Lecture (8): الفصل الثالث ، م. فاطمة المطوي .

Integration by parts:

our primary goal in this section is to develop a general method for attacking integrals of the form: $\int f(x)\, g(x)\, dx$.

Let $G(x)$ is any antiderivative of $g(x)$,

$$G'(x) = g(x).$$

$$\frac{d}{dx}\left[f(x)\, G(x)\right] = f(x)\, G'(x) + f'(x)\, G(x)$$

$$\frac{d}{dx}\left[f(x)\, G(x)\right] = f(x)\, g(x) + f'(x)\, G(x).$$

By integration:

$$f(x)\, G(x) = \int f(x)\, g(x)\, dx + \int f'(x)\, G(x)\, dx.$$

$$\int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx.$$

let: $u = f(x)$, $dv = g(x)\, dx$

$du = f'(x)\, dx$, $v = G(x)$

$-\int$

Then:

$$\boxed{\int u\, dv = uv - \int v\, du}$$ #

Ex ①: Use integration by parts to evaluate:

$$\int x \cos x \, dx .$$

Let: $u = x$ $\qquad$ $dv = \cos x \, dx$

$du = dx$ $\qquad$ $v = \int \cos x \, dx = \sin x$

$-\int$

$\int u \, dv = uv - \int v \, du$

$$\Rightarrow \int x \cos x \, dx = x \sin x - \int \sin x \, dx .$$

$$= x \sin x - (-\cos x) + C$$

$$= x \sin x + \cos x + C .$$

Guidelines for integration by parts:

[I] There is useful strategy for choosing u and d

that can be applied when the integrand is a product of two functions from different categories in the list:

Logarithmic, **I**nverse trignometric, **A**lgebric, **T**rigonometric, **E**xponential.

Ex ②: Evaluate $\int x\, e^x dx$

Let: $u = x$ $\qquad dv = e^x\, dx$

$du = dx$ $\qquad v = e^x$

$-\int$

$\boxed{\int u\,dv = uv - \int v\,du}$

$$\int x e^x\, dx = x e^x - \int e^x dx$$

$$= x e^x - e^x + C.$$

Ex ③: Evaluate $\int \ln x\, dx$.

Let $u = \ln x$ $\qquad dv = dx$

$du = \frac{1}{x}\, dx$ $\qquad v = x$

$-\int$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C.$$

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[2] It is sometimes necessary to use integration by parts more than once in the same problem.

Ex (4): Evaluate $\int x^2 e^{-x}\,dx$

$$u = x^2 \qquad dv = e^{-x}\,dx$$
$$du = 2x\,dx \qquad v = -e^{-x}$$
$$-\int$$

$$\int u\,dv = uv - \int v\,du$$

$$I = \int x^2 e^{-x}\,dx = -x^2 e^{-x} - \int -e^{-x}\,2x\,dx$$
$$= -x^2 e^{-x} + 2\underbrace{\int x e^{-x}\,dx}_{II}.$$

$$II = \int x e^{-x}\,dx$$

let
$$u = x \qquad dv = e^{-x}\,dx$$
$$du = dx \qquad v = -e^{-x}$$
$$-\int$$

$$II = \int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$$
$$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$$

Then:
$$I = -x^2 e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C$$
$$= -e^{-x}\left(x^2 + 2x + 2\right) + C.$$
#

Ex (5): Evaluate $\int e^x \cos x \, dx$

let $u = \cos x$ $\qquad$ $dv = e^x \, dx$

$du = -\sin x \, dx$ $\qquad$ $v = e^x$

$-\int$

$$I = \int e^x \cos x \, dx = e^x \cdot \cos x - \int e^x (-\sin x) \, dx$$

$$= e^x \cos x + \underbrace{\int e^x \sin x \, dx}_{II} .$$

$$II = \int e^x \sin x \, dx$$

$u = \sin x$ $\qquad$ $dv = e^x \, dx$

$du = \cos x \, dx$ $\qquad$ $v = e^x$

$-\int$

$$II = e^x \sin x - \int e^x \cos x \, dx .$$

Then:

$$I = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$I = e^x \cos x + e^x \sin x - I$$

$$I + I = e^x \cos x + e^x \sin x$$

$$2I = e^x \cos x + e^x \sin x$$

$$I = \frac{1}{2} \left(e^x \cos x + e^x \sin x \right) + C .$$

#

Integration by parts for definite integrals:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Ex ⑥: Evaluate $\int_0^1 \tan^{-1}x\,dx$.

Let $u = \tan^{-1}x$ $\qquad$ $dv = dx$

$du = \dfrac{1}{1+x^2}\,dx$ $\qquad$ $v = x$

$-\int$

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^1$$

$$= \tan^{-1}(1) - 0 - \frac{1}{2}\Big[\ln(2) - \underbrace{\cancel{\ln(1)}}_{=0}\Big]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2).$$

$$= \frac{\pi}{4} - \ln(2)^{\frac{1}{2}}$$

$$= \frac{\pi}{4} - \ln\sqrt{2}.$$

H.W: EXC 8.2, P. 520

(3 + 7 + 11 + 19)

Reduction formulas:—

① $\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\, dx.$

② $\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\, dx.$

Ex ⑦:

Evaluate $\int \cos^4 x\, dx$

Using $\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\, dx$ with $n=4$.

$$\int \cos^4 x\, dx = \frac{1}{4}\cos^{4-1}x\sin x + \frac{4-1}{4}\int \cos^{4-2}x\, dx$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\underbrace{\int \cos^2 x\, dx}_{\Downarrow\; n=2}$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left[\frac{1}{2}\cos^{2-1}x\sin x + \frac{2-1}{2}\int \cos^{2-2}x\, dx\right]$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left[\frac{1}{2}\cos x\sin x + \frac{1}{2}\int dx\right]$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C.$$

H.w: Exc 8.2

55 (a, b), P. 521

Lecture (9):

* Trigonometric integrals.

[1] Integrating powers of sine and cosine.

By reduction formulas:

① $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$

② $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$

Remember: The trigonometric identities: ① $\sin^2 x + \cos^2 x = 1$

② $\sin^2 x = \frac{1}{2}(1 - \cos 2x) \Longrightarrow \cos 2x = 1 - 2\sin^2 x$

③ $\cos^2 x = \frac{1}{2}(1 + \cos 2x) \Longrightarrow \cos 2x = 2\cos^2 x - 1.$

④ $\sin 2x = 2 \sin x \cos x.$

Ex: Evaluate $\int \sin^2 x \, dx.$

① By reduction formulas:

with $n = 2$,

$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int \sin^0 x \, dx$

$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx$

$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$

Using: $\sin 2x = 2 \sin x \cos x$

$\Rightarrow \boxed{\sin x \cos x = \frac{1}{2} \sin 2x}$

Then:

$\int \sin^2 x \, dx = -\frac{1}{2}\left(\frac{1}{2} \sin 2x\right) + \frac{1}{2} x + C$

$= -\frac{1}{4} \sin 2x + \frac{1}{2} x + C$

*

② By trigonometric identities:

$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$

$= \frac{1}{2} \int (1 - \cos 2x) \, dx$

$= \frac{1}{2}\left[\int dx - \int \cos 2x \, dx\right]$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$

Ex : $\int \cos^2 x \, dx$.

$$\int \cos^2 x \, dx = \int \frac{1}{2}(1+\cos 2x)\, dx$$
$$= \frac{1}{2}\int (1+\cos 2x)\, dx$$
$$= \frac{1}{2}\left[\int dx + \int \cos 2x \, dx\right]$$
$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

Rules:

① $\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

② $\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$

③ $\int \sin^3 x \, dx = \frac{1}{3}\cos^3 x - \cos x + C.$

④ $\int \cos^3 x \, dx = \sin x - \frac{1}{3}\sin^3 x + C.$

⑤ $\int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

⑥ $\int \cos^4 x \, dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

Ex : Evaluate $\int \sin^3 x \, dx$

$$= \int \sin^2 x \, \sin x \, dx$$
$$= \int (1-\cos^2 x) \cdot \sin x \, dx$$
$$= -\int (1-u^2)\, du$$
$$= -\left[\int du - \int u^2 du\right] = -\left[u - \frac{u^3}{3}\right] + C = -\cos x + \frac{1}{3}\cos^3 x + C$$

$\left\{ \begin{array}{l} \text{let } u = \cos x \\ du = -\sin x \, dx \end{array} \right.$

#

Ex ①: Find the volume V of the solid that is obtained when the region under the curve $y = \sin^2 x$ over the interval $[0, \pi]$ is revolved about the x_axis.

Solution:

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

$$V = \int_0^{\pi} \pi (\sin^2 x)^2 \, dx$$

$$= \pi \int_0^{\pi} \sin^4 x \, dx$$

$$= \pi \left[\frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x \right]_0^{\pi}$$

$$= \pi \left[\frac{3}{8} \pi - \frac{1}{4} \sin 2(\pi) + \frac{1}{32} \sin 4(\pi) - \left(\frac{3}{8}(0) - \frac{1}{4} \sin(0) + \frac{1}{32} \sin(0 \right. \right.$$

$$= \pi \left[\frac{3}{8} \pi - \frac{1}{4}(0) + \frac{1}{32}(0) - 0 \right]$$

$$= \frac{3}{8} \pi^2 .$$

Integrating products of sines and cosines:

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by one of the three procedures stated in this table.

$\int \sin^m x \cos^n x \, dx$	Proce ure	Relevant identity
1 n odd	1 Split off a factor of $\cos x$. 2 Apply the relevant identity. 3 Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$.
2 m odd	1 Split off a factor of $\sin x$. 2 Apply the relevant identity. 3 Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$.
3 m even n even	Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\sin^2 x = \frac{1}{2}[1 - \cos 2x]$ $\cos^2 x = \frac{1}{2}[1 + \cos 2x]$.

Ex ②: Evaluate

ⓐ $\int \sin^4 x \cos^5 x \, dx$.

$m = 4$, $n = 5$ is odd,

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \left(\cos^2 x\right)^2 \cdot \cos x \, dx$$

$$= \int \sin^4 x \left(1 - \sin^2 x\right)^2 \cdot \cos x \, dx.$$

Using the substitution $u = \sin x$
$du = \cos x dx$

$$= \int u^4 (1 - u^2)^2 \cdot du$$

$$= \int u^4 (1 - 2u^2 + u^4) du$$

$$= \int (u^4 - 2u^6 + u^8) du$$

$$= \int u^4 du - 2\int u^6 du + \int u^8 du$$

$$= \frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9} + C.$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

#

(b) $\int \sin^4 x \cdot \cos^4 x \, dx$, $\underbrace{m=4 \ , n=4}_{\text{even}}$

$$I = \int (\sin^2 x)^2 \cdot (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1}{2}[1-\cos 2x]\right)^2 \cdot \left(\frac{1}{2}[1+\cos 2x]\right)^2 dx$$

$$= \int \left(\frac{1}{2}\right)^2 \cdot (1-\cos 2x)^2 \cdot \left(\frac{1}{2}\right)^2 (1+\cos 2x)^2 \, dx$$

$$= \frac{1}{16} \int (1-\cos 2x)^2 \cdot (1+\cos 2x)^2 \, dx$$

$$= \frac{1}{16} \int \left[(1-\cos 2x)(1+\cos 2x)\right]^2 dx$$

Remember: $(x-y)(x+y) = x^2 - y^2$

$$= \frac{1}{16} \int (1-\cos^2 2x)^2 \, dx$$, using: $\sin^2 x = 1 - \cos^2 x$

$$= \frac{1}{16} \int (\sin^2 2x)^2 \, dx$$

$$= \frac{1}{16} \int \sin^4 2x \, dx$$ let $u = 2x \Rightarrow du = 2dx$, $dx = \frac{1}{2} du$

$$= \frac{1}{16} \int \sin^4 u \cdot \frac{1}{2} du = \frac{1}{32} \int \sin^4 u \, du$$

using $\int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$.

Then:

$$I = \frac{1}{32}\left[\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right] + C$$

$$= \frac{1}{32}\left[\frac{3}{8} \cdot 2x - \frac{1}{4}\sin 2(2x) + \frac{1}{32}\sin 4(2x)\right] + C$$

$$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C.$$ #

Integrals of the form:

$\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, $\int \cos mx \cos nx\, dx$

Can be found by using the trigonometric identities:

$\boxed{1}$ $\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$

$\boxed{2}$ $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$\boxed{3}$ $\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$.

Ex 3 : Evaluate

$$\int \sin 7x \cos 3x\, dx = \int \frac{1}{2}\left[\sin(7x-3x) + \sin(7x+3x)\right] dx$$

$$= \frac{1}{2}\int \left(\sin 4x + \sin 10x\right) dx$$

$$= \frac{1}{2}\left[\int \sin 4x\, dx + \int \sin 10x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{-\cos 4x}{4} - \frac{\cos 10x}{10}\right] + C$$

$$= \frac{-1}{8}\cos 4x - \frac{1}{20}\cos 10x + C.$$

Integrating powers of tangent and secant:

Reduction formula: For $n > 1$, (n is a positive integer).

[1] $$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

[2] $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$$

If $n = 1$:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$$
$$= \ln|\cos x|^{-1} + C$$
$$= \ln \frac{1}{|\cos x|} + C$$

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C}$$

$$\int \sec x \, dx = \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} \, dx$$

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

If $n = 2$: using [1]:

$$\int \tan^2 x \, dx = \frac{\tan x}{2-1} - \int \tan^{2-2} x \, dx$$
$$= \tan x - \int dx$$

$$\boxed{\int \tan^2 x \, dx = \tan x - x + C}$$

Also, we can evaluate $\int \tan^2 x \, dx$ by the identity

$1 + \tan^2 x = \sec^2 x$,

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \, dx - \int dx$$

$$\boxed{\int \tan^2 x \, dx = \tan x - x + C}$$

$$\boxed{\int \sec^2 x \, dx = \tan x + C}$$

If $n = 3$:

using $\boxed{1}$

$\int \tan^3 x \, dx = \dfrac{\tan^2 x}{2} - \int \tan x \, dx$

$$\boxed{\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \ln|\sec x| + C}$$

using $\boxed{2}$

$\int \sec^3 x \, dx = \dfrac{\sec x \tan x}{2} + \dfrac{1}{2} \int \sec x \, dx$

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C}$$

* Integrating products of tangents and secants:

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x \, dx$$

can be evaluated by one of the three procedures stated in this table.

$\int \tan^m x . \sec^n x \, dx$	Procedure	Relevant identity
$\boxed{1}$ n even	$\boxed{1}$ Split off a factor of $\sec^2 x$. $\boxed{2}$ Apply the relevant identity. $\boxed{3}$ Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
$\boxed{2}$ m odd	$\boxed{1}$ Split off a factor of $\sec x . \tan x$ $\boxed{2}$ Apply the relevant identity. $\boxed{3}$ Make the substitution $u = \sec x$	$\tan^2 x = \sec^2 x - 1$
$\boxed{3}$ m even n odd	$\boxed{1}$ Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. $\boxed{2}$ Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Ex ④. Evaluate:

ⓐ $\int \tan^2 x \cdot \sec^4 x \, dx$, $n = 4$ (even)

$= \int \tan^2 x \ \sec^2 x \ \sec^2 x \ dx$, using: $\sec^2 x = \tan^2 + 1$

$= \int \tan^2 x \ (\tan^2 x + 1) \ \sec^2 x \ dx$ Let $u = \tan x$

$du = \sec^2 x \ dx$

$= \int u^2 (u^2 + 1) \ du$

$= \int (u^4 + u^2) \ du = \int u^4 du + \int u^2 du = \frac{u^5}{5} + \frac{u^3}{3} + C$

$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.$

ⓑ $\int \tan^3 x \ \sec^3 x \ dx$, $m = 3$ (odd)

$= \int \tan^2 x \ \tan x \ \sec^2 x \ \sec x \ dx$

$= \int \tan^2 x \ . \sec^2 x . (\sec x \ \tan x) \ dx$

$= \int (\sec^2 x - 1) \ \sec^2 x \ (\sec x \ \tan x) \ dx$, using: $\tan^2 x = \sec^2 x - 1$

Let $u = \sec x$

$du = \sec x \ \tan x \ dx$

$= \int (u^2 - 1) \ u^2 . \ du$

$= \int (u^4 - u^2) du$

$= \int u^4 \ du - \int u^2 \ du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.$

(c) $\int \tan^2 x \sec x \, dx$, $m=2$ even , $n=1$ odd

$I = \int (\sec^2 - 1) \sec x \, dx$

$= \int (\sec^3 x - \sec x) \, dx$

$= \int \sec^3 x \, dx - \int \sec x \, dx$

using:

$\int \sec^n x \, dx = \frac{\sec^{n-2} \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

with $n = 3$

$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx$.

Then,

$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx - \int \sec x \, dx$

$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \int \sec x \, dx$

$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$.

H.w: Exc 8.3

(3 + 7 + 51) , p. 529

Lecture (10):

* Trigonomatric substitutions.

We will be concerned with integrals that contain expression of the form:

$$\sqrt{a^2-x^2}\ ,\ \sqrt{x^2+a^2}\ ,\ \sqrt{x^2-a^2}.$$

in which a is a positive constant. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical.

For example:

$\sqrt{a^2-x^2}$, we can make the substitution

$$x = a\sin\theta \quad , \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Then,

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta}$$
$$= a\cos\theta$$

Expression in the integrand	Substitution	Restriction on θ
[1] $\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
[2] $\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
[3] $\sqrt{x^2-a^2}$	$x = a\sec\theta$	if $x \geq a$: $0 \leq \theta < \frac{\pi}{2}$ if $x \leq -a$: $\frac{\pi}{2} < \theta \leq \pi$

Ex ①: Evaluate

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$.

Then,

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta \, d\theta}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}$$

$$= \int \frac{2\cos\theta \quad d\theta}{4\sin^2\theta \sqrt{4-4\sin^2\theta}}$$

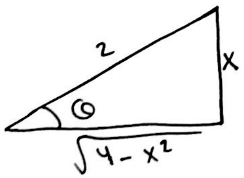
$$= \int \frac{2\cos\theta \ d\theta}{4\sin^2\theta\sqrt{4(1-\sin^2\theta)}}$$

$$= \int \frac{2\cos\theta \quad d\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}}$$

$$= \int \frac{\cancel{2\cos\theta} \ d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}}$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \csc^2\theta \ d\theta = -\frac{1}{4}\cot\theta + C.$$



$x = 2\sin\theta$

$\Rightarrow \sin\theta = \frac{x}{2}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$= -\frac{1}{4}\,\frac{\frac{x}{\cancel{2}}}{\frac{\sqrt{4-x^2}}{\cancel{2}}} + C$$

$$= -\frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}} + C$$

#

Ex ②: Evaluate $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$.

Let $x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}\frac{x}{2}$

$dx = 2\cos\theta \, d\theta$

if $x = 1 \Longrightarrow \theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

$x = \sqrt{2} \Longrightarrow \theta = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$.

Then:

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\cos\theta \, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\cos\theta \quad d\theta}{4\sin^2\theta\sqrt{4(1-\sin^2\theta)}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\cos\theta \, d\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cancel{2\cos\theta} \, d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}}$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc^2\theta \, d\theta$$

$$= -\frac{1}{4}\cot\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= -\frac{1}{4}\left[\cot\left(\frac{\pi}{4}\right) - \cot\frac{\pi}{6}\right]$$

$$= -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}.$$

#

Ex ③: Find the arc length of the curve $y = \frac{x^2}{2}$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}} \cdot \cancel{2}x = x$$

from $x = 0$ to $x = 1$.

<u>solution</u>:
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$L = \int_0^1 \sqrt{1 + x^2}\, dx$$

let: $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$dx = \sec^2\theta\, d\theta$$

if $x = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$

$x = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$.

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\ \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta \cdot d\theta$$

Using $\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$.

$$L = \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \frac{\sec\theta \tan\theta}{2}\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec\theta\, d\theta$$

$$= \frac{\sec\theta \tan\theta}{2}\Big|_0^{\frac{\pi}{4}} + \frac{1}{2} \ln|\sec\theta + \tan\theta|\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\Big[\sec\left(\tfrac{\pi}{4}\right) \cdot \underbrace{\tan\left(\tfrac{\pi}{4}\right)}_{1=} - \sec(0) \cdot \tan(0) + \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln|\sec(0) + \tan$$

$$= \frac{1}{2}\Big[\sqrt{2}(1) - \cancel{(1)(0)}^{0} + \ln|\sqrt{2} + 1| - \underbrace{\cancel{\ln|1 + 0|}}_{0}\Big] = \frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2} + 1)\right].$$

$$\approx 1.148$$

Ex ④: Evaluate $\int \frac{\sqrt{x^2-25}}{x}\,dx$

Let $x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$.

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta-25}}{\cancel{5\sec\theta}}\cdot \cancel{5\sec\theta}\tan\theta\,d\theta$$

$$= \int \sqrt{25(\sec^2\theta-1)}\cdot\tan\theta\,d\theta$$

$$= \int \sqrt{25\tan^2\theta}\;\tan\theta\,d\theta$$

$$= \int 5\tan\theta\cdot\tan\theta\,d\theta$$

$$= 5\int \tan^2\theta\,d\theta$$

$$= 5\int(\sec^2\theta-1)\,d\theta$$

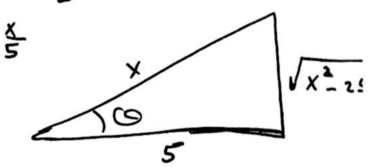
$$= 5\left[\int\sec^2\theta\,d\theta - \int d\theta\right]$$

$$= 5\left[\tan\theta-\theta\right]+C$$

$$= 5\tan\theta - 5\theta + C = 5\cdot\frac{\sqrt{x^2-25}}{5} - 5\sec^{-1}\frac{x}{5} + C$$

$\because x = 5\sec\theta \Rightarrow \sec\theta = \frac{x}{5} \Rightarrow \theta = \sec^{-1}\frac{x}{5}$

$\Rightarrow \frac{1}{\cos\theta} = \frac{x}{5}$



$\tan\theta = \frac{\sqrt{x^2-25}}{5}$. $\left\{\begin{array}{l}\cos\theta = \frac{5}{x}\\ \sin\theta = \frac{\sqrt{x^2-25}}{x}\end{array}\right.$

#

Integrals involving ax^2+bx+C :

Integrals that involve a quadratic experssion ax^2+bx+C. where $a \neq 0$ and $b \neq 0$ often be evaluated by first completing the square, then making an appropriate substitution

Ex 5: Evaluate $\int \frac{x}{x^2-4x+8}\,dx$

Completing the sequare yields :

$$x^2-4x+8 = x^2-4x+4-4+8 = (x^2-4x+4)+4$$
$$= (x-2)^2+4$$

Then :

$$\int \frac{x}{x^2-4x+8}\,dx = \int \frac{x}{(x-2)^2+4}\,dx$$

Let $u = x-2$
$\Rightarrow du = dx$
$x = u+2$

$$= \int \frac{(u+2)}{u^2+4}\,du$$
$$= \int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du$$
$$= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du$$
$$= \frac{1}{2}\ln|u^2+4| + \cancel{2}\cdot\cancel{\tfrac{1}{2}}\tan^{-1}\frac{u}{2} + C$$
$$= \frac{1}{2}\ln\left[(x-2)^2+4\right] + \tan^{-1}\left(\frac{x-2}{2}\right) + C$$

H.w : Exc (8.4)
(5+17), P. 535

Ex ⑥ : Evaluate

$$\int \frac{dx}{\sqrt{5-4x-2x^2}}$$

$$5-4x-2x^2 = 5-2(x^2-2x)$$

$$= 5-2(x^2-2x+1-1)$$

$$= 5-2(x^2-2x+1)+2$$

$$= 7-2(x-1)^2$$

Then :

$$\int \frac{dx}{\sqrt{5-4x-2x^2}} = \int \frac{dx}{\sqrt{7-2(x-1)^2}}$$

Let $u = x+1$

$du = dx$

$$= \int \frac{du}{\sqrt{7-2u^2}}$$

$$= \int \frac{du}{\sqrt{2\left(\frac{7}{2}-u^2\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{\frac{7}{2}-u^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{u}{\sqrt{\frac{7}{2}}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C .$$

Lecture (11):

Integrating rational functions by partial fractions.

partial fractions:

Ex①: Write out the form of the partial fraction decomposition

$$\frac{5x-10}{x^2-3x-4}$$, then evaluate its integral.

$$\frac{5x-10}{x^2-3x-4} = \frac{5x-10}{(x-4)(x+1)}$$

$$\frac{5x-10}{x^2-3x-4} = \frac{A}{(x-4)} + \frac{B}{(x+1)} \longrightarrow ①$$

$$\frac{5x-10}{x^2-3x-4} = \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$

Then:

$$5x-10 = A(x+1)+B(x-4)$$

$$5x-10 = Ax+A+Bx-4B \longrightarrow ②$$

By comparing the coefficients in ②:

x^1 : $5 = A+B$ (*)

x^0 : $-10 = A-4B$ (**)

$$-5 = -\cancel{A}-B$$
$$-10 = \cancel{A}-4B$$
$$-15 = -5B \Longrightarrow \boxed{B=3}$$

(*) : $5 = A+B$

$$5 = A+3 \Longrightarrow \boxed{A=2}$$

From ①:

$$\frac{5x-10}{x^2-3x-4} = \frac{A}{(x-4)} + \frac{B}{(x+1)}$$

$$\frac{5x-10}{x^2-3x-4} = \underbrace{\frac{2}{x-4} + \frac{3}{x+1}}_{\text{partial fractions}}$$

Then:

$$\int \frac{5x-10}{x^2-3x-4} \cdot dx = \int \frac{2}{x-4}\,dx + \int \frac{3}{x+1}\,dx$$

$$= 2\int \frac{1}{x-4}\,dx + 3\int \frac{1}{x+1}\,dx$$

$$= 2\ln|x-4| + 3\ln|x+1| + C.$$

Ex ②: Evaluate $\int \frac{dx}{x^2+x-2}$

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

$$\frac{1}{x^2+x-2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} \longrightarrow ①$$

$$\frac{1}{x^2+x-2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\Rightarrow \quad 1 = A(x+2) + B(x-1)$$

$$1 = Ax + 2A + Bx - B.$$

By comparing:

$x^1 : \quad 0 = A + B \quad (*)$

$x^0 : \quad 1 = 2A - B \quad (**)$

$1 = 3A \Rightarrow \boxed{A = \frac{1}{3}}$

$(*) : B = -A \Rightarrow \boxed{B = \frac{-1}{3}}$

From (1):

$$\frac{1}{(x^2+x-2)} = \frac{\frac{1}{3}}{x-1} + \frac{\frac{-1}{3}}{x+2} .$$

Then:

$$\int \frac{dx}{x^2+x-2} = \int \frac{\frac{1}{3}}{x-1} dx + \int \frac{\frac{-1}{3}}{x+2} dx$$

$$= \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{1}{x+2} dx$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| + C .$$

$$= \frac{1}{3} \left[\ln|x-1| - \ln|x+2| \right] + C$$

$$= \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C .$$

#

* Special Cases:

1 linear factors.

linear factors rule: For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}.$$

where $A_1, A_2, A_3, \ldots, A_m$ are constants to be determined

Ex 3: Evaluate

$$\int \frac{2x+4}{x^3-2x^2}\,dx.$$

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

$$\frac{2x+4}{x^3-2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)} \longrightarrow 1$$

$$\frac{2x+4}{x^3-2x^2} = \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}.$$

$$\Rightarrow 2x+4 = Ax(x-2)+B(x-2)+Cx^2$$

$$2x+4 = Ax^2-2Ax+Bx-2B+Cx^2$$

By comparing :

$$x^2 : \quad 0 = A + C$$

$$x^1 : \quad 2 = -2A + B$$

$$x^0 : \quad 4 = -2B \Rightarrow B = \frac{4}{-2} \Rightarrow \boxed{B=-2}$$

$$2 = -2A + B \Rightarrow 2 = -2A - 2 \Rightarrow A = \frac{4}{-2} \Rightarrow \boxed{A=-2}$$

$$0 = A + C \Rightarrow 0 = -2 + C \Rightarrow \boxed{C=2}$$

From ① :

$$\frac{2x+4}{x^3-2x^2} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

Then :

$$\int \frac{2x+4}{x^3-2x^2}\,dx = \int \frac{-2}{x}\,dx + \int \frac{-2}{x^2}\,dx + \int \frac{2}{x-2}\,dx$$

$$= -2\int \frac{1}{x}\,dx - 2\int x^{-2}\,dx + 2\int \frac{1}{x-2}\,dx$$

$$= -2\ln|x| - 2\,\frac{x^{-1}}{-1} + 2\ln|x-2| + C$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

$$= 2\left[\ln|x-2| - \ln|x|\right] + \frac{2}{x} + C$$

$$= 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C$$

#

② Quadratic factors:

Quadratic factors rule: For each factor of the form (ax^2+bx+c), the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$ are constants to be determined.

Ex ④: Evaluate $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$

$$\frac{x^2+x-2}{3x^3-x^2+3x-1}=\frac{x^2+x-2}{(3x^3+3x)+(-x^2-1)}=\frac{x^2+x-2}{3x(x^2+1)-(x^2+1)}$$

$$\frac{x^2+x-2}{3x^3-x^2+3x-1}=\frac{x^2+x-2}{(3x-1)(x^2+1)}.$$

$$\frac{x^2+x-2}{3x^3-x^2+3x-1}=\frac{A}{3x-1}+\frac{Bx+C}{x^2+1}\longrightarrow (*)$$

$$\frac{x^2+x-2}{3x^3-x^2+3x-1}=\frac{A(x^2+1)+(Bx+c)(3x-1)}{(3x-1)(x^2+1)}.$$

$$\Longrightarrow x^2+x-2=A(x^2+1)+(Bx+c)(3x-1)$$

$$x^2+x-2=Ax^2+A+3Bx^2-Bx+3Cx-C$$

By Comparing:

x^2: $1 = A + 3B \longrightarrow$ ①

x^1: $1 = -B + 3C \longrightarrow$ ②

x^0: $-2 = A - C \longrightarrow$ ③ $\Longrightarrow A = -2 + C$

From ①: $1 = A + 3B \Rightarrow 1 = -2 + C + 3B \Rightarrow 3 = 3B + C \rightarrow$ ①'

①' + ② $\Longrightarrow$

$$3 = 3\cancel{B} + C$$

$$3\times \quad \overset{3}{1} = \overset{-3\cancel{B}}{-B} + \overset{9C}{3C}$$

$$6 = 10C \Rightarrow C = \frac{6}{10} \Rightarrow \boxed{C = \frac{3}{5}}$$

From ③ $\Longrightarrow$

$$-2 = A - \frac{3}{5}$$

$$-2 + \frac{3}{5} = A \Longrightarrow \boxed{A = \frac{-7}{5}}$$

From ② $\Longrightarrow$ $1 = -B + 3\left(\frac{3}{5}\right) \Rightarrow 1 - \frac{9}{5} = -B \Rightarrow -B = \frac{-4}{5}$

$$\Longrightarrow \boxed{B = \frac{4}{5}}$$

From (*):

$$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} = \frac{-\frac{7}{5}}{3x - 1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2 + 1}$$

Then: $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx = -\frac{7}{5}\int \frac{1}{3x-1}\,dx + \int \frac{\frac{4}{5}x + \frac{3}{5}}{x^2+1}\,dx$

$$= -\frac{7}{5}\cdot\frac{1}{3}\int \frac{3}{3x-1}\,dx + \frac{4}{5}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{-7}{15}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

#

3 Integrating improper rational functions.

Ex ⑤: Evaluate $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$

$$
\begin{array}{r|l}
 & 3x^2 \quad +1 \\
\hline
x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\
 & -\left(\cancel{3x^4} \mp \cancel{3x^3} \pm 6x^2\right) \\
\hline
 & x^2+x-1 \\
 & -\left(x^2 \mp x \pm 2\right) \\
\hline
\text{remainder} \leftarrow & 1
\end{array}
$$

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = 3x^2+1+\frac{1}{x^2+x-2}$$

Hence,

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \underbrace{\int \frac{1}{x^2+x-2}\,dx}_{\Downarrow \text{ From Ex ②}}$$

$$= \frac{3x^3}{3} + x + \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

$$= x^3 + x + \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

H.W: Exc (8.5)

(5 + 13 + 25), P. 543 + 544.

Lecture (12):

* Improper integrals:

Improper integrals are integrals with infinite intervals of integration or infinite discontinuitres within the interval of integration.

For examples:

* Improper integrals with infinite intervals of integrati

$$\int_{1}^{+\infty} \frac{dx}{x^2} \quad , \quad \int_{-\infty}^{0} e^x \, dx \quad , \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

* Improper integrals with infinite discontinuities in the interval of integration.

$$\int_{-3}^{3} \frac{dx}{x^2} \quad , \quad \int_{1}^{2} \frac{dx}{x-1} \quad , \quad \int_{0}^{\pi} \tan x \, dx .$$

* Improper integrals with infinite discontinuities and infinite intervals of integration.

$$\int_{0}^{+\infty} \frac{dx}{\sqrt{x}} \quad , \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9} \quad , \quad \int_{1}^{+\infty} \sec x \, dx .$$

Integrals over infinite intervals.

Definition:

[1] The improper integral of f over the interval $[a, +\infty)$ is defined to be:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx .$$

The integral is said to converge if the limit exists and diverge if it does not.

[2] The improper integral of f over the interval $(-\infty, b]$ is defined to be:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

The integral is said to converge if the limit exists and diverge if it does not.

[3] The improper integral of f over the interval $(-\infty, +\infty)$ is defined to be:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

where c is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

Ex ①: Evaluate

ⓐ $\displaystyle\int_1^{+\infty} \frac{dx}{x^3}$ ⓑ $\displaystyle\int_1^{+\infty} \frac{dx}{x}$

ⓐ $$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^3} = \lim_{b \to +\infty} \int_1^b x^{-3}\, dx$$

$$= \lim_{b \to +\infty} \left[\frac{x^{-2}}{-2} \right]_1^b$$

$$= \lim_{b \to +\infty} \left[\frac{-1}{2x^2} \right]_1^b$$

$$= \lim_{b \to +\infty} \left[\frac{-1}{2b^2} + \frac{1}{2} \right]$$

$$= \frac{-1}{2(\infty)^2} + \frac{1}{2}$$

$$= -\frac{1}{\infty} + \frac{1}{2}$$

$$\boxed{\text{Note: } \frac{1}{\infty} = 0}$$

$$= \frac{1}{2}$$

Then, the given integral is convergent to $\frac{1}{2}$.

ⓑ $$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} \left[\ln x \right]_1^b$$

$$= \lim_{b \to +\infty} \left[\ln b - \underbrace{\ln(1)}_{0} \right]$$

$$= \lim_{b \to +\infty} \left[\ln b \right]$$

$$= \ln(\infty)$$

$$= \infty$$

Then, the given integral is divergent.

#

Ex ②: For what values of p does the integr

$$\int_1^{+\infty} \frac{dx}{x^p}$$ Converge ?

solution:

1. If $p = 1$,

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} \left[\ln x\right]_1^b = \lim_{b\to+\infty} \left[\ln b - \ln 1\right] = \ln \infty = \infty$$

The integral is divergent.

2. If $p \neq 1$,

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} \int_1^b x^{-p}\, dx = \lim_{b\to+\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to+\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b .$$

$$= \lim_{b\to+\infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right].$$

We have two cases:

ⓐ If $p > 1$, then

the exponent $1-p < 0$ and $b^{1-p} \to 0$ as $b \to +\infty$

then: $\displaystyle\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} \left[\underbrace{\frac{\cancel{b^{1-p}}}{1-p}}_{0} - \frac{1}{1-p}\right] = \frac{-1}{1-p} = \frac{1}{p-1} \quad (p > 1)$

The integral is Convergent.

ⓑ If $p < 1$, then the exponent $1-p > 0$ and $b^{1-p} \to +\infty$ as $b \to +\infty$

then: $\displaystyle\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} \left[\underbrace{\frac{b^{1-p}}{1-p}}_{\text{"}\infty\text{"}} - \frac{1}{1-p}\right] = +\infty - \frac{1}{1-p} = +\infty$

The integral is divergent. #

Theorem:

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

Ex ③: Evaluate $\int_0^{+\infty} (1-x)e^{-x}\,dx$

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \int_0^{b} (1-x)e^{-x}\,dx$$

$\int (1-x)e^{-x}\,dx$ (Integration by parts).

$u = 1-x \qquad dv = e^{-x}\,dx$

$du = -dx \qquad v = -e^{-x}$

$$\int(1-x)e^{-x}\,dx = -e^{-x}(1-x) - \int e^{-x}\,dx$$

$$= -e^{-x} + xe^{-x} - \frac{e^{-x}}{-1} + C$$

$$= -e^{-x} + xe^{-x} + e^{-x} + C$$

$$= xe^{-x} + C.$$

Then: $\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \left[xe^{-x}\right]_0^b = \lim_{b\to+\infty} \left[\frac{x}{e^x}\right]$

$$= \lim_{b\to+\infty} \left[\frac{b}{e^b} - 0\right]$$

$$= \lim_{b\to+\infty} \left[\frac{b}{e^b}\right], \quad \text{(Apply L'Hopital rule)}$$

$$= \lim_{b\to+\infty} \left[\frac{1}{e^b}\right] = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

Then, the given integral is convergent to $\underline{\underline{0}}$.

Ex ④: Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

By choosing $c=0$,

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}.$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2}.$$

$$= \lim_{a\to-\infty} \left[\tan^{-1}x\right]_{a}^{0} + \lim_{b\to+\infty} \left[\tan^{-1}x\right]_{0}^{b}$$

$$= \lim_{a\to-\infty} \left[\underbrace{\tan^{-1}(0)}_{0=} - \tan^{-1}(a)\right] + \lim_{b\to+\infty} \left[\tan^{-1}(b) - \underbrace{\tan^{-1}(0)}_{0=}\right].$$

$$= \lim_{a\to-\infty} \left[-\tan^{-1}(a)\right] + \lim_{b\to+\infty} \left[\tan^{-1}(b)\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Then, the given integral is Convergent to π.

Integrals whose integrands have infinite discontinuitie

Definition:

① If f is continuous on the interval $[a,b]$ except fo an infinite discontinuity at b, then the improper integral o f over the interval $[a,b]$ is defined as:

$$\int_{a}^{b} f(x)\,dx = \lim_{k\to b^-} \int_{a}^{k} f(x)\,dx$$

The integral is said to Converge if the limit exists and diverge if it does not.

[2] If f is continuous on the interval $[a,b]$ except for an infinite discontinuity at a, then the improper integral of f over the interval $[a,b]$ is defined as:

$$\int_a^b f(x)\,dx = \lim_{k \to a^+} \int_k^b f(x)\,dx$$

The integral is said to converge if the limit exists and diverge if it does not.

[3] If f is continuous on the interval $[a,b]$ except for an infinite discontinuity at a point c in (a,b), then the improper integral of f over the interval $[a,b]$ is defined as:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The improper integral is said to converge if both terms converge and diverge if either term diverges.

Ex (5): Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.

$\frac{1}{\sqrt{1-x}}$ discontinuous at $x=1$

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \int_0^k (1-x)^{-\frac{1}{2}}\,dx$$

$$= \lim_{k \to 1^-} \left[- \int_0^k - (1-x)^{-\frac{1}{2}}\,dx \right]$$

$$= \lim_{k \to 1^-} \left[-2(1-x)^{\frac{1}{2}} \right]_0^k = \lim_{k \to 1^-} \left[-2\sqrt{1-k} + 2 \right]$$

$$= -2\sqrt{1-1} + 2 = 2(0) + 2 = 2$$

Then, the given integral is convergent to 2. #

Ex ⑥: Evaluate

(a) $\displaystyle\int_1^2 \frac{dx}{1-x}$

$\frac{1}{1-x}$ discontinuous at $x=1$

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\left[-\int_k^2 \frac{-1}{1-x}\,dx\right]$$

$$= \lim_{k\to 1^+}\Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k\to 1^+}\Big[-\ln|1-2| + \ln|1-k|\Big]$$

$$= \lim_{k\to 1^+}\Big[\underbrace{\cancel{-\ln(1)}}_{0=} + \ln|1-k|\Big]$$

$$= \ln|1-1| = \ln(0) = -\infty .$$

Then, the given integral is divergent.

(b) $\displaystyle\int_1^4 \frac{dx}{(x-2)^{\frac{2}{3}}}$

$\frac{1}{(x-2)^{\frac{2}{3}}}$ discontinuou at $x=2$

$$\int_1^4 \frac{dx}{(x-2)^{\frac{2}{3}}} = \int_1^2 \frac{dx}{(x-2)^{\frac{2}{3}}} + \int_2^4 \frac{dx}{(x-2)^{\frac{2}{3}}} .$$

$$= \lim_{k\to 2^-}\int_1^k (x-2)^{-\frac{2}{3}}\,dx + \lim_{k\to 2^+}\int_k^4 (x-2)^{\frac{2}{3}}\,dx$$

$$= \lim_{k\to 2^-}\Big[3(x-2)^{\frac{1}{3}}\Big]_1^k + \lim_{k\to 2^+}\Big[3(x-2)^{\frac{1}{3}}\Big]_k^4$$

$$= \lim_{k\to 2^-}\Big[3(k-2)^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}}\Big] + \lim_{k\to 2^+}\Big[3(2)^{\frac{1}{3}} - 3(k-2)^{\frac{1}{3}}\Big]$$

$$= 3(2-2)^{\frac{1}{3}} - 3(-1) + 3\sqrt[3]{2} - 3(2-2)^{\frac{1}{3}}$$

$$= 3(0) + 3 + 3\sqrt[3]{2} - 3(0)$$

$$= 3 + 3\sqrt[3]{2}$$

The given integral is Convergent. #

(c) $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}\,(x+1)}$. $\frac{1}{\sqrt{x}(x+1)}$ discontinuous at $x=0$ $+\infty$

This integral is improper for two reasons:

① The interval of integration is $+\infty$.

② There is an infinite discontinuity at $x=0$.

Then: choosing $c=1$

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}\,(x+1)} + \int_1^{+\infty} \frac{dx}{\sqrt{x}\,(x+1)}.$$

$$= \lim_{k\to 0^+} \int_k^1 \frac{dx}{\sqrt{x}\,(x+1)} + \lim_{b\to+\infty} \int_1^b \frac{dx}{\sqrt{x}\,(x+1)} \to ①$$

$\displaystyle\int \frac{dx}{\sqrt{x}\,(x+1)}$ (Integration by substitution)

let $u=\sqrt{x} \Rightarrow x=u^2$

$du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow \frac{1}{\sqrt{x}}\,dx = 2du$

$$\Rightarrow \int \frac{dx}{\sqrt{x}\,(x+1)} = \int \frac{2du}{u^2+1} = 2\int \frac{du}{u^2+1} = 2\tan^{-1}u + c = 2\tan^{-1}\sqrt{x} + c$$

Then: From ①

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \lim_{k\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_k^1 + \lim_{b\to+\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^b$$

$$= \lim_{k\to 0^+}\left[2\tan^{-1}(1) - 2\tan^{-1}\sqrt{k}\right] + \lim_{b\to+\infty}\left[2\tan^{-1}\sqrt{b} - 2\tan^{-1}(1)\right]$$

$$= 2.\frac{\pi}{4} - 2\tan^{-1}(0) + 2\tan^{-1}(+\infty) - 2.\frac{\pi}{4}.$$

$$= \frac{\pi}{2} - 0 + 2\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = \pi.$$

The given integral is convergent.

H.w: Exc (8.8), (5 + 15). p. 576