CHAPTER 2

TECHNIQUES OF INTEGRATION.

Floating dish at the King Saud University Sports Stadium complex is an ellipse. Find the length of the ellipse bracket
Involves the integration techniques provided in this chapter.

In this section we will give a brief overview of methods for evaluating integrals, and we will review the integration formulas that were discussed in earlier sections.

We will discuss and provide a more systematic procedure for attacking unfamiliar integrals. We will talk more about Integration by Using Trigonometric Identities and Substitutions, Integration by Parts, Integration of Rational and Irrational Functions and Integration by Partial Fractions.

2.1 INTEGRATION BY USING TRIGONOMETRIC IDENTITIES AND SUBSTITUTIONS.

2.1.1 Integrating Products of Sines and Cosines.

Two special substitution rules are useful in a few simple cases:

1. For $\int \sin^m x \cos^n x \, dx$, where m and n are positive integers.

Evaluating the previous integral depending on weather m and n are odd or even.

In fact, if n is odd,
- Split off a factor of cos x.
- Apply the relevant identity $cos^2 x = 1 - sin^2 x$
- Make the substitution $u = \sin x$.

If m is odd,
- Split off a factor of sin x.
- Apply the relevant identity. $\sin^2 x = 1 - cos^2 x$

• Make the substitution $u = \cos x$.

If m and n both are even , then • Use the relevant identities to reduce the powers on sin x and cos x.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

2. For $\int \tan^m x.\sec^n x\,dx$, where m and n are positive integers.

If n is even, • Split off a factor of sec2 x.
• Apply the relevant identity. $\sec^2 x = 1 + tan^2 x$
• Make the substitution $u = \tan x$.

If m is odd, • Split off a factor of sec x tan x.
• Apply the relevant identity. $\tan^2 x = sec^2 x - 1$
• Make the substitution $u = \sec x$.

If m even and n odd,
• Use the relevant identities to reduce the integrand to powers of sec x alone. Then use the reduction formula for powers of sec x

$$\tan^2 x = sec^2 x - 1$$

Example 1. Evaluate:
a) $\int \sin^2 x \cos^3 x\,dx$, **b)** $\int \sin^5 x \cos^4 x\,dx$

Solution:

a) We note that $m = 2$ is even, $n = 3$ is odd:

$$\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cos x\,dx = \int \sin^2 x\left(1 - \sin^2 x\right)\cos x\,dx$$

Substituting u=sin x , du= cos x dx we get:

$$\int \sin^2 x \cos^3 x\,dx = \int u^2\left(1 - u^2\right)du = \int\left(u^2 - u^4\right)du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + c$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$$

b) We note that $m = 5$ is odd, $n = 4$ is even

$$\int \sin^5 x \cos^4 x\, dx = \int \sin^4 x \cos^4 x \sin x\, dx = \int \left(1-\cos^2 x\right)^2 \cos^4 x \sin x\, dx$$
$$= \int \left(\cos^4 x - 2\cos^6 + \cos^8 x\right) \sin x\, dx$$

Substituting u=cos x , du= -sin x dx we get:

$$\int \sin^5 x \cos^4 x\, dx = -\int (u^4 - 2u^6 + u^8) du = -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + c$$
$$= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 + c$$

Examples2. Evaluate:

a) $\int \tan^2 x \sec^4 x\, dx$, **b)** $\int \tan^3 x \sec^3 x\, dx$

Solution: **a)** Note that $n = 4$ is even, the:

$$\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x \sec^2 x\, dx = \int \tan^2 x(\tan^2 x + 1)\sec^2 x\, dx$$

Substituting u= tan x , du= $\sec^2 x\, dx$ we get:

$$\int \tan^2 x \sec^4 x\, dx = \int \left(u^2 + 1\right) u^2 du = \frac{1}{5}u^5 + \frac{1}{3}u^3 + c = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c$$

b) Note that $m = 3$ is odd, we get:

$$\int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x(\sec x \tan x) dx$$
$$= \int (\sec^2 x - 1)\sec^2 x(\sec x \tan x) dx$$

Substituting $u = \sec x$, $du = \sec x \tan x dx$ we have:

$$\int \tan^3 x \sec^3 x\, dx = \int \left(u^2 - 1\right) u^2 du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + c = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c$$

2.1.2 Integrating Powers of Sine and Cosine.

Let's evaluate the two general integrals by the reduction formulas.

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx \qquad (1)$$

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx \qquad (2).$$

Example 3. Evaluate: **a)** $\int \sin^2 x\, dx$, **b)** $\int \cos^3 x\, dx$

Solution: **a**) Using (1), we get:

$$\int \sin^2 x dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int x dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + c$$

b) Using (2), we get:

$$\int \cos^3 x = \frac{1}{3}\cos^2 x\sin x\, dx + \frac{2}{3}\int \cos x\, dx = \frac{1}{3}\cos^2 x\sin x\, dx + \frac{2}{3}\sin x + c$$

$$= \frac{1}{3}\left(1 - \sin^2 x\right)\sin x + \frac{2}{3}\sin x + c = \sin x - \frac{1}{3}\sin^3 x + c$$

2.1.3 Integrating Powers of Tangent and Secant

The procedures for integrating powers of tangent and secant closely parallel those for sine and cosine. The idea is to use the following reduction formulas to reduce the exponent in the integrand until the resulting integral can be evaluated:

$$\int tan^n\, xdx = \frac{tan^{n-1}x}{n-1} - \int tan^{n-2}\, xdx \qquad (3)$$

$$\int sec^n\, xdx = \frac{sec^{n-2}x\, tan\, x}{n-1} + \frac{n-2}{n-1}\int sec^{n-2}\, xdx \qquad (4)$$

In the case where n is odd, the exponent can be reduced to 1, leaving us with the problem of integrating tan x or sec x. These integrals are given by

$$\int tan\, xdx = \ln|\sec x| + C \qquad (5)$$

$$\int sec\, xdx = \ln|\sec x + \tan x| + C \qquad (6)$$

Precedents Formula can be obtained by writing:

$$\int tan\, xdx = \int \frac{\sin x}{\cos x}\, dx = -\ln|cosx| + c = ln|secx| + c$$

Considering :

$$u = cosx\, , du = -sinx\, dx \quad \text{and}$$

$$\int sec\,xdx = \int secx\left(\frac{secx + tanx}{secx + tanx}\right)dx = \int \frac{sec^2x + secx\,tanx}{secx + tanx}$$
$$= \ln|secx + tanx| + c$$

Considering : $u = secx + tanx\,, du = (sec^2x + secx\,tanx)\,dx$

The following basic integrals occur frequently and are worth noting:

$$\int tan^2xdx = tanx - x + c \qquad (7)$$
$$\int sec^2\,xdx = \tan x + c \qquad (8)$$

Formula (8) is already known to us, since the derivative of tan x is sec^2x, formula (7)can be obtained by applying reduction formula (1) with $n = 2$ (verify) or, alternatively,
by using the identity

$$1 + tan^2x = sec^2x$$

to write

$$\int tan^2xdx = \int (sec^2\,x - 1)dx = tanx - x + c \qquad (9)$$

The formulas

$$\int tan^3\,xdx = \frac{1}{2}tan^2x - \ln|secx| + c \qquad (10)$$

$$\int sec^3\,xdx = \frac{1}{2}\quad secx\,tanx + \frac{1}{2}\ \ln|secx + tanx| + c \qquad (11)$$

2.1.4 Integration by using trigonometric identities.

When the integrand involves some trigonometric functions, we make use of known trigonometric identities to evaluate the integral. We classify these integrals as follows:

2.1.4 (A) Integrals of the type $\int \sin^m x\,dx$, $\int \cos^m x\,dx$ **and** $\int \sin^m x \cos^m xdx$ where m is a positive integer less than or equal to 4.

To evaluate these type of integrals we will make use of the following trigonometric identities:

(i) $\sin^2 A = \dfrac{1-\cos 2A}{2}$

(ii) $\cos^2 A = \dfrac{1+\cos 2A}{2}$

(iii) $\sin 3A = 3\sin A - 4\sin^3 A$ or $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

(iv) $\cos 3A = 4\cos^3 A - 3\cos A$ or $\cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

Example 4. Evaluate $\int \sin^3 x \cos^3 x \, dx$.

Solution: $\sin^3 x \cos^3 x = \dfrac{1}{8}(8\sin^3 x \cos^3 x)$

$= \dfrac{1}{8}(2\sin x \cos x)^3 = \dfrac{1}{8}(\sin 2x)^3 = \dfrac{1}{8}\sin^3 2x$ (Using the identity $\sin 2A = 2\sin A \cos A$) $= \dfrac{1}{8}\left(\dfrac{3\sin 2x - \sin 6x}{4}\right)$ (Using identity (iii) above with $A = 2x$)

Therefore

$\int \sin^3 x \cos^3 x \, dx = \dfrac{1}{32}\int (3\sin 2x - \sin 6x)dx = \dfrac{3}{32}\int \sin 2x \, dx - \dfrac{1}{32}\int \sin 6x \, dx$

$= \dfrac{3}{32}\left(\dfrac{-\cos 2x}{2}\right) - \dfrac{1}{32}\left(\dfrac{-\cos 6x}{6}\right) + C$ (Here we have made use of theorem 2.1.1. Since $\int \sin x \, dx = -\cos x$, using theorem 2.1.1 we have $\int \sin 2x \, dx = \dfrac{-\cos 2x}{2}$). $= \dfrac{-3}{64}\cos 2x + \dfrac{1}{192}\cos 6x + C$.

Example 5. $\int \sin^4 x \cos^4 x \, dx$

Solution: $\sin^4 x \cos^4 x = \dfrac{1}{16}(16\sin^4 x \cos^4 x) = \dfrac{1}{16}(2\sin x \cos x)^4$

$= \dfrac{1}{16}\left(\sin^2 2x\right)^2 = \dfrac{1}{16}\left(\dfrac{1-\cos 4x}{2}\right)^2$ (Using identity (i) with $A = 2x$)

$= \dfrac{1}{64}(1 - 2\cos 4x + \cos^2 4x) = \dfrac{1}{64}\left(1 - 2\cos 4x + \dfrac{1+\cos 8x}{2}\right)$ (Using identity (ii) with $A = 2x$) $= \dfrac{1}{128}\left(3 - 4\cos 4x + \cos 8x\right)$

Therefore

$$\int \sin^4 x \cos^4 x\, dx = \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x)\, dx =$$

$$\frac{3}{128}x - \frac{4}{128}.\frac{\sin 4x}{4} + \frac{1}{128}.\frac{\sin 8x}{8} + C \quad \text{(By Theorem 2.1.1)}$$

$$= \frac{1}{128}\left(3x - \sin 4x + \frac{\sin 8x}{8}\right) + C$$

2.1.4 (B) Integrals of the type $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$ **and** $\int \cos mx \cos nx\, dx$.

To evaluate these type of integrals we will make use of the following trigonometric identities:

(v) $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(vi) $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

(vii) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(viii) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

Example 6. Evaluate the following integrals:

a) $\int \sin 3x \cos 2x\, dx$

b) $\int \sin 3x \sin 2x\, dx$

c) $\int \cos 3x \cos 2x\, dx$

Solution:

a. Using the trigonometric rule (v), we get:

$$\int \sin 3x\, cos\, 2x\, d\,x = \frac{1}{2}\int (\sin 5x + \sin x)\, dx = \frac{-1}{10}\cos 5x - \frac{1}{2}\cos x + c$$

b. Using the trigonometric rule (viii), We get:

$$\int \sin 3x \sin 2x \, dx = \frac{1}{2}\int (\cos x - \cos 5x)dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + c$$

c. Using the trigonometric rule (vii) We get:

$$\int \cos 3x \cos 2x \, dx = \frac{1}{2}\int (\cos x + \cos 5x) dx = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + c$$

Example 7. Evaluate $\int \cos 2x \cos 4x \cos 6x \, dx$.

Solution: $\cos 2x \cos 4x \cos 6x = \frac{1}{2}(2\cos 2x \cos 4x)\cos 6x$

$= \frac{1}{2}(\cos 6x + \cos 2x)\cos 6x$ (Using identity (vii) above with $A = 2$, $B = 4$ and also $\cos(-\theta) = \cos\theta$)

$= \frac{1}{2}(\cos^2 6x + \cos 2x \cos 6x) = \frac{1}{4}(2\cos^2 6x + 2\cos 2x \cos 6x)$

$= \frac{1}{4}(1 + \cos 12x + \cos 8x + \cos 4x)$ (Using identities (ii) and (vii) above).

Therefore,

$\int \cos 2x \cos 4x \cos 6x \, dx = \frac{1}{4}\int (1 + \cos 12x + \cos 8x + \cos 4x)dx$

$= \frac{1}{4}\left(x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} \right) + C$.

2.1.4 (C) Integrals of the type $\int \tan^m x \sec^{2n} x \, dx$ and $\int \cot^m x \csc^{2n} x \, dx$

In the above integrals m denotes any non-negative integer and $2n$ denotes any positive even integer. Clearly $\int \sec^{2n} x \, dx$ and $\int \csc^{2n} x \, dx$ are special cases of the above integrals with $m = 0$. To evaluate these type of integrals we make use of the following :

Algorithm 1. To evaluate $\int \tan^m x \sec^{2n} x \, dx$

Step 1: Write the given integral as $I = \int \tan^m x \left(\sec^2 x\right)^{n-1} \sec^2 x \, dx$.

Step 2: Write the given integral as $I = \int \tan^m x\left(1+\tan^2 x\right)^{n-1} \sec^2 x\, dx$ (using the trigonometric identity $\sec^2 x - \tan^2 x = 1$)

Step 3: Put $\tan x = t$. Differentiating with respect to x we get $\sec^2 x\, dx = dt$. The integral in step 2 becomes $I = \int t^m (1+t^2)^{n-1}\, dt$.

Step 4: Expand $(1+t^2)^{n-1}$ of step 3 by binomial theorem and then integrate using the formula $\int t^n\, dt = \dfrac{t^{n+1}}{n+1}$.

Step 5: After integration in step 4, replace t with $\tan x$.

Note: A similar algorithm follows for $\int cot^m x. csc^{2n} x\ dx$ also.

Example 8. Evaluate $\int \cot^2 x \csc^4 x\, dx$.

Solution: We will follow the steps of Algorithm 1 as follows:

Let $I = \int \cot^2 x \csc^4 x\, dx$. Then

$I = \int \cot^2 x \csc^2 x \csc^2 x\, dx$ or $= \int \cot^2 x (1+\cot^2 x) \csc^2 x\, dx$

Put $\cot x = t$ and $\csc^2 x\, dx = dt$. Then $I = \int t^2 (1+t^2)\, dt = \int t^2 dt + \int t^4 dt$

$= \dfrac{t^3}{3} + \dfrac{t^5}{5} + C = \dfrac{\cot^3 x}{3} + \dfrac{\cot^5 x}{5} + C$

2.1.4 (D) Integrals of the type $\int \tan^{2m+1} x \sec^{2n+1} x\, dx$ and $\int \cot^{2m+1} x \csc^{2n+1} x\, dx$

In the above integrals the exponents of $\tan x$ and $\sec x$ are odd positive integer. To evaluate these type of integrals we make use of the following:

Algorithm 2 To evaluate $\int \tan^{2m+1} x \sec^{2n+1} x\, dx$

Step 1: Write the given integral as $I = \int (\tan^2 x)^m \left(\sec^2 x\right)^n \sec x \tan x\, dx$.

Step 2: Write the given integral as $I = \int (\sec^2 x - 1)^m \left(\sec^2 x\right)^n \sec x \tan x\, dx$ (using the trigonometric identity $\sec^2 x - \tan^2 x = 1$)

Step 3: Put $\sec x = t$. Differentiating with respect to x we get. $\sec x \tan x\, dx = dt$

The integral in step 2 becomes $I = \int (t^2 - 1)^m t^n\, dt$.

Step 4 : Expand $(t^2 - 1)^m$ of step 3 by binomial theorem and then integrate using the formula $\int t^n\, dt = \dfrac{t^{n+1}}{n+1}$.

Step 5: After integration in step 4, replace t with $\sec x$.

Note: A similar algorithm follows for $\int \cot^{2m+1} x \csc^{2n+1} x\, dx$ also.

Example 9. Evaluate $\int \tan^3 x \sec^3 x\, dx$

Solution: Let $I = \int \tan^3 x \sec^3 x\, dx$. Then $I = \int \tan^2 x \sec^2 x \sec x \tan x\, dx$ or $I = \int (\sec^2 x - 1)\sec^2 x \sec x \tan x\, dx$.

Now put $\sec x = t$, so that on differentiating with respect to x we get $\sec x \tan x\, dx = dt$. Then we have

$$I = \int (t^2 - 1)t^2\, dt = \int t^4\, dt - \int t^2\, dt = \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.$$

2.1.5 (E) Integrals of the type $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$.

These type of integrals can be reduced to any of the forms given in sections 2.2.1(c) or 2.2.1(d) by using the identities $\sec^2 x - \tan^2 x = 1$ and $\csc^2 x - \cot^2 x = 1$. The method will be clear from the following examples.

Example 10. Evaluate $\int \tan^3 x\, dx$.

Solution: Let $I = \int \tan^3 x\, dx$. Then

$$I = \int \tan^2 x \tan x\, dx = \int (\sec^2 x - 1)\tan x\, dx = \int \sec^2 x \tan x\, dx - \int \tan x\, dx.$$

Now, $\int \tan x\, dx = \ln|\cos x| + C_1$.

Example 11. Evaluate $\int \cot^6 x\, dx$.

Solution: Let $I = \int \cot^6 x\, dx = \int \cot^4 x \cot^2 x\, dx$

$= \int \cot^4 x(\csc^2 x - 1)\,dx = \int \cot^4 x \csc^2 x\,dx - \int \cot^4 x\,dx = I_1 - I_2$. ...(1)

To evaluate this integral put $\cot x = t$, so that $-\csc^2 x dx = dt$ or $\csc^2 x dx = -dt$ Then

$$I_1 = -\int t^4 dt = -\frac{t^5}{5} + C_1 = -\frac{cot^5 x}{5} + C_1. \quad \ldots \quad (2)$$

The integral $I_2 = \int \cot^4 x\,dx$ can be solved by repeating the above process as follows :

$I_2 = \int \cot^4 x\,dx = \int \cot^2 x \cot^2 x\,dx = \int \cot^2 x(\csc^2 x - 1)\,dx$

$= \int \cot^2 x\ \csc^2 x\,dx - \int \cot^2 x\,dx = \int \cot^2 x \csc^2 x\,dx - \int (\csc^2 x - 1)\,dx$

$= \int \cot^2 x \csc^2 x\,dx - \int (\csc^2 x\,dx + \int 1\,dx = I_3 + \cot x + x + C_2$

(since $\int \csc^2 x\,dx = -\cot x$).

Where $I_3 = \int \cot^2 x \csc^2 x\,dx$. To evaluate I_3, put $\cot x = t$ so that $\csc^2 x\,dx = -dt$

and then $I_3 = -\int t^2 dt = \frac{t^3}{3} + C_3 = \frac{\cot^3 x}{3} + C_3$. Thus we have

$$I_2 = \frac{\cot^3 x}{3} + C_3 + \cot x + x + C_2. \quad \ldots \quad (3)$$

Hence from (1), (2) and (3) we have

$$I = \int \cot^6 x\,dx = -\frac{cot^5 x}{5} - \frac{\cot^3 x}{3} - \cot x - x + C_1 - C_2 - C_3$$

$$= -\frac{cot^5 x}{5} - \frac{\cot^3 x}{3} - \cot x - x + C.$$

2.1.4 (F) Integration using miscellaneous trigonometric identities

In many cases of integration of trigonometric functions, the given integral may not be of any of the above said types and also direct substitutions may not be possible to reduce the given integral to any of the standard form. In such cases we have to make use of various known trigonometric identities to reduce the given integral to any of the known types. The trigonometric identity to be used varies from problem to problem depending upon the function in the integrand. Students are advised to see the following examples carefully and try to understand how to analyses a given function and use necessary trigonometric identity to simplify the given function.

Example12. Evaluate $\int \frac{\sin 2x}{(a + b\cos x)^2}\,dx$.

Solution: $\int \frac{\sin 2x}{(a+b\cos x)^2}\,dx = \int \frac{2\sin x\cos x}{(a+b\cos x)^2}\,dx$

(Using $\sin 2A = 2\sin A\cos A$).

Put $a+b\cos x = t$, so that $\cos x = \frac{t-a}{b}$ and $\sin x\,dx = \frac{-dt}{b}$. Then we have

$$\int \frac{\sin 2x}{(a+b\cos x)^2}\,dx = \int \frac{2\sin x\cos x}{(a+b\cos x)^2}\,dx = \frac{-2}{b^2}\int \frac{t-a}{t^2}dt$$

$$= \frac{-2}{b^2}\int \frac{1}{t}\,dt + \frac{2a}{b^2}\int \frac{1}{t^2}\,dt = \frac{-2}{b^2}\ln|t| - \frac{2a}{b^2}\frac{1}{t} + C$$

$$= \frac{-2}{b^2}\left(\ln(|a+b\cos x| + \frac{a}{a+b\cos x}\right) + C.$$

Example 13. Evaluate $\int \frac{\cos x - \sin x}{1+\sin 2x}\,dx$

Solution: $\int \frac{\cos x - \sin x}{1+\sin 2x}\,dx = \int \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + 2\sin x\cos x}\,dx$

(Note the use of identities) $= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}\,dx$

Put $\sin x + \cos x = t$ so that $(\cos x - \sin x)dx = dt$.
Then we have

$$\int \frac{\cos x - \sin x}{1+\sin 2x}\,dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}\,dx = \int \frac{1}{t^2}\,dt = -\frac{1}{t} + C = \frac{-1}{\sin x + \cos x} + C.$$

Example 14. Evaluate $\int \frac{1}{\sin(x-a)\cos(x-b)}\,dx$.

Solution: $\int \frac{1}{\sin(x-a)\cos(x-b)}\,dx = \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\,dx$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos((x-b)-(x-a))}{\sin(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-b)\cos(x-a) + \sin(x-b)\sin(x-a)}{\sin(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\left(\int \cot(x-a)\,dx + \int \tan(x-b)\,dx\right)$$

$$= \frac{1}{\cos(a-b)}\left(\ln|\sin(x-a)| - \ln|\cos(x-b)|\right) + C$$

$$= \frac{1}{\cos(a-b)}\left(\ln\left|\frac{\sin(x-a)}{\cos(x-b)}\right|\right)+C.$$

2.1.5 Trigonometric Substitutions

In this section we will discuss the methods for evaluating integrals in which the integrand contains algebraic functions under radicals and can be simplified by making use of some trigonometric substitutions.

More generally, an integrand that contains one of the forms

$\sqrt{a^2-b^2x^2}$, $\sqrt{a^2+b^2x^2}$ or $\sqrt{b^2x^2-a^2}$

(where a and b are constants) but no other irrational factor may be transformed into another simple trigonometric functions in a new variable by making some trigonometric substitutions. Such trigonometric substitutions are summarized in the following table

Table (2)

Integrand	Substitution	To obtain
$\sqrt{a^2-b^2x^2}$	$x=\frac{a}{b}\sin\theta$	$a\sqrt{1-\sin^2\theta}=a\cos\theta$
$\sqrt{a^2+b^2x^2}$	$x=\frac{a}{b}\tan\theta$	$a\sqrt{1+\tan^2\theta}=a\sec\theta$
$\sqrt{b^2x^2-a^2}$	$x=\frac{a}{b}\sec\theta$	$a\sqrt{\sec^2\theta-1}=a\tan\theta$

In each case, integration yields an expression in variable θ.

For the special case $b=1$, the above table becomes :

Table (3)

Integrand	Substitution	To obtain
$\sqrt{a^2-x^2}$	$x=a\sin\theta; -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$\sqrt{a^2-x^2}=a\cos\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$\sqrt{a^2+x^2}=a\sec\theta$

$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $\begin{cases} o \le \theta \le \frac{\pi}{2} & \text{if} \quad x \ge a \\ \frac{\pi}{2} < \theta \le \pi & \text{if} \quad x \le -a \end{cases}$	$\sqrt{x^2-a^2} = a\tan\theta$

Example 15. Evaluate: $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solution: The integrand **contains** $\sqrt{4-x^2}$, which is of the form $\sqrt{a^2-x^2}$ with a = 2. Then as in table (3) make the substitution

$$x = 2\sin\theta, dx = 2\cos\theta\, d\theta$$

we get

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta} == -\frac{1}{4}\cot\theta + c \quad ....(1)$$

But we have $x = 2\sin\theta$ or $\sin\theta = \frac{x}{2}$. Then

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{4}} = \frac{1}{2}\sqrt{4-x^2}$$

which yields $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{4-x^2}}{x}$

Now substituting in (1), we get

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C.$$

Example 16. Evaluate: $\int \frac{dx}{x\sqrt{9+4x^2}}$

Solution: As in table (2) make the substitution

$$x = \frac{3}{2}\tan\theta \text{ , then } \quad dx = \frac{3}{2}\sec^2\theta\, d\theta \quad \text{and} \quad \sqrt{9+4x^2} = 3\sec\theta$$

$$\int\frac{dx}{x\sqrt{9+4x^2}}=\int\frac{\frac{3}{2}\sec\theta\, d\theta}{\left(\frac{3}{2}\tan\theta\right)(3\sec\theta)}=\frac{1}{3}\int\csc\theta\, d\theta$$

$$=\frac{1}{3}\ln\left|\csc\theta-\cot\theta\right|+C \qquad (1)$$

Now, $x=\frac{3}{2}\tan\theta$ or $\tan\theta=\frac{2x}{3}$. Then $\cot\theta=\frac{3}{2x}$ and

$\operatorname{cosec}\theta=\sqrt{1+\cot^2\theta}=\sqrt{1+\frac{9}{4x^2}}=\frac{\sqrt{9+4x^2}}{2x}$. Substituting in (1), we get

$$\int\frac{dx}{x\sqrt{9+4x^2}}=\frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{2x}\right|+C$$

Example 17. Evaluate $\int\frac{x^2dx}{\sqrt{2x-x^2}}$.

Solution: We have $\int\frac{x^2dx}{\sqrt{2x-x^2}}=\int\frac{x^2dx}{\sqrt{1-(x-1)^2}}$

Make the substitution $x-1=\sin\theta$, $dx=\cos\theta\, d\theta$ and $\sqrt{2x-x^2}=\cos\theta$.

Then we have ,

$$\int\frac{x^2dx}{\sqrt{2x-x^2}}=\int\frac{(1+\sin\theta)^2}{\cos\theta}\cos\theta\, d\theta$$

$$=\int(1+\sin\theta)^2d\theta=\int(1+2\sin\theta+\sin^2\theta)d\theta$$

$$=\int\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos2\theta\right)d\theta$$

$$=\frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin2\theta+C \qquad (1)$$

Also we have $\sin\theta=x-1$, $\theta=\sin^{-1}(x-1)$,

$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-(x-1)^2}=\sqrt{2x-x^2}$, $\sin2\theta=2\sin\theta\cos\theta=2(x-1)\sqrt{2x-x^2}$.

$$\int\frac{x^2dx}{\sqrt{2x-x^2}}=\frac{3}{2}\left(\sin^{-1}(x-1)\right)-2\sqrt{2x-x^2}-\frac{1}{2}(x-1)\sqrt{2x-x^2}+C.$$

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✔ QUICK ACTIVITY EXERCISES 2.1 (*See after exercises for answers.*)

1. Evaluate the integrals:

$a) \int \cos^2 x\, dx$, $b) \int \tan^2 x\, dx$, $c)$, $\int sin^3 x \cos x\; dx$; $u = sinx$

2. In each of the following, state the trigonometric substitution that would be used to evaluate the integral. Do not evaluate the integral:

a) $\int \sqrt{9 - x^2}\, dx$, b) $\int \sqrt{9 + 3x^2}\, dx$, c) $\int \sqrt{1 + 16x^2}\, dx$

✔ EXERCISES SET 2.1

- Evaluate the following integrals :

1) $\int \frac{x^2}{\sqrt{4 - x^2}}\, dx$ 2) $\int \frac{\sqrt{x^2 - 9}}{x}\, dx$ 3) $\int \frac{dx}{x^2\sqrt{9x^2 - 4}}$ 4) $\int \frac{\cos\theta}{\sqrt{2 - sin^2\theta}} d\theta$	(26) $\int \tan^4 x\, dx =$ 27) $\int_0^{\frac{\pi}{2}} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx$ 28) $\int \sin^4 2x\, dx$ 29) $\int \sin^2 x \cos^3 x\, dx$

5) $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}dx$

6) $\int \frac{\sqrt{1+t^2}}{t}\,dt$

7) $\int \frac{\sqrt{2x^2-4}}{x}\,dx$

8) $\int \frac{dx}{\sqrt{x^2-16}}$

9) $\int \frac{dx}{x^2\sqrt{x^2-1}}$

10) $\int 5x^3\sqrt{1-x^2}\,dx$

11) $\int \frac{dx}{x^2\sqrt{x^2+25}}$

12) $\int \frac{dx}{\left(4-x^2\right)^{3/2}}$

13) $\int \sqrt{x^2+4}\,dx$

14) $\int \frac{dx}{x^2\sqrt{9-x^2}}$

15) $\int \frac{dx}{\left(4x-x^2\right)^{3/2}}$

16) $\int \frac{dx}{\left(9+x^2\right)^2}$

17) $\int \frac{dx}{\sqrt{x^2-4x+13}}$

18) $\int x^3\sqrt{4-x^2}$

19) $\int \frac{dx}{\left(9+x^2\right)^{3/2}}$.

30) $\int \sin 2x \cos 4x\,dx$

٣١) $\int \cos 3x \cos 2x\,dx$

٣٢) $\int \tan^3 x\,dx$

٣٣) $\int \tan^3 x \sec x dx$

٣٤) $\int \cot^3 x\,dx$

٣٥) $\int \cot^3 x \csc^3 x\,dx$

36) $\int \sin^4 x\,dx$

37) $\int_0^{\frac{\pi}{6}} \sec^3 2\theta\ tan 2\theta d\theta$

38) $\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$

39) $\int_0^{\frac{\pi}{3}} \sin^6 3x \cos^3 3x\,dx$.

40) $\int \sin^4 x\ cos^4 x\,dx$

41) $\int tan^3 x\ sec^2 x\ \ dx$

42) $\int \sin^3 3x\,dx$

43) $\int \cos^3 4x\,dx$

44) $\int \sec^3 2x\,dx$

45) $\int sin^2 x\ cos\,x\ \ dx$

46) $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}dx$

20) $\int \sin^4 x cos^2 x dx$	47) $\int \cos 2x \cos 4x \cos 6x\, dx$,
21) $\int \frac{\tan^2 x}{\sin^2 x + 3\cos x^2 x} dx$	48) $\int \frac{1-\cos x}{1+\cos x} dx$
22) $\int \sec^3 2x\, dx$	49) $\int \cos^2 x\, dx$
23) $\int \sin^3 3x\, dx$	50) $\int \sin^3 2x\, dx$
24) $\int \sqrt{tanx} sec^4 x\ \ dx$	
25) $\int \sin^3 x cos^2 x dx$	

✔ QUICK ACTIVITY ANSWERS 2.1

1. a) $\frac{x}{2} + \frac{1}{4}\sin 2x + c$

b) $tan\, x - x + C$

c) $\frac{\sin^4 x}{4} + C$

2. a) $3\sec\theta$ **, b)** $\sqrt{3}\tan\theta$ **, c)** $\frac{1}{4}\tan\theta$

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2.2 INTEGRATION BY PARTS.

2.2.1 Introduction

In this section we will discuss the methods of differentiating functions which are expressed as product of two functions. A primary method to integrate product of two functions is called integration by parts. The formula used is given as follows:

Let $f(x)$ and $g(x)$ be any two functions. Then

$$\int f(x).g(x)dx = f(x)\int g(x)dx - \int\left(f'(x)\int g(x)dx\right)dx$$

Students are advised to remember the above formula as follows:

Let $f(x)$ and $g(x)$ be any two functions of x taken as the **first function** and **second function** respectively. Then= $\int first\ function \times second\ function\, dx$

$$first\ function \times \int second\ function\, dx \ - \int\left(\frac{d}{dx} first\ function \times \int second\ function\, dx\right)dx$$

Namely: The integral of product of two functions = (First function) $\times$ (integral of second function) – integral of { (derivative of first function) $\times$ (integral of second function)}.

Result: when we have $\int first\ function \times \sec ond\ function\ dx$ or $\int f(x)\,g(x)\,dx$,then we take

$u = f(x)\ and\ \ dv = g(x)dx,\ \ so\ \ du = f'(x)dx\ and\ v = G(x) = \int g(x)dx,$ then we can write:

$$\int u\,dv = u \cdot v - \int v\,du \qquad (1)$$

When the last relation is to be used in a required integration the given integral must by separated into two parts, one part being u and the other part, together with dx, being dv (for this, integrations is called "by parts". In this way, two general rules can be stated:

1. The part selected as dv must be easily integrable.
2. $\int v\,du$ must be simpler than $\int u\,dv$.

Example 1. Find

$$\int x e^{3x}\,dx$$

Solution: put $u = x$, $du = dx$ and $dv = e^{3x}\,dx$, $v = \frac{e^{3x}}{3}$

$$\int x e^{3x}\,dx = \frac{1}{3} x\, e^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3} x\, e^{3x} - \frac{1}{9} e^{3x} + c$$

2.2.2 We use this method a lot especially in the following case considering cases.

- **The First Case:** For evaluating the integral: $\int x^n\ e^{ax}\ dx$

We take: $\begin{cases} x^n = u \ \Rightarrow \ du = n\,x^{n-1}dx \\ e^{ax}dx = dv \Rightarrow v = \frac{1}{a}e^{ax} \end{cases}$

then: $\int x^n\, e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}\,e^{ax}\,dx$

The new integral is less complicated than the original, repeated the integration by parts in the right integral $(n-1)$ times, we get the required integral.

Example 2. Find $\int x^2\,e^x dx$

Solution: take $u = x^2 \Rightarrow du = 2x\,dx$

$e^x dx = dv \Rightarrow v = e^x$, then

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x\, dx$$

Now we must evaluate the integral in the right side by parts again with:

$$u = x \Rightarrow du = dx \text{ and } e^x dx = dv \Rightarrow v = e^x$$

hence: $\int x\, e^x dx = xe^x - \int e^x\, dx = xe^x - e^x + c_1$.

$$\int x^2\, e^x dx = x^2\, e^x - 2xe^x + 2e^x + c\ ; \quad c = -2c_1$$

Note:

The technique work for integral $\int x^n e^{ax} dx$ in which n is a positive integer and a is a real number, because differentiating x^n will eventually lead to zero and integrating e^{ax} is easy.

- **The Second Case:**

For evaluating the integral: $\int x^n \cos ax\, dx$ or $\int x^n \sin ax\, dx$

We take : $x^n = u \Rightarrow du = n x^{n-1} dx$ and $dv = \cos ax\, dx \Rightarrow v = \frac{1}{a}\sin ax\, dx$, then:

$$\int x^n \cos ax\, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a}\int x^{n-1} \sin ax\, dx$$

The second integral is like the first except that it has sin ax in place of cos ax.

To evaluate the second integral, repeated the integration by parts on the right integral $(n-1)$ times, we get the required integral.

Example 3. find $\int x \cos x\, dx$

Solution: take $u = x \Rightarrow du = dx$ and $dv = \cos x\, dx = v = \sin x$

then:

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c$$

- **The Third Case:**

For evaluating the integral*:* $\int x^n \ln x\, dx$

We take: $\ln x = u \Rightarrow \frac{dx}{x} = du \quad x^n dx = dv \Rightarrow v = \frac{x^{n+1}}{n+1} \;\; ; n \neq -1$

then: $\int x^n \ln x\, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n dx$

It is obvious that the integral in the right side is less difficult of the one in the left.

Example 4. Find $\int x \ln x\, dx$

Solution: take $\ln x = u \Rightarrow \frac{1}{x} dx = du$ and $x\, dx = dv \Rightarrow \frac{x^2}{2} = v$, then:

$$\int x \ln x\, dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x\, dx = \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$$

Remark: The integrals:

$$\int x^n \sin^{-1} x\, dx, \int x^n \cos^{-1} x\, dx, \int x^n \tan^{-1} x\, dx$$

are solving by the same hypotheses, by mean:

$$u = \sin^{-1} x \qquad \left(\cos^{-1} x,\ \tan^{-1} x\right) \text{ and } \ dv = x^n dx$$

- **The Fourth Case:**

For evaluating the integral*:*

$$\int e^{ax} \cos bx\, dx \quad \text{ or } \quad \int e^{ax} \sin bx\, dx$$

We take: $u = e^{ax} \Rightarrow du = a\, e^{ax} dx$ and $dv = \cos bx\, dx \Rightarrow v = \frac{1}{b} \sin bx$

then: $\int e^{ax} \cos bx\, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx\, d$

Easily seen that, the second integral is like the first one except that sin bx in place of cos bx, to find it we use integration by parts again, with $u = e^{ax}$, $dv = \sin bx\, dx$.

then:

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{b} \sin bx - \frac{a}{b} \left\{ \frac{1}{b} e^{ax} \cos bx - \frac{a}{b} \int e^{ax} \cos bx\, dx \right\}$$

thus:

$$\left(1+\frac{a}{b}\right)\int e^{ax}\cos bx\,dx=\frac{e^{ax}\sin bx}{b}+\frac{ae^{ax}\cos bx}{b^2}+c$$

finally:

$$\int e^{ax}\cos bx\,dx=\frac{b}{a+b}\left\{\frac{be^{ax}\sin bx+ae^{ax}\cos bx}{b^2}\right\}+c_1$$

Example 5. find $\int e^x \sin x dx$

Solution:

take: $u=e^x \Rightarrow du=e^x dx$ and $\sin x\,dx=dv\Rightarrow v=\cos x$

$$\int e^x\sin x\,dx=e^x\cos x-\int e^x\cos x\,dx$$

to evaluate the right side integral by integration again, with:

$$u=e^x \quad , \quad dv=\cos x\,dx$$

we get:

$$\int e^x\sin x dx=e^x\cos x-\left\{e^x\sin x-\int e^x\sin x dx\right\}+c_1 \quad \text{, then:}$$

$$2\int e^x\sin x dx=e^x\cos x-e^x\sin x+c_1 , \quad \text{thus:}$$

$$\int e^x\sin x\,dx=\frac{e^x\cos x+e^x\sin x}{2}+c$$

- **The Fifth Case:**

For evaluating the integral :

$$I_n=\int\frac{dx}{\left(1+x^2\right)^n}$$

by reduction method, we evaluate the integral I_{n-1}, then I_{n-2}, thus till evaluate the integral I_1, where:

$$I_1=\int\frac{dx}{1+x^2}=tan^{-1}x+c$$

indeed, applying integration by parts rule on:

$$I_{n-1} = \int \frac{dx}{\left(1+x^2\right)^{n-1}} \quad , \text{ where:}$$

$$\frac{1}{\left(1+x^2\right)^{n-1}} = u \Rightarrow du = \frac{2\left(1-n\right)x\,dx}{\left(1+x^2\right)^{n}} \quad \text{and } dx = dv \Rightarrow v = x$$

we find: $I_{n-1} = \dfrac{x}{\left(1+x^2\right)^{n-1}} + 2\left(n-1\right)\displaystyle\int \frac{x^2dx}{\left(1+x^2\right)^{n}}$ or

$$I_{n-1} = \frac{x}{\left(1+x^2\right)^{n-1}} + 2\left(n-1\right)\int \frac{x^2+1-1}{\left(1+x^2\right)^{n}}dx$$

$$I_{n-1} = \frac{x}{\left(1+x^2\right)^{n-1}} + 2\left(n-1\right)\left\{\int \frac{dx}{\left(1+x^2\right)^{n-1}} - \int \frac{dx}{\left(1+x^2\right)^{n}}\right\}$$

That: $I_{n-1} = \dfrac{x}{\left(1+x^2\right)^{n-1}} + 2\left(n-1\right)\left\{I_{n-1} - I_n\right\}$,then:

$$\boxed{I_n = \frac{2n-1}{2\left(n-1\right)}I_{n-1} + \frac{x}{2\left(n-1\right)\left(1+x^2\right)^{n-1}}} \qquad (2)$$

Example 6. Find $I_3 = \displaystyle\int \frac{dx}{\left(1+x^2\right)^{3}}$

Solution: Using (2): we have: $I_3 = \dfrac{3}{4}I_2 + \dfrac{x}{4\left(1+x^2\right)^{2}}$, where:

$$I_2 = \frac{1}{2}I_1 + \frac{x}{2\left(1+x^2\right)} = \frac{1}{2}\tan^{-1}x + \frac{x}{2\left(1+x^2\right)}$$

Substituting in I_3, we get:

$$I_3 = \frac{x}{4\left(1+x^2\right)^2} + \frac{3x}{8\left(1+x^2\right)} + \frac{3}{8}\tan^{-1}x + c$$

Example 7. Evaluate $\int x^2 \cos x\, dx$.

Solution: Here the integrand is product of two functions x^2 and $\cos x$. So we take $u = x^2 \rightarrow du = 2x dx$ and $dv = \cos x\, dx \rightarrow v = \int \cos x\, dx = \sin x$

Using the formula

$$\int u\, dv = uv - \int v\, du$$

we find

$$\int x^2 \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx$$

(Note that we don't add constant of integration while integrating the first function)

(Again integrating by parts without changing the order of first function and second function) = $x^2 \sin x + 2x\cos x - 2\sin x + C$.

Note : In the above example if we take $\cos x$ as the first function and x^2 as the second function and integrate we get

$\int x^2 \cos x\, dx = \cos x.\frac{x^3}{3} - \int \sin x \frac{x^3}{3} dx$, that is the given integral is expanded to a more complicated integral with more higher powers of x and this will still increase in the next step. Thus proper choice of first function and second function is important.

Usually we follow the following rules to choose the first function :

1. If one function is a polynomial function and the other function is easily integrable then the polynomial function is taken as the first function.
2. The LIATE rule is also very useful in deciding the first function. According to this rule the first function is chosen as the function which comes first in order of letters of the word LIATE where

   L – stands for logarithmic functions.

   I – stands for inverse trigonometric functions.

A – stands for algebraic functions.

T – stands for trigonometric functions.

E – stands for exponential functions.

3. If there is an inverse trigonometric function or logarithmic function and no other function, then we can take 1 as the second function and integrate by parts.

Example 8. Evaluate $\int \ln x\, dx$.

Solution: Note that $\ln x = \ln x.1$, that is $\ln x$ is the product of $\ln x$ and 1. Now 1 being a polynomial of degree zero, to evaluate $\int \ln x\, dx$ we use integration by parts taking $u = \ln|x|$ *and* $dv = dx$, *then we have* $du = \frac{1}{x}$ *and* $v = x$

$$\int \ln x\, dx = x\ln|x| - \int x\frac{1}{x}dx = x\ln|x| - \int dx = x\ln|x| - x + c.$$

Example 9. Evaluate $\int \tan^{-1} x\, dx$.

Solution: Note that $\tan^{-1} x = \tan^{-1} x.1$. So integrating by parts taking $\tan^{-1} x$ as the first function and 1 as the second function, we get

$$\int \tan^{-1} x\, dx = x.\tan^{-1} x - \int x\frac{1}{1+x^2}dx = \tan^{-1} x.x - \int\left(\frac{1}{1+x^2}.x\right)dx =$$

$$x\tan^{-1} x - \int \frac{x}{1+x^2}\, dx.$$

Now to evaluate the integral above, we will use substitution method.

Put $1+x^2 = t$ so that on differentiating with respect to x we get $xdx = \frac{1}{2}dt$.

Then we have

$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int\frac{1}{t}dt = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C$$ .Hence

$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$ (Note that since C is a constant, it doesn't matter whether we take +C or –C).

2.2.3 Integrals of the form $\int e^x \left[f(x) + f'(x) \right] dx$

To evaluate this type of integral we proceed as follows :

1. Write the given integral as
   $\int e^x \left[f(x) + f'(x) \right] dx = \int e^x f(x)dx + \int e^x f'(x)dx$.
2. Now evaluate the first integral by parts taking $f(x)$ as the first function and e^x as the second function.
3. The second integral will get cancelled.

Theorem 1. $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + C$.

Proof : We have $\int e^x \left[f(x) + f'(x) \right] dx = \int e^x f(x)dx + \int e^x f'(x)dx$. Integrating the first integral by parts taking $f(x)$ as the first function and e^x as the second function we get

$= \int e^x \left[f(x) + f'(x) \right] dx = f(x)e^x - \int f'(x)e^x dx + C + \int e^x f'(x)dx = f(x)e^x + C$.

Example 10. Evaluate $\int e^x \left(\frac{2+\sin 2x}{1+\cos 2x} \right) dx$.

Solution: $\int e^x \left(\frac{2+\sin 2x}{1+\cos 2x} \right) dx = \int e^x \left(\frac{2+2\sin x\cos x}{2\cos^2 x} \right) dx$

$= \int e^x \left(\sec^2 x + \tan x \right) dx$ (This integral is of the above discussed form)

$= \int e^x \tan x dx + \int e^x \sec^2 x dx$

$= \tan x.e^x - \int e^x \sec^2 x dx + \int e^x \sec^2 x dx$ (Integrating the first integral by parts taking $\tan x$ as the first function) $= \tan x.e^x$ + C.

Note: Whenever e^x appears in the integrand, students are advised to check whether the given integral can be reduced to the above form.

Example 11. Evaluate $\int e^x \frac{x}{(x+1)^2} dx$

Solution: $\int e^x \frac{x}{(x+1)^2} dx = \int e^x \frac{x+1-1}{(x+1)^2} dx = \int e^x \left(\frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$

(This integral is of the above discussed form) = $\int e^x \frac{1}{x+1} dx - \int e^x \frac{1}{(x+1)^2} dx$

$\frac{1}{x+1} e^x + \int e^x \frac{1}{(x+1)^2} dx - \int e^x \frac{1}{(x+1)^2} dx$

(Integrating the first integral by parts taking $\tan x$ as the first function)

$$= \frac{1}{x+1} e^x .$$

2.2.4 Integrals of the form $\int e^{ax} \sin bx \, dx$ and $\int e^{ax} \cos bx \, dx$.

In this type of integrals we integrate by parts repeatedly taking e^{ax} as the first function. After few steps we get the same given integral on the Right Hand Side. We illustrate the method below.

Example 12. Evaluate $\int e^{2x} \sin 3x \, dx$.

Solution: Let $I = \int e^{2x} \sin 3x \, dx$. Integrating by parts taking e^{2x} as the first function, we get

$$I = e^{2x} \left(\frac{-\cos 3x}{3} \right) - \int 2e^{2x} \left(\frac{-\cos 3x}{3} \right) dx$$

$= \frac{-e^{2x} \cos 3x}{3} + \frac{2}{3} \int e^{2x} \cos 3x dx$. Again integrating by parts, we get

$$I = \frac{-e^{2x} \cos 3x}{3} + \frac{2}{3} \left(e^{2x} \frac{\sin 3x}{3} - \int 2e^{2x} \frac{\sin 3x}{3} dx \right)$$

$$=\frac{-e^{2x}\cos 3x}{3}+\frac{2e^{2x}}{9}\sin 3x-\frac{4}{9}\int e^{2x}\sin 3x dx=\frac{-e^{2x}\cos 3x}{3}+\frac{2e^{2x}}{9}\sin 3x-\frac{4}{9}I$$

Therefore

$$I+\frac{4}{9}I=\frac{-e^{2x}\cos 3x}{3}+\frac{2e^{2x}}{9}\sin 3x, \quad \text{oror} \quad \frac{13}{9}I=\frac{-e^{2x}\cos 3x}{3}+\frac{2e^{2x}}{9}\sin 3x$$

$$I=-\frac{3}{13}e^{2x}\cos 3x+\frac{2e^{2x}}{13}\sin 3x.$$

Example 13. Evaluate $\int e^x \cos^2 x\, dx$.

Solution: $\int e^x \cos^2 x\, dx = \int e^x\left(\frac{1+\cos 2x}{2}\right)dx$

$$=\frac{1}{2}\int e^x\, dx+\frac{1}{2}\int e^x\cos 2x\, dx=\frac{1}{2}e^x+\frac{1}{2}I+C \qquad \ldots. \quad (1)$$

where $I=\int e^x\cos 2x\, dx$, now $I=\int e^x\cos 2x\, dx$

$$=\frac{e^x\sin 2x}{2}-\frac{1}{2}\int e^x\sin 2x\, dx \text{ (integrating by parts taking } e^x \text{ as first function)}$$

$$=\frac{e^x\sin 2x}{2}-\frac{1}{2}\left(\frac{-e^x\cos 2x}{2}-\int\frac{-e^x\cos 2x}{2}dx\right) \text{ (again integrating by parts)}$$

$$=\frac{e^x\sin 2x}{2}+\frac{e^x\cos 2x}{4}-\frac{1}{4}\int e^x\cos 2x dx=\frac{e^x\sin 2x}{2}+\frac{e^x\cos 2x}{4}-\frac{1}{4}I.$$

Therefore

$$I+\frac{1}{4}I=\frac{e^x\sin 2x}{2}+\frac{e^x\cos 2x}{4} \qquad \frac{5}{4}I=\frac{e^x\sin 2x}{2}+\frac{e^x\cos 2x}{4}$$

$I=\frac{2e^x\sin 2x}{5}+\frac{e^x\cos 2x}{5}$. Thus we have from (1)

$$\int e^x\cos^2 x\, dx=\frac{1}{2}e^x+\frac{1}{2}\left(\frac{2e^x\sin 2x}{5}+\frac{e^x\cos 2x}{5}\right)+C.$$

Example 14. Evaluate $\int\left(\sin^{-1}x\right)^2 dx$.

Solution: Put $\sin^{-1}x = t$. Then $x = \sin t$ and $dx = \cos t\, dt$. We get

$\int (\sin^{-1}x)^2\, dx = \int t^2 \cos t\, dt$. Now integrating by parts

$\int (\sin^{-1}x)^2\, dx = t^2 \sin t - \int 2t \sin t\, dt + c = t^2 \sin t + 2t\cos t - 2\int \cos t\, dt + C$

(again integrating by parts)

$$= t^2 \sin t + 2t\cos t - 2\sin t + C = (\sin^{-1}x)^2 x + 2\sin^{-1}x\sqrt{1-x^2} - 2x + C.$$

Example 15. Find $\int x^3 e^{x^2} dx$

Solution: take $u = x^2 \Rightarrow du = 2x\,dx$ and $dv = dx \Rightarrow v = x$,

then $\int x^3 e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \int x e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$

Example 16. Find the integral $\int \ln(x^2+2)\,dx$

Solution: take $u = \ln(x^2+2) \Rightarrow du = \frac{2x\,dx}{x^2+2}$ and $dv = dx \Rightarrow v = x$,then

$$\int \ln(x^2+2)dx = x\ln(x^2+2) - \int \frac{2x^2 du}{x^2+2}$$
$$= x\ln(x^2+2) - 2x + 2\sqrt{2}\, tan^{-1}\frac{x}{\sqrt{2}} + c$$

٢.2.5 A Tabular Method For Repeated Integration By Parts

Integrals of the form *p(x)f(x) dx*, where *p(x)* is a polynomial, can sometimes be evaluated using repeated integration by parts in which *u* is taken to be *p(x)* or one of its derivatives at each stage. Since *du* is
computed by differentiating *u*, the repeated differentiation of *p(x)* will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called ***tabular integration by parts***.

- **The steps we take to find integration in using Tabular Integration by Parts:**

Step 1. Differentiate *p(x)* repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate *f(x)* repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating + and − signs, starting with a +.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by +1 or −1 in accordance with the sign on the arrow.

Example ١7.

Using Tabular method to evaluate the integral: $\int x^2 e^x \, dx$

Solution:

derivation	integration
(+) x^2	e^x
(-) $2x$	e^x
(+)2	e^x
0	e^x

Since ,we find:

$$\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + c$$

Example 18.

Using Tabular method for evaluate the integral:

$$\int x^3 cosx \, dx$$

Solution:

Derivation	Integration
(+) x^3	$\cos x$
(-) $3x^2$	$\sin x$
(+)$6x$	-$\cos x$

(-)6	-sin x
0	cos x

Since we find:

$$\int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x + 6x\sin x - 6 \cos x + c$$

Example 19. $\int (x^2 - 2x + 9) sinx\, dx$

Solution:

derivation	integration
(+)$(x^2 - 2x + 9)$	$sinx$
(-) (2x-2)	$-cosx$
(+) 2	$-sinx$
0	$cosx$

$$\int (x^2 - 2x + 9) sinx\, dx = -(x^2 - 2x + 9) cosx + (2x - 2) sinx + 2cosx + c.$$

****

✔ **QUICK ACTIVITY EXERCISES 2.2 (*See after exercises for answers.*)**

1. Find appropriate choice of first function *u* and second function *v* for integrating by parts the following integral. Do not evaluate the integral.

$a)\ \int x \ln x\, dx\ ;\ \ u = ---, dv = ---$

$b)\ \int (x + 2) \sin x\, dx\ ;\ \ u = ---, dv = ---$

$c)\ \int \cos^{-1} x\, dx\ ;\ \ u = ---, dv = ---$

$d)\ \int \ln(x + 1)\, dx\ ;\ \ \ u = ---, dv = ---$.

2. Use integration by parts to evaluate the integral.

$a)\ \int \ln(x+1)\,dx \quad , \quad b)\ \int x \sin 2x\,dx.$

✔ EXERCISES SET 2.2

a) Evaluate the following integrals :

1) $\int x^2 \sin x\,dx$	2) $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$
3) $\int \frac{(x^2+1)e^x}{(x+1)^2}\,dx$	4) $\int \cos^{-1}2x\,dx$
5) $\int xe^{-2x}\,dx$	6) $\int x\cos 2x dx$
7) $\int e^x \sin 3x\,dx$	8) $\int \frac{(x-3)}{(x-1)^3}e^x\,dx$
9) $\int e^{2x}\sin x\,dx$	10) $\int \sin^{-1}2x\,dx$

11) $\int (x^2+1)\ln x\,dx$	12) $\int (\sin(\ln x)+\cos(\ln x))\,dx$
13) $\int \frac{\ln x}{(1+\ln x)^2}\,dx$	14) $\int xe^{3x}\,dx$
15) $\int x^2e^{-3x}dx$	16) $\int x^2\sqrt{1-x}dx$
17) $\int x\tan^{-1}x\,dx$	18) $\int sin^3x\,dx$
19) $\int sin^{-1}x\,dx$	20) $\int x^2\,e^x dx$
21) $\int \ln(x+2)\,dx$	22) $\int xsec^2x\,dx$
23) $\int xtan^2x\,dx$	24) $\int_0^1 (5x+2)^2dx$
25) $\int e^{3x}sin2x\,dx$	26) $\int x^2lnx\,dx$
27) $\int \frac{lnx}{x^2}dx$	28) $\int \frac{x^2dx}{\sqrt{1+x}}$
29) $\int x^2\,e^{-2x}dx$	30) $\int xe^{4x}\,dx$
31) $\int \sqrt{x}lnx\,dx$	32) $\int cos^3x\,dx$

b) Using Tabular method to evaluate the integral:

1) $\int (x^2+7x-56)\,sinx\,dx$	4) $\int (3x^2-x+2)e^{-x}\,dx$
2) $\int (x^2-2x)\ln x\,dx$	5) $\int e^{ax}\sin bx\,dx.$
3) $\int (x^2+2x)\,cosx\,dx$	6) $\int e^x\cos x\,dx.$

✔ QUICK ACTIVITY ANSWERS 2.2

1.

$a)$ $u = \ln|x|$, $dv = xdx \rightarrow v = \frac{1}{2}x^2$,

$b)$ $u = x + 2$, $dv = \sin x\, dx \rightarrow v = -\cos x$,

$c)$ $u = cos^{-1}x$, $dv = dx \rightarrow v = x$.

$d)$ $u = \ln|(x + 1)|$, $dv = dx \rightarrow v = x.$

2. $a)\ (x + 1)ln(x + 1) - x + c$,

$b) - \frac{x}{2}cos2x + \frac{1}{4}\ sin2x + c$

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2.3 INTEGRATION OF RATIONAL AND IRRATIONAL FUNCTIONS.

2.3.1 Integration of Standard Rational Functions

In this section first we prove some standard integrals for rational algebraic functions and then give some applications of these integrals. The students are advised to remember these integrals as standard integrals.

Standard Rational Integral I: $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

Proof. $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a}\left[\frac{2a}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\therefore \int \frac{1}{x^2-a^2}\,d = \frac{1}{2a}\left[\int \frac{1}{x-a}\,dx - \int \frac{1}{x+a}\,dx\right] = \frac{1}{2a}\left[\ln|(x-a)| - \ln|\,(x+a)|\right] = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

Standard Rational Integral II : $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

Proof. $\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)}$

$$= \frac{1}{2a}\left[\frac{2a}{(a-x)(a+x)}\right] = \frac{1}{2a}\left[\frac{(a-x)+(a+x)}{(a-x)(a+x)}\right] = \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right]$$

$$\therefore \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\left[\int \frac{1}{a+x}\,dx + \int \frac{1}{a-x}\,dx\right] = \frac{1}{2a}\left[\ln|(a+x)| - \ln|(a-x)|\right]$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c.$$

Standard Rational Integral III : $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Proof. Put $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$

$$\therefore x^2+a^2 = a^2\tan^2\theta + a^2 = a^2\sec^2\theta \to \int \frac{1}{x^2+a^2}\,dx = \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c \quad = \quad \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Many rational algebraic functions and rational transcendental functions can be reduced to any of the above said standard forms by proper simplification and substitutions. Although there is no specific method for simplification, the following steps will be helpful:

1. If degree of numerator is greater than or equal to degree of the denominator, then first divide numerator with denominator.
2. If trigonometric functions are involved then make use of the various trigonometric formulas and then some proper substitutions will reduce the given integral into any of the above said standard forms.
3. Make proper substitutions to reduce the given integral into any of the above said standard forms.

The following solved examples will make the method clear.

2.3.2 Solved examples

Example 1. Evaluate $\int \frac{x^3+x}{x^4-9}dx$.

Solution: $\int \frac{x^3+x}{x^4-9}dx = \int \frac{x^3}{x^4-9}dx + \int \frac{x}{x^4-9}dx = I_1 + I_2$

Now , $I_1 = \int \frac{x^3}{x^4-9}dx$. Put $x^4-9=t$, then $4x^3dx = dt$ or $x^3dx = \frac{dt}{4}$.

$$\therefore I_1 = \int \frac{x^3}{x^4-9}dx = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\ln|t| + c_1 = \tfrac{1}{4}\ln\left|(x^4-9)\right| + c_1 .$$

Again $I_2 = \int \frac{x}{x^4-9}dx$. Put $x^2 = t$, then $2xdx = dt$ or $xdx = \frac{dt}{2}$.

$$\therefore I_2 = \int \frac{x}{x^4-9}dx = \frac{1}{2}\int \frac{dt}{t^2-3^2} = \frac{1}{2}.\frac{1}{6}\ln\left|\left(\frac{t-3}{t+3}\right)\right| + c_2 = \frac{1}{12}\ln\left|\left(\frac{x^2-3}{x^2+3}\right)\right| + c_2$$

Hence , $\int \frac{x^3+x}{x^4-9}dx = \int \frac{x^3}{x^4-9}dx + \int \frac{x}{x^4-9}dx$

$$= \tfrac{1}{4}\ln(x^4-9) + c_1 + \frac{1}{12}\ln\left|\left(\frac{x^2-3}{x^2+3}\right)\right| + c_2 = \tfrac{1}{4}\ln\left|(x^4-9)\right| + \frac{1}{12}\ln\left|\left(\frac{x^2-3}{x^2+3}\right)\right| + c$$

Example 2. Evaluate $\dfrac{dx}{\sqrt{x}+x\sqrt{x}}$.

Solution: $\displaystyle\int \frac{dx}{\sqrt{x}+x\sqrt{x}} = \int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{1}{1+x}.\frac{1}{\sqrt{x}}\,dx$.

Put $\sqrt{x}=t$, then $\dfrac{1}{2\sqrt{x}}dx = dt$ or $\dfrac{1}{\sqrt{x}}dx = 2dt$.

$$\therefore \int \frac{dx}{\sqrt{x}+x\sqrt{x}} = \int \frac{1}{1+x}.\frac{1}{\sqrt{x}}dx = \int \frac{1}{1+t^2}.2dt$$
$$= 2\int \frac{1}{1+t^2}dt = 2\tan^{-1} t + c = 2\tan^{-1}\sqrt{x} + c$$

(Using standard rational integral III).

Example 3. Evaluate $\displaystyle\int \frac{x^4}{x^2-4}dx$.

Solution: Note that in the given function, degree of numerator is greater than the degree of denominator. So first we divide x^4 with x^2-4 using long division method to get $\dfrac{x^4}{x^2-4} = x^2+4+\dfrac{16}{x^2-4}$. Then we have

$$\int \frac{x^4}{x^2-4}dx = \int\left(x^2+4+\frac{16}{x^2-4}\right)dx \Rightarrow \int \frac{x^4}{x^2-4}dx = \int x^2dx + \int 4dx + \int \frac{16}{x^2-4}dx$$

$$\Rightarrow \int \frac{x^4}{x^2-4}dx = \int x^2dx + 4\int dx + 16\int \frac{1}{x^2-2^2}dx = \frac{x^3}{3}+4x+16\left(\frac{1}{4}\ln\left|\left(\frac{x-2}{x+2}\right)\right|\right)$$

(Using standard rational integral I with $a=2$) $= \dfrac{x^3}{3}+4x+4\ln\left|\left(\dfrac{x-2}{x+2}\right)\right|$.

Example 4. Evaluate $\displaystyle\int \frac{\sin 2x}{\sin 3x}dx$.

Solution: We have

$\frac{\sin 2x}{\sin 3x} = \frac{2\sin x\cos x}{3\sin x - 4\sin^3 x} = \frac{2\cos x}{3-4\sin^2 x}$ (Using proper trigonometric identities),therefore $\int \frac{\sin 2x}{\sin 3x}dx = \int \frac{2\cos x}{3-4\sin^2 x}dx$.

Now put $\sin x = t$. Then $\cos x dx = dt$. Then we have

$\int \frac{\sin 2x}{\sin 3x}dx = \int \frac{2}{3-4t^2}dt$. Dividing numerator and denominator of R.H.S by 4,

we get $\int \frac{\sin 2x}{\sin 3x}dx = \int \frac{2/4}{3/4 - t^2}dt = \frac{1}{2}\int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 - t^2}dt \quad = \frac{1}{2}\left(\frac{1}{2.\frac{\sqrt{3}}{2}}\ln\left|\left(\frac{\sqrt{3}/2 + t}{\sqrt{3}/2 - t}\right)\right|\right)$ (using standard rational integral II, with $a = \frac{\sqrt{3}}{2}$).**Example 5**. $= \frac{1}{2\sqrt{3}}\ln\left|\left(\frac{\sqrt{3}/2 + \sin x}{\sqrt{3}/2 - \sin x}\right)\right|$

Evaluate $\int \frac{\sin x}{1+\cos^2 x}dx$.

Solution: Put $\cos x = t$, so that on differentiating with respect to x, we get $\sin x dx = -dt$. Then we have

$$\int \frac{\sin x}{1+\cos^2 x}dx = -\int \frac{1}{1+t^2}dt = -\tan^{-1} t + C = -\tan^{-1}(\cos x) + C.$$

2.3.3 Integration of Standard Irrational Functions

In this section first we prove some standard integrals for irrational algebraic functions and then give some applications of these integrals. The students are advised to remember these integrals as standard integrals.

Standard Irrational Integral I : $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a} + c$

Proof. Put $x = a\sin\theta$, then $dx = a\cos\theta d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{a\cos\theta d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \int d\theta \quad = \quad \theta + c \quad = \quad \sin^{-1}\frac{x}{a} + c$$

Standard Irrational Integral II : $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

Proof. Put $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta d\theta$

$$\therefore \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta - 1)}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta a\tan\theta}$$

$$= \frac{1}{a}\int d\theta \quad = \quad \frac{1}{a}\theta + c \quad = \quad \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

Standard Irrational Integral III : $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left|(x + \sqrt{x^2 - a^2})\right| + c$

Proof. Put $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2(\sec^2\theta - 1)}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \int \sec\theta d\theta$$

$$= \quad \ln|(\sec\theta + \tan\theta)| + c_1 = \ln\left|\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)\right| + c_1$$

$$= \ln\left|\left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right)\right| + c$$

$$= \quad \ln\left|\left(x + \sqrt{x^2 - a^2}\right)\right| - \ln|a| + c_1$$

$$= \ln\left|\left(x + \sqrt{x^2 - a^2}\right)\right| + c.$$

Standard Irrational Integral IV: $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left|(x + \sqrt{x^2 + a^2})\right| + c$

Proof. Put $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$.

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2(\tan^2\theta+1)}} = \int \frac{a\sec^2\theta d\theta}{a\sec\theta}$$

$$= \int \sec\theta d\theta \quad = \quad \ln\left|(\sec\theta+\tan\theta)\right| + c_1$$

$$= \ln\left|(\sqrt{1+\tan^2\theta}+\tan\theta)\right| + c_1$$

$$= \ln\left|\left(\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right)\right| + c_1 \quad = \ln\left|\left(x+\sqrt{x^2+a^2}\right)\right| - \ln|a| + c_1$$

$$= \ln\left|\left(x+\sqrt{x^2+a^2}\right)\right| + c$$

We also present the following three standard integrals without proof :

Standard Irrational Integral V: $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

Standard Irrational Integral VI: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|(x+\sqrt{x^2-a^2})\right| + C$

Standard Irrational Integral VII: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|(x+\sqrt{x^2+a^2})\right| + C$

2.3.4 Solved examples

Proper substitutions can reduce many irrational functions to any of the above said standard irrational functions which can further be integrated using the above formulas. The following examples will make the concept clear.

Example 6. Evaluate $\int \frac{1}{\sqrt{(2-x)^2-1}}\,dx$.

Solution: Put $2-x=t$, then $-dx=dt$ or $dx=-dt$.

$$\int \frac{1}{\sqrt{(2-x)^2-1}}\,dx = -\int \frac{dt}{\sqrt{t^2-1}} = -\ln\left|\left(t+\sqrt{t^2-1}\right)\right| + c =$$

$$-\ln\left|\left((2-x)+\sqrt{(2-x)^2-1}\right)\right| + c = -\ln\left|\left((2-x)+\sqrt{x^2-4x+3}\right)\right| + c$$

Example 7. Evaluate $\int \frac{x^2}{\sqrt{x^6-1}}dx$.

Solution: $\int \frac{x^2}{\sqrt{x^6-1}}dx = \int \frac{x^2}{\sqrt{\left(x^3\right)^2-1}}dx$. Now put $x^3 = t$ so that $3x^2dx = dt$ or

$x^2dx = \frac{1}{3}dt$. Then we have

$$\int \frac{x^2}{\sqrt{x^6-1}}\mathrm{dx} = \frac{1}{3}\int \frac{dt}{\sqrt{t^2-1}} = \frac{1}{3}\ln\left|t+\sqrt{t^2-1}\right| + C$$

$$= \frac{1}{3}\ln\left|x^3+\sqrt{x^6-1}\right| + C$$

Example 8. Evaluate $\int x\sqrt{x^4+1}\,dx$.

Solution: $\int x\sqrt{x^4+1}\,dx = \int x\sqrt{\left(x^2\right)^2+1}\,dx$.

Putting $x^2 = t$ so that $2xdx = dt$ or $xdx = \frac{1}{2}dt$, we get

$$= \frac{1}{2}\left(\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right|\right) + c$$

(using standard irrational integral VII with $x = t$ and $a = 1$)

$$= \frac{x^2}{4}\sqrt{x^4+1} + \frac{1}{4}\ln\left|x^2+\sqrt{x^4+1}\right| + C$$

.

2.3.5 Integrals of some special rational algebraic functions

In this section we will discuss various types of special rational algebraic functions and the method of integrating such functions.

2.3.5 (a) Integrals of the type $\int \frac{1}{ax^2+bx+c}dx$.

These type of integrals can be reduced to either of the standard rational integrals discussed in section 2.4.1, as follows :

Algorithm 1

Step 1: Write $ax^2+bx+c=a(x^2+\frac{b}{a}x+\frac{c}{a})$, i.e. from the expression ax^2+bx+c, take out the coefficient of x^2 common.

Step 2: Now add and subtract square of half the coefficient of x in the parenthesis

$$ax^2+bx+c=a(x^2+\tfrac{b}{a}x+\left(\tfrac{b}{2a}\right)^2-\left(\tfrac{b}{2a}\right)^2+\tfrac{c}{a}) = a\left[(x+\tfrac{b}{2a})^2+\left(\sqrt{\tfrac{c}{a}-\left(\tfrac{b}{2a}\right)^2}\right)^2\right]$$

Step 3: Put $x+\frac{b}{2a}=t$. The given integral reduces to either of the first three integrals.

Example 9. Evaluate: $\int\frac{1}{1-6x-9x^2}dx$.

Solution: We have $1-6x-9x^2=-9(x^2+\frac{2}{3}x-\frac{1}{9})$

$$= -9(x^2+\tfrac{2}{3}x+\tfrac{1}{9}-\tfrac{1}{9}-\tfrac{1}{9})$$

(adding and subtracting square of half the coefficient of x, that is $\left(\frac{1}{2}.\frac{2}{3}\right)^2=\frac{1}{9}$)

$=-9\left((x+\frac{1}{3})^2-\frac{2}{9}\right)$. Thus we have $\int\frac{1}{1-6x-9x^2}dx = -\frac{1}{9}\int\frac{1}{(x+\frac{1}{3})^2-\frac{2}{9}}dx$

Now put $x+\frac{1}{3}=t$, Then $dx=dt$.

$$\therefore \int \frac{1}{1-6x-9x^2}dx = -\frac{1}{9}\int \frac{1}{t^2-\left(\frac{\sqrt{2}}{3}\right)^2}dt \quad = \quad \frac{1}{9}\int \frac{1}{\left(\frac{\sqrt{2}}{3}\right)^2-t^2}dt = \frac{1}{9}\frac{1}{2\frac{\sqrt{2}}{3}}\ln\left(\frac{\frac{\sqrt{2}}{3}+t}{\frac{\sqrt{2}}{3}-t}\right)+C \;,$$

(using standard rational integral II with $a=\dfrac{\sqrt{2}}{3}$)

$$= \frac{1}{6\sqrt{2}}\ln\left|\frac{\frac{\sqrt{2}}{3}+x+\frac{1}{3}}{\frac{\sqrt{2}}{3}-x-\frac{1}{3}}\right|+C = \frac{1}{6\sqrt{2}}\ln\left|\frac{\sqrt{2}+3x+1}{\sqrt{2}-3x-1}\right|+C\;.$$

Example 10. Evaluate $\displaystyle\int \frac{x\,dx}{x^2-4x+8}$.

Solution: Completing the square yields

$x^2-4x+8=\left(x^2-4x+4\right)+8-4=\left(x-2\right)^2+4$, now we take

$u=x-2$, $du=dx$ yields

$$\int \frac{x\,dx}{x^2-4x+8} = \int \frac{x\,dx}{\left(x-2\right)^2+4} = \int \frac{\left(u+2\right)du}{u^2+4} = \int \frac{u\,du}{u^2+4}+2\int \frac{du}{u^2+4}$$

We get $\dfrac{1}{2}\ln\left(u^2+4\right)+2\left(\dfrac{1}{2}\right)\tan^{-1}\dfrac{u}{2}+c=\dfrac{1}{2}\ln\left[\left(x-2\right)^2+4\right]+\tan^{-1}\left(\dfrac{x-2}{2}\right)+c$

Example 11. Evaluate $\displaystyle\int \frac{x^2+x-1}{x^2+x-6}dx$.

Solution: $\displaystyle\int \frac{x^2+x-1}{x^2+x-6}dx = \int \frac{x^2+x-6+5}{x^2+x-6}dx = \int \frac{x^2+x-6}{x^2+x-6}dx + 5\int \frac{1}{x^2+x-6}dx$

$$= \int 1dx + 5\int \frac{1}{x^2+x-6}dx \;= x+5\int \frac{1}{x^2+x-6}dx \qquad \text{---} \qquad (1)$$

To evaluate $\displaystyle\int \frac{1}{x^2+x-6}dx$, we will apply Algorithm 1.

Now $x^2+x-6 \;=\; x^2+x+\frac{1}{4}-\frac{1}{4}-6 \;=\; (x+\frac{1}{2})^2-\dfrac{25}{4}$

$$\int \frac{1}{x^2+x-6}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2-\left(\frac{5}{2}\right)^2}dx = \frac{1}{5}\ln\left|\frac{x+\frac{1}{2}-\frac{5}{2}}{x+\frac{1}{2}+\frac{5}{2}}\right| + C$$

(Using standard rational integral I with $x = x+\frac{1}{2}$ and $a=\frac{5}{2}$).

Substituting in (1), we get

$$\int \frac{x^2+x-1}{x^2+x-6}dx = x + \ln\left|\frac{x-2}{x+3}\right| + C$$

2.3.5 (b) Integrals of the type $\int \frac{px+q}{ax^2+bx+c}dx$.

In this type of integrals we find constants *A* and *B* such that

$px+q = A.\frac{d}{dx}(ax^2+bx+c)+B$. This can be done as follows :

Algorithm 2

Step 1: Put $px+q = A.\frac{d}{dx}(ax^2+bx+c)+B$.

Step 2: Simplify the Right Hand Side of the equation in step 1.

Step 3: Now equate the coefficients of *x* and the constant terms on both sides of the equation obtained in step 2.

Step 4: Solve the simultaneous equations obtained in step 3 to get the required values of *A* and *B*.

Step 5: Substitute the values of A and B in the equation in step 1.

Step 6: Substitute the value of $px+q$ obtained from step 5 in the given integral. The integral reduces to $\int \frac{px+q}{ax^2+bx+c}dx = A\int \frac{f'(x)}{f(x)}dx + B\int \frac{1}{ax^2+bx+c}dx$.

Step 7: The first integral can be solved by the substitution $f(x)=t$ and the second integral can be solved by the method discussed in 4.2.1.

Example 12. Evaluate $\int \frac{2x-3}{x^2+3x-18}dx$.

Solution: Put $2x-3 = A.\frac{d}{dx}(x^2+3x-18)+B = A.(2x+3)+B$. Equating the coefficients of x and the constant terms on both sides, we get $2A=2$ and $3A+B=-3$. Solving these two equations simultaneously, we have $A=1$ and $B=-6$. $\therefore 2x-3=1.(2x+3)-6$.
Substituting this value of $2x-3$ in the given integral , we have

$$\int \frac{2x-3}{x^2+3x-18}dx = \int \frac{1.(2x+3)-6}{x^2+3x-18}dx = \int \frac{1.(2x+3)}{x^2+3x-18}dx - 6\int \frac{1}{x^2+3x-18}dx \quad --- \quad (1)$$

Now

$$\int \frac{1(2x+3)}{x^2+3x-18}dx = \ln|x^2+3x-18| + C_1.$$

And now for $\int \frac{1}{x^2+3x-18}dx$, we see that

$$x^2+3x-18 = x^2+3x+\frac{9}{4}-\frac{9}{4}-18 = (x+\tfrac{3}{2})^2-(\tfrac{9}{2})^2.$$

$$\int \frac{1}{x^2+3x-18}dx = \int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}dx = \frac{1}{9}\ln\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C = \frac{1}{9}\ln\left|\frac{x-3}{x+6}\right| + C.$$

Hence from (1), we get

$$\int \frac{(2x+3)}{x^2+3x-18}dx = \ln|x^2+3x-18| + \frac{1}{9}\ln\left|\frac{x-3}{x+6}\right| + C.$$

2.3.5 (c) Integrals of the type $\int \frac{x^2 \pm a^2}{x^4+kx^2+a^4}dx$.

To solve these type of integrals we proceed as follows :

Algorithm 3

Step 1: Divide numerator and denominator of the integrand by x^2, we get

$$\int \frac{x^2 \pm a^2}{x^4+kx^2+a^4}dx = \int \frac{1 \pm \frac{a^2}{x^2}}{x^2+k+\frac{a^4}{x^2}} \quad --- \quad (1)$$

Step 2: Now in case of $\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx$, put $x-\frac{a^2}{x}=t$, so that on differentiating with respect to x we get $\left(1+\frac{a^2}{x^2}\right)dx=dt$.

Step 3: Simplify $\left(x-\frac{a^2}{x}\right)^2=t^2$ to get $x^2-2a^2+\frac{a^4}{x^2}=t^2$ or $x^2+\frac{a^4}{x^2}=t^2+2a^2$. Substitute these values of $\left(1+\frac{a^2}{x^2}\right)dx$ and $x^2+\frac{a^4}{x^2}$ in (1).

Step 4: The given integral reduces to any of the standard rational integrals.

Note: In case of $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$, put $x+\frac{a^2}{x}=t$ in step 2 above and then proceed .

Example 13. Evaluate $\int \frac{x^2+1}{x^4-x^2+1}dx$.

Solution: Dividing numerator and denominator by x^2, we get $\int \frac{x^2+1}{x^4-x^2+1}dx=\int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}dx$. Now put $x-\frac{1}{x}=t$, then $\left(1+\frac{1}{x^2}\right)dx=dt$. Also $\left(x-\frac{1}{x}\right)^2=t^2$ i.e. $x^2-2+\frac{1}{x^2}=t^2$ or $x^2+\frac{1}{x^2}=t^2+2$. Substituting these values in the above integral we get $\int \frac{x^2+1}{x^4-x^2+1}dx=\int \frac{1}{t^2+1}dt=\tan^{-1}t=\tan^{-1}\left(x-\frac{1}{x}\right)+c$.

2.3.5 (d) Integrals of the type $\int \frac{x^2}{x^4+kx^2+a^4}dx$ and $\int \frac{1}{x^4+kx^2+a^4}dx$.

Here we see that $\int \frac{x^2}{x^4+kx^2+a^4}dx=\frac{1}{2}\int \frac{2x^2}{x^4+kx^2+a^4}dx$

$$=\frac{1}{2}\left[\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx+\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx\right].$$

Similarly , $\int \frac{1}{x^4+kx^2+a^4}dx=\frac{1}{2a^2}\int \frac{2a^2}{x^4+kx^2+a^4}dx$

$$=\frac{1}{2a^2}\left[\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx-\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx\right].$$

Thus the given integrals reduces to the one discussed in **2.4.5 (c) .**

Example 14. Evaluate $\int \frac{1}{x^4 + x^2 + 1} dx$.

Solution: $\int \frac{1}{x^4 + x^2 + 1} dx = \frac{1}{2} \int \frac{2}{x^4 + x^2 + 1} dx \quad = \frac{1}{2} \left[\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx - \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx \right]$

$$\frac{1}{2}\left[\frac{1}{\sqrt{3}}\ tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) - \frac{1}{2}\ln\left|\frac{x^2-x+1}{x^2+x+1}\right|\right] + C.$$

{Solving by the method discussed in 4.2.3}.

2.3.5(e) Integrals reducible to any of the above forms

There exist many integrals which can be reduced to any of the above forms by making proper substitutions. We discuss below some of such integrals.

Example 15. Evaluate $\int \frac{x}{x^4 + x^2 + 1} dx$.

Solution: Put $x^2 = t$, so that on differentiating with respect to x we get $xdx = \frac{1}{2} dt$. Then we have $\int \frac{x}{x^4 + x^2 + 1} dx = \frac{1}{2} \int \frac{1}{t^2 + t + 1} dt$. This integral is of the form discussed in section 4.2.1. Now

$t^2 + t + 1 = t^2 + t + \frac{1}{4} + 1 - \frac{1}{4}$ (adding and subtracting square of half the coefficient of t)

$= \left(t + \frac{1}{2}\right)^2 + \frac{3}{4} = \left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$. Therefore $\frac{1}{2} \int \frac{1}{t^2 + t + 1} dt =$

$\frac{1}{2} \int \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt = \frac{1}{2} . \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{t + 1/2}{\sqrt{3}/2}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + C$. Putting back $t = x^2$, we get $\int \frac{x}{x^4 + x^2 + 1} dx = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C$.

Example 16. $\int \frac{e^{2x}}{e^{2x}+6e^{x}+5}dx$.

Solution: $\int \frac{e^{2x}}{e^{2x}+6e^{x}+5}dx = \int \frac{e^{x}.e^{x}}{e^{2x}+6e^{x}+5}dx$.

Now put $e^{x}=t$ so that on differentiating with respect to x we get, $e^{x}dx = dt$. Then we have $\int \frac{e^{2x}}{e^{2x}+6e^{x}+5}dx = \int \frac{t}{t^2+6t+5}dt$. This integral is of the type discussed in section 4.2.2 and to evaluate this integral we follow Algorithm 2.

Let $t = A\frac{d}{dt}(t^2+6t+5)+B$ .... (1)

Then $t = 2At+6A+B$. Equating the coefficients of t and constant terms on both sides we get $2A=1$(coefficients of t) $6A+B=0$ (constant terms)Solving the above equations we get, $A=\frac{1}{2}$ and $B=-3$. Substituting in (1), we get

$t=\frac{1}{2}(2t+6)-3$. Then we have $\int \frac{t}{t^2+6t+5}dt = \int \frac{\frac{1}{2}(2t+6)-3}{t^2+6t+5}dt =$

$\frac{1}{2}\int \frac{(2t+6)}{t^2+6t+5}dt - 3\int \frac{1}{t^2+6t+5}dt = I_1 - I_2$.

Now, $I_1 = \frac{1}{2}\int \frac{(2t+6)}{t^2+6t+5}dt = \frac{1}{2}\ln|t^2+6t+5|+c$

(By Theorem 2.1.2 of section 2.1.2) $=\frac{1}{2}\ln|e^{2x}+6e^{x}+5|+c$.

$I_2 = 3\int \frac{1}{t^2+6t+5}dt$. This integral is of the type discussed in section 4.2.1. We have $t^2+6t+5 = t^2+6t+9-4$, (adding and subtracting square of half the coefficient of t). = $(t+3)^2-2^2$. Then

$$= 3\int \frac{1}{t^2+6t+5}dt = 3\int \frac{1}{(t+3)^2-2^2}dt = \frac{3}{4}\ln\left|\frac{t+1}{t+5}\right|+c$$

(By standard rational integral I with $a = 2$)

$$= \frac{3}{4} \ln \left| \frac{e^x + 1}{e^x + 5} \right| + c.$$

Hence $\int \frac{e^{2x}}{e^{2x} + 6e^x + 5} dx = \frac{1}{2} \ln|e^{2x} + 6e^x + 5| - \frac{3}{4} \ln \left| \frac{e^x+1}{e^x+5} \right| + c.$

Example17. Evaluate $\int \frac{1}{x(x^n + 1)} dx$.

Solution: $\int \frac{1}{x(x^n + 1)} dx = \int \frac{x^{n-1}}{x^n (x^n + 1)} dx$ (Note the method used. We have multiplied x^{n-1} in the numerator and denominator so as to get a proper substitution)

Now put $x^n = t$ so that on differentiating with respect to x, we get $x^{n-1} dx = \frac{1}{n} dt$.

Then we have $\int \frac{x^{n-1}}{x^n (x^n + 1)} dx = \frac{1}{n} \int \frac{1}{t(t+1)} dt$ ……..etc.

2.3.6 Integrals of some special irrational algebraic functions

2.3.6(a) Integrals of the type $\int \frac{1}{(ax+b)^{\frac{1}{n}} \pm (ax+b)^{\frac{1}{m}}} dx$**.**

This type of integrals can be evaluated by putting $ax + b = t^p$, where *p* is the L.C.M of *m* and *n*.

Example 18. Evaluate $\int \frac{1}{(1+x)^{\frac{1}{2}} - (1+x)^{\frac{1}{3}}} dx$.

Solution: L.C.M of 2 and 3 is 6. Put $1 + x = t^6$ and $dx = 6.t^5$. We get

$\int \frac{1}{(1+x)^{\frac{1}{2}}-(1+x)^{\frac{1}{3}}}dx = 6\int \frac{t^5}{t^3+t^2}dt \qquad = 6\int \frac{t^3}{t-1}dt = 6\int\left(t^2+t+1+\frac{1}{t-1}\right)dt$

(Dividing t^3 by $t-1$)$=6\left(\frac{t^3}{3}+\frac{t^2}{2}+t+\ln(t-1)+c\right) = 2t^3+3t^2+6t+6\ln(t-1)+c = 2(1+x)^{\frac{1}{2}}+3(1+x)^{\frac{1}{3}}+6(1+x)^{\frac{1}{6}}+6\ln\left|(1+x)^{\frac{1}{6}}-1\right|+c$.

2.3.6(b) Integrals of the type $\int \frac{f(x)}{(ax+b)\sqrt{px+q}}dx$ **and** $\int \frac{g(x)}{(ax^2+bx+c)\sqrt{px+q}}dx$

, **where** $f(x)$ and $g(x)$ are some linear functions in x. To evaluate these type of integrals, put $px+q=t^2$. Find dx and substitute the value of dx and x in the given integral. The integral reduces to an algebraic function which can be solved by any of the methods discussed in section 2.4.5.

Example 19. Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$.

Solution: Put $x+1=t^2$. Then we have $dx=2tdt$, $x=t^2-1$ and $x^2=t^4-2t^2+1$. Substituting these values in the given integral we get

$\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx = \int \frac{(t^2+1)2tdt}{(t^4+t^2+1)t} = 2\int \frac{(t^2+1)dt}{(t^4+t^2+1)}$. This type of integral is discussed in section 2.4.5(c). Applying the method discussed there, we get

$\int \frac{(t^2+1)dt}{(t^4+t^2+1)} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right)+c = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+c$. Hence

$\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+c$.

2.3.6(c) Integrals of the type $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$ **and** $\int \sqrt{ax^2+bx+c}\,dx$**.**

These integrals can be reduced to any of the standard irrational integral by the method discussed in section 2.4.5(a).

Example 20. Evaluate $\int \frac{1}{\sqrt{2+3x-2x^2}}dx$.

Solution: Applying the method discussed in section 3.2.1, we get

$2+3x-2x^2=-2\left(x^2-\frac{3}{2}x-1\right)$ (Taking out common the coefficient of x^2)

$=-2\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}-1\right)$ (Adding and subtracting square of half the

coefficient of x). $=2\left((\frac{5}{4})^2-(x-\frac{3}{4})^2\right)$ $=-2\left((x-\frac{3}{4})^2-\frac{25}{16}\right)$

Hence, $\int\frac{1}{\sqrt{2+3x-2x^2}}dx=\int\frac{1}{\sqrt{2\left((\frac{5}{4})^2-(x-\frac{3}{4})^2\right)}}dx=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{(\frac{5}{4})^2-(x-\frac{3}{4})^2}}dx$

$=\frac{1}{\sqrt{2}}\sin^{-1}\frac{x-\frac{3}{4}}{\frac{5}{4}}+c=\frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{5}+c$. (Using standard irrational integral I)

2.3.6(d) Integrals of the type $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$ and $\int(px+q)\sqrt{ax^2+bx+c}dx$.

To evaluate this type of integrals , proceed as follows :

Step 1: Put $px+q=A.\frac{d}{dx}(ax^2+bx+c)+B$ $=A.(2ax+b)+B$ and simplify the Right Hand Side.

Step 2: Equate the coefficients of x and the constant terms on both sides.

Step 3: Solve the simultaneous equations thus obtained to get the required values of *A* and *B*.

Step 4: Replace *A* and *B* in step 1, with the corresponding values obtained in step 3 to get $px+q$ in an expanded form.

Step 5: Replace $px+q$ in the given integral with its expanded form obtained in step 4. The given integral reduces to any of the previously known forms.

Example 21. Evaluate $\int\frac{x+3}{\sqrt{x^2+2x+2}}dx$.

Solution: Put $x+3 = A.\frac{d}{dx}(x^2+2x+2)+B = A.(2x+2)+B = 2Ax+2A+2B$.

Equating the coefficients of x and the constant terms on both sides , we get

$2A=1$ and $2A+B=3$. Solving these two equations simultaneously , we have $A=\frac{1}{2}$ and $B=2$. Therefore, $x+3=\frac{1}{2}.(2x+2)+2$. Substituting this value of $x+3$ in the given integral, we have

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}dx = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+2}}dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+2}}dx + \int \frac{1}{\sqrt{x^2+2x+2}}dx \quad \text{--- (1)}$$

Now, $\frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+2}}dx = \frac{1}{2}2\sqrt{x^2+2x+2}+c_1 = \sqrt{x^2+2x+2}+c_1$

{Note: $\int \frac{f'(t)}{\sqrt{f(t)}}dt = 2.\sqrt{f(t)}$ }

To evaluate $\int \frac{1}{\sqrt{x^2+2x+2}}dx$ we apply the method discussed in section 2.4.6(c).Thus we get,

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \ln\left|(x+1)+\sqrt{(x+1)^2+1}\right| + c_2.$$

Hence from (1) we see that

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \sqrt{x^2+2x+2} + \ln\left|(x+1)+\sqrt{(x+1)^2+1}\right| + c\ ;\ c = c_1 + c_2$$

2.3.6(e) Integrals of the type $\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx$ **.**

To evaluate this type of integrals , proceed as follows :

Step 1: Put $px^2+qx+r=A.(ax^2+bx+c)+B.\frac{d}{dx}(ax^2+bx+c)+C$ and simplify the Right Hand Side.

Step 2: Equate the coefficients of x^2 , x and the constant terms on both sides.

Step 3: Solve the simultaneous equations thus obtained to get the required values of A , B and C.

Step 4: Replace A, B and C in step 1, with the corresponding values obtained in step 3 to get px^2+qx+r in an expanded form.

Step 5: Replace px^2+qx+r in the given integral with its expanded form obtained in step 4. The integral reduces to any of the previously known forms.

Example 22. Evaluate $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}dx$

Solution: Put $x^2+2x+3=A.(x^2+x+1)+B.\frac{d}{dx}(x^2+x+1)+C$. We get

$x^2+2x+3=A.x^2+(A+2B)x+A+C$. Equating the coefficients of x^2 , x and the constant terms on both sides , we get $A=1$, $A+2B=2$ and $A+C=3$. Solving these equations simultaneously , we get $A=1$, $B=\frac{1}{2}$ and $C=2$. Hence $x^2+2x+3=1.(x^2+x+1)+\frac{1}{2}.(2x+1)+2$. Substituting this value of x^2+2x+3 in the given integral , we have $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}dx=\int \frac{1.(x^2+x+1)+\frac{1}{2}.(2x+1)+2}{\sqrt{x^2+x+1}}dx$

$$=\int \sqrt{x^2+x+1}dx+\frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}dx+2\int \frac{1}{\sqrt{x^2+x+1}}dx \qquad \text{---} \quad (1)$$

These integrals can be solved by the methods discussed in Type III and IV . Thus we have

$$\int \sqrt{x^2+x+1}\,dx = \frac{1}{2}\left[(x+\tfrac{1}{2})\sqrt{(x+\tfrac{1}{2})^2+\tfrac{5}{4}}+\tfrac{5}{4}\sinh^{-1}\frac{2x+1}{\sqrt{5}}\right]+c_1$$

$$\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx = 2.\sqrt{x^2+x+1}+c_2$$

$$\int \frac{1}{\sqrt{x^2+x+1}}\,dx = \sinh^{-1}\frac{2x+1}{\sqrt{5}}+c_3$$

Substituting the values of these integrals in (1) , we get

$$\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\left[(x+\tfrac{1}{2})\sqrt{(x+\tfrac{1}{2})^2+\tfrac{5}{4}}+\tfrac{5}{4}\sinh^{-1}\frac{2x+1}{\sqrt{5}}\right]+\sqrt{x^2+x+1}+2\sinh^{-1}\frac{2x+1}{\sqrt{5}}+c$$

or $$\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\left[(x+\tfrac{1}{2})\sqrt{(x+\tfrac{1}{2})^2+\tfrac{5}{4}}\right]+\sqrt{x^2+x+1}+\tfrac{21}{8}\sinh^{-1}\frac{2x+1}{\sqrt{5}}+c$$

2.3.6(f) Integrals of the type $\int \frac{1}{(px+q)^r\sqrt{ax^2+bx+c}}\,dx$.

To evaluate this type of integrals, put $px+q=\frac{1}{t}$, $dx=-\frac{1}{pt^2}\,dt$. Substitute the value of $px+q$, dx and $x=\frac{1}{tp}-\frac{q}{p}$, in the given integral. The integral reduces to any of the previously known forms.

Example 23. Evaluate $\int \frac{1}{(2x+1)^2\sqrt{x^2+x+1}}\,dx$.

Solution: Put $2x+1=\frac{1}{t}$, $x=\frac{1}{2t}-\frac{1}{2}$, $dx=-\frac{1}{2t^2}\,dt$.

We get, $\int \frac{1}{(2x+1)^2\sqrt{x^2+x+1}}\,dx = -\int \frac{t}{\sqrt{3t^2+1}}\,dt$

Note that here it is possible to make the substitution $3t^2+1=u^2$, $tdt=\frac{1}{3}udu$. Thus we get $-\int\frac{t}{\sqrt{3t^2+1}}dt=-\frac{1}{3}\int du=-\frac{1}{3}u=-\frac{1}{3}\sqrt{3t^2+1}$. Therefore

$$\int\frac{1}{(2x+1)^2\sqrt{x^2+x+1}}dx=-\frac{2}{3}\frac{\sqrt{x^2+x+1}}{2x+1}+c.$$

2.3.6(g) Integrals of the type $\int\frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$.

To evaluate these type of integrals put $x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$. The given integral becomes $-\int\frac{t}{(a+bt^2)\sqrt{c+dt^2}}dt$. Now put $c+dt^2=u^2$, $tdt=\frac{1}{d}udu$. The given integral reduces to any of the previously known forms.

Example24. Evaluate $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Solution: Put $x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$ in the given integral. We get

$\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}=-\int\frac{tdt}{(1+t^2)\sqrt{t^2-1}}$. Now put $t^2-1=u^2$, $tdt=udu$. We get

$\int\frac{tdt}{(1+t^2)\sqrt{t^2-1}}=\int\frac{du}{u^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}$. Therefore

$$\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{t^2-1}}{\sqrt{2}}=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1-x^2}}{x\sqrt{2}}+c.$$

2.3.6(h) Integrals of the type $\int x^m(a+bx^n)^p\,dx$.

In this type of integrals we discuss the following cases.

Case 1. p is an integer. In this case expand $(a+bx^n)^p$ binomially and then integration of each term is possible.

Case 2. p is not an integer but $\frac{m+1}{n}$ is an integer. Suppose $p=\frac{r}{s}$. Put $a+bx^n=t^s$ and appropriate value of dx in the given integral.

Case 3. p is not an integer but $p+\frac{m+1}{n}$ is an integer. In this case put $x=\frac{1}{t}$ and appropriate value of dx in the given integral. The integral reduces to that of case 2.

Example 25. Evaluate $\int x^{-\frac{2}{3}}(1+x^{\frac{1}{2}})^{-\frac{5}{3}}dx$.

Solution: Here p is not an integer but $p+\frac{m+1}{n}$ = -1 is an integer. So put $x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$ in the given integral. We get $\int x^{-\frac{2}{3}}(1+x^{\frac{1}{2}})^{-\frac{5}{3}}dx=-\int t^{-\frac{1}{2}}(1+t^{\frac{1}{2}})^{-\frac{5}{3}}dt$. Here we see that $p=-\frac{5}{3}$ is not an integer but $\frac{m+1}{n}=1$ is an integer. Now put $1+t^{\frac{1}{2}}=u^3$, $t^{-\frac{1}{2}}dt=6u^2du$. We get $\int t^{-\frac{1}{2}}(1+t^{\frac{1}{2}})^{-\frac{5}{3}}dt=\int u^{-5}6u^2du=3u^{-2}+c$.

Therefore we have $\int x^{-\frac{2}{3}}(1+x^{\frac{1}{2}})^{-\frac{5}{3}}dx=-3(1+x^{-\frac{1}{2}})^{-\frac{2}{3}}+c$.

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✔ **QUICK ACTIVITY EXERCISES 2.3 (*See after exercises for answers.*)**

a) Evaluate the integral :

$$\int \frac{1}{x^2+x-2}dx$$

b. In each part, determine the substitution u

1) $\int \frac{1}{x^2-2x+10}\,dx = \int \frac{1}{u^2+1}du$; **u** =?

2) $\int \sqrt{x^2-6x+8}\,dx = \int \sqrt{u^2-1}du$ **u** =?

✔ EXERCISE SET 2.3

1. Evaluate the following integrals.

1) $\int \frac{3x^5}{1+x^{12}}dx$	16) $\int \frac{dx}{(2x^2+3)\sqrt{x^2-4}}$
2) $\int \frac{\operatorname{cosec}^2 x}{1-\cot^2 x}dx$	17) $\int x^2(1+2x^4)^{-\frac{3}{4}}dx$
3) $\int \frac{dx}{9x^2-1}$	18) $\int x^{\frac{2}{3}}(2+x^{\frac{6}{5}})^2dx$
4) $\int \frac{x}{x^4-9}dx$ v)	19) $\int \frac{1+\sqrt{x}}{1+x^{\frac{1}{4}}}dx$
5) $\int \frac{dx}{e^x+e^{-x}}dx$	20) $\int \frac{x}{x^4-x^2+1}dx$
6) $\int \frac{x^2}{x^2-9}dx$	21) $\int \frac{dx}{x^2-3x-10}$
7) $\int \frac{3x}{1+2x^4}dx$	22) $\int \frac{dx}{x^2-7x-8}$
8) $\int \frac{\sin x}{1-4\cos^2 x}dx$	23) $\int \frac{x^2-7}{x+3}dx$,
9) $\int \frac{x^2}{1-x^6}dx$	24) $\int \frac{3x^2-10}{x^2-4x+4}dx$
10) $\int \frac{x^2-1}{x^2+4}dx$	25) $\int \frac{2x^2+3}{x(x-1)^2}dx$,
11) $\int \frac{e^x}{e^{2x}+1}dx$	26) $\int \frac{dx}{x^2(x^2-1)}$,
12) $\int x\sqrt{x+x^2}\,dx$	27) $\int \frac{x^2+1}{x-1}dx$,

13) $\int \sqrt{3-2x-2x^2}\,dx$

14) $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$

15) $\int \frac{x^3+3}{\sqrt{x^2+1}}dx$

28) $\int \frac{x^2+1}{x^2-3x+2}dx$,

29) $\int \frac{2x^2-10x+4}{(x+1)(x-3)^2}dx$,

30) $\int \frac{1}{x^2+8x+20}dx$

2. Evaluate the following integrals:

1. $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}}$
2. $\int \frac{1+x^{\frac{1}{2}}}{1+x^{\frac{1}{3}}}dx$
3. $\int \frac{dx}{x+\sqrt{x^2-1}}$
4. $\int \frac{x+1}{(x-1)\sqrt{x+2}}dx$
5. $\int \frac{1}{(x^2+1)\sqrt{x}}dx$
6. $\int \frac{xdx}{(x+3)\sqrt{x+1}}$
7. $\int \frac{x+3}{\sqrt{x^2+2x+2}}dx$
8. $\int \frac{x^5}{\sqrt{1+x^3+x^6}}dx$
9. $\int (3x-2)\sqrt{x^2+x+1}dx$
10. $\int \frac{x^2-2}{\sqrt{3-x^2}}dx$
11. $\int \frac{dx}{(x-3)^2\sqrt{x^2-6x+8}}$
12. $\int \frac{dx}{(x^2+1)\sqrt{1-x^2}}$
13. $\int \frac{\sqrt{1+x^2}}{1-x^2}dx$
14. $\int \frac{x+1}{(x^2+4)\sqrt{x^2+9}}dx$
15. $\int x^4(a^3+x^3)^{\frac{1}{3}}dx$

3. Evaluate the following integrals.

1. $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$
2. $\int \frac{x^2+5x+3}{x^2+3x+2}dx$
3. $\int \frac{1}{3x^2+13x-10}dx$
4. $\int \frac{1}{1-6x-9x^2}dx$
5. $\int \frac{x}{x^2+x+1}dx$
6. $\int \frac{3x+1}{2x^2-2x+3}dx$
7. $\int \frac{2x+1}{4-3x-x^2}dx$
8. $\int \frac{x^2+1}{x^4+1}dx$
9. $\int \frac{dx}{x^4+1}$

4. Evaluate the following integrals:.

1) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx$ 2) $\int \frac{x^2}{\sqrt{x^6+a^6}}dx$ 3) $\int \frac{2^x}{\sqrt{1-4^x}}dx$

4) $\int \cos x\sqrt{4-\sin^2 x}\,dx$, 5) $\int \frac{16+(\ln x)^2}{x}dx$ v) , 6) $\int \frac{dx}{\sqrt{1-e^{2x}}}$.

7) $\int \frac{dx}{\sqrt{15-8x^2}}$ 8) $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ 9) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}dx$

10) $\int \frac{\sin x}{\sqrt{4\cos^2 x-1}}dx$ 11) $\int e^x\sqrt{e^{2x}+4}\,dx$ 12) $\int \sqrt{2x^2-3}\,dx$

✔ **QUICK ACTIVITY ANSWERS 2.3**

a) $\frac{1}{3}ln\left|\frac{x-1}{x+2}\right|+\text{c}$.

b) 1) $u=x-1$, 2) $u=x-3$.

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2.4 INTEGRATION BY PARTIAL FRACTIONS.

2.4.1 Introduction

In this section we will discuss the techniques of integrating some rational functions by making use of partial fractions. Recall that a rational function is the ratio of two polynomial functions, or in other words, rational function is a function which can be written in the form of $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x)\neq 0$. If degree of $P(x)$ is greater than degree of $Q(x)$ then $\frac{P(x)}{Q(x)}$ is an improper rational function. If degree of $P(x)$ is less than degree of $Q(x)$ then $\frac{P(x)}{Q(x)}$ is a proper rational function. Every improper rational function can be expressed as the sum of a polynomial and a proper rational function and every rational function can be expressed as the sum of some basic rational functions also known as partial fractions of the given function. Some of the important basic rational functions are $\frac{1}{ax+b}$ (linear), $\frac{1}{(ax+b)^n}$ where n is a positive integer, $\frac{1}{ax^2+bx+c}$ (quadratic) and $\frac{1}{(ax^2+bx+c)^n}$.

2.4.2 Partial Fractions In this section we will discuss various methods of reducing a proper rational function into partial fractions.

Let $\frac{P(x)}{Q(x)}$ be any proper rational function.

Case 1: When the denominator can be expressed as the product of non repeated linear factors. Let $Q(x)=(a_1x+b_1)(a_2x+b_2)...(a_nx+b_n)$.

Write $\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+...+\frac{A_n}{a_nx+b_n}$ $\quad\cdots\quad$ (1)

Where $A_1, A_2, ...A_n$ are constants whose values are to be determined. Simplify the Right Hand Side of the above equation. We get

$\frac{P(x)}{Q(x)} = \frac{F(x)}{(a_1x+b_1)(a_2x+b_2)...(a_nx+b_n)}$, where $F(x)$ is a polynomial function. Since the denominators on the L.H.S and R.H.S are same , the numerators must also be equal. Hence equating the coefficients of like powers of x in $P(x)$ and $F(x)$ we get some linear equations in terms of the unknowns $A_1, A_2, ...A_n$. Solving these equations we find the values of $A_1, A_2, ...A_n$. Substituting these values of $A_1, A_2, ...A_n$ in equation (1) we get the required partial fractions of the given function.

Now we will discuss some more different cases of the denominator. The method will differ only in formation of equation (1) whereas the remaining procedure will be the same as above.

Case 2: When the denominator can be expressed as the product of repeated linear factors.

Let $Q(x)=(ax+b)(ax+b)...(ax+b)=(ax+b)^n$. Note that here $Q(x)$ is the product of the linear factor $ax+b$ repeated n times.

In this case we write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n} \qquad \ldots. \quad (1)$$

Now proceed as in case 1.

Case 3: When the denominator can be expressed as the product of repeated and non-repeated linear factors.

(3A) Let $Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)(ax+b)^m$. In this case we write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_n}{a_nx+b_n} + \frac{B_1}{ax+b} + \frac{B_2}{(ax+b)^2} + \ldots + \frac{B_m}{(ax+b)^m} \quad \ldots..(1)$$

(3B) Let $Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)(ax+b)^m(cx+d)^p$. In this case we write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_n}{a_nx+b_n} + \frac{B_1}{ax+b} + \frac{B_2}{(ax+b)^2} + \ldots + \frac{B_m}{(ax+b)^m}$$
$$+ \frac{C_1}{cx+d} + \frac{C_2}{(cx+d)^2} + \ldots + \frac{C_p}{(cx+d)^p}.$$

Case 4: When the denominator can be expressed as the product of non-repeated quadratic factors which cannot be further factorised to linear factors.

Let $Q(x) = (a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\ldots.(a_nx^2+b_nx+c_n)$.. In this case we write $\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{a_2x^2+b_2x+c_2} + \ldots + \frac{A_nx+B_n}{a_nx^n+b_nx+c_n}$ ….. (1)

Where $A_1, A_2, \ldots A_n$, $B_1, B_2, \ldots B_n$ are all constants whose values are to be determined by following the same procedure as in case 1.

Case 5: When the denominator can be expressed as the product of repeated quadratic factors.

Let $Q(x) = (ax^2+bx+c)(ax^2+bx+c)\ldots.(ax^2+bx+c) = (ax^2+bx+c)^n$. In this case we write $\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$.

2.4.3 Use of Partial Fractions for Integration.

To integrate rational functions of the form $\frac{P(x)}{Q(x)}$ we will proceed as per the following:

Algorithm 1: To evaluate $\int \frac{P(x)}{Q(x)}dx$.

Step 1: If $\frac{P(x)}{Q(x)}$ is a proper rational function, that is if degree of $P(x)$ is less than degree of $Q(x)$, then go to step 3.

Step 2: If $\frac{P(x)}{Q(x)}$ is an improper rational function, that is if degree of $P(x)$ is greater than or equal to degree of $Q(x)$, then divide $P(x)$ by $Q(x)$ and write $\frac{P(x)}{Q(x)}$ as the sum of a polynomial function and a proper rational function $\frac{P_1(x)}{Q(x)}$.

Step 3: Write the proper rational function $\frac{P_1(x)}{Q(x)}$ as sum of its partial fractions by any of the methods discussed in the previous section.

Step 4: Integrate both sides.

Example 1. Evaluate: $\int \frac{\mathrm{dx}}{x^2 + x - 2}$

Solution:

We note that the integrand is a proper rational function that can be written.

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

where A,B are constants to be determined:

$$1 = A(x+2) + B(x-1)$$

Setting $x = 1$ makes the second term drop out and yields $A = \frac{1}{3}$ and setting $x = -2$ makes the first term in drop out and yields $B = -\frac{1}{3}$. The partial fraction decomposition.

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

and:

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2} = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + c$$

Example 2. Evaluate $\int \frac{2x}{x^2+3x+2}dx$.

Solution: $\int \frac{2x}{x^2+3x+2}dx = \int \frac{2x}{(x+1)(x+2)}dx$. Note that the integrand $\frac{2x}{x^2+3x+2}$ is a proper rational function whose denominator can be expressed as product of non-repeated linear factors. So we apply the procedure of case 1 of section 2.4.2 to reduce $\frac{2x}{x^2+3x+2}$ as sum of partial fractions as follows :

Let $\frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ … (1)

$\Rightarrow \frac{2x}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$ (Note that R.H.S is simplified by cross multiplication method)

$\Rightarrow \frac{2x}{(x+1)(x+2)} = \frac{x(A+B)+(2A+B)}{(x+1)(x+2)}$.

Now equating the coefficient of like powers of x in the numerators on both sides we get $A+B=2$ (coefficient of x) and $2A+B=0$ (coefficient of x^0 or the constant terms).

Solving the above two equations simultaneously, we get $A=-2$ and $B=4$.
Substituting these values in (1) we get

$\frac{2x}{(x+1)(x+2)} = \frac{-2}{x+1} + \frac{4}{x+2}$. Therefore $\int \frac{2x}{x^2+3x+2}dx = \int \frac{2x}{(x+1)(x+2)}dx =$

$\int \frac{-2}{x+1}dx + \int \frac{4}{x+2}dx = \ln(x+1)^{-2} + \ln(x+2)^4 + \ln c = ln\left|\frac{c(x+2)^4}{(x+1)^2}\right|$

Example 3. $\int \frac{3x-2}{(x+1)^2(x+3)}dx$

Solution: Note that the integrand $\frac{3x-2}{(x+1)^2(x+3)}$ is a proper rational function whose denominator is expressed as product of repeated and non-repeated linear factors. So we apply the procedure of case 3 of section 2.4.2 to reduce $\frac{3x-2}{(x+1)^2(x+3)}$ as sum of partial fractions as follows :

Let $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$ … (1)

$\Rightarrow \frac{3x-2}{(x+1)^2(x+3)} = \frac{A(x+1)(x+3)+B(x+3)+C(x+1)^2}{(x+1)^2(x+3)}$

$\Rightarrow \frac{3x-2}{(x+1)^2(x+3)} = \frac{A(x^2+4x+3)+B(x+3)+C(x^2+2x+1)}{(x+1)^2(x+3)}$

$\Rightarrow \frac{3x-2}{(x+1)^2(x+3)} = \frac{x^2(A+C)+x(4A+B+2C)+(3A+3B+C)}{(x+1)^2(x+3)}$

Now equating the coefficient of like powers of x in the numerators on both sides we get $A+C=0$ (coefficient of x^2), $4A+B+2C=3$(coefficient of x) and

$3A+3B+C=-2$ (constant terms).

Solving the above three equations simultaneously, we get

$A=\frac{11}{4}$, $B=\frac{-5}{2}$ and $C=\frac{-11}{4}$. Substituting these values in (1) we get

$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)} + \frac{-5}{2(x+1)^2} - \frac{11}{4(x+3)}$

Therefore $\int \frac{3x-2}{(x+1)^2(x+3)}dx = \frac{11}{4}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+1)^2}dx - \frac{11}{4}\int \frac{1}{(x+3)}dx$

$$= \frac{11}{4}\ln\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + c$$

Example 4. Evaluate $\int \frac{x^2+1}{x^2-5x+6}dx$.

Solution: Note that the integrand $\frac{x^2+1}{x^2-5x+6}$ is an improper rational function, since the degree of numerator = degree of denominator = 2. So first we divide the numerator by denominator to reduce the integrand into a proper rational function. Dividing using long division process we get

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6} = 1 + \frac{5x-5}{(x-2)(x-3)} \quad \ldots. \quad (1)$$

Now we follow the method of case1 to reduce $\frac{5x-5}{(x-2)(x-3)}$ into sum of partial fractions.

Let $\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad \ldots. \quad (2)$

$$= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{x(A+B)-3A-2B}{(x-2)(x-3)}$$

Now equating the coefficient of like powers of x in the numerators on both sides we get $-3A-2B=-5$ ‘ $A+B=5$

Solving the above equations, we have $A=-5$ and $B=10$. Substituting these values in (2). Therefore from (1) $\frac{-5}{x-2} + \frac{10}{x-3} = \frac{5x-5}{(x-2)(x-3)}$

$$\int \frac{x^2+1}{x^2-5x+6}dx = \int 1dx - 5\int \frac{1}{x-2}dx + 10\int \frac{1}{x-3}dx$$

$$= x - 5\ln|x-2| + 10\ln|x-3| + \ln|c| = x + ln\left|\frac{(x-3)^{10}}{(x-2)^5}C\right|.$$

Example 5. Evaluate $\int \frac{x}{(x^2+1)(x-1)}dx$.

Solution: The integrand is a proper rational function, so we reduce it as sum of partial fractions. Note that denominator of the integrand is the product of a non-repeated linear factor ($x-1$) and a non-repeated quadratic factor (x^2+1). So let

$$\frac{x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \qquad \ldots. \qquad (1)$$

$$= \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)} = \frac{x^2(A+B)+x(-B+C)+(A-C)}{(x-1)(x^2+1)}$$

Now equating the coefficient of like powers of x in the numerators on both sides we get $A+B=0$, $A-C=0$, $-B+C=1$

Solving the above equations, we have $A=\frac{1}{2}$, $B=\frac{-1}{2}$ and $C=\frac{1}{2}$. Substituting these values in (1)

$\frac{x}{(x^2+1)(x-1)} = \frac{1}{2(x-1)} + \frac{1-x}{2(x^2+1)}$. Therefore

$$\int \frac{x}{(x^2+1)(x-1)}dx = \frac{1}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1-x}{x^2+1}dx =$$

$$= \frac{1}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{2}tan^{-1}x - \frac{1}{4}\ln(x^2+1) + c\ .$$

(Here the third integral on the R.H.S is integrated by substitution method taking $x^2+1=t$).

Example 6. Evaluate $\int \frac{2x+4}{x^3-2x^2}dx$

Solution:

Let $\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$

Then $$2x+4 = Ax(x-2)+B(x-2)+cx^2$$
$$2x+4 = (A+C)x^2+(-2A+B)x-2B$$

Putting x = 0, we get $B=-2$ and putting $x=2$ yields $C=2$. To find the value of A. we equate the coefficients of x^2 on both sides to get

$$A+C=0 \qquad \text{or} \qquad A=-C=-2$$

Thus the partial fraction decomposition:

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x}+\frac{-2}{x^2}+\frac{2}{x-2}$$ and:

$$\int \frac{2x+4}{x^2(x-2)}dx = -2\int\frac{dx}{x}-2\int\frac{dx}{x^2}+2\int\frac{dx}{x-2}$$
$$= -2\ln|x|+\frac{2}{x}+2\ln|x-2|+c$$
$$= 2\ln\left|\frac{x-2}{x}\right|+\frac{2}{x}+1$$

Example 7. Evaluate

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}dx$$

Solution:

We have: $3x^3-x^2+3x-1 = x^2(3x-1)+(3x-1) = (3x-1)(x^2+1)$

Thus, the partial fraction decomposition is

$$\frac{x^2+x-2}{(3x-1)(x^2-1)} = \frac{A}{3x-1}+\frac{Bx+C}{x^2+1}$$

Multiplying by $(3x-1)(x^2+1)$ yields

$$x^2+x-2 = A(x^2+1)+(Bx+C)(3x-1)$$

or: $x^2+x-2 = (A+3B)x^2+(-B+3C)x+(A-C)$

Equating coefficient of like powers of x on both sides gives

$$\begin{cases} A+3B=1 \\ \quad -B+3C=1 \\ A \qquad -C=-2 \end{cases}$$

Solving the above system, we find:

$$A=\frac{-7}{5}\ ,\ B=\frac{4}{5}\ ,\ C=\frac{3}{5}$$

Thus:

$$\int\frac{x^2+x-2}{(3x-1)\left(x^2+1\right)}dx=-\frac{7}{5}\int\frac{dx}{3x-1}+\frac{4}{5}\int\frac{x}{x^2+1}dx+\frac{3}{5}\int\frac{dx}{x^2+1}$$

$$=-\frac{7}{15}\ln|3x-1|+\frac{2}{5}\ln\left(x^2+1\right)+\frac{3}{5}\tan^{-1}x+C$$

Example 8. Evaluate

$$\int\frac{x^5-x^4+4x^3-4x^2+8x-4}{\left(x^2+2\right)^3}dx$$

Solution: We have:

$$\frac{x^5-x^4+4x^3-4x^2+8x-4}{\left(x^2+2\right)^3}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}+\frac{Ex+F}{\left(x^2+2\right)^3}$$

Then:

$$x^5-x^4+4x^3-4x^2+8x-4=Ax^5+Bx^4+\left(4Arc\right)x^3+$$
$$+\left(4B+D\right)x^2+\left(4A+2C+E\right)x+$$
$$+\left(4B+2D+F\right)$$

from which we get $A=1, B=-1, C=0, D=0, E=4, F=0$

thus the given integral becomes

$$I=\int\frac{x-1}{x^2+2}dx+4\int\frac{xdx}{\left(x^2+2\right)^2}=\frac{1}{2}\ln\left(x^2+2\right)-$$
$$-\frac{\sqrt{2}}{2}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{1}{\left(x^2+2\right)^2}+C$$

Example 9. Evaluate $\int\frac{x+1}{x^3-x^2}dx$

Solution:

We have $\frac{x+1}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}$

Then $x+1=A(x-1)+Bx(x-1)+cx^2$

or $x+1=(B+C)x^2+(A-B)x-A$

Equating corresponding coefficient we get $A=-1$, $B=-2$, $C=2$

Thus the given integral yields

$$I=-\int\frac{dx}{x^2}-2\int\frac{dx}{x}+2\int\frac{dx}{x-1}$$

$$=\frac{1}{x}-2\ln|x|+2\ln|x-1|+C$$

$$=\frac{1}{2}x+\ln\left(\frac{x-1}{x}\right)^2+C$$

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✔ QUICK ACTIVITY EXERCISES 2.4 (*See after exercises for answers.*)

- Complete the partial fraction decomposition **:**

1) $\dfrac{1}{x^2+x-2}=\dfrac{1}{(x-1)(x+2)}=\dfrac{A}{x-1}-\dfrac{\frac{1}{3}}{x+2}$.

2) $\dfrac{-3}{(x+1)(2x-1)}==\dfrac{A}{x+1}-\dfrac{2}{2x-1}$

3) $\dfrac{2x^2-3x}{(x^2+1)(3x+2)}=\dfrac{B}{3x+2}-\dfrac{1}{x^2+1}$

- Evaluate the integral.

4) $\displaystyle\int\frac{3}{(x+1)(1-2x)}$, 5) $\displaystyle\int\frac{2x^2-3x}{(x^2+1)(3x+2)}$

✔ EXERCISES SET 2.4

1. Evaluate the following integrals.	
i) $\displaystyle\int\frac{3x-1}{(x-1)(x-2)(x-3)}dx$ **ii)** $\displaystyle\int\frac{3x-1}{(x+2)^2}dx$	**viii)** $\displaystyle\int\frac{x^3+x-1}{\left(x^2+1\right)^2}dx$

iii) $\int \frac{2x-3}{(x^2-1)(2x+3)}dx$ **iv)** $\int \frac{5x}{(x^2-4)(x+1)}dx$ **v)** $\int \frac{x^2-3x-1}{x^3+x^2-2x}dx$ **vi)** $\int \frac{x^2+3x-4}{x^2-2x-8}dx$ **vii)** $\int \frac{x\,dx}{(x-2)^2}$	**ix)** $\int \frac{2x^3}{\left(x^2+1\right)^2}dx$ **x)** $\int \frac{2x^2+3}{\left(x^2+1\right)^2}dx$ **xi)** $\int \frac{3x^2-10}{x^2-4x+4}dx$ **xii)** $\int \frac{2x^2+3}{x(x-1)^2}dx$ **xiii)** $\int \frac{x^5+x^2+2}{x^3-x}\ dx$

2. Evaluate the following integrals .	
i) $\int \frac{2}{(1-x)(1+x^2)}dx$ **ii)** $\int \frac{x^2+x+1}{(x+2)(x^2+1)}dx$	**iii)** $\int \frac{x}{(x-1)^2(x+2)}dx$ **iv)** $\int \frac{1-x^2}{x(1-2x)}dx$

✔QUICK ACTIVITY ANSWERS 2.4

1) $A = 1/3$, 2) A=1 , 3) $B = 2$,

4) $\int \frac{3}{(x+1)(1-2x)} = ln\frac{x+1}{1-2x} + c$

5) $\int \frac{2x^2-3x}{(x^2+1)(3x+2)} = \frac{2}{3}ln|3x+2| - tan^{-1}x + c$.

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