertex is $(6,10)$. Next, evaluate the objective function at each vertex.

Point	Number of people $=6x+20y$	
$(0,0)$	$6(0)+20(0)=0$	
$(10,0)$	$6(10)+20(0)=60$	
$(0,12)$	$6(0)+20(12)=240$	← Maximum
$(10,6)$	$6(10)+20(6)=180$	
$(6,10)$	$6(6)+20(10)=236$	

12 pigs and no geese produce a maximum profit of \$240.

65. $\begin{bmatrix}5 & x+2\\-6y & z\end{bmatrix}=\begin{bmatrix}a & 3x-1\\5y & 9\end{bmatrix}$

$a=5;\ x+2=3x-1 \Rightarrow 3=2x \Rightarrow x=\frac{3}{2};$
$-6y=5y \Rightarrow 0=11y \Rightarrow y=0;\ z=9$

Thus, $a=5,\ x=\frac{3}{2},\ y=0,$ and $z=9.$

66. $\begin{bmatrix}-6+k & 2 & a+3\\-2+m & 3p & 2r\end{bmatrix}+\begin{bmatrix}3-2k & 5 & 7\\5 & 8p & 5r\end{bmatrix}=\begin{bmatrix}5 & y & 6a\\2m & 11 & -35\end{bmatrix}$

$\begin{bmatrix}(-6+k)+(3-2k) & 2+5 & (a+3)+7\\(-2+m)+5 & 3p+8p & 2r+5r\end{bmatrix}=\begin{bmatrix}5 & y & 6a\\2m & 11 & -35\end{bmatrix}$

$\begin{bmatrix}-3-k & 7 & a+10\\m+3 & 11p & 7r\end{bmatrix}=\begin{bmatrix}5 & y & 6a\\2m & 11 & -35\end{bmatrix}$

For these two matrices to be equal, corresponding elements must be equal. Thus, we have the following.

$-3-k=5 \Rightarrow -k=8 \Rightarrow k=-8;\ y=7;$
$a+10=6a \Rightarrow 10=5a \Rightarrow a=2$
$3+m=2m \Rightarrow m=3; 11p=11 \Rightarrow p=1;$
$7r=-35 \Rightarrow r=-5$

Thus, $k=-8,\ y=7,\ a=2,\ m=3,\ p=1,$ and $r=-5.$

67. $\begin{bmatrix}3\\2\\5\end{bmatrix}-\begin{bmatrix}8\\-4\\6\end{bmatrix}+\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}3-8\\2-(-4)\\5-6\end{bmatrix}+\begin{bmatrix}1\\0\\2\end{bmatrix}$
$=\begin{bmatrix}-5\\6\\-1\end{bmatrix}+\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}-5+1\\6+0\\-1+2\end{bmatrix}$
$=\begin{bmatrix}-4\\6\\1\end{bmatrix}$

68. $4\begin{bmatrix}3 & -4 & 2\\5 & -1 & 6\end{bmatrix}+\begin{bmatrix}-3 & 2 & 5\\1 & 0 & 4\end{bmatrix}$
$=\begin{bmatrix}4(3) & 4(-4) & 4(2)\\4(5) & 4(-1) & 4(6)\end{bmatrix}+\begin{bmatrix}-3 & 2 & 5\\1 & 0 & 4\end{bmatrix}$
$=\begin{bmatrix}12 & -16 & 8\\20 & -4 & 24\end{bmatrix}+\begin{bmatrix}-3 & 2 & 5\\1 & 0 & 4\end{bmatrix}$
$=\begin{bmatrix}12+(-3) & -16+2 & 8+5\\20+1 & -4+0 & 24+4\end{bmatrix}$
$=\begin{bmatrix}9 & -14 & 13\\21 & -4 & 28\end{bmatrix}$

69. $\begin{bmatrix}2 & 5 & 8\\1 & 9 & 2\end{bmatrix}-\begin{bmatrix}3 & 4\\7 & 1\end{bmatrix}$

This operation is not possible because one matrix is 2×3 and the other 2×2. Matrices of different sizes cannot be added or subtracted.

70. $\begin{bmatrix}-3 & 4\\2 & 8\end{bmatrix}\begin{bmatrix}-1 & 0\\2 & 5\end{bmatrix}$
$=\begin{bmatrix}-3(-1)+4(2) & -3(0)+4(5)\\2(-1)+8(2) & 2(0)+8(5)\end{bmatrix}$
$=\begin{bmatrix}3+8 & 0+20\\-2+16 & 0+40\end{bmatrix}=\begin{bmatrix}11 & 20\\14 & 40\end{bmatrix}$

71. $\begin{bmatrix}-1 & 0\\2 & 5\end{bmatrix}\begin{bmatrix}-3 & 4\\2 & 8\end{bmatrix}$
$=\begin{bmatrix}-1(-3)+0(2) & -1(4)+0(8)\\2(-3)+5(2) & 2(4)+5(8)\end{bmatrix}$
$=\begin{bmatrix}3+0 & -4+0\\-6+10 & 8+40\end{bmatrix}=\begin{bmatrix}3 & -4\\4 & 48\end{bmatrix}$

72. The product of a 3×2 matrix and a 2×2 matrix is a 3×2 matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1(4)+2(-1) & 1(8)+2(2) \\ 3(4)+0(-1) & 3(8)+0(2) \\ -6(4)+5(-1) & -6(8)+5(2) \end{bmatrix} = \begin{bmatrix} 4+(-2) & 8+4 \\ 12+0 & 24+0 \\ -24+(-5) & -48+10 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 12 & 24 \\ -29 & -38 \end{bmatrix}$$

73. The product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix.

$$\begin{bmatrix} 3 & 2 & -1 \\ 4 & 0 & 6 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3(-2)+2(0)+(-1)(3) & 3(0)+2(2)+(-1)(1) \\ 4(-2)+0(0)+6(3) & 4(0)+0(2)+6(1) \end{bmatrix} = \begin{bmatrix} -6+0-3 & 0+4-1 \\ -8+0+18 & 0+0+6 \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ 10 & 6 \end{bmatrix}$$

74. The product of a 2×4 matrix and a 4×1 matrix is a 2×1 matrix.

$$\begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -1 & 8 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(-1)+(-2)(2)+4(0)+2(1) \\ 0(-1)+1(2)+(-1)(0)+8(1) \end{bmatrix} = \begin{bmatrix} -1+(-4)+(0)+2 \\ 0+2+0+8 \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \end{bmatrix}$$

75. The product of a 3×3 matrix and a 3×3 matrix is a 3×3 matrix.

$$\begin{bmatrix} -2 & 5 & 5 \\ 0 & 1 & 4 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2(1)+5(-1)+5(1) & -2(0)+5(0)+5(1) & -2(-1)+5(0)+5(-1) \\ 0(1)+1(-1)+4(1) & 0(0)+1(0)+4(1) & 0(-1)+1(0)+4(-1) \\ 3(1)+(-4)(-1)+(-1)(1) & 3(0)+(-4)(0)+(-1)(1) & 3(-1)+(-4)(0)+(-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -2+(-5)+5 & 0+0+5 & 2+0+(-5) \\ 0+(-1)+4 & 0+0+4 & 0+0+(-4) \\ 3+4+(-1) & 0+0+(-1) & -3+0+1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -3 \\ 3 & 4 & -4 \\ 6 & -1 & -2 \end{bmatrix}$$

76. Find the inverse of $A = \begin{bmatrix} -4 & 2 \\ 0 & 3 \end{bmatrix}$ if it exists.

We have $[A|I_2] = \left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$, which yields

$$\left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 3 & 0 & 1 \end{array}\right] -\tfrac{1}{4}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right] \tfrac{1}{3}\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right] \tfrac{1}{2}\text{R2}+\text{R1}$$

Thus, $A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix}$

77. Find the inverse of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, if it exists.

We have $[A|I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$, which yields

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \tfrac{1}{2}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] -5\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{array}\right] 2\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right] -\tfrac{1}{2}\text{R2}+\text{R1}$$

Thus, $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

78. Find the inverse of

$$A=\begin{bmatrix}2&3&5\\-2&-3&-5\\1&4&2\end{bmatrix}$$, if it exists. We have

$$\left[A|I_3\right]=\left[\begin{array}{ccc|ccc}2&3&5&1&0&0\\-2&-3&-5&0&1&0\\1&4&2&0&0&1\end{array}\right].$$

Since $\left[\begin{array}{ccc|ccc}2&3&5&1&0&0\\0&0&0&1&1&0\\1&4&2&0&0&1\end{array}\right]$ R1 + R2 has 0 for all the elements of the second row, it will not be possible to complete the required transformations. Therefore, the inverse of the given matrix does not exist.

79. Find the inverse of $A=\begin{bmatrix}2&-1&0\\1&0&1\\1&-2&0\end{bmatrix}$ if it exists. We have

$$\left[A|I_3\right]=\left[\begin{array}{ccc|ccc}2&-1&0&1&0&0\\1&0&1&0&1&0\\1&-2&0&0&0&1\end{array}\right].$$

$$\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\2&-1&0&1&0&0\\1&-2&0&0&0&1\end{array}\right]\begin{matrix}R1\leftrightarrow R2\\ \\ \\ \end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&-1&-2&1&-2&0\\0&-2&-1&0&-1&1\end{array}\right]\begin{matrix}\\-2R1+R2\\-1R1+R3\end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&2&-1&2&0\\0&-2&-1&0&-1&1\end{array}\right]\begin{matrix}\\-1R2\\ \\ \end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&2&-1&2&0\\0&0&3&-2&3&1\end{array}\right]\begin{matrix}\\ \\2R2+R3\end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&2&-1&2&0\\0&0&1&-\frac{2}{3}&1&\frac{1}{3}\end{array}\right]\begin{matrix}\\ \\ \frac{1}{3}R3\end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{2}{3}&0&-\frac{1}{3}\\0&1&0&\frac{1}{3}&0&-\frac{2}{3}\\0&0&1&-\frac{2}{3}&1&\frac{1}{3}\end{array}\right]\begin{matrix}-1R3+R1\\-2R3+R2\\ \\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{2}{3}&0&-\frac{1}{3}\\\frac{1}{3}&0&-\frac{2}{3}\\-\frac{2}{3}&1&\frac{1}{3}\end{bmatrix}$

80. Given $\begin{aligned}2x+y&=5\\3x-2y&=4\end{aligned}$, we have

$$A=\begin{bmatrix}2&1\\3&-2\end{bmatrix},\ X=\begin{bmatrix}x\\y\end{bmatrix},\ \text{and } B=\begin{bmatrix}5\\4\end{bmatrix}.$$

Finding A^{-1}, we have

$$\left[A|I_2\right]=\left[\begin{array}{cc|cc}2&1&1&0\\3&-2&0&1\end{array}\right]\Rightarrow$$

$$\left[\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{2}&0\\3&-2&0&1\end{array}\right]\begin{matrix}\frac{1}{2}R1\\ \\ \end{matrix}\Rightarrow$$

$$\left[\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{2}&0\\0&-\frac{7}{2}&-\frac{3}{2}&1\end{array}\right]\begin{matrix}\\-3R1+R2\end{matrix}\Rightarrow$$

$$\left[\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{2}&0\\0&1&\frac{3}{7}&-\frac{2}{7}\end{array}\right]\begin{matrix}\\-\frac{2}{7}R2\end{matrix}\Rightarrow$$

$$\left[\begin{array}{cc|cc}1&0&\frac{2}{7}&\frac{1}{7}\\0&1&\frac{3}{7}&-\frac{2}{7}\end{array}\right]\begin{matrix}-\frac{1}{2}R2+R1\\ \\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{2}{7}&\frac{1}{7}\\\frac{3}{7}&-\frac{2}{7}\end{bmatrix}$

Finally,

$$X=A^{-1}B=\begin{bmatrix}\frac{2}{7}&\frac{1}{7}\\\frac{3}{7}&-\frac{2}{7}\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}\frac{2}{7}(5)+\frac{1}{7}(4)\\\frac{3}{7}(5)+\left(-\frac{2}{7}\right)(4)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{10}{7}+\frac{4}{7}\\\frac{15}{7}+\left(-\frac{8}{7}\right)\end{bmatrix}=\begin{bmatrix}\frac{14}{7}\\\frac{7}{7}\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

Solution set: $\{(2,1)\}$

81. Given $\begin{aligned}3x+2y+\ \ z&=-5\\x-\ \ y+3z&=-5\\2x+3y+\ \ z&=0\end{aligned}$ we have

$$A=\begin{bmatrix}3&2&1\\1&-1&3\\2&3&1\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix},\ \text{and } B=\begin{bmatrix}-5\\-5\\0\end{bmatrix}.$$

Finding A^{-1}, we have

$$\left[A|I_3\right]=\left[\begin{array}{ccc|ccc}3&2&1&1&0&0\\1&-1&3&0&1&0\\2&3&1&0&0&1\end{array}\right]\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&-1&3&0&1&0\\3&2&1&1&0&0\\2&3&1&0&0&1\end{array}\right]\begin{matrix}R1\leftrightarrow R2\\ \\ \\ \end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&-1&3&0&1&0\\0&5&-8&1&-3&0\\0&5&-5&0&-2&1\end{array}\right]\begin{matrix}\\-3R1+R2\\-2R1+R2\end{matrix}\Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -\frac{8}{5} & \frac{1}{5} & -\frac{3}{5} & 0 \\ 0 & 5 & -5 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{5}\text{R2} \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{8}{5} & \frac{1}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} \text{R2}+\text{R1} \\ \\ -5\text{R2}+\text{R3} \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{8}{5} & \frac{1}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{3}\text{R3} \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{15} & -\frac{7}{15} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{15} & \frac{8}{15} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{l} -\frac{7}{5}\text{R3}+\text{R1} \\ \frac{8}{5}\text{R3}+\text{R2} \\ \\ \end{array}$$

Thus, $A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{15} & -\frac{7}{15} \\ -\frac{1}{3} & -\frac{1}{15} & \frac{8}{15} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

Finally, we need to find $X = A^{-1}B$.

$$X = A^{-1}B = \begin{bmatrix} \frac{2}{3} & -\frac{1}{15} & -\frac{7}{15} \\ -\frac{1}{3} & -\frac{1}{15} & \frac{8}{15} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}(-5)+\left(-\frac{1}{15}\right)(-5)+\left(-\frac{7}{15}\right)(0) \\ -\frac{1}{3}(-5)+\left(-\frac{1}{15}\right)(-5)+\frac{8}{15}(0) \\ -\frac{1}{3}(-5)+\frac{1}{3}(-5)+\frac{1}{3}(0) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{10}{3}+\frac{1}{3}+0 \\ \frac{5}{3}+\frac{1}{3}+0 \\ \frac{5}{3}+\left(-\frac{5}{3}\right)+0 \end{bmatrix} = \begin{bmatrix} -\frac{9}{3} \\ \frac{6}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

Solution set: $\{(-3,2,0)\}$

82. Given $\begin{aligned} x+y+z&=1 \\ 2x-y&=-2 \\ 3y+z&=2 \end{aligned}$ we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \text{and } B = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Finding A^{-1}, we have

$$\left[A \middle| I_3\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2\text{R1}+\text{R2} \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -\frac{1}{3}\text{R2} \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right] \begin{array}{l} -1\text{R2}+\text{R1} \\ \\ -3\text{R2}+\text{R3} \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right] \begin{array}{l} \\ \\ -1\text{R3} \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right] \begin{array}{l} -\frac{1}{3}\text{R3}+\text{R1} \\ -\frac{2}{3}\text{R3}+\text{R2} \\ \\ \end{array}$$

Thus, $A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 2 & -1 & -1 \end{bmatrix}$. Finally, we need to find $X = A^{-1}B$.

$$X = A^{-1}B = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3}(1)+\frac{2}{3}(-2)+\frac{1}{3}(2) \\ -\frac{2}{3}(1)+\frac{1}{3}(-2)+\frac{2}{3}(2) \\ 2(1)+(-1)(-2)+(-1)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3}+\left(-\frac{4}{3}\right)+\frac{2}{3} \\ -\frac{2}{3}+\left(-\frac{2}{3}\right)+\frac{4}{3} \\ 2+2+(-2) \end{bmatrix} = \begin{bmatrix} -\frac{3}{3} \\ \frac{0}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Solution set: $\{(-1,0,2)\}$

Chapter 9: Test

1. $3x-y=9 \quad (1)$

$x+2y=10 \quad (2)$

Solving equation (2) for x, we obtain $x = 10 - 2y$ (3). Substituting this result into equation (1) and solving for y, we obtain

$$3(10-2y)-y=9 \Rightarrow 30-6y-y=9 \Rightarrow$$
$$-7y=-21 \Rightarrow y=3$$

Substituting $y = 3$ back into equation (3) in order to find x, we obtain

$x = 10-2(3) = 10-6 = 4.$

Solution set: $\{(4,3)\}$

2. $6x + 9y = -21 \quad (1)$

$4x + 6y = -14 \quad (2)$

Solving equation (1) for x, we obtain

$6x = -9y - 21 \Rightarrow x = \frac{-9y-21}{6} \Rightarrow x = \frac{-3y-7}{2}$ (3).

Substituting this result into equation (2) and solving for y, we obtain the following.

$$4\left(\frac{-3y-7}{2}\right) + 6y = -14$$
$$2(-3y-7) + 6y = -14$$
$$-6y - 14 + 6y = -14 \Rightarrow -14 = -14$$

This true statement implies there are infinitely many solutions. We can represent the solution set with y as the arbitrary variable.

Solution set: $\left\{\left(\frac{-3y-7}{2}, y\right)\right\}$

3. $\frac{1}{4}x - \frac{1}{3}y = -\frac{5}{12} \quad (1)$

$\frac{1}{10}x + \frac{1}{5}y = \frac{1}{2} \quad (2)$

To eliminate fractions, multiply equation (1) by 12 and equation (2) by 10.

$3x - 4y = -5 \quad (3)$

$x + 2y = 5 \quad (4)$

Multiply equation (4) by 2 and add the result ;to equation (3).

$$\begin{array}{l} 3x - 4y = -5 \\ \underline{2x + 4y = 10} \\ 5x \qquad = 5 \Rightarrow x = 1 \end{array}$$

Substituting $x = 1$ in equation (4) to find y, we obtain $1 + 2y = 5 \Rightarrow 2y = 4 \Rightarrow y = 2$.

Solution set: $\{(1,2)\}$

4. $x - 2y = 4 \quad (1)$

$-2x + 4y = 6 \quad (2)$

Multiply equation (1) by 2 and add the result to equation (2).

$$\begin{array}{l} 2x - 4y = 8 \\ \underline{-2x + 4y = 6} \\ \qquad 0 = 14 \end{array}$$

The system is inconsistent. The solution set is $\emptyset$.

5. $2x + y + z = 3 \quad (1)$

$x + 2y - z = 3 \quad (2)$

$3x - y + z = 5 \quad (3)$

Eliminate z first by adding equations (1) and (2) to obtain $3x + 3y = 6$ (4). Add equations (2) and (3) to eliminate z and obtain $4x + y = 8$ (5). Multiply equation (5) by –3 and add the results to equation (4).

$$\begin{array}{l} 3x + 3y = \ \ 6 \\ \underline{-12x - 3y = -24} \\ -9x \qquad = -18 \Rightarrow x = 2 \end{array}$$

Substituting $x = 2$ in equation (5) to find y, we have $4(2) + y = 8 \Rightarrow 8 + y = 8 \Rightarrow y = 0$.

Substituting $x = 2$ and $y = 0$ in equation (1) to find z, we have $2(2) + 0 + z = 3 \Rightarrow z = -1$.

Solution set: $\{(2,0,-1)\}$

6. Writing $\begin{array}{l} 3x - 2y = 13 \\ 4x - \ \ y = 19 \end{array}$ as an augmented matrix, we have $\left[\begin{array}{cc|c} 3 & -2 & 13 \\ 4 & -1 & 19 \end{array}\right]$.

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{13}{3} \\ 4 & -1 & 19 \end{array}\right] \tfrac{1}{3}\text{R1} \Rightarrow$$
$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{13}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{array}\right] -4\text{R1} + \text{R2} \Rightarrow$$
$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{13}{3} \\ 0 & 1 & 1 \end{array}\right] \tfrac{3}{5}\text{R2} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \end{array}\right] \tfrac{2}{3}\text{R2} + \text{R1}$$

Solution set $\{(5,1)\}$

7. Writing $\begin{array}{l} 3x - 4y + 2z = 15 \\ 2x - \ \ y + \ \ z = 13 \\ \ \ x + 2y - \ \ z = 5 \end{array}$ as an augmented matrix, we have $\left[\begin{array}{ccc|c} 3 & -4 & 2 & 15 \\ 2 & -1 & 1 & 13 \\ 1 & 2 & -1 & 5 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 1 & 13 \\ 3 & -4 & 2 & 15 \end{array}\right] \text{R1} \leftrightarrow \text{R3} \Rightarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 3 & 3 \\ 0 & -10 & 5 & 0 \end{array}\right] \begin{array}{l} \\ -2\text{R1} + \text{R2} \\ -3\text{R1} + \text{R3} \end{array} \Rightarrow$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -\frac{3}{5} & -\frac{3}{5} \\ 0 & -10 & 5 & 0 \end{array}\right] -\tfrac{1}{5}\text{R2} \Rightarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & \frac{31}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & -1 & -6 \end{array}\right] \begin{array}{l} -2\text{R2} + \text{R1} \\ \\ 10\text{R2} + \text{R3} \end{array} \Rightarrow$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & \frac{31}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & 6 \end{array}\right] -1\text{R3} \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} -\frac{1}{5}\text{R3}+\text{R1} \\ \frac{3}{5}\text{R3}+\text{R2} \\ \ \end{array}$$

Solution set: $\{(5,3,6)\}$

8. Since $y = ax^2 + bx + c$, and the points $(1,5),(2,3),$ and $(4,11)$ are on the graph, we have the following equations which are then simplified.

$$5 = a(1)^2 + b(1) + c \Rightarrow \quad a + b + c = 5 \quad (1)$$
$$3 = a(2)^2 + b(2) + c \Rightarrow \quad 4a + 2b + c = 3 \quad (2)$$
$$11 = a(4)^2 + b(4) + c \Rightarrow 16a + 4b + c = 11 \quad (3)$$

First, eliminate c by adding –1 times equation (1) to equation (2).

$$\begin{array}{l} -a - b - c = -5 \\ \underline{4a + 2b + c = \ \ 3} \\ 3a + b \qquad = -2 \ (4) \end{array}$$

Eliminating c again by adding –1 times equation (1) to equation (3), we have the following.

$$\begin{array}{l} -a - b - c = -5 \\ \underline{16a + 4b + c = 11} \\ 15a + 3b \qquad = \ 6 \ (5) \end{array}$$

Next, eliminate b by adding –3 times equation (4) to equation (5).

$$\begin{array}{l} -9a - 3b = 6 \\ \underline{15a + 3b = 6} \\ 6a \qquad = 12 \Rightarrow a = 2 \end{array}$$

Substituting this value into equation (4), we obtain $3(2) + b = -2 \Rightarrow 6 + b = -2 \Rightarrow b = -8.$

Substituting 2 for a and –8 for b into equation (1), we have $2 + (-8) + c = 5 \Rightarrow$
$-6 + c = 5 \Rightarrow c = 11.$

The equation of the parabola is
$y = 2x^2 - 8x + 11.$

9. Let x = number of units from Toronto;
y = number of units from Montreal;
z = number of units from Ottawa.
The information in the problem gives the system

$$\begin{aligned} x + y + z &= 100 \\ 80x + 50y + 65z &= 5990 \\ x &= z \end{aligned}$$

Multiply the first equation by –50 and add to the second equation.

$$\begin{array}{l} -50x - 50y - 50z = -5000 \\ \underline{\ 80x + 50y + 65z = \ \ 5990} \\ \ 30x \qquad\quad + 15z = \quad 990 \end{array}$$

Substitute x for z in this equation to obtain
$30x + 15x = 990 \Rightarrow 45x = 990 \Rightarrow x = 22.$
If $x = 22$, then $z = 22$. Substitute 22 for x and for z in the first equation and solve for y:
$22 + y + 22 = 100 \Rightarrow y = 56$

22 units from Toronto, 56 units from Montreal, and 22 units from Ottawa were ordered.

10. $\begin{vmatrix} 6 & 8 \\ 2 & -7 \end{vmatrix} = 6(-7) - 2(8) = -58$

11. $\begin{vmatrix} 2 & 0 & 8 \\ -1 & 7 & 9 \\ 12 & 5 & -3 \end{vmatrix}$

This determinant may be evaluated by expanding about any row or any column. If we expand by the first row, we have the following.

$$\begin{vmatrix} 2 & 0 & 8 \\ -1 & 7 & 9 \\ 12 & 5 & -3 \end{vmatrix} = 2\begin{vmatrix} 7 & 9 \\ 5 & -3 \end{vmatrix} - 0\begin{vmatrix} -1 & 9 \\ 12 & -3 \end{vmatrix} + 8\begin{vmatrix} -1 & 7 \\ 12 & 5 \end{vmatrix}$$
$$= 2[7(-3) - 5(9)] - 0$$
$$+ 8[(-1)(5) - 12(7)]$$
$$= 2(-21 - 45) + 8(-5 - 84)$$
$$= 2(-66) + 8(-89)$$
$$= -132 - 712 = -844$$

12. $2x - 3y = -33$
$4x + 5y = 11$

$$D = \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 2(5) - 4(-3) = 10 - (-12) = 22$$
$$D_x = \begin{vmatrix} -33 & -3 \\ 11 & 5 \end{vmatrix} = -33(5) - 11(-3)$$
$$= -165 - (-33) = -132$$
$$D_y = \begin{vmatrix} 2 & -33 \\ 4 & 11 \end{vmatrix} = 2(11) - 4(-33)$$
$$= 22 - (-132) = 154$$
$$x = \frac{D_x}{D} = \frac{-132}{22} = -6; \ y = \frac{D_y}{D} = \frac{154}{22} = 7$$

Solution set: $\{(-6,7)\}$

13. $x+y-z=-4$
$2x-3y-z=5$
$x+2y+2z=3$

Adding column 3 to columns 1 and 2, we have

$$D=\begin{vmatrix}1&1&-1\\2&-3&-1\\1&2&2\end{vmatrix}=\begin{vmatrix}0&0&-1\\1&-4&-1\\3&4&2\end{vmatrix}.$$

Expanding by row one, we have

$$D=-1\begin{vmatrix}1&-4\\3&4\end{vmatrix}=-\left[4-(-12)\right]=-16.$$

Adding –1 times row 1 to row 2 and 2 times row 1 to row 3, we have the following.

$$D_x=\begin{vmatrix}-4&1&-1\\5&-3&-1\\3&2&2\end{vmatrix}=\begin{vmatrix}-4&1&-1\\9&-4&0\\-5&4&0\end{vmatrix}$$

Expanding by column three, we have

$$D_x=-1\begin{vmatrix}9&-4\\-5&4\end{vmatrix}=-(36-20)=-16.$$

Adding –1 times row 1 to row 2 and 2 times row 1 to row 3, we have the following.

$$D_y=\begin{vmatrix}1&-4&-1\\2&5&-1\\1&3&2\end{vmatrix}=\begin{vmatrix}1&-4&-1\\1&9&0\\3&-5&0\end{vmatrix}.$$

Expanding by column three, we have

$$D_y=-1\begin{vmatrix}1&9\\3&-5\end{vmatrix}=-1(-5-27)=-(-32)=32.$$

Adding –2 times row 1 to row 2 and -1 times row 1 to row 3, we have

$$D_z=\begin{vmatrix}1&1&-4\\2&-3&5\\1&2&3\end{vmatrix}=\begin{vmatrix}1&1&-4\\0&-5&13\\0&1&7\end{vmatrix}.$$

Expanding by column one, we have

$$D_z=1\begin{vmatrix}-5&13\\1&7\end{vmatrix}=1(-35-13)=-48.$$

Thus we have

$x=\frac{D_x}{D}=\frac{-16}{-16}=1,\ y=\frac{D_y}{D}=\frac{32}{-16}=-2,$
and $z=\frac{D_z}{D}=\frac{-48}{-16}=3.$

Solution set: $\{(1,-2,3)\}$

14. $$\frac{x+2}{x^3+x^2+x}=\frac{x+2}{x\left(x^2+x+1\right)}=\frac{x+2}{x(x+1)^2}$$
$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

Multiply both sides by $x(x+1)^2$ to get

$x+2=A(x+1)^2+Bx(x+1)+Cx$ (1).

Substituting 0 for x in equation (1), we get

$0+2=A(0+1)^2+B(0)(0+1)+C(0)\Rightarrow A=2$

Substituting 2 for A and –1 for x in equation (1), we get the following.

$-1+2=2(-1+1)^2+B(-1)(-1+1)+C(-1)$
$1=-C\Rightarrow C=-1$

We now have

$x+2=2(x+1)^2+Bx(x+1)+(-1)x$ (2).

Now substitute 1 for x (arbitrary choice) in equation (2) to obtain the following.

$1+2=2(1+1)^2+B(1)(1+1)+(-1)(1)$
$3=2(4)+2B-1$
$3=8+2B-1\Rightarrow 3=7+2B$
$-4=2B\Rightarrow B=-2$

Thus, $$\frac{x+2}{x^3+x^2+x}=\frac{x+2}{x\left(x^2+x+1\right)}=\frac{x+2}{x(x+1)^2}$$
$$=\frac{2}{x}+\frac{-2}{x+1}+\frac{-1}{(x+1)^2}.$$

15. The parabola passes through (0, 4), (1, 1), and (2, 0), so those points satisfy the equation $y=ax^2+bx+c$. Substituting each ordered pair into the equation gives the system

$4=a(0)^2+b(0)+c$
$1=a(1)^2+b(1)+c$
$0=a(2)^2+b(2)+c$

which simplifies to

$4=\quad c$ (1)
$1=a+b+c$ (2)
$0=4a+2b+c$ (3)

Substitute $c=4$ into equations (2) and (3), then solve the system consisting of those two equations for a and b:

$1=a+b+4$
$0=4a+2b+4$ $\Rightarrow$
$-3=a+b$ (4)
$-4=4a+2b$ (5)

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From equation (4), we have $a = -b - 3$, so

$-4 = 4(-b-3) + 2b$

$-4 = -4b - 12 + 2b \Rightarrow 8 = -2b \Rightarrow -4 = b$

$a = -b - 3 \Rightarrow a = -(-4) - 3 \Rightarrow a = 1$

Thus, the equation of the parabola is

$y = x^2 - 4x + 4$

The line passes through (−2, 0) and (1, 1), so $m = \frac{1-0}{1-(-2)} = \frac{1}{3}$, and the equation is

$y - 0 = \frac{1}{3}(x+2) \Rightarrow 3y = x + 2 \Rightarrow x - 3y = -2$.

The system is $\begin{aligned} y &= x^2 - 4x + 4 \\ x - 3y &= -2 \end{aligned}$.

16. $2x^2 + y^2 = 6 \quad (1)$

$x^2 - 4y^2 = -15 \quad (2)$

Multiply equation (1) by 4 and add to equation (2), then solve for x.

$$\begin{aligned} 8x^2 + 4y^2 &= 24 \\ x^2 - 4y^2 &= -15 \\ \hline 9x^2 &= 9 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \end{aligned}$$

Substitute these values into equation (1) and solve for y.

If $x = 1$, then $2(1)^2 + y^2 = 6 \Rightarrow 2 + y^2 = 6 \Rightarrow$

$y^2 = 4 \Rightarrow y = \pm 2$.

If $x = -1$, then $2(-1)^2 + y^2 = 6 \Rightarrow$

$2 + y^2 = 6 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$.

Solution set: $\{(1,2), (-1,2), (1,-2), (-1,-2)\}$

17. $x^2 + y^2 = 25 \quad (1)$

$x + y = 7 \quad (2)$

Solving equation (2) for x, we have $x = 7 - y$ (3). Substituting this result into equation (1), we have

$$\begin{aligned} (7-y)^2 + y^2 &= 25 \\ 49 - 14y + y^2 + y^2 &= 25 \\ 2y^2 - 14y + 24 &= 0 \\ y^2 - 7y + 12 &= 0 \\ (y-3)(y-4) &= 0 \Rightarrow y = 3 \text{ or } y = 4 \end{aligned}$$

Substitute these values into equation (3) and solve for x. If $y = 3$, $x = 7 - 3 = 4$. If $y = 4$, $x = 7 - 4 = 3$.

Solution set: $\{(3,4), (4,3)\}$

18. Let x and y be the numbers.

$x + y = -1 \quad (1)$

$x^2 + y^2 = 61 \quad (2)$

Solving equation (1) for y, we have $y = -x - 1$ (3). Substituting this result into equation (2), we have

$x^2 + (-x-1)^2 = 61 \Rightarrow x^2 + x^2 + 2x + 1 = 61$

$2x^2 + 2x - 60 = 0 \Rightarrow x^2 + x - 30 = 0$

$(x+6)(x-5) = 0 \Rightarrow x = -6$ or $x = 5$

Substitute these values in equation (3) to find the corresponding values of y.

If $x = -6$, $y = -(-6) - 1 = 6 - 1 = 5$.

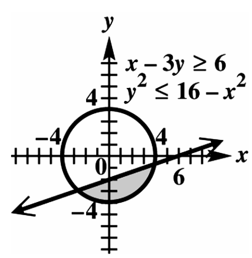
If $x = 5$, $y = -5 - 1 = -6$.

The same pair of numbers results from both cases. The numbers are 5 and –6.

19. $x - 3y \ge 6$

$y^2 \le 16 - x^2$

Graph $x - 3y = 6$ as a solid line with x-intercept 6 and y-intercept of –2. Because the test point $(0,0)$ does not satisfies this inequality, shade the region *below* the line. Graph $y^2 = 16 - x^2$ or $x^2 + y^2 = 16$ as a solid circle with a center at the origin and radius 4. Shade the region, which is the interior of the circle. The solution set is the intersection of these two regions, which is the region shaded in the final graph.

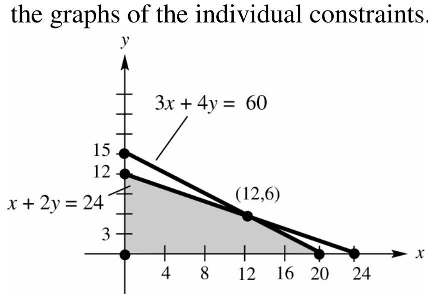


20. Find $x \geq 0$ and $y \geq 0$ such that

$$x + 2y \leq 24$$
$$3x + 4y \leq 60$$

and $2x + 3y$ is maximized.

Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $x + 2y \leq 24,$ draw the line with x-intercept 24 and y-intercept 12 as a solid line. Because the test point $(0, 0)$ satisfies this inequality, shade the region *below* the line. To graph $3x + 4y \leq 60,$ draw the line with x-intercept 20 and y-intercept 15 as a solid line. Because the test point $(0, 0)$ satisfies this inequality, shade the region *below* the line. The constraints $x \geq 0$ and $y \geq 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three vertices are $(0, 0), (0, 12),$ and $(20, 0).$ A fourth vertex is the intersection point of the lines $x + y = 24$ and $3x + 4y = 60.$ To find this point, solve the system
$$\begin{aligned} x + 2y &= 24 \quad (1) \\ 3x + 4y &= 60 \quad (2) \end{aligned}.$$

To eliminate x, multiply equation (1) by –3 and add the result to equation (2).

$$\begin{aligned} -3x - 6y &= -72 \\ 3x + 4y &= 60 \\ \hline -2y &= -12 \Rightarrow y = 6 \end{aligned}$$

Substituting this value into equation (1), we obtain $x + 2(6) = 24 \Rightarrow x + 12 = 24 \Rightarrow x = 12.$

The fourth vertex is $(12, 6).$ Next, evaluate the objective function at each vertex.

Point	Profit $= 2x + 3y.$	
$(0, 0)$	$2(0) + 3(0) = 0$	
$(0, 12)$	$2(0) + 3(12) = 36$	
$(20, 0)$	$2(20) + 3(0) = 40$	
$(12, 6)$	$2(12) + 3(6) = 42$	← Maximum

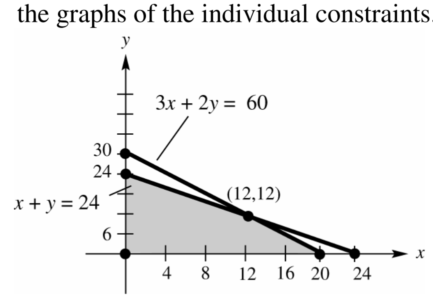
The maximum value is 42 at (12, 6).

21. Let x = the number of VIP rings;
y = the number of SST rings.

The constraints are $x + y \leq 24$
$$3x + 2y \leq 60$$
$$x \geq 0,\ y \geq 0.$$

Maximize profit function, profit $= 30x + 40y.$

Find the region of feasible solutions by graphing the system of inequalities that is made up of the constraints. To graph $x + y \leq 24,$ draw the line with x-intercept 24 and y-intercept 24 as a solid line. Because the test point $(0, 0)$ satisfies this inequality, shade the region *below* the line. To graph $3x + 2y \leq 60,$ draw the line with x-intercept 20 and y-intercept 30 as a solid line. Because the test point $(0, 0)$ satisfies this inequality, shade the region *below* the line. The constraints $x \geq 0$ and $y \geq 0$ restrict the graph to the first quadrant. The graph of the feasible region is the intersection of the regions that are the graphs of the individual constraints.



From the graph, observe that three vertices are $(0, 0), (20, 0),$ and $(0, 24).$ A fourth vertex is the intersection point of the lines $x + y = 24$ and $3x + 2y = 60.$ To find this point, solve the system
$$\begin{aligned} x + y &= 24 \\ 3x + 2y &= 60. \end{aligned}$$

The first equation can be written as $x = 24 - y.$ Substituting this equation into $3x + 2y = 60,$ we have the following.

$$\begin{aligned} 3(24 - y) + \ y = 60 &\Rightarrow 72 - 3y + 2y = 60 \Rightarrow \\ 72 - y = 60 &\Rightarrow -y = -12 \Rightarrow y = 12 \end{aligned}$$

Substituting $y = 12$ into $x = 24 - y,$ we have $x = 24 - 12 = 12.$ Thus, the four vertex is $(12, 12).$ Next, evaluate the objective function at each vertex.

Point	Profit $= 30x + 40y.$	
$(0,0)$	$30(0)+40(0)=0$	
$(20,0)$	$30(20)+40(0)=600$	
$(0,24)$	$30(0)+40(24)=960$	← Maximum
$(12,12)$	$30(12)+40(12)=840$	

0 VIP rings and 24 SST rings should be made daily for a daily profit of \$960.

22. $\begin{bmatrix} 5 & x+6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} y-2 & 4-x \\ 0 & w+7 \end{bmatrix}$

All corresponding elements, position by position, of the two matrices must be equal.

$x+6 = 4-x \Rightarrow 2x = -2 \Rightarrow x = -1;$

$5 = y-2 \Rightarrow y = 7; 4 = w+7 \Rightarrow w = -3$

Thus, $x = -1$, $y = 7$, and $w = -3$.

23. $3\begin{bmatrix} 2 & 3 \\ 1 & -4 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 3(2) & 3(3) \\ 3(1) & 3(-4) \\ 3(5) & 3(9) \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ 3 & -12 \\ 15 & 27 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ -3 & 1 \\ 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 9+(-6) \\ 3+(-3) & -12+1 \\ 15+0 & 27+(-8) \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 0 & -11 \\ 15 & 19 \end{bmatrix}$$

24. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -7 & 5 \end{bmatrix}$

The first two matrices are 2 × 1 and the third is 2 × 2. Only matrices of the same size can be added, so it is not possible to find this sum.

25. The product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix.

$$\begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2(1)+1(2)+(-3)(3) & 2(3)+1(4)+(-3)(-2) \\ 4(1)+0(2)+5(3) & 4(3)+0(4)+5(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+(-9) & 6+4+6 \\ 4+0+15 & 12+0+(-10) \end{bmatrix} = \begin{bmatrix} -5 & 16 \\ 19 & 2 \end{bmatrix}$$

26. $\begin{bmatrix} 2 & -4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$

The first matrix is 2 × 2 and the second is 3 × 1. The first matrix has two columns and the second has three rows, so it is not possible to find this product.

27. There are associative, distributive, and identity properties that apply to multiplication of matrices, but matrix multiplication is not commutative. The correct choice is A.

28. Find the inverse of $A = \begin{bmatrix} -8 & 5 \\ 3 & -2 \end{bmatrix}$, if it exists.

The augmented matrix is

$$[A|I_2] = \left[\begin{array}{cc|cc} -8 & 5 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right].$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{5}{8} & -\frac{1}{8} & 0 \\ 3 & -2 & 0 & 1 \end{array}\right] -\tfrac{1}{8}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{5}{8} & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{8} & \frac{3}{8} & 1 \end{array}\right] -3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{5}{8} & -\frac{1}{8} & 0 \\ 0 & 1 & -3 & -8 \end{array}\right] -8\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & -5 \\ 0 & 1 & -3 & -8 \end{array}\right] \tfrac{5}{8}\text{R2}+\text{R1}$$

Thus, $A^{-1} = \begin{bmatrix} -2 & -5 \\ -3 & -8 \end{bmatrix}$.

29. Find the inverse of $A=\begin{bmatrix}4 & 12\\ 2 & 6\end{bmatrix}$, if it exists.

The augmented matrix is

$$[A|I_2]=\left[\begin{array}{cc|cc}4 & 12 & 1 & 0\\ 2 & 6 & 0 & 1\end{array}\right].$$

$$\left[\begin{array}{cc|cc}1 & 3 & \frac{1}{4} & 0\\ 2 & 6 & 0 & 1\end{array}\right]\tfrac{1}{4}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|cc}1 & 3 & \frac{1}{4} & 0\\ 0 & 0 & -\frac{1}{2} & 1\end{array}\right]-2\text{R1}+\text{R2}$$

The second row, second column element is now 0, so the desired transformation cannot be completed. Therefore, the inverse of the given matrix does not exist.

30. Find the inverse of $A=\begin{bmatrix}1 & 3 & 4\\ 2 & 7 & 8\\ -2 & -5 & -7\end{bmatrix}$, if it exists. Performing row operations on the augmented matrix, we have the following.

$$[A|I_3]=\left[\begin{array}{ccc|ccc}1 & 3 & 4 & 1 & 0 & 0\\ 2 & 7 & 8 & 0 & 1 & 0\\ -2 & -5 & -7 & 0 & 0 & 1\end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1 & 3 & 4 & 1 & 0 & 0\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 1 & 1 & 2 & 0 & 1\end{array}\right]\begin{matrix}\\ -2\text{R1}+\text{R2}\\ 2\text{R1}+\text{R3}\end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 4 & 7 & -3 & 0\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & 4 & -1 & 1\end{array}\right]\begin{matrix}-3\text{R2}+\text{R1}\\ \\ -1\text{R2}+\text{R3}\end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -9 & 1 & -4\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & 4 & -1 & 1\end{array}\right]\begin{matrix}-4\text{R3}+\text{R1}\\ \\ \\ \end{matrix}$$

Thus, $A^{-1}=\begin{bmatrix}-9 & 1 & -4\\ -2 & 1 & 0\\ 4 & -1 & 1\end{bmatrix}$.

31. The system $\begin{matrix}2x+y=-6\\ 3x-y=-29\end{matrix}$ yields the matrix equation $AX = B$ where

$A=\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, and $B=\begin{bmatrix}-6\\ -29\end{bmatrix}$.

Find A^{-1}. The augmented matrix is

$$[A|I_2]=\left[\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 3 & -1 & 0 & 1\end{array}\right].$$

$$\left[\begin{array}{cc|cc}1 & \frac{1}{2} & \frac{1}{2} & 0\\ 3 & -1 & 0 & 1\end{array}\right]\tfrac{1}{2}\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|cc}1 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & -\frac{5}{2} & -\frac{3}{2} & 1\end{array}\right]-3\text{R1}+\text{R2} \Rightarrow$$

$$\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{5} & \frac{1}{5}\\ 0 & -\frac{5}{2} & -\frac{3}{2} & 1\end{array}\right]\tfrac{1}{5}\text{R2}+\text{R1} \Rightarrow$$

$$\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{5} & \frac{1}{5}\\ 0 & 1 & \frac{3}{5} & -\frac{2}{5}\end{array}\right]-\tfrac{2}{5}\text{R2}$$

Thus, $A^{-1}=\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ \frac{3}{5} & -\frac{2}{5}\end{bmatrix}$. Since $X=A^{-1}B$, we have the following.

$$A^{-1}B=\begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ \frac{3}{5} & -\frac{2}{5}\end{bmatrix}\begin{bmatrix}-6\\ -29\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{5}(-6)+\frac{1}{5}(-29)\\ \frac{3}{5}(-6)+\left(-\frac{2}{5}\right)(-29)\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{6}{5}+\left(-\frac{29}{5}\right)\\ -\frac{18}{5}+\frac{58}{5}\end{bmatrix}=\begin{bmatrix}-\frac{35}{5}\\ \frac{40}{5}\end{bmatrix}=\begin{bmatrix}-7\\ 8\end{bmatrix}$$

Solution set: $\{(-7,8)\}$

32. The system $\begin{matrix}x+y=5\\ y-2z=23\\ x+3z=-27\end{matrix}$ yields the matrix equation $AX = B$ where $A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & -2\\ 1 & 0 & 3\end{bmatrix}$,

$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, and $B=\begin{bmatrix}5\\ 23\\ -27\end{bmatrix}$. Find A^{-1}. The augmented matrix is

$$[A|I_3]=\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & 1 & 0\\ 1 & 0 & 3 & 0 & 0 & 1\end{array}\right].$$

$$\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & 1 & 0\\ 0 & -1 & 3 & -1 & 0 & 1\end{array}\right]\begin{matrix}\\ \\ -\text{R1}+\text{R3}\end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & -1 & 0\\ 0 & 1 & -2 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 1 & 1\end{array}\right]\begin{matrix}-1\text{R2}+\text{R1}\\ \\ \text{R2}+\text{R3}\end{matrix} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -3 & -2\\ 0 & 1 & 0 & -2 & 3 & 2\\ 0 & 0 & 1 & -1 & 1 & 1\end{array}\right]\begin{matrix}-2\text{R3}+\text{R1}\\ 2\text{R3}+\text{R2}\\ \\ \end{matrix}$$

Thus, $A^{-1} = \begin{bmatrix} 3 & -3 & -2 \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$. Since

$X = A^{-1}B$, we have the following.

$$A^{-1}B = \begin{bmatrix} 3 & -3 & -2 \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 23 \\ -27 \end{bmatrix}$$
$$= \begin{bmatrix} 3(5)+(-3)(23)+(-2)(-27) \\ -2(5)+3(23)+2(-27) \\ -1(5)+1(23)+1(-27) \end{bmatrix}$$
$$= \begin{bmatrix} 15+(-69)+54 \\ -10+(69)+(-54) \\ -5+23+(-27) \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -9 \end{bmatrix} = X$$

Solution set: $\{(0,5,-9)\}$

Chapter 9: Quantitative Reasoning

1. Solve the system $\begin{matrix} y = 800 + 25x \ (1) \\ y = 700 + 85x \ (2) \end{matrix}$. Substitute $800 + 25x$ for y in equation (2).
$800 + 25x = 700 + 85x \Rightarrow 800 = 700 + 60x \Rightarrow$
$100 = 60x \Rightarrow x = \frac{100}{60} = \frac{5}{3} = 1\frac{2}{3}$ years

2. \$1400 = lump sum payment after one year.
\$1000 + .06(\$1000) = \$1060 = principal plus one year's interest.
The \$1400 lump sum payment in one year is the better deal, by \$1400 – \$1060 = \$340.

Chapter 10

ANALYTIC GEOMETRY

Section 10.1: Parabolas

1. **(a)** The relation $y-2=(x+4)^2$ has vertex $(-4,2)$ and opens up, so the correct choice is D.

(b) The relation $y-4=(x+2)^2$ has vertex $(-2,4)$ and opens up, so the correct choice is B.

(c) The relation $y-2=-(x+4)^2$ has vertex $(-4,2)$ and opens down, so the correct choice is C.

(d) The relation $y-4=-(x+2)^2$ has vertex $(-2,4)$ and opens down, so the correct choice is A.

(e) The relation $x-2=(y+4)^2$ has vertex $(2,-4)$ and opens to the right, so the correct choice is F.

(f) The relation $x-4=(y+2)^2$ has vertex $(4,-2)$ and opens to the right, so the correct choice is H.

(g) The relation $x-2=-(y+4)^2$ has vertex $(2,-4)$ and opens to the left, so the correct choice is E.

(h) The relation $x-4=-(y+2)^2$ has vertex $(4,-2)$ and opens to the left, so the correct choice is G.

2. **(a)** Since the variable *x* is squared, we know that the parabola will either open up or down. Since *y* is isolated and the coefficient of the x^2- term is positive, we know the parabola must open up. Thus, the correct answer is B.

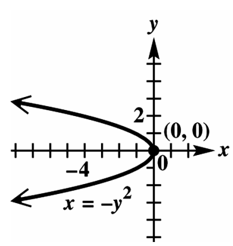
(b) Since the variable *x* is squared, we know that the parabola will either open up or down. Since *y* is isolated and the coefficient of the x^2- term is negative, we know the parabola must open down. Thus, the correct answer is D.

(c) Since the variable *y* is squared, we know that the parabola will either open to the left or to the right. Since *x* is isolated and the coefficient of the y^2- term is positive, we know the parabola must open to the right. Thus, the correct answer is A.

(d) Since the variable *y* is squared, we know that the parabola will either open to the left or to the right. Since *x* is isolated and the coefficient of the y^2- term is negative, we know the parabola must open to the left. Thus, the correct answer is C.

3. $-x=y^2 \Rightarrow x=-y^2 \Rightarrow$
$x=-1(y-0)^2+0 \Rightarrow x-0=-1(y-0)^2$
The vertex is $(0,0)$. The graph opens to the left and has the same shape as $x=y^2$. The domain is $(-\infty,0]$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=0$ (the *x*-axis). Use the vertex and axis and plot a few additional points.

x	y
-4	2
-1	1
0	0
-1	-1
-4	-2

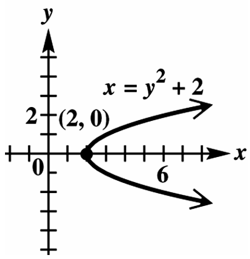


4. $x-2=y^2 \Rightarrow x=y^2+2 \Rightarrow$
$x=(y-0)^2+2 \Rightarrow x-2=(y-0)^2$
The vertex is $(2,0)$. The graph opens to the right and has the same shape as $x=y^2$. It is a translation 2 units to the right of the graph of $x=y^2$. The domain is $[2,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=0$ (the *x*-axis). Use the vertex and axis and plot a few additional points.

(*continued on next page*)

(*continued from page 1027*)

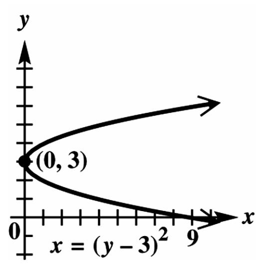
x	y
6	2
3	1
2	0
3	−1
6	−2



5. $x=(y-3)^2 \Rightarrow x-0=(y-3)^2$

The vertex is $(0,3)$. The graph opens to the right and has the same shape as $x=y^2$. It is a translation 3 units up of the graph of $x=y^2$. The domain is $[0,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=3$. Use the vertex and axis and plot a few additional points.

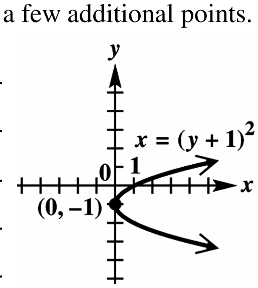
x	y
4	1
1	2
0	3
1	4
4	5



6. $x=(y+1)^2 \Rightarrow x-0=[y-(-1)]^2$

The vertex is $(0,-1)$. The graph opens to the right and has the same shape as $x=y^2$. It is a translation 1 unit down of the graph of $x=y^2$. The domain is $[0,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=-1$. Use the vertex and axis and plot a few additional points.

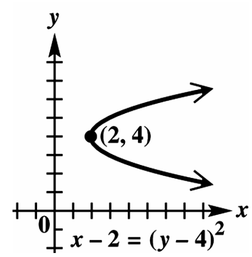
x	y
4	1
1	0
0	−1
1	−2
4	−3



7. $x-2=(y-4)^2$

The vertex is $(2,4)$. The graph opens to the right and has the same shape as $x=y^2$. It is a translation 2 units to the right and 4 units up of the graph of $x=y^2$. The domain is $[2,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=4$. Use the vertex and axis and plot a few additional points.

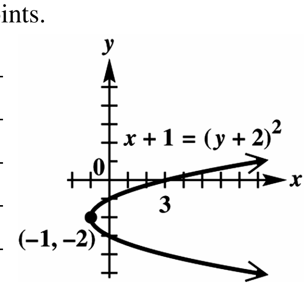
x	y
6	6
3	5
2	4
3	3
6	2



8. $x+1=(y+2)^2 \Rightarrow x-(-1)=[y-(-2)]^2$

The vertex is $(-1,-2)$. The graph opens to the right and has the same shape as $x=y^2$. It is a translation 1 unit to the left and 2 units down of the graph of $x=y^2$. The domain is $[-1,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=-2$. Use the vertex and axis and plot a few additional points.

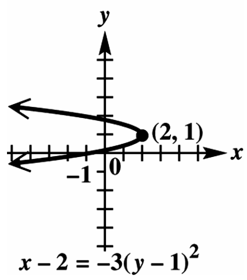
x	y
3	0
0	−1
−1	−2
0	−3
3	−4



9. $x-2=-3(y-1)^2$

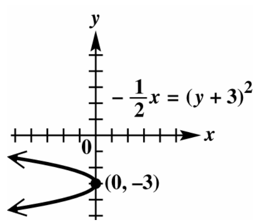
The vertex is $(2,1)$. The graph opens to the left and has the same shape as $x=-3y^2$. It is a translation 1 unit up and 2 units to the right of the graph of $x=-3y^2$. The domain is $(-\infty,2]$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=1$. Use the vertex and axis and plot a few additional points.

x	y
−10	3
−1	2
2	1
−1	0
−10	−1



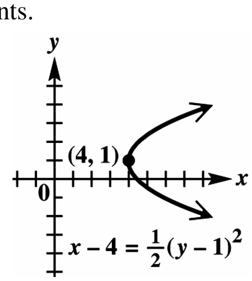
10. $-\frac{1}{2}x = (y+3)^2 \Rightarrow x = -2(y+3)^2 \Rightarrow$
$x - 0 = -2[y-(-3)]^2$
The vertex is $(0,-3)$. The graph opens to the left and has the same shape as $x = -2y^2$. It is a translation 3 units down of the graph of $x = -2y^2$. The domain is $(-\infty, 0]$. The range is $(-\infty, \infty)$. The graph is symmetric about its axis, the horizontal line $y = -3$. Use the vertex and axis and plot a few additional points.

x	y
−8	−1
−2	−2
0	−3
−2	−4
−8	−5



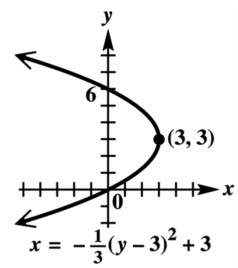
11. $x - 4 = \frac{1}{2}(y-1)^2$
The vertex is $(4,1)$. The graph opens to the right and has the same shape as $x = \frac{1}{2}y^2$. It is a translation 4 units to the right and 1 unit up of the graph of $x = \frac{1}{2}y^2$. The domain is $[4, \infty)$. The range is $(-\infty, \infty)$. The graph is symmetric about its axis, the horizontal line $y = 1$. Use the vertex and axis and plot a few additional points.

x	y
6	3
4.5	2
4	1
4.5	0
6	−1



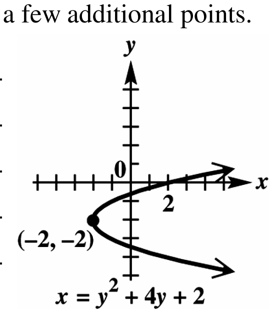
12. $x = -\frac{1}{3}(y-3)^2 + 3 \Rightarrow x - 3 = -\frac{1}{3}(y-3)^2$
The vertex is $(3,3)$. The graph opens to the left and has the same shape as $x = -\frac{1}{3}y^2$, translated 3 units to the right and 3 units up. The domain is $(-\infty, 3]$. The range is $(-\infty, \infty)$. The graph is symmetric about its axis, the horizontal line $y = 3$. Use the vertex and axis and plot a few additional points.

x	y
0	6
$\frac{5}{3}$	5
3	3
$\frac{5}{3}$	1
0	0



13. $x = y^2 + 4y + 2$
Complete the square on y to find the vertex and the axis.
$x = y^2 + 4y + 2 \Rightarrow x = (y^2 + 4y \quad\quad) + 2$
$x = (y^2 + 4y + 4 - 4) + 2$
$x = (y^2 + 4y + 4) - 4 + 2$
$x = (y+2)^2 - 2 \Rightarrow x - (-2) = [y - (-2)]^2$
The vertex is $(-2,-2)$. The graph opens to the right and has the same shape as $x = y^2$, translated 2 units to the left and 2 units down. The domain is $[-2, \infty)$. The range is $(-\infty, \infty)$. The graph is symmetric about its axis, the horizontal line $y = -2$. Use the vertex and axis and plot a few additional points.

x	y
2	0
−1	−1
−2	−2
−1	−3
2	−4



14. $x = 2y^2 - 4y + 6$

Complete the square on y to find the vertex and the axis.

$x = 2y^2 - 4y + 6 \Rightarrow x = 2\left(y^2 - 2y \qquad\right) + 6 \Rightarrow$

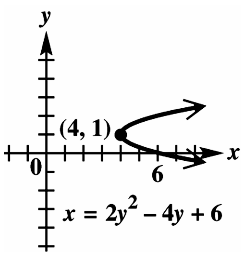
$x = 2\left(y^2 - 2y + 1 - 1\right) + 6$

$x = 2\left(y^2 - 2y + 1\right) + 2(-1) + 6$

$x = 2(y-1)^2 + 4 \Rightarrow x - 4 = 2(y-1)^2$

The vertex is $(4,1)$. The graph opens to the right and has the same shape as $x = 2y^2$, translated 4 units to the right and 1 units up. The domain is $[4,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y = 1$. Use the vertex and axis and plot a few additional points.

x	y
12	3
6	2
4	1
6	0
12	−1



15. $x = -4y^2 - 4y + 3$

Complete the square on y to find the vertex and the axis.

$x = -4y^2 - 4y + 3 \Rightarrow x = -4\left(y^2 + y \qquad\right) + 3$

$x = -4\left(y^2 + y + \frac{1}{4} - \frac{1}{4}\right) + 3$

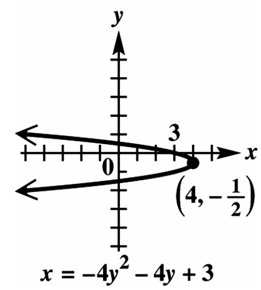
$x = -4\left(y^2 + y + \frac{1}{4}\right) - 4\left(-\frac{1}{4}\right) + 3$

$x = -4\left(y^2 + y + \frac{1}{4}\right) + 1 + 3$

$x = -4\left(y + \frac{1}{2}\right)^2 + 4 \Rightarrow x - 4 = -4\left[y - \left(-\frac{1}{2}\right)\right]^2$

The vertex is $\left(4, -\frac{1}{2}\right)$. The graph opens to the left and has the same shape as $x = -4y^2$. translated 4 units to the right and $\frac{1}{2}$ unit down. The domain is $(-\infty, 4]$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y = -\frac{1}{2}$. Use the vertex and axis and plot a few additional points.

x	y
−12	$1\frac{1}{2}$
0	$\frac{1}{2}$
4	$-\frac{1}{2}$
0	$-1\frac{1}{2}$
−12	$-2\frac{1}{2}$



16. $-x = 2y^2 - 2y + 3$

Complete the square on y to find the vertex and the axis.

$-x = 2y^2 - 2y + 3 \Rightarrow x = -2y^2 + 2y - 3$

$x = -2\left(y^2 - y \qquad\right) - 3$

$x = -2\left(y^2 - y + \frac{1}{4} - \frac{1}{4}\right) - 3$

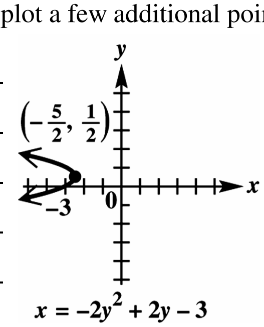
$x = -2\left(y^2 - y + \frac{1}{4}\right) - 2\left(-\frac{1}{4}\right) - 3$

$x = -2\left(y^2 - y + \frac{1}{4}\right) + \frac{1}{2} - 3$

$x = -2\left(y - \frac{1}{2}\right)^2 - \frac{5}{2} \Rightarrow x - \left(-\frac{5}{2}\right) = -2\left(y - \frac{1}{2}\right)^2$

The vertex is $\left(-\frac{5}{2}, \frac{1}{2}\right)$. The graph opens to the left and has the same shape as $x = -2y^2$, translated $\frac{5}{2}$ or $2\frac{1}{2}$ units to the left and $\frac{1}{2}$ unit up. The domain is $\left(-\infty, -\frac{5}{2}\right]$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y = \frac{1}{2}$. Use the vertex and axis and plot a few additional points.

x	y
−7	2
−3	1
$-\frac{5}{2}$	$\frac{1}{2}$
−3	0
−7	−1



17. $x = \frac{1}{2}y^2 - 2y + 3$

Complete the square on y to find the vertex and the axis.

$x = \frac{1}{2}y^2 - 2y + 3 \Rightarrow x = \frac{1}{2}\left(y^2 - 4y \qquad\right) + 3$

$x = \frac{1}{2}\left(y^2 - 4y + 4 - 4\right) + 3$

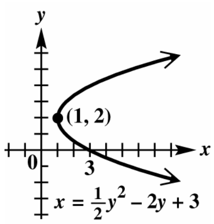
$x = \frac{1}{2}\left(y^2 - 4y + 4\right) + \frac{1}{2}(-4) + 3$

$x = \frac{1}{2}\left(y^2 - 4y + 4\right) - 2 + 3$

$x = \frac{1}{2}(y-2)^2 + 1 \Rightarrow x - 1 = \frac{1}{2}(y-2)^2$

The vertex is $(1,2)$. The graph opens to the right and has the same shape as $x=\frac{1}{2}y^2$, translated 1 unit to the right and 2 units up. The domain is $[1,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=2$. Use the vertex and axis and plot a few additional points.

x	y
3	4
1.5	3
1	2
1.5	1
3	0

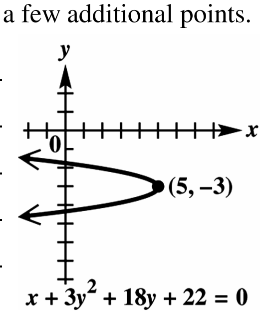


18. $x+3y^2+18y+22=0 \Rightarrow x=-3y^2-18y-22$
Complete the square on y to find the vertex and the axis.

$x=-3y^2-18y-22$
$x=-3\left(y^2+6y\quad\right)-22$
$x=-3\left(y^2+6y+9-9\right)-22$
$x=-3\left(y^2+6y+9\right)-3(-9)-22$
$x=-3\left(y^2+6y+9\right)+27-22$
$x=-3(y+3)^2+5 \Rightarrow x-5=-3\left[y-(-3)\right]^2$

The vertex is $(5,-3)$. The graph opens to the left and has the same shape as $x=-3y^2$, translated 5 units to the right and 3 units down. The domain is $(-\infty,5]$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y=-3$. Use the vertex and axis and plot a few additional points.

x	y
−7	−1
2	−2
5	−3
2	−4
−7	−5



19. The equation $x^2=24y$ has the form $x^2=4py$, so $4p=24$, from which $p=6$. Since the x-term is squared, the parabola is vertical, with focus $(0,p)=(0,6)$ and directrix, $y=-p$, is $y=-6$. The vertex is $(0,0)$, and the axis of the parabola is the y-axis.

20. The equation $x^2=\frac{1}{8}y$ has the form $x^2=4py$, so $4p=\frac{1}{8}$, from which $p=\frac{1}{32}$. Since the x-term is squared, the parabola is vertical, with focus $(0,p)=\left(0,\frac{1}{32}\right)$ and directrix, $y=-p$, is $y=-\frac{1}{32}$. The vertex is $(0,0)$, and the axis of the parabola is the y-axis.

21. The equation $y=-4x^2 \Rightarrow -\frac{1}{4}y=x^2$ has the form $x^2=4py$, so $4p=-\frac{1}{4}$, from which $p=-\frac{1}{16}$. Since the x-term is squared, the parabola is vertical, with focus $(0,p)=\left(0,-\frac{1}{16}\right)$ and directrix, $y=-p$, is $y=\frac{1}{16}$. The vertex is $(0,0)$, and the axis of the parabola is the y-axis.

22. The equation $-9y=x^2$ has the form $x^2=4py$, so $4p=-9$, from which $p=-\frac{9}{4}$. Since the x-term is squared, the parabola is vertical, with focus $(0,p)=\left(0,-\frac{9}{4}\right)$ and directrix, $y=-p$, is $y=\frac{9}{4}$. The vertex is $(0,0)$, and the axis of the parabola is the y-axis.

23. The equation $y^2=-4x$ has the form $y^2=4px$, so $4p=-4$, from which $p=-1$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0)=(-1,0)$ and directrix, $x=-p$, is $x=1$. The vertex is $(0,0)$, and the axis of the parabola is the x-axis.

24. The equation $y^2 = 16x$ has the form $y^2 = 4px$, so $4p = 16$, from which $p = 4$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0) = (4,0)$ and directrix, $x = -p$, is $x = -4$. The vertex is $(0,0)$, and the axis of the parabola is the x-axis.

25. The equation $x = -32y^2 \Rightarrow -\frac{1}{32}x = y^2$ has the form $y^2 = 4px$, so $4p = -\frac{1}{32}$, from which $p = -\frac{1}{128}$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0) = \left(-\frac{1}{128}, 0\right)$ and directrix, $x = -p$, is $x = \frac{1}{128}$. The vertex is $(0,0)$, and the axis of the parabola is the x-axis.

26. The equation $x = -16y^2 \Rightarrow y^2 = -\frac{1}{16}x$ has the form $y^2 = 4px$, so $4p = -\frac{1}{16}$, from which $p = -\frac{1}{64}$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0) = \left(-\frac{1}{64}, 0\right)$ and directrix, $x = -p$, is $x = \frac{1}{64}$. The vertex is $(0,0)$, and the axis of the parabola is the x-axis.

27. $(y-3)^2 = 12(x-1)$ has the form $(y-k)^2 = 4p(x-h)$, with $h = 1, k = 3$, and $4p = 12$, so $p = 3$. The graph of the given equation is a parabola with vertical axis. The vertex (h,k) is $(1,3)$. Because this parabola has a horizontal axis and $p > 0$, the parabola opens right, so the focus is the distance $p = 3$ units to the right of the vertex. Thus the focus is $(4,3)$. The directrix is the vertical line $p = 3$ units to the left of the vertex, so the directrix is the line $x = -2$. The axis is the horizontal line through the vertex, so the equation of the axis is $y = 3$.

28. $(x+2)^2 = 20y$ can be written as $[x-(-2)]^2 = 20(y-0)$. Thus, the parabola has the form $(x-h)^2 = 4p(y-k)$, with $h = -2, k = 0$, and $4p = 20$, so $p = 5$. The graph of the given equation is a parabola with vertical axis. The vertex (h,k) is $(-2,0)$. Because this parabola has a vertical axis and $p > 0$, the parabola opens up, so the focus is the distance $p = 5$ units above the vertex. Thus, the focus is $(-2,5)$. The directrix is the horizontal line $p = 5$ units below the vertex, so the directrix is the line $y = -5$. The axis is the vertical line through the vertex, so the equation of the axis is $x = -2$.

29. $(x-7)^2 = 16(y+5)$ can be written as $(x-7)^2 = 16[y-(-5)]$. Thus, the parabola has the form $(x-h)^2 = 4p(y-k)$, with $h = 7, k = -5$, and $4p = 16$, so $p = 4$. The graph of the given equation is a parabola with vertical axis. The vertex (h,k) is $(7,-5)$. Because this parabola has a vertical axis and $p > 0$, the parabola opens up, so the focus is the distance $p = 4$ units above the vertex. Thus, the focus is $(7,-1)$. The directrix is the horizontal line $p = 4$ units below the vertex, so the directrix is the line $y = -9$. The axis is the vertical line through the vertex, so the equation of the axis is $x = 7$.

30. $(y-2)^2 = 24(x-3)$ has the form $(y-k)^2 = 4p(x-h)$, with $h = 3, k = 2$, and $4p = 24$, so $p = 6$. The graph of the given equation is a parabola with vertical axis. The vertex (h,k) is $(3,2)$. Because this parabola has a horizontal axis and $p > 0$, the parabola opens right, so the focus is the distance $p = 6$ units to the right of the vertex. Thus the focus is $(9,2)$. The directrix is the vertical line $p = 6$ units to the left of the vertex, so the directrix is the line $x = -3$. The axis is the horizontal line through the vertex, so the equation of the axis is $y = 2$.

31. A parabola with focus $(5,0)$ and vertex at the origin is a horizontal parabola. The equation has the form $y^2 = 4px$. Since $p = 5$ is positive, it opens to the right. Substituting 5 for p, we find that an equation for this parabola is $y^2 = 4(5)x \Rightarrow y^2 = 20x$.

32. A parabola with focus $\left(-\frac{1}{2}, 0\right)$ and vertex at the origin is a horizontal parabola. The equation has the form $y^2 = 4px$. Since $p = -\frac{1}{2}$ is negative, it opens to the left. Substituting $-\frac{1}{2}$ for *p*, we find that an equation for this parabola is $y^2 = 4\left(-\frac{1}{2}\right)x \Rightarrow y^2 = -2x.$

33. A parabola with directrix $y = -\frac{1}{4}$ and vertex at the origin is a vertical parabola with focus $\left(0, \frac{1}{4}\right)$. The equation has the form $x^2 = 4py$. Since $p = \frac{1}{4}$ is positive, it opens up. Substituting $\frac{1}{4}$ for *p*, we find that an equation for this parabola is $x^2 = 4\left(\frac{1}{4}\right)y \Rightarrow x^2 = y.$

34. A parabola with directrix $y = \frac{1}{3}$ and vertex at the origin is a vertical parabola with focus $\left(0, -\frac{1}{3}\right)$. The equation has the form $x^2 = 4py$. Since $p = -\frac{1}{3}$ is negative, it opens down. Substituting $-\frac{1}{3}$ for *p*, we find that an equation for this parabola is $x^2 = 4\left(-\frac{1}{3}\right)y \Rightarrow x^2 = -\frac{4}{3}y.$

35. A parabola passing through $\left(\sqrt{3}, 3\right)$, opening up, and vertex at the origin has an equation of the form $x^2 = 4py$. Use this equation with the coordinates of the point $\left(\sqrt{3}, 3\right)$ to find the value of *p*. $x^2 = 4py \Rightarrow \left(\sqrt{3}\right)^2 = 4p \cdot 3 \Rightarrow$

$3 = 12p \Rightarrow \frac{1}{4} = p$

Thus, an equation of the parabola is

$x^2 = 4\left(\frac{1}{4}\right)y \Rightarrow x^2 = y.$

36. A parabola passing through $\left(-2, -2\sqrt{2}\right)$, opening to the left, and vertex at the origin has an equation of the form $y^2 = 4px$. Use this equation with the coordinates of the point $\left(-2, -2\sqrt{2}\right)$ to find the value of *p*.

$y^2 = 4px \Rightarrow \left(-2\sqrt{2}\right)^2 = 4p(-2) \Rightarrow$

$8 = -8p \Rightarrow p = -1$

Thus, an equation of the parabola is $y^2 = -4x.$

37. A parabola through $(3, 2)$, symmetric with respect to the *x*-axis, and vertex at the origin has a horizontal axis (*x*-axis) and the equation is of the form $y^2 = 4px$. Use this equation with the coordinates of the point $(3, 2)$ to find the value of *p*.

$y^2 = 4px \Rightarrow 2^2 = 4p \cdot 3 \Rightarrow 4 = 12p \Rightarrow p = \frac{1}{3}$

Thus, an equation for the parabola is

$y^2 = 4\left(\frac{1}{3}\right)x \Rightarrow y^2 = \frac{4}{3}x.$

38. A parabola through $(2, -4)$, symmetric with respect to the *y*-axis, and vertex at the origin has a horizontal axis (*y*-axis) and the equation is of the form $x^2 = 4py$. Use this equation with the coordinates of the point $(2, -4)$ to find the value of *p*.

$x^2 = 4py \Rightarrow 2^2 = 4p(-4) \Rightarrow$

$4 = -16p \Rightarrow p = -\frac{1}{4}$

Thus, an equation for the parabola is

$x^2 = 4\left(-\frac{1}{4}\right)y \Rightarrow x^2 = -y.$

39. The vertex is $(4, 3)$ and the focus is $(4, 5)$. Since the focus is above the vertex, the axis is vertical and the parabola opens upward. The distance between the vertex and the focus is 5 – 3 = 2. Since the parabola opens upward, choose *p* = 2. The equation will have the form $(x-h)^2 = 4p(y-k)$. Substitute *p* = 2, *h* = 4, and *k* = 3 to find the required equation.

$(x-h)^2 = 4p(y-k)$

$(x-4)^2 = 4(2)(y-3) \Rightarrow (x-4)^2 = 8(y-3)$

40. The vertex is $(-2, 1)$ and the focus is $(-2, -3)$. Since the focus is below the vertex, the axis is vertical and the parabola opens downward. The distance between the vertex and the focus is 1 – (–3) = 4. Since the parabola opens downward, choose *p* = –4. The equation will have the form $(x-h)^2 = 4p(y-k)$. Substitute *p* = –4, *h* = –2, and *k* = 1 to find the required equation.

$$(x-h)^2 = 4p(y-k)$$
$$[x-(-2)]^2 = 4(-4)(y-1)$$
$$(x+2)^2 = -16(y-1)$$

41. The vertex is $(-5,6)$ and the directrix is $x=-12$. Since the directrix is 7 units to the left of the vertex, the focus is 7 units to the right of the vertex, at $(2,6)$ and $p=7$. The focus is to the right of the vertex, so the axis is horizontal and the parabola opens to the right. The equation will have the form $(y-k)^2=4p(x-h)$. Substitute $p=7$, $h=-5$, and $k=6$.

$$(y-k)^2=4p(x-h)$$
$$(y-6)^2=4(7)[x-(-5)]$$
$$(y-6)^2=28(x+5)$$

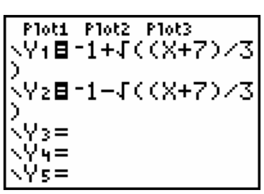
42. The vertex is $(1,2)$ and the directrix is $x=4$. Since the directrix is three units to the right of the vertex, the focus is three units to the left of the vertex, at $(-2,2)$ and $p=-3$

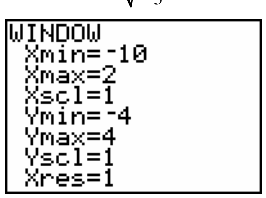
Since the focus is to the left of the vertex, the axis is horizontal and the parabola opens to the left. The equation will have the form $(y-k)^2=4p(x-h)$. Substitute $p=-3$, $h=1$, and $k=2$.

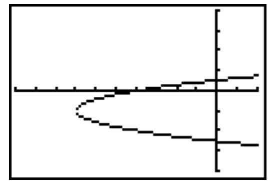
$$(y-k)^2=4p(x-h)$$
$$(y-2)^2=4(-3)(x-1)$$
$$(y-2)^2=-12(x-1)$$

43. Complete the square on y.

$$x=3y^2+6y-4$$
$$3\left(y^2+2y\qquad\right)-4=x$$
$$3\left(y^2+2y+1-1\right)-4=x$$
$$3\left(y^2+2y+1\right)+3(-1)-4=x$$
$$3(y+1)^2-7=x$$
$$3(y+1)^2=x+7$$
$$(y+1)^2=\tfrac{x+7}{3}$$
$$y+1=\pm\sqrt{\tfrac{x+7}{3}}$$
$$y=-1\pm\sqrt{\tfrac{x+7}{3}}$$

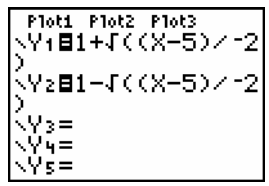


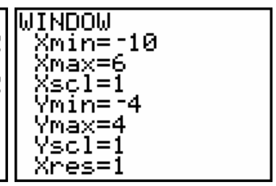


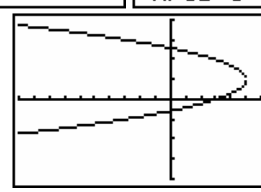


44. Complete the square on y.

$$x=-2y^2+4y+3$$
$$-2\left(y^2-2y\qquad\right)+3=x$$
$$-2\left(y^2-2y+1-1\right)+3=x$$
$$-2\left(y^2-2y+1\right)-2(-1)+3=x$$
$$-2(y-1)^2+5=x$$
$$-2(y-1)^2=x-5$$
$$(y-1)^2=\tfrac{x-5}{-2}$$
$$y-1=\pm\sqrt{\tfrac{x-5}{-2}}$$
$$y=1\pm\sqrt{\tfrac{x-5}{-2}}$$

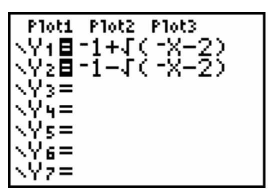


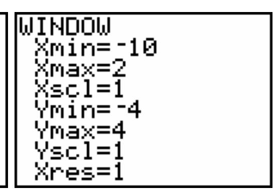


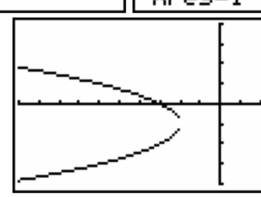


45. Solving for y we have

$$-(y+1)^2=x+2\Rightarrow(y+1)^2=-x-2\Rightarrow$$
$$y+1=\pm\sqrt{-x-2}\Rightarrow y=-1\pm\sqrt{-x-2}$$

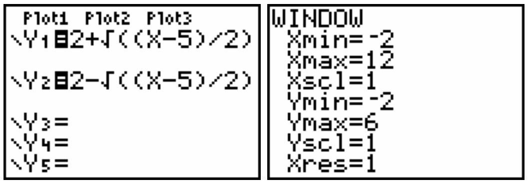


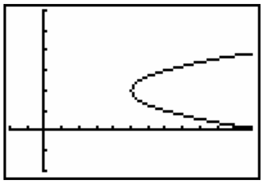




46. Solving for y we have

$$x-5=2(y-2)^2 \Rightarrow (y-2)^2=\frac{x-5}{2} \Rightarrow$$
$$y-2=\pm\sqrt{\frac{x-5}{2}} \Rightarrow y=2\pm\sqrt{\frac{x-5}{2}}$$





47. The equation is of the form $x=ay^2+by+c$.

Substituting $x=-5$, $y=1$, we get

$-5=a(1)^2+b(1)+c \Rightarrow -5=a+b+c \quad (1)$

Substituting $x=-14$, $y=-2$, we get

$-14=a(-2)^2+b(-2)+c \Rightarrow$
$-14=4a-2b+c \quad (2)$

Substituting $x=-10$, $y=2$, we get

$-10=a(2)^2+b(2)+c \Rightarrow$
$-10=4a+2b+c. \quad (3)$

48. The following is the system of three equations.

$$\begin{aligned} a+b+c&=-5 \ (1)\\ 4a-2b+c&=-14 \ (2)\\ 4a+2b+c&=-10 \ (3)\end{aligned}$$

Adding equations (2) and (3), we obtain
$8a+2c=-24$ or $4a+c=-12$. (4) .

Adding 2 times equation (1) to equation (2), we obtain the following.

$$\begin{aligned} 2a+2b+2c&=-10\\ 4a-2b+c&=-14\\ \hline 6a \qquad +3c&=-24 \Rightarrow 2a+c=-8 \ (5)\end{aligned}$$

Add –1 times equation (4) to equation (5).

$$\begin{aligned} -4a-c&=12\\ 2a+c&=-8\\ \hline -2a \qquad &=4 \Rightarrow a=-2\end{aligned}$$

Substitute $a=-2$ into equation (5) to obtain
$2(-2)+c=-8 \Rightarrow -4+c=-8 \Rightarrow c=-4.$

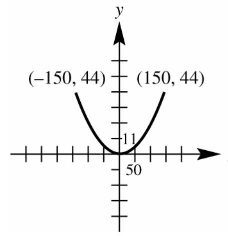
Substitute $a=-2$ and $c=-4$ into equation (1) to obtain
$-2+b-4=-5 \Rightarrow b-6=-5 \Rightarrow b=1.$

Solution set: $\{(-2,1,-4)\}$

49. Since $a=-2<0$, the parabola opens to the left.

50. Substituting $a=-2$, $b=1$, and $c=-4$, the equation of the parabola is $x=-2y^2+y-4$.

51. **(a)** Sketch a cross-section of the dish. Place this parabola on a coordinate system with the vertex at the origin. (This solution will show that the focus lies outside of the dish.)



Since the parabola has vertex (0, 0) and a vertical axis (the y-axis), it has an equation of the form $x^2=4py$. Substitute $x=150$ and $y=44$ to find the value of p.

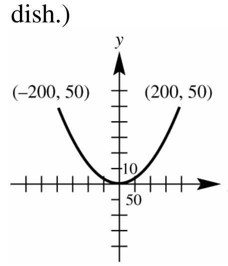
$$x^2=4py \Rightarrow 150^2=4p\cdot 44$$
$$22{,}500=176p \Rightarrow p=\frac{22{,}500}{176}=\frac{5625}{44}$$

The required equation is

$$x^2=4\left(\frac{5625}{44}\right)y \Rightarrow x^2=\frac{5625}{11}y$$
$$y=\frac{11}{5625}x^2$$

(b) Since $p=\frac{5625}{44}\approx 127.8$, the receiver should be placed approximately 127.8 ft from the vertex.

52. **(a)** Sketch a cross-section of the dish. Place this parabola on a coordinate system with the vertex at the origin. (This solution will show that the focus lies outside of the dish.)



Since the parabola has vertex (0, 0) and a vertical axis (the y-axis), it has an equation of the form $x^2=4py$. Substitute $x=200$ and $y=50$ to find the value of p.

$$x^2=4py \Rightarrow 200^2=4p\cdot 50 \Rightarrow$$
$$40{,}000=200p \Rightarrow p=\frac{40{,}000}{200}=200$$

The required equation is

$$x^2=4(200)y \Rightarrow x^2=800y \Rightarrow y=\frac{1}{800}x^2$$

(b) Since $p = 200$, the receiver should be placed 200 ft from the vertex.

53. Place the parabola that represents the arch on a coordinate system with the center of the bottom of the arch at the origin. Then the vertex will be $(0,12)$ and the points $(-6,0)$ and $(6,0)$ will also be on the parabola. Because the axis of the parabola is the y-axis and the vertex is $(0,12)$, the equation will have the form $x^2 = 4p(y-12)$. Use the coordinates of the point $(6,0)$ to find the value of p.

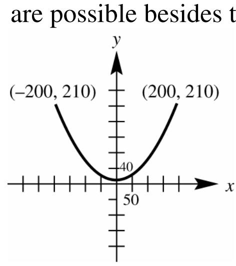
$x^2 = 4p(y-12) \Rightarrow 6^2 = 4p(0-12) \Rightarrow$
$36 = -48p \Rightarrow p = \frac{36}{-48} \Rightarrow p = -\frac{3}{4}$

Thus, the equation of the parabola is
$x^2 = 4\left(-\frac{3}{4}\right)(y-12)$ or $x^2 = -3(y-12)$.

Now find the x-coordinate of point whose y-coordinate is 9 and whose x-coordinate is positive: $x^2 = -3(y-12) \Rightarrow$
$x^2 = -3(9-12) \Rightarrow x^2 = 9 \Rightarrow x = \sqrt{9} = 3$

From the symmetry of the parabola, we see that the width of the arch 9 ft up is 2(3 ft) = 6 ft.

54. Place the parabola on a coordinate system and label points. (Other locations for the parabola are possible besides the one shown below.)



The vertex of the parabola is $(0,10)$. The equation has the form $x^2 = 4p(y-10)$. Use the coordinates of the point $(200,210)$ to find the value of p.

$200^2 = 4p(210-10) \Rightarrow$
$40{,}000 = 800p \Rightarrow p = 50$

The equation of the parabola is
$x^2 = 4(50)(y-10)$ or $x^2 = 200(y-10)$.

The height of the vertical cables we seek are located at 100 ft from the vertex (horizontally). Thus, we need to find the y-coordinate of point whose x-coordinate is 100 (or -100).

$100^2 = 200(y-10)$
$10{,}000 = 200y - 2000$
$12{,}000 = 200y \Rightarrow y = 60$

From the symmetry of the parabola, the height of each of the remaining cables is 60 ft.

55. (a) Locate the cannon at the origin. With $v = 252.982$, the equation becomes

$y = x - \frac{32}{v^2}x^2 \Rightarrow y = x - \frac{32}{252.982^2}x^2 \Rightarrow$
$y \approx x - \frac{1}{2000}x^2$

Complete the square on x.

$y = x - \frac{1}{2000}x^2$
$y = -\frac{1}{2000}\left(x^2 - 2000x \qquad\right)$
$y = -\frac{1}{2000}\left(x^2 - 2000x + 1{,}000{,}000 - 1{,}000{,}000\right)$
$y = -\frac{1}{2000}\left(x^2 - 2000x + 1{,}000{,}000\right) + \left(-\frac{1}{2000}\right)(-1{,}000{,}000)$
$y = -\frac{1}{2000}(x-1000)^2 + 500$
$y - 500 = -\frac{1}{2000}(x-1000)^2$

Thus, the vertex of the parabola is located at $(1000, 500)$. Because of symmetry, the shell then travels an additional 1000 feet for a maximum distance of 2000 feet.

(b) The envelope parabola has x-intercepts located at (–2000, 0) and (2000, 0). The vertex is easily found to be located at $(0,1000)$. Because the axis of the parabola is the y-axis and it opens down, the equation is of the form $x^2 = 4p(y-1000)$. Use the coordinates of the point $(2000,0)$ to find the value of p. $2000^2 = 4p(0-1000) \Rightarrow$
$4{,}000{,}000 = -4000p \Rightarrow p = -1000$

The equation of the parabola is

$x^2 = 4(-1000)(y-1000)$
$x^2 = -4000(y-1000)$
$y - 1000 = -.00025x^2$
$y = 1000 - .00025x^2$

(c) Using the equation of the envelope parabola in part (b), we calculate the maximum possible height of a shell when x is 1500 feet.

$y = 1000 - .00025x^2$

$y = 1000 - .00025(1500)^2$

$y = 1000 - .00025(2,250,000)$

$y = 1000 - 562.5 \Rightarrow y = 437.5$

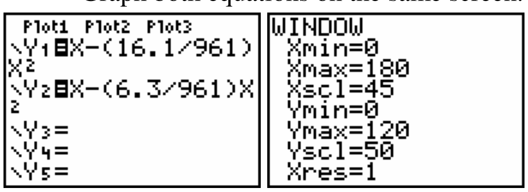
If the helicopter flies at a height of 450 feet, a shell fired by the cannon would never reach the helicopter.

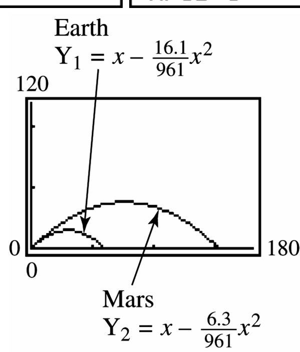
56. (a) $y = x - \frac{g}{1922}x^2$

For Earth, $g = 32.2$, so the equation is $y = x - \frac{32.2}{1922}x^2 = x - \frac{16.1}{961}x^2$.

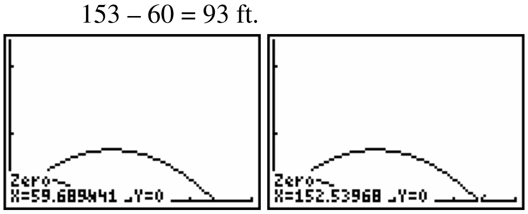
For Mars, $g = 12.6$, so the equation is $y = x - \frac{12.6}{1922}x^2 = x - \frac{6.3}{961}x^2$.

Graph both equations on the same screen.





(b) From the graph, we see that the ball hits the ground (the y-coordinate returns to 0) at $x \approx 153$ ft on Mars and $x \approx 60$ ft on Earth. Therefore, the difference between the horizontal distance traveled between the two balls is approximately $153 - 60 = 93$ ft.

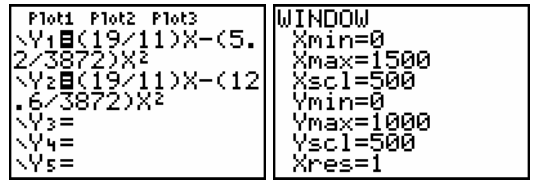


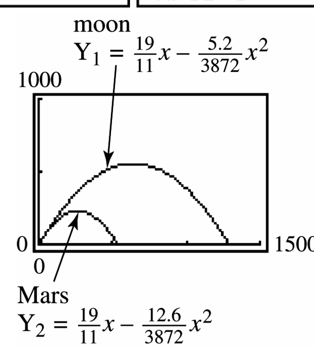
57. (a) $y = \frac{19}{11}x - \frac{g}{3872}x^2$

For the moon, $g = 5.2$, so the equation is $y = \frac{19}{11}x - \frac{5.2}{3872}x^2$.

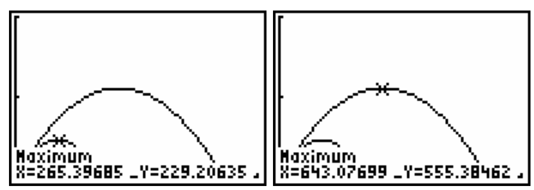
For Mars, $g = 12.6$, so the equation is $y = \frac{19}{11}x - \frac{12.6}{3872}x^2$.

Graph both equations on the same screen.





(b) Using the "maximum" option in the CALC menu, we see that the ball reaches a maximum height of $y \approx 229$ ft on Mars and $y \approx 555$ ft on the moon.

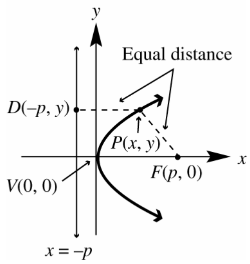


58. Answers will vary. If the variable x is squared, then you will have a parabola that opens either up or down. In either case you will have a vertical axis. If the variable y is squared, then you will have a parabola that opens either to the right or to the left. In either case you will have a horizontal axis.

59. Answers will vary. The proof is similar to the proof given on page 609 for a parabola that opens up (which also applied to parabolas opening down).

(*continued on next page*)

(*continued from page 1037*)



$$d(P,F) = d(P,D)$$
$$\sqrt{(x-p)^2+(y-0)^2} = \sqrt{(x-(-p))^2+(y-y)^2}$$
$$\sqrt{(x-p)^2+y^2} = \sqrt{(x+p)^2}$$
$$x^2-2xp+p^2+y^2 = x^2+2xp+p^2$$
$$-2xp+y^2 = 2xp$$
$$y^2 = 4px$$

60. Answers will vary.

Section 10.2: Ellipses

1. (a) $36x^2+9y^2=324 \Rightarrow \frac{x^2}{9}+\frac{y^2}{36}=1 \Rightarrow \frac{x^2}{3^2}+\frac{y^2}{6^2}=1$

This ellipse has endpoints of the minor axis of $(\pm3,0)$, which are also x-intercepts. It also has vertices of $(0,\pm6)$, which are also y-intercepts. The correct choice is A.

(b) $9x^2+36y^2=324 \Rightarrow \frac{x^2}{36}+\frac{y^2}{9}=1 \Rightarrow \frac{x^2}{6^2}+\frac{y^2}{3^2}=1$

This ellipse has vertices of $(\pm6,0)$, which are also x-intercepts. It also has endpoints of the minor axis of $(0,\pm3)$, which are also y-intercepts. The correct choice is C.

(c) $\frac{x^2}{25}=1-\frac{y^2}{16} \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1 \Rightarrow \frac{x^2}{5^2}+\frac{y^2}{4^2}=1$

This ellipse has vertices of $(\pm5,0)$, which are also x-intercepts. It also has endpoints of the minor axis of $(0,\pm4)$, which are also y-intercepts. The correct choice is D.

(d) $\frac{x^2}{16}=1-\frac{y^2}{25} \Rightarrow \frac{x^2}{16}+\frac{y^2}{25}=1 \Rightarrow \frac{x^2}{4^2}+\frac{y^2}{5^2}=1$

This ellipse has endpoints of the minor axis of $(\pm4,0)$, which are also x-intercepts. It also has vertices of $(0,\pm5)$, which are also y-intercepts. The correct choice is B.

2. (a) This is an ellipse.

(b) This is not an ellipse. It is a circle.

(c) This is not an ellipse. It is a parabola.

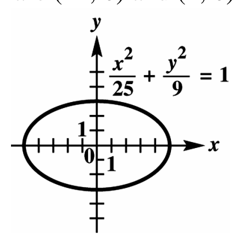
(d) This is not an ellipse. It is a line.

3. $\frac{x^2}{25}+\frac{y^2}{9}=1$

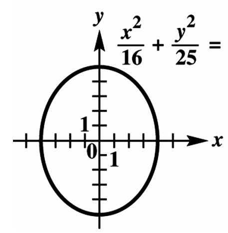
The graph is an ellipse with center (0, 0). Rewriting the given equation, we have $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$. Since $5>3$, we have $a = 5$ and $b = 3$, and the major axis is horizontal. Thus, the vertices are (–5, 0) and (5, 0). The endpoints of the minor axis are (0, –3) and (0, 3). The domain is [–5, 5]. The range is [–3, 3]. To find the foci we need to find c such that $c^2=a^2-b^2$.

$c^2=a^2-b^2 \Rightarrow c^2=25-9 \Rightarrow c^2=16 \Rightarrow c=4$

Since the major axis lies on the x-axis, the foci are (–4, 0) and (4, 0).



Exercise 3



Exercise 4

4. $\frac{x^2}{16}+\frac{y^2}{25}=1$

The graph is an ellipse with center (0, 0). The given equation may be written as $\frac{x^2}{4^2}+\frac{y^2}{5^2}=1$.

Since 5 > 4, we have $a = 5$ and $b = 4$, and the major axis is vertical. Thus, the vertices are (0, –5) and (0, 5). The endpoints of the minor axis are (–4, 0) and (4, 0). The domain is [–4, 4]. The range is [–5, 5]. To find the foci we need to find c such that $c^2=a^2-b^2$.

$c^2=a^2-b^2 \Rightarrow c^2=25-16 \Rightarrow$

$c^2=9 \Rightarrow c=3$

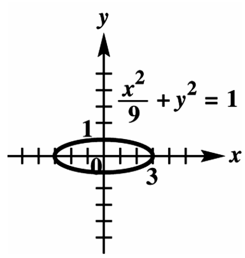
Since the major axis lies on the y-axis, the foci are (0, –3) and (0, 3).

5. $\frac{x^2}{9} + y^2 = 1$

Rewriting the equation, we have $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1.$ The center is (0, 0). Since 3 > 1, we have $a = 3$ and $b = 1$, and the major axis is horizontal. The vertices are (–3, 0) and (3, 0). The endpoints of the minor axis are (0, –1) and (0, 1). The domain is [–3, 3]. The range is [–1, 1]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 9 - 1 \Rightarrow c^2 = 8 \Rightarrow$
$c = \sqrt{8} = 2\sqrt{2}$

Since the major axis lies on the x-axis, the foci are $(-2\sqrt{2}, 0)$ and $(2\sqrt{2}, 0)$.

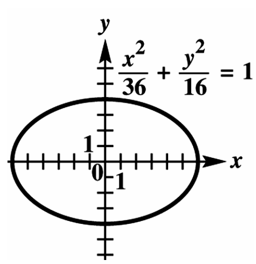


6. $\frac{x^2}{36} + \frac{y^2}{16} = 1$

Rewriting the equation, we have $\frac{x^2}{6^2} + \frac{y^2}{4^2} = 1.$ The center is (0, 0). Since 6 > 4, we have $a = 6$ and $b = 4$, and the major axis is horizontal. The vertices are (–6, 0) and (6, 0). The endpoints of the minor axis are (0, –4) and (0, 4). The domain is [–6, 6]. The range is [–4, 4]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 36 - 16 \Rightarrow c^2 = 20 \Rightarrow$
$c = \sqrt{20} = 2\sqrt{5}$

Since the major axis lies on the x-axis, the foci are $(-2\sqrt{5}, 0)$ and $(2\sqrt{5}, 0)$.

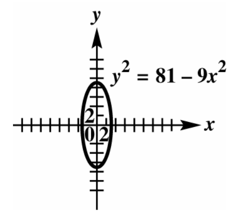


7. $y^2 = 81 - 9x^2 \Rightarrow 9x^2 + y^2 = 81$

Rewriting the equation, we have $\frac{x^2}{9} + \frac{y^2}{81} = 1$ or $\frac{x^2}{3^2} + \frac{y^2}{9^2} = 1.$ The center is (0, 0). Since 9 > 3, we have $a = 9$ and $b = 3$, and the major axis is vertical. The vertices are (0, –9) and (0, 9). The endpoints of the minor axis are (–3, 0) and (3, 0). The domain is [–3, 3]. The range is [–9, 9]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 81 - 9 \Rightarrow c^2 = 72 \Rightarrow$
$c = \sqrt{72} = 6\sqrt{2}$

Since the major axis lies on the y-axis, the foci are $\left(0, -6\sqrt{2}\right)$ and $\left(0, 6\sqrt{2}\right)$.

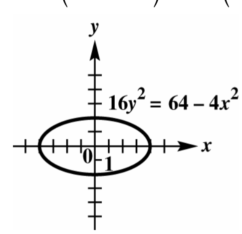


8. $16y^2 = 64 - 4x^2 \Rightarrow 4x^2 + 16y^2 = 64$

Rewriting the equation, we have $\frac{x^2}{16} + \frac{y^2}{4} = 1$ or $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1.$ The center is (0, 0). Since 4 > 2, we have $a = 4$ and $b = 2$, and the major axis is horizontal. The vertices are (–4, 0) and (4, 0). The endpoints of the minor axis are (0, –2) and (0, 2). The domain is [–4, 4]. The range is [–2, 2]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 16 - 4 \Rightarrow c^2 = 12 \Rightarrow$
$c = \sqrt{12} = 2\sqrt{3}$

Since the major axis lies on the x-axis, the foci are $\left(-2\sqrt{3}, 0\right)$ and $\left(2\sqrt{3}, 0\right)$.

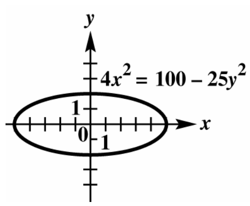


9. $4x^2 = 100 - 25y^2 \Rightarrow 4x^2 + 25y^2 = 100$

Rewriting the equation, we have $\frac{x^2}{25} + \frac{y^2}{4} = 1$ or $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$. The center is (0, 0). Since 5 > 2, we have $a = 5$ and $b = 2$, and the major axis is horizontal. The vertices are (–5, 0) and (5, 0). The endpoints of the minor axis are (0, –2) and (0, 2). The domain is [–5, 5]. The range is [–2, 2]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 25 - 4 \Rightarrow$
$c^2 = 21 \Rightarrow c = \sqrt{21}$

Since the major axis lies on the x-axis, the foci are $\left(-\sqrt{21}, 0\right)$ and $\left(\sqrt{21}, 0\right)$.

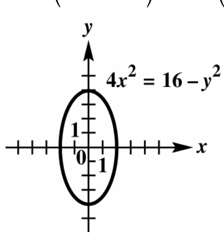


10. $4x^2 = 16 - y^2 \Rightarrow 4x^2 + y^2 = 16$

Rewriting the equation, we have $\frac{x^2}{4} + \frac{y^2}{16} = 1$ or $\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$. The center is (0, 0). Since 4 > 2, we have $a = 4$ and $b = 2$, and the major axis is vertical. The vertices are (0, –4) and (0, 4). The endpoints of the minor axis are (–2, 0) and (2, 0). The domain is [–2, 2]. The range is [–4, 4]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 16 - 4 \Rightarrow c^2 = 12 \Rightarrow$
$c = \sqrt{12} = 2\sqrt{3}$

Since the major axis lies on the y-axis, the foci are $\left(0, -2\sqrt{3}\right)$ and $\left(0, 2\sqrt{3}\right)$.

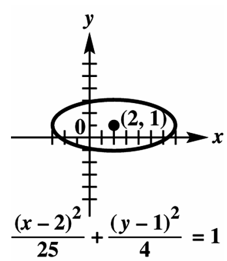


11. $\frac{(x-2)^2}{25} + \frac{(y-1)^2}{4} = 1$

Rewriting the equation, we have $\frac{(x-2)^2}{5^2} + \frac{(y-1)^2}{2^2} = 1$. The center is (2, 1). Since $a = 5$ is associated with x^2, the major axis of the ellipse is horizontal. The vertices are on a horizontal line through (2, 1), while the endpoints of the minor axis are on the vertical line through (2, 1). The vertices are 5 units to the left and right of the center at (–3, 1) and (7, 1). The endpoints of the minor axis are 2 units below and 2 units above the center at (2, –1) and (2, 3). The domain is [–3, 7]. The range is [–1, 3]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 25 - 4 \Rightarrow$
$c^2 = 21 \Rightarrow c = \sqrt{21}$

Since the major axis lies on $y = 1$, the foci are $\left(2 - \sqrt{21}, 1\right)$ and $\left(2 + \sqrt{21}, 1\right)$.

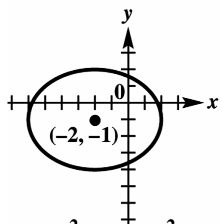


12. $\frac{(x+2)^2}{16} + \frac{(y+1)^2}{9} = 1$

Rewriting the equation, we have $\frac{(x+2)^2}{4^2} + \frac{(y+1)^2}{3^2} = 1$. The center is (–2, –1). We have, $a = 4$ and $b = 3$. Since $a = 4$ is associated with x^2, the vertices are on the horizontal line through the center (–2, –1), that is, the line $y = -1$. Find the vertices by locating two points on this horizontal line, one 4 units to the left of (–2, –1) and one 4 units to the right. The vertices are (–6, –1) and (2, –1). Find the endpoints of the minor axis by locating two points on the vertical line through the center, that is, the line $x = -2$, one 3 units below (–2, –1) and one 3 units above. The endpoints of the minor axis are (–2, –4) and (–2, 2). The domain is [–6, 2]. The range is [–4, 2]. To find the foci, we need to find c such that $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 16 - 9 \Rightarrow$
$c^2 = 7 \Rightarrow c = \sqrt{7}$

Since the major axis lies on $y=-1$, the foci are $\left(-2-\sqrt{7},\,-1\right)$ and $\left(-2+\sqrt{7},\,-1\right)$.



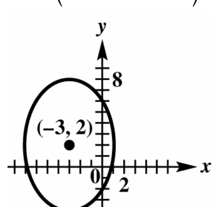
$\frac{(x+2)^2}{16}+\frac{(y+1)^2}{9}=1$

13. $\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$

Rewriting the equation, we have $\frac{(x+3)^2}{4^2}+\frac{(y-2)^2}{6^2}=1.$ The center is (–3, 2). We have $a = 6$ and $b = 4$. Since $a = 6$ is associated with y^2, the major axis of the ellipse is vertical. The vertices are on the vertical line through (–3, 2), and the endpoints of the minor axis are on the horizontal line through (–3, 2). The vertices are 6 units below and 6 units above the center at (–3, –4) and (–3, 8). The endpoints of the minor axis are 4 units to the left and 4 units to the right of the center at (–7, 2) and (1, 2). The domain is [–7, 1]. The range is [–4, 8]. To find the foci, we need to find c such that $c^2=a^2-b^2$.

$c^2=a^2-b^2 \Rightarrow c^2=36-16 \Rightarrow c^2=20 \Rightarrow$
$c=\sqrt{20}=2\sqrt{5}$

Since the major axis lies on $x=-3$, the foci are $\left(-3,\,2-2\sqrt{5}\right)$ and $\left(-3,\,2+2\sqrt{5}\right)$.



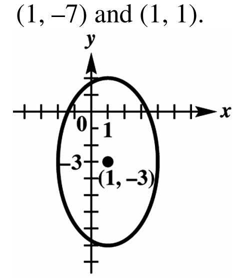
$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$

14. $\frac{(x-1)^2}{9}+\frac{(y+3)^2}{25}=1$

Rewriting the equation, we have $\frac{(x-1)^2}{3^2}+\frac{(y+3)^2}{5^2}=1.$ The center is (1, –3). We have $a = 5$ and $b = 3$. Since $a = 5$ is associated with y^2, the major axis is vertical. The vertices are on the vertical line through (1, –3). The vertices are 5 units below and 5 units above the center at (1, –8) and (1, 2). The endpoints of the minor axis are on the horizontal line through (1, –3). The endpoints of the minor axis are 3 units to the left and 3 units to the right of the center at (–2, –3) and (4, –3). The domain is [–2, 4]. The range is [–8, 2]. To find the foci, we need to find c such that $c^2=a^2-b^2$.

$c^2=a^2-b^2 \Rightarrow c^2=25-9 \Rightarrow$
$c^2=16 \Rightarrow c=\sqrt{16}=4$

Since the major axis lies on $x=1$, the foci are (1, –7) and (1, 1).



$\frac{(x-1)^2}{9}+\frac{(y+3)^2}{25}=1$

15. x-intercepts ± 5; y- intercepts ± 4

From the given information, $a = 5$ and $b = 4$. Since 5 > 4, the x-intercepts represent vertices. The equation has the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ Since $a^2=25$ and $b^2=16,$ the equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1.$

16. x-intercepts $\pm\sqrt{15}$; y-intercepts ± 4

Since the ellipse is centered at (0, 0) and $4>\sqrt{15}$, the y-intercepts represent vertices and thus, we have $a = 4$ and $b=\sqrt{15}$. The equation has the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$ Since $a^2=16$ and $b^2=15,$ the equation of the ellipse is $\frac{x^2}{15}+\frac{y^2}{16}=1.$

17. Major axis with length 6, foci at (0, 2), (0, –2). The length of the major axis is $2a$. Thus, we have $2a=6 \Rightarrow a=3.$ From the foci, we have $c = 2$. Solving for b^2 we have

$c^2=a^2-b^2 \Rightarrow 4=9-b^2 \Rightarrow b^2=5.$

(*continued on next page*)

(*continued from page 1041*)

Since the foci are on the y-axis and the ellipse is centered at the origin, the equation has the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. Since $a=3 \Rightarrow a^2=9$ and and $b^2=5$, the equation of the ellipse is $\frac{x^2}{5}+\frac{y^2}{9}=1$.

18. Minor axis with length 4, foci at (–5, 0) and (5, 0).
The length of the minor axis is $2b$. Thus, we have $2b=4 \Rightarrow b=2$. From the foci, we have $c = 5$. Solving for a^2 we have $c^2=a^2-b^2 \Rightarrow 25=a^2-4 \Rightarrow a^2=29$. Since the foci are on the x-axis and the ellipse is centered at the origin, the equation has the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Since $b=2 \Rightarrow b^2=4$ and $a^2=29$, the equation of the ellipse is $\frac{x^2}{29}+\frac{y^2}{4}=1$.

19. Center at (3, 1); minor axis vertical, with length 8; $c = 3$.
Since the center is (3, 1) and the minor axis is vertical, the equation has the form $\frac{(x-3)^2}{a^2}+\frac{(y-1)^2}{b^2}=1$. The length of the minor axis is $2b$, so $b = 4$. Solving for a^2, we have $c^2=a^2-b^2 \Rightarrow 9=a^2-16 \Rightarrow a^2=25$. Since $b=4 \Rightarrow b^2=16$ and $a^2=25$, the equation of the ellipse is $\frac{(x-3)^2}{25}+\frac{(y-1)^2}{16}=1$.

20. Center at (–2, 7), major axis vertical, with length 10, $c = 2$.
Since the center is (–2, 7) and the major axis is vertical, the equation has the form $\frac{(x+2)^2}{b^2}+\frac{(y-7)^2}{a^2}=1$.
The major axis has length 10, so $a = 5$. Using $a = 5$ and $c = 2$ to find b^2, we have the following.
$c^2=a^2-b^2 \Rightarrow 4=25-b^2 \Rightarrow b^2=21$
Since $a=5 \Rightarrow a^2=25$ and $b^2=21$, the equation of the ellipse is $\frac{(x+2)^2}{21}+\frac{(y-7)^2}{25}=1$.

21. Vertices at (4, 9), (4, 1); minor axis with length 6.
The length of the minor axis is $2b$, so $b = 3$. The distance between the vertices is 9 – 1 = 8, so $2a = 8$ and thus $a = 4$. The center is the midpoint between the vertices. Thus, the center is located at $\left(\frac{4+4}{2},\frac{9+1}{2}\right)=\left(\frac{8}{2},\frac{10}{2}\right)=(4,5)$. Since the major axis is vertical, the equation is of the form $\frac{(x-4)^2}{b^2}+\frac{(y-5)^2}{a^2}=1$. Since $a=4 \Rightarrow a^2=16$ and $b=3 \Rightarrow b^2=9$, the equation of the ellipse is $\frac{(x-4)^2}{9}+\frac{(y-5)^2}{16}=1$.

22. Foci at (–3, –3), (7, –3); (2, –7) on the ellipse.
The distance between the foci is 7 – (–3) = 10, so $2c = 10$ and thus $c = 5$. The center is the midpoint between the foci. Thus, the center is located at $\left(\frac{7+(-3)}{2},\frac{-3+(-3)}{2}\right)=\left(\frac{4}{2},\frac{-6}{2}\right)=(2,-3)$. Since the foci lie on the horizontal line $y = –3$, the major axis is horizontal, so the equation is of the form $\frac{(x-2)^2}{a^2}+\frac{(y+3)^2}{b^2}=1$. The minor axis lies along the vertical line through the center, that is, the line $x = 2$. Since the point (2, –7) is on the ellipse and is also on the minor axis, it must be an endpoint of the minor axis. Since the distance between this point and the center is $|-7-(-3)|=4$, we have $b = 4$. Using $c = 5$ and $b = 4$ to find a^2, we have $c^2=a^2-b^2 \Rightarrow 25=a^2-16 \Rightarrow a^2=41$. Since $a^2=41$ and $b=4 \Rightarrow b^2=16$, the equation of the ellipse is $\frac{(x-2)^2}{41}+\frac{(y+3)^2}{16}=1$.

23. Foci at (0, –3), (0, 3); (8, 3) on the ellipse.
The distance between the foci is 3 – (–3) = 6, so $2c = 6$ and thus $c = 3$. The center is the midpoint between the foci, so the center is (0, 0). Since the foci lie on the y-axis, the major axis is vertical, so the equation is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. Now recall from Figure 15 on page 617 of the text, $d(P,F)+d(P,F')=2a$. If we let $P(8,3)$, $F'(0,-3)$, and $F(0,3)$ represent the point on the ellipse and the foci, respectively, we have the following.

$$\begin{aligned} &d(P,F)+d(P,F') \\ &=\sqrt{(8-0)^2+(3+3)^2}+\sqrt{(8-0)^2+(3-3)^2} \\ &=10+8=18=2a \Rightarrow a=9 \end{aligned}$$

Solving for b^2, we have $c^2=a^2-b^2 \Rightarrow 9=81-b^2 \Rightarrow b^2=72$. Since $a=9 \Rightarrow a^2=81$ and $b^2=72$, the equation of the ellipse is $\frac{x^2}{72}+\frac{y^2}{81}=1$.

24. Foci at (–4, 0), (4, 0); sum of distances from foci to point on ellipse is 9.
From the foci, we have $c = 4$. Let P be any point on the ellipse. Now recall from Figure 15 on page 617 of the text,
$d(P,F)+d(P,F')=2a.$ Since
$9=2a \Rightarrow a=\frac{9}{2}.$ Now since the vertices are points on the ellipse, the equation
$d(P,F)+d(P,F')=2a$ holds when P is one of the vertices. Since the foci lie on the x-axis, the vertices must also lie on the x-axis, so their coordinates are $V(a, 0)$ and $V'(-a, 0)$.
Solving for b^2, we have the following.
$c^2=a^2-b^2 \Rightarrow c^2=\left(\frac{9}{4}\right)^2-b^2 \Rightarrow$
$16=\frac{81}{4}-b^2 \Rightarrow b^2=\frac{81}{4}-16 \Rightarrow$
$b^2=\frac{81}{4}-\frac{64}{4}=\frac{17}{4}$
Since $a=\frac{9}{2} \Rightarrow a^2=\frac{81}{4}$ and $b^2=\frac{17}{4}$, the equation of the ellipse is
$\frac{x^2}{\frac{81}{4}}+\frac{y^2}{\frac{17}{4}}=1$ or $\frac{4x^2}{81}+\frac{4y^2}{17}=1.$

25. Foci at $(0,4)$, $(0,-4)$; sum of distances from foci to point on ellipse is 10.
The center is halfway between the foci, so the center is (0, 0). The distance between the foci is $4-(-4)=8,$ so $2c = 8$ and thus $c = 4$. The sum of the distances from the foci to any point on the ellipse is 10, so $2a=10 \Rightarrow a=5.$ Since the foci lie on the y-axis, the equation is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$ Solving for b^2, we have
$c^2=a^2-b^2 \Rightarrow 4^2=5^2-b^2 \Rightarrow 16=25-b^2 \Rightarrow b^2=9.$
Since $a=5 \Rightarrow a^2=25$ and $b^2=9$, the equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{25}=1.$

26. Eccentricity $\frac{1}{2}$; vertices at (–4, 0), (4, 0).
From the vertices, we have $a = 4$ and $a^2=16.$ Since the vertices lie on the x-axis, the equation is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ Thus, the equation has the form $\frac{x^2}{16}+\frac{y^2}{b^2}=1.$ Use the eccentricity and a to find c.
$e=\frac{c}{a} \Rightarrow \frac{1}{2}=\frac{c}{4} \Rightarrow 4=2c \Rightarrow c=2$
Now solving for b^2, we have
$c^2=a^2-b^2 \Rightarrow 2^2=4^2-b^2 \Rightarrow$
$4=16-b^2 \Rightarrow b^2=12$
The equation is therefore $\frac{x^2}{16}+\frac{y^2}{12}=1.$

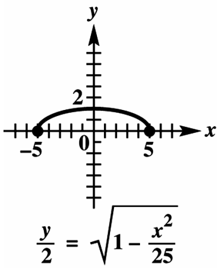
27. Eccentricity $\frac{3}{4}$, foci at (0, –2), (0, 2).
The foci are on the y-axis, so the equation has the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$ From the foci, we have $c = 2$. Use the eccentricity and c to find a.
$e=\frac{c}{a} \Rightarrow \frac{3}{4}=\frac{2}{a} \Rightarrow 3a=8 \Rightarrow a=\frac{8}{3}$
Solving for b^2, we have
$c^2=a^2-b^2 \Rightarrow 2^2=\left(\frac{8}{3}\right)^2-b^2 \Rightarrow$
$4=\frac{64}{9}-b^2 \Rightarrow b^2=\frac{64}{9}-\frac{36}{9}=\frac{28}{9}$
Since $a=\frac{8}{3} \Rightarrow a^2=\frac{64}{9}$ and $b^2=\frac{28}{9}$, the equation is $\frac{x^2}{\frac{28}{9}}+\frac{y^2}{\frac{64}{9}}=1$ or $\frac{9x^2}{28}+\frac{9y^2}{64}=1.$

28. Eccentricity $\frac{2}{3}$, foci at (0, –9), (0, 9).
The foci are on the y-axis, so the equation has the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$ From the given information, we have $e=\frac{2}{3}$ and $c = 9$. Using this information, we have
$e=\frac{c}{a} \Rightarrow \frac{2}{3}=\frac{9}{a} \Rightarrow a=\frac{27}{2}.$ Solving for b^2 we have $c^2=a^2-b^2 \Rightarrow 9^2=\left(\frac{27}{2}\right)^2-b^2 \Rightarrow$
$81=\frac{729}{4}-b^2 \Rightarrow b^2=\frac{729}{4}-\frac{324}{4}=\frac{405}{4}.$ Since $a=\frac{27}{2} \Rightarrow a^2=\frac{729}{4}$ and $b^2=\frac{405}{4}$, the equation is $\frac{x^2}{\frac{405}{4}}+\frac{y^2}{\frac{729}{4}}=1$ or $\frac{4x^2}{405}+\frac{4y^2}{729}=1.$

29. $\frac{y}{2}=\sqrt{1-\frac{x^2}{25}}$
Square both sides to get
$\frac{y^2}{4}=1-\frac{x^2}{25} \Rightarrow \frac{x^2}{25}+\frac{y^2}{4}=1 \Rightarrow \frac{x^2}{5^2}+\frac{y^2}{2^2}=1,$
which is the equation of an ellipse centered at the origin with x-intercepts ± 5 (vertices) and y-intercepts ± 2 (endpoints of minor axis).
Since $\sqrt{1-\frac{x^2}{25}} \ge 0,$ the only possible values of y are those making $\frac{y}{2} \ge 0 \Rightarrow y \ge 0.$ The domain is [–5, 5]. The range is [0, 2]. The graph of the original equation is the upper half of the ellipse. By applying the vertical line test, we see that this is the graph of a function.

(*continued on next page*)

(*continued from page 1043*)



30. $\frac{x}{4}=\sqrt{1-\frac{y^2}{9}}$

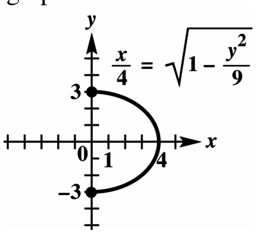
Square both sides to get

$\frac{x^2}{16}=1-\frac{y^2}{9}\Rightarrow\frac{x^2}{16}+\frac{y^2}{9}=1\Rightarrow\frac{x^2}{4^2}+\frac{y^2}{3^2}=1,$

which is the equation of an ellipse centered at the origin with x-intercepts ±4 (vertices) and y-intercepts ±3 (endpoints of minor axis).

Since $\sqrt{1-\frac{y^2}{9}}\geq 0,$ we must have

$\frac{x}{4}\geq 0\Rightarrow x\geq 0,$ so the graph of the original equation is the right half of the ellipse. The domain is [0, 4]. The range is [–3, 3]. The vertical line test shows that this is not the graph of a function.

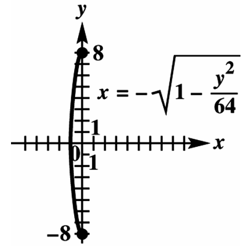


31. $x=-\sqrt{1-\frac{y^2}{64}}$

Square both sides to get

$x^2=1-\frac{y^2}{64}\Rightarrow\frac{x^2}{1}+\frac{y^2}{64}=1\Rightarrow\frac{x^2}{1^2}+\frac{y^2}{8^2}=1,$ the equation of an ellipse centered at the origin with x-intercepts ±1 (endpoints of minor axis) and y-intercepts ±8 (vertices). Since

$-\sqrt{1-\frac{y^2}{64}}\leq 0,$ we must have $x\leq 0$, so the graph of the original equation is the left half of the ellipse. The domain is [–1, 0]. The range is [–8, 8]. The vertical line test shows that this is not the graph of a function.



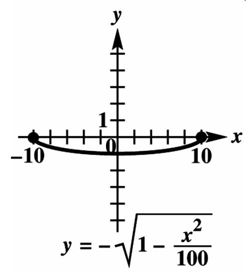
32. $y=-\sqrt{1-\frac{x^2}{100}}$

Square both sides to get

$y^2=1-\frac{x^2}{100}\Rightarrow\frac{x^2}{100}+\frac{y^2}{1}=1\Rightarrow\frac{x^2}{10^2}+\frac{y^2}{1^2}=1,$

which is the equation of an ellipse centered at the origin with x-intercepts ±10 (vertices) and y-intercepts ±1 (endpoints of minor axis).

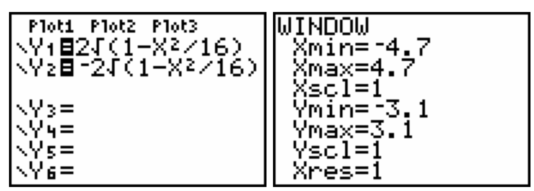
Since $-\sqrt{1-\frac{x^2}{100}}\leq 0,$ we must have $y\leq 0$, so the graph of the original equation is the bottom half of the ellipse. The domain is [–10, 10]. The range is [–1, 0]. The vertical line test shows that this is the graph of a function.

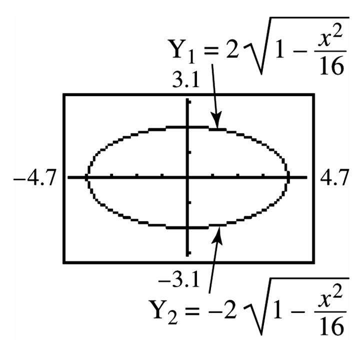


33. Solve for y in the equation of the ellipse.

$\frac{x^2}{16}+\frac{y^2}{4}=1\Rightarrow\frac{y^2}{4}=1-\frac{x^2}{16}\Rightarrow y^2=4\left(1-\frac{x^2}{16}\right)\Rightarrow$

$$y=\pm\sqrt{4\left(1-\frac{x^2}{16}\right)}\Rightarrow y=\pm2\sqrt{1-\frac{x^2}{16}}$$

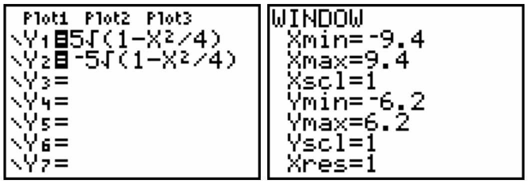


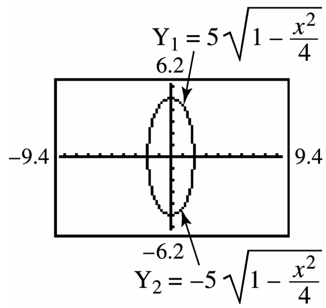


34. Solve for y in the equation of the ellipse.

$$\frac{x^2}{4}+\frac{y^2}{25}=1 \Rightarrow \frac{y^2}{25}=1-\frac{x^2}{4} \Rightarrow y^2=25\left(1-\frac{x^2}{4}\right) \Rightarrow$$

$$y=\pm\sqrt{25\left(1-\frac{x^2}{4}\right)} \Rightarrow y=\pm 5\sqrt{1-\frac{x^2}{4}}$$





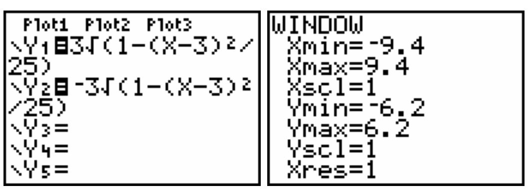
35. Solve for y in the equation of the ellipse.

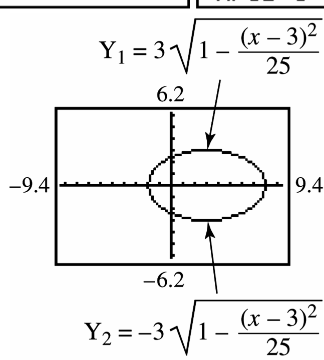
$$\frac{(x-3)^2}{25}+\frac{y^2}{9}=1 \Rightarrow \frac{y^2}{9}=1-\frac{(x-3)^2}{25}$$

$$y^2=9\left(1-\frac{(x-3)^2}{25}\right)$$

$$y=\pm\sqrt{9\left(1-\frac{(x-3)^2}{25}\right)}$$

$$y=\pm 3\sqrt{1-\frac{(x-3)^2}{25}}$$





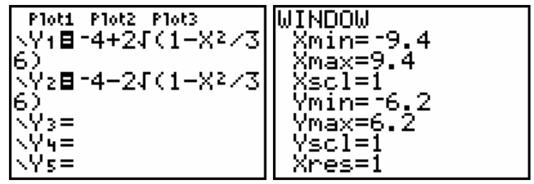
36. Solve for y in the equation of the ellipse.

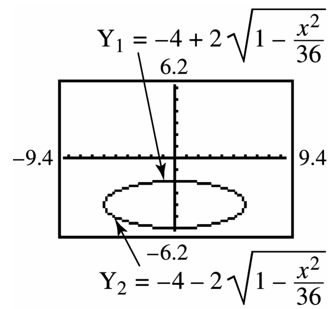
$$\frac{x^2}{36}+\frac{(y+4)^2}{4}=1 \Rightarrow \frac{(y+4)^2}{4}=1-\frac{x^2}{36}$$

$$(y+4)^2=4\left(1-\frac{x^2}{36}\right)$$

$$y+4=\pm\sqrt{4\left(1-\frac{x^2}{36}\right)}$$

$$y=-4\pm 2\sqrt{1-\frac{x^2}{36}}$$





37. $\frac{x^2}{3}+\frac{y^2}{4}=1$

Since $4>3$, $a^2=4$, which gives $a=2$. Also $c=\sqrt{a^2-b^2}=\sqrt{4-3}=\sqrt{1}=1$. Thus, $e=\frac{c}{a}=\frac{1}{2}$.

38. $\frac{x^2}{8}+\frac{y^2}{4}=1$

Since $8>4$, we have $a^2=8$, which gives $a=\sqrt{8}=2\sqrt{2}$. To find c, we determine that $c=\sqrt{a^2-b^2}=\sqrt{8-4}=\sqrt{4}=2$. Thus, we have $e=\frac{c}{a}=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\approx .71$.

39. $4x^2+7y^2=28 \Rightarrow \frac{x^2}{7}+\frac{y^2}{4}=1$

Since $7>4$, $a^2=7$, which gives $a=\sqrt{7}$. Also $c=\sqrt{a^2-b^2}=\sqrt{7-4}=\sqrt{3}$. Thus, $e=\frac{c}{a}=\frac{\sqrt{7}}{\sqrt{3}}=\frac{\sqrt{21}}{3}\approx .65$.

40. $x^2 + 25y^2 = 25 \Rightarrow \frac{x^2}{25} + \frac{y^2}{1} = 1$

Since $25 > 1$, $a^2 = 25$, which gives $a = 5$.

Also $c = \sqrt{a^2 - b^2} = \sqrt{25 - 1} = \sqrt{24} = 2\sqrt{6}$.

Thus, $e = \frac{c}{a} = \frac{2\sqrt{6}}{5} \approx .98$.

41. Answers will vary.

42. Answers will vary. Make the two fixed points, which represent the foci in Exercise 41, a single point. This point becomes the center of the circle and half of the length of the string becomes the radius of the circle when sketched in a similar manner to the ellipse.

43. Place the half-ellipse that represents the overpass on a coordinate system with the center of the bottom of the overpass at the origin. If the complete ellipse were drawn, the center of the ellipse would be $(0,0)$. Then the half-ellipse will include the points $(0,15), (-10,0)$, and $(10,0)$. The equation is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. Thus, for the complete ellipse, we have $a = 15$ and $b = 10$. Thus, we have $\frac{x^2}{10^2} + \frac{y^2}{15^2} = 1 \Rightarrow \frac{x^2}{100} + \frac{y^2}{225} = 1$. To find the equation of the half-ellipse, solve this equation for *y* and use the positive square root since the overpass is represented by the upper half of the ellipse.

$$\frac{x^2}{100} + \frac{y^2}{225} = 1 \Rightarrow \frac{y^2}{225} = 1 - \frac{x^2}{100} \Rightarrow$$
$$y^2 = 225\left(1 - \frac{x^2}{100}\right) \Rightarrow$$
$$y = \sqrt{225\left(1 - \frac{x^2}{100}\right)} \Rightarrow y = 15\sqrt{1 - \frac{x^2}{100}}$$

Find the *y*-coordinate of the point whose *x*-coordinate is $\frac{1}{2}(12) = 6$.

$$y = 15\sqrt{1 - \frac{x^2}{100}} \Rightarrow y = 15\sqrt{1 - \frac{6^2}{100}} \Rightarrow$$
$$y = 15\sqrt{1 - \frac{36}{100}} = 15\sqrt{\frac{64}{100}} = 15\sqrt{\frac{16}{25}} = 15\left(\frac{4}{5}\right) = 12$$

The tallest truck that can pass under the overpass is 12 feet tall.

44. $100x^2 + 324y^2 = 32,400 \Rightarrow$

$$\frac{x^2}{324} + \frac{y^2}{100} = 1 \Rightarrow \frac{x^2}{18^2} + \frac{y^2}{10^2} = 1$$

(a) The height of the arch is the vertical distance from the center, which is 10 m.

(b) The width across the bottom is twice the horizontal distance from the center, which is $2(18) = 36$ m.

45. Referring to Example 6 on page 966 of the text, the greatest distance between the comet and the sun is $a + c = 3281$ million miles. If $a + c = 3281 \Rightarrow c = 3281 - a$ and $e = \frac{c}{a}$, we have $e = \frac{3281-a}{a}$. Since $e = .9673$, we have

$$.9673 = \frac{3281-a}{a} \Rightarrow .9673a = 3281 - a \Rightarrow$$
$$1.9673a = 3281 \Rightarrow a = \frac{3281}{1.9673} \approx 1668.$$

We then have $c = 3281 - 1668 = 1613$, which implies $a - c = 1668 - 1613 = 55$. Thus, the shortest distance between Halley's Comet and the sun is about 55 million miles.

46. **(a)** The graph of the orbit of the satellite is determined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a = 4465$ and $b = 4462$. In order to graph this equation, we must solve the equation for *y*.

$$\frac{x^2}{4465^2} + \frac{y^2}{4462^2} = 1 \Rightarrow \frac{y^2}{4462^2} = 1 - \frac{x^2}{4465^2} \Rightarrow$$
$$y^2 = 4462^2\left(1 - \frac{x^2}{4465^2}\right)$$
$$y = \pm\sqrt{4462^2\left(1 - \frac{x^2}{4465^2}\right)}$$
$$y = \pm 4462\sqrt{1 - \frac{x^2}{4465^2}}$$

From the last equation, we obtain the following two functions.

$Y_1 = 4462\sqrt{1 - \frac{x^2}{4465^2}}$ and

$Y_2 = -4462\sqrt{1 - \frac{x^2}{4465^2}} = -Y_1$

The graph of Earth can be represented by a circle of radius 3960 centered at one focus. To determine the foci of the orbit, we must determine *c*.

$$c^2 = a^2 - b^2 \Rightarrow c^2 = 4465^2 - 4462^2 \Rightarrow$$
$$c^2 = 26,781 \Rightarrow c = \sqrt{26,781} \approx 163.6$$

If the center of Earth is located at (163.6, 0), then the equation of the circle will be

$$(x - 163.6)^2 + y^2 = 3960^2$$

Solving for *y*, we have

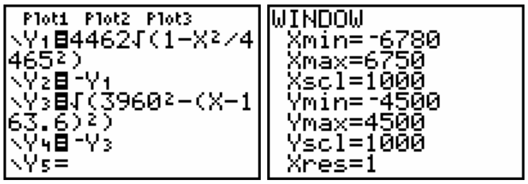
$$(x - 163.6)^2 + y^2 = 3960^2$$
$$y^2 = 3960^2 - (x - 163.6)^2$$
$$y = \pm\sqrt{3960^2 - (x - 163.6)^2}$$

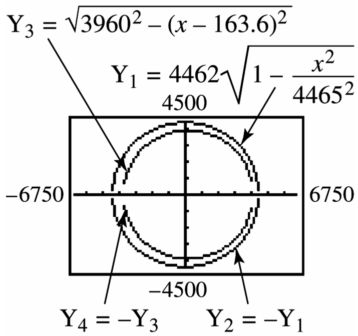
From the last equation, we obtain the following two functions.

$Y_3 = \sqrt{3960^2 - (x-163.6)^2}$ and

$Y_4 = -\sqrt{3960^2 - (x-163.6)^2} = -Y_3$

Graph all four functions on the same screen, using the given window.





(b) From the graph, we can see that the distance is maximized and minimized when the orbits intersect the *x*-axis. The intercepts of the satellite's orbit are ±4465. The *x*-intercepts of Earth's surface occur when $(x-163.6)^2 + y^2 = 3960^2$ and $y = 0$.

$$(x-163.6)^2 + 0^2 = 3960^2$$
$$(x-163.6)^2 = 3960^2$$
$$x - 163.6 = \pm 3960$$
$$x = 163.6 \pm 3960$$
$$x = 4123.6 \text{ or } x = -3796.4$$

The minimum distance is $4465 - 4123.6 = 341.4 \approx 341$ miles, and the maximum distance is $-3796.4 - (-4465) = 668.6 \approx 669$ miles.

47. (a) Use the given values of *e* and *a* to find the value of *c* for each planet. Then use the values of *a* and *c* to find the value of *b*.

Neptune:

$e = \frac{c}{a} \Rightarrow c = ea \Rightarrow c = (.009)(30.1) \Rightarrow c = .2709$

$b^2 = a^2 - c^2 \Rightarrow b^2 = (30.1)^2 - (.2709)^2 \Rightarrow b^2 \approx 905.9366 \Rightarrow b \approx 30.1$

Since $c = .2709$, the graph should be translated .2709 units to the right so that the sun will be located at the origin. It's essentially circular with equation

$\frac{(x-.2709)^2}{30.1^2} + \frac{y^2}{30.1^2} = 1.$

Pluto:

$e = \frac{c}{a} \Rightarrow c = ea \Rightarrow c = (.249)(39.4) \Rightarrow c = 9.8106$

$b^2 = a^2 - c^2 \Rightarrow b^2 = (39.4)^2 - (9.8106)^2 \Rightarrow$

$b^2 \approx 1456.1121 \Rightarrow b \approx 38.16$

As with Neptune, we translate the graph by c units to the right so that the sun will be located at the origin. The equation is

$\frac{(x-9.8106)^2}{39.4^2} + \frac{y^2}{38.16^2} = 1.$

(b) In order to graph these equations on a graphing calculator, we must solve each equation for y. Each equation will be broken down into two functions, so we will need to graph four functions.

Neptune:

$$\frac{(x-.2709)^2}{30.1^2} + \frac{y^2}{30.1^2} = 1$$
$$(x-.2709)^2 + y^2 = 30.1^2$$
$$y = \pm\sqrt{30.1^2 - (x-.2709)^2}$$

Pluto: $\frac{(x-9.8106)^2}{39.4^2} + \frac{y^2}{38.16^2} = 1$

$$\frac{y^2}{38.16^2} = 1 - \frac{(x-9.8106)^2}{39.4^2}$$
$$y^2 = 38.16^2\left(1 - \frac{(x-9.8106)^2}{39.4^2}\right)$$
$$y = \pm\sqrt{38.16^2\left(1 - \frac{(x-9.8106)^2}{39.4^2}\right)}$$
$$y = \pm 38.16\sqrt{1 - \frac{(x-9.8106)^2}{39.4^2}}$$

Graph the following four functions on the same screen.

$Y_1 = \sqrt{30.1^2 - (x-.2709)^2}$

$Y_2 = -\sqrt{30.1^2 - (x-.2709)^2} = -Y_1$

$Y_3 = 38.16\sqrt{1 - \frac{(x-9.8106)^2}{39.4^2}}$

$Y_4 = -38.16\sqrt{1 - \frac{(x-9.8106)^2}{39.4^2}} = -Y_3$

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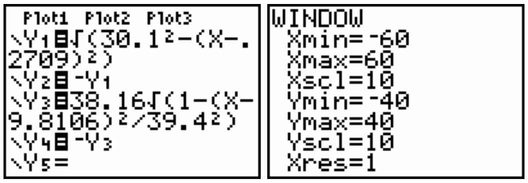
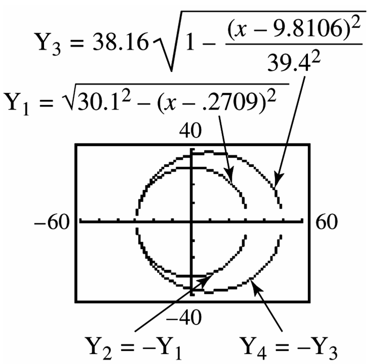




48. **(a)** An ellipse with major axis 620 ft and minor axis 513 ft has $2a = 620$ and $2b = 513$, or $a = 310$ and $b = 256.5$. The distance between the center and a focus is c, where $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2 \Rightarrow c^2 = 310^2 - (256.5)^2 \Rightarrow$

$c^2 = 96{,}100 - 65{,}792.25$

$c^2 = 30{,}307.75 \Rightarrow c \approx 174.1$

(The negative value of c is rejected.) The distance between the two foci of the ellipse is $2c = 2\,(174.1) = 348.2$. There are 348.2 ft between the foci of the Roman Colosseum.

(b) In the relation $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$, let $a = \frac{620}{2} = 310$ and $b = \frac{513}{2} = 256.5$. Thus, we have the following.

$C \approx 2\pi\sqrt{\frac{310^2+(256.5)^2}{2}} \approx 1787.6$

The circumference of the Roman Colosseum is about 1787.6 ft.

49. The stone and the wave source must be placed at the foci, (c, 0) and (–c, 0). Here $a^2 = 36$ and $b^2 = 9$, so $c^2 = a^2 - b^2 \Rightarrow c^2 = 36 - 9 \Rightarrow c^2 = 27 \Rightarrow c = \sqrt{27} = 3\sqrt{3}$. Thus, the kidney stone and source of the beam must be placed $3\sqrt{3}$ units from the center.

50. First, put the equation $9x^2 + 4y^2 = 36$ in standard form. $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$

The stone and the wave source must be placed at the foci, (0, c) and (0, –c). Here $a^2 = 9$ and $b^2 = 4$, so $c^2 = a^2 - b^2 \Rightarrow c^2 = 9 - 4 \Rightarrow c^2 = 5 \Rightarrow c = \sqrt{5}$. Thus, the kidney stone and source of the beam must be placed $\sqrt{5}$ units from the center.

Chapter 10 Quiz
(Sections 10.1–10.2)

1. **(a)** $x + 3 = 4(y-1)^2$

This is the equation of a parabola with vertex (−3, 1). The parabola opens right. The correct choices are B and D.

(b) $(x+3)^2 + (y-1)^2 = 81$

This is the equation of a circle with center (−3, 1). The correct choice is A.

(c) $25(x-2)^2 + (y-1)^2 = 100 \Rightarrow$

$\frac{(x-2)^2}{4} + \frac{(y-1)^2}{100} = 1$

This is the equation of an ellipse with center (2, 1) and major axis vertical. The correct choice is E.

(d) $\frac{(x-2)^2}{16} + \frac{(y-1)^2}{9} = 1$

This is the equation of an ellipse with center (2, 1) and major axis horizontal. The correct choices are C and E.

(e) $-2(x+3)^2 + 1 = y \Rightarrow -2(x+3)^2 = y - 1$

This is the equation of a parabola with vertex (−3, 1). The parabola opens down. The correct choice is D.

2. A parabola with vertex (−1, 2) and focus (2, 2) is a horizontal parabola that opens to the right. The distance between the vertex and the focus is 2 − (−1) = 3, so $p = 3$. The equation has the form $(y-2)^2 = 4p(x+1)$. Substituting 3 for p, we find that an equation for this parabola is $(y-2)^2 = 12(x+1)$.

3. A parabola passing through $\left(\sqrt{10},-5\right)$, opening downward, and vertex at the origin has an equation of the form $y = 4px^2$. Use this equation with the coordinates of the point $\left(\sqrt{10},-5\right)$ to find the value of *p*.

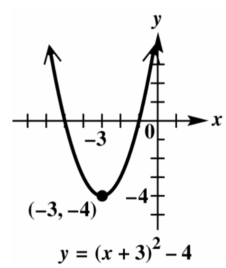
$y = 4px^2 \Rightarrow -5 = 4p\left(\sqrt{10}\right)^2 \Rightarrow$
$-5 = 40p \Rightarrow p = -\frac{1}{8}$
Thus, an equation of the parabola is
$y^2 = 4\left(-\frac{1}{8}\right)x = -\frac{1}{2}x.$

4. Center at (3, −2), $a = 5$, $c = 3$, major axis vertical.
Since the center is (3, −2) and the major axis is vertical, the equation has the form
$\frac{(x-3)^2}{b^2} + \frac{(y+2)^2}{a^2} = 1.$
Using $a = 5$ and $c = 3$ to find b^2, we have the following.
$c^2 = a^2 - b^2 \Rightarrow 9 = 25 - b^2 \Rightarrow b^2 = 16$
Since $a = 5 \Rightarrow a^2 = 25$ and $b^2 = 16$, the equation of the ellipse is $\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$

5. Foci at (−3, 3), (−3, 11), major axis with length 10. The length of the major axis is 2*a*. Thus, we have $2a = 10 \Rightarrow a = 5$. From the foci, we have $c = \frac{11-3}{2} = 4$. Solving for b^2 we have $c^2 = a^2 - b^2 \Rightarrow 16 = 25 - b^2 \Rightarrow b^2 = 9$. The center is the midpoint between the foci. Thus, the center is located at
$\left(\frac{(-3)+(-3)}{2}, \frac{3+11}{2}\right) = \left(\frac{-6}{2}, \frac{14}{2}\right) = (-3,7)$. Since the foci lie on the vertical line $x = -3$, the major axis is vertical, so the equation is of the form $\frac{(x+3)^2}{b^2} + \frac{(y-7)^2}{a^2} = 1$. Since $a = 5 \Rightarrow a^2 = 25$ and and $b^2 = 9$, the equation of the ellipse is
$\frac{(x+3)^2}{9} + \frac{(y-7)^2}{25} = 1.$

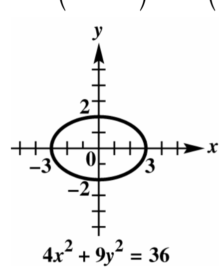
6. $y = (x+3)^2 - 4 \Rightarrow y + 4 = (x+3)^2$
This is a parabola with vertex $(-3,-4)$. The graph opens upward and has the same shape as $y = x^2$. It is a translation 3 units left and 4 units down of the graph of $y = x^2$. Since $y + 4 = (x+3)^2 \Rightarrow y + 4 = 4\left(\frac{1}{4}\right)(x+3)^2$,
$p = \frac{1}{4}$. So the focus is at
$\left(-3,-4+\frac{1}{4}\right) = \left(-3,-\frac{15}{4}\right)$. The directrix is
$y = -4 - \frac{1}{4} = -\frac{17}{4}$. The graph is symmetric about its axis, the vertical line $x = -3$. Use the vertex and axis and plot a few additional points.

x	*y*
−6	5
−5	0
−3	−4
−1	0
0	5



7. $4x^2 + 9y^2 = 36$

Rewriting the equation, we have $\frac{4x^2}{36} + \frac{9y^2}{36} = 1$
or $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$. The center is (0, 0). Since 3 > 2, we have $a = 3$ and $b = 2$, and the major axis is horizontal. The vertices are (−3, 0) and (3, 0). The endpoints of the minor axis are (0, −2) and (0, 2). To find the foci, we need to find *c* such that $c^2 = a^2 - b^2$.
$c^2 = a^2 - b^2 \Rightarrow c^2 = 9 - 4 \Rightarrow c^2 = 5 \Rightarrow c = \sqrt{5}$
Since the major axis lies on the *x*-axis, the foci are $\left(-\sqrt{5},0\right)$ and $\left(\sqrt{5},0\right)$.



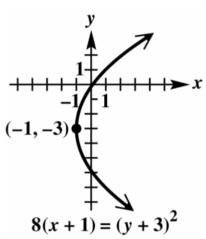
8. $8(x+1)=(y+3)^2$

This is a parabola with vertex $(-1,-3)$. The graph opens to the right and has the same shape as $x=y^2$. It is a translation 1 unit left and 3 units down of the graph of $x=\frac{1}{8}y^2$.

Since $8(x+1)=(y+3)^2 \Rightarrow$

$4(2)(x+1)=(y+3)^2$, $p = 2$. So the focus is at $(-1+2,-3)=(1,-3)$. The directrix is $x=-1-2=-3$. The graph is symmetric about its axis, the horizontal line $y=-3$. Use the vertex and axis and plot a few additional points.

x	y
1	−7
0	−5.83
−1	−2
0	−.17
1	1



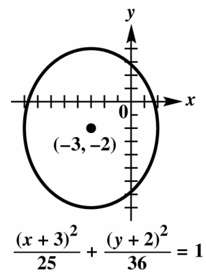
9. $\frac{(x+3)^2}{25}+\frac{(y+2)^2}{36}=1$

Rewriting the equation, we have $\frac{(x+3)^2}{5^2}+\frac{(y+2)^2}{6^2}=1$. The center is (–3, −2). We have $a = 6$ and $b = 5$. Since $a = 6$ is associated with y^2, the major axis of the ellipse is vertical. The vertices are on the vertical line through (–3, −2), and the endpoints of the minor axis are on the horizontal line through (–3, −2). The vertices are 6 units below and 6 units above the center at (–3, 4) and (–3, −8). The endpoints of the minor axis are 5 units to the left and 5 units to the right of the center at (–8, −2) and (2, −2). To find the foci, we need to find c such that $c^2=a^2-b^2$.

$c^2=a^2-b^2 \Rightarrow c^2=36-25 \Rightarrow c^2=11 \Rightarrow$

$c=\sqrt{11}$

Since the major axis lies on $x=-3$, the foci are $\left(-3,\,-2+\sqrt{11}\right)$ and $\left(-3,\,-2-\sqrt{11}\right)$.



10. $x=-4y^2-4y-3$

Complete the square on y to find the vertex and the axis.

$x=-4y^2-4y-3 \Rightarrow x=-4\left(y^2+y\quad\right)-3$

$x=-4\left(y^2+y+\frac{1}{4}-\frac{1}{4}\right)-3$

$x=-4\left(y^2+y+\frac{1}{4}\right)-4\left(-\frac{1}{4}\right)-3$

$x=-4\left(y^2+y+\frac{1}{4}\right)+1-3$

$x=-4\left(y+\frac{1}{2}\right)^2-2 \Rightarrow x+2=-4\left(y+\frac{1}{2}\right)^2$

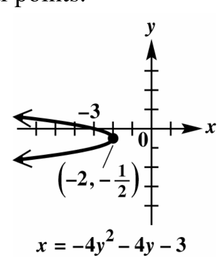
The vertex is $\left(-2,-\frac{1}{2}\right)$. The graph opens to the left and has the same shape as $x=-4y^2$, translated 2 units to the left and $\frac{1}{2}$ unit down.

Since $x+2=-4\left(y+\frac{1}{2}\right)^2 \Rightarrow$

$-\frac{1}{4}(x+2)=\left(y+\frac{1}{2}\right)^2 \Rightarrow$

$4\left(-\frac{1}{16}\right)(x+2)=\left(y+\frac{1}{2}\right)^2$, $p=-\frac{1}{16}$. So the focus is at $\left(-2-\frac{1}{16},-\frac{1}{2}\right)=\left(-\frac{33}{16},-\frac{1}{2}\right)$. The directrix is at $x=-2+\frac{1}{16}=-\frac{31}{16}$. The graph is symmetric about its axis, the horizontal line $y=-\frac{1}{2}$. Use the vertex and axis and plot a few additional points.

x	y
−6	−1.5
−4	−1.2
−2	−.5
−4	.2
−6	.5



Section 10.3: Hyperbolas

Connections (page 977)

From the definition of a hyperbola, we know the difference of the distances equals the constant $2a$. Substituting the given values, we have $d_1 - d_2 = 2a \Rightarrow 80 - 30 = 2a \Rightarrow 50 = 2a \Rightarrow a = 25.$ Since the distance between foci is $2c$, we have $2c = 100 \Rightarrow c = 50.$ Using these values, we solve for b^2. We have

$b^2 = c^2 - a^2 \Rightarrow b^2 = 50^2 - 25^2 \Rightarrow b^2 = 2500 - 625 \Rightarrow b^2 = 1875.$

Substituting these values into the general equation of a hyperbola with a horizontal transverse axis centered at the origin, we get the following.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{625} - \frac{y^2}{1875} = 1$$

Exercises

1. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

This is an equation of an ellipse with x-intercepts ± 5 and y-intercepts ± 3. The correct graph is C.

2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

This is the equation of an ellipse with x-intercepts ± 3 and y-intercepts ± 5. The correct graph is B.

3. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

This is the graph of a hyperbola with a horizontal transverse axis and x-intercepts ± 3 (no y-intercepts). The correct graph is D.

4. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

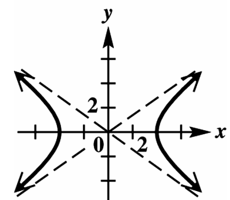
This is the graph of a hyperbola with a horizontal transverse axis and x-intercepts ± 5 (no y-intercepts). The correct graph is A.

5. $\frac{x^2}{16} - \frac{y^2}{9} = 1,$

This equation may be written as $\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1,$

which has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at (–4, 0) and (4, 0) (also x-intercepts ± 4). There are no y-intercepts. The domain is $(-\infty, -4] \cup [4, \infty)$. The range is $(-\infty, \infty)$.

The foci are on the x-axis. Since $c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 9 \Rightarrow c^2 = 25 \Rightarrow c = 5,$ the foci are (–5, 0) and (5, 0). Since $a = 4$ and $b = 3,$ the asymptotes are $y = \pm \frac{b}{a} x = \pm \frac{3}{4} x.$

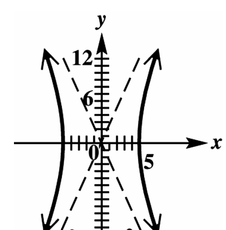


$\frac{x^2}{16} - \frac{y^2}{9} = 1$

6. $\dfrac{x^2}{25} - \dfrac{y^2}{144} = 1$

This equation may be written as $\frac{x^2}{5^2} - \frac{y^2}{12^2} = 1,$

which has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at (–5, 0) and (5, 0) (also x-intercepts ± 5). There are no y-intercepts. The domain is $(-\infty, -5] \cup [5, \infty)$. The range is $(-\infty, \infty)$. The foci are on the x-axis. Since $c^2 = a^2 + b^2 \Rightarrow c^2 = 25 + 144 \Rightarrow c^2 = 169 \Rightarrow c = 13,$ the foci are (–13, 0) and (13, 0). Since $a = 5$ and $b = 12,$ the asymptotes are $y = \pm \frac{b}{a} x = \pm \frac{12}{5} x.$



$\frac{x^2}{25} - \frac{y^2}{144} = 1$

7. $\dfrac{y^2}{25} - \dfrac{x^2}{49} = 1$

This equation may be written as $\frac{y^2}{5^2} - \frac{x^2}{7^2} = 1,$

which has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening upward and downward. The graph has vertices at (0,–5) and (0,5) (also y-intercepts ± 5). There are no x-intercepts. The domain is $(-\infty, \infty)$. The range is $(-\infty, -5] \cup [5, \infty)$. The foci are on the y-axis.

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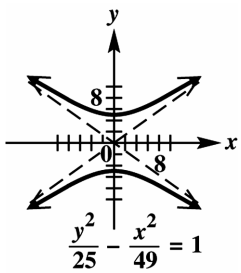
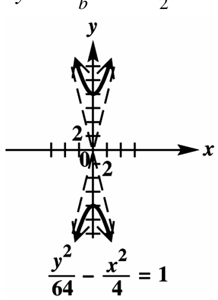
(*continued from page 1051*)

Since $c^2 = a^2 + b^2$, we have
$c^2 = a^2 + b^2 \Rightarrow c^2 = 25 + 49$
$c^2 = 74 \Rightarrow c = \sqrt{74}$
The foci are $\left(0, -\sqrt{74}\right)$ and $\left(0, \sqrt{74}\right)$. Since $a = 5$ and $b = 7$, the asymptotes are $y = \pm\frac{a}{b}x = \pm\frac{5}{7}x.$



8. $\dfrac{y^2}{64} - \dfrac{x^2}{4} = 1$

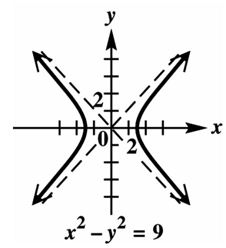
This equation may be written as $\frac{y^2}{8^2} - \frac{x^2}{2^2} = 1,$ which has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening upward and downward. The graph has vertices at (0,–8) and (0,8) (also y-intercepts $\pm$ 8). There are no x-intercepts. The domain is $(-\infty, \infty)$. The range is $\left(-\infty, -8\right] \cup \left[8, \infty\right)$. The foci are on the y-axis. Since $c^2 = a^2 + b^2$, we have $c^2 = a^2 + b^2 \Rightarrow c^2 = 64 + 4 \Rightarrow$
$c^2 = 68 \Rightarrow c = \sqrt{68} = 2\sqrt{17}$
The foci are $\left(0, -2\sqrt{17}\right)$ and $\left(0, 2\sqrt{17}\right)$. Since $a = 8$ and $b = 2$, the asymptotes are $y = \pm\frac{a}{b}x = \pm\frac{8}{2}x = \pm 4x.$



9. $x^2 - y^2 = 9$

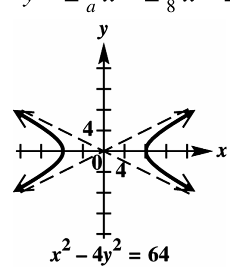
This equation may be written as
$\frac{x^2}{9} - \frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{3^2} - \frac{y^2}{3^2} = 1,$ which has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at $(-3, 0)$ and (3, 0) (also x-intercepts $\pm$ 3). There are no y-intercepts. The domain is $(-\infty, -3] \cup [3, \infty)$. The range is $(-\infty, \infty)$. The foci are on the x-axis. Since $c^2 = a^2 + b^2$, we have $c^2 = a^2 + b^2 \Rightarrow$
$c^2 = 9 + 9 \Rightarrow c^2 = 18 \Rightarrow c = \sqrt{18} = 3\sqrt{2}$. The foci are $\left(-3\sqrt{2}, 0\right)$ and $\left(3\sqrt{2}, 0\right)$. Since $a = 3$ and $b = 3$, the asymptotes are
$y = \pm\frac{b}{a}x = \pm\frac{3}{3}x = \pm x$



10. $x^2 - 4y^2 = 64$

This equation may be written as
$\frac{x^2}{64} - \frac{y^2}{16} = 1 \Rightarrow \frac{x^2}{8^2} - \frac{y^2}{4^2} = 1,$ which has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at $(-8, 0)$ and (8, 0) (also x-intercepts $\pm$ 8). There are no y-intercepts. The domain is $(-\infty, -8] \cup [8, \infty)$. The range is $(-\infty, \infty)$. The foci are on the x-axis. Since $c^2 = a^2 + b^2$, we have
$c^2 = a^2 + b^2 \Rightarrow c^2 = 64 + 16$
$c^2 = 80 \Rightarrow c = \sqrt{80} = 4\sqrt{5}$
The foci are $\left(-4\sqrt{5}, 0\right)$ and $\left(4\sqrt{5}, 0\right)$. Since $a = 8$ and $b = 2$, the asymptotes are
$y = \pm\frac{b}{a}x = \pm\frac{4}{8}x = \pm\frac{1}{2}x$

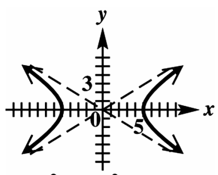


11. $9x^2 - 25y^2 = 225$

This equation may be written as
$\frac{x^2}{25} - \frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{5^2} - \frac{y^2}{3^2} = 1,$ which has the form
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at $(-5, 0)$ and (5, 0) (also x-intercepts ± 5). There are no y-intercepts. The domain is $(-\infty, -5] \cup [5, \infty)$. The range is $(-\infty, \infty)$. The foci are on the x-axis. Since $c^2 = a^2 + b^2,$ we have

$c^2 = a^2 + b^2 \Rightarrow c^2 = 25 + 9 \Rightarrow$
$c^2 = 34 \Rightarrow c = \sqrt{34}$

The foci are $\left(-\sqrt{34}, 0\right)$ and $\left(\sqrt{34}, 0\right).$ Since $a = 5$ and $b = 3,$ the asymptotes are $y = \pm \frac{b}{a} x = \pm \frac{3}{5} x.$



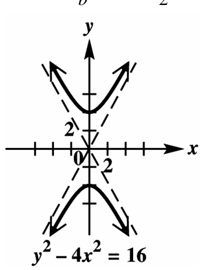
$9x^2 - 25y^2 = 225$

12. $y^2 - 4x^2 = 16$

This equation may be written as
$\frac{y^2}{16} - \frac{x^2}{4} = 1 \Rightarrow \frac{y^2}{4^2} - \frac{x^2}{2^2} = 1,$ which has the form
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening upward and downward. The graph has vertices at (0,–4) and (0,4) (also y-intercepts ± 4). There are no x-intercepts. The domain is $(-\infty, \infty)$. The range is $(-\infty, -4] \cup [4, \infty).$ The foci are on the y-axis. Since $c^2 = a^2 + b^2,$ we have

$c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 4$
$c^2 = 20 \Rightarrow c = \sqrt{20} = 2\sqrt{5}$

The foci are $\left(0, -2\sqrt{5}\right)$ and $\left(0, 2\sqrt{5}\right).$ Since $a = 4$ and $b = 2,$ the asymptotes are $y = \pm \frac{a}{b} x = \pm \frac{4}{2} x = \pm 2x.$



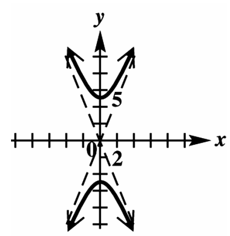
$y^2 - 4x^2 = 16$

13. $4y^2 - 25x^2 = 100$

This equation may be written as
$\frac{y^2}{25} - \frac{x^2}{4} = 1 \Rightarrow \frac{y^2}{5^2} - \frac{x^2}{2^2} = 1,$ which has the form
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening upward and downward. The graph has vertices at (0,–5) and (0,5) (also y-intercepts ± 5). There are no x-intercepts. The domain is $(-\infty, \infty)$. The range is $(-\infty, -5] \cup [5, \infty).$ The foci are on the y-axis. Since $c^2 = a^2 + b^2,$ we have

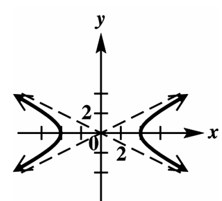
$c^2 = a^2 + b^2 \Rightarrow c^2 = 25 + 4$
$c^2 = 29 \Rightarrow c = \sqrt{29}$

The foci are $\left(0, -\sqrt{29}\right)$ and $\left(0, \sqrt{29}\right).$ Since $a = 5$ and $b = 2,$ the asymptotes are $y = \pm \frac{a}{b} x = \pm \frac{5}{2} x.$



$4y^2 - 25x^2 = 100$

Exercise 13



$\frac{x^2}{4} - y^2 = 4$

Exercise 14

14. $\frac{x^2}{4} - y^2 = 4$

This equation may be written as
$\frac{x^2}{16} - \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{4^2} - \frac{y^2}{2^2} = 1,$ which has the form
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at $(-4, 0)$ and (4, 0) (also x-intercepts ± 4). There are no y-intercepts. The domain is $(-\infty, -4] \cup [4, \infty)$. The range is $(-\infty, \infty)$. The foci are on the x-axis. Since $c^2 = a^2 + b^2,$ we have

$c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 4 \Rightarrow$
$c^2 = 20 \Rightarrow c = \sqrt{20} = 2\sqrt{5}$

The foci are $\left(-2\sqrt{5}, 0\right)$ and $\left(2\sqrt{5}, 0\right).$ Since $a = 4$ and $b = 2,$ the asymptotes are $y = \pm \frac{b}{a} x = \pm \frac{2}{4} x = \pm \frac{1}{2} x.$

15. $9x^2-4y^2=1$

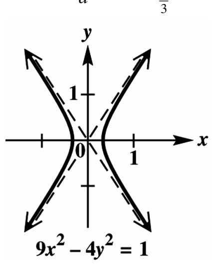
This equation may be written as

$\frac{x^2}{\frac{1}{9}}-\frac{y^2}{\frac{1}{4}}=1 \Rightarrow \frac{x^2}{\left(\frac{1}{3}\right)^2}-\frac{y^2}{\left(\frac{1}{2}\right)^2}=1,$ which has the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ The hyperbola is centered at (0, 0) with branches opening to the left and right. The graph has vertices at $\left(-\frac{1}{3}, 0\right)$ and $\left(\frac{1}{3}, 0\right)$ (also x-intercepts $\pm\ \frac{1}{3}$). There are no y-intercepts. The domain is $\left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, \infty\right).$ The range is $(-\infty, \infty)$.

The foci are on the x-axis. Since $c^2=a^2+b^2$, we have $c^2=a^2+b^2 \Rightarrow c^2=\frac{1}{9}+\frac{1}{4} \Rightarrow$

$c^2=\frac{4+9}{36} \Rightarrow c^2=\frac{13}{36} \Rightarrow c=\sqrt{\frac{13}{36}}=\frac{\sqrt{13}}{6}$. The foci are $\left(-\frac{\sqrt{13}}{6}, 0\right)$ and $\left(\frac{\sqrt{13}}{6}, 0\right).$ Since $a=\frac{1}{3}$ and $b=\frac{1}{2}$, the asymptotes are

$y=\pm\frac{b}{a}x=\pm\frac{\frac{1}{2}}{\frac{1}{3}}x=\pm\frac{3}{2}x.$



16. $25y^2-9x^2=1$

This equation may be written as

$\frac{y^2}{\frac{1}{25}}-\frac{x^2}{\frac{1}{9}}=1 \Rightarrow \frac{y^2}{\left(\frac{1}{5}\right)^2}-\frac{x^2}{\left(\frac{1}{3}\right)^2}=1,$ which has the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$ The hyperbola is centered at (0, 0) with branches opening upward and downward. The graph has vertices at $\left(0, -\frac{1}{5}\right)$ and $\left(0, \frac{1}{5}\right)$ (also y-intercepts $\pm\ \frac{1}{5}$). There are no x-intercepts. The domain is $(-\infty, \infty)$. The range is $\left(-\infty, -\frac{1}{5}\right] \cup \left[\frac{1}{5}, \infty\right).$

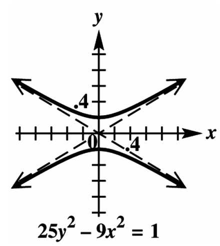
The foci are on the y-axis. Since $c^2=a^2+b^2$, we have

$c^2=a^2+b^2 \Rightarrow c^2=\frac{1}{25}+\frac{1}{9} \Rightarrow c^2=\frac{9+25}{225}$

$c^2=\frac{34}{225} \Rightarrow c=\sqrt{\frac{34}{225}}=\frac{\sqrt{34}}{15}$

The foci are $\left(0, -\frac{\sqrt{34}}{15}\right)$ and $\left(0, \frac{\sqrt{34}}{15}\right).$ Since $a=\frac{1}{5}$ and $b=\frac{1}{3}$, the asymptotes are

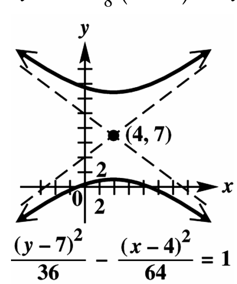
$y=\pm\frac{a}{b}x=\pm\frac{\frac{1}{5}}{\frac{1}{3}}x=\pm\frac{3}{5}x$



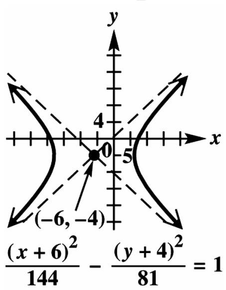
17. $\frac{(y-7)^2}{36}-\frac{(x-4)^2}{64}=1$

Since this equation can be written as $\frac{(y-7)^2}{6^2}-\frac{(x-4)^2}{8^2}=1,$ it has the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ where $h=4$, $k=7$, $a=6$, and $b=8.$ The center is (4, 7). The vertices are 6 units above and below the center (4, 7). These points are (4, 1) and (4, 13). The domain is $(-\infty, \infty)$. The range is $(-\infty, 1] \cup [13, \infty)$. Since $c^2=a^2+b^2 \Rightarrow$ $c^2=36+64 \Rightarrow c^2=100 \Rightarrow c=10,$ the foci are 10 units below and above the center (4, 7). Thus, the foci are (4, –3) and (4, 17). Since $h=4,\ \ k=7,\ \ a=6,\ \ b=8,$ and $y=k\pm\frac{a}{b}(x-h)$, the asymptotes are

$y=7\pm\frac{6}{8}(x-4) \Rightarrow y=7\pm\frac{3}{4}(x-4)$

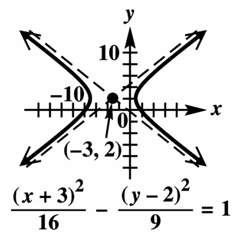


18. $\frac{(x+6)^2}{144}-\frac{(y-4)^2}{81}=1,$
Since this equation can be written as $\frac{[x-(-6)]^2}{12^2}-\frac{[y-(-4)]^2}{9^2}=1,$ it has the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $h=-6,\ k=-4,$ $a = 12$, and $b = 9$. The center is $(-6,-4)$. The vertices are 12 units left and right of the center $(-6,-4)$. These points are $(-18,-4)$ and $(6,-4)$. The domain is $(-\infty,-18] \cup [6,\infty)$. The range is $(-\infty,\infty)$. Since $c^2=a^2+b^2 \Rightarrow c^2=144+81 \Rightarrow c^2=225 \Rightarrow c=15,$ the foci are 15 units left and right of the center $(-6,-4)$. Thus, the foci are $(-21,-4)$ and $(9,-4)$. Since $h=-6,\ k=-4,$ $a=12,\ b=9,$ and $y=k\pm\frac{b}{a}(x-h),$ the asymptotes are
$y=-4\pm\frac{9}{12}[x-(-6)] \Rightarrow y=-4\pm\frac{3}{4}(x+6)$

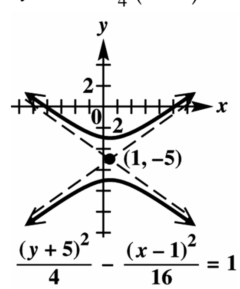


19. $\frac{(x+3)^2}{16}-\frac{(y-2)^2}{9}=1,$
Since this equation can be written as $\frac{[x-(-3)]^2}{4^2}-\frac{(y-2)^2}{3^2}=1,$ it has the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $h=-3,\ k=2,$ $a=4,$ and $b=3$. The center is $(-3,2)$. The vertices are 4 units left and right of the center $(-3,2)$. These points are $(-7,2)$ and $(1,2)$. The domain is $(-\infty,-7] \cup [1,\infty)$. The range is $(-\infty,\infty)$. Since $c^2=a^2+b^2 \Rightarrow c^2=16+9 \Rightarrow c^2=25 \Rightarrow c=5,$ the foci are 5 units left and right of the center $(-3,2)$. Thus, the foci are $(-8,2)$ and $(2,2)$.

Since $h=-3,\ k=2,\ a=4,\ b=3,$ and $y=k\pm\frac{b}{a}(x-h),$ the asymptotes are $y=2\pm\frac{3}{4}[x-(-3)] \Rightarrow y=2\pm\frac{3}{4}(x+3)$.



20. $\frac{(y+5)^2}{4}-\frac{(x-1)^2}{16}=1,$
Since this equation can be written as $\frac{[y-(-5)]^2}{2^2}-\frac{(x-1)^2}{4^2}=1,$ it has the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ where $h=1,\ k=-5,\ a=2,$ and $b=4$. The center is $(1,-5)$. The vertices are 2 units above and below the center $(1,-5)$. These points are $(1,-7)$ and $(1,-3)$. The domain is $(-\infty,\infty)$. The range is $(-\infty,-7] \cup [-3,\infty)$. Since $c^2=a^2+b^2 \Rightarrow c^2=4+16 \Rightarrow c^2=20 \Rightarrow$ $c=\sqrt{20}=2\sqrt{5},$ the foci are $2\sqrt{5}$ units below and above the center $(1,-5)$. Thus, the foci are $(1,-5-2\sqrt{5})$ and $(1,-5+2\sqrt{5})$. Since $h=1,$ $k=-5,\ a=2,\ b=4,$ and $y=k\pm\frac{a}{b}(x-h),$ the asymptotes are
$y=-5\pm\frac{2}{4}(x-1) \Rightarrow y=-5\pm\frac{1}{2}(x-1)$

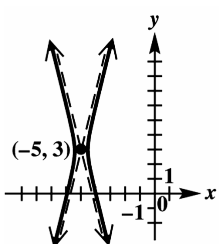


21. $16(x+5)^2-(y-3)^2=1$
Since this equation can be written as $\frac{[x-(-5)]^2}{\left(\frac{1}{4}\right)^2}-\frac{(y-3)^2}{1^2}=1,$ it has the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $h=-5,\ k=3,$ $a=\frac{1}{4},$ and $b=1$. The center is $(-5,3)$.

(*continued on next page*)

(*continued from page 1055*)

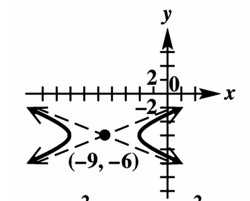
The vertices are $\frac{1}{4}$ unit left and right of the center $(-5,3)$. These points are $\left(-\frac{21}{4},3\right)$ and $\left(-\frac{19}{4},3\right)$. The domain is $\left(-\infty,-\frac{21}{4}\right]\cup\left[-\frac{19}{4},\infty\right)$. The range is $(-\infty,\infty)$. Since $c^2=a^2+b^2 \Rightarrow c^2=\frac{1}{16}+1 \Rightarrow c^2=\frac{1+16}{16} \Rightarrow c^2=\frac{17}{16} \Rightarrow c=\sqrt{\frac{17}{16}}=\frac{\sqrt{17}}{4}$, the foci are $\frac{\sqrt{17}}{4}$ units left and right of the center $(-5,3)$. Thus, the foci are $\left(-5-\frac{\sqrt{17}}{4},3\right)$ and $\left(-5+\frac{\sqrt{17}}{4},3\right)$. Since $h=-5,\ k=3,$ $a=\frac{1}{4},\ b=1,$ and $y=k\pm\frac{b}{a}(x-h)$, the asymptotes are
$y=3\pm\frac{1}{\frac{1}{4}}\left[x+(-5)\right] \Rightarrow y=3\pm 4(x+5)$



$16(x+5)^2-(y-3)^2=1$

22. $4(x+9)^2-25(y+6)^2=100$

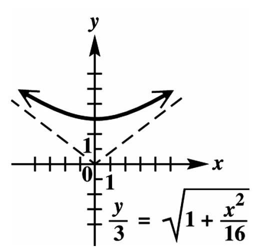
Since this equation can be written as $\frac{\left[x-(-9)\right]^2}{5^2}-\frac{\left[y-(-6)\right]^2}{2^2}=1$, it has the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $h=-9,\ k=-6,$ $a=5,$ and $b=2.$ The center is $(-9,-6)$. The vertices are 5 units left and right of the center $(-9,-6)$. These points are $(-14,-6)$ and $(-4,-6)$. The domain is $(-\infty,-14]\cup[-4,\infty)$. The range is $(-\infty,\infty)$. Since $c^2=a^2+b^2 \Rightarrow c^2=25+4 \Rightarrow c^2=29 \Rightarrow c=\sqrt{29}$, the foci are $\sqrt{29}$ units left and right of the center $(-9,-6)$. Thus, the foci are $\left(-9-\sqrt{29},-6\right)$ and $\left(-9+\sqrt{29},-6\right)$. Since $h=-9,\ k=-6,\ a=5,\ b=2,$ and $y=k\pm\frac{b}{a}(x-h)$, the asymptotes are
$y=-6\pm\frac{2}{5}\left[x+(-9)\right] \Rightarrow y=-6\pm\frac{2}{5}(x+9)$

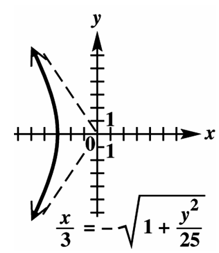


$4(x+9)^2-25(y+6)^2=100$

23. $\frac{y}{3}=\sqrt{1+\frac{x^2}{16}}$

Square both to get $\frac{y^2}{9}=1+\frac{x^2}{16}$ or $\frac{y^2}{9}-\frac{x^2}{16}=1$ or $\frac{y^2}{3^2}-\frac{x^2}{4^2}=1$ This is the equation of a hyperbola with center $(0,0)$ and vertices $(0,-3)$ and $(0,3)$. Since $a=3$ and $b=4$ and $y=\pm\frac{a}{b}x$, we have asymptotes $y=\pm\frac{3}{4}x$. The original equation is the top half of the hyperbola. The domain is $(-\infty,\infty)$. The range is $[3,\infty)$. The vertical line test shows this is a graph of a function.





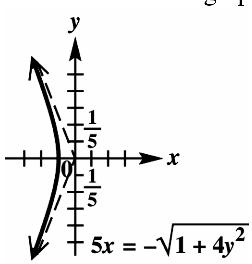
Exercise 23 Exercise 24

24. $\frac{x}{3}=-\sqrt{1+\frac{y^2}{25}}$

Square both to get $\frac{x^2}{9}=1+\frac{y^2}{25}$ or $\frac{x^2}{9}-\frac{y^2}{25}=1$ or $\frac{x^2}{3^2}-\frac{y^2}{5^2}=1$. This is the equation of a hyperbola with center $(0,0)$ and vertices $(-3,0)$ and $(3,0)$. Since $a=3$ and $b=5$ and $y=\pm\frac{b}{a}x$, we have asymptotes $y=\pm\frac{5}{3}x$. The original equation is the left half of the hyperbola. The domain is $(-\infty,-3]$. The range is $(-\infty,\infty)$. The vertical line test shows that this is not the graph of a function.

25. $5x = -\sqrt{1+4y^2}$

Square both to get $25x^2 = 1+4y^2$ or $25x^2 - 4y^2 = 1$ or $\frac{x^2}{\left(\frac{1}{5}\right)^2} - \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$. This is the equation of a hyperbola with center $(0,0)$ and vertices $\left(-\frac{1}{5},0\right)$ and $\left(\frac{1}{5},0\right)$. Since $a = \frac{1}{5}$ and $b = \frac{1}{2}$ and $y = \pm\frac{b}{a}x$, we have asymptotes $y = \pm\frac{\frac{1}{2}}{\frac{1}{5}}x$ or $y = \pm\frac{5}{2}x$. The original equation is the left half of the hyperbola. The domain is $\left(-\infty, -\frac{1}{5}\right]$. The range is $(-\infty, \infty)$. The vertical line test shows that this is not the graph of a function.



26. $3y = \sqrt{4x^2 - 16}$

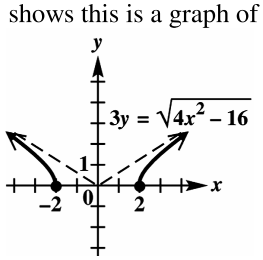
Square both to get $9y^2 = 4x^2 - 16$. This can be rewritten as

$$9y^2 = 4x^2 - 16$$

$$9y^2 - 4x^2 = -16 \Rightarrow \frac{y^2}{\frac{-16}{9}} - \frac{x^2}{\frac{-16}{4}} = 1$$

$$\frac{x^2}{4} - \frac{y^2}{\frac{16}{9}} = 1 \Rightarrow \frac{x^2}{2^2} - \frac{y^2}{\left(\frac{4}{3}\right)^2} = 1$$

This is the equation of a hyperbola with center $(0,0)$ and vertices $(-2,0)$ and $(2,0)$. Since $a = 2$ and $b = \frac{4}{3}$ and $y = \pm\frac{b}{a}x$, we have asymptotes $y = \pm\frac{\frac{4}{3}}{2}x \Rightarrow y = \pm\frac{2}{3}x$. The original equation is the top half of the hyperbola. The domain is $(-\infty, -2] \cup [2, \infty)$. The range is $[0, \infty)$. The vertical line test shows this is a graph of a function.



27. $\frac{x^2}{8} - \frac{y^2}{8} = 1$

Since $a^2 = 8$, we have $a = 2\sqrt{2}$. Also, $c = \sqrt{a^2+b^2} = \sqrt{8+8} = \sqrt{16} = 4$. Thus, $e = \frac{c}{a} = \frac{4}{2\sqrt{2}} = \frac{4\sqrt{2}}{4} = \sqrt{2} \approx 1.4$.

28. $\frac{x^2}{2} - \frac{y^2}{18} = 1$

Since $a^2 = 2$, we have $a = \sqrt{2}$. Also, $c = \sqrt{a^2+b^2} = \sqrt{2+18} = \sqrt{20} = 2\sqrt{5}$. Thus, $e = \frac{c}{a} = \frac{2\sqrt{5}}{\sqrt{2}} = \frac{2\sqrt{10}}{2} = \sqrt{10} \approx 3.2$.

29. $16y^2 - 8x^2 = 16 \Rightarrow \frac{y^2}{1} - \frac{x^2}{2} = 1$

Since $a^2 = 1$, we have $a = 1$. Also, $c = \sqrt{a^2+b^2} = \sqrt{1+2} = \sqrt{3}$. Thus, $e = \frac{c}{a} = \frac{\sqrt{3}}{1} = \sqrt{3} \approx 1.7$.

30. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1$

Since $a^2 = 2$, we have $a = \sqrt{2}$. Also, $c = \sqrt{a^2+b^2} = \sqrt{2+8} = \sqrt{10}$. Thus, $e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5} \approx 2.2$.

31. x-intercepts ± 3; foci at $(-5, 0)$, $(5, 0)$

Since the center is halfway between the foci, the center is $(0,0)$. Since the foci are on a horizontal transverse axis, the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The x-intercepts are also vertices, so $a = 3$ and thus $a^2 = 9$. With the given foci, we have $c = 5$. So given $c^2 = a^2 + b^2$, we have $c^2 = a^2 + b^2 \Rightarrow$
$5^2 = 3^2 + b^2 \Rightarrow 25 = 9 + b^2 \Rightarrow b^2 = 16$
Thus, the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

32. y-intercepts ± 12; foci at (0, –15), (0, 15)
Since the center is halfway between the foci, the center is $(0,0)$. Since the foci are on a vertical transverse axis, the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. The y-intercepts are also vertices, so $a = 12$, and thus, $a^2 = 144$. With the given foci, we have $c = 15$. So given $c^2 = a^2 + b^2$, we have $c^2 = a^2 + b^2 \Rightarrow$
$15^2 = 12^2 + b^2 \Rightarrow 225 = 144 + b^2 \Rightarrow b^2 = 81$
Thus, the equation of the hyperbola is
$\frac{y^2}{144} - \frac{x^2}{81} = 1.$

33. Vertices at (0, 6), (0, –6); asymptotes $y = \pm\frac{1}{2}x$
Since the center is halfway between the vertices, the center is $(0,0)$. Since the vertices are on a vertical transverse axis, the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ and $a = 6$, which implies $a^2 = 36$. Also, since $y = \pm\frac{1}{2}x = \pm\frac{a}{b}x$, we have
$\frac{a}{b} = \frac{1}{2} \Rightarrow \frac{6}{b} = \frac{1}{2} \Rightarrow b = 12 \Rightarrow b^2 = 144$. Thus, the equation of the hyperbola is $\frac{y^2}{36} - \frac{x^2}{144} = 1$

34. Vertices are (–10, 0), (10, 0); asymptotes $y = \pm 5x$
Since the center is halfway between the vertices, the center is $(0,0)$. Since the vertices are on a horizontal transverse axis, the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $a = 10$, which implies $a^2 = 100$. Also, since $y = \pm 5x = \pm\frac{5}{1}x = \pm\frac{b}{a}x$, we have
$\frac{b}{a} = \frac{5}{1} \Rightarrow \frac{b}{10} = \frac{5}{1} \Rightarrow b = 50 \Rightarrow b^2 = 2500$. Thus, the equation of the hyperbola is $\frac{x^2}{100} - \frac{y^2}{2500} = 1$

35. Vertices at (–3, 0), (3, 0); passing through (−6, −1)
Since the center is halfway between the vertices, the center is $(0,0)$. Since the vertices are on a horizontal transverse axis, the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $a = 3$, which implies $a^2 = 9$. Thus, we have
$\frac{x^2}{9} - \frac{y^2}{b^2} = 1.$
Since the hyperbola goes through the point (−6, −1), substitute $x = -6$ and $y = -1$ into the equation and solve for b^2.
$$\frac{x^2}{9} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{(-6)^2}{9} - \frac{(-1)^2}{b^2} = 1 \Rightarrow \frac{36}{9} - \frac{1}{b^2} = 1 \Rightarrow$$
$$4 - \frac{1}{b^2} = 1 \Rightarrow -\frac{1}{b^2} = -3 \Rightarrow b^2 = \frac{1}{3}$$
Thus, the equation is
$\frac{x^2}{9} - \frac{y^2}{\frac{1}{3}} = 1$ or $\frac{x^2}{9} - 3y^2 = 1.$

36. Vertices at (0, 5), (0, –5); passing through (−3, 10)
Since the center is halfway between the vertices, the center is $(0,0)$. Since the vertices are on a vertical transverse axis, the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ and $a = 5$, which implies $a^2 = 25$. Thus, we have $\frac{y^2}{25} - \frac{x^2}{b^2} = 1.$
Since the hyperbola goes through the point (−3, 10), substitute $x = -3$ and $y = 10$ into the equation and solve for b^2.
$$\frac{y^2}{25} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{10^2}{25} - \frac{(-3)^2}{b^2} = 1 \Rightarrow \frac{100}{25} - \frac{9}{b^2} = 1 \Rightarrow$$
$$4 - \frac{9}{b^2} = 1 \Rightarrow -\frac{9}{b^2} = -3 \Rightarrow b^2 = 3$$
Thus, the equation of the hyperbola is
$\frac{y^2}{25} - \frac{x^2}{3} = 1.$

37. Foci at $\left(0, \sqrt{13}\right), \left(0, -\sqrt{13}\right)$; asymptotes $y = \pm 5x$
Since the center is halfway between the foci, the center is $(0,0)$. Since the foci are on a vertical transverse axis, the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ and $c = \sqrt{13}$, which implies $c^2 = 13$. Also, since $y = \pm 5x = \pm\frac{5}{1}x = \pm\frac{a}{b}x$, we have $\frac{a}{b} = \frac{5}{1} \Rightarrow a = 5b$ (1). Since $c^2 = a^2 + b^2$, we have $13 = a^2 + b^2$ (2).
Substituting equation (1) into equation (2) and solving for b^2, we have
$13 = a^2 + b^2 \Rightarrow 13 = (5b)^2 + b^2 \Rightarrow$
$13 = 25b^2 + b^2 \Rightarrow 13 = 26b^2 \Rightarrow b^2 = \frac{1}{2}$. Since $13 = a^2 + b^2$ and $b^2 = \frac{1}{2}$, we have
$13 = a^2 + \frac{1}{2} \Rightarrow 13 - \frac{1}{2} = a^2 \Rightarrow a^2 = \frac{26}{2} - \frac{1}{2} = \frac{25}{2}.$
Thus, the equation of the hyperbola is
$\frac{y^2}{\frac{25}{2}} - \frac{x^2}{\frac{1}{2}} = 1$ or $\frac{2y^2}{25} - 2x^2 = 1.$

38. Foci at $\left(-3\sqrt{5},\, 0\right)$, $\left(3\sqrt{5},\, 0\right)$; asymptotes $y = \pm 2x$

Since the center is halfway between the vertices, the center is $(0,0)$. Since the foci are on a horizontal transverse axis, the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $c = 3\sqrt{5}$, which implies $c^2 = \left(3\sqrt{5}\right)^2 = 45$. Also, since $y = \pm 2x = \pm\frac{2}{1}x = \pm\frac{b}{a}x$, we have $\frac{b}{a} = \frac{2}{1} \Rightarrow b = 2a$ (1). Since $c^2 = a^2 + b^2$, we have $45 = a^2 + b^2$ (2). Substituting equation (1) into equation (2) and solving for b^2, we have $45 = a^2 + b^2 \Rightarrow 45 = a^2 + (2a)^2 \Rightarrow$ $45 = a^2 + 4a^2 \Rightarrow 45 = 5a^2 \Rightarrow a^2 = 9$. Since $a^2 = 9$ and $45 = a^2 + b^2$, we have $45 = 9 + b^2 \Rightarrow b^2 = 36$. Thus, the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{36} = 1$.

39. Vertices at (4, 5), (4,1); asymptotes $y = 3 \pm 7(x-4)$

Since the center is halfway between the vertices, the center is located at $\left(\frac{4+4}{2}, \frac{5+1}{2}\right) = \left(\frac{8}{2}, \frac{6}{2}\right) = (4,3)$. (This could have also been determined from the equation of the asymptotes.) Since the vertices are on a vertical transverse axis, the equation has the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. Since the distance between the vertices is 4, we have $2a = 4 \Rightarrow$ $a = 2$. The slopes of the asymptotes $\pm 7 = \pm\frac{a}{b}$. This yields $\frac{7}{1} = \frac{a}{b} \Rightarrow 7b = a \Rightarrow b = \frac{a}{7}$. Since $a = 2 \Rightarrow b = \frac{2}{7}$. With $a = 2 \Rightarrow a^2 = 4$, $b = \frac{2}{7} \Rightarrow b^2 = \frac{4}{49}$, $h = 4$, and $k = 3$ we have $\frac{(y-3)^2}{4} - \frac{(x-4)^2}{\frac{4}{49}} = 1$ or $\frac{(y-3)^2}{4} - \frac{49(x-4)^2}{4} = 1$.

40. Vertices at (5, –2), (1, –2); asymptotes $y = -2 \pm \frac{3}{2}(x-3)$

Since the center is halfway between the vertices, the center is located at $\left(\frac{5+1}{2}, \frac{-2+(-2)}{2}\right) = \left(\frac{6}{2}, \frac{-4}{2}\right) = (3,-2)$. (This could have also been determined from the equation of the asymptotes.) Since the foci are on a horizontal transverse axis, the equation has the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. Since the distance between the vertices is 4, we have 2a = 4, and thus a = 2. The slopes of the asymptotes $\pm\frac{3}{2} = \pm\frac{b}{a}$. This yields $\frac{3}{2} = \frac{b}{a} \Rightarrow 3a = 2b \Rightarrow$ $b = \frac{3a}{2}$. Since $a = 2 \Rightarrow b = \frac{3(2)}{2} = 3$. With $a = 2 \Rightarrow a^2 = 4$, $b = 3 \Rightarrow b^2 = 9$, $h = 3$, and $k = -2$, we have $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$.

41. Center at (1, –2); focus at (–2, –2); vertex at (−1,–2)

Since the center, focus, and vertices are on a horizontal transverse axis, the equation has the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. The distance from the center to the vertex being 2 implies a = 2. The focus, (–2, –2), is 3 units from the center, so $c = 3$. Given that $c^2 = a^2 + b^2$, $a = 2$, and $c = 3$, we have $c^2 = a^2 + b^2 \Rightarrow 3^2 = 2^2 + b^2 \Rightarrow 9 = 4 + b^2 \Rightarrow b^2 = 5$. With $a = 2 \Rightarrow a^2 = 4$, $b^2 = 5$, $h = 1$, and $k = -2$, we have $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{5} = 1$.

42. Center at (9, –7); focus at (9, −17); vertex at (9, −13)

Since the center, focus, and vertices are on a vertical transverse axis, the equation has the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. The distance from the center to the vertex being 6 implies a = 6. The focus, (9, −17), is 10 units from the center, so $c = 10$. Given that $c^2 = a^2 + b^2$, $a = 6$, and $c = 10$, we have $c^2 = a^2 + b^2 \Rightarrow 10^2 = 6^2 + b^2 \Rightarrow$ $100 = 36 + b^2 \Rightarrow b^2 = 64$. With $a = 6 \Rightarrow a^2 = 36$, $b^2 = 64$, $h = 9$, and $k = -7$, we have $\frac{(y+7)^2}{36} - \frac{(x-9)^2}{64} = 1$.

43. Eccentricity 3; center at (0, 0); vertex at (0, 7)
Since the center and vertex lie on a vertical transverse axis (the y-axis), the equation is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. The distance between the center and a vertex is 7, so $a = 7$. Use the eccentricity to find c: $e=\frac{c}{a}\Rightarrow 3=\frac{c}{7}\Rightarrow c=21$

Now find the value of b^2 given $c^2=a^2+b^2$, $a=7$, and $c=21$.

$c^2=a^2+b^2 \Rightarrow 21^2=7^2+b^2 \Rightarrow$
$441=49+b^2 \Rightarrow b^2=392$

Since $a=7\Rightarrow a^2=49$ and $b^2=392$, the equation of the hyperbola is $\frac{y^2}{49}-\frac{x^2}{392}=1$.

44. Center at (8, 7); focus at (3, 7); eccentricity $\frac{5}{3}$
Since the center and focus lie on a horizontal transverse axis, the equation is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$. The distance between the center and a focus is 5, so $c = 5$. Use the eccentricity to find a:
$e=\frac{c}{a}\Rightarrow \frac{5}{3}=\frac{5}{a}\Rightarrow 5a=15\Rightarrow a=3$

Now find the value of b^2 given $c^2=a^2+b^2$, $a=3$, and $c=5$.

$c^2=a^2+b^2\Rightarrow 5^2=3^2+b^2\Rightarrow$
$25=9+b^2\Rightarrow b^2=16$

Since $a=3\Rightarrow a^2=9$ and $b^2=16$, the equation of the hyperbola is
$\frac{(x-8)^2}{9}-\frac{(y-7)^2}{16}=1$.

45. Foci at (9, –1), (–11, –1); eccentricity $\frac{25}{9}$
Since the foci lie on a horizontal transverse axis, the equation is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$. Since the center is halfway between the foci, the center is located at $\left(\frac{9+(-11)}{2},\frac{(-1)+(-1)}{2}\right)=\left(\frac{-2}{2},\frac{-2}{2}\right)=(-1,-1)$. The distance from the center to each foci is 10, so $c = 10$. Use the eccentricity to find a.
$e=\frac{c}{a}\Rightarrow \frac{25}{9}=\frac{10}{a}\Rightarrow 25a=90\Rightarrow a=\frac{90}{25}=\frac{18}{5}$

Now find the value of b^2 given $c^2=a^2+b^2$, $a=\frac{18}{5}$, and $c=10$.

$c^2=a^2+b^2\Rightarrow 10^2=\left(\frac{18}{5}\right)^2+b^2\Rightarrow$
$100=\frac{324}{25}+b^2\Rightarrow$
$b^2=100-\frac{324}{25}=\frac{2500}{25}-\frac{324}{25}=\frac{2176}{25}$

Since $a=\frac{18}{5}\Rightarrow a^2=\frac{324}{25}$, $b^2=\frac{2176}{25}$, $h=-1$, and $k=2$, we have $\frac{(x+1)^2}{\frac{324}{25}}-\frac{(y+1)^2}{\frac{2176}{25}}=1$ or
$\frac{25(x+1)^2}{324}-\frac{25(y+1)^2}{2176}=1$.

46. Vertices at (2, 10), (2, 2); eccentricity $\frac{5}{4}$
Since the vertices lie on a vertical transverse axis, the equation is of the form
$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.
Since the center is halfway between the vertices, the center is located at
$\left(\frac{2+2}{2},\frac{10+2}{2}\right)=\left(\frac{4}{2},\frac{12}{2}\right)=(2,6)$. The distance from the center to each vertex is 4, so $a = 4$. Use the eccentricity to find c.
$e=\frac{c}{a}\Rightarrow \frac{5}{4}=\frac{c}{4}\Rightarrow 20=4c\Rightarrow c=5$

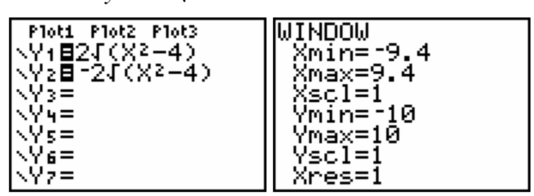
Now find the value of b^2 given $c^2=a^2+b^2$, $a=4$, and $c=5$.

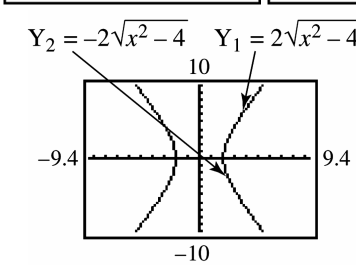
$c^2=a^2+b^2\Rightarrow 5^2=4^2+b^2\Rightarrow$
$25=16+b^2\Rightarrow b^2=9$

Since $a=4\Rightarrow a^2=16$, $b^2=9$, $h=-2$, and $k=6$, we have $\frac{(y-6)^2}{16}-\frac{(x-2)^2}{9}=1$.

47. We solve $\frac{x^2}{4}-\frac{y^2}{16}=1$ for y.

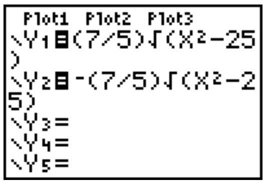
$\frac{x^2}{4}-\frac{y^2}{16}=1\Rightarrow \frac{x^2}{4}-1=\frac{y^2}{16}\Rightarrow$
$y^2=16\left(\frac{x^2}{4}-1\right)\Rightarrow y^2=4x^2-16\Rightarrow$
$y=\sqrt{4x^2-16}\Rightarrow y=\pm\sqrt{4\left(x^2-4\right)}\Rightarrow$
$y=\pm 2\sqrt{x^2-4}$

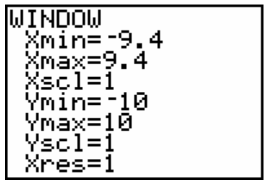


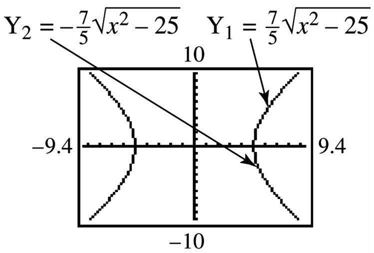


48. We solve $\frac{x^2}{25} - \frac{y^2}{49} = 1$ for y.

$$\frac{x^2}{25} - \frac{y^2}{49} = 1 \Rightarrow \frac{x^2}{25} - 1 = \frac{y^2}{49} \Rightarrow y^2 = 49\left(\frac{x^2}{25} - 1\right) \Rightarrow$$
$$y = \pm 7\sqrt{\frac{x^2}{25} - 1} \Rightarrow y = \pm 7\sqrt{\frac{x^2}{25} - \frac{25}{25}}$$
$$y = \pm 7\sqrt{\frac{x^2 - 25}{25}} \Rightarrow y = \pm \frac{7}{5}\sqrt{x^2 - 25}$$

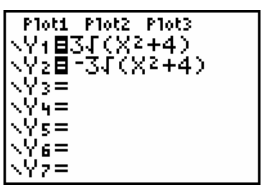


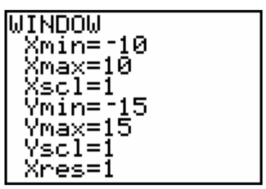


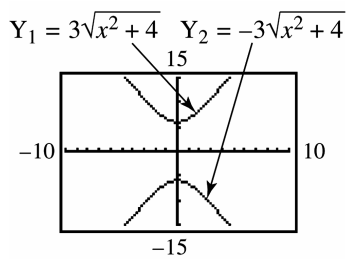


49. We solve $4y^2 - 36x^2 = 144$ for y.

$$4y^2 - 36x^2 = 144 \Rightarrow 4y^2 = 36x^2 + 144 \Rightarrow$$
$$y^2 = 9x^2 + 36 \Rightarrow y = \pm\sqrt{9x^2 + 36}$$
$$y = \pm\sqrt{9\left(x^2 + 4\right)} \Rightarrow y = \pm 3\sqrt{x^2 + 4}$$

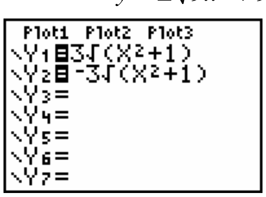


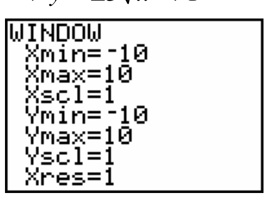


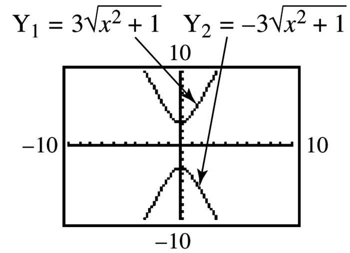


50. We solve $y^2 - 9x^2 = 9$ for y.

$$y^2 - 9x^2 = 9 \Rightarrow y^2 = 9x^2 + 9 \Rightarrow$$
$$y = \pm\sqrt{9x^2 + 9} \Rightarrow y = \pm 3\sqrt{x^2 + 1}$$



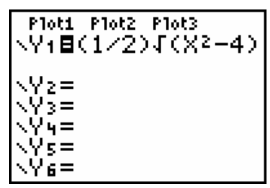


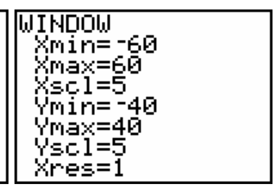


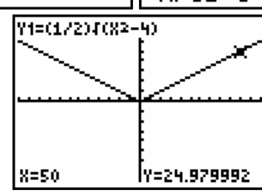
51. Solving $\frac{x^2}{4} - y^2 = 1$ for y, we have $\frac{x^2}{4} - y^2 = 1 \Rightarrow y = \pm\sqrt{\frac{x^2}{4} - 1}$. The positive square root is $y = \sqrt{\frac{x^2}{4} - 1} \Rightarrow y = \sqrt{\frac{x^2}{4} - \frac{4}{4}} \Rightarrow$ $y = \sqrt{\frac{x^2 - 4}{4}} \Rightarrow y = \frac{1}{2}\sqrt{x^2 - 4}$.

52. Writing the equation in the standard form, we have $\frac{x^2}{4} - y^2 = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{1} = 1 \Rightarrow \frac{x^2}{2^2} - \frac{y^2}{1^2} = 1$. Since $a = 2$, $b = 1$, and the slopes of the asymptotes are $\pm\frac{b}{a}$, we have the equation of the asymptote with positive slope is $y = \frac{1}{2}x$.

53. $y = \frac{1}{2}\sqrt{x^2 - 4}$

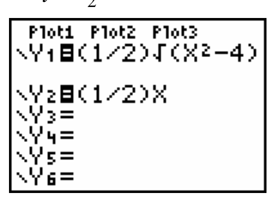


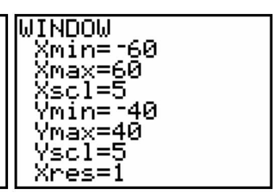


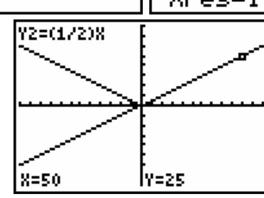


At $x = 50$, $y \approx 24.98$.

54. On the asymptote $y = \frac{1}{2}x$, when $x = 50$, $y = \frac{1}{2} \cdot 50 = 25$.







55. Because 24.98 < 25, the graph of $y=\frac{1}{2}\sqrt{x^2-4}$ lies below the graph of $y=\frac{1}{2}x$ when $x = 50$.

56. If we choose x-values larger than 50, the y-values on the hyperbola will be even closer to (or approach) the y-values on the asymptote.

57. (a) We must determine a and b in the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ The asymptotes are $y = x$ and $y = -x$, which have slopes of 1 and –1, respectively, so $a = b$. Look at the small right triangle that is shown in quadrant III. The line $y = x$ intersects the x-axis to form 45º angles. Since the right angle vertex of the triangle lies on the line $y = x$, we know that this triangle is a 45º–45º–90º triangle (an isosceles right triangle). Thus, both legs of the triangle have length d, and by the Pythagorean theorem, $c^2=d^2+d^2 \Rightarrow c^2=2d^2 \Rightarrow c=d\sqrt{2}$ (1). Thus, the coordinates of N are $\left(-d\sqrt{2},\,0\right)$. Since $a = b$, we have $c^2=a^2+a^2 \Rightarrow c^2=2a^2 \Rightarrow c=a\sqrt{2}$ (2). From equations (1) and (2), we have $d\sqrt{2}=a\sqrt{2} \Rightarrow d=a.$ Thus, $a=b=d=5\times10^{-14}$. The equation of the trajectory of A is given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{\left(5\times10^{-14}\right)^2}-\frac{y^2}{\left(5\times10^{-14}\right)^2}=1$$
$$x^2-y^2=\left(5\times10^{-14}\right)^2$$
$$x^2-y^2=25\times10^{-28}$$
$$x^2-y^2=2.5\times10^{-27}$$
$$x^2=y^2+2.5\times10^{-27}$$
$$x=\sqrt{y^2+2.5\times10^{-27}}$$

(We choose the positive square root since the trajectory occurs only where $x > 0$. This equation represents the right half of the hyperbola, as shown in the figure on text page 979.)

(b) The minimum distance between their centers is $c+a=d\sqrt{2}+d$.

$$c+a=d\sqrt{2}+d$$
$$c+a=\left(5\times10^{-14}\right)\sqrt{2}+\left(5\times10^{-14}\right)$$
$$c+a=\left(\sqrt{2}+1\right)\left(5\times10^{-14}\right)$$
$$c+a\approx 12.07\times10^{-14}\approx 1.2\times10^{-13}\text{ m}$$

58. (a) Rewriting the equation of the hyperbola we have the following.

$$400x^2-625y^2=250{,}000$$
$$\frac{x^2}{625}-\frac{y^2}{400}=1 \Rightarrow \frac{x^2}{25^2}-\frac{y^2}{20^2}=1$$

The two branches of a hyperbola are closest at the vertices and this distance is $2a$. Since $a = 25$, the buildings are 50 m apart at their closest point.

(b) First, solve for y when $x = 50$.

$$\frac{x^2}{25^2}-\frac{y^2}{20^2}=1 \Rightarrow \frac{50^2}{25^2}-\frac{y^2}{20^2}=1 \Rightarrow$$
$$\left(\frac{50}{25}\right)^2-\frac{y^2}{20^2}=1 \Rightarrow 2^2-\frac{y^2}{20^2}=1 \Rightarrow$$
$$4-\frac{y^2}{20^2}=1 \Rightarrow \frac{y^2}{20^2}=3 \Rightarrow$$
$$y^2=20^2\cdot3 \Rightarrow y=20\sqrt{3}$$

Thus,
$$d=2y=2\cdot20\sqrt{3}=40\sqrt{3}\approx 69.3\text{ m.}$$

59. Since the center is the origin and the foci lie on a horizontal transverse axis, the equation has the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ We know from pages 627–628 of the text that $\left|d\left(P,F'\right)-d\left(P,F\right)\right|=2a$ where F and F' are the locations of the foci, in this case $(-c,0)$ and $(c,0)$. Since $d=rt,$ the difference between $d\left(P,F'\right)$ and $d\left(P,F\right)$ is $330t$. Thus, $2a=330t \Rightarrow a=\frac{330t}{2}.$ Since $c^2=a^2+b^2,$ we have the following.

$$c^2=a^2+b^2 \Rightarrow c^2=\left(\frac{330t}{2}\right)^2+b^2 \Rightarrow$$
$$b^2=c^2-\frac{330^2t^2}{4} \Rightarrow b^2=\frac{4c^2-330^2t^2}{4}$$

Thus, the equation of the hyperbola is

$$\frac{x^2}{\left(\frac{330t}{2}\right)^2}-\frac{y^2}{\frac{4c^2-330^2t^2}{4}}=1 \text{ or } \frac{x^2}{330^2t^2}-\frac{y^2}{4c^2-330^2t^2}=\frac{1}{4}.$$

60. In this proof, we have $P(x, y)$, $F'(-c, 0)$, and $F(c, 0)$. Since $\left|d(P, F') - d(P, F)\right| = 2a$, we have

$$\left|\sqrt{\left[x-(-c)\right]^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}\right| = 2a \Rightarrow \left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a$$

Since $\left(|x|\right)^2 = x^2$, we have

$$\left(\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right|\right)^2 = (2a)^2$$

$$\left[\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right]^2 = 4a^2$$

$$(x+c)^2 + y^2 - 2\left(\sqrt{(x+c)^2 + y^2}\right)\left(\sqrt{(x-c)^2 + y^2}\right) + (x-c)^2 + y^2 = 4a^2$$

$$x^2 + 2xc + c^2 + y^2 - 2\sqrt{\left[(x+c)^2 + y^2\right]\left[(x-c)^2 + y^2\right]} + x^2 - 2xc + c^2 + y^2 = 4a^2$$

$$2x^2 + 2c^2 + 2y^2 - 2\sqrt{\left[(x+c)^2 + y^2\right]\left[(x-c)^2 + y^2\right]} = 4a^2$$

$$x^2 + c^2 + y^2 - \sqrt{\left[(x+c)^2 + y^2\right]\left[(x-c)^2 + y^2\right]} = 2a^2$$

$$x^2 + c^2 + y^2 - 2a^2 = \sqrt{\left[(x+c)^2 + y^2\right]\left[(x-c)^2 + y^2\right]}$$

$$\left(x^2 + c^2 + y^2 - 2a^2\right)^2 = \left[(x+c)^2 + y^2\right]\left[(x-c)^2 + y^2\right]$$

$$x^4 + 2x^2y^2 - 4a^2x^2 + 2c^2x^2 + y^4 - 4a^2y^2 + 2c^2y^2 + 4a^4 - 4a^2c^2 + c^4$$
$$= (x+c)^2(x-c)^2 + (x+c)^2 y^2 + (x-c)^2 y^2 + y^4$$

$$x^4 + 2x^2y^2 - 4a^2x^2 + 2c^2x^2 - 4a^2y^2 + 2c^2y^2 + 4a^4 - 4a^2c^2 + c^4$$
$$= \left[(x+c)(x-c)\right]^2 + (x+c)^2 y^2 + (x-c)^2 y^2$$

$$x^4 + 2x^2y^2 - 4a^2x^2 + 2c^2x^2 - 4a^2y^2 + 2c^2y^2 + 4a^4 - 4a^2c^2 + c^4$$
$$= \left(x^2 - c^2\right)^2 + \left(x^2 + 2xc + c^2\right)y^2 + \left(x^2 - 2xc + c^2\right)y^2$$

$$x^4 + 2x^2y^2 - 4a^2x^2 + 2c^2x^2 - 4a^2y^2 + 2c^2y^2 + 4a^4 - 4a^2c^2 + c^4$$
$$= x^4 - 2x^2c^2 + c^4 + x^2y^2 + 2xcy^2 + c^2y^2 + x^2y^2 - 2xcy^2 + c^2y^2$$

$$-4a^2x^2 + 2c^2x^2 - 4a^2y^2 + 4a^4 - 4a^2c^2 = -2x^2c^2$$

$$-4a^2x^2 + 4c^2x^2 - 4a^2y^2 = 4a^2c^2 - 4a^4$$

$$a^2x^2 - c^2x^2 + a^2y^2 = -a^2c^2 + a^4$$

$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right)$$

Since $b^2 = c^2 - a^2$, we have $-b^2x^2 + a^2y^2 = a^2\left(-b^2\right) \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Section 10.4: Summary of the Conic Sections

1. $x^2+y^2=144 \Rightarrow (x-0)^2+(y-0)^2=12^2$
The graph of this equation is a circle with center $(0,0)$ and radius 12. Also, note in our original equation, the x^2 - and y^2-terms have the same positive coefficient.

2. $(x-2)^2+(y+3)^2=25$
$(x-2)^2+\left[y-(-3)\right]^2=5^2$
The graph of the equation is a circle with center $(2,-3)$ and radius 5. Also, note that when expanded, in our original equation, the x^2 - and y^2-terms have the same positive coefficient.

3. $y=2x^2+3x-4 \Rightarrow y=2\left(x^2+\frac{3}{2}x\right)-4$
$y=2\left(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}\right)-4$
$y=2\left(x+\frac{3}{4}\right)^2+2\left(-\frac{9}{16}\right)-4$
$y=2\left(x+\frac{3}{4}\right)^2-\frac{9}{8}-4$
$y=2\left(x+\frac{3}{4}\right)^2-\frac{41}{8} \Rightarrow y-\left(-\frac{41}{8}\right)=2\left[x-\left(-\frac{3}{4}\right)\right]^2$
The graph of this equation is a parabola opening upwards with a vertex of $\left(-\frac{3}{4},-\frac{41}{8}\right)$. Also, note our original equation has an x^2 -term, but no y^2-term.

4. $x=3y^2+5y-6$
$x=3\left(y^2+\frac{5}{3}y+\frac{25}{36}-\frac{25}{36}\right)-6$
$x=3\left(y+\frac{5}{6}\right)^2+3\left(-\frac{25}{36}\right)-6$
$x=3\left(y+\frac{5}{6}\right)^2-\frac{25}{12}-6$
$x=3\left(y+\frac{5}{6}\right)^2-\frac{97}{12} \Rightarrow x-\left(-\frac{97}{12}\right)=3\left[y-\left(-\frac{5}{6}\right)\right]^2$
The graph of this equation is a parabola opening to the right with a vertex of $\left(-\frac{97}{12},-\frac{5}{6}\right)$. Also, note our original equation has a y^2-term, but no x^2 -term.

5. $x-1=-3(y-4)^2$
The graph of this equation is a parabola opening to the left with a vertex of $(1,4)$. Also, note when expanded, our original equation has a y^2-term, but no x^2 -term.

6. $\frac{x^2}{25}+\frac{y^2}{36}=1 \Rightarrow \frac{x^2}{5^2}+\frac{y^2}{6^2}=1$
The graph of this equation is an ellipse centered at the origin and x-intercepts of 5 and –5, and y-intercepts of 6 and –6. Also, note in our original equation, the x^2 - and y^2-terms both have different positive coefficients.

7. $\frac{x^2}{49}+\frac{y^2}{100}=1 \Rightarrow \frac{x^2}{7^2}+\frac{y^2}{10^2}=1$
The graph of this equation is an ellipse centered at the origin and x-intercepts of 7 and –7, and y-intercepts of 10 and –10. Also, note in our original equation, the x^2 - and y^2-terms both have different positive coefficients.

8. $x^2-y^2=1 \Rightarrow \frac{x^2}{1^2}-\frac{y^2}{1^2}=1$
The graph of this equation is a hyperbola centered at the origin with x-intercepts of 1 and –1, and asymptotes of $y=\pm x$. Also, note in our original equation, the x^2 - and y^2-terms have coefficients that are opposite in sign.

9. $\frac{x^2}{4}-\frac{y^2}{16}=1 \Rightarrow \frac{x^2}{2^2}-\frac{y^2}{4^2}=1$
The graph of this equation is a hyperbola centered at the origin with x-intercepts of 2 and –2, and asymptotes of $y=\pm\frac{4}{2}x=\pm 2x$. Also, note in our original equation, the x^2 - and y^2-terms have coefficient that are opposite in sign.

10. $\frac{(x+2)^2}{9}+\frac{(y-4)^2}{16}=1$
$\frac{\left[x-(-2)\right]^2}{3^2}+\frac{(y-4)^2}{4^2}=1$
The graph of this equation is an ellipse centered at $(-2,4)$ and vertices of $(-2,0)$ and $(-2,8)$. The endpoints of the minor axis are $(-5,4)$ and $(1,4)$. Also, note that when expanded, our original equation has x^2 - and y^2-terms with positive coefficients.

11. $\frac{x^2}{25}-\frac{y^2}{25}=1 \Rightarrow \frac{x^2}{5^2}-\frac{y^2}{5^2}=1$

The graph of this equation is a hyperbola centered at the origin with x-intercepts of 5 and –5, and asymptotes of $y=\pm\frac{5}{5}x=\pm x.$

Also, note in our original equation, the x^2 - and y^2 -terms have coefficients that are opposite in sign.

12. $y+7=4(x+3)^2 \Rightarrow y-(-7)=4[x-(-3)]^2$

The graph of this equation is a parabola opening upwards with a vertex of $(-3,-7)$. Also, note when expanded, our original equation has an x^2 - term, but no y^2 -term.

13.
$$\frac{x^2}{4}=1-\frac{y^2}{9} \Rightarrow \frac{x^2}{4}+\frac{y^2}{9}=1$$
$$\frac{(x-0)^2}{2^2}+\frac{(y-0)^2}{3^2}=1$$

The equation is of the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ with $a=3,\ b=2,\ h=0,$ and $k=0,$ so the graph of the given equation is an ellipse.

14.
$$\frac{x^2}{4}=1+\frac{y^2}{9} \Rightarrow \frac{x^2}{4}-\frac{y^2}{9}=1$$
$$\frac{(x-0)^2}{2^2}-\frac{(y-0)^2}{3^2}=1$$

The equation is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ with $a=2,\ b=3,\ h=0,$ and $k=0,$ so the graph of the given equation is a hyperbola.

15.
$$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{16}=1$$
$$(x+3)^2+(y-2)^2=16$$
$$[x-(-3)]^2+(y-2)^2=4^2$$

The equation is of the form $(x-h)^2+(y-k)^2=r^2$ with $r=4,\ h=-3,$ and $k=2,$ so the graph of the given equation is a circle.

16. $x^2=25-y^2 \Rightarrow x^2+y^2=25 \Rightarrow x^2+y^2=5^2$

The equation is of the form $(x-h)^2+(y-k)^2=r^2$ with $r=5,\ h=0,$ and $k=0,$ so the graph of the given equation is a circle.

17.
$$x^2-6x+y=0 \Rightarrow y=-x^2+6x$$
$$y=-(x^2-6x+9-9)$$
$$y=-(x-3)^2+9$$
$$y-9=-(x-3)^2$$

The equation is of the form $y-k=a(x-h)^2$ with $a=-1,\ h=3,$ and $k=9,$ so the graph of the given equation is a parabola.

18.
$$11-3x=2y^2-8y$$
$$11-3x=2(y^2-4y+4-4)$$
$$11-3x=2(y-2)^2-8 \Rightarrow$$
$$-3x+19=2(y-2)^2$$
$$-3\left(x-\tfrac{19}{3}\right)=2(y-2)^2 \Rightarrow x-\tfrac{19}{3}=-\tfrac{2}{3}(y-2)^2$$

The equation is of the form $x-h=a(y-k)^2$ with $a=-\frac{2}{3},\ h=\frac{19}{3},$ and $k=2,$ so the graph of the given equation is a parabola.

19.
$$4(x-3)^2+3(y+4)^2=0$$
$$\frac{4(x-3)^2}{12}+\frac{3(y+4)^2}{12}=0$$
$$\frac{(x-3)^2}{3}+\frac{[y-(-4)]^2}{4}=0$$

The graph is the point $(3,-4)$.

20.
$$2x^2-8x+2y^2+20y=12$$
$$x^2-4x+y^2+10y=6$$
$$(x^2-4x+4-4)+(y^2+10y+25-25)=6$$
$$(x-2)^2-4+(y+5)^2-25=6$$
$$(x-2)^2+(y+5)^2=6+4+25$$
$$(x-2)^2+(y+5)^2=35$$
$$(x-2)^2+[y-(-5)]^2=\left(\sqrt{35}\right)^2$$

The equation is of the form $(x-h)^2+(y-k)^2=r^2$ with $r=\sqrt{35},\ h=2,$ and $k=-5,$ so the graph of the given equation is a circle.

21. $x-4y^2-8y=0 \Rightarrow x=4y^2+8y$

$$x=4\left(y^2+2y+1-1\right)$$
$$x=4(y+1)^2-4$$
$$x-(-4)=4\left[y-(-1)\right]^2$$

The equation is of the form $x-h=a(y-k)^2$ with $a=4,\ h=-4,$ and $k=-1,$ so the graph of the given equation is a parabola.

22. $x^2+2x=-4y \Rightarrow x^2+2x+1-1=-4y$

$$(x+1)^2-1=-4y \Rightarrow (x+1)^2=-4y+1$$
$$(x+1)^2=-4\left(y-\tfrac{1}{4}\right)$$
$$y-\tfrac{1}{4}=-\tfrac{1}{4}\left[x-(-1)\right]^2$$

The equation is of the form $y-k=a(x-h)^2$ with $a=-\frac{1}{4},\ h=-1,$ and $k=\frac{1}{4},$ so the graph of the given equation is a parabola.

23. $4x^2-24x+5y^2+10y+41=0$

$$4\left(x^2-6x+9-9\right)+5\left(y^2+2y+1-1\right)=-41$$
$$4(x-3)^2-36+5(y+1)^2-5=-41$$
$$4(x-3)^2+5(y+1)^2=-41+36+5$$
$$4(x-3)^2+5(y+1)^2=0$$
$$\frac{4(x-3)^2}{20}+\frac{5(y+1)^2}{20}=0$$
$$\frac{(x-3)^2}{5}+\frac{\left[y-(-1)\right]^2}{4}=0$$

The graph is the point $(3,-1)$.

24. $6x^2-12x+6y^2-18y+25=0$

$$6\left(x^2-2x+1-1\right)+6\left(y^2-3y+\tfrac{9}{4}-\tfrac{9}{4}\right)=-25$$
$$6\left(x^2-2x+1\right)-6+6\left(y^2-3y+\tfrac{9}{4}\right)-\tfrac{27}{2}=-25$$
$$6(x-1)^2+6\left(y-\tfrac{3}{2}\right)^2=-25+6+\tfrac{27}{2}$$
$$6(x-1)^2+6\left(y-\tfrac{3}{2}\right)^2=-\tfrac{50}{2}+\tfrac{12}{2}+\tfrac{27}{2}$$
$$6(x-1)^2+6\left(y-\tfrac{3}{2}\right)^2=-\tfrac{11}{2}$$
$$(x-1)^2+\left(y-\tfrac{3}{2}\right)^2=-\tfrac{11}{12}$$

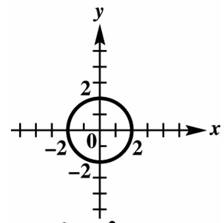
A sum of squares can never be negative. This equation has no graph.

25. $\dfrac{x^2}{4}+\dfrac{y^2}{4}=-1 \Rightarrow x^2+y^2=-4$

A sum of squares can never be negative. This equation has no graph.

26. $\dfrac{x^2}{4}+\dfrac{y^2}{4}=1 \Rightarrow x^2+y^2=4 \Rightarrow x^2+y^2=2^2$

The equation is of the form $(x-h)^2+(y-k)^2=r^2$ with $r=2,\ h=0,$ and $k=0,$ so the graph of the given equation is a circle with center (0, 0) and radius 2.



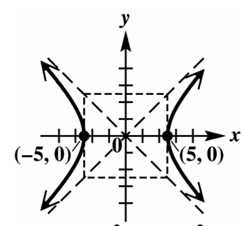
$\dfrac{x^2}{4}+\dfrac{y^2}{4}=1$

$x^2+y^2=4$

27. $x^2=25+y^2 \Rightarrow x^2-y^2=25$

$$\frac{x^2}{25}-\frac{y^2}{25}=1 \Rightarrow \frac{(x-0)^2}{5^2}-\frac{(y-0)^2}{5^2}=1$$

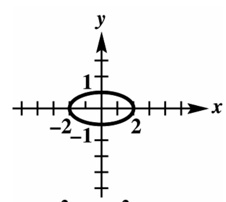
The equation is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ with $a=5,\ b=5,\ h=0,$ and $k=0,$ so the graph of the given equation is a hyperbola with center (0, 0), vertices (−5, 0) and (5, 0), and asymptotes $y=\pm x$.



$x^2=25+y^2$

$\dfrac{x^2}{25}-\dfrac{y^2}{25}=1$

Exercise 27



$9x^2+36y^2=36$

$\dfrac{x^2}{4}+y^2=1$

Exercise 28

28. $9x^2+36y^2=36 \Rightarrow \dfrac{x^2}{4}+\dfrac{y^2}{1}=1$

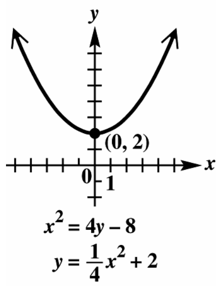
$$\frac{(x-0)^2}{2^2}+\frac{(y-0)^2}{1^2}=1$$

The equation is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ with $a=2,\ b=1,\ h=0,$ and $k=0,$ so the graph of the given equation is an ellipse with center (0, 0), x-intercepts (−2, 0) and (2, 0), and y-intercepts (0, −1) and (0, 1).

29. $x^2 = 4y - 8 \Rightarrow x^2 = 4(y-2) \Rightarrow$
$y - 2 = \frac{1}{4}(x-0)^2$

The equation is of the form $y - k = a(x-h)^2$ with $a = \frac{1}{4}$, $h = 0$, and $k = 2$, so the graph of the given equation is a parabola with vertex (0, 2) and vertical axis $x = 0$ (the y-axis). Use the vertex and axis and plot a few additional points.

x	y
−4	6
−2	3
0	2
2	3
4	6



30. $\frac{(x-4)^2}{8} + \frac{(y+1)^2}{2} = 0$

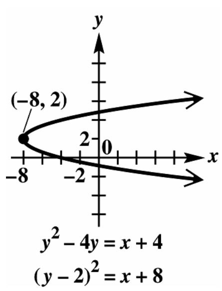
$\frac{(x-4)^2}{8} + \frac{[y-(-1)]^2}{2} = 0$

The graph is the point $(4, -1)$.

31. $y^2 - 4y = x + 4 \Rightarrow y^2 - 4y + 4 - 4 = x + 4$
$(y-2)^2 - 4 = x + 4$
$x + 8 = (y-2)^2 \Rightarrow x - (-8) = (y-2)^2$

The equation is of the form $x - h = a(y-k)^2$ with $a = 1$, $h = -8$, and $k = 2$, so the graph of the given equation is a parabola with vertex (−8, 2) and horizontal axis $y = 2$. Use the vertex and axis and plot a few additional points.

x	y
0	−.8
−4	0
−8	2
−4	4
0	4.8



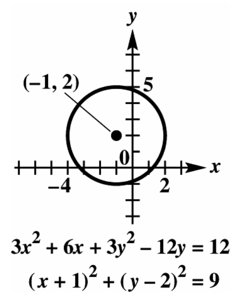
32. $(x+7)^2 + (y-5)^2 + 4 = 0$
$(x+7)^2 + (y-5)^2 = -4$

A sum of squares can never be negative. This equation has no graph.

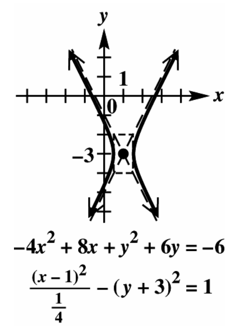
33.

$$3x^2 + 6x + 3y^2 - 12y = 12$$
$$x^2 + 2x + y^2 - 4y = 4$$
$$(x^2 + 2x + 1 - 1) + (y^2 - 4y + 4 - 4) = 4$$
$$(x+1)^2 - 1 + (y-2)^2 - 4 = 4$$
$$(x-1)^2 + (y-2)^2 = 4 + 1 + 4$$
$$(x+1)^2 + (y-2)^2 = 9$$
$$[x-(-1)]^2 + (y-2)^2 = 3^2$$

The equation is of the form $(x-h)^2 + (y-k)^2 = r^2$ with $r = 3$, $h = -1$, and $k = 2$, so the graph of the given equation is a circle with center (−1, 2) and radius 3.



Exercise 33



Exercise 34

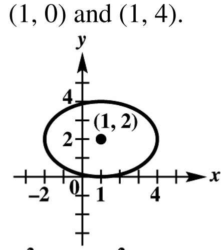
34.

$$-4x^2 + 8x + y^2 + 6y = -6$$
$$4x^2 - 8x - y^2 - 6y = 6$$
$$4(x^2 - 2x \quad) - (y^2 + 6y \quad) = 6$$
$$4(x^2 - 2x + 1 - 1) - (y^2 + 6y + 9 - 9) = 6$$
$$4(x-1)^2 - 4 - (y+3)^2 + 9 = 6$$
$$4(x-1)^2 - (y+3)^2 = 6 + 4 - 9$$
$$4(x-1)^2 - (y+3)^2 = 1$$
$$\frac{(x-1)^2}{\frac{1}{4}} - \frac{(y+3)^2}{1} = 1$$
$$\frac{(x-1)^2}{\left(\frac{1}{2}\right)^2} - \frac{[y-(-3)]^2}{1^2} = 1$$

The equation is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ with $a = \frac{1}{2}$, $b = 1$, $h = 1$, and $k = -3$, so the graph of the given equation is a hyperbola with center (1, −3), vertices $\left(\frac{1}{2}, -3\right)$ and $\left(\frac{3}{2}, -3\right)$, and asymptotes $y = \pm 2x$.

35.
$$4x^2 - 8x + 9y^2 - 36y = -4$$
$$4\left(x^2 - 2x + 1 - 1\right) + 9\left(y^2 - 4y + 4 - 4\right) = -4$$
$$4(x-1)^2 - 4 + 9(y-2)^2 - 36 = -4$$
$$4(x-1)^2 + 9(y-2)^2 = 36$$
$$\frac{4(x-1)^2}{36} + \frac{9(y-2)^2}{36} = 1$$
$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$$
$$\frac{(x-1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1$$

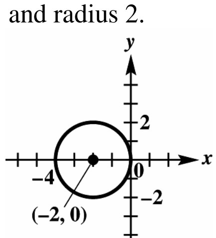
The equation is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ with $a = 3,\ b = 2,\ h = 1,$ and $k = 2,$ so the graph of the given equation is an ellipse with center (1, 2) and vertices (–2, 2), (4, 2), (1, 0) and (1, 4).



$4x^2 - 8x + 9y^2 - 36y = -4$
$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$

36.
$$3x^2 + 12x + 3y^2 = 0$$
$$3\left(x^2 + 4x + 4 - 4\right) + 3y^2 = 0$$
$$3(x+2)^2 - 12 + 3y^2 = 0$$
$$3(x+2)^2 + 3y^2 = 0 + 12$$
$$3(x+2)^2 + 3y^2 = 12$$
$$(x+2)^2 + y^2 = 4$$

The equation is of the form $(x-h)^2 + (y-k)^2 = r^2$ with $r = 2,\ h = -2,$ and $k = 0,$ so the graph of the given equation is a circle with center (–2, 0) and radius 2.



$3x^2 + 12x + 3y^2 = 0$
$(x+2)^2 + y^2 = 4$

37. The definition of an ellipse states "an ellipse is the set of all points in a plane the sum of whose distances from two fixed points is constant." Therefore, the set of all points in a plane for which the sum of the distances from the points (5, 0) and (–5, 0) is 14 is an ellipse with foci (5, 0) and (–5, 0).

38. The definition of a hyperbola states "a hyperbola is the set of all points in a plane such that the absolute value of the difference of the distances from two points is constant." Therefore, the set of all points in a plane for which the absolute value of the difference of the distances from the points (3, 0) and (–3, 0) is 2 is a hyperbola with foci (3, 0) and (–3, 0).

39. Refer to the Geometric Definition of a Conic Section on page 984 of the text. It gives the relation
[distance of P from F] $= e \cdot$ [distance of P from L].
In this exercise, $e = 1\frac{1}{2} = \frac{3}{2}.$ Since $e > 1$, this is a hyperbola.

40. Refer to the Geometric Definition of a Conic Section on page 984 of the text. It gives the relation
[distance of P from F] $= e \cdot$ [distance of P from L].
In this exercise, $e = \frac{1}{3}.$ Since $e < 1$, this is an ellipse.

41. From the graph, the coordinates of P (a point on the graph) are (–3, 8), the coordinates of F (a focus) are (3, 0), the equation of L (the directrix) is $x = 27$. By the distance formula, the distance from P to F is
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[3-(-3)]^2 + (0-8)^2} = \sqrt{6^2 + (-8)^2}$$
$$= \sqrt{36+64} = \sqrt{100} = 10$$
The distance from a point to a line is defined as the perpendicular distance, so the distance from P to L is $|27 - (-3)| = 30.$ Thus,
$$e = \frac{\text{Distance of } P \text{ from } F}{\text{Distance of } P \text{ from } L} = \frac{10}{30} = \frac{1}{3}.$$

42. From the graph, the coordinates of P (a point on the graph) are $\left(4, \frac{10}{3}\right)$, the coordinates of F (a focus) are $(-4, 0)$, the equation of L (the directrix) is $x = -9$. By the distance formula, the distance from P to F is as follows.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-4-4)^2 + \left(0 - \frac{10}{3}\right)^2} = \sqrt{(-8)^2 + \left(-\frac{10}{3}\right)^2}$$
$$= \sqrt{64 + \frac{100}{9}} = \sqrt{\frac{576}{9} + \frac{100}{9}} = \sqrt{\frac{676}{9}} = \frac{26}{3}$$

The distance from a point to a line is defined as the perpendicular distance, so the distance from P to L is $\left|4 - (-9)\right| = 13$. Thus,

$$e = \frac{\text{Distance of } P \text{ from } F}{\text{Distance of } P \text{ from } L} = \frac{\frac{26}{3}}{13} = \frac{26}{3} \cdot \frac{1}{13} = \frac{2}{3}.$$

43. From the graph, we see that $F = \left(\sqrt{2}, 0\right)$ and L is the vertical line $x = -\sqrt{2}$. Choose $(0, 0)$, the vertex of the parabola, as P.

Distance of P from $F = \sqrt{2}$, and distance of P from $L = \sqrt{2}$. Thus, we have

$$e = \frac{\text{Distance of } P \text{ from } F}{\text{Distance of } P \text{ from } L} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

44. From the graph, the coordinates of P are $\left(-27, 48\frac{3}{4}\right)$, the coordinates of F are (27, 0), the equation of L is $x = 4$. By the distance formula, the distance from P to F is as follows.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{\left[27 - (-27)\right]^2 + \left(0 - 48\frac{3}{4}\right)^2}$$
$$= \sqrt{54^2 + \left(-\frac{195}{4}\right)^2} = \sqrt{2916 + \frac{38{,}025}{16}}$$
$$= \sqrt{\frac{46{,}656}{16} + \frac{38{,}025}{16}} = \sqrt{\frac{84{,}681}{16}} = \frac{291}{4}$$

The distance from a point to a line is defined as the perpendicular distance, so the distance from P to L is $\left|4 - (-27)\right| = 31$. Thus,

$$e = \frac{\text{Distance of } P \text{ from } F}{\text{Distance of } P \text{ from } L} = \frac{\frac{291}{4}}{31} = \frac{291}{4} \cdot \frac{1}{31} = \frac{291}{124} \approx 2.3.$$

45. From the graph, we see that $P = (9, -7.5)$, $F = (9, 0)$ and L is the vertical line $x = 4$. Distance of P from $F = 7.5$, and distance of P from $L = 5$. Thus, $e = \frac{\text{Distance of } P \text{ from } F}{\text{Distance of } P \text{ from } L} = \frac{7.5}{5} = 1.5.$

46. From the graph, $P = (5, 20)$, $F = (20,0)$ and L is the vertical line $x = -20$. By the distance formula, the distance from P to F is

$$\sqrt{(20-5)^2 + (0-20)^2} = \sqrt{15^2 + (-20)^2}$$
$$= \sqrt{225 + 400} = \sqrt{625} = 25$$ and distance of P from $L = \left|5 - (-20)\right| = 25$. Thus,

$$e = \frac{\text{Distance of } P \text{ from } F}{\text{Distance of } P \text{ from } L} = \frac{25}{25} = 1.$$

47. If $\frac{k}{\sqrt{D}} = \frac{2.82 \times 10^7}{\sqrt{42.5 \times 10^6}} = \frac{2.82 \times 10^7}{\sqrt{42.5} \times 10^3}$ $\approx .432568 \times 10^4 \approx 4326$ and $V = 2090$, then we have $V < \frac{k}{\sqrt{D}}$. Thus, the shape of the satellite's trajectory was elliptical.

48. If $\frac{k}{\sqrt{D}} = \frac{2.82 \times 10^7}{\sqrt{42.5 \times 10^6}} \approx 4326$, then the velocity must be increased from 2090 m/sec (from Exercise 47) to at least 4326 m/sec. Thus, the minimum increase is, therefore, 4326 – 2090 = 2236 m/sec.

49. Answers will vary.

50. Complete the square on x and on y for $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$
$$\left(Ax^2 + Dx\right) + \left(Cy^2 + Ey\right) = -F$$
$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

Since $\left(\frac{1}{2} \cdot \frac{D}{A}\right)^2 = \frac{D^2}{4A^2}$ and $\left(\frac{1}{2} \cdot \frac{E}{C}\right)^2 = \frac{E^2}{4C^2}$, we have the following.

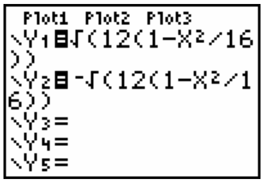
$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2} - \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2} - \frac{E^2}{4C^2}\right) = -F$$
$$A\left(x + \frac{D}{2A}\right)^2 - \frac{D^2}{4A} + C\left(y + \frac{E}{2C}\right)^2 - \frac{E^2}{4C} = -F$$
$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F$$
$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2}{4AC} + \frac{AE^2}{4AC} - \frac{4ACF}{4AC}$$
$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}$$
$$\frac{\left[x - \left(-\frac{D}{2A}\right)\right]^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{\left[y - \left(-\frac{E}{2C}\right)\right]^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1$$

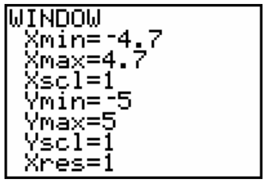
The center of the ellipse is at $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

51. To enter the relation $\frac{x^2}{16}+\frac{y^2}{12}=1$ into the graphing calculator, we must solve for *y*.

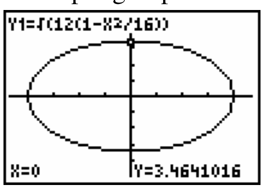
$$\frac{x^2}{16}+\frac{y^2}{12}=1 \Rightarrow \frac{y^2}{12}=1-\frac{x^2}{16}$$

$$y^2=12\left(1-\frac{x^2}{16}\right) \Rightarrow y=\pm\sqrt{12\left(1-\frac{x^2}{16}\right)}$$





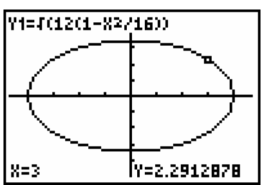
A sampling of points is as follows.



The distance of *P* from

$$(2,0)\approx\sqrt{(2-0)^2+(0-3.4641016)^2}$$
$$=\sqrt{4+(-3.4641016)^2}\approx 3.999999987$$

The distance of *P* from the line $x=8$ is $|0-8|=8$. We have $3.999999987\approx\frac{1}{2}(8)=4.$

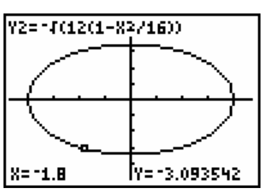


The distance of *P* from

$$(2,0)\approx\sqrt{(2-3)^2+(0-2.2912878)^2}$$
$$=\sqrt{1+(-2.2912878)^2}\approx 2.499999956$$

The distance of *P* from the line $x=8$ is $|3-8|=5.$ We have

$2.499999956\approx\frac{1}{2}(5)=2.5.$

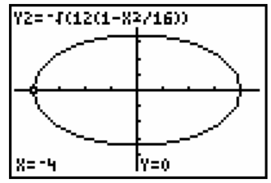


The distance of *P* from

$$(2,0)\approx\sqrt{\left[2-(-1.8)\right]^2+\left[0-(-3.093542)\right]^2}$$
$$=\sqrt{14.44+(3.093542)^2}\approx 4.900000215$$

The distance of *P* from the line $x=8$ is $|-1.8-8|=9.8.$ We have

$4.900000215\approx\frac{1}{2}(9.8)=4.9.$



The distance of *P* from

$$(2,0)\approx\sqrt{\left[2-(-4)\right]^2+(0-0)^2}=\sqrt{36}=6$$

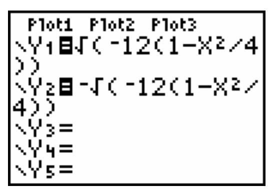
The distance of *P* from the line $x=8$ is $|-4-8|=12.$ We have $6=\frac{1}{2}(12).$

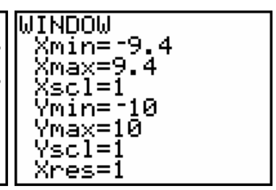
52. To enter the relation $\frac{x^2}{4}-\frac{y^2}{12}=1$ into the graphing calculator, we must solve for *y*.

$$\frac{x^2}{4}-\frac{y^2}{12}=1 \Rightarrow -\frac{y^2}{12}=1-\frac{x^2}{4}$$

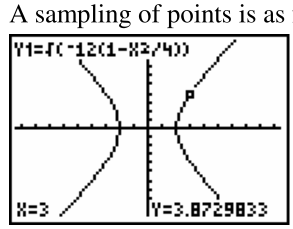
$$y^2=-12\left(1-\frac{x^2}{4}\right)$$

$$y=\pm\sqrt{-12\left(1-\frac{x^2}{4}\right)}$$





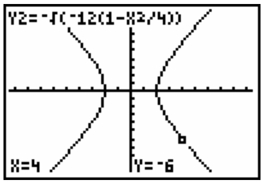
A sampling of points is as follows.



The distance of *P* from

$$(4,0)\approx\sqrt{(4-3)^2+(0-3.8729833)^2}$$
$$=\sqrt{1+(-3.8729833)^2}\approx 3.999999955$$

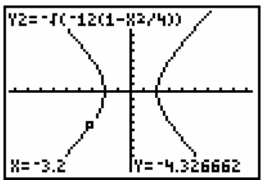
The distance of *P* from the line $x=1$ is $|3-1|=2.$ We have $3.999999955\approx 2(2)=4.$



The distance of *P* from

$(4,0) \approx \sqrt{(4-4)^2 + [0-(-6)]^2} = \sqrt{0+36} = 6$

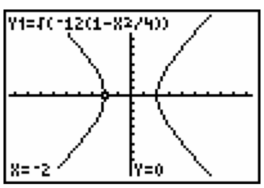
The distance of *P* from the line $x = 1$ is $|4-1| = 3.$ We have $6 \approx 2(3).$



The distance of *P* from

$$(4,0) \approx \sqrt{[4-(-3.2)]^2 + [0-(-4.326662)]^2}$$
$$= \sqrt{51.84 + (4.326662)^2} \approx 8.400000242$$

The distance of *P* from the line $x = 1$ is $|-3.2-1| = 4.2.$ We have $8.400000242 \approx 2(4.2) = 8.4.$



The distance of *P* from

$(4,0) \approx \sqrt{[4-(-2)]^2 + (0-0)^2} = \sqrt{36+0} = 6$

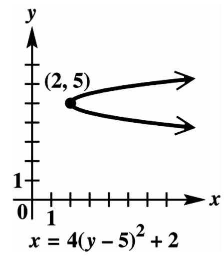
The distance of *P* from the line $x = 1$ is $|-2-1| = 3.$ We have $6 = 2(3).$

Chapter 10: Review Exercises

1. $x = 4(y-5)^2 + 2 \Rightarrow x - 2 = 4(y-5)^2$

The vertex is $(2,5).$ The graph opens to the right and has the same shape as $x = 4y^2$. It is a translation 5 units up and 2 units to the right of the graph of $x = 4y^2$. The domain is $[2,\infty).$ The range is $(-\infty,\infty).$ The graph is symmetric about its axis, the horizontal line $y = 5.$ Use the vertex and axis and plot a few additional points.

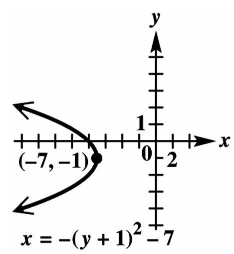
x	y
18	7
6	6
2	5
6	4
18	3



2. $x = -(y+1)^2 - 7 \Rightarrow x - (-7) = -[y-(-1)]^2$

The vertex is $(-7,-1).$ The graph opens to the left and has the same shape as $x = -y^2$. It is a translation 1 unit down and 7 units to the left of the graph of $x = -y^2$. The domain is $(-\infty,-7].$ The range is $(-\infty,\infty).$ The graph is symmetric about its axis, the horizontal line $y = -1.$ Use the vertex and axis and plot a few additional points.

x	y
−11	1
−8	0
−7	−1
−8	−2
−11	−3



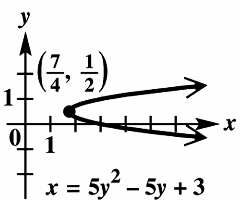
3. $x = 5y^2 - 5y + 3$

Complete the square on *y* to find the vertex and the axis.

$$x = 5y^2 - 5y + 3$$
$$x = 5\left(y^2 - y \qquad\right) + 3$$
$$x = 5\left(y^2 - y + \tfrac{1}{4} - \tfrac{1}{4}\right) + 3$$
$$x = 5\left(y - \tfrac{1}{2}\right)^2 + 5\left(-\tfrac{1}{4}\right) + 3$$
$$x = 5\left(y - \tfrac{1}{2}\right)^2 + \left(-\tfrac{5}{4}\right) + \tfrac{12}{4}$$
$$x = 5\left(y - \tfrac{1}{2}\right)^2 + \tfrac{7}{4} \Rightarrow x - \tfrac{7}{4} = 5\left(y - \tfrac{1}{2}\right)^2$$

The vertex is $\left(\frac{7}{4},\frac{1}{2}\right).$ The graph opens to the right and has the same shape as $x = 5y^2$, translated $\frac{7}{4}$ unit to the right and $\frac{1}{2}$ unit up. The domain is $\left[\frac{7}{4},\infty\right).$ The range is $(-\infty,\infty).$ The graph is symmetric about its axis, the horizontal line $y = \frac{1}{2}.$ Use the vertex and axis and plot a few additional points.

x	y
$\frac{27}{4}$	$\frac{3}{2}$
3	1
$\frac{7}{4}$	$\frac{1}{2}$
3	0
$\frac{27}{4}$	$-\frac{1}{2}$



4. $x = 2y^2 - 4y + 1$

Complete the square on y to find the vertex and the axis.

$x = 2y^2 - 4y + 1$

$x = 2\left(y^2 - 2y \quad\right) + 1$

$x = 2\left(y^2 - 2y + 1 - 1\right) + 1$

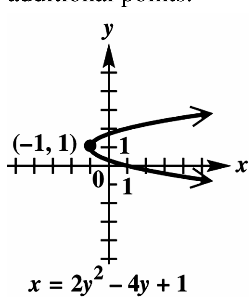
$x = 2(y-1)^2 + 2(-1) + 1$

$x = 2(y-1)^2 + (-2) + 1$

$x = 2(y-1)^2 - 1 \Rightarrow x - (-1) = 2(y-1)^2$

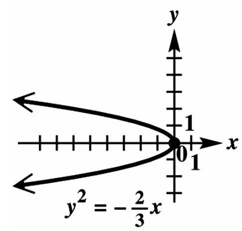
The vertex is $(-1,1)$. The graph opens to the right and has the same shape as $x = 2y^2$, translated 1 unit to the left and 1 unit up. The domain is $[-1,\infty)$. The range is $(-\infty,\infty)$. The graph is symmetric about its axis, the horizontal line $y = 1$. Use the vertex and axis and plot a few additional points.

x	y
7	3
1	2
−1	1
1	0
7	−1



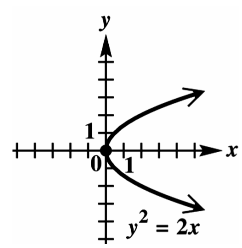
5. The equation $y^2 = -\frac{2}{3}x$ has the form $y^2 = 4px$, so $4p = -\frac{2}{3}$, from which $p = -\frac{1}{6}$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0) = \left(-\frac{1}{6},0\right)$ and directrix, $x = -p$, is $x = \frac{1}{6}$. The vertex is $(0,0)$, and the axis of the parabola is the x-axis. The domain is $(-\infty,0]$. The range is $(-\infty,\infty)$. Use this information and plot a few additional points.

x	y
−6	2
$-\frac{3}{2}$	1
0	0
$-\frac{3}{2}$	−1
−6	−2



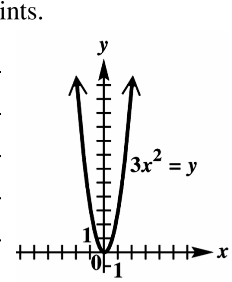
6. The equation $y^2 = 2x$ has the form $y^2 = 4px$, so $4p = 2$, from which $p = \frac{1}{2}$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0) = \left(\frac{1}{2},0\right)$ and directrix, $x = -p$, is $x = -\frac{1}{2}$. The vertex is $(0,0)$, and the axis of the parabola is the x-axis. The domain is $[0,\infty)$. The range is $(-\infty,\infty)$. Use this information and plot a few additional points.

x	y
2	2
$\frac{1}{2}$	1
0	0
$\frac{1}{2}$	−1
2	−2



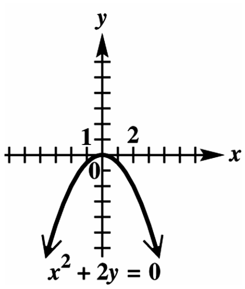
7. The equation $3x^2 = y \Rightarrow x^2 = \frac{1}{3}y$ has the form $x^2 = 4py$, so $4p = \frac{1}{3}$, from which $p = \frac{1}{12}$. Since the x-term is squared, the parabola is vertical, with focus $(0,p) = \left(0,\frac{1}{12}\right)$ and directrix, $y = -p$, is $y = -\frac{1}{12}$. The vertex is $(0,0)$, and the axis of the parabola is the y-axis. The domain is $(-\infty,\infty)$. The range is $[0,\infty)$. Use this information and plot a few additional points.

x	y
2	12
1	3
0	0
−1	3
−2	12



8. The equation $x^2 + 2y = 0 \Rightarrow x^2 = -2y$ has the form $x^2 = 4py$, so $4p = -2$, from which $p = -\frac{1}{2}$. Since the x-term is squared, the parabola is vertical, with focus $(0, p) = \left(0, -\frac{1}{2}\right)$ and directrix, $y = -p$, is $y = \frac{1}{2}$. The vertex is $(0,0)$, and the axis of the parabola is the y-axis. The domain is $(-\infty, \infty)$. The range is $(-\infty, 0]$. Use this information and plot a few additional points.

x	y
2	-2
1	$-\frac{1}{2}$
0	0
-1	$-\frac{1}{2}$
-2	-2



9. A parabola with focus $(4, 0)$ and vertex at the origin is a horizontal parabola. The equation has the form $y^2 = 4px$. Since $p = 4$ is positive, it opens to the right. Substituting 4 for p, we find that an equation for this parabola is $y^2 = 4(4)x \Rightarrow y^2 = 16x.$

10. A parabola with focus $(0, -3)$ and vertex at the origin is a vertical parabola. The equation has the form $x^2 = 4py$. Since $p = -3$ is negative, it opens down. Substituting -3 for p, we find that an equation for this parabola is $x^2 = 4(-3)y \Rightarrow x^2 = -12y.$

11. A parabola passing through $(-3,4)$, opening up, and vertex at the origin has an equation of the form $x^2 = 4py$. Use this equation with the coordinates of the point $(-3,4)$ to find the value of p.

$x^2 = 4py \Rightarrow (-3)^2 = 4p \cdot 4 \Rightarrow 9 = 16p \Rightarrow \frac{9}{16} = p$

Thus, an equation of the parabola is

$x^2 = 4\left(\frac{9}{16}\right)y \Rightarrow x^2 = \frac{9}{4}y.$

12. A parabola passing through $(2,5)$, opening to the right, and vertex at the origin has an equation of the form $y^2 = 4px$. Use this equation with the coordinates of the point $(2,5)$ to find the value of p.

$y^2 = 4px \Rightarrow 5^2 = 4p \cdot 2 \Rightarrow 25 = 8p \Rightarrow p = \frac{25}{8}$

Thus, an equation of the parabola is

$y^2 = 4\left(\frac{25}{8}\right)x \Rightarrow y^2 = \frac{25}{2}x.$

13. $y^2 + 9x^2 = 9 \Rightarrow \frac{x^2}{1} + \frac{y^2}{9} = 1 \Rightarrow \frac{(x-0)^2}{1^2} + \frac{(y-0)^2}{3^2} = 1$

The equation is of the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ with $a = 3, b = 1, h = 0,$ and $k = 0,$ so the graph of the given equation is an ellipse.

14. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow$

$\frac{(x-0)^2}{4^2} - \frac{(y-0)^2}{3^2} = 1$

The equation is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ with $a = 4, b = 3, h = 0,$ and $k = 0,$ so the graph of the given equation is a hyperbola.

15. $3y^2 - 5x^2 = 30 \Rightarrow \frac{y^2}{10} - \frac{x^2}{6} = 1 \Rightarrow$

$\frac{(y-0)^2}{\left(\sqrt{10}\right)^2} - \frac{(x-0)^2}{\left(\sqrt{6}\right)^2} = 1$

The equation is of the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ with $a = \sqrt{10}, b = \sqrt{6}, h = 0,$ and $k = 0,$ so the graph of the given equation is a hyperbola.

16. $y^2 + x = 4 \Rightarrow x - 4 = -y^2 \Rightarrow x - 4 = -(y-0)^2$

The equation is of the form $x - h = a(y-k)^2$ with $a = -1, h = 4,$ and $k = 0,$ so the graph of the given equation is a parabola.

17. $4x^2 - y = 0 \Rightarrow y = 4x^2 \Rightarrow y - 0 = 4(x-0)^2$

The equation is of the form $y - k = a(x-h)^2$ with $a = 4, h = 0,$ and $k = 0,$ so the graph of the given equation is a parabola.

18. $x^2 + y^2 = 25 \Rightarrow (x-0)^2 + (y-0)^2 = 5^2$

The equation is of the form $(x-h)^2 + (y-k)^2 = r^2$ with $r = 5, h = 0,$ and $k = 0,$ so the graph of the given equation is a circle.

19.
$$4x^2-8x+9y^2+36y=-4$$
$$4\left(x^2-2x+1-1\right)+9\left(y^2+4y+4-4\right)=-4$$
$$4(x-1)^2-4+9(y+2)^2-36=-4$$
$$4(x-1)^2+9(y+2)^2=-4+4+36$$
$$4(x-1)^2+9(y+2)^2=36$$
$$\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1$$
$$\frac{(x-1)^2}{3^2}+\frac{\left[y-(-2)\right]^2}{2^2}=1$$

The equation is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ with $a=3,\ b=2,\ h=1,$ and $k=-2,$ so the graph of the given equation is an ellipse.

20.
$$9x^2-18x-4y^2-16y-43=0$$
$$9\left(x^2-2x+1-1\right)-4\left(y^2+4y+4-4\right)=43$$
$$9(x-1)^2-9-4(y+2)^2+16=43$$
$$9(x-1)^2-4(y+2)^2=43+9-16$$
$$9(x-1)^2-4(y+2)^2=36$$
$$\frac{(x-1)^2}{4}-\frac{(y+2)^2}{9}=1$$
$$\frac{(x-1)^2}{2^2}-\frac{\left[y-(-2)\right]^2}{3^2}=1$$

The equation is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ with $a=2,\ b=3,\ h=1,$ and $k=-2,$ so the graph of the given equation is a hyperbola.

21.
$$4x^2+y^2=36\Rightarrow\frac{x^2}{9}+\frac{y^2}{36}=1\Rightarrow$$
$$\frac{(x-0)^2}{3^2}+\frac{(y-0)^2}{6^2}=1$$

The equation is of the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ with $a=6,\ b=3,\ h=0,$ and $k=0,$ so the graph of the given equation is an ellipse. It is centered at the origin with vertices at (0, –6) and (0, 6) and endpoints of the minor axis at (–3, 0) and (3, 0). The correct graph is F.

22. $x=2y^2+3\Rightarrow x-3=2(y-0)^2$

The equation is of the form $x-h=a(y-k)^2$ with $a=2,\ h=3,$ and $k=0,$ so the graph of the given equation is a parabola. This parabola opens to the right since $a>0.$ The vertex is located at (3, 0). The correct graph is C.

23.
$$(x-2)^2+(y+3)^2=36$$
$$(x-2)^2+\left[y-(-3)\right]^2=6^2$$

The equation is of the form $(x-h)^2+(y-k)^2=r^2$ with $r=6,\ h=2,$ and $k=-3,$ so the graph of the given equation is a circle. This circle is centered at (2, –3) and has radius 6. The correct graph is A.

24. $\frac{x^2}{36}+\frac{y^2}{9}=1\Rightarrow\frac{(x-0)^2}{6^2}+\frac{(y-0)^2}{3^2}=1$

The equation is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ with $a=6,\ b=3,\ h=0,$ and $k=0,$ so the graph of the given equation is an ellipse. This ellipse is centered at the origin. It has vertices located at $(-6,0)$ and $(6,0)$. The endpoints of minor axis at (0, –3) and (0, 3). The correct graph is E.

25.
$$(y-1)^2-(x-2)^2=36$$
$$\frac{(y-1)^2}{36}-\frac{(x-2)^2}{36}=1$$
$$\frac{(y-1)^2}{6^2}-\frac{(x-2)^2}{6^2}=1$$

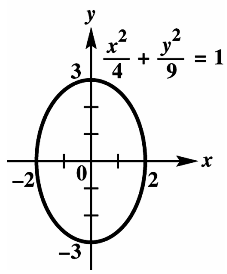
The equation is of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ with $a=6,\ b=6,\ h=2,$ and $k=1,$ so the graph of the given equation is a hyperbola. This hyperbola is centered at (2, 1), opening upward and downward, with vertices at (2, –5) and (2, 7). The correct graph is B.

26.
$$y^2=36+4x^2\Rightarrow y^2-4x^2=36\Rightarrow$$
$$\frac{y^2}{36}-\frac{x^2}{9}=1\Rightarrow\frac{(y-0)^2}{6^2}-\frac{(x-0)^2}{3^2}=1$$

The equation is of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ with $a=6,\ b=3,\ h=0,$ and $k=0,$ so the graph of the given equation is a hyperbola. This hyperbola is centered at the origin, opening upward and downward, with vertices at (0, –6) and (0, 6). The correct graph is D.

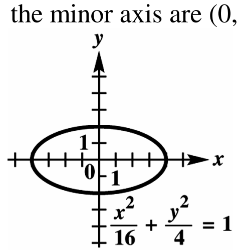
27. $\frac{x^2}{4}+\frac{y^2}{9}=1\Rightarrow\frac{(x-0)^2}{2^2}+\frac{(y-0)^2}{3^2}=1$

The graph is an ellipse centered at the origin with domain [–2, 2], range [–3, 3], and vertices at (0, –3) and (0, 3). The endpoints of the minor axis are (–2, 0) and (2, 0).



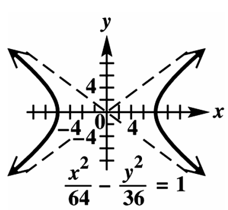
28. $\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow \frac{(x-0)^2}{4^2}+\frac{(y-0)^2}{2^2}=1$

The graph is an ellipse centered at the origin with domain [–4, 4], range [–2, 2], and vertices at (–4, 0) and (4, 0). The endpoints of the minor axis are (0, –2) and (0, 2).



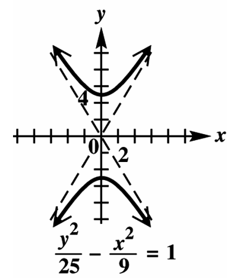
29. $\frac{x^2}{64}-\frac{y^2}{36}=1 \Rightarrow \frac{(x-0)^2}{8^2}-\frac{(y-0)^2}{6^2}=1$

The graph is a hyperbola centered at the origin with domain $(-\infty,-8]\cup[8,\infty)$, range (– ∞, ∞), and vertices (–8, 0) and (8, 0). The asymptotes are the lines $y=\pm\frac{6}{8}x \Rightarrow$ $y=\pm\frac{3}{4}x$.



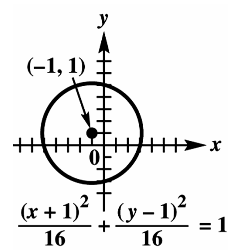
30. $\frac{y^2}{25}-\frac{x^2}{9}=1 \Rightarrow \frac{(y-0)^2}{5^2}-\frac{(x-0)^2}{3^2}=1$

The graph is a hyperbola centered at the origin with domain (– ∞, ∞). The range is $(-\infty,-5]\cup[5,\infty)$. The vertices are (0, –5) and (0, 5). The asymptotes are the lines $y=\pm\frac{5}{3}x$.



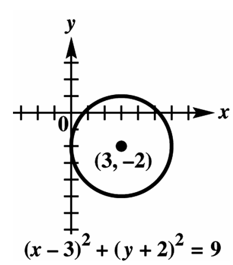
31.
$$\frac{(x+1)^2}{16}+\frac{(y-1)^2}{16}=1$$
$$(x+1)^2+(y-1)^2=16$$
$$[x-(-1)]^2+(y-1)^2=4^2$$

The graph is a circle centered at $(-1,1)$ and radius 4. Its domain is $[-1-4,-1+4]=[-5,3]$. The range is $[1-4,1+4]=[-3,5]$.

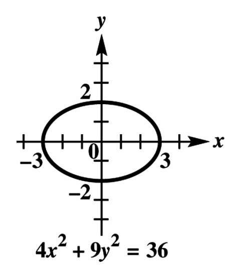


32.
$$(x-3)^2+(y+2)^2=9$$
$$(x-3)^2+[y-(-2)]^2=3^2$$

The graph is a circle centered at $(3,-2)$ and radius 3. Its domain is $[3-3,3+3]=[0,6]$ and range is $[-2-3,-2+3]=[-5,1]$.



Exercise 32



Exercise 33

33. $4x^2+9y^2=36 \Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1 \Rightarrow$

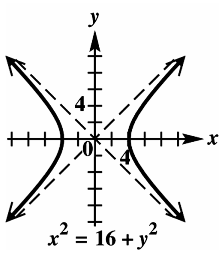
$$\frac{(x-0)^2}{3^2}+\frac{(y-0)^2}{2^2}=1$$

The graph is an ellipse centered at the origin with domain [–3, 3], range [–2, 2], vertices at (–3, 0) and (3, 0). The endpoints of the minor axis at (0, –2) and (0, 2).

34. $x^2 = 16 + y^2 \Rightarrow x^2 - y^2 = 16 \Rightarrow$

$\frac{x^2}{16} - \frac{y^2}{16} = 1 \Rightarrow \frac{(x-0)^2}{4^2} - \frac{(y-0)^2}{4^2} = 1$

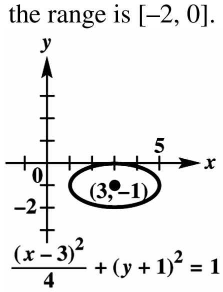
The graph is a hyperbola with domain $(-\infty, -4] \cup [4, \infty)$ and range $(-\infty, \infty)$. The vertices are located at $(-4, 0)$ and $(4, 0)$. The asymptotes have equations $y = \pm\frac{4}{4}x \Rightarrow$ $y = \pm x$.



35. $\frac{(x-3)^2}{4} + (y+1)^2 = 1 \Rightarrow$

$\frac{(x-3)^2}{2^2} + \frac{[y-(-1)]^2}{1^2} = 1$

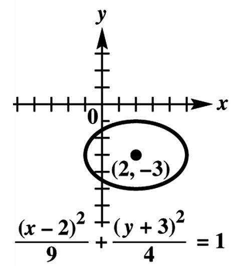
The graph is an ellipse centered at $(3, -1)$. The vertices are $(3-2, -1) = (1, -1)$ and $(3+2, -1) = (5, -1)$. The endpoints of the minor axis are $(3, -1-1) = (3, -2)$ and $(3, -1+1) = (3, 0)$. The domain is [1, 5] and the range is [–2, 0].



36. $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1 \Rightarrow$

$\frac{(x-2)^2}{3^2} + \frac{[y-(-3)]^2}{2^2} = 1$

The graph is an ellipse centered at $(2, -3)$. The vertices are located at $(2-3, -3) = (-1, -3)$ and $(2+3, -3) = (5, -3)$. The endpoints of the minor axis are located at $(2, -3-2) = (2, -5)$ and $(2, -3+2) = (2, -1)$. The domain is [–1, 5] and the range is [–5, –1].

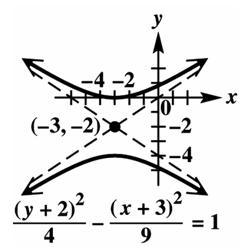


37. $\frac{(y+2)^2}{4} - \frac{(x+3)^2}{9} = 1 \Rightarrow$

$\frac{[y-(-2)]^2}{2^2} - \frac{[x-(-3)]^2}{3^2} = 1$

The graph is a hyperbola centered at $(-3, -2)$. The vertices are located at $(-3, -2-2) = (-3, -4)$ and $(-3, -2+2) = (-3, 0)$. The domain is $(-\infty, \infty)$ and the range is $(-\infty, -4] \cup [0, \infty)$. The asymptotes are the lines

$y - (-2) = \pm\frac{2}{3}[x - (-3)] \Rightarrow$
$y + 2 = \pm\frac{2}{3}(x+3) \Rightarrow y = -2 \pm \frac{2}{3}(x+3).$

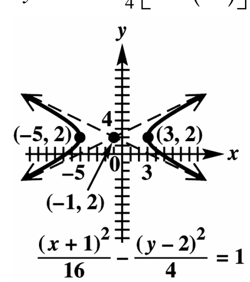


38. $\frac{(x+1)^2}{16} - \frac{(y-2)^2}{4} = 1 \Rightarrow$

$\frac{[x-(-1)]^2}{4^2} - \frac{(y-2)^2}{2^2} = 1$

The graph is a hyperbola centered at $(-1, 2)$. The vertices are located at $(-1-4, 2) = (-5, 2)$ and $(-1+4, 2) = (3, 2)$. The domain is $(-\infty, -5] \cup [3, \infty)$ and the range is $(-\infty, \infty)$. The asymptotes are the lines

$y - 2 = \pm\frac{2}{4}[x - (-1)] \Rightarrow y = 2 \pm \frac{1}{2}(x+1).$

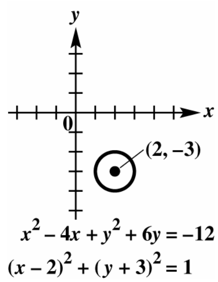


39. $x^2 - 4x + y^2 + 6y = -12$

Complete the squares on x and y:

$$x^2 - 4x + y^2 + 6y = -12$$
$$\left(x^2 - 4x + \quad\right) + \left(y^2 + 6y + \quad\right) = -12$$
$$\left(x^2 - 4x + 4\right) + \left(y^2 + 6y + 9\right) = -12 + 4 + 9$$
$$(x-2)^2 + (y+3)^2 = 1$$

This is the graph of a circle with center (2, −3) and radius 1. The domain is [1, 3], and the range is [−4, −2].

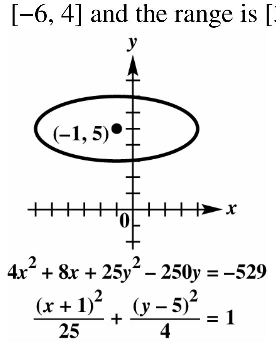


40. $4x^2 + 8x + 25y^2 - 250y = -529$

Complete the squares on x and y:

$$4x^2 + 8x + 25y^2 - 250y = -529$$
$$4\left(x^2 + 2x + \quad\right) + 25\left(y^2 - 10y + \quad\right) = -529$$
$$4\left(x^2 + 2x + 1\right) + 25\left(y^2 - 10y + 25\right) = -529 + 4 + 625$$
$$4(x+1)^2 + 25(y-5)^2 = 100$$
$$\frac{(x+1)^2}{25} + \frac{(y-5)^2}{4} = 1$$

This is an ellipse with center (−1, 5) and vertices (−6, 5) and (4, 5). The domain is [−6, 4] and the range is [3, 7].

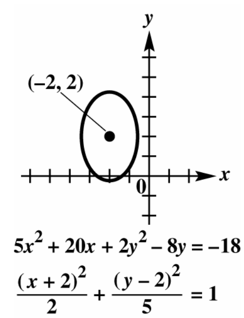


41. $5x^2 + 20x + 2y^2 - 8y = -18$

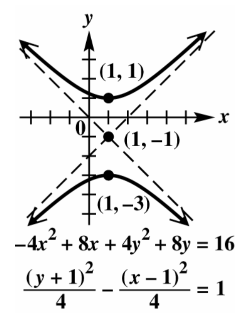
Complete the squares on x and y:

$$5x^2 + 20x + 2y^2 - 8y = -18$$
$$5\left(x^2 + 4x + \quad\right) + 2\left(y^2 - 4y + \quad\right) = -18$$
$$5\left(x^2 + 4x + 4\right) + 2\left(y^2 - 4y + 4\right) = -18 + 20 + 8$$
$$5(x+2)^2 + 2(y-2)^2 = 10$$
$$\frac{(x+2)^2}{2} + \frac{(y-2)^2}{5} = 1$$

$a^2 = 2 \Rightarrow a = \sqrt{2}$ and $b^2 = 5 \Rightarrow b = \sqrt{5}$. The graph is an ellipse with center (−2, 2) and vertices $\left(-2, 2+\sqrt{5}\right)$ and $\left(-2, 2-\sqrt{5}\right)$. The domain is $\left[-2-\sqrt{2}, -2+\sqrt{2}\right]$, and the range is $\left[2-\sqrt{5}, 2+\sqrt{5}\right]$.



Exercise 41



Exercise 42

42. $-4x^2 + 8x + 4y^2 + 8y = 16$

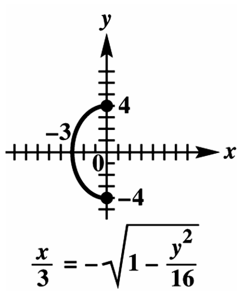
Complete the squares on x and y:

$$-4x^2 + 8x + 4y^2 + 8y = 16$$
$$-4\left(x^2 - 2x + \quad\right) + 4\left(y^2 + 2y + \quad\right) = 16$$
$$-4\left(x^2 - 2x + 1\right) + 4\left(y^2 + 2y + 1\right) = 16 - 4 + 4$$
$$-4(x-1)^2 + 4(y+1)^2 = 16$$
$$-\frac{(x-1)^2}{4} + \frac{(y+1)^2}{4} = 1 \Rightarrow$$
$$\frac{(y+1)^2}{4} - \frac{(x-1)^2}{4} = 1$$

The graph is a hyperbola centered at (1, −1). $a^2 = b^2 = 4 \Rightarrow a = b = 2$, so the vertices are (1, −3) and (1, 1), and the asymptotes are $y + 1 = \pm(x-1) \Rightarrow y = -1 \pm (x-1)$. The domain is $(-\infty, \infty)$, and the range is $(-\infty, -3] \cup [1, \infty)$.

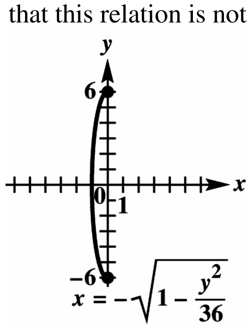
43. $\frac{x}{3}=-\sqrt{1-\frac{y^2}{16}}$

Square both sides to get $\frac{x^2}{9}=1-\frac{y^2}{16}$ or $\frac{x^2}{9}+\frac{y^2}{16}=1$ or $\frac{x^2}{3^2}+\frac{y^2}{4^2}=1$. This is the equation of an ellipse with center $(0,0)$ and vertices $(0,-4)$ and $(0,4)$. It has endpoints of the minor axis of $(-3,0)$ and $(3,0)$. The graph of the original equation is the left half of this ellipse. The domain is [–3, 0] and the range is [–4, 4]. The vertical line test shows that this relation is not a function.



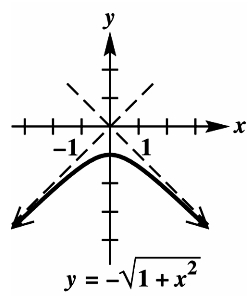
44. $x=-\sqrt{1-\frac{y^2}{36}}$

Square both sides to get $x^2=1-\frac{y^2}{36}$ or $\frac{x^2}{1}+\frac{y^2}{36}=1$ or $\frac{x^2}{1^2}+\frac{y^2}{6^2}=1$. This is the equation of an ellipse with center $(0,0)$ and vertices $(0,-6)$ and $(0,6)$. It has endpoints of the minor axis of $(-1,0)$ and $(1,0)$. The graph of the original equation is the left half of this ellipse. The domain is [–1, 0] and the range is [–6, 6]. The vertical line test shows that this relation is not a function.



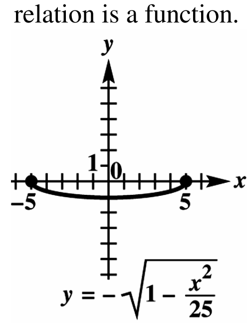
45. $y=-\sqrt{1+x^2}$

Square both sides of the equation to get $y^2=1+x^2$ or $\frac{y^2}{1}-\frac{x^2}{1}=1$ or $\frac{y^2}{1^2}-\frac{x^2}{1^2}=1$. This is the equation of a hyperbola with center $(0,0)$ and vertices $(0,-1)$ and $(0,1)$. Since $a=1$, $b=1$, and $y=\pm\frac{a}{b}x$, we have asymptotes $y=\pm\frac{1}{1}x\Rightarrow y=\pm x$. The original equation is the bottom half of the hyperbola. The domain is $(-\infty,\infty)$. The range is $(-\infty,-1]$. The vertical line test shows this is a graph of a function.



46. $y=-\sqrt{1-\frac{x^2}{25}}$

Square both sides to get $y^2=1-\frac{x^2}{25}$ or $\frac{x^2}{25}+\frac{y^2}{1}=1$ or $\frac{x^2}{5^2}+\frac{y^2}{1^2}=1$. This is the equation of an ellipse with center $(0,0)$ and vertices $(-5,0)$ and $(5,0)$. It has endpoints of the minor axis of $(0,-1)$ and $(0,1)$. The graph of the original equation is the lower half of this ellipse. The domain is [–5, 5] and the range is [–1, 0]. The vertical line test shows that this relation is a function.



47. Ellipse; vertex at (0, −4), focus at (0, −2), center at the origin
Since the vertex is (0, −4) and the focus is (0, −2), we know that the major axis is vertical and thus, the equation is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. Since the ellipse is centered at the origin, we have $a = 4$ which implies $a^2=16$. Since the ellipse is centered at the origin and a focus is at (0, 2), we have $c = 2$ which implies $c^2=4$. Since $c^2=a^2-b^2$, we have $c^2=a^2-b^2 \Rightarrow 4=16-b^2 \Rightarrow b^2=12$. Thus, the equation of the ellipse is $\frac{x^2}{12}+\frac{y^2}{16}=1$.

48. Ellipse; x-intercept 6, focus at (2, 0), center at the origin
Since the ellipse is centered at the origin and a focus is at (2, 0), we know that the major axis is horizontal and thus, the equation is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. With an x-intercept of 6, we have $a = 6$ which implies $a^2=36$. Since the ellipse is centered at the origin and a focus is at $(-2,0)$, we have $c=2$, which implies $c^2=4$. Since $c^2=a^2-b^2$, we have $4=36-b^2 \Rightarrow b^2=32$. Thus, the equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{32}=1$.

49. Hyperbola; focus at (0, 5) transverse axis of length 8, center at the origin
Since the focus is (0, 5) we know that the transverse axis is vertical and thus, the equation is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. Since the length of the transverse axis is 8, we have $2a=8 \Rightarrow a=4$. This implies that $a^2=16$. Since the hyperbola is centered at the origin and a focus is at (0, 5) we have $c=5$ which implies $c^2=25$. Since $c^2=a^2+b^2$, we have $25=16+b^2 \Rightarrow b^2=9$. Thus, the equation of the hyperbola is $\frac{y^2}{16}-\frac{x^2}{9}=1$.

50. Hyperbola; y-intercept –2, passing through (2, 3), center at the origin
Since there is a y-intercept, we know the equation is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.
Since the hyperbola is centered at the origin and the y-intercept is –2, we know that $a=2$. This implies that $a^2=4$. Thus we have the equation $\frac{y^2}{4}-\frac{x^2}{b^2}=1$. Substitute $x = 2$ and $y = 3$ into this equation and solve for b^2.

$$\frac{y^2}{4}-\frac{x^2}{b^2}=1 \Rightarrow \frac{3^2}{4}-\frac{2^2}{b^2}=1 \Rightarrow \frac{9}{4}-\frac{4}{b^2}=1 \Rightarrow$$
$$-\frac{4}{b^2}=\frac{4}{4}-\frac{9}{4} \Rightarrow -\frac{4}{b^2}=-\frac{5}{4} \Rightarrow 16=5b^2 \Rightarrow$$
$$b^2=\frac{16}{5}$$

Thus, the equation of the hyperbola is
$\frac{y^2}{4}-\frac{x^2}{\frac{16}{5}}=1$ or $\frac{y^2}{4}-\frac{5x^2}{16}=1$.

51. Parabola with focus at (3, 2) and directrix $x = -3$
Since the directrix is $x = -3$, a vertical line, the parabola is of the form $(y-k)^2=4p(x-h)$. Since the vertex is halfway between the focus and the directrix, we know that the vertex is $(0,2)=(h,k)$. Because of the orientation of the directrix versus the focus, we know that the parabola opens to the right, so the p value must be positive. By examining the distance between the vertex and the focus or the directrix and the vertex, we know that $p = 3$. Thus, the equation of the parabola is
$(y-2)^2=4\cdot 3(x-0)$ or $(y-2)^2=12x$.

52. Parabola with vertex at (–3, 2) and y-intercepts 5 and –1
In order to pass through the points (–3, 2), (0, 5), and (0, –1), the parabola must open to the right. The equation must be of the form $(y-k)^2=4p(x-h)$. Since the vertex is $(-3,2)=(h,k)$, we have
$(y-2)^2=4p[x-(-3)] \Rightarrow (y-2)^2=4p(x+3)$.
Since the y-intercepts are points on the curve, we can substitute one of them as an ordered pair and solve for p. Substituting $(0,5)$ into our last equation, we have

$$(y-2)^2=4p(x+3)$$
$$(5-2)^2=4p(0+3)$$
$$3^2=4p(3) \Rightarrow 9=12p \Rightarrow p=\frac{9}{12}=\frac{3}{4}.$$

Thus, the equation of the parabola is
$(y-2)^2=4\left(\frac{3}{4}\right)(x+3)$ or $(y-2)^2=3(x+3)$.

53. Ellipse with foci at (–2, 0) and (2, 0) and major axis of length 10
Since the foci are at (–2, 0) and (2, 0), we know that major axis is horizontal and the ellipse is centered at the origin. Thus, the equation is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Since the foci are at (–2, 0) and (2, 0), we know that c = 2, which implies $c^2=4$. Since the major axis is of length 10 we have $2a$ =10. This implies that $a=5$ and $a^2=25$. Since $c^2=a^2-b^2$, we have $c^2=a^2-b^2 \Rightarrow 4=25-b^2 \Rightarrow$ $b^2=21$. Thus, the equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{21}=1$.

54. Ellipse with foci at (0, 3) and (0, –3) and vertex (0, −7)
Since the foci are at (0, 3) and (0, –3), we know that major axis is vertical and the ellipse is centered at the origin. Thus, the equation is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. Since the foci are at (0, 3) and (0, –3), we know that $c=3$, which implies $c^2=9$. With a vertex of (0, −7), we have a = 7. This implies that $a^2=49$. Since $c^2=a^2-b^2$, we have $c^2=a^2-b^2 \Rightarrow\ 9=49-b^2 \Rightarrow b^2=40$. Thus, the equation of the ellipse is $\frac{x^2}{40}+\frac{y^2}{49}=1$.

55. Hyperbola with x-intercepts ± 3; foci at (–5, 0), (5, 0)
Since the center is halfway between the foci, the center is at the origin. Since the foci are on a horizontal transverse axis, the equation has the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The x-intercepts are also vertices, so $a=3$. This implies $a^2=9$. Since the foci at (–5, 0) and (5, 0), we have c = 5. This implies $c^2=25$. Since $c^2=a^2+b^2$, we have $c^2=a^2+b^2 \Rightarrow 25=9+b^2 \Rightarrow b^2=16$. Thus, the equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$.

56. Hyperbola with foci at (0, 12), (0, –12); asymptotes $y=\pm x$
Since the center is halfway between the vertices, the center is at the origin. Since the foci are on a vertical transverse axis, the equation has the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ and c = 12, which implies $c^2=144$. Since $y=\pm x=\pm\frac{1}{1}x=\pm\frac{a}{b}x$, we have $\frac{a}{b}=\frac{1}{1}\Rightarrow$ $a=b\Rightarrow a^2=b^2$. Since $c^2=a^2+b^2$, we have $144=a^2+b^2 \Rightarrow 144=a^2+a^2 \Rightarrow$ $144=2a^2 \Rightarrow a^2=72$. Thus, we also have $b^2=72$. Thus, the equation of the hyperbola is $\frac{y^2}{72}-\frac{x^2}{72}=1$.

57. The points $F'(0,0)$ and $F(4,0)$ are the foci, so the center of the ellipse is $(2,0)=(h,k)$. Since the foci are on a horizontal axis, the equation is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. Since the distance from the center to each of the foci is 2, we have $c=2\Rightarrow c^2=4$. For any point P on the ellipse, $d(P,F)+d(P,F')=2a=8$, so a = 4. Thus, we have $a^2=16$. Since $c^2=a^2-b^2$, we have $c^2=a^2-b^2 \Rightarrow 4=16-b^2 \Rightarrow b^2=12$. Thus, the equation of the ellipse is $\frac{(x-2)^2}{16}+\frac{y^2}{12}=1$.

58. The points $F'(0,0)$ and $F(0,4)$ are the foci, so the center of the hyperbola is $(0,2)=(h,k)$ and $c=2\Rightarrow c^2=4$. Also, the foci are on a vertical transverse axis, and the equation has the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$. Since a represents the distance from the center to a vertex, the vertices are located at $(0,2-a)$ and $(0,2+a)$. For any point P on the hyperbola, we have $\left|d(P,F')-d(P,F)\right|=2a$. Thus, we have $2a=2\Rightarrow a=1$. This implies $a^2=1$. Since $c^2=a^2+b^2$, we have $4=1+b^2 \Rightarrow b^2=3$. Thus, the equation of the hyperbola is $\frac{(y-2)^2}{1}-\frac{(x-0)^2}{3}=1$ or $(y-2)^2-\frac{x^2}{3}=1$.

59. Refer to the chart on page 976 of the text. Graph A is an ellipse, so $0 < e < 1$. Graph B is a parabola, so $e = 1$. Graph C is a circle, so $e = 0$. Graph D is a hyperbola, so $e > 1$. Thus, in increasing order of eccentricity, the graphs are C, A, B, D.

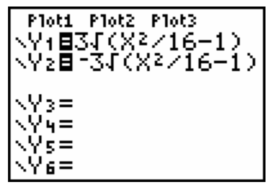
60. Since the major axis has length 134.5, we have $2a = 134.5 \Rightarrow a = 67.25$. Since $e = \frac{c}{a}$, we have $.006775 = \frac{c}{67.25} \Rightarrow$
$c = (.006775)(67.25) = .45561875.$ The smallest distance to the sun is $a - c$. This implies that the smallest distance to the sun is $a - c = 67.25 - .45561875 = 66.79438125 \approx 66.8$ million mi. The greatest distance to the sun is $a + c$. This implies that the greatest distance to the sun is $a + c = 67.25 + .45561875$ $= 67.70561875 \approx 67.7$ million mi.

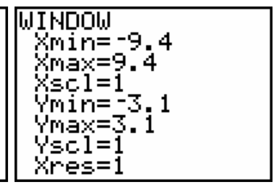
61. Since the eccentricity is .964 and $e = \frac{c}{a}$, we have $.964 = \frac{c}{a} \Rightarrow c = .964a$. Since the closest distance to the sun is 89 million mi, we have $a - c = 89$. Substituting $c = .964a$ into this equation, we have the following.
$a - c = 89 \Rightarrow a - .964a = 89 \Rightarrow .036a = 89 \Rightarrow$
$a = \frac{89}{.036} = \frac{22{,}250}{9}$
Since $c = .964a$, we have
$c = .964\left(\frac{22{,}250}{9}\right) = \frac{21{,}449}{9}$. Since $c^2 = a^2 - b^2$, we have $c^2 = a^2 - b^2 \Rightarrow$
$\left(\frac{21{,}449}{9}\right)^2 = \left(\frac{22{,}250}{9}\right)^2 - b^2 \Rightarrow$
$\frac{460{,}059{,}601}{81} - \frac{495{,}062{,}500}{81} = -b^2 \Rightarrow b^2 = \frac{35{,}002{,}899}{81}$
Since $a^2 = \left(\frac{22{,}250}{9}\right)^2 = \frac{495{,}062{,}500}{81} \approx 6{,}111{,}883$ and $b^2 = \frac{35{,}002{,}899}{81} \approx 432{,}135$, the equation is
$\frac{x^2}{6{,}111{,}883} + \frac{y^2}{432{,}135} = 1.$

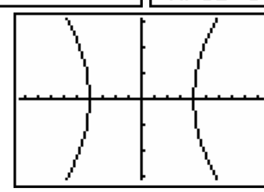
62. The points $F'(-5,0)$ and $F(5,0)$ are the foci, so the hyperbola is centered at the origin and has a horizontal transverse axis. The equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since the points (–5, 0) and (5, 0) are the foci, we have $c = 5$. This implies $c^2 = 25$.

Also, for any point P on the hyperbola we have $\left|d(P,F') - d(P,F)\right| = 2a.$ Thus, we have $2a = 8 \Rightarrow a = 4$. This implies $a^2 = 16$. Since $c^2 = a^2 + b^2$, we have $25 = 16 + b^2 \Rightarrow b^2 = 9.$ Thus, the equation of the hyperbola is
$\frac{x^2}{16} - \frac{y^2}{9} = 1.$
To graph this hyperbola with a graphing calculator, we must solve the equation for y to obtain two functions.

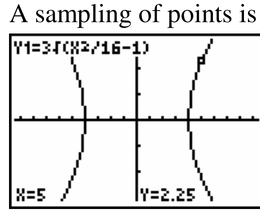
$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow -\frac{y^2}{9} = 1 - \frac{x^2}{16} \Rightarrow$$
$$\frac{y^2}{9} = -\left(1 - \frac{x^2}{16}\right) \Rightarrow \frac{y^2}{9} = \frac{x^2}{16} - 1 \Rightarrow$$
$$y^2 = 9\left(\frac{x^2}{16} - 1\right) \Rightarrow y = \pm\sqrt{9\left(\frac{x^2}{16} - 1\right)} \Rightarrow$$
$$y = \pm 3\sqrt{\frac{x^2}{16} - 1}$$







A sampling of points is as follows.



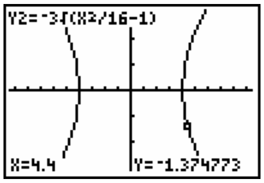
Verify that $\left|d(P,F') - d(P,F)\right| = 8.$

$$\begin{aligned} d(P,F') - d(P,F) &= \sqrt{\left[5-(-5)\right]^2 + (2.25-0)^2} \\ &\quad - \sqrt{(5-5)^2 + (2.25-0)^2} \\ &= \sqrt{10^2 + (2.25)^2} - \sqrt{0^2 + (2.25)^2} \\ &= \sqrt{100 + 5.0625} - \sqrt{0 + 5.0625} \\ &= \sqrt{105.0625} - \sqrt{5.0625} = 10.25 - 2.25 = 8 \end{aligned}$$

Since $\left|d(P,F') - d(P,F)\right| = |8| = 8$, we have the desired results.

(*continued on next page*)

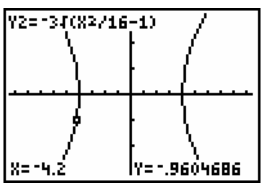
(*continued from page 1081*)



Verify that $\left|d\left(P,F'\right)-d\left(P,F\right)\right|=8.$

$$\begin{aligned}d\left(P,F'\right)-d\left(P,F\right)\\ &=\sqrt{\left[4.4-(-5)\right]^2+(-1.374773-0)^2}\\ &\quad-\sqrt{(4.4-5)^2+(-1.374773-0)^2}\\ &=\sqrt{9.4^2+(-1.374773)^2}\\ &\quad-\sqrt{(-.6)^2+(-1.374773)^2}\\ &\approx 7.999999775\end{aligned}$$

Since $\left|d\left(P,F'\right)-d\left(P,F\right)\right|\approx|7.999999775|$
$= 7.999999775 \approx 8$
we have the desired results.



Verify that $\left|d\left(P,F'\right)-d\left(P,F\right)\right|=8.$

$$\begin{aligned}d\left(P,F'\right)-d\left(P,F\right)\\ &=\sqrt{\left[-4.2-(-5)\right]^2+(-.9604686-0)^2}\\ &\quad-\sqrt{(-4.2-5)^2+(-.9604686-0)^2}\\ &=\sqrt{.8^2+(-.9604686)^2}\\ &\quad-\sqrt{(-9.2)^2+(-.9604686)^2}\\ &\approx -8.000000024\end{aligned}$$

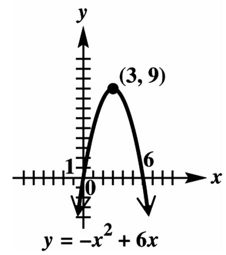
Since $\left|d\left(P,F'\right)-d\left(P,F\right)\right|\approx|-8.000000024|$
$= 8.000000024 \approx 8$
we have the desired results.

Chapter 10 Test

1. $y=-x^2+6x$
Completing the square on *x*, we have
$y=-x^2+6x \Rightarrow y=-\left(x^2-6x+9-9\right)\Rightarrow$
$y=-(x-3)^2+9 \Rightarrow y-9=-(x-3)^2$
The equation has the form $x-h=a(y-k)^2$. The vertex, (h,k), is $(3,9)$. Since the x-term is squared, the axis is the vertical line $x=h$, or $x=3$.

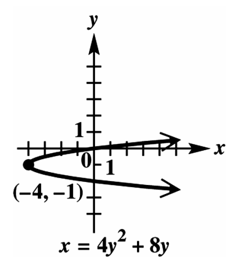
The domain is $(-\infty,\infty)$. The range is $(-\infty,9]$. Using this information and plot a few additional points.

x	y
5	5
4	8
3	9
2	8
1	5



2. $x=4y^2+8y$
Completing the square on *y*, we have
$x=4y^2+8y \Rightarrow x=4\left(y^2+2y+1-1\right)\Rightarrow$
$x=4(y+1)^2-4 \Rightarrow x-(-4)=4\left[y-(-1)\right]^2$
The equation has the form $y-k=a(x-h)^2$. The vertex, (h,k), is $(-4,-1)$. Since the y-term is squared, the axis is the horizontal line $y=k$, or $y=-1$. The domain is $[-4,\infty)$. The range is $(-\infty,\infty)$. Using this information and plot a few additional points.

x	y
12	1
0	0
–4	–1
0	–2
12	–3



3. Since $x=8y^2 \Rightarrow y^2=\frac{1}{8}x$, the parabola is of the form $y^2=4px$. Thus, we have $4p=\frac{1}{8}\Rightarrow p=\frac{1}{32}$. Since the y-term is squared, the parabola is horizontal, with focus $(p,0)=\left(\frac{1}{32},0\right)$ and directrix, $x=-p$, is $x=-\frac{1}{32}$.

4. Parabola; vertex at $(2,3)$, passing through $(-18,1)$, opening to the left
Since the parabola opens to the left, it is of the form $(y-k)^2=4p(x-h)$. Since the vertex is located at $(2,3)$, we have $(y-3)^2=4p(x-2)$.

We can now substitute
$x = -18$ and $y = 1$, and solve for p.

$$(1-3)^2 = 4p(-18-2) \Rightarrow$$
$$(-2)^2 = 4p(-20) \Rightarrow 4 = -80p \Rightarrow p = -\tfrac{1}{20}$$

Thus, the equation is

$$(y-3)^2 = 4\left(-\tfrac{1}{20}\right)(x-2) \Rightarrow (y-3)^2 = -\tfrac{1}{5}(x-2)$$

or alternatively written as $x - 2 = -5(y-3)^2$.

We could have also stated that since the parabola opens to the left, it is of the form $x - h = a(y-k)^2$. With $(h,k) = (2,3)$, we have $x - 2 = a(y-3)^2$. Substituting x = –18 and y = 1, we have the following.

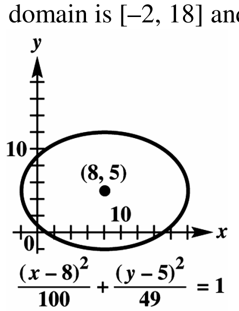
$$-18-2 = a(1-3)^2 \Rightarrow -20 = a(-2)^2 \Rightarrow$$
$$-20 = 4a \Rightarrow a = 5$$

Thus, $x - 2 = -5(y-3)^2$.

5. Answers will vary.

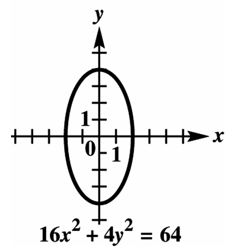
6. $$\frac{(x-8)^2}{100} + \frac{(y-5)^2}{49} = 1$$
$$\frac{(x-8)^2}{10^2} + \frac{(y-5)^2}{7^2} = 1$$

The graph is an ellipse centered at $(8,5)$. We also have the vertices located at $(8-10,5) = (-2,5)$ and $(8+10,5) = (18,5)$. The endpoints of the minor axis are located at $(8,5-7) = (8,-2)$ and $(8,5+7) = (8,12)$. The domain is [–2, 18] and the range is [–2, 12].



7. $$16x^2 + 4y^2 = 64 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1 \Rightarrow$$
$$\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$$

The graph is an ellipse centered at the origin The vertices are located at $(0,-4)$ and $(0,4)$. The endpoints of the minor axis are located at $(-2,0)$ and $(2,0)$. The domain is [–2, 2] and the range is [–4, 4].

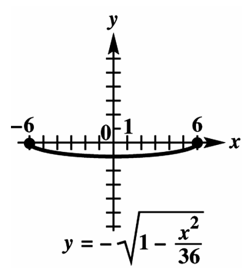


8. $$y = -\sqrt{1 - \frac{x^2}{36}}$$

Square both sides to get

$$y = -\sqrt{1 - \tfrac{x^2}{36}} \Rightarrow y^2 = 1 - \tfrac{x^2}{36} \Rightarrow \tfrac{x^2}{36} + y^2 = 1 \Rightarrow$$
$$\tfrac{x^2}{6^2} + \tfrac{y^2}{1^2} = 1.$$

This is the equation of an ellipse with vertices (–6, 0) and (6, 0). The endpoints of the minor axis are at (0, –1) and (0, 1). The graph of the original equation is the bottom half of the ellipse. The vertical line test shows that this relation is a function.



9. Ellipse; centered at the origin, horizontal major axis with length 6, minor axis with length 4
Since the ellipse is centered at the origin and major axis is horizontal, the ellipse has the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since the major axis has length 6, we have $2a = 6 \Rightarrow a = 3 \Rightarrow a^2 = 9$. Since the minor axis has length 4, we have $2b = 4 \Rightarrow b = 2 \Rightarrow b^2 = 4$. Thus, the equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

10. Place the half-ellipse that represents the overpass on a coordinate system with the center of the bottom of the overpass at the origin. If the complete ellipse were drawn, the center of the ellipse would be $(0,0)$. Then the half-ellipse will include the points $(0,12), (-20,0)$, and $(20,0)$.

The equation is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Thus, for the complete ellipse, we have $a = 20$ and $b=12$. Thus, we have

$\frac{x^2}{20^2}+\frac{y^2}{12^2}=1 \Rightarrow \frac{x^2}{400}+\frac{y^2}{144}=1.$

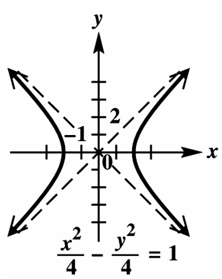
At a distance of 10 ft from the center of the bottom, $x = 10$. Find the positive y-coordinate of the point on the ellipse with x-coordinate 10.

$$\frac{x^2}{400}+\frac{y^2}{144}=1 \Rightarrow \frac{10^2}{400}+\frac{y^2}{144}=1 \Rightarrow$$
$$\frac{100}{400}+\frac{y^2}{144}=1 \Rightarrow \frac{1}{4}+\frac{y^2}{144}=1 \Rightarrow \frac{y^2}{144}=\frac{3}{4} \Rightarrow$$
$$y^2=108 \Rightarrow y=\sqrt{108}\approx 10.39$$

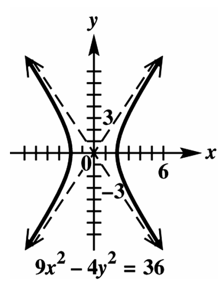
The arch is approximately 10.39 ft high 10 ft from the center of the bottom.

11. $\frac{x^2}{4}-\frac{y^2}{4}=1 \Rightarrow \frac{x^2}{2^2}-\frac{y^2}{2^2}=1$

The graph of the hyperbola is centered at the origin. Since the x^2-term comes first, the branches open to the right and left, with vertices (–2, 0) and (2, 0). The domain is $(-\infty,-2]\cup[2,\infty)$. The range is $(-\infty, \infty)$. The asymptotes are the lines $y=\pm\frac{2}{2}x=\pm x$.



Exercise 11



Exercise 12

12. $9x^2-4y^2=36 \Rightarrow \frac{x^2}{4}-\frac{y^2}{9}=1 \Rightarrow \frac{x^2}{2^2}-\frac{y^2}{3^2}=1$

The graph of the hyperbola is centered at the origin. Since the x^2-term comes first, the branches open to the right and left, with vertices (–2, 0) and (2, 0). The domain is $(-\infty,-2]\cup[2,\infty)$. The range is $(-\infty, \infty)$. The asymptotes are the lines $y=\pm\frac{3}{2}x$.

13. Hyperbola; y-intercepts ± 5, foci at (0, –6) and (0, 6)

Since the center is halfway between the foci, the center is $(0,0)$. Since the foci are on a vertical transverse axis, the equation has the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. The y-intercepts are also vertices, so a = 5, and thus $a^2=25$. With the given foci, we have c = 6. So given $c^2=a^2+b^2$, we have $c^2=a^2+b^2 \Rightarrow$

$6^2=5^2+b^2 \Rightarrow 36=25+b^2 \Rightarrow b^2=11$

Thus, the equation of the hyperbola is

$\frac{y^2}{25}-\frac{x^2}{11}=1.$

14.
$$x^2+8x+y^2-4y+2=0$$
$$\left(x^2+8x+16-16\right)+\left(y^2-4y+4-4\right)=-2$$
$$(x+4)^2-16+(y-2)^2-4=-2$$
$$(x+4)^2+(y-2)^2=-2+16+4$$
$$(x+4)^2+(y-2)^2=18$$
$$\left[x-(-4)\right]^2+(y-2)^2=\left(\sqrt{18}\right)^2$$

The graph of this equation is a circle with center $(-4,2)$ and radius $\sqrt{18}$. Also note in our original equation, the x^2- and y^2- terms have the same positive coefficient.

15.
$$5x^2+10x-2y^2-12y-23=0$$
$$5\left(x^2+2x+1-1\right)-2\left(y^2+6y+9-9\right)=23$$
$$5(x+1)^2-5-2(y+3)^2+18=23$$
$$5(x+1)^2-2(y+3)^2=23+5-18$$
$$5(x+1)^2-2(y+3)^2=10$$
$$\frac{(x+1)^2}{2}-\frac{(y+3)^2}{5}=1$$
$$\frac{\left[x-(-1)\right]^2}{\left(\sqrt{2}\right)^2}-\frac{\left[y-(-3)\right]^2}{\left(\sqrt{5}\right)^2}=1$$

The graph of this equation is a hyperbola centered at $(-1,-3)$ and vertices $\left(-1\pm\sqrt{2},-3\right)$. The asymptotes are

$y-(-3)=\pm\frac{\sqrt{5}}{\sqrt{2}}\left[x-(-2)\right] \Rightarrow y+3=\pm\frac{\sqrt{10}}{2}(x+2).$

Also note in our original equation, the x^2- and y^2-terms have coefficients that are opposite in sign.

16. $3x^2+10y^2-30=0 \Rightarrow 3x^2+10y^2=30 \Rightarrow$

$$\frac{x^2}{10}+\frac{y^2}{3}=1 \Rightarrow \frac{(x-0)^2}{(\sqrt{10})^2}+\frac{(y-0)^2}{(\sqrt{3})^2}=1$$

The graph of this equation is an ellipse centered at the origin and x-intercepts (vertices) of $\pm\sqrt{10}$, and y-intercepts (endpoints of minor axis) of $\pm\sqrt{3}$. Also note in our original equation, the x^2 - and y^2 - terms both have different positive coefficients.

17. $x^2-4y=0 \Rightarrow 4y=x^2 \Rightarrow y=\frac{1}{4}x^2$

This is a parabola with its vertex at the origin, opening upward. Also, note our original equation has an x^2 - term, but no y^2 - term.

18. $(x+9)^2+(y-3)^2=0$

$[x-(-9)]^2+(y-3)^2=0$

This is the equation of a “circle” centered at $(-9,3)$ with radius 0. The graph of this equation is a point.

19. $x^2+4x+y^2-6y+30=0$

$(x^2+4x+4-4)+(y^2-6y+9-9)=-30$

$(x^2+4x+4)-4+(y^2-6y+9)-9=-30$

$(x+2)^2+(y-3)^2=-30+4+9$

$[x-(-2)]^2+(y-3)^2=-17$

This equation has the form of the equation of a circle. However, since r^2 cannot be negative, there is no graph of this equation.

20. Solve the equation $\frac{x^2}{25}-\frac{y^2}{49}=1$ for y.

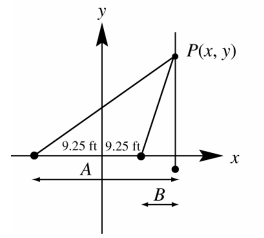
$$\frac{x^2}{25}-\frac{y^2}{49}=1 \Rightarrow -\frac{y^2}{49}=1-\frac{x^2}{25} \Rightarrow \frac{y^2}{49}=\frac{x^2}{25}-1 \Rightarrow$$
$$y^2=49\left(\frac{x^2}{25}-1\right) \Rightarrow y=\pm\sqrt{49\left(\frac{x^2}{25}-1\right)} \Rightarrow$$
$$y=\pm 7\sqrt{\frac{x^2}{25}-1}$$

Thus, the functions

$Y_1=7\sqrt{\frac{x^2}{25}-1}$ and $Y_2=-7\sqrt{\frac{x^2}{25}-1}$ were used to obtain the graph.

Chapter 10 Quantitative Reasoning

1. Locate the goal posts and point $P(x, y)$ as in the figure on the left in the exercise. Label the distances A, B and x as illustrated.



(Here, A corresponds to a and B to b in the above figure. These have been changed to avoid confusion with the same letters in the standard equation for a hyperbola.) From the illustration, we can see that $A=x+9.25$ and $B=x-9.25$. Using Regiomontanus’ result with the reorientation of axes, we have the angle θ is greatest when $y=\sqrt{AB}$.

$$y=\sqrt{AB} \Rightarrow y=\sqrt{(x+9.25)(x-9.25)} \Rightarrow$$
$$y=\sqrt{x^2-9.25^2} \Rightarrow$$
$$y^2=x^2-9.25^2 \Rightarrow x^2-y^2=9.25^2$$

2. If the ball is placed on the line 10 feet to the right of the goal post, then $B = 10$ and $x=10+9.25=19.25$. Thus, we have then following.

$$x^2-y^2=9.25^2 \Rightarrow 19.25^2-y^2=9.25^2 \Rightarrow$$
$$y^2=19.25^2-9.25^2 \Rightarrow y=\sqrt{285}\approx 16.88$$

The ball should be placed about 16.88 feet from the goal.

3. The equations for the asymptotes of the hyperbola $x^2-y^2=9.25^2$ are $y=\pm x$. For the ball located 10 feet to the right of the goal post, $x = 19.25$ and we use the asymptote equation with positive slope, $y=x$, to obtain $y = 19.25$. Thus, the difference between the point on the asymptote and the point on the hyperbola is 19.25 – 16.88 = 2.37, or about 2.4 feet.

Chapter 11

FURTHER TOPICS IN ALGEBRA

Section 11.1: Sequences and Series

Connections (page 998)

1. After the first two terms, each of which is 1, the terms of the sequence are found by adding the two preceding terms. Thus, the third term is $1+1=2$, the fourth is $1+2=3$, and so on.

2. (Assume the original male honeybee is in generation 1.)
$a_1 = 1$
$a_2 = 1$
$a_n = a_{n-1} + a_{n-2}$, if $n \ge 3$

Exercises

1. $a_n = 4n + 10$
Replace n with 1, 2, 3, 4, and 5.
$n = 1: a_1 = 4(1) + 10 = 14$
$n = 2: a_2 = 4(2) + 10 = 18$
$n = 3: a_3 = 4(3) + 10 = 22$
$n = 4: a_4 = 4(4) + 10 = 26$
$n = 5: a_5 = 4(5) + 10 = 30$
The first five terms are 14, 18, 22, 26, and 30.

2. $a_n = 6n - 3$
Replace n with 1, 2, 3, 4, and 5.
$n = 1: a_1 = 6(1) - 3 = 3$
$n = 2: a_2 = 6(2) - 3 = 9$
$n = 3: a_3 = 6(3) - 3 = 15$
$n = 4: a_4 = 6(4) - 3 = 21$
$n = 5: a_5 = 6(5) - 3 = 27$
The first five terms are 3, 9, 15, 21, and 27.

3. $a_n = \frac{n+5}{n+4}$
Replace n with 1, 2, 3, 4, and 5.
$n = 1: a_1 = \frac{1+5}{1+4} = \frac{6}{5}$
$n = 2: a_2 = \frac{2+5}{2+4} = \frac{7}{6}$
$n = 3: a_3 = \frac{3+5}{3+4} = \frac{8}{7}$
$n = 4: a_4 = \frac{4+5}{4+4} = \frac{9}{8}$
$n = 5: a_5 = \frac{5+5}{5+4} = \frac{10}{9}$
The first five terms are $\frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8},$ and $\frac{10}{9}$.

4. $a_n = \frac{n-7}{n-6}$
Replace n with 1, 2, 3, 4, and 5.
$n = 1: a_1 = \frac{1-7}{1-6} = \frac{-6}{-5} = \frac{6}{5}$
$n = 2: a_2 = \frac{2-7}{2-6} = \frac{-5}{-4} = \frac{5}{4}$
$n = 3: a_3 = \frac{3-7}{3-6} = \frac{-4}{-3} = \frac{4}{3}$
$n = 4: a_4 = \frac{4-7}{4-6} = \frac{-3}{-2} = \frac{3}{2}$
$n = 5: a_5 = \frac{5-7}{5-6} = \frac{-2}{-1} = 2$
The first five terms are $\frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2},$ and 2.

5. $a_n = \left(\frac{1}{3}\right)^n (n-1)$
Replace n with 1, 2, 3, 4, and 5.
$n = 1: a_1 = \left(\frac{1}{3}\right)^1 (1-1) = \left(\frac{1}{3}\right)(0) = 0$
$n = 2: a_2 = \left(\frac{1}{3}\right)^2 (2-1) = \left(\frac{1}{9}\right)(1) = \frac{1}{9}$
$n = 3: a_3 = \left(\frac{1}{3}\right)^3 (3-1) = \left(\frac{1}{27}\right)(2) = \frac{2}{27}$
$n = 4: a_4 = \left(\frac{1}{3}\right)^4 (4-1) = \left(\frac{1}{81}\right)(3) = \frac{1}{27}$
$n = 5: a_5 = \left(\frac{1}{3}\right)^5 (5-1) = \left(\frac{1}{243}\right)(4) = \frac{4}{243}$
The first five terms are 0, $\frac{1}{9}, \frac{2}{27}, \frac{1}{27},$ and $\frac{4}{243}$.

6. $a_n = (-2)^n (n)$
Replace n with 1, 2, 3, 4, and 5.
$n = 1: a_1 = (-2)^1 (1) = -2(1) = -2$
$n = 2: a_2 = (-2)^2 (2) = 4(2) = 8$
$n = 3: a_3 = (-2)^3 (3) = -8(3) = -24$
$n = 4: a_4 = (-2)^4 (4) = 16(4) = 64$
$n = 5: a_5 = (-2)^5 (5) = -32(5) = -160$
The first five terms are $-2, 8, -24, 64,$ and -160.

7. $a_n = (-1)^n (2n)$

Replace n with 1, 2, 3, 4, and 5.

$n = 1$: $a_1 = (-1)^1 [2(1)] = -1(2) = -2$

$n = 2$: $a_2 = (-1)^2 [2(2)] = 1(4) = 4$

$n = 3$: $a_3 = (-1)^3 [2(3)] = -1(6) = -6$

$n = 4$: $a_4 = (-1)^4 [2(4)] = 1(8) = 8$

$n = 5$: $a_5 = (-1)^5 [2(5)] = -1(10) = -10$

The first five terms are -2, 4, -6, 8, and -10.

8. $a_n = (-1)^{n-1}(n+1)$

Replace n with 1, 2, 3, 4, and 5.

$n = 1$:
$a_1 = (-1)^{1-1}(1+1) = (-1)^0(2) = 1(2) = 2$

$n = 2$:
$a_2 = (-1)^{2-1}(2+1) = (-1)^1(3) = -1(3) = -3$

$n = 3$:
$a_3 = (-1)^{3-1}(3+1) = (-1)^2(4) = 1(4) = 4$

$n = 4$:
$a_4 = (-1)^{4-1}(4+1) = (-1)^3(5) = -1(5) = -5$

$n = 5$:
$a_5 = (-1)^{5-1}(5+1) = (-1)^4(6) = 1(6) = 6$

The first five terms are 2, -3, 4, -5, and 6.

9. $a_n = \dfrac{4n-1}{n^2+2}$

Replace n with 1, 2, 3, 4, and 5.

$n = 1$: $a_1 = \dfrac{4(1)-1}{(1)^2+2} = \dfrac{4-1}{1+2} = \dfrac{3}{3} = 1$

$n = 2$: $a_2 = \dfrac{4(2)-1}{(2)^2+2} = \dfrac{8-1}{4+2} = \dfrac{7}{6}$

$n = 3$: $a_3 = \dfrac{4(3)-1}{(3)^2+2} = \dfrac{12-1}{9+2} = \dfrac{11}{11} = 1$

$n = 4$: $a_4 = \dfrac{4(4)-1}{(4)^2+2} = \dfrac{16-1}{16+2} = \dfrac{15}{18} = \dfrac{5}{6}$

$n = 5$: $a_5 = \dfrac{4(5)-1}{(5)^2+2} = \dfrac{20-1}{25+2} = \dfrac{19}{27}$

The first five terms are 1, $\frac{7}{6}$, 1, $\frac{5}{6}$, and $\frac{19}{27}$.

10. $a_n = \dfrac{n^2-1}{n^2+1}$

Replace n with 1, 2, 3, 4, and 5.

$n = 1: a_1 = \dfrac{1^2-1}{1^2+1} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0$

$n = 2: a_2 = \dfrac{2^2-1}{2^2+1} = \dfrac{4-1}{4+1} = \dfrac{3}{5}$

$n = 3: a_3 = \dfrac{3^2-1}{3^2+1} = \dfrac{9-1}{9+1} = \dfrac{8}{10} = \dfrac{4}{5}$

$n = 4: a_4 = \dfrac{4^2-1}{4^2+1} = \dfrac{16-1}{16+1} = \dfrac{15}{17}$

$n = 5: a_5 = \dfrac{5^2-1}{5^2+1} = \dfrac{25-1}{25+1} = \dfrac{24}{26} = \dfrac{12}{13}$

The first five terms are 0, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{15}{17}$, and $\frac{12}{13}$.

11. $a_n = \dfrac{n^3+8}{n+2}$

Replace n with 1, 2, 3, 4, and 5.

$n = 1: a_1 = \dfrac{1^3+8}{1+2} = \dfrac{1+8}{3} = \dfrac{9}{3} = 3$

$n = 2: a_2 = \dfrac{2^3+8}{2+2} = \dfrac{8+8}{4} = \dfrac{16}{4} = 4$

$n = 3: a_3 = \dfrac{3^3+8}{3+2} = \dfrac{27+8}{5} = \dfrac{35}{5} = 7$

$n = 4: a_4 = \dfrac{4^3+8}{4+2} = \dfrac{64+8}{6} = \dfrac{72}{6} = 12$

$n = 5: a_5 = \dfrac{5^3+8}{5+2} = \dfrac{125+8}{7} = \dfrac{133}{7} = 19$

The first five terms are 3, 4, 7, 12, and 19.

12. $a_n = \dfrac{n^3+27}{n+3}$

Replace n with 1, 2, 3, 4, and 5.

$n = 1: a_1 = \dfrac{1^3+27}{1+3} = \dfrac{28}{4} = 7$

$n = 2: a_2 = \dfrac{2^3+27}{2+3} = \dfrac{8+27}{5} = \dfrac{35}{5} = 7$

$n = 3: a_3 = \dfrac{3^3+27}{3+3} = \dfrac{27+27}{6} = \dfrac{54}{6} = 9$

$n = 4: a_4 \dfrac{4^3+27}{4+3} = \dfrac{64+27}{7} = \dfrac{91}{7} = 13$

$n = 5: a_5 = \dfrac{5^3+27}{5+3} = \dfrac{125+27}{8} = \dfrac{152}{8} = 19$

The first five terms are 7, 7, 9, 13, and 19.

13. Answers will vary.

14. A sequence is a function with a domain consisting of natural numbers.

15. The sequence of the days of the week has as its domain {1, 2, 3, 4, 5, 6, 7}. Therefore, it is a finite sequence.

16. The sequence of dates in the month of July has as its domain {1, 2, 3, …, 31}. Therefore, it is a finite sequence.

17. The sequence 1, 2, 3, 4, 5 has as its domain {1, 2, 3, 4, 5}. Therefore, it is a finite sequence.

18. The sequence –1, –2, –3, –4, –5 has as its domain {–1, –2, –3, –4, –5}. Therefore, the sequence is finite.

19. The sequence 1, 2, 3, 4, 5, …has as its domain {1, 2, 3, 4, 5, …}. Therefore, the sequence is infinite.

20. The sequence –1, –2, –3, –4, –5, … has as its domain {–1, –2, –3, –4, –5, …}. Therefore, the sequence is infinite.

21. The sequence $a_1 = 4$ and for $2 \le n \le 10$, $a_n = 4 \cdot a_{n-1}$ has as its domain {2, 3, …, 10}. Therefore, the sequence is finite.

22. The sequence $a_1 = 2;\ a_2 = 5;$ and for $n \ge 3$, $a_n = a_{n-1} + a_{n-2}$ has as its domain {3, 4, …}. Therefore, the sequence is infinite.

23. $a_1 = -2,\ a_n = a_{n-1} + 3,$ for $n > 1$
This is an example of a recursive definition. We know $a_1 = -2$. Since $a_n = a_{n-1} + 3$,
$n = 2: a_2 = a_1 + 3 = -2 + 3 = 1$
$n = 3: a_3 = a_2 + 3 = 1 + 3 = 4$
$n = 4: a_4 = a_3 + 3 = 4 + 3 = 7.$
The first four terms are –2, 1, 4, and 7.

24. $a_1 = -1,\ a_n = a_{n-1} - 4$, for $n > 1$
This is an example of a recursive definition. We know $a_1 = -1$. Since $a_n = a_{n-1} - 4$,
$a_2 = a_1 - 4 = -1 - 4 = -5$
$a_3 = a_2 - 4 = -5 - 4 = -9$
$a_4 = a_3 - 4 = -9 - 4 = -13.$
The first four terms are –1, –5, –9, and –13.

25. $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2},$ for $n \ge 3$
$n = 3: a_3 = a_2 + a_1 = 1 + 1 = 2$
$n = 4: a_4 = a_3 + a_2 = 2 + 1 = 3$
The first four terms are 1, 1, 2, and 3.

26. $a_1 = 2,\ a_2 = 5,\ a_n = a_{n-1} + a_{n-2},$ for $n \ge 3$
$n = 3: a_3 = a_2 + a_1 = 5 + 2 = 7$
$n = 4: a_4 = a_3 + a_2 = 7 + 5 = 12$
The first four terms are 2, 5, 7, and 12.

27. $a_1 = 2,\ a_n = n \cdot a_{n-1},$ for $n > 1$
$a_2 = 2 \cdot a_1 = 2 \cdot 2 = 4$
$a_3 = 3 \cdot a_2 = 3 \cdot 4 = 12$
$a_4 = 4 \cdot a_3 = 4 \cdot 12 = 48$
The first four terms are 2, 4, 12, and 48.

28. $a_1 = -3,\ a_n = 2n \cdot a_{n-1},$ for $n > 1$
$a_2 = \left[2(2)\right] \cdot a_1 = 4 \cdot (-3) = -12$
$a_3 = \left[2(3)\right] \cdot a_2 = 6 \cdot (-12) = -72$
$a_4 = \left[2(4)\right] \cdot a_3 = 8 \cdot (-72) = -576$
The first four terms are –3, –12, –72, and –576.

29. $\sum_{i=1}^{5} (2i+1)$
$$\begin{aligned}&= \left[2(1)+1\right] + \left[2(2)+1\right] + \left[2(3)+1\right] \\ &\quad + \left[2(4)+1\right] + \left[2(5)+1\right] \\ &= (2+1) + (4+1) + (6+1) + (8+1) + (10+1) \\ &= 3 + 5 + 7 + 9 + 11 = 35\end{aligned}$$

30. $\sum_{i=1}^{6} (3i-2)$
$$\begin{aligned}&= \left[3(1)-2\right] + \left[3(2)-2\right] + \left[3(3)-2\right] \\ &\quad + \left[3(4)-2\right] + \left[3(5)-2\right] \\ &\quad + \left[3(6)-2\right] \\ &= (3-2) + (6-2) + (9-2) + (12-2) \\ &\quad + (15-2) + (18-2) \\ &= 1 + 4 + 7 + 10 + 13 + 16 = 51\end{aligned}$$

31. $$\sum_{j=1}^{4} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12}{12} + \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{25}{12}$$

32. $$\begin{aligned}\sum_{i=1}^{5} (i+1)^{-1} = \sum_{i=1}^{5} \frac{1}{i+1} &= \frac{1}{(1+1)} + \frac{1}{(2+1)} + \frac{1}{(3+1)} \\ &\quad + \frac{1}{(4+1)} + \frac{1}{(5+1)} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \\ &= \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} + \frac{10}{60} \\ &= \frac{87}{60} = \frac{29}{20}\end{aligned}$$

33. $\sum_{i=1}^{4} i^i = 1^1 + 2^2 + 3^3 + 4^4 = 1 + 4 + 27 + 256 = 288$

34. $\sum_{k=1}^{4} (k+1)^2$

$= (1+1)^2 + (2+1)^2 + (3+1)^2 + (4+1)^2$
$= 2^2 + 3^2 + 4^2 + 5^2$
$= 4 + 9 + 16 + 25 = 54$

35. $\sum_{k=1}^{6} (-1)^k \cdot k$

$= (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4 + (-1)^5 \cdot 5 + (-1)^6 \cdot 6$
$= -1 \cdot 1 + 1 \cdot 2 + (-1) \cdot 3 + 1 \cdot 4 + (-1) \cdot 5 + 1 \cdot 6$
$= -1 + 2 - 3 + 4 - 5 + 6 = 3$

36. $\sum_{i=1}^{7} (-1)^{i+1} \cdot i^2$

$= (-1)^{1+1} \cdot 1^2 + (-1)^{2+1} \cdot 2^2 + (-1)^{3+1} \cdot 3^2 + (-1)^{4+1} \cdot 4^2 + (-1)^{5+1} \cdot 5^2 + (-1)^{6+1} \cdot 6^2 + (-1)^{7+1} \cdot 7^2$
$= 1 + (-4) + 9 + (-16) + 25 + (-36) + 49 = 28$

37. $\sum_{i=2}^{5} (6 - 3i)$

$= [6 - 3(2)] + [6 - 3(3)] + [6 - 3(4)] + [6 - 3(5)]$
$= (6-6) + (6-9) + (6-12) + (6-15)$
$= 0 + (-3) + (-6) + (-9) = -18$

38. $\sum_{i=3}^{7} (5i + 2)$

$= [5(3)+2] + [5(4)+2] + [5(5)+2] + [5(6)+2] + [5(7)+2]$
$= (15+2) + (20+2) + (25+2) + (30+2) + (35+2)$
$= 17 + 22 + 27 + 32 + 37 = 135$

39. $\sum_{i=-2}^{3} 2(3)^i$

$= 2(3)^{-2} + 2(3)^{-1} + 2(3)^0 + 2(3)^1 + 2(3)^2 + 2(3)^3$
$= 2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2(1) + 2(3) + 2(9) + 2(27)$
$= \frac{2}{9} + \frac{2}{3} + 2 + 6 + 18 + 54$
$= \frac{2}{9} + \frac{6}{9} + 80 = 80\frac{8}{9} = \frac{728}{9}$

40. $\sum_{i=-1}^{2} 5(2)^i = 5(2)^{-1} + 5(2)^0 + 5(2)^1 + 5(2)^2$

$= 5\left(\frac{1}{2}\right) + 5(1) + 5(2) + 5(4)$
$= \frac{5}{2} + 5 + 10 + 20$
$= \frac{5}{2} + 35 = \frac{5}{2} + \frac{70}{2} = \frac{75}{2} = 37.5$

41. $\sum_{i=-1}^{5} (i^2 - 2i)$

$= [(-1)^2 - 2(-1)] + [0^2 - 2(0)] + [1^2 - 2(1)] + [2^2 - 2(2)] + [3^2 - 2(3)] + [4^2 - 2(4)] + [5^2 - 2(5)]$
$= [1 - (-2)] + (0 - 0) + (1 - 2) + (4 - 4) + (9 - 6) + (16 - 8) + (25 - 10)$
$= 3 + 0 + (-1) + 0 + 3 + 8 + 15 = 28$

42. $\sum_{i=3}^{6} (2i^2 + 1)$

$= [2(3)^2 + 1] + [2(4)^2 + 1] + [2(5)^2 + 1] + [2(6)^2 + 1]$
$= (2 \cdot 9 + 1) + (2 \cdot 16 + 1) + (2 \cdot 25 + 1) + (2 \cdot 36 + 1)$
$= (18+1) + (32+1) + (50+1) + (72+1)$
$= 19 + 33 + 51 + 73 = 176$

43. $\sum_{i=1}^{5} (3^i - 4) = (3^1 - 4) + (3^2 - 4) + (3^3 - 4) + (3^4 - 4) + (3^5 - 4)$

$= (3-4) + (9-4) + (27-4) + (81-4) + (243-4)$
$= (-1) + 5 + 23 + 77 + 239 = 343$

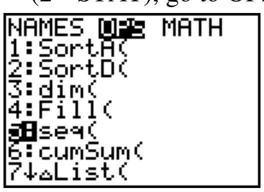
44. $\sum_{i=1}^{4}\left[(-2)^i-3\right]$
$=\left[(-2)^1-3\right]+\left[(-2)^2-3\right]+\left[(-2)^3-3\right]+\left[(-2)^4-3\right]$
$=(-2-3)+(4-3)+(-8-3)+(16-3)$
$=-5+1+(-11)+13=-2$

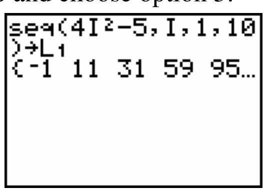
45. $\sum_{i=1}^{3}\left(i^3-i\right)=\left(1^3-1\right)+\left(2^3-2\right)+\left(3^3-3\right)$
$=(1-1)+(8-2)+(27-3)$
$=0+6+24=30$

46. $\sum_{i=1}^{4}\left(i^4-i^3\right)=\left(1^4-1^3\right)+\left(2^4-2^3\right)+\left(3^4-3^3\right)+\left(4^4-4^3\right)$
$=(1-1)+(16-8)+(81-27)+(256-64)$
$=0+8+54+192=254$

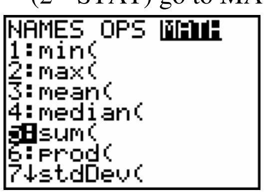
47. $\sum_{i=1}^{10}(4i^2-5)=1490$

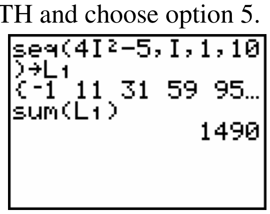
To find the seq(feature, go to the LIST menu (2nd STAT), go to OPS and choose option 5.



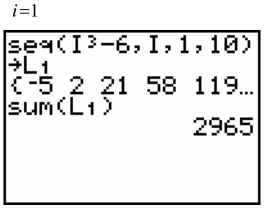


To find the sum(feature, go to the LIST menu (2nd STAT) go to MATH and choose option 5.

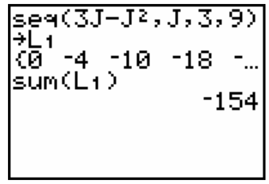




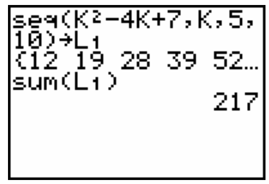
48. $\sum_{i=1}^{10}(i^3-6)=2965$



49. $\sum_{j=3}^{9}(3j-j^2)=-154$



50. $\sum_{k=5}^{10}(k^2-4k+7)=217$



In Exercises 51–59, $x_1=-2,\ x_2=-1,\ x_3=0,\ x_4=1,$ and $x_5=2.$

51. $\sum_{i=1}^{5}x_i=x_1+x_2+x_3+x_4+x_5$
$=-2+(-1)+0+1+2=0$

52. $\sum_{i=1}^{5}-x_i=-x_1+(-x_2)+(-x_3)+(-x_4)+(-x_5)$
$=-(-2)+\left[-(-1)\right]+(-0)+(-1)+(-2)$
$=2+1+0+(-1)+(-2)=0$

53. $\sum_{i=1}^{5}(2x_i+3)$
$=(2x_1+3)+(2x_2+3)+(2x_3+3)+(2x_4+3)+(2x_5+3)$
$=\left[2(-2)+3\right]+\left[2(-1)+3\right]+\left[2(0)+3\right]+\left[2(1)+3\right]+\left[2(2)+3\right]$
$=(-4+3)+(-2+3)+(0+3)+(2+3)+(4+3)$
$=-1+1+3+5+7=15$

54. $\sum_{i=1}^{4}x_i^2=x_1^2+x_2^2+x_3^2+x_4^2$
$=(-2)^2+(-1)^2+0^2+1^1$
$=4+1+0+1=6$

55. $\sum_{i=1}^{3}\left(3x_i - x_i^2\right)$
$= \left(3x_1 - x_1^2\right) + \left(3x_2 - x_2^2\right) + \left(3x_3 - x_3^2\right)$
$= \left[3(-2) - (-2)^2\right] + \left[3(-1) - (-1)^2\right]$
$+ \left[3(0) - 0^2\right]$
$= (-6-4) + (-3-1) + (0-0)$
$= -10 + (-4) + 0 = -14$

56. $\sum_{i=1}^{3}\left(x_i^2 + 1\right) = \left(x_1^2+1\right) + \left(x_2^2+1\right) + \left(x_3^2+1\right)$
$= \left[(-2)^2 + 1\right] + \left[(-1)^2 + 1\right] + \left(0^2 + 1\right)$
$= (4+1) + (1+1) + (0+1)$
$= 5 + 2 + 1 = 8$

57. $\sum_{i=2}^{5}\frac{x_i+1}{x_i+2} = \frac{x_2+1}{x_2+2} + \frac{x_3+1}{x_3+2} + \frac{x_4+1}{x_4+2} + \frac{x_5+1}{x_5+2}$
$= \frac{-1+1}{-1+2} + \frac{0+1}{0+2} + \frac{1+1}{1+2} + \frac{2+1}{2+2}$
$= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$
$= 0 + \frac{6}{12} + \frac{8}{12} + \frac{9}{12} = \frac{23}{12}$

58. $\sum_{i=1}^{5}\frac{x_i}{x_i+3}$
$= \frac{x_1}{x_1+3} + \frac{x_2}{x_2+3} + \frac{x_3}{x_3+3} + \frac{x_4}{x_4+3} + \frac{x_5}{x_5+3}$
$= \frac{-2}{-2+3} + \frac{-1}{-1+3} + \frac{0}{0+3} + \frac{1}{1+3} + \frac{2}{2+3}$
$= \frac{-2}{1} + \frac{-1}{2} + \frac{0}{3} + \frac{1}{4} + \frac{2}{5} = -2 + \frac{-1}{2} + 0 + \frac{1}{4} + \frac{2}{5}$
$= \frac{-40}{20} + \frac{-10}{20} + \frac{5}{20} + \frac{8}{20} = -\frac{37}{20}$

59. $\sum_{i=1}^{4}\frac{x_i^3+1000}{x_i+10}$
$= \frac{x_1^3+1000}{x_1+10} + \frac{x_2^3+1000}{x_2+10} + \frac{x_3^3+1000}{x_3+10}$
$+ \frac{x_4^3+1000}{x_4+10}$
$= \frac{(-2)^3+1000}{(-2)+10} + \frac{(-1)^3+1000}{(-1)+10} + \frac{0^3+1000}{0+10}$
$+ \frac{1^3+1000}{1+10}$
$= \frac{-8+1000}{8} + \frac{-1+1000}{9} + \frac{1000}{10} + \frac{1+1000}{11}$
$= \frac{992}{8} + \frac{999}{9} + \frac{1000}{10} + \frac{1001}{11}$
$= 124 + 111 + 100 + 91 = 426$

In Exercises 61–66, $x_1 = 0,\ x_2 = 2,\ x_3 = 4,\ x_4 = 6,$ and $\Delta x = .5$.

61. $f(x) = 4x - 7 \Rightarrow f(x_i) = 4x_i - 7$
$\sum_{i=1}^{4} f(x_i)\Delta x$
$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$
$= (4x_1 - 7)\Delta x + (4x_2 - 7)\Delta x + (4x_3 - 7)\Delta x$
$+ (4x_4 - 7)\Delta x$
$= \left[4(0) - 7\right](.5) + \left[4(2) - 7\right](.5)$
$+ \left[4(4) - 7\right](.5) + \left[4(6) - 7\right](.5)$
$= (0-7)(.5) + (8-7)(.5)$
$+ (16-7)(.5) + (24-7)(.5)$
$= -7(.5) + 1(.5) + 9(.5) + 17(.5)$
$= -3.5 + .5 + 4.5 + 8.5 = 10$

62. $f(x) = 6 + 2x \Rightarrow f(x_i) = 6 + 2x_i$
$\sum_{i=1}^{4} f(x_i)\Delta x$
$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$
$= (6 + 2x_1)\Delta x + (6 + 2x_2)\Delta x + (6 + 2x_3)\Delta x$
$+ (6 + 2x_4)\Delta x$
$= \left[6 + 2(0)\right](.5) + \left[6 + 2(2)\right](.5)$
$+ \left[6 + 2(4)\right](.5) + \left[6 + 2(6)\right](.5)$
$= (6+0)(.5) + (6+4)(.5)$
$+ (6+8)(.5) + (6+12)(.5)$
$= 6(.5) + 10(.5) + 14(.5) + 18(.5)$
$= 3 + 5 + 7 + 9 = 24$

63. $f(x) = 2x^2 \Rightarrow f(x_i) = 2x_i^2$
$\sum_{i=1}^{4} f(x_i)\Delta x$
$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$
$= 2\left(x_1^2\right)\Delta x + 2\left(x_2^2\right)\Delta x + 2\left(x_3^2\right)\Delta x + 2\left(x_4^2\right)\Delta x$
$= 2\left(0^2\right)(.5) + 2\left(2^2\right)(.5) + 2\left(4^2\right)(.5) + 2\left(6^2\right)(.5)$
$= 2(0)(.5) + 2(4)(.5) + 2(16)(.5) + 2(36)(.5)$
$= 0 + 4 + 16 + 36 = 56$

64. $f(x)=x^2-1 \Rightarrow f(x_i)=x_i^2-1$

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x\\ &= (x_1^2-1)\Delta x + (x_2^2-1)\Delta x + (x_3^2-1)\Delta x + (x_4^2-1)\Delta x\\ &= (0^2-1)(.5)+(2^2-1)(.5)+(4^2-1)(.5)+(6^2-1)(.5)\\ &= (0-1)(.5)+(4-1)(.5)+(16-1)(.5)+(36-1)(.5)\\ &= -1(.5)+3(.5)+15(.5)+35(.5)\\ &= -.5+1.5+7.5+17.5=26\end{aligned}$$

65. $f(x)=\dfrac{-2}{x+1} \Rightarrow f(x_i)=\dfrac{-2}{x_i+1}$

$$\begin{aligned}&\sum_{i=1}^{4} f(x_i)\Delta x\\ &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x\\ &= \left(\frac{-2}{x_1+1}\right)(.5)+\left(\frac{-2}{x_2+1}\right)(.5)+\left(\frac{-2}{x_3+1}\right)(.5)+\left(\frac{-2}{x_4+1}\right)(.5)\\ &= \left(\frac{-2}{0+1}\right)(.5)+\left(\frac{-2}{2+1}\right)(.5)+\left(\frac{-2}{4+1}\right)(.5)+\left(\frac{-2}{6+1}\right)(.5)\\ &= \left(\frac{-2}{1}\right)(.5)+\left(-\frac{2}{3}\right)(.5)+\left(-\frac{2}{5}\right)(.5)+\left(-\frac{2}{7}\right)(.5)\\ &= -1+\left(-\frac{1}{3}\right)+\left(-\frac{1}{5}\right)+\left(-\frac{1}{7}\right)\\ &= \frac{-105}{105}+\left(\frac{-35}{105}\right)+\left(\frac{-21}{105}\right)+\left(\frac{-15}{105}\right)=-\frac{176}{105}\end{aligned}$$

66. $f(x)=\dfrac{5}{2x-1} \Rightarrow f(x_i)=\dfrac{5}{2x_i-1}$

$$\begin{aligned}&\sum_{i=1}^{4} f(x_i)\Delta x\\ &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x\\ &= \left(\frac{5}{2x_1-1}\right)\Delta x+\left(\frac{5}{2x_2-1}\right)\Delta x+\left(\frac{5}{2x_3-1}\right)\Delta x+\left(\frac{5}{2x_4-1}\right)\Delta x\\ &= \left(\frac{5}{2\cdot0-1}\right)(.5)+\left(\frac{5}{2\cdot2-1}\right)(.5)+\left(\frac{5}{2\cdot4-1}\right)(.5)+\left(\frac{5}{2\cdot6-1}\right)(.5)\\ &= \frac{5}{0-1}(.5)+\frac{5}{4-1}(.5)+\frac{5}{8-1}(.5)+\frac{5}{12-1}(.5)\\ &= -5(.5)+\frac{5}{3}(.5)+\frac{5}{7}(.5)+\frac{5}{11}(.5)\\ &= -\frac{5}{2}+\frac{5}{6}+\frac{5}{14}+\frac{5}{22}\\ &= \frac{-1155}{462}+\frac{385}{462}+\frac{165}{462}+\frac{105}{462}=-\frac{500}{462}=-\frac{250}{231}\end{aligned}$$

67. $\displaystyle\sum_{i=1}^{100} 6 = 100(6)=600$

68. $\displaystyle\sum_{i=1}^{20} 5 = 20(5)=100$

69. $$\begin{aligned}\sum_{i=1}^{15} i^2 &= \frac{15(15+1)[2(15)+1]}{6}\\ &= \frac{15(16)(30+1)}{6}=\frac{15(16)(31)}{6}\\ &= \frac{7440}{6}=1240\end{aligned}$$

70. $$\begin{aligned}\sum_{i=1}^{50} 2i^3 &= 2\sum_{i=1}^{50} i^3 = 2\cdot\frac{50^2(50+1)^2}{4}\\ &= 2\cdot\frac{2500(51)^2}{4}=\frac{2500(2601)}{2}\\ &= 1250(2601)=3,251,250\end{aligned}$$

71. $$\begin{aligned}\sum_{i=1}^{5}(5i+3) &= \sum_{i=1}^{5}5i+\sum_{i=1}^{5}3=5\sum_{i=1}^{5}i+5(3)\\ &= 5\cdot\frac{5(5+1)}{2}+15=5\cdot\frac{5(6)}{2}+15\\ &= 5\cdot\frac{30}{2}+15=5(15)+15\\ &= 75+15=90\end{aligned}$$

72. $$\begin{aligned}\sum_{i=1}^{5}(8i-1) &= \sum_{i=1}^{5}8i-\sum_{i=1}^{5}1=8\sum_{i=1}^{5}i-5(1)\\ &= 8\cdot\frac{5(5+1)}{2}-5=8\cdot\frac{5(6)}{2}-5\\ &= 8\cdot\frac{30}{2}-5=8(15)-5\\ &= 120-5=115\end{aligned}$$

73. $\sum_{i=1}^{5}\left(4i^2-2i+6\right)$

$= \sum_{i=1}^{5}4i^2-\sum_{i=1}^{5}2i+\sum_{i=1}^{5}6$

$= 4\sum_{i=1}^{5}i^2-2\sum_{i=1}^{5}i+5(6)$

$= 4\cdot\frac{5(5+1)(10+1)}{6}-2\cdot\frac{5(5+1)}{2}+30$

$= 4\cdot\frac{5(6)(11)}{6}-2\cdot\frac{5(6)}{2}+30$

$= 4\cdot\left[5(11)\right]-2\cdot\left[5(3)\right]+30$

$= 220-30+30=220$

74. $\sum_{i=1}^{6}\left(2+i-i^2\right)=\sum_{i=1}^{6}2+\sum_{i=1}^{6}i-\sum_{i=1}^{6}i^2$

$= 6(2)+\frac{6(6+1)}{2}$

$-\frac{6(6+1)(2\cdot 6+1)}{6}$

$= 12+\frac{6(7)}{2}-\frac{6(7)(12+1)}{6}$

$= 12+3(7)-(7)(13)$

$= 12+21-91=-58$

75. $\sum_{i=1}^{4}\left(3i^3+2i-4\right)$

$= \sum_{i=1}^{4}3i^3+\sum_{i=1}^{4}2i-\sum_{i=1}^{4}4$

$= 3\sum_{i=1}^{4}i^3+2\sum_{i=1}^{4}i-4(4)$

$= 3\cdot\frac{4^2(4+1)^2}{4}+2\cdot\frac{4(4+1)}{2}-16$

$= 3\cdot\frac{16(5)^2}{4}+2\cdot\frac{4(5)}{2}-16$

$= 3\cdot\left[4(25)\right]+2\cdot\left[2(5)\right]-16$

$= 300+20-16=304$

76. $\sum_{i=1}^{6}\left(i^2+2i^3\right)=\sum_{i=1}^{6}i^2+\sum_{i=1}^{6}2i^3=\sum_{i=1}^{6}i^2+2\sum_{i=1}^{6}i^3$

$= \frac{6(6+1)\left[2(6)+1\right]}{6}+2\cdot\frac{6^2(6+1)^2}{4}$

$= \frac{6(7)(12+1)}{6}+2\cdot\frac{36(7)^2}{4}$

$= 7(13)+2\left[9(49)\right]=91+882=973$

In Exercises 77–80, there are other acceptable forms of the answers.

77. $\frac{1}{3(1)}+\frac{1}{3(2)}+\frac{1}{3(3)}+\ldots+\frac{1}{3(9)}=\sum_{i=1}^{9}\frac{1}{3i}$

78. $\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+\ldots+\frac{5}{1+15}=\sum_{i=1}^{15}\frac{5}{1+i}$

79. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots-\frac{1}{128}$

$= \frac{1}{2^0}-\frac{1}{2^1}+\frac{1}{2^2}-\frac{1}{2^3}+\ldots-\frac{1}{2^7}$

$= \left(\frac{1}{2}\right)^0-\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^3+\ldots-\left(\frac{1}{2}\right)^7$

$= \left(\frac{1}{2}\right)^{1-1}-\left(\frac{1}{2}\right)^{2-1}+\left(\frac{1}{2}\right)^{3-1}-\left(\frac{1}{2}\right)^{4-1}$

$+\ldots-\left(\frac{1}{2}\right)^{8-1}$

$= \left(-\frac{1}{2}\right)^{1-1}+\left(-\frac{1}{2}\right)^{2-1}+\left(-\frac{1}{2}\right)^{3-1}+\left(-\frac{1}{2}\right)^{4-1}$

$+\ldots+\left(-\frac{1}{2}\right)^{8-1}$

$= \sum_{k=1}^{8}\left(-\frac{1}{2}\right)^{k-1}$

80. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\ldots-\frac{1}{400}$

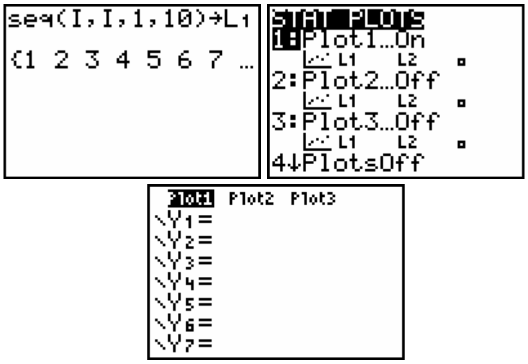
$= \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots-\frac{1}{20^2}$

$= (-1)^{1-1}\frac{1}{1^2}+(-1)^{2-1}\frac{1}{2^2}+(-1)^{3-1}\frac{1}{3^2}$

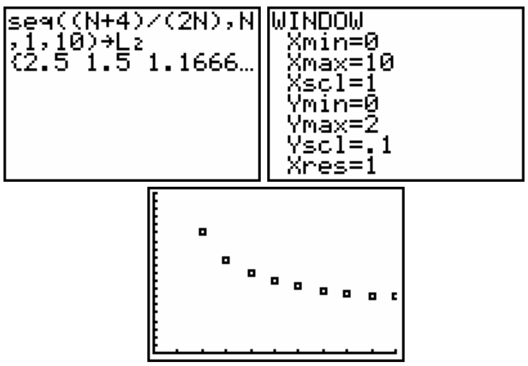
$+(-1)^{4-1}\frac{1}{4^2}+\ldots+(-1)^{20-1}\frac{1}{20^2}$

$= \sum_{j=1}^{20}(-1)^{j-1}\frac{1}{j^2}$

For Exercises 81–86, enter the set $\{1, 2, 3, ..., 10\}$ into L_1. Turn the STAT PLOTS on and clear any functions you have in the $Y =$ screen.

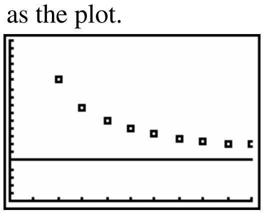


81. $a_n = \dfrac{n+4}{2n}$

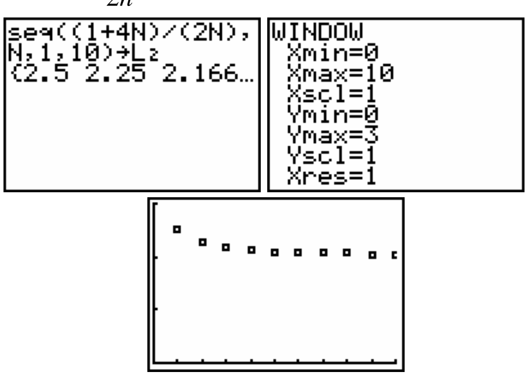


It appears that the sequence converges to $\frac{1}{2}$.

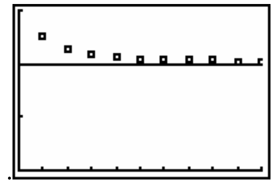
The line $y = \frac{1}{2}$ is graphed in the same screen as the plot.



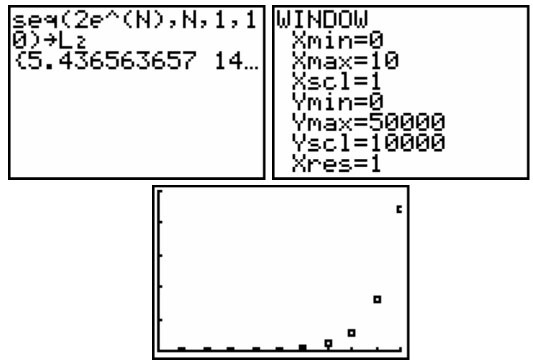
82. $a_n = \dfrac{1+4n}{2n}$



It appears that the sequence converges to 2. The line $y = 2$ is graphed in the same screen as the plot

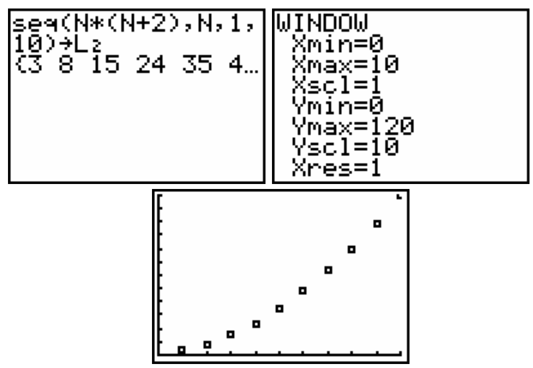


83. $a_n = 2e^n$



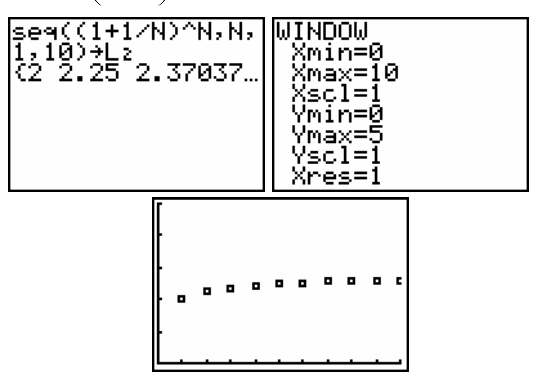
It appears that the sequence diverges.

84. $a_n = n(n+2)$

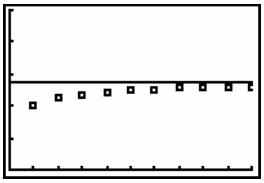


It appears that the sequence diverges.

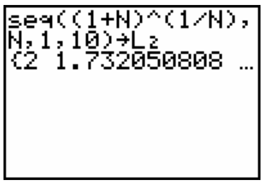
85. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

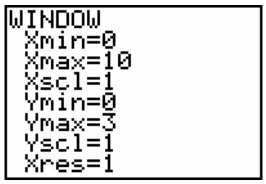


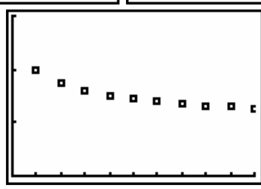
It appears that the sequence converges to $e \approx 2.71828$ The line $y = e$ is graphed in the same screen as the plot.



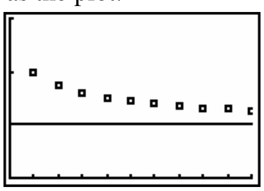
86. $a_n = (1+n)^{1/n}$







It appears that the sequence converges to 1. The line $y = 1$ is graphed in the same screen as the plot.



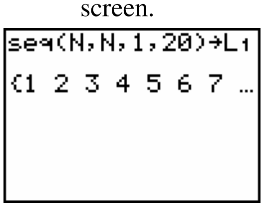
87. **(a)** $a_1 = 8; a_n = 2.9a_{n-1} - .2a_{n-1}^{\ 2}$, for $n > 1$

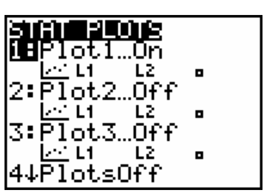
$a_1 = 8$ thousand per acre

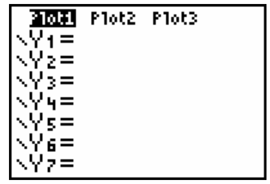
$a_2 = 2.9a_1 - .2a_1^{\ 2} = 2.9(8) - .2(8)^2$
$= 23.2 - 12.8 = 10.4$ thousand per acre

$a_3 = 2.9a_2 - .2a_2^{\ 2} = 2.9(10.4) - .2(10.4)^2$
$= 30.16 - 21.632 = 8.528$ thousand per acre

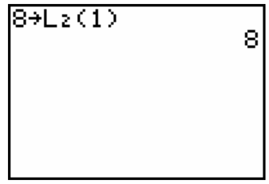
(b) One way to enter the required data into a list is to first enter the set $\{1, 2, 3, ..., 20\}$ into L_1. Turn the STAT PLOTS on and clear any functions you have in the Y = screen.

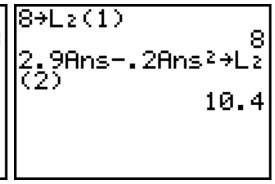




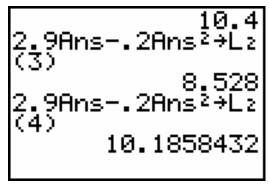


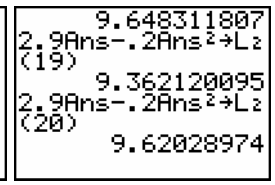
Next, to enter the data from the recursive sequence, we can enter the values in individually as follows.



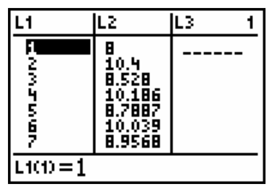


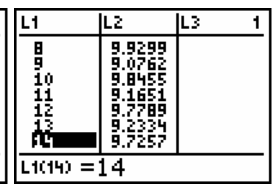
Hit enter, then edit the element in the list until you have completed all 20 elements.

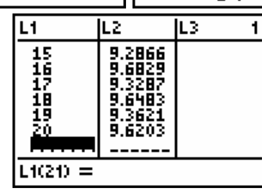




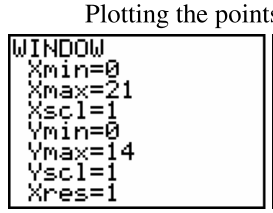
Your lists should look like the following screen shots.

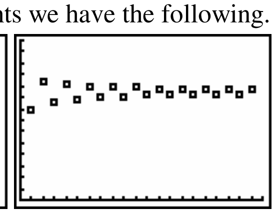




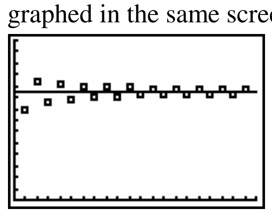


Plotting the points we have the following.

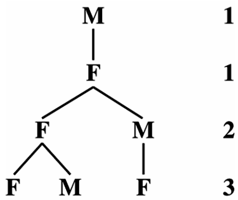




The population density oscillates above and below 9.5 thousand per acre (approximately). The line $y = 9.5$ is graphed in the same screen as the plot.



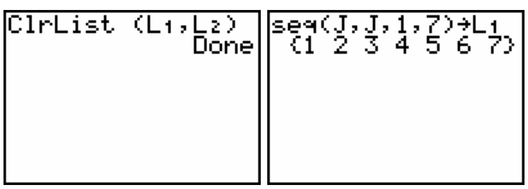
88. Ancestors of a male bee: M is male, F is female



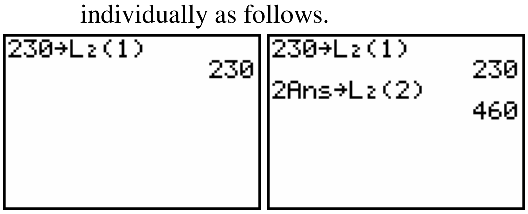
89. **(a)** Since the number of bacteria doubles every 40 minutes, it follows that $N_{j+1} = 2N_j$ for $j \geq 1$.

(b) Two hours is 120 minutes. If $120 = 40(j-1)$, then $3 = j-1 \Rightarrow j = 4$. Since $N_1 = 230$, $N_2 = 460$, $N_3 = 920$, and $N_4 = 1840$, there will be 1840 bacteria after two hours.

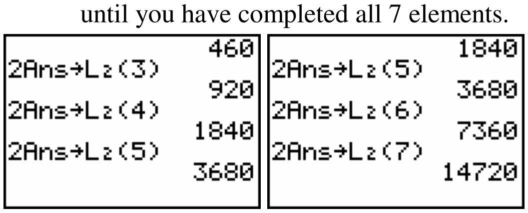
(c) We must graph the sequence $N_{j+1} = 2N_j$ for $j = 1, 2, 3, \ldots, 7$ if $N_1 = 230$. You should clear lists that you may have used in previous exercises and enter the set $\{1, 2, 3, \ldots, 7\}$ into L_1.



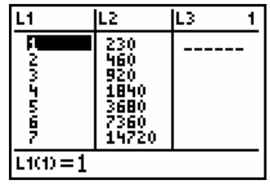
Next, to enter the data from the recursive sequence, we can enter the values in individually as follows.



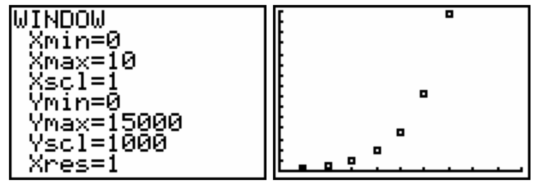
Hit enter, then edit the element in the list until you have completed all 7 elements.



Your lists should look like the following screen shots.



Plotting the points, we have the following.



(d) The growth is very rapid. Since there is a doubling of the bacteria at equal intervals, their growth is exponential.

90. **(a)** Since $N_{j+1} = \left[\dfrac{2}{1+\frac{N_j}{K}}\right]N_j$, we have

$$N_2 = \left[\dfrac{2}{1+\frac{N_1}{K}}\right]N_1 = \left[\dfrac{2}{1+\frac{230}{5000}}\right]230 \approx 439.77.$$

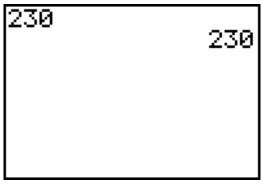
Similarly,

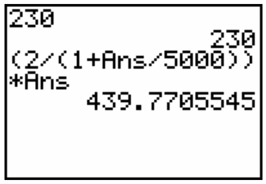
$$N_3 = \left[\dfrac{2}{1+\frac{N_2}{K}}\right]N_2 = \left[\dfrac{2}{1+\frac{439.77}{5000}}\right]439.77 \approx 808.45$$

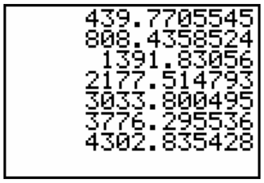
With this process, a table can be generated.

j	N_j	j	N_j
1	230	11	4901
2	440	12	4950
3	808	13	4975
4	1392	14	4987
5	2178	15	4994
6	3034	16	4997
7	3776	17	4998
8	4303	18	4999
9	4625	19	5000
10	4805	20	5000

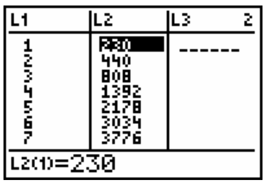
Using the calculator, you could also enter the initial value (first screen) then type in the recursive sequence (second screen) and hit enter. If you keep hitting enter, you will get the desired entries to be rounded (You could also change the mode of the calculator to give rounded answers.).

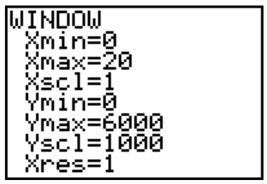


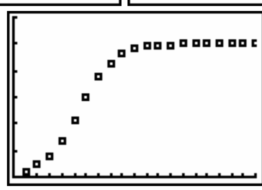




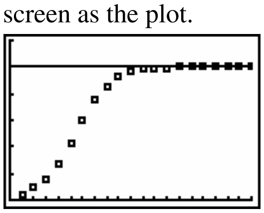
(b) We must graph the sequence given by the table in the first part of this exercise.







(c) The growth is rapid at first but slows and seems to approach 5000 asymptotically. The line $y = 5000$ is graphed in the same screen as the plot.

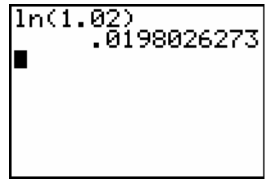


(d) Since the final number of bacteria is 5000, which is equal to K in this exercise, it is reasonable to conjecture that the carrying capacity of the medium is K. That is, K gives the maximum number of bacteria that can be supported on the given medium. When $N_j = K$, the medium becomes saturated. Changing K in the formula would demonstrate that this conjecture is correct.

91. **(a)** $\ln 1.02 = \ln(1+.02)$

$$\approx .02 - \frac{.02^2}{2} + \frac{.02^3}{3} - \frac{.02^4}{4} + \frac{.02^5}{5} - \frac{.02^6}{6}$$

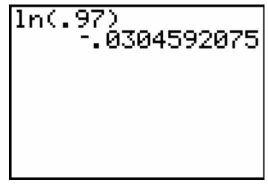
$$\approx .0198026273$$



(b) $\ln 1.97 = \ln(1-.03)$

$$\approx (-.03) - \frac{(-.03)^2}{2} + \frac{(-.03)^3}{3} - \frac{(-.03)^4}{4} + \frac{(-.03)^5}{5} - \frac{(-.03)^6}{6}$$

$$\approx -.0304592075$$

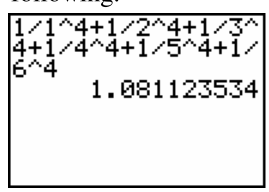


92. The first 6 terms of

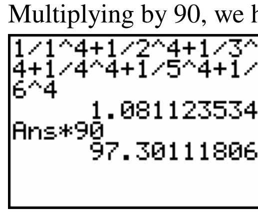
$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots.. + \frac{1}{n^4}$ are

$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4}.$

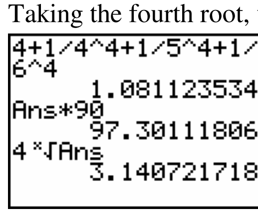
Entering this into the calculator, we have the following.



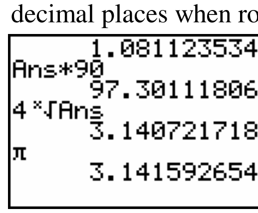
Multiplying by 90, we have the following.



Taking the fourth root, we have the following.



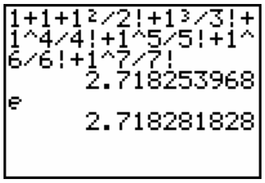
We get $\pi \approx 3.1407$. This is accurate to three decimal places when rounded.



93. $e^a \approx 1 + a + \dfrac{a^2}{2!} + \dfrac{a^3}{3!} + ... + \dfrac{a^n}{n!}$ where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$

(a) Calculate

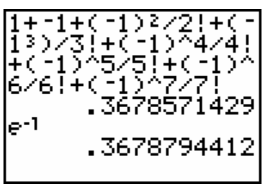
$$e^1 \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \frac{1^6}{6!} + \frac{1^7}{7!} + \frac{1^8}{8!}$$



The approximation is accurate to four decimal places.

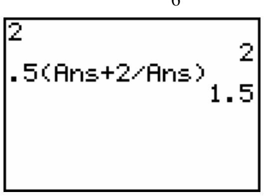
(b) Calculate

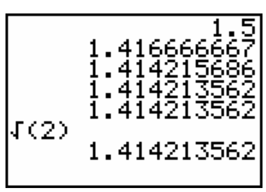
$$e^{-1} \approx 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!} + \frac{(-1)^6}{6!} + \frac{(-1)^7}{7!} + \frac{(-1)^8}{8!}$$



The approximation is accurate to four decimal places.

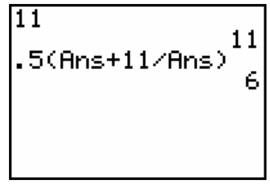
94. **(a)** To find the approximation for $\sqrt{2}$, enter 2 into the calculator and use the previous answer feature of the calculator to generate the recursive sequence. After you obtain 1.5, you have a_2. You will need to press enter four more times to obtain a_6.

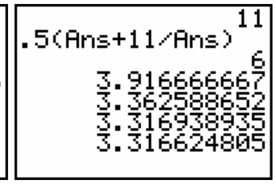




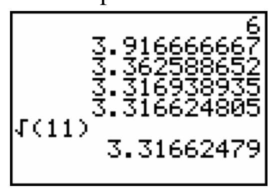
The approximation is accurate up to the accuracy of the calculator (9 decimal places).

(b) To find the approximation for $\sqrt{11}$, enter 11 into the calculator and use the previous answer feature of the calculator to generate the recursive sequence. After you obtain 6, you have a_2. You will need to press enter four more times to obtain a_6.





If we round $\sqrt{11}$ to 7 decimal places, then the calculation is accurate to 7 decimal places.



Section 11.2: Arithmetic Sequences and Series

1. 2, 5, 8, 11, …
$d = a_2 - a_1 = 5 - 2 = 3$

2. 4, 10, 16, 22, …
$d = a_2 - a_1 = 10 - 4 = 6$

3. 3, –2, –7, –12, …
$d = a_2 - a_1 = -2 - 3 = -5$

4. –8, –12, –16, –20, …
$d = a_2 - a_1 = -12 - (-8) = -4$

5. $x + 3y$, $2x + 5y$, $3x + 7y$, …
$d = a_2 - a_1 = (2x + 5y) - (x + 3y)$
$= 2x + 5y - x - 3y = x + 2y$

6. $t^2 + q, \; -4t^2 + 2q, \; -9t^2 + 3q, \ldots$
$d = a_2 - a_1 = (-4t^2 + 2q) - (t^2 + q)$
$= -4t^2 + 2q - t^2 - q = -5t^2 + q$

7. $a_1 = 8$ and $d = 6$
Starting with $a_1 = 8$, add $d = 6$ to each term to get the next term.
$a_2 = 8 + 6 = 14$; $a_3 = 14 + 6 = 20$
$a_4 = 20 + 6 = 26$; $a_5 = 26 + 6 = 32$
The first five terms are 8, 14, 20, 26, and 32.

8. $a_1 = -2$ and $d = 12$

Starting with $a_1 = -2$, add d = 12 to each term to get the next term.

$a_2 = -2 + 12 = 10$
$a_3 = 10 + 12 = 22$
$a_4 = 22 + 12 = 34$
$a_5 = 34 + 12 = 46$

The first five terms are –2, 10, 22, 34, and 46.

9. $a_1 = 5$, $d = -2$

$a_2 = 5 + (-2) = 3$; $a_3 = 3 + (-2) = 1$
$a_4 = 1 + (-2) = -1$
$a_5 = -1 + (-2) = -3$

The first five terms are 5, 3, 1, –1, and –3.

10. $a_1 = 4$, $d = 3$

The first term of the arithmetic sequence is 4, and each succeeding term is found by adding 3 to the preceding term.

$a_2 = 4 + d = 4 + 3 = 7$
$a_3 = 7 + d = 7 + 3 = 10$
$a_4 = 10 + d = 10 + 3 = 13$
$a_5 = 13 + d = 13 + 3 = 16$

The first five terms are 4, 7, 10, 13, and 16.

11. $a_3 = 10$, $d = -2$

$a_4 = 10 + d = 10 + (-2) = 8$
$a_5 = 8 + d = 8 + (-2) = 6$

Subtract the common difference –2 to find the earlier terms.

$a_2 = a_3 - d = 10 - (-2) = 12$
$a_1 = a_2 - d = 12 - (-2) = 14$

The first five terms are 14, 12, 10, 8, and 6.

12. $a_1 = 3 - \sqrt{2}$, $a_2 = 3$

$d = a_2 - a_1 = 3 - (3 - \sqrt{2}) = 3 - 3 + \sqrt{2} = \sqrt{2}$
$a_3 = 3 + \sqrt{2}$
$a_4 = (3 + \sqrt{2}) + \sqrt{2} = 3 + 2\sqrt{2}$
$a_5 = (3 + 2\sqrt{2}) + \sqrt{2} = 3 + 3\sqrt{2}$

The first five terms are $3 - \sqrt{2}$, 3, $3 + \sqrt{2}$, $3 + 2\sqrt{2}$, and $3 + 3\sqrt{2}$.

13. 5, 7, 9, …

$d = a_2 - a_1 = 7 - 5 = 2$

Since $a_n = a_1 + (n-1)d$, we have

$a_8 = 5 + (8-1)2 = 5 + 7 \cdot 2 = 5 + 14 = 19$. Also, we have

$a_n = 5 + (n-1)2 = 5 + 2n - 2 = 3 + 2n.$

14. $-3, -7, -11, \ldots$

$d = a_2 - a_1 = -7 - (-3) = -4$

Since $a_n = a_1 + (n-1)d$, we have

$a_8 = -3 + (8-1)(-4) = -3 + 7 \cdot (-4)$
$= -3 + (-28) = -31$

Also, we have

$a_n = -3 + (n-1)(-4) = -3 - 4n + 4 = 1 - 4n.$

15. $a_1 = 5$, $a_4 = 15$

Since $a_n = a_1 + (n-1)d$, we have

$a_4 = 5 + (4-1)d = 15 \Rightarrow 5 + 3d = 15 \Rightarrow$
$3d = 10 \Rightarrow d = \frac{10}{3}$

We now have

$a_8 = 5 + (8-1)\left(\frac{10}{3}\right) = 5 + 7\left(\frac{10}{3}\right) = \frac{15}{3} + \frac{70}{3} = \frac{85}{3}.$

Also, we have

$a_n = 5 + (n-1)\left(\frac{10}{3}\right)$
$= 5 + \frac{10}{3}n - \frac{10}{3} = \frac{15}{3} + \frac{10}{3}n - \frac{10}{3} = \frac{5}{3} + \frac{10}{3}n$

16. $a_1 = -4$, $a_5 = 16$

Since $a_n = a_1 + (n-1)d$, we have

$a_5 = -4 + (5-1)d = 16 \Rightarrow -4 + 4d = 16 \Rightarrow$
$4d = 20 \Rightarrow d = 5$

We now have

$a_8 = -4 + (8-1)5 = -4 + 7 \cdot 5 = -4 + 35 = 31.$

Also, we have

$a_n = -4 + (n-1)5 = -4 + 5n - 5 = -9 + 5n.$

17. $a_{10} = 6$, $a_{12} = 15$

First find a_1 and d. Since $a_n = a_1 + (n-1)d$, we have the following.

$a_{10} = a_1 + (10-1)d \Rightarrow a_{10} = a_1 + 9d \Rightarrow$
$6 = a_1 + 9d$ (1)

Also, we have

$a_{12} = a_1 + (12-1)d \Rightarrow a_{12} = a_1 + 11d \Rightarrow$
$15 = a_1 + 11d$ (2)

If we solve the system formed by equations (1) and (2) by the substitution method, we can first solve equation (1) for a_1 to obtain $a_1 = 6 - 9d$. Substitute $6 - 9d$ for a_1 in equation (2) and solve for d.

$15 = a_1 + 11d \Rightarrow 15 = (6 - 9d) + 11d \Rightarrow$
$15 = 6 + 2d \Rightarrow 9 = 2d \Rightarrow d = \frac{9}{2}$

Now substitute $d = \frac{9}{2}$ into $a_1 = 6 - 9d$ to find

$a_1 = 6 - 9\left(\frac{9}{2}\right) = 6 - \frac{81}{2} = \frac{12}{2} - \frac{81}{2} = -\frac{69}{2}.$

Now find a_8 and a_n. We have

$$a_8 = -\frac{69}{2} + (8-1)\left(\frac{9}{2}\right) = -\frac{69}{2} + 7\left(\frac{9}{2}\right) = -\frac{69}{2} + \frac{63}{2} = \frac{-6}{2} = -3$$

Also, we have

$$a_n = -\frac{69}{2} + (n-1)\left(\frac{9}{2}\right) = -\frac{69}{2} + \frac{9}{2}n - \frac{9}{2} = -\frac{78}{2} + \frac{9}{2}n = -39 + \frac{9}{2}n$$

18. $a_{15} = 8,\ a_{17} = 2$

First find a_1 and d. Since $a_n = a_1 + (n-1)d$, we have the following.

$a_{15} = a_1 + (15-1)d \Rightarrow a_{15} = a_1 + 14d \Rightarrow$
$8 = a_1 + 14d$ (1)

Also, we have

$a_{17} = a_1 + (17-1)d \Rightarrow a_{17} = a_1 + 16d \Rightarrow$
$2 = a_1 + 16d$ (2)

If we solve the system formed by equations (1) and (2) by the substitution method, we can first solve equation (1) for a_1 to obtain $a_1 = 8 - 14d$. Substitute 8 – 14d for a_1 in equation (2) and solve for d.

$2 = a_1 + 16d \Rightarrow 2 = (8 - 14d) + 16d \Rightarrow$
$2 = 8 + 2d \Rightarrow -6 = 2d \Rightarrow d = -3$

Now substitute $d = -3$ into $a_1 = 8 - 14d$ to find $a_1 = 8 - 14(-3) = 8 - (-42) = 50.$ Now find a_8 and a_n. We have

$$a_8 = 50 + (8-1)(-3) = 50 + 7(-3) = 50 + (-21) = 29$$

Also, we have

$a_n = 50 + (n-1)(-3) = 50 - 3n + 3 = 53 - 3n.$

19. $a_1 = x,\ a_2 = x + 3$

$d = a_2 - a_1 = (x+3) - x = 3$

Since $a_n = a_1 + (n-1)d$, we have $a_8 = x + (8-1)3 = x + 7 \cdot 3 = x + 21.$ Also, we have $a_n = x + (n-1)3 = x + 3n - 3.$

20. $a_2 = y + 1,\ d = -3$

First, find a_1. Since $d = a_2 - a_1$, we have $-3 = (y+1) - a_1 \Rightarrow a_1 = y + 4.$ Since $a_n = a_1 + (n-1)d$, we have

$$a_8 = (y+4) + (8-1)(-3) = y + 4 + 7 \cdot (-3) = y + 4 + (-21) = y - 17$$

Also, we have $a_n = (y+4) + (n-1)(-3) = y + 4 - 3n + 3 = y - 3n + 7$

21. $a_4 = s + 6p, d = 2p$

First, find a_1. Since $a_n = a_1 + (n-1)d$, we have $a_4 = a_1 + 3d \Rightarrow a_1 + 3(2p) = s + 6p \Rightarrow a_1 + 6p = s + 6p \Rightarrow a_1 = s$. Thus,

$a_8 = s + 7(2p) = s + 14p$ and
$a_n = s + (n-1)(2p) = s + 2pn - 2p$

22. $a_2 = a_1 + d$ (1)

$a_3 = a_1 + 2d \Rightarrow \dfrac{a_3 - a_1}{2} = d$ (2)

Substitute the expression for d in equation (2) for into equation (1), then solve for a_2:

$$a_2 = a_1 + \frac{a_3 - a_1}{2} = \frac{a_1 + a_3}{2}$$

23. $a_5 = 27,\ a_{15} = 87$

Since $a_{15} = a_5 + 10d$, it follows that $10d = a_{15} - a_5$. This implies $10d = 87 - 27 \Rightarrow 10d = 60 \Rightarrow d = 6.$

Since $a_5 = a_1 + 4d$, we have

$27 = a_1 + 4(6) \Rightarrow 27 = a_1 + 24 \Rightarrow a_1 = 3.$

24. $a_{12} = 60,\ a_{20} = 84$

Since $a_{20} = a_{12} + 8d$, it follows that $8d = a_{20} - a_{12}$. This implies $8d = 84 - 60 \Rightarrow 8d = 24 \Rightarrow d = 3.$

Since $a_{12} = a_1 + 11d$, we have

$60 = a_1 + 11(3) \Rightarrow 60 = a_1 + 33 \Rightarrow a_1 = 27.$

25. $S_{16} = -160, a_{16} = -25$

Since $S_n = \frac{n}{2}(a_1 + a_n)$, we have

$S_{16} = \frac{16}{2}(a_1 + a_{16}) \Rightarrow -160 = 8[a_1 + (-25)] \Rightarrow$
$-20 = a_1 - 25 \Rightarrow a_1 = 5$

26. $S_{28} = 2926,\ a_{28} = 199$

Since $S_n = \frac{n}{2}(a_1 + a_n)$, we have

$S_{28} = \frac{28}{2}(a_1 + a_{28}) \Rightarrow 2926 = 14(a_1 + 199) \Rightarrow$
$209 = a_1 + 199 \Rightarrow a_1 = 10.$

27. The sequence is comprised of the points $\{(1,-2),(2,-1),(3,0),(4,1),(5,2),(6,3)\}$. If the points were connected, they would form a line with slope $m=\frac{y_2-y_1}{x_2-x_1}=\frac{-1-(-2)}{2-1}=\frac{1}{1}=1$. Using the point-slope form of a line, we have
$y-y_1=m(x-x_1)\Rightarrow y-(-2)=1(x-1)\Rightarrow$
$y+2=x-1\Rightarrow y=x-3$
The *n*th term of the sequence is determined by $a_n=n-3$. The domain is $\{1,2,3,4,5,6\}$ and the range is $\{-2,-1,0,1,2,3\}$.

28. The sequence is comprised of the points $\{(1,1),(2,0),(3,-1),(4,-2),(5,-3)\}$. If the points were connected, they would form a line with slope $m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-1}{2-1}=\frac{-1}{1}=-1$. Using the point-slope form of a line, we have
$y-y_1=m(x-x_1)\Rightarrow y-1=-1(x-1)\Rightarrow$
$y-1=-x+1\Rightarrow y=-x+2$
The *n*th term of the sequence is determined by $a_n=-n+2$ or $a_n=2-n$. The domain is $\{1,2,3,4,5\}$ and the range is $\{-3,-2,-1,0,1\}$.

29. The sequence is comprised of the points $\left\{\left(1,\frac{5}{2}\right),(2,2),\left(3,\frac{3}{2}\right),(4,1),\left(5,\frac{1}{2}\right),(6,0)\right\}$. If the points were connected, they would form a line with slope $m=\frac{y_2-y_1}{x_2-x_1}=\frac{2-\frac{5}{2}}{2-1}=\frac{-\frac{1}{2}}{1}=-\frac{1}{2}$. Using the point-slope form of a line, we have
$y-y_1=m(x-x_1)\Rightarrow y-\frac{5}{2}=-\frac{1}{2}(x-1)\Rightarrow$
$y-\frac{5}{2}=-\frac{1}{2}x+\frac{1}{2}\Rightarrow y=-\frac{1}{2}x+3$
The *n*th term of the sequence is determined by $a_n=-\frac{1}{2}n+3$ or $a_n=3-\frac{1}{2}n$. The domain is $\{1,2,3,4,5,6\}$ and the range is $\left\{0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2}\right\}$ or $\{0,.5,1,1.5,2,2.5\}$.

30. The sequence is comprised of the points $\{(1,-5),(2,0),(3,5),(4,10),(5,15)\}$. If the points were connected, they would form a line with slope $m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-(-5)}{2-1}=\frac{5}{1}=5$. Using the point-slope form of a line, we have
$y-y_1=m(x-x_1)\Rightarrow y-(-5)=5(x-1)\Rightarrow$
$y+5=5x-5\Rightarrow y=5x-10$
The *n*th term of the sequence is determined by $a_n=5n-10$. The domain is $\{1,2,3,4,5\}$ and the range is $\{-5,0,5,10,15\}$.

31. The sequence is comprised of the points $\{(1,10),(2,-10),(3,-30),(4,-50),(5,-70)\}$. If the points were connected, they would form a line with slope
$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-10-10}{2-1}=\frac{-20}{1}=-20$. Using the point-slope form of a line, we have
$y-y_1=m(x-x_1)\Rightarrow y-10=-20(x-1)\Rightarrow$
$y-10=-20x+20\Rightarrow y=-20x+30$
The *n*th term of the sequence is determined by $a_n=-20n+30$ or $a_n=30-20n$. The domain is $\{1,2,3,4,5\}$ and the range is $\{-70,-50,-30,-10,10\}$.

32. The sequence is comprised of the points $\{(1,-2),(2,0),(3,2),(4,4),(5,6)\}$. If the points were connected, they would form a line with slope $m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-(-2)}{2-1}=\frac{2}{1}=2$. Using the point-slope form of a line, we have
$y-y_1=m(x-x_1)\Rightarrow y-(-2)=2(x-1)\Rightarrow$
$y+2=2x-2\Rightarrow y=2x-4$
The *n*th term of the sequence is determined by $a_n=2n-4$. The domain is $\{1,2,3,4,5\}$ and the range is $\{-2,0,2,4,6\}$.

33. 8, 11, 14, …
Since $a_1=8$, $d=a_2-a_1=11-8=3$, and $S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$, we have
$S_{10}=\frac{10}{2}\left[2(8)+(10-1)(3)\right]=5\left[16+9(3)\right]$
$=5(16+27)=5(43)=215$

34. $-9,-5,-1,...$
Since $a_1=-9$, $d=a_2-a_1=-5-(-9)=4$, and $S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$, we have
$S_{10}=\frac{10}{2}\left[2(-9)+(10-1)(4)\right]$
$=5\left[-18+9(4)\right]=5(-18+36)$
$=5(18)=90$

35. 5, 9, 13,…
Since $a_1 = 5$, $d = a_2 - a_1 = 9 - 5 = 4$, and $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have the following.
$S_{10} = \frac{10}{2}\left[2(5) + (10-1)(4)\right] = 5\left[10 + 9(4)\right]$
$= 5(10 + 36) = 5(46) = 230$

36. 8, 6, 4, …
Since $a_1 = 8$, $d = a_2 - a_1 = 6 - 8 = -2$, and $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have the following.
$S_{10} = \frac{10}{2}\left[2(8) + (10-1)(-2)\right] = 5\left[16 + 9(-2)\right]$
$= 5\left[16 + (-18)\right] = 5(-2) = -10$

37. $a_2 = 9$, $a_4 = 13$
To find S_{10}, we need a_1 and d. Since $a_4 = a_2 + 2d$, we have
$13 = 9 + 2d \Rightarrow 4 = 2d \Rightarrow d = 2.$ Also, since $d = a_2 - a_1$, we have $2 = 9 - a_1 \Rightarrow a_1 = 7.$
Thus, since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have
$S_{10} = \frac{10}{2}\left[2(7) + (10-1)(2)\right] = 5\left[14 + 9(2)\right]$
$= 5(14 + 18) = 5(32) = 160$

38. $a_3 = 5$, $a_4 = 8$
To find S_{10}, we need a_1 and d. Since $a_4 = a_3 + d$, we have $8 = 5 + d \Rightarrow d = 3.$
Also, since $a_n = a_1 + (n-1)d$, we have
$a_3 = a_1 + (3-1)3 \Rightarrow 5 = a_1 + 2(3) \Rightarrow$ Thus,
$5 = a_1 + 6 \Rightarrow a_1 = -1$
since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have
$S_{10} = \frac{10}{2}\left[2(-1) + (10-1)(3)\right] = 5\left[-2 + 9(3)\right]$
$= 5(-2 + 27) = 5(25) = 125$

39. $a_1 = 10$, $a_{10} = 5.5$
To find S_{10}, we need a_1 and d. Since $a_{10} = a_1 + 9d$, we have
$5.5 = 10 + 9d \Rightarrow 9d = -4.5 \Rightarrow d = -.5.$ Thus, since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have
$S_{10} = \frac{10}{2}\left[2(10) + (10-1)(-.5)\right] = 5\left[20 + 9(-.5)\right]$
$= 5\left[20 + (-4.5)\right] = 5(15.5) = 77.5$

40. $a_1 = -8$, $a_{10} = -1.25$
To find S_{10}, we need a_1 and d. Since $a_{10} = a_1 + 9d$, we have
$-1.25 = -8 + 9d \Rightarrow 9d = 6.75 \Rightarrow d = .75.$
Thus, since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have
$S_{10} = \frac{10}{2}\left[2(-8) + (10-1)(.75)\right]$
$= 5\left[-16 + 9(.75)\right] = 5(-16 + 6.75)$
$= 5(-9.25) = -46.25$

41. $a_1 = \pi$, $a_{10} = 10\pi$
To find S_{10}, we need a_1 and d. Since $a_{10} = a_1 + 9d$, we have
$10\pi = \pi + 9d \Rightarrow 9\pi = 9d \Rightarrow d = \pi.$ Thus, since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have
$S_{10} = \frac{10}{2}\left[2\pi + (10-1)\pi\right]$
$= 5\left[2\pi + 9\pi\right] = 5(11\pi) = 55\pi$

42. The positive integers form the arithmetic sequence 1, 2, 3, 4, …, so $d = 1$ and $a_1 = 1$.
Thus, $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$
$= \frac{n}{2}\left[2(1) + (n-1)(1)\right] = \frac{n}{2}\left[2 + (n-1)\right]$
$= \frac{n}{2}(n+1) = \frac{n(n+1)}{2}$
The statement is correct.

43. The first 80 positive integers form the arithmetic sequence 1, 2, 3, 4, …80. Thus, in the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we have $n = 80, a_1 = 1,$ and $a_{80} = 80.$ Thus, the sum of the first 80 positive integers is
$S_{80} = \frac{80}{2}(1 + 80) = 40(81) = 3240.$

44. The first 120 positive integers form the arithmetic sequence 1, 2, 3, 4, …120. Thus, in the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we have $n = 120, a_1 = 1,$ and $a_{120} = 120.$ Thus, the sum of the first 120 positive integers is
$S_{120} = \frac{120}{2}(1 + 120) = 60(121) = 7260.$

45. The positive odd integers form an arithmetic sequence 1, 3, 5, 7,… with $a_1 = 1$ and $d = 2.$ Find the sum of the first 50 terms of this sequence. Since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have
$S_{50} = \frac{50}{2}\left[2(1) + (50-1)(2)\right] = 25\left[2 + 49(2)\right]$
$= 25(2 + 98) = 25(100) = 2500$

46. The positive odd integers form an arithmetic sequence 1, 3, 5, 7,… with $a_1 = 1$ and $d = 2$. Find the sum of the first 90 terms of this sequence. Since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$S_{90} = \frac{90}{2}\left[2(1) + (90-1)(2)\right] = 45\left[2 + 89(2)\right]$$
$$= 45(2+178) = 45(180) = 8100$$

47. The positive even integers form an arithmetic sequence 2, 4, 6, 8,… with $a_1 = 2$ and $d = 2$. Find the sum of the first 60 terms of this sequence. Since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$ we have

$$S_{60} = \frac{60}{2}\left[2(2) + (60-1)(2)\right] = 30\left[4 + 59(2)\right]$$
$$= 30(4+118) = 30(122) = 3660$$

48. The positive even integers form an arithmetic sequence 2, 4, 6, 8,… with $a_1 = 2$ and $d = 2$. Find the sum of the first 70 terms of this sequence. Since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$ we have

$$S_{60} = \frac{70}{2}\left[2(2) + (70-1)(2)\right] = 35\left[4 + 69(2)\right]$$
$$= 35(4+138) = 35(142) = 4970$$

49. $S_{20} = 1090,\ a_{20} = 102$
Since the formula for the sum is $S_n = \frac{n}{2}(a_1 + a_n)$, we have $S_{20} = \frac{20}{2}(a_1 + a_{20})$. This yields
$1090 = 10(a_1 + 102) \Rightarrow 109 = a_1 + 102 \Rightarrow a_1 = 7.$
To solve for d, we use the fact that $a_n = a_1 + (n-1)d$.
$a_n = a_1 + (n-1)d \Rightarrow a_{20} = a_1 + (20-1)d \Rightarrow$
$102 = 7 + 19d \Rightarrow 95 = 19d \Rightarrow d = 5$
Thus, $a_1 = 7$ and $d = 5$.

50. $S_{31} = 5580,\ a_{31} = 360$
Since the formula for the sum is $S_n = \frac{n}{2}(a_1 + a_n)$, we have $S_{31} = \frac{31}{2}(a_1 + a_{31})$.
This yields $5580 = \frac{31}{2}(a_1 + 360) \Rightarrow$
$5580 \cdot \frac{2}{31} = a_1 + 360 \Rightarrow 360 = a_1 + 360 \Rightarrow a_1 = 0.$
To solve for d, we use the fact that $a_n = a_1 + (n-1)d$.
$a_n = a_1 + (n-1)d \Rightarrow a_{31} = a_1 + (31-1)d \Rightarrow$
$360 = 0 + 30d \Rightarrow 360 = 30d \Rightarrow d = 12$
Thus, $a_1 = 0$ and $d = 12$.

51. $S_{12} = -108,\ a_{12} = -19$
Since the formula for the sum is $S_n = \frac{n}{2}(a_1 + a_n)$, we have $S_{12} = \frac{12}{2}(a_1 + a_{12})$.
This yields $-108 = 6\left[a_1 + (-19)\right] \Rightarrow$
$-18 = a_1 + (-19) \Rightarrow a_1 = 1.$ To solve for d, we use the fact that $a_n = a_1 + (n-1)d$.
$a_n = a_1 + (n-1)d \Rightarrow a_{12} = a_1 + (12-1)d \Rightarrow$
$-19 = 1 + 11d \Rightarrow -20 = 11d \Rightarrow d = -\frac{20}{11}$
Thus, $a_1 = 1$ and $d = -\frac{20}{11}$.

52. $S_{25} = 650,\ a_{25} = 62$
Since the formula for the sum is $S_n = \frac{n}{2}(a_1 + a_n)$, we have $S_{25} = \frac{25}{2}(a_1 + a_{25})$.
This yields $650 = \frac{25}{2}(a_1 + 62) \Rightarrow$
$650 \cdot \frac{2}{25} = a_1 + 62 \Rightarrow 52 = a_1 + 62 \Rightarrow a_1 = -10.$
To solve for d, we use the fact that $a_n = a_1 + (n-1)d$.
$a_n = a_1 + (n-1)d \Rightarrow a_{25} = a_1 + (25-1)d \Rightarrow$
$62 = -10 + 24d \Rightarrow 72 = 24d \Rightarrow d = 3$
Thus, $a_1 = -10$ and $d = 3$.

53. $\sum_{i=1}^{3}(i+4)$
This is a sum of three terms having a common difference of 1. Thus, in the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we have $a_1 = 1 + 4 = 5$, $n = 3$, and $a_n = a_3 = 3 + 4 = 7$. The sum can, therefore, be determined to be

$$\sum_{i=1}^{3}(i+4) = S_3 = \frac{3}{2}(a_1 + a_3) = \frac{3}{2}(5+7)$$
$$= \frac{3}{2}(12) = 18$$

54. $\sum_{i=1}^{5}(i-8)$
This is the sum of five terms having a common difference of 1. Thus, in the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we have $a_1 = 1 - 8 = -7$, $n = 5$, and $a_n = a_5 = 5 - 8 = -3$. The sum can, therefore, be determined to be

$$\sum_{i=1}^{5}(i-8) = S_5 = \frac{5}{2}\left[-7 + (-3)\right] = \frac{5}{2}(-10) = -25$$

55. $\sum_{j=1}^{10}(2j+3)$

This is the sum of the arithmetic sequence with $a_1 = 2(1)+3 = 5$, $d = 2$, and $n = 10$. Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{j=1}^{10}(2j+3) = S_{10} = \tfrac{10}{2}\left[2(5)+9(2)\right] = 5(10+18) = 5(28) = 140$$

56. $\sum_{j=1}^{15}(5j-9)$

This is the sum of the arithmetic sequence with $a_1 = 5(1)-9 = -4$, $d = 5$, and $n = 15$. Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{j=1}^{15}(5j-9) = S_{15} = \tfrac{15}{2}\left[2(-4)+14(5)\right] = \tfrac{15}{2}(-8+70) = \tfrac{15}{2}(62) = 465$$

57. $\sum_{i=4}^{12}(-5-8i)$

This is the sum of an arithmetic sequence with $d = -8$ and $a_1 = -5-8(4) = -37$. If i started at 1, there would be 12 terms. Since three terms are missing, $n = 9$. Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{i=4}^{12}(-5-8i) = S_9 = \tfrac{9}{2}\left[2(-37)+8(-8)\right] = \tfrac{9}{2}\left[-74+(-64)\right] = \tfrac{9}{2}(-138) = -621$$

58. $\sum_{k=5}^{19}(-3-4k)$

This is the sum of an arithmetic sequence with $d = -4$ and $a_1 = -3-4(5) = -23$. If k started at 1, there would be 19 terms. Since four terms are missing, $n = 15$. Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{k=5}^{19}(-3-4k) = S_{15} = \tfrac{15}{2}\left[2(-23)+14(-4)\right] = \tfrac{15}{2}\left[-46+(-56)\right] = \tfrac{15}{2}(-102) = -765$$

59. $\sum_{i=1}^{1000} i$

This is the sum of 1000 terms having a common difference of 1. Thus, in the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we have $a_1 = 1$, $n = 1000$, and $a_n = a_{1000} = 1000$. The sum can, therefore, be determined as follows.

$$\sum_{i=1}^{1000} i = S_{1000} = \tfrac{1000}{2}(1+1000) = 500(1001) = 500,500$$

60. $\sum_{k=1}^{2000} k$

This is the sum of 2000 terms having a common difference of 1. Thus, in the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we have $a_1 = 1$, $n = 2000$, and $a_n = a_{2000} = 2000$. The sum can, therefore, be determined as follows.

$$\sum_{k=1}^{2000} k = S_{2000} = \tfrac{2000}{2}(1+2000) = 1000(2001) = 2,001,000$$

61. $\sum_{k=1}^{100}(-k)$

This is the sum of 100 terms with $a_1 = -1$, $d = -1$ and $n = 100$. Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{k=1}^{100}(-k) = S_{100} = \tfrac{100}{2}\left[2(-1)+99(-1)\right] = 50(-2-99) = 50(-101) = -5050$$

62. $f(x) = mx+b$
$f(1) = m(1)+b = m+b$
$f(2) = m(2)+b = 2m+b$
$f(3) = m(3)+b = 3m+b$

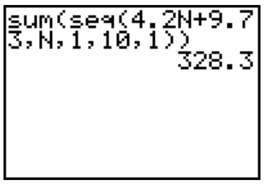
63. The sequence $f(1), f(2), f(3)$ is an arithmetic sequence since the difference between any two adjacent terms is m. Since $d = (3m+b)-(2m+b) = 3m+b-2m-b = m$, the common difference is m.

64. From Exercise 62, we know that $a_1 = m+b$. From Exercise 63, we know that $d = m$. Therefore, we have $a_n = a_1 + (n-1)d \Rightarrow$
$a_n = (m+b)+(n-1)m \Rightarrow$
$a_n = m+b+nm-m \Rightarrow a_n = mn+b.$

For Exercises 65−68, you will need the sum(and seq(features of your graphing calculator. On the TI-83, the sum(feature is located under the LIST then MATH menu. The seq(feature is under the LIST then OPS menu.

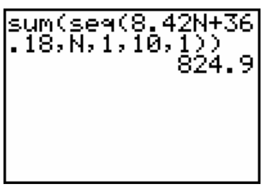
65. $a_n = 4.2n + 9.73$

Using the sequence feature of a graphing calculator, we obtain $S_{10} = 328.3$.



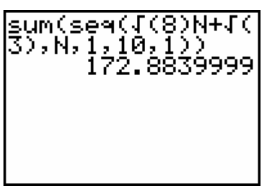
66. $a_n = 8.42n + 36.18$

Using the sequence feature of a graphing calculator, we obtain $S_{10} = 824.9$.



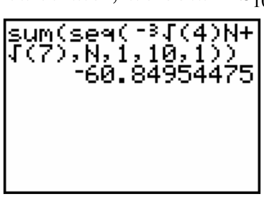
67. $a_n = \sqrt{8}n + \sqrt{3}$

Using the sequence feature of a graphing calculator, we obtain $S_{10} \approx 172.884$.



68. $a_n = -\sqrt[3]{4}n + \sqrt{7}$

Using the sequence feature of a graphing calculator, we obtain $S_{10} \approx -60.850$.



69. Find the sum of all the integers from 51 to 71.

We need to find

$\sum_{i=51}^{71} i = \sum_{i=1}^{71} i - \sum_{i=1}^{50} i = S_{71} - S_{50}$. Since we know $a_1 = 1,\ d = 1,\ a_{50} = 50$, and $a_{71} = 71,$ we have $S_{71} = \frac{71}{2}(1+71) = 71(36) = 2556$ and $S_{50} = \frac{50}{2}(1+50) = 25(51) = 1275.$ Thus, the sum is $S_{71} - S_{50} = 2556 - 1275 = 1281.$

70. To find the sum of all the integers from –8 to 30, we can use the formula $S_n = \frac{n}{2}(a_1 + a_n).$ Since $a_1 = -8$ and $a_n = 30,$ we need to determine the number of terms. Since there are 8 negative integers, 30 positive integers, and 0 we have *n* = 39. Thus,

$S_{39} = \frac{39}{2}(-8+30) = \frac{39}{2}(22) = 39(11) = 429.$

The sum of all the integers from –8 to 30 is 429.

71. In every 12-hour cycle, the clock will chime 1 + 2 + 3 + …+ 12 times. Since $a_1 = 1,\ n = 12,\ a_{12} = 12,$ and $S_n = \frac{n}{2}(a_1 + a_n),$ we have $S_{12} = \frac{12}{2}(1+12) = 6(13) = 78.$ Since there are two 12-hour cycles in 1 day, every day, the clock will chime $2(78) = 156$ times. Since there are 30 days in this month, the clock will chime $156(30) = 4680$ times.

72. In this exercise, we have $a_1 = 30,\ a_n = a_{30} = 1,$ and $n = 30.$ Using $S_n = \frac{n}{2}(a_1 + a_n),$ we have $S_{30} = \frac{30}{2}(30+1) = 15(31) = 465.$ Thus, there are 465 poles in the stack.

73. In this exercise, we have

$a_1 = 49{,}000,\ d = 580,$ and $n = 5.$ Using $a_n = a_1 + (n-1)d,$ we need to find $a_{10}.$ Since

$$\begin{aligned} a_{11} &= 49{,}000 + (11-1)580 = 49{,}000 + 10(580) \\ &= 49{,}000 + 5800 = 54{,}800 \end{aligned}$$

we would expect that five years from now, the population will be 54,800.

74. The longest support is 15 m long, and the shortest support is 2 m long. Since the slide is of uniform slope, the sum of the lengths of these 20 supports can be thought of as the sum of an arithmetic sequence with $a_1 = 15,\ a_n = a_{20} = 2$, and $n = 20$. Since $S_n = \frac{n}{2}(a_1 + a_n)$, we have
$S_{20} = \frac{20}{2}(a_1 + a_{20}) = 10(15+2) = 10(17) = 170$
Thus, the total length is 170 m.

75. The longest rung measures 28 in., and the shortest measures 18 in. Since the rungs are uniformly tapered, the sum of the lengths of these 31 supports can be thought of as the sum of an arithmetic sequence with $a_1 = 28,\ a_n = a_{31} = 18$, and $n = 31$. Since $S_n = \frac{n}{2}(a_1 + a_n)$, we have
$$\begin{aligned} S_{31} &= \tfrac{31}{2}(a_1 + a_{31}) = \tfrac{31}{2}(28+18) \\ &= \tfrac{31}{2}(46) = 713 \end{aligned}$$
Thus, a total of 713 in. of material would be needed.

76. **(a)** $a_1 = 98.2$ and $a_3 = 109.8$
Since $a_3 = a_1 + 2d$, we have
$109.8 = 98.2 + 2d \Rightarrow 11.6 = 2d \Rightarrow d = 5.8.$
The common difference is 5.8 cm

(b) If we have a_1 representing her height at age 3, then her height at age 8 is a_6.
Since $a_6 = a_1 + 5d$, we have
$a_6 = 98.2 + 5(5.8) = 98.2 + 29.0 = 127.2.$
We would expect her height at age 8 to be 127.2 cm.

77. Consider the arithmetic sequence $a_1, a_2, a_3, \ldots$. Then, consider the sequence $a_1, a_3, a_5, \ldots$. We have
$a_3 - a_1 = (a_1 + 2d) - a_1 = 2d$ and
$a_5 - a_3 = (a_1 + 4d) - (a_1 + 2d) = 2d$. In general, we have the following.
$$\begin{aligned} &a_n - a_{n-2} \\ &= \left[a_1 + (n-1)d\right] - \left[a_1 + ((n-2)-1)d\right] \\ &= a_1 + nd - d - \left[a_1 + (n-3)d\right] \\ &= a_1 + nd - d - (a_1 + nd - 3d) \\ &= a_1 + nd - d - a_1 - nd + 3d = 2d \end{aligned}$$
Thus, $a_1, a_3, a_5, \ldots$. is an arithmetic sequence whose first term is a_1 and common difference is $2d$.

78. Since the sequence $\log 2, \log 4, \log 8, \log 16, \ldots$ is the same $\log 2^1, \log 2^2, \log 2^3, \log 2^4, \ldots$, it has a general term of $a_n = \log 2^n$. Notice that we have $a_2 - a_1 = \log 4 - \log 2 = \log \frac{4}{2} = \log 2$ and $a_3 - a_2 = \log 8 - \log 4 = \log \frac{8}{4} = \log 2$. In general, we have
$$\begin{aligned} a_n - a_{n-1} &= \log 2^n - \log 2^{n-1} = \log \tfrac{2^n}{2^{n-1}} \\ &= \log 2^{n-(n-1)} = \log 2^{n-n+1} \\ &= \log 2^1 = \log 2 \end{aligned}$$
Thus, $\log 2, \log 4, \log 8, \log 16, \ldots$ is an arithmetic sequence whose first term is $\log 2$ and common difference is $\log 2$.

Section 11.3: Geometric Sequences and Series

For Exercises 1–4, we are examining the geometric sequence 1, 2, 4, 8,… . As stated on page 673 of the text, the nth term of a geometric sequence is given by $a_n = a_1 r^{n-1}$, where a_1 is the first term and r is the common ratio. For this situation, $a_n = 1 \cdot 2^{n-1} = 2^{n-1}$.

1. day 10

(a) $a_{10} = 2^{10-1} = 2^9 = 512$ cents or \$5.12

(b) Since $S_n = \dfrac{a_1(1-r^n)}{1-r}$ for $r \neq 1$, we have
$S_{10} = \dfrac{1(1-2^{10})}{1-2} = \dfrac{1-1024}{-1} = \dfrac{-1023}{-1} = 1023$
cents or \$10.23.

2. day 12

(a) $a_{12} = 2^{12-1} = 2^{11} = 2048$ cents or \$20.48

(b) Since $S_n = \frac{a_1(1-r^n)}{1-r}$ for $r \neq 1$, we have
$S_{12} = \frac{1(1-2^{12})}{1-2} = \frac{1-4096}{-1} = \frac{-4095}{-1} = 4095$
cents or \$40.95.

3. day 15

(a) $a_{15} = 2^{15-1} = 2^{14} = 16{,}384$ cents or \$163.84

(b) Since $S_n = \frac{a_1(1-r^n)}{1-r}$ for $r \neq 1$, we have $S_{15} = \frac{1(1-2^{15})}{1-2} = \frac{1-32,768}{-1} = \frac{-32,767}{-1} = 32,767$ cents or \$327.67.

4. day 18

(a) $a_{18} = 2^{18-1} = 2^{17} = 131,072$ cents or \$1310.72

(b) Since $S_n = \frac{a_1(1-r^n)}{1-r}$ for $r \neq 1$, we have $S_{18} = \frac{1(1-2^{18})}{1-2} = \frac{1-262,144}{-1} = \frac{-262,143}{-1} = 262,143$ cents or \$2621.43.

5. $a_1 = 5,\ r = -2$

$a_5 = a_1 r^{5-1} = 5(-2)^4 = 5(16) = 80$

$a_n = a_1 r^{n-1} = 5(-2)^{n-1}$

6. $a_1 = 8,\ r = -5$

$a_5 = a_1 r^{5-1} = 8(-5)^4 = 8(625) = 5000$

$a_n = a_1 r^{n-1} = 8(-5)^{n-1}$

7. $a_2 = -4,\ r = 3$

We need to determine a_1. Since

$a_2 = a_1 r^{2-1} \Rightarrow -4 = a_1(3)^1 \Rightarrow$

$-4 = a_1(3) \Rightarrow a_1 = -\frac{4}{3}$, we have

$a_5 = a_1 r^{5-1} = -\frac{4}{3}(3)^4 = -\frac{4}{3}(81) = -108$ and

$a_n = a_1 r^{n-1} = -\frac{4}{3}(3)^{n-1}$

8. $a_3 = -2,\ r = 4$

We need to determine a_1. Since

$a_3 = a_1 r^{3-1} \Rightarrow -2 = a_1(4)^2 \Rightarrow -2 = a_1(16) \Rightarrow$

$a_1 = -\frac{2}{16} = -\frac{1}{8}$, we have

$a_5 = a_1 r^{5-1} = -\frac{1}{8}(4)^4 = -\frac{1}{8}(256) = -32$ and

$a_n = a_1 r^{n-1} = -\frac{1}{8}(4)^{n-1}$

9. $a_4 = 243$, $r = -3$

We need to determine a_1. Since

$a_4 = a_1 r^{4-1} \Rightarrow 243 = a_1(-3)^3 \Rightarrow$

$243 = a_1(-27) \Rightarrow a_1 = -9$, we have

$a_5 = a_1 r^{5-1} = -9(-3)^4 = -9(81) = -729$

$a_n = a_1 r^{n-1} = -9(-3)^{n-1}$

Note that

$(-9)(-3)^{n-1} = -(-3)^2(-3)^{n-1} = (-3)^{2+n-1}$

$= -(-3)^{n+1}$, so $a_n = -(-3)^{n+1}$ is an equivalent formula for the nth term of this sequence.

10. $a_4 = 18,\ r = 2$

We need to determine a_1. Since

$a_4 = a_1 r^{4-1} \Rightarrow 18 = a_1(2)^3 \Rightarrow 18 = a_1(8) \Rightarrow$

$a_1 = \frac{18}{8} = \frac{9}{4}$, we have

$a_5 = a_1 r^{5-1} = \frac{9}{4}(2)^4 = \frac{9}{4}(16) = 36$

$a_n = a_1 r^{n-1} = \frac{9}{4}(2)^{n-1}$

Note that $\frac{9}{4}(2)^{n-1} = \frac{9}{2^2}\cdot(2)^{n-1} = 9\cdot\frac{2^{n-1}}{2^2}$

$= 9\cdot 2^{(n-1)-2} = 9\cdot 2^{n-3}$, so $a_n = 9(2)^{n-3}$ is an equivalent formula for the nth term of this sequence.

11. $-4, -12, -36, -108, \ldots$

First find r. Since $r = \frac{a_2}{a_1}$, we have

$r = \frac{-12}{-4} = 3$. Also, given that $a_1 = -4$, we have

$a_5 = a_1 r^{5-1} = -4(3)^4 = -4(81) = -324$

$a_n = a_1 r^{n-1} = -4(3)^{n-1}$

12. $-2, 6, -18, 54, \ldots$

First find r. Since $r = \frac{a_2}{a_1}$, we have

$r = \frac{6}{-2} = -3$. Also, given that $a_1 = -2$, we have $a_5 = a_1 r^{5-1} = -2(-3)^4 = -2(81) = -162$

$a_n = a_1 r^{n-1} = -2(-3)^{n-1}$

13. $\frac{4}{5}, 2, 5, \frac{25}{2}, \ldots$

First find r. Since $r = \frac{a_2}{a_1}$, we have

$r = \frac{2}{\frac{4}{5}} = 2\left(\frac{5}{4}\right) = \frac{5}{2}$. Also, given that $a_1 = \frac{4}{5}$, we have $a_5 = a_1 r^{5-1} = \frac{4}{5}\left(\frac{5}{2}\right)^4 = \frac{4}{5}\left(\frac{625}{16}\right) = \frac{125}{4}$ and

$a_n = a_1 r^{n-1} = \frac{4}{5}\left(\frac{5}{2}\right)^{n-1}$

Note that $\left(\frac{4}{5}\right)\left(\frac{5}{2}\right)^{n-1} = \frac{2^2}{5^1}\cdot\frac{5^{n-1}}{2^{n-1}} = \frac{5^{n-2}}{2^{n-3}}$, so $a_n = \frac{5^{n-2}}{2^{n-3}}$ is an equivalent formula for the nth term of this sequence.

14. $\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \frac{32}{27}, \ldots$

First find r. Since $r = \frac{a_2}{a_1}$, we have

$r = \frac{\frac{2}{3}}{\frac{1}{2}} = \frac{2}{3}\left(\frac{2}{1}\right) = \frac{4}{3}$. Also, given that $a_1 = \frac{1}{2}$,

we have $a_5 = a_1 r^{5-1} = \frac{1}{2}\left(\frac{4}{3}\right)^4 = \frac{1}{2}\left(\frac{256}{81}\right) = \frac{128}{81}$

$a_n = a_1 r^{n-1} = \frac{1}{2}\left(\frac{4}{3}\right)^{n-1}$

Note that

$\frac{1}{2}\left(\frac{4}{3}\right)^{n-1} = \frac{1}{2} \cdot \frac{4^{n-1}}{3^{n-1}} = \frac{1}{2} \cdot \frac{\left(2^2\right)^{n-1}}{3^{n-1}} = \frac{1}{2} \cdot \frac{2^{2n-2}}{3^{n-1}}$

$= \frac{2^{2n-2-1}}{3^{n-1}} = \frac{2^{2n-3}}{3^{n-1}}$, so $\frac{2^{2n-3}}{3^{n-1}}$ is an equivalent formula for the nth term of this sequence.

15. $10, -5, \frac{5}{2}, -\frac{5}{4}, \ldots$

First find r. Since $r = \frac{a_2}{a_1}$, we have

$r = \frac{-5}{10} = -\frac{1}{2}$. Also, given that $a_1 = 10$, we have $a_5 = a_1 r^{5-1} = 10\left(-\frac{1}{2}\right)^4 = 10\left(\frac{1}{16}\right) = \frac{5}{8}$ and

$a_n = a_1 r^{n-1} = 10\left(-\frac{1}{2}\right)^{n-1}$

16. $3, -\frac{9}{4}, \frac{27}{16}, -\frac{81}{64}, \ldots$

First find r. Since $r = \frac{a_2}{a_1}$, we have

$r = \frac{-\frac{9}{4}}{3} = -\frac{9}{4}\left(\frac{1}{3}\right) = -\frac{3}{4}$. Also, given that $a_1 = 3$, we have

$a_5 = a_1 r^{5-1} = 3\left(-\frac{3}{4}\right)^4 = 3\left(\frac{81}{256}\right) = \frac{243}{256}$ and

$a_n = a_1 r^{n-1} = 3\left(-\frac{3}{4}\right)^{n-1}$

Note that

$3\left(-\frac{3}{4}\right)^{n-1} = \frac{3^1}{1} \frac{3^{n-1}}{(-4)^{n-1}} = \frac{3^{1+n-1}}{(-4)^{n-1}} = \frac{3^n}{(-4)^{n-1}}$, so

$a_n = \frac{3^n}{(-4)^{n-1}}$ is an equivalent formula for the nth term of this sequence.

17. $a_2 = -6$, $a_7 = -192$

Using $a_n = a_1 r^{n-1}$, we have $a_2 = a_1 r = -6$.

This yields $a_1 = \frac{-6}{r}$. Also, $a_7 = a_1 r^6 = -192$.

Substituting $a_1 = \frac{-6}{r}$ into $a_1 r^6 = -192$ equation and solving for r, we have

$a_1 r^6 = -192 \Rightarrow \left(\frac{-6}{r}\right) r^6 = -192 \Rightarrow$

$-6r^5 = -192 \Rightarrow r^5 = 32 \Rightarrow r^5 = 2^5 \Rightarrow r = 2$

Since $a_1 = \frac{-6}{r}$, we have $a_1 = \frac{-6}{2} = -3$.

18. $a_3 = 5$ and $a_8 = \frac{1}{625}$

Using $a_n = a_1 r^{n-1}$, we have $a_3 = a_1 r^2 = 5$.

This yields $a_1 = \frac{5}{r^2}$. Also, $a_8 = a_1 r^7 = \frac{1}{625}$.

Substituting $a_1 = \frac{5}{r^2}$ into $a_1 r^7 = \frac{1}{625}$ equation and solving for r, we have

$a_1 r^7 = \frac{1}{625} \Rightarrow \left(\frac{5}{r^2}\right) r^7 = \frac{1}{625} \Rightarrow 5r^5 = \frac{1}{625} \Rightarrow$

$r^5 = \frac{1}{3125} \Rightarrow r^5 = \left(\frac{1}{5}\right)^5 \Rightarrow r = \frac{1}{5}$

Since $a_1 = \frac{5}{r^2}$ we have

$a_1 = \frac{5}{\left(\frac{1}{5}\right)^2} = \frac{5}{\frac{1}{25}} = 5 \cdot 25 = 125.$

19. $a_3 = 50$, $a_7 = .005$

Using $a_n = a_1 r^{n-1}$, we have $a_3 = a_1 r^2 = 50$.

This yields $a_1 = \frac{50}{r^2}$. Also, $a_7 = a_1 r^6 = .005$.

Substituting $a_1 = \frac{50}{r^2}$ into $a_1 r^6 = .005$ equation and solving for r, we have

$a_1 r^6 = .005 \Rightarrow \left(\frac{50}{r^2}\right) r^6 = .005 \Rightarrow 50r^4 = .005 \Rightarrow$

$r^4 = .0001 \Rightarrow r = \pm .1 \quad a_1 = \frac{50}{(\pm .1)^2} = \frac{50}{.01} = 5000.$

20. $a_4 = -\frac{1}{4}$, $a_9 = -\frac{1}{128}$

Using $a_n = a_1 r^{n-1}$, we have $a_4 = a_1 r^3 = -\frac{1}{4}$.

This yields $a_1 = -\frac{1}{4r^3}$. Also,

$a_9 = a_1 r^8 = -\frac{1}{128}$. Substituting $a_1 = -\frac{1}{4r^3}$ into $a_1 r^8 = -\frac{1}{128}$ equation and solving for r, we have the following.

$a_1 r^8 = -\frac{1}{128} \Rightarrow \left(-\frac{1}{4r^3}\right) r^8 = -\frac{1}{128} \Rightarrow$

$-\frac{1}{4} r^5 = -\frac{1}{128} \Rightarrow r^5 = \frac{1}{32} \Rightarrow r^5 = \left(\frac{1}{2}\right)^5 \Rightarrow r = \frac{1}{2}$

Since $a_1 = -\frac{1}{4r^3}$, we have

$a_1 = -\frac{1}{4\left(\frac{1}{2}\right)^3} = -\frac{1}{4\left(\frac{1}{8}\right)} = -\frac{1}{\frac{1}{2}} = -2.$

21. 2, 8, 32, 128, …

First find r. Since $r = \frac{a_2}{a_1}$, we have $r = \frac{8}{2} = 4$.

Also, given that $a_1 = 2$, $n = 5$, and

$S_n = \frac{a_1\left(1 - r^n\right)}{1 - r}$, we have

$S_n = \frac{a_1\left(1 - r^n\right)}{1 - r} = \frac{2\left(1 - 4^5\right)}{1 - 4} = \frac{2(1 - 1024)}{-3}$

$= \frac{2(-1023)}{-3} = \frac{-2046}{-3} = 682$

22. 4, 16, 64, 256, …

First find r. Since $r = \frac{a_2}{a_1}$, we have $r = \frac{16}{4} = 4$. Also, given that $a_1 = 4$, $n = 5$, and $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{4(1-4^5)}{1-4} = \frac{4(1-1024)}{-3} = \frac{4(-1023)}{-3} = \frac{-4092}{-3} = 1364$$

23. 18, –9, $\frac{9}{2}$, $-\frac{9}{4}$, …

First find r. Since $r = \frac{a_2}{a_1}$, we have $r = \frac{-9}{18} = -\frac{1}{2}$. Also, given that $a_1 = 18$, $n = 5$, and $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{18\left[1-\left(-\frac{1}{2}\right)^5\right]}{1-\left(-\frac{1}{2}\right)} = \frac{18\left[1-\left(-\frac{1}{32}\right)\right]}{\frac{3}{2}} = \frac{18\left(\frac{32}{32}+\frac{1}{32}\right)}{\frac{3}{2}} = 18\left(\frac{33}{32}\right)\left(\frac{2}{3}\right) = 18\left(\frac{11}{16}\right) = \frac{99}{8}$$

24. 12, –4, $\frac{4}{3}$, $-\frac{4}{9}$, …

First find r. Since $r = \frac{a_2}{a_1}$, we have $r = \frac{-4}{12} = -\frac{1}{3}$. Also, given that $a_1 = 12$, $n = 5$, and $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{12\left[1-\left(-\frac{1}{3}\right)^5\right]}{1-\left(-\frac{1}{3}\right)} = \frac{12\left[1-\left(-\frac{1}{243}\right)\right]}{\frac{4}{3}} = \frac{12\left(\frac{243}{243}+\frac{1}{243}\right)}{\frac{4}{3}} = 12\left(\frac{244}{243}\right)\left(\frac{3}{4}\right) = 12\left(\frac{61}{81}\right) = \frac{244}{27}$$

25. $a_1 = 8.423$, $r = 2.859$

Since $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$S_5 = \frac{8.423\left[1-(2.859)^5\right]}{1-2.859} \approx 860.95.$$

26. $a_1 = -3.772$, $r = -1.553$

Since $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$S_5 = \frac{-3.772\left[1-(-1.553)^5\right]}{1-(-1.553)} \approx -14.82.$$

27. $\sum_{i=1}^{5} 3^i$

For this geometric series, $a_1 = 3^1 = 3$, $r = 3$, and $n = 5$. Since $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{i=1}^{5} 3^i = S_5 = \frac{3(1-3^5)}{1-3} = \frac{3(1-243)}{-2} = \frac{3(-242)}{-2} = 3(121) = 363$$

28. $\sum_{i=1}^{4} (-2)^i$

For this geometric series, $a_1 = (-2)^1 = -2$, $r = -2$, and $n = 4$. Since $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{i=1}^{4} (-2)^i = S_4 = \frac{-2\left[1-(-2)^4\right]}{1-(-2)} = \frac{-2(1-16)}{3} = \frac{-2(-15)}{3} = -2(-5) = 10$$

29. $\sum_{j=1}^{6} 48\left(\frac{1}{2}\right)^j$

For this geometric series, $a_1 = 48\left(\frac{1}{2}\right)^1 = 48\left(\frac{1}{2}\right) = 24$, $r = \frac{1}{2}$, and $n = 6$. Since $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{j=1}^{6} 48\left(\frac{1}{2}\right)^j = S_6 = \frac{24\left[1-\left(\frac{1}{2}\right)^6\right]}{1-\frac{1}{2}} = \frac{24\left(1-\frac{1}{64}\right)}{\frac{1}{2}} = \frac{24\left(\frac{64}{64}-\frac{1}{64}\right)}{\frac{1}{2}} = 24\left(\frac{63}{64}\right)(2) = 24\left(\frac{63}{32}\right) = \frac{189}{4}$$

30. $\sum_{j=1}^{5} 243\left(\frac{2}{3}\right)^j$

For this geometric series, $a_1 = 243\left(\frac{2}{3}\right)^1 = 243\left(\frac{2}{3}\right) = 162$, $r = \frac{2}{3}$, and $n = 5$. Since $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{j=1}^{5} 243\left(\frac{2}{3}\right)^j = S_5 = \frac{162\left[1-\left(\frac{2}{3}\right)^5\right]}{1-\frac{2}{3}} = \frac{162\left(1-\frac{32}{243}\right)}{\frac{1}{3}} = \frac{162\left(\frac{243}{243}-\frac{32}{243}\right)}{\frac{1}{3}} = 162\left(\frac{211}{243}\right)(3) = 162\left(\frac{211}{81}\right) = 422$$

31. $\sum_{k=4}^{10} 2^k$

This series is the sum of the fourth through tenth terms of a geometric sequence with $a_1 = 2^1 = 2$ and $r = 2.$ To find this sum, find the difference between the sum of the first ten terms and the sum of the first three terms.

$$\sum_{k=4}^{10} 2^k = \sum_{k=1}^{10} 2^k - \sum_{k=1}^{3} 2^k = \frac{2(1-2^{10})}{1-2} - \frac{2(1-2^3)}{1-2}$$
$$= \frac{2(1-1024)}{-1} - \frac{2(1-8)}{-1} = \frac{2(-1023)}{-1} - \frac{2(-7)}{-1}$$
$$= 2046 - 14 = 2032$$

Note: We could also consider $\sum_{k=4}^{10} 2^k$ as a geometric series with $a_1 = 2^4 = 16$ and $r = 2.$ If the sequence started with $k = 1,$ there would be 10 terms. Since it starts with 4, three of the terms are missing, so there are seven terms and $n = 7.$ Thus we have

$$\sum_{k=4}^{10} 2^k = S_7 = \frac{16(1-2^7)}{1-2} = \frac{16(1-128)}{-1}$$
$$= \frac{16(-127)}{-1} = 2032$$

32. $\sum_{k=3}^{9} (-3)^k$

This series is the sum of the third through ninth terms of a geometric sequence with $a_1 = (-3)^1 = -3$ and $r = -3.$ To find this sum, find the difference between the sum of the first nine terms and the sum of the first two terms.

$$\sum_{k=3}^{9} (-3)^k = \sum_{k=1}^{9} (-3)^k - \sum_{k=1}^{2} (-3)^k$$
$$= \frac{-3\left[1-(-3)^9\right]}{1-(-3)} - \frac{-3\left[1-(-3)^2\right]}{1-(-3)}$$
$$= \frac{-3\left[1-(-19,683)\right]}{4} - \frac{-3(1-9)}{4}$$
$$= \frac{-3(19,684)}{4} - \frac{-3(-8)}{4}$$
$$= -14,763 - 6 = -14,769$$

Note: We could also consider $\sum_{k=3}^{9} (-3)^k$ as a geometric series with $a_1 = (-3)^{-3} = -27$ and $r = -3.$ If the sequence started with $k = 1,$ there would be 9 terms. Since it starts with 3, two of the terms are missing, so there are seven terms and $n = 7.$ Thus we have

$$\sum_{k=3}^{9} (-3)^k = S_7 = \frac{-27\left[1-(-3)^7\right]}{1-(-3)}$$
$$= \frac{-27\left[1-(-2187)\right]}{4} = \frac{-27(2188)}{4}$$
$$= -27(547) = -14,769$$

33. The sum of an infinite geometric series exists if $|r| < 1.$

34. .9 + .09 + .009 + …

Here $S_\infty = \frac{a_1}{1-r} = \frac{.9}{1-.1} = \frac{.9}{.9} = 1.$ The sum is 1, which is contrary to the intuition of most people.

35. 12, 24, 48, 96, …

Since $r = \frac{a_2}{a_1},$ we have $r = \frac{24}{12} = 2.$ The sum of the terms of this infinite geometric sequence would not converge since $r = 2$ is not between –1 and 1.

36. 625, 125, 25, 5, …

Since $r = \frac{a_2}{a_1},$ we have $r = \frac{125}{625} = \frac{1}{5}.$ Since $-1 < r < 1,$ the sum converges.

37. –48, –24, –12, –6, …

Since $r = \frac{a_2}{a_1},$ we have $r = \frac{-24}{-48} = \frac{1}{2}.$ Since $-1 < r < 1,$ the sum converges.

38. 2, –10, 50, –250, …

Since $r = \frac{a_2}{a_1},$ we have $r = \frac{-10}{2} = -5.$ Since $|r| > 1,$ the sum does not converge.

39. $2 + 1 + \frac{1}{2} + \frac{1}{4} + ...$ is a geometric series with $r = \frac{a_2}{a_1} = \frac{1}{2}$ and $a_1 = 2.$ Since $S_n = \frac{a_1(1-r^n)}{1-r},$ we have $S_n = \frac{2\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = \frac{2\left[1-\left(\frac{1}{2}\right)^n\right]}{\frac{1}{2}}.$

Similar to Example 7, we have $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0.$

Thus, we obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2\left[1-\left(\frac{1}{2}\right)^n\right]}{\frac{1}{2}} = \frac{2(1-0)}{\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4.$$

40. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$ is a geometric series with $r=\frac{a_2}{a_1}=\frac{\frac{1}{3}}{1}=\frac{1}{3}$ and $a_1=1$. Since $S_n=\frac{a_1\left(1-r^n\right)}{1-r}$, we have $S_n=\frac{1\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}}=\frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}}$. As in Example 7, we have $\lim_{n\to\infty}\left(\frac{1}{3}\right)^n=0$. Thus, we obtain $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}}=\frac{1-0}{\frac{2}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}$. Using the formula $S_\infty=\frac{a_1}{1-r}$ where $r=\frac{1}{3}$ and $a_1=1$, we obtain $S_\infty=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}$. The results are the same using the two approaches.

41. $18+6+2+\frac{2}{3}+\cdots$
For this geometric series, $a_1=18$ and $r=\frac{6}{18}=\frac{1}{3}$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{18}{1-\frac{1}{3}}=\frac{18}{\frac{2}{3}}=18\cdot\frac{3}{2}=27.$

42. $100+10+1+\cdots$
For this geometric series, $a_1=100$ and $r=\frac{10}{100}=\frac{1}{10}$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{100}{1-\frac{1}{10}}=\frac{100}{\frac{9}{10}}=100\cdot\frac{10}{9}=\frac{1000}{9}.$

43. $\frac{1}{4}-\frac{1}{6}+\frac{1}{9}-\frac{2}{27}+\cdots$
For this geometric series, $a_1=\frac{1}{4}$ and $r=\frac{-\frac{1}{6}}{\frac{1}{4}}=-\frac{1}{6}\cdot\frac{4}{1}=-\frac{2}{3}$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{\frac{1}{4}}{1-\left(-\frac{2}{3}\right)}=\frac{\frac{1}{4}}{\frac{5}{3}}=\frac{1}{4}\cdot\frac{3}{5}=\frac{3}{20}.$

44. $\frac{4}{3}+\frac{2}{3}+\frac{1}{3}+\cdots$
For this geometric series, $a_1=\frac{4}{3}$ and $r=\frac{\frac{2}{3}}{\frac{4}{3}}=\frac{2}{3}\cdot\frac{3}{4}=\frac{1}{2}$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{\frac{4}{3}}{1-\frac{1}{2}}=\frac{\frac{4}{3}}{\frac{1}{2}}=\frac{4}{3}\cdot\frac{2}{1}=\frac{8}{3}.$

45. $\sum_{i=1}^{\infty}3\left(\frac{1}{4}\right)^{i-1}$
For this geometric series, $a_1=3\left(\frac{1}{4}\right)^{1-1}=3\left(\frac{1}{4}\right)^0=3\cdot1=3$ and $r=\frac{1}{4}$. Because $-1<r<1$, this series converges. We have $S_\infty=\frac{a_1}{1-r}=\frac{3}{1-\frac{1}{4}}=\frac{3}{\frac{3}{4}}=3\cdot\frac{4}{3}=4.$

46. $\sum_{i=1}^{\infty}5\left(-\frac{1}{4}\right)^{i-1}$
For this geometric series, $a_1=5\left(-\frac{1}{4}\right)^{1-1}=5\left(-\frac{1}{4}\right)^0=5\cdot1=5$ and $r=-\frac{1}{4}$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{5}{1-\left(-\frac{1}{4}\right)}=\frac{5}{\frac{5}{4}}=5\cdot\frac{4}{5}=4.$

47. $\sum_{k=1}^{\infty}(.3)^k$
For this geometric series, $a_1=(.3)^1=.3$ and $r=.3$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{.3}{1-.3}=\frac{.3}{.7}=\frac{3}{7}.$

48. $\sum_{k=1}^{\infty}10^{-k}=\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k$
For this geometric series, $a_1=\left(\frac{1}{10}\right)^1=\frac{1}{10}$ and $r=\frac{1}{10}$. Because $-1<r<1$, this series converges. We have
$S_\infty=\frac{a_1}{1-r}=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{\frac{1}{10}}{\frac{9}{10}}=\frac{1}{10}\cdot\frac{10}{9}=\frac{1}{9}.$

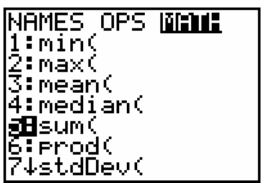
49. $g(x) = ab^x$
$g(1) = ab^1 = ab$
$g(2) = ab^2$
$g(3) = ab^3$

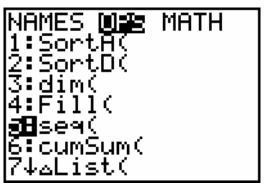
50. The sequence $g(1)$, $g(2)$, $g(3)$ is a geometric sequence because each term after the first is a constant multiple of the preceding term. The common ratio is b.

51. From Exercise 49, $a_1 = ab$. From Exercise 50, $r = b$. Therefore,
$a_n = a_1 r^{n-1} = ab(b)^{n-1} = ab^n$.

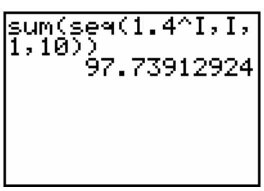
52. Answers will vary.

For Exercises 53−56, you will need the sum(and seq(features of your graphing calculator. On the TI-83, the sum(feature is located under the LIST then MATH menu. The seq(feature is under the LIST then OPS menu.

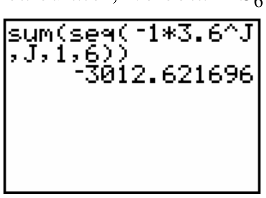




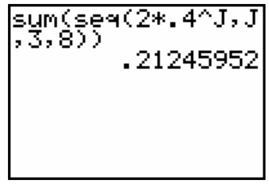
53. $\sum_{i=1}^{10} (1.4)^i$
Using the sequence feature of a graphing calculator, we obtain $S_{10} \approx 97.739$.



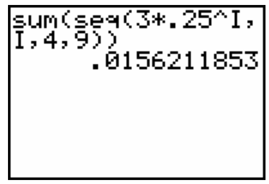
54. $\sum_{j=1}^{6} -(3.6)^j$
Using the sequence feature of a graphing calculator, we obtain $S_6 \approx -3012.622$.



55. $\sum_{j=3}^{8} 2(.4)^j$
Using the sequence feature of a graphing calculator, the sum is approximately .212.



56. $\sum_{i=4}^{9} 3(.25)^i$
Using the sequence feature of a graphing calculator, we find that the sum is approximately .016.



57. **(a)** $a_n = 1276(.916)^n$
$a_1 = 1276(.916)^1 = 1168.816 \approx 1169$
and $r = .916$

(b) $a_n = 1276(.916)^n$
$a_{10} = 1276(.916)^{10} \approx 531$
$a_{20} = 1276(.916)^{20} \approx 221$
This means that a person who is 10 years from retirement should have savings of 531% of his or her annual salary; a person 20 years from retirement should have savings of 221% of his or her annual salary.

58. **(a)** $a_n = 1278(.935)^n$
$a_1 = 1278(.935)^1 = 1194.93 \approx 1195$
and $r = .935$

(b) $a_n = 1278(.935)^n$
$a_{10} = 1278(.935)^{10} \approx 653$
$a_{20} = 1278(.935)^{20} \approx 333$
This means that a person who is 10 years from retirement should have savings of 653% of his or her annual salary; a person 20 years from retirement should have savings of 333% of his or her annual salary.

(c) A conservative investment strategy will accept less risk and therefore, earn a smaller interest rate than a moderate investment strategy. Thus, more needs to be invested to accumulate the same amount.

59. **(a)** Since the first term is a_1 and the common ratio is *r*, we have $a_n = a_1 \cdot 2^{n-1}$.

(b) If $a_1 = 100$, we have $a_n = 100 \cdot 2^{n-1}$. Since 100 = 10^2 and 1,000,000 = 10^6, we need to solve the equation $10^2 \cdot 2^{n-1} = 10^6$. Divide both sides by 10^2 to obtain $2^{n-1} = 10^4$. Take the common logarithm (base 10) of both sides and solve for *n*.

$$\log 2^{n-1} = \log 10^4 \Rightarrow (n-1)\log 2 = 4 \Rightarrow$$
$$n-1 = \frac{4}{\log 2} \Rightarrow n = \frac{4}{\log 2} + 1 \approx 14.28$$

Since the number of bacteria is increasing, the first value of n where $a_n > 1{,}000{,}000$ is 15.

(c) Since a_n represents the number of bacteria after 40(n – 1) minutes, a_{15} represents the number after $40(15-1) = 40 \cdot 14 = 560$ minutes or 9 hours, 20 minutes.

60. If 98% of the fixer is removed within 15 minutes of washing, 2% remains. We will use $a_1 = 1$ for 100% and *r* = .02 for 2%. Because initially be have 100%, we will have after one hour, *n* = 5. Thus, $a_n = a_1 r^{n-1} \Rightarrow$

$$a_5 = 1(.02)^{5-1} = (.02)^4 = .00000016 = .000016\%$$

After one hour, .000016% of the fixer remains.

61. Since there are 200 insects in the first generation, a_1 = 200. The *r* value will be 1.25. Thus, $a_n = a_1 r^{n-1} \Rightarrow$

$$a_5 = 200(1.25)^{5-1} = 200(1.25)^4 = 488.28125.$$

There are approximately 488 fruit flies in the fifth generation.

62. A machine that loses 20% of its value yearly retains 80% of its value. At the end of 6 yr, the value of the machine will be

$$100{,}000(.80)^6 = \$26{,}214.40.$$

63. Use the formula for the sum of an infinite geometric sequence with $a_1 = 1000$ and *r* = .1. Since $S_\infty = \frac{a_1}{1-r} = \frac{1000}{1-.1} = \frac{1000}{.9} = \frac{10{,}000}{9}$, the manager should have ordered $\frac{10{,}000}{9}$ units of sugar.

64. There are two sequences. A term of one is the distance the ball falls each time, and a term of the other is the distance the ball returns bouncing each time. We need the sum of both sequences.
Falling:

$$10 + 10\left(\tfrac{3}{4}\right) + 10\left(\tfrac{3}{4}\right)^2 + \cdots = \frac{10}{1-\frac{3}{4}} = \frac{10}{\frac{1}{4}} = 40$$

Bouncing: $10\left(\tfrac{3}{4}\right) + 10\left(\tfrac{3}{4}\right)^2 + \cdots = \frac{10\left(\frac{3}{4}\right)}{1-\frac{3}{4}} = 30$

Total distance = 40 + 30 = 70
The ball will travel 70 m before coming to rest.

65. Use the formula for the sum of the first *n* terms of a geometric sequence with a_1 = 2 , *r* = 2, and *n* = 5.

$$S_n = \frac{a_1(1-r^n)}{1-r} \Rightarrow$$
$$S_5 = \frac{2(1-2^5)}{1-2} = \frac{2(1-32)}{-1} = \frac{2(-31)}{-1} = 62$$

Going back five generations, the total number of ancestors is 62. Next, use the same formula with a_1 = 2 , *r* = 2, and *n* =10.

$$S_{10} = \frac{2(1-2^{10})}{1-2} = \frac{2(1-1024)}{-1} = 2046$$

Going back ten generations, the total number of ancestors is 2046.

66. Since the body uses 40% of the amount of the drug that is prescribed each day, 60% of the drug is retained.

(a) At the end of 1 day, the number of milligrams of the drug present in the body will be 2(.6).
At the end of 2 days, the amount present will be $[2 + 2(.6)](.6) = 2(.6) + 2(.6)^2$.
At the end of 3 days, the amount present will be $\left[2 + 2(.6) + 2(.6)^2\right](.6)$

$$= 2(.6) + 2(.6)^2 + 2(.6)^3$$

At the end of *n* days, the amount present will be

$$\left[2 + 2(.6) + 2(.6)^2 + \cdots + 2(.6)^{n-1}\right](.6)$$
$$= 2(.6) + 2(.6)^2 + 2(.6)^3 + \cdots + 2(.6)^n$$
$$= \sum_{i=1}^{n} 2(.6)^i$$

(b) This is a geometric series with
$a_1 = 2(.6)^1 = 2(.6) = 1.2$ and $r = .6$.
Since $S_\infty = \frac{a_1}{1-r} = \frac{1.2}{1-.6} = \frac{1.2}{.4} = 3,$ after a long period of time, approximately 3 mg of the drug will be in the body.

67. When the midpoints of the sides of an equilateral triangle are connected, the length of a side of the new triangle is one-half the length of a side of the original triangle. Use the formula for the *n*th term of a geometric sequence with $a_1 = 2$, $r = \frac{1}{2}$, and $n = 8$.

$$a_n = a_1 r^{n-1} \Rightarrow$$
$$a_8 = (2)\left(\tfrac{1}{2}\right)^{8-1} = (2)\left(\tfrac{1}{2}\right)^7 = (2)\left(\tfrac{1}{128}\right) = \tfrac{1}{64}$$

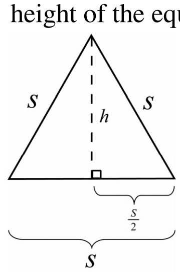
The eighth triangle has sides of length $\frac{1}{64}$ m.

68. The first triangle has sides of length 2 m, so its perimeter is $3(2) = 6$ m. The second triangle has sides of length 1 m, so its perimeter is $3(1) = 3$ m. Thus, as this process continues, we have the following.

$$3(2) + 3(1) + 3\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{4}\right) + \ldots$$
$$= 6 + 3 + \tfrac{3}{2} + \tfrac{3}{4} + \cdots$$

This is the sum of an infinite geometric sequence with $a_1 = 6$ and $r = \frac{1}{2}$. Thus, the total perimeter of all triangles is

$S_\infty = \frac{6}{1-\frac{1}{2}} = \frac{6}{\frac{1}{2}} = 12.$ The sides of these triangles are 2, 1, $\frac{1}{2}, \frac{1}{4}, \ldots$. Since the area of triangle is $\frac{1}{2}bh,$ we need to determine the height of the equilateral triangle with base *s*.



Using the Pythagorean theorem we have
$\left(\frac{s}{2}\right)^2 + h^2 = s^2 \Rightarrow \frac{s^2}{4} + h^2 = s^2 \Rightarrow$ Thus, the
$$h^2 = \tfrac{3}{4}s^2 \Rightarrow h = \tfrac{\sqrt{3}}{2}s$$
area of an equilateral triangle is
$\frac{1}{2} \cdot s \cdot \left(\frac{\sqrt{3}}{2}s\right) = \frac{s^2}{4}\sqrt{3}$, where *s* is the length of the side of the equilateral triangle.

The areas of the triangles would be
$\sqrt{3}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{16}, \ldots$. This is the sum of an infinite geometric sequence with $a_1 = \sqrt{3}$ and $r = \frac{1}{4}$.
Thus, the total area of all triangles is

$$S_\infty = \frac{a_1}{1-r} = \frac{\sqrt{3}}{1-\frac{1}{4}} = \frac{\sqrt{3}}{\frac{3}{4}} = \frac{4\sqrt{3}}{3} \text{ m}^2.$$

69. Option 1 is modeled by the arithmetic sequence $a_n = 5000 + 10{,}000(n-1)$ with the following sum.

$$S_n = \tfrac{n}{2}\left[2a_1 + (n-1)d\right]$$
$$S_{30} = \tfrac{30}{2}\left[2a_1 + (30-1)d\right] = 15\left[2a_1 + 29d\right]$$
$$= 15\left[2(5000) + 29 \cdot 10{,}000\right]$$
$$= 15\left[10{,}000 + 290{,}000\right]$$
$$S_{30} = 15(300{,}000) = 4{,}500{,}000$$

Thus, the first option will pay \$4,500,000 for the month's work. Option 2 is modeled by the sequence $a_n = .01(2)^{n-1}$ with the following sum.

$$S_n = \frac{a_1(1-r^n)}{1-r} \Rightarrow S_{30} = \frac{a_1(1-r^{30})}{1-r} \Rightarrow$$
$$S_{30} = \frac{.01(1-2^{30})}{1-2} = \frac{.01(1-1{,}073{,}741{,}824)}{-1} \Rightarrow$$
$$S_{30} = -.01(-1{,}073{,}741{,}823) = 10{,}737{,}418.23$$

The second will pay \$10,737,418.23. Option 2 pays better.

70. The first generation of ancestors will number 2, your mother and father; the second generation will number 4, your grandparents. Thus, as this process continues, we have $2 + 4 + 8 + 16 + \ldots$ This is the sum of an infinite geometric sequence with $a_1 = 2$ and $r = 2$. The sum of all your ancestors through *n* generations is given by

$S_n = \sum_{i=1}^{n} 2^i$. Thus, for 12 generations we need to find $S_{12} = \sum_{i=1}^{12} 2^i$.

$$S_n = \frac{a_1(1-r^n)}{1-r} \Rightarrow S_{12} = \frac{a_1(1-r^{12})}{1-r} \Rightarrow$$
$$S_{12} = \frac{2(1-2^{12})}{1-2} = \frac{2(1-4096)}{-1}$$
$$= -2(-4095) = 8190$$

So, you have at most 8190 ancestors.

71. The future value of an annuity uses the formula $S_n = \dfrac{a_1(1-r^n)}{1-r}$, where r = 1 + interest rate.
The payments are \$1000 for 9 yr at 3% compounded annually, so $a_1 = 1000$, $r = 1.03$, and $n = 9$.
$$S_9 = \frac{1000\left[1-(1.03)^9\right]}{1-1.03} \approx 10,159.11$$
The future value is \$10,159.11.

```
N=9
I%=3
PV=0
PMT=1000
▪FV=-10159.10613
P/Y=1
C/Y=1
PMT:END BEGIN
```
Exercise 71

```
N=12
I%=2
PV=0
PMT=800
▪FV=-10729.67178
P/Y=1
C/Y=1
PMT:END BEGIN
```
Exercise 72

72. The future value of an annuity uses the formula $S_n = \dfrac{a_1(1-r^n)}{1-r}$, where r = 1 + interest rate.
The payments are \$800 for 12 yr at 2% compounded annually, so $a_1 = 800$, $r = 1.02$, and $n = 12$.
$$S_{12} = \frac{800\left[1-(1.02)^{12}\right]}{1-1.02} \approx 10,729.67$$
The future value is \$10,729.67

73. The future value of an annuity uses the formula $S_n = \dfrac{a_1(1-r^n)}{1-r}$, where r = 1 + interest rate.
The payments are \$2430 for 10 yr at 1% compounded annually, so $a_1 = 2430$, $r = 1.01$, and $n = 10$.
$$S_{10} = \frac{2430\left[1-(1.01)^{10}\right]}{1-1.01} \approx 25,423.18$$
The future value is \$25,423.18.

```
N=10
I%=1
PV=0
PMT=2430
▪FV=-25423.17647
P/Y=1
C/Y=1
PMT:END BEGIN
```
Exercise 73

```
N=6
I%=.5
PV=0
PMT=1500
▪FV=-9113.252818
P/Y=1
C/Y=1
PMT:END BEGIN
```
Exercise 74

74. The future value of an annuity uses the formula $S_n = \dfrac{a_1(1-r^n)}{1-r}$, where r = 1 + interest rate.
The payments are \$1500 for 6 yr at .5% compounded annually, so $a_1 = 1500$, $r = 1.005$, and $n = 6$.
$$S_6 = \frac{1500\left[1-(1.005)^6\right]}{1-1.005} \approx 9113.25$$
The future value is \$9113.25.

75. $S = R\left[\dfrac{(1+i)^n - 1}{i}\right]$ where $R = 2430$, $i = .01$, and $n = 11$.
$$S = 2430\left[\frac{(1.01)^{11}-1}{.01}\right] \approx 28,107.41$$
The balance after 11 years is \$28,107.41.

```
N=11
I%=1
PV=0
PMT=2430
▪FV=-28107.40824
P/Y=1
C/Y=1
PMT:END BEGIN
```
Exercise 75

```
N=7
I%=.5
PV=0
PMT=1500
▪FV=-10658.81908
P/Y=1
C/Y=1
PMT:END BEGIN
```
Exercise 76

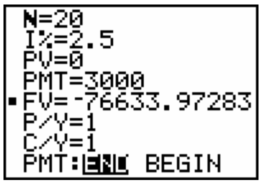
76. $S = R\left[\dfrac{(1+i)^n - 1}{i}\right]$ where $R = 1500$, $i = .005$, and $n = 7$.
$$S = 1500\left[\frac{(1.005)^7-1}{.005}\right] \approx 10,658.82$$
The balance after 7 years is \$10,658.82.

77. For deposits made annually for twenty-five years, we need to find $S = R\left[\dfrac{(1+i)^n - 1}{i}\right]$ where $R = 2000$, $i = .03$, and $n = 25$. Thus, we have $S = 2000\left[\dfrac{(1.03)^{25}-1}{.03}\right] \approx 72,918.53.$
The total amount in Michael's IRA will be \$72,918.53.

```
N=25
I%=3
PV=0
PMT=2000
▪FV=-72918.52864
P/Y=1
C/Y=1
PMT:END BEGIN
```

78. For deposits made annually for 20 years, we need to find $S = R\left[\frac{(1+i)^n - 1}{i}\right]$ where $R = 3000,\ i = .025,$ and $n = 20$. Thus, we have $S = 3000\left[\frac{(1.025)^{20} - 1}{.025}\right] \approx 76,633.97.$

After 20 years, Mort's annuity amounted to \$76,633.97.



79. In the formula $S_n = \frac{a_1(1-r^n)}{1-r}$, we are considering the sum $a_1 + a_1r + a_1r^2 + ... + a_1r^{n-1}$. With an annuity with deposits at the end of each year, this summation is actually the reversed since the last deposit will not receive interest. Reversing the summation we have the following.

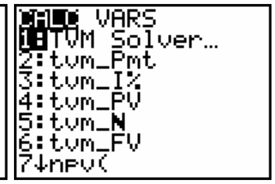
$a_1r^{n-1} + a_1r^{n-2} + ... + a_1r^2 + a_1r + a_1$

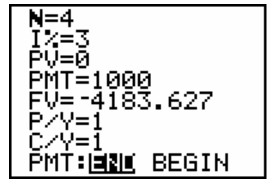
Now the first term of this series represents the first amount deposited having gone through $n-1$ compound periods (since it is deposited at the end of the first year). The value of r is $1+i$, where i is the interest rate in decimal form. The value of a_1 is the equal amounts of money that are deposited annually. If we call this value R, then the formula $S_n = \frac{a_1(1-r^n)}{1-r}$ becomes $S = \frac{R\left(1-(1+i)^n\right)}{1-(1+i)}$. This simplifies as follows.

$$S = \frac{R\left(1-(1+i)^n\right)}{1-(1+i)} \Rightarrow S = \frac{R\left(1-(1+i)^n\right)}{1-1-i} \Rightarrow$$

$$S = R\frac{1-(1+i)^n}{-i} \Rightarrow S = R\frac{(1+i)^n - 1}{i}$$

80. On the TI-83 Plus, to get to this feature, press APPS then ENTER then ENTER again.





To obtain the last screen, enter the information (except for FV, future value) and go to the FV= line and press SOLVE (ALPHA ENTER).

81. If $a_1, a_2, a_3, ...$ is a geometric sequence, then $a_n = a_1{r_a}^{n-1}$ (where r_a is the common ratio for this sequence).

$a_3 = a_1 \cdot r^2, a_5 = a_3 \cdot r^2, a_7 = a_5 \cdot r^2, ...$ So, $a_1, a_3, a_5, \ldots$ is a geometric sequence, with common ratio r^2.

82. The sequence $\log 6,\ \log 36,\ \log 1296,\ \log 1,679,616, \ldots$ can be rewritten as

$$\log 6^1,\ \log 6^2,\ \log 6^4,\ \log 6^8, \ldots$$
$$= \log 6,\ 2 \cdot \log 6,\ 4 \cdot \log 6,\ 8 \cdot \log 6, \ldots$$
$$= \log 6,\ 2^1 \cdot \log 6,\ 2^2 \cdot \log 6,\ 2^3 \cdot \log 6, \ldots$$

It has a general term of $a_n = (\log 6)2^{n-1}$.

Notice that we have $\frac{a_2}{a_1} = \frac{\log 36}{\log 6} = \frac{2\log 6}{\log 6} = 2$ and $\frac{a_3}{a_2} = \frac{\log 1296}{\log 36} = \frac{4\log 6}{2\log 6} = 2$. In general, we have

$$\frac{a_n}{a_{n-1}} = \frac{(\log 6)2^{n-1}}{(\log 6)2^{n-2}} = \frac{2^{n-1}}{2^{n-2}} = 2^{n-1-(n-2)}$$
$$= 2^{n-1-n+2} = 2^1 = 2$$

Thus, $\log 6,\ \log 36,\ \log 1296,\ \log 1,679,616, \ldots$ is a geometric sequence whose first term is $\log 6$ and common ratio is 2.

Summary Exercises on Sequences and Series

1. $2, 4, 8, 16, 32, \ldots$ is a geometric sequence with $r = 2$.
Notice $2(2) = 4,\ 4(2) = 8,\ 8(2) = 16,$ and $16(2) = 32.$

2. $1, 4, 7, 10, 13, \ldots$ is an arithmetic sequence with $d = 3$.
Notice $1 + 3 = 4,\ 4 + 3 = 7,\ 7 + 3 = 10,$ and $10 + 3 = 13.$

3. $3, \frac{1}{2}, -2, -\frac{9}{2}, -7, \ldots$ is an arithmetic sequence with $d = -\frac{5}{2}$.
Notice $3 + \left(-\frac{5}{2}\right) = \frac{1}{2},\ \frac{1}{2} + \left(-\frac{5}{2}\right) = -2,$ $-2 + \left(-\frac{5}{2}\right) = -\frac{9}{2},$ and $-\frac{9}{2} + \left(-\frac{5}{2}\right) = -7.$

4. $1, -2, 3, -4, -5, \ldots$ is neither an arithmetic sequence nor a geometric sequence.
Notice $1 + (-3) = -2$ but $-2 + (-3) \neq 3,$ so the sequence is not arithmetic. Also, $1(-2) = -2$ but $-2(-2) \neq 3,$ so the sequence is not geometric.

5. $\frac{3}{4}, 1, \frac{4}{3}, \frac{16}{9}, \frac{64}{27}, \ldots$ is a geometric sequence with $r = \frac{4}{3}$.
Notice $\frac{3}{4}\left(\frac{4}{3}\right) = 1, 1\left(\frac{4}{3}\right) = \frac{4}{3}, \frac{4}{3}\left(\frac{4}{3}\right) = \frac{16}{9},$ and $\frac{16}{9}\left(\frac{4}{3}\right) = \frac{64}{27}.$

6. $4, -12, 36, -108, 324, \ldots$ is a geometric sequence with $r = -3$.
Notice $4(-3) = -12, -12(-3) = 36,$ $36(-3) = -108,$ and $-108(-3) = 324.$

7. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ is neither an arithmetic sequence nor a geometric sequence.
Notice $\frac{1}{2} + \left(-\frac{1}{6}\right) = \frac{1}{3}$ but $\frac{1}{3} + \left(-\frac{1}{6}\right) \neq \frac{1}{4},$ so the sequence is not arithmetic. Also, $\frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$ but $\frac{1}{3}\left(\frac{2}{3}\right) \neq \frac{1}{4},$ so the sequence is not geometric.

8. $5, 2, -1, -4, -7, \ldots$ is an arithmetic sequence with $d = -3$.
Notice $5 + (-3) = 2,\ 2 + (-3) = -1,$ $-1 + (-3) = -4,$ and $-4 + (-3) = -7.$

9. $1, 9, 10, 19, 29, \ldots$ is neither an arithmetic sequence nor a geometric sequence.
Notice $1 + 8 = 9$, but $9 + 8 \neq 10$, so the sequence is not arithmetic. Also, $1 \cdot 9 = 9$, but $9 \cdot 9 \neq 10$, so the sequence is not geometric.

10. For the arithmetic sequence 5, 5, 5, …, $d = 0$.
For the geometric sequence 5, 5, 5, …, $r = 1$.

11. The sequence $3, 6, 12, 24, 48, \ldots$ is geometric with $a_1 = 3$ and $r = 2.$ Thus, we have

$a_n = a_1 r^{n-1} \Rightarrow a_n = 3(2)^{n-1}$

Since $\sum_{i=1}^{n} a_1 r^{i-1} = S_n = \frac{a_1(1 - r^n)}{1 - r},$ we have

$$\sum_{i=1}^{10} 3(2)^{i-1} = S_{10} = \frac{3(1 - 2^{10})}{1 - 2} = \frac{3(1 - 1024)}{-1} = -3(-1023) = 3069$$

12. The sequence $2, 6, 10, 14, 18, \ldots$ is arithmetic with $a_1 = 2$ and $d = 4.$ Thus, we have $a_n = a_1 + (n-1)d \Rightarrow a_n = 2 + (n-1)4 \Rightarrow$ $a_n = 2 + 4n - 4 \Rightarrow a_n = -2 + 4n$

Since

$$\sum_{i=1}^{n} \left[a_1 + (i-1)d\right] = S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right],$$

we have

$$\sum_{i=1}^{10} -2 + 4i = S_{10} = \frac{10}{2}\left[2(2) + (10-1)4\right] = 5\left[4 + 9 \cdot 4\right] = 5(4 + 36) = 5(40) = 200$$

13. The sequence $4, \frac{5}{2}, 1, -\frac{1}{2}, -2, \ldots$ is arithmetic with $a_1 = 4$ and $d = -\frac{3}{2}.$ Thus, we have

$a_n = a_1 + (n-1)d \Rightarrow a_n = 4 + (n-1)\left(-\frac{3}{2}\right) \Rightarrow$
$a_n = 4 - \frac{3}{2}n + \frac{3}{2} \Rightarrow a_n = \frac{11}{2} - \frac{3}{2}n$

Since

$$\sum_{i=1}^{n} \left[a_1 + (i-1)d\right] = S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right],$$

we have

$$\sum_{i=1}^{10} \frac{11}{2} - \frac{3}{2}i = S_{10} = \frac{10}{2}\left[2(4) + (10-1)\left(-\frac{3}{2}\right)\right] = 5\left[8 + 9\left(-\frac{3}{2}\right)\right] = 5\left[8 + \left(-\frac{27}{2}\right)\right] = 5\left(-\frac{11}{2}\right) = -\frac{55}{2}$$

14. The sequence $\frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$ is geometric with $a_1 = \frac{3}{2}$ and $r = \frac{2}{3}$. Thus, we have

$a_n = a_1 r^{n-1} \Rightarrow a_n = \frac{3}{2}\left(\frac{2}{3}\right)^{n-1}$

Since $\sum_{i=1}^{n} a_1 r^{i-1} = S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{i=1}^{10} \frac{3}{2}\left(\frac{2}{3}\right)^{i-1} = S_{10} = \frac{\frac{3}{2}\left[1-\left(\frac{2}{3}\right)^{10}\right]}{1-\frac{2}{3}} = \frac{\frac{3}{2}\left(1-\frac{1024}{59,049}\right)}{\frac{1}{3}}$$
$$= \frac{9}{2}\left(\frac{58,025}{59,049}\right) = \frac{58,025}{13,122}$$

15. The sequence $3, -6, 12, -24, 48, \ldots$ is geometric with $a_1 = 3$ and $r = -2$. Thus, we have

$a_n = a_1 r^{n-1} \Rightarrow a_n = 3(-2)^{n-1}$

Since $\sum_{i=1}^{n} a_1 r^{i-1} = S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{i=1}^{10} 3(2)^{i-1} = S_{10} = \frac{3\left[1-(-2)^{10}\right]}{1-(-2)}$$
$$= \frac{3(1-1024)}{3} = -1023$$

16. The sequence $-5, -8, -11, -14, -17, \ldots$ is arithmetic with $a_1 = -5$ and $d = -3$. Thus, we have $a_n = a_1 + (n-1)d \Rightarrow$

$a_n = -5 + (n-1)(-3) \Rightarrow a_n = -5 - 3n + 3 \Rightarrow$

$a_n = -2 - 3n$

Since

$\sum_{i=1}^{n} \left[a_1 + (i-1)d\right] = S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$,

we have

$$\sum_{i=1}^{10} -2-3i = S_{10} = \frac{10}{2}\left[2(-5) + (10-1)(-3)\right]$$
$$= 5\left[-10 + 9(-3)\right] = 5\left[-10 + (-27)\right]$$
$$= 5(-37) = -185$$

17. $\sum_{i=1}^{\infty} \frac{1}{3}(-2)^{i-1}$ is an infinite geometric series with $a_1 = \frac{1}{3}$ and $r = -2$. Since $|r| = 2 > 1$, this series diverges.

18. $\sum_{j=1}^{4} 2\left(\frac{1}{10}\right)^{j-1}$ is a geometric series with $a_1 = 2$ and $r = \frac{1}{10}$. Since $\sum_{i=1}^{n} a_1 r^{i-1} = S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{j=1}^{4} 2\left(\frac{1}{10}\right)^{j-1} = S_4 = \frac{2\left[1-\left(\frac{1}{10}\right)^4\right]}{1-\frac{1}{10}} = \frac{2\left(1-\frac{1}{10,000}\right)}{\frac{9}{10}}$$
$$= 2\cdot\left(\frac{10}{9}\right)\left(\frac{9999}{10,000}\right) = \frac{1111}{500}$$

19. $\sum_{i=1}^{25} (4-6i)$ is an arithmetic series with $a_1 = -2$ and $d = -6$. Since the sum of the first n terms of an arithmetic series is $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{i=1}^{25} (4-6i) = S_{25} = \frac{25}{2}\left[2(-2) + (25-1)(-6)\right]$$
$$= \frac{25}{2}\left[-4 + 24(-6)\right]$$
$$= \frac{25}{2}\left[-4 + (-144)\right] = \frac{25}{2}(-148)$$
$$= -1850$$

20. $\sum_{i=1}^{6} 3^i$ is a geometric series with $a_1 = 3$ and $r = 3$. Since the sum of the first n terms of a geometric series is $S_n = \frac{a_1(1-r^n)}{1-r}$, we have

$$\sum_{i=1}^{6} 3^i = S_6 = \frac{3(1-3^6)}{1-3} = \frac{3(1-729)}{-2} = \frac{3(-728)}{-2}$$
$$= 3\cdot 364 = 1092$$

21. $\sum_{i=1}^{\infty} 4\left(-\frac{1}{2}\right)^i$ is an infinite geometric series with $a_1 = -2$ and $r = -\frac{1}{2}$. Since $|r| = \frac{1}{2} < 1$, then the sum of this series is

$$S = \frac{a_1}{1-r} = \frac{-2}{1-\left(-\frac{1}{2}\right)} = \frac{-2}{\frac{3}{2}} = -\frac{4}{3}.$$

22. $\sum_{i=1}^{\infty}(3i-2)$ is an infinite arithmetic series. Since there are an infinite number of terms that get larger as i increments, the series diverges.

23. $\sum_{j=1}^{12}(2j-1)$ is an arithmetic series with $a_1 = 1$ and $d = 2$. To use the formula $S_n = \frac{n}{2}(a_1 + a_n)$, we need find a_{12} (the last term). Since $a_{12} = 2(12)-1 = 24-1 = 23$, we have

$$\sum_{j=1}^{12}(2j-1) = S_{12} = \frac{12}{2}(1+23) = 6(24) = 144$$

24. $\sum_{k=1}^{\infty}3^{-k} = \sum_{k=1}^{\infty}\left(\tfrac{1}{3}\right)^k$ is an infinite geometric series with $a_1 = \frac{1}{3}$ and $r = \frac{1}{3}$. Since $|r| = \frac{1}{3} < 1$, then the sum of this series is

$$S = \frac{a_1}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}.$$

25. $\sum_{k=1}^{\infty}1.001^i$ is an infinite geometric series with $a_1 = 1.001$ and $r = 1.001$ Since $|r| = 1.001 > 1$, this series diverges.

26. $.999\ldots = .9 + .09 + .009 + \cdots$
This geometric series has $a_1 = .9$ and $r = .1$.
Thus $S = \frac{a_1}{1-r} = \frac{.9}{1-.1} = \frac{.9}{.9} = 1$

Section 11.4: The Binomial Theorem

1. $\frac{6!}{3!3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot3\cdot2\cdot1} = \frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 20$

2. $\frac{5!}{2!3!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = \frac{5\cdot4}{2\cdot1} = 10$

3. $\frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$

4. $\frac{8!}{5!3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$

5. $\binom{8}{5} = \frac{8!}{5!3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$

6. $\binom{7}{3} = \frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1} = 35$

7. $\binom{10}{2} = \frac{10!}{2!8!} = \frac{10\cdot9\cdot8!}{2\cdot1\cdot8!} = \frac{10\cdot9}{2\cdot1} = 45$

8. $\binom{9}{3} = \frac{9!}{3!6!} = \frac{9\cdot8\cdot7\cdot6!}{3\cdot2\cdot1\cdot6!} = \frac{9\cdot8\cdot7}{3\cdot2\cdot1} = 84$

9. $\binom{14}{14} = \frac{14!}{14!(14-14)!} = \frac{14!}{14!0!} = \frac{14!}{14!\cdot1} = 1$

10. $\binom{15}{15} = \frac{15!}{15!(15-15)!} = \frac{15!}{15!0!} = \frac{15!}{15!\cdot1} = 1$

11.
$$\binom{n}{n-1} = \frac{n!}{(n-1)![n-(n-1)]!} = \frac{n!}{(n-1)!(n-n+1)!} = \frac{n!}{(n-1)!1!} = \frac{n(n-1)!}{(n-1)!} = n$$

12.
$$\binom{n}{n-2} = \frac{n!}{(n-2)![n-(n-2)]!} = \frac{n!}{(n-2)!(n-n+2)!} = \frac{n!}{(n-1)!2!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2}$$

13. ${}_8C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$

14. ${}_9C_7 = \frac{9!}{7!(9-7)!} = \frac{9!}{7!2!} = \frac{9\cdot8}{2\cdot1} = 36$

15. ${}_{100}C_{98} = \frac{100!}{98!(100-98)!} = \frac{100!}{98!2!} = \frac{100\cdot99}{2\cdot1} = 4950$

16. ${}_{20}C_5 = \frac{20!}{5!(20-5)!} = \frac{20!}{5!15!} = \frac{20\cdot19\cdot18\cdot17\cdot16}{5\cdot4\cdot3\cdot2\cdot1} = 15{,}504$

17. ${}_9C_0 = \frac{9!}{0!(9-0)!} = \frac{9!}{1\cdot9!} = 1$

18. ${}_5C_1 = \frac{5!}{1!(5-1)!} = \frac{5!}{1!4!} = \frac{5}{1} = 5$

19. ${}_{12}C_1 = \dfrac{12!}{1!(12-1)!} = \dfrac{12\cdot 11!}{1!11!} = \dfrac{12}{1} = 12$

20. ${}_{4}C_0 = \dfrac{4!}{0!(4-0)!} = \dfrac{4!}{1\cdot 4!} = 1$

21. The first term in the expansion of $(2x+3y)^4$ is $(2x)^4 = 16x^4$ and the last is $(3y)^4 = 81y^4$.

22. The binomial coefficient for the fifth term in the expansion of $(x+y)^9$ is $\dbinom{n}{k-1}$ where $n=9$ and $k=5$. Thus, we have $\dbinom{n}{k-1} = \dbinom{9}{5-1} = \dbinom{9}{4} = \dfrac{9!}{5!4!}$ as the

$$= \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 126$$

coefficient of the fifth term. The actual term would be $126x^{9-(5-1)}y^{5-1} = 126x^5y^4$.

23.
$$\begin{aligned}(x+y)^6 &= x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + y^6\\ &= x^6 + \frac{6!}{1!5!}x^5y + \frac{6!}{2!4!}x^4y^2 + \frac{6!}{3!3!}x^3y^3 + \frac{6!}{4!2!}x^2y^4 + \frac{6!}{5!1!}xy^5 + y^6\\ &= x^6 + 6x^5y + \frac{6\cdot 5}{2\cdot 1}x^4y^2 + \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}x^3y^3 + \frac{6\cdot 5}{2\cdot 1}x^2y^4 + 6xy^5 + y^6\\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\end{aligned}$$

24.
$$\begin{aligned}(m+n)^4 &= m^4 + \binom{4}{1}m^3n + \binom{4}{2}m^2n^2 + \binom{4}{3}mn^3 + n^4 = m^4 + \frac{4!}{1!3!}m^3n + \frac{4!}{2!2!}m^2n^2 + \frac{4!}{3!1!}mn^3 + n^4\\ &= m^4 + 4m^3n + \frac{4\cdot 3}{2\cdot 1}m^2n^2 + 4mn^3 + n^4 = m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4\end{aligned}$$

25.
$$\begin{aligned}(p-q)^5 &= \left[p+(-q)\right]^5\\ &= p^5 + \binom{5}{1}p^4(-q) + \binom{5}{2}p^3(-q)^2 + \binom{5}{3}p^2(-q)^3 + \binom{5}{4}p(-q)^4 + (-q)^5\\ &= p^5 + \frac{5!}{1!4!}p^4(-q) + \frac{5!}{2!3!}p^3q^2 + \frac{5!}{3!2!}p^2(-q)^3 + \frac{5!}{4!1!}pq^4 + (-q)^5\\ &= p^5 + 5p^4(-q) + \frac{5\cdot 4}{2\cdot 1}p^3q^2 + \frac{5\cdot 4}{2\cdot 1}p^2(-q)^3 + 5pq^4 + (-q)^5\\ &= p^5 - 5p^4q + 10p^3q^2 - 10p^2q^3 + 5pq^4 - q^5\end{aligned}$$

26.
$$\begin{aligned}(a-b)^7 &= \left[a+(-b)\right]^7\\ &= a^7 + \binom{7}{1}a^6(-b) + \binom{7}{2}a^5(-b)^2 + \binom{7}{3}a^4(-b)^3 + \binom{7}{4}a^3(-b)^4 + \binom{7}{5}a^2(-b)^5 + \binom{7}{6}a(-b)^6 + (-b)^7\\ &= a^7 + \frac{7!}{1!6!}a^6(-b) + \frac{7!}{2!5!}a^5b^2 + \frac{7!}{3!4!}a^4(-b)^3 + \frac{7!}{4!3!}a^3b^4 + \frac{7!}{5!2!}a^2(-b)^5 + \frac{7!}{6!1!}ab^6 - b^7\\ &= a^7 + 7a^6(-b) + \frac{7\cdot 6}{2\cdot 1}a^5b^2 + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}a^4(-b)^3 + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}a^3b^4 + \frac{7\cdot 6}{2\cdot 1}a^2(-b)^5 + 7ab^6 - b^7\\ &= a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7\end{aligned}$$

27.
$$\begin{aligned}(r^2+s)^5 &= (r^2)^5 + \binom{5}{1}(r^2)^4s + \binom{5}{2}(r^2)^3s^2 + \binom{5}{3}(r^2)^2s^3 + \binom{5}{4}(r^2)s^4 + s^5\\ &= r^{10} + \frac{5!}{1!4!}r^8s + \frac{5!}{2!3!}r^6s^2 + \frac{5!}{3!2!}r^4s^3 + \frac{5!}{4!1!}r^2s^4 + s^5\\ &= r^{10} + 5r^8s + \frac{5\cdot 4}{2\cdot 1}r^6s^2 + \frac{5\cdot 4}{2\cdot 1}r^4s^3 + 5r^2s^4 + s^5\\ &= r^{10} + 5r^8s + 10r^6s^2 + 10r^4s^3 + 5r^2s^4 + s^5\end{aligned}$$

28. $(m+n^2)^4 = m^4 + \binom{4}{1}m^3n^2 + \binom{4}{2}m^2(n^2)^2 + \binom{4}{3}m(n^2)^3 + (n^2)^4$

$= m^4 + \frac{4!}{1!3!}m^3n^2 + \frac{4!}{2!2!}m^2n^4 + \frac{4!}{3!1!}mn^6 + n^8 = m^4 + 4m^3n^2 + \frac{4\cdot 3}{2\cdot 1}m^2n^4 + 4mn^6 + n^8$

$= m^4 + 4m^3n^2 + 6m^2n^4 + 4mn^6 + n^8$

29. $(p+2q)^4 = p^4 + \binom{4}{1}p^3(2q) + \binom{4}{2}p^2(2q)^2 + \binom{4}{3}p(2q)^3 + (2q)^4$

$= p^4 + \frac{4!}{1!3!}p^3(2q) + \frac{4!}{2!2!}p^2(4q^2) + \frac{4!}{3!1!}p(8q^3) + 16q^4$

$= p^4 + 4p^3(2q) + \frac{4\cdot 3}{2\cdot 1}p^2(4q^2) + 4p(8q^3) + 16q^4 = p^4 + 8p^3q + 24p^2q^2 + 32pq^3 + 16q^4$

30. $(3r-s)^6 = [3r+(-s)]^6$

$= (3r)^6 + \binom{6}{1}(3r)^5(-s) + \binom{6}{2}(3r)^4(-s)^2 + \binom{6}{3}(3r)^3(-s)^3 + \binom{6}{4}(3r)^2(-s)^4 + \binom{6}{5}3r(-s)^5 + (-s)^6$

$= 729r^6 + \frac{6!}{1!5!}(243r^5)(-s) + \frac{6!}{2!4!}(81r^4)s^2 + \frac{6!}{3!3!}(27r^3)(-s)^3 + \frac{6!}{4!2!}(9r^2)s^4 + \frac{6!}{5!1!}(3r)(-s)^5 + s^6$

$= 729r^6 - 6(243)r^5s + \frac{6\cdot 5}{2\cdot 1}(81)r^4s^2 - \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}(27)r^3s^3 + \frac{6\cdot 5}{2\cdot 1}(9)r^2s^4 - 6(3)rs^5 + s^6$

$= 729r^6 - 1458r^5s + 15(81)r^4s^2 - 20(27)r^3s^3 + 15(9)r^2s^4 - 18rs^5 + s^6$

$= 729r^6 - 1458r^5s + 1215r^4s^2 - 540r^3s^3 + 135r^2s^4 - 18rs^5 + s^6$

31. $(7p+2q)^4 = (7p)^4 + \binom{4}{1}(7p)^3(2q) + \binom{4}{2}(7p)^2(2q)^2 + \binom{4}{3}(7p)(2q)^3 + (2q)^4$

$= 2401p^4 + \frac{4!}{1!3!}(343p^3)(2q) + \frac{4!}{2!2!}(49p^2)(4q^2) + \frac{4!}{3!1!}(7p)(8q^3) + 16q^4$

$= 2401p^4 + 4(686p^3q) + \frac{4\cdot 3}{2\cdot 1}(196p^2q^2) + 4(56pq^3) + 16q^4$

$= 2401p^4 + 2744p^3q + 6(196p^2q^2) + 224pq^3 + 16q^4$

$= 2401p^4 + 2744p^3q + 1176p^2q^2 + 224pq^3 + 16q^4$

32. $(4a-5b)^5 = (4a)^5 + \binom{5}{1}(4a)^4(-5b) + \binom{5}{2}(4a)^3(-5b)^2 + \binom{5}{3}(4a)^2(-5b)^3 + \binom{5}{4}(4a)(-5b)^4 + (-5b)^5$

$= 1024a^5 + \frac{5!}{1!4!}(256a^4)(-5b) + \frac{5!}{2!3!}(64a^3)(25b^2) + \frac{5!}{3!2!}(16a^2)(-125b^3)$

$+ \frac{5!}{4!1!}(4a)(625b^4) + (-3125b^5)$

$= 1024a^5 - 5(1280a^4b) + \frac{5\cdot 4}{2\cdot 1}(1600a^3b^2) - \frac{5\cdot 4}{2\cdot 1}(2000a^2b^3) + 5(2500ab^4) - 3125b^5$

$= 1024a^5 - 6400a^4b + 10(1600a^3b^2) - 10(2000a^2b)^3 + 12{,}500ab^4 - 3125b^5$

$= 1024a^5 - 6400a^4b + 16{,}000a^3b^2 - 20{,}000a^2b^3 + 12{,}500ab^4 - 3125b^5$

33. $$\begin{aligned}(3x-2y)^6 &= (3x)^6+\binom{6}{1}(3x)^5(-2y)+\binom{6}{2}(3x)^4(-2y)^2+\binom{6}{3}(3x)^3(-2y)^3\\ &\quad+\binom{6}{4}(3x)^2(-2y)^4+\binom{6}{5}(3x)(-2y)^5+(-2y)^6\\ &= 729x^6+\frac{6!}{1!5!}(243x^5)(-2y)+\frac{6!}{2!4!}(81x^4)(4y^2)+\frac{6!}{3!3!}(27x^3)(-8y^3)\\ &\quad+\frac{6!}{4!2!}(9x^2)(16y^4)+\frac{6!}{5!1!}(3x)(-32y^5)+64y^6\\ &= 729x^6-6(486x^5y)+\frac{6\cdot5}{2\cdot1}(324x^4y^2)-\frac{6\cdot5\cdot4}{3\cdot2\cdot1}(216x^3y^3)\\ &\quad+\frac{6\cdot5}{2\cdot1}(144x^2y^4)-6(96xy^5)+64y^6\\ &= 729x^6-2916x^5y+15(324x^4y^2)-20(216x^3y^3)+15(144x^2y^4)-576xy^5+64y^6\\ &= 729x^6-2916x^5y+4860x^4y^2-4320x^3y^3+2160x^2y^4-576xy^5+64y^6\end{aligned}$$

34. $$\begin{aligned}(7k-9j)^4 &= (7k)^4+\binom{4}{1}(7k)^3(-9j)+\binom{4}{2}(7k)^2(-9j)^2+\binom{4}{3}(7k)(-9j)^3+(-9j)^4\\ &= 2401k^4+\frac{4!}{1!3!}(343k^3)(-9j)+\frac{4!}{2!2!}(49k^2)(81j^2)+\frac{4!}{3!1!}(7k)(-729j^3)+6561j^4\\ &= 2401k^4-4(3087k^3j)+\frac{4\cdot3}{2\cdot1}(3969k^2j^2)-4(5103kj^3)+6561j^4\\ &= 2401k^4-12,348k^3j+6(3969k^2j^2)-20,412kj^3+6561j^4\\ &= 2401k^4-12,348k^3j+23,814k^2j^2-20,412kj^3+6561j^4\end{aligned}$$

35. $$\begin{aligned}\left(\frac{m}{2}-1\right)^6 &= \left(\frac{m}{2}\right)^6+\binom{6}{1}\left(\frac{m}{2}\right)^5(-1)+\binom{6}{2}\left(\frac{m}{2}\right)^4(-1)^2+\binom{6}{3}\left(\frac{m}{2}\right)^3(-1)^3\\ &\quad+\binom{6}{4}\left(\frac{m}{2}\right)^2(-1)^4+\binom{6}{5}\left(\frac{m}{2}\right)(-1)^5+(-1)^6\\ &= \frac{m^6}{64}-\frac{6!}{1!5!}\left(\frac{m^5}{32}\right)+\frac{6!}{2!4!}\left(\frac{m^4}{16}\right)-\frac{6!}{3!3!}\left(\frac{m^3}{8}\right)+\frac{6!}{4!2!}\left(\frac{m^2}{4}\right)-\frac{6!}{5!1!}\left(\frac{m}{2}\right)+1\\ &= \frac{m^6}{64}-6\left(\frac{m^5}{32}\right)+\frac{6\cdot5}{2\cdot1}\left(\frac{m^4}{16}\right)-\frac{6\cdot5\cdot4}{3\cdot2\cdot1}\left(\frac{m^3}{8}\right)+\frac{6\cdot5}{2\cdot1}\left(\frac{m^2}{4}\right)-6\left(\frac{m}{2}\right)+1\\ &= \frac{m^6}{64}-\frac{3m^5}{16}+15\left(\frac{m^4}{16}\right)-20\left(\frac{m^3}{8}\right)+15\left(\frac{m^2}{4}\right)-3m+1\\ &= \frac{m^6}{64}-\frac{3m^5}{16}+\frac{15m^4}{16}-\frac{5m^3}{2}+\frac{15m^2}{4}-3m+1\end{aligned}$$

36. $$\begin{aligned}\left(3+\frac{y}{3}\right)^5 &= 3^5+\binom{5}{1}(3^4)\left(\frac{y}{3}\right)+\binom{5}{2}(3^3)\left(\frac{y}{3}\right)^2+\binom{5}{3}(3^2)\left(\frac{y}{3}\right)^3+\binom{5}{4}(3)\left(\frac{y}{3}\right)^4+\left(\frac{y}{3}\right)^5\\ &= 243+\frac{5!}{1!4!}(81)\left(\frac{y}{3}\right)+\frac{5!}{2!3!}(27)\left(\frac{y^2}{9}\right)+\frac{5!}{3!2!}(9)\left(\frac{y^3}{27}\right)+\frac{5!}{4!1!}(3)\left(\frac{y^4}{81}\right)+\frac{y^5}{243}\\ &= 243+5(27y)+\frac{5\cdot4}{2\cdot1}(3y^2)+\frac{5\cdot4}{2\cdot1}\left(\frac{y^3}{3}\right)+\frac{5y^4}{27}+\frac{y^5}{243}\\ &= 243+135y+10(3y^2)+10\left(\frac{y^3}{3}\right)+\frac{5y^4}{27}+\frac{y^5}{243}\\ &= 243+135y+30y^2+\frac{10y^3}{3}+\frac{5y^4}{27}+\frac{y^5}{243}\end{aligned}$$

37. $\left(\sqrt{2}r+\frac{1}{m}\right)^4=\left(\sqrt{2}r\right)^4+\binom{4}{1}\left(\sqrt{2}r\right)^3\left(\frac{1}{m}\right)+\binom{4}{2}\left(\sqrt{2}r\right)^2\left(\frac{1}{m}\right)^2+\binom{4}{3}\left(\sqrt{2}r\right)\left(\frac{1}{m}\right)^3+\left(\frac{1}{m}\right)^4$

$$=\left(\sqrt{2}r\right)^4+\frac{4!}{1!3!}\left(\sqrt{2}r\right)^3\left(\frac{1}{m}\right)+\frac{4!}{2!2!}\left(\sqrt{2}r\right)^2\frac{1}{m^2}+\frac{4!}{3!1!}\left(\sqrt{2}r\right)\frac{1}{m^3}+\frac{1}{m^4}$$

$$=4r^4+4\left(2\sqrt{2}\right)r^3\left(\frac{1}{m}\right)+\frac{4\cdot 3}{2\cdot 1}\left(2r^2\right)\frac{1}{m^2}+4\left(\sqrt{2}r\right)\frac{1}{m^3}+\frac{1}{m^4}$$

$$=4r^4+\frac{8\sqrt{2}r^3}{m}+(6)\frac{2r^2}{m^2}+\frac{4\sqrt{2}r}{m^3}+\frac{1}{m^4}$$

$$=4r^4+\frac{8\sqrt{2}r^3}{m}+\frac{12r^2}{m^2}+\frac{4\sqrt{2}r}{m^3}+\frac{1}{m^4}$$

38. $\left(\frac{1}{k}-\sqrt{3}p\right)^3=\left(\frac{1}{k}\right)^3+\binom{3}{1}\left(\frac{1}{k}\right)^2\left(-\sqrt{3}p\right)+\binom{3}{2}\left(\frac{1}{k}\right)\left(-\sqrt{3}p\right)^2+\left(-\sqrt{3}p\right)^3$

$$=\frac{1}{k^3}-\frac{3!}{1!2!}\left(\frac{1}{k^2}\right)\left(\sqrt{3}p\right)+\frac{3!}{2!1!}\left(\frac{1}{k}\right)\left(3p^2\right)-3\sqrt{3}p^3$$

$$=\frac{1}{k^3}-3\left(\frac{\sqrt{3}p}{k^2}\right)+3\left(\frac{3p^2}{k}\right)-3\sqrt{3}p^3=\frac{1}{k^3}-\frac{3\sqrt{3}p}{k^2}+\frac{9p^2}{k}-3\sqrt{3}p^3$$

39. $\left(\frac{1}{x^4}+x^4\right)^4=\left(\frac{1}{x^4}\right)^4+\binom{4}{1}\left(\frac{1}{x^4}\right)^3\left(x^4\right)+\binom{4}{2}\left(\frac{1}{x^4}\right)^2\left(x^4\right)^2+\binom{4}{3}\left(\frac{1}{x^4}\right)\left(x^4\right)^3+\left(x^4\right)^4$

$$=\frac{1}{x^{16}}+\frac{4!}{1!3!}\left(\frac{1}{x^{12}}\right)\left(x^4\right)+\frac{4!}{2!2!}\left(\frac{1}{x^8}\right)\left(x^8\right)+\frac{4!}{3!1!}\left(\frac{1}{x^4}\right)\left(x^{12}\right)+x^{16}$$

$$=\frac{1}{x^{16}}+4\left(\frac{1}{x^8}\right)+6+4x^8+x^{16}=\frac{1}{x^{16}}+\frac{4}{x^8}+6+4x^8+x^{16}$$

40. $\left(\frac{1}{y^5}-y^5\right)^5=\left(\frac{1}{y^5}\right)^5-\binom{5}{1}\left(\frac{1}{y^5}\right)^4\left(y^5\right)+\binom{5}{2}\left(\frac{1}{y^5}\right)^3\left(y^5\right)^2-\binom{5}{3}\left(\frac{1}{y^5}\right)^2\left(y^5\right)^3+\binom{5}{4}\left(\frac{1}{y^5}\right)\left(y^5\right)^4-\left(y^5\right)^5$

$$=\frac{1}{y^{25}}-\frac{5!}{1!4!}\left(\frac{1}{y^{20}}\right)\left(y^5\right)+\frac{5!}{2!3!}\left(\frac{1}{y^{15}}\right)\left(y^{10}\right)-\frac{5!}{3!2!}\left(\frac{1}{y^{10}}\right)\left(y^{15}\right)+\frac{5!}{4!1!}\left(\frac{1}{y^5}\right)\left(y^{20}\right)-y^{25}$$

$$=\frac{1}{y^{25}}-5\left(\frac{1}{y^{15}}\right)+10\left(\frac{1}{y^5}\right)-10y^5+5y^{15}-y^{25}=\frac{1}{y^{25}}-\frac{5}{y^{15}}+\frac{10}{y^5}-10y^5+5y^{15}-y^{25}$$

41. $(4h-j)^8$

Using the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n=8$, $k-1=5$, and $n-(k-1)=3$, the sixth term of the expansion is

$$\binom{8}{5}(4h)^3(-j)^5=\frac{8!}{5!3!}\left(64h^3\right)\left(-j^5\right)$$
$$=\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}\left(64h^3\right)\left(-j^5\right)$$
$$=56\left(-64h^3j^5\right)=-3584h^3j^5$$

42. $(2c-3d)^{14}$

Use the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n=14$, $k=8$, $k-1=7$, and $n-(k-1)=7$. The eighth term of the expansion is

$$\binom{14}{7}(2c)^7(-3d)^7$$
$$=\frac{14!}{7!7!}\left(2^7c^7\right)\left(-3^7d^7\right)$$
$$=\frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\left(-6^7c^7d^7\right)$$
$$=-3432\cdot 6^7c^7d^7$$

43. $\left(a^2+b\right)^{22}$

Use the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 22$, $k = 15$, $k - 1 = 14$, and $n - (k - 1) = 8$. The fifteenth term of the expansion is

$$\begin{aligned}\binom{22}{14}\left(a^2\right)^8(b)^{14} &= \frac{22!}{14!8!}\left(a^{16}\right)\left(b^{14}\right)\\ &= \frac{22\cdot21\cdot20\cdot19\cdot18\cdot17\cdot16\cdot15}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}\left(a^{16}b^{14}\right)\\ &= 319{,}770a^{16}b^{14}\end{aligned}$$

44. $\left(2x+y^2\right)^{16}$

Use the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 16$, $k = 12$, $k - 1 = 11$, and $n - (k - 1) = 5$. The twelfth term of the expansion is

$$\begin{aligned}\binom{16}{11}(2x)^5\left(y^2\right)^{11} &= \frac{16!}{11!5!}\left(2^5x^5\right)\left(y^{22}\right)\\ &= \frac{16\cdot15\cdot14\cdot13\cdot12}{5\cdot4\cdot3\cdot2\cdot1}\left(32x^5\right)\left(y^{22}\right)\\ &= 4368\left(32x^5y^{22}\right) = 139{,}776x^5y^{22}\end{aligned}$$

45. $\left(x-y^3\right)^{20}$

Use the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 20$, $k = 15$, $k - 1 = 14$, and $n - (k - 1) = 6$. The fifteenth term of the expansion is

$$\begin{aligned}\binom{20}{14}(x)^6\left(-y^3\right)^{14} &= \frac{20!}{14!6!}\left(x^6\right)\left(y^{42}\right)\\ &= \frac{20\cdot19\cdot18\cdot17\cdot16\cdot15}{6\cdot5\cdot4\cdot3\cdot2\cdot1}\left(x^6y^{42}\right)\\ &= 38{,}760x^6y^{42}\end{aligned}$$

46. $\left(a^3+3b\right)^{11}$

Using the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 11$, $k = 10$, $k - 1 = 9$, and $n - (k - 1) = 2$, the tenth term of the expansion is

$$\begin{aligned}\binom{11}{9}\left(a^3\right)^2(3b)^9 &= \frac{11!}{9!2!}a^6\left(3^9b^9\right)\\ &= \frac{11\cdot10}{2\cdot1}\left(3^9a^6b^9\right) = 55\cdot3^9a^6b^9\end{aligned}$$

47. $\left(3x^7+2y^3\right)^8$

This expansion has nine terms, so the middle term is the fifth term. Using the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 8$, $k = 5$, $k - 1 = 4$, and $n - (k - 1) = 4$, the fifth term is

$$\begin{aligned}\binom{8}{4}\left(3x^7\right)^4\left(2y^3\right)^4 &= \frac{8!}{4!4!}\left(3^4x^{28}\right)\left(2^4y^{12}\right)\\ &= \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}\left(81x^{28}\right)\left(16y^{12}\right)\\ &= (70)\left(1296x^{28}y^{12}\right)\\ &= 90{,}720x^{28}y^{12}\end{aligned}$$

48. The two middle terms of $\left(-2m^{-1}+3n^{-2}\right)^{11}$ are the sixth and seventh terms. Using the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$, we have the following:

For the sixth term we have $n = 11$, $k = 6$, $k - 1 = 5$, and $n - (k - 1) = 6$. Thus, we have

$$\begin{aligned}\binom{11}{5}\left(-2m^{-1}\right)^6\left(3n^{-2}\right)^5 &= \frac{11!}{5!6!}\left(2^6m^{-6}\right)\left(3^5n^{-10}\right)\\ &= \frac{11\cdot10\cdot9\cdot8\cdot7}{5\cdot4\cdot3\cdot2\cdot1}\left(2^6m^{-6}\right)\left(3^5n^{-10}\right)\\ &= 462\left(2^6\right)\left(3^5\right)m^{-6}n^{-10}\end{aligned}$$

For the seventh term we have $n = 11$, $k = 7$, $k - 1 = 6$, and $n - (k - 1) = 5$. Thus, we have

$$\begin{aligned}\binom{11}{6}\left(-2m^{-1}\right)^5\left(3n^{-2}\right)^6 &= \frac{11!}{6!5!}\left(-2^5m^{-5}\right)\left(3^6n^{-12}\right)\\ &= \frac{11\cdot10\cdot9\cdot8\cdot7}{5\cdot4\cdot3\cdot2\cdot1}\left(-2^5m^{-5}\right)\left(3^6n^{-12}\right)\\ &= -462\left(2^5\right)\left(3^6\right)m^{-5}n^{-12}\end{aligned}$$

49. If the coefficients of the fifth and eighth terms in the expansion of $(x+y)^n$ are the same, then the symmetry of the expansion can be used to determine n. There are four terms before the fifth term, so there must be four terms after the eighth term. This means that the last term of the expansion is the twelfth term. This in turn means that $n = 11$, since $(x+y)^{11}$ is the expansion that has twelve terms.

50. Since $\left(\sqrt{x}\right)^8 = x^4$, the term in the expansion of $\left(3+\sqrt{x}\right)^{11}$ that contains x^4 must be

$$\binom{11}{8}(3)^{11-8}\left(x^{1/2}\right)^8 = \frac{11!}{8!3!}\left(3^3\right)x^4 = \frac{11\cdot 10\cdot 9}{3\cdot 2\cdot 1}\left(27x^4\right) = 165\left(27x^4\right) = 4455x^4$$

51. Using a calculator, we obtain the exact value 10! = 3,628,800. Using Stirling's formula, we obtain

$$10! \approx \sqrt{2\pi(10)}\cdot 10^{10}\cdot e^{-10} \approx 3,598,695.619$$

52. Since $\frac{3,628,800-3,598,695.619}{3,628,800} \approx .00830$, the percent error is approximately .830%.

53. Using a calculator, we obtain the exact value 12! = 479,001,600. Using Stirling's formula, we obtain

$$12! \approx \sqrt{2\pi(12)}\cdot 12^{12}\cdot e^{-12} \approx 475,687,486.5$$

Since $\frac{479,001,600-475,687,486.5}{479,001,600} \approx .00692$, the percent error is approximately .692%.

54. Using a calculator, we obtain the exact value 13! = 6,227,020,800. Using Stirling's formula, we obtain

$$13! \approx \sqrt{2\pi(13)}\cdot 13^{13}\cdot e^{-13} \approx 6,187,239,475$$

Since $\frac{6,227,020,800-6,187,239,475}{6,227,020,800} \approx .00639$, the percent error is approximately .639%. As n gets larger, the percent decreases

For Exercises 55−58, we will be using the first four terms of the formula

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

55. $$(1.02)^{-3} = (1+.02)^{-3} = 1+(-3)(.02)+\frac{(-3)(-3-1)}{2!}(.02)^2+\frac{(-3)(-3-1)(-3-2)}{3!}(.02)^3$$
$$= 1-.06+\frac{(-3)(-4)}{2\cdot 1}(.0004)+\frac{(-3)(-4)(-5)}{3\cdot 2\cdot 1}(.000008)$$
$$= 1-.06+6(.0004)-10(.000008) = 1-.06+.0024-.00008 = .94232 \approx .942$$

56. $$(1.04)^{-5} = (1+.04)^{-5} = 1+(-5)(.04)+\frac{(-5)(-5-1)}{2!}(.04)^2+\frac{(-5)(-5-1)(-5-2)}{3!}(.04)^3$$
$$= 1-.2+\frac{(-5)(-6)}{2\cdot 1}(.0016)+\frac{(-5)(-6)(-7)}{3\cdot 2\cdot 1}(.000064)$$
$$= 1-.2+15(.0016)-35(.000064) = 1-.2+.024-.00224 = .82176 \approx .822$$

57. $$(1.01)^{1.5} = (1+.01)^{1.5} = 1+(1.5)(.01)+\frac{(1.5)(1.5-1)}{2!}(.01)^2+\frac{(1.5)(1.5-1)(1.5-2)}{3!}(.01)^3$$
$$= 1+.015+\frac{(1.5)(.5)}{2\cdot 1}(.0001)+\frac{(1.5)(.5)(-.5)}{3\cdot 2\cdot 1}(.000001)$$
$$= 1+.015+.375(.0001)-.0625(.000001)$$
$$= 1+.015+.0000375-.0000000625 = 1.0150374375 \approx 1.015$$

58. $$(1.03)^2 = (1+.03)^2 = 1+(2)(.03)+\frac{(2)(2-1)}{2!}(.03)^2+\frac{(2)(2-1)(2-2)}{3!}(.03)^3$$
$$= 1+.06+\frac{(2)(1)}{2\cdot 1}(.0009)+\frac{(2)(1)(0)}{3\cdot 2\cdot 1}(.000027)$$
$$= 1+.06+.0009+0 = 1.0609 \approx 1.061$$

Section 11.5: Mathematical Induction

1. Let S_n be the statement $1+3+5+\ldots+(2n-1)=n^2$.

$S_1: 1=1^2 \Rightarrow 1=1,$ which is true.

$S_2: 1+3=2^2 \Rightarrow 4=4,$ which is true.

$S_3: 1+3+5=3^2 \Rightarrow 9=9,$ which is true.

$S_4: 1+3+5+7=4^2 \Rightarrow 16=16,$ which is true.

$S_5: 1+3+5+7+9=5^2 \Rightarrow 25=25,$ which is true.

Now prove that S_n is true for every positive integer *n*.

(a) *Verify the statement for* $n=1$.
S_1 is the statement $1=1^2 \Rightarrow 1=1,$ which is true.

(b) *Write the statement for* $n=k$: $S_k=1+3+5+\ldots+(2k-1)=k^2$

(c) *Write the statement for* $n=k+1$: $S_{k+1}=1+3+5+\ldots+(2k-1)+\left[2(k+1)-1\right]=(k+1)^2$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Start with $S_k: 1+3+5+\ldots+(2k-1)=k^2$ and add the $(k+1)$ st term, $\left[2(k+1)-1\right]$, to both sides of this equation.

$$1+3+5+\ldots+(2k-1)=k^2$$
$$1+3+5+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+\left[2(k+1)-1\right]$$
$$1+3+5+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+(2k+2-1)$$
$$1+3+5+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+2k+1$$
$$1+3+5+\ldots+(2k-1)+\left[2(k+1)-1\right]=(k+1)^2$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

2. Let S_n be the statement $2+4+6+\ldots+2n=n(n+1)$.

$S_1: 2=1(1+1) \Rightarrow 2=2,$ which is true.

$S_2: 2+4=2(2+1) \Rightarrow 6=6,$ which is true.

$S_3: 2+4+6=3(3+1) \Rightarrow 12=12,$ which is true.

$S_4: 2+4+6+8=4(4+1) \Rightarrow 20=20,$ which is true.

$S_5: 2+4+6+8+10=5(5+1) \Rightarrow 30=30,$ which is true.

Now prove that S_n is true for every positive integer *n*.

(a) *Verify the statement for* $n=1$.
S_1 is the statement $2=1(1+1) \Rightarrow 2=2,$ which is true.

(b) *Write the statement for* $n=k$: $S_k=2+4+6+\ldots+2k=k(k+1)$

(c) *Write the statement for* $n=k+1$: $S_{k+1}=2+4+6+\ldots+2k+2(k+1)=(k+1)\left[(k+1)+1\right]$

(d) *Assume the statement is true for* $n = k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Start with S_k: $2+4+6+\ldots+2k = k(k+1)$ and add the $(k+1)$ st term, $2(k+1)$, to both sides of this equation.

$$2+4+6+\ldots+2k = k(k+1) \Rightarrow 2+4+6+\ldots+2k+2(k+1) = k(k+1)+2(k+1) \Rightarrow$$
$$2+4+6+\ldots+2k+2(k+1) = k^2+k+2k+2 \Rightarrow 2+4+6+\ldots+2k+2(k+1) = k^2+3k+2 \Rightarrow$$
$$2+4+6+\ldots+2k+2(k+1) = (k+1)(k+2) \Rightarrow 2+4+6+\ldots+2k+2(k+1) = (k+1)[(k+1)+1]$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

3. S_n is the statement $3+6+9+\ldots+3n = \frac{3n(n+1)}{2}$.

(a) *Verify the statement for* $n=1$.
$3(1) = \frac{3(1)(1+1)}{2} \Rightarrow 3 = \frac{6}{2} \Rightarrow 3 = 3,$ which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $3+6+9+\ldots+3k = \frac{3k(k+1)}{2}$

(c) *Write the statement for* $n=k+1$: $3+6+9+\ldots+3k+3(k+1) = \frac{3(k+1)[(k+1)+1]}{2}$

(d) *Assume the statement is true for* $n = k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, $3(k+1)$, to both sides of this equation.

$$3+6+9+\ldots+3k = \frac{3k(k+1)}{2} \Rightarrow 3+6+9+\ldots+3k+3(k+1) = \frac{3k(k+1)}{2}+3(k+1) \Rightarrow$$
$$3+6+9+\ldots+3k+3(k+1) = \frac{3k(k+1)+6(k+1)}{2} \Rightarrow 3+6+9+\ldots+3k+3(k+1) = \frac{3k^2+3k+6k+6}{2} \Rightarrow$$
$$3+6+9+\ldots+3k+3(k+1) = \frac{3k^2+9k+6}{2} \Rightarrow 3+6+9+\ldots+3k+3(k+1) = \frac{3(k^2+3k+2)}{2} \Rightarrow$$
$$3+6+9+\ldots+3k+3(k+1) = \frac{3(k+1)(k+2)}{2} \Rightarrow 3+6+9+\ldots+3k+3(k+1) = \frac{3(k+1)[(k+1)+1]}{2}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

4. S_n is the statement $5+10+15+\ldots+5n = \frac{5n(n+1)}{2}$.

(a) *Verify the statement for* $n=1$.
$5(1) = \frac{5(1)(1+1)}{2} \Rightarrow 5 = \frac{10}{2} \Rightarrow 5 = 5,$ which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $5+10+15+\ldots+5k = \frac{5k(k+1)}{2}$

(c) *Write the statement for* $n=k+1$: $5+10+15+\ldots+5k+5(k+1) = \frac{5(k+1)[(k+1)+1]}{2}$

(d) *Assume the statement is true for* $n = k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, $5(k+1)$, to both sides of this equation.

$$5+10+15+\ldots+5k = \frac{5k(k+1)}{2} \Rightarrow 5+10+15+\ldots+5k+5(k+1) = \frac{5k(k+1)}{2}+5(k+1) \Rightarrow$$
$$5+10+15+\ldots+5k+5(k+1) = \frac{5k(k+1)+10(k+1)}{2} \Rightarrow$$
$$5+10+15+\ldots+5k+5(k+1) = \frac{5k^2+5k+10k+10}{2} \Rightarrow 5+10+15+\ldots+5k+5(k+1) = \frac{5k^2+15k+10}{2} \Rightarrow$$
$$5+10+15+\ldots+5k+5(k+1) = \frac{5(k^2+3k+2)}{2} \Rightarrow 5+10+15+\ldots+5k+5(k+1) = \frac{5(k+1)(k+2)}{2} \Rightarrow$$
$$5+10+15+\ldots+5k+5(k+1) = \frac{5(k+1)[(k+1)+1]}{2}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

5. S_n is the statement $2+4+8+\ldots+2^n = 2^{n+1}-2$.

(a) *Verify the statement for* $n=1$.
$2^1 = 2^{1+1}-2 \Rightarrow 2 = 2^2-2 \Rightarrow 2 = 4-2 \Rightarrow 2 = 2$, which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $2+4+8+\ldots+2^k = 2^{k+1}-2$

(c) *Write the statement for* $n=k+1$: $2+4+8+\ldots+2^k+2^{k+1} = 2^{(k+1)+1}-2$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, 2^{k+1}, to both sides of this equation.

$$2+4+8+\ldots+2^k = 2^{k+1}-2 \Rightarrow 2+4+8+\ldots+2^k+2^{k+1} = 2^{k+1}-2+2^{k+1} \Rightarrow$$
$$2+4+8+\ldots+2^k+2^{k+1} = 2\cdot 2^{k+1}-2 \Rightarrow 2+4+8+\ldots+2^k+2^{k+1} = 2^1\cdot 2^{k+1}-2 \Rightarrow$$
$$2+4+8+\ldots+2^k+2^{k+1} = 2^{(k+1)+1}-2$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

6. S_n is the statement $3+3^2+3^3+\ldots+3^n = \dfrac{3(3^n-1)}{2}$.

(a) *Verify the statement for* $n=1$.
$3^1 = \dfrac{3(3^1-1)}{2} \Rightarrow 3 = \dfrac{3(3-1)}{2} \Rightarrow 3 = \dfrac{3(2)}{2} \Rightarrow 3 = 3$, which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $3+3^2+3^3+\ldots+3^k = \dfrac{3(3^k-1)}{2}$

(c) *Write the statement for* $n=k+1$: $3+3^2+3^3+\ldots+3^k+3^{k+1} = \dfrac{3(3^{k+1}-1)}{2}$

(d) *Assume the statement is true for* $n = k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, 3^{k+1}, to both sides of this equation.

$$3+3^2+3^3+\ldots+3^k=\frac{3\left(3^k-1\right)}{2}\Rightarrow 3+3^2+3^3+\ldots+3^k+3^{k+1}=\frac{3\left(3^k-1\right)}{2}+3^{k+1}\Rightarrow$$
$$3+3^2+3^3+\ldots+3^k+3^{k+1}=\frac{3\left(3^k-1\right)+2\cdot 3^{k+1}}{2}\Rightarrow 3+3^2+3^3+\ldots+3^k+3^{k+1}=\frac{3^{k+1}-3+2\cdot 3^{k+1}}{2}\Rightarrow$$
$$3+3^2+3^3+\ldots+3^k+3^{k+1}=\frac{3\cdot 3^{k+1}-3}{2}\Rightarrow 3+3^2+3^3+\ldots+3^k+3^{k+1}=\frac{3\left(3^{k+1}-1\right)}{2}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

7. S_n is the statement $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

(a) *Verify the statement for* $n=1$.
$1^2=\dfrac{1(1+1)(2\cdot 1+1)}{6}\Rightarrow 1=\dfrac{1(2)(2+1)}{6}\Rightarrow 1=\dfrac{2(3)}{6}\Rightarrow 1=1$, which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $1^2+2^2+3^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$

(c) *Write the statement for* $n=k+1$: $1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\dfrac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6}$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, $(k+1)^2$, to both sides of this equation.

$$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)\left(2k^2+k+6k+6\right)}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)\left(2k^2+7k+6\right)}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+2+1)}{6}$$
$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

8. S_n is the statement $1^3+2^3+3^3+\ldots+n^3=\dfrac{n^2(n+1)^2}{4}$.

(a) *Verify the statement for* $n=1$.

$1^3=\dfrac{1^2(1+1)^2}{4}\Rightarrow 1=\dfrac{1(2)^2}{4}\Rightarrow 1=\dfrac{1(4)}{4}\Rightarrow 1=1,$ which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $1^3+2^3+3^3+\ldots+k^3=\dfrac{k^2(k+1)^2}{4}$

(c) *Write the statement for* $n=k+1$: $1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\dfrac{(k+1)^2\left[(k+1)+1\right]^2}{4}$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, $(k+1)^3$, to both sides of this equation.

$$1^3+2^3+3^3+\ldots+k^3=\frac{k^2(k+1)^2}{4}\Rightarrow 1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3\Rightarrow$$

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{k^2(k+1)^2+4(k+1)^3}{4}\Rightarrow$$

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}\Rightarrow$$

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2(k^2+4k+4)}{4}$$

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2(k+2)^2}{4}\Rightarrow 1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2\left[(k+1)+1\right]^2}{4}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).

Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

9. S_n is the statement $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^n=6(6^n-1)$.

(a) *Verify the statement for* $n=1$.

$5\cdot 6^1=6(6^1-1)\Rightarrow 5\cdot 6=6(6-1)\Rightarrow 30=6(5)\Rightarrow 30=30,$ which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k=6(6^k-1)$

(c) *Write the statement for* $n=k+1$: $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k+5\cdot 6^{k+1}=6(6^{k+1}-1)$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, $5\cdot 6^{k+1}$, to both sides of this equation.

$$5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k=6(6^k-1)$$

$$5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k+5\cdot 6^{k+1}=6(6^k-1)+5\cdot 6^{k+1}$$

$$5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k+5\cdot 6^{k+1}=6\left[(6^k-1)+5\cdot 6^k\right]$$

$$5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k+5\cdot 6^{k+1}=6\left[6\cdot 6^k-1\right]$$

$$5\cdot 6+5\cdot 6^2+5\cdot 6^3+\ldots+5\cdot 6^k+5\cdot 6^{k+1}=6(6^{k+1}-1)$$

(e) *Write a conclusion based on Steps (a) – (d).*
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

10. S_n is the statement $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^n=8\left(8^n-1\right)$.

(a) *Verify the statement for* $n=1$.
$7\cdot 8^1=8\left(8^1-1\right)\Rightarrow 7\cdot 8=8(8-1)\Rightarrow 56=8(7)\Rightarrow 56=56,$ which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k=8\left(8^k-1\right)$

(c) *Write the statement for* $n=k+1$: $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k+7\cdot 8^{k+1}=8\left(8^{k+1}-1\right)$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part (b) to the statement in part (c)*: Add the $(k+1)$ st term, $7\cdot 8^{k+1}$, to both sides of this equation.

$$7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k=8\left(8^k-1\right)$$
$$7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k+7\cdot 8^{k+1}=8\left(8^k-1\right)+7\cdot 8^{k+1}$$
$$7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k+7\cdot 8^{k+1}=8\left[\left(8^k-1\right)+7\cdot 8^k\right]$$
$$7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k+7\cdot 8^{k+1}=8\left[8\cdot 8^k-1\right]$$
$$7\cdot 8+7\cdot 8^2+7\cdot 8^3+\ldots+7\cdot 8^k+7\cdot 8^{k+1}=8\left(8^{k+1}-1\right)$$

(e) *Write a conclusion based on Steps (a) – (d).*
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

11. S_n is the statement $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

(a) *Verify the statement for* $n=1$.
$\frac{1}{(1)(1+1)}=\frac{1}{1+1}\Rightarrow\frac{1}{(1)(2)}=\frac{1}{2}\Rightarrow\frac{1}{2}=\frac{1}{2},$ which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

(c) *Write the statement for* $n=k+1$: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k+1}{(k+1)+1}$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part (b) to the statement in part (c)*: Add the $(k+1)$ st term, $\frac{1}{(k+1)\left[(k+1)+1\right]}$, to both sides of this equation.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k}{k+1}+\frac{1}{(k+1)\left[(k+1)+1\right]}$$
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k(k+2)+1}{(k+1)(k+2)}$$
$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k^2+2k+1}{(k+1)(k+2)}$$
$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{(k+1)^2}{(k+1)(k+2)}$$
$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k+1}{k+2}$$
$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\left[(k+1)+1\right]}=\frac{k+1}{(k+1)+1}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

12. S_n is the statement $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$.

(a) *Verify the statement for* $n=1$.
$\frac{1}{(3\cdot1-2)(3\cdot1+1)}=\frac{1}{3\cdot1+1}\Rightarrow\frac{1}{(3-2)(3+1)}=\frac{1}{3+1}\Rightarrow\frac{1}{(1)(4)}=\frac{1}{4}\Rightarrow\frac{1}{4}=\frac{1}{4}$, which is true. Thus, S_n is true for $n=1$.

(b) *Write the statement for* $n=k$: $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$

(c) *Write the statement for* $n=k+1$:
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{k+1}{3(k+1)+1}$$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Add the $(k+1)$ st term, $\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}$, to both sides of this equation.

$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{k+1}{3k+4}$$
$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{\left[3(k+1)-2\right]\left[3(k+1)+1\right]}=\frac{k+1}{3(k+1)+1}$$

(e) *Write a conclusion based on Steps (a) – (d).*
Since S_n is true for $n = 1$ and $n = k + 1$ when it is true for $n = k$, S_n is true for every positive integer *n*.

13. S_n is the statement $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

(a) *Verify the statement for* $n = 1$.
$\frac{1}{2^1} = 1 - \frac{1}{2^1} \Rightarrow \frac{1}{2} = 1 - \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$, which is true. Thus, S_n is true for $n = 1$.

(b) *Write the statement for* $n = k$: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$

(c) *Write the statement for* $n = k + 1$: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$

(d) *Assume the statement is true for* $n = k$. *Use algebra to change the statement in part (b) to the statement in part (c)*: Add the $(k+1)$ st term, $\frac{1}{2^{k+1}}$, to both sides of this equation.

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \Rightarrow \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \Rightarrow$$
$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1 \cdot 2}{2^k \cdot 2} + \frac{1}{2^{k+1}} \Rightarrow \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} \Rightarrow$$
$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

(e) *Write a conclusion based on Steps (a) – (d).*
Since S_n is true for $n = 1$ and $n = k + 1$ when it is true for $n = k$, S_n is true for every positive integer *n*.

14. S_n is the statement $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^n} = 1 - \frac{1}{5^n}$.

(a) *Verify the statement for* $n = 1$.
$\frac{4}{5^1} = 1 - \frac{1}{5^1} \Rightarrow \frac{4}{5} = 1 - \frac{1}{5} \Rightarrow \frac{4}{5} = \frac{4}{5}$, which is true. Thus, S_n is true for $n = 1$.

(b) *Write the statement for* $n = k$: $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} = 1 - \frac{1}{5^k}$

(c) *Write the statement for* $n = k + 1$: $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{1}{5^{k+1}}$

(d) *Assume the statement is true for* $n = k$. *Use algebra to change the statement in part (b) to the statement in part (c)*: Add the $(k+1)$ st term, $\frac{4}{5^{k+1}}$, to both sides of this equation.

$$\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} = 1 - \frac{1}{5^k} \Rightarrow \frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{1}{5^k} + \frac{4}{5^{k+1}} \Rightarrow$$
$$\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{1 \cdot 5}{5^k \cdot 5} + \frac{4}{5^{k+1}} \Rightarrow \frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{5}{5^{k+1}} + \frac{4}{5^{k+1}} \Rightarrow$$
$$\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{1}{5^{k+1}}$$

(e) *Write a conclusion based on Steps (a) – (d).*
Since S_n is true for $n = 1$ and $n = k + 1$ when it is true for $n = k$, S_n is true for every positive integer *n*.

15. $2^n > 2n$

If $n = 1$, we have $2^1 > 2 \cdot 1$ or $2 > 2$, which is false.

If $n = 2$, we have $2^2 > 2 \cdot 2$ or $4 > 4$, which is false.

If $n = 3$, we have $2^3 > 2 \cdot 3$ or $8 > 6$, which is true.

For $n \geq 3$, the statement is true. The statement is false for $n = 1$ or 2.

16. $3^n > 2n + 1$

If $n = 1$, we have $3^1 > 2 \cdot 1 + 1$ or $3 > 3$, which is false.

If $n = 2$, we have $3^2 > 2 \cdot 2 + 1$ or $9 > 5$, which is true.

For $n \geq 2$, the statement is true. The statement is false for $n = 1$.

17. $2^n > n^2$

If $n = 1$, we have $2^1 > 1^2$ or $2 > 1$, which is true.

If $n = 2$, we have $2^2 > 2^2$ or $4 > 4$, which is false.

If $n = 3$, we have $2^3 > 3^2$ or $8 > 9$, which is false.

If $n = 4$, we have $2^4 > 4^2$ or $16 > 16$, which is false.

If $n = 5$, we have $2^5 > 5^2$ or $32 > 25$, which is true.

For $n \geq 5$, the statement is true. The statement is false for $n = 2$, 3, or 4.

18. $n! > 2n$

If $n = 1$, we have $1! > 2(1)$ or $1 > 2$, which is false.

If $n = 2$, we have $2! > 2(2)$ or $2 > 4$, which is false.

If $n = 3$, we have $3! > 2(3)$ or $6 > 6$, which is false.

If $n = 4$, we have $4! > 2(4)$ or $24 > 8$, which is true.

For $n \geq 4$, the statement is true. The statement is false for $n = 1$, 2, or 3.

19. Let S_n be the statement $(a^m)^n = a^{mn}$.

(Assume that a and m are constant.)

Step 1: Show that the statement is true when $n = 1$. S_1 is the statement

$\left(a^m\right)^1 = a^{m \cdot 1} \Rightarrow a^m = a^m$, which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $\left(a^m\right)^k = a^{mk}$, and S_{k+1} is the statement $\left(a^m\right)^{k+1} = a^{m(k+1)}$.

$$\left(a^m\right)^k = a^{mk} \Rightarrow \left(a^m\right)^k \cdot \left(a^m\right)^1 = a^{mk} \cdot \left(a^m\right)^1 \Rightarrow$$
$$\left(a^m\right)^{k+1} = a^{mk} \cdot a^m \Rightarrow \left(a^m\right)^{k+1} = a^{mk+m} \Rightarrow$$
$$\left(a^m\right)^{k+1} = a^{m(k+1)}$$

It has been shown that S_k implies S_{k+1}. Therefore, the statement S_n is true for every positive integer value of n.

20. Let S_n be the statement $(ab)^n = a^n b^n$.

(Assume that a and b are constant.)

Step 1: Show that the statement is true when $n = 1$. S_1 is the statement

$(ab)^1 = a^1 b^1 \Rightarrow ab = ab$, which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $(ab)^k = a^k b^k$, and S_{k+1} is the statement $(ab)^{k+1} = a^{k+1} b^{k+1}$.

$$(ab)^k = a^k b^k \Rightarrow (ab)^k (ab)^1 = a^k b^k (ab)^1 \Rightarrow$$
$$(ab)^{k+1} = a^k b^k ab \Rightarrow (ab)^{k+1} = \left(a^k a\right)\left(b^k b\right) \Rightarrow$$
$$(ab)^{k+1} = \left(a^k a^1\right)\left(b^k b^1\right) \Rightarrow (ab)^{k+1} = a^{k+1} b^{k+1}$$

It has been shown that S_k implies S_{k+1}. Therefore, the statement S_n is true for every positive integer value of n.

21. Let S_n be the statement $2^n > 2n$, if $n \geq 3$.

Step 1:

Show that the statement is true when $n = 3$: S_3 is the statement $2^3 > 2 \cdot 3 \Rightarrow 8 > 6$, which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $2^k > 2k$, and S_{k+1} is the statement $2^{k+1} > 2(k+1)$.

$$2^k > 2k \Rightarrow 2^k \cdot 2 > 2k \cdot 2 \Rightarrow$$
$$2^k \cdot 2^1 > 4k \Rightarrow 2^{k+1} > 2k + 2k$$

Since $k \geq 3$ we have $2k > 2$.

$$2^{k+1} > 2k + 2k > 2k + 2 \Rightarrow 2^{k+1} > 2k + 2 \Rightarrow$$
$$2^{k+1} > 2(k+1)$$

It has been shown that S_k implies S_{k+1}.
Therefore, the statement S_n is true for every positive integer value of *n* greater than or equal to 3.

22. Let S_n be the statement $3^n > 2n+1$, if $n \geq 2$.
Step 1: Show that the statement is true when $n = 2$: S_2 is the statement
$3^2 > 2\cdot 2+1 \Rightarrow 9 > 5,$ which is true.
Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $3^k > 2k+1,$ and S_{k+1} is the statement $3^{k+1} > 2(k+1)+1.$
$3^k > 2k+1 \Rightarrow 3^k \cdot 3 > (2k+1)\cdot 3 \Rightarrow$
$3^k \cdot 3^1 > 6k+3 \Rightarrow 3^{k+1} > 6k+3$
Since $k \geq 3$ we have $6k > 2k.$
$3^{k+1} > 6k+3 > 2k+3 \Rightarrow 3^{k+1} > 2k+3 \Rightarrow$
$3^{k+1} > 2k+2+1 \Rightarrow 3^{k+1} > 2(k+1)+1$
It has been shown that S_k implies S_{k+1}.
Therefore, the statement S_n is true for every positive integer value of *n* greater than or equal to 2.

23. Let S_n be the statement: If $a > 1$, then $a^n > 1$.
Step 1: Show that the statement is true when $n = 1$: S_1 is the statement, If $a > 1$, then $a^1 > 1,$ which is obviously true since $a = a^1.$
Step 2: Show that S_k implies S_{k+1}, where S_k is the statement: If $a > 1,$ then $a^k > 1.$ S_{k+1} is the statement: If $a > 1,$ then $a^{k+1} > 1.$
If $a > 1$, then $a^k > 1$.
$a^k > 1 \Rightarrow a^k \cdot a > 1\cdot a \Rightarrow a^k \cdot a^1 > a \Rightarrow a^{k+1} > a$
Since $a > 1$ we have the following.
$a^{k+1} > a > 1 \Rightarrow a^{k+1} > 1$
Note: Since $a > 1 > 0,$ we did not need to be concerned about changing the direction of the inequality.
It has been shown that S_k implies S_{k+1}.
Therefore, the statement S_n is true for every positive integer value of *n*.

24. Let S_n be the statement: If $a > 1$, then $a^n > a^{n-1}.$
Step 1: Show that the statement is true when $n = 1$: S_1 is the statement: If $a > 1$, then $a^1 > a^{1-1}.$ Since $a^1 > a^{1-1} \Rightarrow a > a^0 \Rightarrow a > 1,$ which is true.
Step 2: Show that S_k implies S_{k+1}, where S_k is the statement: If $a > 1,$ then $a^k > a^{k-1}.$ S_{k+1} is the statement: If $a > 1,$ then $a^{k+1} > a^{(k+1)-1}.$
If $a > 1$, then $a^k > a^{k-1}.$
$a^k > a^{k-1} \Rightarrow a^k \cdot a^1 > a^{k-1}a^1 \Rightarrow$
$a^{k+1} > a^{(k-1)+1} \Rightarrow a^{k+1} > a^{(k+1)-1}$
Note: Since $a > 1 > 0,$ we did not need to be concerned about changing the direction of the inequality.
It has been shown that S_k implies S_{k+1}.
Therefore, the statement S_n is true for every positive integer value of *n*.

25. Let S_n be the statement: If $0 < a < 1,$ then $a^n < a^{n-1}.$
Step 1: Show that the statement is true when $n = 1$: S_1 is the statement: If $0 < a < 1,$ then $a^1 < a^{1-1}.$ Since $a^1 < a^{1-1} \Rightarrow a < a^0 \Rightarrow a < 1,$ which is true.
Step 2: Show that S_k implies S_{k+1}, where S_k is the statement: If $0 < a < 1,$ then $a^k < a^{k-1}.$ S_{k+1} is the statement: If $0 < a < 1,$ then $a^{k+1} < a^{(k+1)-1}.$
If $0 < a < 1,$ then $a^k < a^{k-1}.$
$a^k < a^{k-1} \Rightarrow a^k \cdot a^1 < a^{k-1}a^1 \Rightarrow$
$a^{k+1} < a^{(k-1)+1} \Rightarrow a^{k+1} < a^{(k+1)-1}$
Note: Since $0 < a < 1,$ we did not need to be concerned about changing the direction of the inequality.
It has been shown that S_k implies S_{k+1}.
Therefore, the statement S_n is true for every positive integer value of *n*.

26. Let S_n be the statement $2^n > n^2$, for $n \geq 5$.
Step 1: Show that the statement is true when $n = 5$: S_5 is the statement $2^5 > 5^2 \Rightarrow 32 > 25,$ which is true.
Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $2^k > k^2,$ and S_{k+1} is the statement $2^{k+1} > (k+1)^2.$
$2^k > k^2 \Rightarrow 2^k \cdot 2 > k^2 \cdot 2 \Rightarrow$
$2^k \cdot 2^1 > 2k^2 \Rightarrow 2^{k+1} > k^2 + k^2$
Since $k > 4$, we have $2k > 8$. Thus, we have $2k > 1$. Also, since $k > 4$, we have $k^2 > 4k.$

(*continued on next page*)

(*continued from page 1135*)

$2^{k+1} > k^2 + k^2 > k^2 + 4k \Rightarrow 2^{k+1} > k^2 + 4k \Rightarrow$
$2^{k+1} > k^2 + 2k + 2k \Rightarrow$
$2^{k+1} > k^2 + 2k + 2k > k^2 + 2k + 1 \Rightarrow$
$2^{k+1} > k^2 + 2k + 1 \Rightarrow 2^{k+1} > (k+1)^2$

It has been shown that S_k implies S_{k+1}. Therefore, the statement S_n is true for every positive integer value of *n* greater than or equal to 5.

27. Let S_n be the statement: If $n \geq 4$, then $n! > 2^n$, where $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.

Step 1: Show that the statement is true when $n = 4$: S_4 is the statement

$4! > 2^4 \Rightarrow 4(3)(2)(1) > (2)(2)(2)(2) \Rightarrow 24 > 16,$

which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $k! > 2^k$, and S_{k+1} is the statement $(k+1)! > 2^{k+1}$.

$k! > 2^k \Rightarrow k!(k+1) > 2^k(k+1) \Rightarrow$
$(k+1)! > 2^k(k+1)$

Since $k+1 > 2$, we have

$(k+1)! > 2^k(k+1) > 2 \cdot 2^k \Rightarrow$
$(k+1)! > 2 \cdot 2^k \Rightarrow (k+1)! > 2^1 \cdot 2^k$
$(k+1)! > 2^{1+k} \Rightarrow (k+1)! > 2^{k+1}$

Note: Since $k+1 > 0$, we did not need to be concerned about changing the direction of the inequality.

It has been shown that S_k implies S_{k+1}. Therefore, the statement S_n is true for every positive integer value of *n* greater than or equal to 4.

28. Let S_n be the statement $4^n > n^4$ for $n \geq 5$.

Step 1: Show that the statement is true when $n = 5$: S_5 is the statement

$4^5 > 5^4 \Rightarrow 1024 > 625,$ which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement $4^k > k^4$, and S_{k+1} is the statement $4^{k+1} > (k+1)^4$.

$4^k > k^4 \Rightarrow 4^k \cdot 4 > 4k^4 \Rightarrow 4^k \cdot 4^1 > 4k^4 \Rightarrow$
$4^{k+1} > 4k^4 \Rightarrow 4^{k+1} > k^4 + k^4 + k^4 + k^4$

Before continuing, we make some observations.

$k \geq 5$, so $k^3 \cdot k > k^3 \cdot 4$ or $k^4 > 4k^3$.

$k > 4$, so
$k^3 > 4^3 \Rightarrow k \cdot k^3 > k \cdot 64 \Rightarrow k^4 > 64k > 4k$
or $k^4 > 4k$

$k > 4$, so
$k^2 > 4^2 \Rightarrow k^2 > 16 \Rightarrow 10k^2 > 10 \cdot 16 \Rightarrow$
$10k^2 > 160 > 1$ or $10k^2 > 1$

$k > 4$, so
$k^2 > 4^2 \Rightarrow k^2 > 16 \Rightarrow k^2 \cdot k^2 > k^2 \cdot 16 \Rightarrow$
$k^4 > 16k^2$

These observations are used in the following.

$4^{k+1} > k^4 + k^4 + k^4 + k^4 > k^4 + 4k^3 + 4k + 16k^2$
$4^{k+1} > k^4 + 4k^3 + 4k + 16k^2$
$4^{k+1} > k^4 + 4k^3 + 4k + 6k^2 + 10k^2$
$4^{k+1} > k^4 + 4k^3 + 4k + 6k^2 + 1$
$4^{k+1} > (k+1)^4$

It has been shown that S_k implies S_{k+1}. Therefore, the statement S_n is true for every positive integer value of *n* greater than or equal to 5.

29. Let S_n be the number of handshakes of the *n* people is $\dfrac{n^2 - n}{2}$.

Since 2 is the smallest number of people who can shake hands, we need to prove this statement for every positive integer $n \geq 2$.

Step 1: Show that the statement is true for $n = 2$. S_2 is the statement that for two people, the number of handshakes is

$\dfrac{2^2 - 2}{2} = \dfrac{4-2}{2} = \dfrac{2}{2} = 1,$ which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement: *k* people shake hands in $\dfrac{k^2 - k}{2}$ ways and S_{k+1} is the statement: $k+1$ people shake hands in $\dfrac{(k+1)^2 - (k+1)}{2}$ ways.

Start with S_k and assume it is a true statement:

For *k* people, there are $\dfrac{k^2 - k}{2}$ handshakes.

If one more person joins the *k* people, this (*k* + 1)st person will shake hands with the previous *k* people one time each.

Thus, there will be k additional handshakes. Thus, the number of handshakes for $k + 1$ people is as follows.

$$\frac{k^2-k}{2}+k=\frac{k^2-k}{2}+\frac{2k}{2}=\frac{k^2+k}{2}$$
$$=\frac{\left(k^2+2k+1\right)-\left(k+1\right)}{2}$$
$$=\frac{\left(k+1\right)^2-\left(k+1\right)}{2}$$

This shows that if S_k is true, S_{k+1} is true. Since both steps for a proof by the generalized principle of mathematical induction have been completed, the given statement is true for every positive integer $n \geq 2$.

30. Look at the figures pictured in the textbook for this exercise (page 698). The first one, the triangle, has $3 = 3 \cdot 4^0$ sides. The second figure has $12 = 3 \cdot 4^1$ sides, and the third figure has $48 = 3 \cdot 4^2$ sides. Generalize the pattern to make the statement that the nth figure has $3 \cdot 4^{n-1}$ sides. Let S_n be the statement that the number of sides in the nth figure is $a_n = 3 \cdot 4^{n-1}$.

Step 1: Show that the statement is true when $n = 1$: S_1 is the statement

$3 \cdot 4^{1-1} = 3 \Rightarrow 3 \cdot 4^0 = 3 \Rightarrow 3 \cdot 1 = 3 \Rightarrow 3 = 3,$

which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement: The number of sides in the kth figure is $3 \cdot 4^{k-1}$. S_{k+1} is the statement: The number of sides in the $(k+1)$ st figure is $3 \cdot 4^{(k+1)-1}$. Assuming that there are $3 \cdot 4^{k-1}$ sides in the kth figure, there are four times as many sides in the $(k+1)$ st figure. Thus, the total number of sides in the $(k+1)$ st figure is

$$\left(3\cdot 4^{k-1}\right)(4)=3\cdot\left(4^{k-1}\cdot 4\right)=3\cdot\left(4^{k-1}\cdot 4^1\right)$$
$$=3\cdot 4^{(k-1)+1}=3\cdot 4^{(k+1)-1}$$

It has been shown that S_k implies S_{k+1}. Therefore, the statement S_n is true for every positive integer value of n.

Note: Without counting, we can state that the fourth figure in the textbook has $3 \cdot 4^3 = 192$ sides.

31. The number of sides of the nth figure is $3 \cdot 4^{k-1}$ (from Exercise 30). To find the perimeter of each figure, multiply the number of sides by the length of each side. In each figure, the lengths of the sides are $\frac{1}{3}$ the lengths of the sides in the preceding figure. To find the perimeter of the nth figure, P_n, we try to find a pattern.

$P_1 = 3(1) = 3$

$P_2 = 3 \cdot 4\left(\frac{1}{3}\right) = 4$

$P_3 = 3 \cdot 4^2\left(\frac{1}{9}\right) = \frac{16}{3}$, and so on.

This gives a geometric sequence with $a_1 = 3$ and $r = \frac{4}{3}$. Thus,

$P_n = a_1 r^{n-1} \Rightarrow P_n = 3\left(\frac{4}{3}\right)^{n-1}$. The result may also be written as

$P_n = \frac{3^1 \cdot 4^{n-1}}{3^{n-1}} \Rightarrow P_n = \frac{4^{n-1}}{3^{-1} \cdot 3^{n-1}} \Rightarrow P_n = \frac{4^{n-1}}{3^{n-2}}$.

32. Recall that the area of an equilateral triangle with sides of length s is $A = \frac{\sqrt{3}}{4}s^2$ (see the solution to Exercise 68 from Section 7.3). The first figure pictured in the textbook is an equilateral triangle with $s = 1$, so it has area $A = \frac{\sqrt{3}}{4}(1)^2 = \frac{\sqrt{3}}{4}$. The second figure consists of the first figure with three smaller triangles added; each of the smaller triangles is equilateral with $s = \frac{1}{3}$ and area $\frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{36}$. The second figure has all the area of the first figure plus an additional $3\left(\frac{\sqrt{3}}{36}\right) = \frac{\sqrt{3}}{12}$. The third figure consists of the second figure with twelve smaller triangles added; each of the smaller triangles is equilateral with $s = \frac{1}{3}\left(\frac{1}{3}\right) = \frac{1}{9}$ and area $\frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2 = \frac{\sqrt{3}}{324}$. The third figure has all the area of the second figure plus an additional $12\left(\frac{\sqrt{3}}{324}\right) = \frac{\sqrt{3}}{27}$.

Likewise, the fourth figure would have an additional area of $\frac{4\sqrt{3}}{243}$. The areas $\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{27}, \frac{4\sqrt{3}}{243}, \ldots$ can be added together to find the area of any of the figures. The sum of the first n numbers in the sequence gives the area of the nth figure. Starting with the second number listed, the sequence is geometric with $a_1 = \frac{\sqrt{3}}{12}$ and $r = \frac{4}{9}$. Given that we have $n-1$ terms to consider, we have

(*continued on next page*)

(*continued from page 1137*)

$$A = \frac{a_1\left(1-r^{n-1}\right)}{1-r} + \frac{\sqrt{3}}{4} = \frac{\frac{\sqrt{3}}{12}\left[1-\left(\frac{4}{9}\right)^{n-1}\right]}{1-\frac{4}{9}} + \frac{\sqrt{3}}{4}$$

$$= \frac{\frac{\sqrt{3}}{12}\left[1-\left(\frac{4}{9}\right)^{n-1}\right]}{\frac{5}{9}} + \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}\left[1-\left(\frac{4}{9}\right)^{n-1}\right]}{12}\cdot\frac{9}{5} + \frac{\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}\left[1-\left(\frac{4}{9}\right)^{n-1}\right]}{20} + \frac{5\sqrt{3}}{20}$$

$$= \frac{3\sqrt{3}\left[1-\left(\frac{4}{9}\right)^{n-1}\right] + 5\sqrt{3}}{20}$$

$$= \frac{3\sqrt{3} - 3\sqrt{3}\left(\frac{4}{9}\right)^{n-1} + 5\sqrt{3}}{20} = \frac{8\sqrt{3} - 3\sqrt{3}\left(\frac{4}{9}\right)^{n-1}}{20}$$

$$= \sqrt{3}\left[\frac{8}{20} - \frac{3}{20}\left(\frac{4}{9}\right)^{n-1}\right] = \sqrt{3}\left[\frac{2}{5} - \frac{3}{20}\left(\frac{4}{9}\right)^{n-1}\right]$$

33. With 1 ring, 1 move is required. With 2 rings, 3 moves are required. Note that $2^2 - 1 = 3 = 2 + 1.$ With 3 rings, 7 moves are required. Note that $2^3 - 1 = 7 = 2^2 + 2 + 1.$ With *n* rings, the number of moves required are $2^{n-1} + 2^{n-2} + \ldots + 2^1 + 1.$ This is a geometric series with $a_1 = 1$ and $r = 2.$ The sum would be $\frac{a_1\left(1-r^n\right)}{1-r} = \frac{1\left(1-2^n\right)}{1-2} = \frac{1-2^n}{-1} = 2^n - 1.$ Thus, S_n is the statement: For *n* rings, the number of required moves is $2^n - 1.$ To prove this statement using mathematical induction, we perform the following steps.

Step 1: Show that the statement is true when $n = 1$.

S_1 is the statement: For 1 ring, the number of required moves is $2^1 - 1 = 2 - 1 = 1,$ which is true.

Step 2: Show that S_k implies S_{k+1}, where S_k is the statement: For k rings, the number of required moves is $2^k - 1.$ S_{k+1} is the statement: For $k+1$ rings, the number of required moves is $2^{k+1} - 1.$

Assume $k + 1$ rings are on the first peg. Since S_k is true, the top *k* rings can be moved to the second peg in $2^k - 1$ moves. Now move the bottom ring to the third peg. Since S_k is true, move the *k* rings on the second peg on top of the largest ring on the third peg in $2^k - 1$ moves. The total number of moves is

$$\left(2^k - 1\right) + 1 + \left(2^k - 1\right) = 2\cdot 2^k - 1 = 2^1\cdot 2^k - 1 = 2^{k+1} - 1.$$

Since S_n is true for $n = 1$ and $n = k + 1$ when is true for $n = k,$ S_n is true for every positive integer *n.*

Chapter 11 Quiz

(Sections 11.1–11.5)

1. $a_n = -4n + 2$

Replace *n* with 1, 2, 3, 4, and 5.

$n = 1: a_1 = -4(1) + 2 = -2$

$n = 2: a_2 = -4(2) + 2 = -6$

$n = 3: a_3 = -4(3) + 2 = -10$

$n = 4: a_4 = -4(4) + 2 = -14$

$n = 5: a_5 = -4(5) + 2 = -18$

The first five terms are −2, −6, −10, −14, and −18. There is a common difference of −4, so the sequence is arithmetic.

2. $a_n = -2\left(-\frac{1}{2}\right)^n$

Replace *n* with 1, 2, 3, 4, and 5.

$n = 1:\ a_1 = -2\left(-\frac{1}{2}\right)^1 = -2\left(-\frac{1}{2}\right) = 1$

$n = 2:\ a_2 = -2\left(-\frac{1}{2}\right)^2 = -2\left(\frac{1}{4}\right) = -\frac{1}{2}$

$n = 3:\ a_3 = -2\left(-\frac{1}{2}\right)^3 = -2\left(-\frac{1}{8}\right) = \frac{1}{4}$

$n = 4:\ a_4 = -2\left(-\frac{1}{2}\right)^4 = -2\left(\frac{1}{16}\right) = -\frac{1}{8}$

$n = 5:\ a_5 = -2\left(-\frac{1}{2}\right)^5 = -2\left(-\frac{1}{32}\right) = \frac{1}{16}$

The first five terms are $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8},$ and $\frac{1}{16}.$

There is a common ratio of $-\frac{1}{2}$, so the sequence is geometric.

3. $a_1 = 5, a_2 = 3, a_n = a_{n-1} + 3a_{n-2},$ for $n \ge 3$

$a_3 = a_2 + 3a_1 = 3 + 3(5) = 18$

$a_4 = a_3 + 3a_2 = 18 + 3(3) = 27$

$a_5 = a_4 + 3a_3 = 27 + 3(18) = 81$

The first five terms are 5, 3, 18, 27, and 81. There is neither a common difference nor a common ratio, so the sequence is neither arithmetic nor geometric.

4. $a_1 = -6, a_9 = 18$

Using the formula $a_n = a_1 + (n-1)d$, we have

$a_9 = a_1 + (9-1)d \Rightarrow 18 = -6 + 8d \Rightarrow 3 = d$

So $a_7 = a_1 + 6d \Rightarrow a_7 = -6 + 6(3) = 12$

5. **(a)** $a_1 = -20, d = 14$

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, so we have

$$S_{10} = \frac{10}{2}\left[2(-20) + (10-1)(14)\right] = 5\left[-40 + 9(14)\right] = 5(86) = 430$$

(b) $a_1 = -20, r = -\frac{1}{2}$

$S_n = \dfrac{a_1(1-r^n)}{1-r}$, so we have

$$S_{10} = \frac{-20\left[1-\left(-\frac{1}{2}\right)^{10}\right]}{1-\left(-\frac{1}{2}\right)} = \frac{-20\left[1-\frac{1}{1024}\right]}{\frac{3}{2}} = \frac{-20\left(\frac{1023}{1024}\right)}{\frac{3}{2}} = -\frac{1705}{128}$$

6. **(a)** $\sum_{i=1}^{30}(-3i+6)$

This is the sum of the arithmetic sequence with $a_1 = -3(1) + 6 = 3$, $d = -3$, and $n = 10$. Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\sum_{j=1}^{30}(-3i+6) = S_{30} = \frac{30}{2}\left[2(3) + 29(-3)\right] = 15(6-87) = 15(-81) = -1215$$

(b) $\sum_{i=1}^{\infty} 2^i$

This is a geometric series with $r = 2 > 1$, so the series diverges and the sum does not exist.

(c) $\sum_{i=1}^{\infty}\left(\frac{3}{4}\right)^i$

For this geometric series, $a_1 = \left(\frac{3}{4}\right)^1 = \frac{3}{4}$ and $r = \frac{3}{4}$. Because $-1 < r < 1$, this series converges. We have

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{\frac{3}{4}}{\frac{1}{4}} = \frac{3}{4}\cdot 4 = 3$$

7. $(x-3y)^5 = \left[x + (-3y)\right]^5$

$$= x^5 + \binom{5}{1}x^4(-3y) + \binom{5}{2}x^3(-3y)^2 + \binom{5}{3}x^2(-3y)^3 + \binom{5}{4}x(-3y)^4 + (-3y)^5$$

$$= x^5 + \frac{5!}{1!4!}x^4(-3y) + \frac{5!}{2!3!}x^3(9)y^2 + \frac{5!}{3!2!}x^2(-27)y^3 + \frac{5!}{4!1!}x(81)y^4 + (-243)y^5$$

$$= x^5 + 5(-3)x^4y + \frac{5\cdot 4}{2\cdot 1}(9)x^3y^2 + \frac{5\cdot 4}{2\cdot 1}(-27)x^2y^3 + 5(81)xy^4 - 243y^5$$

$$= x^5 - 15x^4y + 90x^3y^2 - 270x^2y^3 + 405xy^4 - 243y^5$$

8. $\left(4x - \frac{1}{2}y\right)^5$

Using the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 5$, $k = 5$, $k - 1 = 4$, and $n - (k - 1) = 1$, the fifth term of the expansion is

$$\binom{5}{4}(4x)\left(-\frac{1}{2}y\right)^4 = \frac{5!}{4!1!}(4x)\left(\frac{1}{16}y^4\right) = 5(4x)\left(\frac{1}{16}y^4\right) = \frac{5}{4}xy^4$$

9. **(a)** $9! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 362{,}880$

(b) $$C(10,4) = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$$

10. Let $S_n = 6+12+18+\ldots+6n = 3n(n+1)$

(a) *Verify the statement for* $n=1$.
S_1 is the statement $6 = 3(1)(1+2) \Rightarrow 6 = 6$, which is true.

(b) *Write the statement for* $n=k$: $S_k = 6+12+18+\ldots+6k = 3k(k+1)$

(c) *Write the statement for* $n=k+1$: $S_{k+1} = 6+12+18+\ldots+6k+6(k+1) = 3(k+1)[(k+1)+1]$

(d) *Assume the statement is true for* $n=k$. *Use algebra to change the statement in part* (*b*) *to the statement in part* (*c*): Start with S_k: $6+12+18+\ldots+6k = 3k(k+1)$ and add the $(k+1)$ st term, $7k$, to both sides of this equation.

$$\begin{aligned} 6+12+18+\ldots+6k &= 3k(k+1) \\ 6+12+18+\ldots+6k+6(k+1) &= 3k(k+1)+6(k+1) \\ 6+12+18+\ldots+6k+6(k+1) &= (3k+6)(k+1) \\ 6+12+18+\ldots+6k+6(k+1) &= 3(k+2)(k+1) \\ 6+12+18+\ldots+6k+6(k+1) &= 3(k+1)[(k+1)+1] \end{aligned}$$

(e) *Write a conclusion based on Steps* (*a*) – (*d*).
Since S_n is true for $n=1$ and $n=k+1$ when it is true for $n=k$, S_n is true for every positive integer *n*.

Section 11.6: Counting Theory

1. $P(12,8) = \dfrac{12!}{(12-8)!} = \dfrac{12!}{4!}$
$= \dfrac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1}$
$= 12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5$
$= 19{,}958{,}400$

2. $P(5,\ 5) = \dfrac{5!}{(5-5)!} = \dfrac{5!}{0!} = \dfrac{5\cdot4\cdot3\cdot2\cdot1}{1} = 120$

3. $P(9,\ 2) = \dfrac{9!}{(9-2)!} = \dfrac{9!}{7!} = \dfrac{9\cdot8\cdot7!}{7!} = 9\cdot8 = 72$

4. $P(10,9) = \dfrac{10!}{(10-9)!} = \dfrac{10!}{1!}$
$= \dfrac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1}$
$= 3{,}628{,}800$

5. $P(5,\ 1) = \dfrac{5!}{(5-1)!} = \dfrac{5!}{4!} = \dfrac{5\cdot4!}{4!} = 5$

6. $P(6,\ 0) = \dfrac{6!}{(6-0)!} = \dfrac{6!}{6!} = 1$

7. $C(4,2) = \dbinom{4}{2} = \dfrac{4!}{(4-2)!2!} = \dfrac{4!}{2!2!}$
$= \dfrac{4\cdot3\cdot2!}{2!2!} = \dfrac{4\cdot3}{2\cdot1} = 6$

8. $C(9,3) = \dbinom{9}{3} = \dfrac{9!}{6!3!} = \dfrac{9\cdot8\cdot7\cdot6!}{6!3!} = \dfrac{9\cdot8\cdot7}{3!}$
$= \dfrac{9\cdot8\cdot7}{3\cdot2\cdot1} = 84$

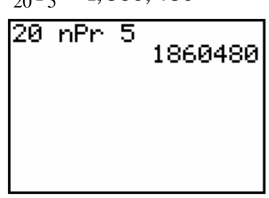
9. $C(6,0) = \dbinom{6}{0} = \dfrac{6!}{(6-0)!0!} = \dfrac{6!}{6!\cdot1} = 1$

10. $C(8,1) = \dbinom{8}{1} = \dfrac{8!}{7!1!} = \dfrac{8\cdot7!}{7!\cdot1} = \dfrac{8}{1} = 8$

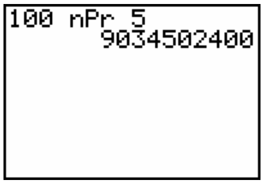
11. $\dbinom{12}{4} = \dfrac{12!}{(12-4)!4!} = \dfrac{12!}{8!4!} = \dfrac{12\cdot11\cdot10\cdot9\cdot8!}{8!4!}$
$= \dfrac{12\cdot11\cdot10\cdot9}{4!} = \dfrac{12\cdot11\cdot10\cdot9}{4\cdot3\cdot2\cdot1} = 495$

12. $\dbinom{16}{3} = \dfrac{16!}{(16-3)!3!} = \dfrac{16!}{13!3!} = \dfrac{16\cdot15\cdot14\cdot13!}{13!3!}$
$= \dfrac{16\cdot15\cdot14}{3\cdot2\cdot1} = 560$

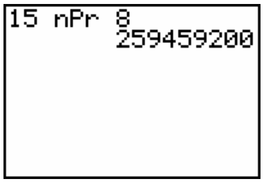
13. ${}_{20}P_5 = 1{,}860{,}480$



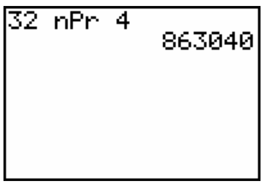
14. ${}_{100}P_5 = 9,034,502,400$



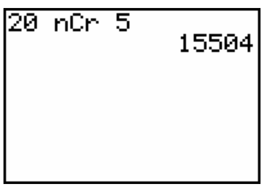
15. ${}_{15}P_8 = 259,459,200$



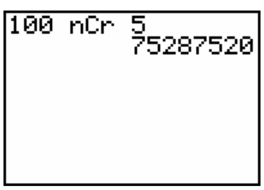
16. ${}_{32}P_4 = 863,040$



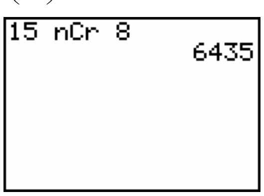
17. ${}_{20}C_5 = 15,504$



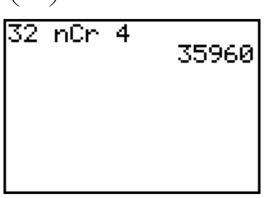
18. ${}_{100}C_5 = 75,287,520$



19. $\binom{15}{8} = C(15,8) = 6435$



20. $\binom{32}{4} = C(32,4) = 35,960$



21. **(a)** Since the order of digits in a telephone number does matter, this involves a permutation.

(b) Since the order of digits in a Social Security number does matter, this involves a permutation.

(c) Since the order of the cards in a poker hand does not matter, this involves a combination.

(d) Since the order of members on a committee of politicians does not matter, this involves a combination.

(e) Since the order of numbers of the "combination" on a combination lock does matter, this involves a permutation.

(f) Since the order does not matter, the lottery choice of six numbers involves a combination.

(g) Since the order of digits and/or letters on a license plate does matter, this involves a permutation.

22. Answers will vary. In determining whether a problem involves a permutation or a combination, you must determine if order matters. If the order is important, then the problem situation involves a permutation. If the order is not of importance, then the problem situation involves a combination.

23. Using the fundamental principle of counting we have $5 \cdot 4 \cdot 2 = 40.$ There are 40 different homes available if a builder offers a choice of 5 basic plans, 4 roof styles, and 2 exterior finishes.

24. Using the fundamental principle of counting we have $7 \cdot 6 \cdot 4 \cdot 5 = 840.$ There are 840 auto varieties if the auto manufacturer offers 7 models, 6 different colors, 4 different upholstery fabrics, and 5 interior colors.

25. **(a)** There are two choices for the first letter, K and W. The second letter can be any of the 26 letters of the alphabet except for the one chosen for the first letter, so there are 25 choices. The third letter can be any of the remaining 24 letters of the alphabet, so there are 24 choices. The fourth letter can be any of the remaining 23 letters of the alphabet, so there are 23 choices.Therefore, by the fundamental principle of counting, the number of possible 4-letter radio-station call letters is $2 \cdot 25 \cdot 24 \cdot 23 = 27,600.$

(b) There are two choices for the first letter. Because repeats are allowed, there are 26 choices for each of the remaining 3 letters.Therefore, by the fundamental principle of counting, the number of possible 4-letter radio-station call letters is $2 \cdot 26 \cdot 26 \cdot 26 = 35{,}152$.

(c) There are two choices for the first letter. There are 24 choices for the second letter since it cannot repeat the first letter and cannot be R. There are 23 choices for the third letter since it cannot repeat any of the first two letters and cannot be R. There is only once choice for the last letter since it must be R. Therefore, by the fundamental principle of counting, the number of possible 4-letter radio-station call letters is $2 \cdot 24 \cdot 23 \cdot 1 = 1104$.

26. Using the fundamental principle of counting, we have $3 \cdot 8 \cdot 5 = 120$.
There are 120 different possible meals if the menu offers 3 salad, 8 main dish, and 5 dessert choices.

27. Since each ordering of the blocks is considered a different arrangement, we have the permutation, $P(7,7)$.

$$P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 5040$$

There are 5040 different arrangements of the 7 blocks. Note: This problem could also be solved by using the fundamental principle of counting. There are 7 choices for the first block, 6 for the second, 5 for the third, etc. The number of arrangements would be $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$.

28. Using the fundamental principle of counting, we have $5 \cdot 3 = 15$ different first- and middle-name arrangements.

29. (a) The first three positions could each be any one of 26 letters, and the second three positions could each be any one of 10 digits. Using the fundamental principle of counting, we have $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$. Thus, 17,576,000 license plates were possible.

(b) Using the fundamental principle of counting, we have $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 17{,}576{,}000$. 17,576,000 additional license plates were made possible by the reversal.

(c) Using the fundamental principle of counting, we have $26 \cdot 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 456{,}976{,}000$. 456,976,000 plates were provided by prefixing the previous pattern with an additional letter.

30. (a) There are 5 single odd digits (1, 3, 5, 7, 9). Each telephone digit can be one of 5 choices. $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^7 = 78{,}125$

(b) The first digit can be one of 9 choices (0 is excluded). The last digit can be 0 only. $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 1 = 900{,}000$

(c) The first digit can be only one of 9 choices (0 is excluded). The last two digits can be 0 only. $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 1 \cdot 1 = 90{,}000$

(d) The first three digits each have only one choice. $1 \cdot 1 \cdot 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$

(e) The first digits can be one of 9 choices (0 is excluded). $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 544{,}320$

31. Since order does matter, we have the permutation,

$$P(9,9) = \frac{9!}{(9-9)!} = \frac{9!}{0!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 362{,}880$$

There are 362,880 ways to seat 9 people in 9 seats in a row.

32. Since the order does matter, we have the following permutation.

$$P(10,\ 7) = \frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 604{,}800$$

There are 604,800 ways to arrange 7 of 10 monkeys.

33. He has 6 choices for the first course, 5 choices for the second, and 4 choices for the third. Thus, the student has $6 \cdot 5 \cdot 4 = 120$ possible schedules according to the fundamental principle of counting.

34. 400 – 20 = 380 of the courses are not mathematics courses; 4 of these are to be arranged in an order to form a schedule. Since order does matter, we have the following permutation.

$$P(380,\ 4)=\frac{380!}{(380-4)!}=\frac{380!}{376!}=\frac{380\cdot 379\cdot 378\cdot 377\cdot 376!}{376!}=380\cdot 379\cdot 378\cdot 377=20,523,714,120$$

There are 20,523,714,120 schedules possible that do not include a math course.

35. The number of ways in which the 3 officers can be chosen from the 15 members is given by the following permutation.

$$P(15,3)=\frac{15!}{(15-3)!}=\frac{15!}{12!}=\frac{15\cdot 14\cdot 13\cdot 12!}{12!}=15\cdot 14\cdot 13=2730$$

36. The number of ways in which the 9 batters can be chosen from 20 players is given by the following permutation.

$$P(20,9)=\frac{20!}{(20-9)!}=\frac{20!}{11!}=\frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11!}{11!}=20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12=60,949,324,800$$

37. **(a)** $P(5,\ 5)=\frac{5!}{(5-5)!}=\frac{5!}{0!}=\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1}=120$

The letters can be arranged in 120 ways.

(b) $P(3,\ 3)=\frac{3!}{(3-3)!}=\frac{3!}{0!}=\frac{3\cdot 2\cdot 1}{1}=6$

The first three letters can be arranged in 6 ways.

38. 5 players can be assigned the 5 positions in

$$P(5,\ 5)=\frac{5!}{(5-5)!}=\frac{5!}{0!}=\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1}=120$$

ways.

10 players can be assigned 5 positions in

$$P(10,5)=\frac{10!}{(10-5)!}=\frac{10!}{5!}=10\cdot 9\cdot 8\cdot 7\cdot 6=30,240 \text{ ways.}$$

39. We want to choose 6 group members out of 40 and the order is not important. The number of possible groups is

$$C(40,6)=\binom{40}{6}=\frac{40!}{(40-6)!6!}=\frac{40!}{34!6!}=\frac{40\cdot 39\cdot 38\cdot 37\cdot 36\cdot 35\cdot 34!}{34!6!}=\frac{40\cdot 39\cdot 38\cdot 37\cdot 36\cdot 35}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=3,838,380$$

40. We want to choose 4 apples from a crate of 25 and order is not important. The number of ways the apples may be sampled is

$$C(25,4)=\binom{25}{4}=\frac{25!}{(25-4)!4!}=\frac{25!}{21!4!}=\frac{25\cdot 24\cdot 23\cdot 22\cdot 21!}{21!4!}=\frac{25\cdot 24\cdot 23\cdot 22}{4\cdot 3\cdot 2\cdot 1}=12,650$$

41. **(a)** We want to choose 4 extras from a choice of 6 extras and order is not important. The number of different hamburgers that can be made is

$$C(6,4)=\binom{6}{4}=\frac{6!}{(6-4)!4!}=\frac{6!}{2!4!}=\frac{6\cdot 5\cdot 4!}{2!4!}=\frac{6\cdot 5}{2\cdot 1}=15$$

(b) Since one of the four choices is fixed, we now want to choose 3 extras from a choice of 5 extras and order is not important. The number of different hamburgers that can be made is given by

$$C(5,3)=\binom{5}{3}=\frac{5!}{(5-3)!3!}=\frac{5!}{2!3!}=\frac{5\cdot 4\cdot 3!}{2!3!}=\frac{5\cdot 4}{2\cdot 1}=10$$

42. 4 planners can be selected from a group of 12 in $C(12,4)$ ways.

$$C(12,4)=\binom{12}{4}=\frac{12!}{(12-4)!4!}=\frac{12!}{8!4!}=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8!}{8!4!}=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}=495$$

Selecting the 8 planners not chosen from the group of 12 can be done $C(12,8)$ ways.

$$C(12,8)=\binom{12}{8}=\frac{12!}{(12-8)!8!}=\frac{12!}{4!8!}=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8!}{4!8!}=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}=495$$

43. This problem involves choosing 2 members from a set of 5 members. There are $C(5,2)$ such subsets.

$$C(5,2)=\binom{5}{2}=\frac{5!}{(5-2)!2!}=\frac{5!}{3!2!}$$
$$=\frac{5\cdot 4\cdot 3!}{3!2!}=\frac{5\cdot 4}{2\cdot 1}=10$$

There are 10 different 2-card combinations.

44. The number of 2-marble samples that can be chosen from a total of 15 marbles is $C(15,2)$.

$$C(15,2)=\binom{15}{2}=\frac{15!}{(15-2)!2!}=\frac{15!}{13!2!}$$
$$=\frac{15\cdot 14\cdot 13!}{13!2!}=\frac{15\cdot 14}{2\cdot 1}=105$$

The number of 4-marble samples that can be chosen from a total of 15 marbles is $C(15,4)$.

$$C(15,4)=\binom{15}{4}=\frac{15!}{(15-4)!4!}=\frac{15!}{11!4!}$$
$$=\frac{15\cdot 14\cdot 13\cdot 12\cdot 11!}{11!4!}=\frac{15\cdot 14\cdot 13\cdot 12}{4\cdot 3\cdot 2\cdot 1}$$
$$=1365$$

45. Since 2 blue marbles are to be chosen and there are 8 blue marbles, this problem involves choosing 2 members from a set of 8 members. There are $C(8,2)$ ways of doing this.

$$C(8,2)=\binom{8}{2}=\frac{8!}{(8-2)!2!}=\frac{8!}{6!2!}$$
$$=\frac{8\cdot 7\cdot 6!}{6!2!}=\frac{8\cdot 7}{2\cdot 1}=28$$

46. There are 25 apples in a crate and 5 of them are rotten.

(a) Since a sample of 3 must all be rotten, there are $C(5,3)$ ways to choose the rotten apples.

$$C(5,3)=\frac{5!}{(5-3)!3!}=\frac{5!}{2!3!}$$
$$=\frac{5\cdot 4\cdot 3!}{2!3!}=\frac{5\cdot 4}{2\cdot 1}=10$$

(b) Draw 1 rotten and 2 good apples in a sample of 3. There are $C(5,1)$ ways to draw one rotten apple and $C(20,2)$ ways to draw one good apple. Therefore, by the fundamental principle of counting, there are $C(5,1)\cdot C(20,2)$ ways to draw the sample.

$$C(5,1)\cdot C(20,2)=\binom{5}{1}\cdot\binom{20}{2}$$
$$=\frac{5!}{(5-1)!1!}\cdot\frac{20!}{(20-2)!2!}$$
$$=\frac{5!}{4!1!}\cdot\frac{20!}{18!2!}$$
$$=\frac{5\cdot 4!}{4!1!}\cdot\frac{20\cdot 19\cdot 18!}{18!2!}$$
$$=\frac{5}{1}\cdot\frac{20\cdot 19}{2\cdot 1}$$
$$=5\cdot 190=950$$

There are 950 ways to choose 1 rotten apple and 2 good apples.

47. There are 5 liberal and 4 conservatives, giving a total of 9 members. Three members are chosen as delegates to a convention.

(a) There are $C(9,3)$ ways of doing this.

$$C(9,3)=\binom{9}{3}=\frac{9!}{(9-3)!3!}=\frac{9!}{6!3!}$$
$$=\frac{9\cdot 8\cdot 7\cdot 6!}{6!3!}=\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}=84$$

84 delegations are possible.

(b) To get all liberals, we must choose 3 members from a set of 5, which can be done $C(5,3)$ ways.

$$C(5,3)=\binom{5}{3}=\frac{5!}{(5-3)!3!}=\frac{5!}{2!3!}$$
$$=\frac{5\cdot 4\cdot 3!}{2!3!}=\frac{5\cdot 4}{2}=10$$

10 delegations could have all liberals.

(c) To get 2 liberals and 1 conservative involves two independent events. First select the liberals. The number of ways to do this is

$$C(5,2)=\binom{5}{2}=\frac{5!}{3!2!}$$
$$=\frac{5\cdot 4\cdot 3!}{3!2!}=\frac{5\cdot 4}{2\cdot 1}=10$$

Now select the conservative. The number of ways to do this is as follows.

$$C(4,1)=\binom{4}{1}=\frac{4!}{(4-1)!1!}=\frac{4!}{3!1!}$$
$$=\frac{4\cdot 3!}{3!1!}=\frac{4}{1}=4$$

To find the number of delegations, use the fundamental principle of counting. The number of delegations with 2 liberals and 1 conservative is $10\cdot 4=40$.

(d) If one particular person must be on the delegation, then there are 2 people left to choose from a set consisting of 8 members. The number of ways to do this is as follows.

$$C(8,2)=\binom{8}{2}=\frac{8!}{(8-2)!2!}=\frac{8!}{6!2!}$$
$$=\frac{8\cdot7\cdot6!}{6!2!}=\frac{8\cdot7}{2\cdot1}=28$$

28 delegations are possible, which includes the mayor.

48. 7 workers send a delegation of 2.

(a) There are

$$C(7,2)=\binom{7}{2}=\frac{7!}{(7-2)!2!}=\frac{7!}{5!2!}$$
$$=\frac{7\cdot6\cdot5!}{5!2!}=\frac{7\cdot6}{2\cdot1}=21$$

possible different delegations.

(b) There are $C(6,1)$ ways to pick the second delegate. The number of ways to do this is as follows.

$$C(6,1)=\binom{6}{1}=\frac{6!}{(6-1)!1!}$$
$$=\frac{6!}{5!1!}=\frac{6\cdot5!}{5!1!}=6$$

Thus, there are $1\cdot\binom{6}{1}=6$ possible delegations that include a certain employee.

(c) 2 women and 5 men are in the group. There are $C(2,1)$ ways to select only one woman, and then $C(5,1)$ ways to select the remaining member, who would be a man. There are $C(2,2)$ ways to select two women and $C(5,0)$ ways to select no men. To find the number of different delegations including at least one woman, add the number of delegations with exactly one woman and the number of delegations containing two women.

$$C(2,1)\cdot C(5,1)+C(2,2)\cdot C(5,0)$$
$$=\binom{2}{1}\cdot\binom{5}{1}+\binom{2}{2}\cdot\binom{5}{0}$$
$$=\frac{2!}{(2-1)!1!}\cdot\frac{5!}{(5-1)!1!}$$
$$+\frac{2!}{(2-2)!2!}\cdot\frac{5!}{(5-0)!0!}$$
$$=\frac{2!}{1!1!}\cdot\frac{5!}{4!1!}+\frac{2!}{0!2!}\cdot\frac{5!}{5!0!}$$
$$=2\cdot5+1\cdot1=10+1=11$$

There are 11 delegations that would include at least one woman.

49. The problem asks how many ways can Dwight arrange his schedule. Therefore, order is important, and this is a permutation problem. There are

$$P(8,4)=\frac{8!}{(8-4)!}=\frac{8!}{4!}=\frac{8\cdot7\cdot6\cdot5\cdot4!}{4!}$$
$$=8\cdot7\cdot6\cdot5=1680$$

ways to arrange his schedule.

50. We are choosing a subset of 9 pineapples from a set containing 12 pineapples, so this is a combination problem. There are

$$C(12,9)=\binom{12}{9}=\frac{12!}{(12-9)!9!}=\frac{12!}{3!9!}$$
$$=\frac{12\cdot11\cdot10\cdot9!}{3!9!}=\frac{12\cdot11\cdot10}{3\cdot2\cdot1}=220$$

samples that can be drawn.

51. The order of the vegetables in the soup is not important, so this is a combination problem. There are

$$C(6,4)=\binom{6}{4}=\frac{6!}{(6-4)!4!}=\frac{6!}{2!4!}$$
$$=\frac{6\cdot5\cdot4!}{2!4!}=\frac{6\cdot5}{2\cdot1}=15$$

different soups she can make.

52. Order is important since the secretaries will be assigned to 3 different managers, so this is a permutation problem. The assignments can be made in the following number of ways.

$$P(7,\ 3)=\frac{7!}{(7-3)!}=\frac{7!}{4!}=\frac{7\cdot6\cdot5\cdot4!}{4!}$$
$$=7\cdot6\cdot5=210$$

53. Order is important in the seating, so this is a permutation problem. All thirteen children will have a specific location; the first twelve will sit down and thirteenth will be left standing.

$$P(13,\ 13)=\frac{13!}{(13-13)!}=\frac{13!}{0!}$$
$$=\frac{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1}$$
$$=6,227,020,800$$

There are 6,227,020,800 seatings possible.

54. Order is unimportant, so these are combination problems. There are a total of $6+3+2=11$ plants.

(a) At random, select 4 of the 11 plants. The number of ways four plants can be chosen from the 11 plants is

$$C(11,4)=\binom{11}{4}=\frac{11!}{(11-4)!4!}=\frac{11!}{7!4!}$$
$$=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7!}{7!4!}=\frac{11\cdot 10\cdot 9\cdot 8}{4\cdot 3\cdot 2\cdot 1}$$
$$=330$$

(b) If 2 of the 4 plants to be selected must be 2 of the 6 wheat plants, then the other 2 plants must be chosen from the remaining $11-6=5$ plants. Since selecting the wheat plants and selecting the other plants are independent events, the fundamental principle of counting can be used.

$$C(6,2)\cdot C(5,2)=\binom{6}{2}\cdot\binom{5}{2}$$
$$=\frac{6!}{(6-2)!2!}\cdot\frac{5!}{(5-2)!2!}$$
$$=\frac{6!}{4!2!}\cdot\frac{5!}{3!2!}$$
$$=\frac{6\cdot 5\cdot 4!}{4!2!}\cdot\frac{5\cdot 4\cdot 3!}{3!2!}$$
$$=\frac{6\cdot 5}{2\cdot 1}\cdot\frac{5\cdot 4}{2\cdot 1}$$
$$=15\cdot 10=150$$

This can be done in 150 ways

55. A club has 8 women and 11 men members. There are a total of $8+11=19$ members, and 5 of them are to be chosen. Order is not important, so this is a combination problem.

(a) Choosing all women implies of the 8 women, choose 5.

$$C(8,5)=\binom{8}{5}=\frac{8!}{(8-5)!5!}=\frac{8!}{3!5!}$$
$$=\frac{8\cdot 7\cdot 6\cdot 5!}{3!5!}=\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}=56$$

56 committees having 5 men can be chosen.

(b) Choose all men implies of the 11 men, choose 5.

$$C(11,5)=\binom{11}{5}=\frac{11!}{(11-5)!5!}=\frac{11!}{6!5!}$$
$$=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!5!}$$
$$=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1}=462$$

462 committees having 5 women can be chosen.

(c) Choosing 3 women and 2 men involves two independent events, each of which involves combinations. First, select the women. There are 8 women in the club, so the number of ways to do this is as follows.

$$C(8,3)=\binom{8}{3}=\frac{8!}{(8-3)!3!}=\frac{8!}{5!3!}$$
$$=\frac{8\cdot 7\cdot 6\cdot 5!}{5!3!}=\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}=56$$

Now, select the men. There are 11 men in the club, so the number of ways to do this is as follows.

$$C(11,2)=\binom{11}{2}=\frac{11!}{(11-2)!2!}=\frac{11!}{9!2!}$$
$$=\frac{11\cdot 10\cdot 9!}{9!2!}=\frac{11\cdot 10}{2\cdot 1}=55$$

To find the number of committees, use the fundamental principle of counting. The number of committees with 3 women and 2 men is $56\cdot 55=3080$.

(d) Choose no more than 3 men.
This means choose 0 men (and 5 women) or choose 1 man (and 4 women) or choose 2 men (and 3 women) or choose 3 men (and 2 women). For each of these choices, we use the fundamental principle of counting. Thus, the number of possible committees with no more than 3 men is

$$C(11,0)\cdot C(8,5)+C(11,1)\cdot C(8,4)$$
$$+C(11,2)\cdot C(8,3)$$
$$+C(11,3)\cdot C(8,2)$$

$$C(11,0)\cdot C(8,5)+C(11,1)\cdot C(8,4)+C(11,2)\cdot C(8,3)+C(11,3)\cdot C(8,2)$$
$$=\binom{11}{0}\cdot\binom{8}{5}+\binom{11}{1}\cdot\binom{8}{4}+\binom{11}{2}\cdot\binom{8}{3}+\binom{11}{3}\cdot\binom{8}{2}$$
$$=\frac{11!}{(11-0)!0!}\cdot\frac{8!}{(8-5)!5!}+\frac{11!}{(11-1)!1!}\cdot\frac{8!}{(8-4)!4!}+\frac{11!}{(11-2)!2!}\cdot\frac{8!}{(8-3)!3!}+\frac{11!}{(11-3)!3!}\cdot\frac{8!}{(8-2)!2!}$$
$$=\frac{11!}{11!0!}\cdot\frac{8!}{3!5!}+\frac{11!}{10!1!}\cdot\frac{8!}{4!4!}+\frac{11!}{9!2!}\cdot\frac{8!}{5!3!}+\frac{11!}{8!3!}\cdot\frac{8!}{6!2!}$$
$$=1\cdot\frac{8\cdot7\cdot6\cdot5!}{3!5!}+\frac{11\cdot10!}{10!1!}\cdot\frac{8\cdot7\cdot6\cdot5\cdot4!}{4!4!}+\frac{11\cdot10\cdot9!}{9!2!}\cdot\frac{8\cdot7\cdot6\cdot5!}{5!3!}+\frac{11\cdot10\cdot9\cdot8!}{8!3!}\cdot\frac{8\cdot7\cdot6!}{6!2!}$$
$$=1\cdot\frac{8\cdot7\cdot6}{3\cdot2\cdot1}+11\cdot\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}+\frac{11\cdot10}{2\cdot1}\cdot\frac{8\cdot7\cdot6}{3\cdot2\cdot1}+\frac{11\cdot10\cdot9}{3\cdot2\cdot1}\cdot\frac{8\cdot7}{2\cdot1}$$
$$=1\cdot56+11\cdot70+55\cdot56+165\cdot28=56+770+3080+4620=8526$$

8526 committees having no more than 3 men can be chosen.

56. Order is important since each person will have a different responsibility, so this is a permutation problem.

$$P(10,\ 4)=\frac{10!}{(10-4)!}=\frac{10!}{6!}=\frac{10\cdot9\cdot8\cdot7\cdot6!}{6!}$$
$$=10\cdot9\cdot8\cdot7=5040$$

The committee can be formed in 5040 ways.

57. For the first lock, there are 10 choices for the first, second, and third digits. Thus, the number of combinations on the first lock is $10\cdot10\cdot10=1000$. This is the same for the second lock, that is, it also has 1000 different combinations. Since each of the locks is independent from the other, we use the fundamental principle of counting. We have a total of $1000\cdot1000=1,000,000$ combinations possible.

58. Since there are 40 numbers when you consider 0 to 39 and a number may be repeated, there would be $40\cdot40\cdot40=64,000$ different combinations.

59. Each of the 12 switches has two possible settings. Since each switch is independent from the others, there is a total of $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2=2^{12}=4096$ codes possible.

60. Each of the 4 numbers has 10 possible choices. Since each choice is independent from the others, there are a total of $10\cdot10\cdot10\cdot10=10,000$ choices possible.

61. Because the keys are arranged in a circle, there is no "first" key. The number of distinguishable arrangements is the number of ways to arrange the other three keys in relation to any one of the keys, which is

$$P(3,3)=\frac{3!}{(3-3)!}=\frac{3!}{0!}=\frac{3!}{1}=3\cdot2\cdot1=6.$$

62. Because the seats are arranged in a circle, there is no "first" seat. The number of distinguishable arrangements is the number of ways to arrange the other seats (people) in relation to any one person, which is

$$P(7,7)=\frac{7!}{(7-7)!}=\frac{7!}{0!}=\frac{7!}{1}$$
$$=7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1=5040$$

In Exercises 63–69, we will be using the formulas $P(n,r)=\frac{n!}{(n-r)!}$ and $\binom{n}{r}=\frac{n!}{(n-r)!r!}$, where $r\le n$.

63. Show $P(n,n-1)=P(n,n)$.

$$P(n,n-1)=\frac{n!}{[n-(n-1)]!}=\frac{n!}{(n-n+1)!}$$
$$=\frac{n!}{1!}=\frac{n!}{1}=\frac{n!}{0!}=\frac{n!}{(n-n)!}=P(n,n)$$

64. Show $P(n,1)=n$.

$$P(n,1)=\frac{n!}{(n-1)!}=\frac{n\cdot(n-1)!}{(n-1)!}=n$$

65. Show $P(n,0)=1$.

$$P(n,0)=\frac{n!}{(n-0)!}=\frac{n!}{n!}=1$$

66. Show $\binom{n}{n} = 1.$

$$\binom{n}{n} = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = \frac{n!}{1 \cdot n!} = \frac{n!}{n!} = 1$$

67. Show $\binom{n}{0} = 1.$

$$\binom{n}{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n!0!} = \frac{n!}{n!1} = \frac{n!}{n!} = 1$$

68. Show $\binom{n}{n-1} = n.$

$$\binom{n}{n-1} = \frac{n!}{[n-(n-1)]!(n-1)!} = \frac{n!}{(n-n+1)!(n-1)!} = \frac{n!}{1!(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!} = n$$

69. Show $\binom{n}{n-r} = \binom{n}{r}.$

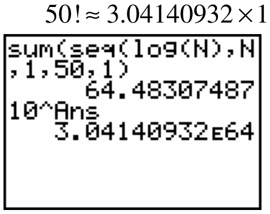
$$\binom{n}{n-r} = \frac{n!}{[n-(n-r)]!(n-r)!} = \frac{n!}{(n-n+r)!(n-r)!} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

70. Answers will vary. If $r > n,$ then $n - r$ would be negative and $(n-r)!$ would not exist.

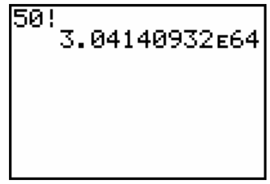
71. **(a)** $\log 50! = \log 1 + \log 2 + \log 3 + \cdots \log 50$

Using a sum and sequence utility on a calculator, we obtain the following.

$$\log 50! \approx 64.48307487$$
$$50! \approx 10^{64.48307487}$$
$$50! \approx 10^{.48307487} \times 10^{64}$$
$$50! \approx 3.04140932 \times 10^{64}$$



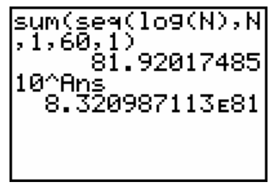
Computing the value directly, we obtain $50! \approx 3.04140932 \times 10^{64}.$



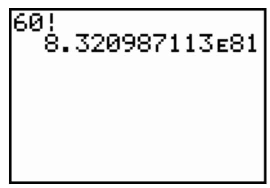
(b) $\log 60! = \log 1 + \log 2 + \log 3 + \cdots \log 60$

Using a sum and sequence utility on a calculator, we obtain the following.

$$\log 60! \approx 81.92017485$$
$$60! \approx 10^{81.92017485}$$
$$60! \approx 10^{.92017485} \times 10^{81}$$
$$60! \approx 8.320987113 \times 10^{81}$$



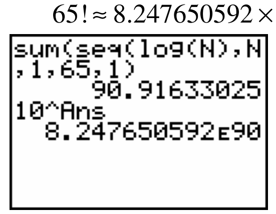
Computing the value directly, we obtain $60! \approx 8.320987113 \times 10^{81}.$



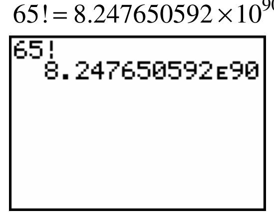
(c) $\log 65! = \log 1 + \log 2 + \log 3 + \cdots \log 65$

Using a sum and sequence utility on a calculator, we obtain the following.

$$\log 65! \approx 90.91633025$$
$$65! \approx 10^{90.91633025}$$
$$65! \approx 10^{.91633025} \times 10^{90}$$
$$65! \approx 8.247650592 \times 10^{90}$$



Computing the value directly, we obtain $65! = 8.247650592 \times 10^{90}$



72. **(a)** $P(47,13)=\dfrac{47!}{(47-13)!}=\dfrac{47!}{34!}$

$=47\cdot 46\cdot 45\cdot\ldots\cdot 35$

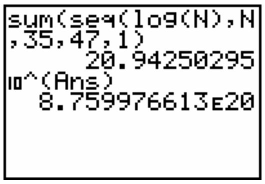
$\log P(47,\ 13)=\log 47+\log 46+\log 45+\ldots+\log 35$

$\log P(47,\ 13)\approx 20.94250295$

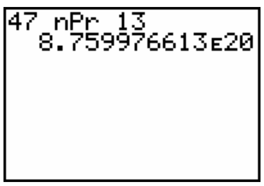
$P(47,\ 13)\approx 10^{20.94250295}$

$P(47,\ 13)\approx 10^{.94250295}\times 10^{20}$

$P(47,\ 13)\approx 8.759976613\times 10^{20}$



Computing the value directly, we obtain $P(47,13)\approx 8.759976613\times 10^{20}$.



(b) $P(50,4)=\dfrac{50!}{(50-4)!}=\dfrac{50!}{46!}$

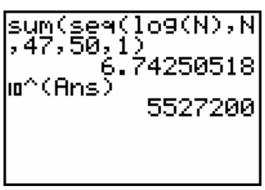
$=50\cdot 49\cdot 48\cdot 47$

$\log P(50,\ 4)=\log 50+\log 49+\log 48+\log 47$

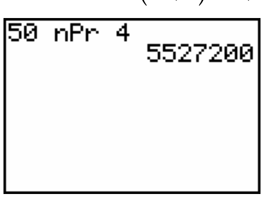
$\log P(50,\ 4)\approx 6.74250518$

$P(50,\ 4))\approx 10^{6.74250518}$

$P(50,\ 4)\approx 5{,}527{,}200$



(This is in fact the exact value of $P(50,4)$.) Computing the value directly, we obtain $P(50,4)=5{,}527{,}200$.



(c) $P(29,21)=\dfrac{29!}{(29-21)!}=\dfrac{29!}{8!}$

$=29\cdot 28\cdot 27\cdot\ldots\cdot 9$

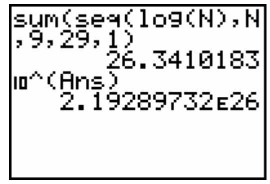
$\log P(29,\ 21)=\log 29+\log 28+\log 27+\ldots+\log 9$

$\log P(29,\ 21)\approx 26.34101830$

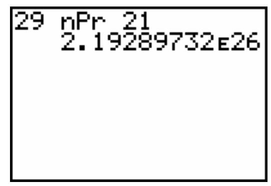
$P(29,\ 21)\approx 10^{26.34101830}$

$P(29,\ 21)\approx 10^{.34101830}\times 10^{26}$

$P(29,\ 21)\approx 2.19289732\times 10^{26}$



Computing the value directly, we obtain $P(29,21)=2.19289732\times 10^{26}$



Section 11.7: Basics of Probability

1. Let H = heads, T = tails. The only possible outcome is a head. Hence, the sample space is $S=\{H\}$.

2. Two coins are tossed. For each coin, either heads, H, or tails, T, lands up. The sample space is $S=\{(H,\ H),\ (H,\ T),\ (T,\ H),\ (T,\ T)\}$.

3. Since each coin can be a head or a tail and there are 3 coins, the sample space is

$S=\{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)\}$

4. Two slips are drawn. For each slip, a number, 1, 2, 3, 4, or 5, shows. The sample space may be written as

$S=\{(1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (1,\ 5),\ (2,\ 3),\ (2,\ 4),\ (2,\ 5),\ (3,\ 4),\ (3,\ 5),\ (4,\ 5)\}$

5. The sample space is

$S=\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3)\}$

6. A die is rolled and a coin tossed. The die shows 1, 2, 3, 4, 5, or 6. The coin shows H or T. The sample space is

$S=\{(1,\ H),\ (2,\ H),\ (3,\ H),\ (4,\ H),\ (5,\ H),\ (6,\ H),\ (1,\ T),\ (2,\ T),\ (3,\ T),\ (4,\ T),\ (5,\ T),\ (6,\ T)\}$

7. **(a)** "The result is heads" is the event $E_1 = \{H\}$. This event is certain to occur, $P(E_1) = 1$.

(b) "The result is tails" is the event $E_2 = \varnothing$. This event is an impossible event, so $P(E_2) = 0$.

8. The sample space is $S = \{(H, H), (H, T), (T, H), (T, T)\}$, so $n(S) = 4$.

(a) The event E_1 of both coins showing the same face is $E_1 = \{(H, H), (T, T)\}$, so $n(E_1) = 2$. The probability is $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{4} = \frac{1}{2}$.

(b) The event E_2 where at least one head appears is $E_2 = \{(H, H), (H, T), (T, H)\}$, so $n(E_2) = 3$. The probability is $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{4}$.

9. The sample space is $S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$, so $n(S) = 9$.

(a) The result is a repeated number" is the event $E_1 = \{(1, 1), (2, 2), (3, 3)\}$, so $n(E_1) = 3$. The probability is $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{9} = \frac{1}{3}$.

(b) "The second number is 1 or 3" is the event $E_2 = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$, so $n(E_2) = 6$. The probability is $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{9} = \frac{2}{3}$.

(c) "The first number is even and the second number is odd" is the event $E_3 = \{(2, 1), (2, 3)\}$, So, $n(E_3) = 2$. The probability is $P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{9}$.

10. The sample space is $S = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$, so $n(S) = 10$.

(a) Event E_1 (both numbers even) = $\{(2, 4)\}$, so $n(E_1) = 1$. The probability is $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{10}$.

(b) Event E_2 (both numbers odd) $= \{(1, 3), (1, 5), (3, 5)\}$, so $n(E_2) = 3$. The probability is $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{10}$

(c) Event E_3 (both numbers the same) $= \varnothing$, so $n(E_3) = 0$. The probability is $P(E_3) = \frac{n(E_3)}{n(S)} = \frac{0}{10} = 0$

(d) Event E_4 (one odd number, one even number) = $\{(1, 2), (1, 4), (2, 3), (2, 5), (3, 4), (4, 5)\}$, so $n(E_4) = 6$. The probability is $P(E_4) = \frac{n(E_4)}{n(S)} = \frac{6}{10} = \frac{3}{5}$.

11. For any event E, $0 \le P(E) \le 1$. Since $\frac{6}{5} > 1$, the student must have made a mistake.

12. Since $P(E) = .857$, $P(E') = 1 - .857 = .143$.

13. **(a)** F **(b)** D **(c)** A **(d)** F **(e)** C **(f)** B **(g)** E

14. A batting average of .300 means for every 10 times at bat (the sample space), the batter will get 3 hits. If the event E is "getting a hit," then $P(E) = .300$. Thus, we have $P(E') = 1 - .300 = .700$. The odds in favor of him getting a hit are $\frac{P(E)}{P(E')} = \frac{.300}{.700} = \frac{3}{7}$ or 3 to 7.

15. There are 15 marbles, so $n(S) = 15$.

(a) E_1 : yellow marbles is drawn
There are three yellow marbles, so $n(E_1) = 3$. The probability is $P(E_1) = \frac{3}{15} = \frac{1}{5}$.

(b) E_2 : black marble is drawn
There are no black marbles so $n(E_2) = 0$. The probability is $P(E_2) = \frac{0}{15} = 0$.

(c) E_3: yellow or white marble is drawn
There are 3 yellow and 4 white marbles so $n(E_3) = 3 + 4 = 7.$ The probability is $P(E_3) = \frac{7}{15}.$

(d) E_4: yellow marble is not drawn
There are 12 non-yellow marbles, so $n(E_4') = 12.$ The probability is $P(E_4') = \frac{12}{15} = \frac{4}{5}.$
The odds in favor of drawing a yellow marble are $\frac{P(E_4)}{P(E_4')} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}$ or 1 to 4.

(e) The probability that a blue marble is drawn is $P(E_2) = \frac{8}{15}$. The probability that a blue marble is not drawn is $P(E_2') = 1 - P(E_2) = \frac{7}{15}.$ The odds against drawing a blue marble are
$$\frac{P(E_2')}{P(E_2)} = \frac{\frac{7}{15}}{\frac{8}{15}} = \frac{7}{8} \text{ or 7 to 8}$$

16. E: a bank will make the loan. We have $P(E) = .002$ and $P(E') = 1 - P(E) = 1 - .002 = .998.$ The odds against such a bank making such a loan are
$$\frac{P(E')}{P(E)} = \frac{.998}{.002} = \frac{499}{1} \text{ or 499 to 1.}$$

17. The sample space for the event of rolling 2 dice is $S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\}$, so $n(S) = 36$ (See Example 5 on pages 715–716 of the text.).

(a) The event in which the sum is at least 10 can be represented by $E_1 = \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$, so $n(E_1) = 6$. The probability of E_1 is $P(E_1) = \frac{6}{36} = \frac{1}{6}.$

(b) If the event in which the sum is 7 is E_2 then $E_2 = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$, so $n(E_2) = 6$. We have $P(E_2) = \frac{6}{36} = \frac{1}{6}.$ The probability of the union of two events, E_2 or E_1, is
$$\begin{aligned} P(E_2 \text{ or } E_1) &= P(E_2 \cup E_1) \\ &= P(E_2) + P(E_1) - P(E_2 \cap E_1) \end{aligned}$$
Since $P(E_2 \cap E_1) = 0$ and $P(E_1) = \frac{1}{6}$ (from part (a)), we have
$P(E_2 \text{ or } E_1) = \frac{1}{6} + \frac{1}{6} - 0 = \frac{2}{6} = \frac{1}{3}.$

(c) The event in which the sum is 2 is $E_3 = \{(1, 1)\}$, so $n(E_3) = 1$, and thus $P(E_3) = \frac{1}{36}$. The event in which both dice show the same number is $E_4 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$, so $n(E_4) = 6$, and $P(E_4) = \frac{6}{36} = \frac{1}{6}$. The intersection of these two events is $E_3 \cap E_4 = \{1, 1\}$, so $n\left(E_3 \cap E_4\right) = 1$, and $P(E_3 \cap E_4) = \frac{1}{36}$. Hence the probability of these two alternate events occurring is
$$\begin{aligned} P(E_3 \text{ or } E_4) &= P(E_3 \cup E_4) \\ &= P(E_3) + P(E_4) \\ &\quad - P(E_3 \cap E_4) \\ &= \tfrac{1}{36} + \tfrac{6}{36} - \tfrac{1}{36} = \tfrac{6}{36} = \tfrac{1}{6} \end{aligned}$$

18. Let E be the event "English is spoken." This corresponds to 100% – 20.4% = 79.6% of the households so $P(E) = .796$. Thus, $P(E') = .204.$ The odds that English is spoken is $\frac{P(E)}{P(E')} = \frac{.796}{.204} = \frac{199}{51}$ or 199 to 51.

19. The total of all types is
$$58{,}930 + 209{,}117 + 43{,}511 + 2{,}744 + 20{,}748 = 335{,}050$$

(a) $P(\text{of Hispanic origin}) = \frac{58{,}930}{335{,}050} \approx .176$

(b) $$\begin{aligned} P(\text{not white}) &= 1 - P(\text{White}) \\ &= 1 - \frac{209{,}117}{335{,}050} = \frac{125{,}933}{335{,}050} \\ &\approx .376 \end{aligned}$$

(c) $$P\big(\text{Indian (Native American) or Black}\big) = \frac{2{,}744 + 43{,}511}{335{,}050} = \frac{46{,}255}{335{,}050} \approx .138$$

(d) Let E be "selected resident was Asian."
We have $P(E) = \frac{20{,}748}{335{,}050}$ and
$$\begin{aligned} P(E') &= 1 - P(E) \\ &= 1 - \frac{20{,}748}{335{,}050} = \frac{314{,}302}{335{,}050}. \end{aligned}$$
The odds that a randomly selected U.S. resident was an Asian is
$$\frac{P(E)}{P(E')} = \frac{\frac{20{,}748}{335{,}050}}{\frac{314{,}302}{335{,}050}} = \frac{10{,}374}{157{,}151}$$
or 10,374 to 157,151 (about 1 to 15).

20. The total population (in millions) in 1995 was $51.4 + 61.8 + 91.8 + 57.7 = 262.7$. The total population (in millions) in 1997 was $51.6 + 62.5 + 94.2 + 59.4 = 267.7$. The total population (in millions) in 2000 was $53.6 + 64.4 + 100.2 + 63.2 = 281.4$.

(a) Let E be "lived in the West in 1997." Then $n(E) = 59.4$. Let S be "total population in 1997." Then $n(S) = 267.7$.
Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{59.4}{267.7} \approx .222$.

(b) Let E be "lived in the Midwest in 1995." Then $n(E) = 61.8$. Let S be "total population in 1995." Then $n(S) = 262.7$
Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{61.8}{262.7} \approx .235$.

(c) Let E be "lived in the Northeast or Midwest in 1997." Then $n(E) = 51.6 + 62.5 = 114.1$. Let ($S$) be "total population in 1997." Then $n(S) = 267.7$.
Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{114.1}{267.7} \approx .426$.

(d) Let E be "lived in the South or West in 1997." Then $n(E) = 94.2 + 59.4 = 153.6$. Let S be "total population in 1997." Then $n(S) = 267.7$.
Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{153.6}{267.7} \approx .574$.

(e) Let E be "from the South in 2000." Then $n(E) = 100.2$. Let S be "total population in 2000." Then $n(S) = 281.4$. Then, $n(E')$, the population not from the South in 2000, is $n(E') = n(S) - n(E)$ $= 281.4 - 100.2 = 181.2$ Thus, the odds that the selected resident in 2000 was not from the South are
$$\frac{n(E')}{n(E)} = \frac{181.2}{100.2} = \frac{1812}{1002} = \frac{302}{167}$$
or 302 to 167.

21. Each suit has thirteen cards, and the probability of choosing the correct card in that suit is $\frac{1}{13}$. Thus, we have P(4 correct choices)
$= \frac{1}{13} \cdot \frac{1}{13} \cdot \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{28,561}$. The probability of getting all four picks correct and winning \$5000 is
$$= \frac{1}{13} \cdot \frac{1}{13} \cdot \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{28,561} \approx .000035.$$

22. There are four choices for the incorrect suit. The probability of picking the incorrect card in a given suit is $\frac{12}{13}$. The probability of picking the correct card in each of the other three suits is $\frac{1}{13}$. Therefore, the probability of getting three picks correct and winning \$200 is as follows.
$$4 \cdot \frac{12}{13} \cdot \frac{1}{13} \cdot \frac{1}{13} \cdot \frac{1}{13} = \frac{48}{28,561} \approx .001681$$

23. **(a)** A 40-year old man who lives 30 more years would be a 70-year old man. Let E be "selected man will live to be 70." Then $n(E) = 66,172$. For this situation, the sample space S is the set of all 40-yr-old men, so $n(S) = 94,558$. Thus, the probability that a 40-yr-old man will live 30 more yr is
$$P(E) = \frac{n(E)}{n(S)} = \frac{66,172}{94,558} \approx .6998.$$

(b) Using the notation and results from part (a), the probability that a 40-yr-old man will not live 30 more yr is
$P(E') = 1 - P(E) = 1 - .6998 = .3002$.

(c) Use the notation and results from parts (a) and (b). In this binomial experiment, we call "a 40-yr-old man survives to age 70" a success. Then $p = P(E) = .6998$ and $1 - p = P(E') = .3002$. There are 5 independent trials and we need the probability of 3 successes, so $n = 5$, $r = 3$. The probability that exactly 3 of the 40-yr-old men survive to age 40 is
$$\begin{aligned}&\binom{n}{r} p^r (1-p)^{n-r} \\ &= \binom{5}{3} (.6998)^3 (.3002)^{5-3} \\ &= \frac{5!}{(5-3)!3!} (.6998)^3 (.3002)^2 \\ &= \frac{5!}{2!3!} (.6998)^3 (.3002)^2 \\ &= \frac{5 \cdot 4}{2 \cdot 1} (.6998)^3 (.3002)^2 \\ &= 10(.6998)^3 (.3002)^2 \approx .3088\end{aligned}$$

(d) Let F be the event "at least one man survives to age 70." The easiest way to find $P(F)$ is to first find the probability of the complement of the event, F'. F' is the event that "neither man survives to age 70." Since $P(F') = P(E')\cdot P(E')$ $= (.3002)^2 \approx .0901,$ we have $P(F) = 1 - P(F') \approx 1 - .0901 = .9099.$

24. **(a)** $P(\text{male}) = 1 - P(\text{female}) = 1 - .28 = .72$

(b) P(less than 5 years)
$= 1 - P(\text{5 or more years}) = 1 - .30 = .70$

(c) $P(\text{retirement plan contributor or female})$
$= P(\text{contributor}) + P(\text{female}) - P(\text{contributor and female})$

We have P(retirement plan contributor) = .65 and P(female) = .28. Since 50% of the female workers contribute to the retirement plan,
$P(\text{contributor and female}) = (.50)(.28) = .14.$

Thus, we have $P(\text{contributor or female})$
$= .65 + .28 - .14 = .79$

25. The amount of growth would be 11,400 – 10,000 = 1400, so the percent growth would be $\dfrac{1400}{10,000} = .14$. Let E be the event "worth at least \$11,400 by the end of the year." This is equivalent to "at least 14 percent growth" which is equivalent to "14 or 18 percent growth." Thus, we have

$$\begin{aligned} P(E) &= P(\text{14\% growth or 18\% growth}) \\ &= P(\text{14\% growth}) + P(\text{18\% growth}) \\ &= .20 + .10 = .30 = .3 \end{aligned}$$

26. From Section 4.2 (page 422 of the text), the compound interest formula is

$$A = P\left(1 + \frac{r}{m}\right)^{tm}.$$

Letting A = 15,000, P = 10,000, m = 1, and t = 3, we solve for r.

$$15,000 = 10,000\left(1 + \frac{r}{1}\right)^{1\cdot 3}$$

$$15,000 = 10,000(1+r)^3 \Rightarrow (1+r)^3 = 1.5$$

$$1 + r = \sqrt[3]{1.5} \Rightarrow r = \sqrt[3]{1.5} - 1 \approx .1447$$

To grow to at least \$15,000, from the table, the growth possibility would have to be 18 percent growth. Thus, the probability this will happen is .10 = .1.

27. In this binomial experiment, we call "smoked less than 10" a success. Then n = 10, r = 4, and p = .45 + .24 = .69.

$$\begin{aligned} &P(\text{4 smoked less than 10}) \\ &= \binom{n}{r} p^r (1-p)^{n-r} \\ &= \binom{10}{4} (.69)^4 (1-.69)^{10-4} \\ &= \frac{10!}{(10-4)!4!} (.69)^4 (.31)^6 \\ &= \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} (.69)^4 (.31)^6 \\ &= 210(.69)^4 (.31)^6 \approx .042246 \end{aligned}$$

28. In this binomial experiment, we call "smoked a pack or more" a success. Then n = 10, r = 5, and p = .11.

$$\begin{aligned} &P(\text{smoked a pack or more}) \\ &= \binom{n}{r} p^r (1-p)^{n-r} \\ &= \binom{10}{5} (.11)^5 (1-.11)^{10-5} \\ &= \frac{10!}{(10-5)!5!} (.11)^5 (.89)^5 \\ &= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} (.11)^5 (.89)^5 \\ &= 252(.11)^5 (.89)^5 \approx .002266 \end{aligned}$$

29. P(smoked between 1 and 19)
$= P$(smoked 1 to 9 or smoked 10 to 19)
$= P$(smoked 1 to 9) + P(smoked 10 to 19)
$= .24 + .20 = .44$

In this binomial experiment, call "smoked between 1 and 19" a success. Then p = .44 and $1 - p$ = .56. "Fewer than 2" means 0 or 1, so P(fewer than 2 smoked between 1 and 19) is

P(0 smoked between 1 and 19)
+ P(1 smoked between 1 and 19)

$$\begin{aligned} &= \binom{10}{0}(.44)^0 (56)^{10} + \binom{10}{1}(.44)^1 (56)^9 \\ &= 1(1)(56)^{10} + 10(.44)(56)^9 \\ &\approx .003033 + .023831 = .026864 \end{aligned}$$

30. In this binomial experiment, we call "smoked less than 1" a success. "No more than 3" corresponds to "0, 1, 2, or 3." Then $n = 10$, $r = 0, 1, 2,$ or 3, and $p = .45$. P(no more than 3 smoked less than 1) is equivalent to
$P(0 \text{ smoked less than } 1) + P(1 \text{ smoked less than } 1) + P(2 \text{ smoked less than } 1) + P(3 \text{ smoked less than } 1)$

This is equivalent to

$$\binom{10}{0}(.45)^0(1-.45)^{10-0} + \binom{10}{1}(.45)^1(1-.45)^{10-1} + \binom{10}{2}(.45)^2(1-.45)^{10-2} + \binom{10}{3}(.45)^3(1-.45)^{10-3}$$

This simplifies as follows.

$$\begin{aligned}
&\frac{10!}{(10-0)!0!}(.45)^0(.55)^{10} + \frac{10!}{(10-1)!1!}(.45)^1(.55)^9 + \frac{10!}{(10-2)!2!}(.45)^2(.55)^8 + \frac{10!}{(10-3)!3!}(.45)^3(.55)^7 \\
&= \frac{10!}{10!0!}(.45)^0(.55)^{10} + \frac{10!}{9!1!}(.45)^1(.55)^9 + \frac{10!}{8!2!}(.45)^2(.55)^8 + \frac{10!}{7!3!}(.45)^3(.55)^7 \\
&= 1(1)(.55)^{10} + 10(.45)(.55)^9 + \frac{10\cdot 9}{2\cdot 1}(.45)^2(.55)^8 + \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}(.45)^3(.55)^7 \\
&= (.55)^{10} + 10(.45)(.55)^9 + 45(.45)^2(.55)^8 + 120(.45)^3(.55)^7 \\
&\approx .002533 + .020724 + .076303 + .166478 = .266038
\end{aligned}$$

31. $P(\text{a student applied to fewer than 4 colleges})$
$= P(\text{a student applied 1 college})$
$\quad + P(\text{a student applied to 2 or 3 colleges})$
$= .20 + .29 = .49$

32. $P(\text{a student applied to at least 2 colleges})$
$= 1 - P(\text{a student applied 1 college})$
$= 1 - .20 = .8$

33. $P(\text{a student applied to more than 3 colleges})$
$= P(\text{a student applied 4 - 6 college})$
$\quad + P(\text{a student applied to 7 or more colleges})$
$= .37 + .14 = .51$

34. Since the sum of the probabilities is
.20 + .29 + .37 + .14 = 1,
P(a student applied to 0 colleges) = 0.

35. In these binomial experiments, we call "the man is color-blind" a success.

a) For "exactly 5 are color-blind", we have $n = 53$, $r = 5$, and $p = .042$.

$$\begin{aligned}
P(\text{exactly 5 are color-blind}) &= \binom{53}{5}(.042)^5(1-.042)^{53-5} = \frac{53!}{(53-5)!5!}(.042)^5(.958)^{48} \\
&= \frac{53!}{48!5!}(.042)^5(.958)^{48} = \frac{53\cdot 52\cdot 51\cdot 50\cdot 49}{5\cdot 4\cdot 3\cdot 2\cdot 1}(.042)^5(.958)^{48} \\
&= 2,869,685(.042)^5(.958)^{48} \approx .047822
\end{aligned}$$

(b) "No more than 5 are color-blind" corresponds to "0, 1, 2, 3, 4, or 5 are color-blind." Then $n = 53$, $r = 0, 1, 2, 3, 4,$ or 5, and $p = .042$.
P(no more than 5 are color-blind) = P(0 are color-blind) + P(1 are color-blind) + P(2 are color-blind) + P(3 are color-blind) + P(4 are color-blind) + P(5 are color-blind)

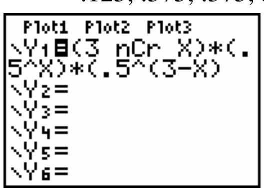
$$\begin{aligned}
&= \binom{53}{0}(.042)^0(1-.042)^{55-0} + \binom{53}{1}(.042)^1(1-.042)^{53-1} + \binom{53}{2}(.042)^2(1-.042)^{53-2} \\
&\quad + \binom{53}{3}(.042)^3(1-.042)^{53-3} + \binom{53}{4}(.042)^4(1-.042)^{53-4} \\
&\quad + \binom{53}{5}(.042)^5(1-.042)^{53-5}
\end{aligned}$$

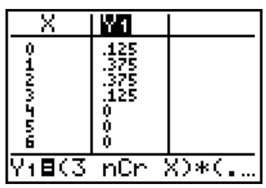
$$= \frac{53!}{(53-0)!0!}(1)(.958)^{53} + \frac{53!}{(53-1)!1!}(.042)^1(.958)^{52} + \frac{53!}{(53-2)!2!}(.042)^2(.958)^{51}$$
$$+ \frac{53!}{(53-3)!3!}(.042)^3(.958)^{50} + \frac{53!}{(53-4)!4!}(.042)^4(.958)^{49}$$
$$+ \frac{53!}{(53-5)!5!}(.042)^5(.958)^{48}$$
$$= \frac{53!}{53!0!}(.958)^{53} + \frac{53\cdot 52!}{52!1!}(.042)^1(.958)^{52} + \frac{53\cdot 52\cdot 51!}{51!2!}(.042)^2(.958)^{51} + \frac{53\cdot 52\cdot 51\cdot 50!}{50!3!}(.042)^3(.958)^{50}$$
$$+ \frac{53\cdot 52\cdot 51\cdot 50\cdot 49!}{49!4!}(.042)^4(.958)^{49} + \frac{53\cdot 52\cdot 51\cdot 50\cdot 49\cdot 48!}{48!5!}(.042)^5(.958)^{48}$$
$$= 1(.958)^{53} + 53(.042)^1(.958)^{52} + 1378(.042)^2(.958)^{51} + 23,426(.042)^3(.958)^{50}$$
$$+ 292,825(.042)^4(.958)^{49} + 2,869,685(.042)^5(.958)^{48}$$
$$\approx .102890 + .239074 + .272514 + .203105 + .111305 + .047822 = .976710$$

(c) Let E be "at least 1 is color-blind." Consider the complementary event E': "0 are color-blind." Thus, we have

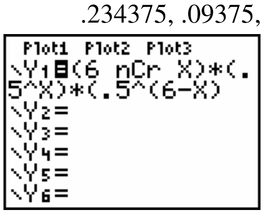
$$P(E) = 1 - P(E') = 1 - \binom{53}{0}(.042)^0(1-.042)^{53-0} = 1 - \frac{53!}{(53-0)!0!}(1)(.958)^{53}$$
$$= 1 - \frac{53!}{53!0!}(1)(.958)^{53} = 1-(1)(.958)^{53} = 1-(.958)^{53}\ 1 - .102890 = .897110$$

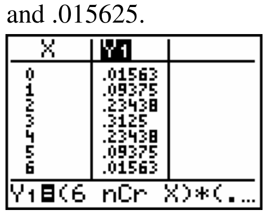
36. (a) Using the TABLE feature of a graphing calculator, the probabilities, in order, are .125, .375, .375, and .125.





(b) Using the TABLE feature of a graphing calculator, the probabilities, in order, are .015625, .09375, .234375, .3125, .234375, .09375, and .015625.





Note: The stated probabilities are the actual values. The rounded values of .01563, .09375, .23438, .3125, .23438, .09375, and .01563 are also acceptable.

37. (a) First compute $q = (1-p)^I = (1-.1)^2 = .9^2 = .81$. Then, with $S = 4$, $k = 3$, and $q = .81$, we have

$$P = \binom{S}{k}q^k(1-q)^{S-k}$$
$$= \binom{4}{3}.81^3(1-.81)^{4-3}$$
$$= \frac{4!}{(4-3)!3!}.81^3(.19)$$
$$= 4(.81^3)(.19) \approx .404$$

There is about a 40.4% chance of exactly 3 people not becoming infected.

(b) Compute $q = (1-p)^I = (1-.5)^2 = .5^2 = .25$ Then, with S = 4, $k = 3$, and $q = .25$, we have

$$P = \binom{S}{k}q^k(1-q)^{S-k}$$
$$= \binom{4}{3}.25^3(1-.25)^{4-3}$$
$$= \frac{4!}{(4-3)!3!}.25^3(.75)$$
$$= 4(.25^3)(.75) \approx .047$$

There is about 4.7 % chance of this occurring when the disease is highly infectious.

(c) Compute $q = (1-p)^I = (1-.5)^1 = .5$.

Then, with $S = 9$, $k = 0$, and $q = .5$, we have

$$P = \binom{S}{k} q^k (1-q)^{S-k} = \binom{9}{0} .5^0 (1-.5)^{9-0}$$
$$= \frac{9!}{(9-0)!0!}(1)(.5)^9 = 1(1)(.5^9) \approx .002$$

There is about a .2% chance of everyone becoming infected. This means that in a large family or group of people, it is highly unlikely that everyone will become sick even though the disease is highly infectious.

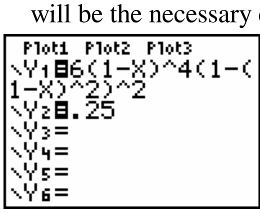
38. First write $q = (1-p)^I = (1-p)^2$.

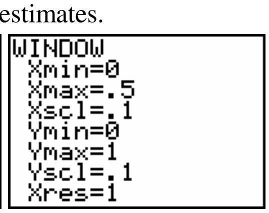
Substituting this expression for q and letting $I = 2$, $S = 4$, and $k = 2$, we can write P as a function of p as follows.

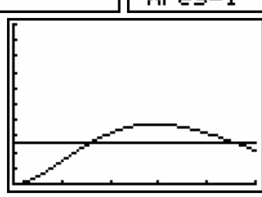
$$P = \binom{S}{k} q^k (1-q)^{S-k}$$
$$= \binom{4}{2}\left[(1-p)^2\right]^2 \left[1-(1-p)^2\right]^{4-2}$$
$$= \frac{4!}{(4-2)!2!}(1-p)^4\left[1-(1-p)^2\right]^2$$
$$= \frac{4!}{2!2!}(1-p)^4\left[1-(1-p)^2\right]^2$$
$$= \frac{4\cdot 3}{2\cdot 1}(1-p)^4\left[1-(1-p)^2\right]^2$$
$$= 6(1-p)^4\left[1-(1-p)^2\right]^2$$

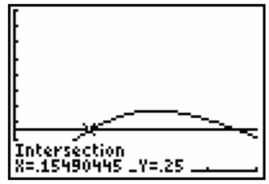
We now have

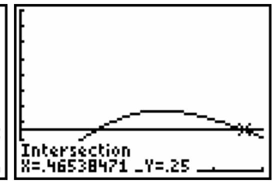
$P(p) = 6(1-p)^4\left[1-(1-p)^2\right]^2$. Graph the equations $Y_1 = 6(1-x)^4\left[1-(1-x)^2\right]^2$ and $Y_2 = .25$. The values of p where they intersect will be the necessary estimates.











Their graphs intersect when $p \approx .1549$ and $p \approx .4654$.

Chapter 11 Review Exercises

1. $a_n = \frac{n}{n+1}$

$a_1 = \frac{1}{1+1} = \frac{1}{2}$; $a_2 = \frac{2}{2+1} = \frac{2}{3}$; $a_3 = \frac{3}{3+1} = \frac{3}{4}$;
$a_4 = \frac{4}{4+1} = \frac{4}{5}$; $a_5 = \frac{5}{5+1} = \frac{5}{6}$

The first five terms are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$.

This sequence does not have a common difference or a common ratio, so the sequence is neither arithmetic nor geometric.

2. $a_n = (-2)^n$

$a_1 = (-2)^1 = -2$; $a_2 = (-2)^2 = 4$; $a_3 = (-2)^3 = -8$;
$a_4 = (-2)^4 = 16$; $a_5 = (-2)^5 = -32$

The first five terms are –2, 4, –8, 16, and –32. There is a common ratio, $r = -2$, so the sequence is geometric.

3. $a_n = 2(n+3)$

$a_1 = 2(1+3) = 2(4) = 8$;
$a_2 = 2(2+3) = 2(5) = 10$;
$a_3 = 2(3+3) = 2(6) = 12$;
$a_4 = 2(4+3) = 2(7) = 14$;
$a_5 = 2(5+3) = 2(8) = 16$

The first five terms are 8, 10, 12, 14, and 16. There is a common difference, $d = 2$, so the sequence is arithmetic.

4. $a_n = n(n+1)$

$a_1 = 1(1+1) = 1(2) = 2$;
$a_2 = 2(2+1) = 2(3) = 6$;
$a_3 = 3(3+1) = 3(4) = 12$;
$a_4 = 4(4+1) = 4(5) = 20$;
$a_5 = 5(5+1) = 5(6) = 30$

The first five terms are 2, 6, 12, 20, and 30. The sequence does not have either a common difference or a common ratio, so the sequence is neither arithmetic nor geometric.

5. $a_1 = 5$; for $n \ge 2$, $a_n = a_{n-1} - 3$

$a_2 = a_{2-1} - 3 = a_1 - 3 = 5 - 3 = 2;$
$a_3 = a_{3-1} - 3 = a_2 - 3 = 2 - 3 = -1;$
$a_4 = a_{4-1} - 3 = a_3 - 3 = -1 - 3 = -4;$
$a_5 = a_{5-1} - 3 = a_4 - 3 = -4 - 3 = -7$

The first five terms are 5, 2, –1, –4, and –7. There is a common difference, $d = -3$, so the sequence is arithmetic.

6. $a_1 = 1, a_2 = 3$; for $n \ge 3$, $a_n = a_{n-2} + a_{n-1}$

$a_3 = a_2 + a_1 = 3 + 1 = 4$
$a_4 = a_3 + a_2 = 4 + 3 = 7$
$a_5 = a_4 + a_3 = 7 + 4 = 11$

The first five terms are 1, 3, 4, 7, and 11. The sequence does not have either a common difference or a common ratio, so the sequence is neither arithmetic nor geometric.

7. $a_1 = 4$; $S_5 = 25$

Using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$ with $n = 5$, we have

$25 = \frac{5}{2}\left[2(4) + (5-1)d\right] \Rightarrow 25 = \frac{5}{2}(8 + 4d) \Rightarrow$
$25 = 5(4 + 2d) \Rightarrow 5 = 4 + 2d \Rightarrow d = \frac{1}{2}$

$a_1 = 4; a_2 = 4 + \frac{1}{2} = 4.5; a_3 = 4 + 2\left(\frac{1}{2}\right) = 5;$
$a_4 = 4 + 3\left(\frac{1}{2}\right) = 5.5; a_5 = 4 + 4\left(\frac{1}{2}\right) = 6$

The five terms are 4, 4.5, 5, 5.5, and 6.

8. Arithmetic, $a_2 = 10$, $d = -2$

$a_2 = a_1 + d \Rightarrow 10 = a_1 + (-2) \Rightarrow 12 = a_1$
$a_3 = a_1 + 2d = 12 + 2(-2) = 12 + (-4) = 8$
$a_4 = a_1 + 3d = 12 + 3(-2) = 12 + (-6) = 6$
$a_5 = a_1 + 4d = 12 + 4(-2) = 12 + (-8) = 4$

The first five terms are 12, 10, 8, 6, and 4.

9. Arithmetic, $a_3 = \pi$, $a_4 = 1$

$d = a_4 - a_3 = 1 - \pi$
$a_3 = a_1 + 2d \Rightarrow \pi = a_1 + 2(1 - \pi) \Rightarrow$
$\pi = a_1 + 2 - 2\pi \Rightarrow 3\pi - 2 = a_1$
$a_2 = a_1 + d \Rightarrow a_2 = (3\pi - 2) + (1 - \pi) \Rightarrow$
$a_2 = 2\pi - 1$
$a_5 = a_1 + 4d \Rightarrow a_5 = (3\pi - 2) + 4(1 - \pi) \Rightarrow$
$a_5 = 3\pi - 2 + 4 - 4\pi \Rightarrow a_5 = -\pi + 2$

The first five terms are $3\pi - 2$, $2\pi - 1$, π, 1, $-\pi + 2$.

10. Geometric, $a_1 = 6$, $r = 2$

$a_2 = a_1 r = 6 \cdot 2 = 12$
$a_3 = a_1 r^2 = 6 \cdot 2^2 = 6 \cdot 4 = 24$
$a_4 = a_1 r^3 = 6 \cdot 2^3 = 6 \cdot 8 = 48$
$a_5 = a_1 r^4 = 6 \cdot 2^4 = 6 \cdot 16 = 96$

The first five terms are 6, 12, 24, 48, and 96.

11. Geometric, $a_1 = -5$, $a_2 = -1$

$r = \frac{a_2}{a_1} = \frac{-1}{-5} = \frac{1}{5}$

$a_3 = a_1 r^2 = -5\left(\frac{1}{5}\right)^2 = -5 \cdot \frac{1}{25} = -\frac{1}{5}$

$a_4 = a_1 r^3 = -5\left(\frac{1}{5}\right)^3 = -5 \cdot \frac{1}{125} = -\frac{1}{25}$

$a_5 = a_1 r^4 = -5\left(\frac{1}{5}\right)^4 = -5 \cdot \frac{1}{625} = -\frac{1}{125}$

The first five terms are –5, –1, $-\frac{1}{5}$, $-\frac{1}{25}$, and $-\frac{1}{125}$.

12. Arithmetic, $a_5 = -3$, $a_{15} = 17$

Since $a_n = a_1 + (n-1)d$, we have

$a_5 = a_1 + 4d \Rightarrow -3 = a_1 + 4d \Rightarrow$
$a_1 = -3 - 4d$ (1) and
$a_{15} = a_1 + 14d \Rightarrow 17 = a_1 + 14d$ (2) Substitute equation (1) into equation (2) and solve for d.
$17 = a_1 + 14d \Rightarrow 17 = (-3 - 4d) + 14d \Rightarrow$
$17 = -3 + 10d \Rightarrow 20 = 10d \Rightarrow d = 2$

Substitute $d = 2$ into equation (1) to obtain

$a_1 = -3 - 4d \Rightarrow a_1 = -3 - 4(2) \Rightarrow$
$a_1 = -3 - 8 \Rightarrow a_1 = -11$

Since $a_n = a_1 + (n-1)d$, we have

$a_n = -11 + (n-1)(2) \Rightarrow a_n = -11 + 2n - 2 \Rightarrow$
$a_n = -13 + 2n$

13. Geometric, $a_1 = -8$ and $a_7 = -\dfrac{1}{8}$

Since $a_n = a_1 r^{n-1}$, we have $a_7 = a_1 r^{7-1} \Rightarrow$
$-\frac{1}{8} = -8r^6 \Rightarrow r^6 = \frac{1}{64} \Rightarrow r = \pm\frac{1}{2}$.

There are two geometric sequences that satisfy the given conditions.

If $r = \frac{1}{2}$, $a_4 = (-8)\left(\frac{1}{2}\right)^3 = (-8)\left(\frac{1}{8}\right) = -1$.

Also, $a_n = -8\left(\frac{1}{2}\right)^{n-1}$ or $a_n = -2^3\left(\frac{1}{2}\right)^{n-1}$

$= -\frac{2^3}{2^{n-1}} = -\frac{1}{2^{(n-1)-3}} = -\frac{1}{2^{n-4}} = -\left(\frac{1}{2}\right)^{n-4}$.

If $r = -\frac{1}{2}$, $a_4 = (-8)\left(-\frac{1}{2}\right)^3 = (-8)\left(-\frac{1}{8}\right) = 1$.

Also, $a_n = -8\left(-\frac{1}{2}\right)^{n-1}$ or $a_n = -2^3\left(-\frac{1}{2}\right)^{n-1}$

$= (-2)^3\left(-\frac{1}{2}\right)^{n-1} = \frac{(-2)^3}{(-2)^{n-1}} = \frac{1}{(-2)^{(n-1)-3}}$

$= \frac{1}{(-2)^{n-4}} = \left(-\frac{1}{2}\right)^{n-4}$.

14. $a_1 = 6$, $d = 2$

$a_n = a_1 + (n-1)d$

$a_8 = a_1 + (8-1)d = 6 + 7(2) = 6 + 14 = 20$

15. $a_1 = 6x - 9$, $a_2 = 5x + 1$

$d = a_2 - a_1 = (5x+1) - (6x-9)$

$= 5x + 1 - 6x + 9 = -x + 10$

$a_n = a_1 + (n-1)d$

$a_8 = a_1 + (8-1)d \Rightarrow a_8 = a_1 + 7d \Rightarrow$

$a_8 = (6x-9) + 7(-x+10)$

$= 6x - 9 - 7x + 70 = -x + 61$

16. $a_1 = 2$, $d = 3$

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$S_{12} = \frac{12}{2}\left[2(2) + (12-1)(3)\right] = 6\left[4 + (11)(3)\right]$

$= 6(4 + 33) = 6(37) = 222$

17. $a_2 = 6$, $d = 10$

To first find a_1, we have

$a_2 = a_1 + d \Rightarrow 6 = a_1 + 10 \Rightarrow a_1 = -4$

Now, to find S_{12}, we use

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$.

$S_{12} = \frac{12}{2}\left[2(-4) + (12-1)(10)\right]$

$= 6\left[-8 + (11)(10)\right] = 6(-8 + 110)$

$= 6(102) = 612$

18. $a_1 = -2$, $r = 3$

$a_n = a_1 r^{n-1} \Rightarrow$

$a_5 = a_1 r^{5-1} = a_1 r^4 = (-2)(3)^4$

$= (-2)(81) = -162$

19. $a_3 = 4$, $r = \frac{1}{5}$

$a_4 = a_3 r = 4 \cdot \frac{1}{5} = \frac{4}{5} \Rightarrow a_5 = a_4 r = \frac{4}{5} \cdot \frac{1}{5} = \frac{4}{25}$

20. $a_1 = 3$, $r = 2$. $S_n = \frac{a_1(1-r^n)}{1-r}$

$S_4 = \frac{a_1(1-r^4)}{1-r} = \frac{3(1-2^4)}{1-2} = \frac{3(1-16)}{-1}$

$= -3(-15) = 45$

21. $a_1 = -1$, $r = 3$. $S_n = \frac{a_1(1-r^n)}{1-r}$

$S_4 = \frac{a_1(1-r^4)}{1-r} = \frac{-1(1-3^4)}{1-3} = \frac{-1(1-81)}{-2}$

$= \frac{1}{2}(-80) = -40$

22. $\frac{3}{4}, -\frac{1}{2}, \frac{1}{3}, \ldots$; $S_n = \frac{a_1(1-r^n)}{1-r}$

This is the geometric sequence with $a_1 = \frac{3}{4}$

and $r = \frac{-\frac{1}{2}}{\frac{3}{4}} = -\frac{2}{3}$.

$S_4 = \frac{a_1(1-r^4)}{1-r} = \frac{\frac{3}{4}\left[1-\left(-\frac{2}{3}\right)^4\right]}{1-\left(-\frac{2}{3}\right)} = \frac{\frac{3}{4}\left(1-\frac{16}{81}\right)}{\frac{5}{3}}$

$= \frac{\frac{3}{4}\left(\frac{65}{81}\right)}{\frac{5}{3}} = \frac{\frac{65}{108}}{\frac{5}{3}} = \frac{65}{108} \cdot \frac{3}{5} = \frac{13}{36}$

23. $\sum_{i=1}^{7} (-1)^{i-1}$; $S_n = \frac{a_1(1-r^n)}{1-r}$

This is a geometric series with

$a_1 = (-1)^{1-1} = (-1)^0 = 1$ and $r = -1$.

$S_7 = \frac{a_1(1-r^7)}{1-r} \Rightarrow$

$S_7 = \frac{1\left[1-(-1)^7\right]}{1-(-1)} = \frac{1\left[1-(-1)\right]}{2} = \frac{1 \cdot 2}{2} = 1$

24. $\sum_{i=1}^{5}\left(i^2+i\right)=\sum_{i=1}^{5}i^2+\sum_{i=1}^{5}i$

Since $\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$ and

$\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$, we have

$$\sum_{i=1}^{5}i^2+\sum_{i=1}^{5}i=\frac{5(5+1)(2\cdot5+1)}{6}+\frac{5(5+1)}{2}$$
$$=\frac{5\cdot6\cdot11}{6}+\frac{5\cdot6}{2}=55+15=70$$

We could have also performed the following calculation.

$$\sum_{i=1}^{5}\left(i^2+i\right)=\left(1^2+1\right)+\left(2^2+2\right)+\left(3^2+3\right)+\left(4^2+4\right)+\left(5^2+5\right)$$
$$=(1+1)+(4+2)+(9+3)+(16+4)+(25+5)$$
$$=2+6+12+20+30=70$$

25. $$\sum_{i=1}^{4}\frac{i+1}{i}=\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}=\frac{24}{12}+\frac{18}{12}+\frac{16}{12}+\frac{15}{12}=\frac{73}{12}$$

26. $$\sum_{j=1}^{10}(3j-4)=\sum_{j=1}^{10}3j-\sum_{j=1}^{10}4=3\sum_{j=1}^{10}j-\sum_{j=1}^{10}4$$

Since $\sum_{i=1}^{n}c=cn$ and $\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$, we have

$$3\sum_{j=1}^{10}j-\sum_{j=1}^{10}4=3\cdot\frac{10(10+1)}{2}-4(10)$$
$$=3\cdot\frac{10(11)}{2}-40=3(5)(11)-40$$
$$=165-40=125$$

27. Since

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2},$$
$$\sum_{j=1}^{2500}j=\frac{2500(2500+1)}{2}$$
$$=1250(2501)=3,126,250$$

28. $\sum_{i=1}^{5}4\cdot2^i$

This is the sum of a geometric series with $a_1=4\cdot2^1=8$ and $r=2$.

Using the formula $S_n=\frac{a_1\left(1-r^n\right)}{1-r}$, we have

$$\sum_{i=1}^{5}4\cdot2^i=S_5=\frac{8\left(1-2^5\right)}{1-2}=\frac{8(1-32)}{-1}$$
$$=-8(-31)=248$$

29. $\sum_{i=1}^{\infty}\left(\frac{4}{7}\right)^i$

This is an infinite geometric series with $a_1=\left(\frac{4}{7}\right)^1=\frac{4}{7}$ and $r=\frac{4}{7}$. Since $|r|=\left|\frac{4}{7}\right|<1$, we can use the formula $S_\infty=\frac{a_1}{1-r}$. Thus, we have

$$\sum_{i=1}^{\infty}\left(\frac{4}{7}\right)^i=S_\infty=\frac{a_1}{1-r}=\frac{\frac{4}{7}}{1-\frac{4}{7}}=\frac{\frac{4}{7}}{\frac{3}{7}}=\frac{4}{7}\cdot\frac{7}{3}=\frac{4}{3}.$$

30. $\sum_{i=1}^{\infty}(-2)\left(\frac{6}{5}\right)^i$

This is the sum of an infinite geometric sequence with $r=\frac{6}{5}$. Since $\frac{6}{5}>1$, the sum does not exist.

31. $\sum_{i=1}^{\infty}2\left(-\frac{2}{3}\right)^i$

This is an infinite geometric series with $a_1=2\left(-\frac{2}{3}\right)^1=-\frac{4}{3}$ and $r=-\frac{2}{3}$. Since $|r|=\left|-\frac{2}{3}\right|<1$, we can use the formula

$S_\infty=\frac{a_1}{1-r}$. Thus, we have

$$\sum_{i=1}^{\infty}2\left(-\frac{2}{3}\right)^i=S_\infty=\frac{a_1}{1-r}=\frac{-\frac{4}{3}}{1-\left(-\frac{2}{3}\right)}$$
$$=\frac{-\frac{4}{3}}{\frac{5}{3}}=-\frac{4}{5}$$

32. $S_\infty = 6, r = \frac{3}{4}$

First find a_1 using the formula $S_\infty = \dfrac{a_1}{1-r}$.

$$S_\infty = \frac{a_1}{1-r} \Rightarrow 6 = \frac{a_1}{1-\frac{3}{4}} \Rightarrow 6 = \frac{a_1}{\frac{1}{4}} \Rightarrow \tfrac{3}{2} = a_1.$$

Thus, the series is

$$\tfrac{3}{2} + \tfrac{3}{2}\left(\tfrac{3}{4}\right)^1 + \tfrac{3}{2}\left(\tfrac{3}{4}\right)^2 + \cdots = \tfrac{3}{2} + \tfrac{9}{8} + \tfrac{27}{32} + \cdots$$

33. $24 + 8 + \dfrac{8}{3} + \dfrac{8}{9} + \ldots$

This is an infinite geometric series with $a_1 = 24$ and $r = \frac{8}{24} = \frac{1}{3}$. Since $|r| = \left|\frac{1}{3}\right| < 1,$ we can use the formula $S_\infty = \dfrac{a_1}{1-r}$. Thus, we have $S_\infty = \dfrac{a_1}{1-r} = \dfrac{24}{1-\frac{1}{3}} = \dfrac{24}{\frac{2}{3}} = 24 \cdot \dfrac{3}{2} = 36.$

34. $-\dfrac{3}{4} + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{2}{9} - \ldots$

This infinite sum has $a_1 = -\dfrac{3}{4}$ and

$r = \dfrac{\frac{1}{2}}{-\frac{3}{4}} = -\dfrac{2}{3}$. Since $|r| = \left|-\dfrac{2}{3}\right| < 1,$ we can use the formula $S_\infty = \dfrac{a_1}{1-r}$. Thus, we have

$$S_\infty = \frac{a_1}{1-r} = \frac{-\frac{3}{4}}{1-\left(-\frac{2}{3}\right)} = \frac{-\frac{3}{4}}{\frac{5}{3}} = -\frac{3}{4} \cdot \frac{3}{5} = -\frac{9}{20}.$$

35. $\dfrac{1}{12} + \dfrac{1}{6} + \dfrac{1}{3} + \dfrac{2}{3} + \ldots$

This is an infinite geometric series with $a_1 = \dfrac{1}{12}$ and $r = \dfrac{\frac{1}{6}}{\frac{1}{12}} = 2.$ Since $|r| > 1$, the series diverges.

36. $.9 + .09 + .009 + .0009 + \ldots$

This infinite sum has $a_1 = .9$ and $r = \dfrac{.09}{.9} = .1.$ Since $|r| = |.1| < 1,$ we can use the formula $S_\infty = \dfrac{a_1}{1-r}$. Thus, we have

$$S_\infty = \frac{a_1}{1-r} = \frac{.9}{1-.1} = \frac{.9}{.9} = 1.$$

In Exercises 37–38, $x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = 3,\ x_5 = 4$ and $x_6 = 5.$

37. $\displaystyle\sum_{i=1}^{4}\left(x_i^2 - 6\right)$

$$\begin{aligned}
&= \left(x_1^2 - 6\right) + \left(x_2^2 - 6\right) + \left(x_3^2 - 6\right) + \left(x_4^2 - 6\right) \\
&= \left(0^2 - 6\right) + \left(1^2 - 6\right) + \left(2^2 - 6\right) + \left(3^2 - 6\right) \\
&= (0-6) + (1-6) + (4-6) + (9-6) \\
&= -6 + (-5) + (-2) + 3 = -10
\end{aligned}$$

38. $f(x) = (x-2)^3, \Delta x = .1$

$$\begin{aligned}
\sum_{i=1}^{6} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\
&= f(0)(\Delta x) + f(1)(\Delta x) + f(2)(\Delta x) + f(3)(\Delta x) + f(4)(\Delta x) + f(5)(\Delta x) \\
&= (0-2)^3(.1) + (1-2)^3(.1) + (2-2)^3(.1) + (3-2)^3(.1) + (4-2)^3(.1) + (5-2)^3(.1) \\
&= (-2)^3(.1) + (-1)^3(.1) + 0^3(.1) + 1^3(.1) + 2^3(.1) + 3^3(.1) \\
&= (-8)(.1) + (-1)(.1) + 0(.1) + 1(.1) + 8(.1) + 27(.1) = -.8 + (-.1) + 0 + .1 + .8 + 2.7 = 2.7
\end{aligned}$$

In Exercises 39–42, other answers are possible.

39. $4 - 1 - 6 - \cdots - 66 = 4 + (-1) + (-6) + \cdots + (-66)$

This series is the sum of an arithmetic sequence with $a_1 = 4$ and $d = -1 - 4 = -5.$ Therefore, the *n*th term is

$$\begin{aligned}
a_n &= a_1 + (n-1)d = 4 + (n-1)(-5) \\
&= 4 - 5n + 5 = -5n + 9
\end{aligned}$$

or, equivalently, $a_i = -5i + 9$.

To find the number of terms in the series, we realize that the last term of the series is –66. Since $a_i = -5i + 9$ we solve $-66 = -5i + 9$ for i: $-66 = -5i + 9 \Rightarrow -75 = -5i \Rightarrow i = 15$

This indicates that the series consists of 15 terms and we have

$$4 - 1 - 6 - \ldots - 66 = \sum_{i=1}^{15}(-5i + 9).$$

40. 10 + 14 + 18 + … + 86
This series is the sum of an arithmetic sequence with $a_1 = 10$ and $d = 14 - 10 = 4$.
Therefore, the nth term is
$$a_n = a_1 + (n-1)d = 10 + (n-1)\cdot 4$$
$$= 10 + 4n - 4 = 4n + 6$$
or, equivalently, $a_i = 4i + 6$. To find the number of terms in the series, we realize that the last term of the series is 86. Since $a_i = 4i + 6$ we solve $86 = 4i + 6$ for i.
$$86 = 4i + 6 \Rightarrow 80 = 4i \Rightarrow i = 20$$
This indicates that the series consists of 20 terms and we have
$$10 + 14 + 18 + \ldots + 86 = \sum_{i=1}^{20}(4i + 6).$$

41. 4 + 12 + 36 + … + 972
This series is the sum of a geometric sequence with $a_1 = 4$ and $r = \frac{12}{4} = 3$.
Therefore, the nth term is
$a_n = a_1 r^{n-1} \Rightarrow a_n = 4(3)^{n-1}$, or, equivalently,
$a_i = 4(3)^{i-1}$. To find the number of terms in the series, we realize that the last term of the series is 972.
Since $a_i = 4(3)^{i-1}$ we solve $972 = 4(3)^{i-1}$ for i. $972 = 4(3)^{i-1} \Rightarrow 243 = 3^{i-1} \Rightarrow$
$$3^5 = 3^{i-1} \Rightarrow 5 = i - 1 \Rightarrow i = 6$$
This indicates that the series consists of 6 terms and we have
$$4 + 12 + 36 + \ldots + 972 = \sum_{i=1}^{6} 4(3)^{i-1}.$$

42. $\frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \ldots + \frac{12}{13}$
This series is neither arithmetic nor geometric, but notice that each denominator is 1 larger than its corresponding numerator. This means $a_n = \frac{n}{n+1}$, or, equivalently, $a_i = \frac{i}{i+1}$. Also, the numerators of the terms of the series begin at $i = 5$ and increase by 1 until $i = 12$. We can conclude that $\frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \ldots + \frac{12}{13} = \sum_{i=5}^{12} \frac{i}{i+1}$

43.
$$\begin{aligned}(x+2y)^4 &= x^4 + \binom{4}{1}x^3(2y)^1 + \binom{4}{2}x^2(2y)^2 + \binom{4}{3}x^1(2y)^3 + (2y)^4 \\ &= x^4 + \frac{4!}{1!(4-1)!}x^3(2y) + \frac{4!}{(4-2)!2!}x^2(4y^2) + \frac{4!}{3!(4-3)!}x(8y^3) + 16y^4 \\ &= x^4 + \frac{4!}{1!3!}x^3(2y) + \frac{4!}{2!2!}x^2(4y^2) + \frac{4!}{3!1!}x(8y^3) + 16y^4 \\ &= x^4 + 4x^3(2y) + \frac{4\cdot 3}{2\cdot 1}x^2(4y^2) + 4x(8y^3) + 16y^4 \\ &= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4\end{aligned}$$

44.
$$\begin{aligned}(3z-5w)^3 &= (3z)^3 + \binom{3}{1}(3z)^2(-5w) + \binom{3}{2}(3z)(-5w)^2 + (-5w)^3 \\ &= 27z^3 + \frac{3!}{1!(3-1)!}(9z^2)(-5w) + \frac{3!}{2!(3-2)!}(3z)(25w^2) + (-125w^3) \\ &= 27z^3 - 3(9z^2)(5w) + 3(3z)(25w^2) - 125w^3 = 27z^3 - 135z^2w + 225zw^2 - 125w^3\end{aligned}$$

45. $\left(3\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^5=\left[3x^{1/2}+\left(-x^{-1/2}\right)\right]^5$

$$=\left(3x^{1/2}\right)^5+\binom{5}{1}\left(3x^{1/2}\right)^4\left(-x^{-1/2}\right)+\binom{5}{2}\left(3x^{1/2}\right)^3\left(-x^{-1/2}\right)^2$$
$$+\binom{5}{3}\left(3x^{1/2}\right)^2\left(-x^{-1/2}\right)^3+\binom{5}{4}\left(3x^{1/2}\right)\left(-x^{-1/2}\right)^4+\left(-x^{-1/2}\right)^5$$
$$=243x^{5/2}+\frac{5!}{1!4!}\left(81x^2\right)\left(-x^{-1/2}\right)+\frac{5!}{2!3!}\left(27x^{3/2}\right)^3\left(x^{-1}\right)$$
$$-\frac{5!}{3!2!}(9x)x^{-3/2}+\frac{5!}{4!1!}\left(3x^{1/2}\right)\left(x^{-2}\right)+\left(-x^{-5/2}\right)$$
$$=243x^{5/2}-5\left(81x^{2+(-1/2)}\right)+\frac{5\cdot 4}{2\cdot 1}\left(27x^{3/2+(-1)}\right)-\frac{5\cdot 4}{2\cdot 1}\left(9x^{1+(-3/2)}\right)+5\left(3x^{1/2+(-2)}\right)-x^{-5/2}$$
$$=243x^{5/2}-405x^{3/2}+270x^{1/2}-90x^{-1/2}+15x^{-3/2}-x^{-5/2}$$

46. $\left(m^3-m^{-2}\right)^4=\left(m^3\right)^4+\binom{4}{1}\left(m^3\right)^3\left(-m^{-2}\right)+\binom{4}{2}\left(m^3\right)^2\left(-m^{-2}\right)^2+\binom{4}{3}\left(m^3\right)\left(-m^{-2}\right)^3+\left(-m^{-2}\right)^4$

$$=m^{12}+\frac{4!}{1!3!}\left(m^9\right)\left(-m^{-2}\right)+\frac{4!}{2!2!}\left(m^6\right)\left(m^{-4}\right)+\frac{4!}{3!1!}\left(m^3\right)\left(-m^{-6}\right)+m^{-8}$$
$$=m^{12}-4\left(m^9\right)\left(m^{-2}\right)+\frac{4\cdot 3}{2\cdot 1}\left(m^6\right)\left(m^{-4}\right)-4\left(m^3\right)\left(m^{-6}\right)+m^{-8}$$
$$=m^{12}-4m^{9+(-2)}+6m^{6+(-4)}-4m^{3+(-6)}+m^{-8}=m^{12}-4m^7+6m^2-4m^{-3}+m^{-8}$$

47. The sixth term of the expansion of $(4x-y)^8$ is $\binom{8}{5}(4x)^3(-y)^5$. This simplifies as follows.

$$\binom{8}{5}(4x)^3(-y)^5=\frac{8!}{5!3!}\left(64x^3\right)\left(-y^5\right)=\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}\left(-64x^3y^5\right)=56\left(-64x^3y^5\right)=-3584x^3y^5$$

48. Seventh term of $(m-3n)^{14}$ is $\binom{14}{6}(m)^8(-3n)^6$. This simplifies as follows.

$$\binom{14}{6}(m)^8(-3n)^6=\frac{14!}{6!8!}\left(m^8\right)(-3)^6n^6=\frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}(-3)^6m^8n^6=3003(-3)^6m^8n^6$$

49. The first four terms of $(x+2)^{12}$ are as follows.

$$(x)^{12}+\binom{12}{1}(x)^{11}(2)+\binom{12}{2}(x)^{10}(2)^2+\binom{12}{3}(x)^9(2)^3=x^{12}+\frac{12!}{1!11!}\left(x^{11}\right)(2)+\frac{12!}{2!10!}\left(x^{10}\right)(4)+\frac{12!}{3!9!}\left(x^9\right)(8)$$
$$=x^{12}+12\left(2x^{11}\right)+\frac{12\cdot 11}{2\cdot 1}\left(4x^{10}\right)+220\left(8x^9\right)$$
$$=x^{12}+24x^{11}+264x^{10}+1760x^9$$

50. Last three terms of $(2a+5b)^{16}$ are as follows.

$$\binom{16}{14}(2a)^2(5b)^{14}+\binom{16}{15}(2a)(5b)^{15}+(5b)^{16}=\frac{16!}{14!2!}\left(4a^2\right)\left(5^{14}\right)b^{14}+\frac{16!}{15!1!}(2a)\left(5^{15}\right)b^{15}+5^{16}b^{16}$$
$$=\frac{16\cdot 15}{2\cdot 1}(4)\left(5^{14}\right)a^2b^{14}+\frac{16!}{15!1!}(2)\left(5^{15}\right)ab^{15}+5^{16}b^{16}$$
$$=120\cdot\left(4a^2\right)\left(5^{14}\right)b^{14}+16(2a)\left(5^{15}\right)b^{15}+5^{16}\cdot b^{16}$$
$$=480\cdot 5^{14}a^2b^{14}+32\cdot 5^{15}ab^{15}+5^{16}b^{16}$$

51. Let S_n be the statement $1+3+5+7+\ldots+(2n-1)=n^2$.

Step 1: Show that the statement is true for $n = 1$.

S_1 is the statement $2(1)-1=1^2 \Rightarrow 2-1=1 \Rightarrow 1=1,$ which is true.

Step 2: Show that if S_k is true, S_{k+1} is also true. S_k is the statement $1+3+5+7+\ldots+(2k-1)=k^2$. and S_{k+1} is the statement $1+3+5+7+\ldots+\left[2(k+1)-1\right]=(k+1)^2$.

Start with S_k: $1+3+5+7+\ldots+(2k-1)=k^2$. and add the $(k+1)$ st term, $\left[2(k+1)-1\right]$, to both sides of this equation.

$$1+3+5+7+\ldots+(2k-1)=k^2$$
$$1+3+5+7+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+\left[2(k+1)-1\right]$$
$$1+3+5+7+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+(2k+2-1)$$
$$1+3+5+7+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+(2k+1)$$
$$1+3+5+7+\ldots+(2k-1)+\left[2(k+1)-1\right]=k^2+2k+1$$
$$1+3+5+7+\ldots+(2k-1)+\left[2(k+1)-1\right]=(k+1)^2$$

It has been shown that S_k implies S_{k+1}. Therefore, by mathematical induction, S_n is true for every positive integer n.

52. Let S_n be the statement $2+6+10+14+\ldots+(4n-2)=2n^2$.

Step 1: Show that the statement is true for $n = 1$.

S_1 is the statement $4(1)-2=2(1^2) \Rightarrow 4-2=2(1) \Rightarrow 2=2,$ which is true.

Step 2: Show that if S_k is true, S_{k+1} is also true. S_k is the statement $2+6+10+14+\ldots+(4k-2)=2k^2$ and S_{k+1} is the statement $2+6+10+14+\ldots+\left[4(k+1)-2\right]=2(k+1)^2$.

Start with S_k: $2+6+10+14+\ldots+(4k-2)=2k^2$ and add the $(k+1)$ st term, $\left[4(k+1)-2\right]$, to both sides of this equation.

$$2+6+10+14+\ldots+(4k-2)=2k^2$$
$$2+6+10+14+\ldots+(4k-2)+\left[4(k+1)-2\right]=2k^2+\left[4(k+1)-2\right]$$
$$2+6+10+14+\ldots+(4k-2)+\left[4(k+1)-2\right]=2k^2+(4k+4-2)$$
$$2+6+10+14+\ldots+(4k-2)+\left[4(k+1)-2\right]=2k^2+(4k+2)$$
$$2+6+10+14+\ldots+(4k-2)+\left[4(k+1)-2\right]=2k^2+4k+2$$
$$2+6+10+14+\ldots+(4k-2)+\left[4(k+1)-2\right]=2\left(k^2+2k+1\right)$$
$$2+6+10+14+\ldots+(4k-2)+\left[4(k+1)-2\right]=2(k+1)^2$$

It has been shown that S_k implies S_{k+1}. Therefore, by mathematical induction, S_n is true for every positive integer n.

53. Let S_n be the statement $2+2^2+2^3+\ldots+2^n=2\left(2^n-1\right)$.

Step 1: Show that the statement is true for $n = 1$.

S_1 is the statement $2^1=2\left(2^1-1\right) \Rightarrow 2=2(2-1) \Rightarrow 2=2(1) \Rightarrow 2=2,$ which is true.

Step 2: Show that if S_k is true, S_{k+1} is also true. S_k is the statement. $2+2^2+2^3+\ldots+2^k=2\left(2^k-1\right)$ S_{k+1} is the statement $2+2^2+2^3+\ldots+2^{k+1}=2\left(2^{k+1}-1\right)$.

Start with S_k: $2+2^2+2^3+\ldots+2^k=2\left(2^k-1\right)$ and add the $(k+1)$ st term, 2^{k+1}, to both sides of this equation.

(*continued on next page*)

(*continued from page 1163*)

$$2+2^2+2^3+\ldots+2^k = 2\left(2^k-1\right)$$
$$2+2^2+2^3+\ldots+2^k+2^{k+1} = 2\left(2^k-1\right)+2^{k+1}$$
$$2+2^2+2^3+\ldots+2^k+2^{k+1} = 2^{k+1}-2+2^{k+1}$$
$$2+2^2+2^3+\ldots+2^k+2^{k+1} = 2\cdot 2^{k+1}-2\cdot 1$$
$$2+2^2+2^3+\ldots+2^k+2^{k+1} = 2\left(2^{k+1}-1\right)$$

It has been shown that S_k implies S_{k+1}. Therefore, by mathematical induction, S_n is true for every positive integer *n*.

54. Let S_n be the statement $1^3+3^3+5^3+\ldots+(2n-1)^3 = n^2\left(2n^2-1\right)$.

Step 1: Show that the statement is true for *n* = 1.

S_1 is the statement $\left[2(1)-1\right]^3 = 1^2\left[2\left(1^2\right)-1\right] \Rightarrow (2-1)^3 = 1\left[2(1)-1\right] \Rightarrow 1^3 = 1(1) \Rightarrow 1=1,$ which is true.

Step 2: Show that if S_k is true, S_{k+1} is also true. S_k is the statement

$1^3+3^3+5^3+\ldots+(2n-1)^3 = n^2\left(2n^2-1\right)$.

S_{k+1} is the statement $1^3+3^3+5^3+\ldots+\left[2(k+1)-1\right]^3 = (k+1)^2\left[2(k+1)^2-1\right]$.

Start with S_k: $1^3+3^3+5^3+\ldots+(2n-1)^3 = n^2\left(2n^2-1\right)$. and add the $(k+1)$ st term, $\left[2(k+1)-1\right]^3$, to both sides of this equation.

$$1^3+3^3+5^3+\ldots+(2k-1)^3 = k^2\left(2k^2-1\right)$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = k^2\left(2k^2-1\right)+\left[2(k+1)-1\right]^3$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = k^2\left(2k^2-1\right)+(2k+2-1)^3$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = k^2\left(2k^2-1\right)+(2k+1)^3$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = k^2\left(2k^2-1\right)+(2k)^3+3(2k)^2+3(2k)+1$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = k^2\left(2k^2-1\right)+8k^3+3\left(4k^2\right)+6k+1$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = 2k^4-k^2+8k^3+12k^2+6k+1$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = 2k^4+8k^3+11k^2+6k+1$$

Our task is to now factor $2k^4+8k^3+11k^2+6k+1$. Since we know that $(k+1)$ should be a factor of this polynomial, we can either use long division or synthetic division. Using synthetic division, we have

$$\begin{array}{r|rrrrr} -1 & 2 & 8 & 11 & 6 & 1 \\ & & -2 & -6 & -5 & -1 \\ \hline & 2 & 6 & 5 & 1 & 0 \end{array}$$

Thus, we have $1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3 = (k+1)\left(2k^3+6k^2+5k+1\right)$.

Now performing synthetic division again on $2k^3+6k^2+5k+1$ we have
$$\begin{array}{r|rrrr} -1 & 2 & 6 & 5 & 1 \\ & & -2 & -4 & -1 \\ \hline & 2 & 4 & 1 & 0 \end{array}$$

Continuing the process, we have

$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3=(k+1)^2\left(2k^2+4k+1\right)$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3=(k+1)^2\left[2\left(k^2+2k+1\right)-2+1\right]$$
$$1^3+3^3+5^3+\ldots+(2k-1)^3+\left[2(k+1)-1\right]^3=(k+1)^2\left[2(k+1)^2-1\right]$$

It has been shown that S_k implies S_{k+1}. Therefore, by mathematical induction, S_n is true for every positive integer *n*.

55. $P(9,\ 2)=\dfrac{9!}{(9-2)!}=\dfrac{9!}{7!}=\dfrac{9\cdot 8\cdot 7!}{7!}=72$

56. $P(6,\ 0)=\dfrac{6!}{(6-0)!}=\dfrac{6!}{6!}=1$

57. $\dbinom{8}{3}=C(8,3)=\dfrac{8!}{(8-3)!3!}=\dfrac{8!}{5!3!}$
$=\dfrac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 1\cdot 5!}=56$

58. $9!=9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=362{,}880$

59. $C(10,\ 5)=\dfrac{10!}{(10-5)!5!}=\dfrac{10!}{5!5!}$
$=\dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!5!}$
$=\dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1}=252$

60. $10\cdot 9!=10!=10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$
$=3{,}628{,}800$

61. Answers will vary. If the order is important, then the problem situation involves a permutation. If the order is not of importance, then the problem situation involves a combination.

62. Using the fundamental principle of counting, we have $2\cdot 4\cdot 3\cdot 2=48$.
There are 48 different wedding arrangements possible as they can select from 2 different chapels, 4 soloists, 3 organists, and 2 ministers.

63. Three independent events are involved. There are 5 choices of style, 3 choices of fabric, and 6 choices of color. Therefore, using the fundamental principle of counting, there are $5\cdot 3\cdot 6=90$ different couches.

64. There are 4 choices for the first job, 3 choices for the second job, and so on. Since $4\cdot 3\cdot 2\cdot 1=24$, there are 24 ways in which the jobs can be assigned.

65. There are 6 people on the student body council.

(a) In selecting a 3-member delegation, $C(6,3)$ choices are possible.
$C(6,3)=\dbinom{6}{3}=\dfrac{6!}{(6-3)!3!}=\dfrac{6!}{3!3!}$
$=\dfrac{6\cdot 5\cdot 4\cdot 3!}{3!3!}=\dfrac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}=20$
20 delegations are possible.

(b) If the president must attend, then there are two slots available for the remaining 5 members, which means there are $C(5,2)$ choices.
$C(5,2)=\dbinom{5}{2}=\dfrac{5!}{2!(5-2)!}=\dfrac{5!}{2!3!}$
$=\dfrac{5\cdot 4\cdot 3!}{2!3!}=\dfrac{5\cdot 4}{2\cdot 1}=10$
10 delegations are possible if the president must attend.

66. Order is important, so this is a permutation problem.
$P(9,\ 3)=\dfrac{9!}{(9-3)!}=\dfrac{9!}{6!}=\dfrac{9\cdot 8\cdot 7\cdot 6!}{6!}$
$=9\cdot 8\cdot 7=504$
The winners can be determined 504 ways.

67. There are 4 spots to be filled by 26 letters and 3 spots by 10 digits. If repeats are allowed, then $26\cdot 10\cdot 10\cdot 10\cdot 26\cdot 26\cdot 26=456{,}976{,}000$ different license plates can be formed. If no repeats are allowed, then
$26\cdot 10\cdot 9\cdot 8\cdot 25\cdot 24\cdot 23=258{,}336{,}000$
different license plates can be formed.

68. **(a)** Order is important, so this is a permutation problem.
$P(9,\ 2)=\dfrac{9!}{(9-2)!}=\dfrac{9!}{7!}=\dfrac{9\cdot 8\cdot 7!}{7!}$
$=9\cdot 8=72$
There are 72 different exacta bets that can be placed.

(b) Order is not important, so this is a combination problem.

$$C(9, 2) = \frac{9!}{(9-2)!2!} = \frac{9!}{7!2!} = \frac{9 \cdot 8 \cdot 7!}{7!2!} = \frac{9 \cdot 8}{2 \cdot 1} = 36$$

There are 36 different quinella bets that can be placed.

69. **(a)** Let E be the event "picking a green marble."

$n(E) = 4$ and $n(S) = 4 + 5 + 6 = 15$ (total number of marbles) Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{15}$$

(b) Let E be the event "picking a black marble."

$n(E) = 5$ and $n(S) = 15$ (from part (a)).

Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{5}{15} = \frac{1}{3}.$

The probability that the marble is not black is given by

$$P(E') = 1 - P(E) = 1 - \frac{1}{3} = \frac{2}{3}.$$

(c) Let E be the event "picking a blue marble."

$n(E) = 0$ since there are no blue marbles and $n(S) = 15$ (from part (a)). Thus, we have $P(E) = \frac{n(E)}{n(S)} = \frac{0}{15} = 0.$

(d) Let E be the event "picking a marble that is not white." We have

$P(E) = \frac{n(E)}{n(S)} = \frac{9}{15}$ and

$P(E') = \frac{n(E')}{n(S)} = \frac{6}{15}.$ Thus, the odds in favor of E are

$$\frac{P(E)}{P(E')} = \frac{\frac{9}{15}}{\frac{6}{15}} = \frac{9}{15} \cdot \frac{15}{6} = \frac{3}{2} \text{ or 3 to 2.}$$

70. **(a)**

$$P(\text{black king}) = \frac{n(\text{black kings})}{n(\text{cards in deck})} = \frac{2}{52} = \frac{1}{26}$$

(b) Let F be the event "face card is drawn" and A be the event "ace is drawn." There are 12 face cards, so $n(F) = 12$ and

$P(F) = \frac{12}{52} = \frac{3}{13}.$ There are 4 aces, so

$n(A) = 4$ and $P(A) = \frac{4}{52} = \frac{1}{13}.$ There are no cards which are both face cards and aces, so $n(F \cap A) = 0$ and $P(F \cap A) = 0.$

Thus, we have

$$\begin{aligned} P(F \text{ or } A) &= P(F \cup A) \\ &= P(F) + P(A) - P(F \cap A) \\ &= \frac{3}{13} + \frac{1}{13} - 0 = \frac{4}{13} \end{aligned}$$

(c) Let A be the event "ace is drawn" and D be the event "diamond is drawn." There are 4 aces, so

$n(A) = 4$ and $P(A) = \frac{4}{52} = \frac{1}{13}.$ There are 13 diamonds, so

$n(D) = 13$ and $P(D) = \frac{13}{52} = \frac{1}{4}.$ There is one card which is both an ace and a diamond, so

$n(A \cap D) = 1$ and $P(A \cap D) = \frac{1}{52}.$ Thus we have

$$\begin{aligned} P(A \text{ or } D) &= P(A \cup D) \\ &= P(A) + P(D) - P(A \cap D) \\ &= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13} \end{aligned}$$

(d) Let D be the event "diamond is drawn." There are 13 diamonds, so $n(D) = 13$ and

$P(D) = \frac{13}{52} = \frac{1}{4}.$ The probability that the card drawn is not a diamond is

$$P(D') = 1 - P(D) = 1 - \frac{1}{4} = \frac{3}{4}$$

(e) Let A be the event "ace is drawn." As shown in the solution to part (b), $P(A)=\frac{1}{13}$. Now, $P(A')=1-P(A)$ $=1-\frac{1}{13}=\frac{12}{13}$. Thus, the odds in favor of drawing an ace are $\frac{P(A)}{P(A')}=\frac{\frac{1}{13}}{\frac{12}{13}}$ $=\frac{1}{13}\cdot\frac{13}{12}=\frac{1}{12}$ or 1 to 12

71. The total number of students polled is $\frac{282{,}200}{1000}=282.2$ (in thousands). Thus, $n(S)=282.2$.

(a) Let E be the event "selected student is conservative politically." Thus, we have

$$P(E)=\frac{n(E)}{n(S)}=\frac{56.51}{282.2}\approx .2002$$

(b) Let L be the event "selected student is on the far left" and R be the event "selected student is on the far right." We have

$P(L)=\frac{n(L)}{n(S)}=\frac{7.06}{282.2}\approx .0250$ and

$P(R)=\frac{n(E)}{n(S)}=\frac{3.673}{282.2}\approx .0130$ There are no students which are both on the far left and far right, so $n(L\cap R)=0$ and $P(L\cap R)=0.$ Thus we have

$$\begin{aligned}P(L\text{ or }R)&=P(L\cup R)\\&=P(L)+P(R)-P(L\cap R)\\&=.0250+.0130-0=.0380\end{aligned}$$

(c) Let M be the event "selected student is middle of the road." Then

$$P(M)=\frac{n(M)}{n(S)}=\frac{143.5}{282.2}\approx .5085.$$

Thus,
$P(M)=1-P(M')=1-.5085=.4915.$

72. (a) "No more than 3" means "0 or 1 or 2 or 3."

$$\begin{aligned}&P(\text{no more than }3)\\&=P(0\text{ or }1\text{ or }2\text{ or }3)\\&=P(0)+P(1)+P(2)+P(3)\\&=.31+.25+.18+.12=.86\end{aligned}$$

(b)

$$\begin{aligned}P(\text{at least }2)&=P(2\text{ or }3\text{ or }4\text{ or }5)\\&=P(2)+P(3)+P(4)+P(5)\\&=.18+.12+.08+.06=.44\end{aligned}$$

(c) In a sample of 5 toaster ovens, it is impossible for the number of defective toaster ovens to be more than 5. The probability of an impossible event is 0.

73. In this binomial experiment, we call rolling a five a success. Then $n = 12$, $r = 2$, and $p = \frac{1}{6}$.

$$\begin{aligned}P(2\text{ fives})&=\binom{n}{r}p^r(1-p)^{n-r}\\&=\binom{12}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{12-2}\\&=\frac{12!}{(12-2)!2!}\left(\frac{1}{6^2}\right)\left(\frac{5}{6}\right)^{10}\\&=\frac{12!}{10!2!}\left(\frac{1}{6^2}\right)\left(\frac{5^{10}}{6^{10}}\right)\\&=\frac{12\cdot11\cdot10!}{10!2!}\left(\frac{5^{10}}{6^{12}}\right)\\&=\frac{12\cdot11}{2\cdot1}\left(\frac{5^{10}}{6^{12}}\right)=66\left(\frac{5^{10}}{6^{12}}\right)\approx .296\end{aligned}$$

74. In this binomial experiment, we call tossing a tail a success. Then, $n = 10$, $r = 4$, and $p = \frac{1}{2}$.

$$\begin{aligned}P(4\text{ tails})&=\binom{n}{r}p^r(1-p)^{n-r}\\&=\binom{10}{4}\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^{10-4}\\&=\frac{10!}{(10-4)!4!}\left(\frac{1}{2^4}\right)\left(\frac{1}{2}\right)^6\\&=\frac{10!}{6!4!}\left(\frac{1}{2^4}\right)\left(\frac{1}{2^6}\right)\\&=\frac{10\cdot9\cdot8\cdot7\cdot6!}{6!4!}\left(\frac{1}{2^{10}}\right)\\&=\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\left(\frac{1}{1024}\right)\\&=210\left(\frac{1}{1024}\right)=\frac{105}{512}\approx .205\end{aligned}$$

Chapter 11 Test

1. $a_n = (-1)^n(n^2+2)$

$n=1: a_1 = (-1)^1(1^2+2) = (-1)(1+2) = -3$
$n=2: a_2 = (-1)^2(2^2+2) = (1)(4+2) = 6$
$n=3: a_3 = (-1)^3(3^2+2) = (-1)(9+2) = -11$
$n=4: a_4 = (-1)^4(4^2+2) = (1)(16+2) = 18$
$n=5: a_5 = (-1)^5(5^2+2) = (-1)(25+2) = -27$

The first five terms are –3, 6, –11, 18, and –27. This sequence does not have either a common difference or a common ratio, so the sequence is neither arithmetic nor geometric.

2. $a_n = -3\cdot\left(\frac{1}{2}\right)^n$

$n=1: a_1 = -3\left(\frac{1}{2}\right)^1 = -3\left(\frac{1}{2}\right) = -\frac{3}{2}$
$n=2: a_2 = -3\left(\frac{1}{2}\right)^2 = -3\left(\frac{1}{4}\right) = -\frac{3}{4}$
$n=3: a_3 = -3\left(\frac{1}{2}\right)^3 = -3\left(\frac{1}{8}\right) = -\frac{3}{8}$
$n=4: a_4 = -3\left(\frac{1}{2}\right)^4 = -3\left(\frac{1}{16}\right) = -\frac{3}{16}$
$n=5: a_5 = -3\left(\frac{1}{2}\right)^5 = -3\left(\frac{1}{32}\right) = -\frac{3}{32}$

The first five terms are $-\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}, -\frac{3}{16}$, and $-\frac{3}{32}$. This sequence has a common ratio, $r = \frac{1}{2}$, so the sequence is geometric.

3. $a_1 = 2,\ a_2 = 3,\ a_n = a_{n-1} + 2a_{n-2},$ for $n \ge 3$

$n=3: a_3 = a_2 + 2a_1 = 3 + 2(2) = 3 + 4 = 7$
$n=4: a_4 = a_3 + 2a_2 = 7 + 2(3) = 7 + 6 = 13$
$n=5: a_5 = a_4 + 2a_3 = 13 + 2(7) = 13 + 14 = 27$

The first five terms are 2, 3, 7, 13, and 27. There is no common difference or common ratio, so the sequence is neither arithmetic nor geometric.

4. $a_1 = 1$ and $a_3 = 25$

$a_n = a_1 + (n-1)d \Rightarrow a_3 = a_1 + 2d \Rightarrow$
$25 = 1 + 2d \Rightarrow 2d = 24 \Rightarrow d = 12$
$a_5 = a_1 + (5-1)d \Rightarrow$
$a_5 = a_1 + 4d = 1 + 4(12) = 1 + 48 = 49$

5. $a_1 = 81$ and $r = -\frac{2}{3}$

$a_n = a_1r^{n-1} \Rightarrow a_6 = 81\left(-\frac{2}{3}\right)^5 = 81\left(-\frac{32}{243}\right) = -\frac{32}{3}$

6. Arithmetic, with $a_1 = -43$ and $d = 12$

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \Rightarrow$
$S_{10} = \frac{10}{2}\left[2a_1 + (10-1)d\right] = 5\left[2(-43) + 9(12)\right]$
$= 5(-86 + 108)$
$= 5(22) = 110$

7. Geometric, with $a_1 = 5$ and $r = -2$

$S_n = \dfrac{a_1(1-r^n)}{1-r} \Rightarrow$
$S_{10} = \dfrac{5\left[1-(-2)^{10}\right]}{1-(-2)} = \dfrac{5(1-1024)}{3} = \dfrac{5(-1023)}{3}$
$= 5(-341) = -1705$

8. $\sum_{i=1}^{30}(5i+2)$

This sum represents the sum of the first 30 terms of the arithmetic sequence having $a_1 = 5\cdot 1 + 2 = 7$ and $a_n = a_{30} = 5\cdot 30 + 2 = 150 + 2 = 152.$ Thus, we have

$\sum_{i=1}^{30}(5i+2) = S_{30} = \frac{n}{2}(a_1 + a_n) = \frac{30}{2}(a_1 + a_{30})$
$= 15(7 + 152) = 15(159) = 2385$

We could have also

$\sum_{i=1}^{30}(5i+2) = \sum_{i=1}^{30}5i + \sum_{i=1}^{30}2 = 5\sum_{i=1}^{30}i + \sum_{i=1}^{30}2$

Since $\sum_{i=1}^{n}c = cn$ and $\sum_{i=1}^{n}i = \dfrac{n(n+1)}{2},$ we have

$5\sum_{i=1}^{30}i + \sum_{i=1}^{30}2 = 5\cdot\dfrac{30(30+1)}{2} + 2(30)$
$= 5\cdot\dfrac{30(31)}{2} + 60$
$= 5(15)(31) + 60$
$= 2325 + 60 = 2385$

9. $\sum_{i=1}^{5}(-3 \cdot 2^i)$

This sum represents the sum of the first five terms of the geometric sequence having $a_1 = -3 \cdot 2^1 = -6$ and $r = 2$. Thus, we have

$$\sum_{i=1}^{5} -3 \cdot 2^1 = S_5 = \frac{a_1\left(1-r^n\right)}{1-r} = \frac{-6\left(1-2^5\right)}{1-2}$$
$$= \frac{-6(1-32)}{-1} = 6(-31) = -186$$

10. $\sum_{i=1}^{\infty}(2^i) \cdot 4$

This is the sum of an infinite geometric sequence with $r = 2$. Since $|r| > 1$, the sum does not exist.

11. $\sum_{i=1}^{\infty} 54\left(\frac{2}{9}\right)^i$

This is the sum of the infinite geometric sequence with $a_1 = 54\left(\frac{2}{9}\right)^1 = 54\left(\frac{2}{9}\right) = 12$ and $r = \frac{2}{9}$. Since $\left|\frac{2}{9}\right| < 1$, we can use the summation formula $S_\infty = \frac{a_1}{1-r}$.

$$\sum_{i=1}^{\infty} 54\left(\tfrac{2}{9}\right)^i = S_\infty = \frac{a_1}{1-r} = \frac{12}{1-\frac{2}{9}} = \frac{12}{\frac{7}{9}}$$
$$= 12 \cdot \frac{9}{7} = \frac{108}{7}$$

12.
$$(x+y)^6 = x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + y^6$$
$$= x^6 + \frac{6!}{1!5!}x^5y + \frac{6!}{2!4!}x^4y^2 + \frac{6!}{3!3!}x^3y^3 + \frac{6!}{4!2!}x^2y^4 + \frac{6!}{5!1!}xy^5 + y^6$$
$$= x^6 + 6x^5y + \frac{6\cdot5}{2\cdot1}x^4y^2 + \frac{6\cdot5\cdot4}{3\cdot2\cdot1}x^3y^3 + \frac{6\cdot5}{2\cdot1}x^2y^4 + 6xy^5 + y^6$$
$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

13.
$$(2x-3y)^4 = (2x)^4 + \binom{4}{1}(2x)^3(-3y) + \binom{4}{2}(2x)^2(-3y)^2 + \binom{4}{3}(2x)(-3y)^3 + (-3y)^4$$
$$= 16x^4 + \frac{4!}{1!3!}\left(8x^3\right)(-3y) + \frac{4!}{2!2!}\left(4x^2\right)\left(9y^2\right) + \frac{4!}{1!3!}(2x)\left(-27y^3\right) + 81y^4$$
$$= 16x^4 - 4\left(24x^3y\right) + \frac{4\cdot3}{2\cdot1}\left(36x^2y^2\right) - 4\left(54xy^3\right) + 81y^4$$
$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

14. To find the third term in the expansion of $(w-2y)^6$, use the formula $\binom{n}{k-1}x^{n-(k-1)}y^{k-1}$ with $n = 6$ and $k = 3$.

Then $k - 1 = 2$ and $n - (k - 1) = 4$. Thus, the third term is as follows.

$$\binom{n}{k-1}x^{n-(k-1)}y^{k-1} = \binom{6}{2}w^{6-2}(-2y)^2$$
$$= \frac{6!}{2!4!}w^4\left(4y^2\right)$$
$$= \frac{6\cdot5\cdot4!}{2!4!}\left(4w^4y^2\right)$$
$$= \frac{6\cdot5}{2\cdot1}\left(4w^4y^2\right) = 60w^4y^2$$

15. $8! = 8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 40{,}320$

16.
$$C(10,\ 2) = \frac{10!}{(10-2)!2!} = \frac{10!}{8!2!}$$
$$= \frac{10\cdot9\cdot8!}{8!2!} = \frac{10\cdot9}{2\cdot1} = 45$$

17.
$$\binom{7}{3} = C(7,3) = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!}$$
$$= \frac{7\cdot6\cdot5\cdot4!}{4!3!} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$$

18.
$$P(11,\ 3) = \frac{11!}{(11-3)!} = \frac{11!}{8!} = \frac{11\cdot10\cdot9\cdot8!}{8!}$$
$$= 11\cdot10\cdot9 = 990$$

19. Let S_n be the statement $1+7+13+\ldots+(6n-5)=n(3n-2)$.

Step 1: Show that the statement is true for $n = 1$.

S_1 is the statement $6(1)+2=3(1^2)+5(1) \Rightarrow 6+2=3(1)+5 \Rightarrow 8=3+5 \Rightarrow 8=8$, which is true.

Step 2: Show that if S_k is true, S_{k+1} is also true. S_k is the statement. $1+7+13+\ldots+(6k-5)=k(3k-2)$

S_{k+1} is the statement $1+7+13+\ldots+\left[6(k+1)-5\right]=(k+1)\left[3(k+1)-2\right]$.

Start with S_k: $1+7+13+\ldots+(6k-5)=k(3k-2)$ and add the $(k+1)$ st term, $\left[6(k+1)-5\right]$, to both sides of this equation.

$$1+7+13+\ldots+(6k-5)=k(3k-2)$$
$$1+7+13+\ldots+(6k-5)+\left[6(k+1)-5\right]=k(3k-2)=k(3k-2)+\left[6(k+1)-5\right]$$
$$1+7+13+\ldots+(6k-5)+\left[6(k+1)-5\right]=3k^2-2k+(6k+6-5)$$
$$1+7+13+\ldots+(6k-5)+\left[6(k+1)-5\right]=3k^2+4k+1$$
$$1+7+13+\ldots+(6k-5)+\left[6(k+1)-5\right]=(k+1)(3k+1)$$
$$1+7+13+\ldots+(6k-5)+\left[6(k+1)-5\right]=(k+1)\left[(3k+3)-2\right]$$
$$1+7+13+\ldots+(6k-5)+\left[6(k+1)-5\right]=(k+1)\left[3(k+1)-2\right]$$

It has been shown that S_k implies S_{k+1}. Therefore, by mathematical induction, S_n is true for every positive integer n.

20. Using the fundamental principle of counting, we have $4\cdot3\cdot2=24$.
There are 24 different kinds of shoes if there are 4 styles, 3 different colors, and 2 different shades.

21. We are choosing three people from a group of ten without regard to order, so there are $C(10, 3)$ ways to do this. Since

$$C(10,\ 3)=\frac{10!}{(10-3)!3!}=\frac{10!}{7!3!}=\frac{10\cdot9\cdot8\cdot7!}{7!3!}=\frac{10\cdot9\cdot8}{3\cdot2\cdot1}=120$$

there are 120 ways 3 officers can be selected to attend a seminar.
If one black and exactly one Asian officer must be included, we have 1 way to select the black officer, 2 ways to select the Asian officer, and 7 ways to select the remaining officer. Altogether, then, we have $1\cdot2\cdot7=14$ ways to do this.

22. The two women can be selected from the four women in

$$C(4,\ 2)=\frac{4!}{(4-2)!2!}=\frac{4!}{2!2!}=\frac{4\cdot3}{2\cdot1}=6 \text{ ways.}$$

The two men can be selected from the six men in $C(6,\ 2)=\frac{6!}{(6-2)!2!}=\frac{6!}{4!2!}=\frac{6\cdot5}{2\cdot1}=15$ ways. Altogether, then, we have $6\cdot15=90$ ways to do this.

23. **(a)** $P(\text{red three})=\frac{n(\text{red three cards})}{n(\text{cards in deck})}=\frac{2}{52}=\frac{1}{26}$

(b) Let F be the event "draw a face card." Each suit contains 3 face cards (jack, queen, and king), so the deck contains 12 face cards. Thus $n(F) = 12$ and $P(F)=\frac{12}{52}=\frac{3}{13}$ The probability of drawing a card that is not a face card is $P(F')=1-P(F)=1-\frac{3}{13}=\frac{10}{13}$.

(c) Let K be the event "king is drawn" and S be the event "spade is drawn." There are 4 kings, so $n(K) = 4$ and $P(K)=\frac{4}{52}=\frac{1}{13}$
There are 13 spades, so $n(S) = 13$ and $P(S)=\frac{13}{52}=\frac{1}{4}$. There is one card which is both a king and a spade, so $n(K\cap S)=1$ and $P(K\cap S)=\frac{1}{52}$. Thus we have

$$P(K \text{ or } S)=P(K\cup S)=P(K)+P(S)-P(K\cap S)=\frac{1}{13}+\frac{1}{4}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$$

(d) Consider the event F: "draw a face card." As shown in the solution to part (b), $P(F) = \frac{3}{13}$ and $P(F') = \frac{10}{13}$. Thus, the odds in favor of drawing a face card are

$$\frac{P(F)}{P(F')} = \frac{\frac{3}{13}}{\frac{10}{13}} = \frac{3}{13} \cdot \frac{13}{10} = \frac{3}{10} \text{ or 3 to 10}$$

24. "At most 2" means "0 or 1 or 2."

$$\begin{aligned} P(\text{at most } 2) &= P(0 \text{ or } 1 \text{ or } 2) \\ &= P(0) + P(1) + P(2) \\ &= .19 + .43 + .30 = .92 \end{aligned}$$

The probability that at most 2 light bulbs are defective is .92.

25. In this binomial experiment, we call rolling a five a success. Then $n = 6$, $r = 2$, and $p = \frac{1}{6}$.

$$\begin{aligned} P(2 \text{ fives}) &= \binom{n}{r} p^r (1-p)^{n-r} \\ &= \binom{6}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{6-2} \\ &= \frac{6!}{(6-2)!2!}\left(\frac{1}{6^2}\right)\left(\frac{5}{6}\right)^4 \\ &= \frac{6!}{4!2!}\left(\frac{1}{6^2}\right)\left(\frac{5^4}{6^4}\right) = \frac{6 \cdot 5 \cdot 4!}{4!2!}\left(\frac{5^4}{6^6}\right) \\ &= \frac{6 \cdot 5}{2 \cdot 1}\left(\frac{5^4}{6^6}\right) = 15\left(\frac{5^4}{6^6}\right) \approx .201 \end{aligned}$$

Chapter 11 Quantitative Reasoning

1. For the median person who joins the work force immediately after high school, we have $a_1 = 26,364$, $a_2 = 26,364 + 534$, $a_3 = 26,364 + 534(2)$, and so on

Thus,

$$\begin{aligned} a_n &= 26,364 + 534(n-1) \\ &= 26,364 + 534n - 534 = 25,830 + 534n \end{aligned}$$

The person will work for $65 - 18 = 47$ years after entering the work force, and the sum of his/her earnings will be $\sum_{i=1}^{47}[25,830 + 534n]$.

Since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\begin{aligned} \sum_{i=1}^{47}[25,830 + 534n] = S_{47} &= \tfrac{47}{2}\left[2a_1 + (47-1)d\right] \\ &= \tfrac{47}{2}\left[2(26,364) + 46(534)\right] \\ &= \tfrac{47}{2}(52,728 + 24,564) \\ &= \tfrac{47}{2}(77,292) = 47(38,646) \\ &= 1,816,362 \end{aligned}$$

Thus, the person will earn $\$1,816,362$ until retirement.

2. For the median person who attends four years of college before joining the work force, we have

$b_1 = 43,368$, $b_2 = 43,368 + 994$, $b_3 = 43,368 + 994(2)$, and so on

Thus,

$$\begin{aligned} b_n &= 43,368 + 994(n-1) \\ &= 43,368 + 994n - 994 = 42,374 + 994n \end{aligned}$$

The person will work for $65 - 22 = 43$ years after college and after entering the work force and the sum of his/her earnings will be $\sum_{i=1}^{43}[42,374 + 994n]$. Since $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, we have

$$\begin{aligned} \sum_{i=1}^{43}[42,374 + 994n] = S_{43} &= \tfrac{43}{2}\left[2a_1 + (43-1)d\right] \\ &= \tfrac{43}{2}\left[2(43,368) + 42(994)\right] \\ &= \tfrac{43}{2}(86,736 + 41,748) \\ &= \tfrac{43}{2}(128,484) = 43(64,242) \\ &= 2,762,406 \end{aligned}$$

Thus, the person will earn $\$2,762,406$ until retirement.

3. Since $2,762,406 - 1,816,362 = 946,044$, the difference is $\$946,044$, which is much greater than \$130,000. Additional answers will vary.

Chapter R

REVIEW OF BASIC CONCEPTS

Section R.1: Sets

1. The elements of the set {12, 13, 14, …, 20} are all the natural numbers from 12 to 20 inclusive. There are 9 elements in the set, {12, 13, 14, 15, 16, 17, 18, 19, 20}.

2. The elements of the set {8, 9, 10, …, 17} are all the natural numbers from 8 to 17 inclusive. There are 10 elements in the set, {8, 9, 10, 11, 12, 13, 14, 15, 16, 17}.

3. Each element of the set $\left\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}\right\}$ after the first is found by multiplying the preceding number by $\frac{1}{2}$. There are 6 elements in the set, $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\right\}$.

4. Each element of the set {3, 9, 27, …, 729} after the first is found by multiplying the preceding number by 3. There are 6 elements in the set, {3, 9, 27, 81, 243, 729}.

5. To find the elements of the set {17, 22, 27 …, 47}, start with 17 and add 5 to find the next number. There are 7 elements in the set, {17, 22, 27, 32, 37, 42, 47}.

6. To find the elements of the set {74, 68, 62, …, 38}, start with 74 and subtract 6 (or add –6) to find the next number. There are 7 elements in the set, {74, 68, 62, 56, 50, 44, 38}.

7. When you list all elements in the set {all natural numbers greater than 7 and less than 15}, you obtain {8, 9, 10, 11, 12, 13, 14}.

8. When you list all elements in the set {all natural numbers not greater than 4} you obtain {1, 2, 3, 4}.

9. The set {4, 5, 6, …, 15} has a limited number of elements, so it is a finite set.

10. The set {4, 5, 6, …} has an unlimited number of elements, so it is an infinite set.

11. The set $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$ has an unlimited number of elements, so it is an infinite set.

12. The set {0, 1, 2, 3, 4, 5, …, 75} has a limited number of elements, so it is a finite set.

13. The set $\{x \mid x$ is a natural number larger than 5}, which can also be written as {6, 7, 8, 9, …}, has an unlimited number of elements, so it is an infinite set.

14. The set $\{x \mid x$ is a person alive now} has a limited number of elements (although it is a very large number), so it is a finite set.

15. There are an infinite number of fractions between 0 and 1, so $\{x \mid x$ is a fraction between 0 and 1} is an infinite set.

16. The set $\{x \mid x$ is an even natural number} has no largest element. Since it has an unlimited number of elements, this is an infinite set.

17. 6 is an element of the set {3, 4, 5, 6}, so we write $6 \in \{3, 4, 5, 6\}$.

18. 9 is an element of the set {3, 2, 5, 9, 8}, so we write $9 \in \{3, 2, 5, 9, 8\}$.

19. –4 is not an element of {4, 6, 8, 10}, so we write $-4 \notin \{4, 6, 8, 10\}$.

20. –12 is not an element of the set {3, 5, 12, 14}, so we write $-12 \notin \{3, 5, 12, 14\}$.

21. 0 is an element of {2, 0, 3, 4}, so we write $0 \in \{2, 0, 3, 4\}$.

22. 0 is not an element of the set {5, 6, 7, 8, 10}, so we write $0 \notin \{5, 6, 7, 8, 10\}$.

23. {3} is a subset of {2, 3, 4, 5}, not an element of {2, 3, 4, 5}, so we write $\{3\} \notin \{2, 3, 4, 5\}$.

24. {5} is a subset of {3, 4, 5, 6, 7}, not an element of {3, 4, 5, 6, 7}, so we write $\{5\} \notin \{3, 4, 5, 6, 7\}$.

25. {0} is a subset of {0, 1, 2, 5}, not an element of {0, 1, 2, 5}, so we write $\{0\} \notin \{0, 1, 2, 5\}$.

26. {2} is a subset of {2, 4, 6, 8}, not an element of {2, 4, 6, 8}, so we write $\{2\} \notin \{2, 4, 6, 8\}$.

27. 0 is not an element of $\varnothing$, since the empty set contains no elements. Thus, $0 \notin \varnothing$.

28. $\varnothing$ is a subset of $\varnothing$, not an element of $\varnothing$. The empty set contains no elements. Thus we write, $\varnothing \notin \varnothing$.

29. $3 \in \{2, 5, 6, 8\}$
Since 3 is not one of the elements in $\{2, 5, 6, 8\}$, the statement is false.

30. $6 \in \{-2, 5, 8, 9\}$
Since 6 is not one of the elements of $\{-2, 5, 8, 9\}$, the statement is false.

31. $1 \in \{3, 4, 5, 11, 1\}$
Since 1 is one of the elements of $\{3, 4, 5, 11, 1\}$, the statement is true.

32. $12 \in \{18, 17, 15, 13, 12\}$
Since 12 is one of the elements of $\{18, 17, 15, 13, 12\}$, the statement is true.

33. $9 \notin \{2, 1, 5, 8\}$
Since 9 is not one of the elements of $\{2, 1, 5, 8\}$, the statement is true.

34. $3 \notin \{7, 6, 5, 4\}$
Since 3 is not an element of $\{7, 6, 5, 4\}$, the statement is true.

35. $\{2, 5, 8, 9\} = \{2, 5, 9, 8\}$
This statement is true because both sets contain exactly the same four elements.

36. $\{3, 0, 9, 6, 2\} = \{2, 9, 0, 3, 6\}$
This statement is true because both sets contain exactly the same five elements.

37. $\{5, 8, 9\} = \{5, 8, 9, 0\}$
These two sets are not equal because $\{5, 8, 9, 0\}$ contains the element 0, which is not an element of $\{5, 8, 9\}$. Therefore, the statement is false.

38. $\{3, 7, 12, 14\} = \{3, 7, 12, 14, 0\}$
These two sets are not equal because $\{3, 7, 12, 14, 0\}$ contains the element 0, which is not an element of $\{3, 7, 12, 14\}$. Therefore, the statement is false.

39. $\{x \mid x \text{ is a natural number less than } 3\} = \{1, 2\}$
Since 1 and 2 are the only natural numbers less than 3, this statement is true.

40. $\{x \mid x \text{ is a natural number greater than } 10\}$
$= \{11, 12, 13, \ldots\}$
Since both sets describe the same elements, the statement is true.

41. $\{5, 7, 9, 19\} \cap \{7, 9, 11, 15\} = \{7, 9\}$
Since 7 and 9 are the only elements belonging to both sets, the statement is true.

42. $\{8, 11, 15\} \cap \{8, 11, 19, 20\} = \{8, 11\}$
Since 8 and 11 are the only elements belonging to both sets, the statement is true.

43. $\{2, 1, 7\} \cup \{1, 5, 9\} = \{1\}$
$\{2, 1, 7\} \cup \{1, 5, 9\} = \{1, 2, 5, 7, 9\}$, while $\{2, 1, 7\} \cap \{1, 5, 9\} = \{1\}$. Therefore, the statement is false.

44. $\{6, 12, 14, 16\} \cup \{6, 14, 19\} = \{6, 14\}$
$\{6, 12, 14, 16\} \cup \{6, 14, 19\}$
$= \{6, 12, 14, 16, 19\}$, while
$\{6, 12, 14, 16\} \cap \{6, 14, 19\} = \{6, 14\}$.
Therefore, the statement is false.

45. $\{3, 2, 5, 9\} \cap \{2, 7, 8, 10\} = \{2\}$
Since 2 is the only element belonging to both sets, the statement is true.

46. $\{8, 9, 6\} \cup \{9, 8, 6\} = \{8, 9\}$
The sets $\{8, 9, 6\}$ and $\{9, 8, 6\}$ are equal since they contain exactly the same three elements. Their union contains the same elements, namely 8, 9, and 6. Thus, the statement is false.

47. $\{3, 5, 9, 10\} \cap \varnothing = \{3, 5, 9, 10\}$
In order to belong to the intersection of two sets, an element must belong to both sets. Since the empty set contains no elements, $\{3, 5, 9, 10\} \cap \varnothing = \varnothing$, so the statement is false.

48. $\{3, 5, 9, 10\} \cup \varnothing = \{3, 5, 9, 10\}$
For any set A, $A \cup \varnothing = A$. Thus, the statement is true.

49. $\{1, 2, 4\} \cup \{1, 2, 4\} = \{1, 2, 4\}$
Since the two sets are equal, their union contains the same elements, namely 1, 2, and 4. Thus, the statement is true.

50. $\{1, 2, 4\} \cap \{1, 2, 4\} = \varnothing$
This statement is false, since
$\{1, 2, 4\} \cap \{1, 2, 4\} = \{1, 2, 4\}$

51. $\varnothing \cup \varnothing = \varnothing$
Since the empty set contains no elements, the statement is true.

52. $\varnothing \cap \varnothing = \varnothing$
This statement is true.

For Exercises 53–64, $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{2, 4, 8, 10\}$, $C = \{4, 10, 12\}$, $D = \{2, 10\}$, and $U = \{2, 4, 6, 8, 10, 12, 14\}$.

53. $A \subseteq U$
This statement says "A is a subset of U." Since every element of A is also an element of U, the statement is true.

54. $C \subseteq U$
This statement says "*C* is a subset of *U*." Since every element of *C* is also an element of *U*, the statement is true.

55. $D \subseteq B$
Since both elements of *D*, 2 and 10, are also elements of *B*, *D* is a subset of *B*. The statement is true.

56. $D \subseteq A$
Since both elements of *D*, 2, and 10, are also elements of *A*, *D* is a subset of *A*. The statement is true.

57. $A \subseteq B$
Set *A* contains two elements, 6 and 12, that are not elements of *B*. Thus, *A* is not a subset of *B*. The statement is false.

58. $B \subseteq C$
Set *B* contains two elements, 2 and 8, that are not elements of *C*. Thus, *B* is not a subset of *C*. The statement is false.

59. $\varnothing \subseteq A$
The empty set is a subset of every set, so the statement is true.

60. $\varnothing \subseteq \varnothing$
The empty set is a subset of every set, so the statement is true.

61. $\{4, 8, 10\} \subseteq B$
Since 4, 8, and 10 are all elements of *B*, {4, 8, 10} is a subset of *B*. The statement is true.

62. $\{0, 2\} \subseteq D$
Since 0 is not an element of *D*, {0, 2} is not a subset of *D*. The statement is false.

63. $B \subseteq D$
Since *B* contains two elements, 4 and 8, that are not elements of *D*, *B* is not a subset of *D*. The statement is false.

64. $A \not\subset C$
There are three elements of *A* (2, 6, and 8) that are not elements of *C*, so *A* is not a subset of *C*. The statement is true.

65. Every element of {2, 4, 6} is also an element of {3, 2, 5, 4, 6}, so {2, 4, 6} is a subset of {3, 2, 5, 4, 6}.
We write $\{2, 4, 6\} \subseteq \{3, 2, 5, 4, 6\}$.

66. Every element of {1, 5} is also an element of {0, –1, 2, 3, 1, 5}, so {1, 5} is a subset of the set {0, –1, 2, 3, 1, 5}. We write $\{1, 5\} \subseteq \{0, -1, 2, 3, 1, 5\}$.

67. Since 0 is an element of {0, 1, 2}, but is not an element of {1, 2, 3, 4, 5}, {0, 1, 2} is not a subset of {1, 2, 3, 4, 5}. We write $\{0, 1, 2\} \not\subseteq \{1, 2, 3, 4, 5\}$.

68. Since 8 is an element of {5, 6, 7, 8}, but is not an element of {1, 2, 3, 4, 5, 6, 7}, {5, 6, 7, 8} is not a subset of {1, 2, 3, 4, 5, 6, 7}. We write $\{5, 6, 7, 8\} \not\subseteq \{1, 2, 3, 4, 5, 6, 7\}$.

69. The empty set is a subset of every set, so $\varnothing \subseteq \{1, 4, 6, 8\}$.

70. The empty set is a subset of every set, including itself, so $\varnothing \subseteq \varnothing$.

For Exercises 71–94,
$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$,
$M = \{0, 2, 4, 6, 8\}$, $N = \{1, 3, 5, 7, 9, 11, 13\}$,
$Q = \{0, 2, 4, 6, 8, 10, 12\}$, and $R = \{0, 1, 2, 3, 4\}$.

71. $M \cap R$
The only elements belonging to both *M* and *R* are 0, 2, and 4, so $M \cap R = \{0, 2, 4\}$.

72. $M \cup R$
The union *M* and *R* is made up of elements which belong to *M* or to *R* (or to both).
$M \cup R = \{0, 1, 2, 3, 4, 6, 8\}$

73. $M \cup N$
The union of two sets contains all elements that belong to either set or to both sets.
$M \cup N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$

74. $M \cap U$
Since $M \subseteq U$, the intersection of *M* and *U* will contain the same elements as *M*. $M \cap U = M$ or {0, 2, 4, 6, 8}.

75. $M \cap N$
There are no elements which belong to both *M* and *N*, so $M \cap N = \varnothing$. *M* and *N* are disjoint sets.

76. $M \cup Q$
Since $M \subseteq Q$, the elements belonging to *M* or *Q* are all the elements belonging to *Q*.
$M \cup Q = Q$ or {0, 2, 4, 6, 8, 10, 12}

77. $N \cup R = \{0, 1, 2, 3, 4, 5, 7, 9, 11, 13\}$

78. $U \cap N$
Since $N \subseteq U$, the elements belonging to *U* and *N* are all the elements belonging to *N*,
$U \cap N = N$ or {1, 3, 5, 7, 9, 11, 13}

79. N'
The set N' is the complement of set N, which means the set of all elements in the universal set U that do not belong to N.
$N' = Q$ or $\{0, 2, 4, 6, 8, 10, 12\}$

80. Q'
The set Q' is the complement of set Q, which means the set of all elements in the universal set U that do not belong to Q.
$Q' = N$ or $\{1, 3, 5, 7, 9, 11, 13\}$

81. $M' \cap Q$
First form M', the complement of M. M' contains all elements of U that are not elements of M. Thus,
$M' = \{1, 3, 5, 7, 9, 10, 11, 12, 13\}$. Now form the intersection of M' and Q. Thus, we have $M' \cap Q = \{10, 12\}$.

82. $Q \cap R'$
First form R', the complement of R. R' contains all elements of U the are not elements of R. Thus,
$R' = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$. Now form the intersection of Q and R'. Thus, we have $Q \cap R' = \{6, 8, 10, 12\}$

83. $\varnothing \cap R$
Since the empty set contains no elements, there are no elements belonging to both $\varnothing$ and R. Thus, $\varnothing$ and R are disjoint sets, and $\varnothing \cap R = \varnothing$.

84. $\varnothing \cap Q$
Since the empty set contains no elements, there are no elements belonging to both $\varnothing$ and Q. Thus, $\varnothing$ and Q are disjoint sets, and $\varnothing \cap Q = \varnothing$.

85. $N \cup \varnothing$
Since $\varnothing$ contains no elements, the only elements belonging to N or $\varnothing$ are the elements of N. Thus, $\varnothing$ and N are disjoint sets, and $N \cup \varnothing = N$ or $\{1, 3, 5, 7, 9, 11, 13\}$.

86. $R \cup \varnothing$
Since $\varnothing$ contains no elements, the only elements belonging to R or $\varnothing$ are the elements of R. Thus, $\varnothing$ and R are disjoint sets, and $R \cup \varnothing = R$ or $\{0, 1, 2, 3, 4\}$.

87. $(M \cap N) \cup R$
First form the intersection of M and N. Since M and N have no common elements (they are disjoint), $M \cap N = \varnothing$. Thus, $(M \cap N) \cup R = \varnothing \cup R$. Now, since $\varnothing$ contains no elements, the only elements belonging to R or $\varnothing$ are the elements of R. Thus, $\varnothing$ and R are disjoint sets, and $\varnothing \cup R = R$ or $\{0, 1, 2, 3, 4\}$.

88. $(N \cup R) \cap M$
First form the union of N and R. We have $N \cup R = \{0, 1, 2, 3, 4, 5, 7, 9, 11, 13\}$. Now form the intersection of this set with M. We have $(N \cup R) \cap M = \{0, 2, 4\}$.

89. $(Q \cap M) \cup R$
First form the intersection of Q and M. We have $Q \cap M = \{0, 2, 4, 6, 8\} = M$. Now form the union of this set with R. We have $(Q \cap M) \cup R = M \cup R = \{0, 1, 2, 3, 4, 6, 8\}$.

90. $(R \cup N) \cap M'$
First form the union of R and N. We have $R \cup N = \{0, 1, 2, 3, 4, 5, 7, 9, 11, 13\}$. Now find the complement of M. We have $M' = \{1, 3, 5, 7, 9, 10, 11, 12, 13\}$. Now, find the intersection of these two sets. We have $(R \cup N) \cap M' = N$ or $\{1, 3, 5, 7, 9, 11, 13\}$.

91. $(M' \cup Q) \cap R$
First, find M', the complement of M. We have $M' = \{1, 3, 5, 7, 9, 10, 11, 12, 13\}$. Next, form the union of M' and Q. We have $M' \cup Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} = U$. Thus, we have $(M' \cup Q) \cap R = U \cap R = R$ or $\{0, 1, 2, 3, 4\}$.

92. $Q \cap (M \cup N)$
First, form the union of M and N. We have $M \cup N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$. Now form the intersection of Q with this set. We have $Q \cap (M \cup N) = M$ or $\{0, 2, 4, 6, 8\}$.

93. $Q' \cap (N' \cap U)$
First, find Q', the complement of Q. We have $Q' = \{1, 3, 5, 7, 9, 11, 13\} = N$. Now find N', the complement of N. We have $N' = \{0, 2, 4, 6, 8, 10, 12\} = Q$. Next, form the intersection of N' and U. We have $N' \cap U = Q \cap U = Q$ Finally, we have $Q' \cap (N' \cap U) = Q' \cap Q = \varnothing$ Since the intersection of Q' and $(N' \cap U)$ is $\varnothing$, Q' and $(N' \cap U)$ are disjoint sets.

94. $(U \cap \varnothing') \cup R$
Since $\varnothing' = U$, and $U \cap U = U$, we have $(U \cap \varnothing') \cup R = (U \cap U) \cup R = U \cup R = U$ or $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$.

95. M' is the set of all students in this school who are not taking this course.

96. $M \cup N$ is the set of all students in this school who are taking this course or calculus or both.

97. $N \cap P$ is the set of all students in this school who are taking both calculus and history.

98. $N' \cap P'$ is the set of all students in this school who are not taking calculus and are not taking history.

99. $M \cup P$ is the set of all students in this school who are taking this course or history or both.

100. $P' \cup M'$ is the set of all students in this school who are not taking history or not taking this course or are taking neither course.

Section R.2: Real Numbers and Their Properties

1. **(a)** 0 is a whole number. Therefore, it is also an integer, a rational number, and a real number. 0 belongs to B, C, D, F.

(b) 34 is a natural number. Therefore, it is also a whole number, an integer, a rational number, and a real number. 34 belongs to A, B, C, D, F.

(c) $-\frac{9}{4}$ is a rational number and a real number. $-\frac{9}{4}$ belongs to D, F.

(d) $\sqrt{36} = 6$ is a natural number. Therefore, it is also a whole number, an integer, a rational number, and a real number. $\sqrt{36}$ belongs to A, B, C, D, F.

(e) $\sqrt{13}$ is an irrational number and a real number. $\sqrt{13}$ belongs to E, F.

(f) $2.16 = \frac{216}{100} = \frac{54}{25}$ is a rational number and a real number. 2.16 belongs to D, F.

2. Answers will vary. Possible answers include that *D. Rational Numbers and E. Irrational numbers* are two sets that have no common elements. Hence, a number could not belong to both sets.

3. False. Positive integers are whole numbers, but negative integers are not.

4. True. Every natural number is an integer.

5. False. No irrational numbers are integers.

6. True. Every integer is a rational number.

7. True. Every natural number is a whole number.

8. False. No rational numbers are irrational.

9. True. Some rational numbers are whole numbers.

10. True. Some real numbers are integers.

11. 1 and 3 are natural numbers.

12. 0, 1, and 3 are whole numbers.

13. -6, $-\frac{12}{4}$ (or -3), 0, 1, and 3 are integers.

14. $-6, -\frac{12}{4}(\text{or } -3), -\frac{5}{8}, 0, \frac{1}{4}, 1, \text{and } 3$ are rational numbers.

15. $-2^4 = -(2\cdot 2\cdot 2\cdot 2) = -16$

16. $-3^5 = -(3\cdot 3\cdot 3\cdot 3\cdot 3) = -243$

17. $(-2)^4 = (-2)\cdot(-2)\cdot(-2)\cdot(-2) = 16$

18. $-2^6 = -(2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2) = -64$

19. $(-3)^5 = (-3)\cdot(-3)\cdot(-3)\cdot(-3)\cdot(-3) = -243$

20. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

21. $-2\cdot 3^4 = -2\cdot(3\cdot 3\cdot 3\cdot 3) = -2\cdot 81 = -162$

22. $$\begin{aligned}-4(-5)^3 &= -4\cdot[(-5)\cdot(-5)\cdot(-5)]\\ &= -4\cdot(-125) = 500\end{aligned}$$

23. $$\begin{aligned}-2\cdot 5 + 12 \div 3 &= -10 + 12 \div 3\\ &= -10 + 4 = -6\end{aligned}$$

24. $$\begin{aligned}9\cdot 3 - 16 \div 4 &= 27 - 16 \div 4\\ &= 27 - 4 = 23\end{aligned}$$

25. $$\begin{aligned}-4(9-8) + (-7)(2)^3 &= -4(1) + (-7)(2)^3\\ &= -4(1) + (-7)\cdot 8\\ &= -4 + (-7)\cdot 8\\ &= -4 + (-56) = -60\end{aligned}$$

26. $6(-5)-(-3)(2)^4 = 6(-5)-(-3)\cdot 16$
$= -30-(-3)\cdot 16$
$= -30-(-48)$
$= -30+48 = 18$

27. $(4-2^3)(-2+\sqrt{25}) = (4-8)(-2+5)$
$= (-4)(3) = -12$

28. $\left[-3^2-(-2)\right]\left[\sqrt{16}-2^3\right] = \left[-9-(-2)\right]\left[4-2^3\right]$
$= [-9+2][4-8]$
$= (-7)(-4) = 28$

29. $\left(-\frac{2}{9}-\frac{1}{4}\right)-\left[-\frac{5}{18}-\left(-\frac{1}{2}\right)\right]$
$= \left(-\frac{8}{36}-\frac{9}{36}\right)-\left(-\frac{5}{18}+\frac{9}{18}\right)$
$= \left(-\frac{17}{36}\right)-\left(\frac{4}{18}\right) = -\frac{17}{36}-\frac{8}{36} = -\frac{25}{36}$

30. $\left(-\frac{5}{8}-\left(-\frac{2}{5}\right)\right)-\left[\frac{3}{2}-\frac{11}{10}\right]$
$= \left(-\frac{25}{40}+\frac{16}{40}\right)-\left(\frac{15}{10}-\frac{11}{10}\right) = \left(-\frac{9}{40}\right)-\left(\frac{4}{10}\right)$
$= -\frac{9}{40}-\frac{16}{40} = -\frac{25}{40} = -\frac{5}{8}$

31. $\frac{-8+(-4)(-6)\div 12}{4-(-3)} = \frac{-8+24\div 12}{4+3}$
$= \frac{-8+2}{7}$
$= \frac{-6}{7} = -\frac{6}{7}$

32. $\frac{15\div 5\cdot 4\div 6-8}{-6-(-5)-8\div 2} = \frac{3\cdot 4\div 6-8}{-6-(-5)-4}$
$= \frac{12\div 6-8}{-6+5-4} = \frac{2-8}{-1-4}$
$= \frac{-6}{-5} = \frac{6}{5}$

33. Let $p = -4$, $q = 8$, and $r = -10$.
$2p-7q+r^2 = 2(-4)-7\cdot 8+(-10)^2$
$= 2(-4)-7\cdot 8+100$
$= -8-7\cdot 8+100$
$= -8-56+100$
$= -64+100 = 36$

34. Let $p = -4$, $q = 8$, and $r = -10$.
$-p^3-2q+r = -(-4)^3-2\cdot 8+(-10)$
$= -(-64)-2\cdot 8+(-10)$
$= -(-64)-16+(-10)$
$= 64-16+(-10)$
$= 48+(-10) = 38$

35. Let $p = -4$, $q = 8$, and $r = -10$.
$\frac{q+r}{q+p} = \frac{8+(-10)}{8+(-4)} = \frac{-2}{4} = -\frac{1}{2}$

36. Let $p = -4$, $q = 8$, and $r = -10$.
$\frac{3q}{3p-2r} = \frac{3\cdot 8}{3(-4)-2(-10)} = \frac{24}{-12-2(-10)}$
$= \frac{24}{-12-(-20)} = \frac{24}{-12+20} = \frac{24}{8} = 3$

37. Let $p = -4$, $q = 8$, and $r = -10$.
$\frac{3q}{r}-\frac{5}{p} = \frac{3\cdot 8}{-10}-\frac{5}{-4} = \frac{24}{-10}-\frac{5}{-4} = -\frac{12}{5}-\frac{5}{-4}$
$= -\frac{12}{5}+\frac{5}{4} = -\frac{48}{20}+\frac{25}{20} = -\frac{23}{20}$

38. Let $p = -4$, $q = 8$, and $r = -10$.
$\frac{\frac{q}{4}-\frac{r}{5}}{\frac{p}{2}+\frac{q}{2}} = \frac{\frac{8}{4}-\frac{-10}{5}}{\frac{-4}{2}+\frac{8}{2}} = \frac{2-\frac{-10}{5}}{-2+\frac{8}{2}}$
$= \frac{2-(-2)}{-2+4} = \frac{2+2}{2} = \frac{4}{2} = 2$

39. Let $p = -4$, $q = 8$, and $r = -10$.
$\frac{-(p+2)^2-3r}{2-q} = \frac{-(-4+2)^2-3(-10)}{2-8}$
$= \frac{-(-2)^2-3(-10)}{-6}$
$= \frac{-4-3(-10)}{-6} = \frac{-4-(-30)}{-6}$
$= \frac{-4+30}{-6} = \frac{26}{-6} = -\frac{13}{3}$

40. Let $p = -4$, $q = 8$, and $r = -10$.
$\frac{5q+2(1+p)^3}{r+3} = \frac{5\cdot 8+2\left[1+(-4)\right]^3}{-10+3}$
$= \frac{5\cdot 8+2[-3]^3}{-7} = \frac{5\cdot 8+2(-27)}{-7}$
$= \frac{40+2(-27)}{-7} = \frac{40+(-54)}{-7}$
$= \frac{-14}{-7} = 2$

41. $A = 451,\ C = 281,\ Y = 3049,\ T = 22,\ I = 6$
Passing Rating
$$= 85.68\left(\frac{C}{A}\right) + 4.31\left(\frac{Y}{A}\right) + 326.42\left(\frac{T}{A}\right) - 419.07\left(\frac{I}{A}\right)$$
$$\approx 85.68\left(\frac{281}{451}\right) + 4.31\left(\frac{3049}{451}\right) + 326.42\left(\frac{22}{451}\right) - 419.07\left(\frac{6}{451}\right)$$
$$\approx 53.38 + 29.14 + 15.92 - 5.58 \approx 92.9$$

42. $A = 470,\ C = 287,\ Y = 3690,\ T = 26,\ I = 13$
Passing Rating
$$= 85.68\left(\frac{C}{A}\right) + 4.31\left(\frac{Y}{A}\right) + 326.42\left(\frac{T}{A}\right) - 419.07\left(\frac{I}{A}\right)$$
$$\approx 85.68\left(\frac{287}{470}\right) + 4.31\left(\frac{3690}{470}\right) + 326.42\left(\frac{26}{470}\right) - 419.07\left(\frac{13}{470}\right)$$
$$\approx 52.32 + 33.84 + 18.06 - 11.59 \approx 92.6$$

43. $A = 610,\ C = 375,\ Y = 4359,\ T = 24,\ I = 15$
Passing Rating
$$= 85.68\left(\frac{C}{A}\right) + 4.31\left(\frac{Y}{A}\right) + 326.42\left(\frac{T}{A}\right) - 419.07\left(\frac{I}{A}\right)$$
$$\approx 85.68\left(\frac{375}{610}\right) + 4.31\left(\frac{4359}{610}\right) + 326.42\left(\frac{24}{610}\right) - 419.07\left(\frac{15}{610}\right)$$
$$\approx 52.67 + 30.80 + 12.84 - 10.31 = 86.0$$

44. $A = 591,\ C = 392,\ Y = 4200,\ T = 27,\ I = 19$
Passing Rating
$$= 85.68\left(\frac{C}{A}\right) + 4.31\left(\frac{Y}{A}\right) + 326.42\left(\frac{T}{A}\right) - 419.07\left(\frac{I}{A}\right)$$
$$\approx 85.68\left(\frac{392}{591}\right) + 4.31\left(\frac{4200}{591}\right) + 326.42\left(\frac{27}{591}\right) - 419.07\left(\frac{19}{591}\right)$$
$$\approx 56.83 + 30.63 + 14.91 - 13.47 = 88.9$$

45. $\text{BAC} = 48 \times 3.2 \times .075 \div 190 - 2 \times .015 \approx .031$

46. $\text{BAC} = 36 \times 4.0 \times .075 \div 135 - 3 \times .015 = .035$

47. Exercise 45:
$\text{BAC} = 48 \times 3.2 \times .075 \div 215 - 2 \times .015 \approx .024$
Exercise 46:
$\text{BAC} = 36 \times 4.0 \times .075 \div 160 - 3 \times .015 = 0.023$
The increased weight results in lower BACs.

48. Decreased weight will result in higher BACs.
Exercise 45:
$\text{BAC} = 48 \times 3.2 \times .075 \div 165 - 2 \times .015 \approx .040$
Exercise 46:
$\text{BAC} = 36 \times 4.0 \times .075 \div 110 - 3 \times .015 = .053$

49. distributive

50. commutative

51. inverse

52. inverse

53. identity

54. closure

55. No; in general $a - b \neq b - a$. Examples will vary, i.e. if $a = 15$ and $b = 0$, then $a - b = 15 - 0 = 15$, but $b - a = 0 - 15 = -15$.

56. No; in general $(a-b)-c \neq a-(b-c)$.
Examples will vary, i.e. if $a = 15$, $b = 0$, and $c = 3$, then
$(a-b)-c = (15-0)-3 = 15-3 = 12$, but
$a-(b-c) = 15-(0-3) = 15-(-3) = 18$

57. $8p - 14p = (8-14)p = -6p$

58. $15x - 10x = (15-10)x = 5x$

59. $-4(z-y) = -4z - (-4y) = -4z + 4y$

60. $-3(m+n) = -3m + (-3n) = -3m - 3n$

61. $\frac{10}{11}(22z) = \left(\frac{10}{11} \cdot 22\right)z = 20z$

62. $\left(\frac{3}{4}r\right)(-12) = (-12)\left(\frac{3}{4}r\right) = \left(-12 \cdot \frac{3}{4}\right)r = -9r$

63. $(m+5)+6 = m+(5+6) = m+11$

64. $8+(a+7) = 8+(7+a) = (8+7)+a = 15+a$ (or $a+15$)

65. $\frac{3}{8}\left(\frac{16}{9}y+\frac{32}{27}z-\frac{40}{9}\right)$
$=\frac{3}{8}\left(\frac{16}{9}y\right)+\frac{3}{8}\left(\frac{32}{27}z\right)-\frac{3}{8}\left(\frac{40}{9}\right)$
$=\left(\frac{3}{8}\cdot\frac{16}{9}\right)y+\left(\frac{3}{8}\cdot\frac{32}{27}\right)z-\frac{5}{3}$
$=\frac{2}{3}y+\frac{4}{9}z-\frac{5}{3}$

66. $-\frac{1}{4}(20m+8y-32z)$
$=-\frac{1}{4}(20m)+\left(-\frac{1}{4}\right)(8y)-\left(-\frac{1}{4}\right)(32z)$
$=\left(-\frac{1}{4}\cdot 20\right)m+\left(-\frac{1}{4}\cdot 8\right)y-\left(-\frac{1}{4}\cdot 32\right)z$
$=-5m+(-2y)-(-8z)=-5m-2y+8z$

67. The process in your head should be the following:
$72\cdot 17+28\cdot 17=(72+28)(17)$
$=(100)(17)=1700$

68. The process in your head should be the following:
$32\cdot 80+32\cdot 20=32(80+20)$
$=32(100)=3200$

69. The process in your head should be the following:
$123\frac{5}{8}\cdot 1\frac{1}{2}-23\frac{5}{8}\cdot 1\frac{1}{2}=\left(123\frac{5}{8}-23\frac{5}{8}\right)\left(1\frac{1}{2}\right)$
$=(100)\left(1\frac{1}{2}\right)=(100)(1.5)$
$=150$

70. The process in your head should be the following:
$17\frac{2}{5}\cdot 14\frac{3}{4}-17\frac{2}{5}\cdot 4\frac{3}{4}=17\frac{2}{5}\left(14\frac{3}{4}-4\frac{3}{4}\right)$
$=17\frac{2}{5}(10)=17.4\cdot 10$
$=174$

71. This statement is false since $|6-8|=|-2|=2$ and $|6|-|8|=6-8=-2$. A corrected statement would be $|6-8|\neq|6|-|8|$ or $|6-8|=|8|-|6|$

72. This statement is false since $\left|(-3)^3\right|=|-27|=27$ and $-\left|3^3\right|=-|27|=-27$. A corrected statement would be $\left|(-3)^3\right|\neq-\left|3^3\right|$ or $\left|(-3)^3\right|=3^3$.

73. This statement is true since $|-5|\cdot|6|=5\cdot 6=30$ and $|-5\cdot 6|=|-30|=30$.

74. This statement is true since $\frac{|-14|}{|2|}=\frac{14}{2}=7$ and $\left|\frac{-14}{2}\right|=|-7|=7$.

75. This statement is false. For example if you let $a=2$ and $b=6$ then $|2-6|=|-4|=4$ and $|a|-|b|=|2|-|6|=2-6=-4$. A corrected statement is $|a-b|=|b|-|a|$, if $b>a>0$.

76. This statement is true by the algebraic definition of absolute value stated on page 14 in the text.

77. $|-10|=10$

78. $|-15|=15$

79. $-\left|\frac{4}{7}\right|=-\frac{4}{7}$

80. $-\left|\frac{7}{2}\right|=-\frac{7}{2}$

81. Let $x=-4$ and $y=2$.
$|x-y|=|-4-2|=|-6|=6$

82. Let $x=-4$ and $y=2$.
$|2x+5y|=|2(-4)+5(2)|=|-8+10|=|2|=2$

83. Let $x=-4$ and $y=2$.
$|3x+4y|=|3(-4)+4(2)|$
$=|-12+8|=|-4|=4$

84. Let $x=-4$ and $y=2$.
$|-5y+x|=|-5(2)+(-4)|$
$=|-10+(-4)|=|-14|=14$

85. Let $x=-4$ and $y=2$.
$\frac{2|y|-3|x|}{|xy|}=\frac{2|2|-3|-4|}{|-4(2)|}$
$=\frac{2(2)-3(4)}{|-8|}=\frac{4-12}{8}=\frac{-8}{8}=-1$

86. Let $x = -4$ and $y = 2$.

$$\frac{4|x|+4|y|}{|x|} = \frac{4|-4|+4|2|}{|-4|} = \frac{4(4)+4(2)}{4} = \frac{16+8}{4} = \frac{24}{4} = 6$$

87. Let $x = -4$ and $y = 2$.

$$\frac{|-8y+x|}{-|x|} = \frac{|-8(2)+(-4)|}{-|-4|} = \frac{|-16+(-4)|}{-|-4|} = \frac{|-20|}{-(4)} = \frac{20}{-4} = -5$$

88. Let $x = -4$ and $y = 2$.

$$\frac{|x|+2|y|}{5+x} = \frac{|-4|+2|2|}{5+(-4)} = \frac{4+2(2)}{1} = \frac{4+4}{1} = \frac{8}{1} = 8$$

89. Property 2

90. Property 1

91. Property 3

92. Since $|k-m| = |k+(-m)|$, $|k-m| \le |k| + |-m|$ by property 5, the triangle inequality.

93. Property 1

94. Property 4

95. Since $|-3-5| = |-8| = 8$ and $|5-(-3)| = |8| = 8$, the number of strokes between their scores is 8.

96. $12,243 + 5411 + (-6) = 17,648$ yards.
No, it is not the same, because the sum of the absolute values is

$$|12,243| + |5411| + |-6| = 12,243 + 5411 + 6 = 17,660$$

The fact that $|-6| = 6$ changes the two answers.

97. $P_d = |P-125| = |116-125| = |-9| = 9$
The P_d value for a woman whose actual systolic pressure is 116 and whose normal value should be 125 is 9.

98. We need to consider the relation, $P_d = |P-130| = 17$. Since 113 and 147 both differ 130 by 17, these are the two possible values for the patient's systolic blood pressure.

99. The absolute value of the difference in wind-chill factors for wind at 15 mph with a 30°F temperature and wind at 10 mph with a –10°F temperature is $|19° - (-28°)| = |47°| = 47°$ F.

100. The absolute value of the difference in wind-chill factors for wind at 20 mph with a –20°F temperature and wind at 5 mph with a 30°F temperature is $|-48° - 25°| = |-73°| = 73°$ F.

101. The absolute value of the difference in wind-chill factors for wind at 30 mph with a –30°F temperature and wind at 15 mph with a –20°F temperature is
$|-67° - (-45°)| = |-22°| = 22°$ F.

102. The absolute value of the difference in wind-chill factors for wind at 40 mph with a 40°F temperature and wind at 25 mph with a –30°F temperature is $|27° - (-64°)| = |91°| = 91°$ F.

103. $d(P,Q) = |-1-(-4)| = |-1+4| = |3| = 3$ or
$d(P,Q) = |-4-(-1)| = |-4+1| = |-3| = 3$

104. $d(P,R) = |8-(-4)| = |8+4| = |12| = 12$ or
$d(P,R) = |-4-8| = |-12| = 12$

105. $d(Q,R) = |8-(-1)| = |8+1| = |9| = 9$ or
$d(Q,R) = |-1-8| = |-9| = 9$

106. $d(Q,S) = |12-(-1)| = |13| = 13$ or
$d(Q,S) = |-1-12| = |-13| = 13$

107. $xy > 0$ if x and y have the same sign.

108. $x^2 y > 0$ if y is positive, because x^2 is positive for any x.

109. $\frac{x}{y} < 0$ if x and y have different signs.

110. $\frac{y^2}{x} < 0$ if x is negative, because y^2 is positive for any y.

111. Since x^3 has the same sign as x, $\frac{x^3}{y} > 0$ if x and y have the same sign.

112. $-\frac{x}{y} > 0$ if x and y have different signs.

Section R.3: Polynomials

1. Incorrect: $(mn)^2 = m^2n^2$

2. Correct: $y^2 \cdot y^5 = y^{2+5} = y^7$

3. Incorrect: $\left(\frac{k}{5}\right)^3 = \frac{k^3}{5^3} = \frac{k^3}{125}$

4. Incorrect: $3^0 y = 1 \cdot y = y$

5. Incorrect: $4^5 \cdot 4^2 = 4^{5+2} = 4^7$

6. Incorrect: $\left(a^2\right)^3 = a^{2\cdot3} = a^6$

7. Incorrect: $cd^0 = c \cdot 1 = c$

8. Incorrect: $(2b)^4 = 2^4 \cdot b^4 = 16b^4$

9. Correct: $\left(\frac{1}{4}\right)^5 = \frac{1}{4^5}$

10. Correct: $x^3 \cdot x \cdot x^4 = x^{3+1+4} = x^8$

11. $9^3 \cdot 9^5 = 9^{3+5} = 9^8$

12. $4^2 \cdot 4^8 = 4^{2+8} = 4^{10}$

13. $\left(-4x^5\right)\left(4x^2\right) = (-4 \cdot 4)\left(x^5x^2\right)$
$= -16x^{5+2} = -16x^7$

14. $\left(3y^4\right)\left(-6y^3\right) = [3(-6)]\left(y^4y^3\right)$
$= -18y^{4+3} = -18y^7$

15. $n^6 \cdot n^4 \cdot n = n^{6+4+1} = n^{11}$

16. $a^8 \cdot a^5 \cdot a = a^{8+5+1} = a^{14}$

17. $\left(-3m^4\right)\left(6m^2\right)\left(-4m^5\right)$
$= [(-3)(6)(-4)]\left(m^4m^2m^5\right)$
$= 72m^{4+2+5} = 72m^{11}$

18. $\left(-8t^3\right)\left(2t^6\right)\left(-5t^4\right)$
$= [(-8)(2)(-5)]\left(t^3t^6t^4\right)$
$= 80t^{3+6+4} = 80t^{13}$

19. $(2^2)^5 = 2^{2\cdot5} = 2^{10}$

20. $(6^4)^3 = 6^{4\cdot3} = 6^{12}$

21. $\left(-6x^2\right)^3 = (-6)^3\left(x^2\right)^3 = (-6)^3 x^6 = -216x^6$

22. $\left(-2x^5\right)^5 = (-2)^5\left(x^5\right)^5 = (-2)^5 x^{25} = -32x^{25}$

23. $-(4m^3n^0)^2 = -\left[4^2(m^3)^2(n^0)^2\right]$
$= -4^2m^{3\cdot2}n^{0\cdot2}$
$= -4^2m^6n^0 = -\left(4^2\right)m^6 \cdot 1$
$= -4^2m^6 = -16m^6$

24. $(2x^0y^4)^3 = 2^3(x^0)^3(y^4)^3$
$= 2^3x^{0\cdot3}y^{4\cdot3} = 2^3x^0y^{12}$
$= 8 \cdot 1 \cdot y^{12} = 8y^{12}$

25. $\left(\frac{r^8}{s^2}\right)^3 = \frac{(r^8)^3}{(s^2)^3} = \frac{r^{8\cdot3}}{s^{2\cdot3}} = \frac{r^{24}}{s^6}$

26. $-\left(\frac{p^4}{q}\right)^2 = -\frac{(p^4)^2}{q^2} = -\frac{p^{4\cdot2}}{q^2} = -\frac{p^8}{q^2}$

27. $\left(\frac{-4m^2}{t}\right)^4 = \frac{(-4)^4(m^2)^4}{t^4} = \frac{(-4)^4m^8}{t^4} = \frac{256m^8}{t^4}$

28. $\left(\frac{-5n^4}{r^2}\right)^3 = \frac{(-5)^3(n^4)^3}{(r^2)^3} = \frac{(-5)^3n^{12}}{r^6} = -\frac{125n^{12}}{r^6}$

29. **(a)** $6^0 = 1$; B **(b)** $-6^0 = -1$; C

(c) $(-6)^0 = 1$; B **(d)** $-(-6)^0 = -1$; C

30. **(a)** $3p^0 = 3 \cdot 1 = 3$; D

(b) $-3p^0 = -3 \cdot 1 = -3$; E

(c) $(3p)^0 = 1$; B

(d) $(-3p)^0 = 1$; B

31. Answers will vary.
$x^2 + x^2 = 2x^2$

32. Answers will vary
$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$
Note: The square of a binomial is always a trinomial.

33. $-5x^{11}$ is a polynomial. It is a monomial since it has one term. It has degree 11 since 11 is the highest exponent.

34. $9y^{12}+y^2$ is a polynomial. It is a binomial since it has two terms. It has degree 12 since 12 is the highest exponent.

35. $18p^5q+6pq$ is a polynomial. It is a binomial since it has two terms. It has degree 6 because 6 is the sum of the exponents in the term $18p^5q$, and this term has a higher degree than the term $6pq$.

36. $2a^6+5a^2+4a$ is a polynomial. It is a trinomial since it has three terms. It has degree 6 since 6 is the highest exponent.

37. $\sqrt{2}x^2+\sqrt{3}x^6$ is a polynomial. It is a binomial since it has two terms. It has degree 6 since 6 is the highest exponent.

38. $-\sqrt{7}m^5n^2+2\sqrt{3}m^3n^2$ is a polynomial. It is a binomial since it has two terms. It has degree 7 since the sum of the exponents in the term is 7, and this is the highest degree term.

39. $\frac{1}{3}r^2s^2-\frac{3}{5}r^4s^2+rs^3$ is a polynomial. It is a trinomial since it has three terms. It has degree 6 because the sum of the exponents in the term $-\frac{3}{5}r^4s^2$ is 6, and this term has the highest degree.

40. $\frac{13}{10}p^7-\frac{2}{7}p^5$ is a polynomial of degree 7 and is a binomial.

41. $\frac{5}{p}+\frac{2}{p^2}+\frac{5}{p^3}$ is not a polynomial since positive exponents in the denominator are equivalent to negative exponents in the numerator.

42. $-5\sqrt{z}+2\sqrt{z^3}-5\sqrt{z^5}=5z^{1/2}+2z^{3/2}-5z^{5/2}$ is not a polynomial since the exponents are not integers.

43.
$$\begin{aligned}&\left(5x^2-4x+7\right)+\left(-4x^2+3x-5\right)\\&=\left[5+(-4)\right]x^2+(-4+3)x+\left[7+(-5)\right]\\&=1\cdot x^2+(-1)x+2=x^2-x+2\end{aligned}$$

44.
$$\begin{aligned}&\left(3m^3-3m^2+4\right)+\left(-2m^3-m^2+6\right)\\&=\left[3+(-2)\right]m^3+\left[-3+(-1)\right]m^2+(4+6)\\&=1\cdot m^3+(-4)m^2+10=m^3-4m^2+10\end{aligned}$$

45.
$$\begin{aligned}&2\left(12y^2-8y+6\right)-4\left(3y^2-4y+2\right)\\&=2\left(12y^2\right)-2(8y)+2(6)-4\left(3y^2\right)\\&\qquad-4(-4y)-4\cdot 2\\&=24y^2-16y+12-12y^2+16y-8\\&=12y^2+4\end{aligned}$$

46.
$$\begin{aligned}&3\left(8p^2-5p\right)-5\left(3p^2-2p+4\right)\\&=3\left(8p^2\right)-3(5p)-5\left(3p^2\right)-5(-2p)-5\cdot 4\\&=24p^2-15p-15p^2+10p-20\\&=9p^2-5p-20\end{aligned}$$

47.
$$\begin{aligned}&\left(6m^4-3m^2+m\right)-\left(2m^3+5m^2+4m\right)+\left(m^2-m\right)\\&=6m^4-3m^2+m-2m^3-5m^2-4m+m^2-m\\&=6m^4-2m^3+(-3-5+1)m^2+(1-4-1)m\\&=6m^4-2m^3+(-7)m^2+(-4)m\\&=6m^4-2m^3-7m^2-4m\end{aligned}$$

48.
$$\begin{aligned}&-\left(8x^3+x-3\right)+\left(2x^3+x^2\right)-\left(4x^2+3x-1\right)\\&=-8x^3-x+3+2x^3+x^2-4x^2-3x+1\\&=(-8+2)x^3+(1-4)x^2+(-1-3)x+(3+1)\\&=-6x^3+(-3)x^2+(-4)x+(3+1)\\&=-6x^3-3x^2-4x+4\end{aligned}$$

49.
$$\begin{aligned}(4r-1)(7r+2)&=4r(7r)+4r(2)-1(7r)-1(2)\\&=28r^2+8r-7r-2\\&=28r^2+r-2\end{aligned}$$

50.
$$\begin{aligned}&(5m-6)(3m+4)\\&=(5m)(3m)+(5m)(4)-6(3m)-6(4)\\&=15m^2+20m-18m-24\\&=15m^2+2m-24\end{aligned}$$

51. $x^2\left(3x-\frac{2}{3}\right)\left(5x+\frac{1}{3}\right)$
$=x^2\left[\left(3x-\frac{2}{3}\right)\left(5x+\frac{1}{3}\right)\right]$
$=x^2\left[(3x)(5x)+(3x)\left(\frac{1}{3}\right)-\frac{2}{3}(5x)-\frac{2}{3}\left(\frac{1}{3}\right)\right]$
$=x^2\left(15x^2+x-\frac{10}{3}x-\frac{2}{9}\right)$
$=x^2\left(15x^2+\frac{3}{3}x-\frac{10}{3}x-\frac{2}{9}\right)$
$=x^2\left(15x^2-\frac{7}{3}x-\frac{2}{9}\right)=15x^4-\frac{7}{3}x^3-\frac{2}{9}x^2$

52. $\left(2m-\frac{1}{4}\right)\left(3m+\frac{1}{2}\right)$
$=(2m)(3m)+2m\left(\frac{1}{2}\right)-\frac{1}{4}\cdot 3m-\frac{1}{4}\left(\frac{1}{2}\right)$
$=6m^2+m-\frac{3}{4}m-\frac{1}{8}=6m^2+\frac{4}{4}m-\frac{3}{4}m-\frac{1}{8}$
$=6m^2+\frac{1}{4}m-\frac{1}{8}$

53. $4x^2\left(3x^3+2x^2-5x+1\right)$
$=4x^2\left(3x^3\right)+4x^2\left(2x^2\right)-4x^2(5x)+4x^2\cdot 1$
$=12x^5+8x^4-20x^3+4x^2$

54. $2b^3\left(b^2-4b+3\right)=2b^3\left(b^2\right)-2b^3(4b)+2b^3(3)$
$=2b^5-8b^4+6b^3$

55. $(2z-1)\left(-z^2+3z-4\right)$
$=(2z-1)\left(-z^2\right)+(2z-1)(3z)-(2z-1)(4)$
$=2z\left(-z^2\right)-1\left(-z^2\right)+2z(3z)-1(3z)$
$-(2z)(4)-(-1)(4)$
$=-2z^3+z^2+6z^2-3z-8z-(-4)$
$=-2z^3+7z^2-11z+4$

We may also multiply vertically.

$$\begin{array}{rl} -z^2+3z-4 & \\ 2z-1 & \\ \hline z^2-3z+4 & \leftarrow -1(-z^2+3z-4) \\ -2z^3+6z^2-8z & \leftarrow 2z(-z^2+3z-4) \\ \hline -2z^3+7z^2-11z+4 & \end{array}$$

56. $(k+2)\left(12k^3-3k^2+k+1\right)$
$=(k+2)\left(12k^3\right)-(k+2)\left(3k^2\right)$
$+(k+2)(k)+(k+2)(1)$
$=k\left(12k^3\right)+2\left(12k^3\right)-k\left(3k^2\right)-2\left(3k^2\right)$
$+k\cdot k+2k+k+2$
$=12k^4+24k^3-3k^3-6k^2+k^2+3k+2$
$=12k^4+21k^3-5k^2+3k+2$

We may also multiply vertically.

$$\begin{array}{rl} 12k^3-3k^2+k+1 & \\ k+2 & \\ \hline 24k^3-6k^2+2k+2 & \leftarrow 2(12k^3-3k^2+k+1) \\ 12k^4-3k^3+k^2+k & \leftarrow k(12k^3-3k^2+k+1) \\ \hline 12k^4+21k^3-5k^2+3k+2 & \end{array}$$

57. $(m-n+k)(m+2n-3k)$
$=(m-n+k)(m)+(m-n+k)(2n)$
$-(m-n+k)(3k)$
$=m^2-mn+km+2mn-2n^2$
$+2kn-3km+3kn-3k^2$
$=m^2+mn-2n^2-2km+5kn-3k^2$

We may also multiply vertically.

$$\begin{array}{r} m-n+k \\ m+2n-3k \\ \hline -3km+3kn-3k^2 \\ 2mn-2n^2+2kn \\ m^2-mn+km \\ \hline m^2+mn-2n^2-2km+5kn-3k^2 \end{array}$$

58. $(r-3s+t)(2r-s+t)$
$=(r-3s+t)(2r)-(r-3s+t)(s)$
$+(r-3s+t)(t)$
$=2r^2-6rs+2rt-rs+3s^2-st+rt-3st+t^2$
$=2r^2-7rs+3s^2+3rt-4st+t^2$

We may also multiply vertically.

$$\begin{array}{r} 2r-s+t \\ r-3s+t \\ \hline 2rt-st+t^2 \\ -6rs+3s^2-3st \\ 2r^2-rs+rt \\ \hline 2r^2-7rs+3s^2+3rt-4st+t^2 \end{array}$$

59. $(2m+3)(2m-3)=(2m)^2-3^2$
$=4m^2-9$

60. $(8s-3t)(8s+3t)=(8s)^2-(3t)^2$
$=64s^2-9t^2$

61. $(4x^2-5y)(4x^2+5y)=(4x^2)^2-(5y)^2$
$=16x^4-25y^2$

62. $(2m^3+n)(2m^3-n)=(2m^3)^2-n^2$
$= 4m^6-n^2$

63. $(4m+2n)^2=(4m)^2+2(4m)(2n)+(2n)^2$
$=16m^2+16mn+4n^2$

64. $(a-6b)^2=a^2-2(a)(6b)+(6b)^2$
$=a^2-12ab+36b^2$

65. $(5r-3t^2)^2=(5r)^2-2(5r)(3t^2)+(3t^2)^2$
$=25r^2-30rt^2+9t^4$

66. $(2z^4-3y)^2=(2z^4)^2-2(2z^4)(3y)+(3y)^2$
$=4z^8-12z^4y+9y^2$

67. $\left[(2p-3)+q\right]^2$
$=(2p-3)^2+2(2p-3)(q)+q^2$
$=(2p)^2-2(2p)(3)+(3)^2+4pq-6q+q^2$
$=4p^2-12p+9+4pq-6q+q^2$

68. $[(4y-1)+z]^2$
$=(4y-1)^2+2(4y-1)(z)+z^2$
$=(4y)^2-2(4y)(1)+(1)^2+8yz-2z+z^2$
$=16y^2-8y+1+8yz-2z+z^2$

69. $[(3q+5)-p][(3q+5)+p]$
$=(3q+5)^2-p^2$
$=[(3q)^2+2(3q)(5)+5^2]-p^2$
$=9q^2+30q+25-p^2$

70. $[(9r-s)+2][(9r-s)-2]=(9r-s)^2-2^2$
$=81r^2-18rs+s^2-4$

71. $[(3a+b)-1]^2=(3a+b)^2-2(3a+b)(1)+1^2$
$=(9a^2+6ab+b^2)-2(3a+b)+1$
$=9a^2+6ab+b^2-6a-2b+1$

72. $[(2m+7)-n]^2$
$=(2m+7)^2-2(2m+7)(n)+n^2$
$=(4m^2+28m+49)-2(2m+7)(n)+n^2$
$=4m^2+28m+49-4mn-14n+n^2$

73. $(y+2)^3=(y+2)^2(y+2)$
$=(y^2+4y+4)(y+2)$
$=y^3+4y^2+4y+2y^2+8y+8$
$=y^3+6y^2+12y+8$

74. $(z-3)^3=(z-3)^2(z-3)=(z^2-6z+9)(z-3)$
$=z^3-6z^2+9z-3z^2+18z-27$
$=z^3-9z^2+27z-27$

75. $(q-2)^4=(q-2)^2(q-2)^2$
$=(q^2-4q+4)(q^2-4q+4)$
$=q^4-4q^3+4q^2-4q^3+16q^2$
$-16q+4q^2-16q+16$
$=q^4-8q^3+24q^2-32q+16$

76. $(r+3)^4=(r+3)^2(r+3)^2$
$=(r^2+6r+9)(r^2+6r+9)$
$=r^4+6r^3+9r^2+6r^3+36r^2$
$+54r+9r^2+54r+81$
$=r^4+12r^3+54r^2+108r+81$

77. $(p^3-4p^2+p)-(3p^2+2p+7)$
$=p^3-4p^2+p-3p^2-2p-7$
$=p^3-7p^2-p-7$

78. $(2z+y)(3z-4y)$
$=(2z)(3z)-(2z)(4y)+y(3z)-y(4y)$
$=6z^2-8zy+3zy-4y^2=6z^2-5zy-4y^2$

79. $(7m+2n)(7m-2n)=(7m)^2-(2n)^2$
$=49m^2-4n^2$

80. $(3p+5)^2=(3p)^2+2(3p)(5)+5^2$
$=9p^2+30p+25$

81. $-3(4q^2-3q+2)+2(-q^2+q-4)$
$=-12q^2+9q-6-2q^2+2q-8$
$=-14q^2+11q-14$

82. $2(3r^2+4r+2)-3(-r^2+4r-5)$
$=6r^2+8r+4+3r^2-12r+15$
$=9r^2-4r+19$

83. $p(4p-6)+2(3p-8)=4p^2-6p+6p-16$
$=4p^2-16$

84. $m(5m-2)+9(5-m)=5m^2-2m+45-9m$
$=5m^2-11m+45$

85. $-y(y^2-4)+6y^2(2y-3)$
$=-y^3+4y+12y^3-18y^2$
$=11y^3-18y^2+4y$

86. $-z^3(9-z)+4z(2+3z)$
$$= -9z^3 + z^4 + 8z + 12z^2$$
$$= z^4 - 9z^3 + 12z^2 + 8z$$

87.
$$\begin{array}{r} 2x^5 + 7x^4 - 5x^2 + 7 \\ -2x^2\overline{)-4x^7 - 14x^6 + 10x^4 - 14x^2} \\ \underline{-4x^7} \qquad\qquad\qquad\qquad \\ -14x^6 \qquad\qquad\qquad \\ \underline{-14x^6} \qquad\qquad\qquad \\ 10x^4 \qquad\qquad \\ \underline{10x^4} \qquad\qquad \\ -14x^2 \\ \underline{-14x^2} \\ 0 \end{array}$$

$$\frac{-4x^7 - 14x^6 + 10x^4 - 14x^2}{-2x^2} = 2x^5 + 7x^4 - 5x^2 + 7$$

88.
$$\begin{array}{r} -2r^2 - 3rs + 5s^2 \\ 4rs\overline{)-8r^3s - 12r^2s^2 + 20rs^3} \\ \underline{-8r^3s} \qquad\qquad\qquad\qquad \\ -12r^2s^2 \qquad\qquad \\ \underline{-12r^2s^2} \qquad\qquad \\ 20rs^3 \\ \underline{20rs^3} \\ 0 \end{array}$$

$$\frac{-8r^3s - 12r^2s^2 + 20rs^3}{4rs} = -2r^2 - 3rs + 5s^2$$

89.
$$\begin{array}{r} -5x^2 + 8 \qquad\qquad \\ -2x^6\overline{)10x^8 - 16x^6 - 4x^4} \\ \underline{10x^8} \qquad\qquad\qquad\quad \\ -16x^6 \qquad\quad \\ \underline{-16x^6} \qquad\quad \\ -4x^4 \end{array}$$

$$\frac{10x^8 - 16x^6 - 4x^4}{-2x^6} = -5x^2 + 8 - \frac{4x^4}{-2x^6}$$
$$= -5x^2 + 8 + \frac{2}{x^2}$$

90.
$$\begin{array}{r} 3x^2 + 9x + 25 \quad \\ x-3\overline{)3x^3 + 0x^2 - 2x + 5} \\ \underline{3x^3 - 9x^2} \qquad\qquad\quad \\ 9x^2 - 2x \qquad \\ \underline{9x^2 - 27x} \qquad \\ 25x + 5 \\ \underline{25x - 75} \\ 80 \end{array}$$

$$\frac{3x^3 - 2x + 5}{x-3} = 3x^2 + 9x + 25 + \frac{80}{x-3}$$

91.
$$\begin{array}{r} 2m^2 + m - 2 \\ 3m+2\overline{)6m^3 + 7m^2 - 4m + 2} \\ \underline{6m^3 + 4m^2} \qquad\qquad\quad \\ 3m^2 - 4m \qquad \\ \underline{3m^2 + 2m} \qquad \\ -6m + 2 \\ \underline{-6m - 4} \\ 6 \end{array}$$

$$\frac{6m^3 + 7m^2 - 4m + 2}{3m+2} = 2m^2 + m - 2 + \frac{6}{3m+2}$$

92.
$$\begin{array}{r} 2x^2 + 3x + 2 \qquad\qquad \\ 3x^2 + 0x - 2\overline{)6x^4 + 9x^3 + 2x^2 - 8x + 7} \\ \underline{6x^4 + 0x^3 - 4x^2} \qquad\qquad\qquad \\ 9x^3 + 6x^2 - 8x \qquad \\ \underline{9x^3 + 0x^2 - 6x} \qquad \\ 6x^2 - 2x + 7 \\ \underline{6x^2 + 0x - 4} \\ -2x + 11 \end{array}$$

$$\frac{6x^4 + 9x^3 + 2x^2 - 8x + 7}{3x^2 - 2}$$
$$= 2x^2 + 3x + 2 + \frac{-2x + 11}{3x^2 - 2}.$$

93.

$$\begin{array}{r|l} & x^3 - x^2 - x + 4 \\ 3x+3 & 3x^4 - 0x^3 - 6x^2 + 9x - 5 \\ & \underline{3x^4 + 3x^3} \\ & -3x^3 - 6x^2 \\ & \underline{-3x^3 - 3x^2} \\ & -3x^2 + 9x \\ & \underline{-3x^2 - 3x} \\ & 12x - 5 \\ & \underline{12x + 12} \\ & -17 \end{array}$$

$$\frac{3x^4 - 6x^2 + 9x - 5}{3x+3} = x^3 - x^2 - x + 4 + \frac{-17}{3x+3}$$

or $x^3 - x^2 - x + 4 - \dfrac{17}{3x+3}$.

94. Both polynomials have missing terms. Insert each missing term with a 0 coefficient.

$$\begin{array}{r|l} & k^2 - 5 \\ k^2 + 0k + 1 & k^4 + 0k^3 - 4k^2 + 2k + 5 \\ & \underline{k^4 + 0k^3 + k^2} \\ & -5k^2 + 2k + 5 \\ & \underline{-5k^2 + 0k - 5} \\ & 2k + 10 \end{array}$$

$$\frac{k^4 - 4k^2 + 2k + 5}{k^2 + 1} = k^2 - 5 + \frac{2k+10}{k^2+1}.$$

95. $99 \times 101 = (100-1)(100+1) = 100^2 - 1^2$
$= 10,000 - 1 = 9999$

96. $63 \times 57 = (60+3)(60-3) = 60^2 - 3^2$
$= 3600 - 9 = 3591$

97. $102^2 = (100+2)^2 = 100^2 + 2(100)(2) + 2^2$
$= 10,000 + 400 + 4 = 10,404$

98. $71^2 = (70+1)^2 = 70^2 + 2(70)(1) + 1^2$
$= 4900 + 140 + 1 = 5041$

99. **(a)** The area of the largest square is $s^2 = (x+y)^2$.

(b) The areas of the two squares are x^2 and y^2. The area of each rectangle is xy. Therefore, the area of the largest square can be written as $x^2 + 2xy + y^2$.

(c) Answers will vary. The total area must equal the sum of the four parts.

(d) It reinforces the special product for squaring a binomial:
$(x+y)^2 = x^2 + 2xy + y^2$.

100. Answers will vary.
The total area of the largest rectangle is $x(y+z)$. The area of the two smaller rectangles are xy and xz. Since the whole rectangle is the sum of the parts, we have $x(y+z) = xy + xz$.

101. **(a)** The volume is

$$V = \frac{1}{3}h(a^2 + ab + b^2)$$
$$= \frac{1}{3}(200)(314^2 + 314 \times 756 + 756^2)$$
$$\approx 60,501,000 \text{ ft}^3$$

(b) The shape becomes a rectangular box with a square base. Its volume is given by length × width × height or $b^2 h$.

(c) If we let $a = b$, then

$V = \frac{1}{3}h(a^2 + ab + b^2)$ becomes

$V = \frac{1}{3}h(b^2 + bb + b^2)$, which simplifies

to $V = hb^2$. Yes, the Egyptian formula gives the same result.

102. **(a)** When $a = 0$, $V = \frac{1}{3}h(a^2 + ab + b^2)$

becomes $V = \frac{1}{3}hb^2$, which is the correct formula for the volume of a pyramid with a square base.

(b)

$$V = \frac{1}{3}hb^2 = \frac{1}{3} \cdot 481 \cdot 756^2$$
$$= \frac{481 \cdot 571,536}{3} = \frac{274,908,816}{3}$$
$$= 91,636,272 \text{ ft}^3 \approx 91.6 \text{ million ft}^3$$

The pyramid is slightly smaller when compared to the Superdome in New Orleans.

(c) The base area of the Great Pyramid is $b^2 = 756^2 = 571,536 \text{ ft}^2$. Since 1 acre = $43,560 \text{ ft}^2$, the Great Pyramid will cover $\dfrac{571,536}{43,560} \approx 13.1$ acres.

103. $x = 1940$

$$\begin{aligned}&.000020591075(1940)^3\\&\quad -.1201456829(1940)^2\\&\quad +233.5530856(1940)\\&\quad -151{,}249.8184 \approx 6.2\end{aligned}$$

The formula is .1 high.

104. $x = 1959$

$$\begin{aligned}&.000020591075(1959)^3\\&\quad -.1201456829(1959)^2\\&\quad +233.5530856(1959)\\&\quad -151{,}249.8184 \approx 3.9\end{aligned}$$

The formula is .2 high.

105. $x = 1978$

$$\begin{aligned}&.000020591075(1978)^3\\&\quad -.1201456829(1978)^2\\&\quad +233.5530856(1978)\\&\quad -151{,}249.8184 \approx 2.3\end{aligned}$$

The formula is exact.

106. $x = 1997$

$$\begin{aligned}&.000020591075(1997)^3\\&\quad -.1201456829(1997)^2\\&\quad +233.5530856(1997)\\&\quad -151{,}249.8184 \approx 2.1\end{aligned}$$

The formula is exact.

107. $$\left(.25^3\right)\left(400^3\right) = \left[(.25)(400)\right]^3 = 100^3 = 1{,}000{,}000$$

108. $$\left(24^2\right)\left(.5^2\right) = \left[(24)(.5)\right]^2 = 12^2 = 144$$

109. $$\frac{4.2^5}{2.1^5} = \left(\frac{4.2}{2.1}\right)^5 = 2^5 = 32$$

110. $$\frac{15^4}{5^4} = \left(\frac{15}{5}\right)^4 = 3^4 = 81$$

Section R.4: Factoring Polynomials

1. The greatest common factor is 12.

$$12m + 60 = 12(m) + 12(5) = 12(m+5)$$

2. The greatest common factor is 3.

$$15r - 27 = 3(5r) - 3(9) = 3(5r-9)$$

3. The greatest common factor is $8k$.

$$8k^3 + 24k = 8k\left(k^2\right) + 8k(3) = 8k\left(k^2+3\right)$$

4. The greatest common factor is $9z$.

$$9z^4 + 81z = 9z\left(z^3\right) + 9z(9) = 9z\left(z^3+9\right)$$

5. The greatest common factor is xy.

$$xy - 5xy^2 = xy\cdot 1 - xy(5y) = xy(1-5y)$$

6. The greatest common factor is hj.

$$5h^2j + hj = hj(5h) + hj\cdot 1 = hj(5h+1)$$

7. The greatest common factor is $-2p^2q^4$.

$$\begin{aligned}&-4p^3q^4 - 2p^2q^5\\&\quad = \left(-2p^2q^4\right)(2p) + \left(-2p^2q^4\right)(q)\\&\quad = -2p^2q^4(2p+q)\end{aligned}$$

8. The greatest common factor is $-3z^3w^2$.

$$\begin{aligned}&-3z^5w^2 - 18z^3w^4\\&\quad = \left(-3z^3w^2\right)\left(z^2\right) + \left(-3z^3w^2\right)\left(6w^2\right)\\&\quad = -3z^3w^2\left(z^2+6w^2\right)\end{aligned}$$

9. The greatest common factor is $4k^2m^3$.

$$\begin{aligned}&4k^2m^3 + 8k^4m^3 - 12k^2m^4\\&\quad = \left(4k^2m^3\right)(1) + \left(4k^2m^3\right)\left(2k^2\right) - \left(4k^2m^3\right)(3m)\\&\quad = 4k^2m^3\left(1+2z^2-3m\right)\end{aligned}$$

10. The greatest common factor is $7r^3s$.

$$\begin{aligned}&28r^4s^2 + 7r^3s - 35r^4s^3\\&\quad = 7r^3s(4rs) + 7r^3s(1) - 7r^3s(5rs^2)\\&\quad = 7r^3s(4rs+1-5rs^2)\end{aligned}$$

11. The greatest common factor is $2(a+b)$

$$\begin{aligned}&2(a+b) + 4m(a+b)\\&\quad = [2(a+b)](1) + [2(a+b)](2m)\\&\quad = 2(a+b)(1+2m)\end{aligned}$$

12. $$\begin{aligned}&4(y-2)^2 + 3(y-2)\\&\quad = (y-2)\cdot 4(y-2) + (y-2)\cdot 3\\&\quad = (y-2)[4(y-2)+3]\\&\quad = (y-2)(4y-8+3) = (y-2)(4y-5)\end{aligned}$$

13. $$\begin{aligned}&(5r-6)(r+3) - (2r-1)(r+3)\\&\quad = (r+3)[(5r-6)-(2r-1)]\\&\quad = (r+3)[5r-6-2r+1] = (r+3)(3r-5)\end{aligned}$$

14. $$\begin{aligned}&(3z+2)(z+4) - (z+6)(z+4)\\&\quad = (z+4)[(3z+2)-(z+6)]\\&\quad = (z+4)(3z+2-z-6)\\&\quad = (z+4)(2z-4) = 2(z+4)(z-2)\end{aligned}$$

15. $2(m-1)-3(m-1)^2+2(m-1)^3$
$=(m-1)\left[2-3(m-1)+2(m-1)^2\right]$
$=(m-1)\left[2-3m+3+2(m^2-2m+1)\right]$
$=(m-1)(2-3m+3+2m^2-4m+2)$
$=(m-1)(2m^2-7m+7)$

16. $5(a+3)^3-2(a+3)+(a+3)^2$
$=(a+3)[5(a+3)^2-2+(a+3)]$
$=(a+3)[5(a^2+6a+9)-2+(a+3)]$
$=(a+3)(5a^2+30a+45-2+a+3)$
$=(a+3)(5a^2+31a+46)$

17. The completely factored form of $4x^2y^5-8xy^3$ is $4xy^3\left(xy^2-2\right)$.

18. $10ab-6b+35a-21$
$=(10ab-6b)+(35a-21)$
$=2b(5a-3)+7(5a-3)=(5a-3)(2b+7)$

19. $6st+9t-10s-15=(6st+9t)-(10s+15)$
$=3t(2s+3)-5(2s+3)$
$=(2s+3)(3t-5)$

20. $15-5m^2-3r^2+m^2r^2$
$=(15-5m^2)-(3r^2-m^2r^2)$
$=5(3-m^2)-r^2(3-m^2)$
$=(3-m^2)(5-r^2)$

21. $2m^4+6-am^4-3a$
$=(2m^4+6)-(am^4+3a)$
$=2(m^4+3)-a(m^4+3)=(m^4+3)(2-a)$

22. $20z^2-8x+5pz^2-2px$
$=\left(20z^2-8x\right)+\left(5pz^2-2px\right)$
$=4\left(5z^2-2x\right)+p\left(5z^2-2x\right)$
$=\left(5z^2-2x\right)\left(4+p\right)$

23. $p^2q^2-10-2q^2+5p^2$
$=p^2q^2-2q^2+5p^2-10$
$=q^2(p^2-2)+5(p^2-2)$
$=(p^2-2)(q^2+5)$

24. Both answers are correct.
$(8a-3)(2a-5)=16a^2-40a-6a+15$ and
$(3-8a)(5-2a)=15-6a-40a+16a^2$
$=16^2-40a-6a+15$

25. The positive factors of 6 could be 2 and 3, or 1 and 6. Since the middle term is negative, we know the factors of 4 must both be negative. As factors of 4, we could have -1 and -4, or -2 and -2. Try different combinations of these factors until the correct one is found.
$6a^2-11a+4=\left(2a-1\right)\left(3a-4\right)$

26. The positive factors of 8 could be 2 and 4, or 1 and 8. As factors of -21, we could have -3 and 7, 3 and -7, 1 and -21, or -1 and 21. Try different combinations of these factors until the correct one is found.
$8h^2-2h-21=\left(4h-7\right)\left(2h+3\right)$

27. The positive factors of 3 are 1 and 3. Since the middle term is positive, we know the factors of 8 must both be positive. As factors of 8, we could have 1 and 8, or 2 and 4. Try different combinations of these factors until the correct one is found.
$3m^2+14m+8=\left(3m+2\right)\left(m+4\right)$

28. The positive factors of 9 are 3 and 3, or 1 and 9. Since the middle term is negative, we know the factors of 8 must both be negative. As factors of 8, we could have -1 and -8, or -2 and -4. Try different combinations of these factors until the correct one is found.
$9y^2-18y+8=\left(3y-4\right)\left(3y-2\right)$

29. The positive factors of 15 are 1 and 15, or 3 and 5. Since the middle term is positive, we know the factors of 8 must both be positive. As factors of 8, we could have 1 and 8, or 2 and 4. Trying different combinations of these factors we find that $15p^2+24p+8$ is prime.

30. The positive factors of 9 are 1 and 9, or 3 and 3. As factors of -2, we could have -1 and 2, or 1 and -2. Trying different combinations of these factors we find that $9x^2+4x-2$ is prime.

31. Factor out the greatest common factor, $2a$:
$12a^3+10a^2-42a=2a(6a^2+5a-21)$. Now factor the trinomial by trial and error:
$6a^2+5a-21=(3a+7)(2a-3)$. Thus,
$12a^3+10a^2-42a=2a(3a+7)(2a-3)$.

32. Factor out the greatest common factor, $2x$:
$36x^3 + 18x^2 - 4x = 2x(18x^2 + 9x - 2)$. Now factor the trinomial by trial and error:
$18x^2 + 9x - 2 = (6x-1)(3x+2)$. Thus,
$36x^3 + 18x^2 - 4x = 2x(6x-1)(3x+2)$.

33. The positive factors of 6 could be 2 and 3, or 1 and 6. As factors of –6, we could have –1 and 6, -6 and 1, –2 and 3, or –3 and 2. Try different combinations of these factors until the correct one is found.
$6k^2 + 5kp - 6p^2 = (2k+3p)(3k-2p)$

34. The positive factors of 14 could be 2 and 7, or 1 and 14. As factors of –15, we could have 3 and –5, –3 and 5, 1 and –15, or –1 and 15. Try different combinations of these factors until the correct one is found.
$14m^2 + 11mr - 15r^2 = (7m-5r)(2m+3r)$

35. The positive factors of 5 can only be 1 and 5. As factors of –6, we could have –1 and 6, –6 and 1, –2 and 3, or –3 and 2. Try different combinations of these factors until the correct one is found.
$5a^2 - 7ab - 6b^2 = (5a+3b)(a-2b)$

36. The positive factors of 12 could be 4 and 3, 2 and 6, or 1 and 12. As factors of –5 we could have –1 and 5 or –5 and 1. Try different combinations of these factors until the correct one is found.
$12s^2 + 11st - 5t^2 = (4s+5t)(3s-t)$

37. The positive factors of 12 could be 4 and 3, 2 and 6, or 1 and 12. The factors of $-y^2$ are y and $-y$. Try different combination of these factors until the correct one is found.
$12x^2 - xy - y^2 = (4x+y)(3x-y)$

38. The positive factors of 30 could be 5 and 6, 3 and 10, 2 and 15, or 1 and 30. The only factors of –1 (the coefficient of the last term) are 1 and –1. Try different combinations of these factors until the correct one is found.
$30a^2 + am - m^2 = (5a+m)(6a-m)$

39. Factor out the greatest common factor, $2a^2$:
$24a^4 + 10a^3b - 4a^2b^2 = 2a^2\left(12a^2 + 5ab - 2b^2\right)$
Now factor the trinomial by trial and error:
$12a^2 + 5ab - 2b^2 = (4a-b)(3a+2b)$
Thus,
$$\begin{aligned}24a^4 + 10a^3b - 4a^2b^2 &= 2a^2\left(12a^2 + 5ab - 2b^2\right)\\ &= 2a^2(4a-b)(3a+2b)\end{aligned}$$

40. First, factor out the greatest common factor, $3x^3$:
$18x^5 + 15x^4z - 75x^3z^2 = 3x^3\left(6x^2 + 5xz - 25z^2\right)$
Now factor the trinomial by trial and error: $6x^2 + 5xz - 25z^2 = (3x-5z)(2x+5z)$
Thus,
$$\begin{aligned}18x^5 + 15x^4z - 75x^3z^2 &= 3x^3\left(6x^2 + 5xz - 25z^2\right)\\ &= 3x^3(3x-5z)(2x+5z)\end{aligned}$$

41. $$\begin{aligned}9m^2 - 12m + 4 &= (3m)^2 - 12m + 2^2\\ &= (3m)^2 - 2(3m)(2) + 2^2\\ &= (3m-2)^2\end{aligned}$$

42. $$\begin{aligned}16p^2 - 40p + 25 &= (4p)^2 - 40p + 5^2\\ &= (4p)^2 - 2(4p)(5) + 5^2\\ &= (4p-5)^2\end{aligned}$$

43. $$\begin{aligned}&32a^2 + 48ab + 18b^2\\ &= 2(16a^2 + 24ab + 9b^2)\\ &= 2[(4a)^2 + 24ab + (3b)^2]\\ &= 2[(4a)^2 + 2(4a)(3b) + (3b)^2] = 2(4a+3b)^2\end{aligned}$$

44. $$\begin{aligned}&20p^2 - 100pq + 125q^2\\ &= 5(4p^2 - 20pq + 25q^2)\\ &= 5[(2p)^2 - 20pq + (5q)^2]\\ &= 5[(2p)^2 - 2(2p)(5q) + (5q)^2] = 5(2p-5q)^2\end{aligned}$$

45. $$\begin{aligned}4x^2y^2 + 28xy + 49 &= (2xy)^2 + 28xy + 7^2\\ &= (2xy)^2 + 2(2xy)(7) + 7^2\\ &= (2xy+7)^2\end{aligned}$$

46. $$\begin{aligned}9m^2n^2 + 12mn + 4 &= (3mn)^2 + 12mn + 2^2\\ &= (3mn)^2 + 2(3mn)(2) + 2^2\\ &= (3mn+2)^2\end{aligned}$$

47. $$\begin{aligned}&(a-3b)^2 - 6(a-3b) + 9\\ &= (a-3b)^2 - 6(a-3b) + 3^2\\ &= (a-3b)^2 - 2(a-3b)(3) + 3^2\\ &= \left[(a-3b) - 3\right]^2 = (a-3b-3)^2\end{aligned}$$

48. $(2p+q)^2-10(2p+q)+25$
$=(2p+q)^2-10(2p+q)+5^2$
$=(2p+q)^2-2(2p+q)(5)+5^2$
$=\left[(2p+q)-5\right]^2=(2p+q-5)^2$

49. **(a)** Since $(x+5y)^2=x^2+10xy+25y^2$, a matches B.

(b) Since $(x-5y)^2=x^2-10xy+25y^2$, b matches C.

(c) Since $(x+5y)(x-5y)=x^2-25y^2$, c matches A.

(d) Since $(5y+x)(5y-x)=25y^2-x^2$, d matches D.

50. **(a)** Since $(2x-3)(4x^2+6x+9)=8x^3-27$, a matches B.

(b) Since $(2x+3)(4x^2-6x+9)=8x^3+27$, b matches C.

(c) Since $(3-2x)(9+6x+4x^2)=27-8x^3$, c matches A.

51. $9a^2-16=(3a)^2-4^2$
$=(3a+4)(3a-4)$

52. $16q^2-25=(4q)^2-5^2$
$=(4q+5)(4q-5)$

53. $36x^2-\frac{16}{25}=\left(6x-\frac{4}{5}\right)\left(6x+\frac{4}{5}\right)$

54. $100y^2-\frac{4}{49}=\left(10y+\frac{2}{7}\right)\left(10y-\frac{2}{7}\right)$

55. $25s^4-9t^2=(5s^2)^2-(3t)^2$
$=(5s^2+3t)(5s^2-3t)$

56. $36z^2-81y^4=9(4z^2-9y^4)$
$=9\left[(2z)^2-(3y^2)^2\right]$
$=9(2z+3y^2)(2z-3y^2)$

57. $(a+b)^2-16=(a+b)^2-4^2$
$=[(a+b)+4][(a+b)-4]$
$=(a+b+4)(a+b-4)$

58. $(p-2q)^2-100$
$=(p-2q)^2-10^2$
$=[(p-2q)+10][(p-2q)-10]$
$=(p-2q+10)(p-2q-10)$

59. $p^4-625=(p^2)^2-25^2=(p^2+25)(p^2-25)$
$=(p^2+25)(p^2-5^2)$
$=(p^2+25)(p+5)(p-5)$

Note that p^2+25 is a prime factor.

60. $m^4-81=(m^2)^2-9^2=(m^2+9)(m^2-9)$
$=(m^2+9)(m^2-3^2)$
$=(m^2+9)(m+3)(m-3)$

Note that m^2+9 is a prime factor.

61. $8-a^3=2^3-a^3=(2-a)(2^2+2\cdot a+a^2)$
$=(2-a)(4+2a+a^2)$

62. $r^3+27=r^3+3^3=(r+3)(r^2-3r+9)$

63. $125x^3-27=(5x)^3-3^3$
$=(5x-3)\left[(5x)^2+5x\cdot 3+3^2\right]$
$=(5x-3)(25x^2+15x+9)$

64. $8m^3-27n^3$
$=(2m)^3-(3n)^3$
$=(2m-3n)\left[(2m)^2+(2m)(3n)+(3n)^2\right]$
$=(2m-3n)(4m^2+6mn+9n^2)$

65. $27y^9+125z^6$
$=(3y^3)^3+(5z^2)^3$
$=(3y^3+5z^2)\left[(3y^3)^2-(3y^3)(5z^2)+(5z^2)^2\right]$
$=(3y^3+5z^2)(9y^6-15y^3z^2+25z^4)$

66. $27z^3+729y^3=27(z^3+27y^3)=27\left[z^3+(3y)^3\right]$
$=27(z+3y)(z^2-3zy+9y^2)$

67. $(r+6)^3-216$
$=(r+6)^3-6^3$
$=\left[(r+6)-6\right]\left[(r+6)^2+(r+6)(6)+6^2\right]$
$=\left[(r+6)-6\right]\left[r^2+12r+36+(r+6)(6)+6^2\right]$
$=\left[r+6-6\right]\left[r^2+12r+36+6r+36+36\right]$
$=r(r^2+18r+108)$

68. $(b+3)^3-27$
$=(b+3)^3-3^3$
$=\left[(b+3)-3\right]\left[(b+3)^2+(b+3)(3)+3^2\right]$
$=\left[(b+3)-3\right]\left[b^2+6b+9+(b+3)(3)+3^2\right]$
$=\left[b+3-3\right]\left[b^2+6b+9+3b+9+9\right]$
$=b(b^2+9b+27)$

69. $27-(m+2n)^3$
$$=3^3-(m+2n)^3$$
$$=\left[3-(m+2n)\right]\cdot\left[3^2+(3)(m+2n)+(m+2n)^2\right]$$
$$=\left[3-(m+2n)\right]\cdot\left[3^2+(3)(m+2n)+m^2+4mn+4n^2\right]$$
$$=(3-m-2n)\left(9+3m+6n+m^2+4mn+4n^2\right)$$

70. $125-(4a-b)^3$
$$=5^3-(4a-b)^3$$
$$=\left[5-(4a-b)\right]\left[5^2+(5)(4a-b)+(4a-b)^2\right]$$
$$=\left[5-(4a-b)\right]\cdot\left[5^2+(5)(4a-b)+16a^2-8ab+b^2\right]$$
$$=(5-4a+b)\left(25+20a-5b+16a^2-8ab+b^2\right)$$

71. The correct complete factorization of x^4-1 is choice B: $(x^2+1)(x+1)(x-1)$. Choice A is not a complete factorization, since x^2-1 can be factored as $(x+1)(x-1)$. The other choices are not correct factorizations of x^4-1.

72. The correct complete factorization of x^3+8 is choice C: $(x+2)(x^2-2x+4)$. Use the pattern for factoring the sum of cubes, $x^3+y^3=(x+y)(x^2-xy+y^2)$. We have $x^3+8=x^3+2^3$, so substitute 2 for y.

73. $x^6-1=(x^3)^2-1^2$
$$=(x^3+1)(x^3-1) \text{ or } (x^3-1)(x^3+1)$$
Use the patterns for the difference of cubes and sum of cubes to factor further. Since $x^3-1=(x-1)(x^2+x+1)$ and $x^3+1=(x+1)(x^2-x+1)$, we obtain the factorization
$$x^6-1=(x^3-1)(x^3+1)$$
$$=(x-1)(x^2+x+1)(x+1)(x^2-x+1)$$

74. $x^6-1=(x^2)^3-1^3$
$$=(x^2-1)\left[(x^2)^2+x^2\cdot 1+1^2\right]$$
$$=(x^2-1)(x^4+x^2+1)$$
$$=(x-1)(x+1)(x^4+x^2+1)$$

75. From Exercise 73, we have
$x^6-1=(x-1)(x^2+x+1)(x+1)(x^2-x+1)$.
From Exercise 74, we have
$x^6-1=(x-1)(x+1)(x^4+x^2+1)$.
Comparing these answers, we see that
$x^4+x^2+1=(x^2-x+1)(x^2+x+1)$.

76. $x^4+x^2+1=x^4+2x^2+1-x^2$ by the additive inverse property (0 in the form x^2-x^2)
$=(x^4+2x^2+1)-x^2$ by the associative property of addition
$=(x^2+1)^2-x^2$ by factoring a perfect square trinomial
$=(x^2+1-x)(x^2+1+x)$ by factoring the difference of squares
$=(x^2-x+1)(x^2+x+1)$ by the commutative property of addition

77. The answer in Exercise 75 and the final line in Exercise 76 are the same.

78. $x^8+x^4+1=x^8+2x^4+1-x^4$
$$=(x^8+2x^4+1)-x^4$$
$$=(x^4+1)^2-(x^2)^2$$
$$=(x^4+1-x^2)(x^4+1+x^2)$$
$$=(x^4-x^2+1)(x^4+x^2+1)$$
$$=(x^4-x^2+1)(x^2+x+1)(x^2-x+1)$$
Use result from Exercise 76.

79. m^4-3m^2-10
Let $x=m^2$, so that $x^2=(m^2)^2=m^4$.
$x^2-3x-10=(x-5)(x+2)$.
Replacing x with m^2 gives
$m^4-3m^2-10=(m^2-5)(m^2+2)$.

80. a^4-2a^2-48
Let $z=a^2$, so that $z^2=(a^2)^2=a^4$.
$z^2-2z-48=(z-8)(z+6)$
Replacing z with a^2 gives
$a^4-2a^2-48=(a^2-8)(a^2+6)$.

81. Let $x = 3k - 1$. This substitution gives

$$7(3k-1)^2 + 26(3k-1) - 8 = 7x^2 + 26x - 8 = (7x-2)(x+4)$$

Replacing x with $3k - 1$ gives

$$\begin{aligned}&7(3k-1)^2 + 26(3k-1) - 8\\ &= [7(3k-1)-2][(3k-1)+4]\\ &= (21k-7-2)(3k-1+4)\\ &= (21k-9)(3k+3) = 3(7k-3)(3)(k+1)\\ &= 9(7k-3)(k+1).\end{aligned}$$

82. Let $a = 4z - 3$. This substitution gives

$$6(4z-3)^2 + 7(4z-3) - 3 = 6a^2 + 7a - 3 = (2a+3)(3a-1)$$

Replacing a with $4z - 3$ gives

$$\begin{aligned}&6(4z-3)^2 + 7(4z-3) - 3\\ &= [2(4z-3)+3][3(4z-3)-1]\\ &= (8z-6+3)(12z-9-1)\\ &= (8z-3)(12z-10) = 2(8z-3)(6z-5)\end{aligned}$$

83. Let $x = a - 4$. This substitution gives

$$\begin{aligned}&9(a-4)^2 + 30(a-4) + 25\\ &= 9x^2 + 30x + 25\\ &= (3x)^2 + 2(3x)(5) + 5^2 = (3x+5)^2\end{aligned}$$

Replacing x by $a - 4$ gives

$$\begin{aligned}&9(a-4)^2 + 30(a-4) + 25\\ &= [3(a-4)+5]^2\\ &= (3a-12+5)^2 = (3a-7)^2\end{aligned}$$

84. Let $x = 4 - p$. This substitution gives

$$20(4-p)^2 - 3(4-p) - 2 = 20x^2 - 3x - 2 = (5x-2)(4x+1)$$

Replacing x with $4 - p$ gives

$$\begin{aligned}&20(4-p)^2 - 3(4-p) - 2\\ &= [5(4-p)-2][4(4-p)+1]\\ &= [20-5p-2][16-4p+1]\\ &= (18-5p)(17-4p).\end{aligned}$$

85. $$\begin{aligned}&4b^2 + 4bc + c^2 - 16\\ &= (4b^2 + 4bc + c^2) - 16\\ &= \left[(2b)^2 + 2(2b)(c) + c^2\right] - 16\\ &= (2b+c)^2 - 4^2 = [(2b+c)+4][(2b+c)-4]\\ &= (2b+c+4)(2b+c-4)\end{aligned}$$

86. Let $x = 2y - 1$. This substitution gives

$$\begin{aligned}(2y-1)^2 - 4(2y-1) + 4 &= x^2 - 4x + 4\\ &= x^2 - 2(x)(2) + 2^2\\ &= (x-2)^2\end{aligned}$$

Replacing x by $2y - 1$ gives

$$\begin{aligned}(2y-1)^2 - 4(2y-1) + 4 &= [(2y-1)-2]^2\\ &= (2y-1-2)^2\\ &= (2y-3)^2\end{aligned}$$

87. $$\begin{aligned}x^2 + xy - 5x - 5y &= \left(x^2 + xy\right) - \left(5x + 5y\right)\\ &= x\left(x+y\right) - 5\left(x+y\right)\\ &= \left(x+y\right)\left(x-5\right)\end{aligned}$$

88. $8r^2 - 3rs + 10s^2$

Try to factor this trinomial by trial and error. Trying all the possible combinations of factors will show that none of them are correct.

$$\begin{aligned}&(8r-10s)(r-s) \ne 8r^2 - 3rs + 10s^2;\\ &(8r-s)(r-10s) \ne 8r^2 - 3rs + 10s^2;\\ &(8r-2s)(r-5s) \ne 8r^2 - 3rs + 10s^2;\\ &(8r-5s)(r-2s) \ne 8r^2 - 3rs + 10s^2;\\ &(4r-10s)(2r-s) \ne 8r^2 - 3rs + 10s^2;\\ &(4r-s)(2r-10s) \ne 8r^2 - 3rs + 10s^2;\\ &(4r-2s)(2r-5s) \ne 8r^2 - 3rs + 10s^2;\\ &(4r-5s)(2r-2s) \ne 8r^2 - 3rs + 10s^2\end{aligned}$$

Therefore, the polynomial is prime.

89. $p^4(m-2n) + q(m-2n) = (m-2n)(p^4+q)$

90. $$\begin{aligned}36a^2 + 60a + 25 &= (6a)^2 + 2(6a)(5) + 5^2\\ &= (6a+5)^2\end{aligned}$$

91. $$\begin{aligned}4z^2 + 28z + 49 &= (2z)^2 + 2(2z)(7) + 7^2\\ &= (2z+7)^2\end{aligned}$$

92. $6p^4 + 7p^2 - 3$

Let $x = p^2$, so that $x^2 = (p^2)^2 = p^4$.

$6x^2 + 7x - 3 = (3x-1)(2x+3).$

Replacing x with p^2 gives

$6p^4 + 7p^2 - 3 = (3p^2 - 1)(2p^2 + 3).$

93. $$\begin{aligned}&1000x^3 + 343y^3\\ &= (10x)^3 + (7y)^3\\ &= (10x+7y)\left[(10x)^2 - (10x)(7y) + (7y)^2\right]\\ &= (10x+7y)(100x^2 - 70xy + 49y^2)\end{aligned}$$

94. $$\begin{aligned}&b^2 + 8b + 16 - a^2\\ &= (b^2 + 8b + 16) - a^2\\ &= \left[b^2 + 2(b)(4) + 4^2\right] - a^2\\ &= (b+4)^2 - a^2 = [(b+4)+a][(b+4)-a]\\ &= (b+4+a)(b+4-a)\end{aligned}$$

95. $125m^6 - 216 = (5m^2)^3 - 6^3$
$= (5m^2 - 6)\left[(5m^2)^2 + 5m^2(6) + 6^2\right]$
$= (5m^2 - 6)(25m^4 + 30m^2 + 36)$

96. $q^2 + 6q + 9 - p^2 = (q^2 + 6q + 9) - p^2$
$= \left[q^2 + 2(q)(3) + 3^2\right] - p^2$
$= (q+3)^2 - p^2$
$= [(q+3) + p][(q+3) - p)]$
$= (q + 3 + p)(q + 3 - p)$

97. $64 + (3x+2)^3$
$= 4^3 + (3x+2)^3$
$= \left[4 + (3x+2)\right]\left[4^2 - (4)(3x+2) + (3x+2)^2\right]$
$= \left[4 + (3x+2)\right]\left[\begin{array}{l} 4^2 - (4)(3x+2) \\ \quad + 9x^2 + 12x + 4 \end{array}\right]$
$= (4 + 3x + 2)(16 - 12x - 8 + 9x^2 + 12x + 4)$
$= (3x+6)(9x^2+12)$
$= 3(x+2)(3)(3x^2+4) = 9(x+2)(3x^2+4)$

98. $216p^3 + 125q^3$
$= (6p)^3 + (5q)^3$
$= (6p + 5q)\left[(6p)^2 - (6p)(5q) + (5q)^2\right]$
$= (6p + 5q)(36p^2 - 30pq + 25q^2)$

99. $\frac{4}{25}x^2 - 49y^2 = \left(\frac{2}{5}x + 7y\right)\left(\frac{2}{5}x - 7y\right)$

100. $100r^2 - 169s^2 = (10r)^2 - (13s)^2$
$= (10r + 13s)(10r - 13s)$

101. $144z^2 + 121$
The sum of squares cannot be factored.
$144z^2 + 121$ is prime.

102. Let $w = (3a + 5)$. This substitution gives
$(3a+5)^2 - 18(3a+5) + 81 = w^2 - 18w + 81$
$= w^2 - 2(w)(9) + 9^2$
$= (w-9)^2$
Replacing w with $3a + 5$ gives
$(3a+5)^2 - 18(3a+5) + 81 = [(3a+5) - 9]^2$
$= (3a + 5 - 9)^2$
$= (3a - 4)^2$

103. $(x+y)^2 - (x-y)^2$
$= [(x+y) + (x-y)][(x+y) - (x-y)]$
$= (x + y + x - y)(x + y - x + y)$
$= (2x)(2y) = 4xy$

104. Let $t = z^2$, so that $t^2 = (z^2)^2 = z^4$.
$4z^4 - 7z^2 - 15 = 4t^2 - 7t - 15$. Factor
$4t^2 - 7t - 15$ by trial and error;
$4t^2 - 7t - 15 = (4t + 5)(t - 3)$.
Replacing t with z^2 gives
$4z^4 - 7z^2 - 15 = (4z^2 + 5)(z^2 - 3)$.

105. Answers will vary.

106. Answers will vary.

107. $4z^2 + bz + 81 = (2z)^2 + bz + 9^2$ will be a perfect trinomial if $bz = \pm 2(2z)(9) \Rightarrow$
$bz = \pm 36z \Rightarrow b = \pm 36$.
If $b = 36$, $4z^2 + 36z + 81 = (2z+9)^2$.
If $b = -36$, $4z^2 - 36z + 81 = (2z-9)^2$.

108. $9p^2 + bp + 25 = (3p)^2 + bp + 5^2$ will be a perfect trinomial if $bp = \pm 2(3p)(5) \Rightarrow$
$bp = \pm 30p \Rightarrow b = \pm 30$.
If $b = 30$, $9p^2 + 30p + 25 = (3p+5)^2$.
If $b = -30$, $9p^2 - 30p + 25 = (3p-5)^2$.

109. $100r^2 - 60r + c = (10r)^2 - \underbrace{2(10r)(3)}_{60r} + c$
will be a perfect trinomial if $c = 3^2 = 9$.
If $c = 9$, $100r^2 - 60r + 9 = (10r - 3)^2$.

110. $49x^2 + 70x + c = (7x)^2 + \underbrace{2(7x)(5)}_{70x} + c$
will be a perfect trinomial if $c = 5^2 = 25$.
If $c = 25$, $49x^2 + 70x + 25 = (7x+5)^2$.

Section R.5: Rational Expressions

1. In the rational expression $\frac{x+3}{x-6}$, the solution to the equation $x - 6 = 0$ is excluded from the domain.
$x - 6 = 0 \Rightarrow x = 6$
The domain is $\{x | x \neq 6\}$.

2. In the rational expression $\frac{2x-4}{x+7}$, the solution to the equation $x + 7 = 0$ is excluded from the domain.
$x + 7 = 0 \Rightarrow x = -7$
The domain is $\{x | x \neq -7\}$.

3. In the rational expression $\frac{3x+7}{(4x+2)(x-1)}$, the solution to the equation $(4x+2)(x-1)=0$ is excluded from the domain.

$(4x+2)(x-1)=0$

$4x+2=0$ or $x-1=0$

$4x=-2$ $\quad x=1$

$x=-\frac{1}{2}$

The domain is $\left\{x \,\middle|\, x \neq -\frac{1}{2}, 1\right\}$.

4. In the rational expression $\frac{9x+12}{(2x+3)(x-5)}$, the solution to the equation $(2x+3)(x-5)=0$ is excluded from the domain.

$(2x+3)(x-5)=0$

$2x+3=0$ or $x-5=0$

$2x=-3$ $\quad x=5$

$x=-\frac{3}{2}$

The domain is $\left\{x \,\middle|\, x \neq -\frac{3}{2}, 5\right\}$.

5. In the rational expression $\frac{12}{x^2+5x+6}$, the solution to the equation $x^2+5x+6=0$ is excluded from the domain.

$x^2+5x+6=0$

$(x+3)(x+2)=0$

$x+3=0$ or $x+2=0$

$x=-3$ $\quad x=-2$

The domain is $\{x \mid x \neq -3, -2\}$.

6. In the rational expression $\frac{3}{x^2-5x-6}$, the solution to the equation x^2-5x-6 is excluded from the domain.

$x^2-5x-6=0$

$(x+1)(x-6)=0$

$x+1=0$ or $x-6=0$

$x=-1$ $\quad x=6$

The domain is $\{x \mid x \neq -1, 6\}$.

7. $x=4,\ y=2$

(a) $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}+\frac{1}{2}=\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$

(b) $\frac{1}{x+y}=\frac{1}{4+2}=\frac{1}{6}$.

8. No; $\frac{1}{x}+\frac{1}{y}\neq\frac{1}{x+y}$.

9. If $x=3,\ y=5$

(a) $\frac{1}{x}-\frac{1}{y}=\frac{1}{3}-\frac{1}{5}=\frac{5}{15}-\frac{3}{15}=\frac{2}{15}$

(b) $\frac{1}{x-y}=\frac{1}{3-5}=\frac{1}{-2}=-\frac{1}{2}$.

10. No; $\frac{1}{x}-\frac{1}{y}\neq\frac{1}{x-y}$.

11. $\frac{8k+16}{9k+18}=\frac{8(k+2)}{9(k+2)}=\frac{8}{9}$

12. $\frac{20r+10}{30r+15}=\frac{10(2r+1)}{15(2r+1)}=\frac{2\cdot5(2r+1)}{3\cdot5(2r+1)}=\frac{2}{3}$

13. $\frac{3(3-t)}{(t+5)(t-3)}=\frac{3(3-t)(-1)}{(t+5)(t-3)(-1)}=\frac{-3}{t+5}$

14. $\frac{-8(y+4)}{(y+2)(y+4)}=\frac{-8}{y+2}$

15. $\frac{8x^2+16x}{4x^2}=\frac{8x(x+2)}{4x^2}=\frac{2\cdot4x(x+2)}{x\cdot4x}$
$=\frac{2(x+2)}{x}=\frac{2x+4}{x}$

16. $\frac{36y^2+72y}{9y}=\frac{36y(y+2)}{9y}=\frac{4\cdot9y(y+2)}{9y}$
$=4(y+2)=4y+8$

17. $\frac{m^2-4m+4}{m^2+m-6}=\frac{(m-2)(m-2)}{(m-2)(m+3)}=\frac{m-2}{m+3}$

18. $\frac{r^2-r-6}{r^2+r-12}=\frac{(r+2)(r-3)}{(r+4)(r-3)}=\frac{r+2}{r+4}$

19. $\frac{8m^2+6m-9}{16m^2-9}=\frac{(2m+3)(4m-3)}{(4m+3)(4m-3)}=\frac{2m+3}{4m+3}$

20. $\frac{6y^2+11y+4}{3y^2+7y+4}=\frac{(2y+1)(3y+4)}{(3y+4)(y+1)}=\frac{2y+1}{y+1}$

21. $\frac{15p^3}{9p^2} \div \frac{6p}{10p^2} = \frac{15p^3}{9p^2} \cdot \frac{10p^2}{6p} = \frac{150p^5}{54p^3}$
$= \frac{25 \cdot 6p^5}{9 \cdot 6p^3} = \frac{25p^2}{9}$

22. $\frac{3r^2}{9r^3} \div \frac{8r^3}{6r} = \frac{3r^2}{9r^3} \cdot \frac{6r}{8r^3}$
$= \frac{18r^3}{72r^6} = \frac{18r^3}{18 \cdot 4r^6} = \frac{1}{4r^3}$

23. $\frac{2k+8}{6} \div \frac{3k+12}{2} = \frac{2k+8}{6} \cdot \frac{2}{3k+12}$
$= \frac{2(k+4)(2)}{6(3)(k+4)} = \frac{2}{9}$

24. $\frac{5m+25}{10} \cdot \frac{12}{6m+30} = \frac{5(m+5)}{10} \cdot \frac{12}{6(m+5)}$
$= \frac{60(m+5)}{60(m+5)} = 1$

25. $\frac{x^2+x}{5} \cdot \frac{25}{xy+y} = \frac{x(x+1)}{5} \cdot \frac{25}{y(x+1)}$
$= \frac{25x(x+1)}{5y(x+1)} = \frac{5x}{y}$

26. $\frac{3m-15}{4m-20} \cdot \frac{m^2-10m+25}{12m-60}$
$= \frac{3(m-5)}{4(m-5)} \cdot \frac{(m-5)(m-5)}{12(m-5)} = \frac{m-5}{16}$

27. $\frac{4a+12}{2a-10} \div \frac{a^2-9}{a^2-a-20} = \frac{4a+12}{2a-10} \cdot \frac{a^2-a-20}{a^2-9}$
$= \frac{4(a+3)}{2(a-5)} \cdot \frac{(a-5)(a+4)}{(a+3)(a-3)}$
$= \frac{2(a+4)}{a-3} = \frac{2a+8}{a-3}$

28. $\frac{6r-18}{9r^2+6r-24} \cdot \frac{12r-16}{4r-12}$
$= \frac{6(r-3)}{3(3r^2+2r-8)} \cdot \frac{4(3r-4)}{4(r-3)}$
$= \frac{6(r-3)}{3(3r-4)(r+2)} \cdot \frac{4(3r-4)}{4(r-3)} = \frac{2}{r+2}$

29. $\frac{p^2-p-12}{p^2-2p-15} \cdot \frac{p^2-9p+20}{p^2-8p+16}$
$= \frac{(p-4)(p+3)}{(p-5)(p+3)} \cdot \frac{(p-5)(p-4)}{(p-4)(p-4)} = 1$

30. $\frac{x^2+2x-15}{x^2+11x+30} \cdot \frac{x^2+2x-24}{x^2-8x+15}$
$= \frac{(x-3)(x+5)}{(x+6)(x+5)} \cdot \frac{(x+6)(x-4)}{(x-3)(x-5)} = \frac{x-4}{x-5}$

31. $\frac{m^2+3m+2}{m^2+5m+4} \div \frac{m^2+5m+6}{m^2+10m+24}$
$= \frac{m^2+3m+2}{m^2+5m+4} \cdot \frac{m^2+10m+24}{m^2+5m+6}$
$= \frac{(m+2)(m+1)}{(m+4)(m+1)} \cdot \frac{(m+6)(m+4)}{(m+3)(m+2)} = \frac{m+6}{m+3}$

32. $\frac{y^2+y-2}{y^2+3y-4} \div \frac{y^2+3y+2}{y^2+4y+3}$
$= \frac{y^2+y-2}{y^2+3y-4} \cdot \frac{y^2+4y+3}{y^2+3y+2}$
$= \frac{(y-1)(y+2)}{(y-1)(y+4)} \cdot \frac{(y+1)(y+3)}{(y+2)(y+1)} = \frac{y+3}{y+4}$

33. $\frac{xz-xw+2yz-2yw}{z^2-w^2} \cdot \frac{4z+4w+xz+wx}{16-x^2}$
$= \frac{x(z-w)+2y(z-w)}{(z+w)(z-w)} \cdot \frac{4(z+w)+x(z+w)}{(4+x)(4-x)}$
$= \frac{(z-w)(x+2y)}{(z+w)(z-w)} \cdot \frac{(z+w)(4+x)}{(4+x)(4-x)} = \frac{x+2y}{4-x}$

34. $\frac{ac+ad+bc+bd}{a^2-b^2} \cdot \frac{a^3-b^3}{2a^2+2ab+2b^2}$
$= \frac{a(c+d)+b(c+d)}{(a+b)(a-b)} \cdot \frac{(a-b)(a^2+ab+b^2)}{2(a^2+ab+b^2)}$
$= \frac{(c+d)(a+b)}{(a+b)(a-b)} \cdot \frac{(a-b)(a^2+ab+b^2)}{2(a^2+ab+b^2)} = \frac{c+d}{2}$

35. $\frac{x^3+y^3}{x^3-y^3} \cdot \frac{x^2-y^2}{x^2+2xy+y^2}$
$= \frac{(x+y)(x^2-xy+y^2)}{(x-y)(x^2+xy+y^2)} \cdot \frac{(x+y)(x-y)}{(x+y)(x+y)}$
$= \frac{x^2-xy+y^2}{x^2+xy+y^2}$

36. $\frac{x^2-y^2}{(x-y)^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \div \frac{x^3+y^3}{(x-y)^4}$
$= \frac{x^2-y^2}{(x-y)^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \cdot \frac{(x-y)^4}{x^3+y^3}$
$= \frac{(x+y)(x-y)}{(x-y)^2} \cdot \frac{x^2-xy+y^2}{(x-y)(x-y)}$
$\cdot \frac{(x-y)^4}{(x+y)(x^2-xy+y^2)} = x-y$

37. Expressions (B) and (C) are both equal to –1, since the numerator and denominator are additive inverses.

B. $\frac{-x-4}{x+4}=\frac{-1(x+4)}{x+4}=-1$

C. $\frac{x-4}{4-x}=\frac{-1(4-x)}{4-x}=-1$

38. Answers will vary.

39. $\frac{3}{2k}+\frac{5}{3k}=\frac{3\cdot 3}{2k\cdot 3}+\frac{5\cdot 2}{3k\cdot 2}=\frac{9}{6k}+\frac{10}{6k}=\frac{19}{6k}$

40. $\frac{8}{5p}+\frac{3}{4p}=\frac{8\cdot 4}{5p\cdot 4}+\frac{3\cdot 5}{4p\cdot 5}$
$=\frac{32}{20p}+\frac{15}{20p}=\frac{47}{20p}$

41. $\frac{1}{6m}+\frac{2}{5m}+\frac{4}{m}=\frac{1\cdot 5}{6m\cdot 5}+\frac{2\cdot 6}{5m\cdot 6}+\frac{4\cdot 30}{m\cdot 30}$
$=\frac{5}{30m}+\frac{12}{30m}+\frac{120}{30m}=\frac{137}{30m}$

42. $\frac{8}{3p}+\frac{5}{4p}+\frac{9}{2p}=\frac{8\cdot 4}{3p\cdot 4}+\frac{5\cdot 3}{4p\cdot 3}+\frac{9\cdot 6}{2p\cdot 6}$
$=\frac{32}{12p}+\frac{15}{12p}+\frac{54}{12p}=\frac{101}{12p}$

43. $\frac{1}{a}-\frac{b}{a^2}=\frac{1\cdot a}{a\cdot a}-\frac{b}{a^2}=\frac{a}{a^2}-\frac{b}{a^2}=\frac{a-b}{a^2}$

44. $\frac{3}{z}+\frac{x}{z^2}=\frac{3\cdot z}{z\cdot z}+\frac{x}{z^2}=\frac{3z}{z^2}+\frac{x}{z^2}=\frac{3z+x}{z^2}$

45. $\frac{5}{12x^2y}-\frac{11}{6xy}=\frac{5}{12x^2y}-\frac{11\cdot 2x}{6xy\cdot 2x}$
$=\frac{5}{12x^2y}-\frac{22x}{12x^2y}=\frac{5-22x}{12x^2y}$

46. $\frac{7}{18a^3b^2}-\frac{2}{9ab}=\frac{7}{18a^3b^2}-\frac{2\cdot 2a^2b}{9ab\cdot 2a^2b}$
$=\frac{7}{18a^3b^2}-\frac{4a^2b}{18a^3b^2}$
$=\frac{7-4a^2b}{18a^3b^2}$

47. $\frac{17y+3}{9y+7}-\frac{-10y-18}{9y+7}=\frac{(17y+3)-(-10y-18)}{9y+7}$
$=\frac{17y+3+10y+18}{9y+7}$
$=\frac{27y+21}{9y+7}$
$=\frac{3(9y+7)}{9y+7}=3$

48. $\frac{7x+8}{3x+2}-\frac{x+4}{3x+2}=\frac{(7x+8)-(x+4)}{3x+2}$
$=\frac{7x+8-x-4}{3x+2}=\frac{6x+4}{3x+2}$
$=\frac{2(3x+2)}{3x+2}=2$

49. $\frac{1}{x+z}+\frac{1}{x-z}=\frac{1\cdot(x-z)}{(x+z)(x-z)}+\frac{1\cdot(x+z)}{(x-z)(x+z)}$
$=\frac{(x-z)+(x+z)}{(x-z)(x+z)}$
$=\frac{x-z+x+z}{(x+z)(x-z)}=\frac{2x}{(x+z)(x-z)}$

50.
$\frac{m+1}{m-1}+\frac{m-1}{m+1}=\frac{(m+1)(m+1)}{(m-1)(m+1)}+\frac{(m-1)(m-1)}{(m+1)(m-1)}$
$=\frac{\left(m^2+2m+1\right)+\left(m^2-2m+1\right)}{(m+1)(m-1)}$
$=\frac{m^2+2m+1+m^2-2m+1}{(m+1)(m-1)}$
$=\frac{2m^2+2}{(m+1)(m-1)}$

51. Since $a-2=(-1)(2-a)$ we have
$\frac{3}{a-2}-\frac{1}{2-a}=\frac{3}{a-2}-\frac{1(-1)}{(2-a)(-1)}$
$=\frac{3}{a-2}-\frac{-1}{a-2}=\frac{3-(-1)}{a-2}$
$=\frac{4}{a-2}$

We may also use $2-a$ as the common denominator.
$\frac{3}{a-2}-\frac{1}{2-a}=\frac{3(-1)}{(a-2)(-1)}-\frac{1}{2-a}$
$=\frac{-3}{2-a}-\frac{1}{2-a}=\frac{-4}{2-a}$

The two results, $\frac{4}{a-2}$ and $\frac{-4}{2-a}$, are equivalent rational expressions.

52. Since $p - q = (-1)(q - p)$

$$\frac{q}{p-q} - \frac{q}{q-p} = \frac{q}{p-q} - \frac{q(-1)}{(q-p)(-1)} = \frac{q}{p-q} - \frac{-q}{p-q} = \frac{q-(-q)}{p-q} = \frac{2q}{p-q}$$

We may also use $q - p$ as the common denominator. In this case, our result will be $\frac{-2q}{q-p}$. The two results are equivalent rational expressions.

53. Since $2x - y = (-1)(y - 2x)$

$$\frac{x+y}{2x-y} - \frac{2x}{y-2x} = \frac{x+y}{2x-y} - \frac{2x(-1)}{(y-2x)(-1)} = \frac{x+y}{2x-y} - \frac{-2x}{2x-y} = \frac{x+y-(-2x)}{2x-y} = \frac{x+y+2x}{2x-y} = \frac{3x+y}{2x-y}$$

We may also use $y - 2x$ as the common denominator. In this case, our result will be $\frac{-3x-y}{y-2x}$. The two results are equivalent rational expressions.

54. Since $3m - 4 = (-1)(4 - 3m)$

$$\frac{m-4}{3m-4} + \frac{3m+2}{4-3m} = \frac{m-4}{3m-4} + \frac{(3m+2)(-1)}{(4-3m)(-1)} = \frac{m-4}{3m-4} + \frac{-3m-2}{3m-4} = \frac{m-4-3m-2}{3m-4} = \frac{-2m-6}{3m-4}$$

We may also use $4 - 3m$ as the common denominator. In this case, our result will be $\frac{2m+6}{4-3m}$.
The two results are equivalent rational expressions.

55.
$$\frac{4}{x+1} + \frac{1}{x^2-x+1} - \frac{12}{x^3+1}$$
$$= \frac{4}{x+1} + \frac{1}{x^2-x+1} - \frac{12}{(x+1)(x^2-x+1)}$$
$$= \frac{4(x^2-x+1)}{(x+1)(x^2-x+1)} + \frac{1(x+1)}{(x+1)(x^2-x+1)} - \frac{12}{(x+1)(x^2-x+1)}$$
$$= \frac{4(x^2-x+1)+(x+1)-12}{(x+1)(x^2-x+1)}$$
$$= \frac{4x^2-4x+4+x+1-12}{(x+1)(x^2-x+1)}$$
$$= \frac{4x^2-3x-7}{(x+1)(x^2-x+1)}$$
$$= \frac{(4x-7)(x+1)}{(x+1)(x^2-x+1)} = \frac{4x-7}{x^2-x+1}$$

56.
$$\frac{5}{x+2} + \frac{2}{x^2-2x+4} - \frac{60}{x^3+8}$$
$$= \frac{5}{x+2} + \frac{2}{x^2-2x+4} - \frac{60}{(x+2)(x^2-2x+4)}$$
$$= \frac{5(x^2-2x+4)}{(x+2)(x^2-2x+4)} + \frac{2(x+2)}{(x+2)(x^2-2x+4)} - \frac{60}{(x+2)(x^2-2x+4)}$$
$$= \frac{5(x^2-2x+4)+2(x+2)-60}{(x+2)(x^2-2x+4)}$$
$$= \frac{5x^2-10x+20+2x+4-60}{(x+2)(x^2-2x+4)}$$
$$= \frac{5x^2-8x-36}{(x+2)(x^2-2x+4)} = \frac{(x+2)(5x-18)}{(x+2)(x^2-2x+4)}$$
$$= \frac{5x-18}{x^2-2x+4}$$

57.
$$\frac{3x}{x^2+x-12} - \frac{x}{x^2-16} = \frac{3x}{(x-3)(x+4)} - \frac{x}{(x-4)(x+4)} = \frac{3x(x-4)}{(x-3)(x+4)(x-4)} - \frac{x(x-3)}{(x-4)(x+4)(x-3)}$$
$$= \frac{3x(x-4)-x(x-3)}{(x-3)(x+4)(x-4)} = \frac{3x^2-12x-x^2+3x}{(x-3)(x+4)(x-4)} = \frac{2x^2-9x}{(x-3)(x+4)(x-4)}$$

58. $$\frac{p}{2p^2-9p-5}-\frac{2p}{6p^2-p-2}=\frac{p}{(2p+1)(p-5)}-\frac{2p}{(3p-2)(2p+1)}$$
$$=\frac{p(3p-2)}{(2p+1)(p-5)(3p-2)}-\frac{2p(p-5)}{(2p+1)(3p-2)(p-5)}$$
$$=\frac{p(3p-2)-2p(p-5)}{(2p+1)(p-5)(3p-2)}=\frac{3p^2-2p-2p^2+10p}{(2p+1)(p-5)(3p-2)}$$
$$=\frac{p^2+8p}{(2p+1)(p-5)(3p-2)}$$

59. $$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{x\left(1+\frac{1}{x}\right)}{x\left(1-\frac{1}{x}\right)}=\frac{x\cdot 1+x\left(\frac{1}{x}\right)}{x\cdot 1-x\left(\frac{1}{x}\right)}=\frac{x+1}{x-1}$$

60. $$\frac{2-\frac{2}{y}}{2+\frac{2}{y}}=\frac{y\left(2-\frac{2}{y}\right)}{y\left(2+\frac{2}{y}\right)}=\frac{2y-y\left(\frac{2}{y}\right)}{2y+y\left(\frac{2}{y}\right)}=\frac{2y-2}{2y+2}=\frac{2(y-1)}{2(y+1)}=\frac{y-1}{y+1}$$

61. $$\frac{\frac{1}{x+1}-\frac{1}{x}}{\frac{1}{x}}=\frac{x(x+1)\left(\frac{1}{x+1}-\frac{1}{x}\right)}{x(x+1)\left(\frac{1}{x}\right)}=\frac{x(x+1)\left(\frac{1}{x+1}\right)-x(x+1)\left(\frac{1}{x}\right)}{x(x+1)\left(\frac{1}{x}\right)}=\frac{x-(x+1)}{x+1}=\frac{x-x-1}{x+1}=\frac{-1}{x+1}$$

62. $$\frac{\frac{1}{y+3}-\frac{1}{y}}{\frac{1}{y}}=\frac{y(y+3)\left(\frac{1}{y+3}-\frac{1}{y}\right)}{y(y+3)\left(\frac{1}{y}\right)}=\frac{y(y+3)\left(\frac{1}{y+3}\right)-y(y+3)\left(\frac{1}{y}\right)}{y(y+3)\left(\frac{1}{y}\right)}=\frac{y-(y+3)}{y+3}=\frac{y-y-3}{y+3}=\frac{-3}{y+3}$$

63. $$\frac{1+\frac{1}{1-b}}{1-\frac{1}{1+b}}=\frac{(1-b)(1+b)\left(1+\frac{1}{1-b}\right)}{(1-b)(1+b)\left(1-\frac{1}{1+b}\right)}=\frac{(1-b)(1+b)(1)+(1-b)(1+b)\left(\frac{1}{1-b}\right)}{(1-b)(1+b)(1)-(1-b)(1+b)\left(\frac{1}{1+b}\right)}=\frac{(1-b)(1+b)+(1+b)}{(1-b)(1+b)-(1-b)}$$
$$=\frac{(1+b)[(1-b)+1]}{(1-b)[(1+b)-1]}=\frac{(1+b)(2-b)}{(1-b)b}\text{ or }\frac{(2-b)(1+b)}{b(1-b)}$$

64. $$m-\frac{m}{m+\frac{1}{2}}=m-\frac{2\cdot m}{2\left(m+\frac{1}{2}\right)}$$
$$=m-\frac{2m}{2m+1}=\frac{m(2m+1)}{(2m+1)}-\frac{2m}{(2m+1)}=\frac{2m^2+m-2m}{2m+1}=\frac{2m^2-m}{2m+1}\text{ or }\frac{m(2m-1)}{2m+1}$$

65. $$\frac{m-\frac{1}{m^2-4}}{\frac{1}{m+2}}=\frac{m-\frac{1}{(m+2)(m-2)}}{\frac{1}{m+2}}$$

Multiply both numerator and denominator by the LCD of all the fractions, $(m+2)(m-2)$.

$$\frac{m-\frac{1}{m^2-4}}{\frac{1}{m+2}}=\frac{(m+2)(m-2)\left(m-\frac{1}{(m+2)(m-2)}\right)}{(m+2)(m-2)\left(\frac{1}{m+2}\right)}=\frac{(m+2)(m-2)(m)-(m+2)(m-2)\left(\frac{1}{(m+2)(m-2)}\right)}{(m+2)(m-2)\left(\frac{1}{m+2}\right)}$$
$$=\frac{m(m+2)(m-2)-1}{m-2}=\frac{m(m^2-4)-1}{m-2}=\frac{m^3-4m-1}{m-2}$$

66. $\dfrac{\frac{3}{p^2-16}+p}{\frac{1}{p-4}}=\dfrac{\frac{3}{(p+4)(p-4)}+p}{\frac{1}{p-4}}$

Multiply both numerator and denominator by the LCD of all the fractions, $(p+4)(p-4)$.

$$\dfrac{\frac{3}{p^2-16}+p}{\frac{1}{p-4}}=\dfrac{(p+4)(p-4)\left[\frac{3}{(p+4)(p-4)}+p\right]}{(p+4)(p-4)\left[\frac{1}{p-4}\right]}=\dfrac{(p+4)(p-4)\left(\frac{3}{(p+4)(p-4)}\right)+(p+4)(p-4)p}{p+4}$$
$$=\dfrac{3+(p^2-16)p}{p+4}=\dfrac{3+p^3-16p}{p+4}=\dfrac{p^3-16p+3}{p+4}$$

67. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}=\dfrac{x(x+h)\left(\frac{1}{x+h}-\frac{1}{x}\right)}{x(x+h)(h)}$

$$=\dfrac{x(x+h)\left(\frac{1}{x+h}\right)-x(x+h)\left(\frac{1}{x}\right)}{x(x+h)(h)}=\dfrac{x-(x+h)}{x(x+h)(h)}=\dfrac{x-x-h}{x(x+h)(h)}=\dfrac{-h}{x(x+h)(h)}=\dfrac{-1}{x(x+h)}$$

68. $\dfrac{\frac{1}{(x+h)^2+9}-\frac{1}{x^2+9}}{h}=\dfrac{\left[(x+h)^2+9\right]\left(x^2+9\right)\left[\frac{1}{(x+h)^2+9}-\frac{1}{x^2+9}\right]}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}$

$$=\dfrac{\left[(x+h)^2+9\right]\left(x^2+9\right)\left[\frac{1}{(x+h)^2+9}\right]-\left[(x+h)^2+9\right]\left(x^2+9\right)\left[\frac{1}{x^2+9}\right]}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}$$
$$=\dfrac{\left(x^2+9\right)-\left[(x+h)^2+9\right]}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}=\dfrac{x^2+9-\left[x^2+2xh+h^2+9\right]}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}$$
$$=\dfrac{x^2+9-x^2-2xh-h^2-9}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}=\dfrac{-2xh-h^2}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}$$
$$=\dfrac{h(-2x-h)}{\left[(x+h)^2+9\right]\left(x^2+9\right)h}=\dfrac{-2x-h}{\left[(x+h)^2+9\right]\left(x^2+9\right)}$$

69. $\dfrac{\frac{y+3}{y}-\frac{4}{y-1}}{\frac{y}{y-1}+\frac{1}{y}}=\dfrac{y(y-1)\left(\frac{y+3}{y}-\frac{4}{y-1}\right)}{y(y-1)\left(\frac{y}{y-1}+\frac{1}{y}\right)}=\dfrac{y(y-1)\left(\frac{y+3}{y}\right)-y(y-1)\left(\frac{4}{y-1}\right)}{y(y-1)\left(\frac{y}{y-1}\right)+y(y-1)\left(\frac{1}{y}\right)}=\dfrac{(y-1)(y+3)-4y}{y^2+y-1}$

$$=\dfrac{y^2+2y-3-4y}{y^2+y-1}=\dfrac{y^2-2y-3}{y^2+y-1}$$

70. Altitude of 7000 feet, $x=7$ (thousand)
The distance from the origin is
$\dfrac{7-7}{0.639(7)+1.75}=0,$
which represents 0 miles.

71. Altitude of 1200 feet, $x=1.2$ (thousand)
The distance from the origin is
$\dfrac{7-1.2}{0.639(1.2)+1.75}\approx 2.305,$
which represents about 2305 miles.

72. **(a)** $y=\dfrac{6.7x}{100-x}$; Let $x = 75$ (75%)

$y=\dfrac{6.7(75)}{100-75}=\dfrac{502.5}{25}=20.1$

The cost of removing 75% of the pollutant is 20.1 thousand dollars.

(b) $y=\dfrac{6.7x}{100-x}$; Let $x = 95$ (95%).

$y=\dfrac{6.7(95)}{100-95}=\dfrac{636.5}{5}=127.3$

The cost of removing 95% of the pollutant is 127.3 thousand dollars.

Section R.6: Rational Exponents

1. **(a)** $4^{-2}=\dfrac{1}{4^2}=\dfrac{1}{16}$; B

(b) $-4^{-2}=-\left(4^{-2}\right)=-\dfrac{1}{4^2}=-\dfrac{1}{16}$; D

(c) $\left(-4\right)^{-2}=\dfrac{1}{\left(-4\right)^2}=\dfrac{1}{16}$; B

(d) $-\left(-4\right)^{-2}=-\dfrac{1}{\left(-4\right)^2}=-\dfrac{1}{16}$; D

2. **(a)** $5^{-3}=\dfrac{1}{5^3}=\dfrac{1}{125}$; C

(b) $-5^{-3}=-\left(5^{-3}\right)=-\dfrac{1}{5^3}=-\dfrac{1}{125}$; D

(c) $\left(-5\right)^{-3}=\dfrac{1}{\left(-5\right)^3}=\dfrac{1}{-125}=-\dfrac{1}{125}$; D

(d) $-\left(-5\right)^{-3}=-\dfrac{1}{\left(-5\right)^3}=-\dfrac{1}{-125}=\dfrac{1}{125}$; C

3. $\left(-4\right)^{-3}=\dfrac{1}{\left(-4\right)^3}=\dfrac{1}{-64}=-\dfrac{1}{64}$

4. $\left(-5\right)^{-2}=\dfrac{1}{\left(-5\right)^2}=\dfrac{1}{25}$

5. $-\left(-5\right)^{-4}=-\dfrac{1}{\left(-5\right)^4}=-\dfrac{1}{625}$

6. $-7^{-2}=-\left(7^{-2}\right)=-\dfrac{1}{7^2}=-\dfrac{1}{49}$

7. $\left(\dfrac{1}{3}\right)^{-2}=\dfrac{1}{\left(\frac{1}{3}\right)^2}=\dfrac{1}{\frac{1}{9}}=9$, also

$\left(\dfrac{1}{3}\right)^{-2}=\dfrac{1}{\left(\frac{1}{3}\right)^2}=\left(\dfrac{3}{1}\right)^2=3^2=9$

8. $\left(\dfrac{4}{3}\right)^{-3}=\dfrac{1}{\left(\frac{4}{3}\right)^3}=\dfrac{1}{\frac{64}{27}}=\dfrac{27}{64}$, also

$\left(\dfrac{4}{3}\right)^{-3}=\dfrac{1}{\left(\frac{4}{3}\right)^3}=\left(\dfrac{3}{4}\right)^3=\dfrac{27}{64}$

9. $(4x)^{-2}=\dfrac{1}{(4x)^2}=\dfrac{1}{16x^2}$

10. $(5t)^{-3}=\dfrac{1}{(5t)^3}=\dfrac{1}{125t^3}$

11. $4x^{-2}=4\cdot x^{-2}=4\cdot\dfrac{1}{x^2}=\dfrac{4}{x^2}$

12. $5t^{-3}=5\cdot t^{-3}=5\cdot\dfrac{1}{t^3}=\dfrac{5}{t^3}$

13. $-a^{-3}=-\dfrac{1}{a^3}$

14. $-b^{-4}=-\dfrac{1}{b^4}$

15. $\dfrac{4^8}{4^6}=4^{8-6}=4^2=16$

16. $\dfrac{5^9}{5^7}=5^{9-7}=5^2=25$

17. $\dfrac{x^{12}}{x^8}=x^{12-8}=x^4$

18. $\dfrac{y^{14}}{y^{10}}=y^{14-10}=y^4$

19. $\dfrac{r^7}{r^{10}}=r^{7-10}=r^{-3}=\dfrac{1}{r^3}$

20. $\dfrac{y^8}{y^{12}}=y^{8-12}=y^{-4}=\dfrac{1}{y^4}$

21. $\dfrac{6^4}{6^{-2}} = 6^{4-(-2)} = 6^6$

Because 6^6 is a relatively large number, it is generally acceptable not to simplify it to be 46,656.

22. $\dfrac{7^5}{7^{-3}} = 7^{5-(-3)} = 7^8$

Because 7^8 is a relatively large number, it is generally acceptable not to simplify it to be 5,764,801.

23. $\dfrac{4r^{-3}}{6r^{-6}} = \dfrac{4}{6} \cdot \dfrac{r^{-3}}{r^{-6}} = \dfrac{2}{3} r^{-3-(-6)} = \dfrac{2}{3} r^3 = \dfrac{2r^3}{3}$

24. $\dfrac{15s^{-4}}{5s^{-8}} = \dfrac{15}{5} \cdot \dfrac{s^{-4}}{s^{-8}} = 3s^{-4-(-8)} = 3s^4$

25. $\dfrac{16m^{-5}n^4}{12m^2n^{-3}} = \dfrac{16}{12} \cdot \dfrac{m^{-5}}{m^2} \cdot \dfrac{n^4}{n^{-3}} = \dfrac{4}{3} m^{-5-2} n^{4-(-3)}$
$= \dfrac{4}{3} m^{-7} n^7 = \dfrac{4n^7}{3m^7}$

26. $\dfrac{15a^{-5}b^{-1}}{25a^{-2}b^4} = \dfrac{15}{25} \cdot \dfrac{a^{-5}}{a^{-2}} \cdot \dfrac{b^{-1}}{b^4} = \dfrac{3}{5} a^{-5-(-2)} b^{-1-4}$
$= \dfrac{3}{5} a^{-3} b^{-5} = \dfrac{3}{5a^3b^5}$

27. $-4r^{-2}\left(r^4\right)^2 = -4r^{-2}r^8 = -4r^{-2+8} = -4r^6$

28. $-2m^{-1}\left(m^3\right)^2 = -2m^{-1}m^6 = -2m^{-1+6} = -2m^5$

29. $\left(5a^{-1}\right)^4\left(a^2\right)^{-3} = 5^4 a^{-4} a^{-6} = 5^4 a^{-4+(-6)}$
$= 5^4 a^{-10} = \dfrac{5^4}{a^{10}}$

Because 5^4 is a relatively large number, it is generally acceptable not to simplify to $\dfrac{625}{a^{10}}$.

30. $\left(3p^{-4}\right)^2\left(p^3\right)^{-1} = 3^2 p^{-8} p^{-3} = 3^2 p^{-8+(-3)}$
$= 3^2 p^{-11} = \dfrac{3^2}{p^{11}}$ or $\dfrac{9}{p^{11}}$

31. $\dfrac{\left(p^{-2}\right)^0}{5p^{-4}} = \dfrac{p^0}{5p^{-4}} = \dfrac{1}{5} \cdot \dfrac{p^0}{p^{-4}}$
$= \dfrac{1}{5} p^{0-(-4)} = \dfrac{1}{5} p^4 = \dfrac{p^4}{5}$

32. $\dfrac{\left(m^4\right)^0}{9m^{-3}} = \dfrac{m^0}{9m^{-3}} = \dfrac{1}{9} \cdot \dfrac{m^0}{m^{-3}}$
$= \dfrac{1}{9} m^{0-(-3)} = \dfrac{1}{9} m^3 = \dfrac{m^3}{9}$

33. $\dfrac{(3pq)q^2}{6p^2q^4} = \dfrac{3pq^{1+2}}{6p^2q^4} = \dfrac{3pq^3}{6p^2q^4}$
$= \dfrac{3}{6} \cdot \dfrac{p}{p^2} \cdot \dfrac{q^3}{q^4} = \dfrac{1}{2} p^{1-2} q^{3-4}$
$= \dfrac{1}{2} p^{-1} q^{-1} = \dfrac{1}{2pq}$

34. $\dfrac{(-8xy)y^3}{4x^5y^4} = \dfrac{-8xy^{1+3}}{4x^5y^4} = \dfrac{-8xy^4}{4x^5y^4}$
$= \dfrac{-8}{4} \cdot \dfrac{x}{x^5} \cdot \dfrac{y^4}{y^4} = -2x^{1-5} y^{4-4}$
$= -2x^{-4} y^0 = -2x^{-4} \cdot 1 = -\dfrac{2}{x^4}$

35. $\dfrac{4a^5\left(a^{-1}\right)^3}{\left(a^{-2}\right)^{-2}} = \dfrac{4a^5a^{-3}}{a^4} = \dfrac{4a^{5+(-3)}}{a^4}$
$= \dfrac{4a^2}{a^4} = 4a^{2-4} = 4a^{-2} = \dfrac{4}{a^2}$

36. $\dfrac{12k^{-2}\left(k^{-3}\right)^{-4}}{6k^5} = \dfrac{12k^{-2}k^{12}}{6k^5} = \dfrac{12k^{-2+12}}{6k^5}$
$= \dfrac{12k^{10}}{6k^5} = 2k^{10-5} = 2k^5$

37. $169^{1/2} = 13$, because $13^2 = 169$.

38. $121^{1/2} = 11$, because $11^2 = 121$.

39. $16^{1/4} = 2$, because $2^4 = 16$.

40. $625^{1/4} = 5$, because $5^4 = 625$.

41. $\left(-\dfrac{64}{27}\right)^{1/3} = -\dfrac{4}{3}$, because $\left(-\dfrac{4}{3}\right)^3 = -\dfrac{64}{27}$.

42. $\left(\dfrac{8}{27}\right)^{1/3} = \dfrac{2}{3}$, because $\left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27}$.

43. $(-4)^{1/2}$ is not a real number, because no real number, when squared, will yield a negative quantity.

44. $(-64)^{1/4}$ is not a real number, because no real number, when raised to the fourth power, will yield a negative quality.

45. (a) $\left(\frac{4}{9}\right)^{3/2}=\left[\left(\frac{4}{9}\right)^{1/2}\right]^3=\left(\frac{2}{3}\right)^3=\frac{8}{27}$; E

(b) $\left(\frac{4}{9}\right)^{-3/2}=\left(\frac{9}{4}\right)^{3/2}=\left[\left(\frac{9}{4}\right)^{1/2}\right]^3$

$=\left(\frac{3}{2}\right)^3=\frac{27}{8}$; G

(c) $-\left(\frac{9}{4}\right)^{3/2}=-\left[\left(\frac{9}{4}\right)^{1/2}\right]^3$

$=-\left(\frac{3}{2}\right)^3=-\frac{27}{8}$; F

(d) $-\left(\frac{4}{9}\right)^{-3/2}=-\left(\frac{9}{4}\right)^{3/2}=-\left[\left(\frac{9}{4}\right)^{1/2}\right]^3$

$=-\left(\frac{3}{2}\right)^3=-\frac{27}{8}$; F

46. (a) $\left(\frac{8}{27}\right)^{2/3}=\left[\left(\frac{8}{27}\right)^{1/3}\right]^2=\left(\frac{2}{3}\right)^2=\frac{4}{9}$; D

(b) $\left(\frac{8}{27}\right)^{-2/3}=\left(\frac{27}{8}\right)^{2/3}=\left[\left(\frac{27}{8}\right)^{1/3}\right]^2$

$=\left(\frac{3}{2}\right)^2=\frac{9}{4}$; A

(c) $-\left(\frac{27}{8}\right)^{2/3}=-\left[\left(\frac{27}{8}\right)^{1/3}\right]^2$

$=-\left(\frac{3}{2}\right)^2=-\frac{9}{4}$; B

(d) $-\left(\frac{27}{8}\right)^{-2/3}=-\left(\frac{8}{27}\right)^{2/3}=-\left[\left(\frac{8}{27}\right)^{1/3}\right]^2$

$=-\left(\frac{2}{3}\right)^2=-\frac{4}{9}$; C

47. $8^{2/3}=\left(8^{1/3}\right)^2=2^2=4$

48. $27^{4/3}=\left(27^{1/3}\right)^4=3^4=81$

49. $100^{3/2}=\left(100^{1/2}\right)^3=10^3=1000$

50. $64^{3/2}=\left(64^{1/2}\right)^3=8^3=512$

51. $-81^{3/4}=-\left(81^{1/4}\right)^3=-3^3=-27$

52. $(-32)^{-4/5}=\left(-\frac{1}{32}\right)^{4/5}=\left[\left(-\frac{1}{32}\right)^{1/5}\right]^4$

$=\left(-\frac{1}{2}\right)^4=\frac{1}{16}$

53. $\left(\frac{27}{64}\right)^{-4/3}=\left(\frac{64}{27}\right)^{4/3}=\left[\left(\frac{64}{27}\right)^{1/3}\right]^4$

$=\left(\frac{4}{3}\right)^4=\frac{256}{81}$

54. $\left(\frac{121}{100}\right)^{-3/2}=\left(\frac{100}{121}\right)^{3/2}=\left[\left(\frac{100}{121}\right)^{1/2}\right]^3$

$=\left(\frac{10}{11}\right)^3=\frac{1000}{1331}$

55. $3^{1/2}\cdot 3^{3/2}=3^{1/2+3/2}=3^{4/2}=3^2=9$

56. $6^{4/3}\cdot 6^{2/3}=6^{4/3+2/3}=6^{6/3}=6^2=36$

57. $\frac{64^{5/3}}{64^{4/3}}=64^{5/3-4/3}=64^{1/3}=4$

58. $\frac{125^{7/3}}{125^{5/3}}=125^{7/3-5/3}=125^{2/3}$

$=\left(125^{1/3}\right)^2=5^2=25$

59. $y^{7/3}\cdot y^{-4/3}=y^{7/3+(-4/3)}=y^{3/3}=y^1=y$

60. $r^{-8/9}\cdot r^{17/9}=r^{-8/9+17/9}=r^{9/9}=r^1=r$

61. $\frac{k^{1/3}}{k^{2/3}\cdot k^{-1}}=\frac{k^{1/3}}{k^{2/3+(-1)}}=\frac{k^{1/3}}{k^{2/3+(-3/3)}}$

$=\frac{k^{1/3}}{k^{-1/3}}=k^{1/3-(-1/3)}=k^{2/3}$

62. $\frac{z^{3/4}}{z^{5/4}\cdot z^{-2}}=\frac{z^{3/4}}{z^{5/4+(-2)}}=\frac{z^{3/4}}{z^{5/4+(-8/4)}}$

$=\frac{z^{3/4}}{z^{-3/4}}=z^{3/4-(-3/4)}=z^{6/4}=z^{3/2}$

63. $\dfrac{\left(x^{1/4}y^{2/5}\right)^{20}}{x^2} = \dfrac{x^5y^8}{x^2} = x^{5-2}y^8 = x^3y^8$

64. $\dfrac{\left(r^{1/5}s^{2/3}\right)^{15}}{r^2} = \dfrac{r^3s^{10}}{r^2} = r^{3-2}s^{10} = r^1s^{10} = rs^{10}$

65. $\dfrac{\left(x^{2/3}\right)^2}{\left(x^2\right)^{7/3}} = \dfrac{x^{4/3}}{x^{14/3}} = x^{4/3-14/3} = x^{-10/3} = \dfrac{1}{x^{10/3}}$

66. $\dfrac{\left(p^3\right)^{1/4}}{\left(p^{5/4}\right)^2} = \dfrac{p^{3/4}}{p^{10/4}} = p^{3/4-10/4}$
$= p^{-7/4} = \dfrac{1}{p^{7/4}}$

67. $\left(\dfrac{16m^3}{n}\right)^{1/4}\left(\dfrac{9n^{-1}}{m^2}\right)^{1/2} = \dfrac{16^{1/4}m^{3/4}}{n^{1/4}} \cdot \dfrac{9^{1/2}n^{-1/2}}{m}$
$= \dfrac{2m^{3/4}}{n^{1/4}} \cdot \dfrac{3n^{-1/2}}{m}$
$= (2\cdot 3)\dfrac{m^{3/4}}{m} \cdot \dfrac{n^{-1/2}}{n^{1/4}}$
$= 6m^{(3/4)-1}n^{(-1/2)-(1/4)}$
$= 6m^{(3/4)-(4/4)}n^{(-2/4)-(1/4)}$
$= 6m^{-1/4}n^{-3/4} = \dfrac{6}{m^{1/4}n^{3/4}}$

68. $\left(\dfrac{25^4a^3}{b^2}\right)^{1/8}\left(\dfrac{4^2b^{-5}}{a^2}\right)^{1/4}$
$= \dfrac{25^{1/2}a^{3/8}}{b^{1/4}} \cdot \dfrac{4^{1/2}b^{-5/4}}{a^{1/2}}$
$= \dfrac{5a^{3/8}}{b^{1/4}} \cdot \dfrac{2b^{-5/4}}{a^{1/2}} = (5\cdot 2)\dfrac{a^{3/8}}{a^{1/2}} \cdot \dfrac{b^{-5/4}}{b^{1/4}}$
$= 10a^{(3/8)-(1/2)}b^{(-5/4)-(1/4)}$
$= 10a^{(3/8)-(4/8)}b^{-6/4} = 10a^{-1/8}b^{-3/2}$
$= \dfrac{10}{a^{1/8}b^{3/2}}$

69. $\dfrac{p^{1/5}p^{7/10}p^{1/2}}{\left(p^3\right)^{-1/5}} = \dfrac{p^{1/5+7/10+1/2}}{p^{-3/5}}$
$= \dfrac{p^{2/10+7/10+5/10}}{p^{-6/10}} = \dfrac{p^{14/10}}{p^{-6/10}}$
$= p^{(14/10)-(-6/10)}$
$= p^{20/10} = p^2$

70. $\dfrac{z^{1/3}z^{-2/3}z^{1/6}}{\left(z^{-1/6}\right)^3} = \dfrac{z^{1/3+(-2/3)+1/6}}{z^{-3/6}}$
$= \dfrac{z^{2/6-4/6+1/6}}{z^{-3/6}} = \dfrac{z^{-1/6}}{z^{-3/6}}$
$= z^{(-1/6)-(-3/6)} = z^{2/6} = z^{1/3}$

71. **(a)** Let $w = 25$
$t = \dfrac{31,293}{w^{1.5}} = \dfrac{31,293}{(25)^{3/2}} = \dfrac{31,293}{(25^{1/2})^3}$
$= \dfrac{31,293}{(5)^3} = \dfrac{31,293}{125} \approx 250$ sec

(b) To double the weight, replace w with $2w$ to get $\dfrac{31,293}{(2w)^{1.5}} = \dfrac{31,293}{2^{3/2}w^{1.5}} = \dfrac{1}{2^{3/2}}(t)$; so the holding time changes by a factor of $\dfrac{1}{2^{3/2}} \approx .3536$.

72. **(a)** Let $D = 4$
$T = .07D^{3/2} = .07(4)^{3/2}$
$= .07(4^{1/2})^3 = .07(2)^3 = .07(8) = .56$
The storm will last approximately 0.56 hours or almost 34 minutes $(.56\cdot 60 = 33.6)$.

(b) Let $D = 7$
$T = .07D^{3/2} = .07(7)^{3/2} \approx 1.296$
No; since 1.296 hours < 1.5 hours.

73. $y^{5/8}(y^{3/8} - 10y^{11/8}) = y^{5/8}y^{3/8} - 10y^{5/8}y^{11/8}$
$= y^{5/8+3/8} - 10y^{5/8+11/8}$
$= y - 10y^2$

74. $p^{11/5}(3p^{4/5} + 9p^{19/5})$
$= 3p^{4/5}\cdot p^{11/5} + 9\cdot p^{19/5}\cdot p^{11/5}$
$= 3p^{15/5} + 9p^{30/5} = 3p^3 + 9p^6$

75. $-4k(k^{7/3} - 6k^{1/3}) = -4k^1k^{7/3} + 24k^1k^{1/3}$
$= -4k^{10/3} + 24k^{4/3}$

76. $-5y(3y^{9/10} + 4y^{3/10}) = -15y^{19/10} - 20y^{13/10}$

77. $(x + x^{1/2})(x - x^{1/2})$ has the form $(a-b)(a+b) = a^2 - b^2$.
$(x + x^{1/2})(x - x^{1/2}) = x^2 - (x^{1/2})^2 = x^2 - x$

78. $(2z^{1/2} + z)(z^{1/2} - z) = 2z - 2z^{3/2} + z^{3/2} - z^2$
$= 2z - z^{3/2} - z^2$

79. $\left(r^{1/2}-r^{-1/2}\right)^2$

$=\left(r^{1/2}\right)^2-2\left(r^{1/2}\right)\left(r^{-1/2}\right)+\left(r^{-1/2}\right)^2$

$=r-2r^{(1/2)+(-1/2)}+r^{-1}=r-2r^0+r^{-1}$

$=r-2\cdot 1+r^{-1}=r-2+r^{-1}=r-2+\frac{1}{r}$

80. $(p^{1/2}-p^{-1/2})(p^{1/2}+p^{-1/2})$ has the form $(a-b)(a+b)=a^2-b^2$.

$(p^{1/2}-p^{-1/2})(p^{1/2}+p^{-1/2})$

$=\left(p^{1/2}\right)^2-\left(p^{-1/2}\right)^2=p-p^{-1}$ or $p-\frac{1}{p}$

81. Factor $4k^{-1}+k^{-2}$, using the common factor k^{-2}: $4k^{-1}+k^{-2}=k^{-2}(4k+1)$

82. Factor $y^{-5}-3y^{-3}$, using the common factor y^{-5}: $y^{-5}-3y^{-3}=y^{-5}(1-3y^2)$

83. Factor $4t^{-2}+8t^{-4}$, using the common factor $4t^{-4}$: $4t^{-2}+8t^{-4}=4t^{-4}(t^2+2)$

84. Factor $5r^{-6}-10r^{-8}$, using the common factor $5r^{-8}$: $5r^{-6}-10r^{-8}=5r^{-8}(r^2-2)$

85. Factor $9z^{-1/2}+2z^{1/2}$, using the common factor $z^{-1/2}$:

$9z^{-1/2}+2z^{1/2}=z^{-1/2}(9+2z)$

86. Factor $3m^{2/3}-4m^{-1/3}$, using the common factor $m^{-1/3}$:

$3m^{2/3}-4m^{-1/3}=m^{-1/3}(3m^{3/3}-4)$

$=m^{-1/3}(3m-4)$

87. Factor $p^{-3/4}-2p^{-7/4}$, using the common factor $p^{-7/4}$:

$p^{-3/4}-2p^{-7/4}=p^{-7/4}(p^{4/4}-2)$

$=p^{-7/4}(p-2)$

88. Factor $6r^{-2/3}-5r^{-5/3}$, using the common factor $r^{-5/3}$:

$6r^{-2/3}-5r^{-5/3}=r^{-5/3}(6r^{3/3}-5)$

$=r^{-5/3}(6r-5)$

89. Factor $-4a^{-2/5}+16a^{-7/5}$, using the common factor $4a^{-7/5}$:

$-4a^{-2/5}+16a^{-7/5}=4a^{-7/5}(-a+4)$

90. Factor: $-3p^{-3/4}-30p^{-7/4}$, using the common factor $-3p^{-7/4}$:

$-3p^{-3/4}-30p^{-7/4}=-3p^{-7/4}(p+10)$

91. Factor $(p+4)^{-3/2}+(p+4)^{-1/2}+(p+4)^{1/2}$, using the common factor $(p+4)^{-3/2}$.

$(p+4)^{-3/2}+(p+4)^{-1/2}+(p+4)^{1/2}$

$=(p+4)^{-3/2}\cdot[1+(p+4)+(p+4)^2]$

$=(p+4)^{-3/2}\cdot(1+p+4+p^2+8p+16)$

$=(p+4)^{-3/2}(p^2+9p+21)$

92. Factor $(3r+1)^{-2/3}+(3r+1)^{1/3}+(3r+1)^{4/3}$, using the common factor $(3r+1)^{-2/3}$.

$(3r+1)^{-2/3}+(3r+1)^{1/3}+(3r+1)^{4/3}$

$=(3r+1)^{-2/3}\cdot[1+(3r+1)^{3/3}+(3r+1)^{6/3}]$

$=(3r-1)^{-2/3}\cdot[1+(3r+1)+(3r+1)^2]$

$=(3r+1)^{-2/3}\cdot(1+3r+1+9r^2+6r+1)$

$=(3r+1)^{-2/3}(9r^2+9r+3)$

$=3(3r+1)^{-2/3}(3r^2+3r+1)$

93. Factor

$2(3x+1)^{-3/2}+4(3x+1)^{-1/2}+6(3x+1)^{1/2}$

using the common factor $2(3x+1)^{-3/2}$:

$2\left(3x+1\right)^{-3/2}+4\left(3x+1\right)^{-1/2}+6\left(3x+1\right)^{1/2}$

$=2\left(3x+1\right)^{-3/2}\left[1+2\left(3x+1\right)+3\left(3x+1\right)^2\right]$

$=2\left(3x+1\right)^{-3/2}\left[1+\left(6x+2\right)+3\left(9x^2+6x+1\right)\right]$

$=2\left(3x+1\right)^{-3/2}\left[1+6x+2+27x^2+18x+3\right]$

$=2\left(3x+1\right)^{-3/2}\left(27x^2+24x+6\right)$

$=2\left(3x+1\right)^{-3/2}\cdot 3\left(9x^2+8x+2\right)$

$=6\left(3x+1\right)^{-3/2}\left(9x^2+8x+2\right)$

94. Factor
$7(5t+3)^{-5/3}+14(5t+3)^{-2/3}-21(5t+3)^{1/3}$ using the common factor $7(5t+3)^{-5/3}$:

$$\begin{aligned}&7(5t+3)^{-5/3}+14(5t+3)^{-2/3}-21(5t+3)^{1/3}\\&=7(5t+3)^{-5/3}\left[1+2(5t+3)-3(5t+3)^2\right]\\&=7(5t+3)^{-5/3}\left[1+(10t+6)-3(25t^2+30t+9)\right]\\&=7(5t+3)^{-5/3}\left[1+10t+6-75t^2-90t-27\right]\\&=7(5t+3)^{-5/3}\left[-75t^2-80t-20\right]\\&=7(5t+3)^{-5/3}\cdot(-5)\left[15t^2+16t+4\right]\\&=-35(5t+3)^{-5/3}(15t^2+16t+4)\end{aligned}$$

95. Factor
$4x(2x+3)^{-5/9}+6x^2(2x+3)^{4/9}-8x^3(2x+3)^{13/9}$ using the common factor $2x(2x+3)^{-5/9}$:

$$\begin{aligned}&4x(2x+3)^{-5/9}+6x^2(2x+3)^{4/9}-8x^3(2x+3)^{13/9}\\&=2x(2x+3)^{-5/9}\left[2+3x(2x+3)-4x^2(2x+3)^2\right]\\&=2x(2x+3)^{-5/9}\\&\quad\cdot\left[2+(6x^2+9x)-4x^2(4x^2+12x+9)\right]\\&=2x(2x+3)^{-5/9}\\&\quad\cdot\left[2+6x^2+9x-16x^4-48x^3-36x^2\right]\\&=2x(2x+3)^{-5/9}\left[-16x^4-48x^3-30x^2+9x+2\right]\end{aligned}$$

96. Factor
$6y^3(4y-1)^{-3/7}-8y^2(4y-1)^{4/7}+16y(4y-1)^{11/7}$ using the common factor $2y(4y-1)^{-3/7}$:

$$\begin{aligned}&6y^3(4y-1)^{-3/7}-8y^2(4y-1)^{4/7}+16y(4y-1)^{11/7}\\&=2y(4y-1)^{-3/7}\left[3y^2-4y(4y-1)+8(4y-1)^2\right]\\&=2y(4y-1)^{-3/7}\\&\quad\cdot\left[3y^2-16y^2+4y+8(16y^2-8y+1)\right]\\&=2y(4y-1)^{-3/7}\\&\quad\cdot\left[3y^2-16y^2+4y+128y^2-64y+8\right]\\&=2y(4y-1)^{-3/7}\left[115y^2-60y+8\right]\end{aligned}$$

97. $$\frac{a^{-1}+b^{-1}}{(ab)^{-1}}=\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{ab}}=\frac{\frac{1\cdot b}{a\cdot b}+\frac{1\cdot a}{b\cdot a}}{\frac{1}{ab}}=\frac{\frac{b+a}{ab}}{\frac{1}{ab}}=\frac{b+a}{ab}\cdot\frac{ab}{1}=b+a$$

98. $$\frac{p^{-1}-q^{-1}}{(pq)^{-1}}=\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{pq}}=\frac{\frac{1\cdot q}{p\cdot q}-\frac{1\cdot p}{q\cdot p}}{\frac{1}{pq}}=\frac{\frac{q-p}{pq}}{\frac{1}{pq}}=\frac{q-p}{pq}\cdot\frac{pq}{1}=q-p$$

99. $$\begin{aligned}\frac{r^{-1}+q^{-1}}{r^{-1}-q^{-1}}\cdot\frac{r-q}{r+q}&=\frac{\frac{1}{r}+\frac{1}{q}}{\frac{1}{r}-\frac{1}{q}}\cdot\frac{r-q}{r+q}\\&=\frac{rq\left(\frac{1}{r}+\frac{1}{q}\right)}{rq\left(\frac{1}{r}-\frac{1}{q}\right)}\cdot\frac{r-q}{r+q}\\&=\frac{q+r}{q-r}\cdot\frac{r-q}{r+q}=\frac{r-q}{q-r}\\&=\frac{r-q}{-1(r-q)}=-1\end{aligned}$$

100. $$\begin{aligned}\frac{xy^{-1}+yx^{-1}}{x^2+y^2}&=\frac{\frac{x}{y}+\frac{y}{x}}{x^2+y^2}=\frac{\frac{x^2}{xy}+\frac{y^2}{xy}}{x^2+y^2}=\frac{\frac{x^2+y^2}{xy}}{x^2+y^2}\\&=\frac{x^2+y^2}{xy}\cdot\frac{1}{x^2+y^2}=\frac{1}{xy}\end{aligned}$$

101. $$\begin{aligned}\frac{x-9y^{-1}}{(x-3y^{-1})(x+3y^{-1})}&=\frac{x-\frac{9}{y}}{\left(x-\frac{3}{y}\right)\left(x+\frac{3}{y}\right)}\\&=\frac{x-\frac{9}{y}}{x^2-\frac{9}{y^2}}=\frac{y^2\left(x-\frac{9}{y}\right)}{y^2\left(x^2-\frac{9}{y^2}\right)}\\&=\frac{y^2x-9y}{y^2x^2-9}\text{ or }\frac{y(xy-9)}{x^2y^2-9}\end{aligned}$$

102. $$\begin{aligned}\frac{(m+n)^{-1}}{m^{-2}-n^{-2}}&=\frac{\frac{1}{m+n}}{\frac{1}{m^2}-\frac{1}{n^2}}=\frac{\frac{1}{m+n}}{\frac{n^2}{m^2n^2}-\frac{m^2}{m^2n^2}}=\frac{\frac{1}{m+n}}{\frac{n^2-m^2}{m^2n^2}}\\&=\frac{1}{m+n}\cdot\frac{m^2n^2}{n^2-m^2}\\&=\frac{1}{m+n}\cdot\frac{m^2n^2}{(n-m)(n+m)}\\&=\frac{m^2n^2}{(n-m)(n+m)^2}\end{aligned}$$

103. $$\frac{(x^2+1)^4(2x)-x^2(4)(x^2+1)^3(2x)}{(x^2+1)^8}$$
$$=\frac{(2x)(x^2+1)^3\left[(x^2+1)-4x^2\right]}{(x^2+1)^8}$$
$$=\frac{(2x)\left[(x^2+1)-4x^2\right]}{(x^2+1)^5}=\frac{(2x)(1-3x^2)}{(x^2+1)^5}$$

104. $$\frac{(y^2+2)^5(3y)-y^3(6)(y^2+2)^4(3y)}{(y^2+2)^7}$$
$$=\frac{(3y)(y^2+2)^4\left[(y^2+2)-6y^3\right]}{(y^2+2)^7}$$
$$=\frac{3y(y^2+2-6y^3)}{(y^2+2)^3}$$

105. $$\frac{4(x^2-1)^3+8x(x^2-1)^4}{16(x^2-1)^3}$$
$$=\frac{4(x^2-1)^3\left[1+2x(x^2-1)\right]}{16(x^2-1)^3}$$
$$=\frac{1+2x(x^2-1)}{4}=\frac{1+2x^3-2x}{4}$$

106. $$\frac{10(4x^2-9)^2-25x(4x^2-9)^3}{15(4x^2-9)^6}$$
$$=\frac{5(4x^2-9)^2\left[2-5x(4x^2-9)\right]}{15(4x^2-9)^6}$$
$$=\frac{2-5x(4x^2-9)}{3(4x^2-9)^4}=\frac{2-20x^3+45x}{3(4x^2-9)^4}$$

107. $$\frac{2(2x-3)^{1/3}-(x-1)(2x-3)^{-2/3}}{(2x-3)^{2/3}}$$
$$=\frac{(2x-3)^{1/3}\left[2-(x-1)(2x-3)^{-1}\right]}{(2x-3)^{2/3}}$$
$$=\frac{2-(x-1)(2x-3)^{-1}}{(2x-3)^{1/3}}=\frac{2-\frac{x-1}{2x-3}}{(2x-3)^{1/3}}$$
$$=\frac{2(2x-3)-(x-1)}{(2x-3)^{4/3}}=\frac{3x-5}{(2x-3)^{4/3}}$$

108. $$\frac{7(3t+1)^{1/4}-(t-1)(3t+1)^{-3/4}}{(3t+1)^{3/4}}$$
$$=\frac{(3t+1)^{1/4}\left[7-(t-1)(3t+1)^{-1}\right]}{(3t+1)^{3/4}}$$
$$=\frac{7-(t-1)(3t+1)^{-1}}{(3t+1)^{1/2}}=\frac{7-\frac{t-1}{3t+1}}{(3t+1)^{1/2}}$$
$$=\frac{7(3t+1)-(t-1)}{(3t+1)^{3/2}}=\frac{20t+8}{(3t+1)^{3/2}}=\frac{4(5t+2)}{(3t+1)^{3/2}}$$

109. $a^7=30\Rightarrow(a^7)^3=30^3\Rightarrow a^{21}=27,000$

110.
$$a^{-3}=.2$$
$$(a^{-3})^{-2}=(.2)^{-2}$$
$$a^{(-3)(-2)}=\frac{1}{.2^2}\Rightarrow a^6=\frac{1}{.04}\Rightarrow a^6=25$$

111. Let x = length of side of cube. Then $3x$ = length of side of bigger cube (side tripled). x^3 is the volume of the cube, and $(3x)^3=3^3x^3=27x^3$ is the volume of the bigger cube. The volume will change by a factor of 27.

112. Let r = radius of circle. Then $2r$ = radius of the bigger circle (radius doubled). πr^2 is the area of the circle, and $\pi(2r)^2=\pi\cdot 2^2r^2=\pi\cdot 4r^2=4\pi r^2$ is the area of the bigger circle. The area will change by a factor of 4.

113. $.2^{2/3}\cdot 40^{2/3}=(.2\cdot 40)^{2/3}=(8^{1/3})^2=2^2=4$

114. $.1^{3/2}\cdot 90^{3/2}=(.1\cdot 90)^{3/2}=9^{3/2}$
$=(9^{1/2})^3=3^3=27$

115. $$\frac{2^{2/3}}{2000^{2/3}}=\left(\frac{2}{2000}\right)^{2/3}$$
$$=\left(\frac{1}{1000}\right)^{2/3}\left[\left(\frac{1}{1000}\right)^{1/3}\right]^2$$
$$=\left(\frac{1}{10}\right)^2=\frac{1}{100}$$

116. $$\frac{20^{3/2}}{5^{3/2}}=\left(\frac{20}{5}\right)^{3/2}=4^{3/2}=(4^{1/2})^3=2^3=8$$

Section R.7: Radical Expressions

1. **(a)** F; $(-3x)^{1/3} = \sqrt[3]{-3x}$

   (b) H; $(-3x)^{-1/3} = \dfrac{1}{(-3x)^{1/3}} = \dfrac{1}{\sqrt[3]{-3x}}$

   (c) G; $(3x)^{1/3} = \sqrt[3]{3x}$

   (d) C; $(3x)^{-1/3} = \dfrac{1}{\sqrt[3]{3x}}$

2. **(a)** B; $-3x^{1/3} = -3\sqrt[3]{x}$

   (b) D; $-3x^{-1/3} = \dfrac{-3}{x^{1/3}} = \dfrac{-3}{\sqrt[3]{x}}$

   (c) A; $3x^{-1/3} = \dfrac{3}{x^{1/3}} = \dfrac{3}{\sqrt[3]{x}}$

   (d) E; $3x^{1/3} = 3\sqrt[3]{x}$

3. $m^{2/3} = \sqrt[3]{m^2}$ or $\left(\sqrt[3]{m}\right)^2$

4. $p^{5/4} = \sqrt[4]{p^5}$ or $\left(\sqrt[4]{\text{p}}\right)^5$

5. $(2m+p)^{2/3} = \sqrt[3]{(2m+p)^2}$ or $\left(\sqrt[3]{2m+p}\right)^2$

6. $(5r+3t)^{4/7} = \sqrt[7]{(5r+3t)^4}$ or $\left(\sqrt[7]{5r+3t}\right)^4$

7. $\sqrt[5]{k^2} = k^{2/5}$

8. $-\sqrt[4]{z^5} = -z^{5/4}$

9. $-3\sqrt{5p^3} = -3(5p^3)^{1/2} = -3 \cdot 5^{1/2} p^{3/2}$

10. $m\sqrt{2y^5} = m(2y^5)^{1/2} = 2^{1/2} my^{5/2}$

11. A

12. 3, 5, 7, … (odd positive integers greater than or equal to 3)

13. It is true for all $x \geq 0$.

14. Expression (D) is not simplified, because the radicand is a fraction. This expression may be simplified by using the rule $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$:

    $\sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{\sqrt{4}} = \dfrac{\sqrt{3}}{2}$

15. $\sqrt{(-5)^2} = |-5| = 5$

16. $\sqrt[6]{x^6} = |x|$

17. $\sqrt{25k^4m^2} = \sqrt{\left(5k^2m\right)^2} = \left|5k^2m\right| = 5k^2\,|m|$

18. $\sqrt[4]{81p^{12}q^4} = \sqrt[4]{\left(3p^3q\right)^4} = \left|3p^3q\right| = 3\left|p^3q\right|$

19. $\sqrt{(4x-y)^2} = |4x-y|$

20. $\sqrt[4]{(5+2m)^4} = |5+2m|$

21. $\sqrt[3]{125} = 5$

22. $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$

23. This is not a real number since no real number raised to the fourth power will yield a negative quantity.

24. This is not a real number since no real number raised to the sixth power will yield a negative quantity.

25. $\sqrt[3]{81} = \sqrt[3]{27 \cdot 3} = \sqrt[3]{27} \cdot \sqrt[3]{3} = 3\sqrt[3]{3}$

26. $\sqrt[3]{250} = \sqrt[3]{125 \cdot 2} = \sqrt[3]{125} \cdot \sqrt[3]{2} = 5\sqrt[3]{2}$

27. $-\sqrt[4]{32} = -\sqrt[4]{16 \cdot 2} = -\sqrt[4]{16} \cdot \sqrt[4]{2} = -2\sqrt[4]{2}$

28. $-\sqrt[4]{243} = -\sqrt[4]{81 \cdot 3} = -3\sqrt[4]{3}$

29. $\sqrt{14} \cdot \sqrt{3pqr} = \sqrt{14 \cdot 3pqr} = \sqrt{42pqr}$

30. $\sqrt{7} \cdot \sqrt{5xt} = \sqrt{7 \cdot 5xt} = \sqrt{35xt}$

31. $\sqrt[3]{7x} \cdot \sqrt[3]{2y} = \sqrt[3]{7x \cdot 2y} = \sqrt[3]{14xy}$

32. $\sqrt[3]{9x} \cdot \sqrt[3]{4y} = \sqrt[3]{9x \cdot 4y} = \sqrt[3]{36xy}$

33. $-\sqrt{\dfrac{9}{25}} = -\dfrac{\sqrt{9}}{\sqrt{25}} = -\dfrac{3}{5}$

34. $-\sqrt{\dfrac{12}{49}} = -\dfrac{\sqrt{12}}{\sqrt{49}} = -\dfrac{\sqrt{4 \cdot 3}}{7}$

    $= -\dfrac{\sqrt{4} \cdot \sqrt{3}}{7} = -\dfrac{2\sqrt{3}}{7}$

35. $-\sqrt[3]{\dfrac{5}{8}} = -\dfrac{\sqrt[3]{5}}{\sqrt[3]{8}} = -\dfrac{\sqrt[3]{5}}{2}$

36. $\sqrt[4]{\frac{3}{16}} = \frac{\sqrt[4]{3}}{\sqrt[4]{16}} = \frac{\sqrt[4]{3}}{2}$

37. $\sqrt[4]{\frac{m}{n^4}} = \frac{\sqrt[4]{m}}{\sqrt[4]{n^4}} = \frac{\sqrt[4]{m}}{n}$

38. $\sqrt[6]{\frac{r}{s^6}} = \frac{\sqrt[6]{r}}{\sqrt[6]{s^6}} = \frac{\sqrt[6]{r}}{s}$

39. $3\sqrt[5]{-3125} = 3\sqrt[5]{(-5)^5} = 3(-5) = -15$

40. $5\sqrt[3]{343} = 5\sqrt[3]{7^3} = 5(7) = 35$

41. $\sqrt[3]{16(-2)^4(2)^8} = \sqrt[3]{2^4 \cdot (-2)^4 2^8} = \sqrt[3]{2^4 \cdot 2^4 \cdot 2^8}$
$= \sqrt[3]{2^{15} \cdot 2} = \sqrt[3]{2^{15}} \cdot \sqrt[3]{2}$
$= 2^5 \cdot \sqrt[3]{2} = 32\sqrt[3]{2}$

42. $\sqrt[3]{25(3)^4(5)^3} = \sqrt[3]{5^2 \cdot 3^3 \cdot 3 \cdot 5^3}$
$= \sqrt[3]{3^3 \cdot 5^3} \cdot \sqrt[3]{5^2 \cdot 3} = 3 \cdot 5 \cdot \sqrt[3]{5^2 \cdot 3}$
$= 15\sqrt[3]{25 \cdot 3} = 15\sqrt[3]{75}$

43. $\sqrt{8x^5z^8} = \sqrt{2 \cdot 4 \cdot x^4 \cdot x \cdot z^8}$
$= \sqrt{4x^4z^8} \cdot \sqrt{2x} = 2x^2z^4\sqrt{2x}$

44. $\sqrt{24m^6n^5} = \sqrt{2^2 \cdot 6 \cdot m^6 \cdot n^4 \cdot n}$
$= \sqrt{2^2m^6 \cdot n^4} \cdot \sqrt{6n} = 2m^3n^2\sqrt{6n}$

45. $\sqrt[4]{x^4 + y^4}$ cannot be simplified further.

46. $\sqrt[3]{27 + a^3}$ cannot be simplified further.

47. $\sqrt{\frac{2}{3x}} = \frac{\sqrt{2}}{\sqrt{3x}} = \frac{\sqrt{2}}{\sqrt{3x}} \cdot \frac{\sqrt{3x}}{\sqrt{3x}} = \frac{\sqrt{2 \cdot 3x}}{\sqrt{9x^2}} = \frac{\sqrt{6x}}{3x}$

48. $\sqrt{\frac{5}{3p}} = \frac{\sqrt{5}}{\sqrt{3p}} = \frac{\sqrt{5} \cdot \sqrt{3p}}{\sqrt{3p} \cdot \sqrt{3p}} = \frac{\sqrt{5 \cdot 3p}}{\sqrt{9p^2}} = \frac{\sqrt{15p}}{3p}$

49. $\sqrt{\frac{x^5y^3}{z^2}} = \frac{\sqrt{x^5y^3}}{\sqrt{z^2}} = \frac{\sqrt{x^4xy^2y}}{z}$
$= \frac{\sqrt{x^4y^2} \cdot \sqrt{xy}}{z} = \frac{x^2y\sqrt{xy}}{z}$

50. $\sqrt{\frac{g^3h^5}{r^3}} = \frac{\sqrt{g^3h^5}}{\sqrt{r^3}} = \frac{\sqrt{g^3h^5}}{\sqrt{r^3}} \cdot \frac{\sqrt{r}}{\sqrt{r}} = \frac{\sqrt{g^3h^5r}}{\sqrt{r^4}}$
$= \frac{\sqrt{g^2h^4} \cdot \sqrt{ghr}}{r^2} = \frac{gh^2\sqrt{ghr}}{r^2}$

51. $\sqrt[3]{\frac{8}{x^2}} = \frac{\sqrt[3]{8}}{\sqrt[3]{x^2}} = \frac{2}{\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x}}{x}$

52. $\sqrt[3]{\frac{9}{16p^4}} = \frac{\sqrt[3]{9}}{\sqrt[3]{16p^4}} = \frac{\sqrt[3]{9}}{\sqrt[3]{8p^3} \cdot \sqrt[3]{2p}} = \frac{\sqrt[3]{9}}{2p\sqrt[3]{2p}}$
$= \frac{\sqrt[3]{9}}{2p\sqrt[3]{2p}} \cdot \frac{\sqrt[3]{4p^2}}{\sqrt[3]{4p^2}} = \frac{\sqrt[3]{36p^2}}{2p\sqrt[3]{8p^3}}$
$= \frac{\sqrt[3]{36p^2}}{2p \cdot 2p} = \frac{\sqrt[3]{36p^2}}{4p^2}$

53. $\sqrt[4]{\frac{g^3h^5}{9r^6}} = \frac{\sqrt[4]{g^3h^5}}{\sqrt[4]{9r^6}} = \frac{\sqrt[4]{h^4(g^3h)}}{\sqrt[4]{r^4(9r^2)}} = \frac{h\sqrt[4]{g^3h}}{r\sqrt[4]{9r^2}}$
$= \frac{h\sqrt[4]{g^3h}}{r\sqrt[4]{3^2r^2}} \cdot \frac{\sqrt[4]{3^2r^2}}{\sqrt[4]{3^2r^2}} = \frac{h\sqrt[4]{3^2g^3hr^2}}{r\sqrt[4]{3^4r^4}}$
$= \frac{h\sqrt[4]{9g^3hr^2}}{3r^2}$

54. $\sqrt[4]{\frac{32x^5}{y^5}} = \frac{\sqrt[4]{32x^5}}{\sqrt[4]{y^5}} = \frac{\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}}{\sqrt[4]{y^4} \cdot \sqrt[4]{y}} = \frac{2x\sqrt[4]{2x}}{y\sqrt[4]{y}}$
$= \frac{2x\sqrt[4]{2x}}{y\sqrt[4]{y}} \cdot \frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}} = \frac{2x\sqrt[4]{2xy^3}}{y^2}$

55. $\sqrt[8]{3^4} = 3^{4/8} = 3^{1/2} = \sqrt{3}$

56. $\sqrt[9]{5^3} = 5^{3/9} = 5^{1/3} = \sqrt[3]{5}$

57. $\sqrt[3]{\sqrt{4}} = \sqrt[3]{4^{1/2}} = (4^{1/2})^{1/3} = 4^{1/6}$
$= (2^2)^{1/6} = 2^{2/6} = 2^{1/3} = \sqrt[3]{2}$

58. $\sqrt[4]{\sqrt{25}} = \sqrt[4]{25^{1/2}} = (25^{1/2})^{1/4} = 25^{1/8}$
$= (5^2)^{1/8} = 5^{2/8} = 5^{1/4} = \sqrt[4]{5}$

59. $\sqrt[4]{\sqrt[3]{2}} = \sqrt[4]{2^{1/3}} = (2^{1/3})^{1/4} = 2^{1/12} = \sqrt[12]{2}$

60. $\sqrt[5]{\sqrt[3]{9}} = \sqrt[5]{9^{1/3}} = (9^{1/3})^{1/5} = 9^{1/15} = \sqrt[15]{9}$

61. Cannot be simplified further

62. Cannot be simplified further

63. $8\sqrt{2x} - \sqrt{8x} + \sqrt{72x}$
$= 8\sqrt{2x} - \sqrt{4 \cdot 2x} + \sqrt{36 \cdot 2x}$
$= 8\sqrt{2x} - 2\sqrt{2x} + 6\sqrt{2x} = 12\sqrt{2x}$

64. $4\sqrt{18k}-\sqrt{72k}+\sqrt{50k}$
$=4\sqrt{9\cdot 2k}-\sqrt{36\cdot 2k}+\sqrt{25\cdot 2k}$
$=4(3)\sqrt{2k}-6\sqrt{2k}+5\sqrt{2k}$
$=12\sqrt{2k}-6\sqrt{2k}+5\sqrt{2k}=11\sqrt{2k}$

65. $2\sqrt[3]{3}+4\sqrt[3]{24}-\sqrt[3]{81}$
$=2\sqrt[3]{3}+4\sqrt[3]{8\cdot 3}-\sqrt[3]{27\cdot 3}$
$=2\sqrt[3]{3}+4(2)\sqrt[3]{3}-3\sqrt[3]{3}$
$=2\sqrt[3]{3}+8\sqrt[3]{3}-3\sqrt[3]{3}=7\sqrt[3]{3}$

66. $\sqrt[3]{32}-5\sqrt[3]{4}+2\sqrt[3]{108}$
$=\sqrt[3]{8\cdot 4}-5\sqrt[3]{4}+2\sqrt[3]{27\cdot 4}$
$=2\sqrt[3]{4}-5\sqrt[3]{4}+2(3)\sqrt[3]{4}$
$=2\sqrt[3]{4}-5\sqrt[3]{4}+6\sqrt[3]{4}=3\sqrt[3]{4}$

67. $\sqrt[4]{81x^6y^3}-\sqrt[4]{16x^{10}y^3}$
$=\sqrt[4]{(81x^4)x^2y^3}-\sqrt[4]{(16x^8)x^2y^3}$
$=3x\sqrt[4]{x^2y^3}-2x^2\left(\sqrt[4]{x^2y^3}\right)$

68. $\sqrt[4]{256x^5y^6}-\sqrt[4]{625x^9y^2}$
$=\sqrt[4]{(256x^4y^4)xy^2}-\sqrt[4]{(625x^8)xy^2}$
$=4xy\sqrt[4]{xy^2}-5x^2\left(\sqrt[4]{xy^2}\right)$

69. This product has the pattern $(a+b)(a-b)=a^2-b^2,$ the difference of squares.
$\left(\sqrt{2}+3\right)\left(\sqrt{2}-3\right)=\left(\sqrt{2}\right)^2-3^2=2-9=-7$

70. This product has the pattern $(a+b)(a-b)=a^2-b^2,$ the difference of squares.
$\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{2}\right)=\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2$
$=5-2=3$

71. This product has the pattern $(a-b)(a^2+ab+b^2)=a^3-b^3,$ the difference of cubes.
$\left(\sqrt[3]{11}-1\right)\left(\sqrt[3]{11^2}+\sqrt[3]{11}+1\right)=\left(\sqrt[3]{11}\right)^3-1^3$
$=11-1=10.$

72. This product has the pattern $(a+b)(a^2-ab+b^2)=a^3+b^3,$ the sum of two cubes.
$\left(\sqrt[3]{7}+3\right)\left(\sqrt[3]{7^2}-3\sqrt[3]{7}+9\right)=\left(\sqrt[3]{7}\right)^3+3^3$
$=7+27=34$

73. This product has the pattern $(a+b)^2=a^2+2ab+b^2,$ the square of a binomial.
$\left(\sqrt{3}+\sqrt{8}\right)^2=\left(\sqrt{3}\right)^2+2\left(\sqrt{3}\right)\left(\sqrt{8}\right)+\left(\sqrt{8}\right)^2$
$=3+2\sqrt{24}+8=11+2\sqrt{4\cdot 6}$
$=11+2(2)\sqrt{6}=11+4\sqrt{6}$

74. $\left(\sqrt{2}-1\right)^2=\left(\sqrt{2}\right)^2-2\left(\sqrt{2}\right)+1^2$
$=2-2\sqrt{2}+1=3-2\sqrt{2}$

75. This product can be found by using the FOIL method.
$\left(3\sqrt{2}+\sqrt{3}\right)\left(2\sqrt{3}-\sqrt{2}\right)$
$=3\sqrt{2}\left(2\sqrt{3}\right)-3\sqrt{2}\left(\sqrt{2}\right)+\sqrt{3}\left(2\sqrt{3}\right)-\sqrt{3}\sqrt{2}$
$=6\sqrt{6}-3\cdot 2+2\cdot 3-\sqrt{6}=6\sqrt{6}-6+6-\sqrt{6}$
$=5\sqrt{6}$

76. $\left(4\sqrt{5}-1\right)\left(3\sqrt{5}+2\right)$
$=4\sqrt{5}\left(3\sqrt{5}\right)+4\sqrt{5}(2)-1\left(3\sqrt{5}\right)-1(2)$
$=12\cdot 5+8\sqrt{5}-3\sqrt{5}-2$
$=60+5\sqrt{5}-2=58+5\sqrt{5}$

77. $\dfrac{\sqrt[3]{mn}\cdot\sqrt[3]{m^2}}{\sqrt[3]{n^2}}=\sqrt[3]{\dfrac{mnm^2}{n^2}}=\sqrt[3]{\dfrac{m^3}{n}}$
$=\dfrac{\sqrt[3]{m^3}}{\sqrt[3]{n}}\cdot\dfrac{\sqrt[3]{n^2}}{\sqrt[3]{n^2}}=\dfrac{m\sqrt[3]{n^2}}{n}$

78. $\dfrac{\sqrt[3]{8m^2n^3}\cdot\sqrt[3]{2m^2}}{\sqrt[3]{32m^4n^3}}=\dfrac{\sqrt[3]{16m^4n^3}}{\sqrt[3]{32m^4n^3}}=\sqrt[3]{\dfrac{16m^4n^3}{32m^4n^3}}$
$=\sqrt[3]{\dfrac{1}{2}}=\dfrac{\sqrt[3]{1}}{\sqrt[3]{2}}=\dfrac{1}{\sqrt[3]{2}}\cdot\dfrac{\sqrt[3]{4}}{\sqrt[3]{4}}=\dfrac{\sqrt[3]{4}}{2}$

79. $\sqrt[3]{\dfrac{2}{x^6}}-\sqrt[3]{\dfrac{5}{x^9}}=\dfrac{\sqrt[3]{2}}{\sqrt[3]{x^6}}-\dfrac{\sqrt[3]{5}}{\sqrt[3]{x^9}}=\dfrac{\sqrt[3]{2}}{x^2}-\dfrac{\sqrt[3]{5}}{x^3}$
$=\dfrac{\sqrt[3]{2}\cdot x}{x^2\cdot x}-\dfrac{\sqrt[3]{5}}{x^3}=\dfrac{x\sqrt[3]{2}}{x^3}-\dfrac{\sqrt[3]{5}}{x^3}$
$=\dfrac{x\sqrt[3]{2}-\sqrt[3]{5}}{x^3}$

80. $\sqrt[4]{\frac{7}{t^{12}}}+\sqrt[4]{\frac{9}{t^4}}=\frac{\sqrt[4]{7}}{\sqrt[4]{t^{12}}}+\frac{\sqrt[4]{9}}{\sqrt[4]{t^4}}=\frac{\sqrt[4]{7}}{t^3}+\frac{\sqrt[4]{9}}{t}$
$=\frac{\sqrt[4]{7}}{t^3}+\frac{\sqrt[4]{9}\cdot t^2}{t\cdot t^2}=\frac{\sqrt[4]{7}}{t^3}+\frac{t^2\sqrt[4]{9}}{t^3}$
$=\frac{\sqrt[4]{7}+t^2\sqrt[4]{9}}{t^3}$

81. $\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{8}}+\frac{1}{\sqrt{32}}=\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{4\cdot 2}}+\frac{1}{\sqrt{16\cdot 2}}$
$=\frac{1}{\sqrt{2}}+\frac{3}{2\sqrt{2}}+\frac{1}{4\sqrt{2}}$
$=\frac{4\cdot 1}{4\sqrt{2}}+\frac{3\cdot 2}{2\cdot 2\sqrt{2}}+\frac{1}{4\sqrt{2}}$
$=\frac{4}{4\sqrt{2}}+\frac{6}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}$
$=\frac{4+6+1}{4\sqrt{2}}=\frac{11}{4\sqrt{2}}$
$=\frac{11\sqrt{2}}{4\sqrt{2}\cdot\sqrt{2}}=\frac{11\sqrt{2}}{4\cdot 2}=\frac{11\sqrt{2}}{8}$

82. $\frac{2}{\sqrt{12}}-\frac{1}{\sqrt{27}}-\frac{5}{\sqrt{48}}=\frac{2}{\sqrt{4\cdot 3}}-\frac{1}{\sqrt{9\cdot 3}}-\frac{5}{\sqrt{16\cdot 3}}$
$=\frac{2}{2\sqrt{3}}-\frac{1}{3\sqrt{3}}-\frac{5}{4\sqrt{3}}$
$=\frac{6\cdot 2}{6\cdot 2\sqrt{3}}-\frac{4\cdot 1}{4\cdot 3\sqrt{3}}-\frac{3\cdot 5}{3\cdot 4\sqrt{3}}$
$=\frac{12}{12\sqrt{3}}-\frac{4}{12\sqrt{3}}-\frac{15}{12\sqrt{3}}$
$=\frac{12-4-15}{12\sqrt{3}}=-\frac{7}{12\sqrt{3}}$
$=-\frac{7\sqrt{3}}{12\sqrt{3}\cdot\sqrt{3}}$
$=-\frac{7\sqrt{3}}{12\cdot 3}=-\frac{7\sqrt{3}}{36}$

83. $\frac{-4}{\sqrt[3]{3}}+\frac{1}{\sqrt[3]{24}}-\frac{2}{\sqrt[3]{81}}$
$=\frac{-4}{\sqrt[3]{3}}+\frac{1}{\sqrt[3]{8\cdot 3}}-\frac{2}{\sqrt[3]{27\cdot 3}}=\frac{-4}{\sqrt[3]{3}}+\frac{1}{2\sqrt[3]{3}}-\frac{2}{3\sqrt[3]{3}}$
$=\frac{-4\cdot 6}{\sqrt[3]{3}\cdot 6}+\frac{1\cdot 3}{2\sqrt[3]{3}\cdot 3}-\frac{2\cdot 2}{3\sqrt[3]{3}\cdot 2}$
$=\frac{-24}{6\sqrt[3]{3}}+\frac{3}{6\sqrt[3]{3}}-\frac{4}{6\sqrt[3]{3}}=\frac{-24+3-4}{6\sqrt[3]{3}}=\frac{-25}{6\sqrt[3]{3}}$
$=\frac{-25}{6\sqrt[3]{3}}\cdot\frac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}}=\frac{-25\sqrt[3]{9}}{6\cdot 3}=\frac{-25\sqrt[3]{9}}{18}$

84. $\frac{5}{\sqrt[3]{2}}-\frac{2}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{54}}=\frac{5}{\sqrt[3]{2}}-\frac{2}{\sqrt[3]{8\cdot 2}}+\frac{1}{\sqrt[3]{27\cdot 2}}$
$=\frac{5}{\sqrt[3]{2}}-\frac{2}{2\sqrt[3]{2}}+\frac{1}{3\sqrt[3]{2}}$
$=\frac{5\cdot 3}{3\sqrt[3]{2}}-\frac{1\cdot 3}{3\sqrt[3]{2}}+\frac{1}{3\sqrt[3]{2}}$
$=\frac{15-3+1}{3\sqrt[3]{2}}=\frac{13}{3\sqrt[3]{2}}$
$=\frac{13}{3\sqrt[3]{2}}\cdot\frac{\sqrt[3]{4}}{\sqrt[3]{4}}=\frac{13\sqrt[3]{4}}{3\sqrt[3]{8}}$
$=\frac{13\sqrt[3]{4}}{3\cdot 2}=\frac{13\sqrt[3]{4}}{6}$

85. $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$
$=\frac{\sqrt{3}\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$
$=\frac{\sqrt{3}\sqrt{5}-\sqrt{3}\sqrt{3}}{5-3}=\frac{\sqrt{15}-3}{2}$

86. $\frac{\sqrt{7}}{\sqrt{3}-\sqrt{7}}=\frac{\sqrt{7}}{\sqrt{3}-\sqrt{7}}\cdot\frac{\sqrt{3}+\sqrt{7}}{\sqrt{3}+\sqrt{7}}$
$=\frac{\sqrt{7}\left(\sqrt{3}+\sqrt{7}\right)}{3-7}=\frac{\sqrt{21}+7}{-4}$
$=\frac{-\left(\sqrt{21}+7\right)}{4}=\frac{-\sqrt{21}-7}{4}$

87. $\frac{\sqrt{7}-1}{2\sqrt{7}+4\sqrt{2}}=\frac{\sqrt{7}-1}{2\sqrt{7}+4\sqrt{2}}\cdot\frac{2\sqrt{7}-4\sqrt{2}}{2\sqrt{7}-4\sqrt{2}}$
$=\frac{\left(\sqrt{7}-1\right)\left(2\sqrt{7}-4\sqrt{2}\right)}{\left(2\sqrt{7}+4\sqrt{2}\right)\left(2\sqrt{7}-4\sqrt{2}\right)}$
$=\frac{\sqrt{7}\cdot 2\sqrt{7}-\sqrt{7}\cdot 4\sqrt{2}-1\cdot 2\sqrt{7}+1\cdot 4\sqrt{2}}{\left(2\sqrt{7}\right)^2-\left(4\sqrt{2}\right)^2}$
$=\frac{2\cdot 7-4\sqrt{14}-2\sqrt{7}+4\sqrt{2}}{4\cdot 7-16\cdot 2}$
$=\frac{14-4\sqrt{14}-2\sqrt{7}+4\sqrt{2}}{28-32}$
$=\frac{14-4\sqrt{14}-2\sqrt{7}+4\sqrt{2}}{-4}$
$=\frac{-2\left(-7+2\sqrt{14}+\sqrt{7}-2\sqrt{2}\right)}{-4}$
$=\frac{-7+2\sqrt{14}+\sqrt{7}-2\sqrt{2}}{2}$

.

88. $\frac{1+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}} = \frac{1+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}} \cdot \frac{3\sqrt{5}-2\sqrt{3}}{3\sqrt{5}-2\sqrt{3}}$

$= \frac{(1+\sqrt{3})(3\sqrt{5}-2\sqrt{3})}{(3\sqrt{5})^2-(2\sqrt{3})^2}$

$= \frac{3\sqrt{5}-2\sqrt{3}+\sqrt{3}(3\sqrt{5})-\sqrt{3}(2\sqrt{3})}{45-12}$

$= \frac{3\sqrt{5}-2\sqrt{3}+3\sqrt{15}-6}{33}$

89. $\frac{p}{\sqrt{p}+2} = \frac{p}{\sqrt{p}+2} \cdot \frac{\sqrt{p}-2}{\sqrt{p}-2}$

$= \frac{p(\sqrt{p}-2)}{(\sqrt{p})^2-2^2} = \frac{p(\sqrt{p}-2)}{p-4}$

90. $\frac{\sqrt{r}}{3-\sqrt{r}} = \frac{\sqrt{r}}{3-\sqrt{r}} \cdot \frac{3+\sqrt{r}}{3+\sqrt{r}}$

$= \frac{\sqrt{r}(3+\sqrt{r})}{3^2-(\sqrt{r})^2} = \frac{\sqrt{r}(3+\sqrt{r})}{9-r}$

91. $\frac{5\sqrt{x}}{2\sqrt{x}+\sqrt{y}} = \frac{5\sqrt{x}(2\sqrt{x}-\sqrt{y})}{(2\sqrt{x}+\sqrt{y})(2\sqrt{x}-\sqrt{y})}$

$= \frac{5\sqrt{x}(2\sqrt{x}-\sqrt{y})}{4x-y}$

92. $\frac{a}{\sqrt{a+b}-1} = \frac{a}{\sqrt{a+b}-1} \cdot \frac{\sqrt{a+b}+1}{\sqrt{a+b}+1}$

$= \frac{a(\sqrt{a+b}+1)}{(\sqrt{a+b})^2-1^2} = \frac{a(\sqrt{a+b}+1)}{a+b-1}$

93. $\frac{3m}{2+\sqrt{m+n}} = \frac{3m}{2+\sqrt{m+n}} \cdot \frac{2-\sqrt{m+n}}{2-\sqrt{m+n}}$

$= \frac{3m(2-\sqrt{m+n})}{2^2-(\sqrt{m+n})^2}$

$= \frac{3m(2-\sqrt{m+n})}{4-(m+n)}$

$= \frac{3m(2-\sqrt{m+n})}{4-m-n}$

94. $S = 15.18\sqrt[9]{n} = 15.18\sqrt[9]{4} \approx 17.7$

The speed of the boat with a four-man crew is approx. 17.7 ft/sec

95. $S = 15.18\sqrt[9]{n} = 15.18\sqrt[9]{8} \approx 19.1$

The speed of the boat with an eight-man crew is approx. 19.1 ft/sec.

96. Windchill temperature

$= 35.74 + .6215T - 35.75V^{.16} + .4275TV^{.16}$

Windchill temperature

$= 35.74 + .6215(30) - 35.75(15^{.16})$

$+ .4275(30)(15^{.16}) \approx 19.0°$

The table gives $19°$.

97. Windchill temperature

$= 35.74 + .6215T - 35.75V^{.16} + .4275TV^{.16}$

Windchill temperature

$= 35.74 + .6215(10) - 35.75(30^{.16})$

$+ .4275(10)(30^{.16}) \approx -12.3°$

The table gives $-12°$.

98. $\frac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\frac{54}{2}} = \sqrt[3]{27} = 3$

99. $\sqrt[4]{8} \cdot \sqrt[4]{2} = \sqrt[4]{8 \cdot 2} = \sqrt[4]{16} = 2$

100. $\sqrt{.1} \cdot \sqrt{40} = \sqrt{.1 \cdot 40} = \sqrt{4} = 2$

101. $\frac{\sqrt[5]{320}}{\sqrt[5]{10}} = \sqrt[5]{\frac{320}{10}} = \sqrt[5]{32} = 2$

102. $\sqrt[6]{2} \cdot \sqrt[6]{4} \cdot \sqrt[6]{8} = \sqrt[6]{2 \cdot 4 \cdot 8} = \sqrt[6]{64} = 2$

103. $\frac{\sqrt[3]{15}}{\sqrt[3]{5}} \cdot \sqrt[3]{9} = \sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{27} = 3$

104. $\frac{377}{120} = 3.141\overline{6}$ and $\pi \approx 3.14159\ldots$, so it first differs in the fourth decimal place.

105. $\frac{355}{113} = 3.1415929\ldots$ and $\pi \approx 3.1415926\ldots$, so it gives six decimal places of accuracy.

106. $\frac{3927}{1250} = 3.1416$ and $\pi \approx 3.1415926\ldots$, so it first differs in the fourth decimal place.

Chapter R: Review Exercises

1. The elements of the set {6, 8, 10, …, 20} are the even numbers from 6 to 20 inclusive. The elements in the set are {6, 8, 10, 12, 14, 16, 18, 20}.

2. The set $\{x \mid x$ is a decimal between 0 and 1$\}$ has an unlimited number of elements, so it is infinite.

3. True. The set of negative integers = {…, –4, –3, –2, –1}, while the set of whole numbers = {0, 1, 2, 3, …}. The two sets do not intersect, and so they are disjoint.

4. False. 6 is not an element in {1, 2, 3, 4, 5}.

5. True

6. False. 7 is an element in {1, 3, 5, 7}

7. False. The two sets are not equal because they do not have the same elements.

8. False. *D* is not a subset of *A* because *D* includes 2 as an element while *A* does not.

9. True

10. True

11. True

12. False. All of the elements in *E* are also elements in *A*, so *E* is a subset of *A*.

13. $A' = \{2, 6, 9, 10\}$

14. $B \cap A = \{4, 8\}$

15. $B \cap E = \varnothing$

16. $C \cup E = \{1, 3, 5, 7\} = C$

17. $D \cap \varnothing = \varnothing$

18. $B \cup \varnothing = \{2, 4, 6, 8\} = B$

19. $(C \cap D) \cup B = \{1, 3\} \cup B = \{1, 2, 3, 4, 6, 8\}$

20.
$$\begin{aligned}(D' \cap U) \cup E &= (\{4, 5, 6, 7, 8, 9, 10\} \cap U) \cup E \\ &= \{4, 5, 6, 7, 8, 9, 10\} \cup E \\ &= \{3, 4, 5, 6, 7, 8, 9, 10\}\end{aligned}$$

21. –12, –6, $-\sqrt{4}$ (or –2), 0, and 6 are integers.

22. The rational numbers are –12, –6, –.9, $-\sqrt{4}$ (or –2), 0, $\frac{1}{8}$, and 6.

23. 0 is a whole number, an integer, a rational number, and a real number.

24. $-\sqrt{36} = -6$ is an integer, a rational number, and a real number.

25. $\frac{4\pi}{5}$ is an irrational number and a real number.

26. Answers will vary. Sample answer: The multiple of a difference is the difference of the multiples.

27. Answers will vary. Sample answer: The reciprocal of a product is the product of the reciprocals.

28. Answers will vary. Sample answer: The difference of the squares of two numbers is the product of the sum of the numbers and the difference of the numbers.

29. Answers will vary. Sample answer: A product raised to a power is the product of each factor raised to the power.

30. Answers will vary. Sample answer: The absolute value of a product is the product of the absolute values of the factors.

31. commutative

32. distributive

33. associative

34. inverse

35. identity

36. **(a)**
$$\begin{aligned}3x^2(4y+5) &= 3x^2 \cdot 4y + 3x^2 \cdot 5 \\ &= (3 \cdot 4)x^2 y + (3 \cdot 5)x^2 \\ &= 12x^2 y + 15x^2\end{aligned}$$

(b)
$$\begin{aligned}4m^2 n + xm^2 n &= m^2 n \cdot 4 + m^2 n \cdot x \\ &= m^2 n(4 + x)\end{aligned}$$

37. In a sample of 5000 students, 38% + 21% + 16% or 75% are expected to be over 19, or 5000(0.75) = 3750 students.

38.
$$\begin{aligned}(6-9)(-2-7) \div (-4) &= (-3)(-9) \div (-4) \\ &= 27 \div (-4) = -\tfrac{27}{4}\end{aligned}$$

39.
$$\begin{aligned}(-4-1)(-3-5) - 2^3 &= (-5)(-8) - 8 \\ &= 40 - 8 = 32\end{aligned}$$

40.
$$\begin{aligned}\left(-\frac{2^3}{5} - \frac{3}{4}\right) - \left(-\frac{1}{2}\right) &= \left(-\frac{8}{5} - \frac{3}{4}\right) - \left(-\frac{1}{2}\right) \\ &= \left(-\frac{8 \cdot 4}{5 \cdot 4} - \frac{3 \cdot 5}{4 \cdot 5}\right) - \left(-\frac{1}{2}\right) \\ &= \left(-\frac{32}{20} - \frac{15}{20}\right) - \left(-\frac{1}{2}\right) \\ &= \frac{-47}{20} + \frac{1}{2} = \frac{-47}{20} + \frac{1 \cdot 10}{2 \cdot 10} \\ &= \frac{-47}{20} + \frac{10}{20} = -\frac{37}{20}\end{aligned}$$

41. $\left(-\frac{5}{9}-\frac{2}{3}\right)-\frac{5}{6}=\left(-\frac{5}{9}-\frac{6}{9}\right)-\frac{5}{6}$
$=\frac{-11}{9}-\frac{5}{6}=-\frac{22}{18}-\frac{15}{18}=-\frac{37}{18}$

42. $\frac{(-7)(-3)-(-2^3)(-5)}{(-2^2-2)(-1-6)}=\frac{(-7)(-3)-(-8)(-5)}{(-4-2)(-1-6)}$
$=\frac{21-40}{(-6)(-7)}=-\frac{19}{42}$

43. $\frac{6(-4)-3^2(-2)^3}{-5[-2-(-6)]}=\frac{6(-4)-9(-8)}{-5[-2+6]}$
$=\frac{6(-4)-9(-8)}{-5(4)}$
$=\frac{-24-(-72)}{-20}=\frac{-24+72}{-20}$
$=\frac{48}{-20}=-\frac{12}{5}$

44. Let $a=-1$, $b=-2$, $c=4$.
$(a-2)\div 5\cdot b+c=(-1-2)\div 5(-2)+4$
$=-3\div 5(-2)+4=\frac{-3}{5}(-2)+4$
$=\frac{6}{5}+4=\frac{6}{5}+\frac{20}{5}=\frac{26}{5}$

45. Let $a=-1$, $b=-2$, $c=4$.
$-c(2a-5b)=-4[2(-1)-5(-2)]$
$=-4[-2-(-10)]=-4(-2+10)$
$=-4(8)=-32$

46. Let $a=-1$, $b=-2$, $c=4$.
$\frac{3|b|-4|c|}{|ac|}=\frac{3|-2|-4|4|}{|-1\cdot 4|}=\frac{3\cdot 2-4\cdot 4}{|-4|}$
$=\frac{6-16}{4}=\frac{-10}{4}=-\frac{5}{2}$

47. Let $a=-1$, $b=-2$, $c=4$.
$\frac{9a+2b}{a+b+c}=\frac{9(-1)+2(-2)}{-1+(-2)+4}$
$=\frac{-9+(-4)}{-3+4}=\frac{-13}{1}=-13$

48. $|x|=3.5$ for $x=\pm 3.5$

49. $(3q^3-9q^2+6)+(4q^3-8q+3)$
$=3q^3-9q^2+6+4q^3-8q+3$
$=7q^3-9q^2-8q+9$

50. $2(3y^6-9y^2+2y)-(5y^6-10y^2-4y)$
$=6y^6-18y^2+4y-5y^6+10y^2+4y$
$=y^6-8y^2+8y$

51. $(8y-7)(2y+7)=16y^2+56y-14y-49$
$=16y^2+42y-49$

52. $(2r+11s)(4r-9s)=8r^2-18rs+44rs-99s^2$
$=8r^2+26rs-99s^2$

53. $(3k-5m)^2=(3k)^2-2(3k)(5m)+(5m)^2$
$=9k^2-30km+25m^2$

54. $(4a-3b)^2=(4a)^2-2(4a)(3b)+(-3b)^2$
$=16a^2-24ab+9b^2$

55. **(a)** 51 million

(b) Evaluate
$.146x^4-2.54x^3+11.0x^2+16.6x+51.5$
when $x=0$.
$.146\cdot 0^4-2.54\cdot 0^3+11.0\cdot 0^2+16.6\cdot 0+51.5$
$=.146\cdot 0-2.54\cdot 0+11.0\cdot 0+16.6\cdot 0+51.5$
$=0-0+0+0+51.5$
$=51.5$ or approximately 52 million

(c) The approximation is 1 million high.

56. **(a)** 106 million

(b) Evaluate
$.146x^4-2.54x^3+11.0x^2+16.6x+51.5$
when $x=2$.
$.146\cdot 2^4-2.54\cdot 2^3+11.0\cdot 2^2+16.6\cdot 2+51.5$
$=.146\cdot 16-2.54\cdot 8+11.0\cdot 4+16.6\cdot 2+51.5$
$=2.336-20.32+44+33.2+51.5$
$=110.716$ or approximately 111 million

(c) The approximation is 5 million high.

57. **(a)** 183 million

(b) Evaluate
$.146x^4-2.54x^3+11.0x^2+16.6x+51.5$
when $x=5$.
$.146\cdot 5^4-2.54\cdot 5^3+11.0\cdot 5^2+16.6\cdot 5+51.5$
$=.146\cdot 625-2.54\cdot 125+11.0\cdot 25$
$+16.6\cdot 5+51.5$
$=91.25-317.5+275+83+51.5$
$=183.25$ or approximately 183 million

(c) They are the same.

58. $\dfrac{72r^2+59r+12}{8r+3}$

$$\begin{array}{r|l} & 9r+4 \\ \hline 8r+3 & 72r^2+59r+12 \\ & \underline{72r^2+27r} \\ & \quad 32r+12 \\ & \quad \underline{32r+12} \\ & \qquad 0 \end{array}$$

Thus,
$$\frac{72r^2+59r+12}{8r+3}=9r+4.$$

59. $\dfrac{30m^3-9m^2+22m+5}{5m+1}$

$$\begin{array}{r|l} & 6m^2-3m+5 \\ \hline 5m+1 & 30m^3-9m^2+22m+5 \\ & \underline{30m^3+6m^2} \\ & \quad -15m^2+22m \\ & \quad \underline{-15m^2-\ 3m} \\ & \qquad 25m+5 \\ & \qquad \underline{25m+5} \\ & \qquad\quad 0 \end{array}$$

Thus,
$$\frac{30m^3-9m^2+22m+5}{5m+1}=6m^2-3m+5.$$

60. $\dfrac{5m^3-7m^2+14}{m^2-2}$

Insert each missing term with a zero coefficient.

$$\begin{array}{r|l} & 5m-7 \\ \hline m^2+0m-2 & 5m^3-7m^2+0m+14 \\ & \underline{5m^3+0m^2-10m} \\ & \quad -7m^2+10m+14 \\ & \quad \underline{-7m^2+\ 0m+14} \\ & \qquad 10m \end{array}$$

Thus,
$$\frac{5m^3-7m^2+14}{m^2-2}=5m-7+\frac{10m}{m^2-2}.$$

61. $\dfrac{3b^3-8b^2+12b-30}{b^2+4}$

Insert the missing term in the divisor with a 0 coefficient.

$$\begin{array}{r|l} & 3b-8 \\ \hline b^2+0b+4 & 3b^3-8b^2+12b-30 \\ & \underline{3b^3+0b^2+12b} \\ & \quad -8b^2+0b-30 \\ & \quad \underline{-8b^2+0b-32} \\ & \qquad 2 \end{array}$$

Thus,
$$\frac{3b^3-8b^2+12b-30}{b^2+4}=3b-8+\frac{2}{b^2+4}.$$

62. $7z^2-9z^3+z=z(7z-9z^2+1)$

63. $3(z-4)^2+9(z-4)^3=3(z-4)^2[1+3(z-4)]$
$=3(z-4)^2(1+3z-12)$
$=3(z-4)^2(3z-11)$

64. $r^2+rp-42p^2$

Find two numbers whose product is –42 and whose sum is 1. They are 7 and –6. Thus,
$r^2+rp-42p^2=(r+7p)(r-6p)$.

65. $z^2-6zk-16k^2=(z-8k)(z+2k)$

66. $6m^2-13m-5$

The positive factors of 6 could be 2 and 3 or 1 and 6. As factors of –5, we could have –1 and 5 or –5 and 1. Try different combinations of these factors until the correct one is found.
$6m^2-13m-5=(3m+1)(2m-5)$

67. $48a^8-12a^7b-90a^6b^2=6a^6(8a^2-2ab-15b^2)$
$=6a^6(4a+5b)(2a-3b)$

68. $169y^4-1=(13y^2)^2-1^2=(13y^2+1)(13y^2-1)$

69. $49m^8-9n^2=(7m^4)^2-(3n)^2$
$=(7m^4+3n)(7m^4-3n)$

70. $8y^3-1000z^6=8(y^3-125z^6)=8[y^3-(5z^2)^3]$
$=8(y-5z^2)[y^2+y(5z^2)+(5z^2)^2]$
$=8(y-5z^2)(y^2+5yz^2+25z^4)$

71. $6(3r-1)^2+(3r-1)-35$

Let $x = 3r – 1$. With this substitution,
$6(3r-1)^2+(3r-1)-35$ becomes
$6x^2+x-35$.
Factor the trinomial by trial and error.
$6x^2+x-35=(3x-7)(2x+5)$
Replacing x with $3r – 1$ gives
$[3(3r-1)-7][2(3r-1)+5]$
$=(9r-3-7)(6r-2+5)=(9r-10)(6r+3)$
$=3(9r-10)(2r+1).$

72. $15mp+9mq-10np-6nq$
$=(15mp+9mq)+(-10np-6nq)$
$=3m(5p+3q)-2n(5p+3q)$
$=(5p+3q)(3m-2n)$

73. $(3x-4)^2+(x-5)(2)(3x-4)(3)$
$=(3x-4)[(3x-4)+(x-5)(2)(3)]$
$=(3x-4)[3x-4+6x-30]=(3x-4)(9x-34)$

74. $(5-2x)(3)(7x-8)^2(7)+(7x-8)^3(-2)$
$=(7x-8)^2[(5-2x)(3)(7)+(7x-8)(-2)]$
$=(7x-8)^2[(5-2x)(21)+(7x-8)(-2)]$
$=(7x-8)^2[105-42x-14x+16]$
$=(7x-8)^2(121-56x)$

75. $\frac{k^2+k}{8k^3}\cdot\frac{4}{k^2-1}=\frac{k(k+1)(4)}{8k^3(k+1)(k-1)}$
$=\frac{4k}{8k^3(k-1)}=\frac{1}{2k^2(k-1)}$

76. $\frac{3r^3-9r^2}{r^2-9}\div\frac{8r^3}{r+3}=\frac{3r^3-9r^2}{r^2-9}\cdot\frac{r+3}{8r^3}$
$=\frac{3r^2(r-3)}{(r+3)(r-3)}\cdot\frac{(r+3)}{8r^3}=\frac{3}{8r}$

77. $\frac{x^2+x-2}{x^2+5x+6}\div\frac{x^2+3x-4}{x^2+4x+3}$
$=\frac{x^2+x-2}{x^2+5x+6}\cdot\frac{x^2+4x+3}{x^2+3x-4}$
$=\frac{(x+2)(x-1)}{(x+3)(x+2)}\cdot\frac{(x+3)(x+1)}{(x+4)(x-1)}=\frac{x+1}{x+4}$

78. $\frac{27m^3-n^3}{3m-n}\div\frac{9m^2+3mn+n^2}{9m^2-n^2}$
$=\frac{27m^3-n^3}{3m-n}\cdot\frac{9m^2-n^2}{9m^2+3mn+n^2}$
$=\frac{(3m)^3-n^3}{3m-n}\cdot\frac{(3m)^2-n^2}{9m^2+3mn+n^2}$
$=\frac{(3m-n)[(3m)^2+3mn+n^2]}{3m-n}\cdot\frac{(3m+n)(3m-n)}{9m^2+3mn+n^2}$
$=\frac{(3m-n)(9m^2+3mn+n^2)(3m+n)(3m-n)}{(3m-n)(9m^2+3mn+n^2)}$
$=(3m+n)(3m-n)$

79. $\frac{p^2-36q^2}{(p-6q)^2}\cdot\frac{p^2-5pq-6q^2}{p^2-6pq+36q^2}\div\frac{5p}{p^3+216q^3}$
$=\frac{p^2-36q^2}{(p-6q)^2}\cdot\frac{p^2-5pq-6q^2}{p^2-6pq+36q^2}\cdot\frac{p^3+216q^3}{5p}$
$=\frac{(p+6q)(p-6q)}{(p-6q)^2}\cdot\frac{(p-6q)(p+q)}{p^2-6pq+36q^2}$
$\cdot\frac{(p+6q)(p^2-6pq+36q^2)}{5p}$
$=\frac{(p+q)(p+6q)^2}{5p}$

80. $\frac{1}{4y}+\frac{8}{5y}=\frac{1\cdot5}{4y\cdot5}+\frac{8\cdot4}{5y\cdot4}=\frac{5}{20y}+\frac{32}{20y}$
$=\frac{5+32}{20y}=\frac{37}{20y}$

81. $\frac{m}{4-m}+\frac{3m}{m-4}=\frac{m(-1)}{(4-m)(-1)}+\frac{3m}{m-4}$
$=\frac{-m}{m-4}+\frac{3m}{m-4}=\frac{2m}{m-4}$

We may also use 4 – m as the common denominator. In this case, the result will be $\frac{-2m}{4-m}$. The two results are equivalent rational expressions.

82. $\frac{3}{x^2-4x+3}-\frac{2}{x^2-1}$
$=\frac{3}{(x-3)(x-1)}-\frac{2}{(x+1)(x-1)}$
$=\frac{3(x+1)}{(x-3)(x-1)(x+1)}-\frac{2(x-3)}{(x+1)(x-1)(x-3)}$
$=\frac{3(x+1)-2(x-3)}{(x-3)(x-1)(x+1)}$
$=\frac{3x+3-2x+6}{(x-3)(x-1)(x+1)}=\frac{x+9}{(x-3)(x-1)(x+1)}$

83. $\frac{\frac{1}{p}+\frac{1}{q}}{1-\frac{1}{pq}}=\frac{pq\left(\frac{1}{p}+\frac{1}{q}\right)}{pq\left(1-\frac{1}{pq}\right)}$
$=\frac{pq\left(\frac{1}{p}\right)+pq\left(\frac{1}{q}\right)}{pq(1)-pq\left(\frac{1}{pq}\right)}=\frac{q+p}{pq-1}$

84. $\frac{3+\frac{2m}{m^2-4}}{\frac{5}{m-2}}=\frac{3+\frac{2m}{(m+2)(m-2)}}{\frac{5}{m-2}}$
$=\frac{(m-2)(m+2)\left(3+\frac{2m}{(m-2)(m+2)}\right)}{(m-2)(m+2)\left(\frac{5}{(m-2)}\right)}$
$=\frac{3(m-2)(m+2)+2m}{5(m+2)}$
$=\frac{3m^2-12+2m}{5m+10}=\frac{3m^2+2m-12}{5m+10}$
$=\frac{3m^2+2m-12}{5(m+2)}$

85. $2^{-6}=\frac{1}{2^6}=\frac{1}{64}$

86. $-3^{-2}=-(3^{-2})=-\left(\frac{1}{3^2}\right)=-\frac{1}{9}$

87. $\left(-\frac{5}{4}\right)^{-2} = \left(-\frac{4}{5}\right)^{2} = \frac{16}{25}$

88. $3^{-1} - 4^{-1} = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$

89. $(5z^3)(-2z^5) = -10z^{3+5} = -10z^8$

90. $(8p^2q^3)(-2p^5q^{-4}) = -16p^{2+5}q^{3-4}$
$= -16p^7q^{-1} = -\frac{16p^7}{q}$

91. $(-6p^5w^4m^{12})^0 = 1$ Definition of a^0

92. $(-6x^2y^{-3}z^2)^{-2} = (-6)^{-2}x^{-4}y^6z^{-4}$
$= \frac{y^6}{(-6)^2x^4z^4} = \frac{y^6}{36x^4z^4}$

93. $\frac{-8y^7p^{-2}}{y^{-4}p^{-3}} = -8y^{7-(-4)}p^{(-2)-(-3)} = -8y^{11}p$

94. $\frac{a^{-6}(a^{-8})}{a^{-2}(a^{11})} = \frac{a^{-6+(-8)}}{a^{-2+11}}$
$= \frac{a^{-14}}{a^9} = a^{-14-9} = a^{-23} = \frac{1}{a^{23}}$

95. $\frac{(p+q)^4(p+q)^{-3}}{(p+q)^6} = (p+q)^{4+(-3)-6}$
$= (p+q)^{-5} = \frac{1}{(p+q)^5}$

96. $\frac{[p^2(m+n)^3]^{-2}}{p^{-2}(m+n)^{-5}} = \frac{p^{-4}(m+n)^{-6}}{p^{-2}(m+n)^{-5}}$
$= p^{-4-(-2)}(m+n)^{-6-(-5)}$
$= p^{-2}(m+n)^{-1} = \frac{1}{p^2(m+n)}$

97. $(7r^{1/2})(2r^{3/4})(-r^{1/6}) = -14r^{1/2+3/4+1/6}$
$= -14r^{17/12}$

98. $(a^{3/4}b^{2/3})(a^{5/8}b^{-5/6}) = a^{3/4}\cdot a^{5/8}\cdot b^{2/3}\cdot b^{-5/6}$
$= a^{3/4+5/8}\cdot b^{2/3+(-5/6)}$
$= a^{6/8+5/8}\cdot b^{4/6+(-5/6)}$
$= a^{11/8}b^{-1/6} = \frac{a^{11/8}}{b^{1/6}}$

99. $\frac{y^{5/3}\cdot y^{-2}}{y^{-5/6}} = \frac{y^{5/3+(-2)}}{y^{-5/6}} = \frac{y^{5/3+(-6/3)}}{y^{-5/6}} = \frac{y^{-1/3}}{y^{-5/6}}$
$= y^{-1/3-(-5/6)} = y^{-2/6+5/6}$
$= y^{3/6} = y^{1/2}$

100. $\left(\frac{25m^3n^5}{m^{-2}n^6}\right)^{-1/2} = \left(25m^{3-(-2)}n^{5-6}\right)^{-1/2}$
$= \left(25m^5n^{-1}\right)^{-1/2} = \left(\frac{25m^5}{n}\right)^{-1/2}$
$= \left(\frac{n}{25m^5}\right)^{1/2} = \frac{n^{1/2}}{(25)^{1/2}m^{5/2}}$
$= \frac{n^{1/2}}{5m^{5/2}}$

101. $2z^{1/3}(5z^2-2) = 2z^{1/3}(5z^2) - 2z^{1/3}(2)$
$= 10z^{1/3+2} - 4z^{1/3}$
$= 10z^{1/3+6/3} - 4z^{1/3}$
$= 10z^{7/3} - 4z^{1/3}$

102. $-m^{3/4}(8m^{1/2} + 4m^{-3/2})$
$= -m^{3/4}(8m^{1/2}) - m^{3/4}(4m^{-3/2})$
$= -8m^{3/4+1/2} - 4m^{3/4+(-3/2)}$
$= -8m^{3/4+2/4} - 4m^{3/4+(-6/4)}$
$= -8m^{5/4} - 4m^{-3/4}$ or $-8m^{5/4} - \frac{4}{m^{3/4}}$

103. $\sqrt{200} = \sqrt{100\cdot 2} = \sqrt{100}\cdot\sqrt{2} = 10\sqrt{2}$

104. $\sqrt[3]{16} = \sqrt[3]{8\cdot 2} = \sqrt[3]{8}\cdot\sqrt[3]{2} = 2\sqrt[3]{2}$

105. $\sqrt[4]{1250} = \sqrt[4]{625\cdot 2} = \sqrt[4]{625}\cdot\sqrt[4]{2} = 5\sqrt[4]{2}$

106. $-\sqrt{\frac{16}{3}} = -\frac{\sqrt{16}}{\sqrt{3}} = -\frac{4}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}$

107. $-\sqrt[3]{\frac{2}{5p^2}} = -\frac{\sqrt[3]{2}}{\sqrt[3]{5p^2}} = -\frac{\sqrt[3]{2}}{\sqrt[3]{5p^2}}\cdot\frac{\sqrt[3]{25p}}{\sqrt[3]{25p}}$
$= -\frac{\sqrt[3]{2\cdot 25p}}{\sqrt[3]{125p^3}} = -\frac{\sqrt[3]{50p}}{5p}$

108. $\sqrt{\frac{2^7y^8}{m^3}} = \frac{\sqrt{2^7y^8}}{\sqrt{m^3}} = \frac{\sqrt{2^6y^8\cdot 2}}{\sqrt{m^2m}} = \frac{\sqrt{2^6y^8}\cdot\sqrt{2}}{\sqrt{m^2}\cdot\sqrt{m}}$
$= \frac{2^3y^4\sqrt{2}}{m\sqrt{m}} = \frac{2^3y^4\sqrt{2}}{m\sqrt{m}}\cdot\frac{\sqrt{m}}{\sqrt{m}}$
$= \frac{8y^4\sqrt{2m}}{m\sqrt{m^2}} = \frac{8y^4\sqrt{2m}}{m\cdot m} = \frac{8y^4\sqrt{2m}}{m^2}$

109. $\sqrt[4]{\sqrt[3]{m}} = \left(\sqrt[3]{m}\right)^{1/4} = (m^{1/3})^{1/4} = m^{1/3\cdot 1/4}$
$= m^{1/12} = \sqrt[12]{m}$

110. $\dfrac{\sqrt[4]{8p^2q^5}\cdot\sqrt[4]{2p^3q}}{\sqrt[4]{p^5q^2}} = \sqrt[4]{\dfrac{8p^2q^5\cdot 2p^3q}{p^5q^2}}$
$= \sqrt[4]{\dfrac{16p^5q^6}{p^5q^2}} = \sqrt[4]{16q^{6-2}}$
$= \sqrt[4]{16q^4} = 2q$

111. This product has the pattern $(a+b)(a^2-ab+b^2) = a^3+b^3$, the sum of two cubes.
$\left(\sqrt[3]{2}+4\right)\left(\sqrt[3]{2^2}-4\sqrt[3]{2}+16\right)$
$= \left(\sqrt[3]{2}+4\right)\left[\left(\sqrt[3]{2}\right)^2 - \sqrt[3]{2}\cdot 4 + 4^2\right]$
$= \left(\sqrt[3]{2}\right)^3 + 4^3 = 2+64 = 66$

112. $\dfrac{3}{\sqrt{5}} - \dfrac{2}{\sqrt{45}} + \dfrac{6}{\sqrt{80}} = \dfrac{3}{\sqrt{5}} - \dfrac{2}{\sqrt{9\cdot 5}} + \dfrac{6}{\sqrt{16\cdot 5}}$
$= \dfrac{3}{\sqrt{5}} - \dfrac{2}{3\sqrt{5}} + \dfrac{6}{4\sqrt{5}}$
$= \dfrac{3\cdot 12}{\sqrt{5}\cdot 12} - \dfrac{2\cdot 4}{3\sqrt{5}\cdot 4} + \dfrac{6\cdot 3}{4\sqrt{5}\cdot 3}$
$= \dfrac{36}{12\sqrt{5}} - \dfrac{8}{12\sqrt{5}} + \dfrac{18}{12\sqrt{5}}$
$= \dfrac{36-8+18}{12\sqrt{5}} = \dfrac{46}{12\sqrt{5}} = \dfrac{23}{6\sqrt{5}}$
$= \dfrac{23}{6\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{23\sqrt{5}}{6\cdot 5} = \dfrac{23\sqrt{5}}{30}$

113. $\sqrt{18m^3} - 3m\sqrt{32m} + 5\sqrt{m^3}$
$= \sqrt{9m^2\cdot 2m} - 3m\sqrt{16\cdot 2m} + 5\sqrt{m^2m}$
$= 3m\sqrt{2m} - 4\cdot 3m\sqrt{2m} + 5m\sqrt{m}$
$= 3m\sqrt{2m} - 12m\sqrt{2m} + 5m\sqrt{m}$
$= -9m\sqrt{2m} + 5m\sqrt{m}$ or $m\left(-9\sqrt{2m}+5\sqrt{m}\right)$

114. $\dfrac{2}{7-\sqrt{3}} = \dfrac{2}{7-\sqrt{3}}\cdot\dfrac{7+\sqrt{3}}{7+\sqrt{3}} = \dfrac{2\left(7+\sqrt{3}\right)}{7^2-\left(\sqrt{3}\right)^2}$
$= \dfrac{14+2\sqrt{3}}{49-3} = \dfrac{14+2\sqrt{3}}{46}$
$= \dfrac{2\left(7+\sqrt{3}\right)}{46} = \dfrac{7+\sqrt{3}}{23}$

115. $\dfrac{6}{3-\sqrt{2}} = \dfrac{6}{3-\sqrt{2}}\cdot\dfrac{3+\sqrt{2}}{3+\sqrt{2}}$
$= \dfrac{6\left(3+\sqrt{2}\right)}{9-2} = \dfrac{6\left(3+\sqrt{2}\right)}{7}$

116. $\dfrac{k}{\sqrt{k}-3} = \dfrac{k}{\sqrt{k}-3}\cdot\dfrac{\sqrt{k}+3}{\sqrt{k}+3} = \dfrac{k\left(\sqrt{k}+3\right)}{k-9}$

117. $x(x^2+5) = x\cdot x^2 + x\cdot 5 = x^3+5x$

118 $-3^2 = -(3^2) = -9$

119. $(m^2)^3 = m^{2\cdot 3} = m^6$

120. $3x\cdot 3y = 3\cdot 3\cdot x\cdot y = 9xy$

121. $\dfrac{\left(\frac{a}{b}\right)}{2} = \left(\dfrac{a}{b}\right)\cdot\left(\dfrac{1}{2}\right) = \dfrac{a}{2b}$

122. $\dfrac{m}{r}\cdot\dfrac{n}{r} = \dfrac{m\cdot n}{r\cdot r} = \dfrac{mn}{r^2}$

123. One possible answer is
$\dfrac{1}{\sqrt{a}+\sqrt{b}} = \dfrac{1}{\sqrt{a}+\sqrt{b}}\cdot\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{a}-\sqrt{b}}{a-b}.$

124. $\dfrac{(2x)^3}{2y} = \dfrac{2^3x^3}{2y} = \dfrac{2^2x^3}{y} = \dfrac{4x^3}{y}$

125. $4-(t+1) = 4-t-1$ or $3-t$

126. $\dfrac{1}{(-2)^3} = \dfrac{1}{-8} = (-2)^{-3}$ or
$\dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8} = -2^{-3}$

127. $(-5)^2 = (-5)(-5) = 25$ or 5^2

128. $\left(\dfrac{8}{7}+\dfrac{a}{b}\right)^{-1} = \left(\dfrac{8b}{7b}+\dfrac{7a}{7b}\right)^{-1}$
$= \left(\dfrac{8b+7a}{7b}\right)^{-1} = \dfrac{7b}{8b+7a}$

Chapter R: Test

1. False. $B' = \{2,4,6,7,8\}$
2. True
3. False. $D\cap\varnothing = \varnothing$
4. False. $(B\cap C)\cup D = \{1\}\cup D = \{1,4\}$
5. True
6. **(a)** -13, $-\dfrac{12}{4}$ (or -3), 0, and $\sqrt{49}$ (or 7) are integers.

(b) $-13,\ -\frac{12}{4}$ (or –3), 0, $\frac{3}{5}$, 5.9 (or $\frac{59}{10}$), and $\sqrt{49}$ (or 7) are rational numbers.

(c) All numbers in the set are real numbers.

7. Let $x = -2$, $y = -4$, $z = 5$.

$$\left|\frac{x^2+2yz}{3(x+z)}\right| = \left|\frac{(-2)^2+2(-4)(5)}{3(-2+5)}\right| = \left|\frac{4+(-40)}{3(3)}\right| = \left|\frac{-36}{9}\right| = |-4| = 4$$

8. **(a)** associative property

(b) commutative

(c) distributive

(d) inverse

9. $A = 419$, $C = 267$, $Y = 3075$, $T = 15$, $I = 10$

Rating

$$\approx 85.68\left(\frac{C}{A}\right)+4.31\left(\frac{Y}{A}\right)+326.42\left(\frac{T}{A}\right)-419.07\left(\frac{I}{A}\right)$$
$$= 85.68\left(\frac{267}{419}\right)+4.31\left(\frac{3075}{419}\right)+326.42\left(\frac{15}{419}\right)-419.07\left(\frac{10}{419}\right)$$
$$\approx 87.9$$

Matt Hasselbeck's rating is 87.9.

10. $(x^2-3x+2)-(x-4x^2)+3x(2x+1)$
$= x^2-3x+2-x+4x^2+6x^2+3x$
$= 11x^2-x+2$

11. $(6r-5)^2 = (6r)^2-2(6r)(5)+5^2$
$= 36r^2-60r+25$

12. $(t+2)(3t^2-t+4)$

$$\begin{array}{r} 3t^2 - t + 4 \\ t + 2 \\ \hline 6t^2 - 2t + 8 \\ 3t^3 - t^2 + 4t \quad\quad \\ \hline 3t^3 + 5t^2 + 2t + 8 \end{array}$$

13. $\dfrac{2x^3-11x^2+28}{x-5}$

$$\begin{array}{r|l} & 2x^2 - x - 5 \\ x-5 & 2x^3 - 11x^2 + 0x + 28 \\ & \underline{2x^3 - 10x^2} \\ & -x^2 + 0x \\ & \underline{-x^2 + 5x} \\ & -5x + 28 \\ & \underline{-5x + 25} \\ & 3 \end{array}$$

Thus,

$$\frac{2x^3-11x^2+28}{x-5} = 2x^2-x-5+\frac{3}{x-5}.$$

14. $x = 3$

Adjusted poverty threshold
$\approx 5.476x^2+154.3x+7889$
$= 5.476\cdot 3^2+154.3\cdot 3+7889$
$= 49.284+462.9+7889 = 8401.184$
approximately \$8401

15. $x = 5$

Adjusted poverty threshold
$\approx 5.476x^2+154.3x+7889$
$= 5.476\cdot 5^2+154.3\cdot 5+7889$
$= 136.9+771.5+7889 = 8797.4$
approximately \$8797

16. $6x^2-17x+7 = (3x-7)(2x-1)$

17. $x^4-16 = (x^2)^2-4^2$
$= (x^2+4)(x^2-4)$
$= (x^2+4)(x^2-2^2)$
$= (x^2+4)(x+2)(x-2)$

18. $24m^3-14m^2-24m = 2m(12m^2-7m-12)$
$= 2m(4m+3)(3m-4)$

19. $x^3y^2-9x^3-8y^2+72$
$= (x^3y^2-9x^3)-(8y^2-72)$
$= x^3(y^2-9)-8(y^2-9) = (x^3-8)(y^2-9)$
$= (x^3-2^3)(y^2-3^2)$
$= (x-2)(x^2+2x+4)(y+3)(y-3)$

20. $\dfrac{5x^2-9x-2}{30x^3+6x^2}\cdot\dfrac{2x^8+6x^7+4x^6}{x^4-3x^2-4}$
$=\dfrac{(5x+1)(x-2)}{6x^2(5x+1)}\cdot\dfrac{2x^6(x^2+3x+2)}{(x^2-4)(x^2+1)}$
$=\dfrac{(5x+1)(x-2)}{6x^2(5x+1)}\cdot\dfrac{2x^6(x+2)(x+1)}{(x+2)(x-2)(x^2+1)}$
$=\dfrac{2x^6(x+1)}{6x^2(x^2+1)}=\dfrac{x^4(x+1)}{3(x^2+1)}$

21. $\dfrac{x}{x^2+3x+2}+\dfrac{2x}{2x^2-x-3}$
$=\dfrac{x}{(x+2)(x+1)}+\dfrac{2x}{(2x-3)(x+1)}$
The least common denominator is $(x+2)(x+1)(2x-3)$.
$=\dfrac{x(2x-3)}{(x+2)(x+1)(2x-3)}+\dfrac{2x(x+2)}{(2x-3)(x+1)(x+2)}$
$=\dfrac{2x^2-3x}{(x+2)(x+1)(2x-3)}+\dfrac{2x^2+4x}{(x+2)(x+1)(2x-3)}$
$=\dfrac{2x^2-3x+2x^2+4x}{(x+2)(x+1)(2x-3)}=\dfrac{4x^2+x}{(x+2)(x+1)(2x-3)}$
$=\dfrac{x(4x+1)}{(x+2)(x+1)(2x-3)}$

22. $\dfrac{a+b}{2a-3}-\dfrac{a-b}{3-2a}=\dfrac{a+b}{2a-3}-\dfrac{(a-b)(-1)}{(3-2a)(-1)}$
$=\dfrac{a+b}{2a-3}+\dfrac{a-b}{2a-3}=\dfrac{2a}{2a-3}$
If $3-2a$ is used as the common denominator, the result will be $\dfrac{-2a}{3-2a}$. The rational expressions $\dfrac{2a}{2a-3}$ and $\dfrac{-2a}{3-2a}$ are equivalent.

23. $\dfrac{y-2}{y-\frac{4}{y}}=\dfrac{y(y-2)}{y\left(y-\frac{4}{y}\right)}=\dfrac{y^2-2y}{y^2-4}$
$=\dfrac{y(y-2)}{(y+2)(y-2)}=\dfrac{y}{y+2}$

24. $\left(\dfrac{x^{-2}y^{-1/3}}{x^{-5/3}y^{-2/3}}\right)^3=\dfrac{x^{-6}y^{-1}}{x^{-5}y^{-2}}$
$=x^{-6-(-5)}y^{-1-(-2)}=x^{-1}y=\dfrac{y}{x}$

25. $\left(-\dfrac{64}{27}\right)^{-2/3}=\left(-\dfrac{27}{64}\right)^{2/3}=\left[\left(-\dfrac{27}{64}\right)^{1/3}\right]^2$
$=\left(-\dfrac{3}{4}\right)^2=\dfrac{9}{16}$

26. $\sqrt{18x^5y^8}=\sqrt{(9x^4y^8)(2x)}$
$=\sqrt{9x^4y^8}\cdot\sqrt{2x}=3x^2y^4\sqrt{2x}$

27. $\sqrt{32x}+\sqrt{2x}-\sqrt{18x}$
$=\sqrt{16\cdot 2x}+\sqrt{2x}-\sqrt{9\cdot 2x}$
$=4\sqrt{2x}+\sqrt{2x}-3\sqrt{2x}=2\sqrt{2x}$

28. $\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)=\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2$
$=x-y$

29. $\dfrac{14}{\sqrt{11}-\sqrt{7}}=\dfrac{14}{\sqrt{11}-\sqrt{7}}\cdot\dfrac{\sqrt{11}+\sqrt{7}}{\sqrt{11}+\sqrt{7}}$
$=\dfrac{14\left(\sqrt{11}+\sqrt{7}\right)}{11-7}=\dfrac{14\left(\sqrt{11}+\sqrt{7}\right)}{4}$
$=\dfrac{7\left(\sqrt{11}+\sqrt{7}\right)}{2}$

30. Let $L=3.5$.
$t=2\pi\sqrt{\dfrac{L}{32}}=2\pi\sqrt{\dfrac{3.5}{32}}\approx 2.1$
The period of a pendulum 3.5 ft long is approximately 2.1 seconds.

Chapter R: Quantitative Reasoning

The area of the 10-in pizza is
$25\pi\ \text{in}^2\left(A=\pi r^2 \text{ where } r=\dfrac{10}{2}=5\text{ in}\right)$, and the area of the 15-in pizza is
$56.25\pi\ \text{in}^2\left(A=\pi r^2 \text{ where } r=\dfrac{15}{2}=7.5\text{ in}\right)$. Since
$56.25\pi=2.25\cdot 25\pi$, the cost of the larger pizza should be $\$2.25(4)=\$9.00$. Thus, the owner is overcharging $.25 for the larger pizza.