السعدي ALSAADI

نسخة جديدة منقحة

1432/33

9

Chapter Three
Limits and Continuity

3.2

Calculating Limits Using The Limits Laws

MATH-110

جمال السعدي
رياضيات - إحصاء

[illegible]ing Limits
Using the Li[illegible]s

In this section:

To calculate limits we use the following properties of limits called " The limits Laws "

Limits Laws:

suppose that : $\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a} g(x) = m$

and c is constant .

1. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = L \pm M$

* توزيع النهاية على الجمع والطرح

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2. $\lim\limits_{x \to a} [f(x)\, g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = L . M$

* توزيع النهايه على الضرب

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3. $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{L}{M}$ (If: $\lim\limits_{x \to a} g(x) \neq 0$)

* توزيع النهايه على القسمه

4 $\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x) = c \cdot L$

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5 $\lim_{x \to a} c = c$ نهايه الثابت نفس الثابت

Where c is constant

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6 $\lim_{x \to a} x = a$ التعويض عن x بـ a

---

7 $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n = L^n$

---

8 $\lim_{x \to a} x^n = a^n$ التعويض عن x بـ a

---

9 $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If: n is even (زوجى) a must be positive (موجب))

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10 $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$

(If : n is even L must be positive)

Example:

Evaluate the following limits

1. $\lim\limits_{x \to 5} (2x^2 - 3x + 4)$ [*by direct substitution*]

بالتعويض المباشر

$= 2 \lim\limits_{x \to 5} x^2 - 3 \lim\limits_{x \to 5} x + \lim\limits_{x \to 5} 4 \longleftarrow$ * ممكن عدم كتابه هذه الخطوه

$= 2 (5)^2 - 3(5) + 4$

$= 50 - 15 + 4$

$= 39$

---

2. $\lim\limits_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

$= \dfrac{\lim\limits_{x \to -2} x^3 + 2 \lim\limits_{x \to -2} x^2 - \lim\limits_{x \to -2} 1}{\lim\limits_{x \to -2} 5 - 3 \lim\limits_{x \to -2} x} \longleftarrow$ *ممكن عدم كتابه هذه الخطوه

$= \dfrac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$

$= \dfrac{-8 + 8 - 1}{5 + 6} = \dfrac{-1}{11}$

Example:

Use the limit laws and graphs of f and g in figure

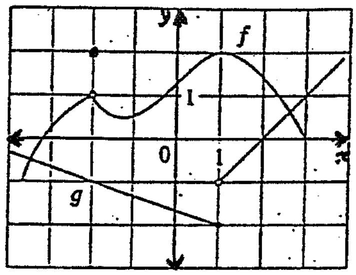
to evaluate the following limits (if they exist) .

1 $\lim\limits_{x \to -2} [\, f(x) + 5\, g(x) \,]$

$= \lim\limits_{x \to -2} f(x) + 5 \lim\limits_{x \to -2} g(x)$

$= \quad 1 \quad + 5\,(-1)$

$= 1 - 5 = \boxed{-4}$



من الرسم نجد أن :

* $\lim\limits_{x \to -2} f(x) = 1$
* $\lim\limits_{x \to -2} g(x) = -1$

---

2 $\lim\limits_{x \to 1} [\, f(x)\, g(x) \,]$

Does not exist

because:

the $\lim\limits_{x \to 1} g(x)$ is not exist

where $\lim\limits_{x \to 1^+} g(x) \neq \lim\limits_{x \to 1^-} g(x)$

$(-1) \qquad (-2)$

* $\lim\limits_{x \to 1} f(x) = 2$
* $\lim\limits_{x \to 1} g(x)$

does not exist

---

* $\lim\limits_{x \to 2} f(x) \approx 1.4$
* $\lim\limits_{x \to 2} g(x) = 0$

3 $\lim\limits_{x \to 2} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to 2} f(x)}{\lim\limits_{x \to 2} g(x)} = \dfrac{1.4}{0}$ ➡ Does not exist

لأن المقام zero

النهايه فى حاله الدوال المعرفه بأكثر من قاعده (فرعين أو أكثر)

ى : وجود العلامات > ، <

- نوجد النهايه اليمنى بالتعويض فى الفرع الذى يحتوى على علامه أكبر من
- نوجد النهايه اليسرى بالتعويض فى الفرع الذى يحتوى على علامه أصغر من

اذا كانت : النهايه اليمنى = النهايه اليسرى تكون النهايه موجوده.

اذا كانت : النهايه اليمنى ≠ النهايه اليسرى تكون النهايه غير موجوده .

(Does not exit)

Example:

$$\text{If: } f(x) = \begin{cases} \sqrt{x-4} & , x > 4 \\ 8-2x & , x < 4 \end{cases}$$

Find the $\lim\limits_{x \to 4} f(x)$?

- نوجد النهايه اليمنى من الفرع الذى يحتوى على أكبر من

$$\lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = \sqrt{0} = \boxed{0}$$

- نوجد النهايه اليسرى من الفرع الذى يحتوى على أصغر من

$$\lim_{x \to 4^-} (8 - 2x) = 8 - 2(4) = 8 - 8 = \boxed{0}$$

∵ النهايه اليمنى = النهايه اليسرى = Zero

∴ النهايه موجوده *Exist* وتساوى *Zero* $\quad \therefore \lim\limits_{x \to 4} f(x) = 0$

الحاله الثانيه : وجود علامه ≠ ، =

Example:

$$f(x) = \begin{cases} 2x+1 & , x \neq 3 \\ x+5 & , x = 3 \end{cases} \qquad \text{find: } \lim_{x \to 3} f(x) \text{ ?}$$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x+1) = 2(3)+1 = \boxed{7}$$

نستخدم الفرع الذى يحتوي على ≠

في حالة وجود دالة القيمة المطلقة | | = f (x) لابد من إعادة تعريف المطلق ثم إيجاد النهاية اليمنى من عند أكبر من والنهاية اليسرى من عند أصغر من.

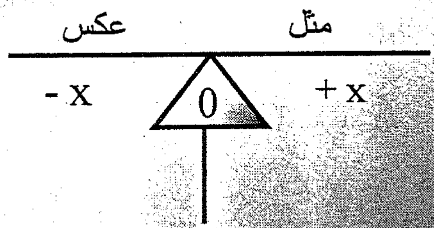
Example:

Find : $\lim\limits_{x \to 0} \dfrac{|x|}{x}$?

لابد من إعادة تعريف المطلق:

$|x|$

نضع ما بداخل المطلق ⬅ $x = 0$



النهايه اليمنى $* \lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{x \to 0^+} \dfrac{x}{x} = \lim\limits_{x \to 0^+} (1) = \boxed{1}$

النهايه اليمنى $* \lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} \dfrac{-x}{x} = \lim\limits_{x \to a^-} (-1) = \boxed{-1}$

$\because \lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0^-} f(x)$

$\therefore \lim\limits_{x \to 0} f(x)$ *Does not exist.*

أى أن النهايه غير موجوده

لعدم تساوى النهايتين

اليمنى واليسرى

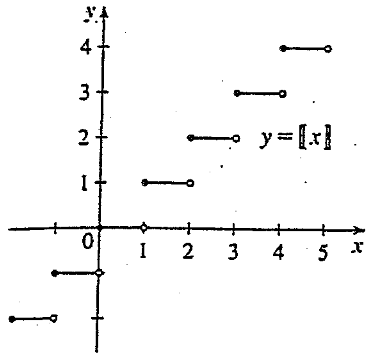
- *The greatest integer function* (داله الصحيح)

is defind by

Other notations for ⟦ x ⟧ are [x] and ⌊x⌋ the greatest integer function sometimes called the floor function

$[\![x]\!]$ = *the largest integer that is less than or equal to x*

$$[\![x]\!] = a \quad \text{for} \quad a \le x < a + 1$$



Note:

⟦ عدد صحيح ⟧ = نفس العدد الصحيح   **Greatest integer function**

⟦ عدد غير صحيح ⟧ = العدد الصحيح الأقل من العدد الغير صحيح (الموجود على يساره)

Example: *find the value of :*

$[\![2]\!] = 2$, $[\![-2]\!] = -2$, $[\![2.9]\!]$, $[\![-2.9]\!] = -3$

$[\![\pi]\!] = 3$, $[\![e]\!] = 2$, $[\![\sqrt{3}]\!] = 1$, $[\![-\sqrt{3}]\!] = -2$

$\downarrow$ 3.14 $\downarrow$ 2.7 $\downarrow$ 1.7 $\downarrow$ -1.7

Example: *find* $\lim\limits_{x \to 3} [\![x]\!]$

* $\lim\limits_{x \to 3^+} [\![x]\!] = \boxed{3}$ * $\lim\limits_{x \to 3^-} [\![x]\!] = \boxed{2}$ $\Longrightarrow$ اليمنى ≠ اليسرى

كأنه تعويض مباشر   كأنه تعويض مباشر –

$\Rightarrow$ النهاية اليمنى ≠ النهاية اليسرى

$\Rightarrow$ $\lim\limits_{x \to 3} [\![x]\!]$ *Does not exist.*

نظريات هامة

1. If : $f(x) \le g(x) \Longrightarrow \lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$

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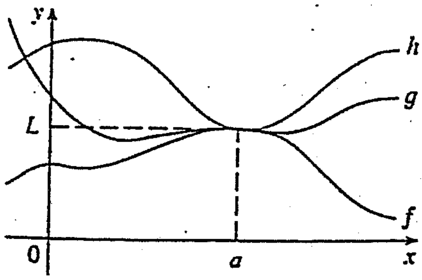
2. (squeeze theorem)

If : $f(x) \le g(x) \le h(x)$

and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$

then $\lim\limits_{x \to a} g(x) = L$



Example:

If : $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$

Find : $\lim\limits_{x \to 4} f(x)$?

* $\lim\limits_{x \to 4} (4x - 9) = 16 - 9 = \boxed{7}$

* $\lim\limits_{x \to 4} (x^2 - 4x + 7) = \cancel{16} - \cancel{16} + 7 = \boxed{7}$

$\boxed{\lim\limits_{x \to 4} f(x) = 7}$

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3. * فى حاله ايجاد نهايه حاصل ضرب دالتين احدهما نهايتها صفر والأخرى محدوده يكون الناتج zero

Example :

1. $\lim\limits_{x \to 0} \underbrace{x^2}_{\text{نهايتها صفر}} . \underbrace{\sin \frac{1}{x}}_{\text{داله محدوده}} = \boxed{0}$

2. $\lim\limits_{x \to 0} \underbrace{x^4}_{\text{نهايتها صفر}} . \underbrace{\cos \frac{2}{x}}_{\text{داله محدوده}} = \boxed{0}$

$-1 \le \sin \le 1$
$\quad\;\;\cos$

أى أن دوال cos, sin
دوال محدودة دائما
بين 1 , -1

مهم جداً :

عند إيجاد النهايه

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad f$$

⬇

تعويض مباشر

عن x بـــ a

الناتج : عدد أو ∞ أو $-\infty$

نتوقف

Stop

الناتج : $\frac{0}{0}$ أو $\frac{\infty}{\infty}$ أو $\infty - \infty$

- نحلل البسط والمقام
- نختصر المتشابه بين البسط والمقام
- نعوض بعد الأختصار عن **x** بـــ **a**
- فى حاله وجود

($\sqrt{\;}$ − عدد) أو (عدد − $\sqrt{\;}$) أو ($\sqrt{\;} - \sqrt{\;}$)

نضرب فى المرافق ⟵ **conjugate**

- فى حاله وجود كسور ⟵ توحيد مقامات

⬇ ⬇ ⬇

** هناك تصرف أسرع وأسهل (بدلاً من التحليل ....)

وهو استخدام قاعده لوبيتال ⟵ *L'Hpoital Rule*

بأن نشتق البسط والمقام **كلاً على حده** ثم التعويض بعد الاشتقاق عن x بـ a

اذا كان الناتج بعد الاشتقاق :

1. عدد أو ∞ أو $-\infty$ ، نتوقف *stop*
2. $\frac{0}{0}$ أو $\frac{\infty}{\infty}$ .... نشتق مره آخرى ونعوض عن x بـــ a ـــ حتى نحصل على الناتج عدد

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Exercises:

1) $\lim_{x \to 2} (2x + 1) = 2(2) + 1 = \boxed{5}$ تعويض مباشر

عدد Stop.

2) $\lim_{y \to 5} \frac{y^2}{5 - y} = \frac{(5)^2}{5 - 5} = \frac{25}{0} = \boxed{\infty}$ تعويض مباشر

Stop.

3) $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \frac{4 + 2 - 6}{2 - 2} = \frac{0}{0}$ (I . f.) ممكن بالتحليل

حاله عدم تعيين

$$\lim_{x \to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x \to 2} (x + 3) = 2 + 3 = \boxed{5}$$

* او ممكن باستخدام قاعده لوبيتال (by L . H . R)

بأن نشتق البسط والمقام كلا على حده

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} \frac{2x+1}{1} = 2(2) + 1 = \boxed{5}$$

4) $\lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \frac{16 - 20 + 4}{16 - 12 - 4} = \frac{0}{0}$ (I . f.) ممكن بالتحليل

حاله عدم تعيين

$$\lim_{x \to -4} \frac{\cancel{(x+4)}(x+1)}{\cancel{(x+4)}(x-1)} = \lim_{x \to -4} \frac{x+1}{x-1} = \frac{-4+1}{-4-1} = \frac{-3}{-5} = \boxed{\frac{3}{5}}$$

* ممكن باستخدام قاعده لوبيتال (by L . H . R) أسهل وأسرع

$$\lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x \to -4} \frac{2x + 5}{2x + 3} = \frac{-8 + 5}{-8 + 3} = \frac{-3}{-5} = \boxed{\frac{3}{5}}$$

5 $\lim\limits_{x \to 4} \dfrac{x^2 - 4x}{x^2 - 3x - 4} = \dfrac{16 - 16}{16 - 12 - 4} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{x \to 4} \dfrac{2x - 4}{2x - 3} = \dfrac{8 - 4}{8 - 3} = \boxed{\dfrac{4}{5}}$

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6 $\lim\limits_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3} = \dfrac{9 - 9}{18 - 21 + 3} = \dfrac{0}{0}$ (I . f) (by L . H . R)

$= \lim\limits_{t \to -3} \dfrac{2t}{4t + 7} = \dfrac{-6}{-12 + 7} = \dfrac{\cancel{-}6}{\cancel{-}5} \boxed{\dfrac{6}{5}}$

---

7 $\lim\limits_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right) = \dfrac{1}{0} - \dfrac{1}{0} = \infty - \infty$ (I . f)

- فى حاله الكسور توحيد مقامات أولاً

$= \lim\limits_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t(t+1)} \right) = \lim\limits_{t \to 0} \left(\dfrac{t+1}{t(t+1)} - \dfrac{1}{t(t+1)} \right)$

$= \lim\limits_{t \to 0} \left(\dfrac{t \cancel{+1} \cancel{-1}}{t(t+1)} \right) = \lim\limits_{t \to 0} \dfrac{t}{t(t+1)}$

$= \lim\limits_{t \to 0} \dfrac{1}{t+1}$

$= \dfrac{1}{0+1} = \dfrac{1}{1} = \boxed{1}$

8 $\lim\limits_{h \to 0} \dfrac{(4+h)^2-16}{h} = \dfrac{(4+0)^2-16}{0} = \dfrac{16-16}{0} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{h \to 0} \dfrac{2(4+h)^1 .1}{1} = \lim\limits_{h \to 0} 2\,(4+h) = 2(4+0) = \boxed{8}$

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9 $\lim\limits_{x \to 1} \dfrac{x^3-1}{x^2-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{x \to 1} \dfrac{3x^{\cancel{2}}}{2\cancel{x}} = \lim\limits_{x \to 1} \dfrac{3x}{2} = \dfrac{3(1)}{2} = \boxed{\dfrac{3}{2}}$

---

10 $\lim\limits_{x \to -2} \dfrac{x+2}{x^3+8} = \dfrac{-2+2}{-8+8} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{x \to -2} \dfrac{1}{3\,x^2} = \dfrac{1}{3(-2)^2} = \dfrac{1}{3\,(4)} = \boxed{\dfrac{1}{12}}$

---

11 $\lim\limits_{t \to 9} \dfrac{9-t}{3-\sqrt{t}} = \dfrac{9-9}{3-3} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{t \to 9} \dfrac{-1}{-\dfrac{1}{2\sqrt{t}}} = \lim\limits_{t \to 9} 1 \cdot \dfrac{2\sqrt{t}}{1} = 2\sqrt{9} = 2\,(3) = \boxed{6}$

12 $\lim\limits_{x \to 7} \dfrac{\sqrt{x+2}-3}{x-7} = \dfrac{\sqrt{9}-3}{7-7} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$$= \lim_{x \to 7} \frac{\frac{1}{2\sqrt{x+2}}}{1} = \lim_{x \to 7} \frac{1}{2\sqrt{x+2}} = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$$

---

13 $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h}-1}{h} = \dfrac{\sqrt{1+0}-1}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$$= \lim_{h \to 0} \frac{\frac{1}{2\sqrt{1+h}}}{1} = \lim_{h \to 0} \frac{1}{2\sqrt{1+h}} = \frac{1}{2\sqrt{1+0}} = \boxed{\frac{1}{2}}$$

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14 $\lim\limits_{x \to 2} \dfrac{x^4-16}{x-2} = \dfrac{16-16}{2-2} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$$= \lim_{x \to 2} \frac{4x^3}{1} = 4(2)^3 = 4(8) = \boxed{32}$$

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15 $\lim\limits_{x \to -4} \dfrac{\frac{1}{4}+\frac{1}{x}}{4+x} = \dfrac{\frac{1}{4}+\frac{1}{-4}}{4+(-4)} = \dfrac{\frac{1}{4}-\frac{1}{4}}{4-4} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$$= \lim_{x \to -4} \frac{0+\left(\frac{-1}{x^2}\right)}{0+1} = \lim_{x \to -4} -\frac{1}{x^2} = -\frac{1}{(-4)^2} = \boxed{-\frac{1}{16}}$$

مشتقه

هى $\frac{}{x^2}$

16 $\lim\limits_{x \to 9} \dfrac{x^2 - 81}{\sqrt{x} - 3} = \dfrac{81 - 81}{3 - 3} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{x \to 9} \dfrac{2x}{\dfrac{1}{2\sqrt{x}}} = \lim\limits_{x \to 9} 2x . 2\sqrt{x}$

$= \lim\limits_{x \to 9} 4x\sqrt{x} = 4(9)\sqrt{9} = 36(3)$ $\boxed{108}$

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17 $\lim\limits_{h \to 0} \dfrac{(3+h)^{-1} - 3^{-1}}{h} = \dfrac{3^{-1} - 3^{-1}}{0} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{h \to 0} \dfrac{-1(3+h)^{-2} . 1}{1} = \lim\limits_{h \to 0} \dfrac{-1}{(3+h)^2}$

$= \dfrac{-1}{(3+0)^2} = \dfrac{-1}{(3)^2} = \boxed{\dfrac{-1}{9}}$

---

18 $\lim\limits_{x \to -4} \dfrac{\sqrt{x^2 + 9} - 5}{x + 4} = \dfrac{\sqrt{16 + 9} - 5}{-4 + 4} = \dfrac{0}{0}$ (I . f .) (by L . H . R)

$= \lim\limits_{x \to -4} \dfrac{\dfrac{2x}{2\sqrt{x^2 + 9}}}{1} = \lim\limits_{x \to -4} \dfrac{x}{\sqrt{x^2 + 9}}$

$= \dfrac{-4}{\sqrt{16 + 9}} = \dfrac{-4}{\sqrt{25}} = \boxed{\dfrac{-4}{5}}$

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19 $\lim\limits_{t\to 0}\left(\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right)=\dfrac{1}{0}-\dfrac{1}{0}=\infty-\infty\ \ (I\ .\ f\ .)$

توحيد المقامات

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{\sqrt{1+t}}{t\sqrt{1+t}}\right)=\lim_{t\to 0}\left(\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\right)=\frac{0}{0}\ \ (I\ .\ f\ .)$$

بالضرب فى مرافق البسط (بسطاً ومقاماً)

$$=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}$$

$$=\lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}\ (1+\sqrt{1+t})}=\lim_{t\to 0}\frac{\cancel{1}-\cancel{1}-t}{t\sqrt{1+t}\ (1+\sqrt{1+t})}$$

$$=\lim_{t\to 0}\frac{-\cancel{t}}{\cancel{t}\sqrt{1+t}\ (1+\sqrt{1+t})}=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\ (1+\sqrt{1+t})}$$

$$=\frac{-1}{\sqrt{1+0}\ (1+\sqrt{1+0})}=\frac{-1}{1\ (1+1)}=\boxed{\frac{-1}{2}}$$

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20 $\lim\limits_{x\to 0}\underbrace{\sqrt{x^3+x^2}}_{\text{داله نهايتها zero}}\ \ \underbrace{\sin\frac{\pi}{x}}_{\text{داله محدوده بين 1- ، 1}}=\boxed{0}$

نظريه

3 Page 8

21 $\lim\limits_{x\to 3}\left(2x+|x-3|\right)$

تعريف المطلق

$x-3=0$

$x=3$

عكس	مثل
3	
$-(x-3) = -x+3$	$(x-3)$

النهايه اليمنى

$* \lim\limits_{x\to 3^+}(2x+x-3)$

$= \lim\limits_{x\to 3^+}(3x-3) = 9-3 = \boxed{6}$

النهايه اليسرى

$* \lim\limits_{x\to 3^-}(2x-x+3)$

$= \lim\limits_{x\to 3^-}(x+3) = 3+3 = \boxed{6}$ النهايه اليمنى= النهايه اليسرى

$\because \lim\limits_{x\to 3^+} = \lim\limits_{x\to 3^-} = 6 \rightarrow \lim\limits_{x\to 3}(2x+|x-3|) = \boxed{6}$

---

22 $\lim\limits_{x\to -6}\dfrac{2x+12}{|x+6|}$

اعاده تعريف المطلق

$x+6=0$

$x=-6$

عكس	مثل
-6	
$-(x+6)$	$(x+6)$

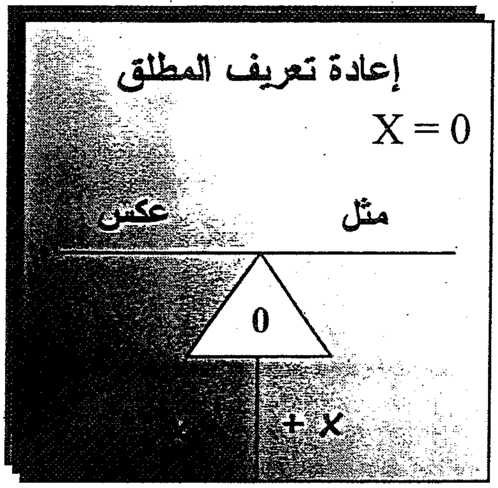
$* \lim\limits_{x\to -6^+}\dfrac{2(x+\cancel{6})}{(x+\cancel{6})} = \boxed{2}$

$* \lim\limits_{x\to -6^-}\dfrac{2(x+\cancel{6})}{-(x+\cancel{6})} = \dfrac{2}{-1} = \boxed{-2}$

$\therefore \lim\limits_{x\to -6^+} \neq \lim\limits_{x\to -6^-}$ النهايه اليمنى $\neq$ النهايه اليسرى

$\therefore \lim\limits_{x\to -6}\dfrac{2x+12}{|x+6|}$ *does not exist*

ALSAADI



23 $\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

$= \lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{-x}\right)$

$= \lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{x}\right)$

$= \lim_{x\to 0^-}\left(\frac{2}{x}\right)=\frac{2}{0}=\boxed{\infty}$

---

24 $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

$= \lim_{x\to 0}\left(\cancel{\frac{1}{x}}-\cancel{\frac{1}{x}}\right)$

$= \lim_{x\to 0^+}(0)=\boxed{0}$

---

25 $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ *dose not exist*

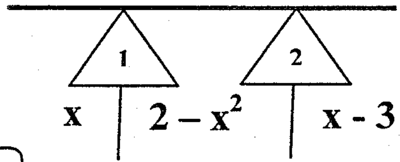
لأن النهايه اليمنى ≠ النهايه اليسرى.

Because: $\lim_{x\to 0^-} \neq \lim_{x\to 0^+}$

26 let : $g(x) = \begin{cases} x & ; x < 1 \\ 3 & ; x = 1 \\ 2 - x^2 & ; 1 < x \le 2 \\ x - 3 & ; x \quad x > 2 \end{cases}$

Evaluat each of the following limits if it exists

1 $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} (x) = \boxed{1}$



2 $\lim\limits_{x \to 1^+} g(x) \lim\limits_{x \to 1^+} (2 - x^2) = 2 - 1 = \boxed{1}$

3 $\lim\limits_{x \to 1} g(x) = 1$ (1 = النهايه اليسرى = النهايه اليمنى)

4 $g(1) = 3$

---

5 $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} (2 - x^2) = 2 - 4 = \boxed{-2}$

6 $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} (x - 3) = 2 - 3 = \boxed{-1}$

7 $\lim\limits_{x \to 2} g(x)$ does not exist

لأن النهايه اليمنى $\neq$ النهايه اليسرى .

Because: $\lim\limits_{x \to 2^-} \neq \lim\limits_{x \to 2^+}$

27 If n is integer عدد صحيح

Find

1. $\lim\limits_{x \to n^-} [x] = n - 1$ فى النهايه اليسرى تعويض (مباشر-1)

2. $\lim\limits_{x \to n^+} [x] = n$ فى النهايه اليمنى تعويض (مباشر فقط)

3. $\lim\limits_{x \to n} [x]$ does not exist لأن: النهايه اليسرى ≠ النهايه اليمنى

---

28 If : $f(x) = [x] + [-x]$

Find : $\lim\limits_{x \to 2} f(x)$? and $f(2)$?

* $\lim\limits_{x \to 2^+} ([x] + [-x]) = (2) + (-3) = 2 - 3 \boxed{-1}$ 2.1 -2.1

* $\lim\limits_{x \to 2^-} ([x] + [-x]) = (1) + (-2) = 1 - 2 = \boxed{-1}$ 1.9 -1.9

∴ النهايه اليمنى = النهايه اليسرى

$\therefore \lim\limits_{x \to 2} f(x) = \boxed{-1}$

* $f(2) = [2] + [-2]$

$= 2 + (-2) = 2 - 2 = 0$

ALSAADI

29 If : $\lim\limits_{x\to 1} \dfrac{f(x)-8}{x-1} = 10$

Find $\lim\limits_{x\to 1} f(x)$?

$$\because \lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$$

$$\Rightarrow \frac{\lim\limits_{x\to 1} f(x) - \lim\limits_{x\to 1} 8}{\lim\limits_{x\to 1} x - \lim\limits_{x\to 1} 1} = 10$$

$$\Rightarrow \lim_{x\to 1} f(x) - \lim_{x\to 1} 8 = 10\left(\lim_{x\to 1} x - \lim_{x\to 1} 1\right)$$

$$\Rightarrow \lim_{x\to 1} f(x) - 8 = 10(1-1)$$

$$\Rightarrow \lim_{x\to 1} f(x) = \underbrace{10(0)}_{0} + 8$$

$$\therefore \lim_{x\to 1} f(x) = 8$$

31 $\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}=\frac{\sqrt{6-2}-2}{\sqrt{3-2}-1}=\frac{\sqrt{4}-2}{\sqrt{1}-1}=\frac{2-2}{1-1}=\frac{0}{0}$ (I. f.)

- ممكن الضرب فى المرافق مره للبسط ومره للمقام

لكن الأسهل والأسرع استخدام لوبيتال

(by L . H . R)

$$\lim_{x\to 2}\frac{\dfrac{-1}{2\sqrt{6-x}}}{\dfrac{-1}{2\sqrt{3-x}}}$$

$$=\lim_{x\to 2}\frac{\cancel{-1}}{\cancel{2}\sqrt{6-x}}\cdot\frac{\cancel{2}\sqrt{3-x}}{\cancel{-1}}$$

$$=\lim_{x\to 2}\frac{\sqrt{3-x}}{\sqrt{6-x}}=\frac{\sqrt{3-2}}{\sqrt{6-2}}=\frac{\sqrt{1}}{\sqrt{4}}=\boxed{\frac{1}{2}}$$

ALSAADI

32 If there a number a such that :

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$ exist موجوده

1 Find the value of a .

2 Find the value of the limit .

Solution

1 $\therefore \lim_{x \to -2} \frac{3(-2)^2 + a(-2) + a + 3}{(-2)^2 + (-2) - 2}$

$$= \frac{12 - 2a + a + 3}{4 - 2 - 2} = \frac{15 - a}{0}$$

$\because$ النهايه exist موجوده

$\therefore$ لابد أن البسط = 0

$15 - a = 0 \longrightarrow - a = -15 \longrightarrow \boxed{a = 15}$

2 $\lim_{x \to -2} \frac{3x^2 + 15x + 15 + 3}{x^2 + x - 2}$

$$= \lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \frac{12 - 30 + 18}{4 - 2 - 2} = \frac{0}{0} \quad (\text{I . f .})$$

(by L . H > R) $\lim_{x \to -2} \frac{6x + 15}{2x+1} = \frac{-12 + 15}{-4 + 1} = \frac{3}{-3} = \boxed{-1}$

A L S A A D I

33 Find $\lim_{x \to 0} \underbrace{\sqrt{x}}_{\text{داله نهايتها zero}} \quad \underbrace{e^{\sin(\frac{\pi}{x})}}_{\text{داله محدوده}}$

داله نهايتها zero

داله محدوده

حيث أن

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$$

$$e^{-1} \le e^{\sin\left(\frac{\pi}{x}\right)} \le e^{1}$$

$\therefore e^{\sin\left(\frac{\pi}{x}\right)}$ داله محدوده بين e^{-1} ، e

$$\therefore \lim_{x \to 0} \sqrt{x} \quad e^{\sin\left(\frac{\pi}{x}\right)} = 0$$

الداله
الأولى نهايتها صفر

داله محدوده
بين e^{-1} ، e

نظريه

3 Page 8

كل التمنيات بالنجاح والتوفيق

السعدى

السعدي ALSAADI

نسخة جديدة منقحة

1432/33

10

Chapter Three
Limits and Continuity

3.3 3.4

MATH-110

جمال السعدي
رياضيات - إحصاء

• limits at infinity
النهاية عند اللا نهاية
• Horizontal asymptotes
خطوط التقارب الأفقية

Note that

- $(\infty)^n = \infty$ حيث n موجب

- $(-\infty)^n = \begin{cases} \infty & \longrightarrow \text{حيث } n \text{ زوجي} \\ -\infty & \longrightarrow \text{حيث } n \text{ فردي} \end{cases}$

- $(\pm\infty)^n = zero$ حيث n سالب

- $\dfrac{\text{عدد}}{\pm\infty} = 0$

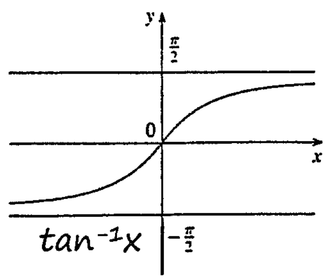
- $\dfrac{\pm\infty}{\text{عدد}} = \pm\infty$

- $\left(\dfrac{a}{b}\right)^{\infty} = 0$ إذا كانت a اصغر من b ➡ $\left(\dfrac{2}{3}\right)^{\infty} = 0$

- $\left(\dfrac{a}{b}\right)^{\infty} = \infty$ إذا كانت a أكبر من b ➡ $\left(\dfrac{3}{2}\right)^{\infty} = \infty$

- $e^{\infty} = \infty$ • $e^{-\infty} = 0$

- $\tan^{-1}\infty = \dfrac{\pi}{2}$ • $\tan^{-1} -\infty = \dfrac{-\pi}{2}$



• **أسهل طريقة لإيجاد النهاية** $\lim\limits_{x \to \pm\infty} \dfrac{\text{بسط}}{\text{مقام}}$

إذا كانت درجة البسط = درجة المقام

يكون الناتج

$$\frac{\text{معامل أكبر أس ل } x \text{ في البسط}}{\text{معامل أكبر أس ل } x \text{ في المقام}}$$

1. $\lim\limits_{x \to \pm\infty} \dfrac{2x^2 - x}{3x^2 - 1} = \dfrac{2}{3}$

إذا كانت درجة البسط أصغر من درجة المقام

يكون الناتج zero

2. $\lim\limits_{x \to \pm\infty} \dfrac{2x + 1}{x^2 - x} = 0$

إذا كانت درجة البسط أكبر من درجة المقام الناتج ∞

ولتحديد إشارته.

نعوض في الحد الذي يحتوي على أكبر أس في البسط

والحد الذي يحتوي على أكبر أس في المقام عن x

3.
• $\lim\limits_{x \to \pm\infty} \dfrac{x^3 - 2x^2}{x + 1} = \dfrac{+}{+}\infty = \infty$

• $\lim\limits_{x \to -\infty} \dfrac{2x^3 - x - 2}{2x + x^2} = \dfrac{-}{+}\infty = -\infty$

• $\lim\limits_{x \to -\infty} \dfrac{2x^3 - x - 2}{2x - x^2} = \dfrac{-}{-}\infty = \infty$

Find the limits:

1 $\lim_{x \to \infty} \left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^\infty = 0$

البسط أصغر من المقام الناتج 0

2 $\lim_{x \to \infty} \left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^\infty = \infty$

البسط أكبر من المقام الناتج

3 $\lim_{x \to -\infty} \left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-\infty} = \left(\frac{3}{2}\right)^\infty = \infty$

البسط 3 أكبر من المقام 2 الناتج

4 $\lim_{x \to -\infty} \left(\frac{\pi}{e}\right)^x = \left(\frac{\pi}{e}\right)^{-\infty} = \left(\frac{e}{\pi}\right)^\infty = 0$

البسط اصغر من المقام الناتج 0

$\pi \approx 3.14$ أصغر من $e \approx 2.7$

5 $\lim_{x \to \pm\infty} \frac{1}{2 + \frac{1}{x}} = \frac{1}{2 + \frac{1}{\pm\infty}} = \frac{1}{2+0} = \frac{1}{2}$

6 $\lim_{x \to \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

درجة البسط ← -1

أكبر من

درجة المقام ← -2

$= \frac{+}{+} \infty = \infty$

7 $\lim_{x \to \infty} (\sqrt{x+2} - \sqrt{x}) = \infty - \infty \ (I.F.)$

حالة عدم تعيين الضرب في المرافق

$$= \lim_{x \to \infty} \sqrt{x+2} - \sqrt{x} \ . \ \frac{\sqrt{x+2} + \sqrt{x}}{\sqrt{x+2} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\cancel{x} + 2 - \cancel{x}}{\sqrt{x+2} + \sqrt{x}} = \lim_{x \to \infty} \frac{2}{\sqrt{x+2} + \sqrt{x}} = \frac{2}{\infty} = 0$$

---

8 $\lim_{x \to \infty} \frac{\sqrt{1 + 4x^2}}{4 + x} = \frac{2x}{x} = 2$

عندما $x \to \infty$

فإن $\sqrt{4x^2} = 2|x| = 2x$

---

9 $\lim_{x \to -\infty} \frac{\sqrt{1 + 4x^2}}{4 + x} = \frac{-2x}{x} = -2$

عندما $x \to -\infty$

فإن $\sqrt{4x^2} = 2|x| = -2x$

---

10 $\lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \frac{\cos\left(\frac{1}{\infty}\right)}{1 + \frac{1}{\infty}} = \frac{\cos 0}{1 + 0} = \frac{1}{1} = 1$

نظرية هامة جداً

$$\lim_{x \to a} f(x) \cdot g(x) = 0$$

نهاية حاصل ضرب دالتين **أحدهما نهايتها 0** و **الأخرى محدودة** يكون الناتج 0

ملحوظة

دالة sin ، cos دوال محدودة دائماً

$$-1 \le \begin{matrix} \text{Sin} \\ \cos \end{matrix} \le 1$$

Example:

Find the limits:

1) $\lim_{x \to \infty} \frac{1}{x} \cos x = 0$

$\frac{1}{\infty} = 0$ $\qquad -1 \le \cos x \le 1$

محدودة

حاصل ضرب دالتين

* الأولى نهايتها 0

$$\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

* الثانية cos x

محدودة بين 1 - , 1

* يكون ناتج النهاية بـ zero

---

2) $\lim_{x \to \infty} \frac{\sin 2x}{x}$

$$= \lim_{x \to \infty} \frac{1}{x} \cdot \sin 2x = 0$$

$\frac{1}{\infty} = 0$ $\qquad -1 \le \sin 2x \le 1$

محدودة

Find the limits:

1 $\lim\limits_{x \to \infty} \sqrt{x^2 + x} - \sqrt{x^2 - x} = \infty - \infty \quad (I.F.)$

حالة عدم تعيين

بالضرب في المرافق

$$= \lim_{x \to \infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x}\right) \cdot \frac{\sqrt{x^2 + x} + \sqrt{x^2 - x}}{\sqrt{x^2 + x} + \sqrt{x^2 - x}}$$

$$= \lim_{x \to \infty} \frac{\left(x^2 + x\right) - \left(x^2 - x\right)}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{\cancel{x^2} + x - \cancel{x^2} + x}{\sqrt{x^2 + x} + \sqrt{x^2 - x}}$$

$$= \lim_{x \to \infty} \frac{2x}{\sqrt{x^2} + x + \sqrt{x^2} - x}$$

$\sqrt{x^2}$ تعني x عندما $x \to \infty$

وتعني $-x$ عندما $x \to -\infty$

∴ درجة البسط = درجة المقام

$$= \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{1 + 1} = \frac{2}{2} = \boxed{1}$$

---

2 $\lim\limits_{x \to \infty} \left(\frac{3}{x^2} - \cos\frac{1}{x}\right)\left(1 + \sin\frac{1}{x}\right)$

تعويض مباشر

$\dfrac{\text{عدد}}{\infty} = 0$

$$= \left(\frac{3}{\infty} - \cos\frac{1}{\infty}\right)\left(1 + \sin\frac{1}{\infty}\right)$$

$$= (0 - \cos 0)(1 + \sin 0)$$

$$= (0 - 1)(1 + 0) = (-1)(1) = \boxed{-1}$$

3 $\lim_{x \to \infty} \dfrac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}}$

تعويض مباشر

$$= \frac{\cos\left(\frac{1}{\infty}\right)}{1 + \frac{1}{\infty}} = \frac{\cos 0}{1 + 0} = \frac{1}{1} = \boxed{1}$$

قاعدة هامة

* $\lim_{x \to \pm\infty} \dfrac{\sin\ ax}{bx} = 0 \longrightarrow \lim_{x \to \pm\infty} \dfrac{\sin\ x}{3x} = 0$

* $\lim_{x \to \pm\infty} \dfrac{\cos\ ax}{bx} = 0 \longrightarrow \lim_{x \to \pm\infty} \dfrac{\cos\ 3x}{5x} = 0$

Example:

Find: $\lim_{x \to -\infty} \dfrac{2 - x + \sin x}{x + \cos\ x}$

للوصول لشكل القاعدة السابقة نقسم بسطاً ومقاماً على X

$$= \lim_{x \to -\infty} \frac{\frac{2}{x} - 1 + \frac{\sin\ x}{x}}{1 + \frac{\cos\ x}{x}} = \frac{0 - 1 + 0}{1 + 0} = \frac{-1}{1} = \boxed{-1}$$

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11 *Guess the value of the limit:*

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

* عندما تزداد x حتى تقترب من ∞ فإن المقام
يصل أسرع إلى ∞ قبل البسط أي أن :

$$\frac{\text{عدد}}{\infty} = 0$$

x :	1	2	..... 10..... ➡	∞
x^2 :	1	4	... 100.... ➡	عدد
2^x :	2	4	... 1024... ➡	∞

$$\therefore \lim_{x \to \infty} \frac{x^2}{2^x} = \frac{\text{عدد}}{\infty} = 0$$

الدالة الأسية أسرع (أكبر) من دالة القوى

ملحوظة

$$\lim_{x \to \infty} \frac{2^x}{x^2} = \frac{\infty}{\text{عدد}} = \infty$$

---

28 $\lim_{x \to \infty} \cos x$

* عند ∞ تكون cos x

أما $\boxed{1}$ أو $\boxed{-1}$ ⟵ لها نهايتان مختلفتان

$\therefore \lim_{x \to \infty} \cos x$ Does Not Exist

---

35 $\lim_{x \to \infty} (e^{-2x} \quad \cos x) = 0$

أحدهما دالة محدودة

بين 1- , 1

والأخرى

دالة نهايتها 0

حيث $e^{-\infty} = 0$

حاصل ضرب دالتين

---

31 $\lim_{x \to -\infty} \left(x^4 + x^5\right)$

$= \lim_{x \to -\infty} x^4 \cdot (1 + x)$

x^4 عامل مشترك

$= (-\infty)^4 . (1 - \infty)$

تعويض مباشر

$= (\infty) . (-\infty) = \boxed{-\infty}$

---

33 $\lim_{x \to \infty} \dfrac{1 - e^x}{1 + 2e^x} = \dfrac{-\infty}{\infty}$

بالقسمة على e^x بسطاً ومقاماً

$$= \lim_{x \to \infty} \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 2}$$

$$= \frac{\frac{1}{\infty} - 1}{\frac{1}{\infty} + 2} = \frac{0 - 1}{0 + 2} = \boxed{\frac{-1}{2}}$$

حل أخر أسهل وأسرع

$$\lim_{x \to \infty} \frac{1 - \boxed{1} e^x}{1 + \boxed{2} e^x} = \frac{-1}{2}$$

34 $\lim\limits_{x \to \infty} \tan^{-1}(x^2 - x^4)$

عامل مشترك x^2

$= \lim\limits_{x \to \infty} \tan^{-1}\left[x^2\,(1-x)^2\right]$

$= \tan^{-1}\left[\infty \,.\, (1 - \infty)\right]$

$= \tan^{-1}\left[\infty \,.\, -\infty\right] = \tan^{-1}\left[-\infty\right] = \boxed{-\dfrac{\pi}{2}}$

---

36 $\lim\limits_{x \to (\pi/2)^+} e^{\tan x} \qquad = e^{-\infty} \quad = \boxed{0}$

الربع الثاني

* tan سالبة

* $\tan 90 = \infty$

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57 find : $\lim\limits_{x \to \infty} f(x) \qquad$ if $\dfrac{10e^x - 21}{2\,e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x} - 1}$

By: sandwich theorem

باستخدام نظرية الساندويتش

* $\lim\limits_{x \to \infty} \dfrac{\boxed{10}\,e^x - 21}{\boxed{2}\,e^x} = \dfrac{10}{2} = \boxed{5}$

* $\lim\limits_{x \to \infty} \dfrac{\boxed{5}\sqrt{x}}{\boxed{\sqrt{x}} - 1} = \dfrac{5}{1} = \boxed{5} \longrightarrow \lim\limits_{x \to \infty} f(x) = \boxed{5}$

Horizontal asymptotes
خطوط التقارب الأفقية

$$y = \lim_{x \to \pm\infty} f(x)$$

- L
  $\therefore y = L$ is h. asymptote.
- $\pm L$
  $\therefore y = \pm L$ are h. asymptotes.
- $+\infty$ أو $-\infty$
  $\therefore$ No h. asymptotes.

Example:

Find horizontal asymptotes:

1 $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4}$ درجة البسط = درجة المقام

$$y = \lim_{x \to \pm\infty} \frac{x^2 - 5x + 6}{x^2 - 4} = \frac{1}{1} = 1$$

$\therefore y = 1$ is horizontal asymptote

→ ($y = 1$ is h. asym.)

2 $f(x) = \dfrac{2x-1}{\sqrt{x^2+1}}$

لاحظ أن

$\sqrt{x^2} = |x| = \begin{cases} x & \text{عندما } x \to \infty \\ -x & \text{عندما } x \to -\infty \end{cases}$

* $y = \lim\limits_{x \to \infty} \dfrac{2x - 1}{\sqrt{x^2} + 1} = \dfrac{+2}{+1} = \boxed{2}$

* $y = \lim\limits_{x \to -\infty} \dfrac{2x - 1}{\sqrt{x^2} + 1} = \dfrac{+2}{-1} = \boxed{-2}$

➡ $y = 2$, $y = -2$ are h. asymptotes

---

3 $f(x) = \dfrac{|x+2|}{x+4}$

لابد من إعادة تعريف المطلق

$x + 2 = 0 \to x = -2$

عكس	-2	مثل
$-(x+2)$		$+(x+2)$

* $y = \lim\limits_{x \to \infty} \dfrac{x+2}{x+4} = \dfrac{1}{1} = \boxed{1}$

* $y = \lim\limits_{x \to -\infty} \dfrac{-(x+2)}{x+4} = \dfrac{-1}{1} = \boxed{-1}$

➡ $y = 1$, $y = -1$ are h. asymptotes.

---

4 $f(x) = \dfrac{x^4}{|x|}$

$|x| = \begin{cases} x & \text{عندما } x \to \infty \\ -x & \text{عندما } x \to -\infty \end{cases}$

* $y = \lim\limits_{x \to \infty} \dfrac{x^4}{x} = \lim\limits_{x \to \infty} x^3 = \boxed{\infty}$

* $y = \lim\limits_{x \to -\infty} \dfrac{x^4}{-x} = \lim\limits_{x \to -\infty} -x^3 = -(-\infty) = \boxed{\infty}$

➡ f (x) has not h. asymptotes ليس لها خطوط تقارب أفقية

vertical asymptotes
خطوط التقارب الرأسية

v. asymptotes * خطوط إيجاد خطوط التقارب الرأسية

* نوجد أصفار مقام الدالة المعطاه $F(x)$ ولتكن a, b
* نعوض بـ a , b في الدالة نفسها

$F(b) = \frac{0}{0}$

$\therefore x = b$

Not V. asymptotes

$x = b$ لايمثل خط تقارب رأسي

$F(a) = \frac{\text{عدد}}{0}$

$\therefore x = a$

Is V. asymptote

$x = a$ يمثل خط تقارب رأسي

Example:

Find the vertical asymptotes:

1

$$f(x) = \frac{2x - 1}{x - 2}$$

$$f(2) = \frac{2(2) - 1}{2 - 2} = \frac{3}{0} = \frac{\text{عدد}}{0}$$

* أصفار المقام

$x - 2 = 0$

$x = 2$

➡ $x = 2$ is V. asymptote.

$$f(x) = \frac{x^2 - 5x + 6}{x^2 - 4}$$

* أصفار المقام

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

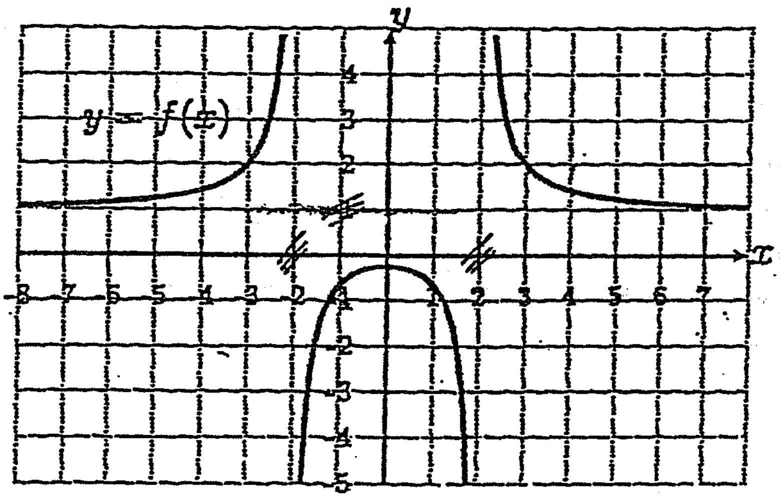
$$* \; f(2) = \frac{4 - 10 + 6}{4 - 4} = \frac{0}{0}$$

➡ x = 2 is not V. asym.

$$* \; f(-2) = \frac{4 + 10 + 6}{4 - 4} = \frac{20}{0} = \frac{\text{عدد}}{0}$$

➡ x = −2 is V. asym.

The horizontal and vertical asymptotes of f are



(a) $y = -2$, $y = 2$ $x = 1$

√ (b) $x = -2$, $x = 2$ $y = 1$

(c) $x = -2$, $x = 0$ $y = 1$

(d) $x = 0$, $x = 2$ $y = 1$

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For the function (g) whose graph is given, state the following

(a) $\lim_{x \to \infty} g(x) = 2$

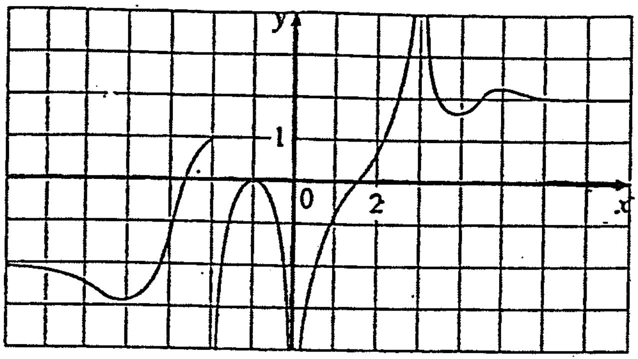
(b) $\lim_{x \to -\infty} g(x) = -2$

(c) $\lim_{x \to 3} g(x) = \infty$

(d) $\lim_{x \to 0} g(x) = -\infty$

(e) $\lim_{x \to -2^+} g(x) = -\infty$

(f) the equations of the asymptotes



* V. asymptotes

$X = -2$, $x = 0$, $x = 3$

* H . asymptotes

$y = -2$, $y = 2$

For the function (f) whose graph is given, state the following

(a) $\lim_{x \to 2} f(x) = \infty$

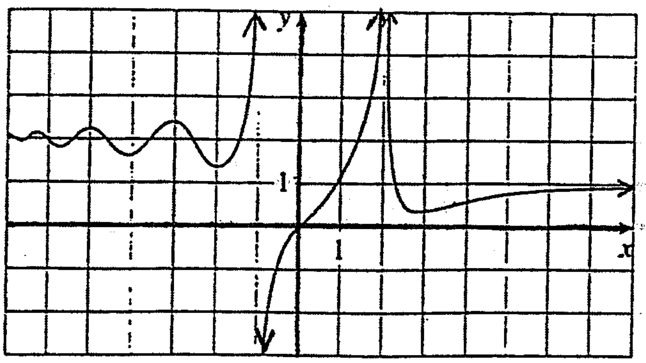
(b) $\lim_{x \to -1^-} f(x) = \infty$

(c) $\lim_{x \to -1^+} f(x) = -\infty$

(d) $\lim_{x \to \infty} f(x) = 1$

(e) $\lim_{x \to -\infty} f(x) = 2$

(f) the equations of the asymptotes



* V. asymptotes

$X = -1$, $x = 2$

* H . asymptotes

$y = 1$, $y = 2$

الموضوع الثالث: نظريات على الدوال المثلثية

في حالة $x \to 0$

1. $$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b} \rightarrow \lim_{x \to 0} \frac{\sin 2x}{3x} = \frac{2}{3}$$

2. $$\lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b} \rightarrow \lim_{x \to 0} \frac{\tan 3x}{5x} = \frac{3}{5}$$

3. $$\lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b} \rightarrow \lim_{x \to 0} \frac{\sin 2x}{\tan 7x} = \frac{2}{7}$$

* مقلوبات الصور السابقة صحيحة

$$\lim_{x \to 0} \frac{3x}{\sin 2x} = \frac{3}{2}$$

4. * $$\lim_{x \to 0} \frac{1 - \cos ax}{bx} = 0$$ * $$\lim_{x \to 0} \frac{\cos ax - 1}{bx} = 0$$

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Find the limits:

$$\lim_{x \to 0} x \cot x = \lim_{x \to 0} x \cdot \frac{1}{\tan x} = \lim_{x \to 0} \frac{x}{\tan x} = \frac{1}{1} = \boxed{1}$$

Page 196

Find the limits

39 $\lim_{x \to 0} \dfrac{\sin 3x}{x} = \dfrac{3}{1} = 3$

40 $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 6x} = \dfrac{4}{6} = \dfrac{2}{3}$

41 $\lim_{x \to 0} \dfrac{\tan 6x}{\sin 2x} = \dfrac{6}{2} = 3$

42 $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\sin \theta}$

بالقسمة بسطاً ومقاماً على θ
وذلك للوصول إلى شكل النظريات

$$= \lim_{\theta \to 0} \dfrac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \dfrac{0}{\dfrac{1}{1}} = \dfrac{0}{1} = 0$$

43 $\lim_{\theta \to 0} \dfrac{\sin(\cos\theta)}{\sec\theta}$

* تعويض مباشر
لأنها ليست صورة لأي نظرية

$$= \dfrac{\sin(\cos 0)}{\sec 0} = \dfrac{\sin(1)}{\dfrac{1}{\cos 0}} = \dfrac{\sin 1}{\dfrac{\cancel{1}}{\cancel{1}}} = \sin 1$$

44 $\lim_{t \to 0} \dfrac{\sin^2 3t}{t^2} = \lim_{t \to 0} \dfrac{\sin 3t}{t} \cdot \dfrac{\sin 3t}{t} = \dfrac{3}{1} \cdot \dfrac{3}{1} = 9$

فك التربيع

45 $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta + \tan\theta}$

بالقسمة بسطاً ومقاماً على θ وذلك للحصول على صورة النظرية

$$= \lim_{\theta \to 0} \dfrac{\dfrac{\sin\theta}{\theta}}{1 + \dfrac{\tan\theta}{\theta}}$$

$$= \dfrac{\dfrac{1}{1}}{1 + \dfrac{1}{1}} = \dfrac{1}{1+1} = \dfrac{1}{2}$$

ممكن الحل بمجرد النظر بأخذ المعاملات لـ θ

$$\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta + \tan\theta} = \dfrac{1}{1+1} = \dfrac{1}{2}$$

---

46 $\lim_{x \to 0} \dfrac{\sin(x^2)}{x}$

بالضرب بسطاً ومقاماً في x للحصول على صورة النظرية

$$= \lim_{x \to 0} x \cdot \dfrac{\sin x^2}{x^2} = 0 \cdot \dfrac{1}{1} = 0$$

---

47 $\lim_{x \to \frac{\pi}{4}} \dfrac{1 - \tan x}{\sin x - \cos x}$

لاحظ أن x تؤول إلى عدد وليس zero
∴ تعويض مباشر

$$= \dfrac{1 - \tan 45}{\sin 45 - \cos 45} = \dfrac{1 - 1}{\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}} = \dfrac{0}{0} \quad (I.f.)$$

باستخدام لوبيتال by (L . H . R)

$$\lim_{x \to \frac{\pi}{4}} \dfrac{-\sec^2 x}{\cos x + \sin x} = \dfrac{-\dfrac{1}{\cos^2 45}}{\cos 45 + \sin 45}$$

$$= \dfrac{-\dfrac{1}{(\sqrt{2}/2)^2}}{\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}} = \dfrac{-\dfrac{1}{2/4}}{\dfrac{\cancel{2}\sqrt{2}}{\cancel{2}}} = -\dfrac{1/2}{\sqrt{2}} = -\dfrac{2}{\sqrt{2}}$$

مع التمنيات بالنجاح والتوفيق
السعدي

السعدي ALSAADI

نسخة جديدة منقحة

1432/33

11

Chapter Three
Limits and Continuity

3.5

Continuity

MATH-110

جمال السعدي
رياضيات - إحصاء

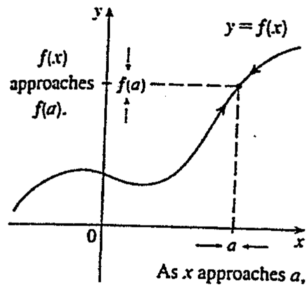
6.15

Continuity
الأتصال

- Continuous at the number $x = a$ ← الأتصال عند عدد

requires three things
يتطلب ثلاثة اشياء

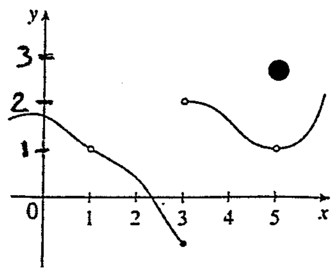
1. $f(a)$ is defined (الداله معرفه عند a)
2. $\lim\limits_{x \to a} f(x)$ exist (النهايه موجوده)
3. $\lim\limits_{x \to a} f(x) = f(a)$ (النهايه = قيمه الداله)



- اذا لم تتحقق الشروط الثلاثه السابقه معاً

تكون غير متصله عند $x = a$ (discontinuous)

Example: from the figure



** $\boxed{\text{at } x = 1}$

$f(1)$ is not defined

$\therefore f(X)$ is discontinuous at $x = 1$

** $\boxed{\text{at } x = 3}$

$\lim\limits_{x \to 3} f(x)$ does not exist (Jump)

$\therefore f(X)$ is discontinuous at $x = 3$

** $\boxed{\text{at } x = 5}$

$f(5) = 3$

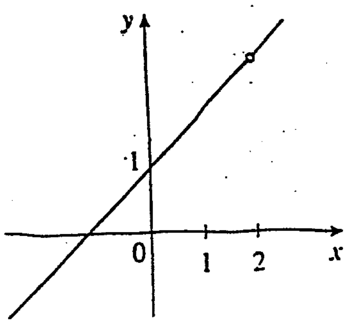
$\lim\limits_{x \to 5} f(x) = 1$

$\therefore f(X)$ is discontinuous at $x = 5$

Example:

where are each of the following functions discontinuous?

1 $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

F (2) is not defined

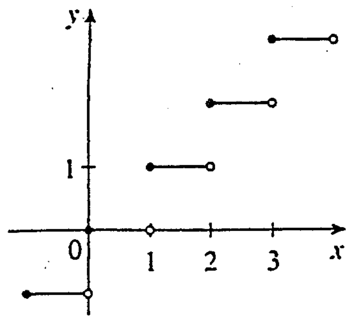
So f (x) is discontinuous at x = 2

Or f (x) is continuous on R − { 2 }

بشكل آخر f (x) is continuous on (−∞, 2) U (2, ∞)

---

2 $f(x) = [x]$



$f(x) = [\![x]\!]$

f (x) is discontinuous

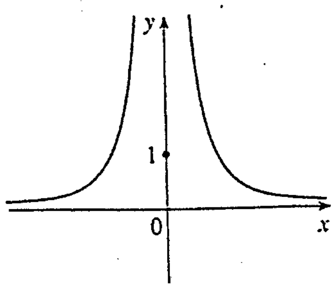
at all of the integers

where the $\lim\limits_{x \to n} [x]$

does not exist (where n is integer)

---

3 $f(x) = \begin{cases} \dfrac{1}{x^2} & if \;\; x \neq 0 \\ 1 & if \;\; x = 0 \end{cases}$



$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{1}{x^2}$ does not exist

so f (x) is discontinuous at x = 0

Definition

If : $\lim_{x \to a^+} f(x) = f(a) \Longrightarrow$ f is continuous from the right a

متصله على يمين a

If : $\lim_{x \to a^-} f(x) = f(a) \Longrightarrow$ f is continuous from the left a

متصله على يسار a

Example:

If : $f(x) = [x]$

at each integer a

- $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} [x] = a \quad * \; f(a) = a$

$\therefore$ f (x) is **continuous** from the right — متصله على يمين العدد a

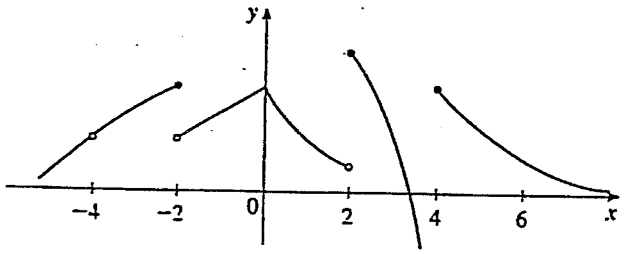
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} [x] = a - 1 \quad * \; f(a) = a$

$\therefore$ f (x) is **discontinuous** from the left — غير متصله على يسار العدد a

3

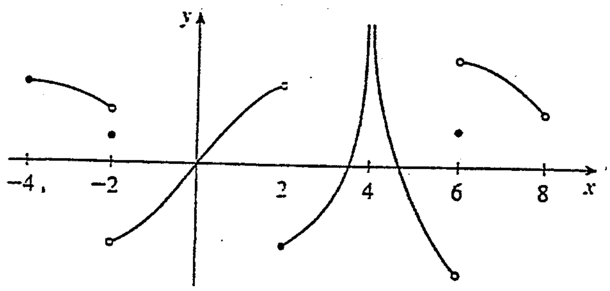
a From the graph of f, state the numbers at which f is discontinuous and explain why.

b For each of the numbers stated in part (a), determine whether f is continuous form right or from the left or neither.



- at x = − 4 f (x) is discontinuous
  where f (− 4) undefined
  f (x) neither continuous from right nor from left. * at x = −4
- at x = − 2 f (x) is discontinuous (Jump) → قفزه
  f (x) is continuous from the left ($\lim_{x \to -2^-} f(x) = f(-2)$)
- at x = 2 f (x) is discontinuous (Jump)
  f (x) is continuous from the right ($\lim_{x \to 2^+} f(x) = f(2)$)
- at x = 4 f (x) is discontinuous (Jump)
  f (x) is continuous from the right ($\lim_{x \to 4^+} f(x) = f(4)$)

4 From the graph of g, state the intervals on which g is continuous.



g (x) is continuous on :

* [− 4 , − 2)   * (− 2 , 2)
* [2 , 4)   * (4 , 6)   * (6 , 8)

Continuous on the interval

الأتصال على فتره

1. اذا كانت $f(x)$ كثيره حدود (*polynomial*)
تكون متصله على $R = (-\infty, \infty)$

2. اذا كانت $f(x)$ كسريه (*rational*)
تكون متصله على R – { اصفار المقام }

3. اذا كانت $f(x)$ جذريه (*root function*)

$$F(x) = \sqrt[n]{\ }$$

n is odd
دليل الجذر فردى

- الجذر فى المقام
تكون الداله
متصله على
R-{اصفار المقام}

  $F(x) = \dfrac{1}{\sqrt[3]{x-2}}$
  متصله على
  $R - \{2\}$

- الجذر فى البسط
تكون الداله
متصله على
$R = (\infty, \infty)$

  $F(x) = \sqrt[3]{x-2}$
  متصله على
  $(-\infty, \infty)$

n is even
دليل الجذر زوجي

- الجدر فى المقام
تكون الداله
متصله
على المفترات
الموجبه مفتوحه
من عند العدد

  $F(x) = \dfrac{1}{\sqrt{x-2}}$
  متصله على
  $(2, \infty)$

- الجذر فى البسط
تكون الداله
متصله
على الفترات
الموجبه مغلقه
من عند العدد

  $F(x) = \sqrt{x-2}$
  متصله على
  $[2, \infty)$

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Note that

The following types of function

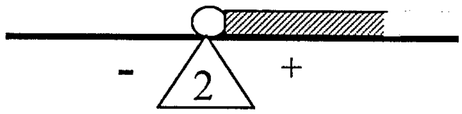
متصله　　　　　　مجالها

are **continuous** on their **domain**.

- trigonometric functions. الدوال المثلثيه
- Inverse trigonometric functions. الدوال المثلثيه العكسيه
- exponential functions. الدوال الأسيه
- logarithmic functions. الدوال اللوغاريتميه

Example:

1. $f(x) = \ln(x-2)$　　　　دالة $\ln$ متصلة على



الفترات الموجبه مفتوحه من عند العدد 2

$\therefore f(x)$ is continuous on $(2, \infty)$

2. $f(x) = \tan^{-1}x$　　　　داله $\tan^{-1}x$ متصله على مجالها

$\therefore f(x)$ is continuous on $(-\infty, \infty)$

3. $f(x) = \ln(x-2) + \tan^{-1}x$　　　　متصله على المجال المشترك

$\therefore f(x)$ is continuous

on

$(2, \infty) \cap (-\infty, \infty) = (2, \infty)$

Example:

Where is the function f (x) continuous ?

1 $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$

*الداله $f(x)$ متصله على المجال المشترك لداله $\ln x$ ، $\tan^{-1} x$

باستبعاد اصفار المقام

$(-\infty,\infty)$ $(0,\infty)$

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

$\therefore$ f (x) is continuous

on $(-\infty,\infty) \cap (0,\infty) - \{-1, 1\}$

$= (0,\infty) - \{-1, 1\}$

$= (0, 1) \cup (1, \infty)$

---

2 $f(x) = 2x^3 - x^2 + 1 \longrightarrow$ polynomial كثيره حدود

f (x) is continuous on $(-\infty,\infty) = R$

---

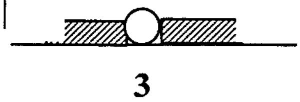
3 $*f(x) = 2$ $\quad * f(x) = \sqrt{5}$ $\quad * f(x) = -\dfrac{2}{3}$ $\quad * f(x) = 0$

Are continuous on $(-\infty,\infty) = R$

---

4 $f(x) = |x-3|$ continuous on $(-\infty,\infty)$

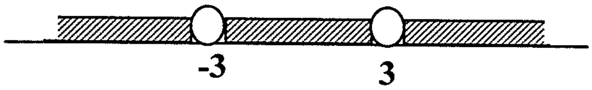
5 $f(x) = \dfrac{1}{|x-3|}$ متصلة على $R-$ {أصفار المقام}



$\therefore f(x)$ continuous on $R-\{3\} = (-\infty,3)\cup(3,\infty)$

---

6 $f(x) = \dfrac{1}{|x|-3}$ اصفار المقام $|x| - 3 = 0$



$|x| = 3 \Rightarrow x = \pm 3$

$\therefore f(x)$ continuous on $R-\{-3,3\} = (-\infty,-3)\cup(-3,3)\cup(3,\infty)$

---

7 $f(x) = \dfrac{1}{|x|+3}$ المقام ليس له أصفار

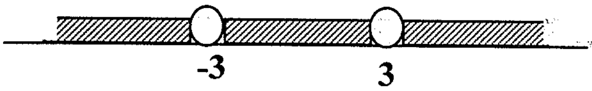
لأن $|x| + 3 = 0$

$|x| = -3$ مرفوض (discard)

$\therefore f(x)$ continuous on R .

---

8 $f(x) = \dfrac{3x}{x^2-9} \Rightarrow$ أصفار المقام $x^2 - 9 = 0$



$x^2 = 9 \Rightarrow x \pm 3$

$\therefore f(x)$ continuous on $R-\{-3,3\} = (-\infty,-3)\cup(-3,3)\cup(3,\infty)$

---

9 $f(x) = \dfrac{3x}{x^2+9}$ المقام ليس له اصفار

لأن $x^2 + 9$ لايمكن أن تساوى zero

$\therefore f(x)$ continuous on R .

---

10 $f(x) = \sqrt[3]{x^2-4}$ جذر تكعيبى فى البسط

$\therefore f(x)$ continuous on $R = (-\infty,\infty)$

---

11 $f(x) = \dfrac{2x-1}{\sqrt[3]{x^2-4}}$ جذر تكعيبى فى المقام

$\therefore$ الدالة متصله على $R-$ {اصفار المقام}

$f(x)$ continuous on $R-\{-2, 2\}$

12 $f(x) = \sqrt[5]{x^2 - x}$ الجذر الخامس (دليل الجذر n فردى)

والجذر فى البسط

$\therefore f(x)$ is continuous on R

$\therefore f(x)$ متصله على R.

---

13 $f(x) = \dfrac{2}{x}$

داله كسريه

متصله على

R - { أصفار المقام }

$\therefore f(x)$ continuous on $R - \{0\}$

oR $f(x)$ is discontinuous at $x = 0$

---

14 $f(x) = \dfrac{2x-1}{x^2-5x+6} - 5x$

داله كسريه

اصفار المقام

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$x = 3 \quad , \quad x = 2$

$f(x)$ is continuous

on $R - \{2, 3\}$

---

15 $f(x) = \dfrac{2x-1}{x^2-5x+6} + \dfrac{2x^2}{3}$

discontinuous at x = اصفار المقام

$x = 2, 3$

$\therefore f(x)$ is discontinuous at $x = 2, 3$

16 *find the interval*

on which (1) $f(x)=\sqrt{|x|-2}$ is continuous

- جذر تربيعي فى البسط

∴ الداله متصله على الفترات الموجبه مغلقه من عند العدد

$\Rightarrow \quad |x|-2\geq 0$

$|x|-2\geq$

$x\geq 2$

$x\geq 2 \quad or \quad x\leq -2$ نظريه

$\therefore f(x)$ is continuous on $(-\infty,-2]\cup[2,\infty)$

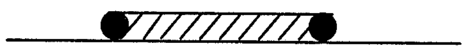
---

(2) $f(x)=\sqrt{2-|x|}$

$\Rightarrow \quad 2-|x|\geq 0$

$\Rightarrow -|x|\geq -2 \Rightarrow |x|\leq 2$

$-2\leq x\leq 2$ نظرية



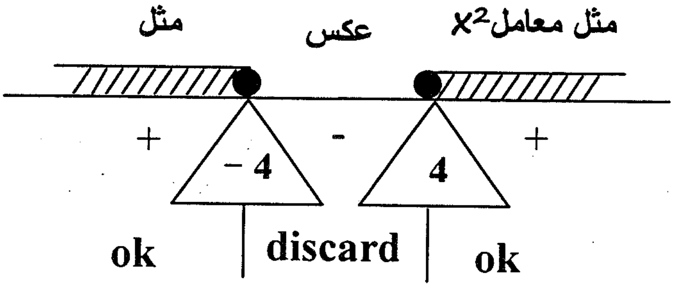
$\therefore f(x)$ is continuous on $[-2,2]$

17 $f(x) = \sqrt{x^2-16}$

* جذر تربيعى فى البسط

∴ متصله على الفترات الموجبه مغلقه

من عند العدد



$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

$f(x)$ continuous on $(-\infty,-4] \cup [4, \infty)$

** $f(x)$ discontinuous on $(-4, 4)$

18 $f(x) = \dfrac{2x}{\sqrt{x^2-16}}$

* جذر تربيعى فى المقام

∴ متصله على الفترات الموجبه

مفتوحه

من عند العدد

نفس المثال السابق

$f(x)$ continuous on $(-\infty,-4) \cup (4, \infty)$

Note that :

$$f(x) = \begin{cases} g(x) ; & x \geq a \\ h(x) ; & x < a \end{cases}$$

الداله المعرفه باكثر من قاعده فى حاله وجود أكبر وأقل من لكى تكون الداله متصله عند $x = a$

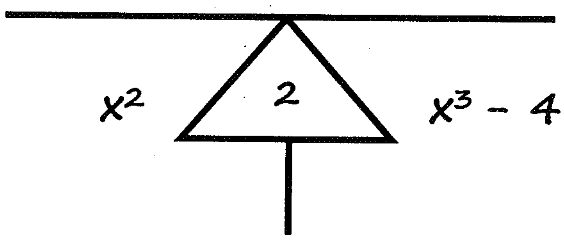
لابد أن:

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^-} h(x) = g(a)$$

قيمة الداله = النهايه اليسرى = النهايه اليمنى ⟵ أى أن

التعويض فى الطرف الموجود به علامه المساواه | التعويض فى طرف أصغر من | التعويض فى طرف اكبر من

$$f(x) = \begin{cases} x^3 - 4 \; ; & x \geq 2 \\ x^2 \; ; & x < 2 \end{cases}$$



$f(x)$ is continuous on $(-\infty, 2)$ and $(2, \infty)$

because : it is polynomial كثيره حدود

at $x = 2$ نقطة فاصله

∴ لابد من ايجاد النهايه اليمنى، اليسرى ، قيمه الداله

- $\lim\limits_{x \to 2^+} (x^3 - 4) = 8 - 4 = 4$ * النهايه اليمنى

التعويض فى طرف اكبر من

- $\lim\limits_{x \to 2^-} x^2 = 4$ *النهايه اليسرى

التعويض في طرف أصغر من

- $f(2) = (2^3 - 4) = 8 - 4 = 4$ * قيمه الداله

التعويض فى الطرف الذى به علامه المساواه

$\therefore \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) = f(2)$

$\therefore f(x)$ is continuous at $x = 2$

→ ** $f(x)$ is continuous on $(-\infty, \infty)$

Example:

Find the value of c

which makes

$$f(x) = \begin{cases} c\,x + 5 \;;\; x < 2 \\ c\,x^2 + 1 \;;\; x \geq 2 \end{cases}$$

is continuous at $x = 2$

$$\lim_{x \to 2^+} (c\,x^2 + 1) = \lim_{x \to 2^-} (c\,x + 5)$$

$$c(2^2) + 1 = c(2) + 5$$

$$4c + 1 = 2c + 5$$

$$4c - 2c = 5 - 1$$

$$2c = 4 \longrightarrow c = 2$$

Note that

فى حاله الداله المعرفه بقاعدتين

احدهما تحتوى على $(x \neq a)$

والأخرى تحتوى على $(x = a)$

* نوجد النهايه من عند $\neq$ ، قيمه الداله من عند $=$

اذا تساوى الناتجين تكون الداله متصله عند $x = a$

وإلا تكون الداله غير متصله عند $x = a$

Example:

$$If : f(x) = \begin{cases} \dfrac{x^2-16}{x-4} ; & x \neq 4 \\ 7 ; & x = 4 \end{cases}$$

Is $f(x)$ continuous at $x = 4$?

*** نوجد النهايه من عند $x \neq 4$**

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = \frac{16-16}{4-4} = \frac{0}{0} \; (I.f.) \; (by\ L.H.R)$$

$$= \lim_{x \to 4} \frac{2x}{1} = \boxed{8}$$

*** نوجد قيمه الداله من عند $x = 4$**

$$f(4) = \boxed{7}$$

$$\because \lim_{x \to 4} f(x) \neq f(4)$$

*** نهايه $\neq$ قيمه الداله**

$\therefore f(x)$ is discontinuous at $x = 4$

ALSAADI

Intermediate value theorem

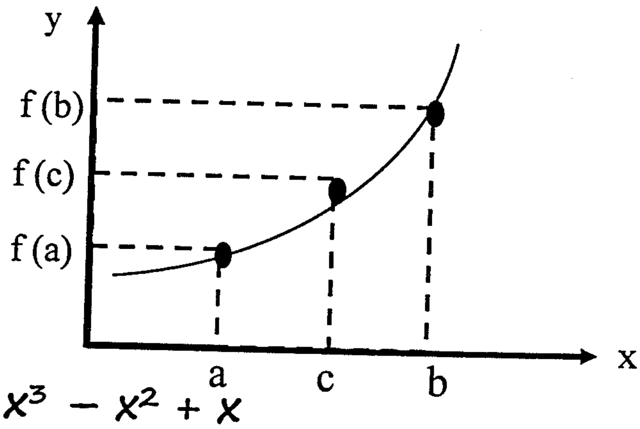
نظريه القيمه الوسطى

* اذا كانت الداله $f(x)$ متصله على $[a, b]$

فإن يوجد عدد واحد على الأقل c

Where $a \le c \le b$

$\Rightarrow f(a) \le f(c) \le f(b)$



Example:

For the function $f(x) = x^3 - x^2 + x$

there is a number $c \in [1, 3]$

such that $f(c) =$ ------

(A) 0 (B) 10 (C) -1 (D) 40

Solution

$\because c \subset [1, 3]$

$1 \le c \le 3$

$f(1) \le f(c) \le f(3)$

عوض ب 1 في الداليه $f(x)$ ↓ عوض ب 3 في الداليه $f(x)$ ↓

$1 \le f(c) \le 21$

الأختيار المناسب هو 10 (B) لأنه الوحيد المحصور بين 1 ، 21

Removable discontinuity → ازاله عدم الأتصال

اعاده تعريف الداله الغير متصله بحيث تصبح متصله

اذا كانت $f(x)$ داله غير متصله عند a

فإنه يمكن اعاده تعريف الداله $f(x)$ بشكل آخر (داله آخرى $g(x)$)

تكون متصله عند a كما يلى:

$$g(x) = \begin{cases} f(x) & \text{قاعده الداله } f \; ; x \neq a \\ \lim\limits_{x \to a} f(x) & \text{نهايه الداله } f \; ; x = a \end{cases}$$

* مع العلم أن الداله $f(x)$: *يمكن اعاده تعريفها اذا كانت نهايتها $\boxed{\frac{0}{0}}$

* لا يمكن اعاده تعريفها اذا كانت نهايتها $\boxed{\frac{\text{عدد}}{0}}$

Example:

Which of the following functions f has removable discontinuity ?

1. $f(x) = \dfrac{x^4+1}{x-1}$ discontinuous at $x = 1$

$$\lim_{x \to 1} \frac{x^4+1}{x-1} = \frac{1+1}{1-1} = \frac{2}{0} = \boxed{\frac{\text{عدد}}{0}} = \infty$$

$\therefore$ discontinuity is not removable لا يمكن ازاله عدم الأتصال

(Infinite discontinuous)

2 $f(x) = \dfrac{x^4 - 1}{x - 1}$ داله كسريه

∴ غير متصله عند اصفار المقام

$f(x)$ is discontinuous at $x = 1$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \frac{1-1}{1-1} = \boxed{\frac{0}{0}} \ (I.f.)$$

→ we can removable discontinuity

$$(\text{by L.H.R}) \lim_{x \to 1} \frac{4x^3}{1} = 4(1^3) = 4$$

$$\rightarrow g(x) \begin{cases} \dfrac{x^4 - 1}{x - 1} & f \text{ قاعده الداله} \ ; x \neq 1 \\ 4 & f \text{ نهايه الداله} \ ; x = 1 \end{cases}$$

---

a page 128

If : f and g are continuous function

With $f(3) = 5$ and $\lim [2f(x) - g(x)] = 4$

Find g (3) ?

$$\lim_{x \to 3} [2f(x) - g(x)] = 4$$

$$2f(3) - g(3) = 4$$

$$2(5) - g(3) = 4 \rightarrow g(3) = 10 - 4 = \boxed{6}$$

Example: page 125

Evaluate

$$\lim_{x \to 1} \arcsin \left(\frac{1-\sqrt{x}}{1-x} \right)$$

$$= \arcsin \left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x} \right)$$

تعويض مباشر $\frac{0}{0}$

by L . H . R لوبتيال

$$= \arcsin \left(\lim_{x \to 1} \frac{-\frac{1}{2\sqrt{x}}}{-1} \right)$$

$$= \arcsin \left(\lim_{x \to 1} \frac{1}{2\sqrt{x}} \right)$$

$$= \arcsin \left(\frac{1}{2\sqrt{1}} \right)$$

$$= \sin^{-1} \left(\frac{1}{2} \right) = 30 = \frac{\pi}{6}$$

كل التمنيات بالنجاح والتوفيق

السعدي

THOMAS' CALCULUS

MEDIA UPGRADE

Chapter 3

Differentiation

3.1

The Derivative as a Function

DEFINITION **Derivative Function**

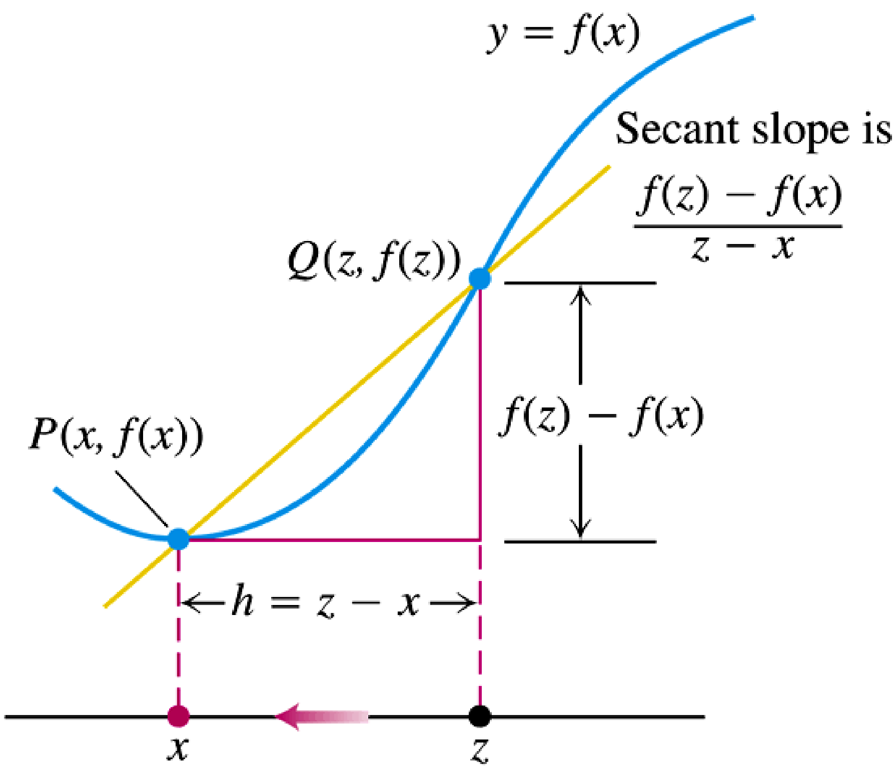
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

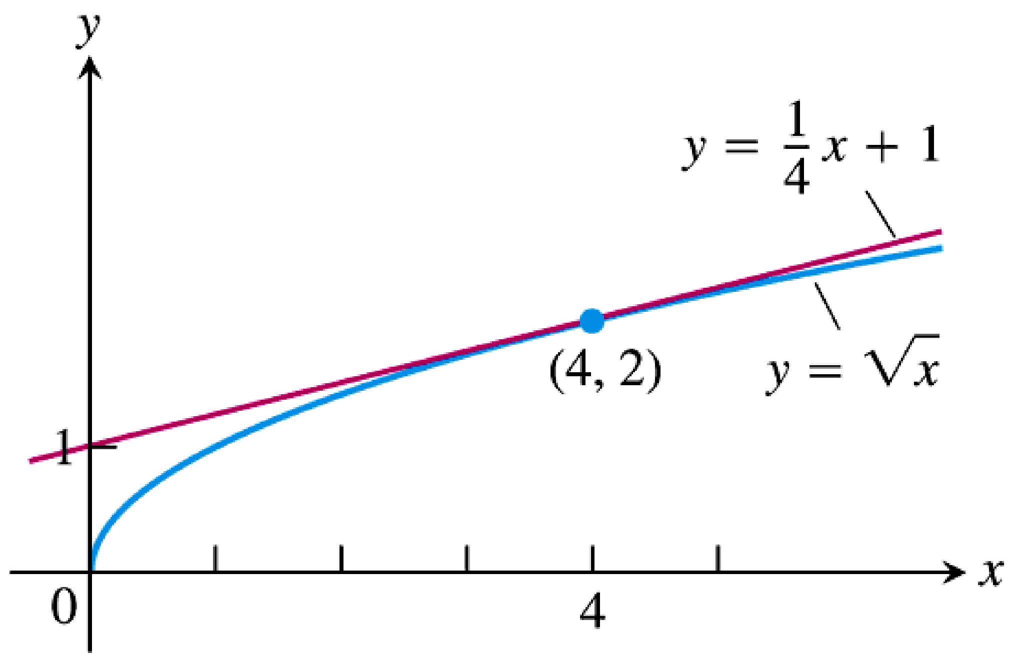


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

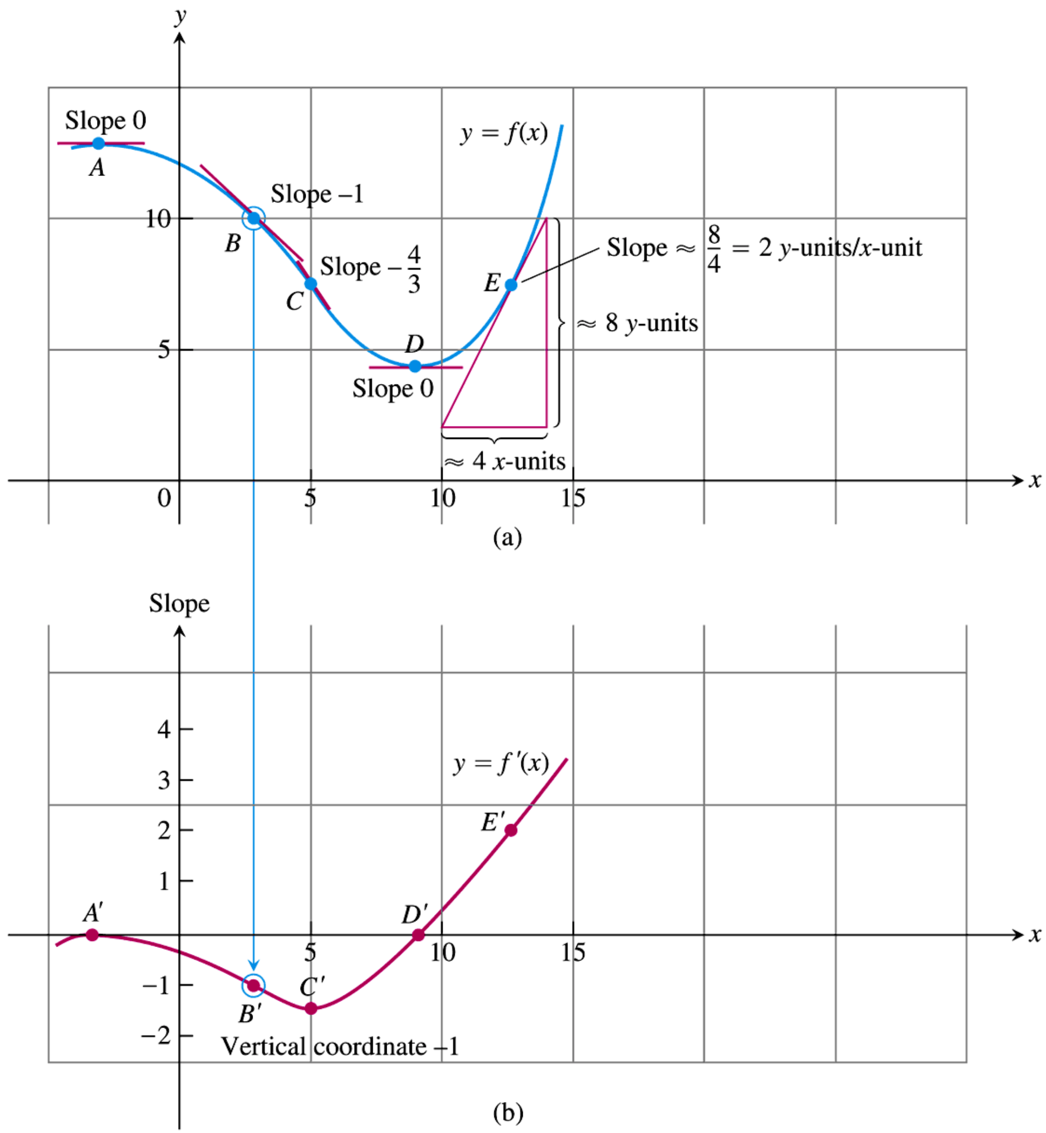
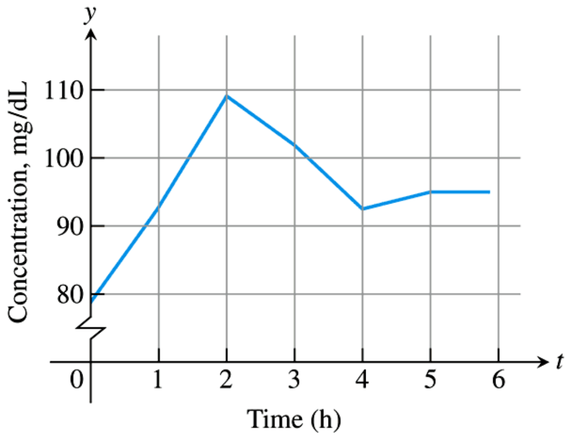
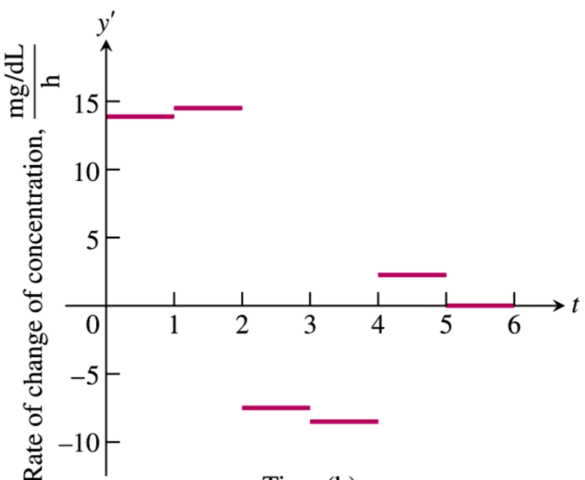


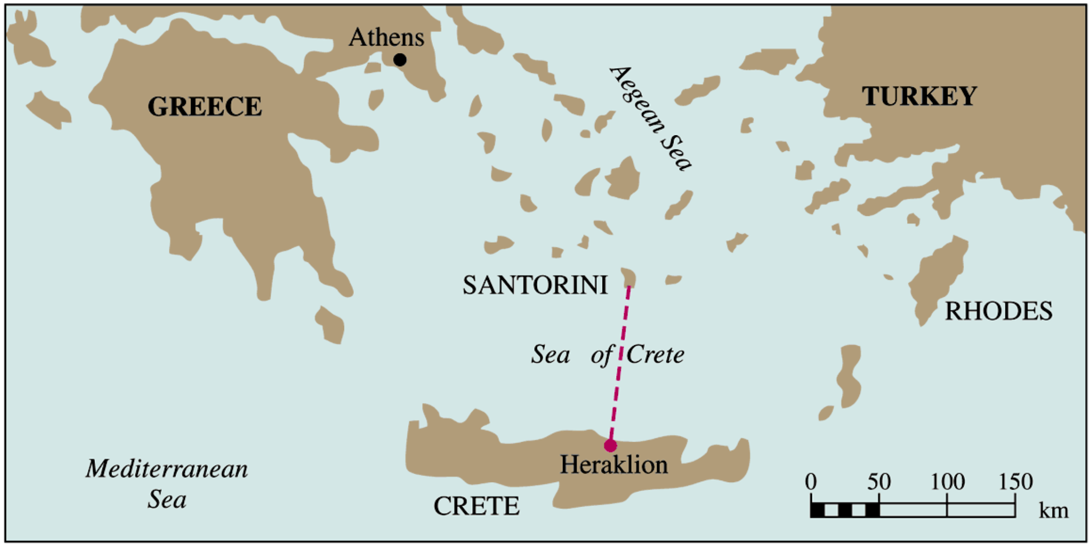
FIGURE 3.3 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B and so on. The graph of f' is a visual record of how the slope of f changes with x.



(a)



(b)



Daedalus's flight path on April 23, 1988

FIGURE 3.4 (a) Graph of the sugar concentration in the blood of a *Daedalus* pilot during a 6-hour preflight endurance test. (b) The derivative of the pilot's blood-sugar concentration shows how rapidly the concentration rose and fell during various portions of the test.

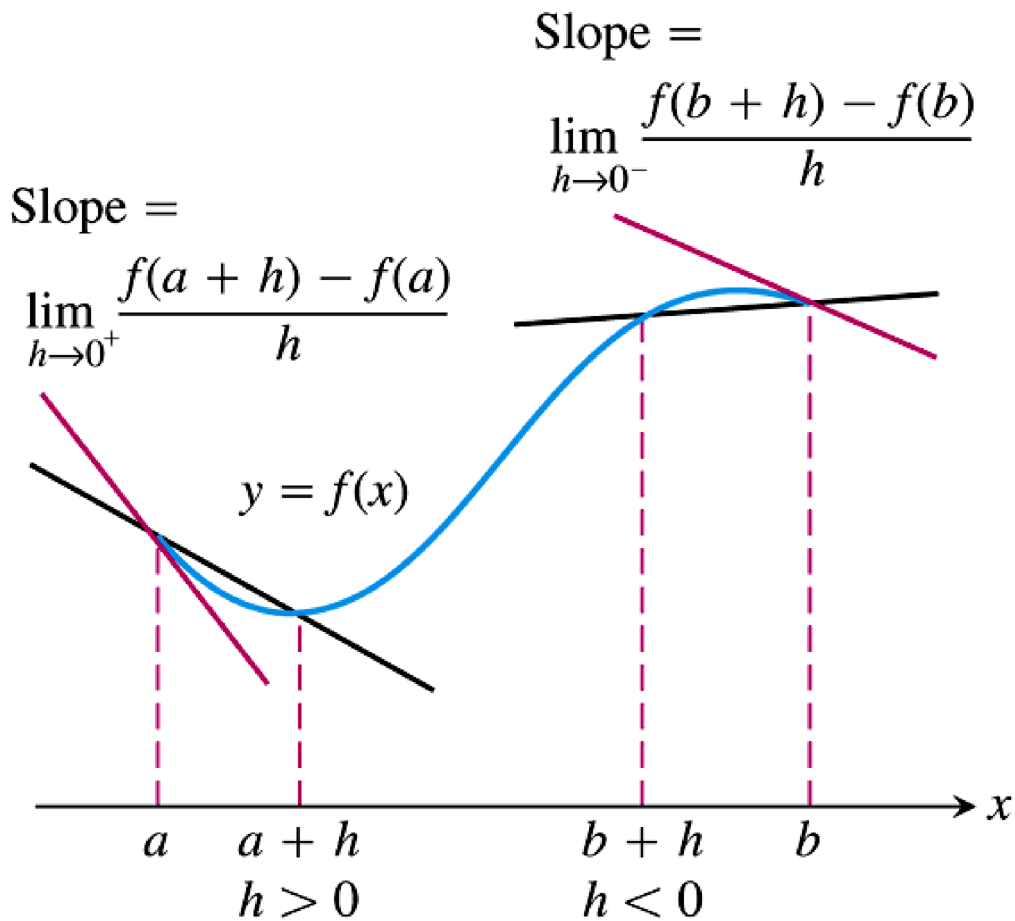


FIGURE 3.5 Derivatives at endpoints are one-sided limits.

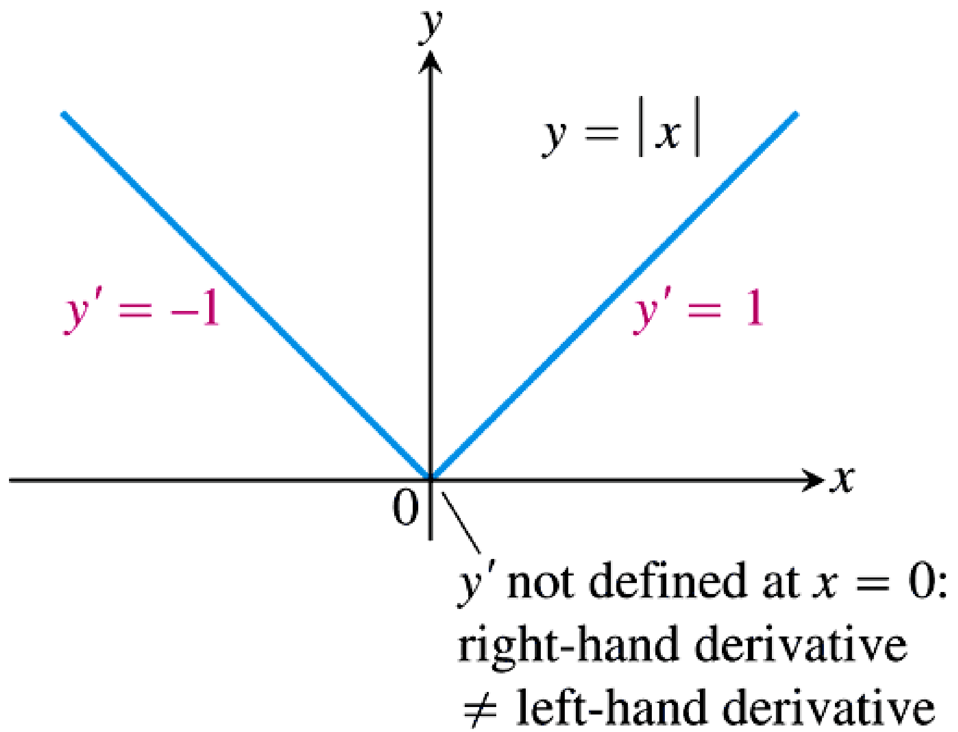
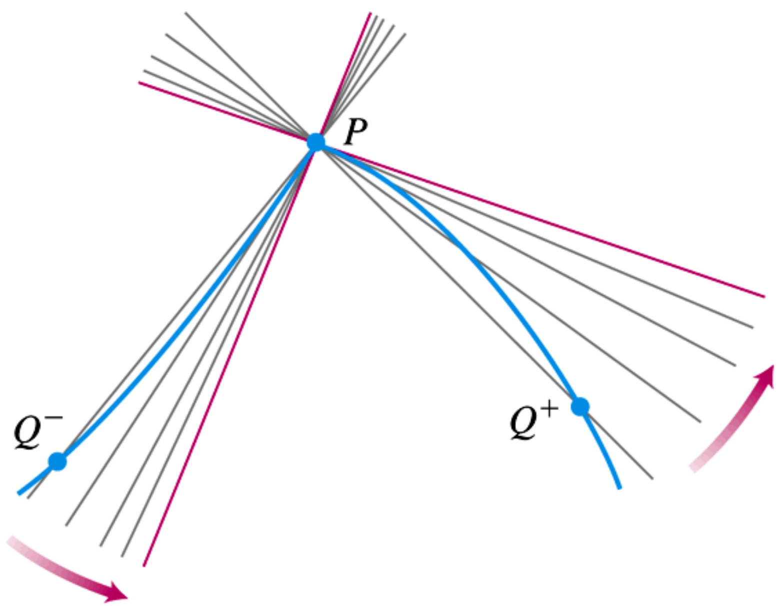
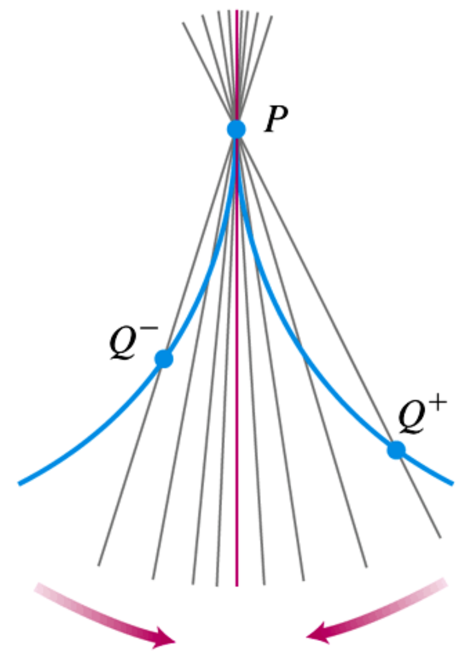


FIGURE 3.6 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

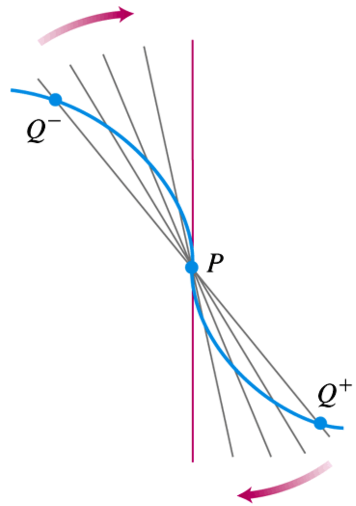
1. a *corner*, where the one-sided derivatives differ.



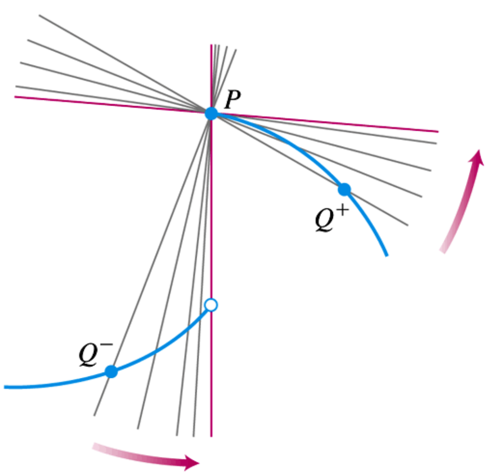
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.

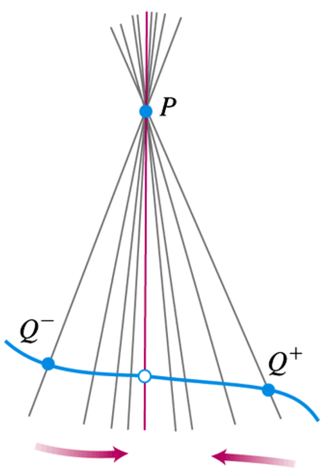


3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.





THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

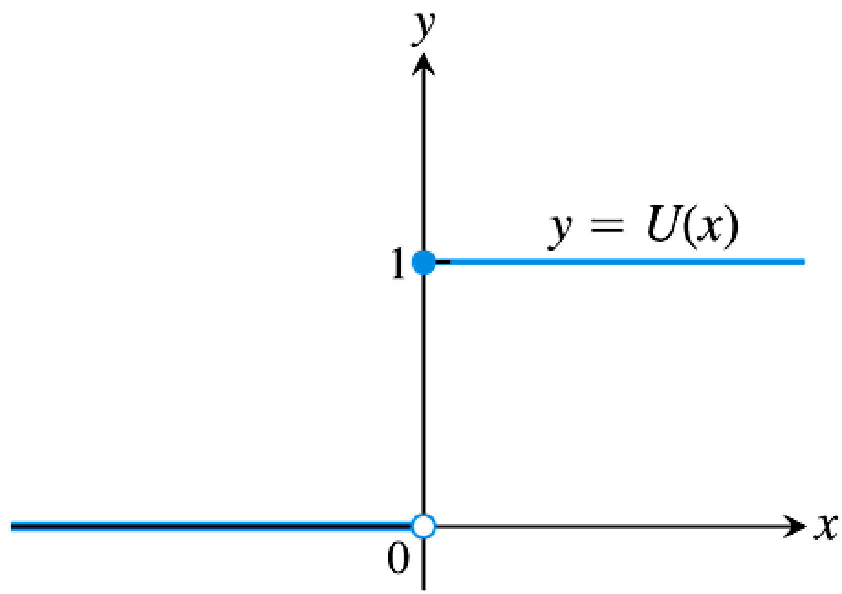


FIGURE 3.7 The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

THEOREM 2 **Darboux's Theorem**

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

3.2

Differentiation Rules

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

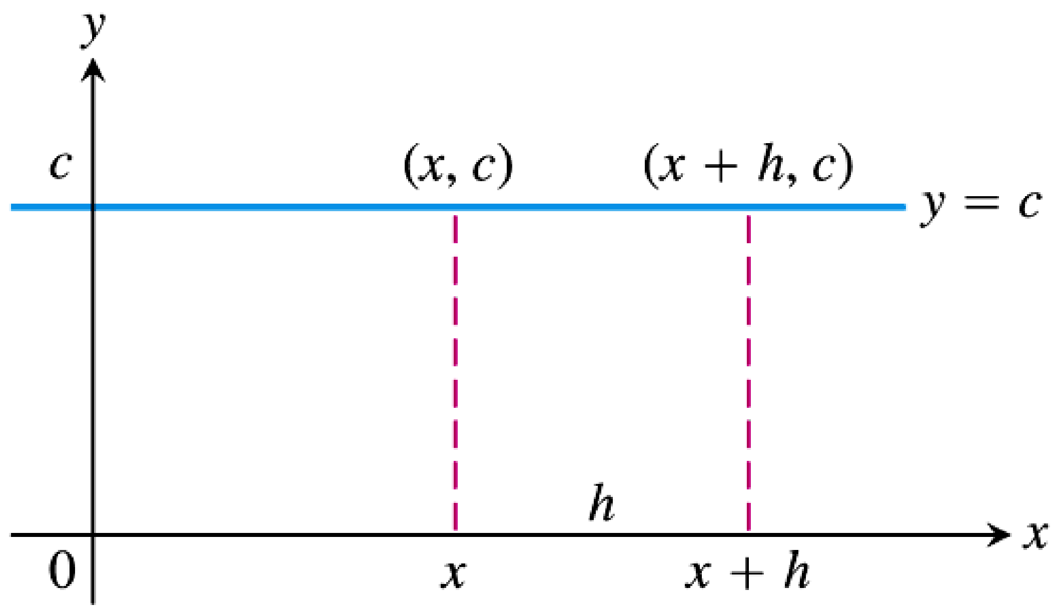


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

RULE 2 **Power Rule for Positive Integers**

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

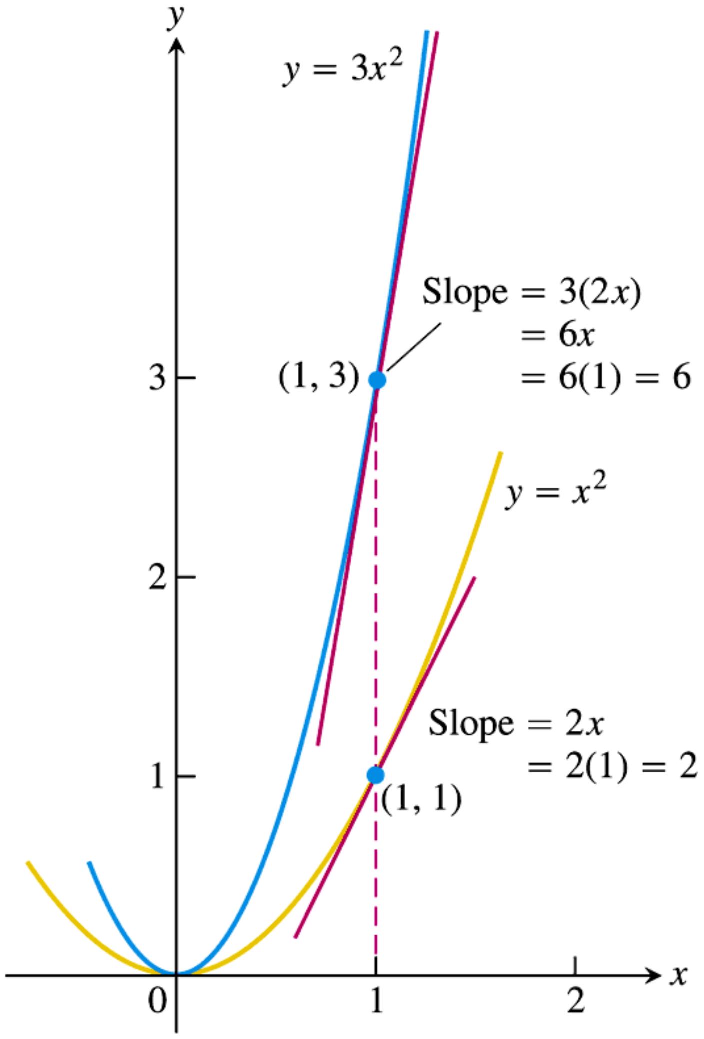


FIGURE 3.9 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

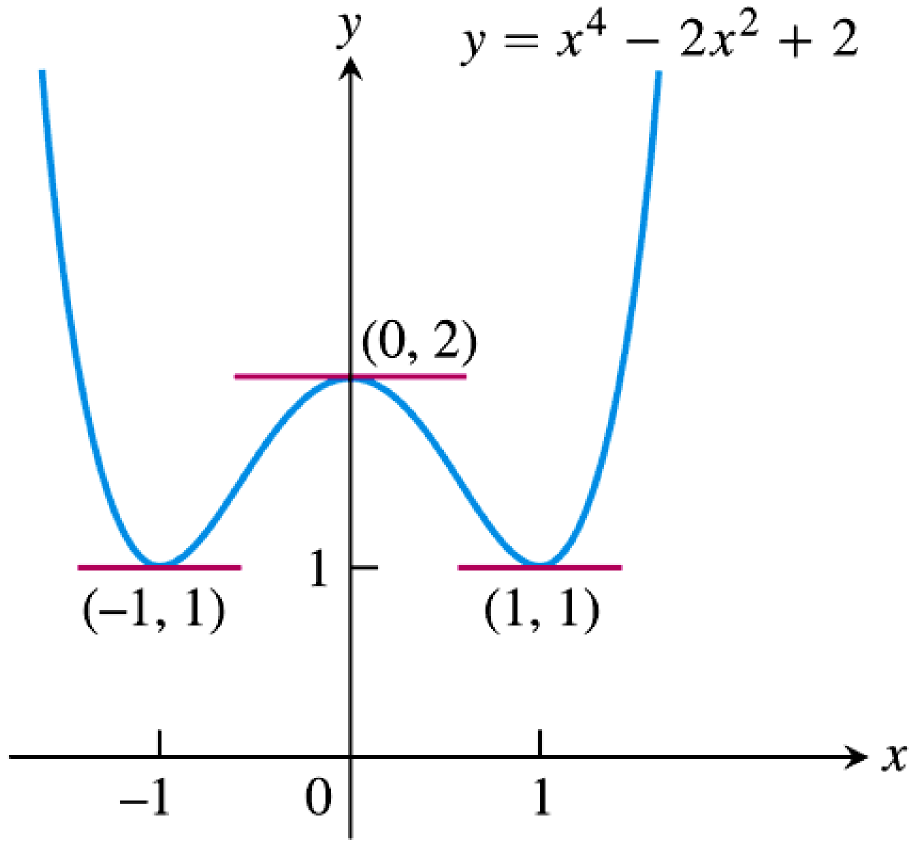


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

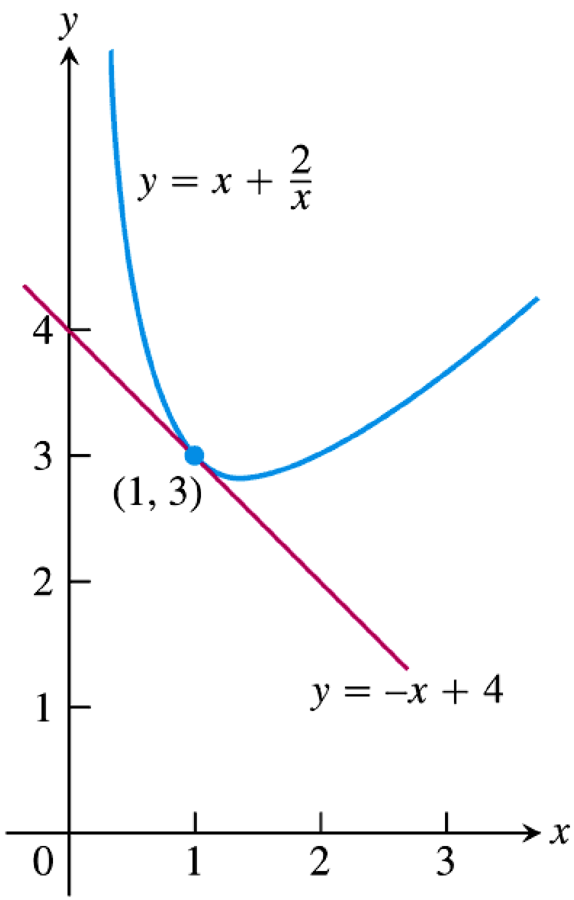


FIGURE 3.11 The tangent to the curve $y = x + (2/x)$ at $(1, 3)$ in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

3.3

The Derivative as a Rate of Change

DEFINITION **Instantaneous Rate of Change**

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

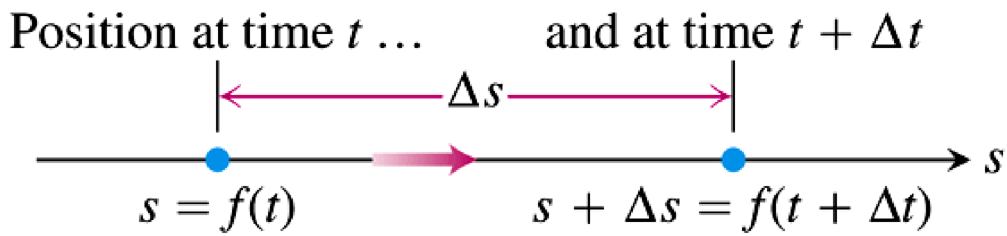


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

DEFINITION Velocity

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

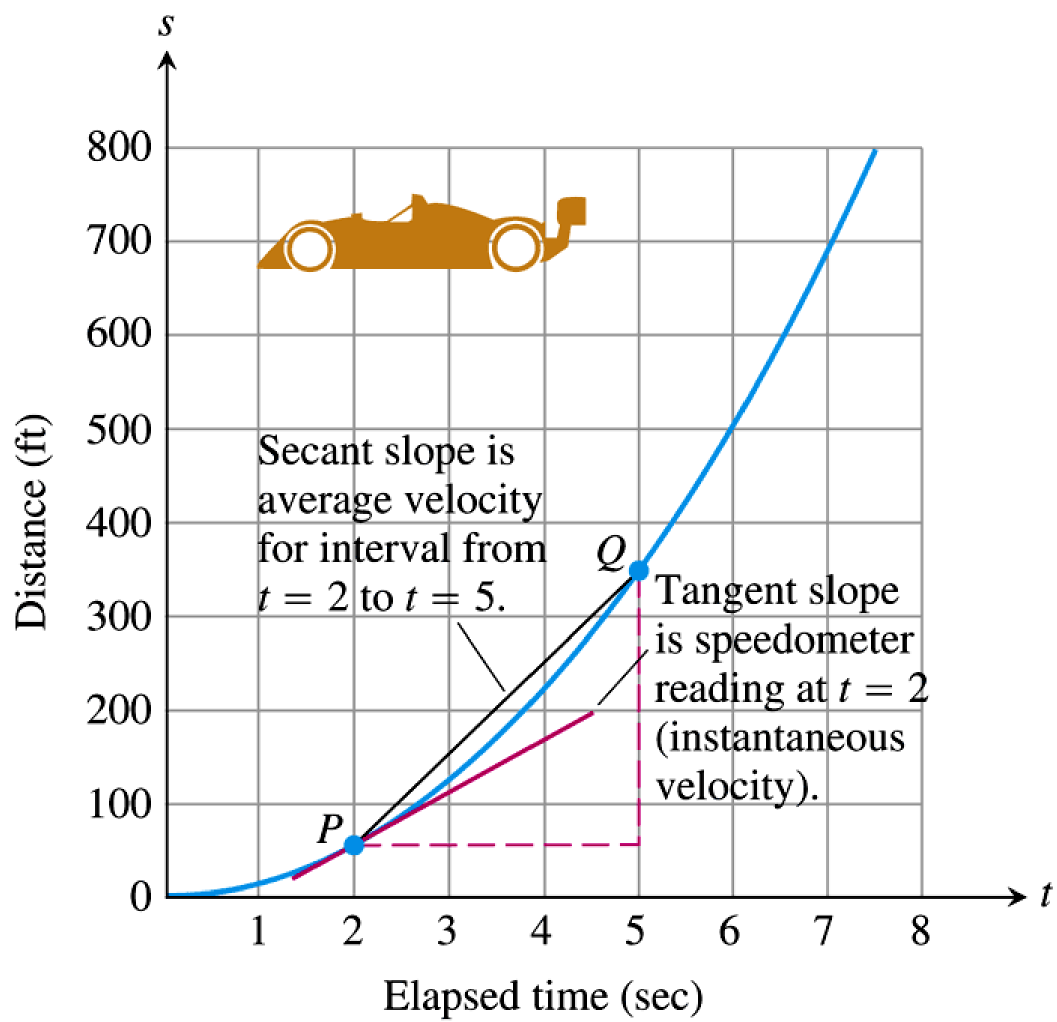


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

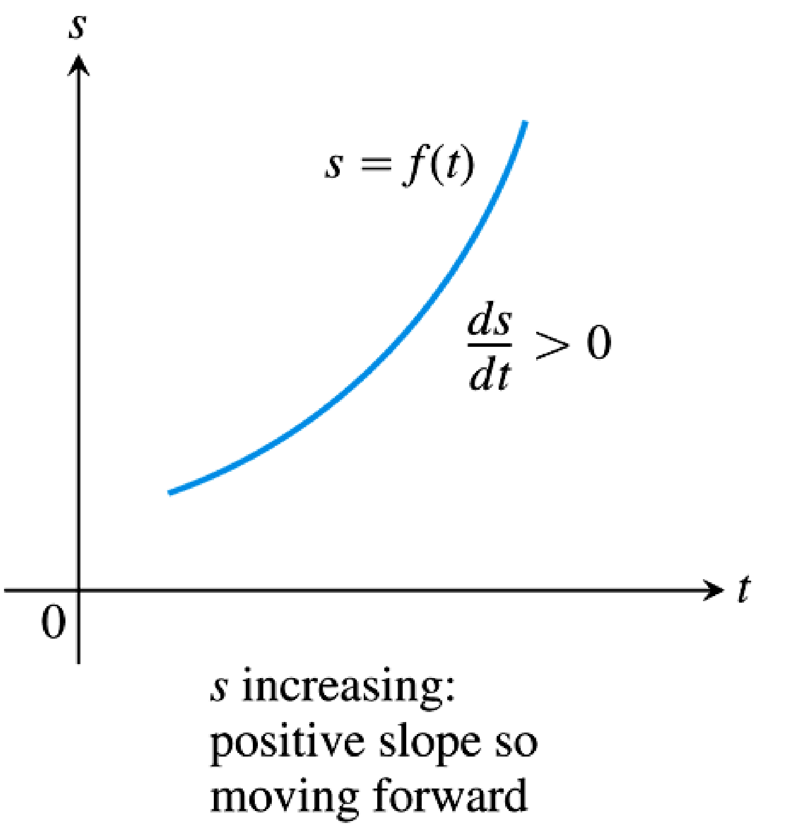


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

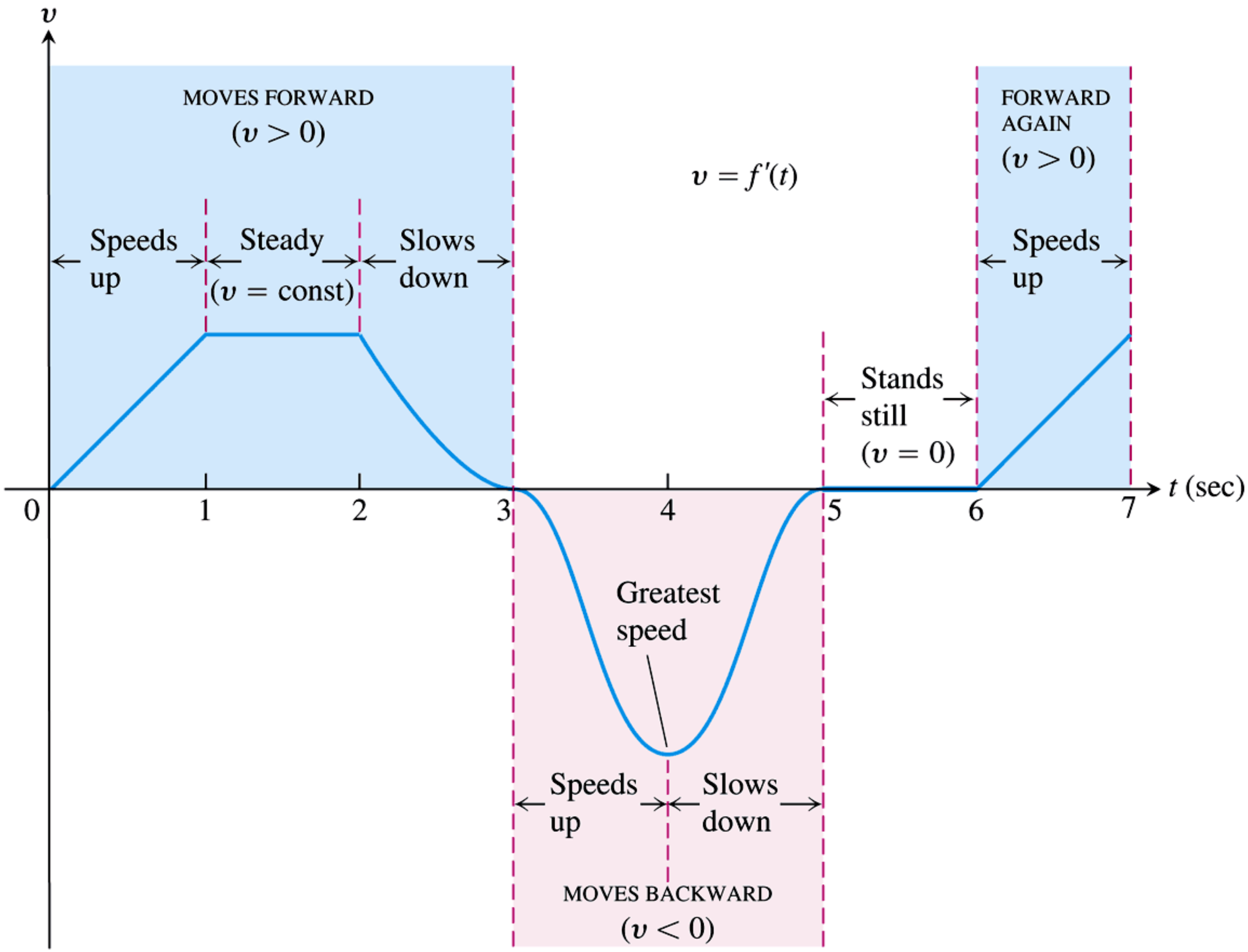


FIGURE 3.15 The velocity graph for Example 3.

DEFINITIONS Acceleration, Jerk

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

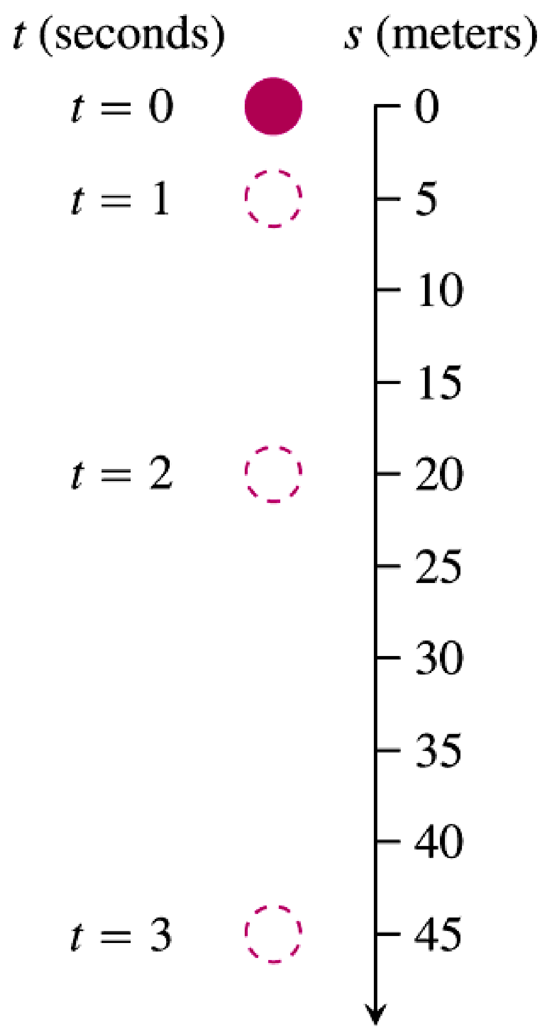
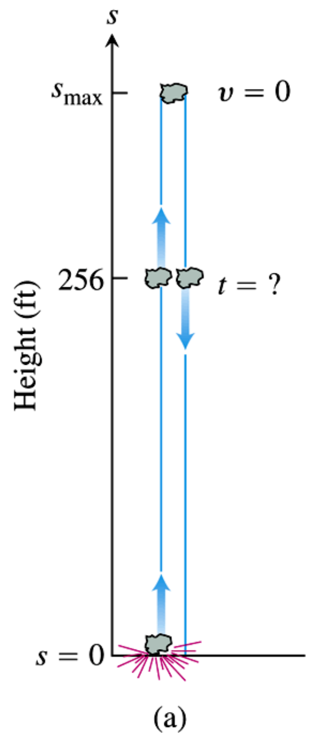


FIGURE 3.16 A ball bearing falling from rest (Example 4).



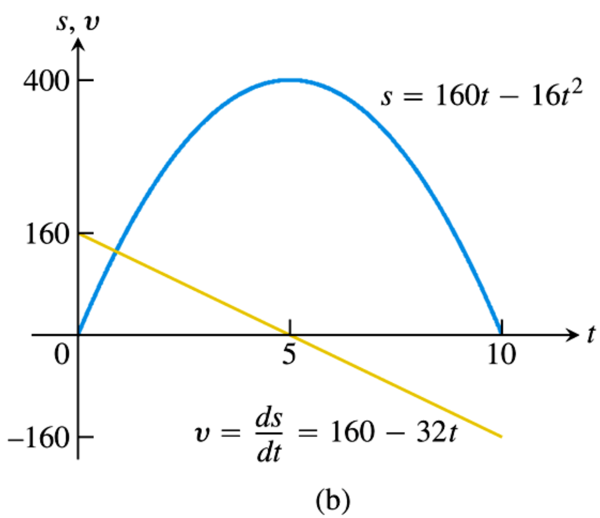


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

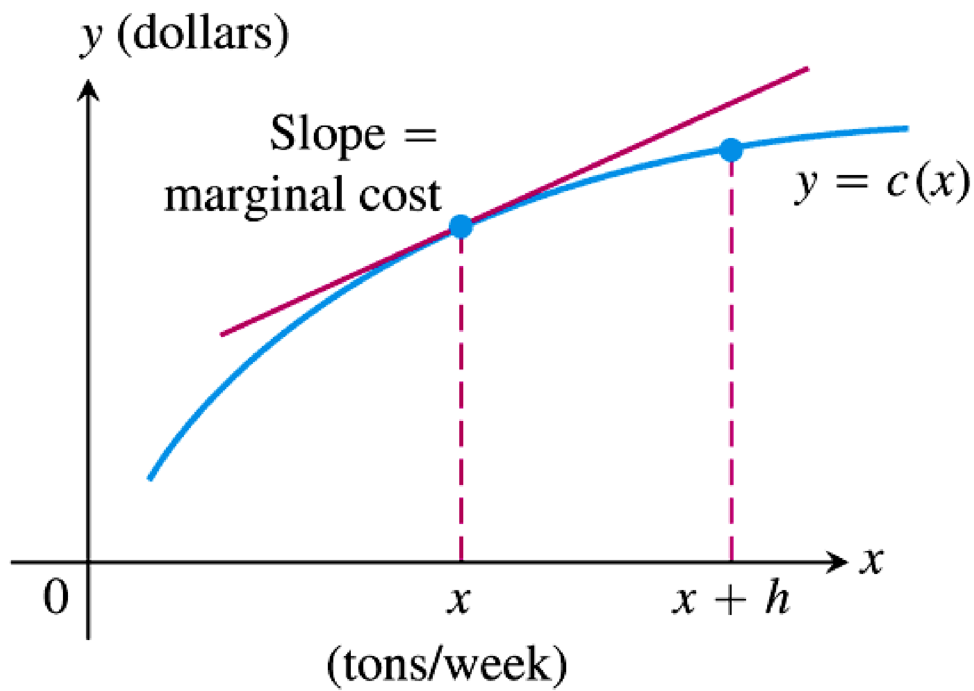


FIGURE 3.18 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

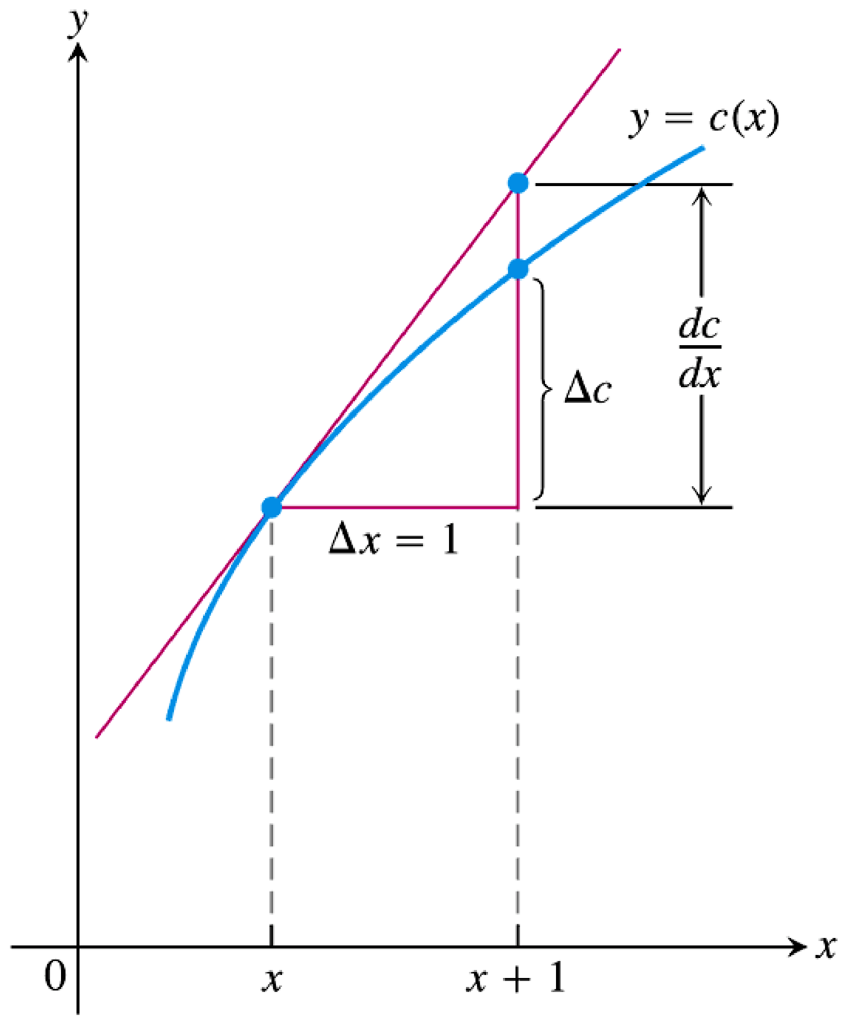


FIGURE 3.19 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

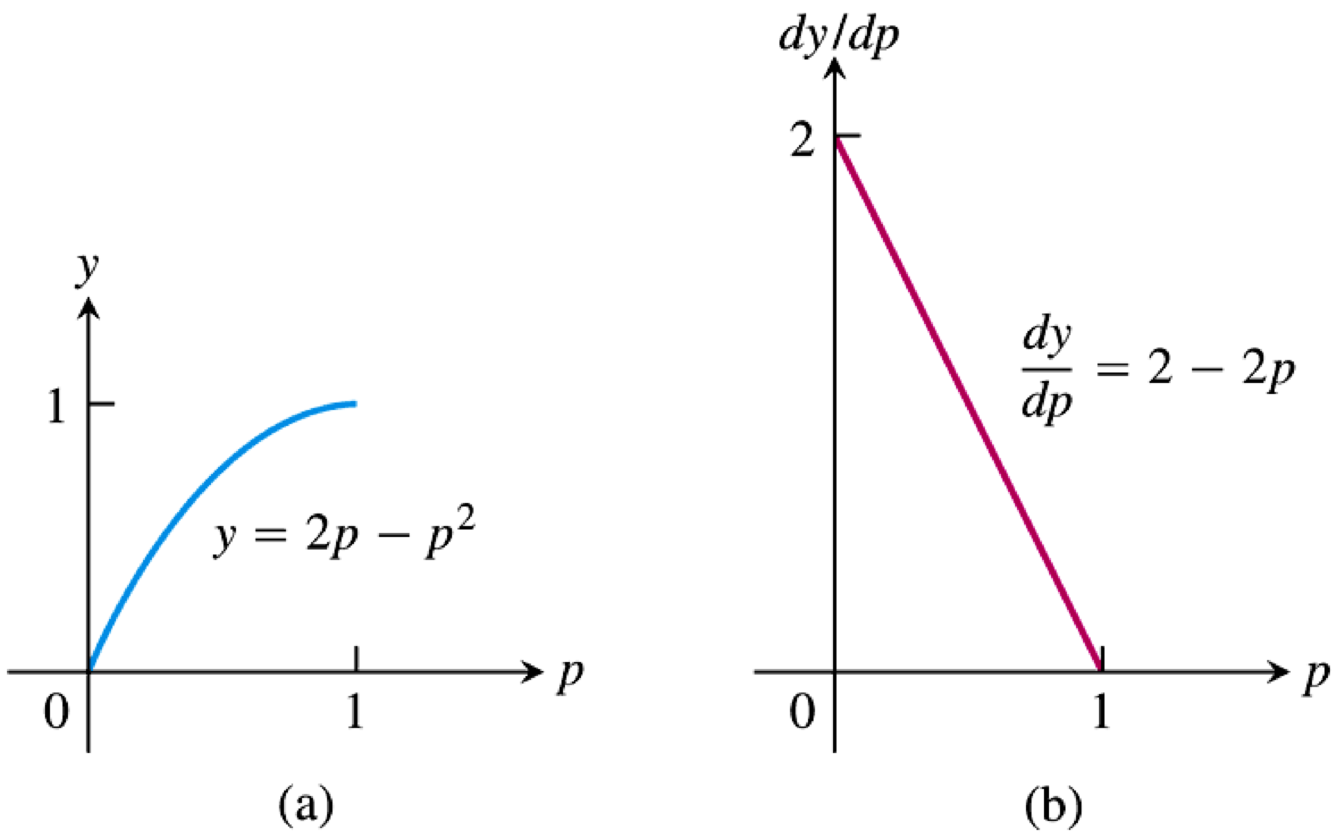


FIGURE 3.20 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas. (b) The graph of dy/dp (Example 8).

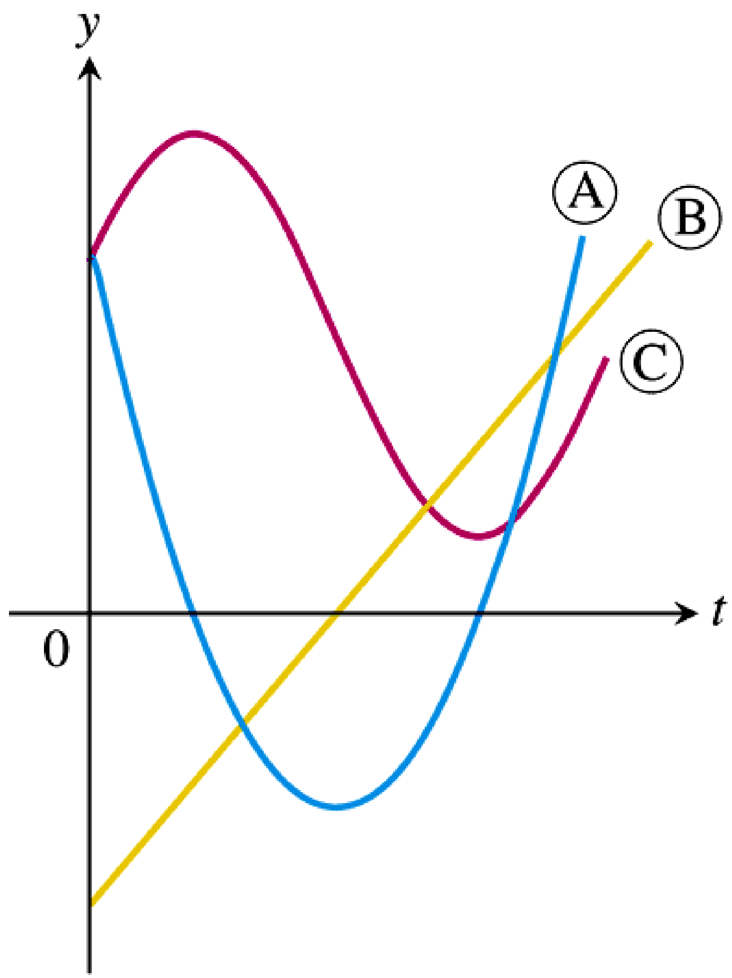


FIGURE 3.21 The graphs for Exercise 21.

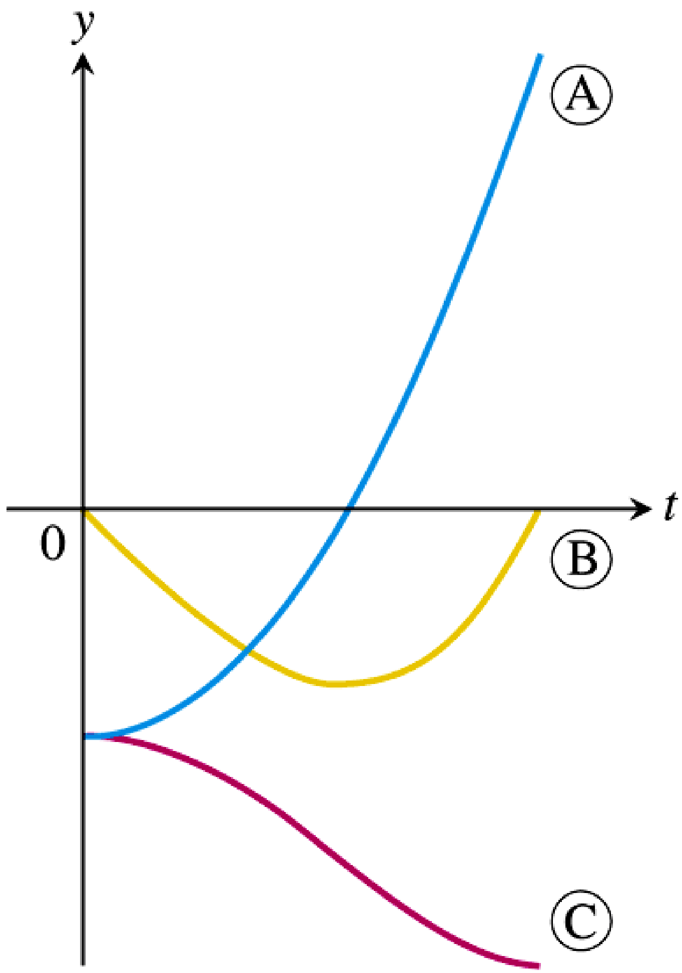


FIGURE 3.22 The graphs for Exercise 22.

3.4

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

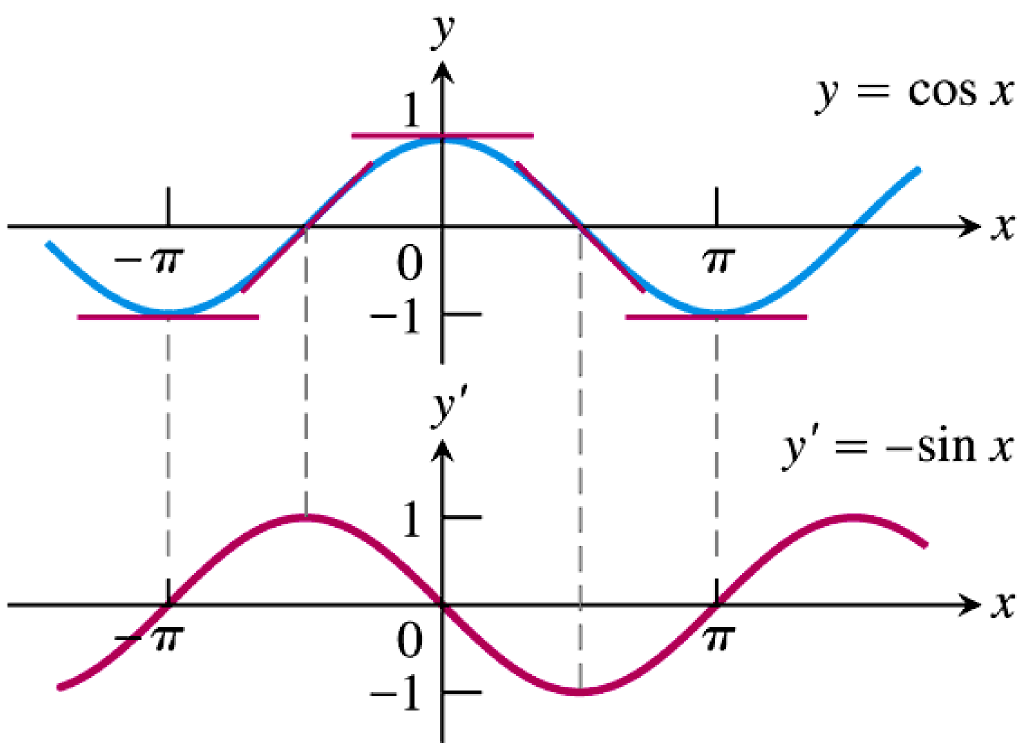


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

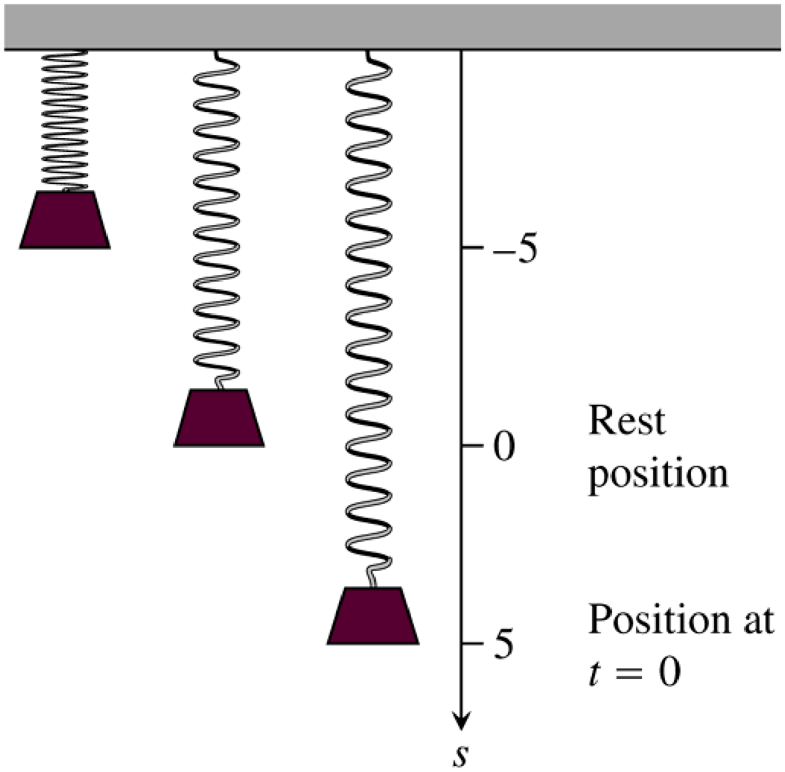


FIGURE 3.24 A body hanging from a vertical spring and then displaced oscillates above and below its rest position. Its motion is described by trigonometric functions (Example 3).

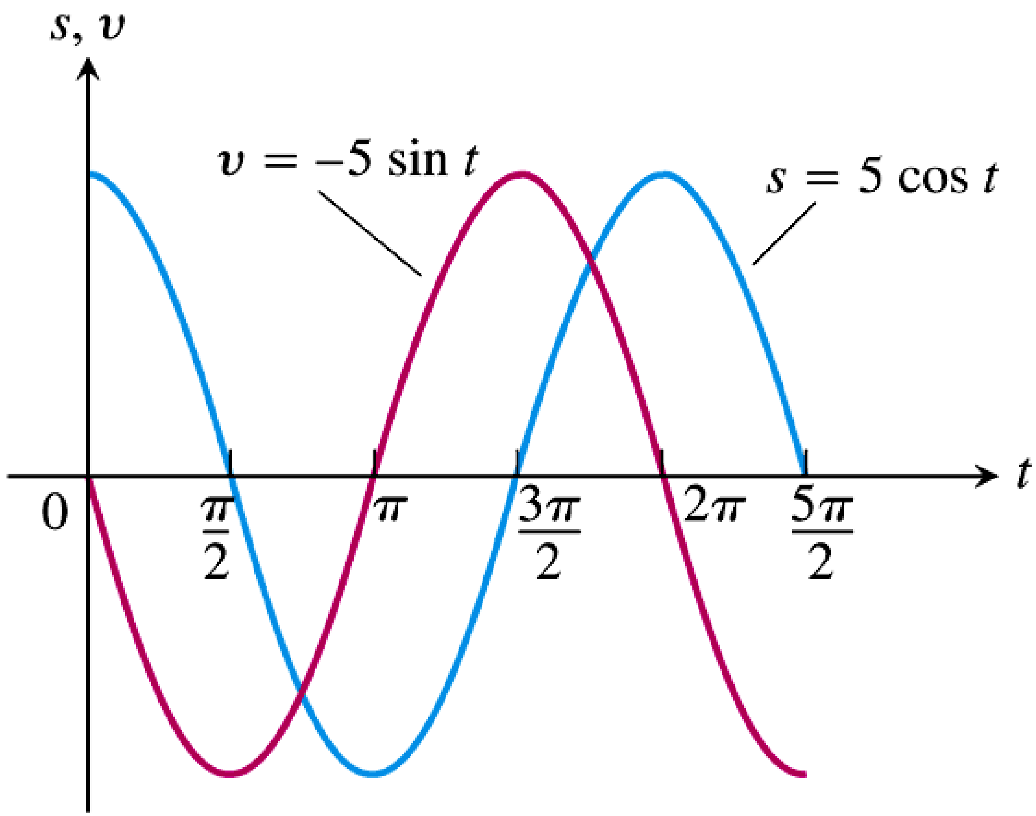


FIGURE 3.25 The graphs of the position and velocity of the body in Example 3.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

3.5

The Chain Rule and Parametric Equations

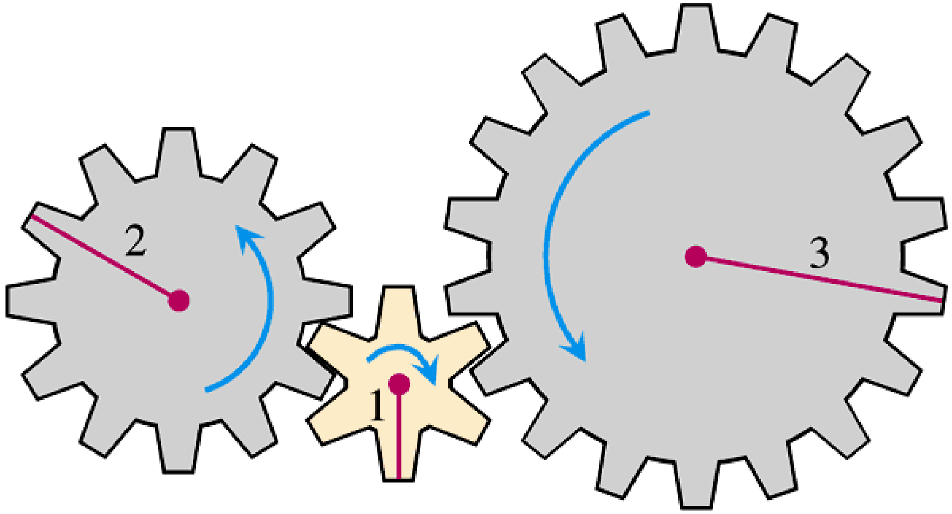


FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

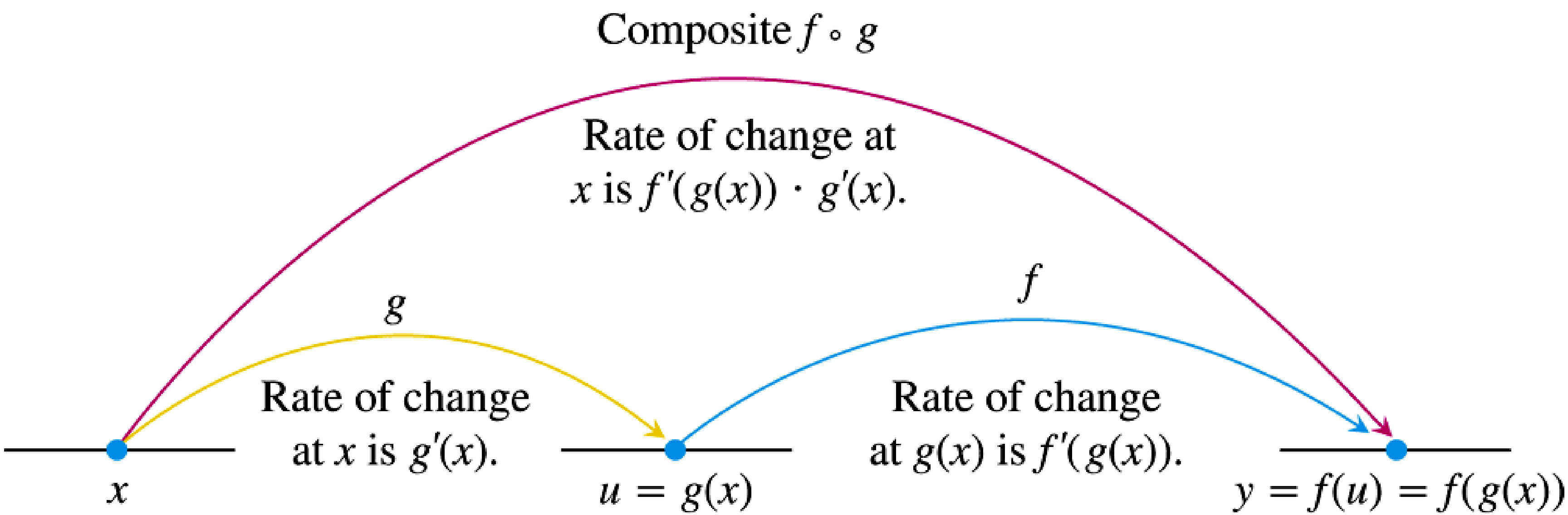


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

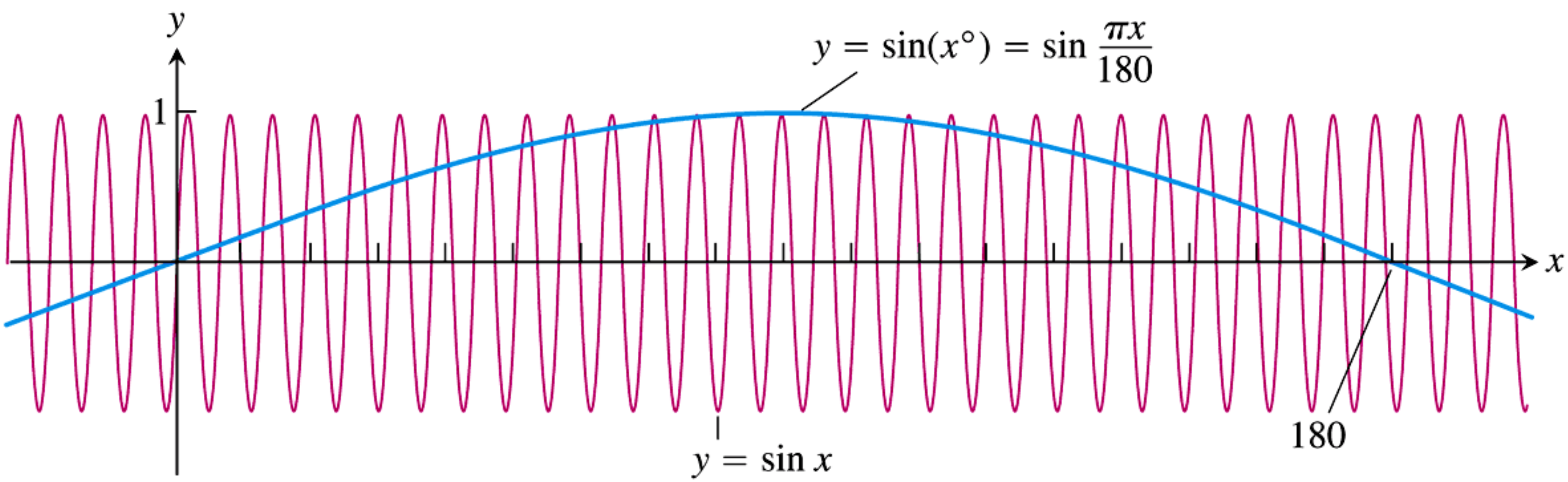


FIGURE 3.28 Sin $(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 8).

DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

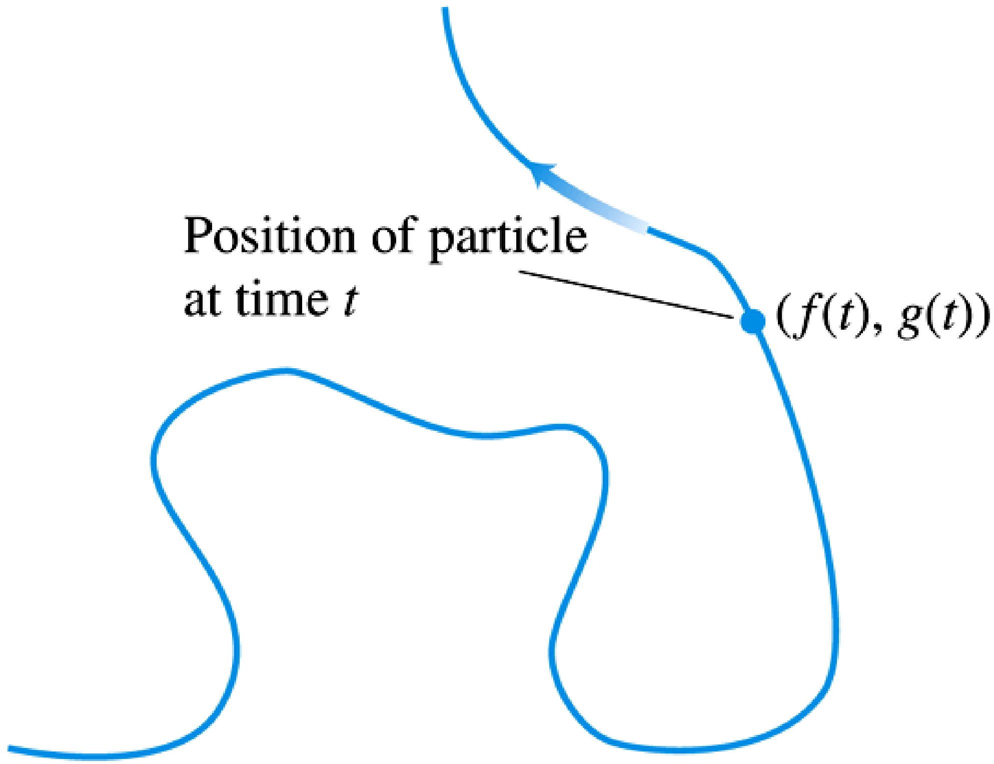


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

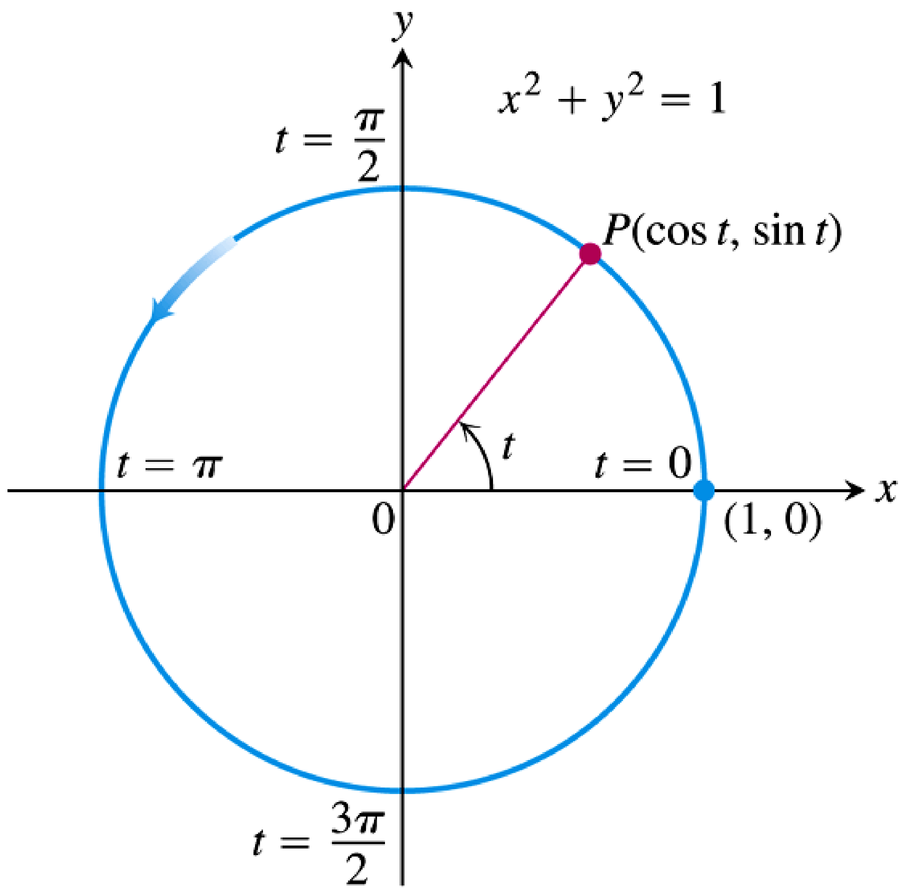


FIGURE 3.30 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 9).

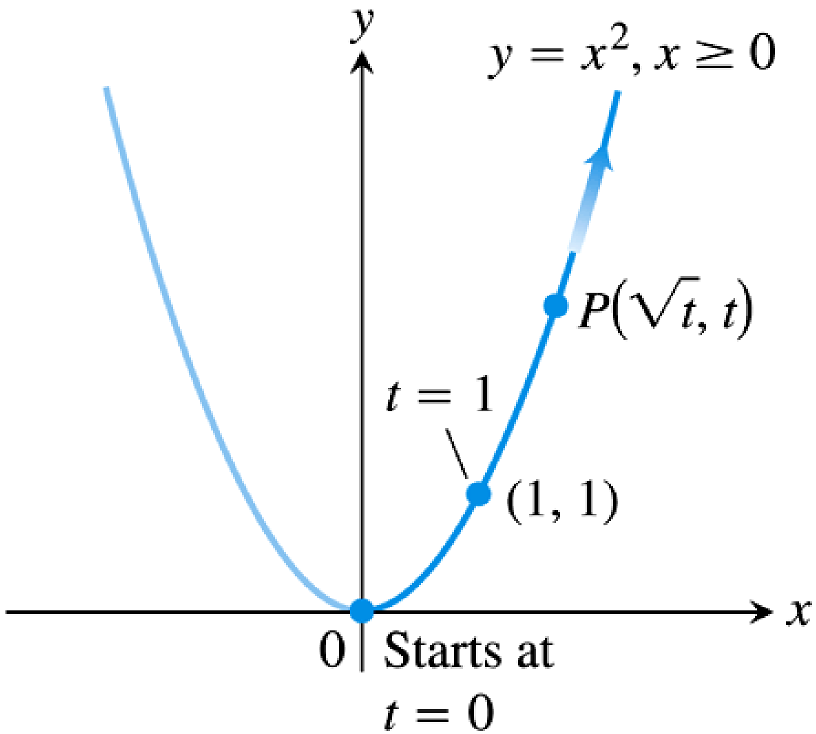


FIGURE 3.31 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{2}$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{3}$$

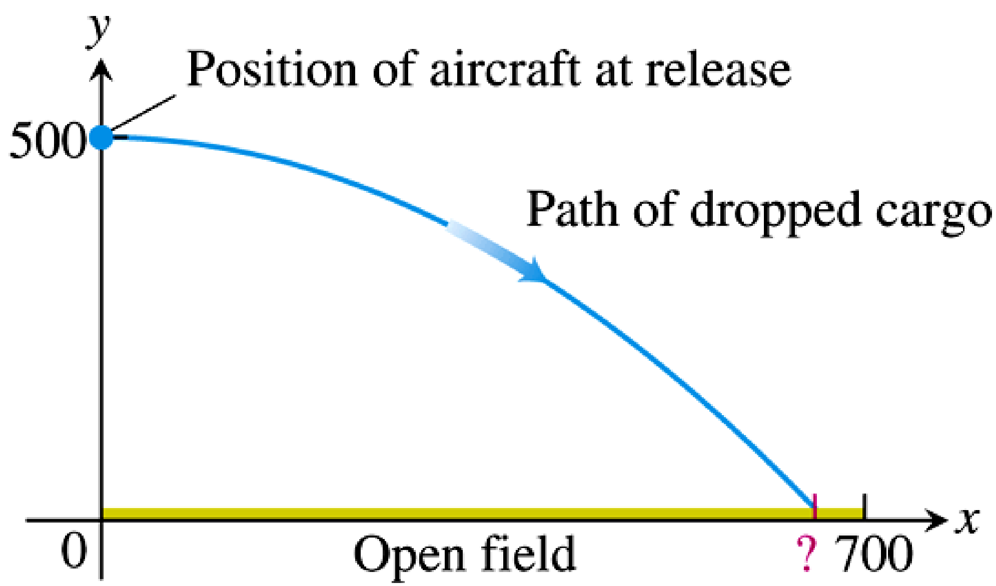


FIGURE 3.32 The path of the dropped cargo of supplies in Example 15.

Standard Parametrizations and Derivative Rules

CIRCLE $x^2 + y^2 = a^2$:

$$x = a\cos t$$
$$y = a\sin t$$
$$0 \le t \le 2\pi$$

ELLIPSE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$:

$$x = a\cos t$$
$$y = b\sin t$$
$$0 \le t \le 2\pi$$

FUNCTION $y = f(x)$:

$$x = t$$
$$y = f(t)$$

DERIVATIVES

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

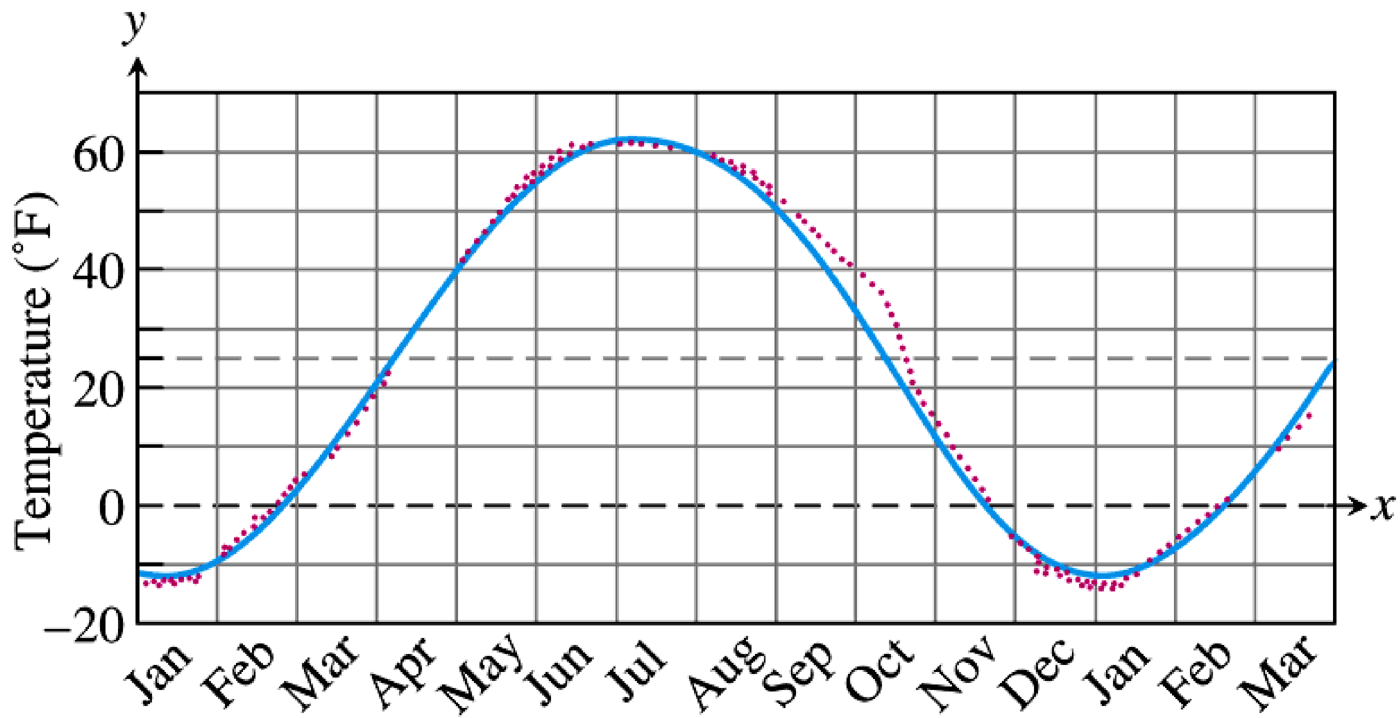


FIGURE 3.33 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 96).

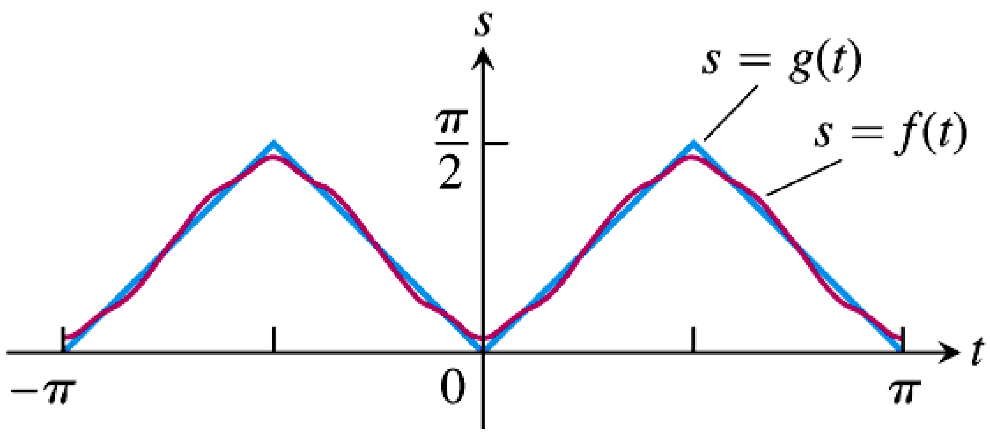


FIGURE 3.34 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 111).

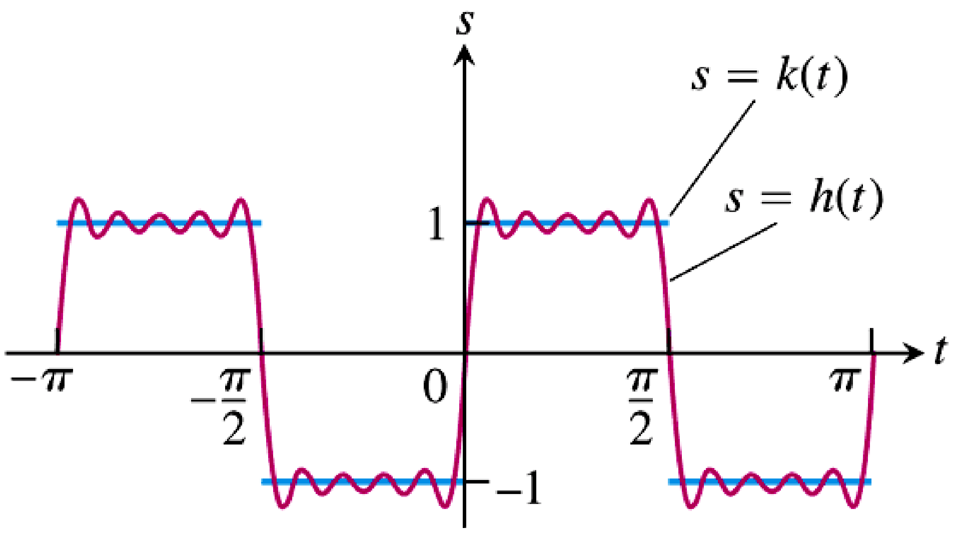


FIGURE 3.35 The approximation of a step function by a trigonometric "polynomial" (Exercise 112).

3.6

Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

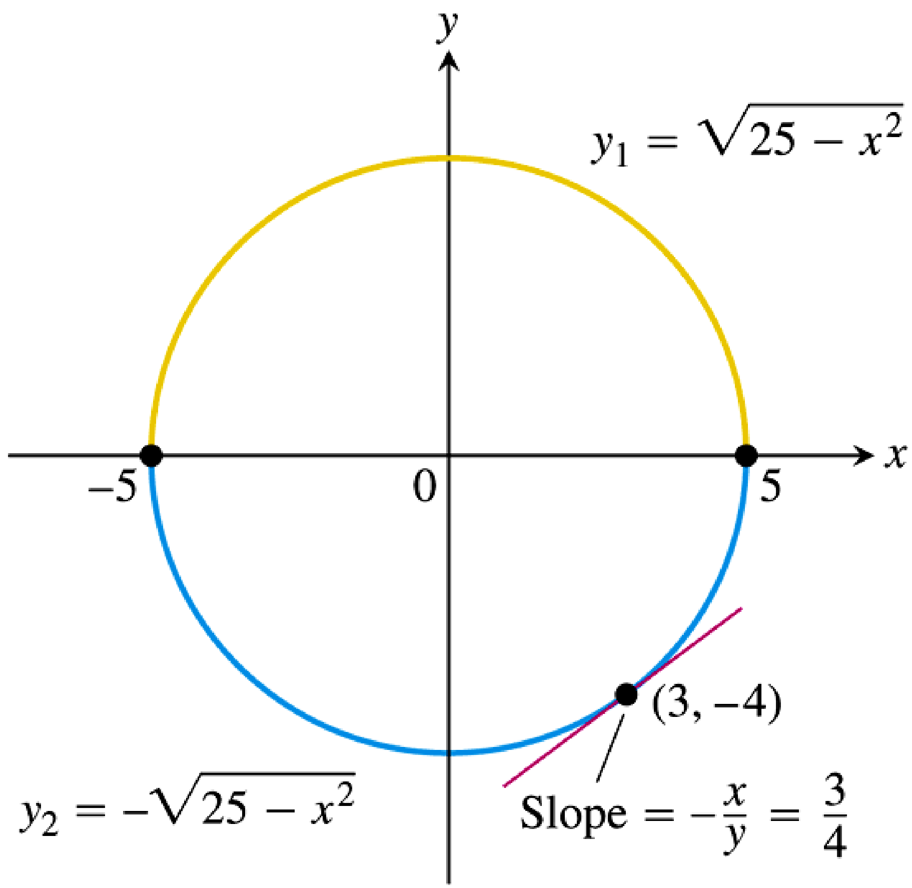


FIGURE 3.36 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

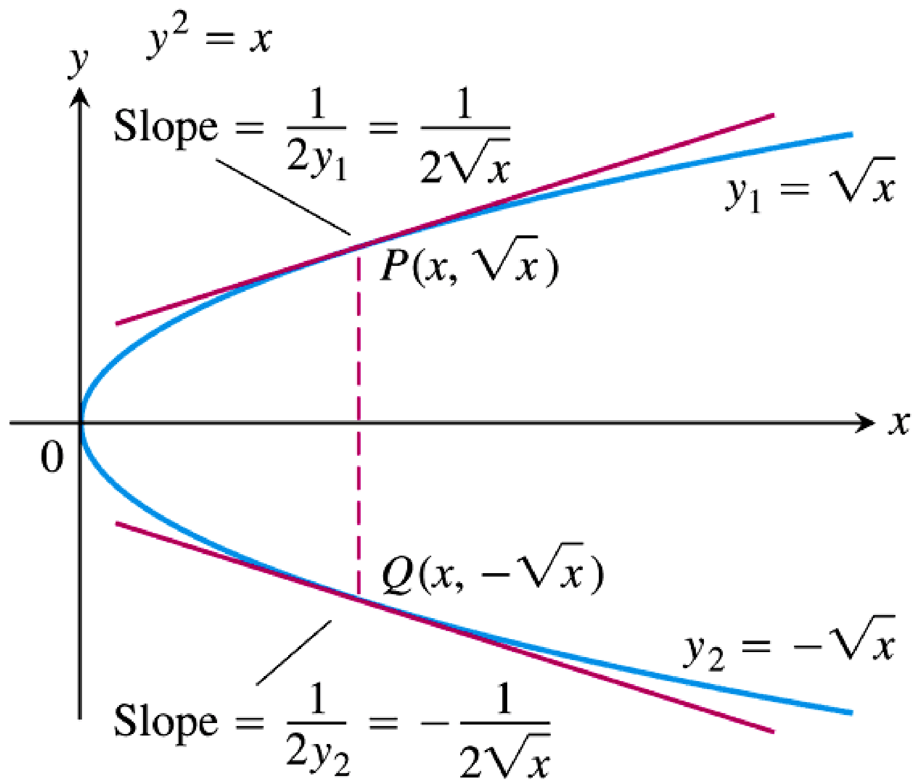


FIGURE 3.37 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

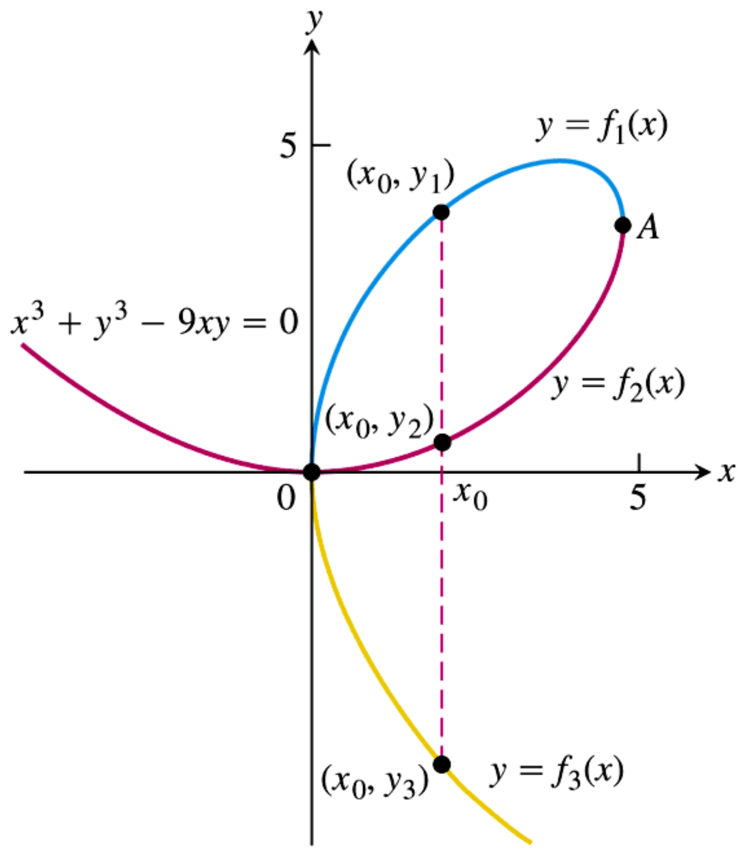


FIGURE 3.38 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

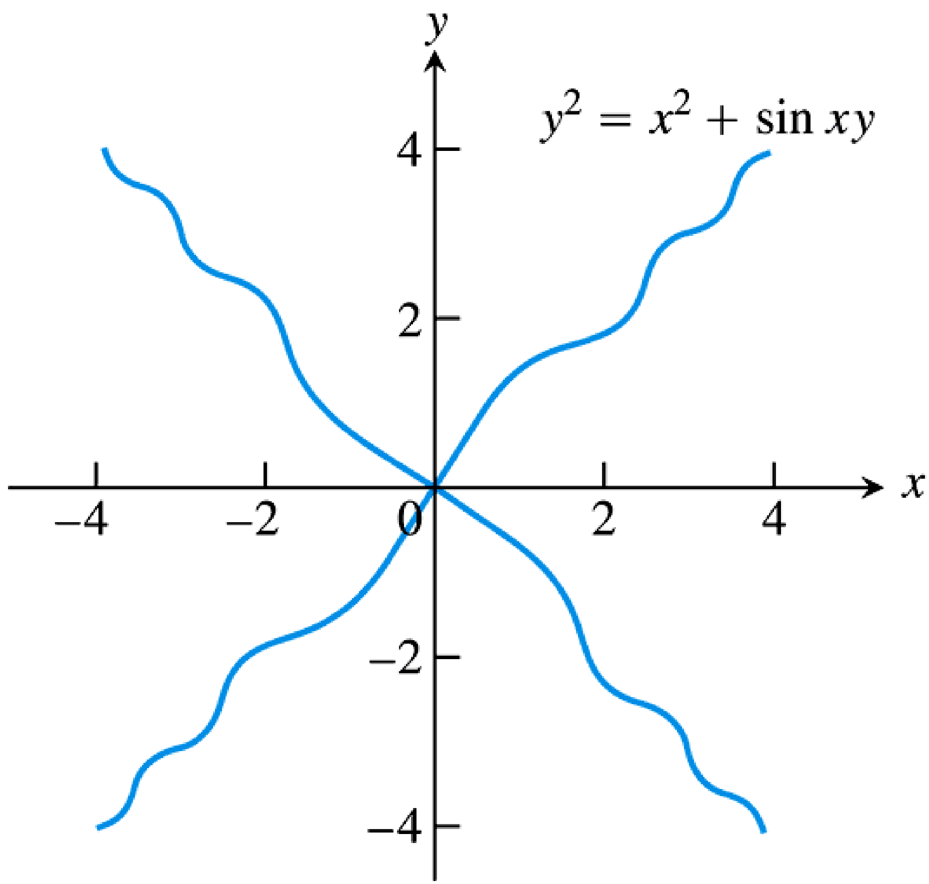


FIGURE 3.39 The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

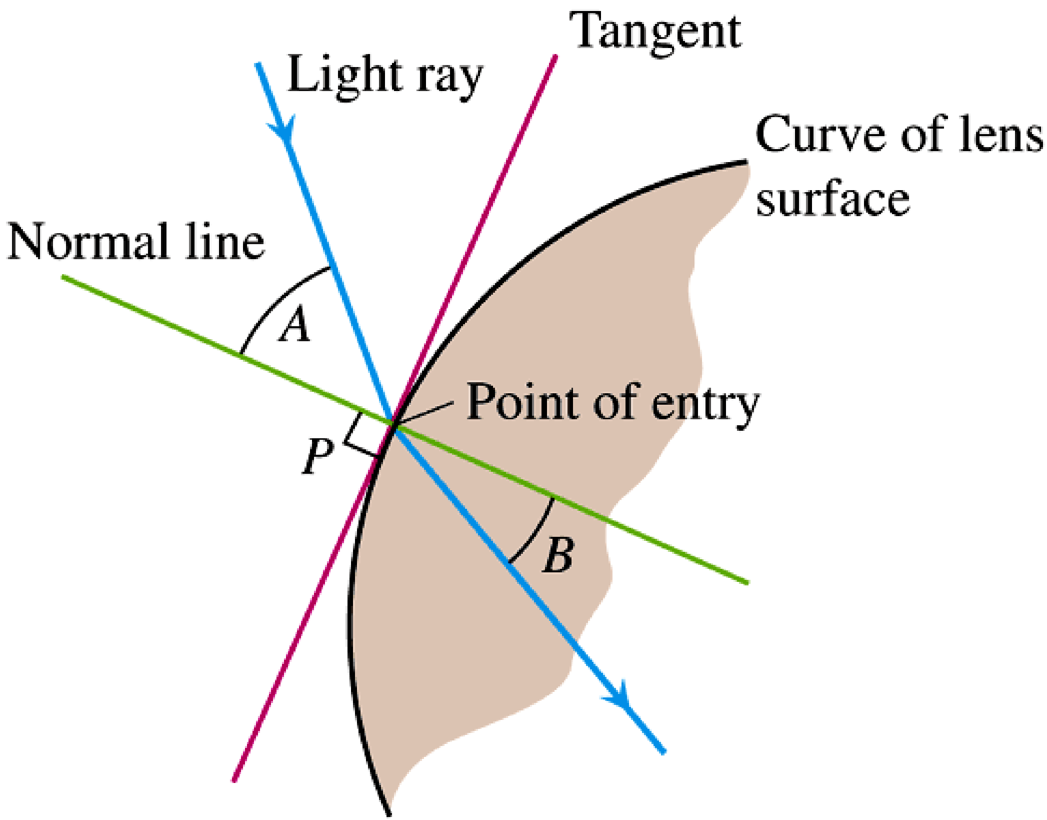


FIGURE 3.40 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

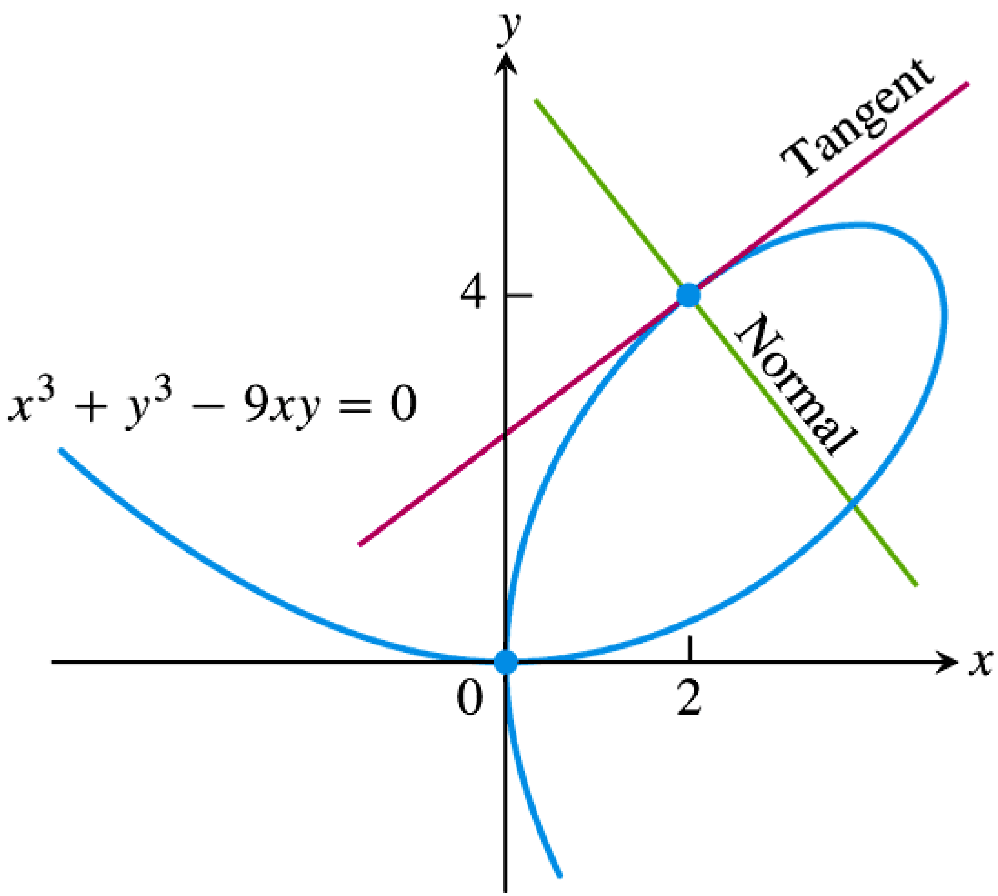


FIGURE 3.41 Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

THEOREM 4 Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{(p/q)-1}.$$

3.7

Related Rates

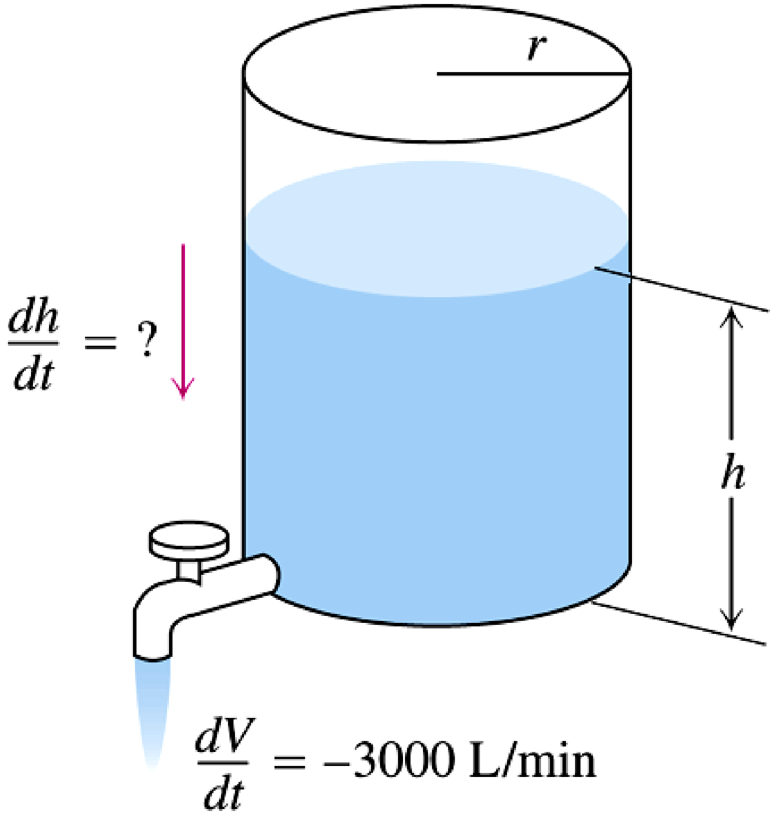


FIGURE 3.42 The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

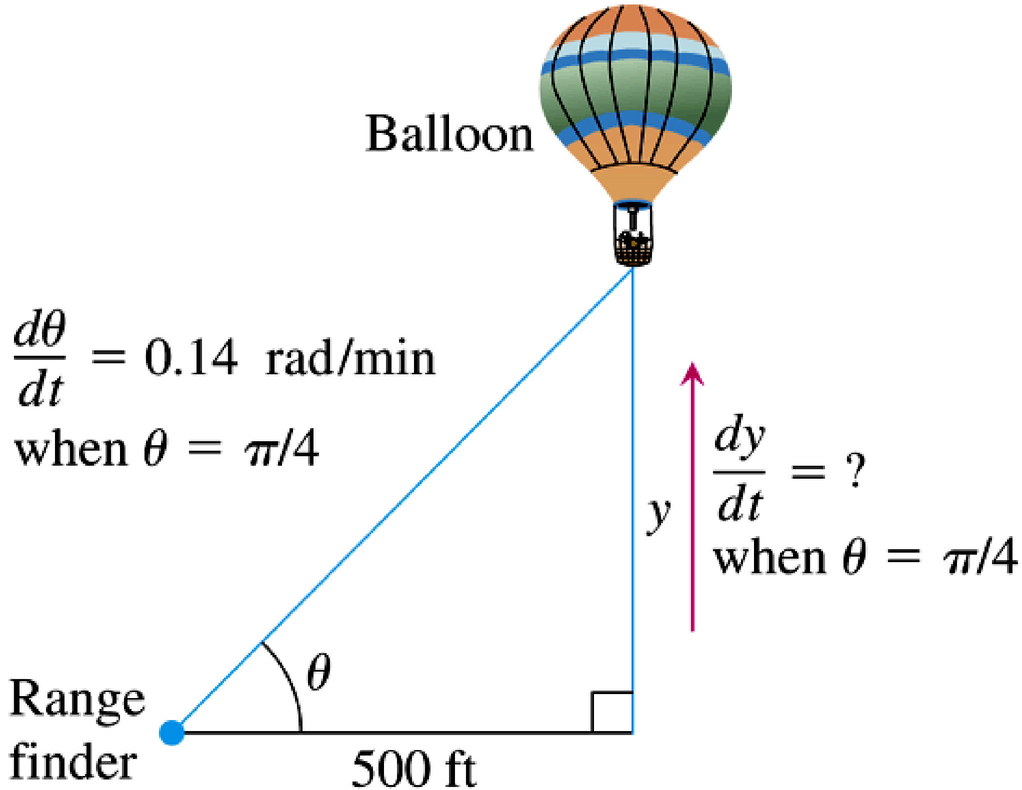


FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

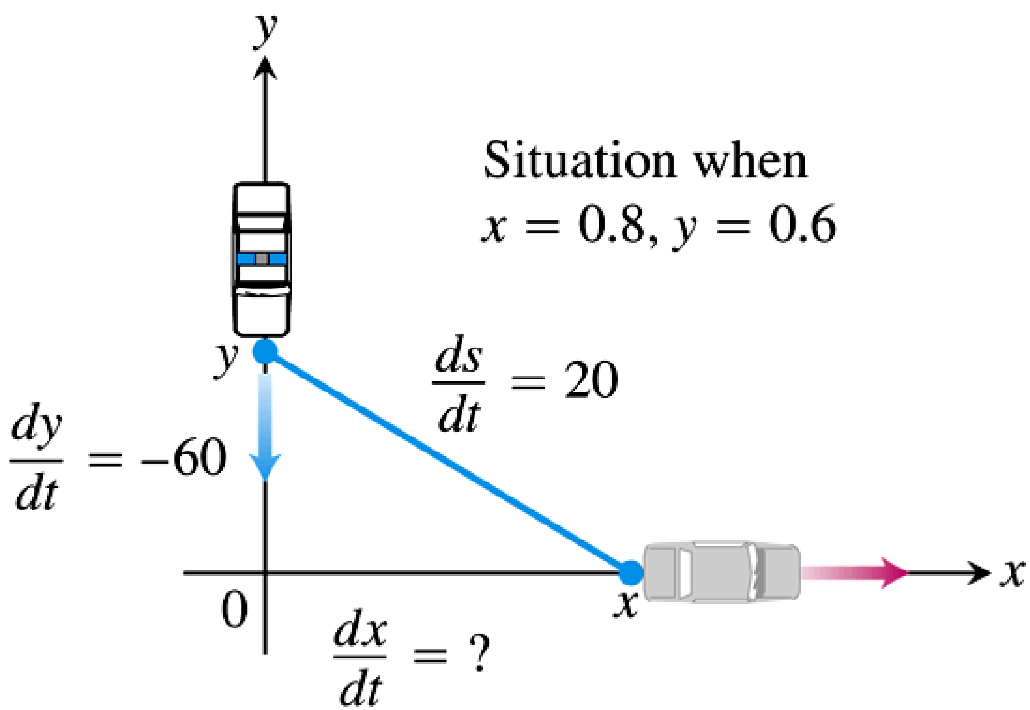


FIGURE 3.44 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

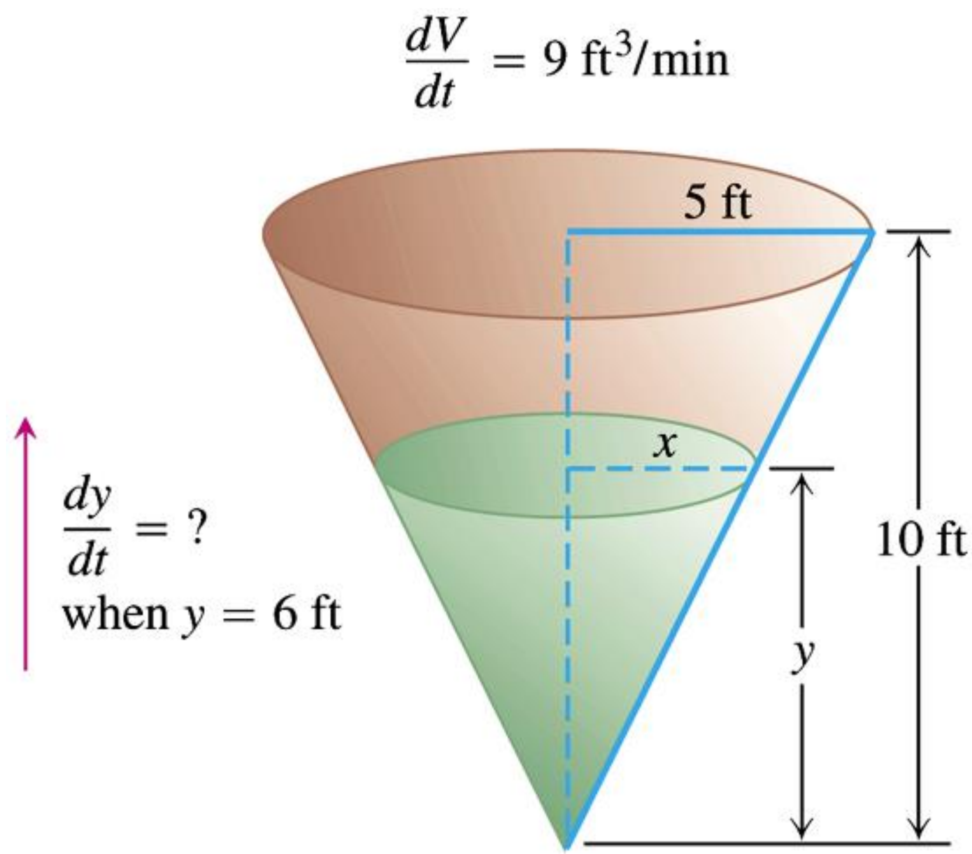
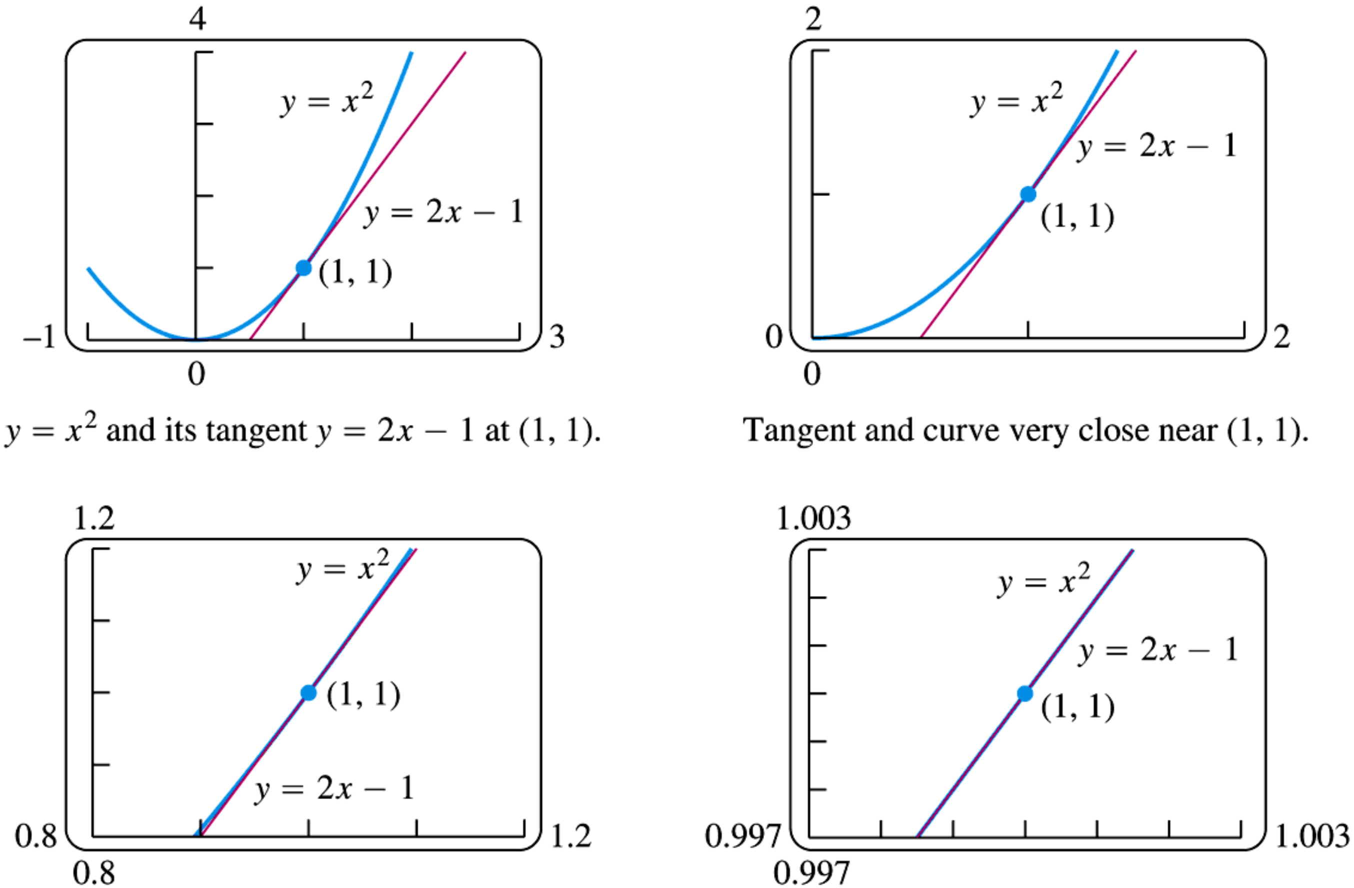


FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 4).

3.8

Linearization and Differentials



$y = x^2$ and its tangent $y = 2x - 1$ at $(1, 1)$.

Tangent and curve very close near $(1, 1)$.

Tangent and curve very close throughout entire x-interval shown.

Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

FIGURE 3.46 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

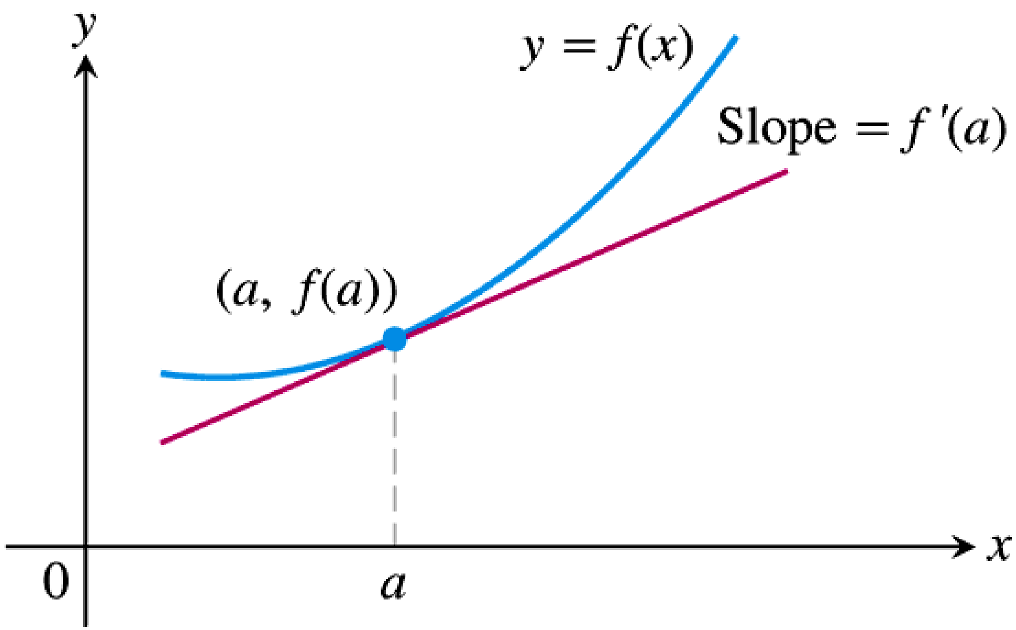


FIGURE 3.47 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

DEFINITIONS Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

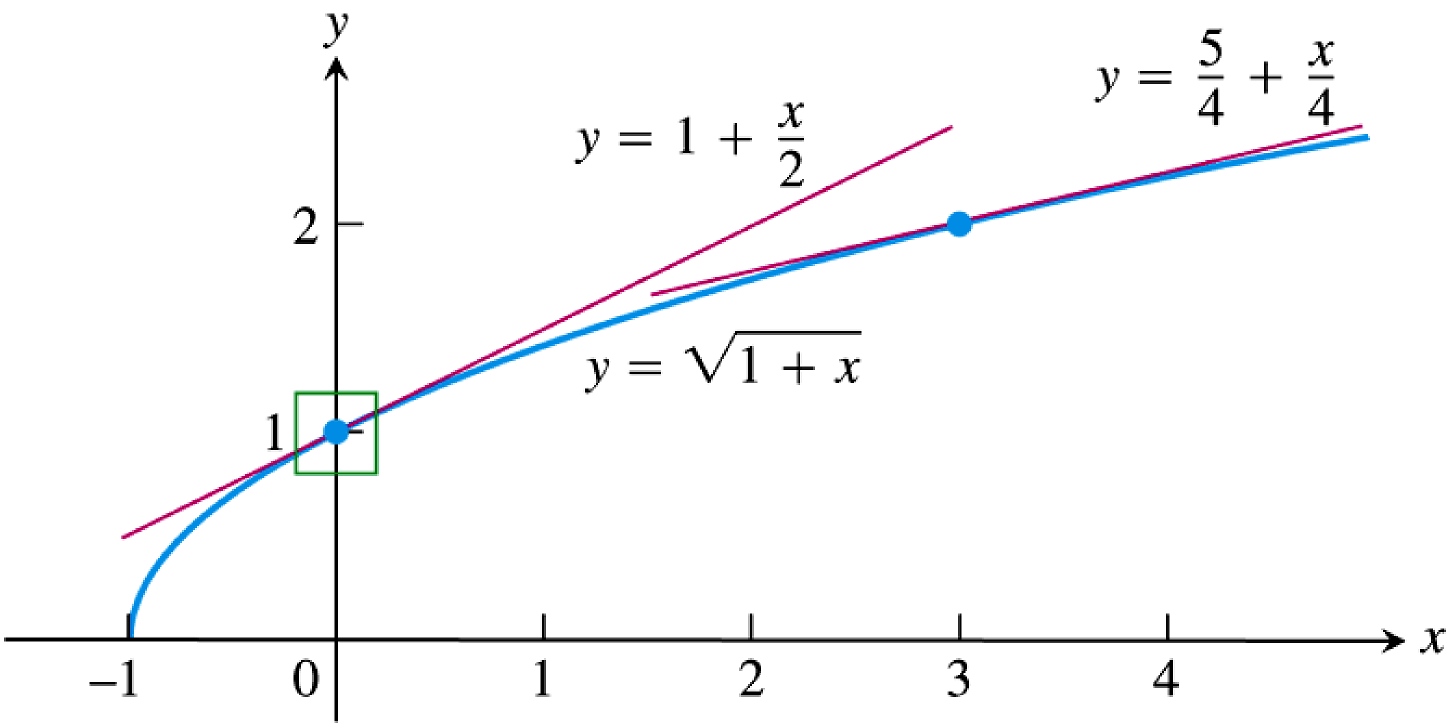


FIGURE 3.48 The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.49 shows a magnified view of the small window about 1 on the y-axis.

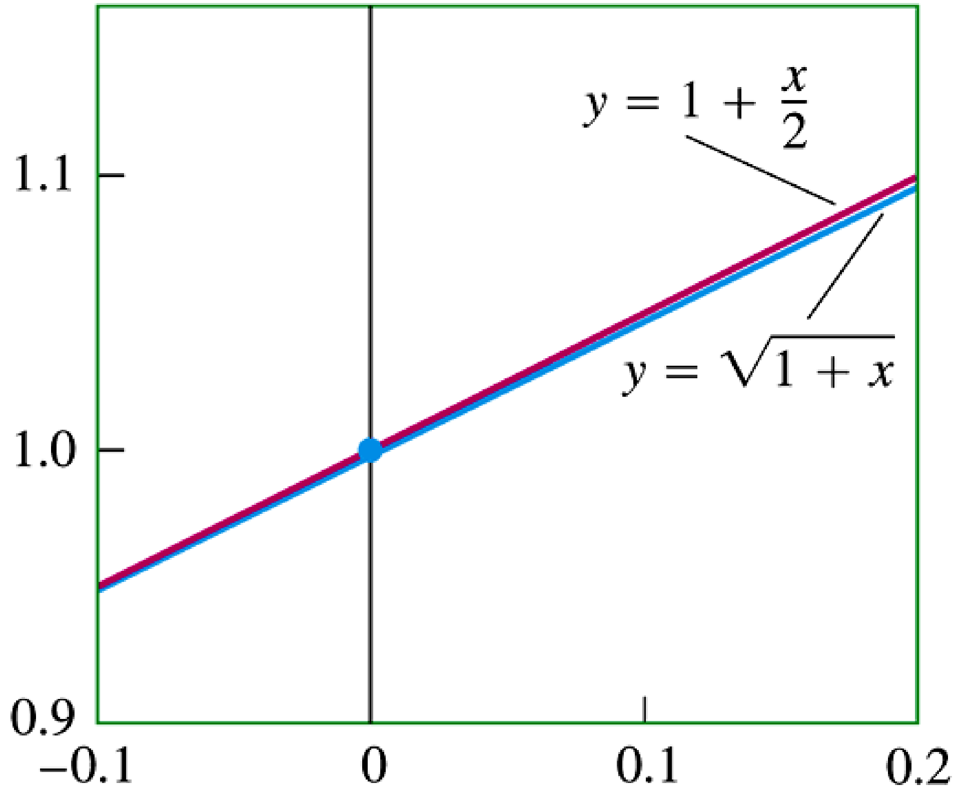


FIGURE 3.49 Magnified view of the window in Figure 3.48.

Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

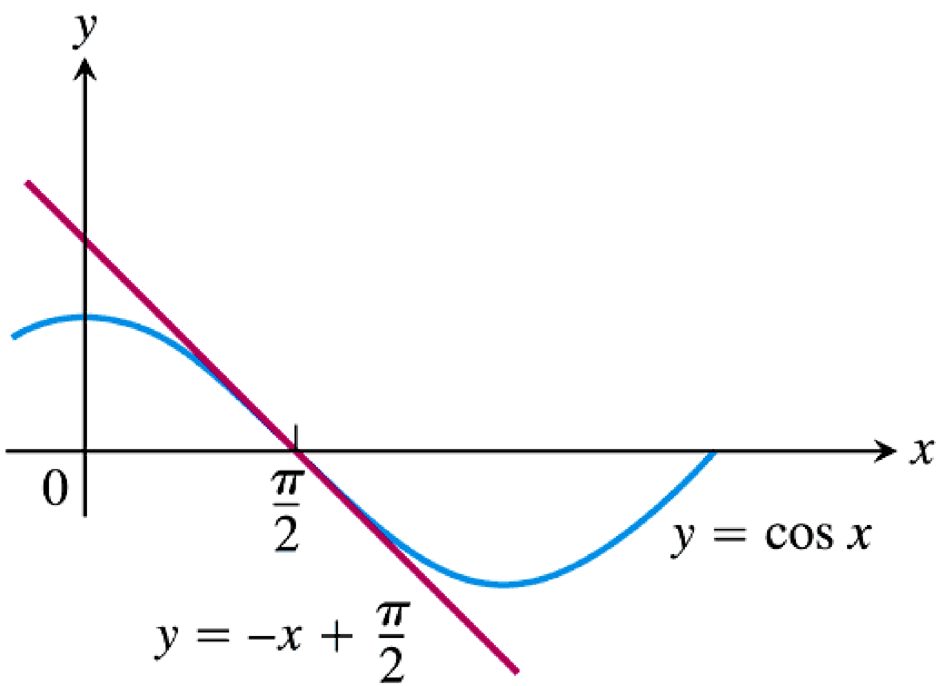


FIGURE 3.50 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

DEFINITION Differential

Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$dy = f'(x)\,dx.$$

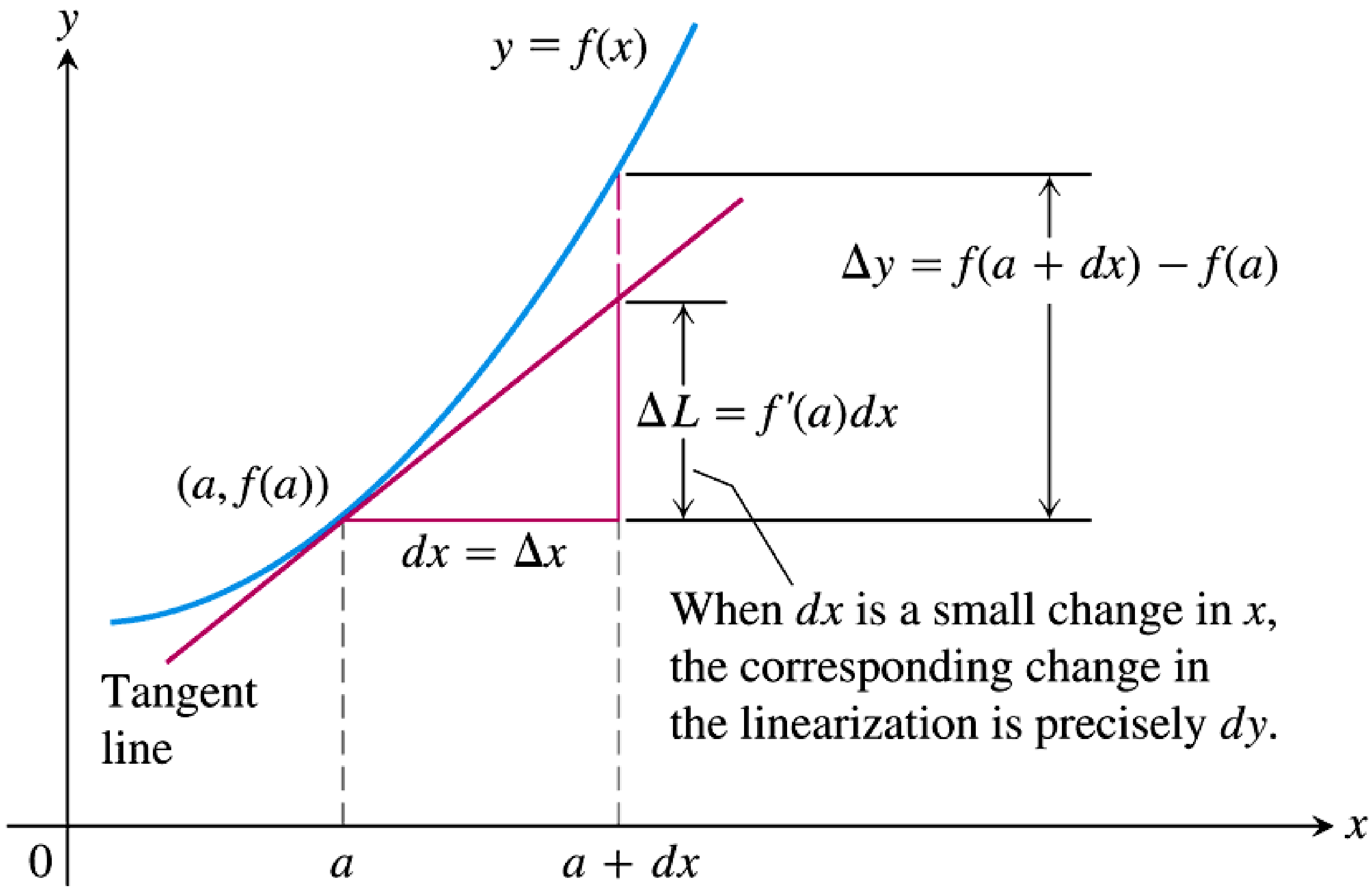


FIGURE 3.51 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

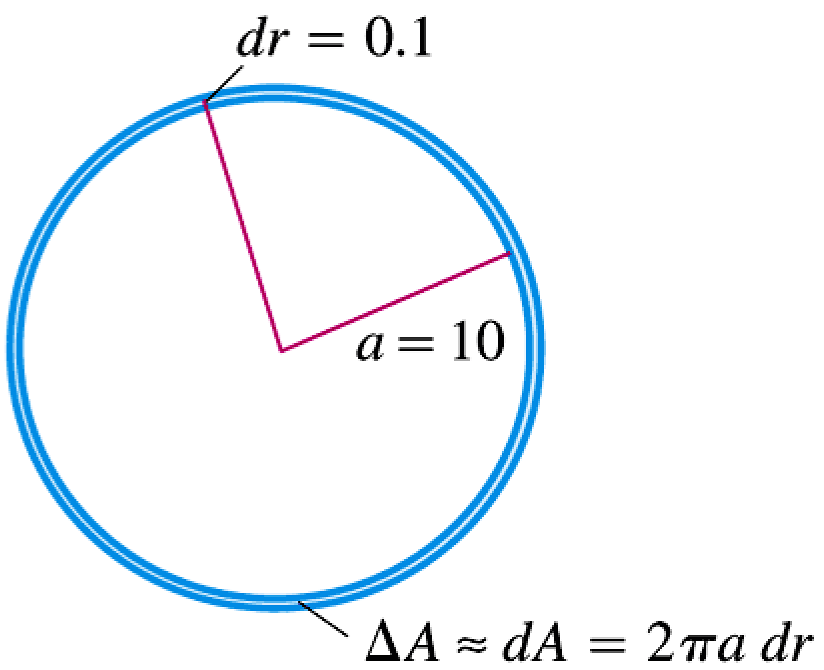


FIGURE 3.52 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a way to estimate the area of the circle with radius $r = a + dr$ (Example 6).

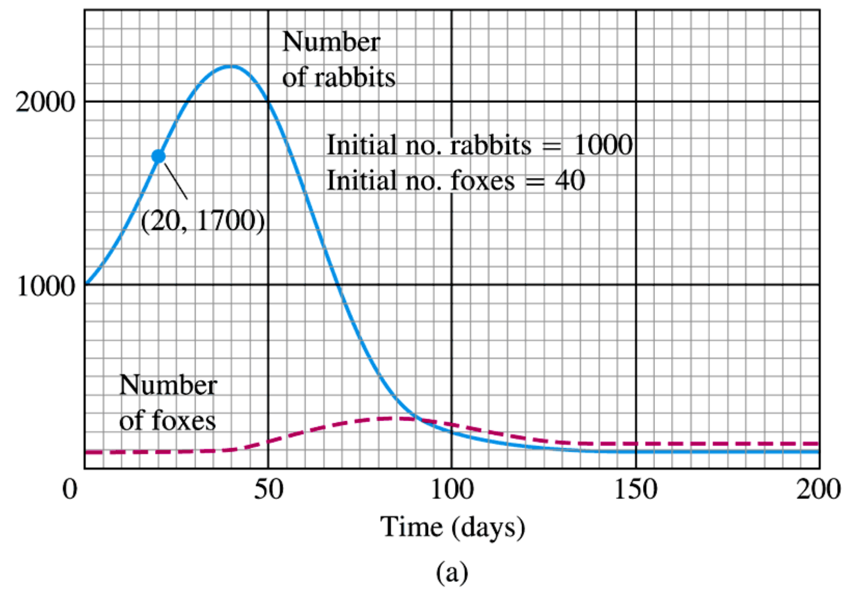
Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by an equation of the form

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$



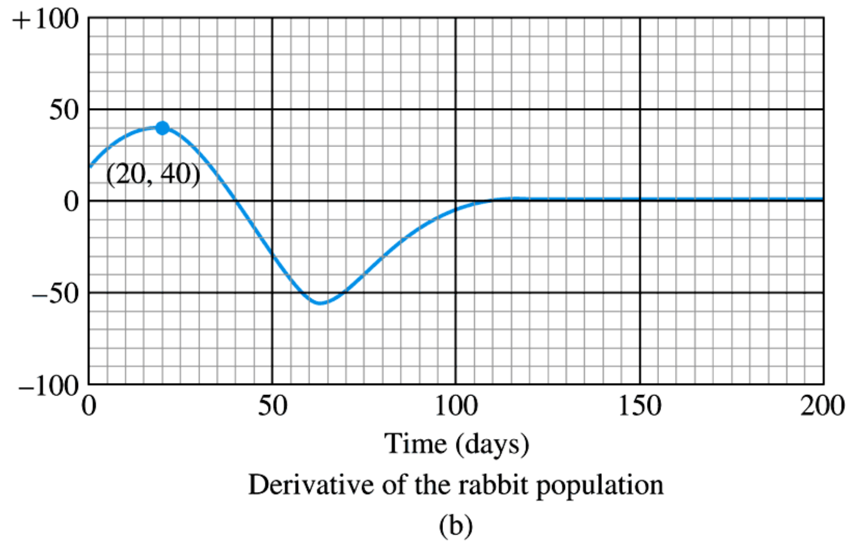


FIGURE 3.53 Rabbits and foxes in an arctic predator-prey food chain.

Workshop Solutions to Chapter 4

1) If $f(x)$ is a differentiable function, then $f'(x) =$ Solution: $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$	2) If $f(x) = 4x^2$, then $f'(x) =$ Solution: $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{4(x+h)^2-4x^2}{h}$$
3) If $f(x) = x^2 - 3$, then $f'(x) =$ Solution: $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ $$= \lim_{h\to 0}\frac{[(x+h)^2-3]-[x^2-3]}{h}$$	4) If $f(x) = \sqrt{x}$, $x \geq 0$, then $f'(x) =$ Solution: $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$
5) If f is a differentiable function at a, then f is a continuous function at a.	6) If f is a continuous function at a, then f is a differentiable function at a. Solution: False
7) If $y = x^4 + 5x^2 + 3$, then $y' =$ Solution: $$y' = 4x^3 + 10x$$	8) If $y = x^4 - 5x^2 + 3$, then $y' =$ Solution: $$y' = 4x^3 - 10x$$
9) If $y = x^{-5/2}$, then $y' =$ Solution: $$y' = -\frac{5}{2}x^{-\frac{5}{2}-1} = -\frac{5}{2}x^{-7/2}$$	10) If $y = \frac{1}{3x^3} + 2\sqrt{x} = \frac{1}{3}x^{-3} + 2x^{1/2}$, then $y' =$ Solution: $$y' = (-3)\left(\frac{1}{3}\right)x^{-3-1} + \left(\frac{1}{2}\right)(2)x^{\frac{1}{2}-1}$$ $$= -x^{-4} + x^{-1/2} = -\frac{1}{x^4} + \frac{1}{x^{1/2}} = -\frac{1}{x^4} + \frac{1}{\sqrt{x}}$$
11) If $y = (x-3)(x-2)$, then $y' =$ Solution: $$y = (x-3)(x-2) = x^2 - 5x + 6$$ $$y' = 2x - 5$$	12) If $y = (x^3+3)(x^2-1)$, then $y' =$ Solution: $$y = (x^3+3)(x^2-1) = x^5 - x^3 + 3x^2 - 3$$ $$y' = 5x^4 - 3x^2 + 6x$$
13) If $y = \sqrt{x}(2x+1)$, then $y' =$ Solution: $$y = \sqrt{x}(2x+1) = 2x\sqrt{x} + \sqrt{x} = 2x^{\frac{3}{2}} + x^{\frac{1}{2}}$$ $$y' = \left(\frac{3}{2}\right)(2)x^{\frac{3}{2}-1} + \left(\frac{1}{2}\right)x^{\frac{1}{2}-1} = 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$ $$= 3\sqrt{x} + \frac{1}{2\sqrt{x}}$$ **OR** Use the rule $(f.g)' = f'g + fg'$ $$y' = (2)(\sqrt{x}) + \left(\frac{1}{2\sqrt{x}}\right)(2x+1) = 2\sqrt{x} + \frac{2x+1}{2\sqrt{x}}$$	14) If $y = \frac{x+3}{x-2}$, then $y' =$ Solution: Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$ $$y' = \frac{(1)(x-2)-(x+3)(1)}{(x-2)^2} = \frac{x-2-x-3}{(x-2)^2} = \frac{-5}{(x-2)^2}$$ $$= -\frac{5}{(x-2)^2}$$
15) If $y = \frac{x+3}{x-2}$, then $y'\|_{x=4} =$ Solution: $$y' = \frac{(1)(x-2)-(x+3)(1)}{(x-2)^2} = \frac{x-2-x-3}{(x-2)^2}$$ $$= \frac{-5}{(x-2)^2} = -\frac{5}{(x-2)^2}$$ $$y'\|_{x=4} = -\frac{5}{(4-2)^2} = -\frac{5}{4}$$	16) If $y = \frac{x-1}{x+2}$, then $y' =$ Solution: Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$ $$y' = \frac{(1)(x+2)-(x-1)(1)}{(x+2)^2} = \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

17) If $y = \sqrt{3x^2 + 6x}$, then $y' =$

Solution:

Use the rule $\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y' = \frac{6x + 6}{2\sqrt{3x^2 + 6x}} = \frac{6(x+1)}{2\sqrt{3x^2 + 6x}} = \frac{3(x+1)}{\sqrt{3x^2 + 6x}}$$

18) If $y = \sqrt{3x^2 + 6x}$, then $y'|_{x=1} =$

Solution:

$$y' = \frac{6x + 6}{2\sqrt{3x^2 + 6x}} = \frac{6(x+1)}{2\sqrt{3x^2 + 6x}} = \frac{3(x+1)}{\sqrt{3x^2 + 6x}}$$

$$y'|_{x=1} = \frac{3((1)+1)}{\sqrt{3(1)^2 + 6(1)}} = \frac{6}{\sqrt{9}} = \frac{6}{3} = 2$$

19) The tangent line equation to the curve $y = x^2 + 2$ at the point $(1,3)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = 2x$$

Thus, the slope at $x = 1$ is

$$y'|_{x=1} = 2(1) = 2$$

Hence, the tangent line equation passing through the point $(1,3)$ with slope $m = 2$ is

$$y - 3 = 2(x - 1)$$
$$y - 3 = 2x - 2$$
$$y = 2x - 2 + 3$$
$$y = 2x + 1$$

20) The tangent line equation to the curve $y = \frac{2x}{x+1}$ at the point $(0,0)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = \frac{(2)(x+1) - (2x)(1)}{(x+1)^2} = \frac{2x + 2 - 2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Thus, the slope at $x = 0$ is

$$y'|_{x=0} = \frac{2}{(0+1)^2} = 2$$

Hence, the tangent line equation passing through the point $(0,0)$ with slope $m = 2$ is

$$y - 0 = (2)(x - 0)$$
$$y = 2x$$

21) The tangent line equation to the curve $y = 3x^2 - 13$ at the point $(2,-1)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = 6x$$

Thus, the slope at $x = 2$ is

$$y'|_{x=2} = 6(2) = 12$$

Hence, the tangent line equation passing through the point $(2,-1)$ with slope $m = 12$ is

$$y - (-1) = 12(x - 2)$$
$$y + 1 = 12x - 24$$
$$y = 12x - 24 - 1$$
$$y = 12x - 25$$

22) The tangent line equation to the curve $y = 3x^2 + 2x + 5$ at the point $(0,5)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = 6x + 2$$

Thus, the slope at $x = 2$ is

$$y'|_{x=0} = 6(0) + 2 = 2$$

Hence, the tangent line equation passing through the point $(0,5)$ with slope $m = 2$ is

$$y - 5 = 2(x - 0)$$
$$y - 5 = 2x$$
$$y = 2x + 5$$

23) If $y = xe^x$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$ and $(e^u) = e^u.u'$

$$y' = (1)(e^x) + (x)(e^x) = e^x + xe^x = e^x(1 + x)$$

24) If $y = x - e^x$, then $y'' =$

Solution:

Use the rules $(f - g)' = f' - g'$ and $(e^u) = e^u.u'$

$$y' = 1 - e^x$$
$$y'' = -e^x$$

25) If $x^2 - y^2 = 4$, then $y' =$

Solution:

$$2x - 2yy' = 0$$
$$-2yy' = -2x$$
$$y' = \frac{-2x}{-2y}$$
$$y' = \frac{x}{y}$$

26) If $x^2 + y^2 = 4$, then $y' =$

Solution:

$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$y' = \frac{-2x}{2y}$$
$$y' = -\frac{x}{y}$$

27) If $y = \frac{x+1}{x+2}$, then $y' =$

Solution:

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x+2) - (x+1)(1)}{(x+2)^2} = \frac{x + 2 - x - 1}{(x+2)^2}$$
$$= \frac{1}{(x+2)^2}$$

28) If $y = \frac{1}{\sqrt[2]{x^5}} + \sec x$, then $y' =$

Solution:

Use the rules

$(f + g)' = f' + g'$ and $(\sec u)' = \sec u \tan u \, . u'$

$$y = \frac{1}{\sqrt[2]{x^5}} + \sec x = x^{-\frac{5}{2}} + \sec x$$

$$y' = \left(-\frac{5}{2}\right)x^{-\frac{5}{2}-1} + \sec x \tan x = -\frac{5}{2}x^{-7/2} + \sec x \tan x$$

29) If $y = \tan^{-1}(x^3)$, then $y' =$

Solution:

Use the rule $(\tan^{-1} u)' = \frac{u'}{1+u^2}$

$$y' = \frac{1}{1+(x^3)^2} \, .(3x^2) = \frac{3x^2}{1+x^6}$$

30) If $y = \tan x - x$, then $y' =$

Solution:

Use the rules

$$(f-g)' = f' - g' \quad \text{and} \quad (\tan u)' = \sec^2 u \ .u'$$

$$y' = \sec^2 x - 1$$

31) If $y = \sec^2 x - 1$, then $y' =$

Solution:

Use the rules $(f-g)' = f' - g'$, $(u)^n = n(u)^{n-1}.u'$ and $(\sec u)' = \sec u \tan u \, .u'$

$$y' = 2\sec x \, . \, \sec x \tan x = 2\sec^2 x \tan x$$

32) If $y = x^{\sin x}$, then $y' =$

Solution:

Use the rule $(\sin u)' = \cos u \ .u'$

$$y = x^{\sin x}$$
$$\ln y = \ln x^{\sin x}$$
$$\ln y = \sin x \, . \ln x$$
$$\frac{y'}{y} = \cos x \, . \, \ln x + \sin x \, . \frac{1}{x} = \cos x \, . \, \ln x + \frac{\sin x}{x}$$
$$y' = y\left(\cos x \, . \, \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \, . \, \ln x + \frac{\sin x}{x}\right)$$

33) If $y = x^{\cos x}$, then $y' =$

Solution:

Use the rule $(\cos u)' = -\sin u \ .u'$

$$y = x^{\cos x}$$
$$\ln y = \ln x^{\cos x}$$
$$\ln y = \cos x \, . \ln x$$
$$\frac{y'}{y} = -\sin x \, . \, \ln x + \cos x \, . \frac{1}{x} = -\sin x \, . \, \ln x + \frac{\cos x}{x}$$
$$y' = y\left(-\sin x \, . \, \ln x + \frac{\cos x}{x}\right)$$
$$= x^{\cos x}\left(\frac{\cos x}{x} - \sin x \, . \, \ln x\right)$$

34) If $y = (2x^2 + \csc x)^9$, then $y' =$

Solution:

Use the rules

$$(u)^n = n(u)^{n-1}.u' \quad \text{and} \quad (\csc u)' = -\csc u \cot u \, .u'$$

$$y' = 9(2x^2 + \csc x)^8.(4x - \csc x \cot x)$$

35) If $y = \frac{5^x}{\cot x}$, then $y' =$

Solution:

Use the rules

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad (a^u)' = a^u.\ln a \, .u'$$

and $(\csc u)' = -\csc u \cot u \, .u'$

$$y' = \frac{(5^x \ln 5)(\cot x) - (5^x)(-\csc^2 x)}{(\cot x)^2}$$
$$= \frac{5^x(\ln 5 \cot x + \csc^2 x)}{\cot^2 x}$$

36) If $y = e^{2x}$, then $y^{(6)} =$

Solution:

Use the rule $(e^u)' = e^u.u'$

$$y' = 2e^{2x}$$
$$y'' = 4e^{2x}$$
$$y''' = 8e^{2x}$$
$$y^{(4)} = 16e^{2x}$$
$$y^{(5)} = 32e^{2x}$$
$$y^{(6)} = 64e^{2x}$$

37) If $y = x^{-2}e^{\sin x}$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$, $(e^u) = e^u.u'$ and $(\sin u)' = \cos u \ .u'$

$$y' = (-2x^{-3})\left(e^{\sin x}\right) + (x^{-2})\left(e^{\sin x}.\cos x\right)$$
$$= -2x^{-3}e^{\sin x} + x^{-2}\cos x \, e^{\sin x}$$
$$= x^{-3}e^{\sin x}(-2 + x\cos x)$$
$$= x^{-3}e^{\sin x}(x\cos x - 2)$$

38) If $y = 5^{\tan x}$, then $y' =$

Solution:

Use the rules

$$(a^u)' = a^u.\ln a \, .u' \quad \text{and} \quad (\tan u)' = \sec^2 u \, .u'$$

$$y' = 5^{\tan x}. \, \ln 5 \, . \sec^2 x$$

39) If $x^2 + y^2 = 3xy + 7$, then $y' =$

Solution:

$$2x + 2yy' = 3y + 3xy'$$
$$2yy' - 3xy' = 3y - 2x$$
$$y'(2y - 3x) = 3y - 2x$$
$$y' = \frac{3y - 2x}{2y - 3x}$$

40) If $y = \sin^3(4x)$, then $y^{(6)} =$

Solution:

Use the rules

$$(u)^n = n(u)^{n-1}.u' \quad \text{and} \quad (\sin u)' = \cos u \, .u'$$

$$y' = 3\sin^2(4x).\cos(4x).(4)$$
$$= 12\sin^2(4x).\cos(4x)$$

41) If $y = 3^x \cot x$, then $y' =$
Solution:
Use the rules $(f.g)' = f'g + fg'$, $(a^u)' = a^u . \ln a . u'$ and $(\cot u)' = -\csc^2 u \; . u'$

$$y' = (3^x . \ln 3)(\cot x) + (3^x)(-\csc^2 x)$$
$$= 3^x \ln 3 \; \cot x - 3^x \csc^2 x$$
$$= 3^x (\ln 3 \; \cot x - \csc^2 x)$$

42) If $y = (2x^2 + \sec x)^7$, then $y' =$
Solution:
Use the rules
$(u)^n = n(u)^{n-1} . u'$ and $(\sec u)' = \sec u \tan u . u'$

$$y' = 7(2x^2 + \sec x)^6 . (4x + \sec x \tan x)$$

43) If $f(x) = \cos x$, then $f^{(45)}(x) =$
Solution:
$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
Note: $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4.
Hence,
$$f^{(44)}(x) = \cos x$$
$$f^{(45)}(x) = -\sin x$$

44) If $D^{47}(\sin x) =$
Solution:
$$D(\sin x) = \cos x$$
$$D^2(\sin x) = -\sin x$$
$$D^3(\sin x) = -\cos x$$
$$D^4(\sin x) = \sin x$$
Note: $D^n(\sin x) = \sin x$ whenever n is a multiple of 4.
Hence,
$$D^{44}(\sin x) = \sin x$$
$$D^{45}(\sin x) = \cos x$$
$$D^{46}(\sin x) = -\sin x$$
$$D^{47}(\sin x) = -\cos x$$

45) If $y = x^x$, then $y' =$
Solution:
Use the rule $(\ln u)' = \frac{u'}{u}$

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{y'}{y} = (1)(\ln x) + (x)\left(\frac{1}{x}\right)$$
$$\frac{y'}{y} = \ln x + 1$$
$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

46) If $f(x) = \frac{\ln x}{x^2}$, then $f'(1) =$
Solution:
Use the rules $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ and $(\ln u)' = \frac{u'}{u}$

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x^2) - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x \ln x}{x^4}$$
$$= \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3}$$

$$\therefore \quad f'(1) = \frac{1 - 2\ln(1)}{(1)^3} = \frac{1 - 2(0)}{1} = 1$$

47) If $y = \cot^{-1}(e^x)$, then $y' =$
Solution:
Use the rules $(\cot^{-1} u)' = -\frac{u'}{1+u^2}$ and $(e^u) = e^u . u'$

$$y' = -\frac{1}{1 + (e^x)^2} . e^x = -\frac{e^x}{1 + e^{2x}}$$

48) If $y = \tan^{-1}(e^x)$, then $y' =$
Solution:
Use the rules $(\tan^{-1} u)' = \frac{u'}{1+u^2}$ and $(e^u) = e^u . u'$

$$y' = \frac{1}{1 + (e^x)^2} . e^x = \frac{e^x}{1 + e^{2x}}$$

49) If $y = \sin^{-1}(e^x)$, then $y' =$
Solution:
Use the rules $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$ and $(e^u) = e^u . u'$

$$y' = \frac{1}{\sqrt{1 - (e^x)^2}} . e^x = \frac{e^x}{\sqrt{1 - e^{2x}}}$$

50) If $y = \cos^{-1}(e^x)$, then $y' =$
Solution:
Use the rules $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$ and $(e^u) = e^u . u'$

$$y' = -\frac{1}{\sqrt{1 - (e^x)^2}} . e^x = -\frac{e^x}{\sqrt{1 - e^{2x}}}$$

51) If $y = \cos(2x^3)$, then $y' =$
Solution:
Use the rule $(\cos u)' = -\sin u \; . u'$

$$y' = -\sin(2x^3) . (6x^2) = -6x^2 \sin(2x^3)$$

52) If $y = \csc x \cot x$, then $y' =$
Solution:
Use the rules $(f.g)' = f'g + fg'$,
$(\csc u)' = -\csc u \cot u . u'$ and $(\cot u)' = -\csc^2 u \; . u'$

$$y' = (-\csc x \cot x)(\cot x) + (\csc x)(-\csc^2 x)$$
$$= -\csc x \cot^2 x - \csc^3 x = -\csc x(\cot^2 x + \csc^2 x)$$

53) If $y = \sqrt{x^2 - 2\sec x}$, then $y' =$
Solution:
Use the rules

$$\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}} \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = \frac{2x - 2\sec x \tan x}{2\sqrt{x^2 - 2\sec x}} = \frac{2(x - \sec x \tan x)}{2\sqrt{x^2 - 2\sec x}}$$
$$= \frac{x - \sec x \tan x}{\sqrt{x^2 - 2\sec x}}$$

54) If $y = (3x^2 + 1)^6$, then $y' =$
Solution:
Use the rule $(u)^n = n(u)^{n-1}.u'$

$$y' = 6(3x^2 + 1)^5.(6x) = 36x(3x^2 + 1)^5$$

55) If $xy + \tan x = 2x^3 + \sin y$, then $y' =$
Solution:

$$[(1)(y) + (x)(y')] + \sec^2 x = 6x^2 + \cos y \,.\, y'$$
$$y + xy' + \sec^2 x = 6x^2 + y' \cos y$$
$$xy' - y' \cos y = 6x^2 - y - \sec^2 x$$
$$y'(x - \cos y) = 6x^2 - y - \sec^2 x$$
$$y' = \frac{6x^2 - y - \sec^2 x}{x - \cos y}$$

56) If $y = x^{-1} \sec x$, then $y' =$
Solution:
Use the rules
$(f.g)' = f'g + fg'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = (-x^{-2})(\sec x) + (x^{-1})(\sec x \tan x)$$
$$= x^{-1} \sec x \tan x - x^{-2} \sec x$$
$$= x^{-2} \sec x\,(x \tan x - 1)$$

57) If $y = \sin^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$

$$y' = \frac{1}{\sqrt{1 - (x^3)^2}}\,.3x^2 = \frac{3x^2}{\sqrt{1 - x^6}}$$

58) If $y = \cos^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$

$$y' = -\frac{1}{\sqrt{1 - (x^3)^2}}\,.3x^2 = -\frac{3x^2}{\sqrt{1 - x^6}}$$

59) If $y = \sec^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $(\sec^{-1} u)' = \frac{u'}{|u|\sqrt{u^2-1}}$

$$y' = \frac{1}{x^3\sqrt{(x^3)^2 - 1}}\,.3x^2 = \frac{3x^2}{x^3\sqrt{x^6 - 1}} = \frac{3}{x\sqrt{x^6 - 1}}$$

60) If $y = \csc^{-1}(x^3)$, then $y' =$
Solution:
Use the rule $(\csc^{-1} u)' = -\frac{u'}{|u|\sqrt{u^2-1}}$

$$y' = -\frac{1}{x^3\sqrt{(x^3)^2 - 1}}\,.3x^2 = -\frac{3x^2}{x^3\sqrt{x^6 - 1}} = -\frac{3}{x\sqrt{x^6 - 1}}$$

61) If $y = \ln(x^3 - 2\sec x)$, then $y' =$
Solution:
Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = \frac{1}{x^3 - 2\sec x}\,.(3x^2 - 2\sec x \tan x)$$
$$= \frac{3x^2 - 2\sec x \tan x}{x^3 - 2\sec x}$$

62) If $y = \ln(\cos x)$, then $y' =$
Solution:
Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\cos u)' = -\sin u \,.\, u'$$

$$y' = \frac{1}{\cos x}\,.(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

63) If $y = \ln(\sin x)$, then $y' =$
Solution:
Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\sin u)' = \cos u \,.\, u'$$

$$y' = \frac{1}{\sin x}\,.(\cos x) = \frac{\cos x}{\sin x} = \cot x$$

64) If $y = \ln\sqrt{3x^2 + 5x}$, then $y' =$
Solution:
Use the rules $(\ln u)' = \frac{u'}{u}$ and $\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y' = \frac{1}{\sqrt{3x^2 + 5x}}\,.\left(\frac{6x + 5}{2\sqrt{3x^2 + 5x}}\right) = \frac{6x + 5}{2(3x^2 + 5x)}$$

65) If $y = \log_5(x^3 - 2\csc x)$, then $y' =$

Solution:

Use the rules

$$(\log_a u)' = \frac{u'}{u \ln a} \quad \text{and} \quad (\csc u)' = -\csc u \cot u \,.\, u'$$

$$y' = \frac{1}{(x^3 - 2\csc x)(\ln 5)} \,.\, [3x^2 - 2(-\csc x \cot x)]$$
$$= \frac{3x^2 + 2\csc x \cot x}{(x^3 - 2\csc x)(\ln 5)}$$

66) If $y = \ln \frac{x-1}{\sqrt{x+2}}$, then $y' =$

Solution:

Use the rules

$$(\ln u)' = \frac{u'}{u} \,,\quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \text{and} \quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

$$y' = \frac{1}{\frac{x-1}{\sqrt{x+2}}} \cdot \left(\frac{(1)\left(\sqrt{x+2}\right) - (x-1)\left(\frac{1}{2\sqrt{x+2}}\right)}{\left(\sqrt{x+2}\right)^2}\right)$$
$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\sqrt{x+2} - \frac{x-1}{2\sqrt{x+2}}}{x+2}\right)$$
$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\frac{2(x+2)-(x-1)}{2\sqrt{x+2}}}{x+2}\right)$$
$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\frac{x+5}{2\sqrt{x+2}}}{x+2}\right)$$
$$= \frac{\sqrt{x+2}}{x-1}\left(\frac{x+5}{2(x+2)\sqrt{x+2}}\right)$$
$$= \frac{x+5}{2(x-1)(x+2)}$$

67) If $y = 2x^3 - \sin x$, then $y' =$

Solution:

Use the rule $\quad (\sin u)' = \cos u \,.\, u'$

$$y' = 6x^2 - \cos x$$

68) If $y = x^3 \cos x$, then $y' =$

Solution:

Use the rules

$(f.g)' = f'g + fg'$ and $(\cos u)' = -\sin u \,.\, u'$

$$y' = (3x^2)(\cos x) + (x^3)(-\sin x)$$
$$= 3x^2 \cos x - x^3 \sin x$$

69) If $y = x^{\sqrt{x}}$, then $y' =$

Solution:

Use the rule $\quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y = x^{\sqrt{x}}$$
$$\ln y = \ln x^{\sqrt{x}}$$
$$\ln y = \sqrt{x} \ln x$$
$$\frac{y'}{y} = \left(\frac{1}{2\sqrt{x}}\right)(\ln x) + \left(\sqrt{x}\right)\left(\frac{1}{x}\right)$$
$$\frac{y'}{y} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{x\ln x + 2x}{2x\sqrt{x}} = \frac{x(\ln x + 2)}{2x\sqrt{x}}$$
$$= \frac{\ln x + 2}{2\sqrt{x}}$$
$$y' = y\left(\frac{\ln x + 2}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)$$

70) If $y = (\sin x)^x$, then $y' =$

Solution:

Use the rule $\quad (\sin u)' = \cos u \,.\, u'$

$$y = (\sin x)^x$$
$$\ln y = \ln(\sin x)^x$$
$$\ln y = x \ln(\sin x)$$
$$\frac{y'}{y} = (1)(\ln(\sin x)) + (x)\left(\frac{\cos x}{\sin x}\right)$$
$$\frac{y'}{y} = \ln(\sin x) + \frac{x\cos x}{\sin x} = \ln(\sin x) + x\cot x$$
$$y' = y(\ln(\sin x) + x\cot x)$$
$$= (\sin x)^x(\ln(\sin x) + x\cot x)$$

71) If $y = \log_7(x^3 - 2)$, then $y' =$

Solution:

Use the rule $\quad (\log_a u)' = \frac{u'}{u \ln a}$

$$y' = \frac{1}{(x^3 - 2)(\ln 7)} \,.\, (3x^2) = \frac{3x^2}{(x^3 - 2)(\ln 7)}$$

72) If $y = \cos(x^5)$, then $y' =$

Solution:

Use the rule $\quad (\cos u)' = -\sin u \,.\, u'$

$$y' = -\sin(x^5) \,.\, (5x^4) = -5x^4 \sin(x^5)$$

73) If $y = \sec x \tan x$, then $y' =$

Solution:

$(f.g)' = f'g + fg'$, $(\sec u)' = \sec u \tan u \,.\, u'$ and $(\tan u)' = \sec^2 u \,.\, u'$

$$y' = (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x)$$
$$= \sec x \tan^2 x + \sec^3 x = \sec x(\tan^2 x + \sec^2 x)$$

74) If $D^{99}(\cos x) =$

Solution:

$$D(\cos x) = -\sin x$$
$$D^2(\cos x) = -\cos x$$
$$D^3(\cos x) = \sin x$$
$$D^4(\cos x) = \cos x$$

Note: $D^n(\cos x) = \cos x$ whenever n is a multiple of 4. Hence,

$$D^{96}(\cos x) = \cos x$$
$$D^{97}(\cos x) = -\sin x$$
$$D^{98}(\cos x) = -\cos x$$
$$D^{99}(\cos x) = \sin x$$

75) If $y = (x + \sec x)^3$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = 3(x + \sec x)^2.(1 + \sec x \tan x)$$

76) If $x^2 = 5y^2 + \sin y$, then $y' =$

Solution:

$$2x = 10yy' + \cos y \,.\, y'$$
$$y'(10y + \cos y) = 2x$$
$$y' = \frac{2x}{10y + \cos y}$$

77) If $x^2 - 5y^2 + \sin y = 0$, then $y' =$

Solution:

$$2x - 10yy' + \cos y \,.\, y' = 0$$
$$y'(-10y + \cos y) = -2x$$
$$y' = \frac{-2x}{-10y + \cos y} = \frac{2x}{10y - \cos y}$$

78) If $y = \sin x \sec x$, then $y' =$

Solution:

$(f.g)' = f'g + fg'$, $(\sin u)' = \cos u \,.\, u'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = (\cos x)(\sec x) + (\sin x)(\sec x \tan x)$$
$$= 1 + \sin x \,.\, \frac{1}{\cos x} \,.\, \frac{\sin x}{\cos x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$
$$= \sec^2 x$$

79) If $f(x) = \sin^2(x^3 + 1)$, then $f'(x) =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sin u)' = \cos u \,.\, u'$

$$f'(x) = 2\sin(x^3 + 1) \,.\, (\cos(x^3 + 1)) \,.\, (3x^2)$$
$$= 6x^2 \sin(x^3 + 1) \cos(x^3 + 1)$$

80) If $y = (x + \cot x)^3$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\cot u)' = -\csc^2 u \,.\, u'$

$$y' = 3(x + \cot x)^2.(1 - \csc^2 x)$$

81) If $y = \tan^{-1}\left(\frac{x}{2}\right)$, then $y' =$

Solution:

Use the rule $(\tan^{-1} u)' = \frac{u'}{1+u^2}$

$$y' = \frac{1}{1 + \left(\frac{x}{2}\right)^2} \,.\, \frac{1}{2} = \frac{1}{2\left(1 + \frac{x^2}{4}\right)} = \frac{1}{2\left(\frac{4 + x^2}{4}\right)} = \frac{2}{4 + x^2}$$

82) If $y = \cot^{-1}\left(\frac{x}{2}\right)$, then $y' =$

Solution:

Use the rule $(\cot^{-1} u)' = -\frac{u'}{1+u^2}$

$$y' = -\frac{1}{1 + \left(\frac{x}{2}\right)^2} \,.\, \frac{1}{2} = -\frac{1}{2\left(1 + \frac{x^2}{4}\right)} = -\frac{1}{2\left(\frac{4 + x^2}{4}\right)}$$
$$= -\frac{2}{4 + x^2}$$

83) If $y = \sin^{-1}\left(\frac{x}{3}\right)$, then $y' =$

Solution:

Use the rule $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$

$$y' = \frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \,.\, \frac{1}{3} = \frac{1}{3\sqrt{1 - \frac{x^2}{9}}} = \frac{1}{3\sqrt{\frac{9 - x^2}{9}}}$$
$$= \frac{1}{\sqrt{9 - x^2}}$$

84) If $y = \cos^{-1}\left(\frac{x}{3}\right)$, then $y' =$

Solution:

Use the rule $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$

$$y' = -\frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \,.\, \frac{1}{3} = -\frac{1}{3\sqrt{1 - \frac{x^2}{9}}} = -\frac{1}{3\sqrt{\frac{9 - x^2}{9}}}$$
$$= -\frac{1}{\sqrt{9 - x^2}}$$

85) If $D^{99}(\sin x) =$ <u>Solution:</u> $D(\sin x) = \cos x$ $D^2(\sin x) = -\sin x$ $D^3(\sin x) = -\cos x$ $D^4(\sin x) = \sin x$ **Note:** $D^n(\sin x) = \sin x$ whenever n is a multiple of 4. Hence, $D^{96}(\sin x) = \sin x$ $D^{97}(\sin x) = \cos x$ $D^{98}(\sin x) = -\sin x$ $D^{99}(\sin x) = -\cos x$	

3.1

3.2

Notes

14

- التركيز على المفاهيم الاساسية.
- شرح أبواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

رياضيات ١١٠
Math. 110

جمال السعدي

استاذ الرياضيات والإحصاء للمرحلة الجامعية

3.1 and 3.2

3.1 Derivatives of polynomials and exponential fun.

3.2 The product and quotient Rules.

Differentiation Rules

(1) $F(x) = c$ where c is constant. (ثابت)
$F'(x) = 0$ (مشتقة الثابت بـ zero)

(2) $F(x) = ax$ where a is constant.
$F'(x) = a$ (نأخذ المعامل فقط)

(3) $F(x) = ax^n$
$F'(x) = n \cdot a x^{n-1}$ (نضرب الأس في المعامل و ننقص من الأس ١)

(4) $F(x) = g(x) \cdot h(x)$ * قاعده مشتقة حاصل ضرب دالتين

مشتقة الأولى . الثانيه + مشتقة الثانيه . الأولى

$$F'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$$

(5) $F(x) = \frac{(x)}{h(x)}$ * قاعده مشتقة خارج قسمه دالتين

$$F'(x) = \frac{g' \cdot h - h' \cdot g}{(h)^2}$$

= → (المقام . مشتقة البسط − البسط . مشتقة المقام) / $(\text{المقام})^2$

(6) $F(x) = (\quad)^{n}$

$F'(x) = n(\quad)^{n-1}$. مشتقة ما بداخل القوس

(7) $F(x) = \sqrt{\quad}$ (مشتقة الجذر التربيعي)

$$F'(x) = \frac{\text{مشتقة ما تحت الجذر}}{2\sqrt{\quad}}$$

(8) $F(x) = \frac{a}{x^{n}}$

$F'(x) = \frac{-a \cdot n}{x^{n+1}}$ *(نعكس اشارة العدد a ثم نضرب في n ثم نقسم على x^{n+1})

(9) $F(x) = \frac{1}{x}$ $\Longrightarrow F(x) = \frac{a}{x}$

$F'(x) = \frac{-1}{x^{2}}$ $F'(x) = \frac{-a}{x^{2}}$

(10) معادلة المماس (equation of tangent line)

$$y = m(x - x_1) + y_1$$

* حيث (x_1 , y_1) النقطة المعطاة * m slope مشتقة الدالة عند النقطة المعطاة

(11) معادلة العمودي (equation of normal line)

or perpendicular line

$$y = -\frac{1}{m}(x - x_1) + y_1$$

(12) * اذا كان المقام يتكون من حد واحد فقط يتم توزيع حدود البسط على نفس المقام ثم الاختصار ثم الاشتقاق.

(13) $\frac{d}{dx}[f \pm g] = \frac{df}{dx} \pm \frac{dg}{dx}$ * المشتقة تتوزع على جمع و طرح الدوال

(14) $F(x) = e^{h(x)}$ مشتقة الدالة الأسية

$F'(x) = e^{h(x)} \cdot h'(x)$

$e^{h(x)}$ ← الدالة كما هي

$h'(x)$ ← مشتقة الأس

Example: $F(x) = e^{3x^2-2x}$

$F'(x) = e^{3x^2-2x} \cdot (6x-2)$

الدالة كما هي . مشتقة الأس

Note :

- $F(x) = \sqrt{x} \longrightarrow F'(x) = \frac{1}{2\sqrt{x}}$
- $F(x) = \frac{1}{x} \longrightarrow F'(x) = \frac{-1}{x^2}$
- $F(x) = e^{x} \longrightarrow F'(x) = e^{x}$

Differentiate the following functions:

Find y' or f' ?

$y = \sqrt{5} \longrightarrow y' = 0$

$y = e^2 \longrightarrow y' = 0$

$y = \pi^4 \longrightarrow y' = 0$

* مشتقة الثابت zero *

$$y = \sqrt{x^2-2x} \longrightarrow y' = \frac{\text{مشتقة ما تحت الجذر}}{2\sqrt{}} = \frac{2x-2}{2\sqrt{x^2-2x}}$$

$$y' = \frac{2(x-1)}{2\sqrt{x^2-2x}} = \frac{x-1}{\sqrt{x^2-2x}}$$

$$y = \sqrt[3]{x^2-2x}$$

* أى جذر غير التربيعي يحول إلى قوس

$$y = (x^2-2x)^{\frac{1}{3}} \longrightarrow y' = \frac{1}{3}(x^2-2x)^{\frac{1}{3}-1} \cdot \underbrace{(2x-2)}_{\text{مشتقة ما بداخل القوس}}$$

$$\therefore y' = \frac{1}{3}(x^2-2x)^{-\frac{2}{3}} \cdot (2x-2) = \frac{1 \cdot (2x-2)}{3(x^2-2x)^{2/3}}$$

$$\Rightarrow y' = \frac{2(x-1)}{3\sqrt[3]{(x^2-2x)^2}}$$

$$y = \frac{2x^3 - 6x^4}{2x^2}$$

(توزيع)

$$y = \frac{2x^3}{2x^2} - \frac{6x^4}{2x^2}$$

(اختصار)

$$y = x - 3x^2$$

(اشتقاق)

$$y' = 1 - 6x$$

* المقام يتكون من حد واحد
توزع حدود البسط
على نفس المقام
ثم الاختصار
ثم الاشتقاق .

If: $f(x) = \dfrac{x^{3/2} + x^{5/2}}{x^{\frac{1}{2}}}$ Find $f'(1)$?

* نفس طريقة التمرين السابق .

$$F(x) = \frac{x^{3/2}}{x^{\frac{1}{2}}} + \frac{x^{5/2}}{x^{\frac{1}{2}}}$$

(عند القسمة نطرح الأسس)

* $\frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1$
* $\frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$

$$F(x) = x + x^2$$

$$F'(x) = 1 + 2x$$

$$\Rightarrow F'(1) = 1 + 2(1) = 1 + 2 = \boxed{3}$$

* تحويل الجذر إلى صوره أسيه ثم رفعه للبسط بإشاره سالبه ثم الاشتقاق.

$$y = \frac{3}{\sqrt[3]{x^2}}$$

$$y = \frac{3}{x^{2/3}}$$

$$y = 3x^{-\frac{2}{3}} \Rightarrow y' = 3 \cdot \frac{-2}{3} x^{-\frac{2}{3}-1} = -2x^{-\frac{5}{3}} = \frac{-2}{x^{5/3}}$$

$$= \frac{-2}{\sqrt[3]{x^5}} = \frac{-2}{x\sqrt[3]{x^2}}$$

---

* قاعده :

$$\text{اشتقاق } \frac{\text{عدد}}{x^n} = \frac{\text{نعكس اشاره العدد ثم نضرب في } n}{x^{n+1}}$$

$$y = \frac{4}{x^5}$$

$$y' = \frac{-4 \cdot (5)}{x^{5+1}} = \frac{-20}{x^6}$$

$$y = -2x^5 + 3x^{-5} + \frac{1}{x} - \sqrt{x} + e^{3x}$$

مشتقه الأس في الداله نفسها

$$y' = -10x^4 - 15x^{-6} - \frac{1}{x^2} - \frac{1}{2\sqrt{x}} + 3e^{3x}$$

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(7) $g(x) = \dfrac{3x - 1}{2x + 1}$ Find $g'(x)$?

$$g'(x) = \frac{(3)(1) - (2)(-1)}{(2x+1)^2}$$

$$= \frac{3 + 2}{(2x+1)^2}$$

$$= \frac{5}{(2x+1)^2}$$

قاعده

تستخدم هذه القاعده
اذا كان البسط والمقام
من الدرجه الأولى

$$F(x) = \frac{ax + b}{cx + d}$$

$$F'(x) = \frac{(a \cdot d) - (c \cdot b)}{(cx + d)^2}$$

(13) $y = \dfrac{x^3}{1 - x^2}$

$$y' = \frac{\underset{\text{مشتقة البسط}}{(3x^2)} \cdot \underset{\text{المقام}}{(1 - x^2)} - \underset{\text{مشتقة المقام}}{(-2x)} \cdot \underset{\text{البسط}}{x^3}}{(1 - x^2)^2}$$

$$= \frac{3x^2 - 3x^4 + 2x^4}{(1 - x^2)^2} = \frac{3x^2 - x^4}{(1 - x^2)^2}$$

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② $F(x) = \dfrac{x - 3x\sqrt{x}}{\sqrt{x}}$ Find $F'(x)$?

$F(x) = \dfrac{\boxed{x}}{\sqrt{x}} - \dfrac{3x\sqrt{x}}{\sqrt{x}}$

* المقام يتكون من حد واحد
∴ نوزع حدود البسط
على نفس المقام

$= \dfrac{\boxed{\sqrt{x}\cdot\cancel{\sqrt{x}}}}{\cancel{\sqrt{x}}} - \dfrac{3x\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}$

$= \sqrt{x} - 3x$

$\Rightarrow F'(x) = \dfrac{1}{2\sqrt{x}} - 3$ ممكن توحيد مقامات $= \dfrac{1 - 6\sqrt{x}}{2\sqrt{x}}$.

$y = \dfrac{x^3}{3} + \dfrac{2}{x^2}$

$y' = \dfrac{3x^2}{3} + \dfrac{-2\cdot(2)}{x^3}$

$\Rightarrow y' = x^2 - \dfrac{4}{x^3}$

- $F(x) = x \cdot (\sqrt{x} + 3)$ Find $F'(x)$?

يمكن:
* مشتقة حاصل ضرب دالتين
* يمكن فك الأقواس أولاً ثم الاشتقاق ثانياً وهذا هو الأسرع.

$$F(x) = x\sqrt{x} + 3x$$

$$F(x) = x^{3/2} + 3x$$

$$\Rightarrow F'(x) = \frac{3}{2} x^{\frac{1}{2}} + 3 = \frac{3}{2}\sqrt{x} + 3$$

- $y = \dfrac{5}{(5x-1)^3} \Rightarrow y = 5(5x-1)^{-3}$

$$\Rightarrow y' = -15(5x-1)^{-4} \cdot \underbrace{5}_{\text{مشتقة ما بداخل القوس}} = \frac{-75}{(5x-1)^4}$$

- $y = x\sqrt{x}$

$$y = x \cdot x^{\frac{1}{2}} \Rightarrow y = x^{3/2} \Rightarrow y' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

- $y = \sqrt{x} - 2e^x$

$$y' = \frac{1}{2\sqrt{x}} - 2e^x$$

* قاعده : بمجرد النظر :
انظر Page 6

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(16) $R(x) = \dfrac{\sqrt{10}}{x^7} \Rightarrow R'(x) = \dfrac{-7\sqrt{10}}{x^8}$

(13) $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = \frac{4}{3}\pi \cdot 3r^2 = \underline{\underline{4\pi r^2}}$

(18) $y = \sqrt[3]{x} \Rightarrow y = x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3x^{2/3}}$

$$\Rightarrow y' = \frac{1}{3\sqrt[3]{x^2}}$$

(20) $F(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}}$

$$\boxed{t^{3/2} = t\sqrt{t}}$$

$\Rightarrow F(t) = \sqrt{t} - t^{-\frac{1}{2}}$

$\Rightarrow F'(t) = \dfrac{1}{2\sqrt{t}} - \left(-\frac{1}{2}\right) t^{-\frac{3}{2}} = \dfrac{1}{2\sqrt{t}} + \dfrac{1}{2t^{3/2}}$

$$= \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}$$

(28) $y = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2}$

$$y' = ae^v + \frac{-b}{v^2} + \frac{-2c}{v^3} = ae^v - \frac{b}{v^2} - \frac{2c}{v^3}$$

* قاعده : بمجرد النظر :
انظر Page 6 ،

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(16) $R(x) = \dfrac{\sqrt{10}}{x^7} \Rightarrow R'(x) = \dfrac{-7\sqrt{10}}{x^8}$

(13) $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = \frac{4}{3}\pi \cdot 3r^2 = \underline{\underline{4\pi r^2}}$

(18) $y = \sqrt[3]{x} \Rightarrow y = x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3x^{2/3}}$

$$\Rightarrow y' = \frac{1}{3\sqrt[3]{x^2}}$$

(20) $F(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}}$

$\boxed{t^{3/2} = t\sqrt{t}}$

$\Rightarrow F(t) = \sqrt{t} - t^{-\frac{1}{2}}$

$\Rightarrow F'(t) = \dfrac{1}{2\sqrt{t}} - \left(-\frac{1}{2}\right)t^{-\frac{3}{2}} = \dfrac{1}{2\sqrt{t}} + \dfrac{1}{2t^{3/2}}$

$$= \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}$$

(28) $y = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2}$

$y' = ae^v + \dfrac{-b}{v^2} + \dfrac{-2c}{v^3} = ae^v - \dfrac{b}{v^2} - \dfrac{2c}{v^3}$

(31) $z = \frac{A}{y^{10}} + Be^{y}$

$z' = \frac{-10A}{y^{11}} + Be^{y}$.

(32) $y = e^{x+1} + 1$

$y' = e^{x+1} \cdot \underbrace{1}_{\text{مشتقة الأس}} = e^{x+1}$

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Find the equation of the tangent line and the normal line

to (33) $y = \underline{2x}\ \underline{e^{x}}$ at $(\underset{x_1}{0}, \underset{y_1}{0})$

$y' = 2 \cdot e^{x} + e^{x} \cdot 2x$

$m = 2e^{0} + e^{0} \cdot \underbrace{2(0)}_{0} \Rightarrow m = 2e^{0} = 2(1) = \boxed{2}$

- eq. of tangent line: $y = m(x - x_1) + y_1$

$y = 2(x - 0) + 0$

$\boxed{y = 2x}$

- eq. of normal line : $y = \frac{-1}{m}(x - x_1) + y_1$

$y = \frac{-1}{2}(x - 0) + 0$

$\Rightarrow \boxed{y = -\frac{1}{2}x}$

(32) $y = \dfrac{e^x}{x}$ at $(1, e)$

$\qquad\qquad\qquad\quad\; x_1 \;\; y_1$

البسط . مشتقة المقام   المقام . مشتقة البسط

$$y' = \frac{e^x \cdot x - 1 \cdot e^x}{x^2}$$

لإيجاد m
عوض عن x بـ 1
في y'

$$m = \frac{e^1 \cdot 1 - 1 \cdot e^1}{1^2} = e - e = \boxed{0}$$

معادلة المماس

eq. of tangent line : $y = m(x - x_1) + y_1$

$$y = \boxed{0}(x - 1) + e$$

$$\boxed{y = e}$$

∵ النقطة ← $(1, e)$
∴ معادلة المماس $y = e$
∴ معادلة العمودي $x = 1$

∴ معادلة العمودي

eq. of normal line

$$\boxed{x = 1}$$

(23) $F(x) = \dfrac{A}{B + ce^x} \Longrightarrow = \dfrac{0 \cdot (B + ce^x) - ce^x \cdot A}{(B + ce^x)^2}$

$\Longrightarrow F'(x) = \dfrac{-Ace^x}{(B + ce^x)^2}$

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(51) Find the points on the curve

$$y = 2x^3 + 3x^2 - 12x + 1$$

where the $\underbrace{\overset{\text{المماس}}{\text{tangent}} \text{ is } \overset{\text{أفقي}}{\text{horizontal}}}$.

$\Rightarrow$ ∵ المماس أفقي

$\therefore y' = 0$

$6x^2 + 6x - 12 = 0$ $\quad (\div 6)$ لسهولة التحليل*

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x + 2 = 0$	$x - 1 = 0$
$\boxed{x = -2}$	$\boxed{x = 1}$
لإيجاد y نعوض عن x بـ -2 في الداله الأصليه	لإيجاد y نعوض عن x بـ 1 في الداله الأصليه
$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1$	$y = 2(1)^3 + 3(1)^2 - 12(1) + 1$
$= -16 + 12 + 24 + 1$	$= 2 + 3 - 12 + 1$
$= \boxed{21}$	$= \boxed{-6}$

∴ The tangent is horizontal

at the points: $(-2, 21)$ and $(1, -6)$

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معادله

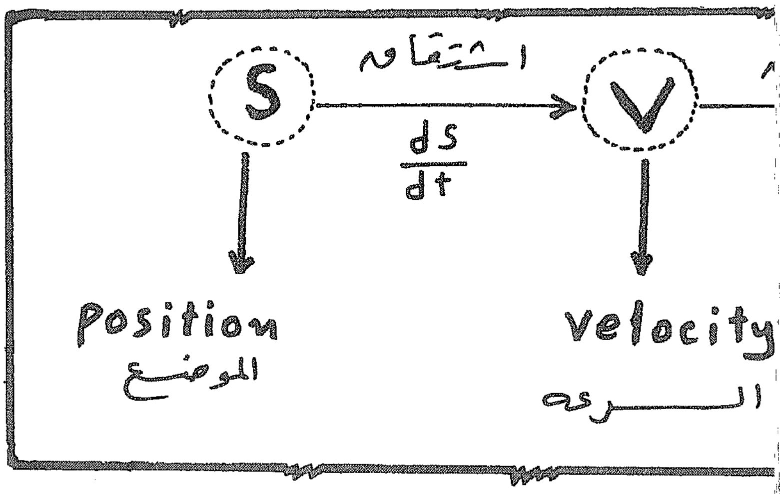
(50) The equation of mot

is: $\boxed{S = t^3 - 3t}$ wher

and

السرعه

(a) Find the velocity and



* velocity: $v = \frac{ds}{dt} = 3t^2$

* acceleration: $a = \frac{dv}{dt} =$

(b) Find the acceleration

↳ $a(1) = 6(1) = 6$

$x\sqrt{x} = x \cdot x^{\frac{1}{2}} = x^{3/2}$

$\Rightarrow$ مشتقة $\underline{x\sqrt{x}}$ $\longrightarrow \frac{3}{2}x^{\frac{1}{2}} = \underline{\frac{3}{2}\sqrt{x}}$

page 181

(54) Find an eq. of the tangent line

to the curve $\boxed{y = x\sqrt{x}}$

that is parallel (يوازي) to the line $\boxed{y = 1 + 3x}$

شرط التوازي هو

$$m_1 = m_2$$

$\Rightarrow$ ميل المماس للمنحنى = ميل المستقيم الموازي له

(مشتقة المنحنى) (مشتقة المستقيم)

$$\frac{3}{2}\sqrt{x} = 3$$ (بالضرب في $\frac{2}{3}$ للتخلص من معامل الجذر)

$\cancel{\frac{2}{3}} \cdot \cancel{\frac{3}{2}} \sqrt{x} = \frac{2}{\cancel{3}} \cdot \cancel{3} \Rightarrow \sqrt{x} = 2 \xrightarrow{\text{بالتربيع}} \boxed{x = 4}$

عوض في معادلة المنحنى للحصول على y

$y = 4\sqrt{4} \longrightarrow \boxed{y = 8}$

$\therefore$ نقطة التماس $(\underset{x_1}{4}, \underset{y_1}{8})$ ، الميل (slope) $3 = m$

eq. of tangent line :

$$y = m(x - x_1) + y_1$$
$$y = 3(x - 4) + 8$$
$$y = 3x - 12 + 8$$
$$\boxed{y = 3x - 4}$$

* للتأكد من صحة الحل
استخدم النقطة
(4 , 8)
في المعادلة الأخيرة
عوض عن x بـ 4
يكون الناتج y = 8
مما يؤكد صحة الحل.

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(75) let $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx+b & \text{if } x > 2 \end{cases}$

Find the values of m and b

that make f differentiable every

قابلة للاشتقاق
معناه لابد من
تحقق شرطين

$(m) \quad (2x)$

المشتقة اليمنى عند $x=2$ = المشتقة اليسرى عند $x=2$

$m = 2(2)$

$\therefore \boxed{m = 4}$

الدالة متصلة عند $x=2$

$\lim_{x \to 2^+} (mx+b) = \lim_{x \to 2^-} x^2$

$4(2) + b = 4$

$8 + b = 4$

$b = 4 - 8$

$\therefore \boxed{b = -4}$

الرشفة فوق ...

$*(1)$ If: $y = x^3 + 3(\pi^2 + x^2)$ Find y'' ?

$*(2)$ If: $y = \sin^2 x + \cos^2 x$ Find y' ?

$*(3)$ Find: $\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$

أفترض

- Suppose u and v are differentiable functions where:

$$\boxed{\begin{array}{ll} u(1) = 2 , & u'(1) = 0 \\ v(1) = 5 , & v'(1) = -1 \end{array}}$$

Find:

1. $\frac{d}{dx}(uv) = u' \cdot v + v' \cdot u$

at $(x = 1)$ $= 0 \cdot (5) + (-1) \cdot (2) = \boxed{-2}$

2. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u' \cdot v - v' \cdot u}{v^2}$

at $(x = 1)$ $= \frac{(0)(5) - (-1)(2)}{(5)^2} = \boxed{\frac{2}{25}}$

3. $\frac{d}{dx}(7v - 2u^2) = 7v' - 4uu'$

at $(x = 1)$ $= 7(-1) - 4(2)(0)$

$= -7 - 0 = \boxed{-7}$

- $y = e^x - \underbrace{3x^4}$ Find $y^{(5)}$?

* درجه هذا الحد $(4 = \text{degree})$
رتبه المشتقه $(5 = \text{order})$
∴ المشتقه الخامسه لهذا الحد zero
* اما e^x فمشتقتها دائماً e^x
مهما كان عدد مرات الاشتقاق

$$\Rightarrow y^{(5)} = e^x - 0 = e^x$$

- $y = \dfrac{e^{3x} + e^{2x}}{e^{2x}}$ Find y' ?

* المقام يتكون من حد واحد فقط
∴ توزيع حدود البسط على نفس المقام ثم الاختصار ثم الاشتقاق

$$y = \frac{e^{3x}}{e^{2x}} + \frac{\cancel{e^{2x}}}{\cancel{e^{2x}}}$$

$$y = e^x + 1$$

$$y' = e^x + 0 \Rightarrow y' = e^x$$

- $y = x + \dfrac{1}{x}$ Find y' ?

$$y' = 1 - \frac{1}{x^2}$$

مشتقة $\frac{1}{x}$
هي $-\frac{1}{x^2}$

كل التمنيات بالنجاح والتوفيق

السعدي

Chapter (3)

(3.1) Differentation Rules:

① Find the first and higher derivative

② Find the point on the curve where The tangent is horizontal

$\boxed{\text{Example (6) in 178}}$ اوجدي النقطه التي يكون عندها المماس أفقي مثال عليها ?

(3.2) The product and Quotient Rule:-

① Find the first and higher dirvative اوجدي المشتقة الأولى او المشتقات العليا لدالتين مضروبتين ببعض او مقسومتين على بعض. ?

② Find $y'(a)$ if $y = f \cdot g$ or $y = f \pm g$
(where $f(a)$, $f'(a)$, $g(a)$, $g'(a)$ are exist)

Example:- If $f(x) = \sqrt{x}\, g(x)$, $g(4) = 2$, $g'(4) = 3$
Find $f'(4)$??

الحل:

$$f'(x) = \sqrt{x}\; g'(x) + \frac{1}{2\sqrt{x}}\, g(x)$$

$$\Longrightarrow f'(4) = \sqrt{4}\; g'(4) + \frac{1}{2\sqrt{4}}\, g(4)$$

$$= 2(3) + \frac{1}{4}(2) = 6 + \frac{1}{2} = \frac{12+1}{2} = \frac{13}{2}$$

③ Find the equation of the tangent line.

كمان بيعطيه داله, وتطلب معادلة المماس ?

Example $y = \dfrac{e^x}{1+x^2}$ at $(1, \frac{1}{2}e)$

الحل:-

the eqation of the tangen is $y - y_1 = m(x - x_1)$

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$$m = y' = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(x^2-2x+1)}{(1+x^2)^2}$$

$$\Rightarrow m = y'\Big|_{x=1} = \frac{e(1-2+1)}{(1+1)^2} = \frac{e(2-2)}{4} = 0$$

$$\therefore y - \tfrac{1}{2}e = 0(x-1)$$

$$y - \tfrac{1}{2}e = 0$$

$$y = \tfrac{1}{2}e$$

؟ مشتقة نقطة ... أو ... محاور ... حلول

④ Find the dirvateve at the point.

Ex. Find the dirvative of $y = x\sqrt{x}$ at $x=1$

الحل:

$$y = x\sqrt{x} = x\,x^{1/2} = x^{1/2+1} = x^{3/2}$$

$$\therefore y' = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$$

$$\Rightarrow y'(1) = \frac{3}{2}\sqrt{1} = \frac{3}{2}$$

(3.3) Derivative of trigonometric functions:-

؟ الدالة المثلثية مشتقة أو ...

① Find the derivative of trigonometric function

Ex: $\boxed{a}$ If $f(x) = \sin^3(2x) \Rightarrow y' = 3(\sin(2x))^2(\cos(2x))\cdot 2$

$$= 6\cos(2x)\sin^2(2x)$$

$\boxed{b}$ If $f(x) = 3^x \cot x \Rightarrow f'(x) = 3^x(-\csc^2 x) + 3^x \ln 3 \cot x$

$$= 3^x \ln 3 \cot x - 3^x \csc^2 x$$

(3)

ثم الرتبة الثانية

مشتقة الدوال المثلثية؟

③ Find the second derivative of Trigonometric functions.

④ Find the n th derivative of trigonometric functions.

EX : Find 27th derivative of $\cos x$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

$\therefore f^{(n)}(x) = \cos x$ where n is a multiple of 4

$4 \times 1 = 4$

$4 \times 2 = 8$

$4 \times 3 = 12$

$\vdots$

$\boxed{4 \times 7 = 28}$

$\therefore f^{(28)}(x) = \cos x$

$f^{(25)}(x) = -\sin x$

$f^{(26)}(x) = -\cos x$

$\boxed{f^{(27)}(x) = \sin x}$

⑤ Find the equation of the tangent line

Example : $f(x) = \sec x$ at $(\frac{\pi}{3}, 2)$

$\therefore y - y_1 = m(x - x_1)$

$m = y' = \sec x \tan x \Longrightarrow m = \sec \frac{\pi}{3} \cdot \tan \frac{\pi}{3}$

$= (2)(\sqrt{3}) = 2\sqrt{3}$

$\therefore y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$

$\longrightarrow y = 2\sqrt{3}x - \frac{2\sqrt{3}\pi}{3} + 2$

$\sec 60$

$\cos 60 = \frac{1}{2}$

(3.4) The chain rule :- قاعدة التسلسل

اشتقاق وحده

① Find the derivative,

Example : ⓐ $y = \sqrt{x^2+1} \longrightarrow y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$

$= \frac{x}{\sqrt{x^2+1}}$.

$\boxed{b}$ $y = \sin(x^2) \Longrightarrow y' = \cos(x^2) \cdot 2x$

$= 2x \cos(x^2)$

هناك أمثلة كثيرة في الكتاب وتمارين الواجب .

الاشتقاق الثانية وحده

② Find y''.

Example: $y = \cos(x^2)$

$y' = -\sin(x^2) \cdot 2x = -2x \sin(x^2)$

$y'' = -2x(\cos(x^2) \cdot 2x) + (-2) \sin(x^2)$

$= -4x \cos(x^2) - 2 \sin(x^2)$

(3.5) Implicit differentation:- الاشتقاق الضمني

① Find y' or y'' ?

وحدي معادلة خط المماس ?

② Find the equation of the tangent line.

Example ② in page 211

الدالة المثلثية معكوس مشتقة وحدي ?

③ Find the derivative of inverse trigonometric function.

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(3.6) Derivative of logarithmic function.

① Find y' or y''

② Find the derivative by using Logarithmic differentation.

نستخدم هذه الطريقة في حالتين :-

① عندما تكون الداله معقده كما في مثال 7 في الصفحه 221

② اذا اعطاني داله مرفوعه لداله كما في مثال 8 في الصفحه 221

(6)

حل مثال ⑦ في صفحة 295

$y = x^4 - 4x^3$

① $D_f = \mathbb{R}$

② $f'(x) = 4x^3 - 12x^2$

③ $f' = 0$ | f' not defined ✗

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$\boxed{x=0}$, $\boxed{x=3}$

		0		3	
$4x^2$	+		+		+
$x-3$	−		−		+
f' إشارة	−		−		+
	decreasing		decreasing	local min	increasing

∴ * f is decreasing in $(-\infty, 3)$
and ~~decr~~ increasing in $(3, \infty)$

* f has a local minimum at $(3, f(3))$

$f'(x) = 4x^3 - 12x^2$

① $D_{f'} = \mathbb{R}$

② $f''(x) = 12x^2 - 24x$

③ $f'' = 0$ | f'' not defined

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$\boxed{x=0}$, $\boxed{x=2}$

		0		2	
$12x$	−		+		+
$x-2$	−		−		+
f'' إشارة	+		−		+
	Concave up	inflection	Concave down	inflection	Concave up

* f is Concave up in $(-\infty, 0) \cup (2, \infty)$
and Concave down in $(0, 2)$

* f has an inflection points at $(0, f(0))$, $(2, f(2))$