THOMAS' CALCULUS

MEDIA UPGRADE

Chapter 3

Differentiation

3.1

The Derivative as a Function

DEFINITION Derivative Function

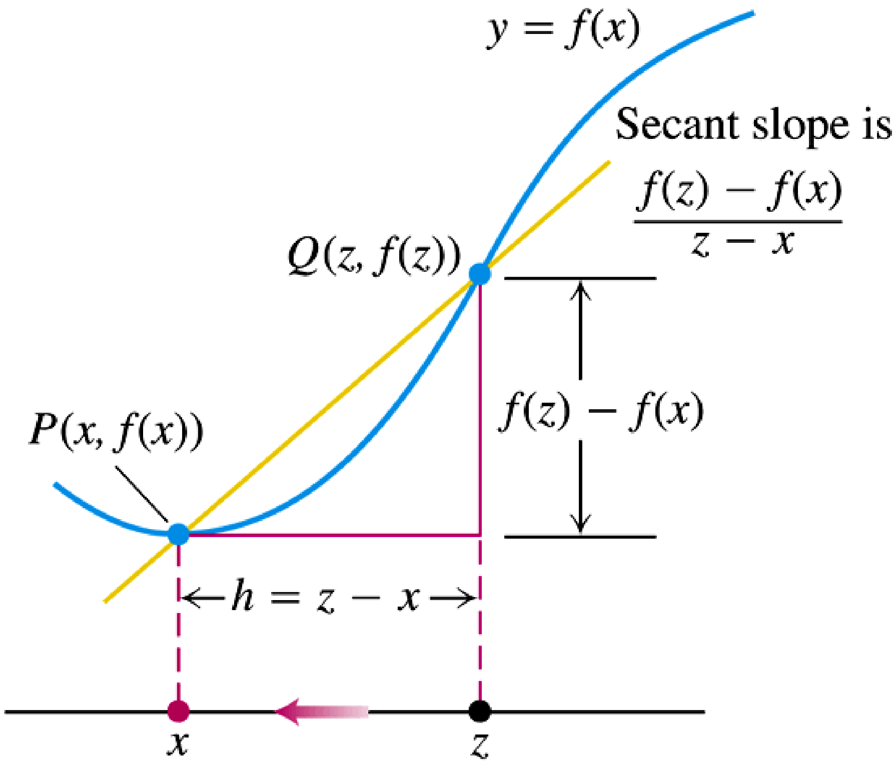
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

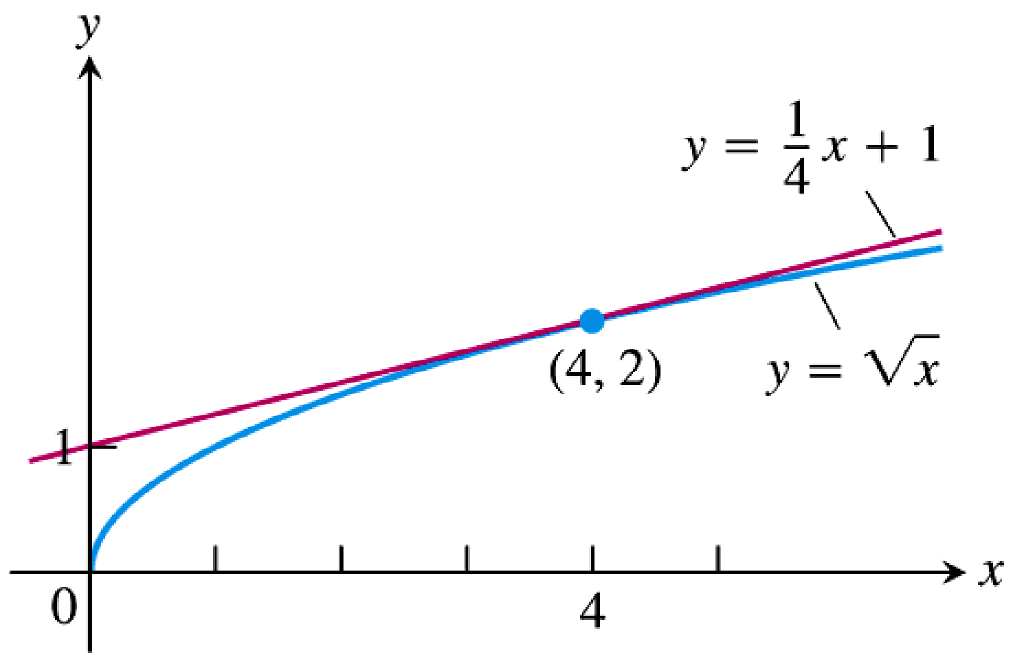


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

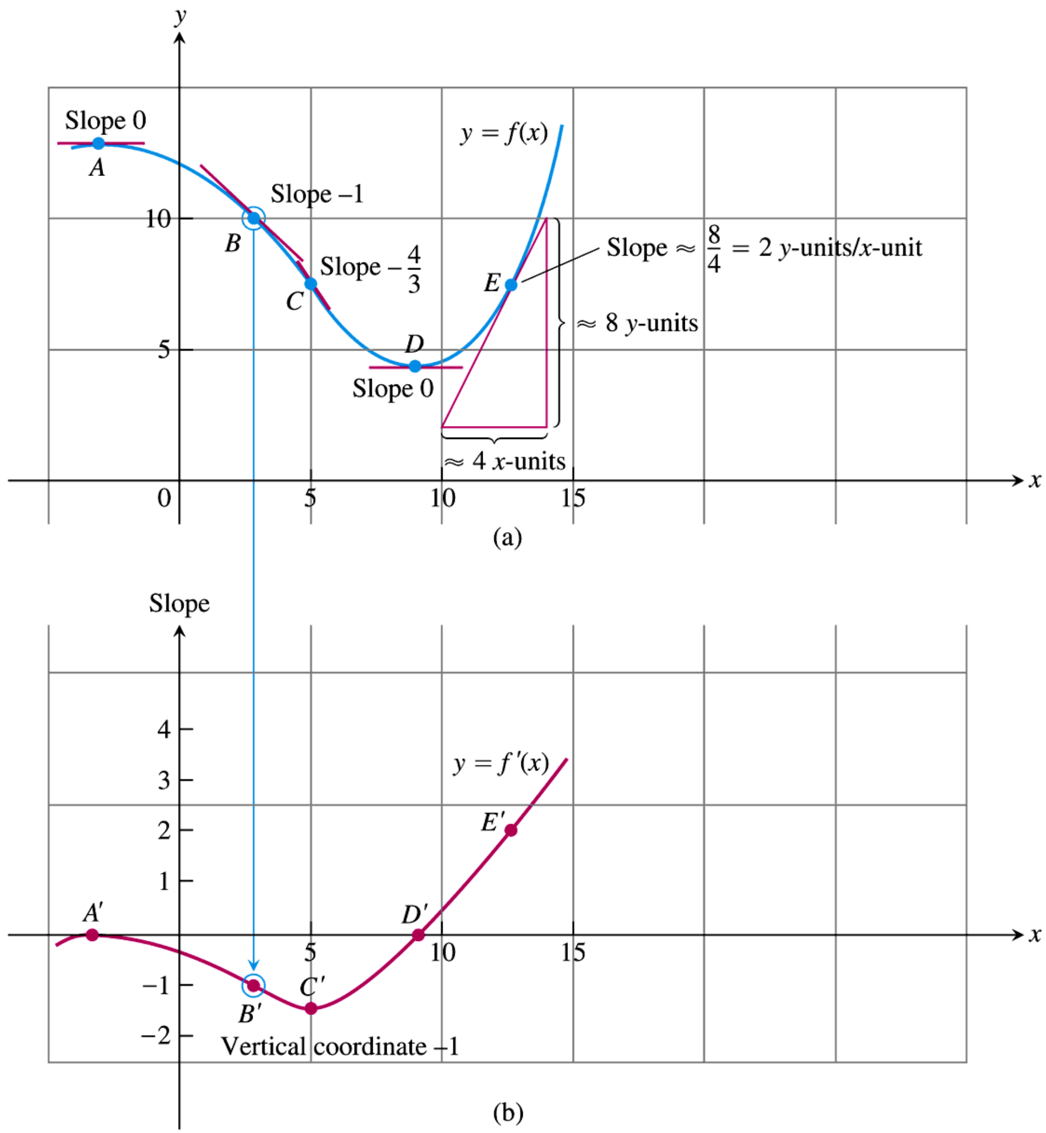
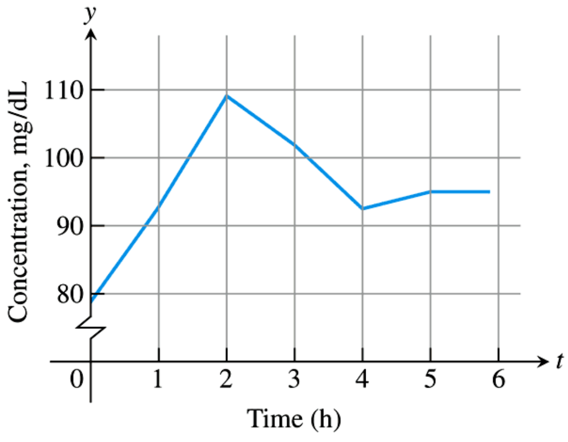
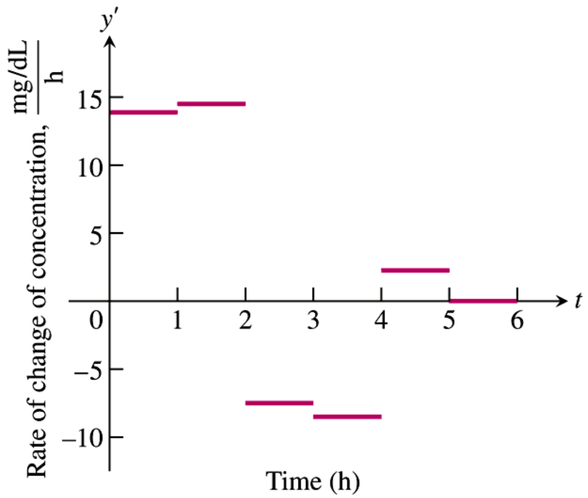


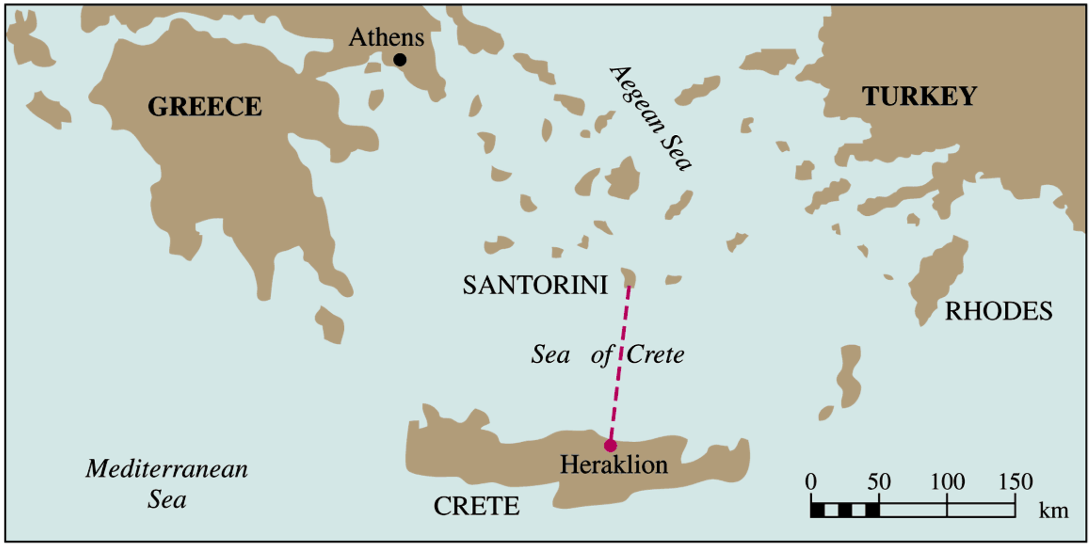
FIGURE 3.3 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B and so on. The graph of f' is a visual record of how the slope of f changes with x.



(a)



(b)



Daedalus's flight path on April 23, 1988

FIGURE 3.4 (a) Graph of the sugar concentration in the blood of a *Daedalus* pilot during a 6-hour preflight endurance test. (b) The derivative of the pilot's blood-sugar concentration shows how rapidly the concentration rose and fell during various portions of the test.

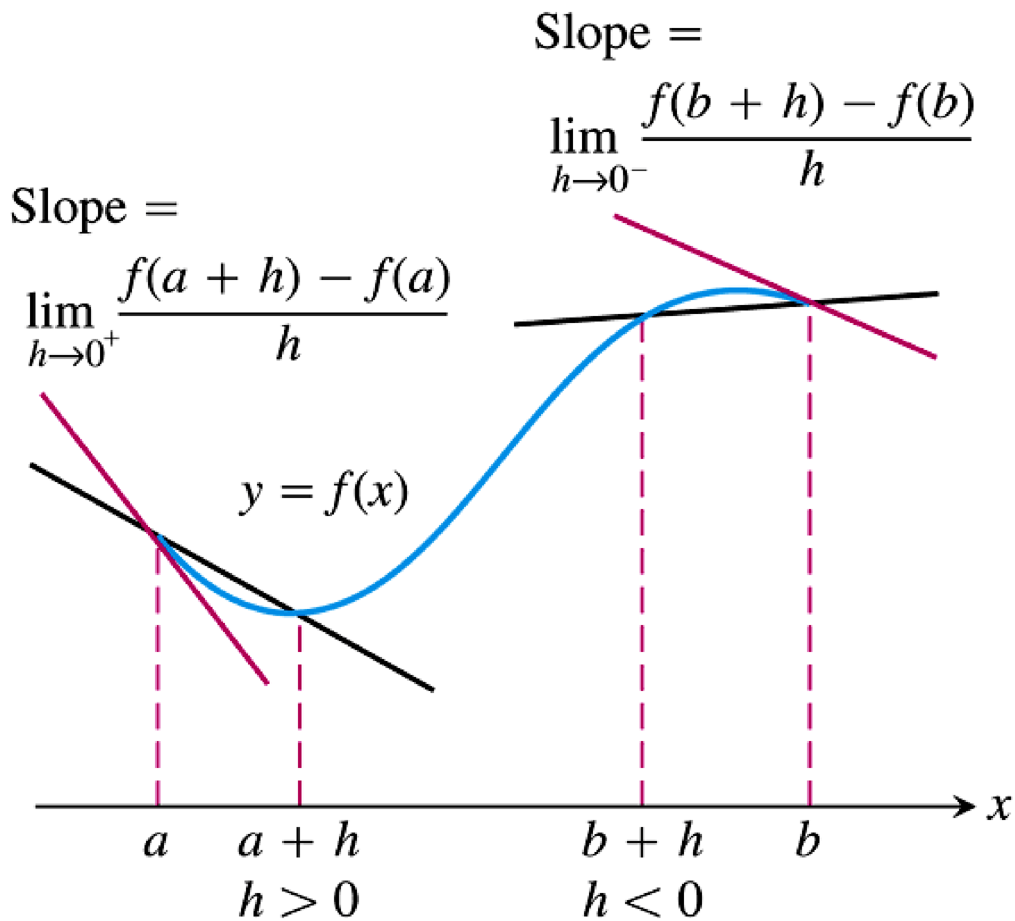


FIGURE 3.5 Derivatives at endpoints are one-sided limits.

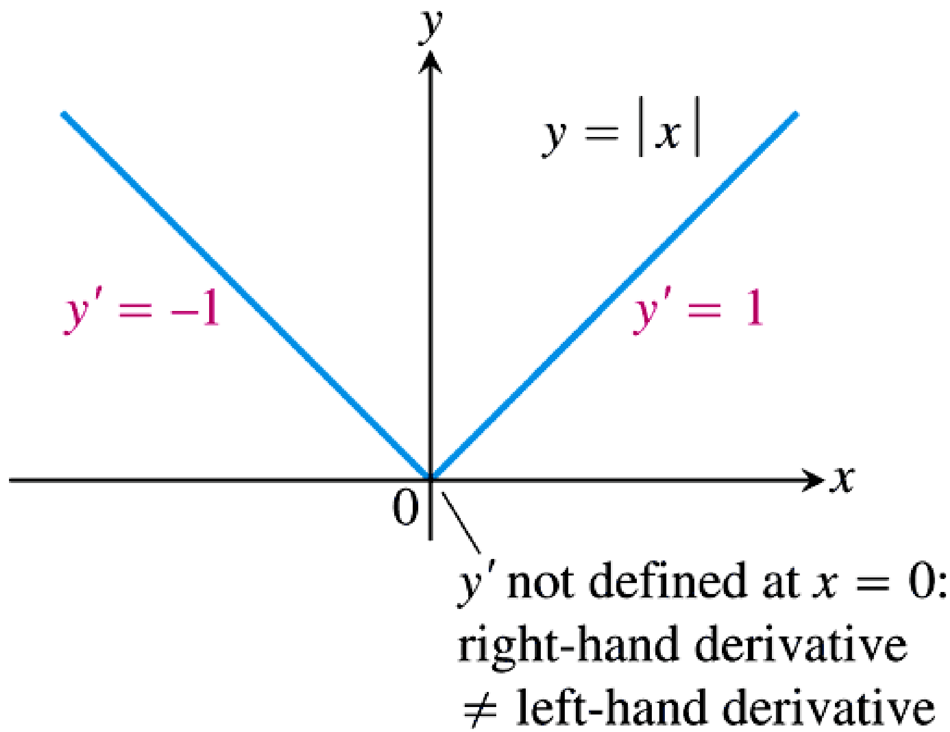
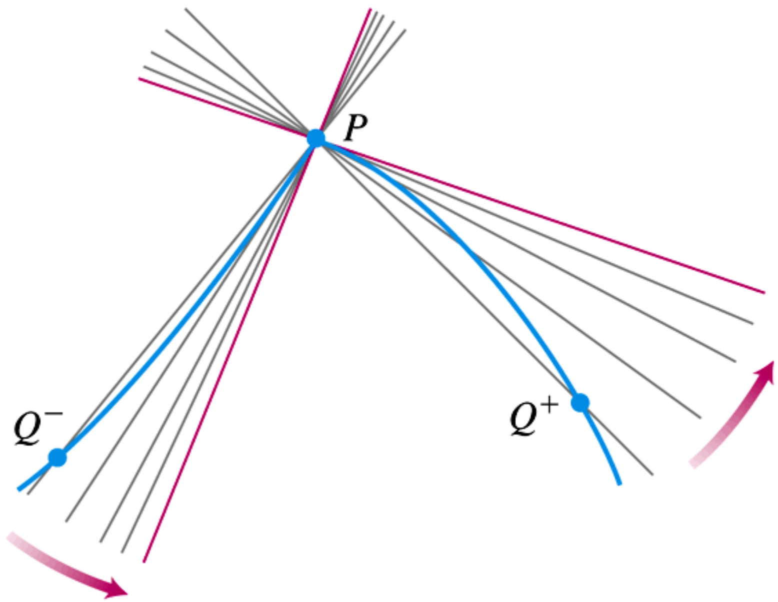
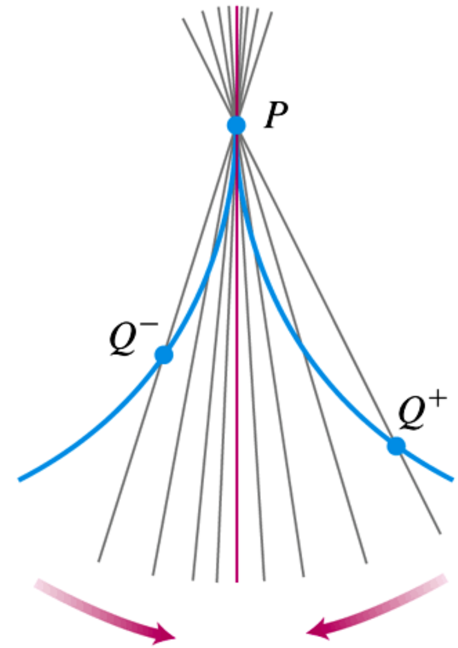


FIGURE 3.6 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

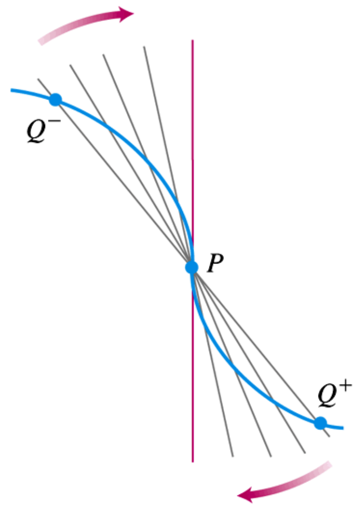
1. a *corner*, where the one-sided derivatives differ.



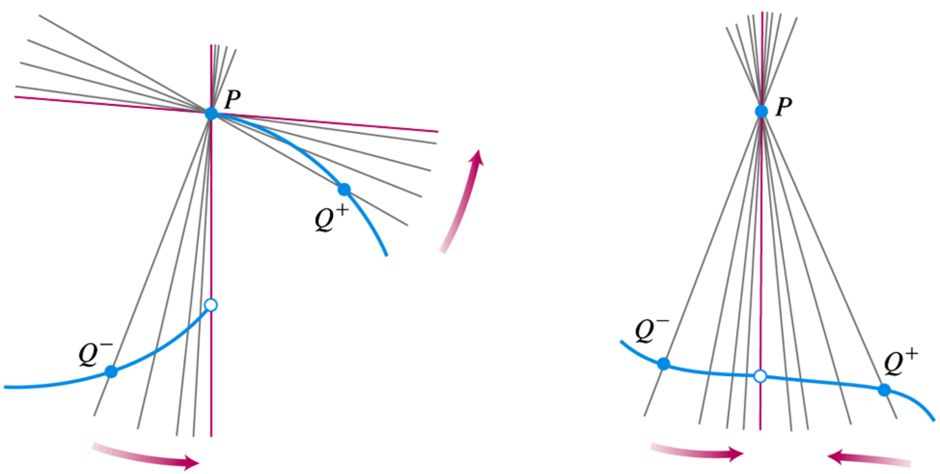
2. a *cusp*, where the slope of *PQ* approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.



THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

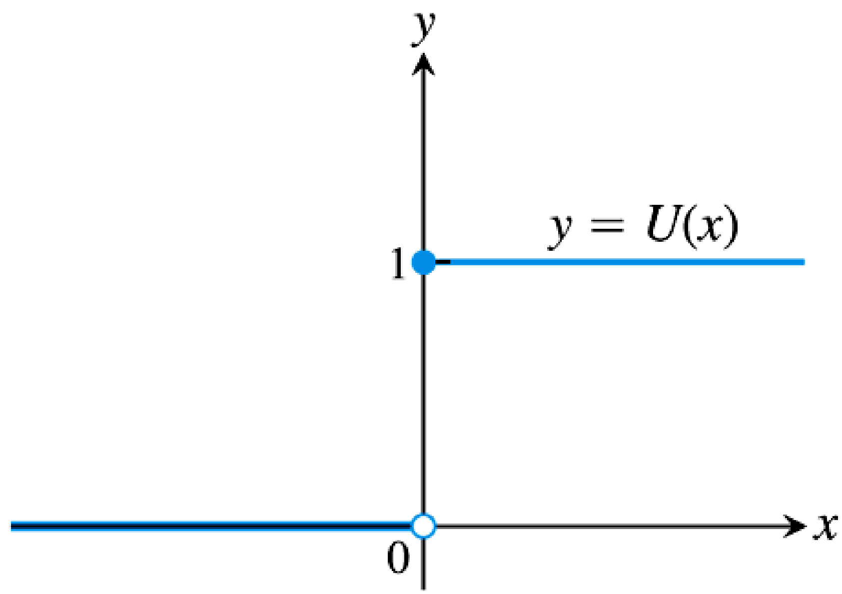


FIGURE 3.7 The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

THEOREM 2 Darboux's Theorem

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

3.2

Differentiation Rules

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

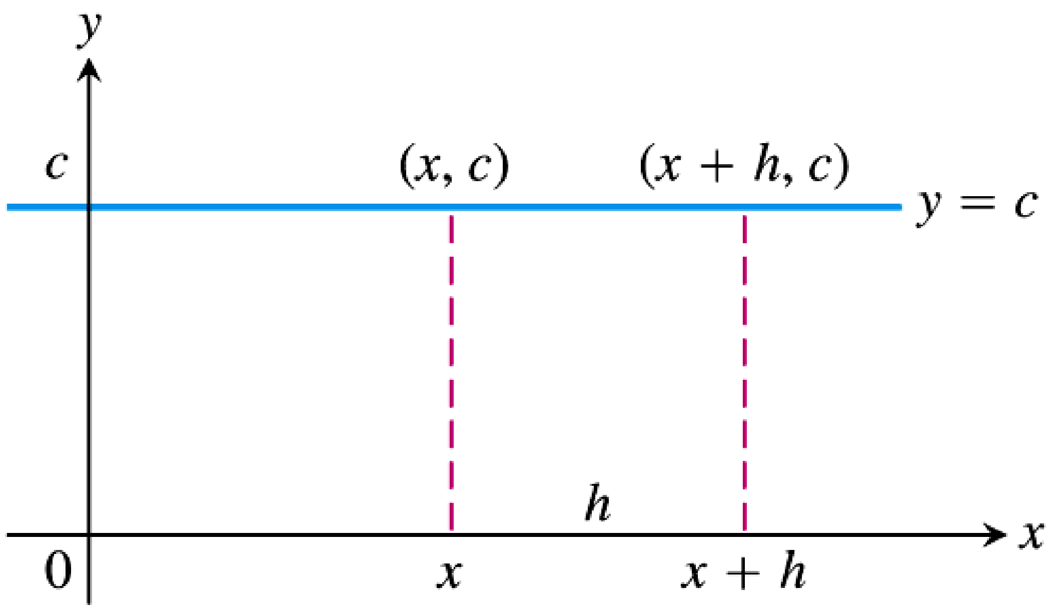


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

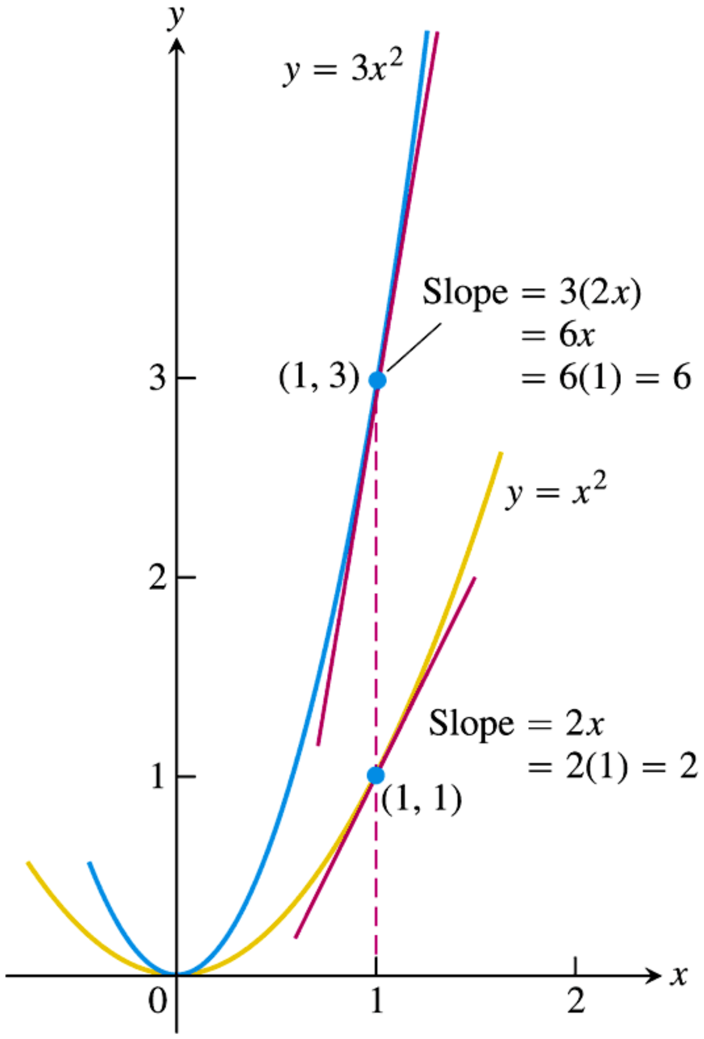


FIGURE 3.9 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

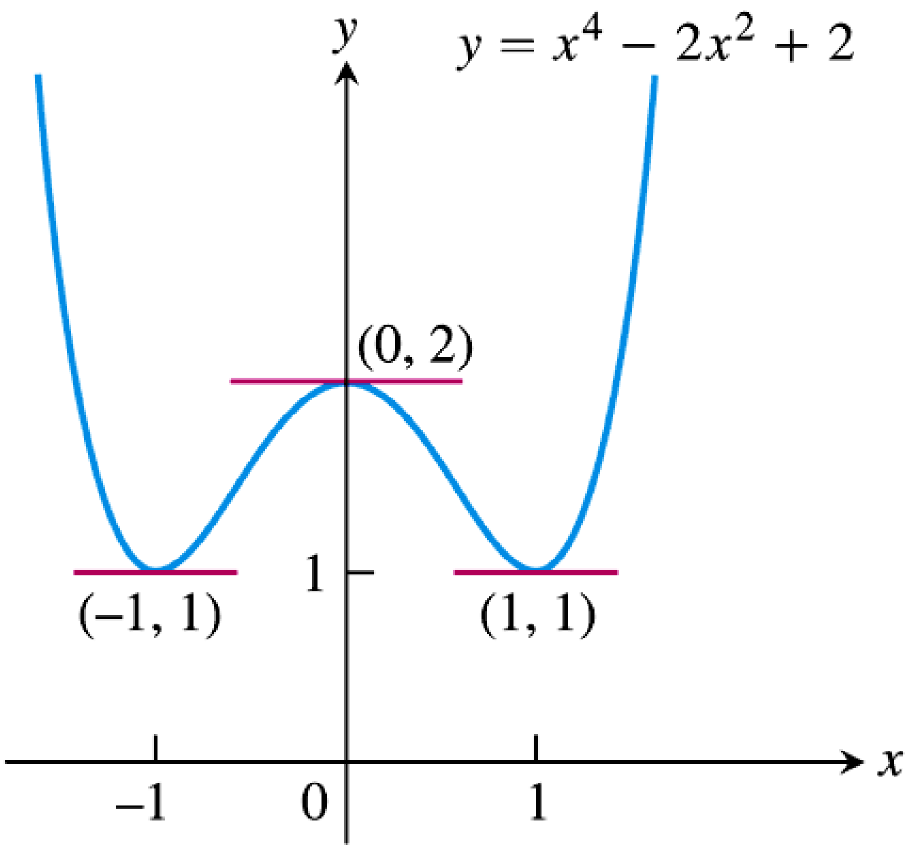


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

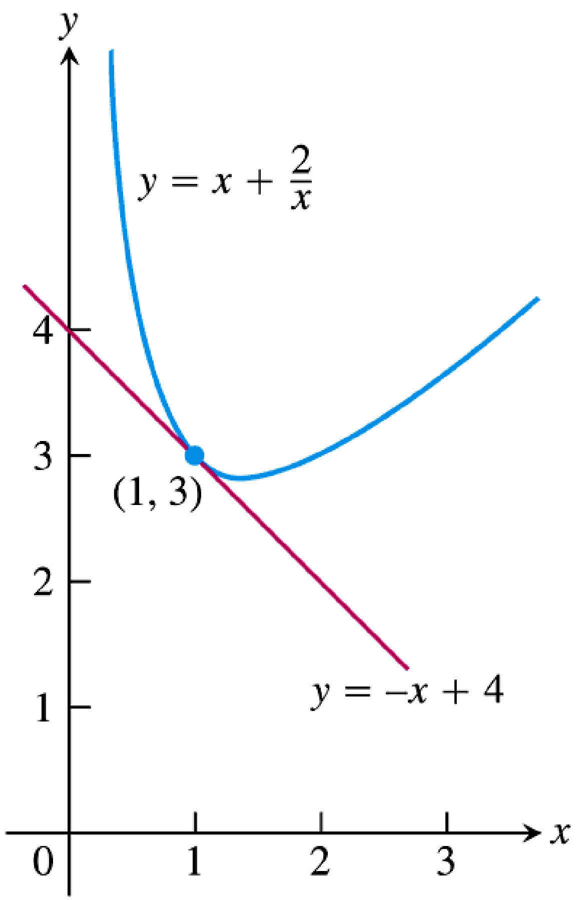


FIGURE 3.11 The tangent to the curve $y = x + (2/x)$ at $(1, 3)$ in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

3.3

The Derivative as a Rate of Change

DEFINITION **Instantaneous Rate of Change**

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

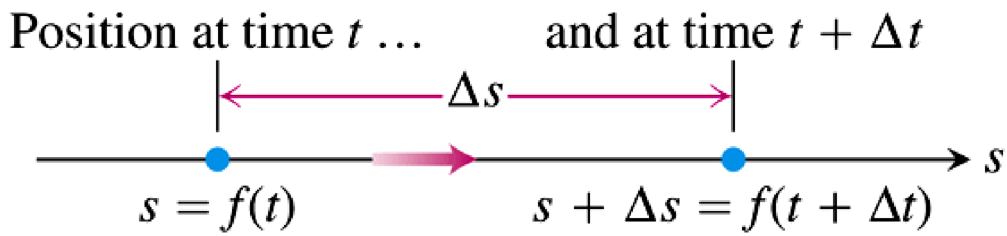


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

DEFINITION Velocity

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

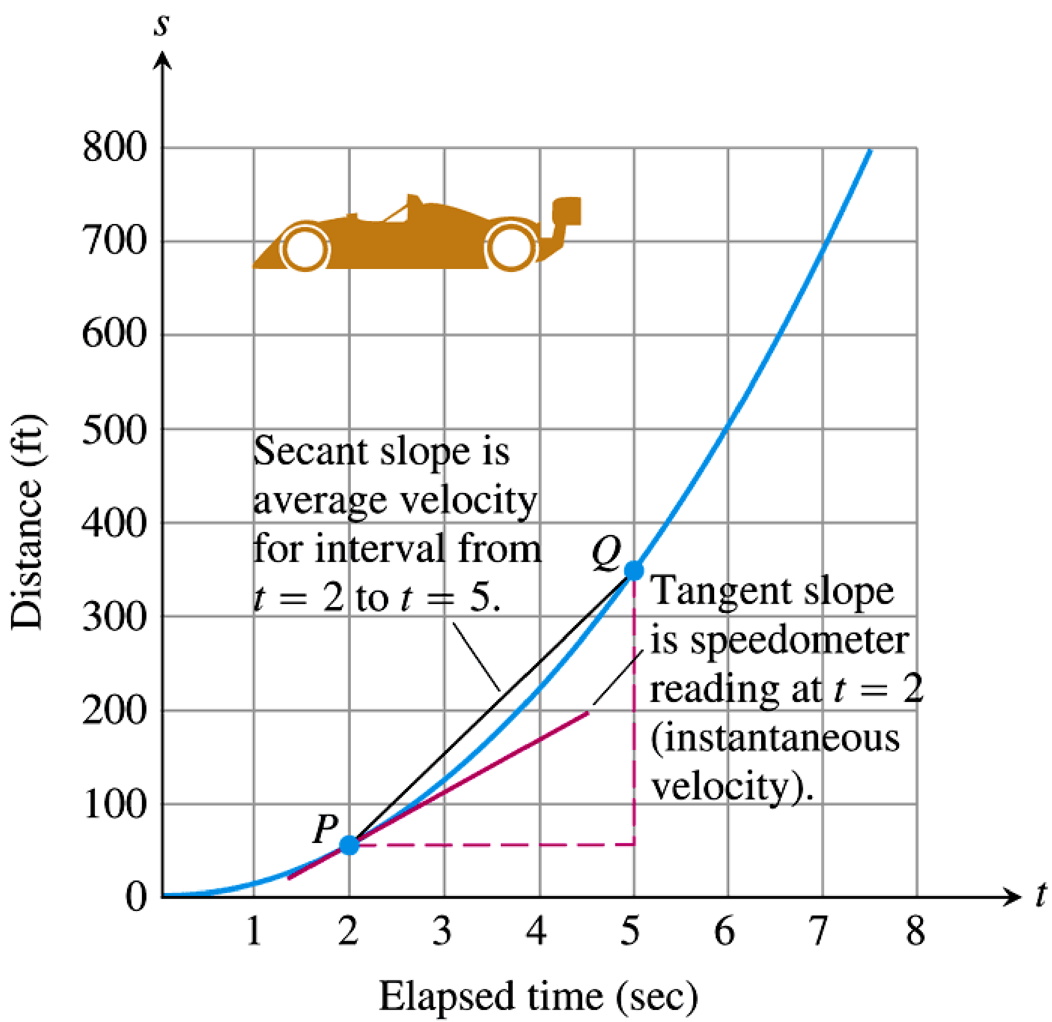


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

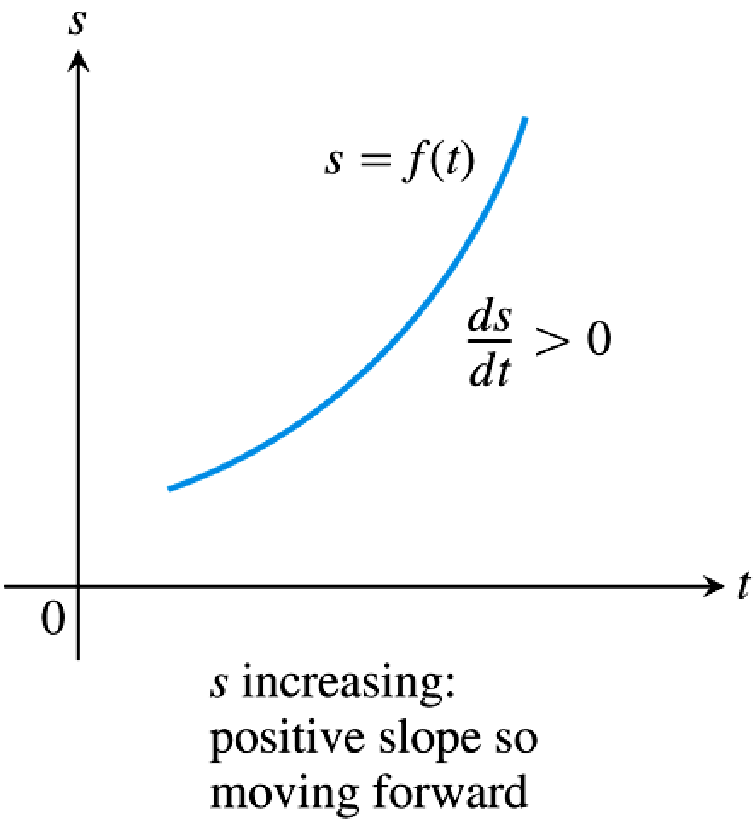


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

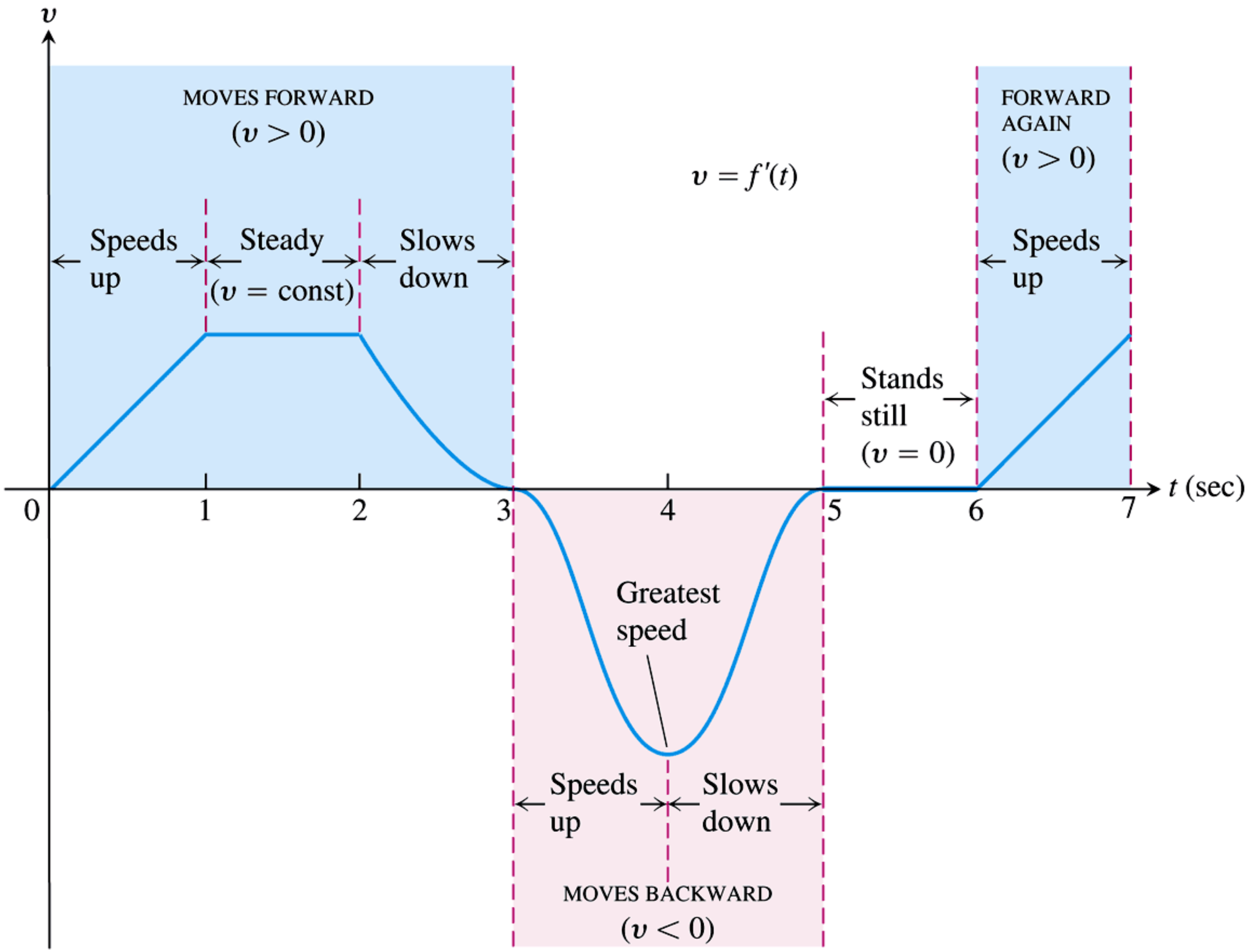


FIGURE 3.15 The velocity graph for Example 3.

DEFINITIONS Acceleration, Jerk

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

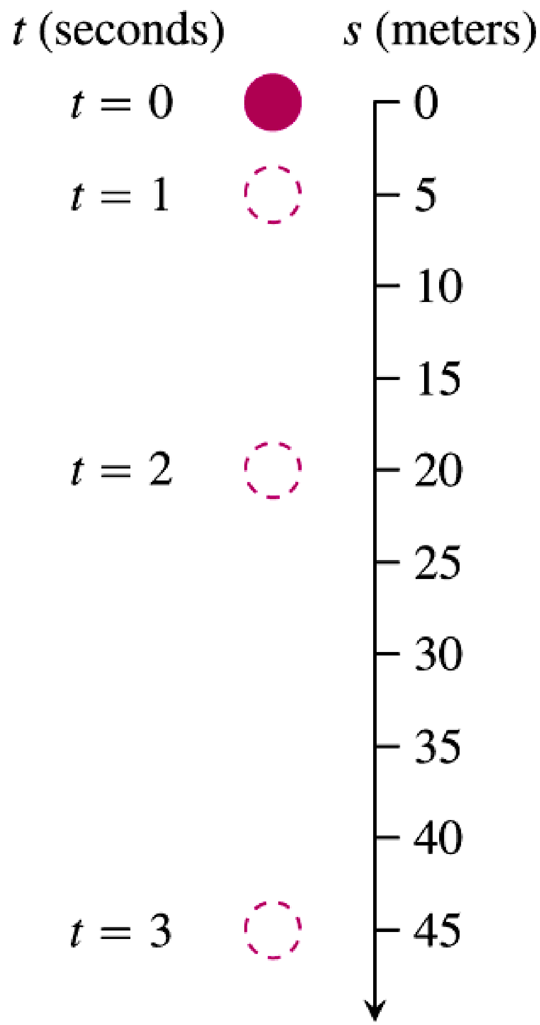
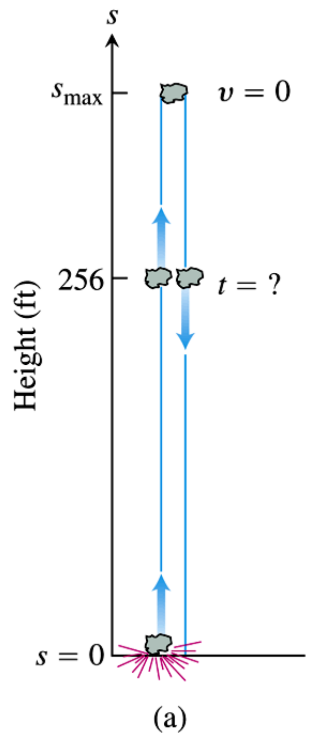


FIGURE 3.16 A ball bearing falling from rest (Example 4).



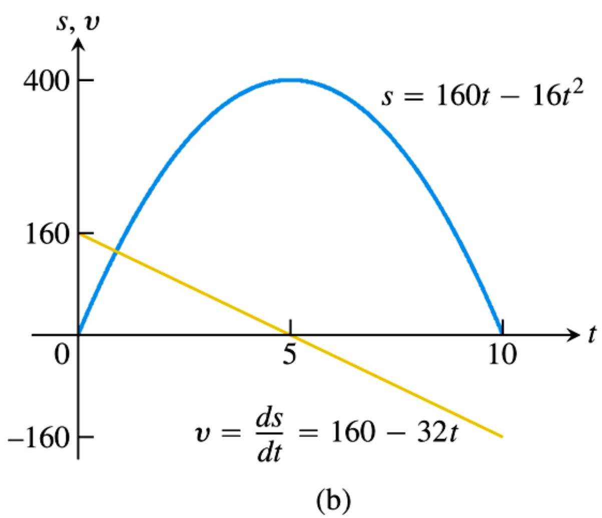


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

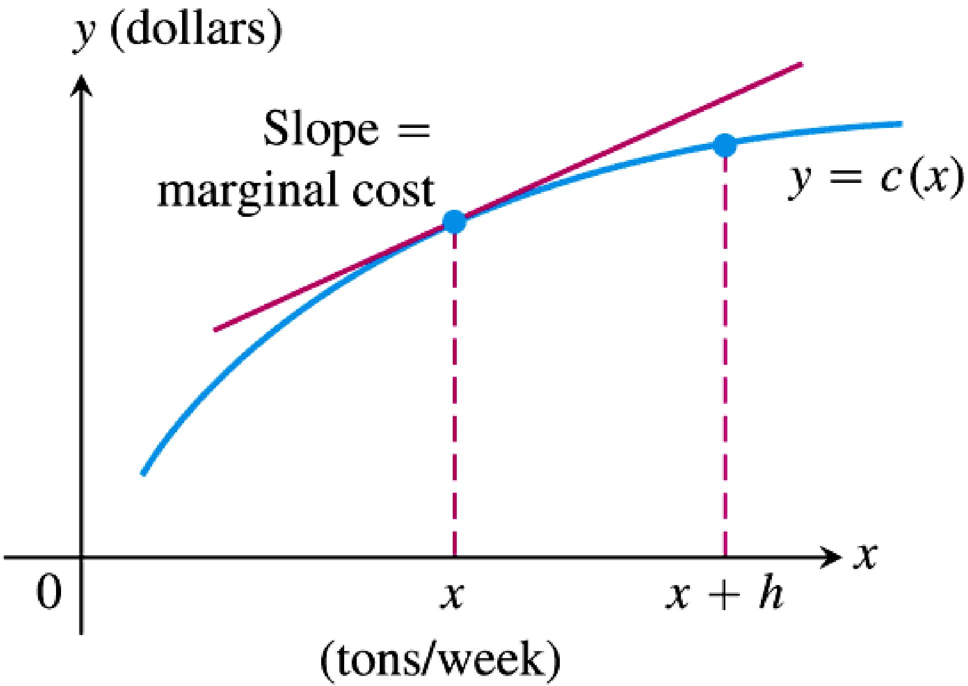


FIGURE 3.18 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

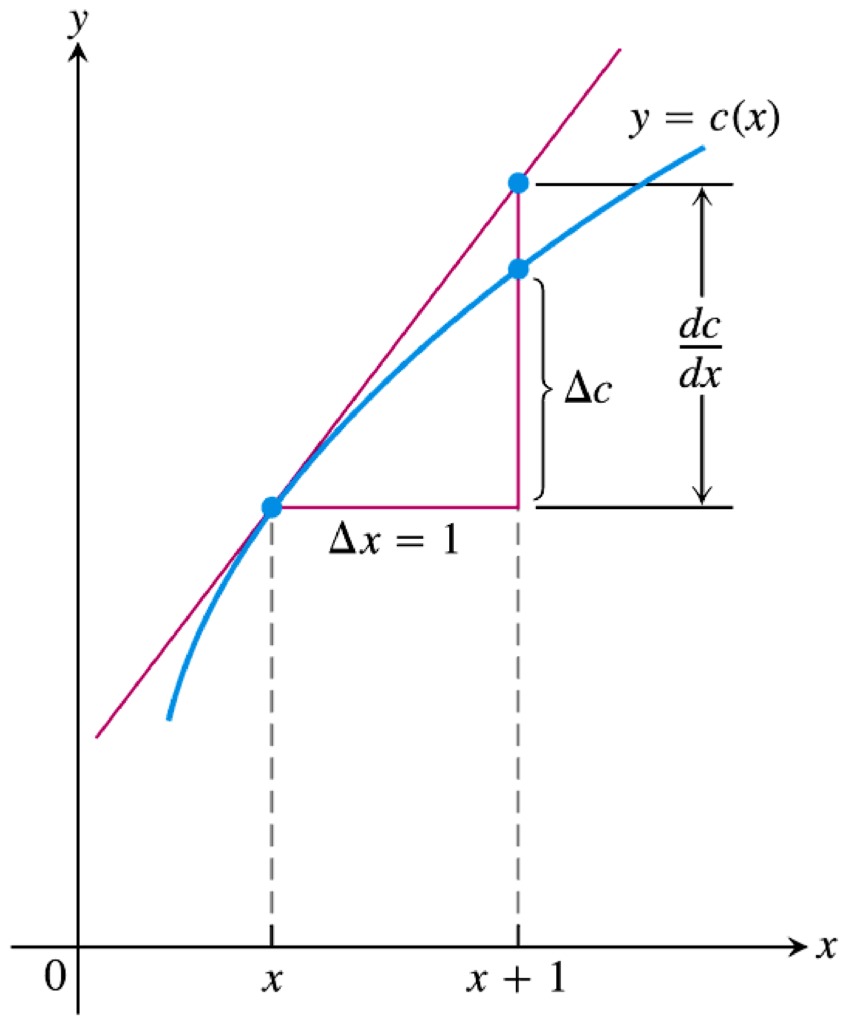


FIGURE 3.19 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

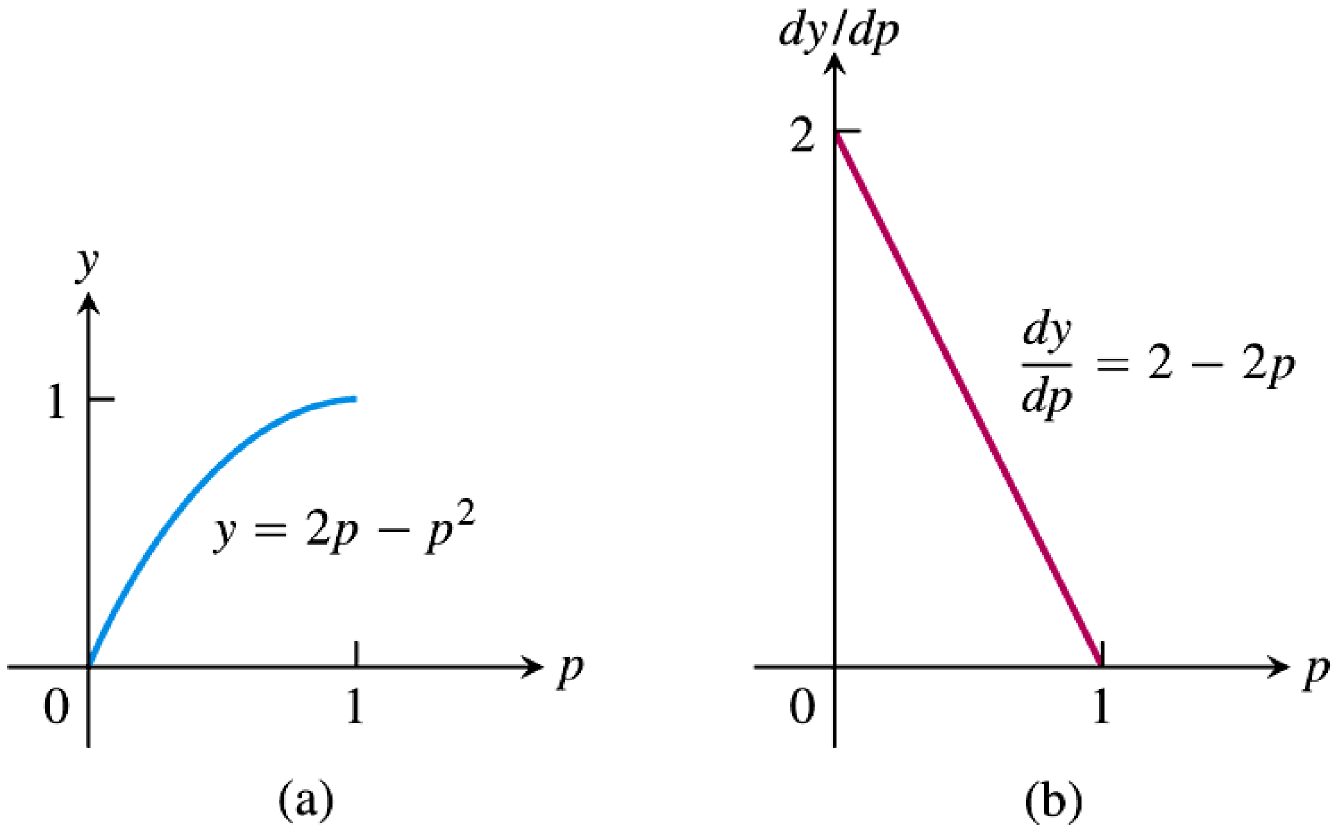


FIGURE 3.20 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas. (b) The graph of dy/dp (Example 8).

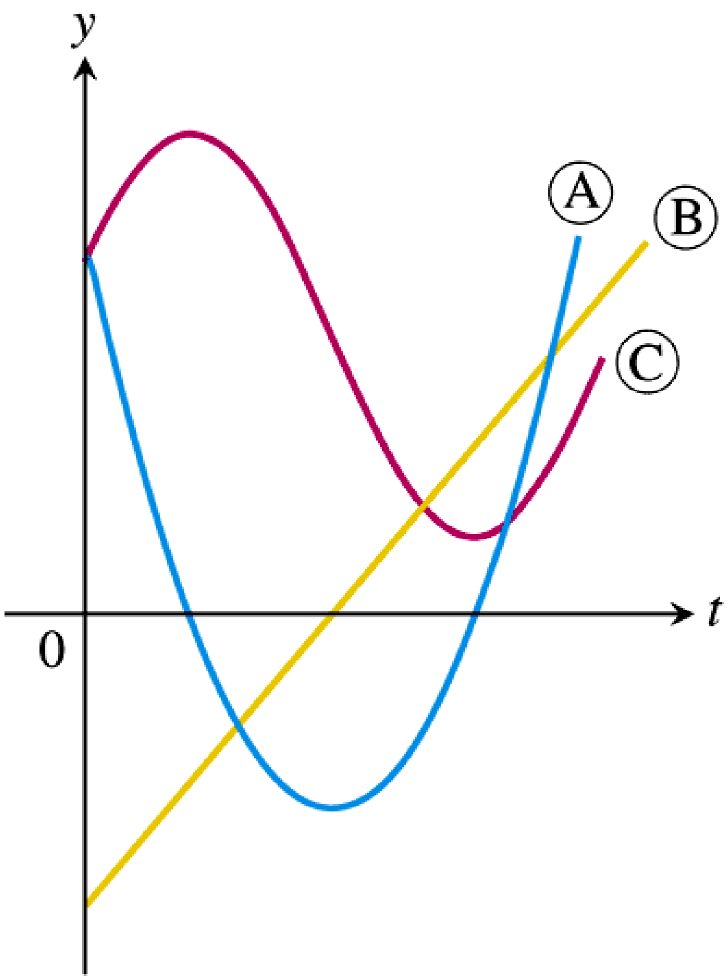


FIGURE 3.21 The graphs for Exercise 21.

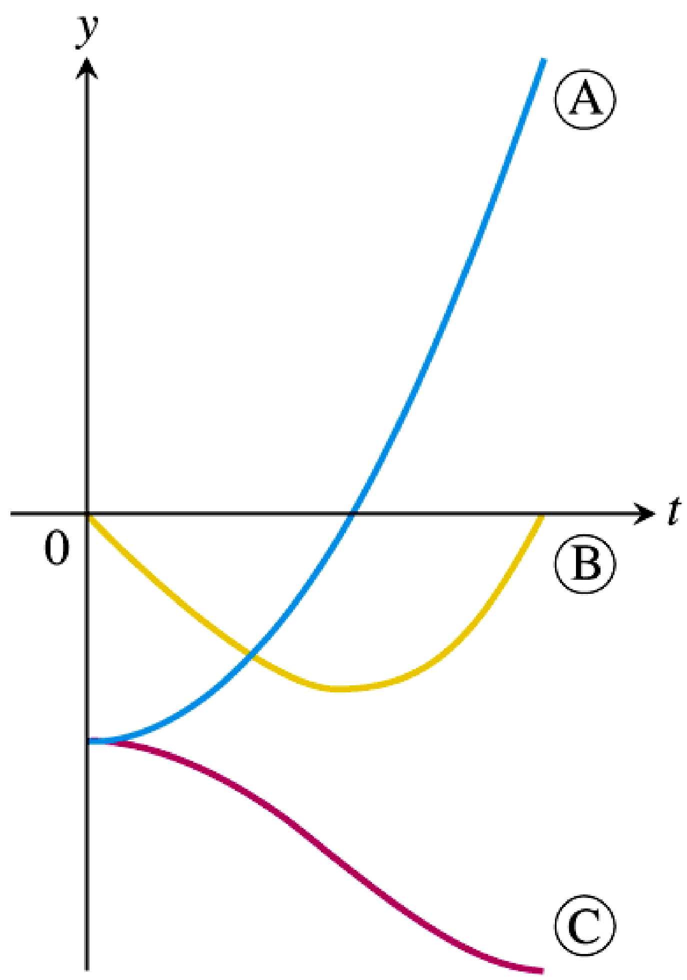


FIGURE 3.22 The graphs for Exercise 22.

3.4

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

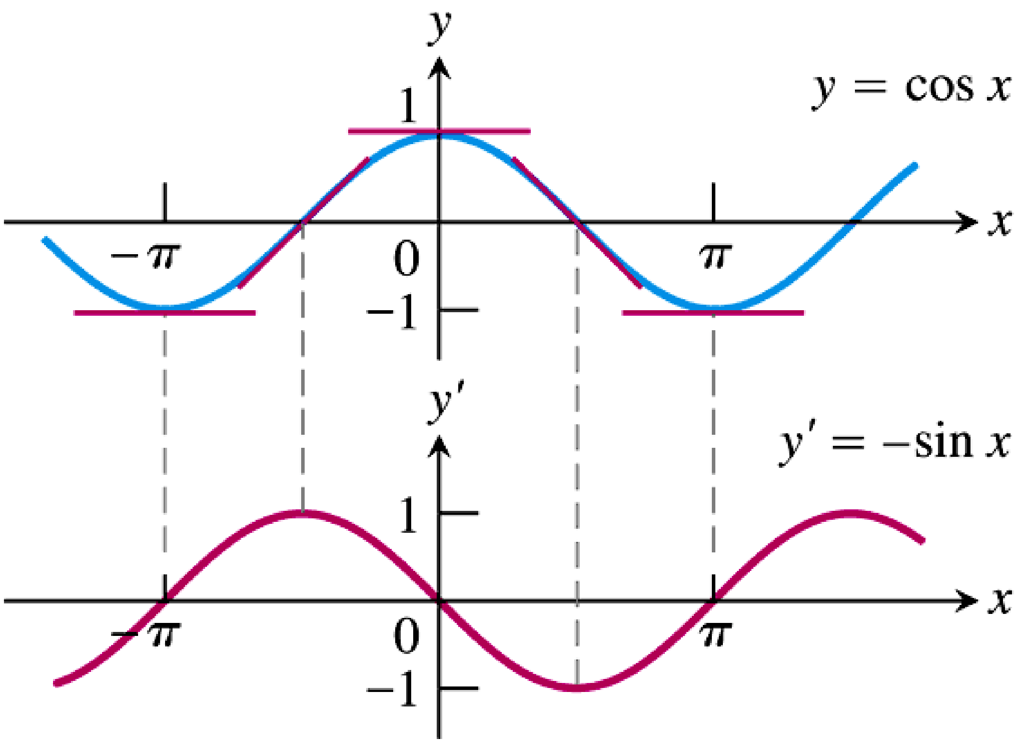


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

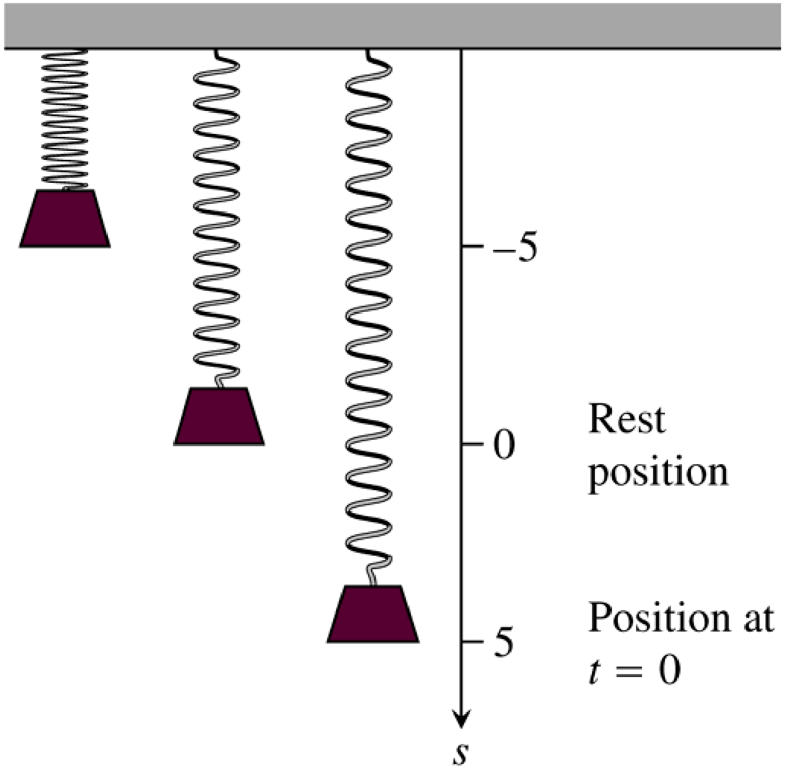


FIGURE 3.24 A body hanging from a vertical spring and then displaced oscillates above and below its rest position. Its motion is described by trigonometric functions (Example 3).

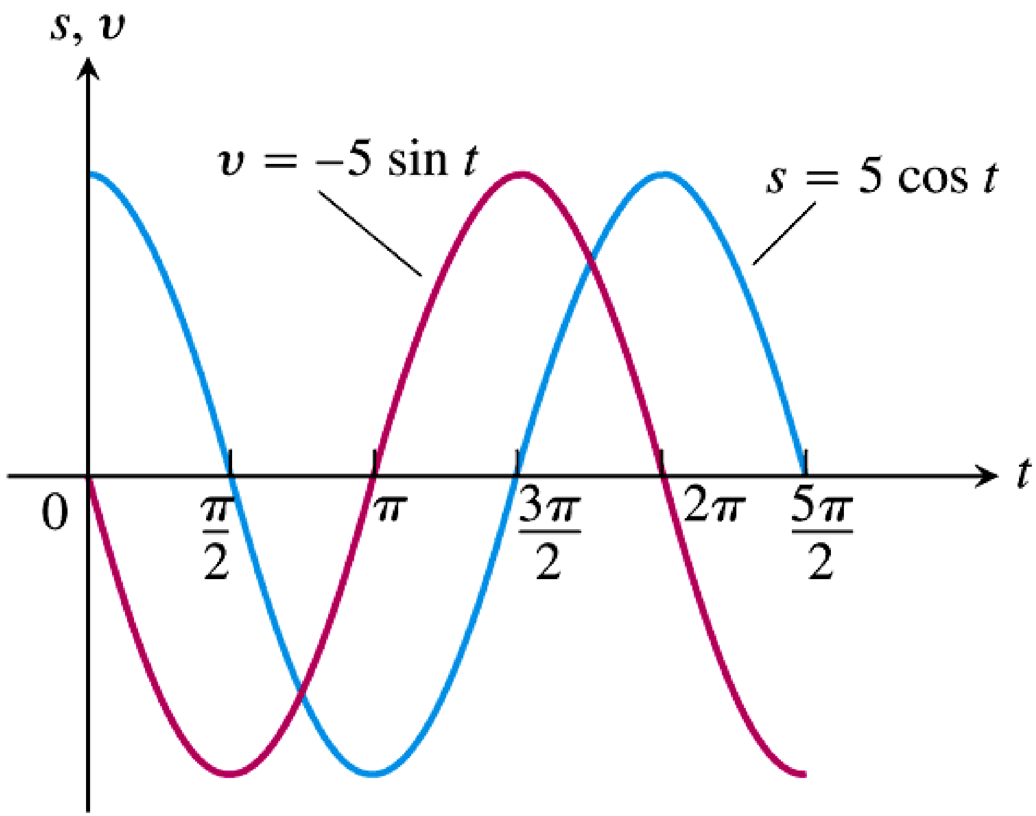


FIGURE 3.25 The graphs of the position and velocity of the body in Example 3.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

3.5

The Chain Rule and Parametric Equations

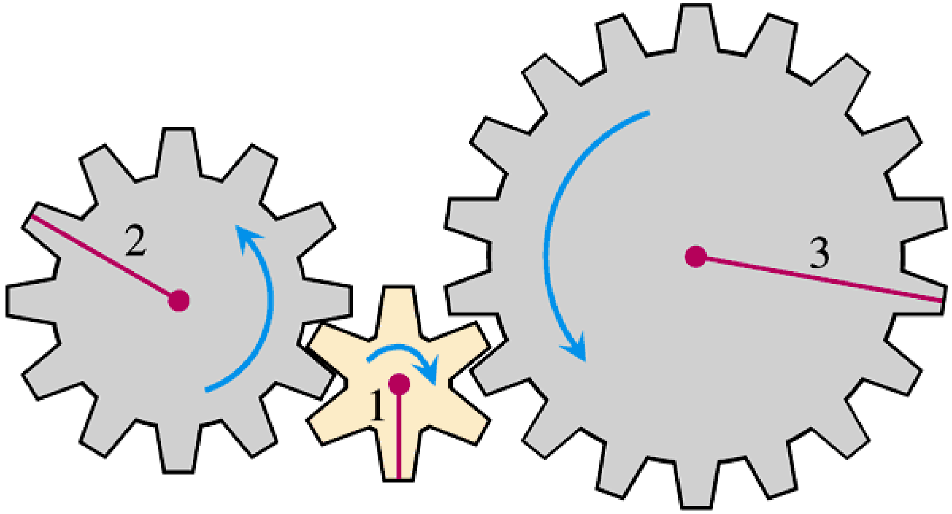


FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

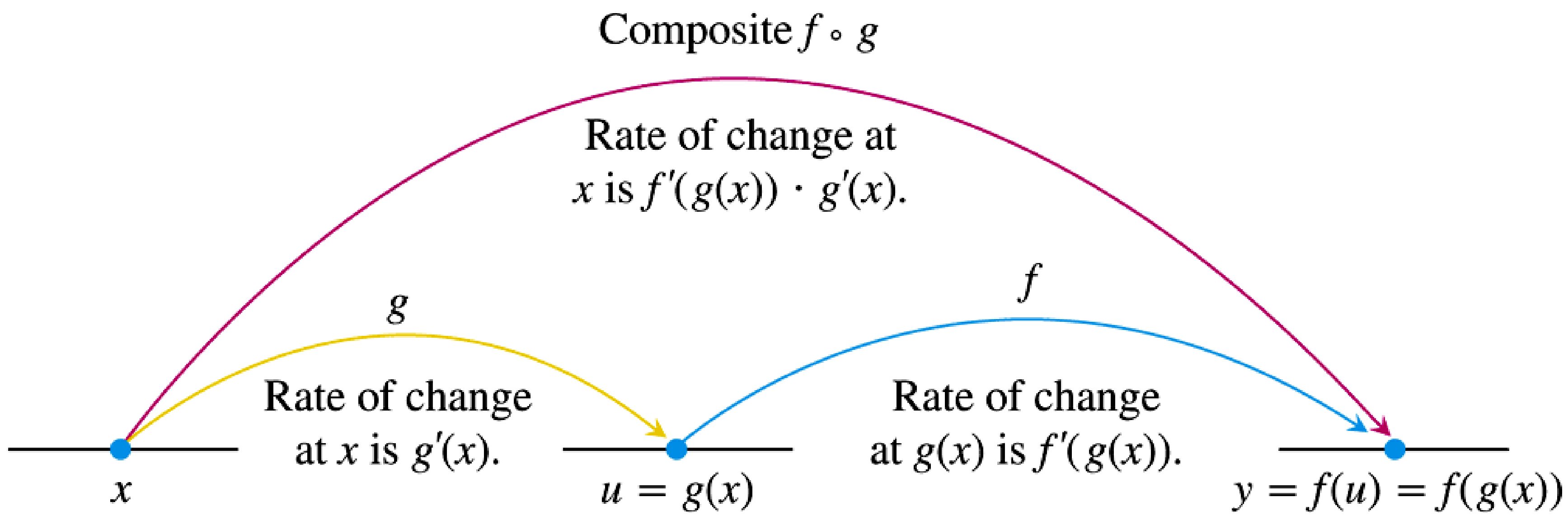


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

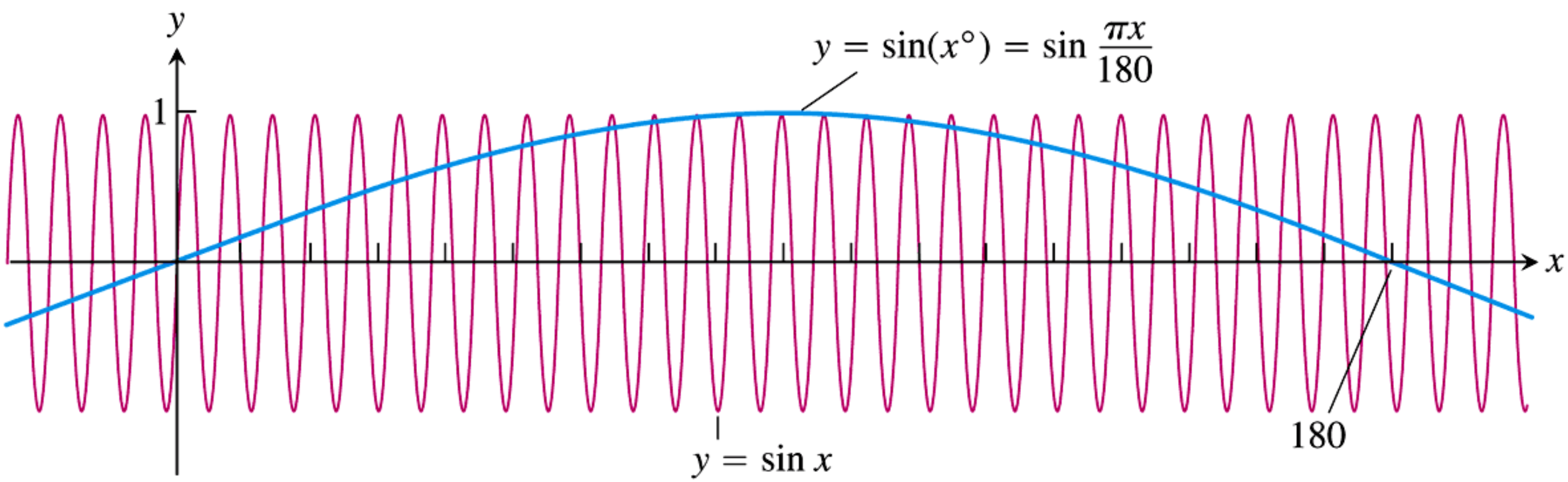


FIGURE 3.28 Sin $(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 8).

DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

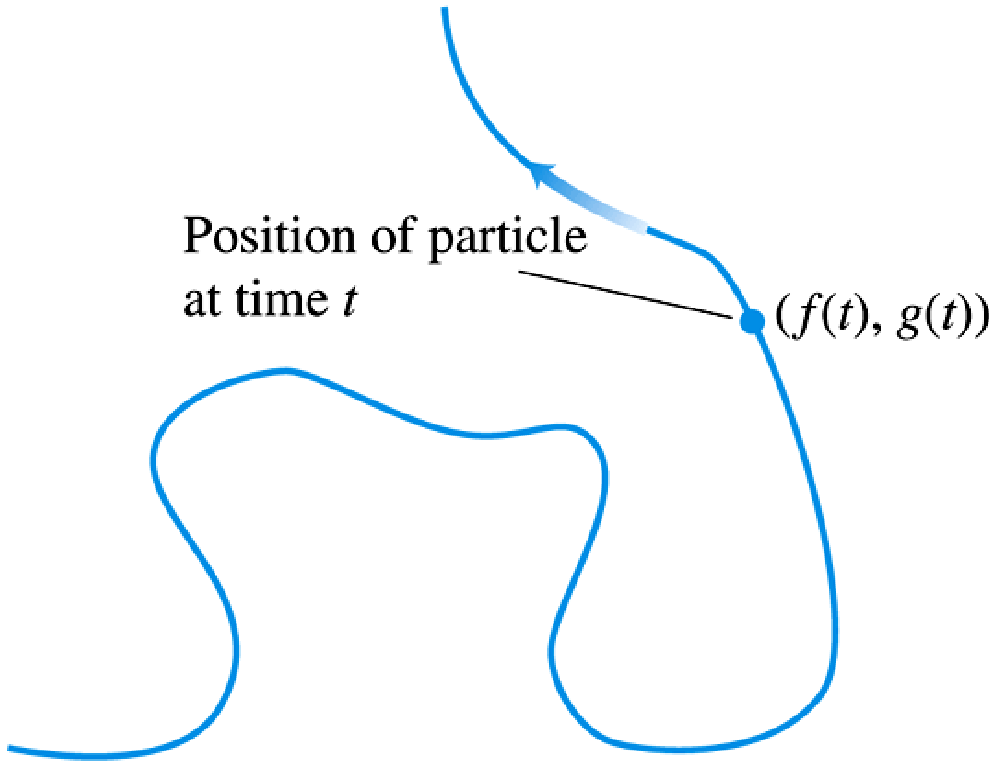


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

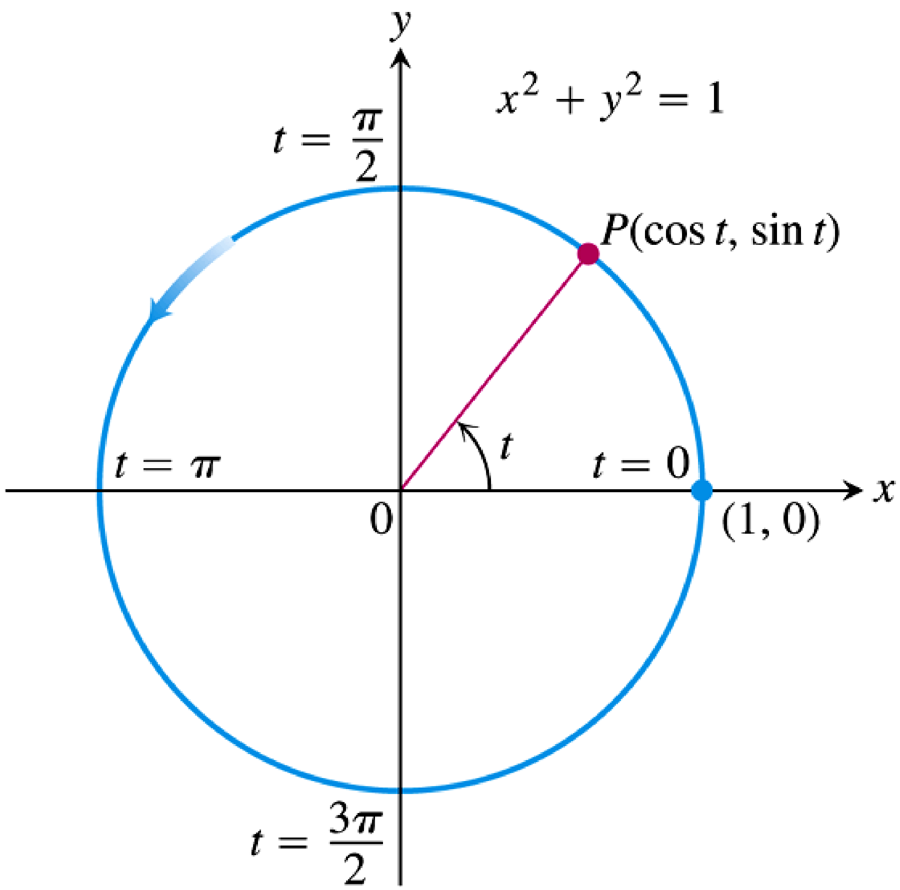


FIGURE 3.30 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 9).

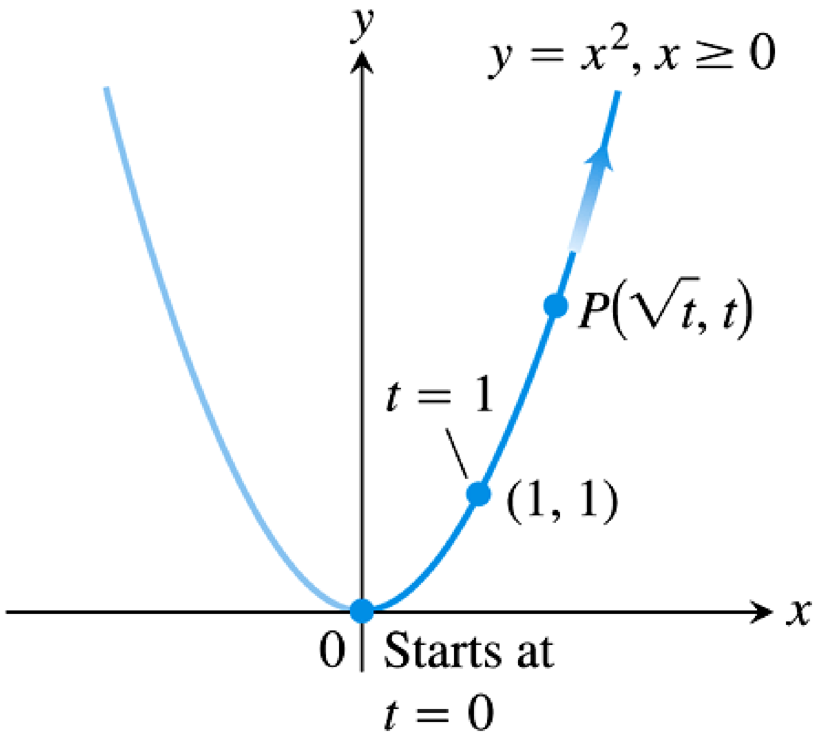


FIGURE 3.31 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{2}$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \quad (3)$$

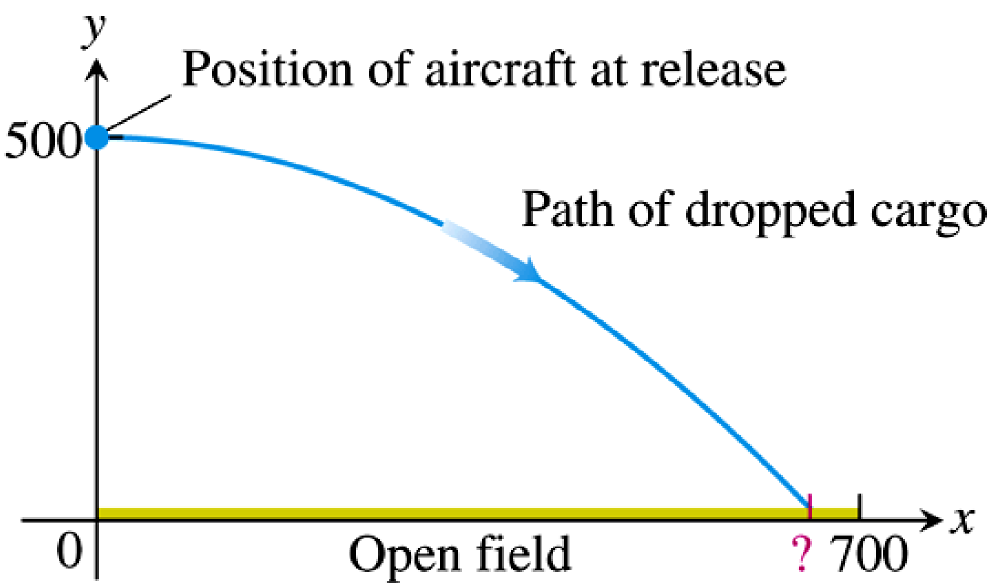


FIGURE 3.32 The path of the dropped cargo of supplies in Example 15.

Standard Parametrizations and Derivative Rules

CIRCLE $x^2 + y^2 = a^2$:

$$x = a \cos t$$
$$y = a \sin t$$
$$0 \le t \le 2\pi$$

ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

$$x = a \cos t$$
$$y = b \sin t$$
$$0 \le t \le 2\pi$$

FUNCTION $y = f(x)$:

$$x = t$$
$$y = f(t)$$

DERIVATIVES

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

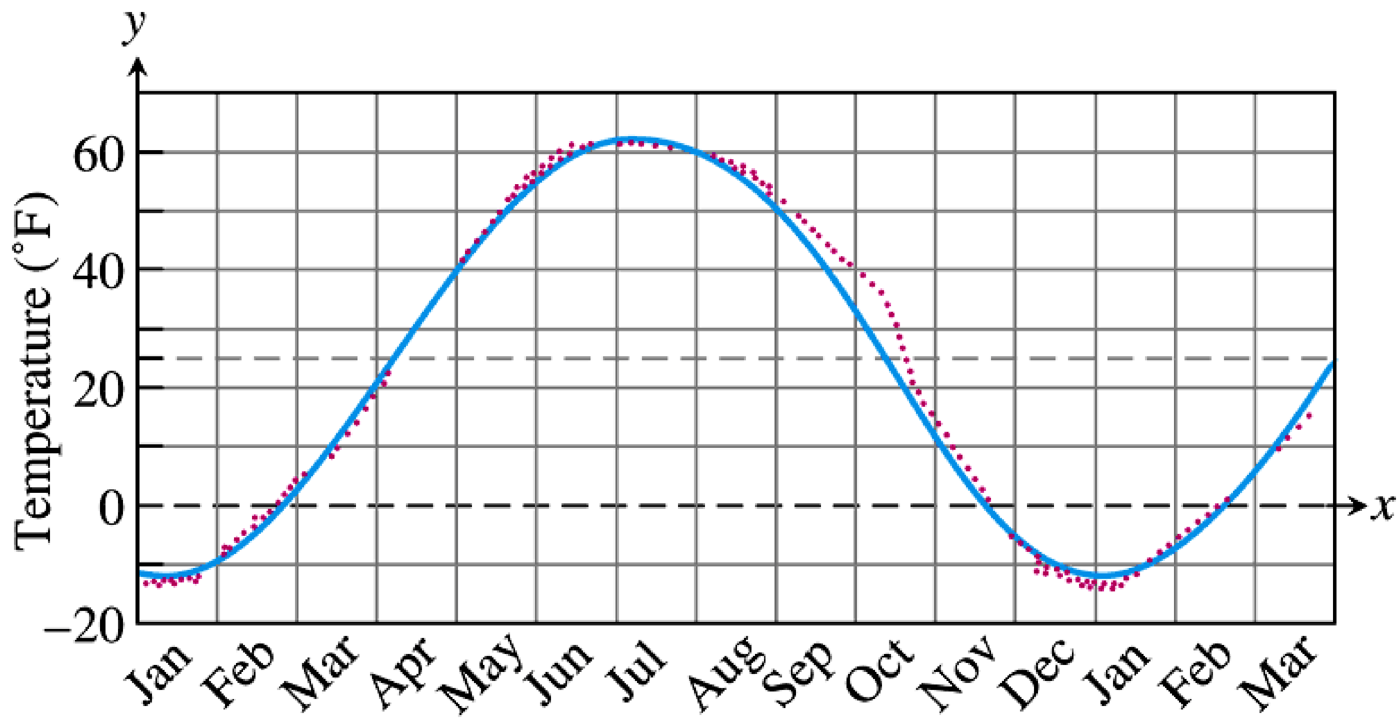


FIGURE 3.33 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 96).

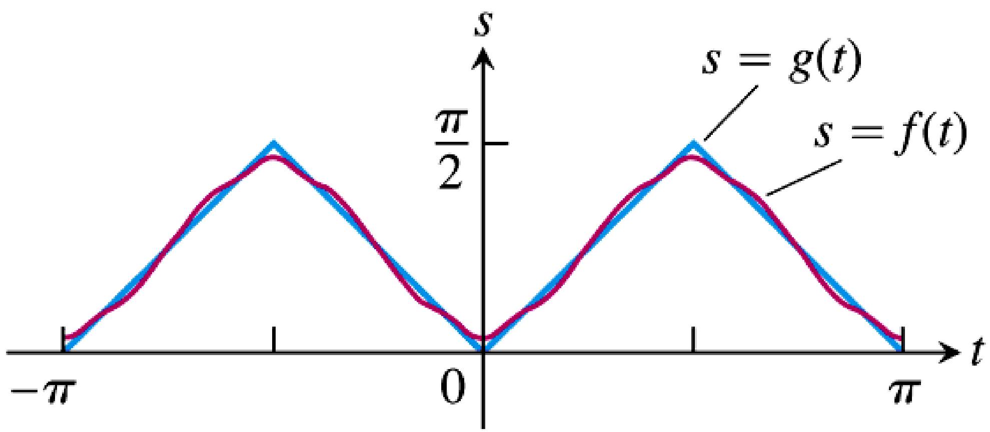


FIGURE 3.34 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 111).

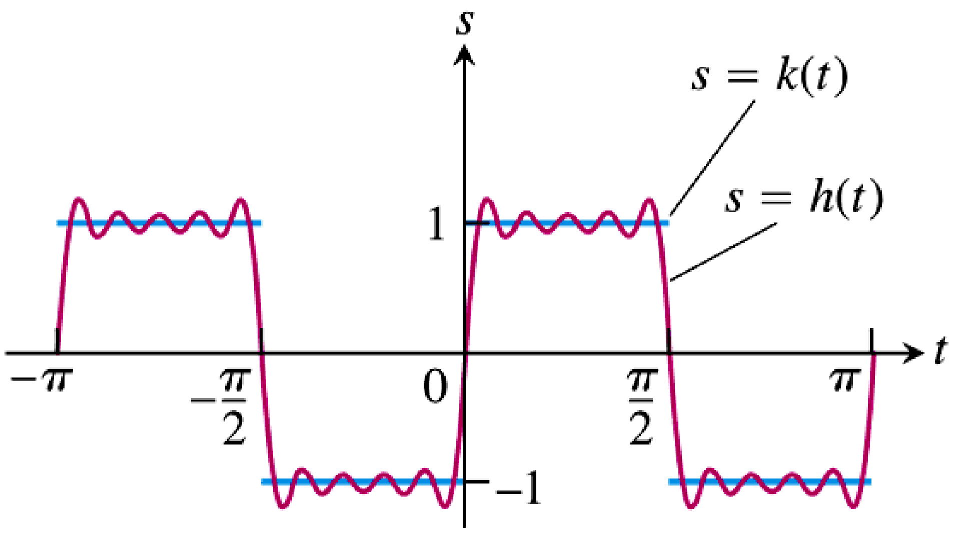


FIGURE 3.35 The approximation of a step function by a trigonometric "polynomial" (Exercise 112).

3.6

Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

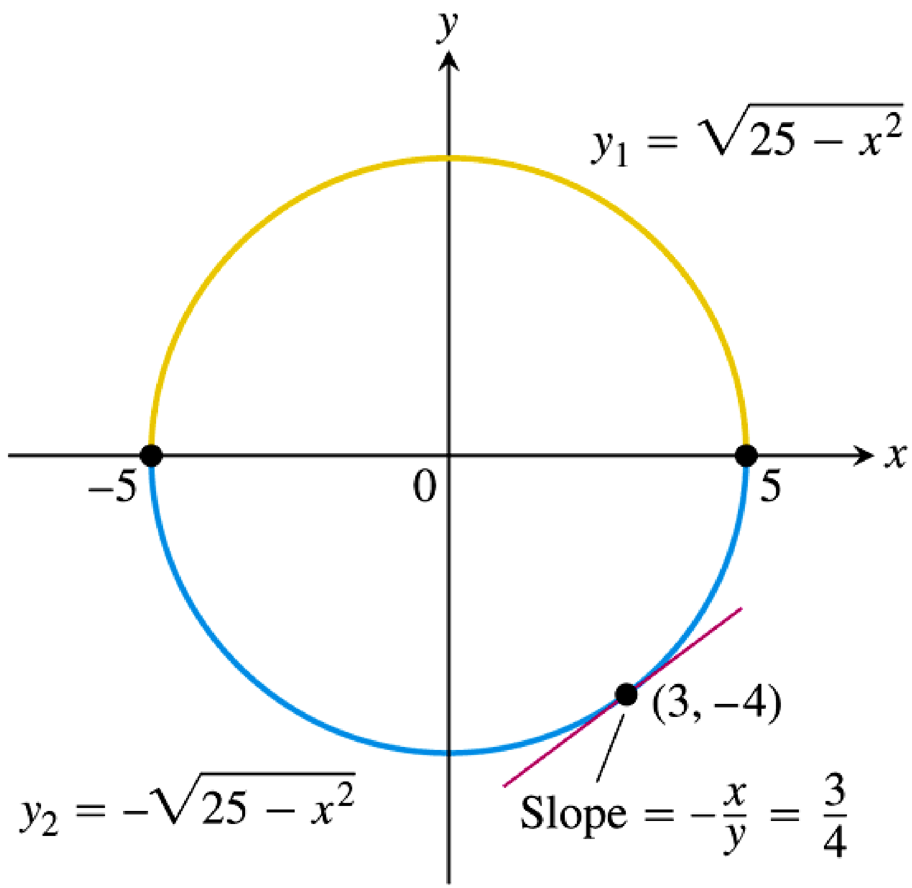


FIGURE 3.36 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

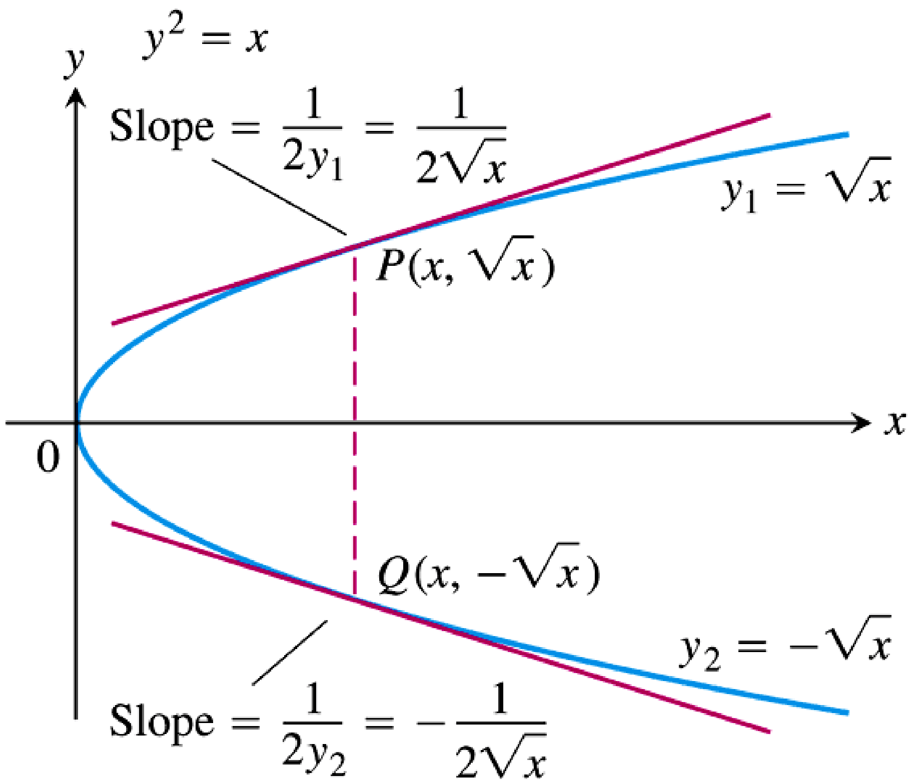


FIGURE 3.37 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

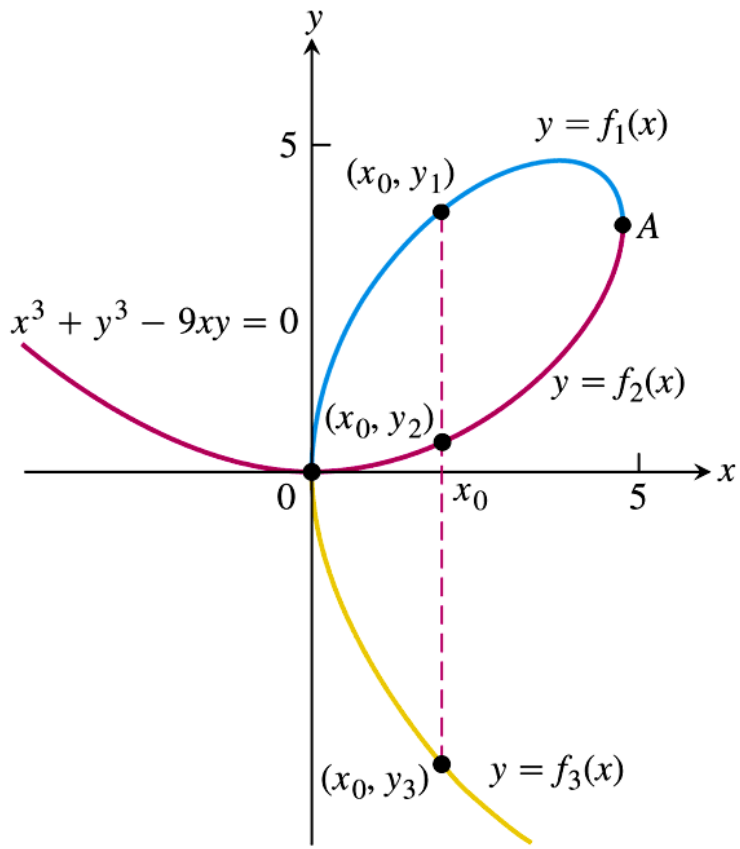


FIGURE 3.38 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

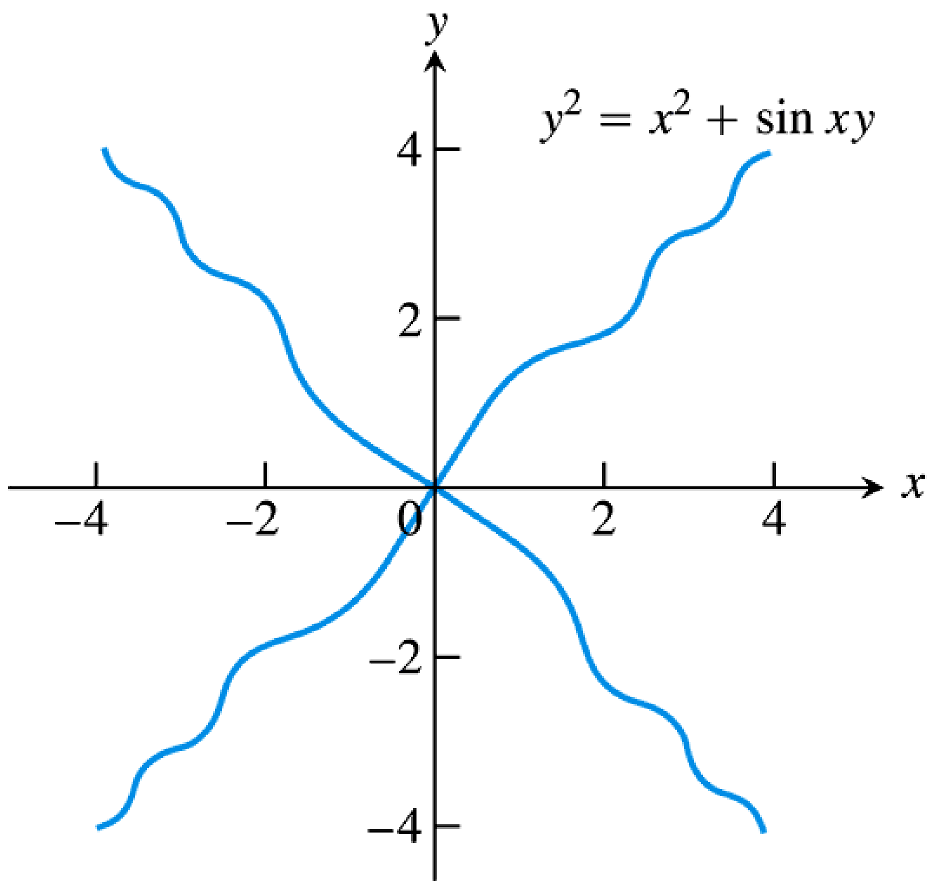


FIGURE 3.39 The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

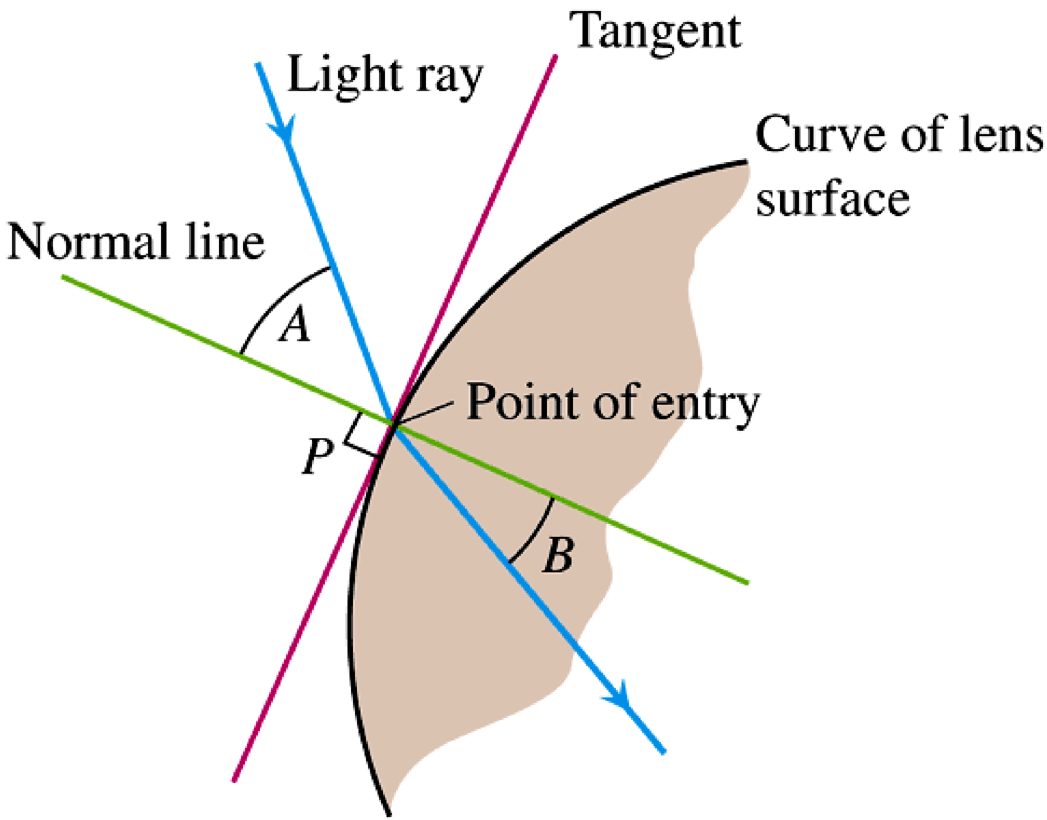


FIGURE 3.40 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

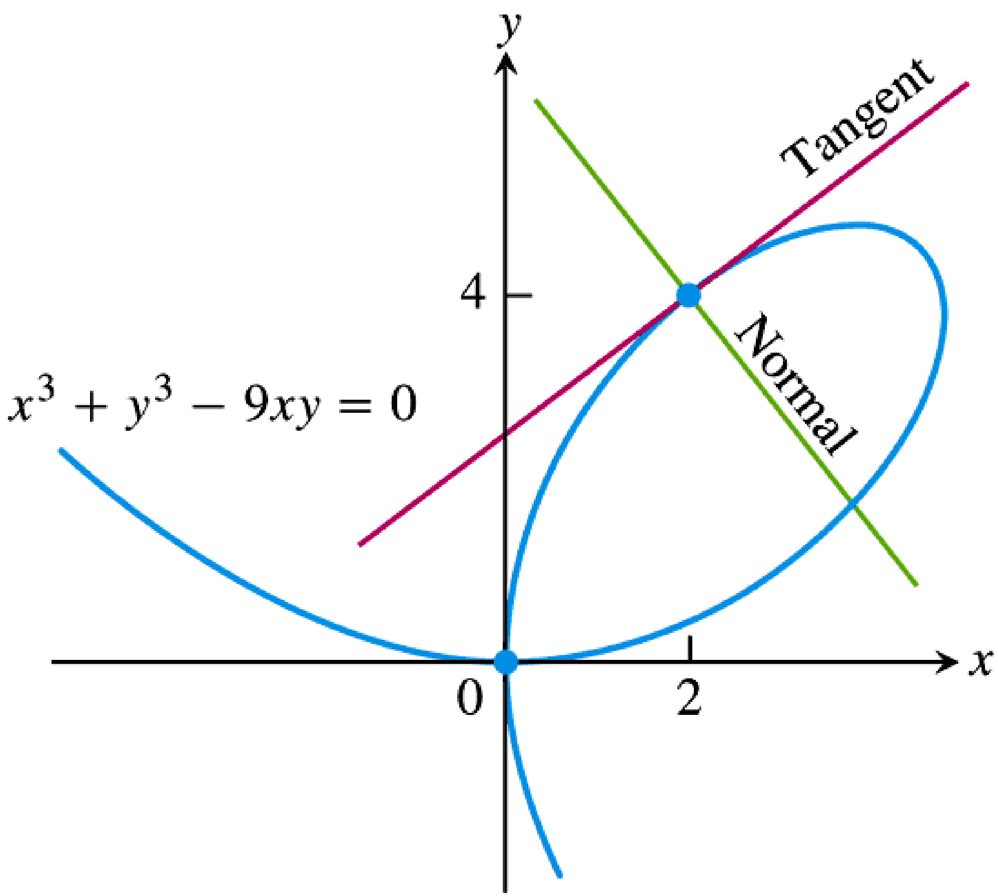


FIGURE 3.41 Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

THEOREM 4 Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

3.7

Related Rates

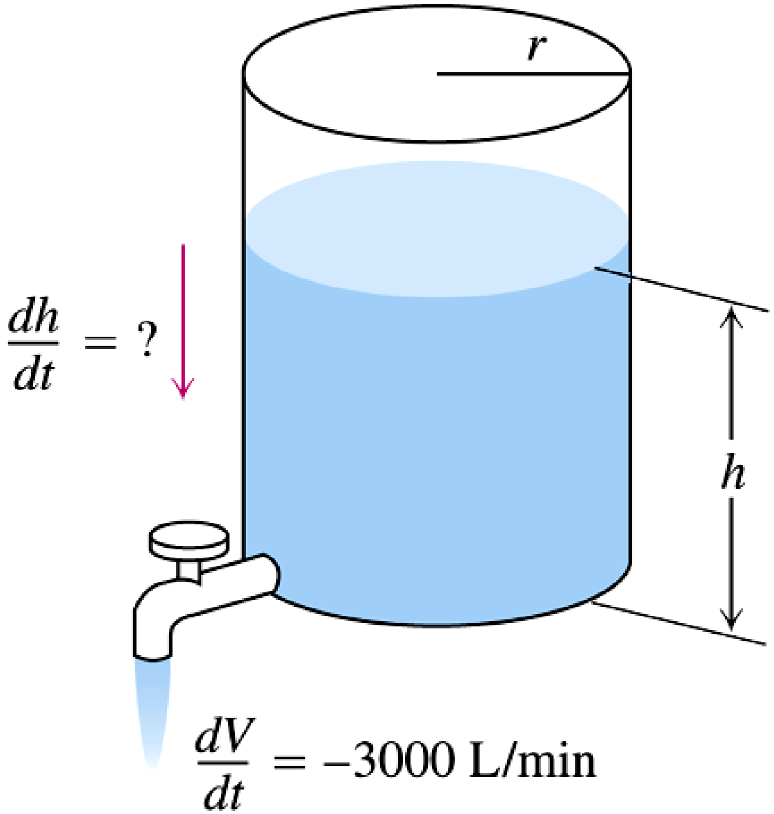


FIGURE 3.42 The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

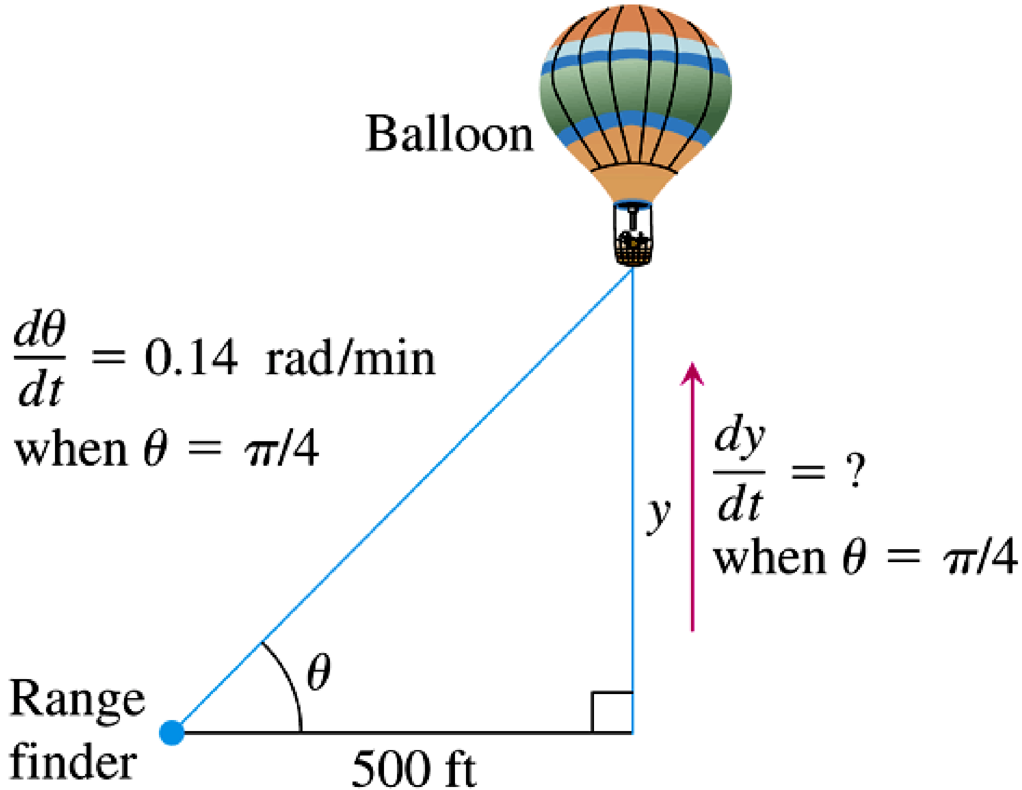


FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

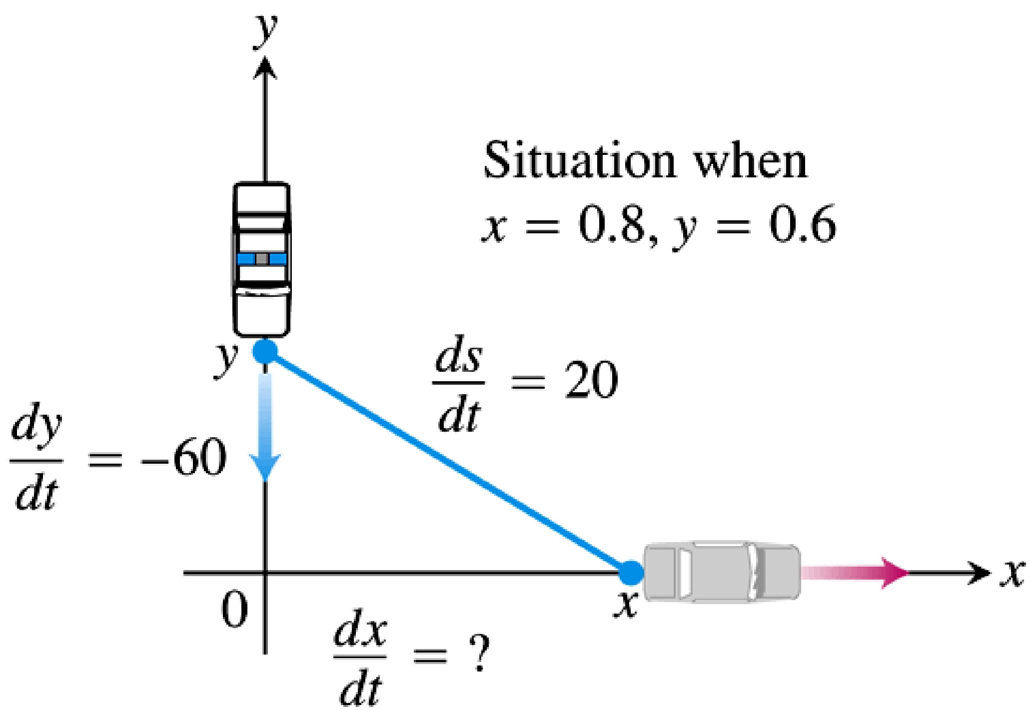


FIGURE 3.44 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

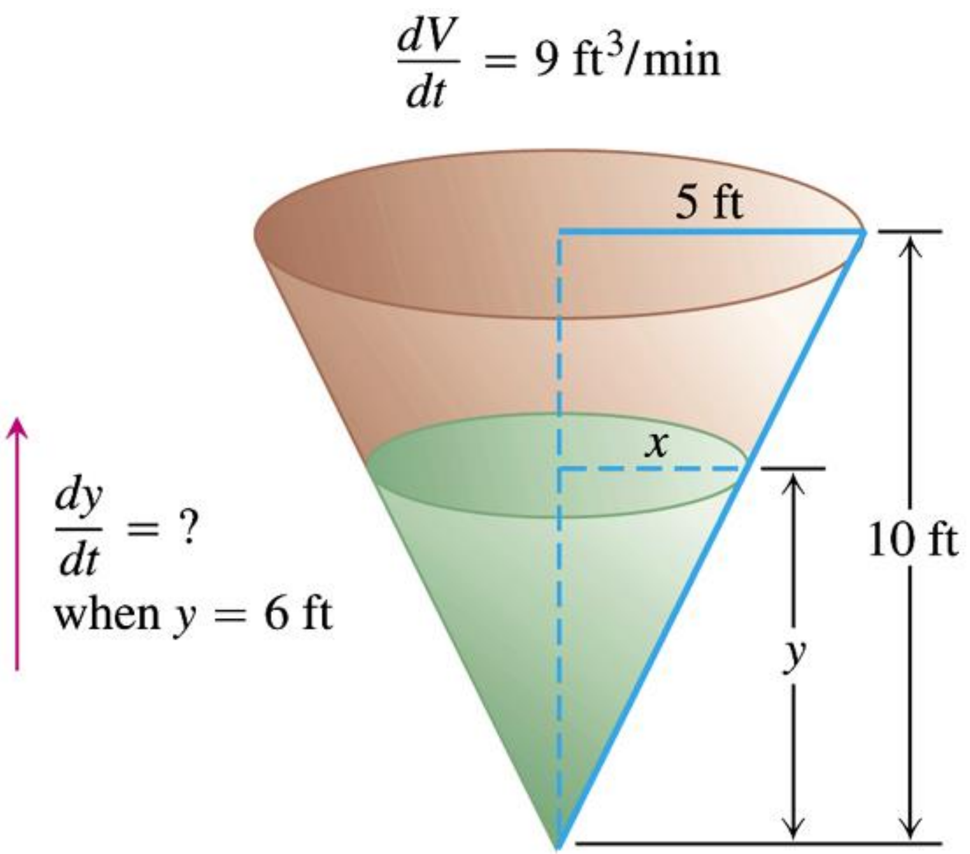
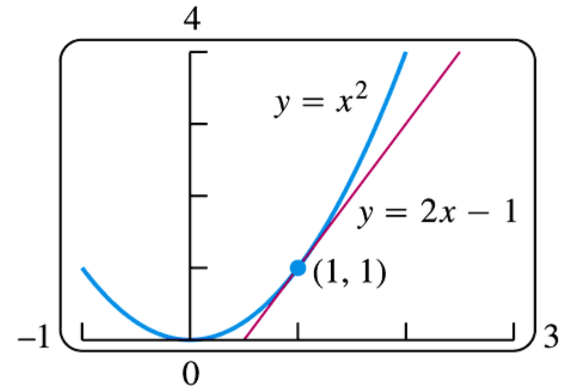


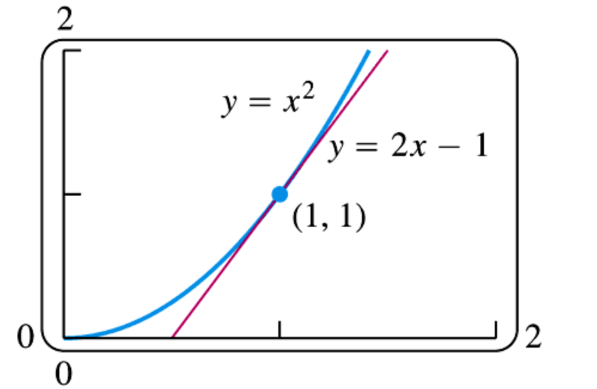
FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 4).

3.8

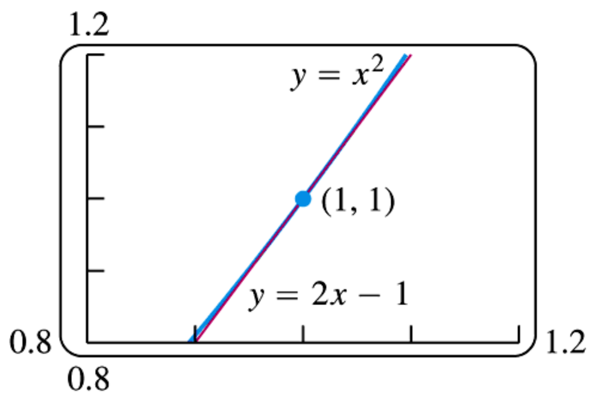
Linearization and Differentials



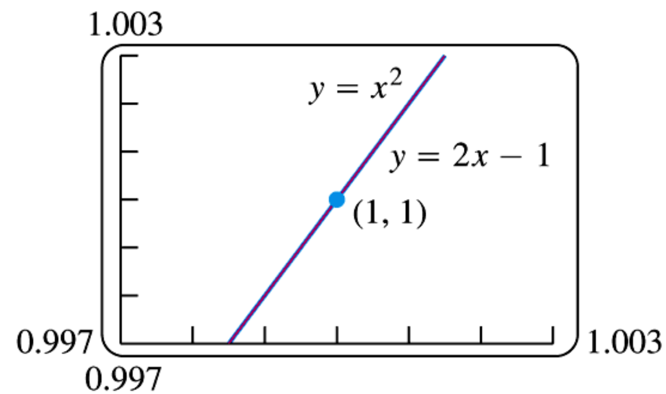
$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).



Tangent and curve very close near (1, 1).



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

FIGURE 3.46 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

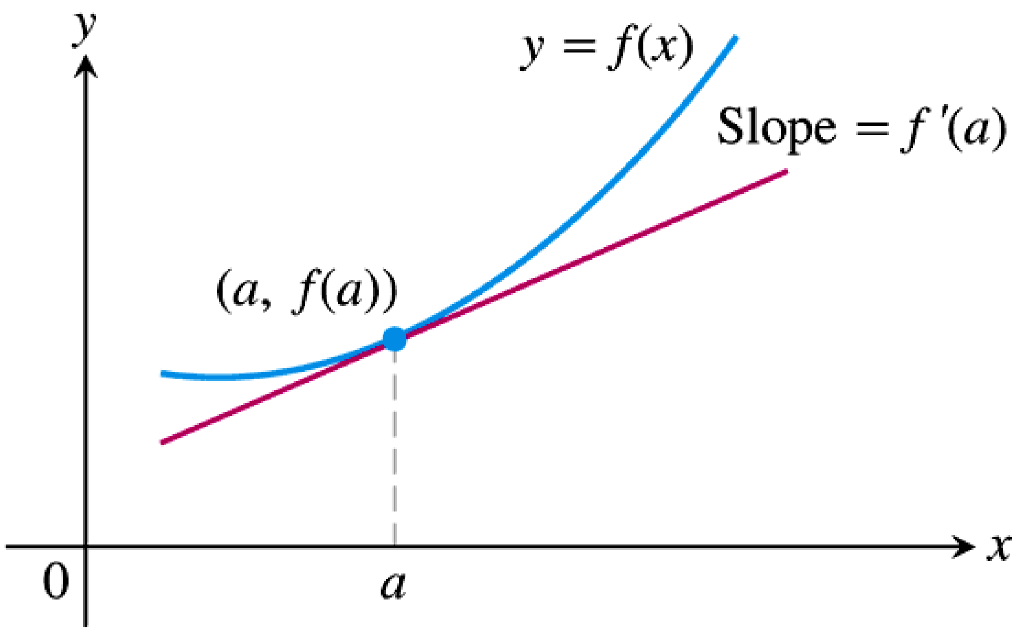


FIGURE 3.47 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

DEFINITIONS Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

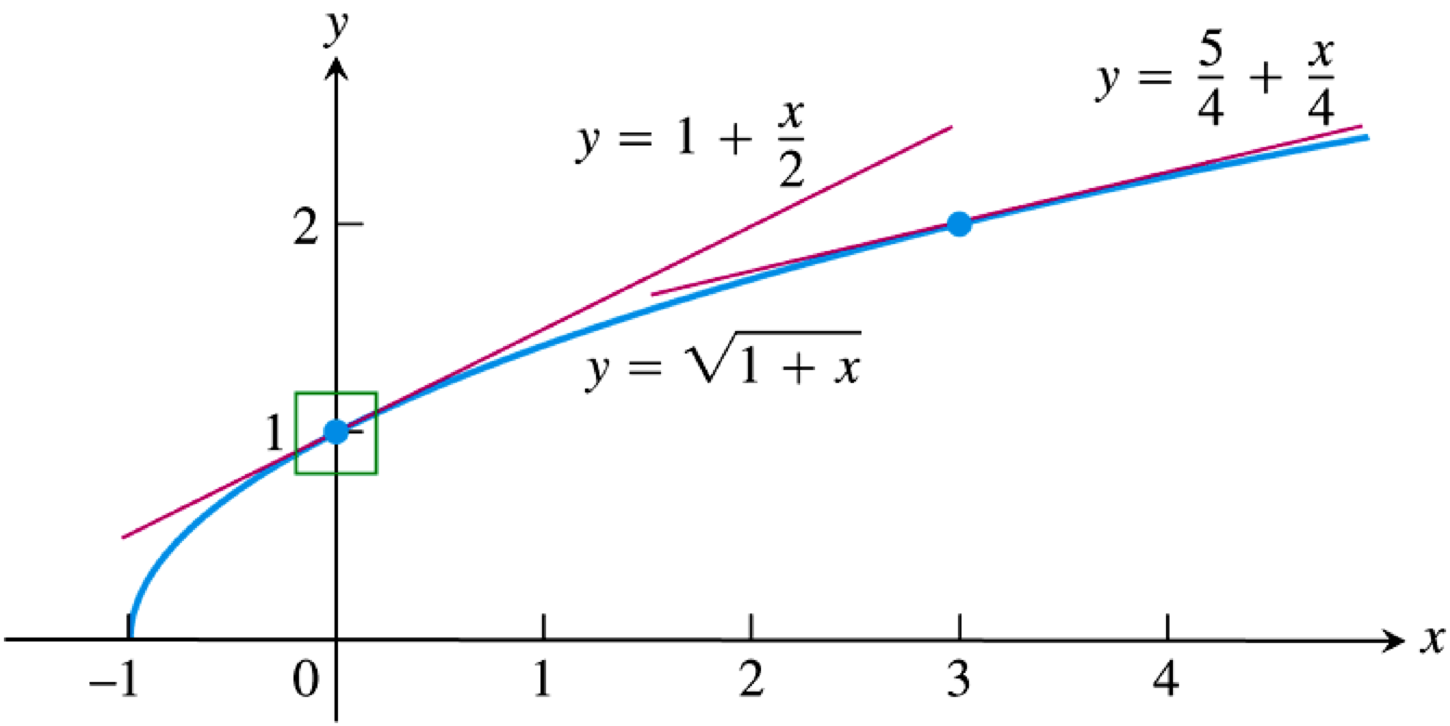


FIGURE 3.48 The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.49 shows a magnified view of the small window about 1 on the y-axis.

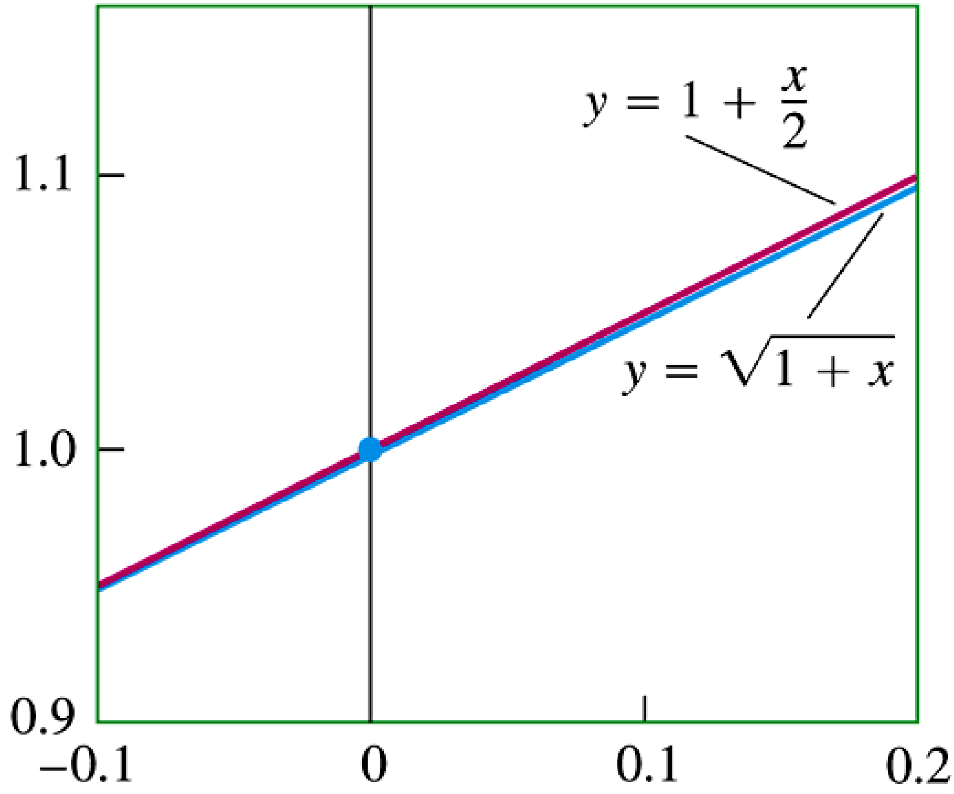


FIGURE 3.49 Magnified view of the window in Figure 3.48.

Approximation	True value	$\lvert \text{True value} - \text{approximation} \rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

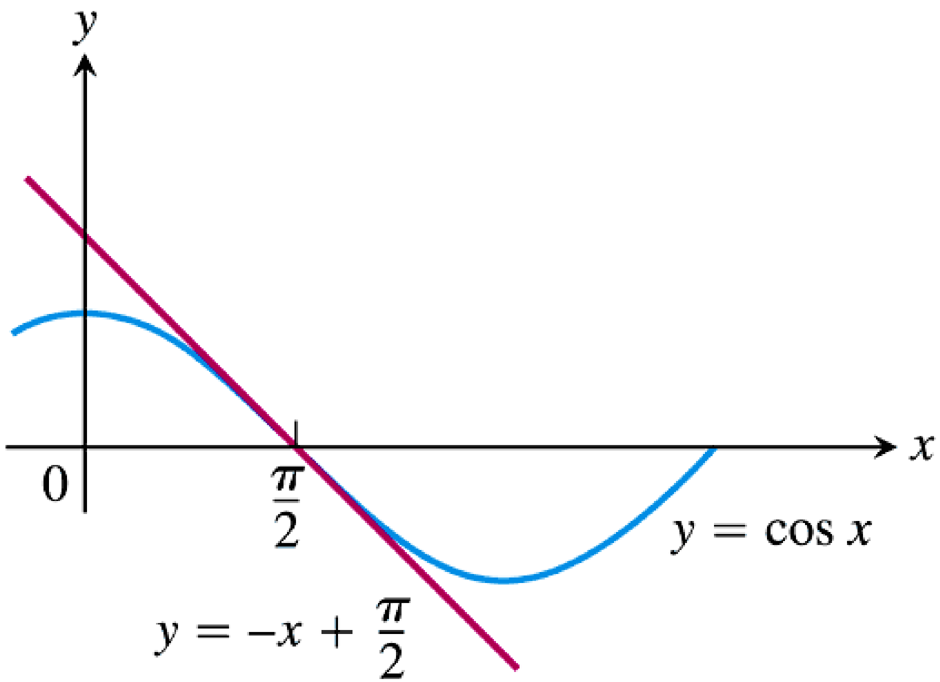


FIGURE 3.50 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

DEFINITION **Differential**

Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$dy = f'(x)\,dx.$$

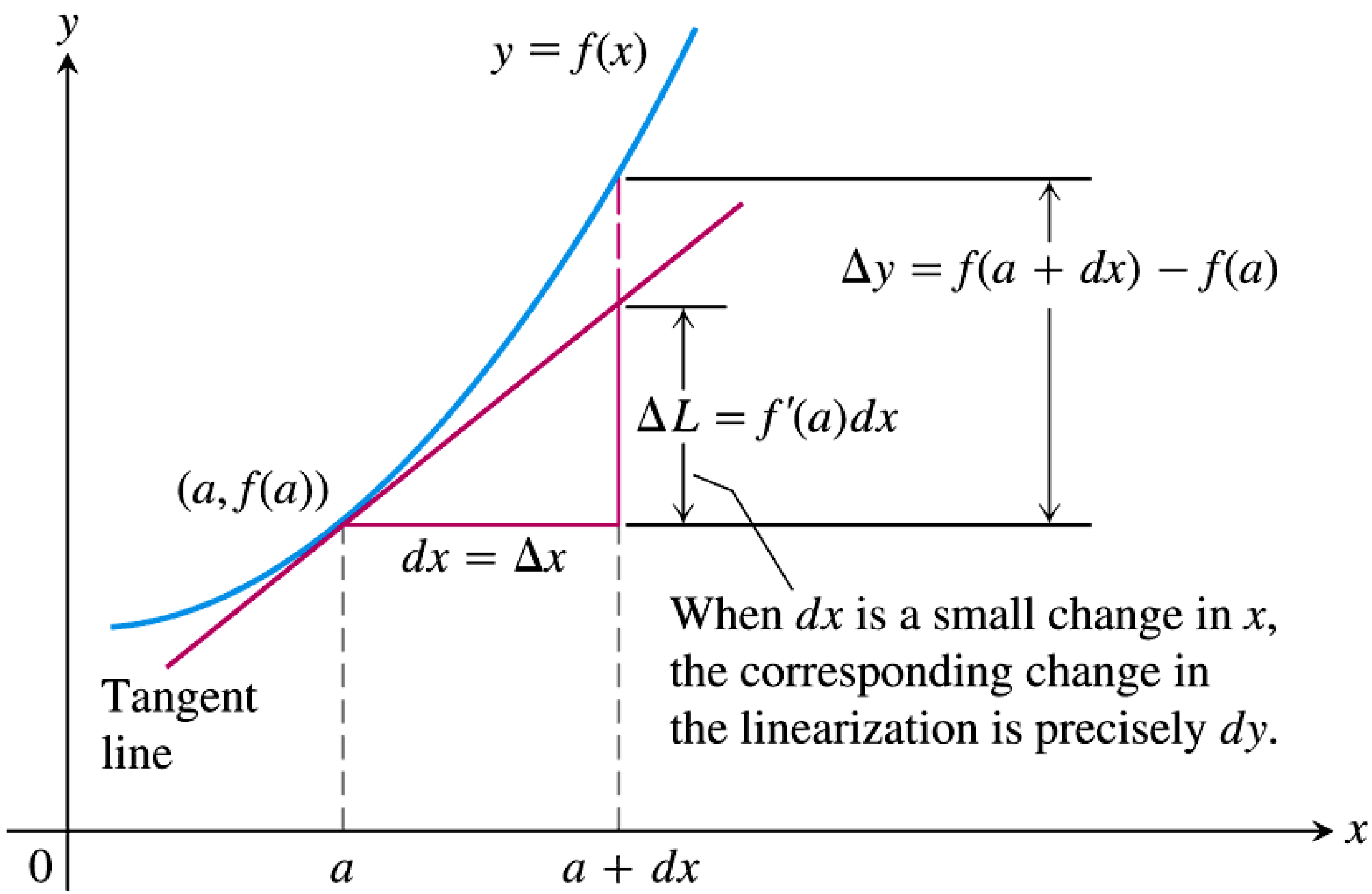


FIGURE 3.51 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

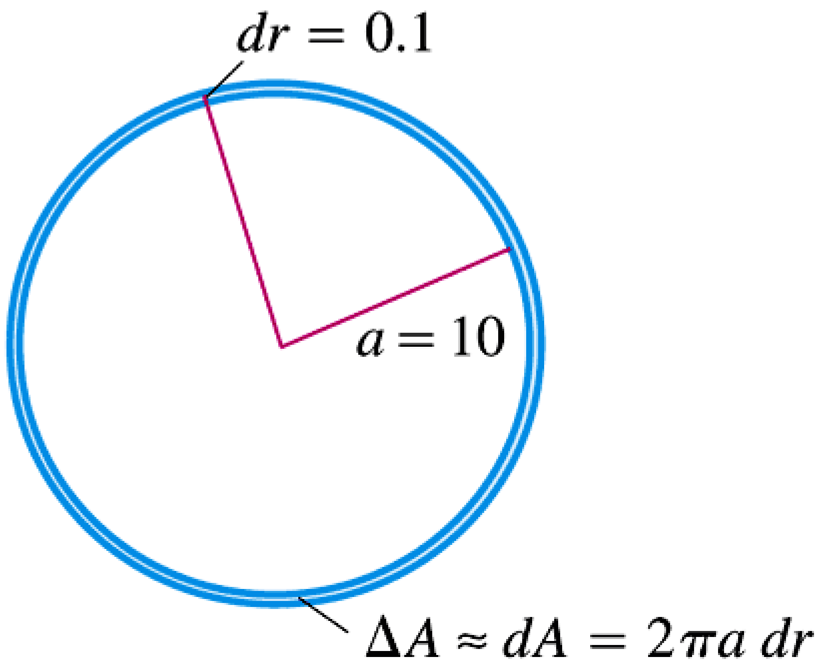


FIGURE 3.52 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a way to estimate the area of the circle with radius $r = a + dr$ (Example 6).

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by an equation of the form

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

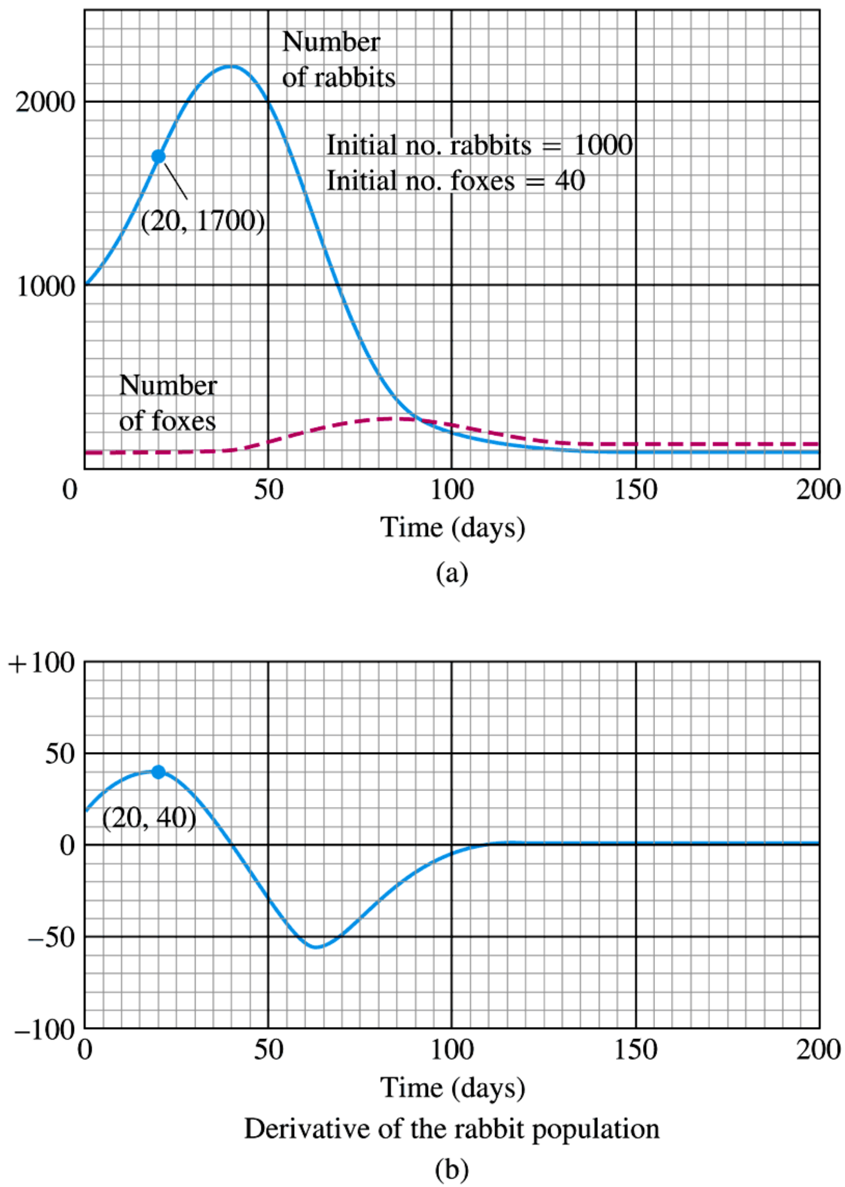


FIGURE 3.53 Rabbits and foxes in an arctic predator-prey food chain.