lecture (13):

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Infinite Series

Sequences:

An infinite sequence, or more simply sequence, is an unending succession of numbers, called terms.

A sequence be written as:

$$\underbrace{a_1}_{\text{First term}},\ \underbrace{a_2}_{\text{second term}},\ a_3,\ a_4,\ \dots,\ \underbrace{a_n}_{\substack{\text{nth term} \\ \text{(general term)}}},\ \dots = \{a_n\}_{n=1}^{\infty}$$

n is called index for the sequence.

For Example:

$1, 2, 3, 4, \dots$

$2, 4, 6, 8, \dots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

$1, -1, 1, -1, \dots$

Ex ①: In each part, find the general term of the sequence,

ⓐ $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

$$a_n = \frac{n}{n+1}.$$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

$$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots, \frac{1}{2^n}, \dots$$

$$a_n = \frac{1}{2^n}.$$

(c) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots$

$$a_n = (-1)^{n+1} \frac{n}{n+1}$$

(d) $1, 3, 5, 7, \dots$

$$a_n = 2n - 1.$$

<u>Remark</u>: It isn't essential to start the index at 1, Sometimes it is more Convenient to start it at 0.

For example:

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots = \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$$

Definition: A sequence is a function whose domain is a set of integers.

Definition: A sequence $\{a_n\}$ is said to converge to the limit L if given $\varepsilon > 0$, there is a positive integer N such that $|a_n - L| < \varepsilon$ for $n \geq N$. In this Case we write

$$\lim_{n \to \infty} a_n = L$$

A sequence that does not Converge to some finite limit is said to diverge.

Theorem: Suppose that the sequence $\{a_n\}$ and $\{b_n\}$ Converge to limit L_1 and L_2, respectively and C is a constant. Then:

[1] $\lim_{n\to\infty} c = c$

[2] $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cL_1$.

[3] $\lim_{n\to\infty} (a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L_1 + L_2$.

[4] $\lim_{n\to\infty} (a_n-b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L_1 - L_2$.

[5] $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = L_1 L_2$.

[6] $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{L_1}{L_2}$, $L_2 \neq 0$

Ex②: In each part, Determine whether the sequence Converges or diverges. If it Converges, find the limit.

ⓐ $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$.

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$$

$$= \frac{1}{2+\frac{1}{\infty}}$$

$$\boxed{\begin{array}{c}\text{* Note} \\ \frac{1}{\infty} = 0\end{array}}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2}$$

Then, the sequence Converges to $\frac{1}{2}$. #

(b) $\left\{(-1)^{n+1}\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$

n odd

$$\lim_{n\to\infty}\frac{n}{2n+1}$$

$$=\lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2}$$

n even

$$\lim_{n\to\infty}(-1)\frac{n}{2n+1}$$

$$=\lim_{n\to\infty}(-1)\frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}}$$

$$=\lim_{n\to\infty}(-1)\frac{1}{2+\frac{1}{n}}=\frac{-1}{2}.$$

$$\therefore \lim_{n\to\infty}\frac{n}{2n+1}\neq\lim_{n\to\infty}(-1)\frac{n}{2n+1}$$

Then, the sequence diverges.

(c) $\left\{(-1)^{n+1}\dfrac{1}{n}\right\}_{n=1}^{+\infty}$

n odd

$$\lim_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0$$

$=$

n even

$$\lim_{n\to\infty}(-1)\frac{1}{n}=-1\cdot\frac{1}{\infty}=-1.0=$$

Then, the sequence converges to $\underline{\underline{0}}$.

(d) $\{8-2n\}_{n=1}^{+\infty}$

$$\lim_{n\to\infty}(8-2n)=8-2(\infty)=8-\infty=-\infty$$

Then, the sequence diverges.

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Ex:

1. $\{n+1\}_{n=1}^{\infty}$

2. $\{(-1)^{n+1}\}_{n=1}^{\infty}$

3. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

4. $\left\{1+\left(-\frac{1}{2}\right)^{n}\right\}_{n=1}^{\infty}$

Ex ③: In each part, determine wheather the sequence converges and if so find its limit.

ⓐ $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\right) = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$$

Then, the sequence convergent to $\underline{\underline{0}}$.

ⓑ $1, 2, 2^2, 2^3, 2^4, \dots, 2^n, \dots$

$$\lim_{n\to\infty} 2^n = 2^\infty = \infty.$$

Then, the sequence divergent.

Ex ④: Find the limit of the sequence $\left\{\frac{n}{e^n}\right\}_{n=1}^{+\infty}$

$$\lim_{n\to\infty}\frac{n}{e^n} = \frac{\infty}{e^\infty} = \frac{\infty}{\infty} \text{ (is an indetermine).}$$

Using LHopital rule:

$$\lim_{n\to\infty}\frac{n}{e^n} = \lim_{n\to\infty}\frac{1}{e^n}$$

$$= \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

#

Ex (5): Show that $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

$$\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{\frac{1}{n}} = \infty^{\frac{1}{\infty}} = \infty^{0} \text{ (is an indetermine)}$$

$$\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{\frac{1}{n}} = \lim_{n \to \infty} e^{\ln n^{\frac{1}{n}}}$$

$$= \lim_{n \to \infty} e^{\frac{1}{n} \ln n}$$

$$= \lim_{n \to \infty} e^{\frac{\ln n}{n}} \quad \text{(using L'Hopital rule)}$$

$$= \lim_{n \to \infty} e^{\frac{\frac{1}{n}}{1}}$$

$$= e^{\frac{1}{\infty}} = e^{0} = 1$$

Theorem: A sequence converges to a limit **L** if and only if the sequences of even-numbered terms and odd-numbered terms both converge to **L**.

Ex (7): Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$

odd-numbered terms

$$\lim_{n \to \infty} \left(\frac{1}{2^n} \right) = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0$$

even numbered term

$$\lim_{n \to \infty} \left(\frac{1}{3^n} \right) = \frac{1}{3^{\infty}} = \frac{1}{\infty} = 0$$

Then, the given sequence converges to $\underline{\underline{0}}$.

#

(b) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \cdots$

odd ~~even~~ numbered terms

even ~~odd~~ numbered terms

$\lim_{n \to \infty} \{1\} = 1 \quad \neq \quad \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$

Then, the given sequence diverges.

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* Theorem (the squeezing theorem for sequences).

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences such that:

$$a_n \leq b_n \leq c_n$$

If the sequences $\{a_n\}$ and $\{b_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

Theorem:

If $\lim_{n\to+\infty} |a_n| = 0$, then $\lim_{n\to+\infty} a_n = 0$.

Ex (8): $\left\{(-1)^n \frac{1}{2^n}\right\}_{n=0}^{+\infty}$

$$\left\{(-1)^n \frac{1}{2^n}\right\}_{n=0}^{+\infty} = 1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \ldots, (-1)^n\frac{1}{2^n}, \ldots$$

If we take the absolute value of each term, we obtain the sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$$

Then:

$$\lim_{n \to \infty} \left| (-1)^n \frac{1}{2^n} \right| = \lim_{n \to \infty} \frac{1}{2^n} = \frac{1}{\infty} = 0$$

Therefore,

$$\lim_{n \to \infty} \left\{ (-1)^n \frac{1}{2^n} \right\} = 0 .$$

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H.w: Exc (10.1)

(7 + 23), p. 633 + 634

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Monotone sequences:

Definition: A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

increasing if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$

strictly decreasing if $a_1 > a_2 > a_3 > \cdots > a_n > \cdots$

decreasing if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

A sequence that is either increasing or decreasing is said to be <u>monotone</u>.

A sequence that is either strictly increasing or strictly decreasing is said to be <u>strictly monotone</u>.

Example:

Sequence	Description
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$	strictly increasing.
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	strictly decreasing.
$1, 1, 2, 2, 3, 3, \ldots$	increasing.
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$	decreasing.
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	Neither increasing or decreasing.

Testing for monotonicity:

$$a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$$

Difference test	Ratio test	Derivative test	Conclusion
$a_{n+1} - a_n > 0$	$\frac{a_{n+1}}{a_n} > 1$	$f'(x) > 0$	strictly increasing.
$a_{n+1} - a_n \geq 0$	$\frac{a_{n+1}}{a_n} \geq 1$	$f'(x) \geq 0$	increasing.
$a_{n+1} - a_n < 0$	$\frac{a_{n+1}}{a_n} < 1$	$f'(x) < 0$	strictly decreasing.
$a_{n+1} - a_n \leq 0$	$\frac{a_{n+1}}{a_n} \leq 1$	$f'(x) \leq 0$	decreasing.

Ex ①: Show that the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$$

is a strictly increasing.

I Difference test: $a_n = \frac{n}{n+1}$,

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}.$$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= \frac{\cancel{n^2} + \cancel{2n} + 1 - \cancel{n^2} - \cancel{2n}}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)} > 0 \text{ for } n \geq 1$$

Then, the given sequence is strictly increasing.

$\boxed{2}$ Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{n+1/n+2}{n/n+1} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1,$$

for $n \geq 1$.

The given sequence is strictly increasing.

$\boxed{3}$ Derivative test:

$$a_n = \frac{n}{n+1}.$$

Let $f(x) = \dfrac{x}{x+1}$,

then: $f'(x) = \dfrac{1.(x+1) - x.(1)}{(x+1)^2} = \dfrac{x+1-x}{(x+1)^2} = \dfrac{1}{(x+1)^2} > 0.$

for $x \geq 1$.

Then, the given sequence is strictly increasing.

#

Properties that hold eventually:

Ex: the sequence $9, -8, -17, 12, 1, 2, 3, 4, \ldots$

is strictly increasing from the fifth term, then it is eventually strictly increasing.

Definition:

If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property, then the original sequence is said to have that property eventually.

Ex ②: show that the sequence $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ is eventually strictly decreasing.

$$a_n = \frac{10^n}{n!} \quad , \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}.$$

Using ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{10^n \cdot 10}{(n+1)\,n!} \cdot \frac{n!}{10^n} = \frac{10}{n+1} < 1$$

for $n \geq 10$.

Then, the given sequence is eventually strictly decreasing. #

H.w: Exc (10.2): 7 + 19
(P. 641 + 642)

* Convergence of monotone sequences.

Theorem: If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilites:

ⓐ There is a constant M (upper bound for the sequence) such that $a_n \leq M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$.

ⓑ No upper bound exists, in which case $\lim\limits_{n\to+\infty} a_n = +\infty$.

If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilites:

ⓐ There is a constant M (lower bound for the sequence) such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$.

ⓑ No lower bound exists, in which case $\lim\limits_{n\to+\infty} a_n = -\infty$.

Ex ③: show that the sequence $\left\{ \frac{10^n}{n!} \right\}_{n=1}^{+\infty}$ Converges and find its limit.

From Ex ②: $\quad \frac{a_{n+1}}{a_n} = \frac{10}{n+1} \implies \boxed{a_{n+1} = \frac{10}{n+1} a_n}$

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \frac{10}{n+1} \cdot a_n = \lim_{n\to\infty} \frac{10}{n+1} \lim_{n\to\infty} a_n$$

$$L = \lim_{n\to+\infty} a_{n+1} = \frac{10}{\infty} \cdot L = 0 \cdot L = 0$$

Then, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$ #

using the fact $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$

Lecture (14):

Infinite series.

Definition: An infinite series is an expression that can be written in the form:

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the terms of the series.

For example: the infinite series

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots = \sum_{k=1}^{\infty} \frac{3}{10^k}.$$

The finite sums of this series are:

$$S_1 = \frac{3}{10} = 0.3$$

$$S_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33$$

$$S_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333$$

$$S_4 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} = 0.3333$$

$\vdots$

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} \longrightarrow (1)$$

$$\frac{1}{10} S_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^{n+1}} \longrightarrow (2)$$

and then subtract ② from ① to obtain:

$$S_n - \frac{1}{10} S_n = \frac{3}{10} + \cancel{\frac{3}{10^2}} + \cancel{\frac{3}{10^3}} + \cdots + \cancel{\frac{3}{10^n}}$$

$$- \cancel{\frac{3}{10^2}} - \cancel{\frac{3}{10^3}} - \cancel{\frac{3}{10^4}} - \cdots - \frac{3}{10^{n+1}}$$

$$S_n - \frac{1}{10} S_n = \frac{3}{10} - \frac{3}{10^{n+1}}.$$

$$\left(1 - \frac{1}{10}\right) S_n = \frac{3}{10} - \frac{3}{10^n \cdot 10}$$

$$\frac{9}{10} S_n = \frac{3}{10}\left[1 - \frac{1}{10^n}\right]$$

$$S_n = \frac{10}{9} \cdot \frac{3}{10}\left[1 - \frac{1}{10^n}\right]$$

$$\boxed{S_n = \frac{1}{3}\left[1 - \frac{1}{10^n}\right]}.$$

closed form of S_n

Then, $\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}\left(1 - \frac{1}{10^\infty}\right)$

$$= \frac{1}{3}(1 - 0)$$

$$= \frac{1}{3}.$$

Therefore, the given series converges to $\frac{1}{3}$ and has sum equal $\frac{1}{3}$.

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots$$

#

S_n is called the nth partial sum of the series, and the sequence $\{S_n\}_{n=1}^{+\infty}$ is called the sequence of partial sums.

Definition:

Let $\{S_n\}$ be the sequence of partial sums of the series $u_1 + u_2 + u_3 + \cdots + u_k + \cdots$

* If the sequence $\{S_n\}$ converges to a limit S, then the series is said to be converge to S and S is called the sum of the series. We doenote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

* If the sequence $\{S_n\}$ diverges, then the series is said to diverge. A divergent series has no sum.

Ex ①: Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Converges or diverges. If it converges find the sum.

Solution:

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_4 = 1 - 1 + 1 - 1 = 0$

Then, the sequence of partial sums is:

$1, 0, 1, 0, \ldots$

This sequence is divergent, then the given series is divergent and has no sum. #

* Geometric series:

Definition: A geometric series

$$\sum_{k=0}^{\infty} a r^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \quad (a \neq 0).$$

is called ratio for the series

Converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If the series converges, then the sum is

$$\sum_{k=1}^{\infty} a r^k = \frac{a}{1-r}.$$

* Examples for geometric series :

$\boxed{1}$ $1+2+4+8+\cdots+2^k+\cdots$, $a=1$, $r=2$

$\boxed{2}$ $\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\cdots+\frac{3}{10^k}+\cdots$, $a=\frac{3}{10}$, $r=\frac{1}{10}$

$\boxed{3}$ $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\cdots+(-1)^{k+1}\frac{1}{2^k}+\cdots$, $a=\frac{1}{2}$, $r=\frac{-1}{2}$.

$\boxed{4}$ $1+1+1+\cdots+1+\cdots$, $a=1$, $r=1$

$\boxed{5}$ $1-1+1-1+\cdots+(-1)^{k+1}+\cdots$, $a=1$, $r=-1$

$\boxed{6}$ $1+x+x^2+x^3+\cdots+x^k+\cdots$, $a=1$, $r=x$.

Ex ②: In each part, determine whether the series Converges, and if so find its sum.

ⓐ $\sum_{k=0}^{\infty} \frac{5}{4^k}$

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^k} + \cdots$$

is a geometric series with $a=5$, $r=\frac{1}{4}$.

Since $|r| = |\frac{1}{4}| = \frac{1}{4} < 1$,

Then, the given series is convergent and the sum is :
$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{5}{\frac{3}{4}} = 5 \cdot \frac{4}{3} = \frac{20}{3} .$$

#

ⓑ $\sum_{k=1}^{\infty} 3^{2k}\, 5^{1-k}$

$$\sum_{k=1}^{\infty} 3^{2k}\, 5^{1-k} = \sum_{k=1}^{\infty} \frac{(3^2)^k}{5^{-(1-k)}} = \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} \frac{9 \cdot 9^k \cdot 9^{-1}}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} 9 \, \frac{9^{k-1}}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} 9 \left(\frac{9}{5}\right)^{k-1}$$

The given series is geometric with $a=9$ and

$r = \frac{9}{5}$, since $|r| = \left|\frac{9}{5}\right| > 1$,

Then the given series is divergent.

ⓒ $\sum_{k=0}^{\infty} x^k$

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$$

This series is a geometric series with

$a = 1$, $r = x$.

So, it converges if $|x| < 1$ and diverges if $|x| \geq 1$

When the series converges its sum is:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

#

Ex ③: Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$$

Converges or diverges. If it converges find the sum.

<u>solution:</u> The nth partial sum of the series is:

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots + \frac{1}{n(n+1)}.$$

To calculate $\lim_{n\to\infty} S_n$ we will write S_n in closed. form.

Using the method of partial fraction:

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} \longrightarrow ①$$

$$1 = A(k+1) + Bk$$

$$1 = Ak + A + Bk.$$

Comparing:

k^1 : $0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B=-1}$

k^0 : $1 = A$ $\boxed{A=1}$

Then :

From ①: $$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1}$$

$$= 1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}}$$

$$- \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cdots - \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} .$$

Then :

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{\infty} = 1 - 0 = 1$$

#

* Harmonic series :

is a divergent series of the form :

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

#

H.w : Exc (10.3)

(3+7) p. 649 + 650

Convergence tests:

1 The divergence test:

Theorem (The divergence test):–

(a) If $\lim\limits_{k\to\infty} u_k \neq 0$, then the series $\sum u_k$ diverges.

(b) If $\lim\limits_{k\to\infty} u_k = 0$, then the series $\sum u_k$ may either converge or diverge.

Theorem:

If the series $\sum u_k$ converges, then $\lim\limits_{k\to\infty} u_k = 0$

Ex 1: Determine whether the series $\sum\limits_{k=1}^{\infty} \frac{k}{k+1}$ converges or diverges.

Using the divergence test:

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{\frac{k}{k}}{\frac{k}{k}+\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}}$$

$$= \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1 \neq 0$$

$\therefore$ Then, the given series is divergent.

#

Algebraic properties of infinite series :–

Theorem :

(a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum (u_k + v_k)$ and $\sum (u_k - v_k)$ are convergent series and the sums of these series are related by:

$$\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) If c is non zero constant, then the series $\sum u_k$ and $\sum c u_k$ both converge or both diverge. In the case of convergence, the sums are related by:

$$\sum_{k=1}^{\infty} c u_k = c \sum_{k=1}^{\infty} u_k$$

(c) Convergence or divergence is unaffected by deleting a finite number of terms from a series, in particular, for any positive integer K, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

Ex ②: Find the sum of the series

$$\sum_{k=1}^{\infty} \left(\frac{3}{4^k} - \frac{2}{5^{k-1}} \right)$$

solution:

$$\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is ageometric series with $a = \frac{3}{4}$ and $r = \frac{1}{4}$,

since $|r| = \left|\frac{1}{4}\right| = \frac{1}{4} < 1$, then $\sum_{k=1}^{\infty} \frac{3}{4^k}$ is Convergent.

$$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

is ageometric Series with $a = 2$ and $r = \frac{1}{5}$,

since $|r| = \frac{1}{5} < 1$, then $\sum_{k=1}^{\infty} \frac{2}{5^{k-1}}$ is Convergent series.

Thus:

$$\sum_{k=1}^{\infty} \left(\frac{3}{4^k} - \frac{2}{5^{k-1}} \right) = \sum_{k=1}^{\infty} \frac{3}{4^k} - \sum_{k=1}^{\infty} \frac{2}{5^{k-1}}$$

$$= \frac{\frac{3}{4}}{1 - \frac{1}{4}} - \frac{2}{1 - \frac{1}{5}}$$

$$= \frac{\cancel{\frac{3}{4}}}{\cancel{\frac{3}{4}}} - \frac{2}{\frac{4}{5}}$$

$$= 1 - \cancel{2} \cdot \frac{5}{\cancel{4}\,2}$$

$$= 1 - \frac{5}{2} = \frac{-3}{2}.$$

#

Ex ③: Determine whether the following series converge or diverge.

ⓐ $\sum_{k=1}^{\infty} \frac{5}{k} = 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \cdots$

$= \sum_{k=1}^{\infty} 5 \cdot \frac{1}{k}$ is a divergent series because

$\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent harmonic series.

ⓑ $\sum_{k=10}^{\infty} \frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$

is a divergent series because the series

$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent harmonic series.

$\boxed{2}$ The integral test:

<u>Theorem</u> (The integral test): Let $\sum u_k$ be a series with positive terms, If f is a function that is decreasing and continuous on an interval $[a, +\infty)$ and such that $u_k = f(k)$ for all $k \geq a$, then

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{+\infty} f(x)\,dx$$

both converge or diverge.

Ex ④: Use the integral test to determine whether the following series converge or diverge.

ⓐ $\sum_{k=1}^{\infty} \frac{1}{k}$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to+\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \big[\ln b - \ln 1\big]$$

$$= \ln\infty = +\infty.$$

The integral diverges, Then the given series is divergent.

ⓑ $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{b} + 1\right]$$

$$= \frac{-1}{\infty} + 1 = 0 + 1 = 1$$

The integral converges, then the given series converges.

#

3 p-series:

A p-series is an infinite series of the form:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{k^p} + \cdots$$

where $p > 0$.

Examples of p-series are:

① $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} + \cdots, \quad p = 1$

② $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{k^2} + \cdots, \quad p = 2$

③ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{k}} + \cdots, \quad p = \frac{1}{2}.$

Theorem (Convergence of p-series):

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{k^p} + \cdots$$

Converges if $p > 1$ and diverges if $0 < p \leq 1$.

Ex ⑤: Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

This series is a p-series with $p = \frac{1}{3} < 1$,

Therefore, it is divergent series. #

H.w (Exc 10.4): (5+9) P. 657 + 658.

؟؟

The Comparison, ratio and root tests:

[1] The Comparison test:

Theorem (the Comparison test): —

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms and suppose that:

$$a_1 \leq b_1 \,,\; a_2 \leq b_2 \,,\; a_3 \leq b_3 \,,\; \dots \,,\; a_k \leq b_k \,,\; \dots$$

ⓐ If the bigger series $\sum b_k$ Converges, then the smaller series $\sum a_k$ also Converges.

ⓑ If the smaller series $\sum a_k$ diverges, then the bigger series $\sum b_k$ diverges.

Informal principle:

[1] Constant term in the denominator of u_k Can usually be deleted without affecting the Convergence or divergence of the series.

[2] If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the Leading term in the polynomial can usually be discarded without affecting the Convergence or divergence of the Series.

Ex ①: Use the Comparison test to determine whether the following series Converge or diverge.

ⓐ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}} = \sum_{k=1}^{\infty} a_k$

According to principle $\boxed{I}$:

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (is a p-series),

with $p = \frac{1}{2} < 1$, then $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a divergent series.

since, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}} > \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}} = 2 + 1.09 + 0.81 + \dots$ (bigger series)

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + 0.71 + 0.58 + \dots$ (smaller series)

$$\underbrace{\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}}_{\text{bigger series}} > \underbrace{\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} b_k}_{\text{smaller series}}$$

is divergent $\Longleftarrow$ is divergent

Then, the given series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$ is a divergent series.

$\boxed{2}$ $$\sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \sum_{k=1}^{\infty} a_k$$

According to principle $\boxed{2}$:

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}$$ is a p-series

with $p = 2 > 1$, then $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ is Convergent.

$$\sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \cdots \text{ (smaller series)}$$

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} + \frac{1}{8} + \frac{1}{18} + \cdots \text{ (bigger series)}$$

Then,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2k^2+k} < \sum_{k=1}^{\infty} \frac{1}{2k^2} = \sum_{k=1}^{\infty} b_k$$

smaller series is convergent $\Longleftarrow$ bigger series is Convergent

Then, the given series $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ is

a Convergent series.

[2] The limit comparison test:

Theorem (the limit comparison test):—

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that:

$$\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$$

If ρ is finite and $\rho > 0$, then the series both converge or diverge.

Ex②: Use the limit comparison test to determine whether the following series converge or diverge.

ⓐ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1} = \sum_{k=1}^{\infty} a_k$

According to priciple $\boxed{1}$: $\displaystyle\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$$\rho = \lim_{k \to \infty} \frac{a_k}{b_k}, \quad a_k = \frac{1}{\sqrt{k}+1}, \quad b_k = \frac{1}{\sqrt{k}}$$

$$= \lim_{k \to \infty} \frac{\frac{1}{\sqrt{k}+1}}{\frac{1}{\sqrt{k}}} = \lim_{k \to \infty} \left[\frac{1}{\sqrt{k}+1} \cdot \frac{\sqrt{k}}{1}\right]$$

$$= \lim_{k \to \infty} \frac{\sqrt{k}}{\sqrt{k}+1} = \frac{1}{1} = 1 > 0$$

Then, both series converge or diverge.

since, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is p - series with $p = \frac{1}{2} < 1$, then: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a divergent series.

Therefore, The given series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ is divergent.

#

(b) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \sum_{k=1}^{\infty} a_k$

According principle $\boxed{2}$: $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2k^2}$,

$$p = \lim_{k\to\infty} \frac{a_k}{b_k}, \quad a_k = \frac{1}{2k^2+k}, \quad b_k = \frac{1}{2k^2}.$$

$$= \lim_{k\to\infty} \frac{\frac{1}{2k^2+k}}{\frac{1}{2k^2}} = \lim_{k\to\infty} \left[\frac{1}{2k^2+k} \cdot \frac{2k^2}{1}\right]$$

$$= \lim_{k\to\infty} \left[\frac{2k^2}{2k^2+k}\right] = \frac{2}{2} = 1 > 0$$

Then, both series Converge or diverge.

since, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2}$ is Convergent,

because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$.

Therefore, The given series $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ is a convergent series.

#

(c) $\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2} = \sum_{k=1}^{\infty} a_k$

According principle $\boxed{2}$: $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4}$,

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}, \quad a_k = \frac{3k^3-2k^2+4}{k^7-k^3+2}, \quad b_k = \frac{3}{k^4}$$

$$= \lim_{k\to\infty} \frac{\frac{3k^3-2k^2+4}{k^7-k^3+2}}{\frac{3}{k^4}} = \lim_{k\to\infty}\left[\frac{3k^3-2k^2+4}{k^7-k^3+2}\cdot\frac{k^4}{3}\right]$$

$$= \lim_{k\to\infty} \frac{3k^7-2k^6+4k^4}{3k^7-3k^3+6}$$

$$= \frac{3}{3} = 1 > 0.$$

Then, both series converge or diverge.

since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{3}{k^4} = 3\sum_{k=1}^{\infty} \frac{1}{k^4}$ is Convergent,

because $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is a p-series with $p=4>1$ (convergent)

Therefore, the given series $\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$

is Convergent.

#

3 The ratio test:

Theorem (the ratio test) :-

Let $\sum u_k$ be a series with positive terms and suppose that:

$$\rho = \lim_{k \to \infty} \frac{u_{k+1}}{u_k}$$

1. If $\rho < 1$, the series Converges.
2. If $\rho > 1$ or $\rho = +\infty$, the series diverges.
3. If $\rho = 1$, the series may converges or diverges, So that another test must be tried.

Ex ③: Use the ratio test to determine whether the following series Converge or diverge.

ⓐ $\sum_{k=1}^{\infty} \frac{1}{k!}$

$u_k = \frac{1}{k!}$, $u_{k+1} = \frac{1}{(k+1)!}$

$$\rho = \lim_{k \to \infty} \frac{u_{k+1}}{u_k} = \lim_{k \to \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}$$

$$= \lim_{k \to \infty} \left[\frac{1}{(k+1)!} \div \frac{k!}{1} \right]$$

$$= \lim_{k \to \infty} \left[\frac{k!}{(k+1)!} \right] = \lim_{k \to \infty} \frac{\cancel{k!}}{(k+1)\cancel{k!}}$$

Then, the given Series is Convergent.

$$= \lim_{k \to \infty} \frac{1}{k+1} = \frac{1}{\infty} = 0 < 1$$

(b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

$$u_k = \frac{k}{2^k} \quad , \quad u_{k+1} = \frac{k+1}{2^{k+1}}$$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k\to\infty} \left[\frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} \right] = \lim_{k\to\infty} \left[\frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} \right]$$

$$= \lim_{k\to\infty} \left[\frac{k+1}{\cancel{2^k} \cdot 2^1} \cdot \frac{\cancel{2^k}}{k} \right]$$

$$= \lim_{k\to\infty} \left[\frac{k+1}{2k} \right] = \frac{1}{2} < 1$$

Then, the given series is Convergent

(c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

$$u_k = \frac{k^k}{k!} \quad , \quad u_{k+1} = \frac{(k+1)^{k+1}}{(k+1)!}$$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k\to\infty} \left[\frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} \right]$$

$$= \lim_{k\to\infty} \left[\frac{(k+1)^k \cdot \cancel{(k+1)}^1}{\cancel{(k+1)}\,\cancel{k!}} \cdot \frac{\cancel{k!}}{k^k} \right]$$

$$= \lim_{k\to\infty} \left[\frac{(k+1)^k}{k^k} \right]$$

$$P = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k$$

Since, $\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{\infty} = \infty^{\infty}$,

$$= \lim_{k\to\infty} e^{\ln\left(\frac{k+1}{k}\right)^k}$$

$$= \lim_{k\to\infty} e^{k\ln\left(\frac{k+1}{k}\right)}$$

$$= \lim_{k\to\infty} e^{k\ln\left(1+\frac{1}{k}\right)}$$

$$= \lim_{k\to\infty} e^{\frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}}}$$

$$= \lim_{k\to\infty} e^{\frac{\left(\frac{\cancel{\frac{-1}{k^2}}}{1+\frac{1}{k}}\right)}{\cancel{\frac{-1}{k^2}}}}$$, (Using LHopital rule)

$$= \lim_{k\to\infty} e^{\left(\frac{1}{1+\frac{1}{k}}\right)}$$

$$= e^{\frac{1}{1+\frac{1}{\infty}}} = e^{\frac{1}{1+0}} = e^1 \approx 2.7 > 1$$

Then, the given series is divergent.

#

ⓓ $\displaystyle\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$

$$u_k = \frac{(2k)!}{4^k}, \quad u_{k+1} = \frac{[2(k+1)]!}{4^{k+1}} = \frac{(2k+2)!}{4^{k+1}}$$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \left[\frac{(2k+2)!}{4^{k+1}} \cdot \frac{4^k}{(2k)!}\right]$$

$$= \lim_{k\to\infty} \left[\frac{(2k+2).(2k+1).\cancel{(2k)!}}{\cancel{4^k}.4^1} \cdot \frac{\cancel{4^k}}{\cancel{(2k)!}}\right]$$

$$= \lim_{k\to\infty} \left[\frac{(2k+2)(2k+1)}{4}\right]$$

$$= \frac{1}{4} \lim_{k\to\infty} (2k+2)(2k+1)$$

$$= \frac{1}{4} \left[(2(\infty)+2)(2(\infty)+1)\right]$$

$$= \frac{1}{4}(\infty) = \infty .$$

Then, the given series is divergent.

ⓔ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}$

$$u_k = \frac{1}{2k-1}, \quad u_{k+1} = \frac{1}{2(k+1)-1} = \frac{1}{2k+2-1} = \frac{1}{2k+1}$$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \left[\frac{1}{2k+1} \cdot \frac{2k-1}{1}\right]$$

$$= \lim_{k\to\infty} \left[\frac{2k-1}{2k+1}\right]$$

$$= \frac{2}{2} = 1$$

Then, the ratio test is of no help.

We use the integral test:

$$\int_1^{\infty} \frac{1}{2x-1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{2x-1}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_1^b \frac{2}{2x-1}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\ln(2x-1)\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\ln(2b-1) - \underbrace{\ln(1)}_{=0}\right]$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\ln(2b-1)\right]$$

$$= \frac{1}{2}\ln(2\cdot\infty - 1)$$

$$= \frac{1}{2}\ln \infty)$$

$$= \infty .$$

Then, the given series is divergent . #

$\boxed{4}$ The root test:

Theorem (The root test): Let $\sum u_k$ be a series with positive terms and suppose that:

$$\rho = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} (u_k)^{\frac{1}{k}} .$$

$\boxed{1}$ If $\rho < 1$, the series Converges.

$\boxed{2}$ If $\rho > 1$ or $\rho = +\infty$, the series diverges.

$\boxed{3}$ If $\rho = 1$, the series may converges or diverges, So that an other test must be tried.

Ex (4): Use the root test to determine whether the following series converges or diverges.

(a) $\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$

$u_k = \left(\frac{4k-5}{2k+1}\right)^k$,

$\rho = \lim_{k\to\infty} [u_k]^{\frac{1}{k}} = \lim_{k\to\infty} \left[\left(\frac{4k-5}{2k+1}\right)^k\right]^{\frac{1}{k}}$

$= \lim_{k\to\infty} \left(\frac{4k-5}{2k+1}\right)$

$= \frac{4}{2} = 2 > 1$

Then, the given series is divergent.

(b) $\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$

$\left\{ u_k = \frac{1}{(\ln(k+1))^k} = \left(\frac{1}{\ln(k+1)}\right)^k \right.$

$\rho = \lim_{k\to\infty} (u_k)^{\frac{1}{k}}$

$= \lim_{k+1} \left[\left(\frac{1}{\ln(k+1)}\right)^k\right]^{\frac{1}{k}}$

$= \lim_{k\to\infty} \frac{1}{\ln(k+1)} = \frac{1}{\ln(\infty)} = \frac{1}{\infty} = 0 < 1$,

Then, the given series is convergent.

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H.w : Exc (10.5),

(3+ 5+11) : P. 664

Maclaurin and Taylor polynomials.

* The Local linear approximation of a function $f(x)$ at $x = x_0$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

* The local linear approximation of a function $f(x)$ at $x = 0$

$$f(x) \approx f(0) + f'(0)x.$$

* The Quadratic approximation of a function $f(x)$ at $x = x_0$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

* The Quadratic approximation of a function $f(x)$ at $x = 0$

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

Ex ①: Find the Local linear and quadratic approximation of e^x at $x = 0$.

* local linear approximation of e^x at $x = 0$:

$$f(x) \approx f(0) + f'(0)X.$$

$$e^x \approx 1 + X$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$
$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$
$f''(x) = e^x \Rightarrow f''(0) = 1$

* Local quadratic approximation of e^x at $x = 0$:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

$$e^x \approx 1 + x + \frac{1}{2}x^2$$

#

Maclaurin polynomial:

Definition: If f can be differentiated n times at $\underline{\underline{0}}$ (zero), then we define nth Maclaurin Polynomial for f to be,

$$\boxed{P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n}$$

Ex②: Find the Maclaurin polynomial P_0, P_1, P_2, P_3 and P_n for e^x.

Solution:

$f(x) = e^x$ $\qquad$ $f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad$ $f'(0) = e^0 = 1$

$f''(x) = e^x$ $\qquad$ $f''(0) = e^0 = 1$

$f'''(x) = e^x$ $\qquad$ $f'''(0) = e^0 = 1$

$\vdots$ $\qquad$ $\vdots$

$f^{(n)}(x) = e^x$ $\qquad$ $f^{(n)}(0) = e^0 = 1$

Therefore,

$P_0(x) = f(0) = 1$

$P_1(x) = f(0) + f'(0)\,x$

$P_1(x) = 1 + x$.

$$P_2(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2$$

$$P_2(x) = 1 + x + \frac{1}{2!}\,x^2.$$

$$P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3.$$

$$P_3(x) = 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3.$$

$$\vdots$$

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n$$

$$P_n(x) = 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n.$$

\#

Taylor polynomial:

Definition: If f can be differentiated n times at x_0, then we define the nth Taylor polynomial for f about $x = x_0$ to be:

$$\boxed{P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}$$

Ex ③: Find the first four Taylor polynomial for $\ln x$ about $x = 2$.

Solution:

$f(x) = \ln x$ $\qquad$ $f(2) = \ln 2$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(2) = \frac{1}{2}$.

$f''(x) = \frac{-1}{x^2}$ $\qquad$ $f''(2) = \frac{-1}{4}$.

$f'''(x) = \frac{0 \cdot x^2 - (-1)2x}{x^4}$ $\qquad$ $f'''(2) = \frac{2}{8} = \frac{1}{4}$.

$= \frac{2x}{x^4} = \frac{2}{x^3}$

Then,

$P_0(x) = f(2) = \ln 2$

$P_1(x) = f(2) + f'(2)(x-2)$

$= \ln 2 + \frac{1}{2}(x-2)$

$P_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$

$= \ln 2 + \frac{1}{2}(x-2) + \frac{\frac{-1}{4}}{2}(x-2)^2$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$

$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{\frac{1}{4}}{6}(x-2)^3$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$ #

Sigma notation for Taylor and Maclaurin polynomials:—

We can write the nth Maclaurin polynomial for f(x) a

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Also, we can write the nth Taylor polynomial for f(x) at x_0:

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

Ex ④: Find nth Maclaurin polynomials for

ⓐ $\sin x$

Solution:

$f(x) = \sin x$	$f(0) = \sin(0) = 0$
$f'(x) = \cos x$	$f'(0) = \cos(0) = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos(0) = -1$
$f''''(x) = \sin x$	$f''''(0) = \sin(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = \cos(0) = 1$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = -\sin(0) = 0$
$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -\cos(0) = -1$

Then,

$$P_0(x) = f(0) = 0$$

$$P_1(x) = f(0) + \underbrace{f'(0)x} = 0 + 1 \cdot x = 0 + x$$

$$P_2(x) = f(0) + f'(0)x + \underbrace{\frac{f''(0)}{2!}} x^2 = 0 + x + \frac{0}{2!}x^2 = 0 + x + 0$$

$$P_3(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + x + 0 - \frac{1}{3!}x^3$$

$$P_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + \frac{f^{(4)}(0)x^4}{4!} = 0 + x + 0 - \frac{1}{3!}x^3 + 0.$$

$$P_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{f^{(5)}(x_0)}{5!}x^5 = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5$$

$$P_6(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6$$

$$= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0$$

$$P_7(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7.$$

$\Longrightarrow$ The even terms $= 0$.

Therefore, the Maclaurin polynomial for $\sin x$ is:

$$P_{2k+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad (k = 0, 1, 2, \ldots)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

-- #

(b) $\cos x$

Solution:

$f(x) = \cos x$ $\qquad$ $f(0) = \cos(0) = 1$

$f'(x) = -\sin x$ $\qquad$ $f'(0) = -\sin(0) = 0$

$f''(x) = -\cos x$ $\qquad$ $f''(0) = -\cos(0) = -1$

$f'''(x) = \sin x$ $\qquad$ $f'''(0) = \sin(0) = 0$

$f^{(4)}(x) = \cos x$ $\qquad$ $f^{(4)}(0) = \cos(0) = 1$

$f^{(5)}(x) = -\sin x$ $\qquad$ $f^{(5)}(0) = -\sin(0) = 0$

$f^{(6)}(x) = -\cos x$ $\qquad$ $f^{(6)}(0) = -\cos(0) = -1$

$f^{(7)}(x) = \sin x$ $\qquad$ $f^{(7)}(0) = \sin(0) = 0$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)x = 1 + 0 \cdot x = 1 + 0$$

$$P_2(x) = 1 + 0 + \frac{f''(0)}{2!}x^2 = 1 + 0 - \frac{1}{2!}x^2.$$

$$P_3(x) = 1 + 0 - \frac{1}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + 0 - \frac{1}{2!}x^2 + 0$$

$$P_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{f^{(4)}(0)}{4!}x^4 = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4$$

$$P_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0$$

$$P_6(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 - \frac{1}{6!}x^6$$

$$P_7(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 - \frac{1}{6!}x^6 + 0.$$

The odd terms $= 0$

$$P_{2k}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^k x^{2k}}{(2k)!}, \ (k=0,1,2,\ldots$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

#

(c) $\dfrac{1}{1-x}$.

solution:

$$f(x) = \frac{1}{1-x} \qquad f(0) = \frac{1}{1-0} = 1 = 0!$$

$$f'(x) = \frac{\cancel{0(1-x)} - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \qquad f'(0) = \frac{1}{(1-0)^2} = 1 = 1!$$

$$f''(x) = \frac{\cancel{0\cdot(1-x)^2} - 1.[2(1-x)]}{(1-x)^4}$$

$$= \frac{-2\cancel{(1-x)}}{\cancel{(1-x)^4}} = \frac{2}{(1-x)^3} \qquad f''(0) = \frac{2}{(1-0)^3} = 2 = 2!$$

$$f'''(x) = \frac{\cancel{0\cdot(1-x)^3} - (2).3(1-x)^2(-1)}{(1-x)^6}$$

$$= \frac{6(1-x)^2}{(1-x)^6} = \frac{6}{(1-x)^4} \qquad f'''(0) = \frac{6}{(1-0)^4} = 6 = 3!$$

$$f^{(4)}(x) = \frac{\cancel{0(1-x)^4} - 6.4(1-x)^3(-1)}{(1-x)^8}$$

$$= \frac{24(1-x)^3}{(1-x)^8} = \frac{24}{(1-x)^5} \qquad f^{(4)}(0) = \frac{24}{(1-0)^5} = 24 = 4!$$

$\vdots$

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}} \qquad f^{(k)}(0) = k!$$

Then, the Maclaurin polynomial for $f(x) = \frac{1}{1-x}$ is:

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

$$= 1 + x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{n!}{n!}x^n.$$

$$= 1 + x + x^2 + x^3 + \cdots + x^n$$

$$= \sum_{k=0}^{n} x^k.$$

#

Ex ⑤: Find the nth Taylor polynomial for $\frac{1}{x}$ about $x=1$

$f(x) = \frac{1}{x}$ $\qquad f(1) = 1$

$f'(x) = \frac{-1}{x^2}$ $\qquad f'(1) = -1$

$f''(x) = \frac{0\cdot x^2 - (-1)\cdot 2x}{x^4} = \frac{2}{x^3}$ $\qquad f''(1) = 2$

$f'''(x) = \frac{0\cdot x^3 - 2\cdot 3x^2}{x^6} = \frac{-6}{x^4}$ $\qquad f'''(1) = -6$

$f^{(4)}(x) = \frac{0\cdot x^4 - (-6)\cdot 4x^3}{x^8} = \frac{24}{x^5}$ $\qquad f^{(4)}(1) = 24.$

$\vdots$

$f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}$ $\qquad f^{(k)}(1) = (-1)^k k!.$

Then,

The Taylor polynomial for $f(x) = \frac{1}{1-x}$ about $x = 1$:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= 1 - (x-1) + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n$$

$$= \sum_{k=0}^{n} (-1)^k (x-1)^k .$$

#

H.w: Exc (10.7):

(11 + 19), p. 684