3. Integration by Substitution

If F is any antiderivative of f (i.e. $F' = f$), and $u = u(x)$

then from the chain rule, we have

$$\frac{d}{dx}[F(u)] = \frac{dF}{du}.\frac{du}{dx} = F'(u).\frac{du}{dx} = f(u).\frac{du}{dx}$$

From this, we have that

$$\int f(u).\frac{du}{dx}.dx = \int \frac{d}{dx}[F(u)].dx = F(u) + c = \int f(u).du$$

i.e. $\int f(u).du = F(u) + c$

so, if we cannot compute the integral $\int h(x)\,dx$ directly, we often look for a new variable u and function $f(u)$ for which

$$\int h(x)\,dx = \int f(u(x)).\frac{du}{dx}.dx = \int f(u)\,du$$

Where the second integral is easier than the first.

EX.3.1

Evaluate $\int (x^3 + 5)^{100}\,(3x^2)dx$

Solution you might observe that $\frac{d}{dx}(x^3 + 5) = 3x^2$

This leads us to make the substitution:

$u = x^3 + 5$, so that

$du = 3x^2 dx$

This gives us

$$\int \underbrace{(x^3+5)^{100}}_{u^{100}} \underbrace{(3x^2)dx}_{du} = \int u^{100}\,du = \frac{u^{101}}{101} + c = \frac{(x^3+5)^{101}}{101} + c$$

INTECRATION BY PARTS

The technique of integration by substitution illustrated in above example **consists of the following general steps**

- *Choose an expression for u* : a common choice is the innermost expression or "inside" term of a composition of function (In above example، note that $x^3 + 5$ is the inside term of $(x^3 + 5)^{100}$)
- *Compute* $du = \frac{du}{dx}\,.dx$
- *Replace all terms* in the original integral with expressions involving u and du.
- *Evaluate* the resulting (u) integral. If you still can't evaluate the integral, you may need to try different choice of u.
- *Replace each occurrence* of u in the antiderivative with the corresponding expression in x .

EX.3.2 (A power function inside a cosine)

Evaluate $\int x.\cos x^2\,dx$

Solution Notice that $\frac{d}{dx}x^2 = 2x.dx$

so, if we substitute

$u = x^2$, so that $du = 2xdx$

we can rewrite the given integral as

$$\int x.\cos x^2\,dx = \frac{1}{2}\int \underbrace{\cos x^2}_{\cos u}\,.\,\underbrace{2x\,dx}_{du}$$

$$= \frac{1}{2}\int \cos u\,.\,du$$

$$= \frac{1}{2}\sin u + c$$

$$= \frac{1}{2}\sin x^2 + c$$

Again, as a check, observe that

$$\frac{d}{dx}\left(\frac{1}{2}\sin x^2\right) = \frac{1}{2}\cos x^2\,(2x) = x\cos x^2$$

Which is the original integrand ■

EX.3.3 (A Root function inside a sine)

Find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Solution Letting $u = \sqrt{x}$ (the "inside"), we get

$$du = \frac{1}{2\sqrt{x}}\,dx$$

Thus, we can rewrite the original integral as

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \underbrace{\sin\sqrt{x}}_{\sin u}\,.\underbrace{\left(\frac{1}{2\sqrt{x}}\right)dx}_{du}$$

$$= 2\int \sin u\,.\,du$$

$$= -2\cos u + c$$

$$= -2\cos\sqrt{x} + c$$

■

EX.3.4 (Where the numerator is the derivative of the denominator)

Evaluate $\int \frac{x^2}{x^3+5}\,dx$

Solution since $\frac{d}{dx}(x^3+5) = 3x^2$, you might notice that we should let $u = x^3+5$, so that $du = 3x^2\,dx$

We now have

$$\int \frac{x^2}{x^3+5}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3+5}\,dx = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + c$$

$$= \frac{1}{3}\ln|x^3+5| + c$$

■

> ***Recall that:***
>
> For any continuous function f
>
> $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$
>
> Provided $f(x) \neq 0$

EX.3.5 (An antiderivative for the tangent function)

Evaluate $\int \tan x \, dx$

Solution

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \overset{u=\cos x}{=\!=\!=} -\int \frac{u'(x)}{u(x)} dx = -ln|u| + c$$

$$= -ln|\cos x| + c$$

EX.3.6 (A substitution for an inverse tangent)

Evaluate $\int \frac{(tan^{-1}x)^2}{1+x^2} dx$

Solution Again, the key is to look for a substitution.

since $\frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2}$

we let $u = tan^{-1}x$, so that $du = \frac{1}{1+x^2} dx$

we now have

$$\int \frac{(tan^{-1}x)^2}{1+x^2} dx = \int \underbrace{(tan^{-1}x)^2}_{u^2} . \underbrace{\frac{1}{1+x^2} dx}_{du}$$

$$= \int u^2 \, du$$

$$= \frac{1}{3} u^3 + c$$

$$= \frac{1}{3} (tan^{-1}x)^3 + c \qquad \blacksquare$$

EX.3.7 (A substitution that lets you Expand the Integral)

Evaluate $\int x.\sqrt{2-x}\,dx$

Solution Letting $u = 2 - x$, we get

$$du = -dx \quad \text{and} \quad x = 2 - u$$

Making these substitutions in the integral, we get

$$\int x.\sqrt{2-x}\,dx = (-1)\int \underbrace{x}_{2-u}.\underbrace{\sqrt{2-x}}_{\sqrt{u}}\ \underbrace{(-1)dx}_{du}$$

$$= -\int (2-u).\sqrt{u}\ du$$

$$= -\int (2u^{\frac{1}{2}} - u^{\frac{3}{2}})du$$

$$= -2\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= -\frac{4}{3}(2-x)^{\frac{3}{2}} + \frac{2}{5}(2-x)^{\frac{5}{2}} + c \qquad \blacksquare$$

- ***Substitution in definite integrals***

There is only one slight difference in using substitution for evaluating a definite integral:

"If you change variables, you must also change the limits of integration to correspond to the new variable"

i.e. $\int_a^b f\big(u(x)\big).u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)du$

EX.3.8

Evaluate $\int_1^2 x^3 \sqrt{x^4+5}\, dx$

Solution Notice that $\frac{d}{dx}(x^4+5) = 4x^3$

This suggests the substitution

$u = x^4 + 5$ so that $du = 4x^3 dx$

For the limits of integration, we have the table

x	1	2
$u = x^4 + 5$	6	21

we now have

$$\int_1^2 x^3 \sqrt{x^4+5}\, dx = \frac{1}{4}\int_1^2 \underbrace{\sqrt{x^4+5}}_{\sqrt{u}} . \underbrace{(4x^3)\; dx}_{du}$$

CAUTION
You must change the limits of integration as soon as you change variables!

$$= \frac{1}{4}\int_6^{21} \sqrt{u}.du = \frac{1}{4}\int_6^{21} u^{\frac{1}{2}}.du$$

$$= \frac{1}{4}.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big]_6^{21} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)[21^{\frac{3}{2}} - 6^{\frac{3}{2}}]$$

EX.3.9

Compute $\int_0^{15} t\, e^{\frac{-t^2}{2}} dt$

Solution Letting $u = \frac{-t^2}{2}$, we get $du = -tdt$ and

t	0	15
$u = \frac{-t^2}{2}$	0	$-\frac{225}{2}$

This gives us

$$\int_0^{15} t\, e^{-\frac{t^2}{2}} dt = -\int_0^{15} \underbrace{e^{-\frac{t^2}{2}}}_{e^u} \underbrace{(-t)\, dt}_{du} = -\int_0^{-\frac{225}{2}} e^u\, du = -e^u\Big]_0^{-\frac{225}{2}}$$

$$= -e^{-112.5} + 1$$

■

Exercises

- In exercises 1-6, use the given substitution to evaluate the indicated integral .

1. $\int (3x^4-1)^{13}\, x^3 dx$, $u = 3x^4 - 1$
2. $\int \frac{dx}{\sqrt{2x+5}}$, $u = 2x + 5$
3. $\int 2x^5 . \sqrt{x^2+2}\ dx$, $u = x^2 + 2$
4. $\int \frac{dx}{x\sqrt{x^6-4}}$, $u = x^3$
5. $\int \frac{2+cos^3\theta}{sin^2\theta} d\theta$, $u = \sin\theta$
6. $\int_0^4 \frac{x}{\sqrt{x^2+9}} dx$, $u = x^2 + 9$

- IN exercises 7-12, evaluate the indicated integral

7. $\int \frac{x^2}{1+x^6} dx$
8. $\int_0^2 x^2\, e^{x^3} dx$
9. $\int_1^e \frac{\ln x}{x}\ dx$
10. $\int_{-2}^{-1} e^{\ln(x^2+1)} dx$
11. $\int \frac{x+1}{x^2+2x+4} dx$
12. $\int_0^2 \frac{4x}{(x^2+1)} dx$

THE END